The Languages of Strings and Arrows

LANGUAGES OF STRINGS & ARROWS TABLE OF CONTENTS

Two fundamental modes of thought for understanding the world around us are the classification of objects into sets and the study of relationships among objects. In everyday life, we classify cars by brand (Ford, Chevrolet, Toyota, and so on) and we study relationships among people (Sally is Mark's sister, Nancy is Mark's cousin). Chemists classify elements by properties, and zoologists study predator-prey relationships. Similarly, mathematicians sort numbers by primeness, and they employ functions to model predicted relationships, for example, between inflation and unemployment.

Many of a child's earliest learning experiences involve attempts to classify and to discern relationships. A child classifies people by roles (the teacher, the doctor), and creates relationships between the smell and taste of foods. Part of language development depends on a child's repeated attempts to sort objects by function, and to relate words with things or events.

The role of sets and relations is so pervasive in mathematics, that perhaps the simplest definition of mathematics is "the study of sets and relations principally involving numbers and geometrical objects." Given the equally pervasive presence of these two notions in everyday life and in a child's experiences, it is natural that they should play a key role in an elementary mathematics curriculum. Yet the inclusion of classification and the study of relations require an appropriate language for representing and studying them. For that reason, *CSMP* develops the non-verbal languages of strings and of arrows.

The pictorial language of strings represents the grouping together of objects into sets. The pictorial language of arrows represents relations among objects of the same or different sets. Each of these languages permeates the different content strands of the *CSMP* curriculum, providing unity both pedagogically and mathematically. With continual use, the languages become versatile student tools for modeling situations, for posing and solving problems, and for investigating mathematical concepts.

The general aim of this strand is to present situations that are inherently interesting and thought provoking, and that involve classification or the analysis of relations. The activities emphasize the role of logical thinking in problem solving rather than the development of specific problem-solving techniques.

Classification: The Language of Strings

As the word implies, classifying means putting things into classes, or as the mathematician says, sets. The mathematics of sets can help students to understand and use the ideas of classification. The basic idea is simple: Given a set S and any object x, either x belongs to S (x is in S) or x does not belong to S (x is not in S). We represent this simple act of sorting—in or out—by using pictures to illustrate in and out in a dramatic way. Objects to be sorted are represented by dots, and the sets into which they are sorted are represented by drawing strings around dots. A dot inside the region delineated by a set's string is for an object in the set, and a dot outside a set's string is for an object not in the set.

This language of strings and dots provides a precise (and nonverbal) way of recording and communicating thoughts about classification. The ability to classify, to reason about classification, and to extract information from a classification are important skills for everyday life, for intellectual activity in general, and for the pursuit and understanding of mathematics in particular. The unique quality of the language of strings is that it provides a nonverbal language that is particularly suited to the mode of thinking involved in classification. It frees young minds to think logically and creatively about classes, and to report their thinking long before they have extensive verbal skills.

Relations: The Language of Arrows

Relations are interesting and important to us in our everyday lives, in our careers, in school, and in scientific pursuits. We are always trying to establish, explore, and understand relations. In mathematics it is the same; to study mathematics means to study the relations among mathematical objects like numbers or geometric shapes. The tools we use to understand the everyday world are useful to understand the world of mathematics. Conversely, the tools we develop to help us think about mathematical things often serve us in non-mathematical situations.

A serious study of anything requires a language for representing the things under investigation. The language of arrows provides an apt language for studying and talking about relations. Arrow diagrams are a handy graphic representation of a relation, somewhat the same way that a blueprint is a handy graphic representation of a house. By means of arrow diagrams, we can represent important facts about a given relation in a simple, suggestive, pictorial way—usually more conveniently than the same information could be presented in words.

The convenience of arrow diagrams has important pedagogical consequences for introducing children to the study of relations in the early grades. A child can read—and also draw—an arrow diagram of a relation long before he or she can read or present the same information in words. The difficulty of presenting certain ideas to children lies not in their intellectual inability to grasp the ideas; rather, the limitations are often mechanical. Arrow diagrams have all the virtues of a good notation: they present information in a clear, natural way; they are attractive, colorful things to look at; they are easy and fun for children to draw. Students may use arrow pictures to study, test, and explain their thinking about concepts or situations under consideration. Discussion about an arrow picture often aids the teacher in clarifying a student's solution or misunderstanding of a problem.

Another educational bonus occurs when an arrow diagram spurs students' curiosity to investigate variations or extensions of the original problem. A minor change in an arrow picture sometimes reinforces a pattern already discussed and at other times suggests new problems to explore.

One of the purposes of this strand is to use the power of the language of arrows to help children think logically about relations. Again it must be remembered that our goals concern the thinking process and not the mechanism. The ability to draw prescribed arrows is not the objective in itself, nor is viewing an arrow diagram just another format for drill problems in arithmetic.

The general aim of the Languages of Strings and Arrows strand is to suggest situations that are inherently interesting and thought provoking, and to give children modes of thinking and appropriate languages with which they can organize, classify, and analyze. In addition to a varied assortment of lessons concerning sets and relation, this strand includes lessons involving systematic methods for [illegible] problems.

Content Overview

The String Game

This semester, students play *The String Game* with numbers frequently, independently developing strategies as they gradually become more familiar with the numerical concepts. Regularly playing the game for fun, without forced analysis, respects individual rates of learning and maintains student enthusiasm. Yet at this stage, many students begin to play more carefully, considering the possible attributes for each set. This shift in behavior indicates a desire and ability to begin to analyze the game situations more formally. Therefore, several activities involve discussing the deductions that can be made from the original clues in a string game. These analyses lead to a consideration of plays in the game that are "safe" versus plays that yield additional information. A valuable effect of the analysis is that students begin to realize that even plays in a wrong region often allow useful deductions.

In order to better understand the numerical concepts involved in the game, students explore number theory concepts involving divisors and multiples. For example, they search for numbers with exactly three divisors or for numbers with exactly four odd divisors. These problems challenge students and support work in the World of Numbers strand, as well as improve student's skill in playing *The String Game*.

Lessons: L1, 3, 5, 7, 9, and 10

Logical Thinking

The role of the Language of Strings in the *CSMP* curriculum is not limited to string games. Several lessons in this strand use string pictures as a vehicle to present problems that stimulate logical thinking. For example, given a set of objects, students must determine the truth or falsity of statements about the set. The statements place particular emphasis on the precise meaning of terms such as *every*, *at least*, *at most*, and *none*. Disagreements often lead to lively student discussions that result in a clearer understanding of the terms involved. Such lessons encourage reasoning and clear explanations on the part of students, and they draw upon the students' numerical and non-numerical knowledge.

Lessons: L4 and 8

Combinatorics

Two lesson in this strand involve students in generating systematic methods of counting. This experience leads to solving combinatorics problems or problems that involve counting all possible outcomes in a situation. For example:

> An ice cream store offers 31 flavors of ice cream.
> How many different choices of double dip cones are possible?

A direct answer might be to list all combinations. But listing presents two difficulties: inefficiency and the possibility of missing some combinations. A primary goal in the study of combinatorics is to develop efficient, systematic techniques for counting. The lessons lead students to use pictorial or graphic techniques in these problems such as tree diagrams and arrow roads.

THE LANGUAGES OF STRINGS & ARROWS INTRODUCTION

Combinatorics is a basic element of probability as well as a source of challenging recreational mathematics problems. Several lesson in the Probability and Statistics strand further apply the techniques developed in these lessons.

Lessons: L2 and 6

L1 STRING GAME WITH NUMBERS #1

Capsule Lesson Summary

Investigate numbers with exactly four (three, two, or five) positive divisors. Find a sequence of numbers all of which have 1 and 5 as their only odd positive divisors. Find another sequence of numbers all of which have 3 as their only prime divisor. Play *The String Game* with numbers.

Materials

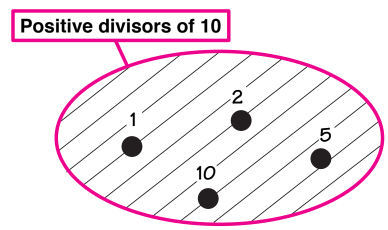
Teacher
- Colored chalk
- Numerical String Game kit

Student
- Paper

Description of Lesson

Exercise 1

Ask the class to name the positive divisors of 10; put them in a string picture on the board as they are given.



Note: The hatching in this picture indicates that nothing else is inside this string. You may comment that this use of hatching is to show that 10 has exactly four positive divisors. Remove the dot labels and 10 from the string label as you ask,

T: ***Are there other numbers like 10 with exactly four positive divisors?***

S: ***6 (positive divisors are 1, 2 , 3, and 6).***

S: ***8 (positive divisors are 1, 2, 4, and 8).***

S: ***15 (positive divisors are 1, 3, 5, and 15).***

Check each number that is suggested and list solutions on the board.

Note: For your information, there are two types of numbers with exactly four positive divisors:

- numbers that are products of two distinct prime numbers, such as 10 (2 x 5), 6 (2 x 3), 15 (3 x 5), and 77 (7 x 11); and
- numbers that are cubes of prime numbers, such as 8 (2 x 2 x 2), 27 (3 x 3 x 3), and 125 (5 x 5 x 5).

Students may begin to see some patterns, but do not expect a complete characterization of the numbers with exactly four positive divisors at this time. Leave the problem when you have a list of five or six possibilities.

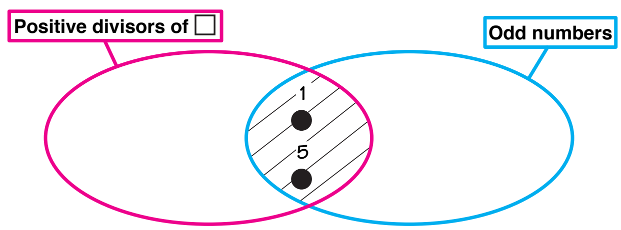
Repeat this exercise for numbers with exactly three, then two, and finally five positive divisors. Each time, alter the string picture and allow students to find several solutions before continuing. Check answers in the string picture and make lists on the board.

Note: For your information, a characterization of each type of number is shown below. Do not expect the class to find such characterizations, although they may make some pattern observations, and they should recognize numbers with exactly two positive divisors as primes.

I		II	III	IV
Numbers with exactly four positive divisors (products of two distinct primes or cubes of prime numbers)		Numbers with exactly three positive divisors (squares of prime numbers)	Numbers with exactly two positive divisors (prime numbers)	Numbers with exactly five positive divisors (fourth powers of prime numbers)
6	8	4	2	16
10	27	9	3	81
15	125	25	5	625
21	⋮	49	7	⋮
22		121	11	
77		⋮	⋮	
⋮				

Exercise 2

Draw this string picture on the board.



T: *Let's find some numbers that could be in the box. What does the picture tell us?*

S: *1 and 5 are positive divisors of the number in the box.*

S: *Only multiples of 5 could go in the box because 5 is a positive divisor.*

S: *The hatching means that 1 and 5 are the only numbers in the middle.*

S: *A number in the box can have other positive divisors but no other odd positive divisor.*

T: *What are some possibilities for a number to put in the box?*

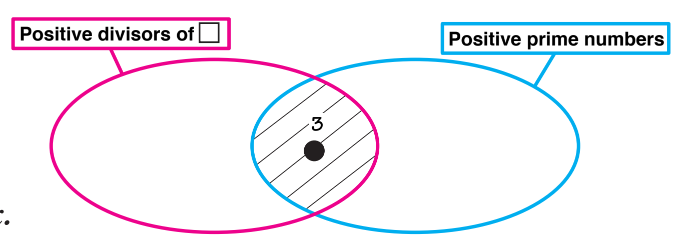
S: *5 (only positive divisors are 1 and 5).*

S: *10 (positive divisors are 1, 2, 5, and 10).*

S: *20 (positive divisors are 1, 2, 4, 5, 10, and 20).*

As you check possibilities with the picture, students should begin to make other observations, such as the only odd number for the box is 5. After generating a list of five to ten possibilities, they may be able to describe a pattern; for example, the possibilities in numerical order start with 5 and double each time to find the next number (5, 10, 20, 40, 80, 160, …).

Repeat the exercise with this problem.



S: *The number in the box is a multiple of 3.*

S: *The hatching means 3 is the only prime divisor of the number in the box.*

Students may be concerned about where 1 belongs in the picture. Ask someone to locate 1 (inside the red string but outside the blue string). Emphasize that 1 is a divisor of any number, but 1 is not a prime number.

S: ***3 could be in the box (only positive divisors are 1 and 3).***

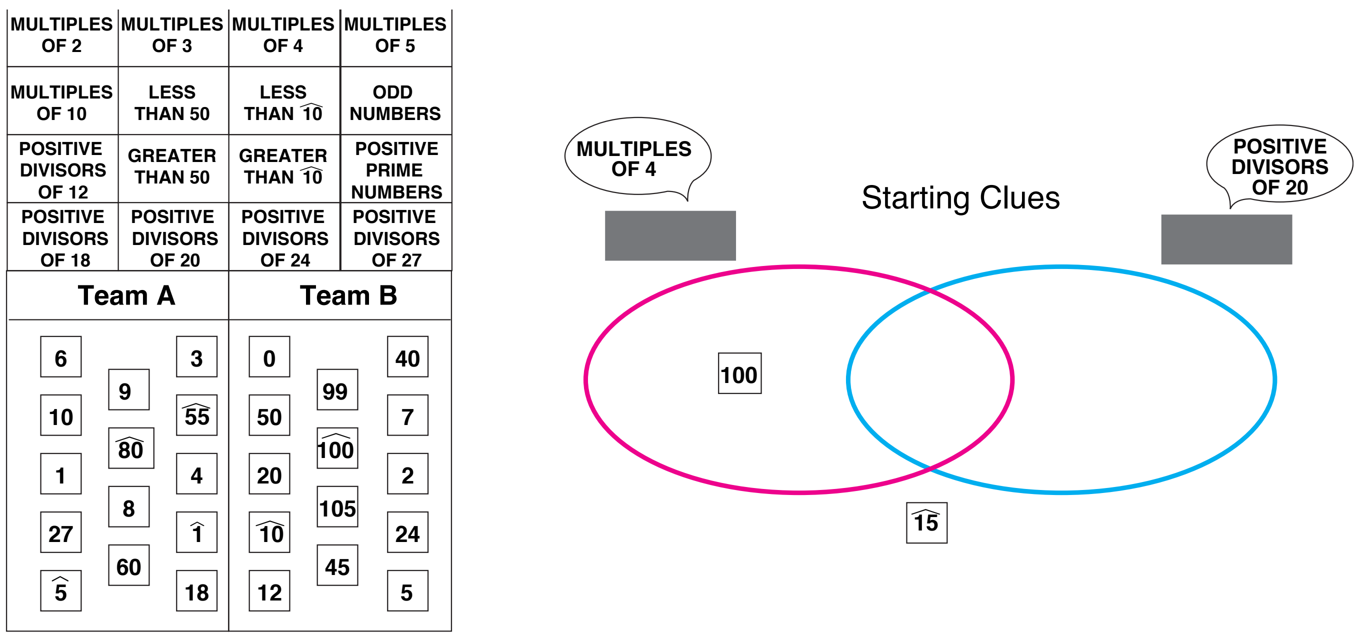
S: ***9 (only positive divisors are 1, 3, and 9).***

S: ***27 (only positive divisors are 1, 3, 9, and 27).***

Check suggested possibilities with the picture. Soon students should make other observations, such as an even number cannot be in the box (otherwise 2 would be a divisor and 2 is a prime number). They may observe a pattern in the list of possibilities, especially if it is presented in numerical order, i.e., start the list with 3 and multiply by 3 each time to find the next number (3, 9, 27, 81, …).

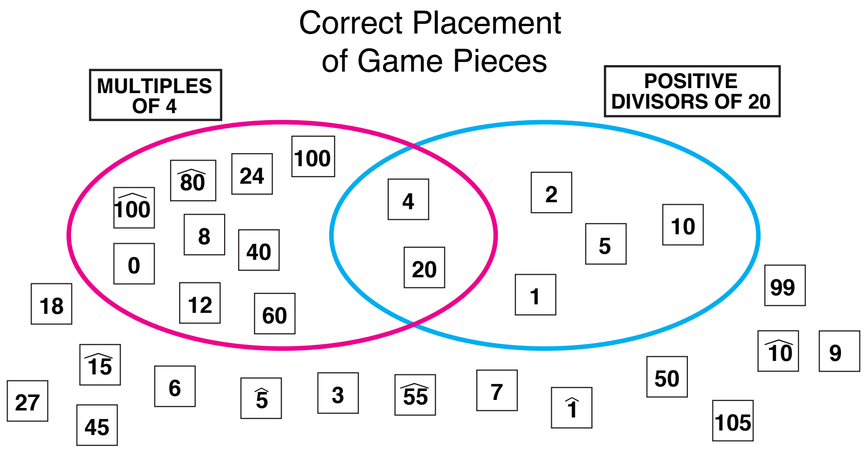
Exercise 3

Prepare to play *The String Game* with numbers by setting up a team board and taping the Numerical String Game poster above it. Divide the class into teams and distribute the game pieces on the team board. The illustration below shows a possible game with one piece from each side of the team board placed correctly as starting clues.



