Geometry & Measurement

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GEOMETRY & MEASUREMENT INTRODUCTION

Geometry is useful and interesting, an integral part of mathematics. Geometric thinking permeates mathematics and relates to most of the applications of mathematics. Geometric concepts provide a vehicle for exploring the rich connections between arithmetic concepts (such as numbers and calculations) and physical concepts (such as length, area, symmetry, and shape). Even as children experience arithmetic notions in their daily lives, they encounter geometric notions. Common opinion recognizes that each child should begin developing a number sense early. There is no sound argument to postpone the development of geometric sense. These are compelling reasons why the systematic development of geometric thinking should be a substantial component of an elementary school curriculum.

A benefit of early development of geometric thinking is that it provides opportunities for all learners. Children develop concepts at individual rates. We are careful to allow students a great deal of time to explore quantitative relationships among physical objects before studying arithmetic. In the same spirit, students should be allowed to explore geometric relationships through free play and guided experiences. Before studying geometric properties of lines and shapes, they should have free play with models and with drawing instruments, using both to create new figures and patterns. Before considering parallelism and perpendicularity as relations on geometric lines, they should have free play in creating patterns with parallel and perpendicular lines. Before learning formulas for finding measurements, they should have informal experiences to establish and sharpen their intuitive notions. These early experiences are aesthetically rewarding, and just as stimulating and suggestive of underlying concepts as early experiences leading to arithmetic. Such experiences are a crucial first step in the development of a geometric sense.

A variety of constructions forms the basis for the geometry of *CSMP Mathematics for the Intermediate Grades*. Students use tools to explore geometric concepts, directly discovering their properties and interrelationships. The tools include a straightedge, a compass, mirrors, angle templates, and a translator for drawing parallel lines. The constructions foster insights into the properties of shapes, independent of the measurement of those properties. Only after students are familiar with the shapes do they begin to use rulers and protractors to measure lengths and angles. In this sequential development of geometric ideas, the measurement is viewed as the intersection of geometric concepts and arithmetic concepts.

The focus of this strand is experience. The measurement activities guide the students to refine their ability to accurately measure lengths of line segments and areas of polygons. Another sequence of lessons allows students to explore symmetry through use of a common tool, a mirror. As a natural consequence of their involvement in these activities, the students develop their knowledge and skills in geometry. The effects of this informal approach should be judged by the long-term effects on the students' knowledge, confidence, intuition, and interest in the world of geometry and measurement.

Content Overview

Measurement

Consensus calls for measurement activities in the elementary curriculum, but with no agreement on the form or scope of these activities. Rather than stress mastery of formulas for area and perimeter, or for comparison of standard units, the lessons of this strand provide open-ended experiences within rich problem-solving situations. Measurement becomes a means for investigating problems and developing concepts, rather than an end in itself. Direct experiences with the concepts and tools is central. The carefully designed problems do lead to insights concerning accurate measurement and involving area and perimeter of rectangles. The emphasis is on the development of ideas and understanding rather than on the memorization of rules. The measurement activities become the means for investigating problems and developing concepts rather than an end in themselves.

One lesson in this strand extends earlier experiences with linear measure. Students review equivalences between meter, decimeter, centimeter, and millimeter measurements. They practice using a ruler to measure the length of line segments and to verify estimates of length. The problems require first estimating measurements and then finding actual measurements for a variety of common objects. Further, students compare estimates and actual measurements in a graphing activity.

Two lessons in this strand pose the problem of finding a polygon of largest perimeter with sides lying along some of the grid lines of a square or rectangular region of a grid. Students discover that the maximum perimeter is related to the number of grid points (corners) and is always an even number.

A series of three lessons provide experience in calculating the area of a rectangle by multiplying the lengths of adjacent sides. In these lessons, the problems arise in the context of several farmers who will construct rectangular pens under differing constraints and who each wants a solution of largest possible area. Each of the problems has a best solution, in the search for which a student will incidentally find the areas of many rectangles.

Lessons: G1, 2, 3, 10, 11, and 12

Symmetry

Suggestions of reflective symmetry are everywhere; the human body; the design of an airplane or an automobile; a soaring sea gull; a candelabra; a snowflake. The architectural embodiments of symmetry indicate its role even in the earliest geometry. A rich and fascinating study of symmetry and related ideas has developed over many centuries. Observations of symmetry provide powerful problem-solving techniques in many areas, not just those involving geometrical notions.

Four lessons in this strand are devoted to an exploration of reflective symmetry making use of a single mirror. Students should have many opportunities to play with the idea; to discover properties of a reflection; to investigate the lines of symmetry of different designs; and to create designs with specified line of symmetry.

GEOMETRY & MEASUREMENT INTRODUCTION

Following several activities using a single mirror to study symmetry, students are introduced to a double mirror. Two lessons in this strand involve using a double mirror with a variable angle between the mirrors to explore compositions of reflections. Students attempt to create a variety of designs using the double mirror and, thereby, begin to understand how it works. In a more directed activity, students examine the types of reflective symmetry created by using the double mirror placed on a square corner and classify letters of the alphabet having one or more of four types of symmetry.

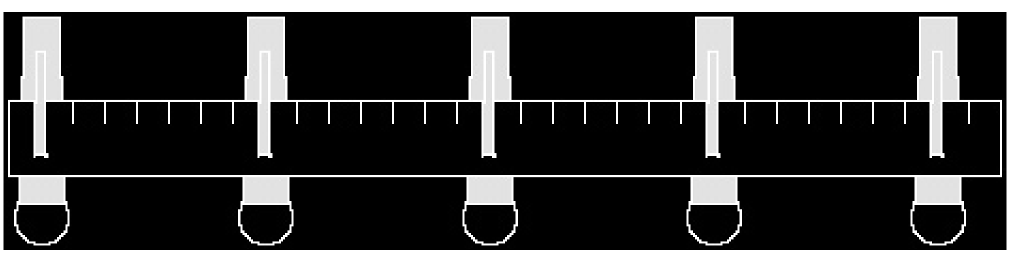
In previous semesters, lessons making use of the translator developed some geometrical concepts using parallelism. They explored the motion, or transformation, geometry ideas of a translation and a parallel projection. The *IG-III* lessons on symmetry provide experience with another motion geometry idea, that of a reflection. In keeping with the spirit of *CSMP* geometry activities, these concepts from transformation geometry build from experience.

Lessons: G4, 5, 6, 7, 8, and 9

Note on Grids

Several lesson call for demonstration on a grid board. A portable, commercially made, permanent grid chalkboard is ideal. However, if such a grid board is not available, try one of these alternatives.

- Make a grid transparency for use on an overhead projector and use erasable overhead markers. The Blackline section includes several size grids to use in making a grid transparency.
- Printed grids on large sheets of paper are provided in classroom sets of materials for *CSMP*. Tape one of these sheets to the board, use thick felt-tipped pens for drawings, and discard the sheet after use. You may prefer to laminate or mount one of the sheets on cardboard and cover it with clear contact paper to provide a protective surface. Then use erasable markers or wax crayons for drawings. These grid sheets also are useful as graphing mats.
- For each lesson, carefully draw a grid on the chalkboard using an easy-to-make tool. Either (1) use a music staff drawer with chalk in the first, third, and fifth positions; or (2) clip five pen caps, each having the diameter to hold a piece of chalk tightly, to a ruler at intervals of 4–8 centimeters, and secure them with tape.



G1 LENGTH

Capsule Lesson Summary

Review equivalences between meter, decimeter, centimeter, and millimeter measurements. Estimate and find actual measurements for the length and perimeters of a variety of objects. Make a graphic comparison of estimates and actual measurements.

Materials

Teacher		Student	
	• Cardboard strips • Blacklines G1(a) and (b) • *IG-III Geometry Poster* #1		• Collection of objects • Measurement recording chart • Centimeter grid paper

Advance Preparation: Prepare the following cardboard strips: one strip 1 m long; ten strips 1 dm long; and ten strips 1 cm long. For Exercise 2 you will divide the class into cooperative groups of 4–5 students. For each group, you will need a collection of 10–15 objects, some suitable for measuring length (e.g., toothpicks, small and big paper clips, new pencils or pieces of chalk, staplers, keys, spoons, straws, markers) and others for measuring perimeter (e.g., postage stamps, ID cards, playing cards, business cards). Also, make copies of the student measurement recording chart from Blackline G1(a) and the grid from Blackline G1(b), one copy for each group.

Description of Lesson

Exercise 1

Hold up the 1-m cardboard strip you prepared.

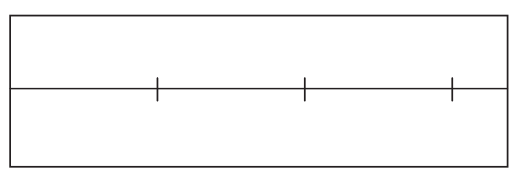
T: ***How long do you think this strip is?***

Let students estimate its length before telling them you measured it to be 1 m long.

T: ***About how many meters long is our chalkboard from end to end?***

In this lesson description, the chalkboard is between 338 cm and 339 cm long. If your chalkboard is very close to being an even number of meters long, measure something else in the room of comparable length.

Let students estimate the length of the chalkboard, and then invite students to use your 1-m cardboard strip to measure it. Draw meter marks as a student moves the cardboard strip.



S: ***The chalkboard is between 3 m and 4 m long, but it is closer to 3 m.***

Record the inequality on the board.

$$3 \text{ m} < \text{length of the board} < 4 \text{ m}$$

Hold up your ten 1-dm cardboard strips as you announce,

T: ***I cut a 1-m cardboard strip into ten pieces of the same length as close as I could. What should the length of each piece be?***

S: ***A decimeter.***

G1

Record the equivalences on the board.

1 meter = 10 decimeters
1 m = 10 dm

Distribute the decimeter strips.

T: ***About how long is a decimeter? Find some length on your body that is about 1 dm long.***

Let students share their body referents; for example, the distance between a student's eyes might be very close to a decimeter.

T: ***About how many decimeters long is the chalkboard?***

S: ***33 dm.***

S: ***35 dm.***

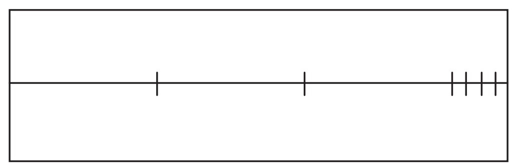
T: ***Why are your estimates between 30 dm and 40 dm?***

S: ***Each meter is 10 decimeters. The board's length is between 3 m and 4 m, so it is between 30 dm and 40 dm.***

T: ***To find about how many decimeters long the board is, what do we need to measure?***

S: ***From the 3-m mark to the end of the board.***

Invite students to measure the board with a decimeter strip. Draw decimeter marks as a student moves the cardboard strip.



T: ***So the length of the board is between 33 dm and 34 dm.***

3 m < length of the board < 4 m
33 dm < length of the board < 34 dm

Hold up your ten 1-cm cardboard strips as you announce,

T: ***I cut a 1-dm cardboard strip into ten pieces of the same length as carefully as I could. What is the length of each piece?***

S: ***A centimeter.***

T: ***How many centimeters is a meter?***

S: ***100 cm; 10 x 10 = 100.***

Ask students to name things that are about 1 cm long, for example, the width of a fingernail.

1 m = 10 dm = 100 cm
1 dm = 10 cm

T: ***How long do you think the chalkboard is in centimeters?***

S: ***335 cm.***

S: ***338 cm.***

T: ***Why are your estimates between 330 cm and 340 cm?***

S: ***33 dm is 330 cm, 34 dm is 340 cm, and we know the chalkboard is longer than 33 dm and shorter than 34 dm.***

T: ***Do we need to measure the entire board with these little pieces to find out its length in centimeters?***

S: ***No, we just need to measure from the last decimeter mark to the edge of the board.***

Invite a student to make the measurement. Record the results.

$$3\text{ m} < \text{length of the board} < 4\text{ m}$$
$$33\text{ dm} < \text{length of the board} < 34\text{ dm}$$
$$338\text{ cm} < \text{length of the board} < 339\text{ cm}$$

S: ***We could keep cutting up the little strips and measuring.***

S: ***No, they'd get too small.***

T (holding up one of the centimeter cardboard strips)**:** ***Suppose I did cut this 1-cm strip into ten pieces of the same length. Do you remember what we call the length of such a piece?***

S: ***1 millimeter.***

T: ***About how long do you think a millimeter is?***

Let students give referents, for example, the width of the point of a lead pencil.

T: ***10 mm = 1 cm. How many millimeters equal 1 meter?***

S: ***1000 mm. 100*** x ***10 = 1000.***

$$1\text{ m} = 10\text{ dm} = 100\text{ cm} = 1000\text{ mm}$$

Pose several problems involving metric equivalents, such as 2 m = – dm = – cm = – mm. Ask students for the numbers to put in the boxes. The illustration below gives suggested problems with an order indicated by the red numerals on the left.

