Geometry & Measurement

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GEOMETRY & MEASUREMENT INTRODUCTION

Geometry is useful and interesting, an integral part of mathematics. Geometric thinking permeates mathematics and relates to most of the applications of mathematics. Geometric concepts provide a vehicle for exploring the rich connections between arithmetic concepts (such as numbers and calculations) and physical concepts (such as length, area, symmetry, and shape). Even as children experience arithmetic notions in their daily lives, they encounter geometric notions. Common opinion recognizes that each child should begin developing a number sense early. There is no sound argument to postpone the development of geometric sense. These are compelling reasons why the systematic development of geometric thinking should be a substantial component of an elementary school curriculum.

A benefit of early development of geometric thinking is that it provides opportunities for all learners. Children develop concepts at individual rates. We are careful to allow students a great deal of time to explore quantitative relationships among physical objects before studying arithmetic. In the same spirit, students should be allowed to explore geometric relationships through free play and guided experiences. Before studying geometric properties of lines and shapes, they should have free play with models and with drawing instruments, using both to create new figures and patterns. Before considering parallelism and perpendicularity as relations on geometric lines, they should have free play in creating patterns with parallel and perpendicular lines. Before learning length and area formulas, they should have informal experiences to establish and sharpen intuitive notions of length and area. These early experiences are aesthetically rewarding, and just as stimulating and suggestive of underlying concepts as early experiences leading to arithmetic. Such experiences are a crucial first step in the development of a geometric sense.

A variety of constructions forms the basis for the geometry of *CSMP Mathematics for the Intermediate Grades*. Students use tools to explore geometric concepts, directly discovering their properties and interrelationships. The tools include a straightedge, a compass, mirrors, angle templates, and a new tool called a *translator*, used for drawing parallel lines. The constructions foster insights into the properties of shapes, independent of the measurement of those properties. Only after students are familiar with the shapes do they begin to use rulers and protractors to measure lengths and angles. In this sequential development of geometric ideas, the measurement is viewed as the intersection of geometric concepts and arithmetic concepts.

The focus of this strand is experience. The measurement activities guide the students to refine their ability to accurately measure lengths of line segments and areas of polygons. Another sequence of lessons allows students to explore parallelism through use of a translator. As a natural consequence of their involvement in these activities, the students develop their knowledge and skills in geometry. The effects of this informal approach should be judged by the long-term effects on the students' knowledge, confidence, intuition, and interest in the world of geometry and measurement.

Content Overview

Measurement

Consensus calls for measurement activities in the elementary curriculum, but with no agreement on the form or scope of these activities. Rather than stress mastery of formulas for area and perimeter, or for comparison of standard units, the lessons of this strand provide open-ended experiences within rich problem-solving situations. Measurement becomes a means for investigating problems and developing concepts, rather than an end in itself. Direct experiences with the concepts and tools is central. Mastery of measurement skills will develop naturally from these experiences.

Two lessons in this strand extend earlier experiences with linear measure. Students review conversions such as 13.6 cm = 136 mm. The drawing of zig-zags of specified lengths between two circles requires not only the measurement of lengths but also requires the students to use estimation and trial-and-error techniques to find solutions. Another activity compares student performances in identical measurement activities. The variations in the students' answers reveals the inherent error in measurement activities. The median is introduced as a technique for choosing a representative measurement.

A series of four lessons explores the area of triangles. Through a sequences of activities on grid paper, students discover that any triangle can be surrounded by a rectangle that has twice its area. This discovery leads to a construction that suggests the area formula for a triangle: base x height x ½. With this technique in hand, students are able to find the area of other polygonal regions by decomposing them into triangles and rectangles. All of these activities develop and reinforce the concept of conservation of area.

Lessons: G1, 2, 3, 7, 8, and 12

Parallelism

Suggestions of parallel lines are everywhere: railroad tracks across the Great Plains, the flight of a pair of mallards, the opposite sides of a desk, the ruled lines on notebook paper. The architectural embodiments of parallelism indicate its role even in the earliest geometry. From its formal debut as a central concept of Euclid's *Elements*, a rich and fascinating study of parallel lines and related objects has developed over many centuries.

Euclid's development of geometry focused on the circle and the line. All the concepts and constructions in *Elements* require only a compass and straightedge. Analogously, the geometry lessons this semester in the *CSMP* curriculum illustrate a variety of geometrical concepts that develop using only parallelism. Constructions require a rolling straightedge called a *translator* and a compelling, dynamic definition of parallel line segments. Although traditionally postponed to secondary school, the notions of parallelism appeal directly to one's intuition.

In *IG-I* students developed familiarity with the translator by constructing parallelogram trains. In these constructions, certain sets of line segments stand out. The line segments within a set are parallel and appear to have the same length. This observation serves to introduce *equipollence*. Roughly speaking, two couples of points are equipollent if they define opposite sides of a parallelogram.

GEOMETRY & MEASUREMENT INTRODUCTION

The lessons this semester introduce the concept of parallel projection. Given two lines (a "wall" and a "floor") on a plane, students use a translator to project points parallel to the wall and onto the floor. Combining the technique of parallel projection with the notion of equipollence provides the means for making several constructions: find the midpoint of a given line segment, partition a line segment into five segments of equal length, and graphically convert between Celsius and Fahrenheit temperature scales. These constructions dynamically involve the students with many geometrical concepts, concepts that are built from their own experiences. One lesson this semester is a three-dimensional experience with the idea of parallel projection. This lesson does not involve constructions but does provide the beginnings of extensions of the concepts in an everyday application.

Hesitation and unease at the unusual character of the lessons should be treated by remembering the spirit of a spiral curriculum and of development of understanding through experience. Remember, the focus of this strand is experience. Lead your students' development both of their physical adeptness with the translator and of their concept of parallelism.

Before using the translator in the classroom, practice using it yourself. Take some time to practice using a translator at a desk as well as at the chalkboard (occasionally a construction at the board requires an odd posture). Provide a few small pieces of masking tape to each student so that paper can be anchored to the desk. Instruct the class in the proper manipulation of the translator: lightly hold the plastic tube at the center and gently push or pull to move it across the paper or desk.

Lessons: G4, 5, 6, 9, 10, and 11

Networks

One lesson during the semester deals with situations from an area of mathematics called *graph theory* or *network analysis*. The setting is the two-dimensional plane, and the problems depend on planar characteristics.

The notion of a graph, a collection of points connected by segments (arrows), is one that can depict a variety of interesting, real situations. Graphs are used in a wide range of disciplines; for example, psychology (sociograms), economics (organizational charts), anthropology (family trees), chemistry (structure), engineering (circuit diagrams).

The pictorial nature of much of graph theory, and its association with many intriguing puzzles and problems suggest its appropriateness at the elementary school level. The inductive nature of some of the work provides useful experiences for students. The problem-solving techniques used in analysis of these simple situations is important for students in many areas of study.

Lesson: G13

Note on Grids

Several lesson call for a demonstration on the grid board. A portable, commercially made, permanent grid chalkboard is ideal. However, if such a grid board is not available, try one of these alternatives.

- Make a grid transparency for use on an overhead projector and use erasable overhead markers. The Blackline section includes several size grids to use in making a grid transparency.
- Printed grids on large sheets of paper are provided in classroom sets of materials for *CSMP*. Tape one of these sheets to the board, use thick felt-tipped pens for drawings, and discard the sheet after use. You may prefer to laminate or mount one of the sheets on cardboard and cover it with clear contact paper to provide a protective surface. Then use erasable markers or wax crayons for drawings. These grid sheets also are useful as graphing mats.
- For each lesson, carefully draw a grid on the chalkboard using an easy-to-make tool. Either (1) use a music staff drawer with chalk in the first, third, and fifth positions; or (2) clip five pen caps, each having the diameter to hold a piece of chalk tightly, to a ruler at intervals of 4–8 centimeters, and secure them with tape.

Capsule Lesson Summary

Practice using the ruler by constructing zigzags meeting certain specifications.

Materials

Teacher		Student	
	• Meter stick		• Metric ruler • Worksheets G1*, **, and ***

Description of Lesson

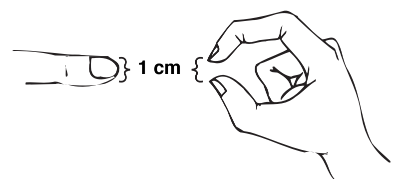
Exercise 1

Begin with a quick review of the use of the ruler, its markings, and the centimeter and millimeter units.

T: ***Everyone hold up two fingers about 1 centimeter apart.***

Look around the room to verify and commend those who are approximately correct.

Note: This illustration shows an easy way to estimate 1 cm.



T: ***1 centimeter is how many millimeters?***

S: ***10 millimeters.***

Record the equivalence on the board. Review the spelling and abbreviation for each of the two words.

centimeter (cm)
millimeter (mm)
1 cm = 10 mm

T: ***About how big is a millimeter?***

S: ***About the width of the lead in my pencil.***

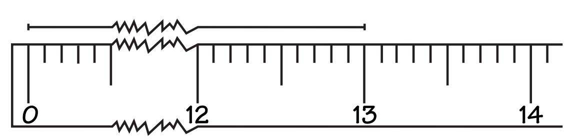
S: ***About the thickness of the cover of my book.***

S: ***Very small!***

Review the use of the decimal point in writing lengths as follows:

T: ***Suppose that you measure a line segment and find that its length is 7 centimeters and 4 millimeters. You might write 7 cm and 4 mm for its length, but we usually indicate lengths in a shorter form. We write 7.4 cm instead. We use the same idea when we write amounts of money. 10 dimes equal 1 dollar; 10 millimeters equal 1 centimeter. 7 dollars and 4 dimes is written $7.40 or 7.4 dollars, and 7 centimeters and 4 millimeters is written 7.4 cm.***

Draw this enlargement of a ruler on the board. Tell the class that the zigzag in the drawing indicates that you have not drawn a section of the ruler, and that this is also true of the line segment being measured above it.



T: ***Pretend that the longest marks on this ruler are 1 cm apart.***

Run your finger along the portion of the line segment above the ruler, starting at 0 and ending at its right endpoint.

T: ***One endpoint of the line segment is at the mark for 0 on the ruler. How long is the line segment?***

S: ***13 cm.***

T: ***How long is it in millimeters?***

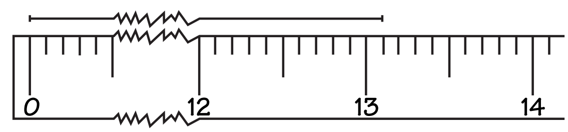
S: ***130 mm.***

Write this equivalence on the board.

13.0 cm = 130 mm

Note: Of course, 13.0 cm can be more naturally written as 13 cm. Use of the 0 serves only to emphasize the relationship.

Change the length of the line segment.



T: ***How long is this line segment now?***

S: ***13 cm and 6 mm.***

S: ***13.6 cm.***

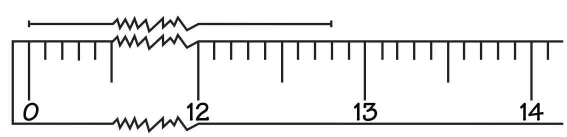
T: ***How long is it in millimeters?***

S: ***136 mm.***

Record the equivalence on the board.

13.0 cm = 130 mm
13.6 cm = 136 mm

Repeat the activity for a line segment 12.8 cm long.



12.8 cm = 128 mm

Exercise 2

Draw two circles on the board, each with a diameter of about 10 cm and with centers about 60 cm apart. Write 50 cm and 40 cm near the picture (see the next illustration).

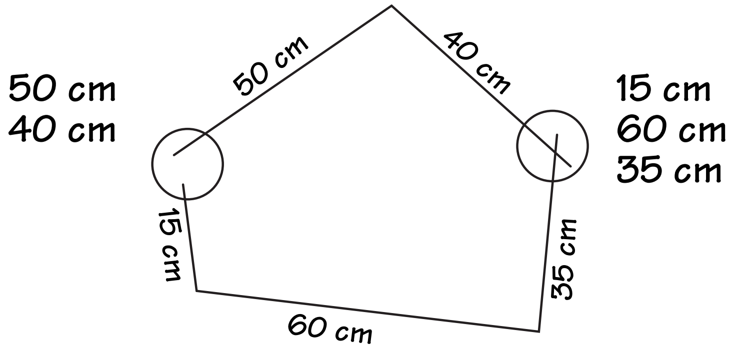
T: ***Let's draw a zigzag starting anywhere inside one circle and ending inside the other circle. One piece of the zigzag must be 50 cm long and the other piece 40 cm long. Who would like to start by drawing one of the segments?***

Let a volunteer draw a segment of one of the two lengths using a meter stick. Invite another student to try and complete the task by drawing a segment of the other length. Perhaps the student will not be able to draw such a segment and will comment that the first segment is too low or too high. In this case, erase the first segment and begin again.

When the problem has been solved, list three more lengths on the board.

15 cm
60 cm
35 cm

Lead the class to construct a three-piece zigzag connecting the two circles. (It is possible that the two zigzags on the board will cross.)



Note: You may let students use the listed lengths in any order.

Worksheets G1*, **, and *** are available for individual work. Students will need metric rulers that have both centimeter and millimeter graduations. When many students have found solutions to the problems on Worksheet G1*, you might wish to discuss these problems collectively. Such a discussion could suggest techniques to those students who are having difficulties.

Writing Activity

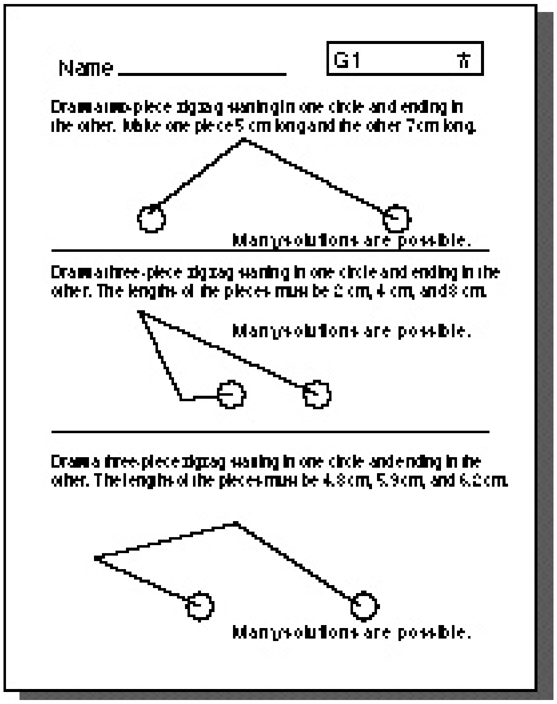
You may like students to take lesson notes on some, most, or even all their math lessons. The "Lesson Notes" section in Notes to the Teacher gives some suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students may note metric length equivalences such as 10 mm = 1 cm and 100 cm = 1 m. They may also note the length measurement of objects, such as their pencils, in both centimeters and millimeters.

Additional Practice

Zigzag construction problems like the ones in this lesson can be posed with measurement in inches. In this case, however, you might want lengths such as 3¼ inches.

Home Activity

Send home a problem similar to one on the worksheets for students to work with a family member. You may like to invite students to make up a problem for their family members.



Name ____________ G1 ☆

Draw a two-piece zigzag starting in one circle and ending in the other. Make one piece 5 cm long and the other 7 cm long.

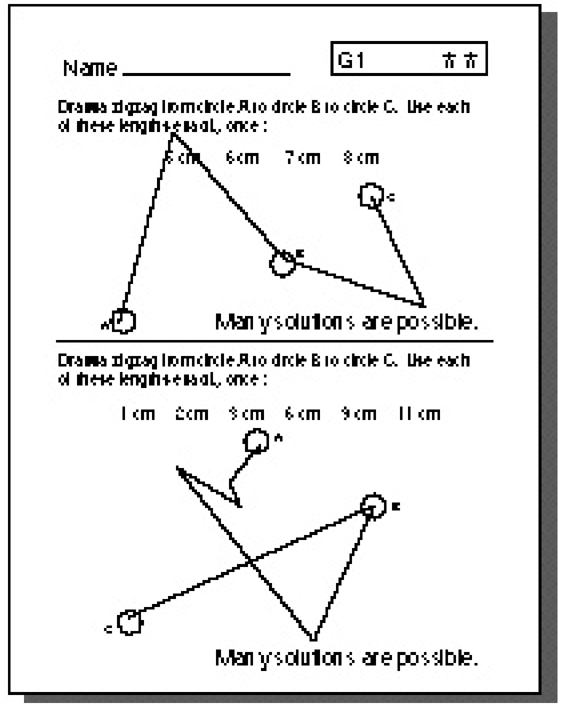
Many solutions are possible.

Draw a three-piece zigzag starting in one circle and ending in the other. The lengths of the pieces must be 2 cm, 4 cm, and 8 cm.

Many solutions are possible.

Draw a three-piece zigzag starting in one circle and ending in the other. The lengths of the pieces must be 4.8 cm, 5.9 cm, and 6.2 cm.

Many solutions are possible.



Name ____________ G1 ☆☆

Draw a zigzag from circle A to circle B to circle C. Use each of these lengths exactly once:

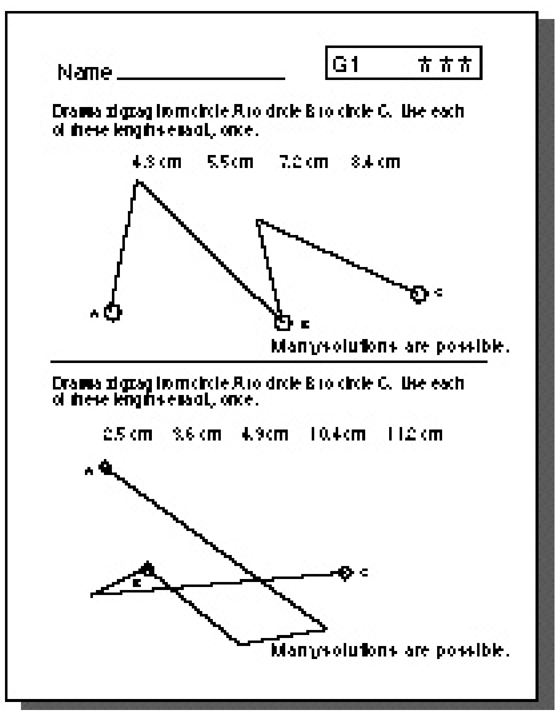
5 cm 6 cm 7 cm 8 cm

Many solutions are possible.

Draw a zigzag from circle A to circle B to circle C. Use each of these lengths exactly once:

1 cm 2 cm 3 cm 6 cm 9 cm 11 cm

Many solutions are possible.



Name ____________ G1 ☆☆☆

Draw a zigzag from circle A to circle B to circle C. Use each of these lengths exactly once.

4.3 cm 5.5 cm 7.2 cm 8.4 cm

Many solutions are possible.

Draw a zigzag from circle A to circle B to circle C. Use each of these lengths exactly once.

2.5 cm 3.6 cm 4.9 cm 10.4 cm 11.2 cm

Many solutions are possible.

Capsule Lesson Summary

With several exercises, lead to the useful fact that any triangle can be surrounded by a rectangle that has twice its area.

Materials

Teacher
- Grid board†
- *IG-II Geometry Poster* #1
- Tape
- Colored markers or chalk
- Piece of cardboard
- Meter stick or straightedge

Student
- Worksheets G2(a) and (b)
- Centimeter grid paper
- Colored pencils, pens, or crayons
- Rulers or straightedge
- Scissors

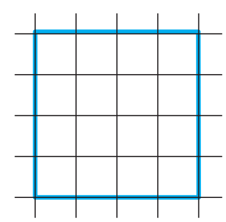
Advance Preparation: Use a Blackline, if necessary, to make centimeter grid paper for students.

Description of Lesson

Note: This lesson assumes that Lessons G10 and G11 from *IG-I* have been taught before.

Review with the class that the perimeter of a shape is the length of its border, and that the area of a shape is the amount of cover or surface enclosed by its border.

Draw a 4 by 4 square on a grid board.



T: ***Let's pretend that each of the little grid squares measure 1 cm on a side and, therefore, that the area of a little grid square is 1 cm^2.***

You may wish to show your class how big such a square actually is.

T: ***What is the perimeter of the blue square?***

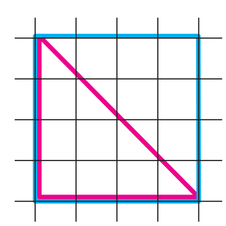
S: ***16 cm. I multiplied 4 x 4 because each side is 4 cm long and there are four sides.***

T: ***What is the area of the blue square?***

S: ***16 cm^2; 4 x 4 = 16. There are four rows with four squares in each row, so there are 16 squares altogether.***

T: ***Not too long ago†† we divided a square like this into two pieces of the same size and shape. We found several ways to do this; one way was like this.***

Draw a diagonal of the blue square and then complete this red triangle inside the square.