Note: If you decide to play the game with three or more teams, create a team board with sections for more teams.

The following illustration shows correct placement of all 30 numbers and may be used by you as a crib sheet during the play of the game.



Writing Activity

You may like students to take lesson notes on some, most, or even all their math lessons. The "Lesson Notes" section in the Notes to the Teacher gives suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students may note several facts and patterns about numbers with a specific number of divisors, about numbers that have a particular prime number as a divisor, or about numbers with particular odd numbers as divisors.

Home Activity

Suggest that parents work with their child to make a list of numbers that have exactly three or exactly six positive divisors.

L2 HOW MANY DESIGNS?

Capsule Lesson Summary

Given specific restrictions, let students draw circles, squares, and triangles to create a simple geometric design. Construct a tree to count the number of different designs that can be drawn. Use another tree to assist in building all of the red-blue arrow roads with exactly three arrows.

Materials

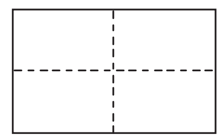
Teacher
- Colored chalk

Student
- Unlined paper
- Colored pencils, pens, or crayons
- Worksheets L2*, **, and ***

Description of Lesson

Exercise 1

Each student needs a piece of plain white paper. Show students how to fold their papers into fourths to get creases as indicated here.



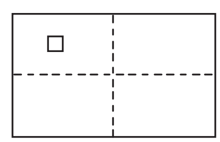
T: ***Today we are going to draw some simple geometric designs using squares, circles, and triangles. You may draw one design in each quarter of your paper, but the designs must follow rules.***

Write the following information on the board.

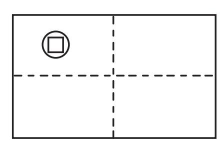
First: □ or ○ or △
Second: Different from first
Third: □ or ○ or △

Model by drawing a sample design while you explain the rules.

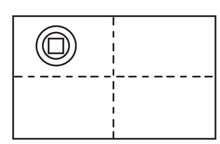
T: ***A design first has a small square, circle, or triangle in its middle. I'll begin a design with a square.***



Second, it has a bigger shape around the first shape. This second shape can be a square, circle, or triangle, but it must be different from the first shape. In my design I'll draw a circle.

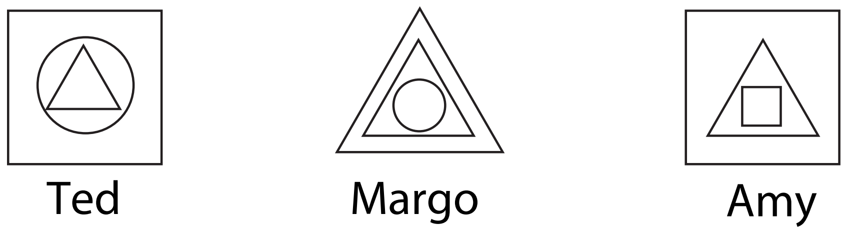


Third, it has a large square, circle, or triangle around its first two shapes. This time we can choose any of the three shapes again. I'll draw a circle again .



Direct students to draw different designs in the four regions of their papers. Emphasize that for each design, they must follow the same three rules. You may like to let students work with partners so that a pair of students might generate eight different designs.

As you observe students working, invite some of them to draw their favorite designs on the board. Ask them to sign their names by their designs.



Also, draw a couple designs that do not follow the rules. For example:

Continue a class discussion when most students have finished their designs.

T: ***Ted's, Margo's, and Amy's designs are all good, but these last two designs are not. What is wrong?***

S: ***The one on the left is not finished; there are only two shapes. Draw a square (or a circle or a triangle) on the outside.***

S: ***A design cannot have three circles—the second shape cannot be the same as the first. We could make the second (middle) shape a square (or triangle) rather than a circle.***

Complete, change, or erase the incorrect designs.

T: ***How many different designs do you think we could draw following the rules?***

Record students' estimates on the board.

T: ***How could we count the total number of different designs?***

Accept students' suggestions, especially those that use systematic techniques. Remind students of the use of trees in Lesson L7 *How Many Disguises?* from *IG-I*.

T: ***Do you remember the story about the detective with many disguises?***

Let students explain how the detective disguises himself with hair pieces, mustaches, and beards.

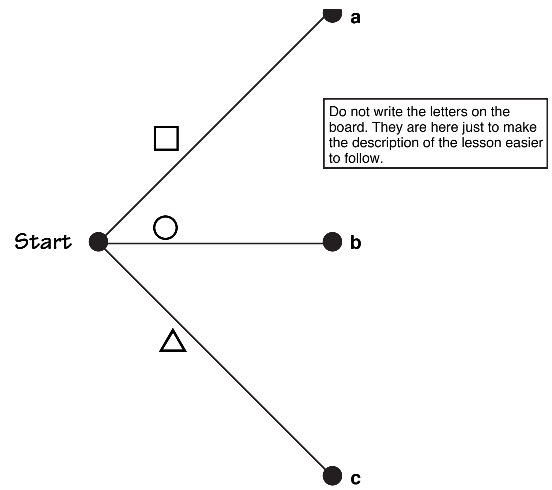
T: ***What problem did we solve in that story? How did we solve it?***

S: ***We drew a tree to help count the number of different disguises Spike could wear.***

T: ***Let's use a tree to count the number of different designs we could draw. Our first choice is whether to draw a square, a circle, or a triangle.***

Begin this tree on the board.

Refer to the designs on the board, and trace appropriate branches in the tree.



T: ***In my design the first shape is a square, so it is here*** (point to **a**)***.***
How does Ted's design fit in the tree?

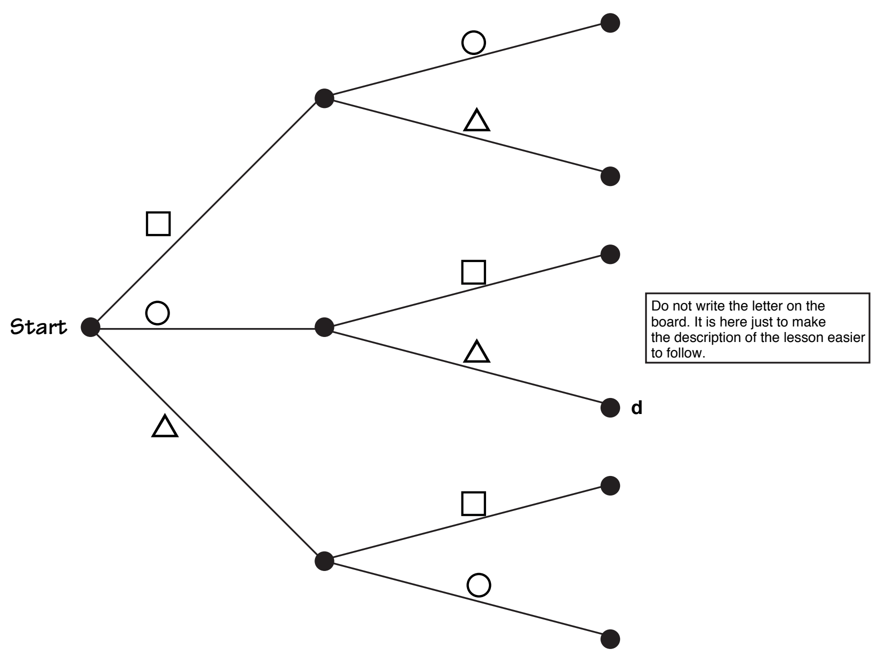
S: ***The first shape is a triangle, so it is here*** (at **c**)***.***

Check other designs in a similar manner.

T: ***For the second shape, you must draw a shape different from the first. How can I show these choices in the tree?***

S: ***Here*** (at **a**) ***the first shape was a □ so the second shape could be a circle or a triangle. Draw two branches from there—one for ○ and one for △.***

Follow student suggestions to extend the tree. Choose some designs to see where they fit now in the tree.



Point to the ending point of a second branch in the tree, for example at **d**, and ask,

T: ***What do we know about a design here?***

S: ***It could be Margo's.***

S: ***It has a circle first, then second a triangle.***

Emphasize that at this point you know the first two shapes in a design.

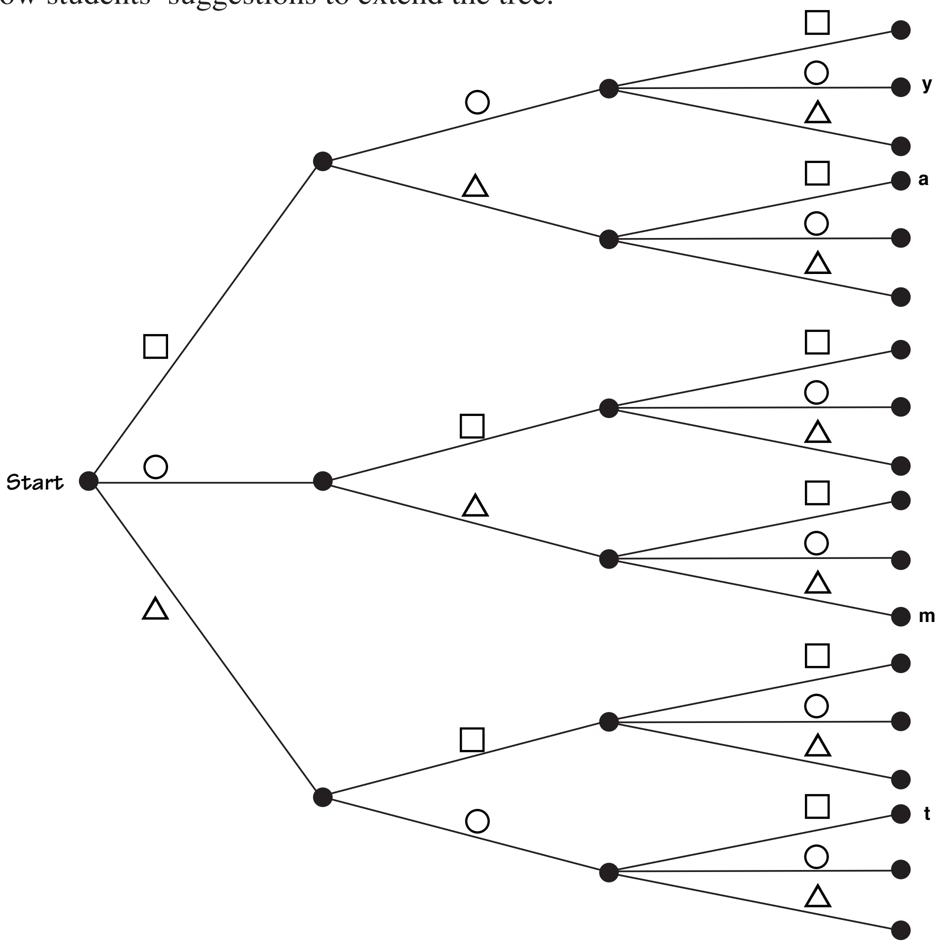
T: ***Do any of you have a design that could fit here?***

Check a few examples.

T: ***How do you complete the tree to show the choices for the third shape?***

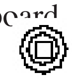
S: ***The third shape could be a square, a circle, or a triangle. So at each point draw three branches—one for □, one for ○, and one for △.***

Follow students' suggestions to extend the tree.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

On the board, locate the designs in the tree. For example:

-  is at **y**.
- Ted's is at **t**.
- Margo's is at **m**.
- Amy's is at **a**.

Ask other students to locate one of their designs in the tree, and let the class check. You can also point to an ending dot and ask if anyone has a design that fits there. If not, invite someone to draw such a design.

As students respond to the preceding questions, occasionally invite a students to trace the appropriate path from the starting dot along the three branches to an ending dot. Emphasize that each path from the starting dot to an ending dot represents three choices of shapes and therefore one complete design.

T: ***How many different designs can we draw following our rules?***

S: ***18 designs. There are 18 dots on the right. Each path from the starting dot to one of them represents a complete design.***

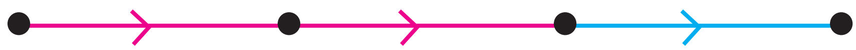
If students suggest counting all of the dots in the tree, explain that the dots in the middle of the tree refer to incomplete designs; that is, only to the first shape, or to the first and second shapes.

Compare the number of possible designs (18) to the students' predictions. Erase the board before going on to Exercise 2.

Exercise 2

Explain to the class that you have another counting problem.

T: ***This time we want to find all of the arrow roads with exactly three arrows, red or blue. In a road you can use all red arrows, all blue arrows, or some of each. I'll draw one such road on the board as an example.***



How many roads with exactly three arrows (red or blue) do you think we could draw?

Accept and record students' estimates.

T: ***Let's try to draw all of these roads. Tell me another road I could draw.***

Draw the roads as students describe them.

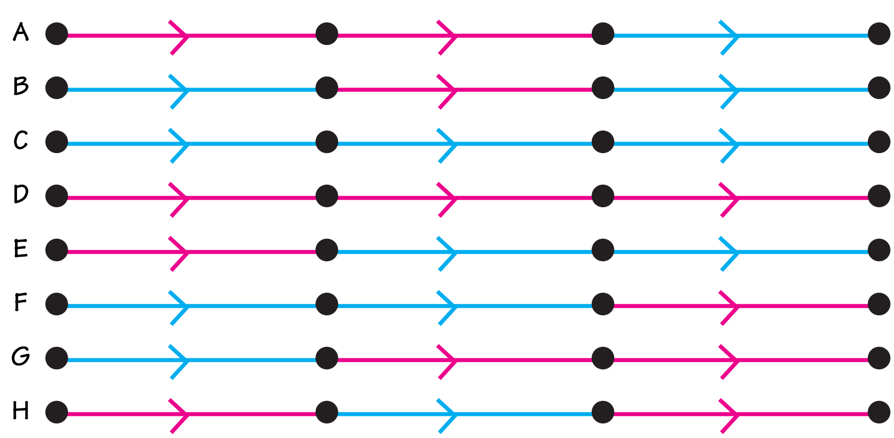
It is unlikely, and not necessary, that students will find all eight roads. If they find fewer than eight roads, adapt the following dialogue. The procedure described will reveal any missing roads.

T: ***We've found eight different roads, but how can we be sure that there are not other possibilities?***

Encourage students to comment.

T: ***Drawing a tree could reveal whether we found all of the roads. First I'll give a name to each road on the board.***

Label the roads **A** to **H** from top to bottom. In your class, of course, the correspondence between letters and roads may be different from that given here. Adapt the dialogue accordingly.



Begin a tree as you say,

T: ***Let's start our tree here. In building a road what is the first choice we make?***

S: ***To draw a red arrow or to draw a blue arrow.***

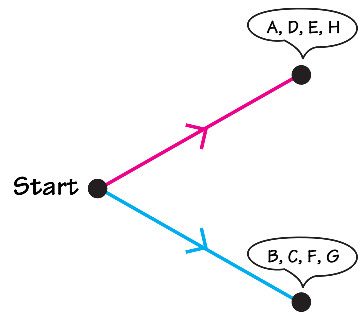
T: ***How should I start the tree?***

S: ***Draw two arrows from the starting dot, one red and one blue.***

T (pointing to the end of the red arrow)**:** ***Which of the eight roads could we be building at this time?***

S: **A**, **D**, **E**, *and* **H**. ***Each of these roads has a red arrow first.***

S: ***Roads*** **B**, **C**, **F**, ***and*** **G** ***all have a blue arrow first.***



T: ***What are our second choices as we build an arrow road?***

S: ***We can draw either a red arrow or a blue arrow for the second arrow on the road.***

T: ***How do we show that on the tree?***

S: ***Draw one red arrow and one blue arrow from each of the two ending dots.***

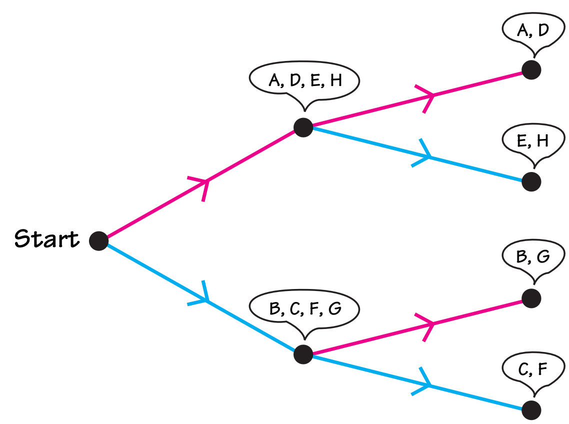
Extend the tree according to students' suggestions.