Encourage students to notice 10x and ÷10 patterns as you move to the right or to the left. Also, encourage using other answers to find the one being considered. For example, one can use the second and fourth problems to do the fifth. 15 dm + 20 dm = 35 dm, so the other corresponding measurements can be added also.

	m	dm	cm	mm
③	0.5 m =	$\boxed{5}$ dm =	$\boxed{50}$ cm =	$\boxed{500}$ mm
①	1 m =	10 dm =	100 cm =	1000 mm
④	1.5 m =	$\boxed{15}$ dm =	$\boxed{150}$ cm =	$\boxed{1500}$ mm
②	2 m =	$\boxed{20}$ dm =	$\boxed{200}$ cm =	$\boxed{2000}$ mm
⑤	$\boxed{3.5}$ m =	35 dm =	$\boxed{350}$ cm =	$\boxed{3500}$ mm
⑥	$\boxed{18}$ m =	180 dm =	$\boxed{1800}$ cm =	$\boxed{18000}$ mm
⑦	$\boxed{5.4}$ m =	$\boxed{54}$ dm =	$\boxed{540}$ cm =	5400 mm
⑧	$\boxed{23}$ m =	$\boxed{230}$ dm =	$\boxed{2300}$ cm =	$\boxed{23000}$ mm

Exercise 2

Organize the class into cooperative groups of 4–6 students in a group and provide each group with the student materials: a collection of 10–15 objects for measuring length and perimeter; centimeter grid from *IG-III Geometry Poster* #1 (If you prefer, make copies of Blackline G1(b) with a half-centimeter grid); a metric ruler; measurement recording chart.

First, direct the groups to estimate either the length or perimeter of their objects, and to record group agreed estimates in the recording chart.

When a group has made and recorded all their estimates, direct them to make actual measurements and record them in the recording chart.

Finally, instruct the groups to make a graph to compare estimates with actual measurements. You will need to demonstrate a graphing technique; do this either with each group or as a full class presentation.

T: ***Choose one of your objects.***

S: ***The pencil.***

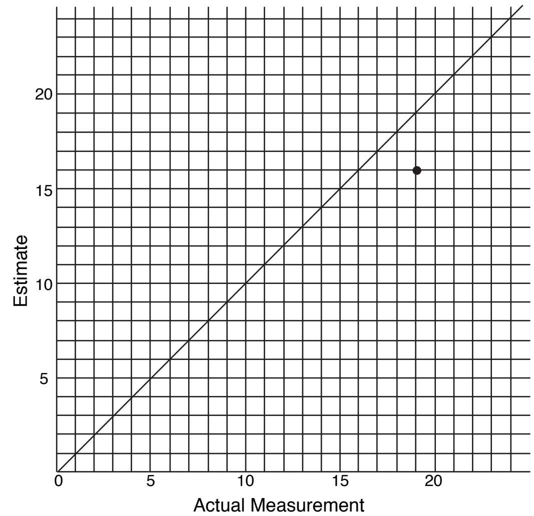
T: ***What did you estimate its length to be?***

S: ***16 cm.***

T: ***Find its actual measurement.***

S: ***19 cm.***

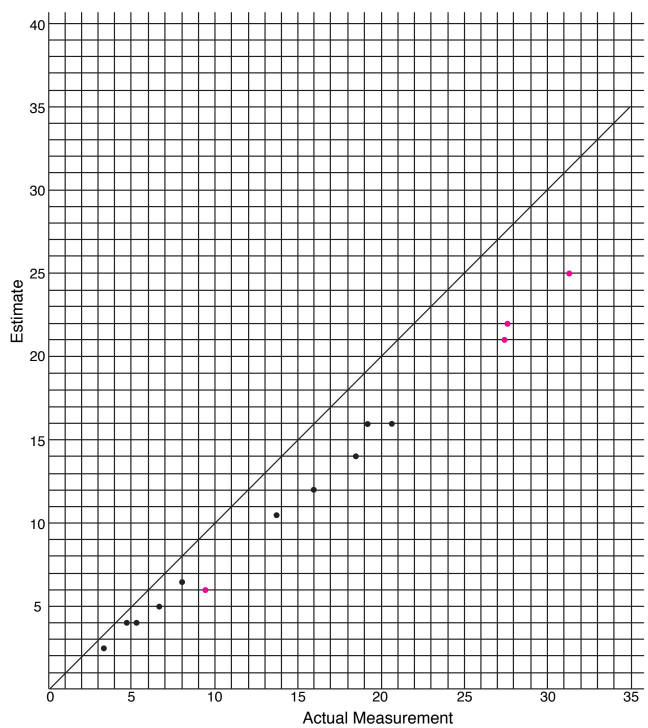
Demonstrate finding the estimate vertically and the actual measurement horizontally. Then draw a dot for the pair in the graph.



Note: The dot will be below the diagonal line in the graph if the estimate is low and above the diagonal line if the estimate is high. A dot will be on the diagonal line if an estimate is exact.

The following illustration is an example of a group's completed work.

Object	Estimate	Actual Measurement
Length of toothpick	5 cm	6.7 cm
Length of small paper clip	2.5 cm	3.2 cm
Length of big paper clip	4 cm	4.9 cm
Length of pencil	16 cm	19.1 cm
Length of chalk	6.5 cm	8 cm
Length of stapler	14 cm	18.5 cm
Length of key	4 cm	5.2 cm
Length of spoon	12 cm	16 cm
Length of straw	16 cm	20.7 cm
Length of marker	10.5 cm	13.8 cm
Perimeter of stamp	6 cm	9.4 cm
Perimeter of ID card	21 cm	27.4 cm
Perimeter of playing card	25 cm	31.2 cm
Perimeter of business card	22 cm	27.6 cm
Perimeter of	cm	cm



Writing Activity

You may like students to take lesson notes on some, most, or even all their math lessons. The "Lesson Notes" section in Notes to the Teacher gives some suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students may note metric length equivalences such as 10 mm = 1 cm and 100 cm = 1 m. They may also write about their length estimates compared to actual measurements, noting, for example, whether they consistently overestimate or underestimate; whether they make better estimates of short things or long things; and whether they are more or less accurate in estimations of perimeter.

Home Activity

Send home a copy of Blackline G1(a) and suggest that parents/guardians look for things at home to measure, first estimating length and then finding an actual measurement. You can suggest that they do the measurements either in centimeters or in inches.

Capsule Lesson Summary

In the context of a story about designs for an aquarium with special restrictions, find a polygon that has sides along some of the grid lines of a 5-by-5 square region and that has the longest perimeter possible. Discover and informally justify that the number of corners is the same as the number of sides.

Materials

Teacher
- Grid board
- Colored chalk
- A strip of paper or cardboard (see illustration)
- Box with a square bottom (optional)

Student
- Colored pencils, pens, or crayons
- Worksheets G2(a), (b), and (c)

Advance Preparation: Using paper or cardboard, prepare a strip that looks similar to this:

If you use cardboard, bend it along the black bars so it folds easily into a rectangle (see page G-14 in the lesson description).

Description of Lesson

Begin the lesson with a story about a project that results when a town decides to build an aquarium.

T: ***The town thinks that an aquarium would be nice for the people living there and also for tourists who visit. As with all construction projects, it is important for the designers and the builders to discuss the plans for the building. So one morning the designers and the builders of the aquarium meet.***

The builders say that the first thing the designers should understand is that the walls will be heavy glass panels. The panels that would be best for the aquarium come in only one size and cannot be cut. Looking over the property, the builders conclude that the floor of the aquarium can be as large as a square five panels on each side, but no longer than five panels in any direction.

Draw a large blue square on the board to represent the possible shape of the floor of the aquarium.

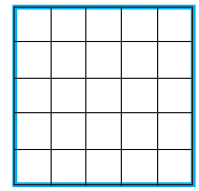
T: ***The designers say that an aquarium of this size would really be too big for the number of fish they are planning to put in it. They wonder if the shape could be changed so that more people could stand along the walls.***

Trace around the square.

T: ***"Sure!" say the builders. "But we think that it would be best to use only square corners, and remember, we cannot cut any panels." One of the builders draws this picture and says that they can put panels anywhere along these line segments.***

G2

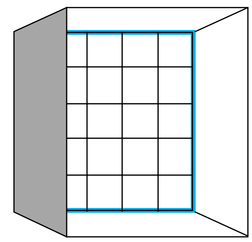
Draw grid lines inside the blue square to get this picture.



T: ***The builders leave and the designers are left to draw up the plans. Let's help.***

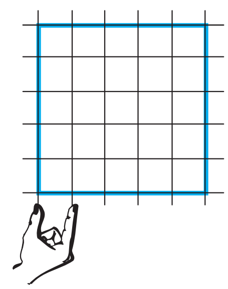
Display your grid board and draw a blue square 5 units by 5 units.

Note: You may wish to locate a box with a square bottom and to draw a 5-by-5 grid on the inside bottom to help students visualize the floor of the aquarium. Hold the box up to the picture on the grid board to insure proper perspective.



T: ***Let's measure the length of the border of the floor of the aquarium, in other words the* perimeter *of the floor, by counting the number of panels along the edges.***

Using a thumb and forefinger, count the number of panels along an edge of the blue square on the grid board.



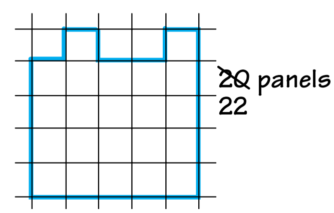
T: ***One side of the aquarium, if the floor is a square shape, would require five panels. So what would the perimeter of the floor be in panels?***

S: ***20 panels; 4* x *5 = 20.***

Write 20 panels near the blue square.

T: ***Yes, the perimeter of this shape on the board is 20 grid lengths. Can you think of a way to make the border longer so that more people could stand along the walls of the aquarium?***

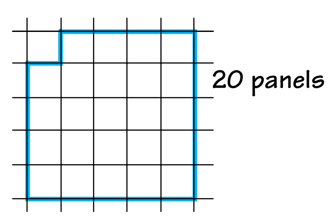
Invite a student to alter the shape on the grid board. The following are only examples of what a student might do. Suppose a student changes the shape but not its perimeter, as illustrated here.



T: ***Oliver took out two panels and put in two. What has this done to the perimeter?***

S: ***Nothing. The perimeter is still 20 panels.***

Invite another student to alter the picture again.



T: ***Nicole took out two panels and put in four*** (trace them)**.**

S: ***She increased the perimeter of the floor by 2. Her design would require 22 panels.***

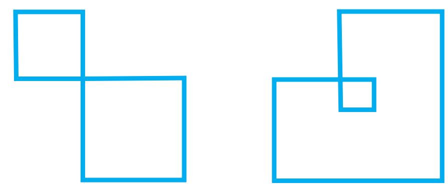
S: ***More people could stand around it.***

Cross out "20 panels" and write the new number of panels under it, in this case, "22 panels."

T: ***I'd like you to help the aquarium designers. But before you start, let me remind you that you can put panels only along the dotted segments on the worksheet*** (hold up a copy of Worksheet G2(a)) ***and that you want the floor of the aquarium to have the longest possible perimeter to maximize viewing space.***

Also, here are two kinds of shapes the designers wish to avoid.

Draw these two shapes on the chalkboard.



T: ***Why do you think the designers wish to avoid shapes like these?***

S: ***In the one on the left, there would be two compartments for fish.***

T: ***Right. The fish in one compartment could not swim into the other.***

S: ***In the one on the right, if the small square were not for fish, they would waste standing space. The people could not get into that small square space.***

S: ***Unless there were underground stairs.***

S: ***If the center square were for fish, they'd have two compartments like in the other shape.***

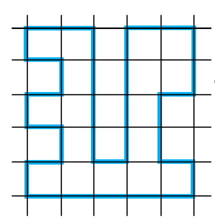
T: ***The designers do not want special stairs or compartments like this.***

Distribute copies of Worksheet G2(a) and erase the grid board.

T: ***You can draw two different designs for the aquarium on this worksheet. Try to get the most viewing space possible. We have one on the board that has 22 panels; see if you can design one that uses more panels. Next to each of your designs, mark how many panels it would use.***

As you observe students' work, ask questions to make them aware of problems. For example, if students draw shapes that are not closed, ask them where the fish are and to trace around their shapes. If students put panels inside or outside their closed shapes, point out that they serve no purpose.

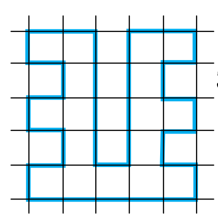
After most students have drawn one or two designs, choose one that has between 30 and 35 panels. Ask the student with that design to draw it on the grid board. For example:



T: ***Can anyone see a way to change this design to include more panels?***

Note: You may need to remind students that the total length or width of the aquarium cannot exceed five panels; that is, the design must not extend beyond the 5-by-5 square.

Perhaps a student will see an alteration that gives a longer perimeter. From the preceding example, such a change in the design is easy to make.



G2

If a student does not see a way to make the border longer in the particular picture displayed and neither do you (this is not always easy to do), ask if someone has a design requiring more panels than the one displayed. Let several students answer, and then choose a student with a 36-panel design to display the design on the board. The preceding illustration shows such a design.

T: ***Wherever two panels come together, the builders will need to put a support post. Who would like to draw dots to show where the posts would go for this design*** (point to the design on the grid board)***?***

While a volunteer is drawing dots for posts, ask other students how many posts are required. Lead to the idea that exactly one post is needed for every panel of glass. A sample dialogue follows.