T: ***What is the area of the red triangle?***

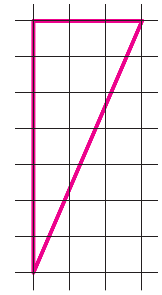
S: ***8 cm^2.***

†See the "Note on Grids" section in the introduction to this strand.
††See Lesson G11 of *IG-I*.

T: ***How do you know?***

S: ***It is half the area of the blue square. ½ x 16 = 8.***

Draw a 3 by 7 right triangle on the grid board.



T: ***How could we find the area of this red triangle?***

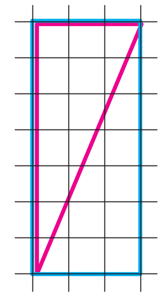
S: ***We could count the squares.***

T (pointing to the partial squares inside the triangle)**:** ***Yes, but we'd have to estimate the areas of these pieces. Is there another way?***

S: ***First, draw a rectangle around it.***

T: ***Come to the board and trace the rectangle that you are thinking will help us.***

If the students does not trace the smallest rectangle that surrounds the triangle and that follows grid lines, ask if anyone can trace another rectangle that could be helpful. Draw this blue rectangle when it is suggested.



T: ***How can this rectangle help us to find the area of the triangle?***

S: ***We just need to find the rectangle's area and then calculate half of it.***

T (counting segments along adjacent sides of the rectangle)**:** ***The rectangle is … 7 cm long and … 3 cm wide. What is its area?***

S: ***21 cm²; 3 x 7 = 21.***

S: ***So the area of the red triangle is 10.5 cm²; ½ x 21 = 10.5.***

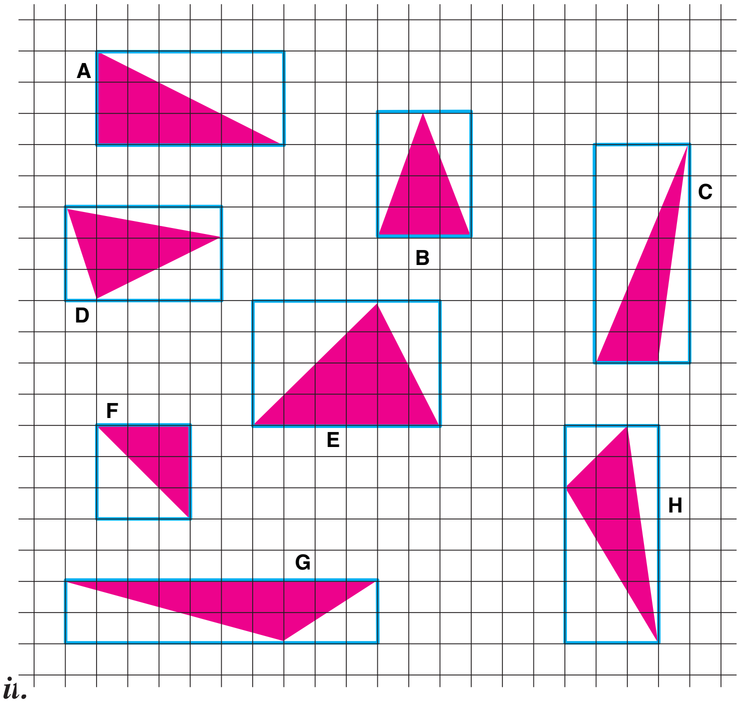
Distribute copies of Worksheet G2(a). You may like to let students work in pairs on this worksheet.

T: ***Draw rectangles that follow grid lines and are as small as possible to surround these triangles. Then consider each triangle and the rectangle surrounding it; decide whether the area of the triangle is half of the area of the rectangle. Write "yes" or "no" near the triangle.***

As you monitor students' work, you may need to help individual students as follows.

- To help a student with the rectangles, trace an overly large rectangle surrounding one of the triangles and following grid lines. Ask, "Do we need a rectangle this big?" (No) Trace another rectangle a little smaller and ask if that size is necessary. Continue until the student understands what is being asked.
- Some students may write "yes" only by shapes **A** and **F** because in each of these cases one side of the triangle is a diagonal of the rectangle. Tell these students that there are other triangles that should have "yes" by them. You may check triangle **B** by asking the student to suppose that you have a cake (trace the rectangle around it) and that you want the part with red icing (trace triangle **B**). Ask whether, if you give the student the rest of the cake, you would both get the same amount of cake. (Yes) Then ask the student to reconsider other triangles.

When many students have drawn most of the rectangles on the worksheet, display *Geometry Poster* #1. This poster is a large copy of the worksheet with the rectangles drawn in blue. You may prefer to put all the triangles from the worksheet on the grid board and invite students to draw the rectangles there.



T: ***These are the smallest rectangles that follow grid lines and that surround the triangles. Do you notice anything else about the rectangles and triangles?***

S: ***Some rectangles have a side along one or two sides of the triangle.***

S: ***The corners of a triangle touch sides of the rectangle surrounding it.***

You may need to point this out yourself.

T: ***Is there any triangle where it is easy to see its area is half of the area of the rectangle surrounding it?***

S: ***A (or F).***

Write "yes" by triangle **A** and by triangle **F**.

T (referring to the rectangle surrounding triangle **A**)**:** ***What is the area of this rectangle? It is 6 cm wide and 3 cm long.***

S: ***18 cm²***.

T: ***So what is the area of triangle A?***

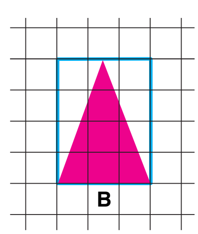
S: ***9 cm²***.

Similarly, find the area of triangle **F** (4.5 cm^2).

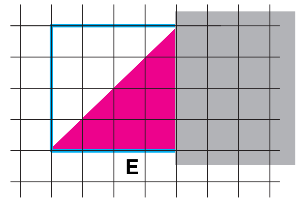
T: ***Is there another triangle that has half of the area of its surrounding rectangle?***

Your students should identify triangles **B**, **E**, and **G**. Write "yes" by each and ask for explanations. You may find that a piece of cardboard or paper is useful to cover part of a shape, especially in the following or similar arguments. Here the interior of a blue rectangle is viewed as consisting of red and white pieces.

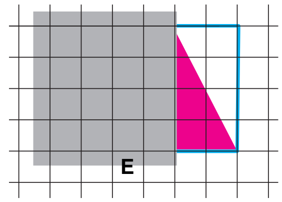
S: ***The area of triangle B is half of the area of the rectangle that surrounds it. The two white pieces will exactly cover the red piece.***



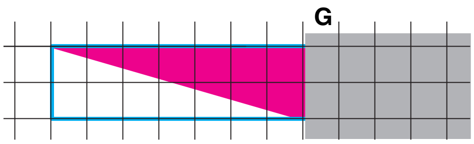
S: ***The area of triangle* E *is half the area of the rectangle that surrounds it. The red piece has the same area as the white piece in this part of the picture …***

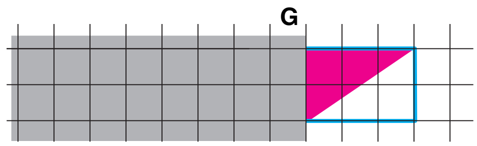


S: ***… and this red piece has the same area as this white piece. So together, the red triangle is half the area of the blue rectangle.***



(Similar argument to the preceding one for **E**)

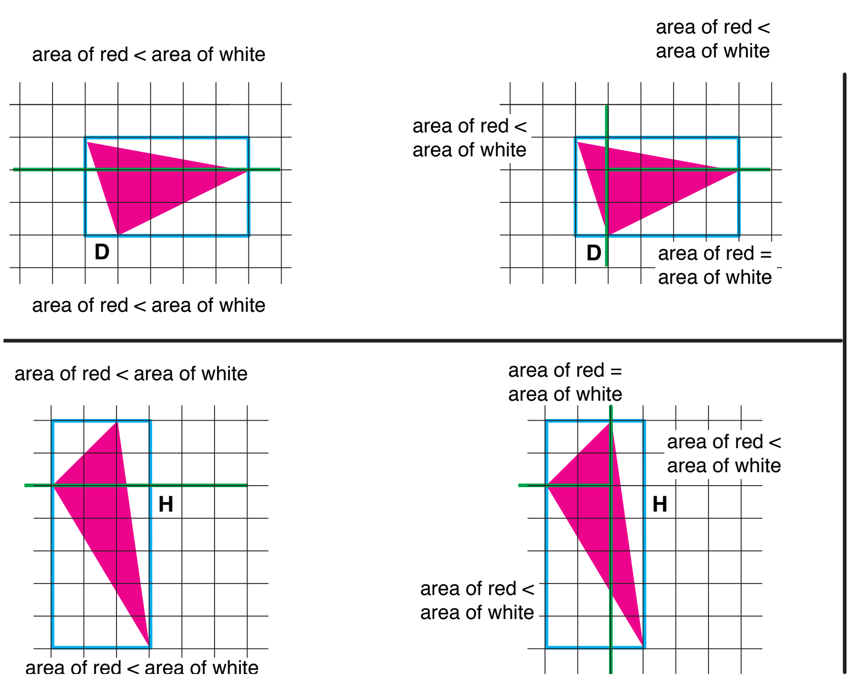


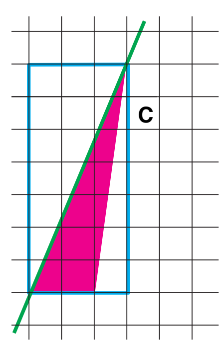


T: ***What about triangles* C*,* D*, and* H*?***

Students may not agree on these at first. Ask for explanations. Students could divide pictures into two
or three sections, using a piece of cardboard as a cover. Then they would consider each section of a picture and compare the areas of the red and the white pieces. Examples of possible divisions are shown in the following illustration. Follow the suggestions of students as there are many correct explanations of why you write "no" by triangles **C**, **D**, and **H**.

Note: Although there may be several natural ways to divide a picture, all of them cannot necessarily be used for the comparison. If there is a section in which the red piece has more area than the white piece(s) and another section in which the red piece has less area than the white piece(s), no conclusion can be reached.





The green segment divides the rectangular region in half. The top half is all white and the lower half is partially white and partially red.

T: *Do you notice any similarities among the triangles we wrote "yes" by and those we wrote "no" by?*

S: *A triangle with "yes" by it has area half that of the rectangle around it. Others do not.*

S: *A triangle with "yes" by it has a side along a side of the rectangle around it.*

T: *Let's check. Does* **A** *have a side lying along the side of its rectangle?*

S (tracing the two perpendicular sides of **A**)**:** *Yes, it has two.*

T: *What about* **B***?*

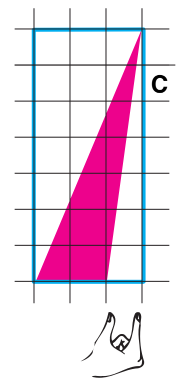
S (tracing the bottom side of **B**)**:** *It has one.*

T: *What about the other triangles?*

S: **D** *and* **H** *have "no" by them, and neither of them has a side along a side of its rectangle.*

T: *What about* **C** (trace its bottom edge)*?*

S: *It has a side along a side of its rectangle, but the rectangle's edge is longer.*

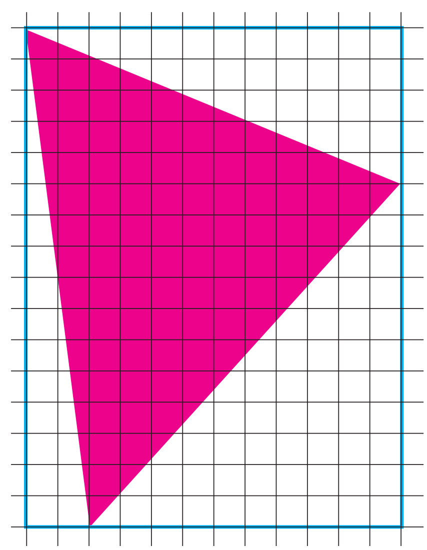


T (pointing to triangles **A**, **B**, **E**, **F**, and **G**)**:** *So each triangle with "yes" by it has a complete side in common with its corresponding rectangle.*

Distribute copies of Worksheet G2(b), scissors, straightedges (rulers), and grid paper.

T: *How could we find this triangle's area?*

S: *Let's draw a rectangle around it.*



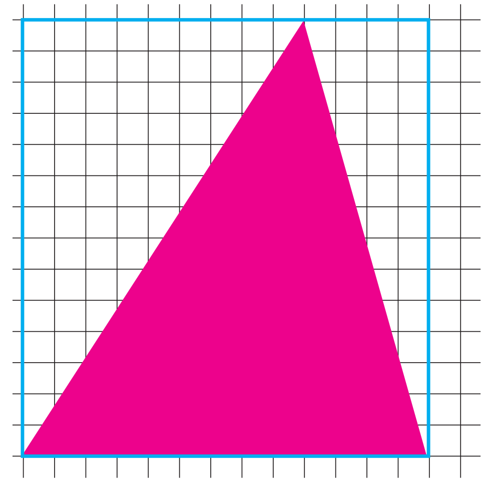
Most likely your students will draw the smallest rectangle that surrounds the triangle and that follows grid lines.

S: *The triangle and the rectangle do not have a side in common.*

S: *The triangle does not have half of the area of this rectangle.*

S: *It's crooked in the rectangle.*

T: *Maybe changing its position on the grid will help. Cut out the triangle, and on your grid paper try to find a rectangle that does have exactly twice as much area as the triangle has.*



Encourage students to turn the triangle over once it is cut out so that the grid lines on the triangle do not confuse them. Instruct students to line up one side of the triangle with a grid line. Then they can trace and shade the triangle on the grid paper, and draw a rectangle around it. The triangle will line up so that the smallest rectangle surrounding it and lying along grid lines measures 13 cm by 14 cm.

Do not criticize students who have the triangle lined up on a different side; you will discuss other orientations in a few minutes. Show the class the paper of a student who has the triangle lined up as shown in this illustration.

T: ***If you lined up the triangle like this, you found that it fits exactly inside a rectangle whose sides fall along grid lines. That makes the area easy to find. How can we find the area of this rectangle? It measures 13 cm by 14 cm.***

S: ***Multiply 13* x *14.***

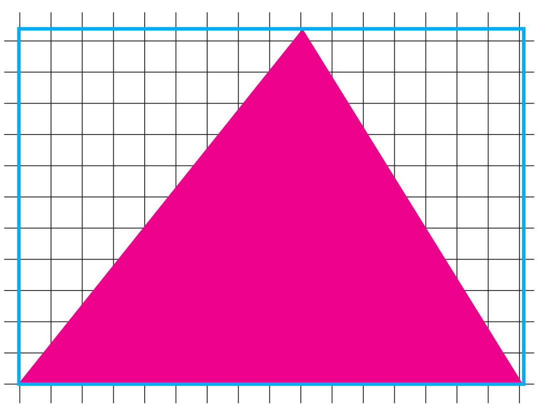
T: ***What is 10* x *14?*** (140)
What is 3* x *14? (42)
So what is 13* x *14? (182)

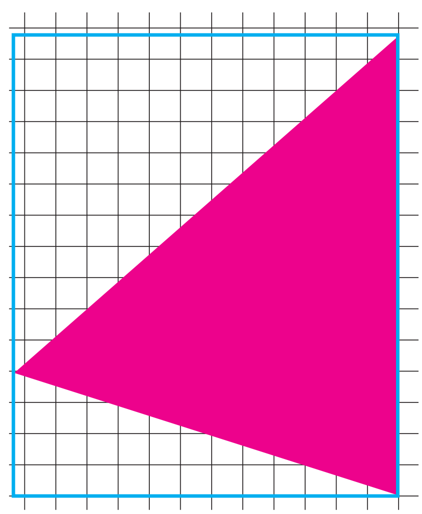
$$\begin{array}{r} 10 \times 14 = 140 \\ 3 \times 14 = 42 \\ \hline 13 \times 14 = 182 \end{array}$$

T: ***The triangle has area half of this rectangle.***
What is half of 182 cm²?

S: ***91 cm².***

Tell students that if they have the triangle turned differently, they can still get good estimates of its area by surrounding it with a rectangle with sides parallel to grid lines and by then determining the number of whole squares and estimating the total area of the partial squares of the rectangle. The triangle in each of the other two positions is shown below.





Keep the work of one student who has the triangle surrounded by a rectangle 13 cm by 14 cm for use in Lesson G3.

Capsule Lesson Summary

Review finding the area of a triangle as done in Lesson G2. Find the area of an arbitrary triangle by constructing a rectangle around it that has twice its area.

Materials

Teacher

- Grid board
- *IG-II Geometry Poster #1*
- Tape
- Student's work from Lesson G2
- Meter stick
- Colored chalk
- Calculator
- Translator (optional)

Student

- Worksheet G3
- Unlined paper
- Metric ruler
- Colored pencils, pens, or crayons
- Calculator
- Translator (optional)

Description of Lesson

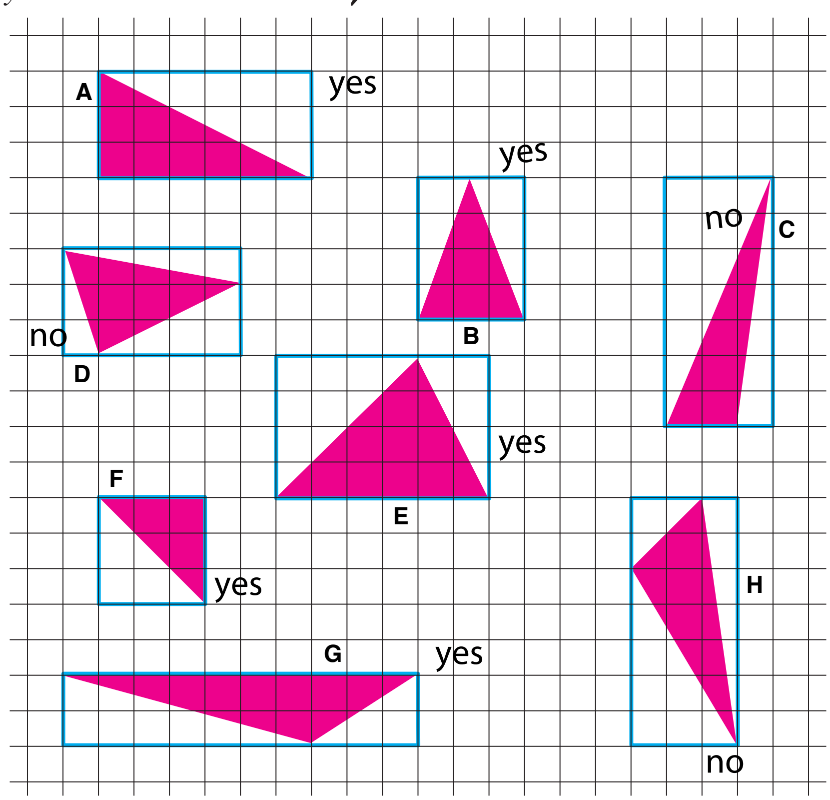
Ask the class to find the area and perimeter of several rectangles drawn on a grid board, pretending each of the little grid squares measures 1 cm on a side. Then ask,

T: ***I'm thinking of a rectangle 50 cm long and 20 cm wide. The grid board is not large enough to draw such a rectangle, but how can we find its area?***

S: ***Multiply 20 x 50.***

S: ***It's area is 1 000 cm^2; 2 x 50 = 100, so 20 x 50 = 1 000.***

Display *Geometry Poster* #1 with notations *yes* or *no* from Lesson G2.



T: ***Last week when we looked at these red triangles, we wrote "yes" by some and "no" by others. What were we deciding?***

S: ***We were deciding whether a red triangle had half of the area of the blue rectangle around it.***

T: ***And what did we notice about the triangles with "yes" by them?***

S: ***Each of them had a side in common with the rectangle around it.***

T: ***What about* C** (trace its lower edge)***? Does it have a side in common with the rectangle around it?***

S: ***No, the rectangle's side is longer than the triangle's side.***

You may need to point this out yourself.

Hold up the red triangle from a student's Worksheet G2(b) used in Lesson G2.

T: ***We wanted to find the area of this large red triangle.***
What was the problem? How did we find its area?

S: ***We cut out the triangle and lined it up on another piece of grid paper.***

T: ***What did we line up?***

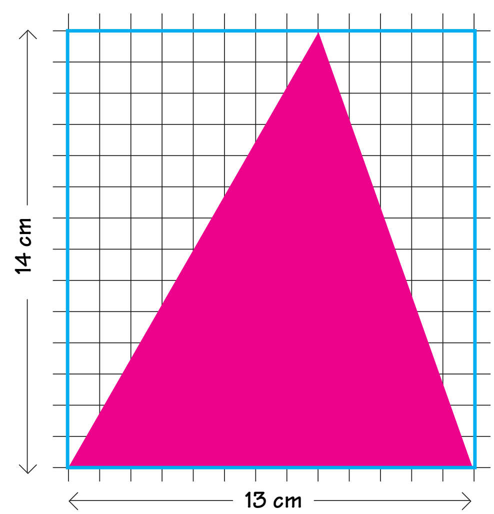
S: ***We lined up a side of the triangle with a grid line.***

S: ***Then we drew a rectangle around it.***

T: ***If you lined up a particular side of the triangle with a grid line, the rectangle you got measured 13 cm by 14 cm.***

Show the completed work (done in Lesson G2) of a student with the triangle surrounded by a rectangle 13 cm by 14 cm.