T (pointing to the top-most dot)**:** ***Which roads could we be building at this point?***

S: **A** ***and*** **D*****; each of those roads starts with two red arrows.***

T: ***Where could we be building the*** **B** ***road?***

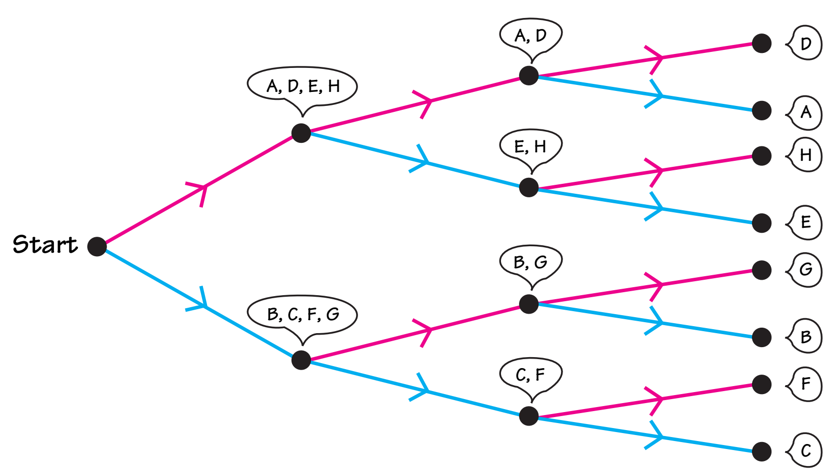
S: ***Here (third dot down), because road*** **B** ***starts with a blue arrow followed by a red arrow.***



Continue until all eight roads are accounted for.

In a similar manner, let students explain how to complete the tree.

- Draw one red arrow and one blue arrow from each ending dot.
- Identify the roads by their letter names.



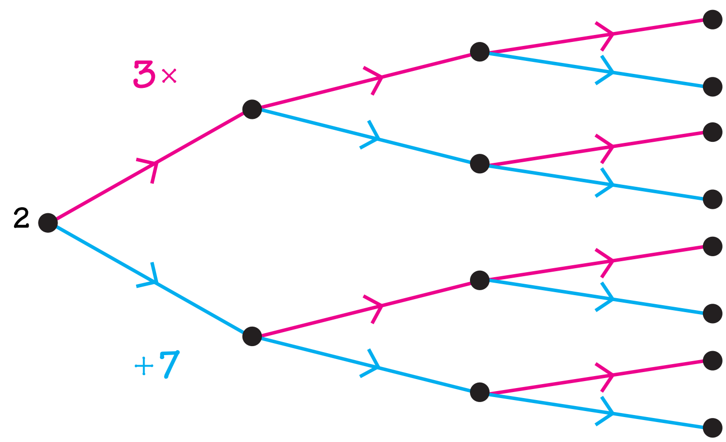
T: ***How many different arrow roads with exactly three arrows, red or blue, are there?***

S: ***Eight, there are eight ending dots on the tree. Therefore, there are eight distinct three-arrow roads from the starting dot.***

T: ***Did we find all eight arrow roads?***

If your students did not find all eight roads, use the tree to determine which roads were missed. In either case, point out the usefulness of the tree in listing all the possibilities.

Erase the letters from the arrow picture. Label the starting dot and key the arrows, as shown below.



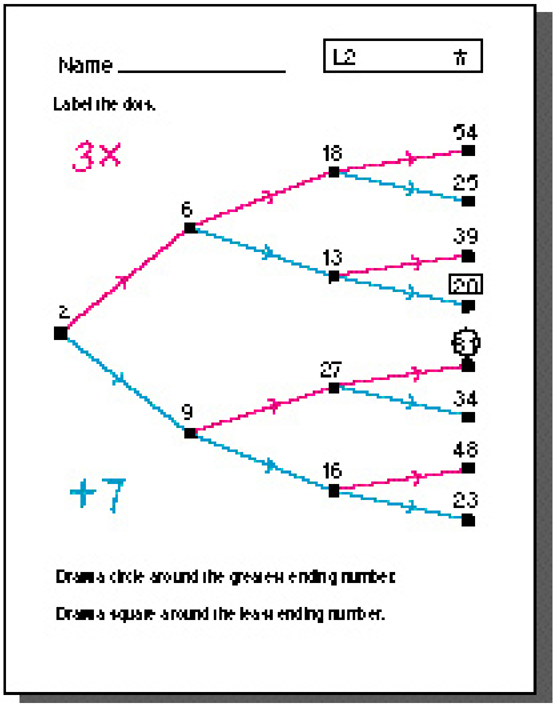
T: ***Before doing any calculations, can you predict which ending dot is for the greatest number? For the least number?***

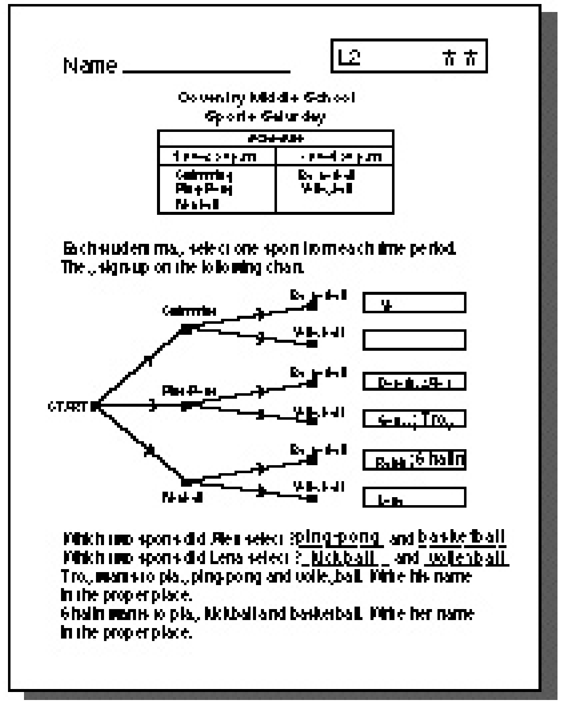
Let several students respond and explain their answers if they wish. Refer to Worksheet L2* as you announce,

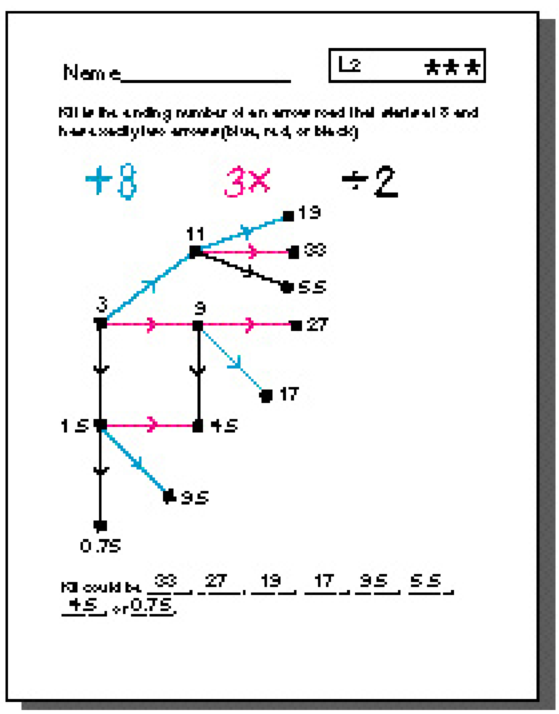
T: ***The picture on this worksheet is the same as the one on the board. Label the dots.***

Worksheets L2*, **, and *** are available for individual work. Suggest that students check their predictions for the locations of the greatest number and the least number.

L2


Name
L2
Label the dots.
3x
+7
2
6
9
18
13
27
16
54
26
39
20
61
34
48
23
Draw a circle around the greatest ending number.
Draw a square around the least ending number.


Name
L2
START
Troy
Shalin
ping-pong
basketball
kickball
volleyball
in the proper place.
in the proper place.


Name
L2
(blue, red, or black)
+8
3x
÷2
19
11
33
5.5
3
9
27
17
1.5
4.5
9.5
0.75
33
27
19
17
9.5
5.5
4.5
or
0.75

Capsule Lesson Summary

Investigate relationships in the respective lists of positive divisors for 10, 20, 50, and 100. Review the notion of a prime number and list the primes less than 30. Find a sequence of numbers all of which have exactly two multiples of 3 as positive divisors. Find another sequence of numbers all of which have exactly two positive divisors less than 5. Play *The String Game* with numbers.

Materials

Teacher	• Numerical String Game kit • Colored chalk	Student	• Unlined paper • Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

On their papers, ask students to list the positive divisors of 10, 20, 50, and 100 in turn. Then solicit the lists from the class in numerical order, and encourage the observation that the positive divisors of 10 are all positive divisors of 20, 50, and 100 as well. Further, the positive divisors of 20 and 50 are also positive divisors of 100.

Positive divisors of 10 : 1, 2, 5, 10
Positive divisors of 20 : 1, 2, 5, 10, 4, 20
Positive divisors of 50 : 1, 2, 5, 10, 25, 50
Positive divisors of 100: 1, 2, 5, 10, 4, 20, 25, 50, 100

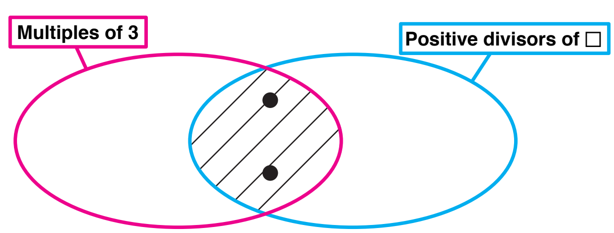
T: ***What are the prime divisors of these numbers?***

S: ***2 and 5.***

Ask students to recall that a prime number is one with exactly two positive divisors, 1 and itself. Call on students to generate a list of the positive prime numbers less than 30 (2, 3, 5, 7, 11, 13, 17, 19, 23, and 29).

Exercise 2

Draw this string picture on the board.



T: ***Let's find some numbers that could be in the box. What does the picture tell us?***

S: ***The hatching tells us that there are exactly two numbers in the middle.***

S: ***The numbers we are looking for have exactly two multiples of 3 as positive divisors.***

At this point, students may make observations based on further analysis. Accept them without trying to get full class awareness. Examples of numbers for the box will suggest other restrictions.

T: ***What are some possibilities for a number to put in the box?***

S: ***6 (positive divisors are 1, 2, 3, and 6).***

S: ***9 (positive divisors are 1, 3, and 9).***

S: ***33 (positive divisors are 1, 3, 11, and 33).***

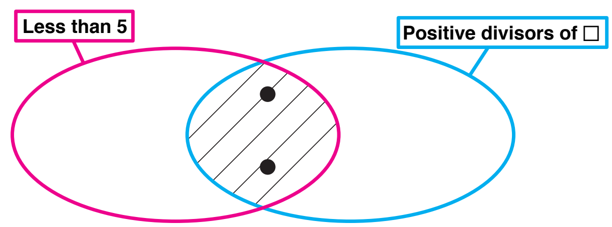
After checking several possibilities, encourage additional observations about the picture.

S: ***The number in the box must be a multiple of 3.***

S: ***One of the numbers in the middle is 3, because 3 must be a positive divisor of any number that has a multiple of 3 as a positive divisor.***

Note: For your information, the number in the box could be 3x any prime number. Make a list of five to ten possibilities. Students may not be able to verbalize a characterization, but they may be able to observe patterns enough to generate many possibilities.

Repeat this exercise with this similar problem.



S: ***The numbers that could be in the box have exactly two positive divisors less than 5.***

S: ***1 is in the middle region.***

S: ***2 (positive divisors are 1 and 2, both less than 5).***

S: ***9 (positive divisors are 1, 3, and 9).***

S: ***10 (positive divisors are 1, 2, 5, and 10).***

S: ***21 (positive divisors are 1, 3, 7, and 21).***

Again students may make observations based on further analysis. Let the class become aware of other restrictions on the number in the box through looking at examples and checking them with the picture. Allow students to comment as they find new ways to describe the possible numbers or as they observe patterns. Do not expect a complete characterization.

Note: For your information, the number in the box could be:

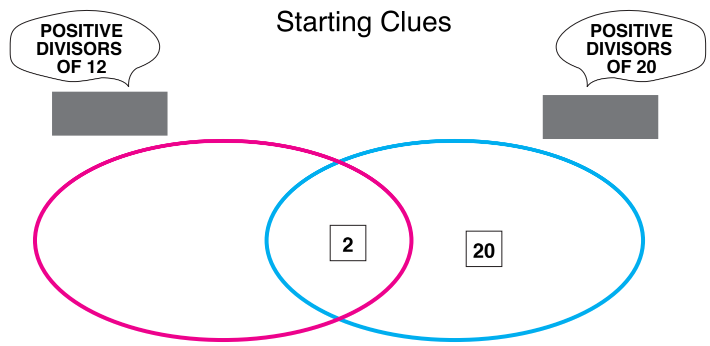
- 2 or 3
- 2x a prime number greater than 3
- 3x a prime number greater than 2

Exercise 3

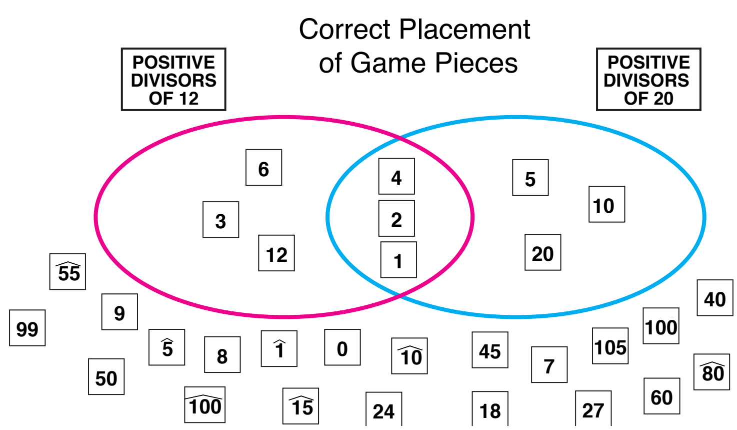
Prepare to play *The String Game* with numbers. The illustration below shows a possible game with two pieces placed correctly as starting clues.

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN 10	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN 10	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27

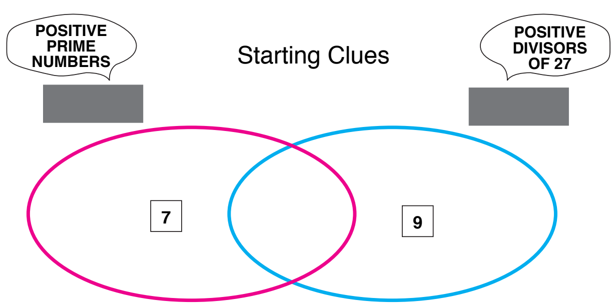
Team A			Team B		
6	9	3	0	99	40
10	80	55	50	100	7
1	8	4	20	105	2
27	60	1	10	45	24
5		18	12		5

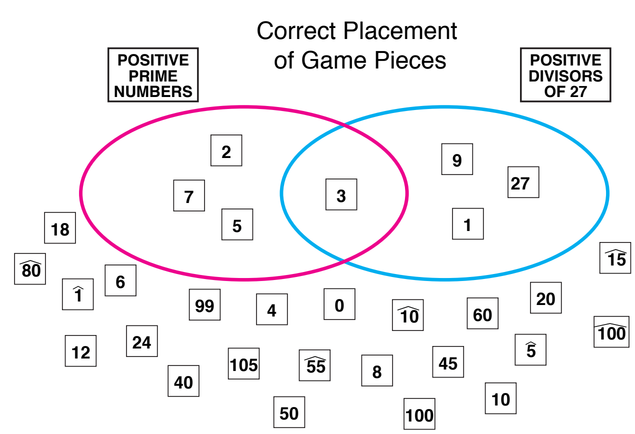


The following illustration shows correct placement of all 30 numbers and may be used by you as a crib sheet during the play of the game.



If time permits, play a second game such as the one illustrated below.