T: ***How many posts are needed for this design?***

S: ***72 posts; two posts are needed for each panel.***

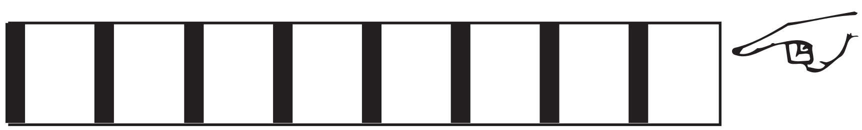
T: ***But that's too many! The panels are not held up separately.***

S: ***36 posts are needed: one post, one panel; one post, one panel; and so on.***

S: ***I think 37 posts are needed, because we still need an extra post for the last panel.***

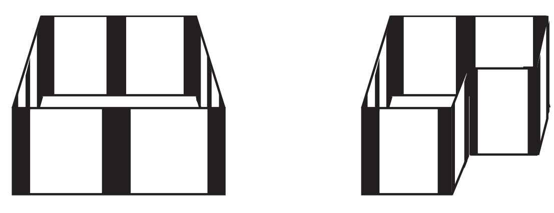
Show the class the strip of paper or cardboard that you made before the lesson began.

T: ***This cut-out shows only eight panels, but I think it will help us. There is one post, one panel; one post, one panel; and so on. Do we need an extra post at the end? What do you think?***



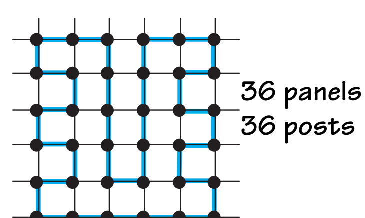
S: ***That glass panel will connect to the starting post.***

Fold the cut-out into a square or other closed aquarium shape to show how the starting post in the strip can support the last panel. For example:



T: ***So we have exactly one post for each panel of glass.***

Refer back to the picture on the grid board.

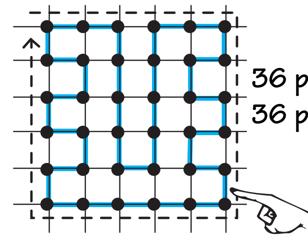


T: ***There is an easy way to look at the picture on the grid board and to decide how many dots for posts there are. Does anyone see it?***

There are 36 dots for posts; there are 6 rows of 6 dots.

If no one makes this observation, do so yourself.

Record the number of posts. With your finger, trace the 5-by-5 square that encloses the design.



T: ***Can we put anymore posts in the square region?***

S: ***No.***

T: ***So can we have more than 36 panels of glass?***

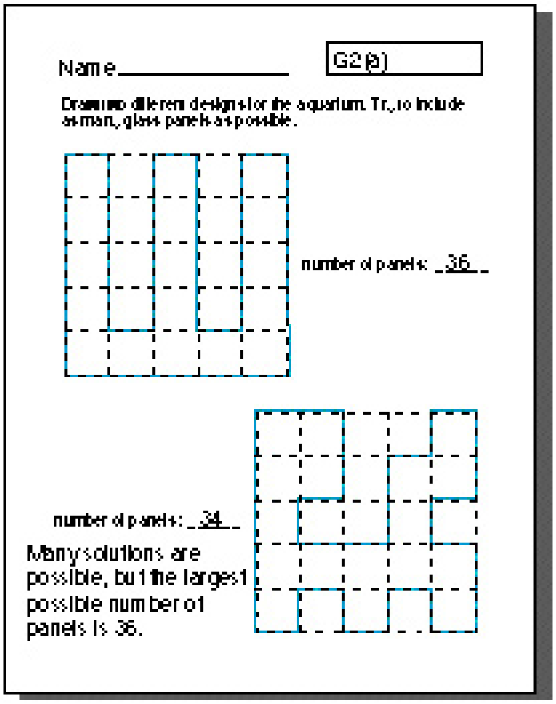
S: ***No; otherwise, we'd need to put in more posts and there is nowhere to put them.***

Distribute copies of Worksheets G2(b) and (c). Direct students first to decide how many possible places there are to put in posts in a 3-by-3 square and in a 4-by-4 square, and then to design aquariums that would fit in each size. Again, remind students that they want to find the maximum perimeter of the floor in terms of panels of glass.

At the end of the class, collect the worksheets for your review and for use again in Lesson G3.

Extension/Home Activity

Suggest that students take home the problems on worksheets G2(b) and (c) to describe to a family member. Then they can work on similar problems for a 6-by-6 square and a 7-by-7 square.



Name

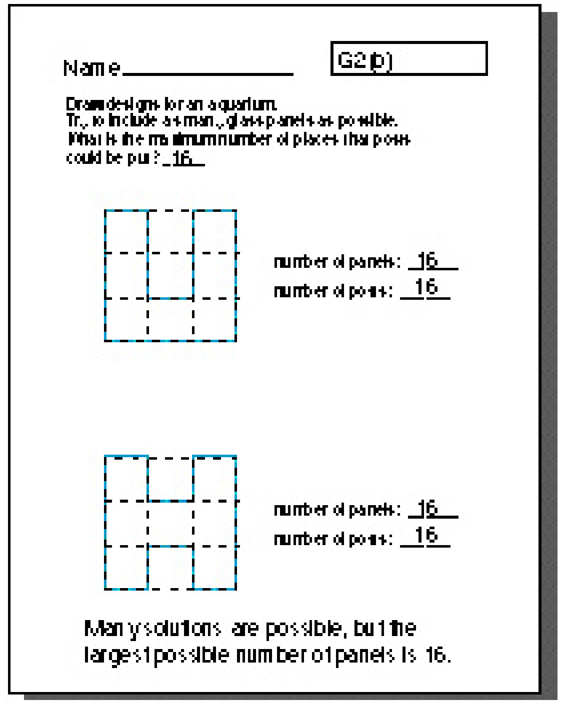
G2(a)

Draw two different designs for the aquarium. Try to include as many glass panels as possible.

number of panels: 36

number of panels: 34

Many solutions are possible, bu1 the larges1 possible number o1 panels is 36.



Name

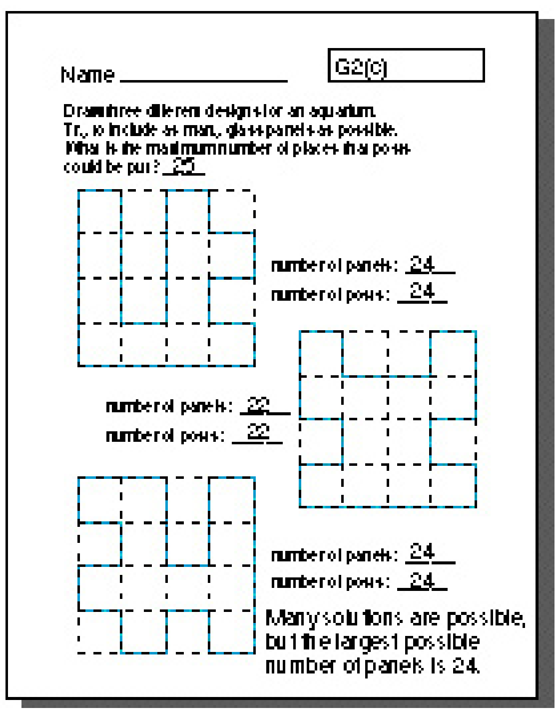
G2(b)

Draw designs for an aquarium. Try to include as many glass panels as possible. What is the maximum number of places that posts could be put? 16

number of panels: 16
number of posts: 16

number of panels: 16
number of posts: 16

Many solutions are possible, bu1 the larges1 possible number o1 panels is 16.



Name

G2(c)

Draw three different designs for an aquarium. Try to include as many glass panels as possible. What is the maximum number of places that posts could be put? 25

number of panels: 24
number of posts: 24

number of panels: 22
number of posts: 22

number of panels: 24
number of posts: 24

Many solutions are possible, bu1 the larges1 possible number o1 panels is 24.

Capsule Lesson Summary

Extend the experiences of Lesson G2 to include rectangles of various sizes that are not in squares. Determine and informally justify that the perimeter of a polygon meeting the necessary requirements is an even number of units. Therefore, if an odd number of grid points is involved, there cannot be a polygon that touches every one. In cases with an odd number of grid points, find a polygon that touches all but one of the grid points.

Materials

Teacher
- Grid board
- Colored chalk

Student
- Worksheets G3(a) and (b)
- Colored pencils, pens, or crayons

Description of Lesson

Review the story about designing aquariums from Lesson G2. Recall that the longest possible perimeter† of the floor of the aquarium in terms of panels of glass is 36 panels and that one post is needed for each panel in any design.

Return the worksheets from Lesson G2 and go over the results. Ask for the number of possible places to put posts, and survey the class for the longest possible perimeter (greatest number of panels) of the aquarium in each problem. As the class arrives at the following results, display them on the board.

Note: If some students worked on the problem for a 6-by-6 square or a 7-by-7 square, they may like to include their results in the table.

Dimensions of the available space	The number of possible places to put posts	Longest possible perimeter
3-by-3	16	16 panels
4-by-4	25	24 panels
5-by-5	36	36 panels
6-by-6	49	48 panels
7-by-7	64	64 panels

Let students comment on the 4-by-4 (or the 6-by-6) case. They should notice that although there are 25 (49) possible places to put posts, in this shape no one designed an aquarium with a perimeter of more than 24 (48) glass panels.

Distribute copies of Worksheets G3(a) and (b) and refer to the worksheets as you say,

T: ***Let's suppose the designers have to find a shape for the floor with the longest possible perimeter that will fit in regions like these. The regions are rectangles, but they are not squares. Look at A, the 7-by-1 rectangle. How many possible places are there to put posts?***

S: ***16 places.***

T: ***How would the designers best use the space?***

S: ***They would use the whole rectangle for the floor.***

†In Lesson G2, the constraints on the design of the aquarium and on the maximum amount of space available were given.

If this is not obvious to the class, copy rectangle **A** onto the grid board and let students try other designs. Rule out designs that have more than one compartment for the fish or that do not stay within the confines of the rectangle.

T: ***And what would the perimeter be in panels of glass?***

S: ***16 panels; seven on each long side and one on each short side.***

Record the information in the chart on the board.

Organize the class into six groups and assign each of the groups one of the rectangles on the worksheets (other than the 7-by-1 picture). Each group should decide how many possible places there are to put posts in their assigned rectangle and then decide the longest possible perimeter. Groups that finish quickly can work with other rectangles on the worksheet.

After about 10–15 minutes, ask each group to report its results and record the results in the chart.

Dimensions of the available space	The number of possible places to put posts	Longest possible perimeter
3-by-3	16	16 panels
4-by-4	25	24 panels
5-by-5	36	36 panels
7-by-1	16	16 panels
6-by-2	21	20 panels
2-by-4	15	14 panels
4-by-5	30	30 panels
3-by-4	20	20 panels
5-by-3	24	24 panels
6-by-4	35	34 panels

T: ***What patterns do you notice in the chart?***

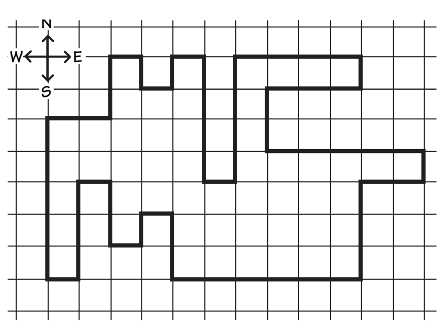
S: ***Sometimes the number of possible places to put posts is the same as the number of panels needed; other times the numbers differ by one.***

S: ***All of the designs with longest perimeter involve an even number of panels.***

S: ***When there is an odd number of possible places to put posts, we cannot get a floor with a perimeter of that many panels.***

The remainder of the lesson leads students to verify these last two observations. Keep the chart on the board until the end of the lesson.

Display a grid board. Indicate compass directions in one corner of the grid or chalkboard. Draw an elaborate design using white chalk for an aquarium floor with a perimeter of reasonable length. For example:

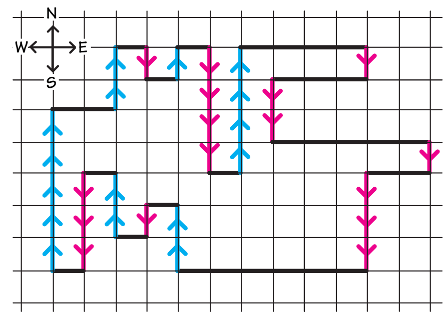


T: ***We've noticed that in some of the designs we use all of the possible places for posts and sometimes we don't.***

Ask a student to choose a grid point as a starting point on the polygon. Divide the class into two groups, for example, girls and boys.

T: ***I'll trace along the edge of this design. Girls, when I trace a segment going north, say, "North," and I'll draw a blue arrow. Boys, when I trace a segment going south, say, "South," and I'll draw a red arrow. Don't say anything if I go east or west.***

Trace the entire polygon on the grid board from the starting point suggested, moving in a clockwise direction. Draw blue and red arrows when the students indicate you are moving north and south, respectively.



T: ***How many blue arrows do we have?*** (16)
Red arrows? (16)

Record the number of each color arrow on the board.

North: 16 arrows
South: 16 arrows

Leave the arrows on the grid board. Point to another starting place, and begin tracing in a clockwise direction around the polygon, again asking the students indicate when your finger moves north and when it moves south. The class should see that the number of red arrows is the same as the number of blue arrows; the choice of a starting place does not affect the numbers.