T: ***If you lined up one of the other two edges of the triangle with a grid line, you still could draw a rectangle around it and estimate its area.***

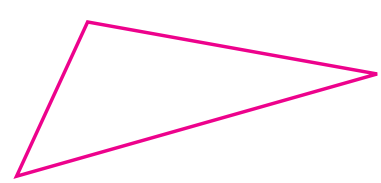


On the board, draw a triangle with no right angles, similar to the one here.

T: ***How could we find the area of this triangle?***

S: ***First draw a rectangle around it.***

T: ***Any rectangle? What do we want to be true?***

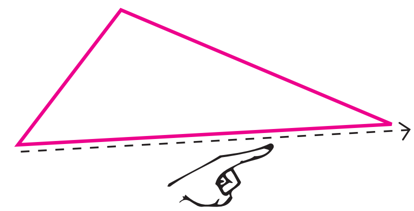


S: ***We want the triangle to have half of the area of the rectangle.***

T: ***Can someone trace such a rectangle?***

Note: In this case, because the triangle has one angle larger than a right angle, there is only one rectangle that surrounds the triangle and has twice its area. For some triangles, there are three different such rectangles.

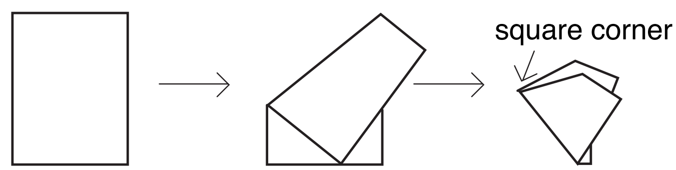
T: ***Let's construct the rectangle. The rectangle has this side in common with the triangle.***



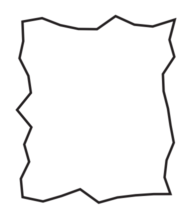
Draw a blue line segment along that side of the triangle.

T: ***We know that a rectangle has four square corners. I'll make a square corner that we can use to help us draw the rectangle. Each of you take out a piece of paper and make a square corner with me. You will not need yours until later.***

Fold the paper any way you want, … and then fold it again so that the folded edges line up.



Note: It might impress your students more if you use an irregular-shaped piece of paper for your demonstration.



T: ***Now what should we do?***

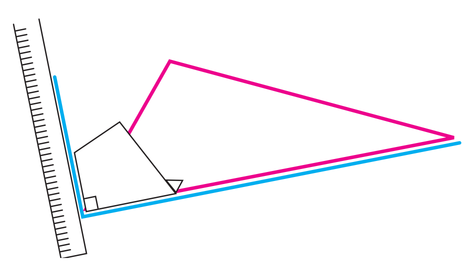
S: ***Put the square corner at either end of the blue line segment, and draw along one of its edges to make a side of the rectangle.***

Invite a student to place the square corner and indicate how to draw another side of the rectangle.

Use a straightedge or meter stick to draw the side since the paper square corner will not be big or sturdy enough.

Construct a third side of the rectangle similarly.

Note: Students may like to use a translator to construct this. Align the translator with the second side and roll until it touches the corner of the triangle at the opposite end of the first side of the rectangle.



T: ***How can we complete the triangle?***

S: ***Draw the fourth side. It must touch the top corner (vertex) of the triangle.***

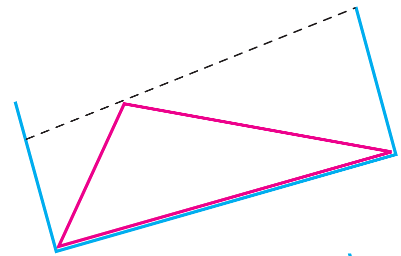
T: ***Any which way?***

Trace the fourth side incorrectly.

S: ***That's crooked; you don't get a rectangle.***

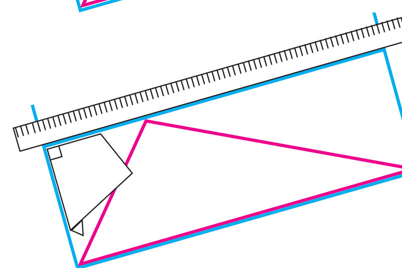
S: ***We need square corners.***

T: ***How can we use the paper square corner to help us draw the fourth side?***



With student help, construct the fourth side of the rectangle as shown here.

Note: Another good method students may suggest is to use a translator. Align the translator on the first side of the rectangle (matching a side of triangle) and roll up until it just touches the top vertex of the triangle.



Then tidy the picture by erasing pieces of segments that were too long.

T: ***The red triangle has half of the area of the blue rectangle. So once we find the area of the rectangle, we can easily compute the area of the triangle. How can we find the area of the rectangle?***

S: ***Multiply its length times its width.***

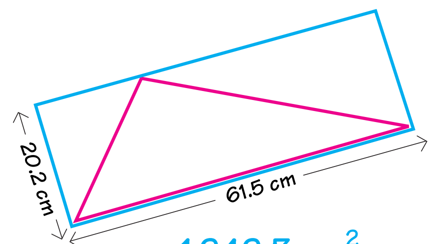
Invite students to measure in centimeters the length and width of the blue rectangle. Record the measurements near the picture. Here we suppose that the rectangle measures 61.5 cm by 20.2 cm. Allow students to use calculators to compute the area.

S: ***The area of the rectangle is 1 242.3 cm²***.

T: ***Now, what is the area of the red triangle?***

S: ***621.15 cm²; ½ x 1 242.3 = 1 242.3 ÷ 2 = 621.15.***

Record both areas on the board.



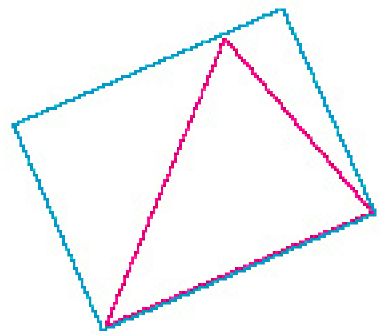
Distribute copies of Worksheet G3, rulers or straightedges, and calculators. Ask students to find the area of the red triangle in the same way as you found the area of the triangle on the board. There are three different rectangles that your students can use to compute the area of the triangle.

Home Activity

Invite students to draw any triangle they like on a piece of paper. Then suggest they take the rectangle home and show a family member how to find its area using the method of surrounding it with a rectangle having twice the area.

Name ____________ G3

Find the area of this triangle. Use the method of drawing a rectangle around it that has twice its area.



Area of the rectangle is 108 sq cm.

Area of the triangle is 54 sq cm.

Answers may vary slightly.

Capsule Lesson Summary

Use a falling ball to demonstrate parallel projections. Project several sets of points. Introduce and play *Follow My Finger*.

Materials

Teacher
- Small rubber ball with a good bounce
- Translator
- Colored chalk

Student
- Translator
- Colored pencils, pens, or crayons
- Tape
- Worksheets G4* and **

Description of Lesson

Exercise 1

Begin the lesson by dropping a ball on the floor from shoulder height. Each time, let it bounce only once before you catch it. Continue dropping and catching the ball occasionally throughout the ensuing discussion.

T: ***Think about the path of the ball as it leaves my hand until it first hits the floor. What do you notice?***

S: ***The ball goes straight down.***

T: ***Where does the ball hit the floor?***

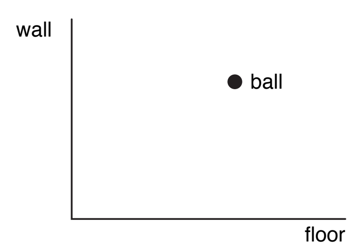
S: ***Directly under your hand.***

T: ***Think about the path of the ball and think about how this path is related to the wall.***

S: ***They are both straight.***

S: ***The ball falls and bounces up in a line parallel to the wall.***

When an observation similar to the last remark is made, illustrate the situation on the board.



T: ***Let's draw a line segment in this picture to show the path of the ball and the place where it hits the floor.***

Invite students to trace the path until someone does so accurately.

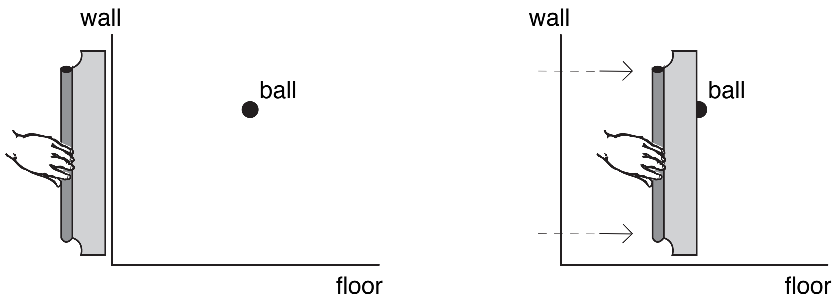
T: ***How can we draw a line segment accurately to show the ball's path?***

S: ***Use a translator.***

T: ***Tell me what to do.***

S: ***Align the translator with the wall and roll it to the ball.***

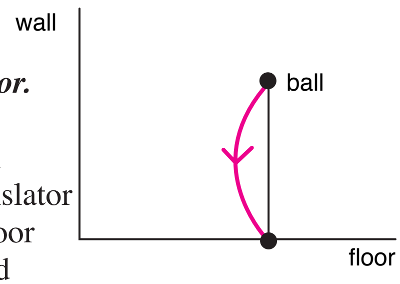
Repeat the instructions as you perform the actions.



S: ***Now draw a line segment from the ball to the floor.***

S: ***The ball will hit the floor where the line crosses the floor.***

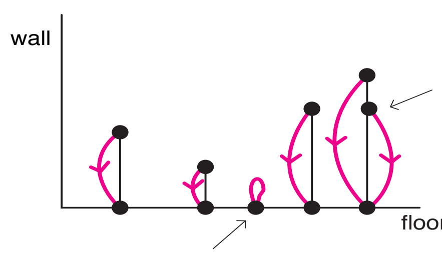
Draw a dot at the intersection of the floor line† and the path line.



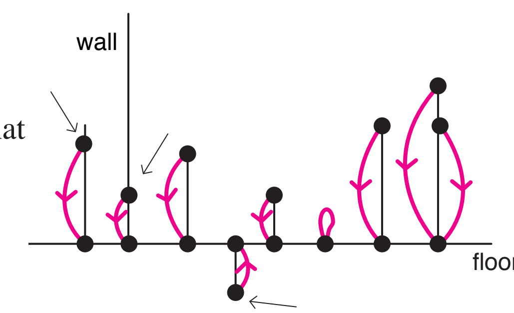
T: ***A red arrow will indicate the ball hits this point on the floor.***

Then draw a number of other dots for balls in the picture. For each one, ask a student to trace its path to the floor and then use the translator to draw the path accurately. Put a dot where the path crosses the floor line to show the place where the ball hits the floor; finish with a red arrow from the ball to the point where it hits the floor.

Next provide several special situations.



- First, draw a dot for a ball on one of the paths already drawn. Elicit discussion to observe that no new line segments need to be drawn and that the image point is already determined. Draw an arrow.
- Second, draw a dot for a ball on the floor line. Elicit discussion to show that the ball does not fall, that no path line need be drawn, and that a red loop describes the situation.



- Third, draw a dot for a ball on the other side of the wall (left side in the illustration). Elicit discussion to observe that the floor line must be extended but that the rest of the construction is as before.
- Fourth, draw a dot for a ball on the wall line and observe that the image point is already determined. Add an arrow.
- Fifth, draw a dot below the floor line. Students may observe that a ball would fall down and not up to the floor. Respond that instead of a ball, this dot represents a gas-filled balloon. The problem is the same; where will the balloon hit the floor? The technique of the construction is the same.

†In projection lessons, we will refer to the floor line as the line determined by the line segment labeled "floor." The same is true of the wall line.

T: ***This is an example of* parallel projection*. The balls are projected onto the floor parallel to the wall.***

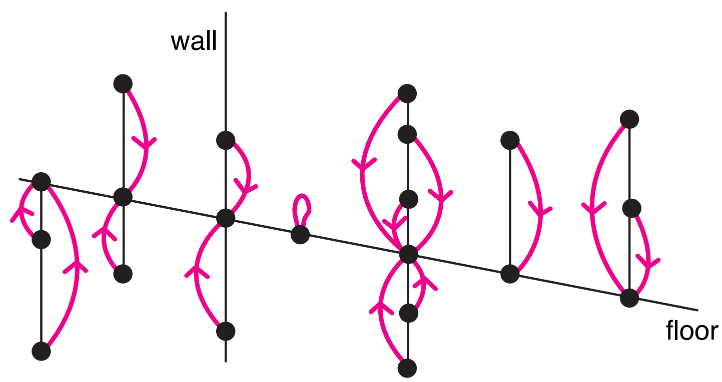
Write the words *parallel projection* and ask students to repeat several times that a ball is "projected onto the floor parallel to the wall."

Exercise 2

T: ***Suppose we have a slanted floor.***

Repeat Exercise 1 using a drawing similar to the one below.

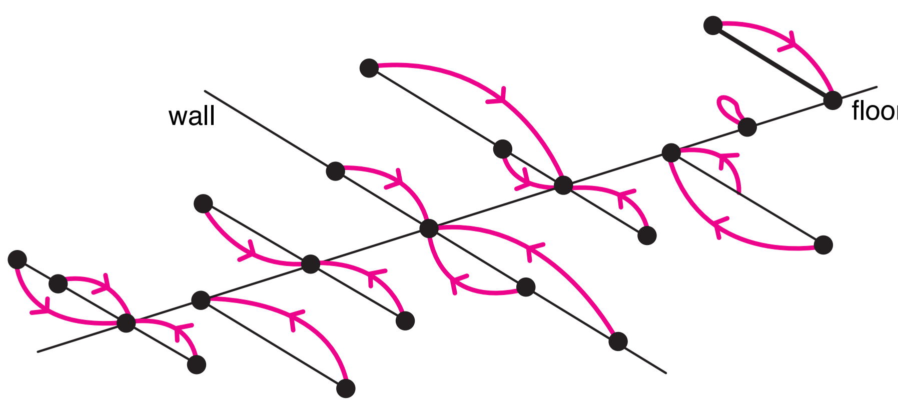
Emphasize that the projection is onto the floor and parallel to the wall. Use a large number of points for this exercise; in particular, several times use a set of points all of which lie on a single line, parallel to the wall line, as shown in the illustration. The points on the wall line require no special treatment. In fact, the wall line determines the projection of any point on it to be its intersection with the floor line.



Exercise 3

T: ***Now suppose that both the wall and the floor are slanted.***

Repeat the exercise creating a drawing similar to the one below. Sometimes locate a dot and ask students to point to its image; other times point to a dot on the floor and ask students to locate some point for which it is the image.

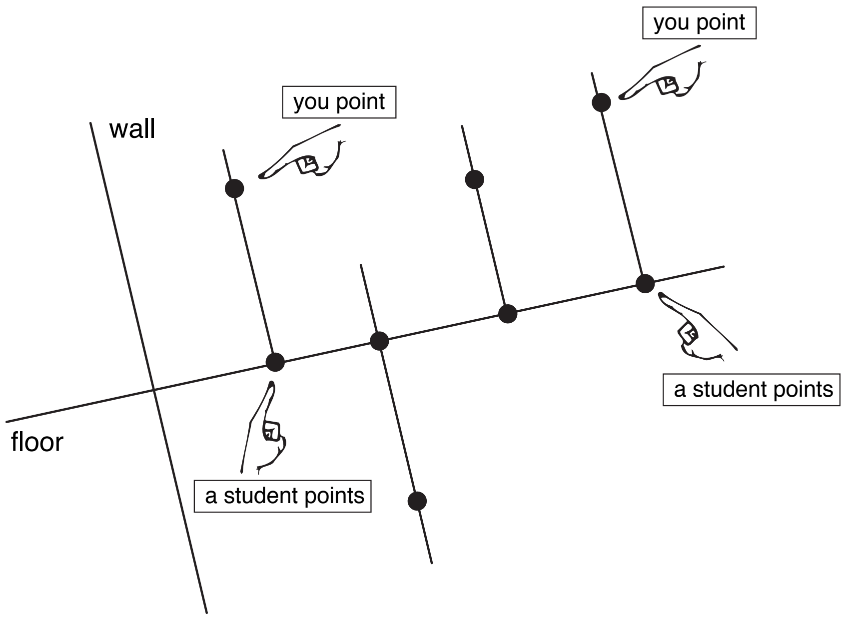


Some discussion may arise now that the orientation of the wall in the drawing does not correspond to that of the wall in the classroom. Emphasize that the projection is parallel to the wall in the drawing.

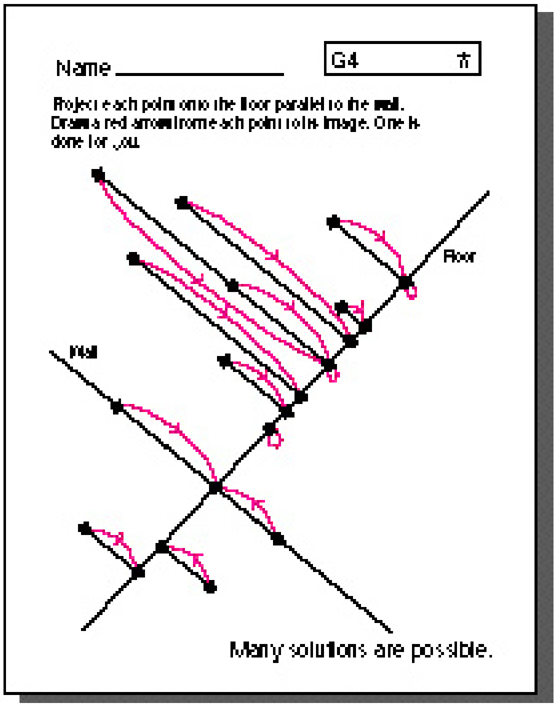
Exercise 4

G4

Play a game ~~called~~ ~~*Follow My*~~ *Finger*. Point to one of the dots in the drawing. A student plays opposite you and points to the image of the dot to which you are pointing. As you move your finger from dot to dot, the student follows by pointing to the corresponding images. When the game is well understood, start to mix in points whose images are not explicitly shown in the drawing. This technique provides you with a check on the students' grasp of the idea of parallel projection. If there is any dispute over the location of the image of a point, perform the construction with a translator.



Distribute translators and copies of Worksheets G4* and ** for individual work. You may suggest that students tape the worksheets to their desks. Monitor students' work to check on proper use of the translator in finding image points. Also check that projections are being made onto the floor and parallel to the wall and not the reverse. On each of the worksheets, the problem is printed on both sides to give a second chance for students who go astray in their constructions. If there is general difficulty with the problem on the * worksheet, stop for a few moments of collective work. Using the last drawing on the board, indicate a point on the floor line and ask a student to find a point whose image is the one indicated by you. Demonstrate the construction with a translator and then return to the individual work.



Name ____________

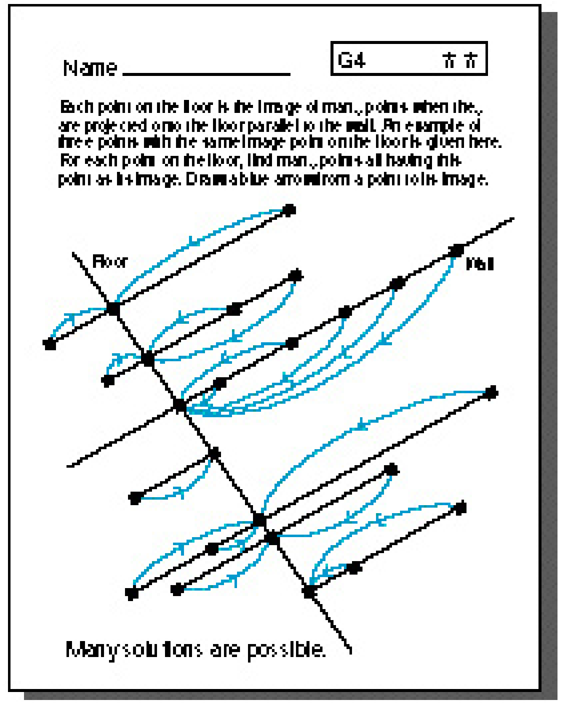
G4 ☆

Project each point onto the floor parallel to the wall. Draw a red arrow from each point to its image. One is done for you.

Floor

Wall

Many solutions are possible.



Name ____________

G4 ☆☆

Each point on the floor is the image of many points when they are projected onto the floor parallel to the wall. An example of three points with the same image point on the floor is given here. For each point on the floor, find many points all having this point as its image. Draw a blue arrow from a point to its image.