Home Activity

L3

Send home a definition of prime numbers. Ask parents/guardians to list the following with their child:

- the prime numbers less than 50
- the prime numbers between 120 and 140

Capsule Lesson Summary

Use string pictures to provide a context for examining statements to find if they are true, false, or indeterminate. In these statements, practice using terms of comparison: *at most*, *exactly,* and *at least*. Draw or modify pictures to reflect given statements' truth or falsity.

Materials

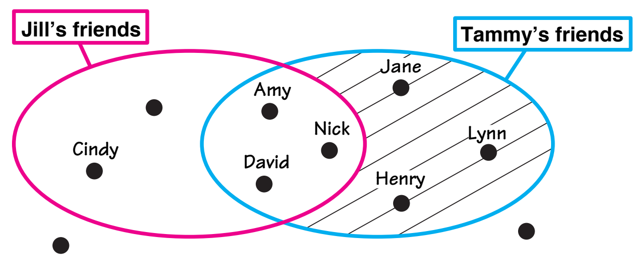
Teacher
- Colored chalk

Student
- Unlined paper
- Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

Draw this string picture on the board and ask the class to make observations. You may receive a variety of comments, for example:



S: ***Amy is both Jill's friend and Tammy's friend.***

S: ***Jill and Tammy share three friends.***

S: ***Tammy has exactly three friends who are not friends of Jill's.***

S: ***Some dots do not have names.***

S: ***Jill has at least five friends.***

Encourage students to make accurate statements, and let the class discuss statements about which there is some disagreement.

T: ***Now I will make some statements. You tell me whether a statement is true or false or whether you cannot tell. You will need to explain your answers. My first statement is this: Tammy has exactly six friends.***

S: ***False. There could be more dots in the middle, and they would be for more of Tammy's friends.***

S: ***Yes, but really we can't tell. More dots could be in the middle, but we don't know for sure that there are more.***

T: ***Very good. You can't tell about this statement. Could you change my statement so that it is true?***

S: ***Tammy has at least six friends.***

Someone might suggest "Tammy has at most six friends," but this is also a statement the truth of which one cannot decide from the picture. As necessary, review the meanings of *at least* and *at most*.

L4

Note: The phrases *at most* and *at least* are difficult. Do not worry if some students are still having problems with them at this time; remind students of their meanings often. These phrases occur regularly in many contexts throughout *CSMP Mathematics for the Intermediate Grades* and will eventually become a natural and useful part of your students' vocabulary.

In the following dialogue, the left side has a variety of other statements you can use for this exercise. On the right are correct responses. Your students may at first make quick incorrect responses; encourage them to think before responding and to discuss each other's answers.

T: ***Jill has at least four friends.***

S: ***True. Jill has more than four friends.***

T: ***Cindy is a friend of both Jill's***
not

S: ***False. Cindy is a friend of Jill's but is***
and Tammy's. ***a friend of Tammy's.***

T: ***Jill has at most one friend***
who is not a friend of Tammy's.

S: ***False. There are two dots for friends of***
Jill's who are not friends of Tammy's.

T: ***Jill and Tammy have at most***
or
more than three friends in common.
be in the middle, but

S: ***Cannot tell. They could have three***
three friends in common.
More dots could
we are not sure.

T: ***Tammy has exactly three friends***

S: ***True. The hatching means there are***
who are not friends of Jill's. ***exactly***

three dots in that part of the
picture.

T: ***Brad is a friend of Tammy's but***
is not a friend of Jill's.

S: ***False. Jane, Henry, and Lynn are the***
only friends of Tammy's who are not
friends of Jill's.

T: ***Brad is a friend of Jill's but is not***
a friend of Tammy's.

S: ***Cannot tell. Brad could be, but we don't***
know for sure where he is in the
picture.

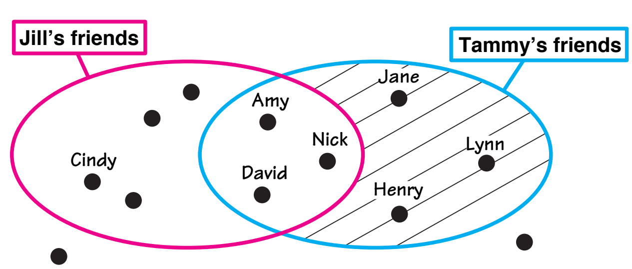
Encourage students to discuss the possibilities for Brad's location in the picture.

You can change the exercise here and ask students for some statements of particular types. Be specific. Ask for a true statement, a false statement, and a "cannot tell" statement. For any suggested statements, ask the rest of the class to determine whether it is true, false, or "cannot tell." Then continue.

T: ***Jill has more friends than Tammy has.***

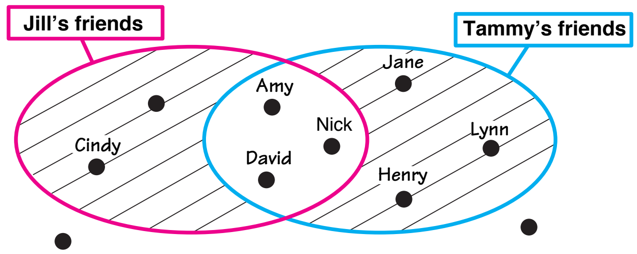
S: ***Cannot tell. Jill could have more***
friends, but we are not sure.

Ask students to change the picture so that the last statement is true. Let students experiment and discuss the problem. They should discover that more dots must be added to the left region inside the red string, as illustrated here.



With additional dots placed in the left region, discuss why Jill certainly has more friends than Tammy. (Any other friends of Tammy must be in the center and so also are friends of Jill.) Some students might want to do some more hatching. This would be correct, but not necessary.

You could also discuss whether the original picture could be changed to assure that Tammy has more friends than Jill. One solution is to hatch the left part inside the red string; it has just two dots in it.



Return to the original picture.

T: ***Now I will tell you a true statement. Sam is a friend of Tammy's. Can you locate Sam in the picture?***

The volunteer should draw a dot to locate Sam in the middle region.

T: ***What else do we know about Sam?***

S: ***Sam is also a friend of Jill's.***

T: ***Here is another true statement. Sally is a friend of Jill's. Can you locate Sally?***

Let students discuss the situation. They should decide that they cannot be sure where to locate Sally. She could be a friend of both Jill's and Tammy's and hence be in the middle, or she could be a friend of Jill's but not a friend of Tammy's and hence be in the left region inside the red string.

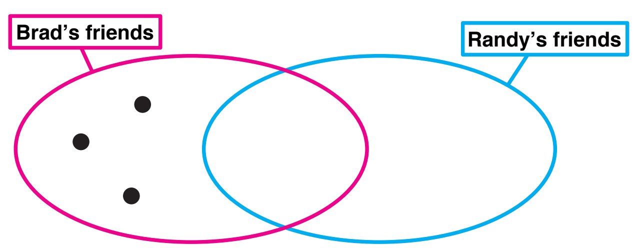
Exercise 2

On the board, draw two overlapping strings, one for Brad's friend and one for Randy's friends.

T: ***Let me tell you about Brad's and Randy's friends. Then you help me fix the picture so that certain statements are true. Here is the first one: Brad has at least two friends.***

Call on a volunteer to put dots in the picture; the placement of two or more dots anywhere inside the red string will make this statement true.

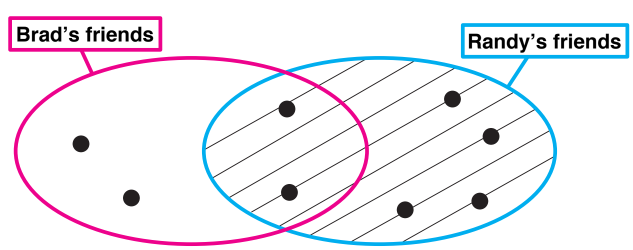
Suppose three dots are added to represent Brad's friends as shown here.



T: ***Randy has exactly six friends and at least two of them are also friends of Brad's.***

Note: If your students put more than six dots inside the blue string, some dots will need to be erased in order to make this statement true.

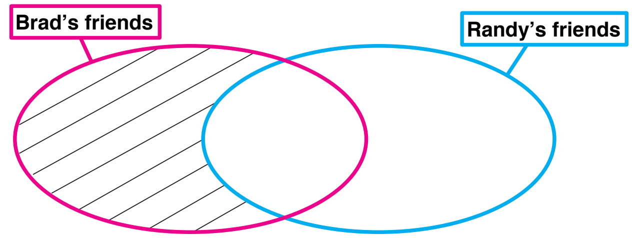
Let students experiment and discuss the problem. One solution is shown here.



Return to the original picture (with no dots).

T: ***Let's fix the picture so that this statement is true: Every friend of Brad's is also a friend of Randy's.***

Again, let students experiment and discuss each other's attempts. You may get several correct solutions. The one that gives the minimum information to make the above statement true is shown here.



T: ***Can we fix the picture so that it is true that Brad has at least two friends?***

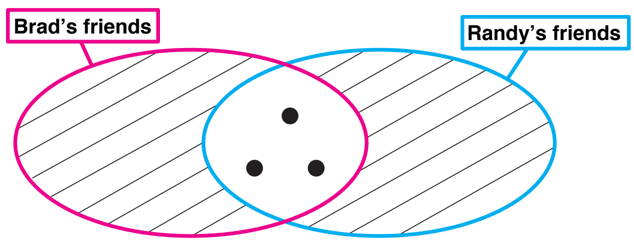
Students should decide that two or more dots should be in the middle region.

T: ***What other true statement can we make now?***

S: ***Randy has at least two friends.***

T: ***Fix the picture so that this statement is false: Randy has a friend who is not a friend of Brad's.***

After some discussion, your class should conclude that this means that the statement "Every friend of Randy's is also a friend of Brad's" must be true. Do not insist that this restatement be made precisely, but rather help the class discover that the right region inside the blue string should be hatched.



Perhaps students will notice that the picture now shows that Brad and Randy have the same friends.

Exercise 3

Students will need paper and colored pencils for this exercise. Write these statements on the board.

Mark has at least five friends.

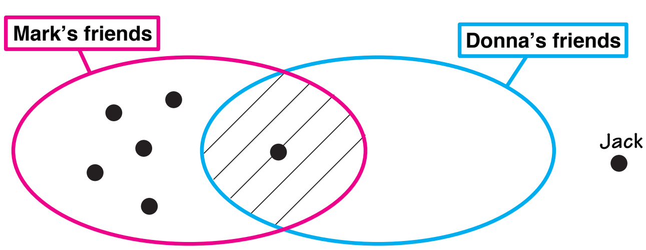
Mark and Donna have exactly one friend in common.

Jack is not a friend of Mark's and is not a friend of Donna's.

Instruct students to draw a string picture so that all of these statements are true. To help students get started, draw the labeled strings on the board.

Ask students who finish quickly to write some other true statements, false statements, and "cannot tell" statements about their pictures.

When many students have a picture, invite several students to draw their solutions on the board. Here is one example:



Extension/Home Activity

L5 STRING GAME WITH NUMBERS #3

Capsule Lesson Summary

Find pairs of numbers whose positive divisors satisfy the conditions imposed by given string pictures. Play *The String Game* with numbers.

Materials

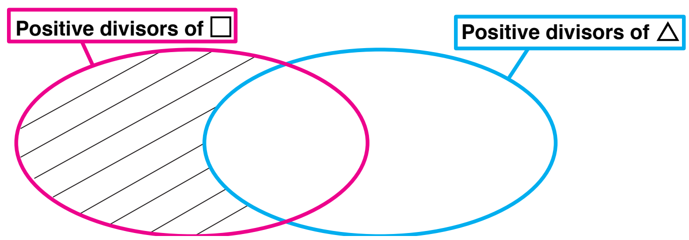
Teacher
- Numerical String Game kit
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

Draw this string picture on the board.



T: ***The problem here is to find two numbers, one for the box and the other for the triangle. What information does the picture give us?***

S: ***All of the positive divisors of a number for the box are also positive divisors of a number for the triangle.***

S: ***The number in the triangle can have more positive divisors than the number in the box.***

T: ***What are some possibilities for the numbers in the box and the triangle?***

Check each suggestion with the picture and record them in a two-column table.

□	△
6	12
10	100
2	4
2	28
4	28
10	20
15	35
15	45
⋮	⋮

S: ***6 (positive divisors are 1, 2, 3, and 6) for the box and 12 (positive divisors are 1, 2, 3, 4, 6, and 12) for the triangle.***

S: ***10 (positive divisors are 1, 2, 5, and 10) for the box and 100 (positive divisors are 1, 2, 4, 5, 10, 20, 25, 50, and 100) for the triangle.***

After finding five to ten possibilities, students may comment on patterns.

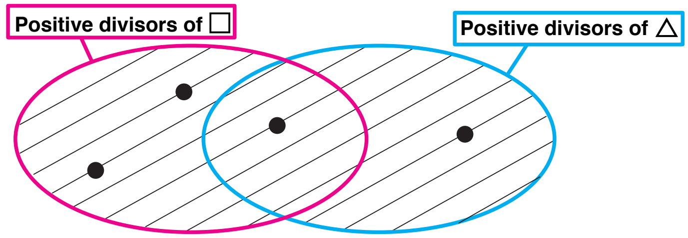
S: ***The number in the triangle is a multiple of the number in the box.***

S: ***The number in the box is a divisor of the number in the triangle.***

S: ***The numbers in the box and the triangle could be the same; then all of the positive divisors would be in the middle.***

Exercise 2

Erase the possibilities found for □ and △ in Exercise 1 and change the string picture as shown here.



T: *I've changed the problem. Again we need to find two numbers, one for the box and the other for the triangle. What information does the picture give us now?*

S: *The number in the box has exactly three positive divisors, and the number in the triangle has exactly two positive divisors.*

S: *1 is in the middle, because 1 is a positive divisor of every number.*

S: *The number in the triangle is prime, because prime numbers have exactly two positive divisors.*

T: *What are some possibilities for the numbers in the box and the triangle?*

Check each suggestion with the picture and make a table to record the possibilities. In this problem, students should find that the number in the box must be the square of a prime number (for example, 4, 9, 25, 49, 121, …) and that the number in the triangle is a prime number. The only further restriction is that when a certain prime number is given for the triangle, the number for the box cannot be the square of that prime number.

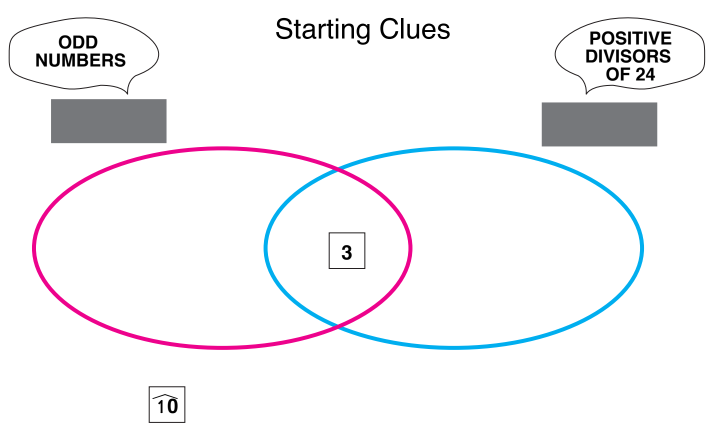
□	△
~~4~~, 9, 25, 49, …	2
4, ~~9~~, 25, 49, …	3
4, 9, ~~25~~, 49, …	5
4, 9, 25, ~~49~~, …	7
	⋮

Exercise 3

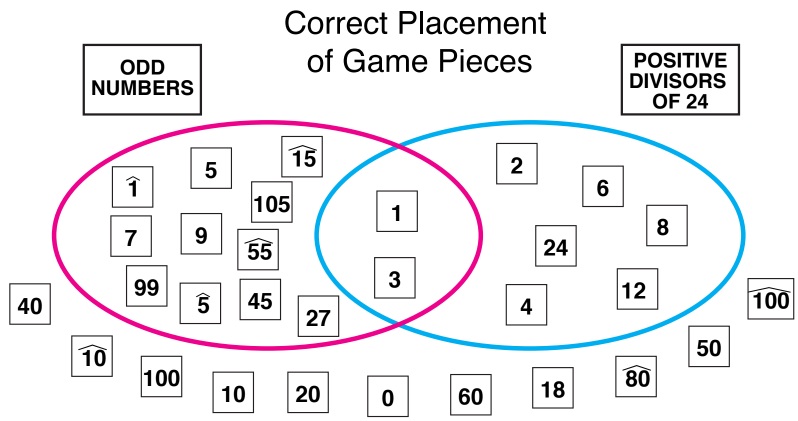
Prepare to play *The String Game* with numbers. The illustration below shows a possible game with two pieces placed correctly as starting clues.