Begin at another grid point on the polygon and ask what happens if you trace counterclockwise (in the opposite direction). The class should observe that the arrows change color, red to blue and vice versa, but the number of blue arrows is still the same as the number of red arrows.

T: ***Do you think that we could ever start somewhere on an aquarium design, go all the way around it, returning to where we started, and find that we went north more than we went south?*** (No)

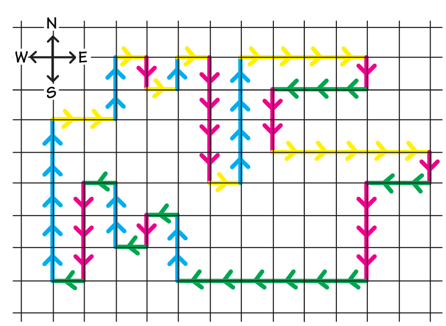
This may be obvious to some students and not so obvious to others.

T: ***If we go north as much as we go south, the number of red and blue arrows together must be what kind of number?***

S: ***Even; the double of any number is even.***

S: ***If you add an even number to itself, you get an even number. If you add an odd number to itself, you get an even number.***

Repeat the activity, using yellow and green arrows for east and west. Do not erase the red and blue arrows. Your picture might look similar to this one.



Your students should conclude that wherever you start tracing an entire aquarium design (in a clockwise or counterclockwise direction), that the number of green and yellow arrows is the same and that the total number of these arrows is even. Record the number of each color arrow on the board.

North: 16 arrows	East: 15 arrows
South: 16 arrows	West: 15 arrows

T: ***To find how many panels are needed for this design, what computation should we do?***

S: ***Add 16 + 16 + 15 + 15.***

S: ***Add 16 + 15 and then double.***

S: ***Add 32 + 30.***

T: ***For any aquarium design, we could do the same. We could find the numbers of north, south, east, and west arrows and add them.***

Since the total number of blue (north) and red (south) arrows is even, and the total number of yellow (east) and green (west) arrows is even, what kind of number do we get?

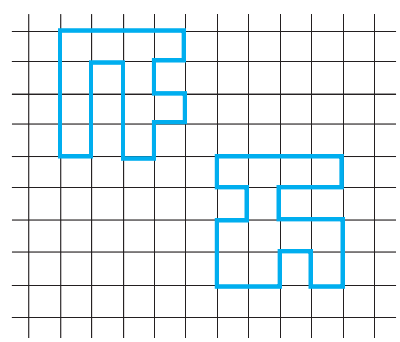
S: ***Even; an even number plus an even number is even.***

S: ***In this case, we get 62 panels. 16 + 16 = 32; 15 + 15 = 30; 32 + 30 = 62. 62 is an even number.***

T: ***What does this tell us about the floor design of an aquarium?***

S: ***It must have an even number of panels.***

T (referring back to the chart)**:** ***So even though in the 4-by-4 case there are 25 places to put posts, we can only use an even number of them. We found we could use 24 of them. We could say similar things about the 6-by-2 case, the 2-by-4 case, and the 6-by-4 case.***

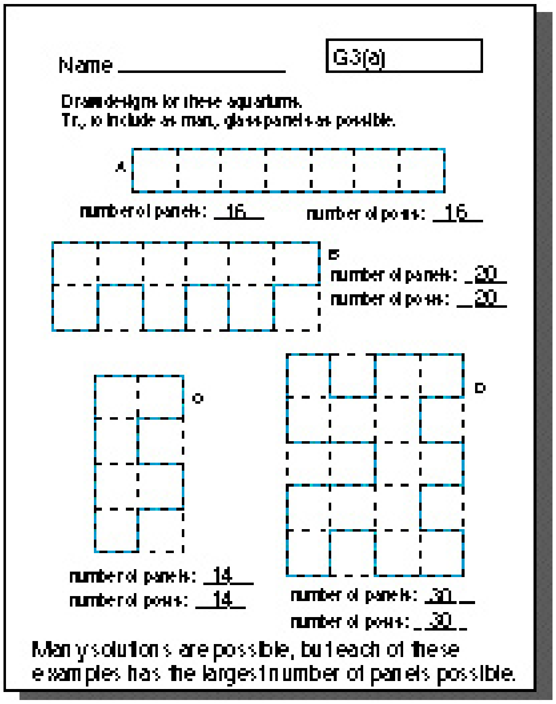


Erase the grid board. If time allows, show these two designs for the 4-by-4 case. Each has the maximum perimeter of 24 panels.

Invite two students to find the perimeters of the two designs.

T: ***These designs both use 24 panels of glass. Do you think the designers would prefer one over the other or does it matter?***

Allow students to comment. If no one mentions that the amount of space differs for the fish in the two picture, (i.e., they have different areas), lead to this observation yourself.



Name ______________ G3(a)

Draw designs for these aquariums.
Try to include as many glass panels as possible.

A

number of panels: 16 number of posts: 16

B

number of panels: 20
number of posts: 20

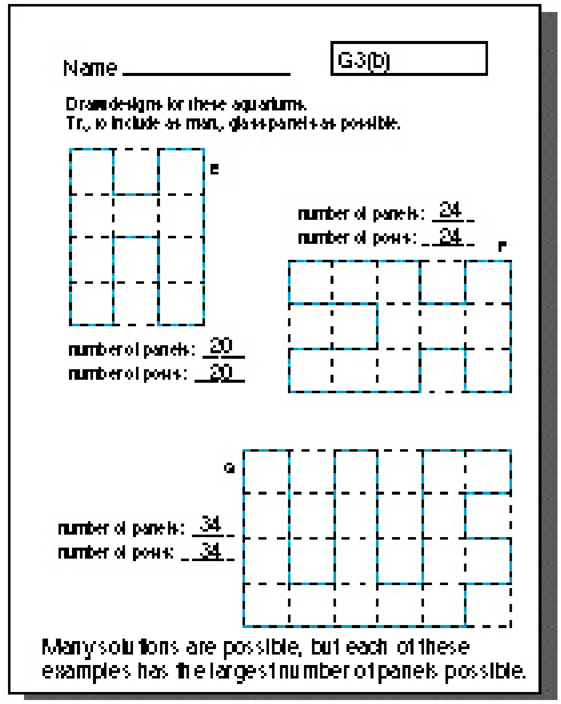
C

number of panels: 14
number of posts: 14

D

number of panels: 30
number of posts: 30

Many solutions are possible, but each of these examples has the largest number of panels possible.



Name ______________ G3(b)

Draw designs for these aquariums.
Try to include as many glass panels as possible.

E

number of panels: 20
number of posts: 20

number of panels: 24
number of posts: 24

F

G

number of panels: 34
number of posts: 34

Many solutions are possible, but each of these examples has the largest number of panels possible.

Capsule Lesson Summary

Envision certain shapes created by a mirror placed on or by a picture. Discover the characteristics of a reflection by finding designs that can and cannot be made using a triangle and a mirror. Write a message so that it is legible when viewed in a mirror.

Materials

Teacher
- Teacher triangle set
- Mirror

Student
- Mirror
- Worksheets G4(a), (b), (c), (d), (e), (f), and (g)

Advance Preparation: Prepare a teacher's demonstration triangle set by carefully cutting the triangles out and putting magnetic material or some kind of adhesive material on their backs.

Description of Lesson

Exercise 1

Distribute mirrors and copies of Worksheets G4(a) and (b) to students.

T: ***Look at Worksheet G4(a). Do you see a full moon?***

S: ***No.***

T: ***Can you use the mirror to see a full moon?***

Instruct students to answer the questions on Worksheets G4(a), and to be ready to demonstrate their answers. Observe students' efforts and give assistance when necessary. When most students have answered all of the questions, focus their attention on Worksheet G4(b).

T: ***Can you use the mirror to find a rabbit in Figure 1?***
Can you find a butterfly in Figure 2? a lady bug in Figure 3?

Distribute copies of Worksheet G4(c).

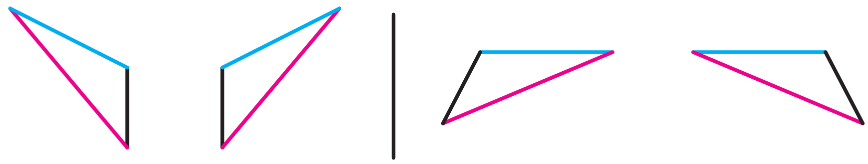
T: ***Here are some shapes you can see using the mirror and the figures on Worksheet G4(b). See how many you can find. Circle the shapes on Worksheet G4(c) as you find them.***

Give students several minutes to work individually. When some students have found all or most of the shapes, direct the class's attention to Worksheet G4(d) on the back of G4(c). Ask students to describe the triangle pictured.

T: ***See if you can use the triangle and your mirror to make this design.***

Place two demonstration triangles on the board, as shown here.

Check the work of a few students; then continue the activity with the following designs.



Distribute copies of Worksheets G4(e), (f), and (g).

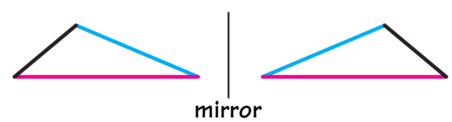
T: ***Which of these designs can be seen using a mirror and the triangle? Circle the numbers of the designs you can see.***

Allow several minutes for students to work on these problems. Your students will discover that all of the designs are possible except those numbered 4, 5, 8, 10, and 11. When most students have completed Worksheets G4(e) and (f), discuss them collectively. Ask students to explain why particular designs cannot be seen.

Place two demonstration triangles on the board as shown here.

T: ***We saw this design before with the triangle and a mirror. Place the triangle and your mirror so you can see this design again. Who can draw a line segment to show where the mirror would be?***

Invite a student to draw the line segment.
Accept any reasonable approximation.



T: ***How can we check that this line[†] shows us where to place the mirror?***

S: ***Put a mirror on the line and see if you get the same design.***

Direct students to return to Worksheets G4(e), (f), and (g) and to draw line segments to show where the mirror would be in the designs they were able to make. Ask them to check their work in each picture by placing the mirror on the line segment. What they see with and without the mirror should appear to be the same.

Exercise 2

Instruct everyone in the class to face the back of the room. Then write the following message on the board.

CAN YOU READ THIS?

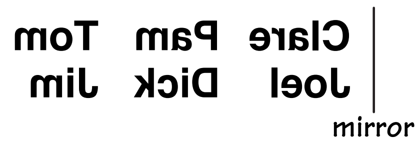
Note: If you have difficulty mirror writing, prepare this message on a large piece of paper before class, or write **CAN YOU READ THIS?** on a piece of tracing paper and tape it face down on the board.

Now direct students to hold up their mirrors (while still facing the back of the room) and read what is on the board. Then tell students to turn around and look at the board.

[†]Since the line segment can be extended infinitely, we will refer to this segment in pictures as the "mirror line."

Your students should be intrigued by this function of the mirror. You may want to mention that several people, including Leonardo DaVinci and Lewis Carroll, often wrote this way, DaVinci to protect his notes and Carroll just for fun. A class project could be to find some samples of these writings.

In the remaining time, ask students to write their names using mirror writing. Some examples are shown below.

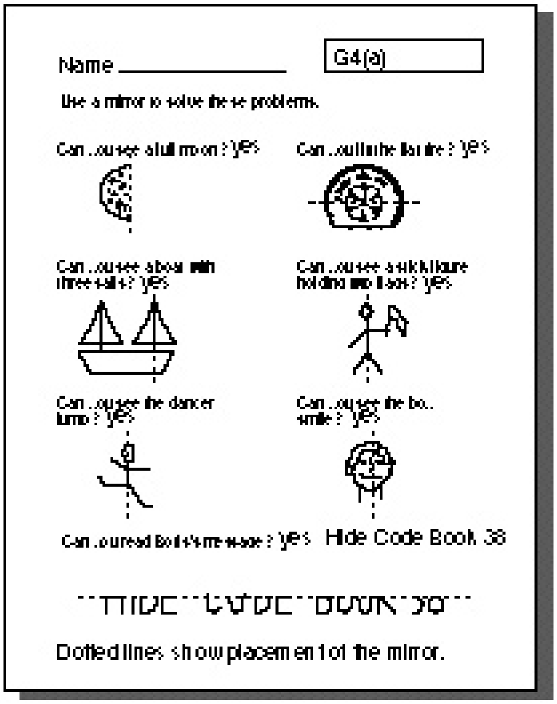


Writing Activity

Suggest that students write a short message in mirror writing and ask a friend to read it.

Extension Activity

Find some samples of writings by Leonardo DaVinci or Lewis Carroll that require a mirror to decipher.



Name ____________ G4(a)

Use a mirror to solve these problems.