Floor

Wall

Many solutions are possible.

Capsule Lesson Summary

Project line segments and other curves. Play *Follow My Finger*.

Materials

Teacher
- Small rubber ball
- Translator
- Colored chalk

Student
- Translator
- Colored pencils, pens, or crayons
- Tape
- Worksheets G5*, **, ***, and ****

Description of Lesson

Begin the lesson by dropping the rubber ball on the floor, each time catching it after one bounce.

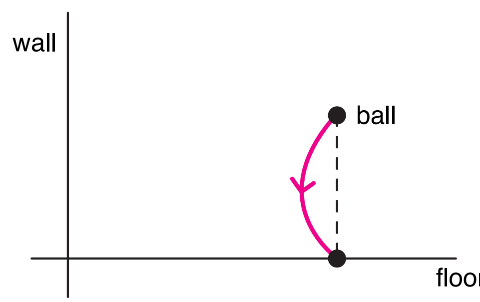
T: ***What is the path of the ball as it leaves my hand until it hits the floor?***

S: ***When you drop the ball, it falls straight to the floor.***

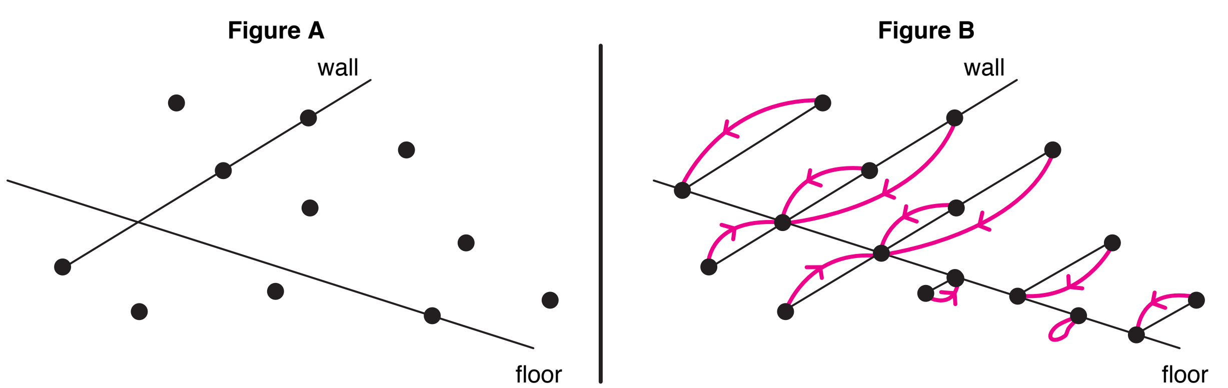
S: ***Parallel to the wall.***

Illustrate the path of the ball.

T: ***The red arrow is from a point to its image when it is projected onto the floor parallel to the wall. The red arrows are for parallel projection.***



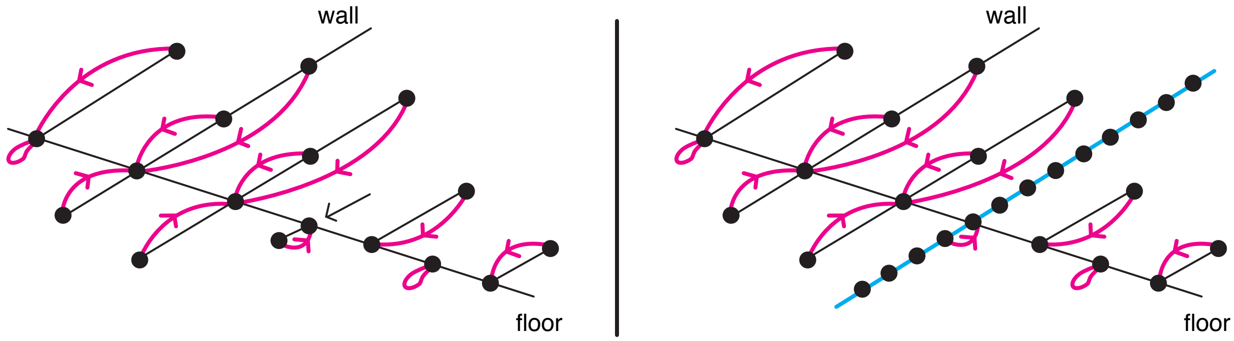
Erase the board and then draw the wall and floor lines on a slant, similar to those in the next illustrations. Draw dots for many points in various locations (see Figure A), and call on students to indicate the image of each point on the floor after parallel projection. Then, for each point, use your translator to find its image point and draw a red arrow to indicate the parallel projection (see Figure B).



Choose one of the image points on the floor line and point to it.

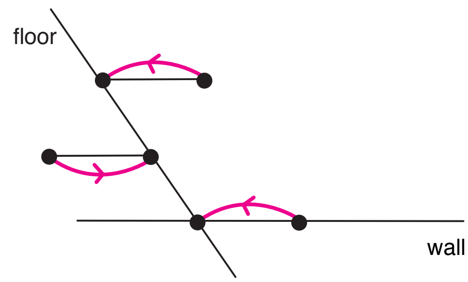
T: ***Who can find other points whose image is this point?***

For example, if you pick the point at the black arrow (below left), then any point on the line indicated by the blue segment (below right) is a correct answer.



Direct the discussion to a complete solution of the problem. Use a translator to check the parallel projection of each suggested point, and continue to ask for more solutions until you have received some on both sides of the floor and the one on the floor itself. Ask how many other points project to this point on the floor to informally raise the question of an infinite number of points on a line.

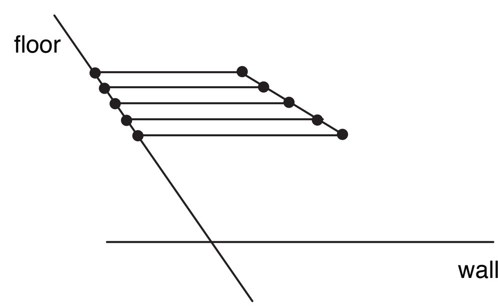
Erase the board, and draw wall and floor lines as shown here. Locate two or three points, and ask students to find their images by projecting onto the floor parallel to the wall. You may have to point out which is the floor line and which is the wall line. Use a translator to check the location of each image point.



Erase everything on the board except the floor and wall lines. Draw a line segment that does not intersect either the floor line or the wall line (see next illustration).

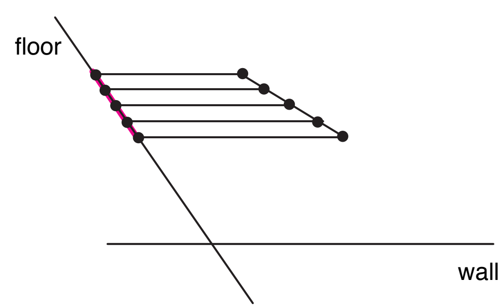
T: ***There are many points on this line segment. Let's find the projection of this line segment onto the floor parallel to the wall.***

Identify the endpoints of the segment, and call on a student to project them onto the floor, or do so yourself. Use a translator to accurately locate the image points. Also, find the images of a few other points of the segment between its endpoints.



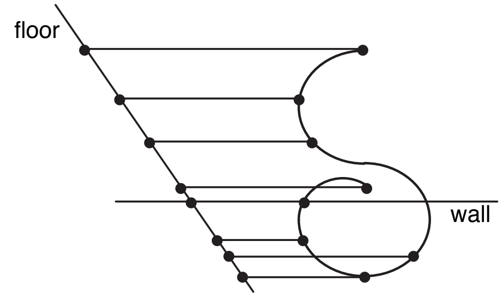
T: ***Imagine that we could find the image of each point on the line segment.***

Play *Follow My Finger* by moving your finger continuously over the line segment from one endpoint to the other. A student must move continuously along the floor line to follow you. Pause at the points whose images are constructed so that the student can check orientation on the floor line. After playing the game, mark all of the image points with colored chalk.

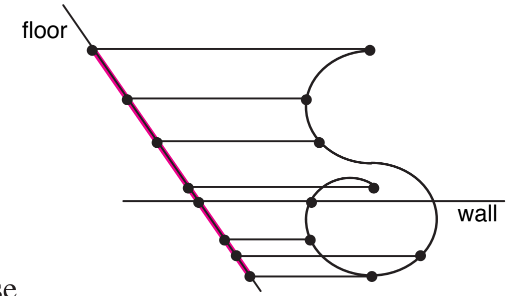


T: ***The image of this line segment is also line segment.***

Again erase everything on the board except the floor and wall lines. Draw a curve something like the one here.

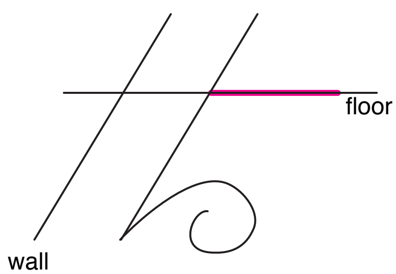


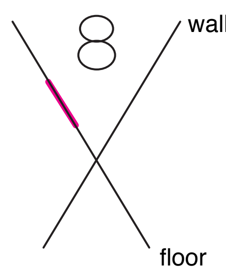
Ask students to suggest points on the curve to project. Ask other students to indicate the images of those points on the floor. Locate image points accurately with a translator.



Now play *Follow My Finger* with several students in turn. The curve provides a challenge to students in that it requires the follower to reverse directions twice over its length. Begin by jumping among the points whose images are drawn. Mix in some other points on the spiral. Then take each player in a continuous motion along the curve. Concentrate on the spiral at the lower end. Occasionally pause at one of the points whose image is drawn on the board to allow the student to orient properly. Eventually many students will arrive at an understanding of the situation. Then indicate the image of the curve with colored chalk.

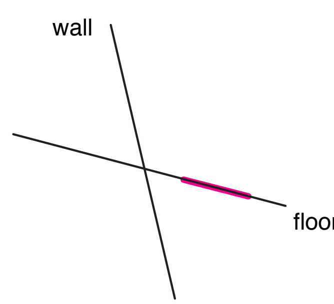
If interest in the game remains, play *Follow My Finger* with a few more curves as in the following illustrations, or use your own or a student's creation. The image for each curve is shown by the red segment.





For the remainder of the lesson, let students work independently on Worksheets G5*, **, ***, and ****. You may suggest that students tape the worksheets to their desks. On each worksheet the problem is printed on both sides to give a second chance to those who go astray. Encourage students who finish all four worksheets to create similar problems for each other.

If some students want additional work, ask them to draw a wall, a floor, and a line segment on the floor, as in this illustration.

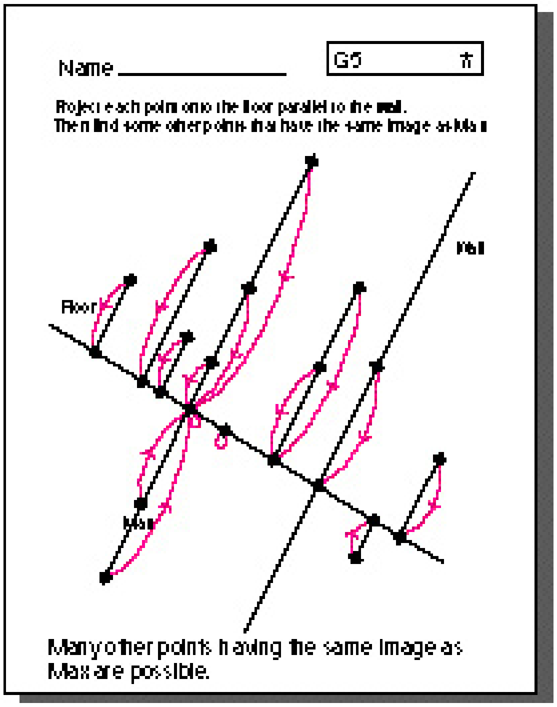


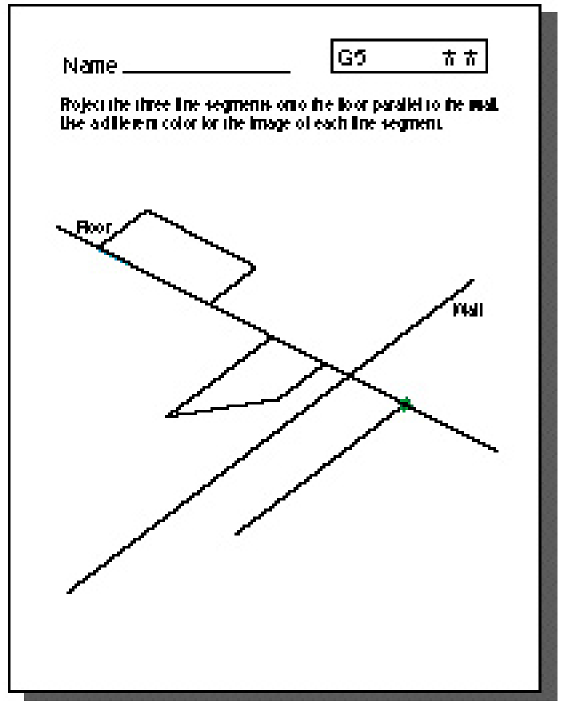
Ask students to find a fancy figure whose image is the line segment.

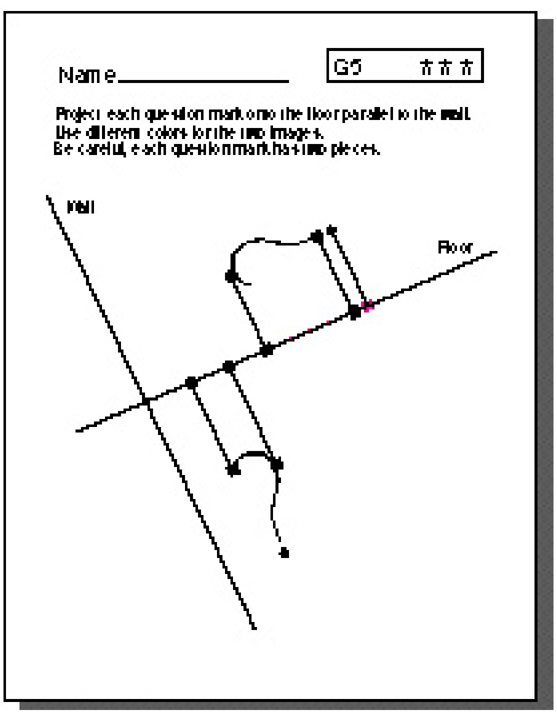
Home Activity

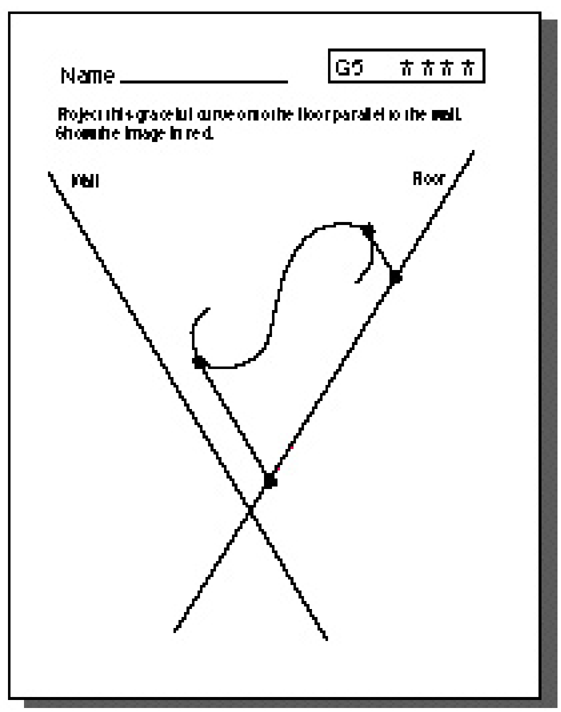
Suggest that students take home a picture (for example, Worksheet G5*** or ****) to play *Follow My Finger* with a family member.

G5


Name
G5 ☆
Project each point onto the floor parallel to the wall.
Then find some other points that have the same image as Max.
Floor
Wall
Max
Many other points having the same image as Max are possible.


Name
G5 ☆☆
Project the three line segments onto the floor parallel to the wall.
Use a different color for the image of each line segment.
Floor
Wall


Name
G5 ☆☆☆
Project each question mark onto the floor parallel to the wall.
Use different colors for the two images.
Be careful, each question mark has two pieces.
Wall
Floor


Name
G5 ☆☆☆☆
Project this graceful curve onto the floor parallel to the wall.
Show the image in red.
Wall
Floor

G6 PARALLEL PROJECTION #3

Capsule Lesson Summary

Review *equipollent couples*. Project equipollent couples onto a floor parallel to a wall, and observe that the images are equipollent couples.

Materials

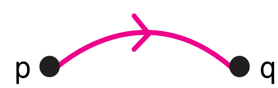
Teacher
- Translator
- Colored chalk

Student
- Translator
- Tape
- Worksheets G6 (no star), G6*, **, and ***

Description of Lesson

Exercise 1

With the class, review the idea of a *couple* as introduced in *IG-I* Lesson G6. A couple is an ordered pair of points; an arrow can be used to show the starting (first) and ending (second) points in a couple.



If the starting point is **p** and the ending point is **q**, then the couple is called (**p**, **q**).

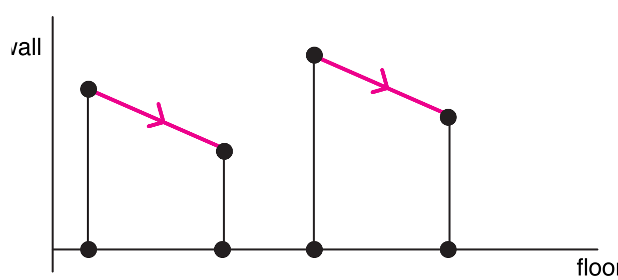
Distribute translators and copies of Worksheet G6 (no star). Instruct students on how to check for equipollent couples. Roughly speaking, couples are equipollent if they define opposite sides of a parallelogram.

Exercise 2

On the board, draw a pair of equipollent couples together with wall and floor lines (as in the next illustration).

T: ***Let's see what happens when equipollent couples are projected onto a floor parallel to a wall.***

Invite students to do the projections at the board.



T: ***What do you notice about the points on the floor?***

S: ***If you drew two arrows, then you would have two more couples.***

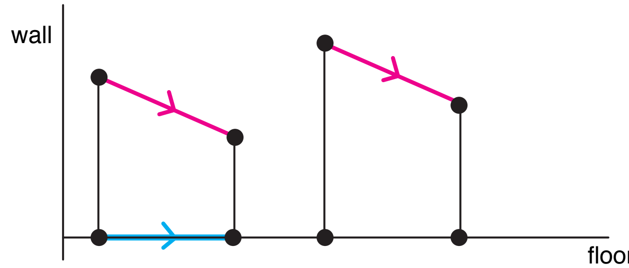
T (pointing to the two left-most points)**:** ***Can I draw a red arrow here?***

S: ***No, that couple is not equipollent to the red couples.***

S: ***Use blue.***

G6

Add a blue arrow.

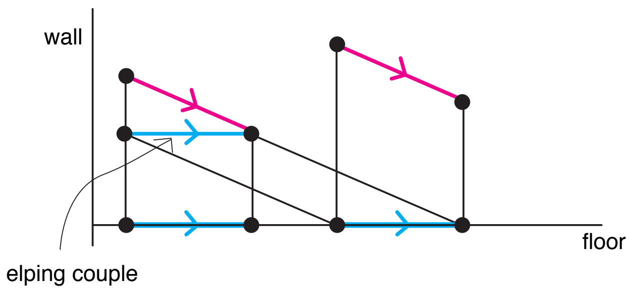


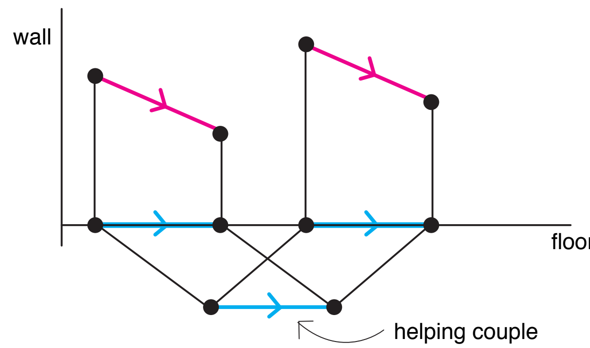
T: ***What about this other pair of points?***

S: ***The couple is equipollent to the blue couple.***

T: ***Let's check.***

Do the construction at the board. Two different constructions are suggested below. Use a construction that your class seems to prefer.