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN 10	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN 10	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27

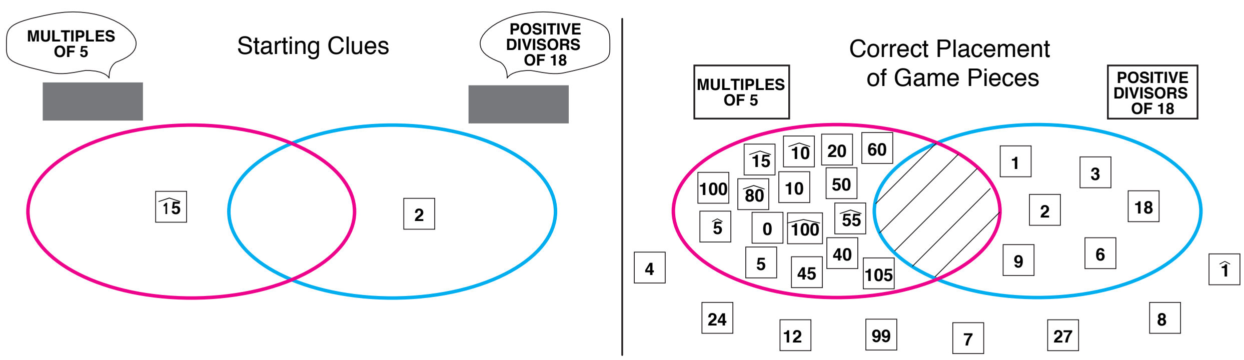
Team A	Team B
6, 10, 1, 27, 5, 9, 80, 8, 60, 3, 55, 4, 1, 18	0, 50, 20, 10, 12, 99, 100, 105, 45, 40, 7, 2, 24, 5



The following illustration shows correct placement of all 30 numbers and may be used by you as a crib sheet during the play of the game.



If time permits, play a second game such as the one illustrated below.



Capsule Lesson Summary

Find the number of ways (via one-way roads) to go between certain cities visiting intermediate cities. Begin to recognize combinatorial strategies for solving such problems.

Materials

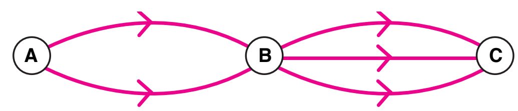
Teacher
- Colored chalk

Student
- Paper
- Worksheets L6*, **, ***, and ****

Description of Lesson

Exercise 1

Draw this picture on the board.



T: ***This is a map showing some one-way roads between cities* A*,* B*, and* C*. Who can trace a way to go from city* A *to city* C *through city* B*?***

Let several students trace paths from **A** to **C**.

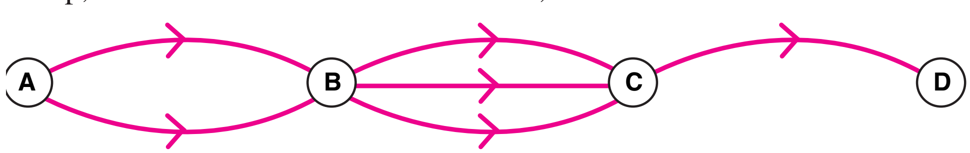
T: ***How many ways are there to go from* A *to* C *using these one-way roads?***

Get estimates from several students and then ask,

T: ***Can someone trace all the ways and organize them so we easily can tell that none was missed?***

One method is to consider each road from **A** to **B** separately and trace all the ways from **A** to **C** using that road. Another method is to consider each road from **B** to **C** separately, and to trace all of the ways to go from **A** to **C** using that road from **B** to **C**. With either method there are six possibilities, two groups of three or three groups of two. You may suggest the first method by tracing the upper road from **A** to **B** and asking how many ways there are to finish the trip to **C**. (Three) Do the same with the lower road from **A** to **B**.

On the arrow map, add a road from **C** to another town, **D**.



T: ***Who can show a way to go from* A *to* D*?***

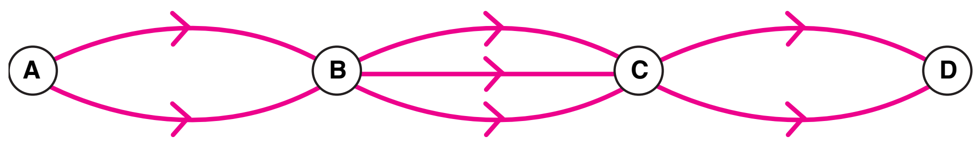
Let several students trace paths; then ask,

T: ***Who can trace all of the ways to go from A to D and organize them so we are sure that none was missed?***

Since there is only one road from **C** to **D**, the systematic tracing of paths from **A** to **C** can be easily extended to a systematic tracing of paths from **A** to **D**. Trace paths in the same order, but add traveling the road from **C** to **D** to each.

S: ***There are still six ways.***

Draw another arrow from **C** to **D**.



T: ***Now how many ways are there to go from A to D?***

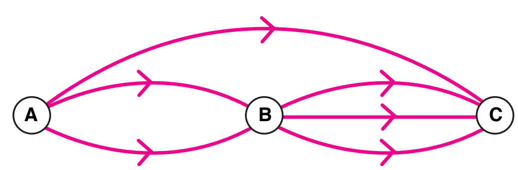
Accept estimates but do not comment.

Let one student start a path going from **A** to **C**, and then invite another student to finish by going to **D**. Students should see that there are two ways to finish any path. Repeat several times and then ask,

T: ***How many ways are there to go from A to C?*** (Six ways)
For each way to go from A to C, there are two ways to continue to D.
Six 2s are …? (12)
So there are 12 ways to go from A to D.

Perhaps a student will observe that you can multiply the numbers of roads from **A** to **B**, from **B** to **C**, and from **C** to **D** to get 12 ways: 2 x 3 x 2 = 12. Let students discover this during the lesson rather than pointing it out yourself.

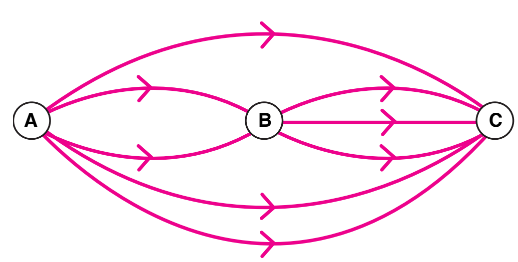
Erase the arrows from **C** to **D**, and add one arrow from **A** to **C**.



T: ***Now how many ways are there to go from A to C?***

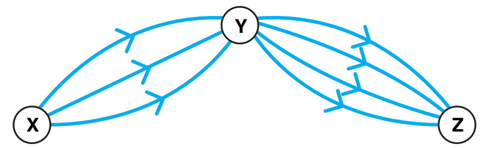
S: ***One more than before; seven ways.***

Draw two more arrows from **A** to **C**. After each is drawn, ask how many ways there are to go from **A** to **C**. (Eight and nine, respectively)



Exercise 2

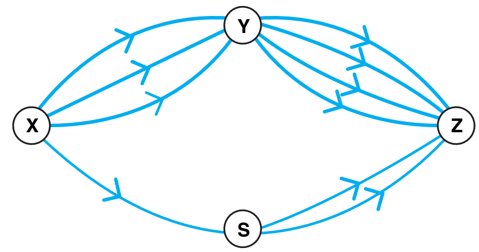
Erase the board and draw a new arrow map with cities **X**, **Y**, and **Z**.



T: ***How many ways are there to go from* X *to* Y *going through* Y*? Can you trace all of them?***

Encourage students to note that once you go from **X** to **Y**, there are four ways to complete the trip to **Z**. Since there are three ways to go from **X** to **Y**, there are 12 (4 + 4 + 4 = 3 x 4 = 12) ways to go from **X** to **Z**.

Add city **S** to your picture and some one-way roads from **X** to **S** and **S** to **Z**.



T: ***To go from* X *to* Z*, there is another city we can go through instead of* Y*, city* S*.***
How many ways are there to go from* X *to* Z *through* S*?

S: ***Two ways.***

Invite a student to trace them.

T: ***So how many ways are there altogether to go from* X *to* Z*?***

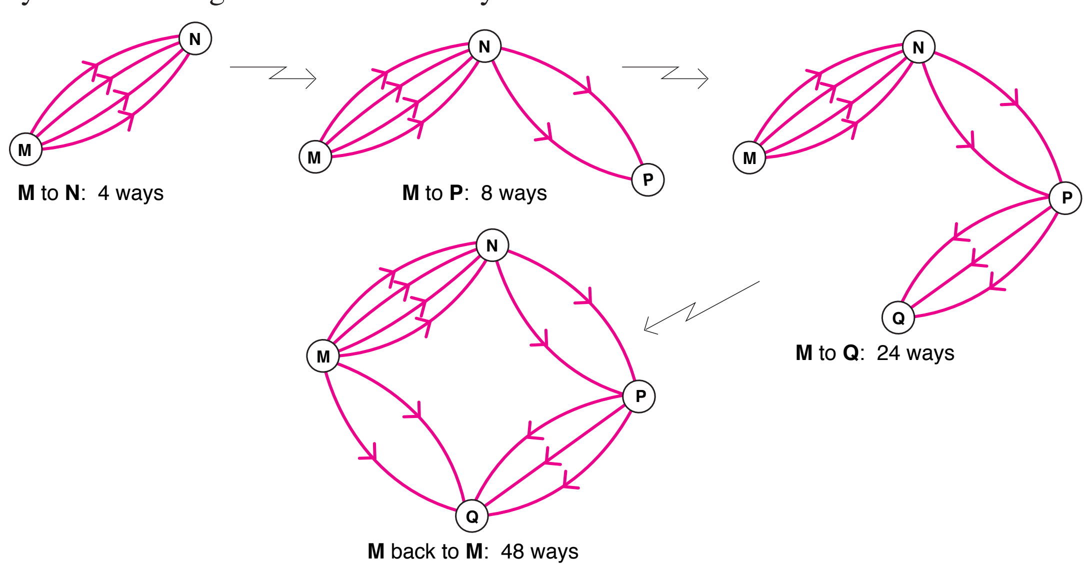
S: ***14 ways; 12 + 2 = 14.***

Exercise 3

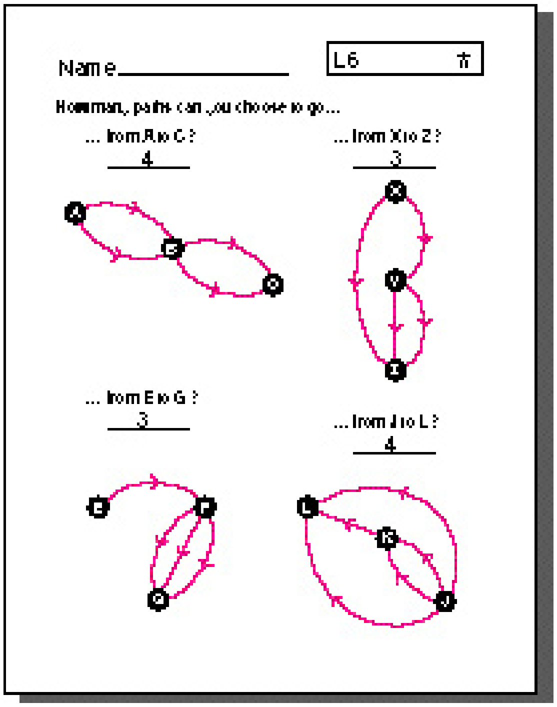
Erase the board before starting a new picture.

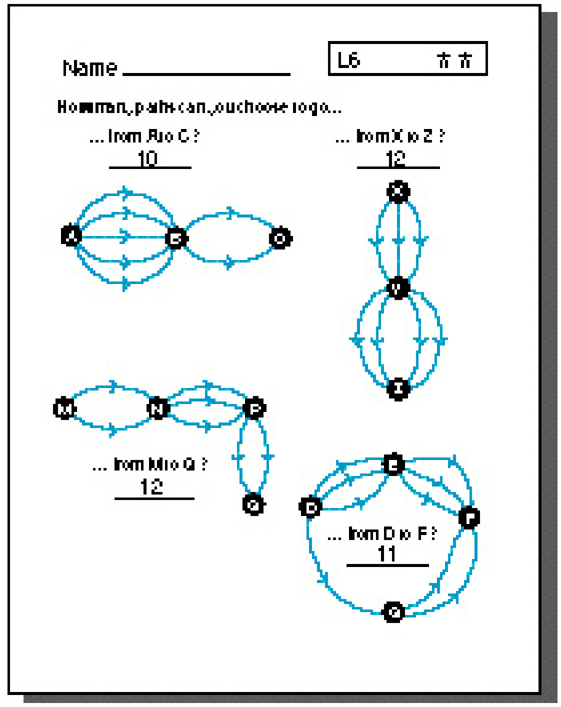
T: ***Suppose we are planning a trip from city* M *to cities* N*,* P*, and* Q*, and then back to city* M*.***

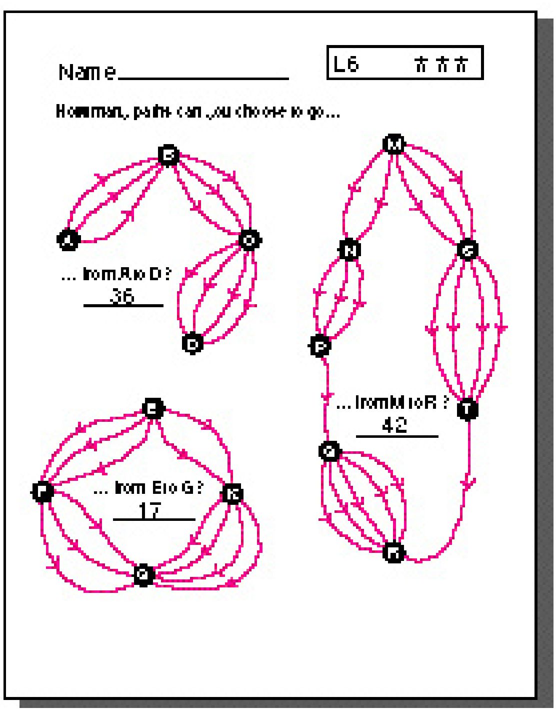
Progressively build an arrow map as shown below. Each time the roads to a city are added, ask how many ways there are to get from **M** to that city.

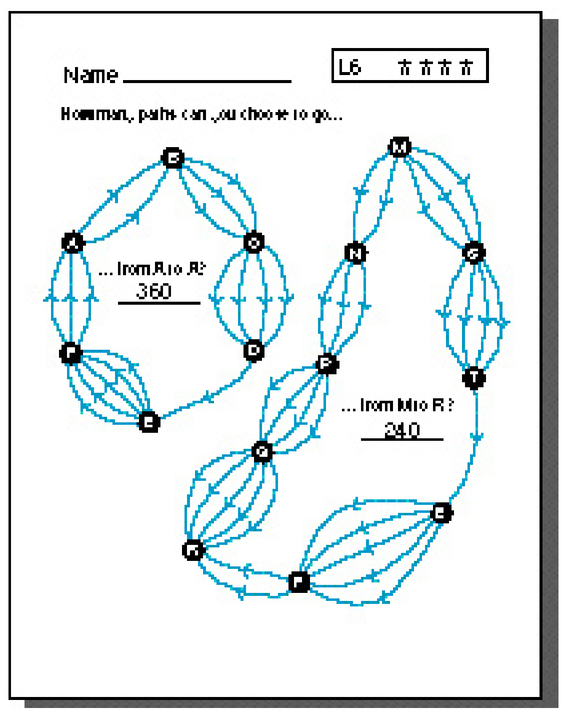


Worksheets L6*, **, ***, and **** are available for individual work.


Name
L6 ☆
How many paths can you choose to go...
... from A to C ?
4
... from X to Z ?
3
... from E to G ?
3
... from J to L ?
4


Name
L6 ☆☆
How many paths can you choose to go...
... from A to C ?
10
... from X to Z ?
12
... from M to Q ?
12
... from D to F ?
11


Name
L6 ☆☆☆
How many paths can you choose to go...
... from A to D ?
36
... from M to R ?
42
... from E to G ?
17


Name
L6 ☆☆☆☆
How many paths can you choose to go...
... from A to R ?
360
... from M to R ?
240

Capsule Lesson Summary

One at a time, correctly put some numbers (game pieces) in the string picture for a string game. Analyze the information obtained from each of these clues and determine the string labels. Play *The String Game* with numbers.