Can you see a full moon? yes

Can you fix the flat tire? yes

Can you see a boat with three sails? yes

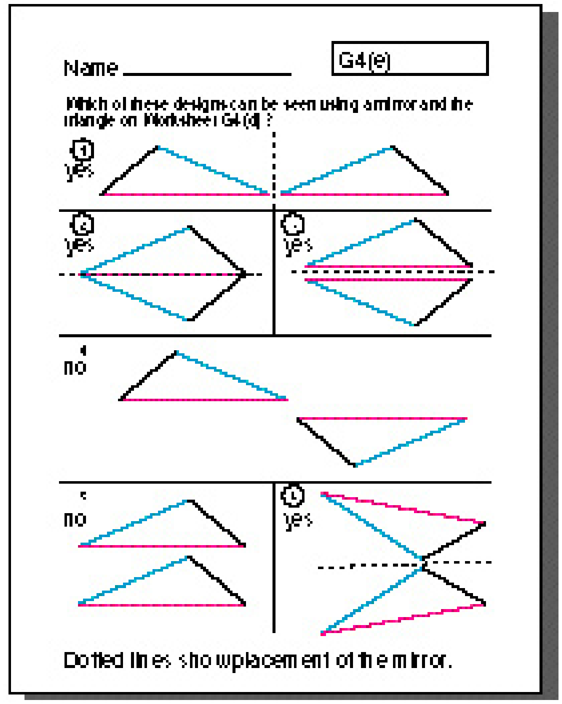
Can you see a stick figure holding two flags? yes

Can you see the dancer jump? yes

Can you see the boy smile? yes

Can you read Bob's message? yes Hide Code Book 38

Dotted lines show placement of the mirror.



Name ____________ G4(e)

Which of these designs can be seen using a mirror and the triangle on Worksheet G4(d)?

1 yes

2 yes

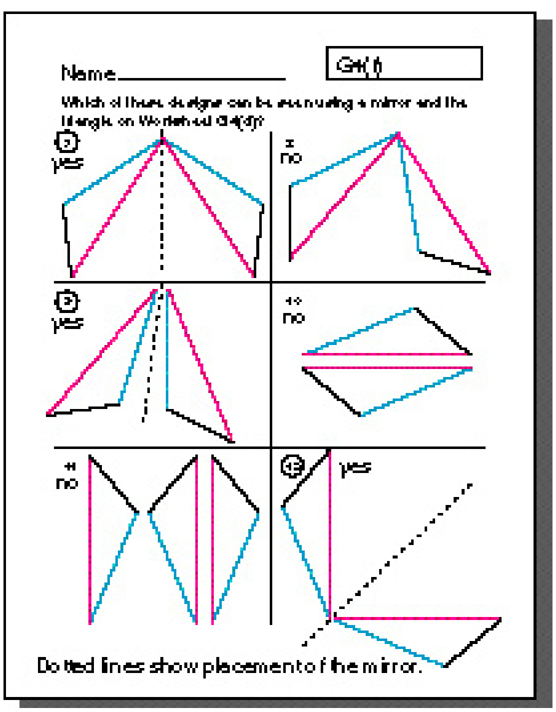
3 yes

4 no

5 no

6 yes

Dotted lines show placement of the mirror.



Name ____________ G4(f)

Which of these designs can be seen using a mirror and the triangle on Worksheet G4(d)?

7 yes

8 no

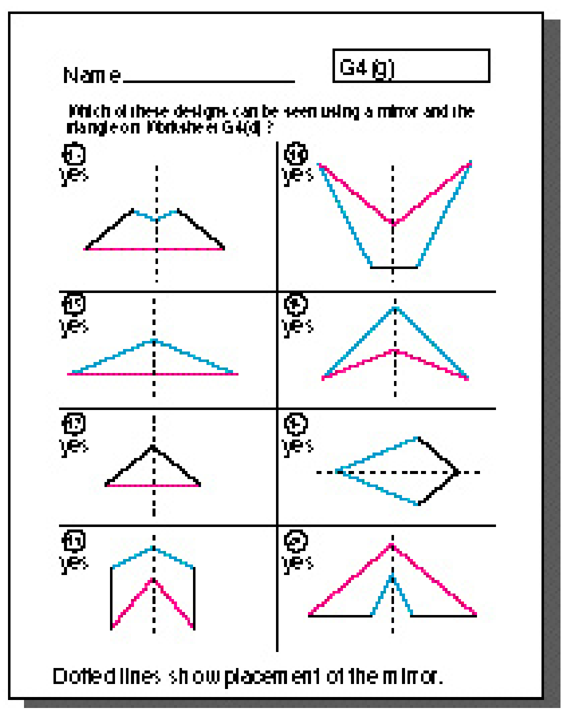
9 yes

10 no

11 no

12 yes

Dotted lines show placement of the mirror.



Name ____________ G4(g)

Which of these designs can be seen using a mirror and the triangle on Worksheet G4(d)?

13 yes

14 yes

15 yes

16 yes

17 yes

18 yes

19 yes

20 yes

Dotted lines show placement of the mirror.

Capsule Lesson Summary

Build a shape using triangles. Find all of the different quadrilaterals that can be built using exactly two of the triangles. Play *The Reflection Game* with three different designs.

Materials

Teacher
- Teacher triangle set
- Mirror

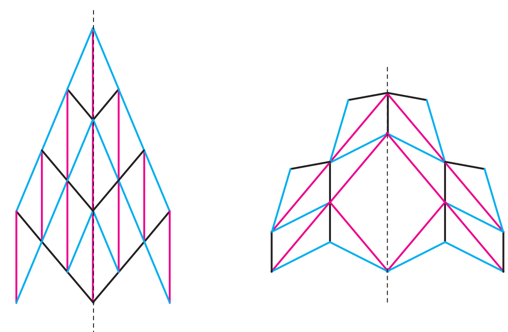
Student
- Student triangle set
- Mirror
- Worksheets G5*, **, and ***
- Straightedge

Description of Lesson

You may like to let students work in pairs during most of this lesson.

Exercise 1

Distribute student triangle sets and mirrors. Suggest students use the triangles to make a design and that they show you their favorites. As you observe their work, you are likely to see many students building designs with a line of symmetry similar to those illustrated here.

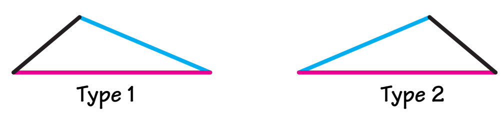


In these designs, when you place a mirror on the dotted lines, you see the same design in the mirror. You may wish to use a mirror to show students this property of their designs.

Exercise 2

Display the 20 demonstration triangles (teacher triangle set) on one side of the board.

T: ***Sort your triangles into two piles, each pile containing the same type of triangles.***



Ask students to explain how these two types of triangles are the same and how they are different. Students should notice that these triangles are all the same shape and size, and that the same color sides are the same length.

T: ***How are they different?***

S: ***They are opposites of each other.***

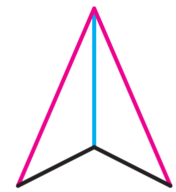
T: ***Let's see what that means. Look at a triangle from one of the piles in your mirror. What do you see?***

S: ***A triangle that belongs in the other pile.***

T: ***What is another way of saying that they are opposites?***

S: ***They are reflections of each other.***

Place two demonstration triangles together on the board, as shown here.



T: ***How many sides does this shape have?***

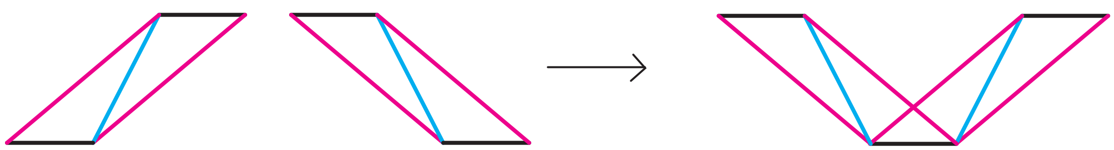
If students have difficulty seeing that the shape has four sides, trace the shape with chalk and then remove the triangles.

T: ***What is another name for a four-sided shape?***

S: ***Quadrilateral.***

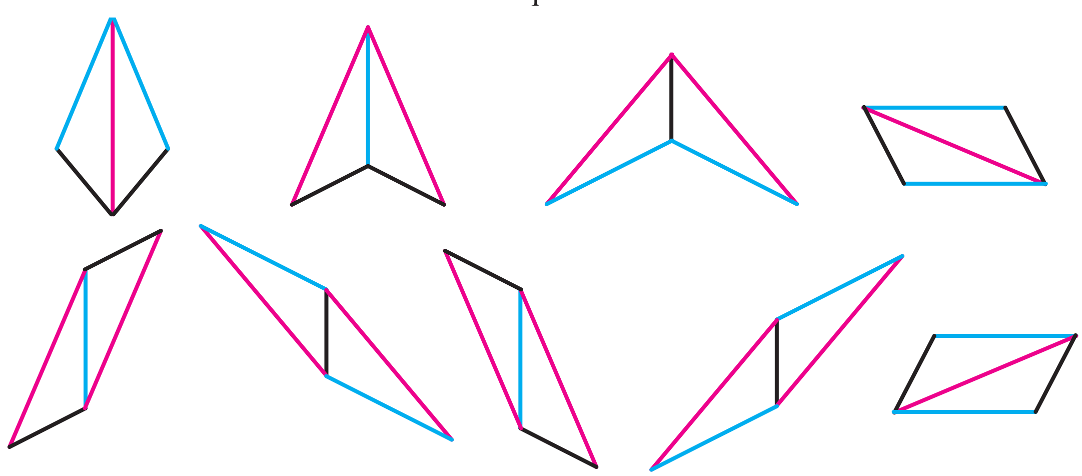
T: ***Use exactly two triangles and see how many different quadrilaterals you can make.***

Allow several minutes for individual or partner work; then call on students to form different quadrilaterals on the board with the demonstration triangles. Each time a new quadrilateral is displayed, ask the class if it is different from the ones displayed so far. If students feel two quadrilaterals are the same, check by placing one on top of the other (without flipping). If they match, remove one of them from the board and continue. Caution: some students may think that the following two quadrilaterals are the same.



However, placing one on top of the other (without flipping) reveals that they do not match. Students may notice though that the quadrilaterals are reflections of each other.

Continue until the class discovers nine different quadrilaterals.



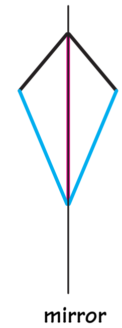
Exercise 3

Draw a vertical line segment approximately one meter long on the board and label it **mirror**.

T: ***Today we are going to play a game called The Reflection Game. In this game, we pretend that this line segment is a mirror.***

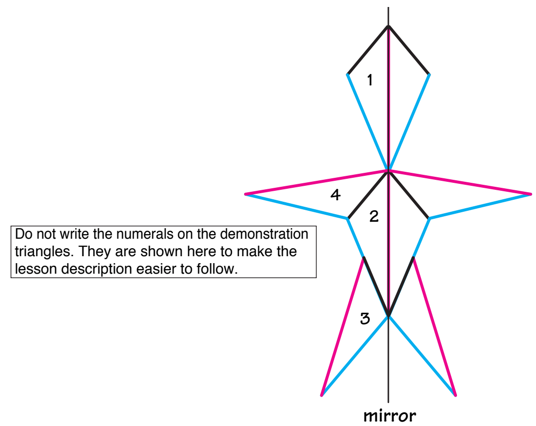
Place a demonstration triangle with its red side flush on the line segment as shown (mirror line).

T: ***If this line segment represents the edge of a mirror, who can show me what we would see in the mirror? Use your triangles and mirror to help you.***

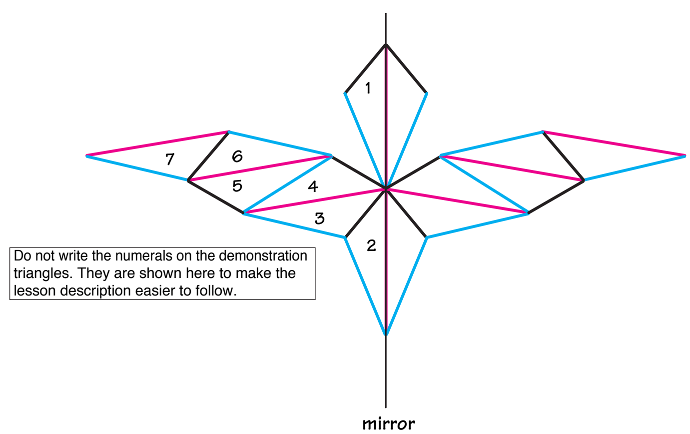


Invite a student to place a demonstration triangle on the board to show the reflection of the original triangle.

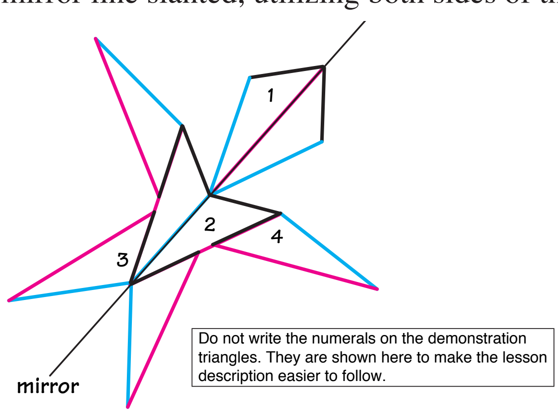
Ask for class agreement, and then continue placing triangles on the board in the order indicated by the numbered triangles here. The correct student responses are shown as triangles without numbers.