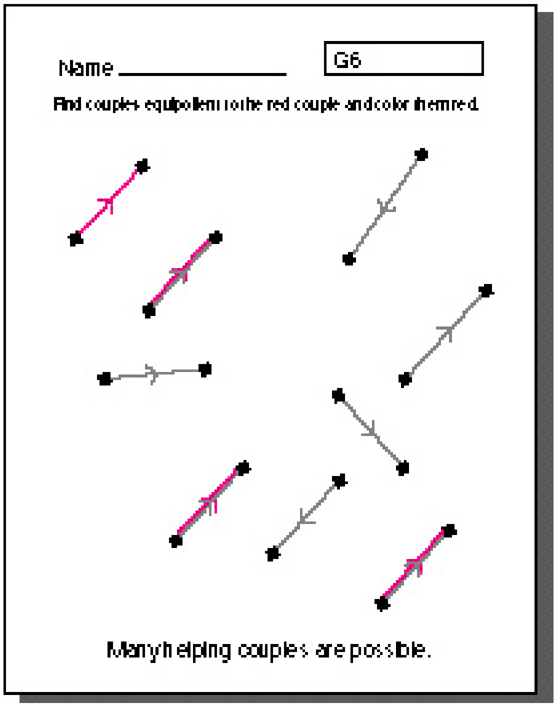


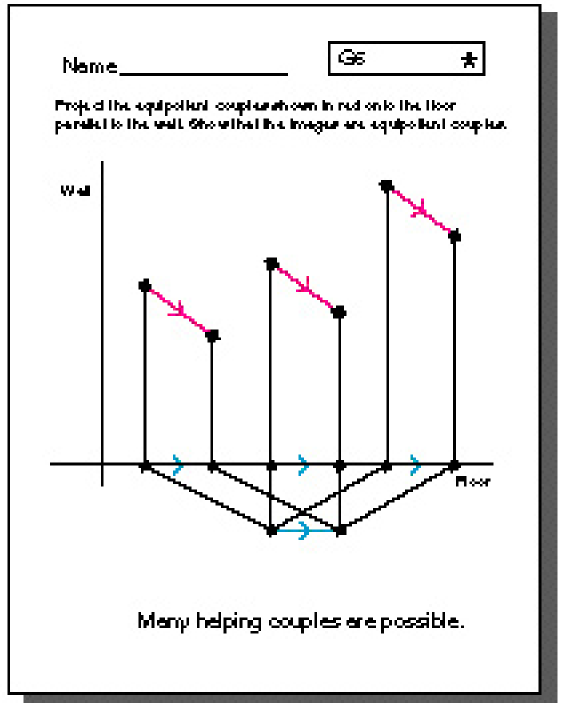
T: ***Suppose more red couples were added to my picture and projected onto the floor parallel to the wall. What would happen?***

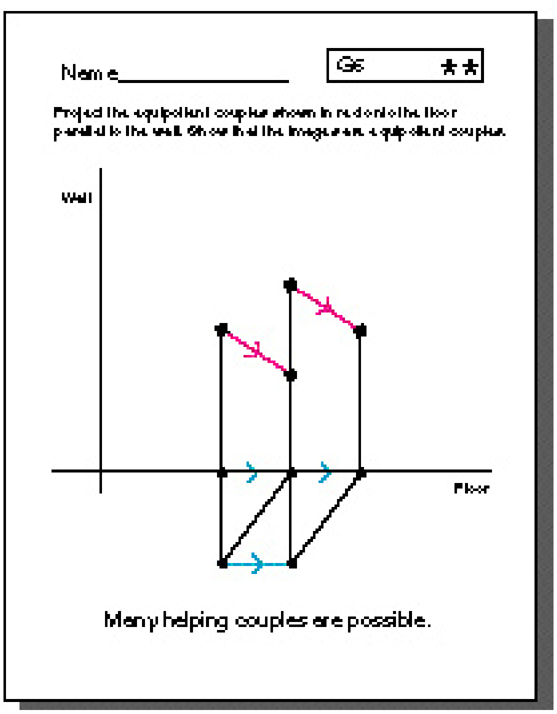
S: ***There would be more blue couples.***

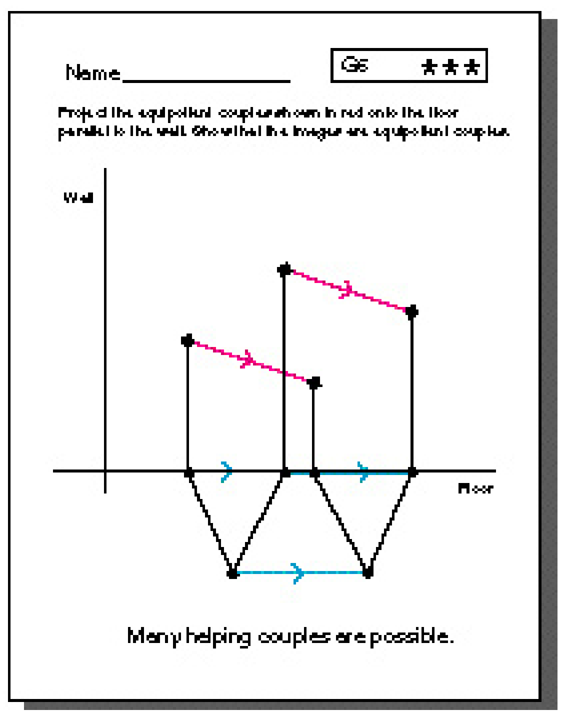
Encourage the class to conclude that the parallel projection of equipollent couples yields equipollent couples.

Worksheets G6*, **, and *** are available for individual work. Provide tape for students to attach the worksheets to their desks.


Name
G6
Find couples equipollent to the red couple and color them red.
Many helping couples are possible.


Name
G6 ★
Project the equipollent couples shown in red onto the floor parallel to the wall. Show that the images are equipollent couples.
Wall
Floor
Many helping couples are possible.


Name
G6 ★★
Project the equipollent couples shown in red onto the floor parallel to the wall. Show that the images are equipollent couples.
Wall
Floor
Many helping couples are possible.


Name
G6 ★★★
Project the equipollent couples shown in red onto the floor parallel to the wall. Show that the images are equipollent couples.
Wall
Floor
Many helping couples are possible.

Capsule Lesson Summary

Find the areas of shapes on a grid by decomposing them into triangles and rectangles.

Materials

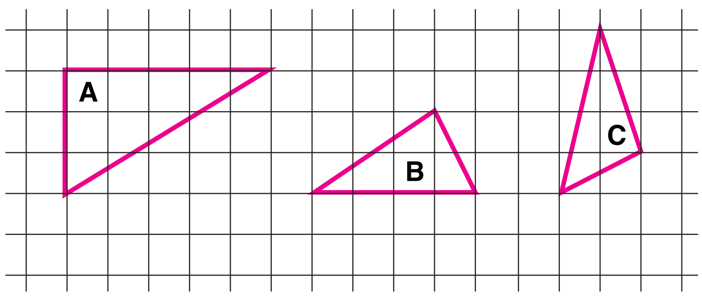
Teacher
- Grid board
- Colored chalk
- Straightedge or meter stick

Student
- Colored pencils, pens, or crayons
- Centimeter grid paper
- Worksheets G7*, **, and ***

Description of Lesson

Exercise 1

Display a grid board and ask the class to pretend that each of the grid squares measures 1 cm by 1 cm. Provide centimeter grid paper for student use. Draw several triangles on the grid board similar to the ones here.

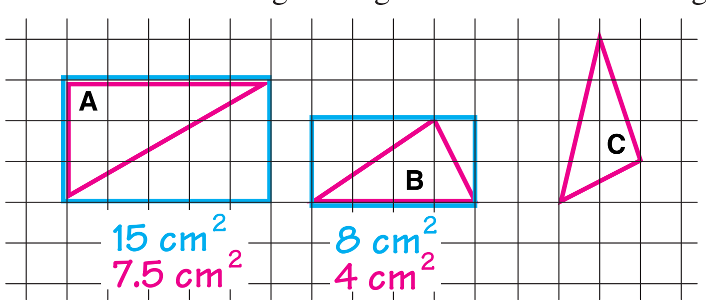


T: ***How have we been finding the areas of triangles?***

S: ***We draw rectangles around them that have twice their areas.***

Invite students to do this on the board. In this case, they are likely to draw smallest rectangles surrounding **A** and **B** because they can do so following grid lines.

For **A** and **B** find the areas of the surrounding rectangles and then of the triangles.



T: ***This method is more complicated to use on triangle* C*. Why?***

S: ***We cannot use the grid lines to draw a rectangle that has twice its area.***

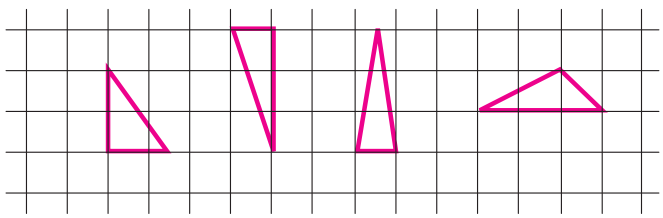
S: ***It's like the large red triangle that we cut out and lined up on a separate piece of grid paper.***

S: ***We could use a square corner and meterstick to draw a rectangle with twice its area.***

You may like to let students estimate the area of triangle **C**, or some students may suggest other methods. For the time being, tell the class that they do not need to find the area of **C** now, but later they will learn a method. Erase the grid board.

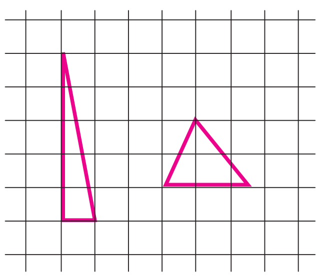
T: ***Let's draw a triangle that has an area of 1½ cm^2?***

Instruct students to draw such a triangle on their grid papers and call on several volunteers to do so on the grid board. Ask the volunteers to explain why their triangles have an area of 1½ cm^2. The following illustration gives several possibilities. You may want to ask students to think first about the area of a surrounding rectangle.

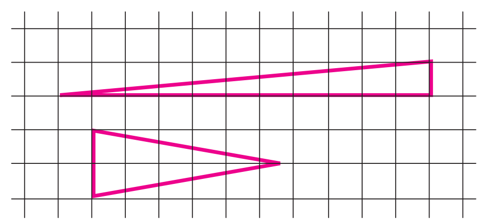


Continue the activity by asking for triangles with areas of 2½ cm^2, 3 cm^2, 5½ cm^2, and 6 cm^2. A couple examples of each are shown below.

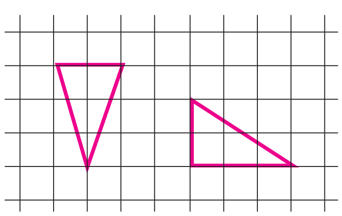
$^1/_2$ cm^2



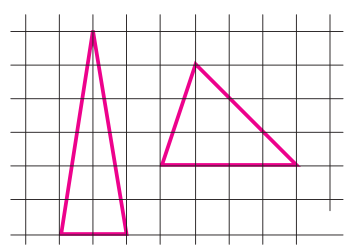
5 $^1/_2$ cm^2



3 cm^2

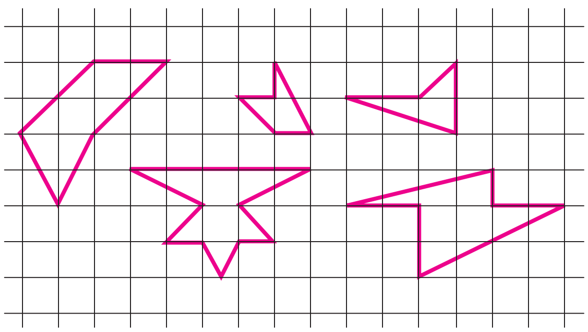


6 cm^2

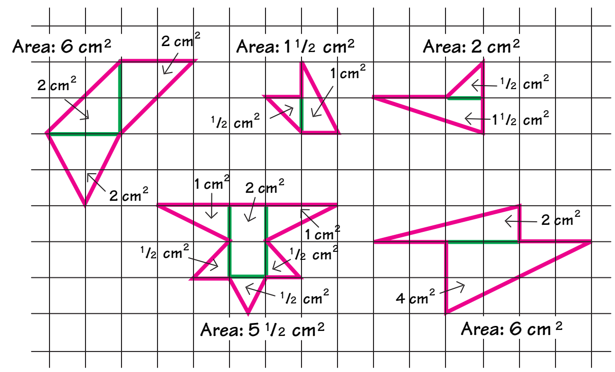


Exercise 2

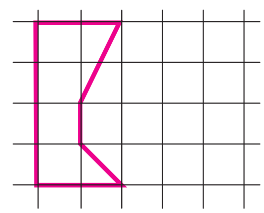
On the grid board, present the shapes shown below one at a time. Let the class discuss methods of finding the areas of the shapes. When someone suggests decomposing them into constituent triangles and rectangles, lead the class to use that method. For each shape, one possible decomposition is indicated with green segments in the key that follows.



Solutions



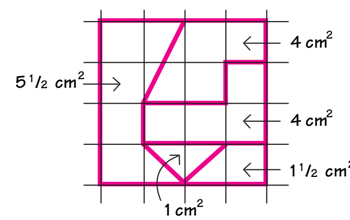
Draw this shape on the grid board.



T: ***What is the area of this shape?***

S: ***5½ cm².***

Ask a student to convince the class that the area is 5½ cm². Then continue by adding four more shapes, one at a time, and asking for their areas. The five shapes fill a 4 by 4 square. Check that the total area of the five shapes is 16 cm², the area of the square.



Worksheets G7*, **, and *** are available for individual work. Each picture is printed on both sides of a worksheet to give students an extra copy in case they need to start over.

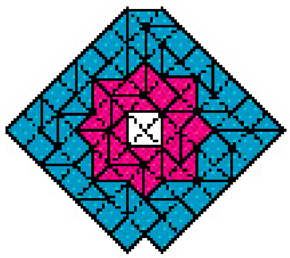
Home Activity

Students may like to take one of the worksheets home to do with a family member.

Name ______________ G7 ★

Color all shapes of area 1.5 cm² with the same color.

Use another color to color all shapes of area 2.5 cm².

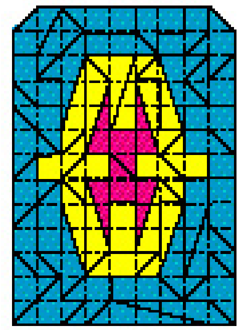


Name ______________ G7 ★★

Use one color to color all shapes of area 1.5 cm².

Use another color to color all shapes of area 2 cm².

Use a third color to color all shapes of area 2.5 cm².

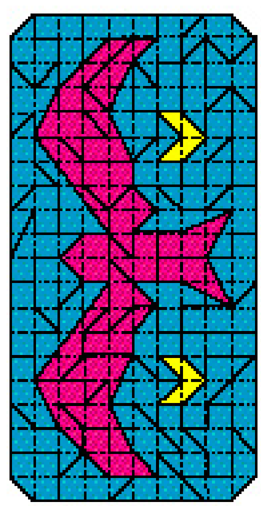


Name ______________ G7 ★★★

Use one color to color all shapes of area 2 cm².

Use another color to color all shapes of area 3 cm².

Use a third color to color all shapes of area 3.5 cm².



Capsule Lesson Summary

Given a rectangle, draw many triangles inside it that have half of its area. Look at a rectangle divided into pieces colored red and blue. Find the area of the rectangle and the area of the pieces in one color, and use the results to find the area of the pieces in the other color.

Materials

Teacher
- Blackline G8
- Grid board
- Colored chalk
- Meter stick or straightedge
- Piece of cardboard

Student
- Page of 4 cm by 7 cm rectangles
- Colored pencils, pens, or crayons
- Ruler or straightedge
- Worksheets G8*, **, and ***

Advance Preparation: Use Blackline G8 to make copies of 4 cm by 7 cm rectangles for students.

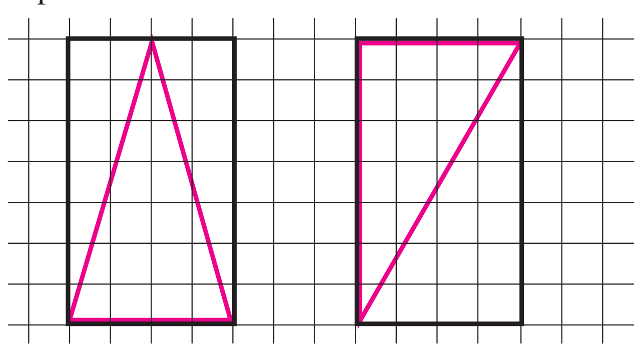
Description of Lesson

Exercise 1

Display several 4 by 7 rectangles on a grid board or use Blackline G8 to make a transparency for overhead projection. Ask students to imagine the grid squares are 1 cm by 1 cm, just like the ones on their pages of rectangles.

T: ***These rectangles are all the same size. Who can trace within one of the rectangles a triangle that has half the area of the rectangle.***

Let several students trace such triangles. Draw two of the suggested triangles in the rectangles on the board, one that has a long side of the rectangle in common and one that has a short side of the rectangle in common. Examples are shown below.

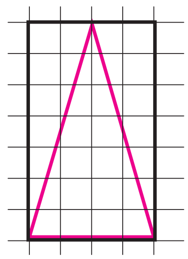


T: ***On your paper there are several rectangles like these, 4 cm by 7 cm. In each rectangle, draw a different triangle with half the area of the rectangle.***

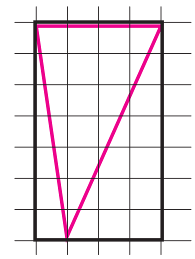
G8

Any correct triangle will have a side in common with these rectangles and a vertex touching the opposite side. Observe students' work and call the class's attention to individual solutions by drawing some of them in one of the rectangles on the board. Draw any triangle having a side in common with a long side of the rectangle in one picture, and draw any triangle having a side in common with a short side of the rectangle in the other picture. For example:

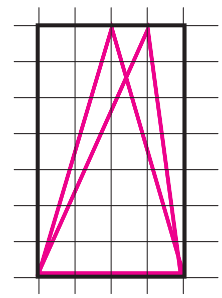
Suppose that this triangle is on the board …



…and a student shows you this one:

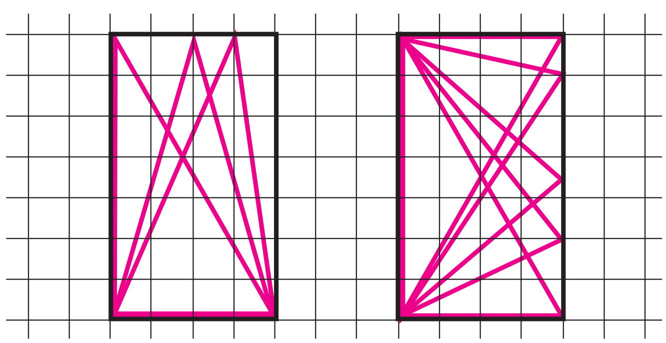


Rotate the student's paper 180° and tell the student you will show that triangle on the board. Then superimpose the triangle on the other one in the rectangle on the board.



Note: You may wish to first let the student explain why the area of the triangle is half of the area of the rectangle as was done in Lesson G2 *Area #1*. Do this by displaying the triangle first in a separate rectangle picture; then erase the picture as you superimpose the triangle on one already on the board.

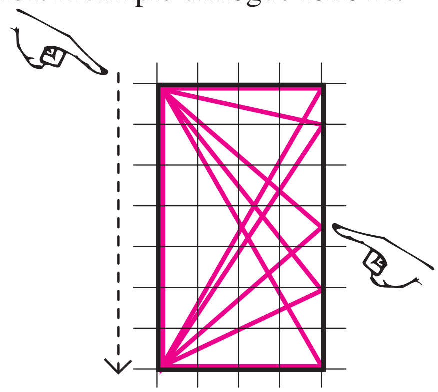
Continue until there are several triangles in each rectangle. Your pictures might look similar to these:



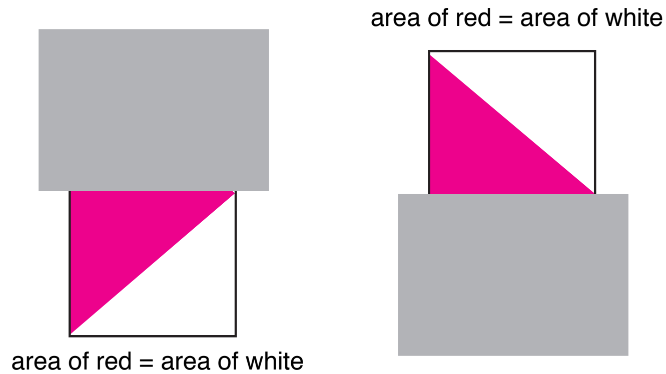
Lead the class to observe that any triangle having a side in common with the rectangle and a vertex anywhere on the opposite side has half of the rectangle's area. A sample dialogue follows.

T: ***Would a triangle with this side in common with the rectangle…***

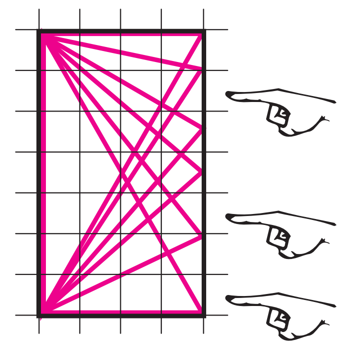
… and with a vertex here have half of the rectangle's area?