Materials

Teacher
- Numerical String Game kit
- Colored chalk
- Colored markers or crayons

Student
- String Game analysis sheet

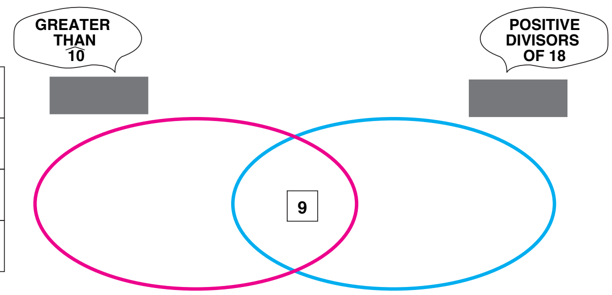
Description of Lesson

Exercise 1

Before the lesson begins, prepare your board as illustrated below. Bubbles indicate what is on the hidden labels. Display two Numerical String Game posters (included in the Numerical String Game kit), one to the left of the red string and one to the right of the blue string.

Red

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN 10	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN 10	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27



Blue

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN 10	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN 10	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27

T: *We are going to play The String Game today, but first let's look at what information we get from some clues. This can help us be better players. Playing the game is like solving a detective story; we want to find out what labels the red and blue strings have, and each piece that is played is a clue. Let's play a game together.*

Our first clue is that 9 is in the center region; 9 is inside both strings. What information does this clue give us about the strings?

S: *The red string cannot be for* MULTIPLES OF 2*, because 9 is not a multiple of 2.*

T: *Could the blue string be for* MULTIPLES OF 2*?*

S: *No, because 9 is also inside the blue string.*

Encourage students to eliminate as many labels as they can from both lists (posters). Each time they suggest crossing out a label on one list because the corresponding string cannot have that label, they should see that the same label should be crossed out on the other list—a number in the center region gives the same information about both strings. Do not allow a label to be crossed out until an adequate explanation has been given. For example:

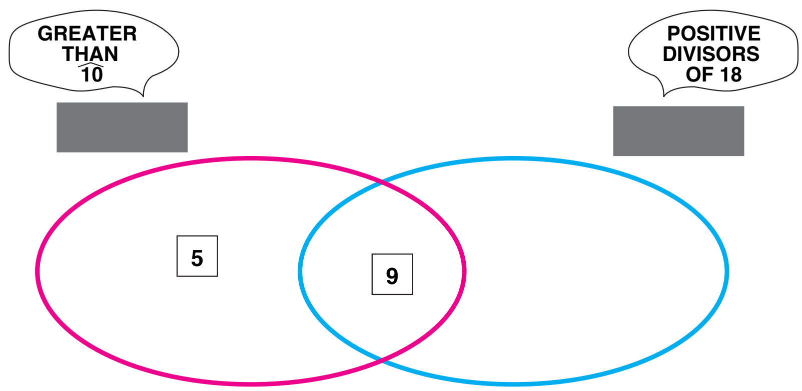
S: *The red (blue) string cannot be for* POSITIVE DIVISORS OF 12*, because 9 is not a positive divisor of 12.*

When all of the information from this clue has been discussed, there should be only six remaining possibilities for each string.

Red			
~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	LESS THAN 50	~~LESS THAN $\widehat{10}$~~	ODD NUMBERS
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	GREATER THAN $\widehat{10}$	~~POSITIVE PRIME NUMBERS~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 27

Blue			
~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	LESS THAN 50	~~LESS THAN $\widehat{10}$~~	ODD NUMBERS
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	GREATER THAN $\widehat{10}$	~~POSITIVE PRIME NUMBERS~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 27

T: ***Now, I'll give you a second clue. 5 is in the red string but not in the blue sting.*** (Place 5 correctly in the string picture.) ***What new information does this clue give us?***



S: ***The red string cannot be for* MULTIPLES OF 3*, because 5 is not a multiple of 3.***

T: ***Could the blue string be for* MULTIPLES OF 3*?*** (Yes)

If it is not mentioned, lead the class to realize that with this clue whatever remaining label the red string cannot have, the blue string can have, and vice versa. For example:

S: ***The blue string cannot be for* ODD NUMBERS*, because 5 is an odd number and is outside of the blue string.***

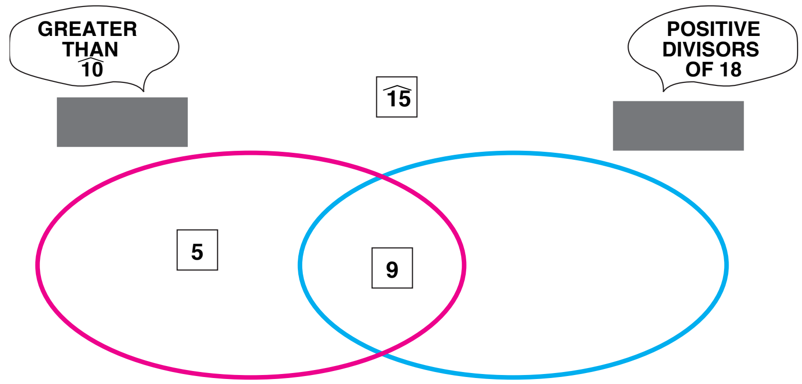
T: ***Could the red string be for* ODD NUMBERS*?*** (Yes)

When all of the information from this clue has been discussed, there should be only three remaining possibilities on each list.

Red			
~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	LESS THAN 50	~~LESS THAN $\widehat{10}$~~	ODD NUMBERS
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	GREATER THAN $\widehat{10}$	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

Blue			
~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN $\widehat{10}$~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	~~GREATER THAN $\widehat{10}$~~	~~POSITIVE PRIME NUMBERS~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 27

T: ***My third clue is that $\widehat{15}$ is outside of both strings.*** (Place $\widehat{15}$ correctly in the string picture.) ***What new information does this clue give us?***



S: ***The red string cannot be for* ODD NUMBERS*, because $\widehat{15}$ is odd and it is outside of the red string.***

S: ***The blue string cannot be for* MULTIPLES OF 3*, because $\widehat{15}$ is a multiple of 3 and it is outside of the blue string.***

These three clues determine the red string, and there are two possibilities remaining for the blue string.

Red

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN $\widehat{10}$~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	GREATER THAN $\widehat{10}$	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

Blue

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN $\widehat{10}$~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	~~GREATER THAN $\widehat{10}$~~	~~POSITIVE PRIME NUMBERS~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 27

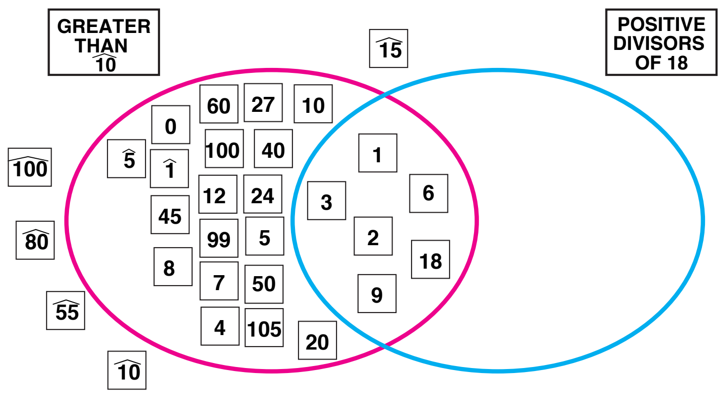
T: ***Now we know that the red string is for numbers* GREATER THAN $\widehat{10}$*. There are still two possibilities for the blue string. We need another clue. Can someone suggest a number to play? I'll put it in the correct place. Try to choose a number so that once we know where it belongs in the picture, we can determine the blue string.***

This problem is difficult; allow students to make several trials, if necessary. If a student chooses $\widehat{100}$, $\widehat{80}$, $\widehat{55}$, $\widehat{10}$, $\hat{5}$, $\hat{1}$, 0, 1, 3, 4, 7, 8, 10, 12, 20, 24, 40, 45, 50, 60, 99, 100, or 105, ask him or her to place it. Where these numbers belong in the picture is already determined, so playing any one of them gives no new information. If someone chooses 2, 6, 18, or 27, place the number correctly yourself. If, after several trials, the class does not see that 2, 6, 18, or 27 are the only numbers that can determine the blue string, suggest one of these numbers yourself.

The following dialogue assumes that 27 is placed correctly in the string picture.

S: *The blue string cannot be for* **POSITIVE DIVISORS OF 27***, because 27 is outside of the blue string. So the blue string must be for* **POSITIVE DIVISORS OF 18.**

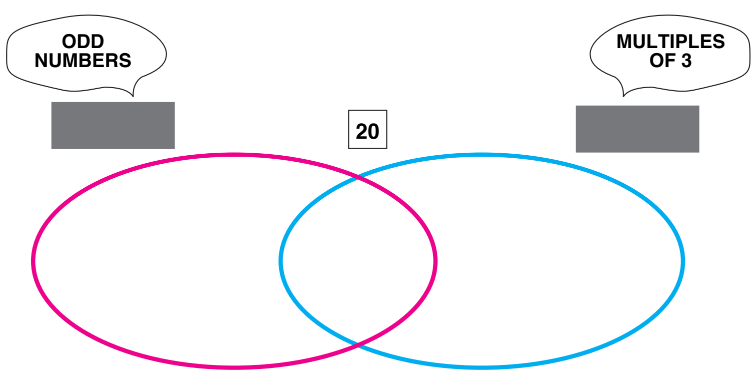
Correct Placement
of Game Pieces



Exercise 2†

Repeat Exercise 1 with the situation below. You can move more quickly by encouraging one student to eliminate ~~several similar la~~bels simultaneously.

Clue 1



These are the labels that can be crossed out after the first clue.

Red

~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	LESS THAN 10	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	~~GREATER THAN 10~~	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27

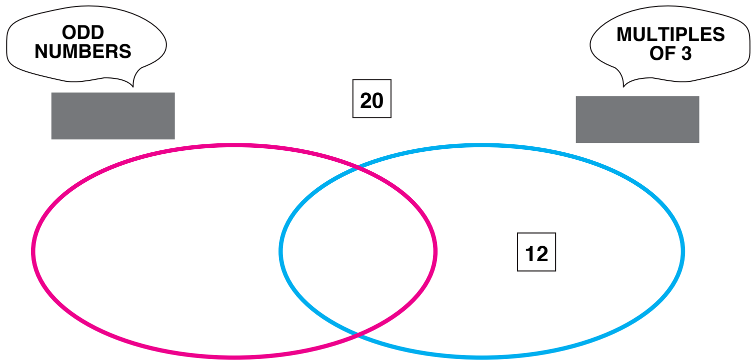
Blue

~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	LESS THAN 10	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	~~GREATER THAN 10~~	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27

Note: The same labels are crossed out on both lists because this clue gives the same information about both strings; 20 is outside of both strings.

Clue 2

†If Exercise 1 took more than half of your class time, you may wish to skip this exercise and go on to Exercise 3, playing *The String Game* in the time that remains.



These are the labels that can be crossed out after the second clue.

Red

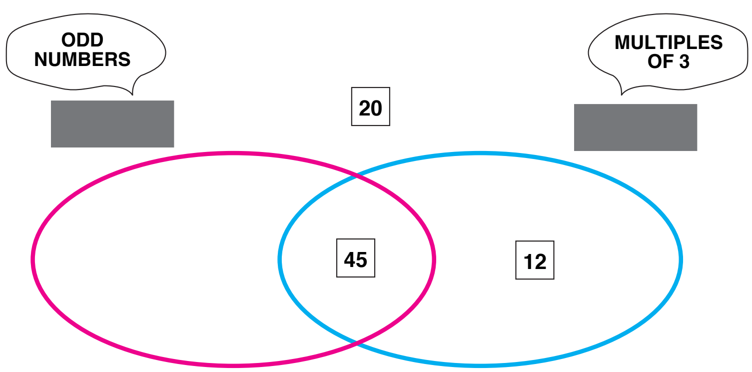
~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	LESS THAN 10	ODD NUMBERS
~~POSITIVE DIVISORS OF 12~~	GREATER THAN 50	~~GREATER THAN 10~~	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 27

Blue

~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN 10~~	~~ODD NUMBERS~~
POSITIVE DIVISORS OF 12	~~GREATER THAN 50~~	~~GREATER THAN 10~~	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~

Note: Of the possibilities remaining after the first clue, those that get crossed out on one list do not get crossed out on the other; 12 is inside one string and outside of the other.

Clue 3



These three clues determine both the red string and the blue string.

Red

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN 10~~	ODD NUMBERS
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	~~GREATER THAN 10~~	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

Blue

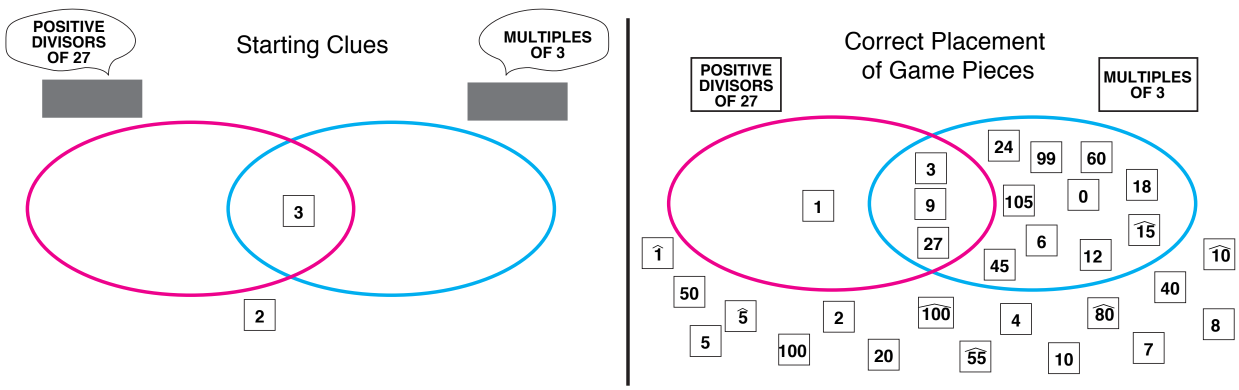
~~MULTIPLES OF 2~~	MULTIPLES OF 3	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN 10~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	~~GREATER THAN 10~~	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

Exercise 3

Distribute String Game analysis sheets to students, and point out that now they have their own lists of possibilities for the red and blue strings. Suggest that students try to use these lists during the game to help discover the string labels.

Prepare to play *The String Game* in the usual way. The illustration below shows a possible game with two pieces placed correctly as starting clues.

In order to give students time to do some of their own analysis between plays, you may pause briefly between turns. Such pauses in the game should encourage students to use the clues to cross out labels on their individual lists.



Capsule Lesson Summary

Use string pictures to provide a context for examining statements to find if they are true, false, or indeterminate. Analyze situations from *The String Game* with numbers to determine certainty or uncertainty for placement of numbers in various game situations.

Materials

Teacher	• Colored chalk • Numerical String Game kit	Student	• None

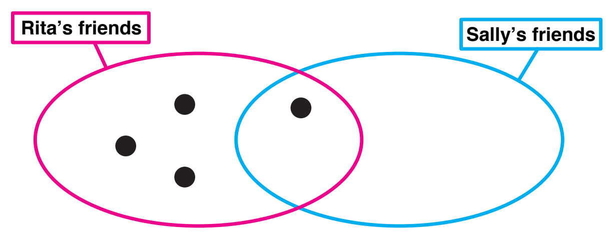
Description of Lesson

Exercise 1

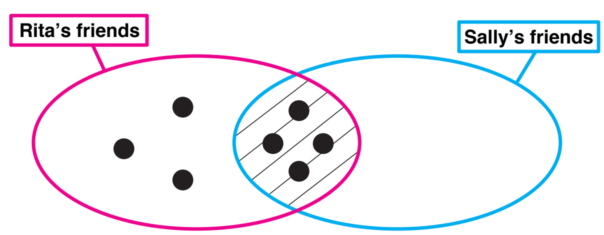
On the board, draw two overlapping strings, one for Rita's friends and one for Sally's friends.

For each of the following statements, ask students to adjust the picture on the board so that the statement is true. After each response, let the class discuss its correctness and other possible solutions. If necessary, restore the original picture before proceeding from one statement to the next. However, this may not always be necessary, as shown in the pictures on the right.