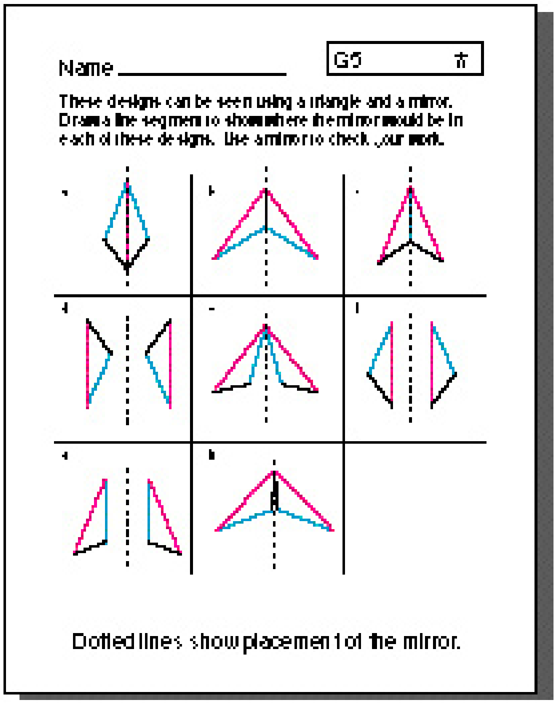
Remove all of the triangles from the board and play the game again, generating the following picture.



Repeat the activity with the mirror line slanted, utilizing both sides of the line.



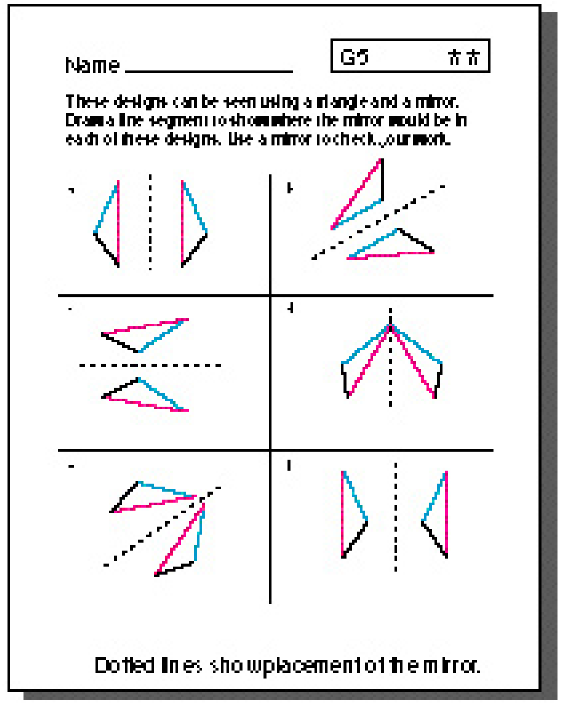
Worksheets G5*, **, and *** are available for individual work.



Name ______________ G5 ☆

These designs can be seen using a triangle and a mirror. Draw a line segment to show where the mirror would be in each of these designs. Use a mirror to check your work.

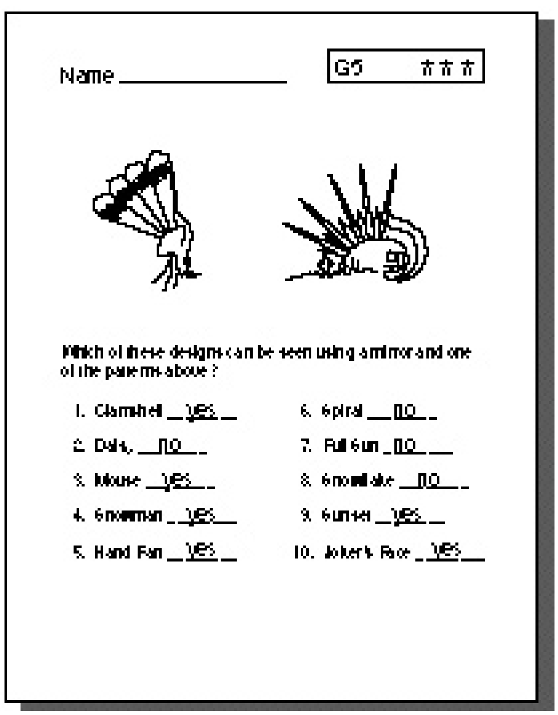
Dotted lines show placement of the mirror.



Name ______________ G5 ☆☆

These designs can be seen using a triangle and a mirror. Draw a line segment to show where the mirror would be in each of these designs. Use a mirror to check your work.

Dotted lines show placement of the mirror.



Name ______________ G5 ☆☆☆

Which of these designs can be seen using a mirror and one of the patterns above?

1. Clamshell __YES__
2. Daisy __NO__
3. Mouse __YES__
4. Snowman __YES__
5. Hand Fan __YES__
6. Spiral __NO__
7. Full Sun __NO__
8. Snowflake __NO__
9. Sunset __YES__
10. Joker's Face __YES__

Capsule Lesson Summary

Discover the characteristics of the reflection of a point. Play *The Reflection Game* using magnetic checkers. Draw the reflection of a picture by reflecting and connecting several points in the picture.

Materials

Teacher
- Large ball
- Magnetic checkers

Student
- Mirror
- Straightedge
- Worksheets G6*, **, ***, and ****

Description of Lesson

Exercise 1

Distribute mirrors. Direct students to place the mirrors flat on their desks with the shiny side faceup. Hold a large ball in front of you.

T: ***Hold the eraser of your pencil over the mirror. Where does the reflection of your eraser appear to be?***

S: ***Directly below the eraser.***

T: ***Now pretend the floor is a mirror. Where would the reflection of this ball appear to be?***

S: ***Directly below the ball and below the floor.***

Let a student point to a spot on the floor directly below the ball.

T: ***Suppose we dropped this ball. What direction would it fall?***

S: ***Down.***

T: ***What direction would the reflection of the ball appear to be moving?***

S: ***Up.***

Encourage students to test this conjecture by moving their pencils up and down while observing the pencil's reflection. Then ask students to hold their mirrors in a vertical position.

T: ***Now let's pretend that the wall is a mirror.*** (Stand close to the wall.) ***Who can show where the reflection of the ball appears to be?***

Let a student point to the location on the wall.

T: ***If we were holding the ball 1.5 meters off the ground, how far above the floor would the reflection of the ball appear to be?***

S: ***1.5 meters, because it would appear to be the same distance from the floor.***

T: ***If we drop the ball, in which direction will it go?***

S: ***Down.***

T: ***In which direction will its reflection go?***

S: ***Down.***

Encourage students to test this with their pencils and mirrors.

T: ***Why did the reflection fall in the same direction this time and in a different direction before?***

S: ***Because in the first experiment the ball was moving toward the mirror, and this time the ball is always the same distance away from the mirror.***

S: ***When the ball moves toward the mirror, the reflection appears to get closer. The ball and its reflection go in opposite directions.***

T: ***Let's pretend that the floor, the wall, and the ceiling are mirrors and that we can throw the ball up or down. What should we do and where should we look so that the reflection of the ball will appear to be falling downward?***

S: ***Throw the ball up and look at the floor mirror.***

S: ***Throw the ball down and look at the wall mirror.***

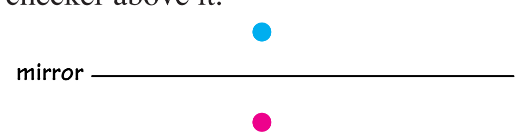
S: ***Throw the ball up and look at the ceiling mirror.***

Students should verify each of theses situations for themselves, using their pencils and mirrors.

Exercise 2

Draw a line segment on the board and place (draw) a blue checker above it.

T: ***If this line segment represents the edge of a mirror and the blue checker a ball, who can show us with a red checker where the reflection would be?***

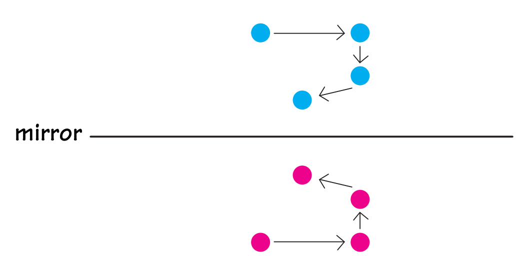


Move only the blue checker up several centimeters.

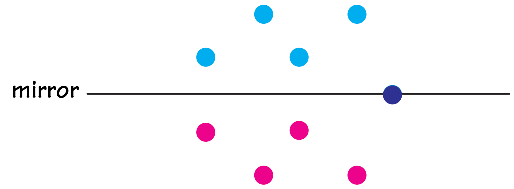
T: ***Is the red checker still the reflection of the blue checker?***

S: ***No, the red checker should be the same distance from the mirror line as the blue checker is.***

Ask a student to move the red checker to the correct position. Refer to the mirror-and-eraser model if some students have difficulty with this exercise.

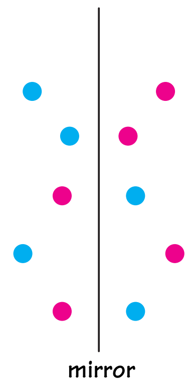


Continue in the same manner, moving the blue checker as indicated here. Student responses are indicated by red checkers and arrows.

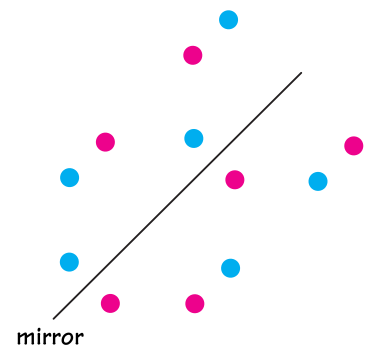


Next, place several checkers one at a time above the mirror line, and ask students to place red checkers to show the reflection of each blue checker.

Repeat the exercise with the mirror line in a vertical position, placing blue checkers on either side of it.



Repeat the exercise again with a slanted line segment.



Exercise 3

Distribute copies of Worksheet G6*.

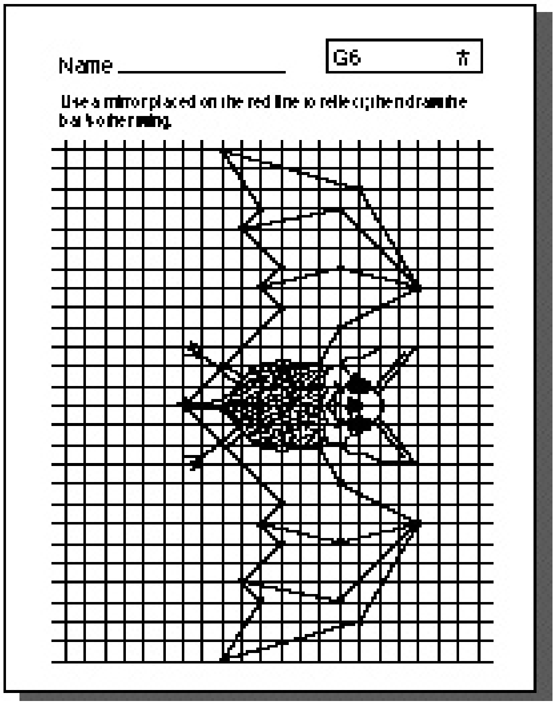
T: ***What do you see on this worksheet?***

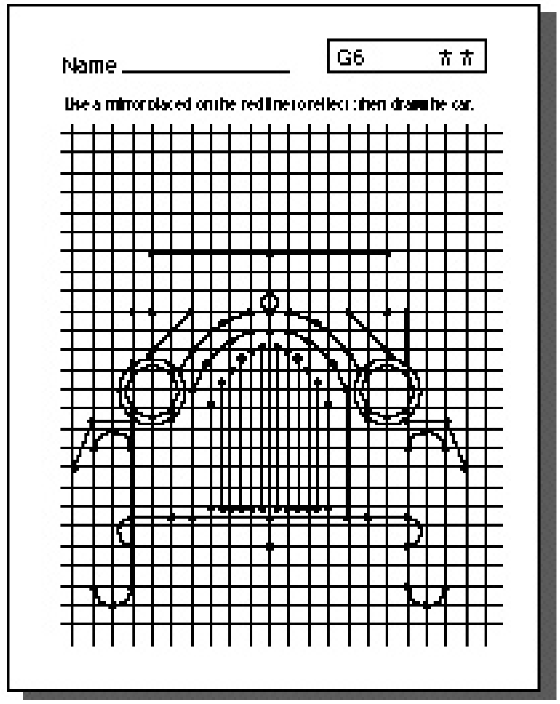
S: ***A bat with a missing wing.***

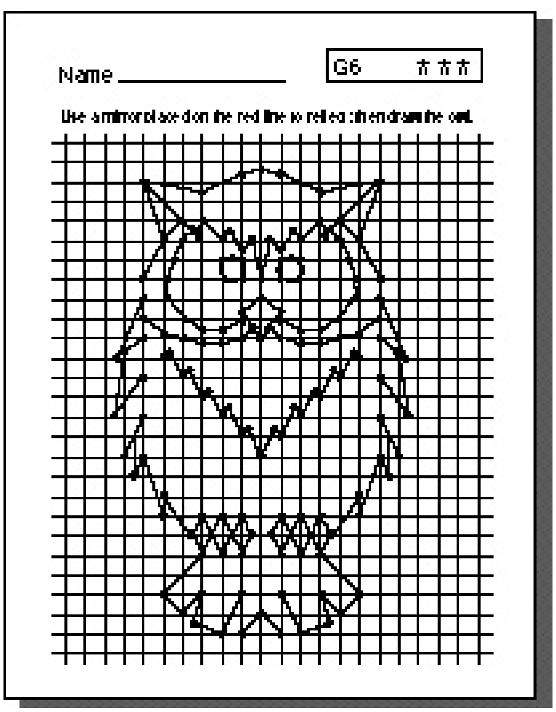
T: ***Place your mirror on the red line. Now what do you see?***