Let students comment. Then, on the chalkboard, draw a rectangle of the same size as the one on the grid board, and draw the red triangle you were indicating. Shade the interior of the triangle red for emphasis. Ask a student to explain how to check that this red triangle has half of the rectangle's area. Have a piece of cardboard available to use as a cover.



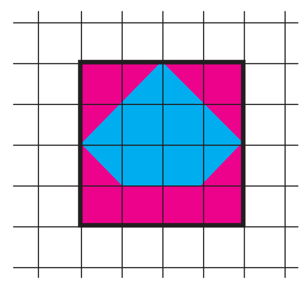
Draw the triangle in the picture on the board. Point to several other places along the same side of that rectangle, and each time ask if a vertex of such a triangle could be there. Your class should conclude that the third vertex could be anywhere along that side.



Examine the triangles in the other rectangle in a similar way. Determine that each has a side in common with one side of the rectangle and that each has a third vertex on the opposite side of the rectangle. Erase the board before going on to Exercise 2.

Exercise 2

Draw a 4 by 4 square on the grid board.



T: ***What is the area of this square?***

S: ***16 cm²***.

Color the interior of the square as shown here.

T: ***How much area is colored blue?*** (7 cm^2)

Let a student convince the class that the blue shape has area 7 cm^2.

T: ***How much area is colored red? How do you know?***

S: ***9 cm^2; I counted the squares and the half-squares.***

S: ***9 cm^2; I just figured out what number you add to 7 to get 16.***

Write the indicated number sentence on the board.

$$7 + \boxed{9} = 16$$

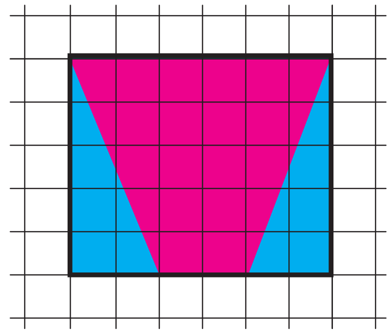
T: ***That's a very good method also. The rectangle's area is 16 cm^2, 7 cm^2 of which is colored blue and the rest of which is colored red. 7 plus 9 equals 16.***

G8

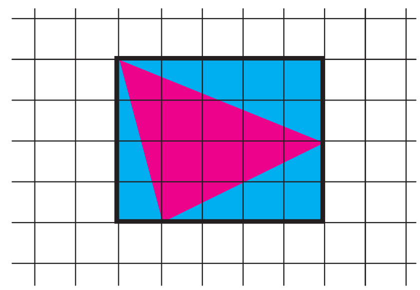
Repeat the activity with this picture. Have a piece of cardboard available for students to use as a cover. Let the class find either the area of the red piece or the area of the blue pieces first. Students should suggest both methods (counting and missing addend) of finding the area in the other color.

Area of the rectangle: 30 cm^2
Area of the red piece: 20 cm^2
Area of blue pieces: 10 cm^2

$10 + 20 = 30$



Erase the grid board and then draw this picture centered on it. Let students comment on the picture.

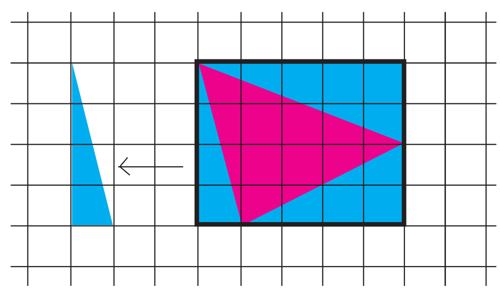


S: ***That's one of those crooked triangles.***

S: ***It's like the large red triangle we cut out.***

S: ***The red triangle does not have half of the rectangle's area. The area in red is not the same as the area in blue.***

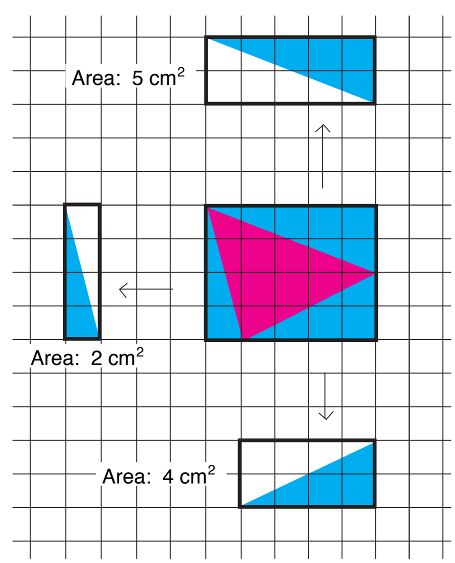
T: ***The area of the red triangle is not simple to find. Maybe we could find the areas of the blue pieces. What is the area of this blue triangle*** (point to the one on the left)***? I'll redraw the triangle away from the picture so we can get a better look at it.***



S: ***Its area is 2 cm^2.***

S: ***It has half of the area of a rectangle 1 cm by 4 cm.***

Invite a student to highlight the rectangle to see that the area of that blue triangle is 2 cm^2. Likewise, ask the class to find the areas of the other two blue triangles.



T: ***So what is the total area of the blue pieces?***

S: ***11 cm^2.***

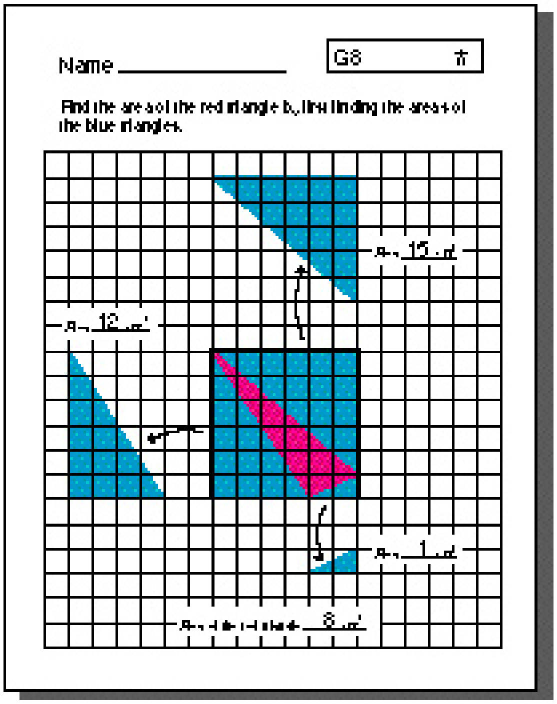
T: ***Can we use the total area of the blue triangles to find the area of the red triangle?***

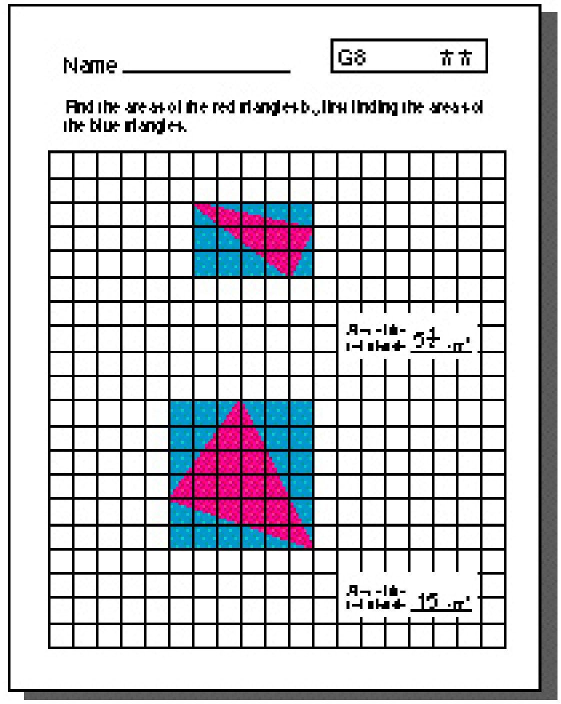
S: ***Yes. The blue and red together make a rectangle with area 20 cm^2. The blue triangles have 11 cm^2 in area, so the red triangle must have area 9 cm^2. 11 + 9 = 20.***

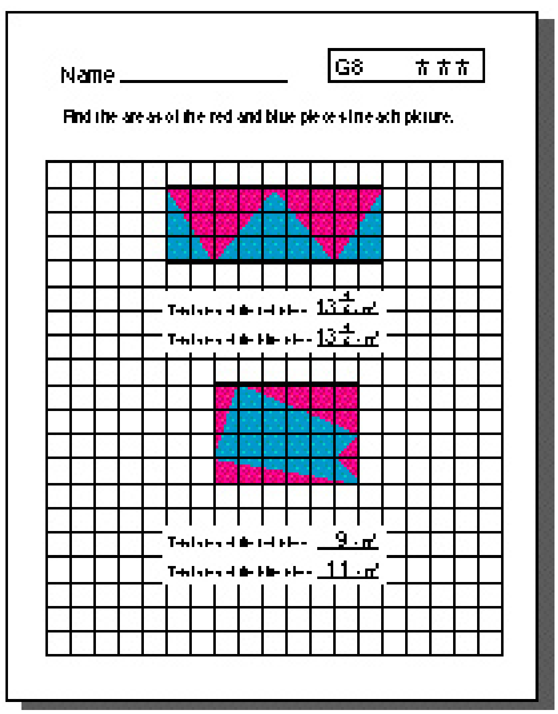
Write the number sentence on the board.

$11 + 9 = 20$

Similar problems are available on Worksheets G8*, **, and *** for students to do individually or with a partner.







Capsule Lesson Summary

Project midpoints onto the floor line parallel to the wall line. Using parallel projection, find the midpoint of a pair of points.

Materials

Teacher

- *IG-II Geometry Posters* #2 and #3
- Translator
- Colored chalk
- Tape

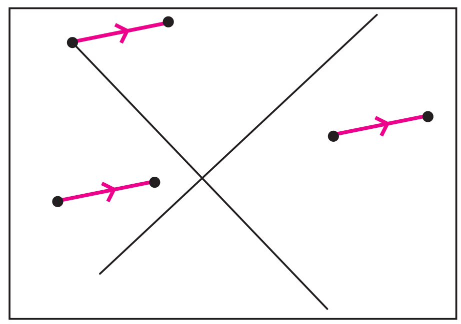
Student

- Translator
- Unlined paper
- Colored pencils, pens, or crayons
- Tape

Description of Lesson

Exercise 1

Display *IG-II Geometry Poster* #2.

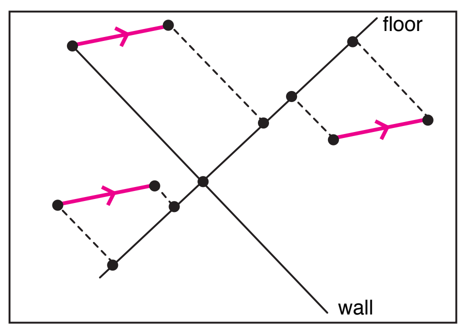


T: ***Why are these arrows the same color?***

S: ***The couples are equipollent.***

If necessary, verify that the couples are equipollent using a translator. Try to keep construction lines on the poster to a minimum as they will only clutter the work to follow.

Invite a student to project each couple onto a floor parallel to a wall. Let the student choose which is the floor line and which is the wall line. Try to keep construction lines to a minimum, as the floor and wall lines will be switched later in this exercise. One of the two possible pictures is shown here.



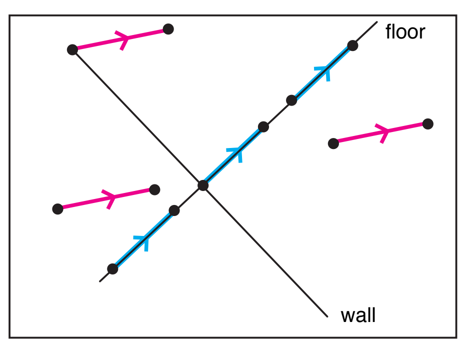
T: ***What about the points on the floor?***

S: ***They form equipollent couples.***

T: ***Yes. Are the couples equipollent to the red couples?***

S: ***No. The points form a new family of equipollent couples.***

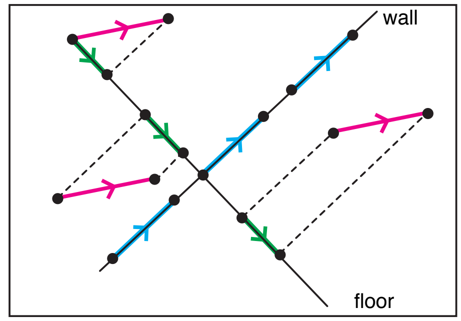
Add blue arrows to indicate these equipollent couples.



T: ***Let's reverse the floor and wall lines, and again project the red couples onto the floor parallel to the wall. What will happen?***

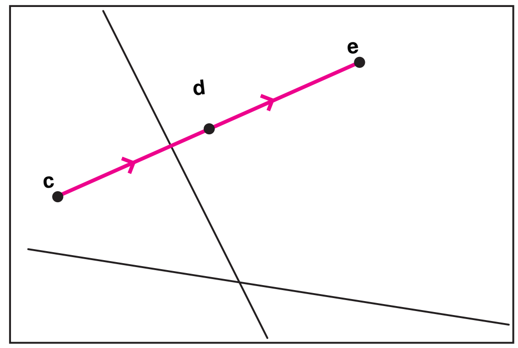
S: ***We will get more equipollent couples.***

Label the new floor and wall lines for reference, and ask students to do the projection. Upon completion, add arrows of a third color in the appropriate places.



Exercise 2

Display *Geometry Poster #3*.



Do not write the letters on the poster. They are here just to make the description of the lesson easier to follow.

T (pointing to **d**)**:** ***What can you tell me about this point?***

S: ***It is a starting point and an ending point.***

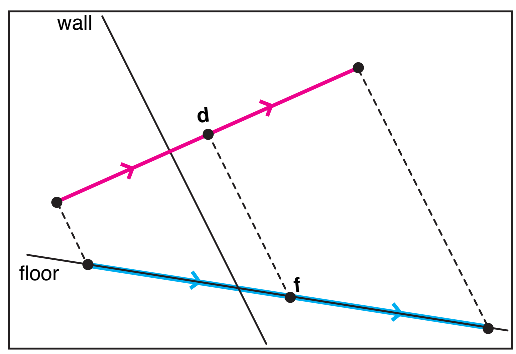
S: ***It is midway between the other two points.***

T: ***Yes, it is the midpoint of this segment*** (trace from **c** to **e**)***. What happens when equipollent couples are projected onto a floor parallel to a wall?***

S: ***We get more equipollent couples.***

T: ***Let's see what happens when midpoints are projected onto a floor parallel to a wall.***

Invite a student to project the couples onto a floor. As in the first exercise, let the student choose which is the floor line and which is the wall line.



Do not write the letters on the poster. They are here just to make the description of the lesson easier to follow.

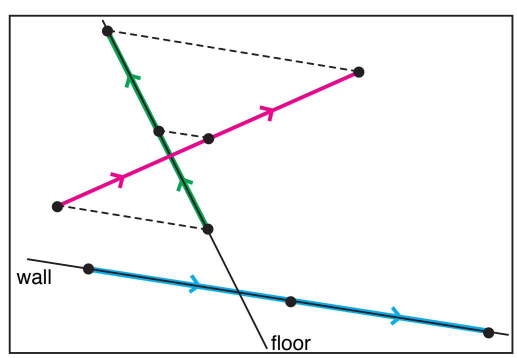
T (pointing to **d**)**:** ***Point to the image of this point.***

A student should point to **f**.

T (pointing to **f**)**:** ***What can we say about this point?***

S: ***It is a midpoint, too.***

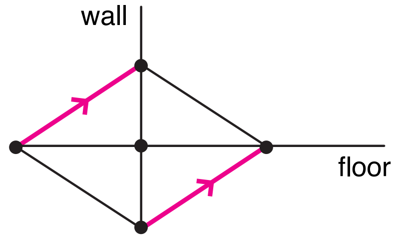
Exchange the floor and wall lines, and then repeat the activity.



These two activities should convince the class that the image of a midpoint under parallel projection is a midpoint.

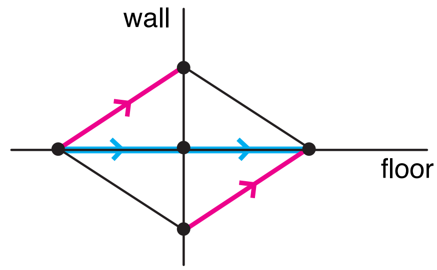
Exercise 3

Draw this picture of a parallelogram on the board.



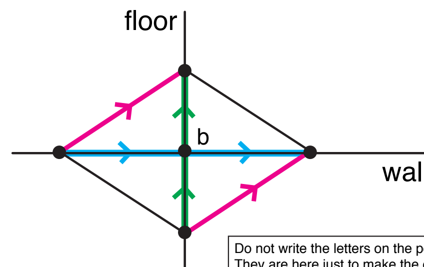
T: ***The red couples are equipollent. Let's project the red couples onto the one diagonal (floor line) parallel to the other diagonal (wall line).***

Invite a student to color the image couples in blue.



S: ***The blue couples are equipollent.***

Exchange the floor and wall lines, and project the red couples again. Color the image couples this time in green.



Do not write the letters on the poster. They are here just to make the description of the lesson easier to follow.

S: ***The green couples are equipollent.***

T (pointing to **b**)**:** ***What can we say about this point?***

S: ***It is a midpoint with the blue couples.***

S: ***It is a midpoint with the green couples.***

S: ***It is the midpoint of both diagonals.***

T: ***Yes, the diagonals of a parallelogram intersect each other at their midpoints.***

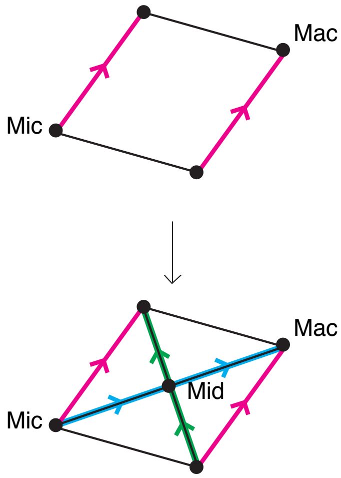
You may wish to leave this picture on the board as a hint for Exercise 4.

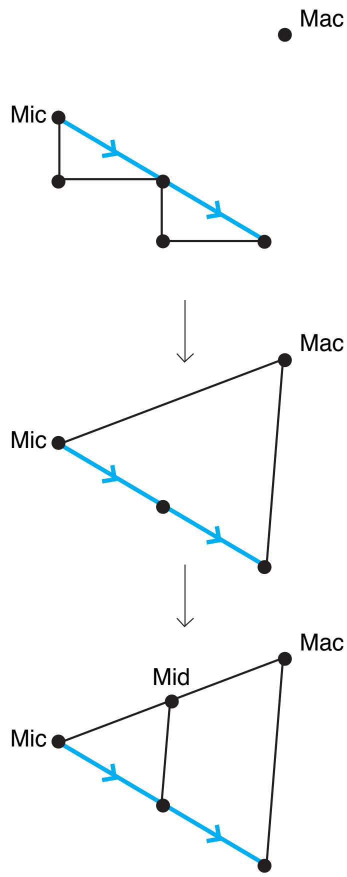
Exercise 4

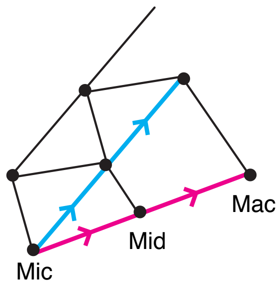
Draw two dots on the board and label them Mic and Mac. Distribute translators, tape, and paper to the class.

T: ***Mid is the midpoint of Mic and Mac. Find Mid using only your translator.***

Let students work independently on this problem for a few minutes. Three solutions are given below. Your students are likely to find the first two solutions but need to be shown the third one.


Mac
Mic
Mac
Mid
Mic


Mac
Mic
Mac
Mic
Mac
Mid
Mic


Mac
Mid
Mic

G10 PARALLEL PROJECTION #5

Capsule Lesson Summary

Using parallel projection, divide a line segment into three pieces of equal length and five pieces of equal length. Construct a number line and label some points. Convert Fahrenheit temperatures to Celsius temperatures and vice versa.

Materials

Teacher
- Translator
- *IG-II Geometry Poster #4*
- Colored chalk
- Tape

Student
- Translator
- Unlined paper
- Colored pencils, pens, or crayons
- Tape

Description of Lesson

Exercise 1

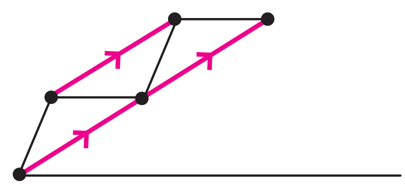
Each student needs a translator and two sheets of paper. Provide tape for students to tape their papers to their desks.