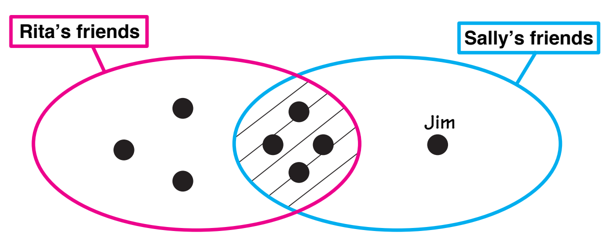
T: ***Rita has at least three friends.***



T: ***Rita and Sally have exactly four friends in common.***



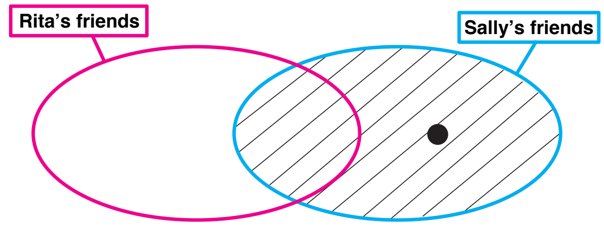
T: ***Jim is Sally's friend but is not Rita's friend.***



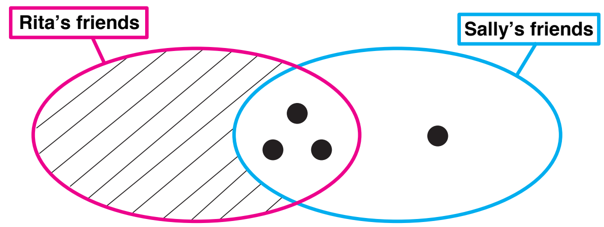
L8

Reestablish the original picture before going on to each of the following statements.

T: ***Sally has at most two friends.***



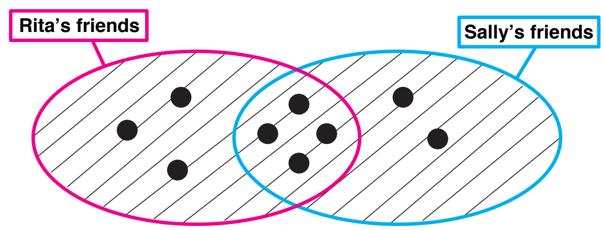
T: ***Every friend of Rita's is a friend of Sally's, but not every friend of Sally's is a friend of Rita's.***



You may want to ask the class to find the least number of dots needed in the picture to make this statement true. (One; inside the blue string and outside the red string)

Reestablish the original picture before going on to the next statement.

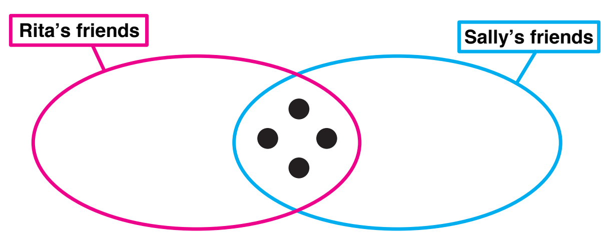
T: ***Rita has exactly seven friends; Sally has exactly six friends; and altogether they have exactly nine friends.***



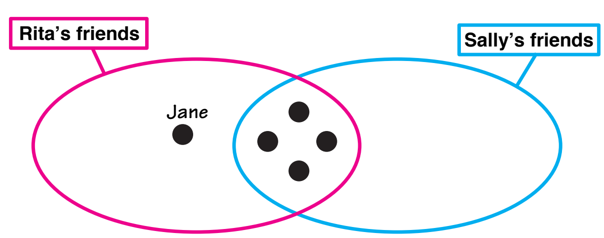
For this statement, allow students to use trial-and-error methods. They may not immediately find the solution; however, by attempting to fix mistakes, they will discover a correct picture.

Return to the original picture. For each of the following statements, ask students to adjust the picture to make the statement false. Again, it may be necessary to restore the original picture before proceeding from one statement to the next. Correct responses are shown at the right.

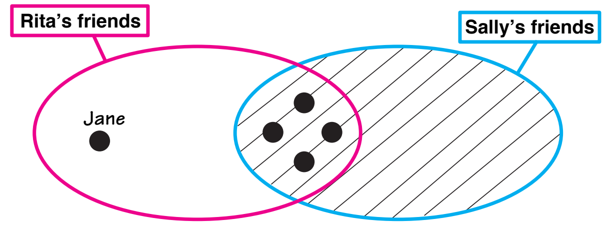
T: ***Rita and Sally have at most three friends in common.***



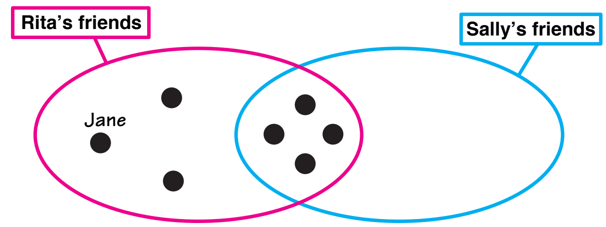
T: ***Jane is Sally's friend.***



T: *Sally has at least five friends.*



T: *Rita has exactly one friend that is not a friend of Sally's.*



Exercise 2

Draw a red string on the board, and put the string game cards ODD NUMBERS and POSITIVE PRIME NUMBERS next to it.

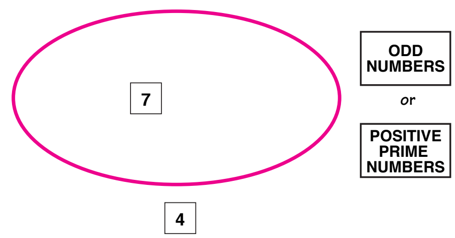
T: *Imagine that we are playing The String Game and that we know from other clues that the red string must be for* **ODD NUMBERS** *or for* **POSITIVE PRIME NUMBERS.**

Let's see if we can place numbers in this picture. Where does 7 belong?

S: *7 is odd and 7 is a prime number; 7 belongs inside the red string no matter which label the red string has.*

T: *Where does 4 belong?*

S: *4 is not odd and not prime; 4 belongs outside the red string with either label.*



T: *Where does 27 belong?*

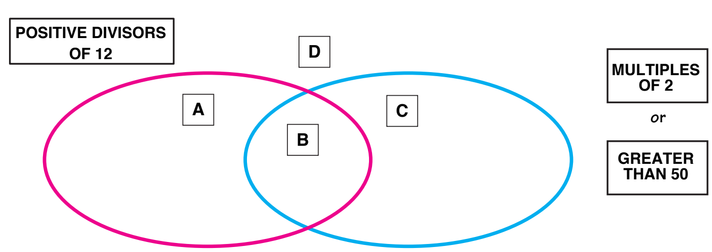
S: *27 is odd but not prime. If the red string is for* **ODD NUMBERS***, then 27 belongs inside. If the red string is for* **POSITIVE PRIME NUMBERS***, then 27 belongs outside. So we cannot tell.*

Continue the discussion with several other numbers.

0 (outside)	$\hat{2}$ (outside)	$\widehat{55}$ (cannot tell)
2 (cannot tell)	3 (inside)	40 (outside)

Exercise 3

Prepare this picture on the board.



T: *Again, imagine that we are playing The String Game. Suppose we know the red string's label is* **POSITIVE DIVISORS OF 12***, and we know that the blue string's label is either* **MULTIPLES OF 2** *or* **GREATER THAN 50** *(assume that all other possibilities have been eliminated).*

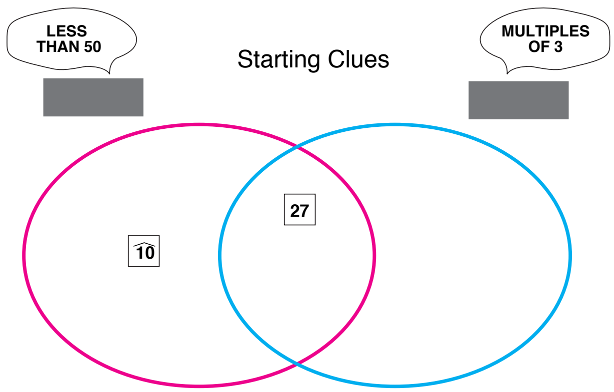
L8

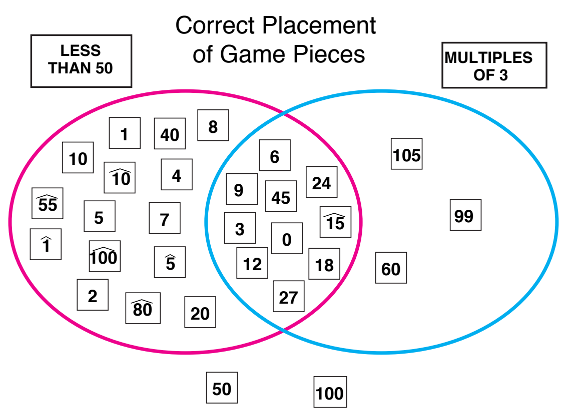
Ask students to decide which region (**A**, **B**, **C**, or **D**) each of the following numbers would be in or to determine that you cannot tell. If you cannot tell for sure, ask for the possible locations.

9	(**D**)	1	(**A**)	$\widehat{55}$	(**D**)
$\widehat{80}$	(**C** or **D**)	12	(**A** or **B**)	3	(**A**)
5	(**D**)	105	(**C** or **D**)	0	(**C** or **D**)
		60	(**C**)		

Exercise 4 (optional)

If time permits, play *The String Game* with numbers in the usual way. The following illustration is for a possible game.





Capsule Lesson Summary

Using correct plays in a string game as clues, see how quickly the class can identify the string labels in a cooperative game. Play *The String Game* with numbers.

Materials

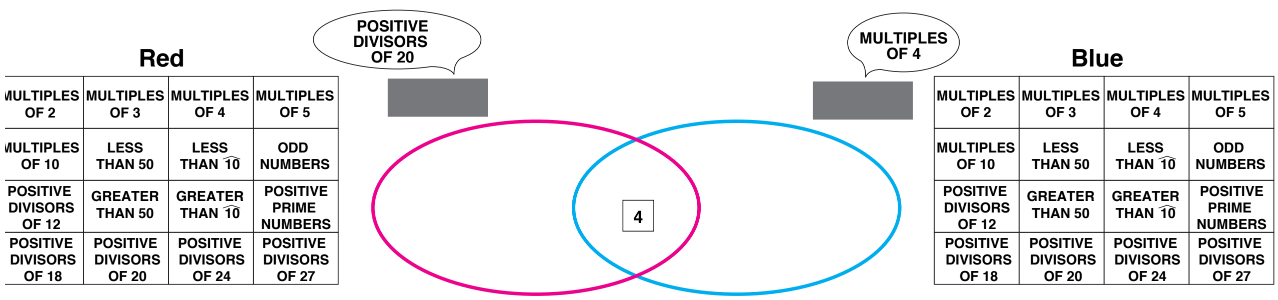
Teacher
- Numerical String Game kit
- Colored chalk
- Colored markers or crayons

Student
- String Game analysis sheet
- Worksheets L9*, **, ***, and ****

Description of Lesson

Exercise 1

Distribute String Game analysis sheets to students. Using two Numerical String Game posters from the Numerical String Game kit, prepare your board as illustrated below. Bubbles indicate what is on the hidden labels.



T: ***Today we are going to play The String Game. First, let's play together as a class and see how quickly we can discover the strings' labels. The first clue is that 4 is in the center region. What information does this give us about the strings?***

S: *The red string cannot be for* MULTIPLES OF 3 *because 4 is not a multiple of 3.*

S: *Also, the blue string cannot be for* MULTIPLES OF 3.

Cross out labels from the two lists as they are mentioned. Encourage students to recognize that the same labels are crossed out on both lists because 4 is inside both strings, thus giving the same information about both strings. This clue eliminates all but seven possibilities on each list.

Red

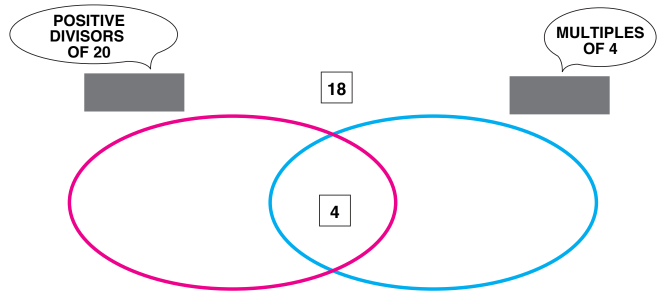
MULTIPLES OF 2	~~MULTIPLES OF 3~~	MULTIPLES OF 4	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	LESS THAN 50	~~LESS THAN 10~~	~~ODD NUMBERS~~
POSITIVE DIVISORS OF 12	~~GREATER THAN 50~~	GREATER THAN 10	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~

Blue

MULTIPLES OF 2	~~MULTIPLES OF 3~~	MULTIPLES OF 4	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	LESS THAN 50	~~LESS THAN 10~~	~~ODD NUMBERS~~
POSITIVE DIVISORS OF 12	~~GREATER THAN 50~~	GREATER THAN 10	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~

Place 18 in the picture as you announce a second clue.

T: ***The second clue is that 18 is in the outside region. What new information does this clue give us about the strings?***



S: ***The red string cannot be for* MULTIPLES OF 2*, because 18 is a multiple of 2 and it is outside of the red string.***

S: ***Also, the blue string cannot be for* MULTIPLES OF 2.**

Again, cross out labels from the two lists as they are mentioned. Encourage students to recognize again that this clue gives the same information about both strings; 18 is outside of both strings. After using information from this clue, only four possibilities remain for each of the strings.

Red

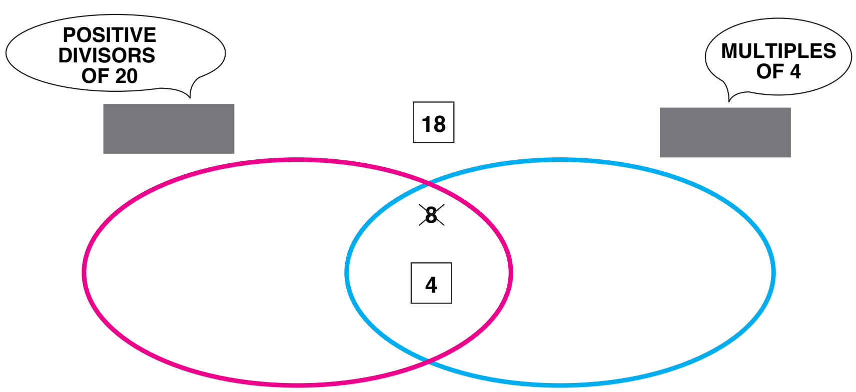
~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	MULTIPLES OF 4	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN 10~~	~~ODD NUMBERS~~
POSITIVE DIVISORS OF 12	~~GREATER THAN 50~~	~~GREATER THAN 10~~	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~

Blue

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	MULTIPLES OF 4	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN 10~~	~~ODD NUMBERS~~
POSITIVE DIVISORS OF 12	~~GREATER THAN 50~~	~~GREATER THAN 10~~	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~

T: ***Now, let's play the game together in the usual way and see now quickly we can discover the strings' labels. Who would like to make a first play?***

Call on students to make plays, keeping count of how many plays are made. Accept any play, answering "yes" if the piece is played correctly and "no" if it is played incorrectly. Whenever a piece is played correctly, stop and analyze what new information is obtained. Cross out the appropriate labels on the two lists. Whenever a piece is played incorrectly, record the information in the picture. For example, if 8 is played in the center, indicate on your picture that it was put in the wrong region (see picture below).

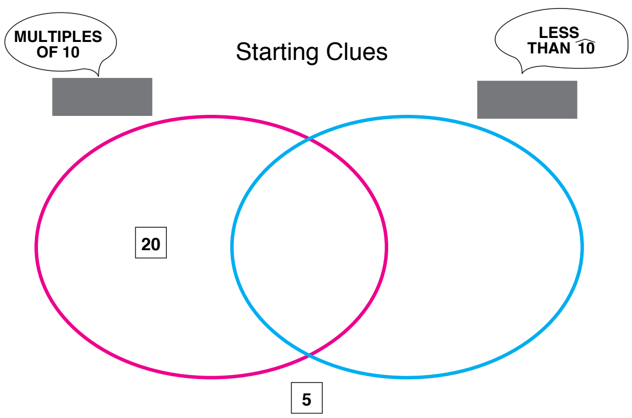


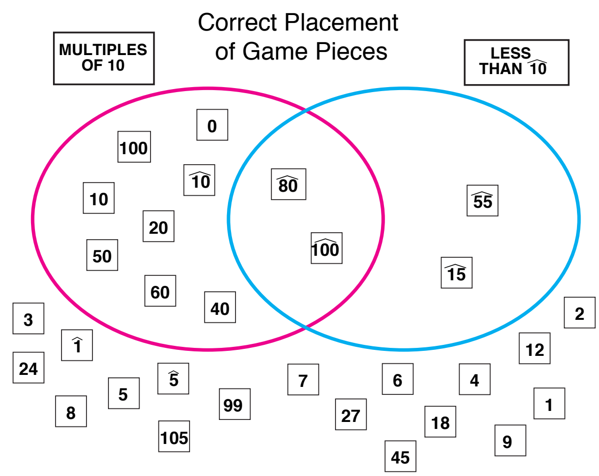
Caution: Students might suggest that this play tells them that the red string cannot, for example, be for **POSITIVE DIVISORS OF 24**. Remind the class that they still do not know where 8 belongs; it could still be inside the blue string, inside the red string, or outside of both strings.