T: ***Today we are going to use a translator to subdivide line segments into shorter segments of equal length.***

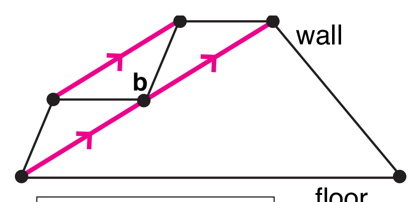
As you draw a line segment on the board, instruct students to do likewise on their papers. Encourage students to draw the segment in the middle of the sheet of paper.

T: ***First, we will divide this segment into two smaller segments of equal length.***

Perform the construction at the board while the students follow you at their seats. First construct a hook-up of two equipollent couples starting at one end point of the segment.



Next draw a segment from the ending point of the couples to the endpoint of the segment. Indicate floor and wall lines as shown here.



Do not write the letter **b** on the board. It is here just to make the description of the lesson easier to follow.

T (pointing to **b**)**:** ***What can you tell me about this point?***

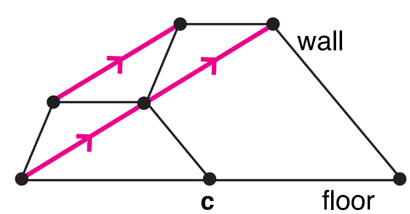
S: ***It's a midpoint.***

T: ***Let's project this midpoint onto the floor parallel to the wall.***

(Point to **c**.) ***What about this point?***

S: ***It is the midpoint of the line segment we started with.***

T: ***Yes, midpoints project to midpoints.***



Do not write the letter **c** on the board. It is here just to make the description of the lesson easier to follow.

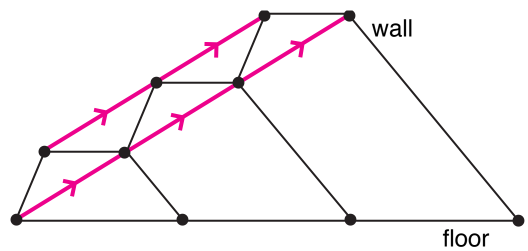
Direct students to draw another line segment and to practice the construction just demonstrated. Observe students' work and offer assistance when necessary. Do not spend more than ten minutes on this activity.

Draw another segment near the first, and ask how the segment might be subdivided into three shorter segments of equal lengths.

S: ***Draw (construct) a hook-up of three couples and then project them.***

Do not be concerned if the above suggestion is not forthcoming. Perform the construction collectively.

Emphasize that the wall segment is drawn between the ending point of the hook-up of couples and the endpoint of the segment.



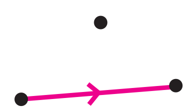
T: ***Suppose I wanted to subdivide a line segment into four shorter segments of equal length. What could I do?***

S: ***Draw (construct) a hook-up of four couples and then project them.***

Allow about five minutes for students to try this construction. Encourage faster students to subdivide a segment into more equal pieces (five to ten).

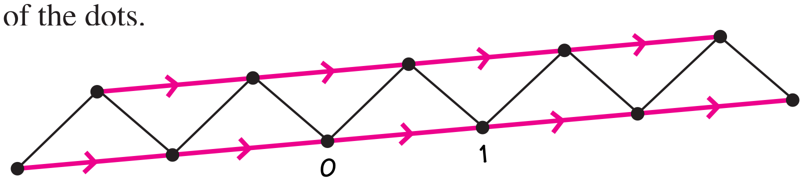
Exercise 2

Ask students to draw a couple and a third dot a few centimeters away on their papers.



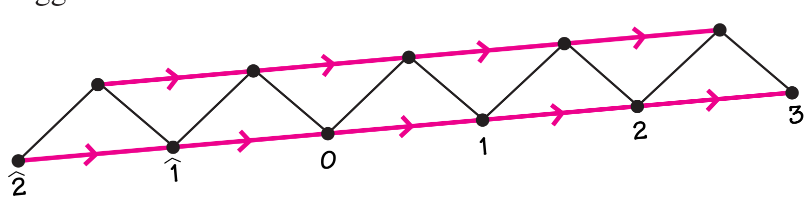
Next, construct a parallelogram train† while students follow your lead. The larger their drawings, the easier the following work will be for students.

Put 0 and 1 at two of the dots.



T: ***How could we label the other dots?***

Many students will suggest that the bottom row of dots can be labeled like a number line.



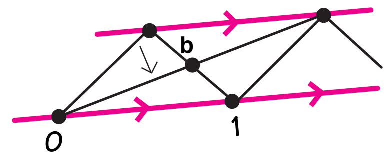
T: ***Where is ½ on this part of the number line?***

†Review Lesson G9 *Parallelism #4* in *IG-I* on how to construct this parallelogram train.

Let students express their ideas freely. Many students will come to the board and point to a point that is about halfway between 0 and 1. Encourage students to consider a construction that would more accurately locate ½. Some students might try to use parallel projection.

T: ***Let me show you something from a previous lesson.***

Add this line and point to your picture.



Do not write the letter **b** on the board. It is here just to make the description of the lesson easier to follow.

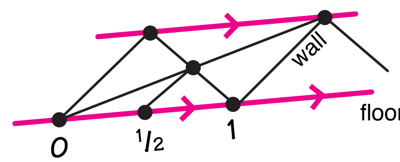
T (pointing to **b**)**:** ***What can you tell me about this point?***

S: ***This point is the midpoint of each diagonal.***

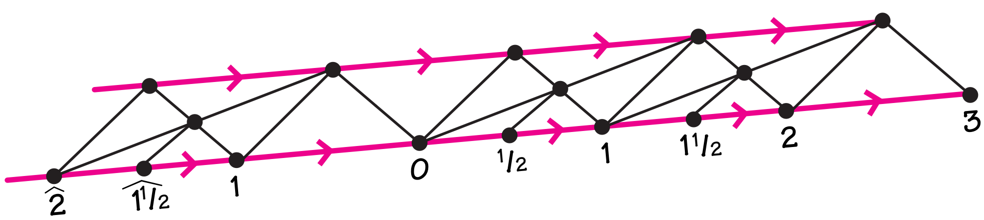
T: ***Yes. And what if a midpoint is projected onto a floor parallel to a wall?***

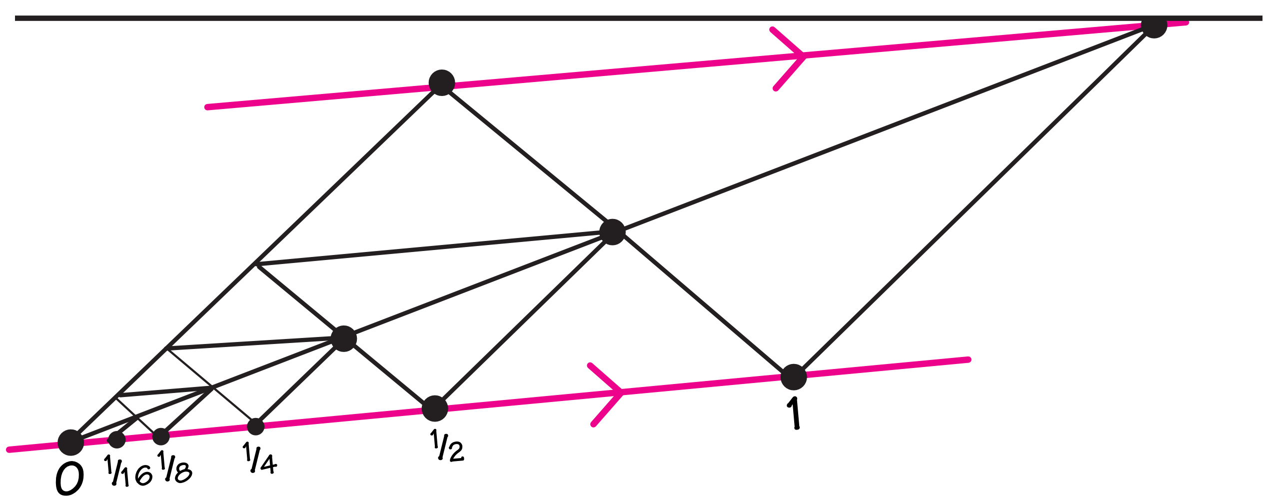
S: ***We get another midpoint.***

Invite a student to do the projection.



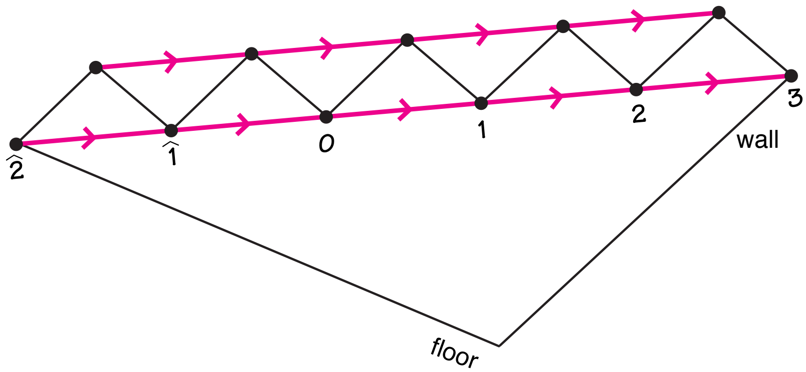
Ask students to locate $\widehat{1\frac{1}{2}}$ and 1½. You may challenge some students to try to locate ¼. In this case students may observe a pattern to the construction and go on to locate ⅛ or 1⁄16. See the drawings below.





Exercise 3

Erase and then redraw the number line by creating a parallelogram train. Add floor and wall lines, as shown here.

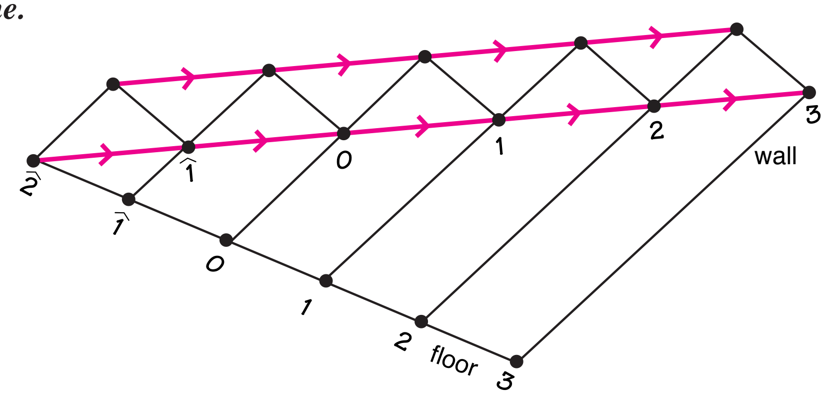


T: ***Let's project each point of this part of the number line onto the floor parallel to the wall.***

Call on students to do the projection.

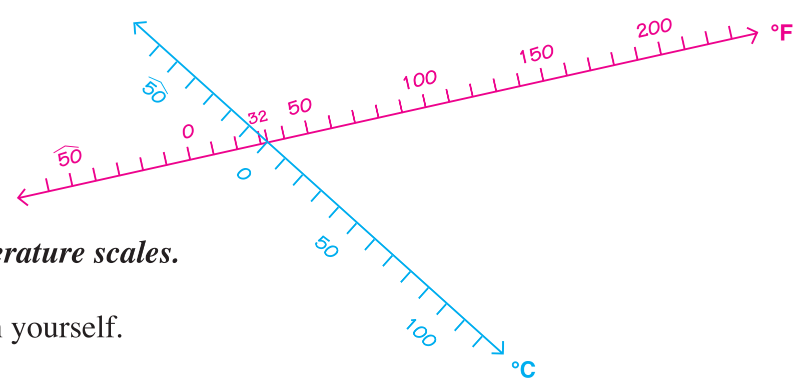
T: ***How could we label these new points?***

S: ***We get a new number line.***



Exercise 4

Display *IG-II Geometry Poster #4.*



T: ***Here are two number lines that intersect at 0 and 32. Do these number lines remind you of anything?***

S: ***Fahrenheit and Celsius temperature scales.***

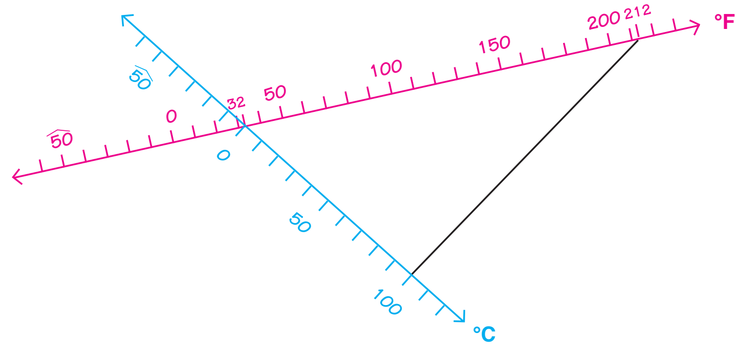
You may need to give this information yourself.

T: ***What can you tell us about these two scales?***

Let students recall information, such as water freezes at 0°C or at 32°F, and water boils at 100°C or at 212°F.

T: ***Since water boils at 212°F or 100°C, let's connect those two points with a segment.***

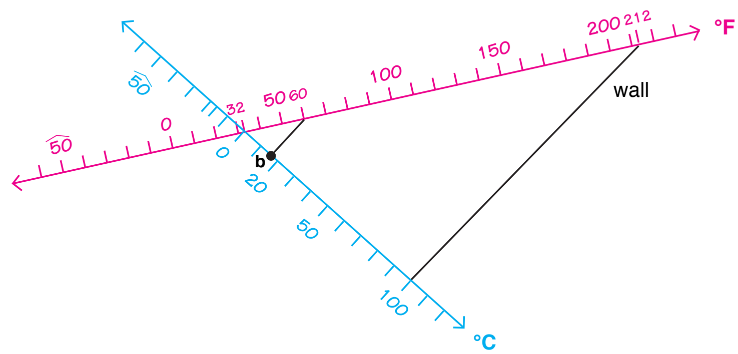
The freezing points of water are connected and so are the boiling points. This morning, I heard on the radio that the outside temperature was 60°F. Who can locate this point?



Invite a student to locate 60 on the °F-line.

T: *How can we use these lines to help us find the Celsius temperature that corresponds to 60°F?*

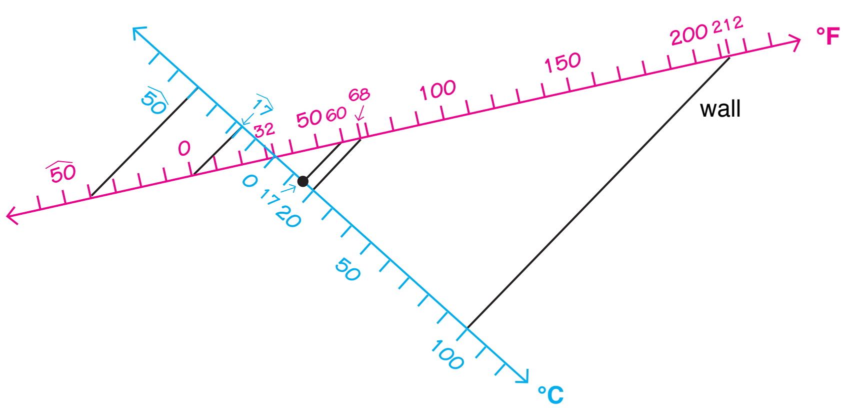
S: *Use the line connecting 212 and 100 for a wall. Then project 60 onto the °C-line (floor).*



T (pointing to **b**)**:** *What is your estimate of this number?*

S: *60°F is about 17°C.*

Ask students to find the Fahrenheit temperature corresponding to 20°C and the Celsius temperature corresponding to 0°F. If time permits, ask about $\widehat{40}$°F. $\widehat{40}$°F is the same temperature as $\widehat{40}$°C.



This poster can be left up and used whenever it is convenient to convert temperatures.

Home Activity

Make a copy of Blackline G10 for students to take home. They can show their parents how to convert °F to °C (geometrically) or vice versa.

Capsule Lesson Summary

Investigate the kinds of shadows that can result from solid and plane figures.

Materials

Teacher
- Collection of solids
- Flashlight
- Overhead projector (optional)
- Screen
- Collection of plane figures

Student
- Collection of solids
- Flashlight
- Unlined paper
- Collection of plane figures
- Pencil or straw

Advance Preparation: Prepare collections of solids such as cubes, cylinders, cones, rectangular and hexagonal prisms, and different pyramids. Prepare collections of plane figures such as rectangles, squares, circles, hexagons, and various triangles. The screen can be a board, a large sheet of white paper, or a projection screen.

Description of Lesson

Begin the lesson with a discussion of shadows. This discussion might include how shadows are made (light source, projection of object onto a plane); what happens to a shadow as an object moves and the light stays still, or vice-versa; play with shadows such as hand "puppet" shadows; and so on.

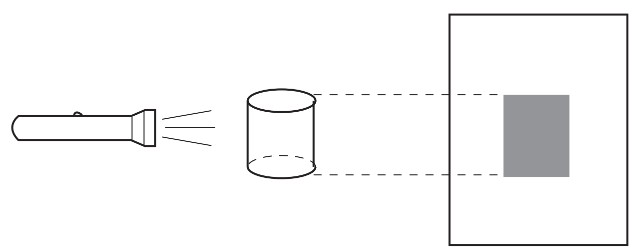
Note: This lesson is a three dimensional experience with parallel projections. Students should recognize shadows as similar to parallel projection images. The activities in this lesson will give students experience recognizing shapes in a variety of positions in space.

Exercise 1

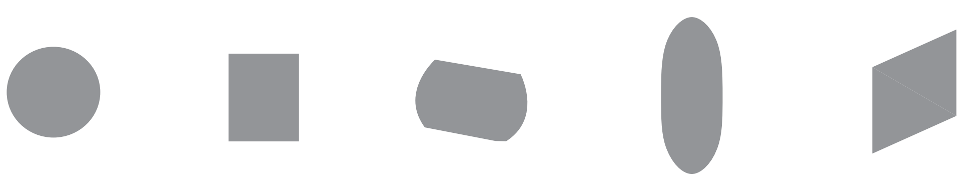
Display a collection of solids, a flashlight or other bright light, and a screen. If available, you may like to use an overhead projector. Hold up one of the solids, for example, a cylinder.

T: ***What shadow shape could we get from this cylinder?***

Let students first make predictions, and then invite several students to show how to get various shadow shapes. As students demonstrate how to get a particular shadow shape, ask them to describe what they did to get that shape and to draw a sketch of the shape on the board.



From a cylinder, for example, they may find they can get some of the following shadow shapes. If one of these shapes is not found by the class, you may sketch it on the board and challenge students to create a shadow that shape.



Note: Students may have difficulty holding the cylinder so that their hands or fingers do not obscure the shape of the shadow. You may need to give assistance.

Organize the class into cooperative groups and provide each group with a collection of solids, a flashlight, and a large piece of unlined white paper. Direct the groups to explore the possible shadow shapes they can get from each of the solids. Suggest that the groups draw pictures or otherwise describe all the different shadow shapes they find for each solid and prepare to present some of their findings to the entire class.

During a collective discussion of the groups' findings, you may like to display a picture of a possible shadow—for example, a rectangle or a hexagon—and ask,

T: ***I drew this shape as the shadow from one of the solids, but I forgot which solid I was using. Can you figure out which solid I could have been using?***

If, for example, your shape is a hexagon, it could be a shadow from a cube or a hexagonal prism. The possibilities here, of course, will depend on the collection of solids available.

Exercise 2

Repeat Exercise 1 with a collection of plane figures such as rectangles, squares, circles, several nonsimilar triangles, and hexagons.

In this case, you may suggest that students tape a cut-out figure to a pencil or a straw to make it easier to hold when casting a shadow.

Home Activity

Suggest that students take home some shadow shapes and ask family members to guess what figures they used to make each shadow.

Capsule Lesson Summary

Review rounding-off to the nearest centimeter and to the nearest millimeter. Ask students to each cut a strip of cardboard of a specified length. Then notice the differences in lengths and discuss which one a person might choose as representative. Make lists of the students' measurements of a line segment and of a zigzag and discuss the variations in each list and between lists.

Materials

Teacher
- Meter stick

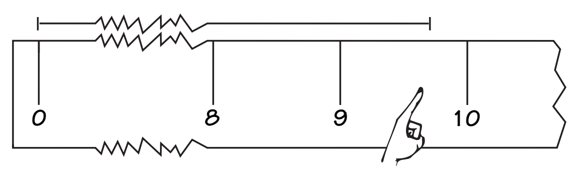
Student
- Cardboard strip
- Metric ruler
- Scissors
- Worksheets G12(a) and (b)

Advance Preparation: Prepare strips of cardboard about 1 cm wide and between 20 and 30 cm long. The cardboard backs of tablets will make strips about 28 cm long.

Description of Lesson

Exercise 1

Draw an enlargement of part of a metric ruler on the board. Use a zigzag to indicate that you are not drawing the section of the ruler between 1 cm and 8 cm. Above the ruler, show a line segment measuring 9.7 cm.



T: ***To the nearest whole centimeter, how long is a line segment with one end at the 0-mark of the ruler and the other end here.***

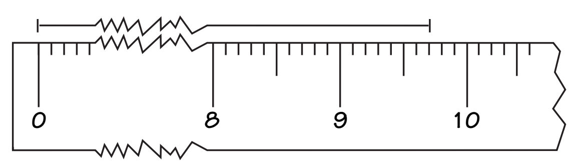
S: ***10 cm; it is closer to 10 cm long than it is to 9 cm.***

T: ***About how much longer is it than 9 cm?***

S: ***7 or 8 millimeters.***

T: ***Let's indicate millimeters in our picture. How many millimeters in 1 centimeter?***

S: ***10 mm.***



T: ***So I'll divide each centimeter in our picture into 10 pieces of the same size.***

How long is the line segment in centimeters?

S: ***9.7 cm.***

T: ***How long is it in millimeters?***

S: ***97 mm.***

T: ***What if the line segment's endpoint were between 9.7 cm and 9.8 cm?***

S: ***It could be 9.75 cm.***

S: ***It could also be 9.73 cm or 9.78 cm.***

T: ***What could we do to our picture?***

S: ***Divide each millimeter into 10 pieces.***

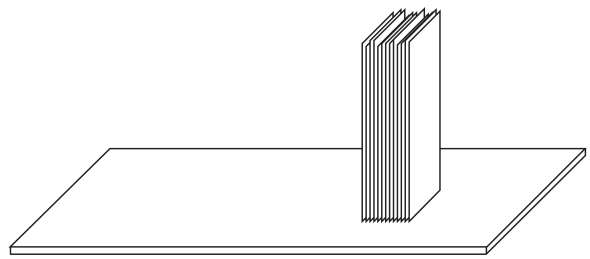
T: ***Can you imagine how tiny those pieces would be? For our purposes, we will measure to the nearest centimeter or to the nearest millimeter. If the end of this line segment were between 9.7 cm and 9.8 cm, what would its length to the nearest millimeter be?***

S: ***97 mm or 98 mm.***

Exercise 2

Give each student a strip of cardboard, scissors, and a metric ruler. Ask students to each cut a piece 16.7 cm† long as accurately as possible. Have extra strips of cardboard available for second tries since you want students to make strips that they really believe are 16.7 cm long. Do not help students with the measurement nor let students help one another. Cut such a strip yourself.

Collect the strips of cardboard from students and include your own. Stack them, but do not arrange them in any order so the differences in lengths will be more dramatic. Tap one end of the stack of cardboard strips against your desk, and then let the students compare their lengths. Walk around the room or invite students to your desk to see that the lengths vary.



T: ***Are there any that are obviously incorrect?***

You may have a few students who do not measure from the 0-mark on the ruler, but measure either from an edge or from the 1-mark and do not correct for it. Errors such as these may be spotted quickly, and those strips can be removed from the pile after discussing what might have been done incorrectly.

T: ***If you needed a piece of cardboard 16.7 cm long, which one of these would you choose? For what reason?***

Students may want you to tell them which is correct. Convey the idea that no one person's measurement can be said to be absolutely accurate, including yours. Measurement involves estimation and error, but we wish to minimize the error.

Some other possible student responses are given below. Mention the terms *mode* and *median* should the opportunity present itself. Average (mean) is less meaningful here because the strips are all supposed to measure 16.7 cm. Balancing lengths to find an (mean) average is not practical nor accurate.

S: ***We could measure them again.***

S: ***Several look like they have the same length. I'd find which length occurs most often.***

T: ***You'd find the mode.***

†Adjust this length if your strips of cardboard are not between 20 and 30 cm long.

S: ***Some are too long and some are too short.***

T: ***Suppose we line them up from shortest to longest; which one would you select?***

S: ***One in the middle.***

T: ***So you'd find the median.***

Mention to the class that even with precise procedures and instruments, two surveyors will not necessarily get exactly the same measurements.

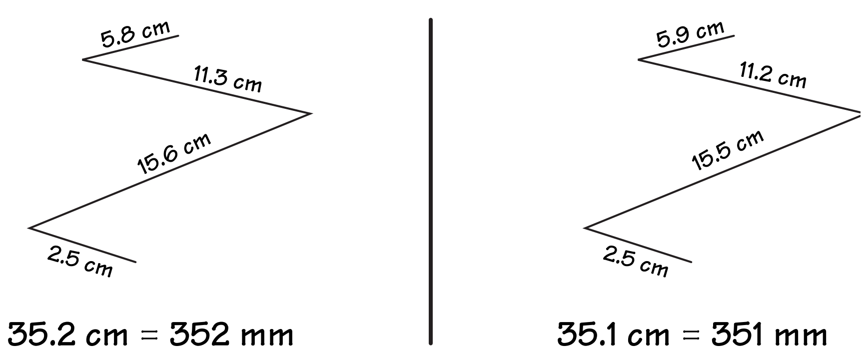
Exercise 3

Distribute copies of Worksheets G12(a) and (b). Ask students to measure the zigzags as accurately as possible. After everyone has measured those on the (a) worksheet, collect results and record them on the board. If a student agrees with the length of a zigzag given by another student, record that length again.

Let the class discuss variations in and between the lists, and what could account for them. There most likely will be more variation in the list for the four-piece zigzag than for the one-piece zigzag. These are some reasons for variations in and between the lists that your students might mention:

- Finding the length of the four-piece zigzag involves adding lengths, and some students may not have added correctly.
- Some students may not be measuring correctly; perhaps they are using the end of the ruler instead of the 0-mark and not correcting for the difference.
- Some students may not be reading the ruler by looking directly down at the place where the end of a segment meets the ruler.
- Rounding-off can account for a difference in two people's measurements of the same line segment.
- Each time a measurement is made, there is a likelihood of some error; the more measurements, the greater possibility of error. (Errors might cancel one another, but on the other hand they may add to one another.)

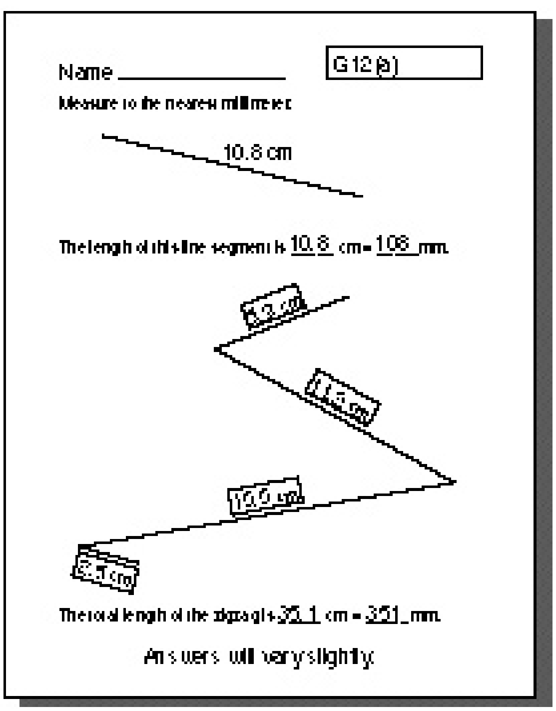
The last two reasons here should be emphasized. Ask two students who have lengths for the four-piece zigzag that differ by 1 or 2 mm to compare their measurements of each of the pieces for the class. For example:

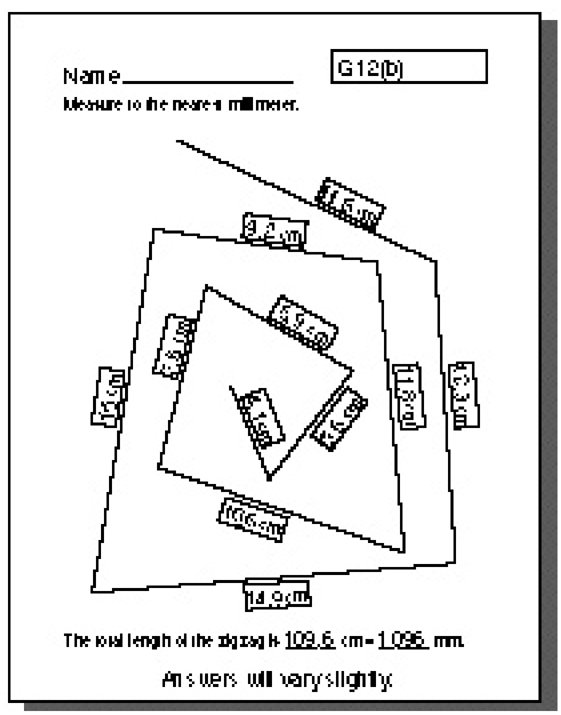


Two of the measurements on the left are more than the measurements on the right of corresponding segments; one is less; and one is the same. (0.1 + 0.1 – 0.1 = 0.1).

Ask students what they think the lengths of the two zigzags are. At no time imply that one can know for sure what the exact lengths are. The class should rule out obvious errors, and you should erase those entries in the lists. For each list, students may suggest taking the (mean) average of the entries, taking the entry that occurs most frequently (the mode), or arranging the entries from shortest to longest and taking the one that occurs in the middle (the median[†]).

If there is time, collect results on the (b) worksheet and report them to the class.





[†]If there are an even number of entries, one finds the median by taking the average of the middle two entries.

Capsule Lesson Summary

Review the concept of a graph, and of paths through connected graphs that cross each dot exactly once. Present the idea of giving weights to the edges of a graph. Then look for shortest paths or round trips that cross each of the dots of the graph once.

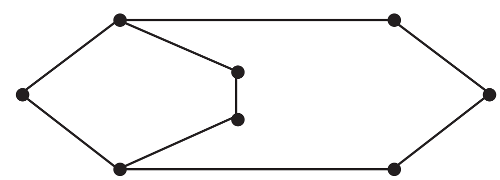
Materials

Teacher		Student	
	• Colored chalk		• Colored pencils, pens, or crayons • Worksheets G13*, **, ***, and ****

Description of Lesson

Exercise 1

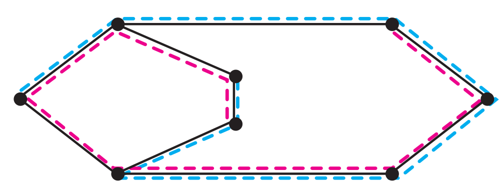
Draw this picture on the board.



Let students discuss what the picture could represent; for example, they may well suggest it is a kind of map with towns, and roads between the towns. Each time a representation is suggested, check the interpretation of dots and edges, counting them. There are eight dots (towns) and nine edges (roads). Point out that there are always eight dots and nine edges no matter what the interpretation (e.g., dots for towns and edges for roads between the towns).

T: ***Let's imagine that this picture is a kind of map of towns and roads between the towns. Can you trace a trip (route) visiting each town exactly once? You may start and end wherever you please.***

Dotted red and blue paths illustrate two different routes.



T: ***How many of the roads do you use in the trip?***

S: ***Seven.***

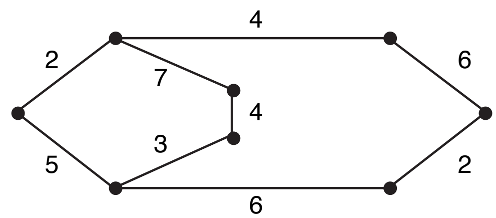
T: ***Now suppose we would like to find a round trip visiting each town exactly once and returning to our starting point. (Such a trip must start and end at the same town and go through each other town exactly once.) Can we find a round trip?*** (No)

Let students experiment and discuss what goes wrong with each attempt. They may observe that a round trip could be found if the left-most dot (town) were omitted or if a new edge (road) were added between, for example, the starting point and the ending point of a route in the above illustration that visits each town exactly once.

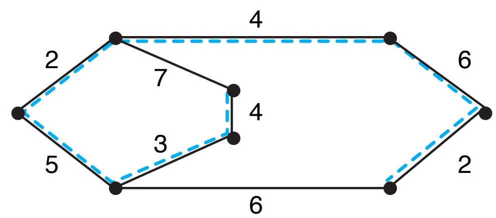
Exercise 2

Add weights to the edges of the graphs from Exercise 1.

T: ***Suppose these are the lengths of the roads between the towns. We cannot find a round trip, but we can find a trip that starts and ends in different towns and that visits each town exactly once. Let's try to find the shortest such trip. Remember that we may start and end wherever we wish, but we must visit each town exactly once—and now we want to try to make our trip as short as possible.***

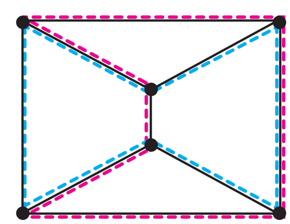


Examine several trips that students suggest. Each time, find the total length by adding the lengths of the roads used. Each time, also look at the lengths of the two roads not used in the trip. Since such a trip always uses seven of the nine roads in the map, there will always be two not used. In this way, students may begin to discover a problem-solving strategy: try to maximize the lengths of the unused roads. The shortest trip is one that does not use roads of lengths 6 and 7.



Exercise 3

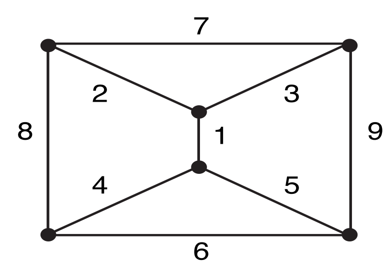
Repeat Exercise 1, using this graph with six dots and nine edges. In this case, it is possible to find round trips visiting each town exactly once and returning to the starting point.



Observe that each such round trip uses six of the nine edges.

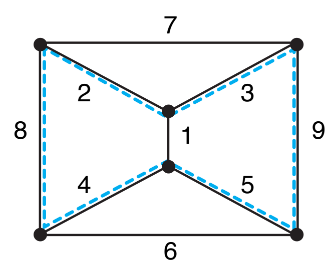
Exercise 4

Give weights to the edges of the graph in Exercise 3 and then repeat Exercise 2 finding shortest round trips.

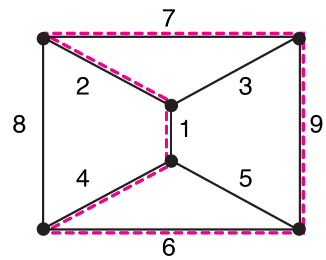


In this case, there is no round trip that omits the three edges of greatest weight. But there are two types of round trips—one that does not use the middle road (length 1) and ones that do use the middle road.

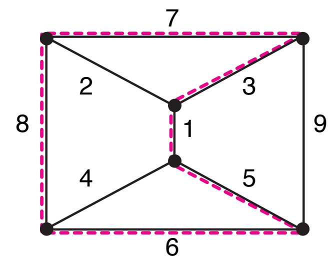
a) A round trip not using the middle road omits the roads of lengths 1, 6, and 7. It has length 31.



b) A round trip using the middle road can omit either the 8-road or the 9-road but not both. Omitting the 8-road forces also the omission of the 3-road and the 5-road. Similarly, omitting the 9-road forces the omission of the 2-road and the 4-road.



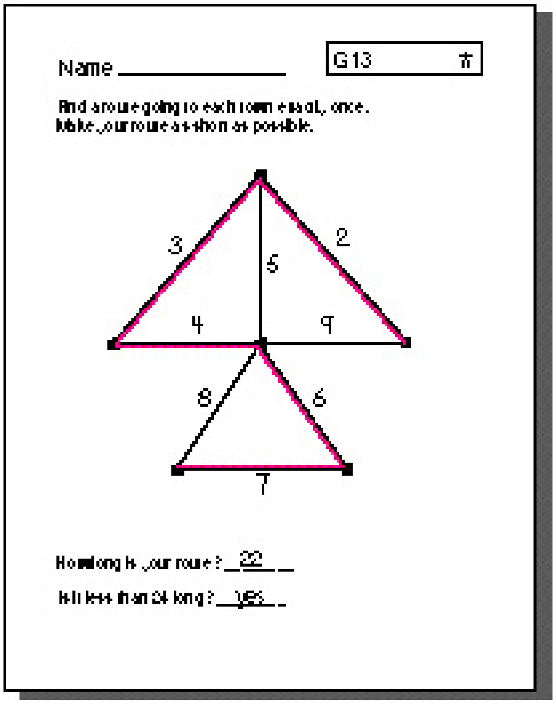
Length: 29



Length: 30

So there are three possible round trips, the shortest having length 29.

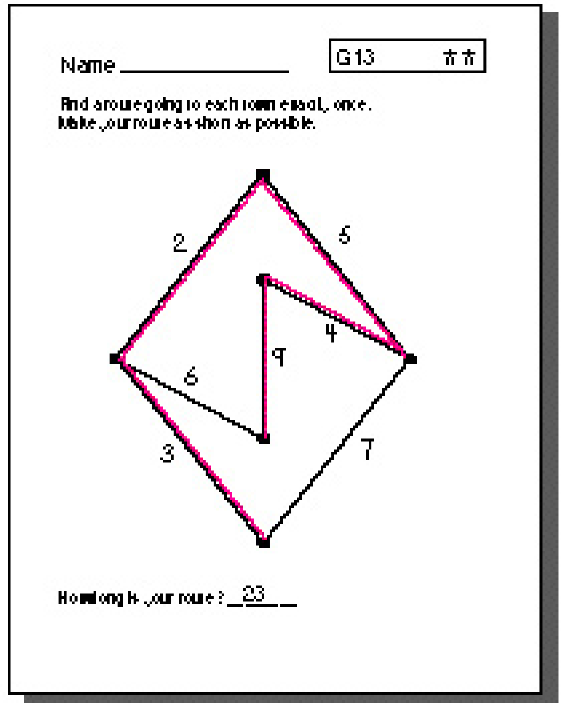
Worksheets G13*, **, ***, and **** are available for individual work.


Name ________ G 13 ☆

Find a route going to each town exactly once.
Make your route as short as possible.

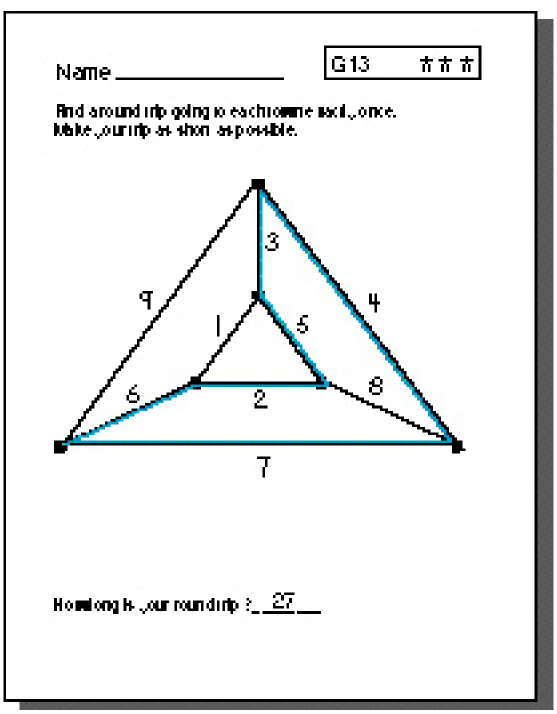
How long is your route? 22

Is it less than 24 long? yes


Name ________ G 13 ☆☆

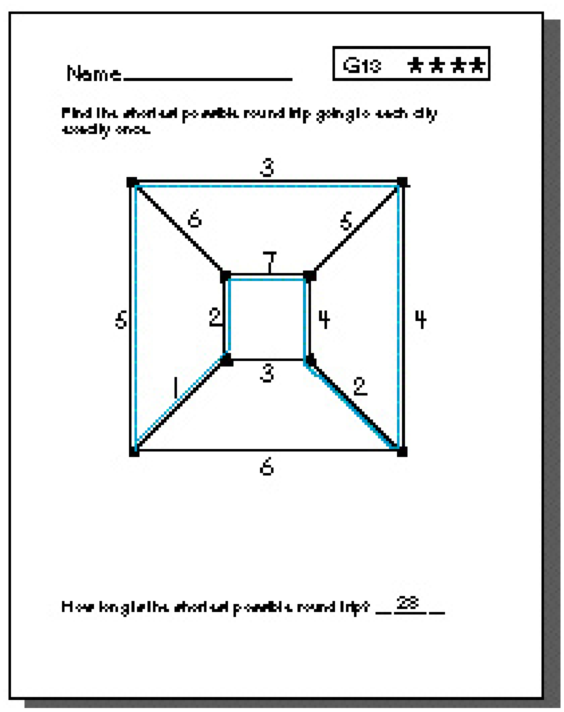
Find a route going to each town exactly once.
Make your route as short as possible.

How long is your route? 23


Name ________ G 13 ☆☆☆

Find a round trip going to each town exactly once.
Make your trip as short as possible.

How long is your round trip? 27


Name ________ G 13 ★★★★

Find the shortest possible round trip going to each city exactly once.

How long is the shortest possible round trip? 28