Continue playing the game as a class until the strings are determined. You may wish to ask students to make plays even after the string labels are known just for the practice of recognizing positive divisors of 20 and multiples of 4.

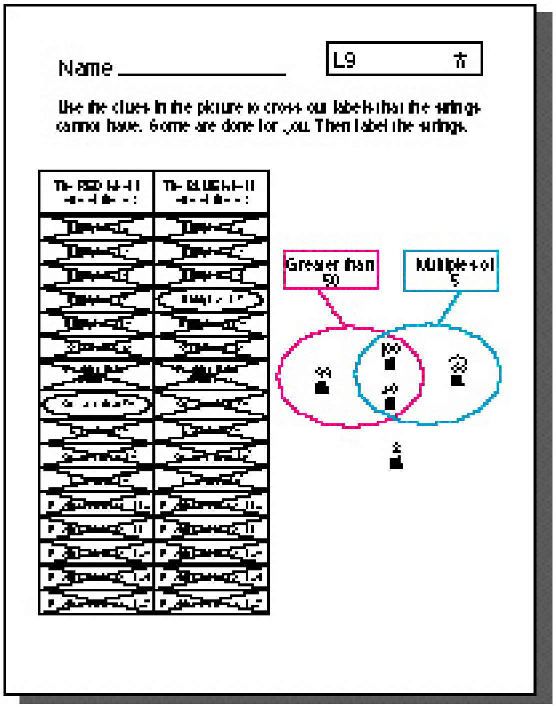
Exercise 2

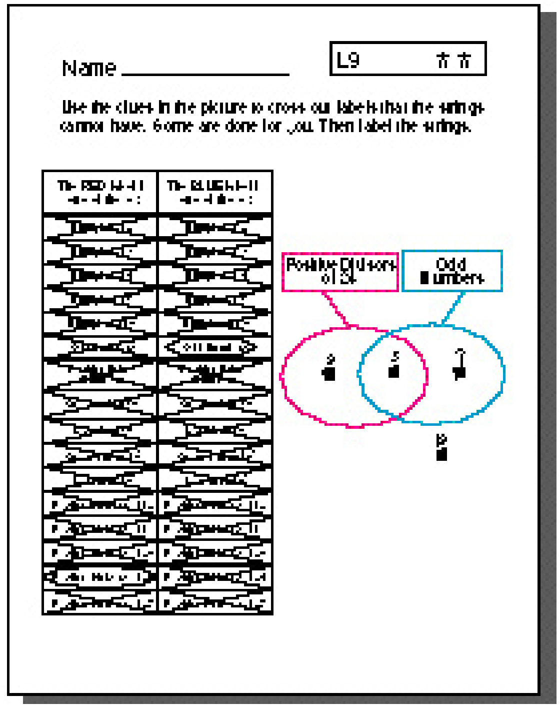
Play *The String Game* in the usual way. The illustration below shows a possible game with two starting clues. Encourage students to use their lists (String Game analysis sheets) to eliminate possibilities for string labels during the game. Suggest that when they are sure what the strings are for, they should circle the correct label on each list. This will help them avoid mistakes when playing.

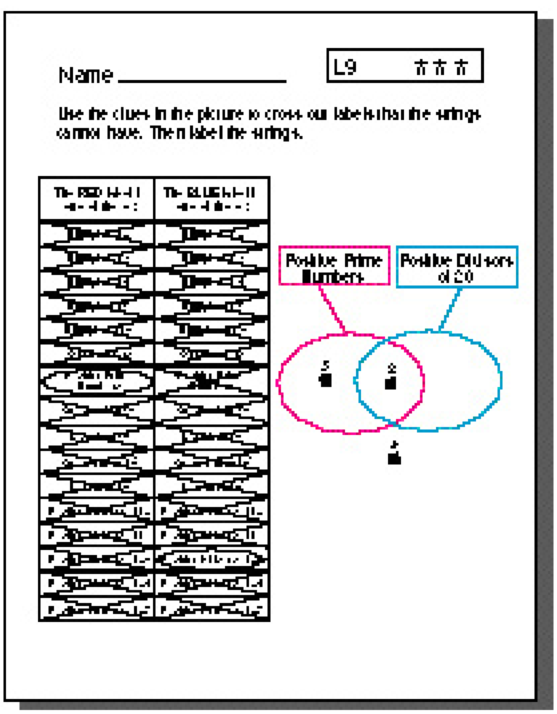


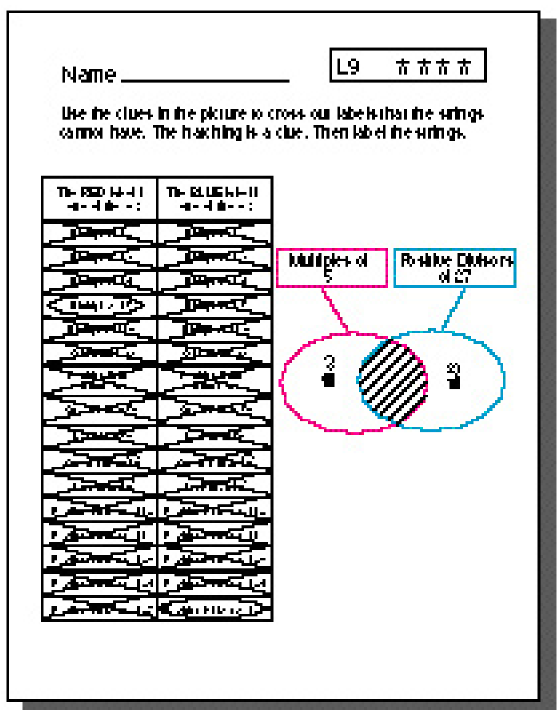


Worksheets L9*, **, ***, and **** are available for individual work.


Name
L9 ☆
Use the clues in the picture to cross out labels that the strings cannot have. Some are done for you. Then label the strings.
Greater than 50
Multiples of 5


Name
L9 ☆☆
Use the clues in the picture to cross out labels that the strings cannot have. Some are done for you. Then label the strings.
Positive Divisors of 24
Odd Numbers


Name
L9 ☆☆☆
Use the clues in the picture to cross out labels that the strings cannot have. Then label the strings.
Positive Prime Numbers
Positive Divisors of 20


Name
L9 ☆☆☆☆
Use the clues in the picture to cross out labels that the strings cannot have. The hatching is a clue. Then label the strings.
Multiples of 5
Positive Divisors of 27

Capsule Lesson Summary

Analyze a situation where the starting clues of a string game determine the strings. Begin to discuss the strategy of certain plays in a string game. Play *The String Game* with numbers.

Materials

Teacher
- Numerical String Game kit
- Colored chalk
- Colored markers or crayons

Student
- String Game analysis sheet

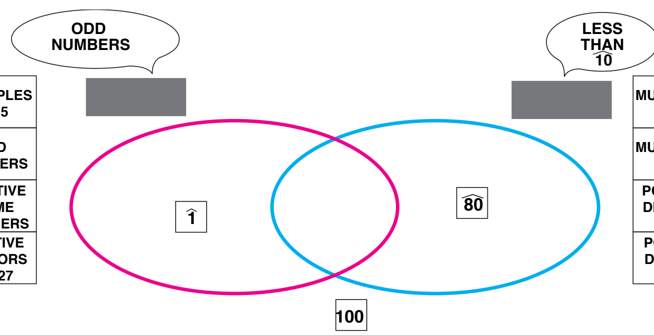
Description of Lesson

Exercise 1

Using two Numerical String Game posters, prepare your board as illustrated below. Bubbles indicate what is on the hidden labels. In this situation the clues do, in fact, determine the string labels.

Red

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN $\widehat{10}$	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN $\widehat{10}$	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27



Blue

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN $\widehat{10}$	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN $\widehat{10}$	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27

T: ***What information do these clues give us about the strings?***

Let students suggest labels to cross out on the lists. Each time, ask for an explanation as to why a label can be crossed out. For example:

S: ***The red string cannot be for numbers* LESS THAN 50 *because $\widehat{80}$ is less than 50 and it is outside of the red string.***

S: ***The blue string cannot be for positive divisors of any number because $\widehat{80}$ is negative.***

On the two lists, cross out the labels that the strings cannot have as verified by students. All labels except one on each list should be crossed out. That is, these clues determine the strings; the red string is for **ODD NUMBERS** and the blue string is for numbers **LESS THAN $\widehat{10}$**.

Red

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN $\widehat{10}$~~	ODD NUMBERS
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	~~GREATER THAN $\widehat{10}$~~	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

Blue

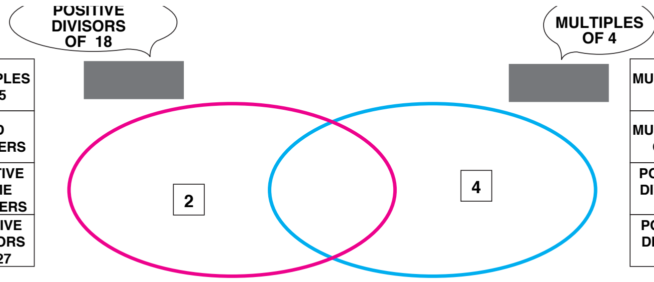
~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	LESS THAN $\widehat{10}$	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	~~GREATER THAN $\widehat{10}$~~	~~POSITIVE PRIME NUMBERS~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

Exercise 2

Prepare your board for *The String Game*, again taping Numerical String Game posters near the strings. Distribute String Game analysis sheets to students. Start with two clues as shown here.

Red

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN $\hat{10}$	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN $\hat{10}$	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27



Blue

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN $\hat{10}$	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN $\hat{10}$	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27

Direct students to cross out as many labels as they can on their lists. After a few minutes, collectively do the analysis with these clues. The class should find that there are two remaining possibilities for the red string and that the blue string is determined.

Red

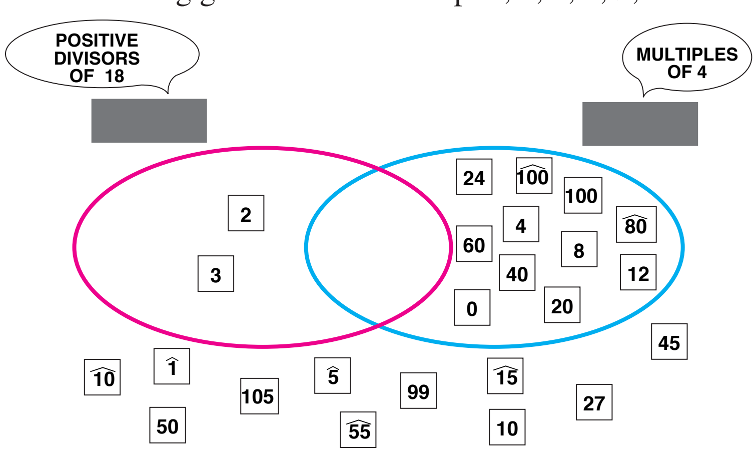
~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN $\hat{10}$~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	~~GREATER THAN $\hat{10}$~~	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

Blue

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	MULTIPLES OF 4	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN $\hat{10}$~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	~~GREATER THAN $\hat{10}$~~	~~POSITIVE PRIME NUMBERS~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

T: *Now we know that the red string is for either* **POSITIVE DIVISORS OF 18** *or for* **POSITIVE PRIME NUMBERS.** *We also know that the blue string is for* **MULTIPLES OF 4.** *Let's see which numbers we can put in the picture correctly even though we are not yet sure what label the red string has.*

Let students place as many of *The String Game* numbers as they can. This activity should allow everyone a chance to participate. Encourage discussion about why, for example, you are sure that 10 is in the outside region (10 is not a positive divisor of 18 and not a positive prime number, and 10 is not a multiple of 4); or why you are sure that 3 is inside the red string but outside the blue string (3 is a positive divisor of 18 and a positive prime number, but 3 is not a multiple of 4); and so on. Students should put all of the string game numbers except 1, 5, 6, 7, 9, and 18 in the picture.



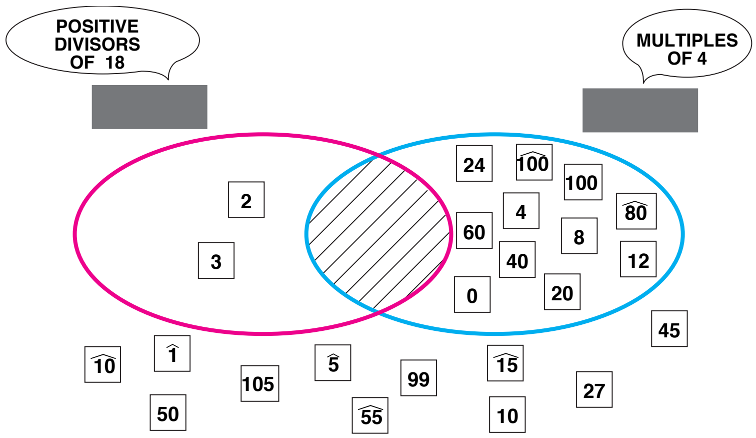
Discuss possibilities for the numbers 1, 5, 6, 7, 9, and 18. Each of these numbers could be in the outside region, or inside the red string and outside of the blue string.

T: ***Do you see a region that could be hatched because it is empty?***

S: ***The center region.***

T: ***Why?***

S: ***No number is both a positive divisor of 18 and a multiple of 4, and no number is both prime and a multiple of 4.***



Remove all of the game pieces except $\boxed{2}$ and $\boxed{4}$; that is, return to the original situation.

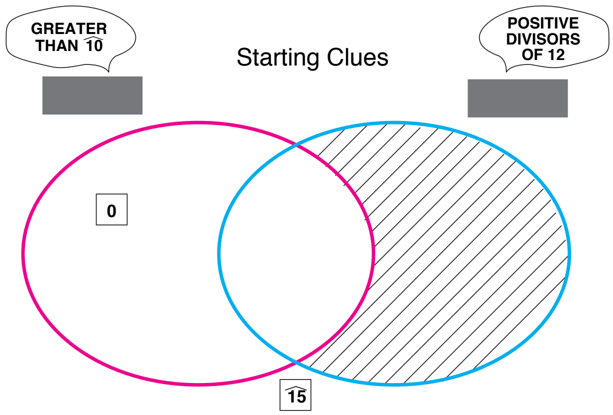
T: ***Suppose it is your turn to play in The String Game, and you would like to determine the red string as quickly as possible. Which piece would you play?***

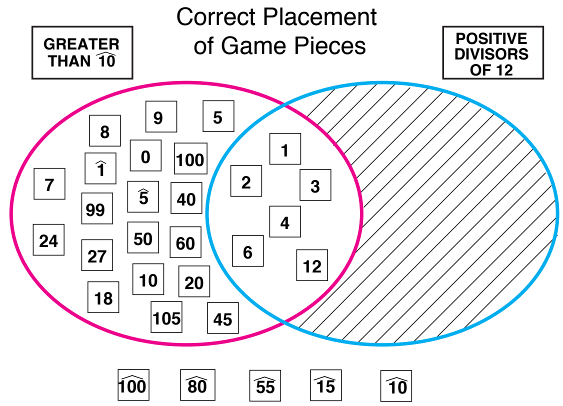
Give students a few minutes to think about what play would be helpful. They should, after some trial and error, find that playing any of the numbers 1, 5, 6, 7, 9, or 18 would give them the necessary information (as long as they did not make a careless play). For example, if 5 were played in the outside region and given a "no" answer, then it would be certain that 5 belonged inside the red string. So the red string would be for **POSITIVE PRIME NUMBERS**. If 5 were played in the outside region and given a "yes" answer, then it would be certain that the red string was for the **POSITIVE DIVISORS OF 18**. A similar analysis applies to each of the numbers 1, 6, 7, 9, and 18.

Let students make one of these plays to determine the red string.

Exercise 3

Play *The String Game* in the usual way. The illustration below shows a possible game with starting clues. Encourage students to use their lists (String Game analysis sheet) to eliminate possibilities for string labels during the game.


GREATER THAN 10
Starting Clues
POSITIVE DIVISORS OF 12
0
15


GREATER THAN 10
Correct Placement of Game Pieces
POSITIVE DIVISORS OF 12
8
9
5
0
100
1
7
5
40
99
24
50
60
27
10
20
18
105
45
1
2
3
4
6
12
100
80
55
15
10