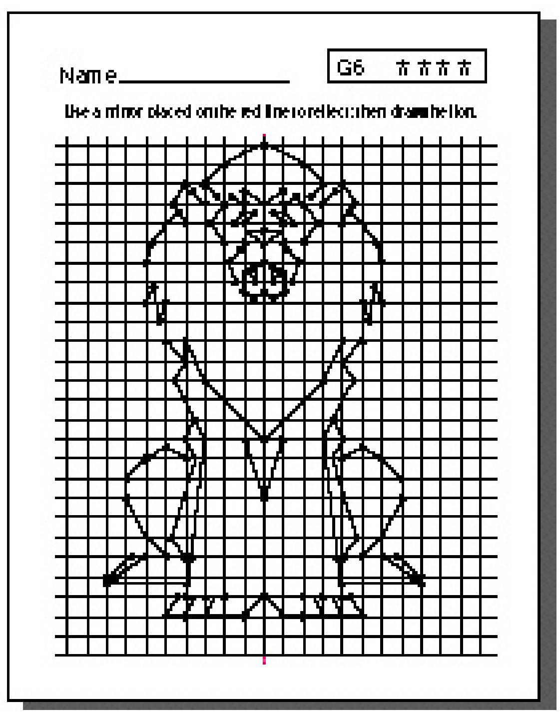
S: ***A bat with two wings.***

T: ***Draw the bat's other wing by drawing the reflections of the dots and then connecting them with a straightedge. Use your mirror to help you and also to check your work.***


Name
G6 ☆
Use a mirror placed on the red line to reflect; then draw the bat's other wing.


Name
G6 ☆☆
Use a mirror placed on the red line to reflect; then draw the car.


Name
G6 ☆☆☆
Use a mirror placed on the red line to reflect; then draw the owl.


Name
G6 ☆☆☆☆
Use a mirror placed on the red line to reflect; then draw the lion.

Capsule Lesson Summary

Find the reflections of several triangles that are not touching the mirror. Play *The Reflection Game*. Define the concept "line of symmetry," and find lines of symmetry for several shapes.

Materials

Teacher	• Teacher triangle set	• Student triangle set • Worksheets G7 (no star), G7*, **, and *** • Paper
Student	• Mirror	

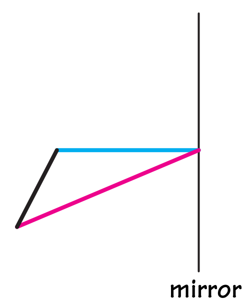
Description of Lesson

Distribute mirrors and student triangle sets. Encourage students to use these tools to mimic the triangle placement in the collective discussion.

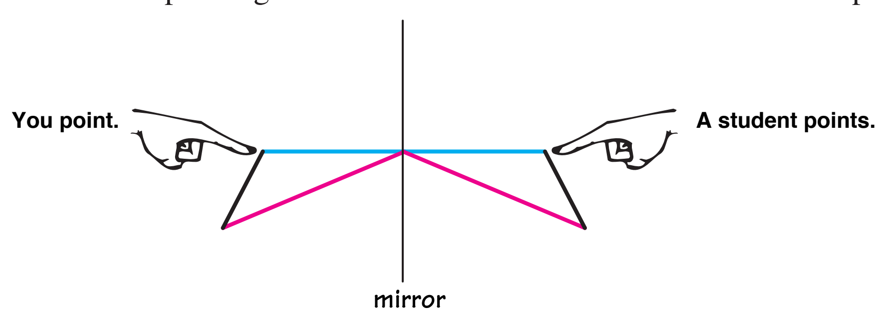
Exercise 1

On the board, draw a vertical line segment as the mirror line and place a demonstration triangle to the left of it, as shown here.

T: ***Who can show us the reflection of this triangle?***



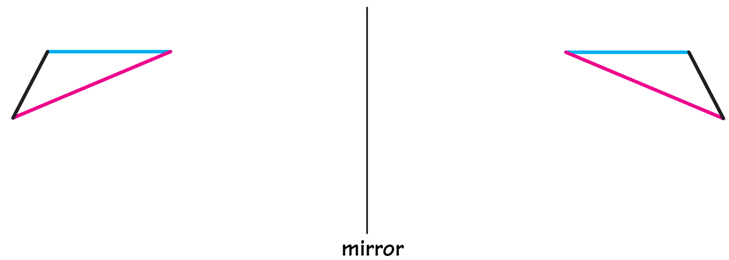
Invite a student to place another demonstration triangle to represent the reflection of the triangle. After the reflection is placed, point to each of the corners of the original triangle, and ask students to point to where the corresponding corners of the reflection should be. For example:



This activity should bring out any errors in the placement of the reflection triangle. Allow students to make modifications. Then move the original triangle several centimeters to the left.

T: ***Is the other triangle still the reflection of this triangle?*** (No)

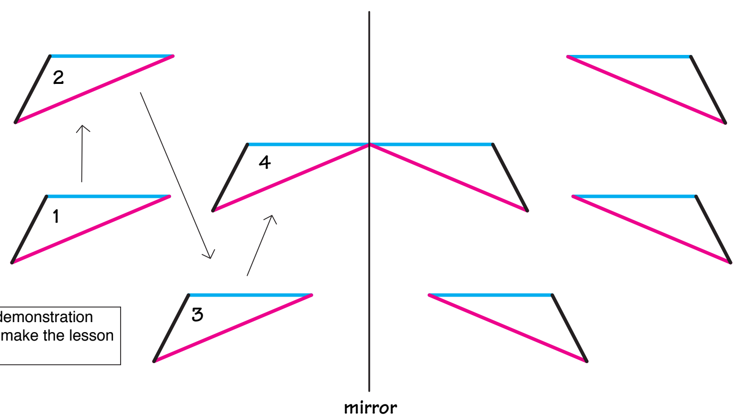
Invite a student to move the reflection triangle to its proper location.



T: ***How can we check to see if the reflection triangle is correctly placed?***

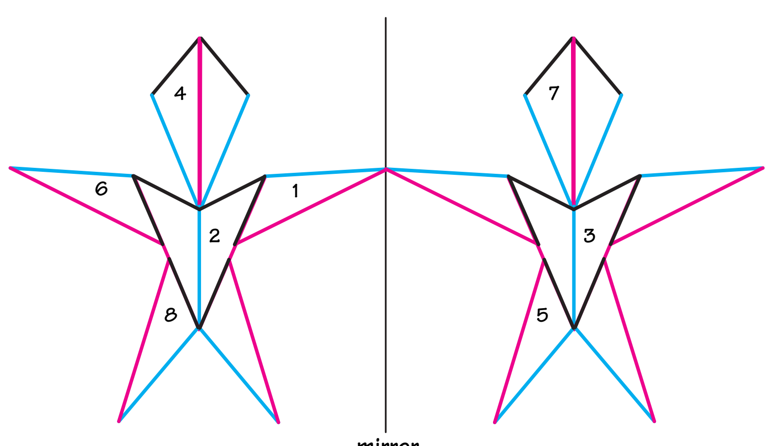
S: ***The reflection triangle should be the same distance from the line and directly across from the other triangle.***

Let students come to the board and check the placement. Direct the checking by pointing to each of the corners and asking where the reflection of that corner should be. Allow students to use string, a ruler, or any other measuring device to measure distances. Continue in the same manner, moving the original triangle as indicated by the arrows below. The reflections of the triangle are shown below on the right of the mirror line.



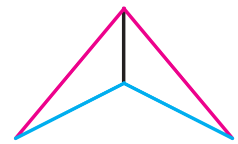
Do not write the numerals on the demonstration triangles. They are shown here to make the lesson description easier to follow.

Play *The Reflection Game* with the following picture. Place triangles on the board in the order indicated by the numbered triangles in the picture below. The reflections of these triangles are show without numerals.



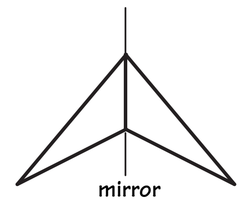
Do not write the numerals on the demonstration triangles. They are shown here to make the lesson description easier to follow.

Place two demonstration triangles on the board, as shown here. Instruct students to make this shape with their triangles.



T: ***Who can draw a line segment to show us where we could place a mirror on this shape and still see the whole shape?***

Outline the shape with chalk, and remove the triangles from the board.



T: ***What would happen now if we placed a mirror on this line?***

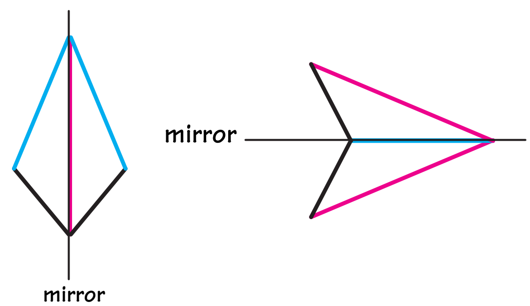
S: ***You would still see the whole shape.***

S: ***Why?***

S: ***Because what is on one side of the line is the reflection of the other side.***

Repeat this exercise with the following two shapes.

T: ***A line on which we can place a mirror and still see the whole shape has a special name. It is called a* line of symmetry.**

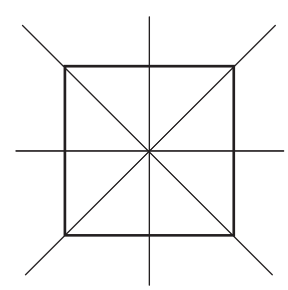


Write "line of symmetry" on the board. Distribute copies of Worksheet G7 (no star) and draw a large square on the board.

T: ***There is a square on Worksheet G7. Where can we draw a line of symmetry for a square?***

Let students come to the board and draw lines of symmetry for the square. Encourage all students to draw the lines on their worksheets and to verify each solution with a mirror.

The class should find four lines of symmetry for the square.



Instruct students to draw lines of symmetry for the other shapes on the worksheet (front and back). After a few minutes, discuss some of the shapes collectively.

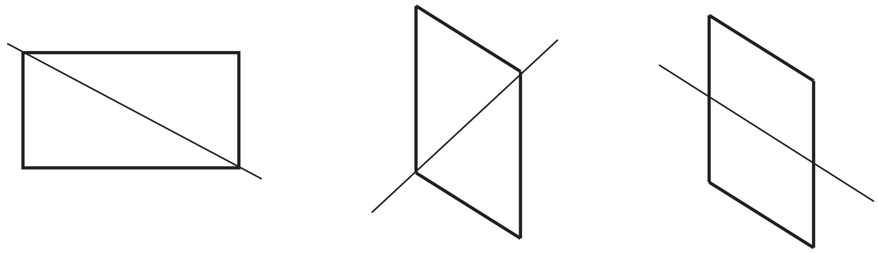
Some students may think that any line bisecting a shape is a line of symmetry.

T: ***What do you notice about these lines of symmetry?***

S: ***They each cut the shape in half.***

T: ***Is any line that cuts a shape in half a line of symmetry? Try to find a line that cuts a shape in half but is not a line of symmetry.***

There are several examples that students might find. For example:

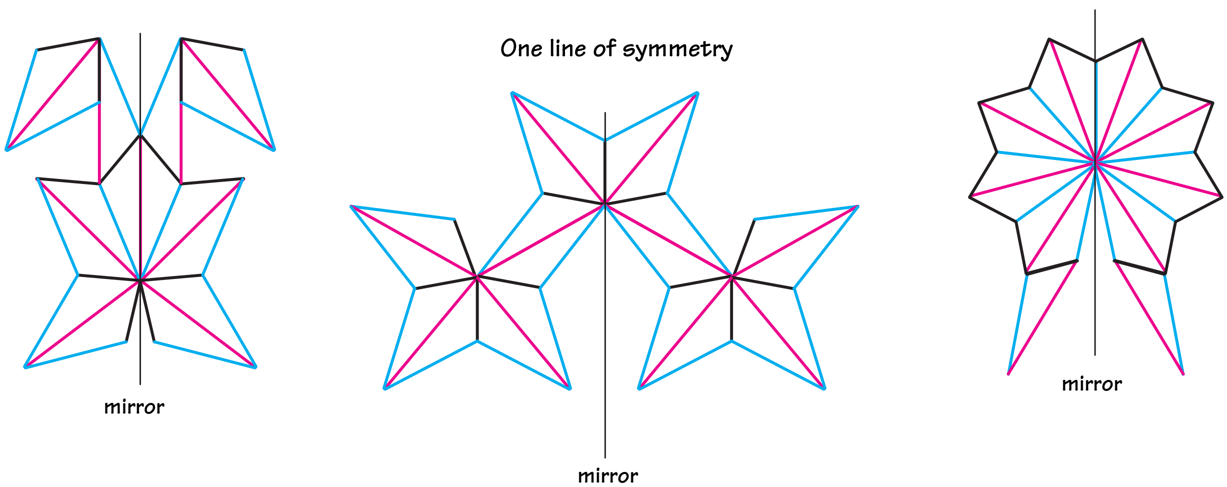


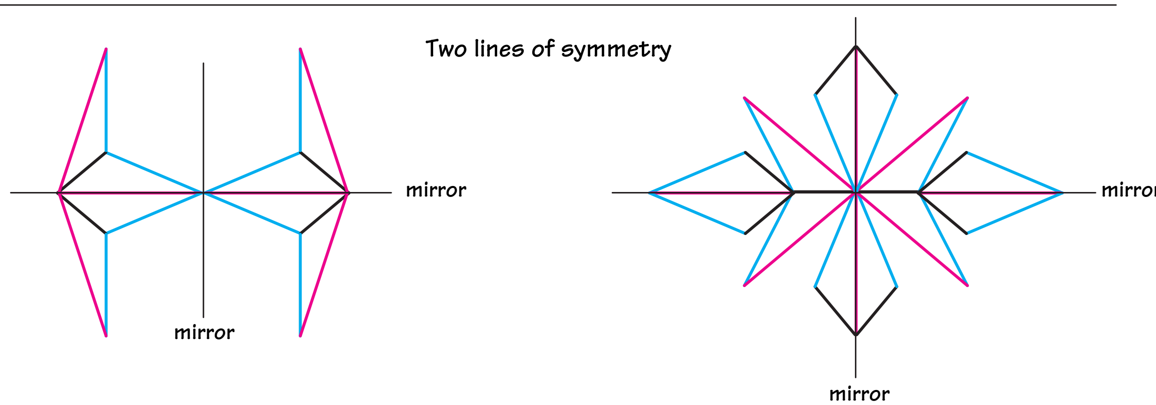
T: ***Why are these not lines of symmetry?***

S: ***Because you don't see the original shape when a mirror is put on the line.***

S: ***The part of the shape on one side of the line is not the reflection of the part on the other side of the line.***

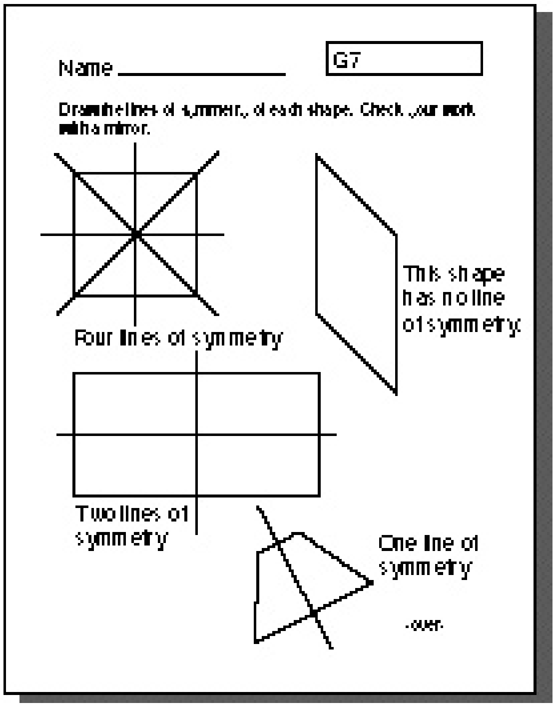
Direct students to use their triangles to build a shape with exactly one line of symmetry and another shape with exactly two lines of symmetry. There are, of course, many different solutions to this problem. The following are examples of shapes the students may create.

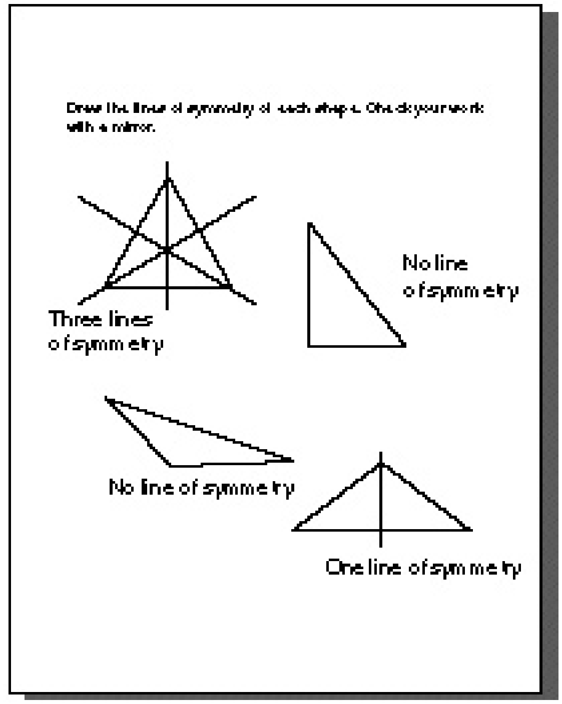


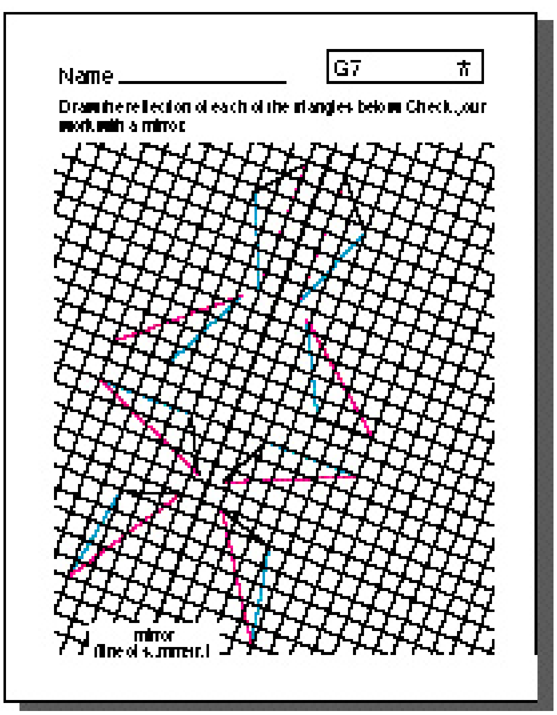


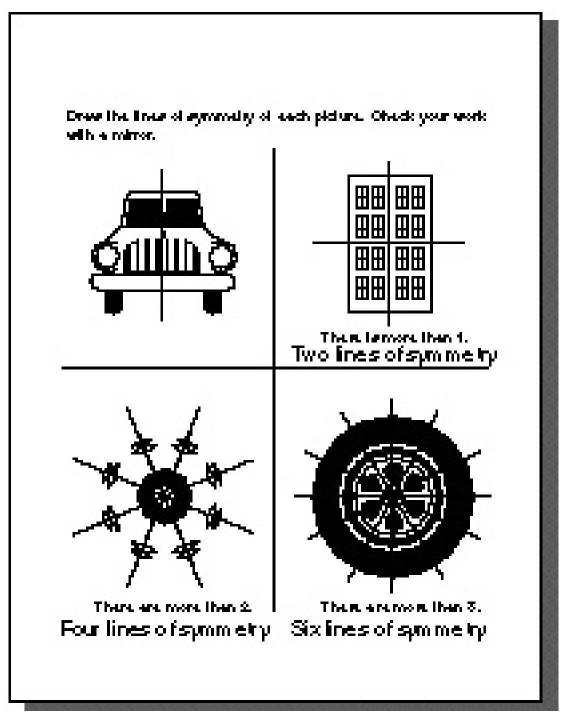
Ask students to indicate the lines of symmetry in their shapes and to use a mirror to verify the symmetry.

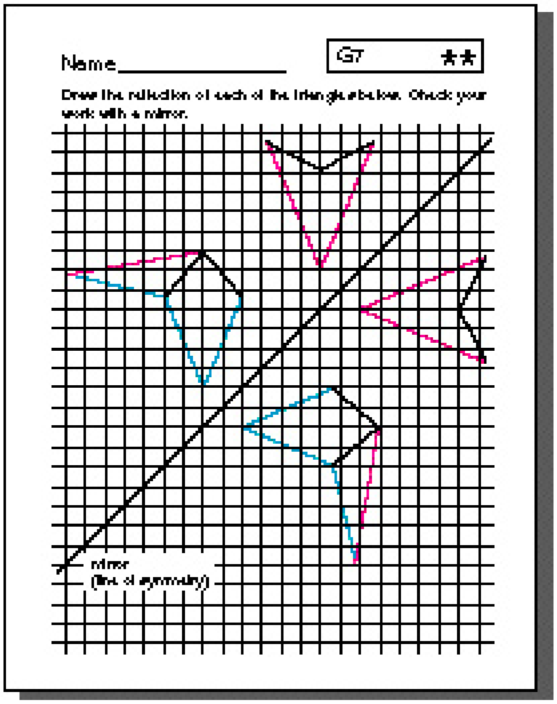
Worksheets G7*, **, and *** are available for individual work.

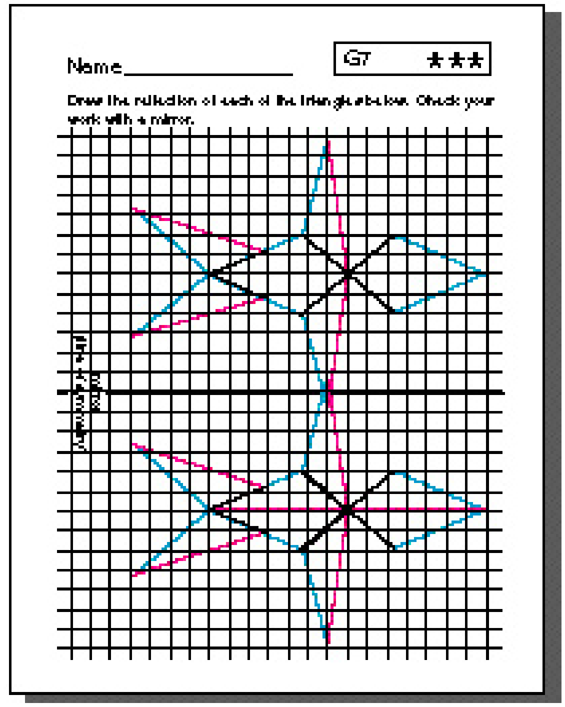

Name
G7
Draw the lines of symmetry of each shape. Check your work with a mirror.
Four lines of symmetry
This shape has no line of symmetry.
Two lines of symmetry
One line of symmetry
-over-


Draw the lines of symmetry of each shape. Check your work with a mirror.
Three lines of symmetry
No line of symmetry
No line of symmetry
One line of symmetry


Name
G7
Draw the reflection of each of the triangles below. Check your work with a mirror.
mirror
(line of symmetry)


Draw the lines of symmetry of each picture. Check your work with a mirror.
There are more than 1.
Two lines of symmetry
There are more than 2.
Four lines of symmetry
There are more than 3.
Six lines of symmetry


Name
G7


Name
G7

Capsule Lesson Summary

Make specified designs using a double mirror and a column of five dots. Determine which of four types of symmetry is characteristic of each of the letters of the alphabet. Find words that have these types of symmetry.

Materials

Teacher	• Colored chalk	Student	• Double mirror • Worksheets G8(a), (b), (c), (d), and (e)

Advance Preparation: Prepare a double mirror for each student. Lay two mirrors facedown and touching side-to-side. Place a piece of sturdy packing tape across them to form a hinge.

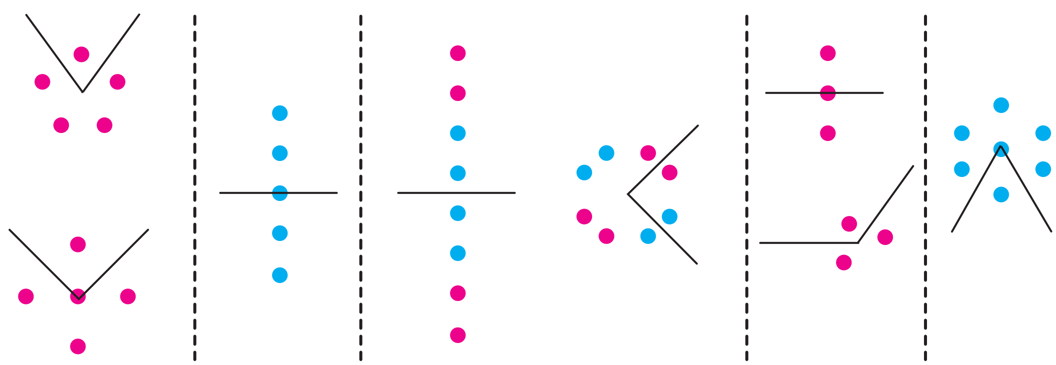
Description of Lesson

Exercise 1

Distribute double mirrors and allow a few minutes for students to investigate how a double mirror works. Then distribute Worksheet G8(a).

T: ***Use a double mirror and Worksheet G8(a) to see a design with five red dots; a design with five blue dots; a design with four red dots and four blue dots; a design with three red dots; and a design with seven blue dots.***

Allow several minutes for independent work, and assist students as needed. Do not expect all students to find a solution to each problem, and recognize that many solutions are possible. Some possible solutions are show below with black line segments showing placement of the double mirror.



Distribute copies of Worksheets G8(b), (c), and (d).

T: ***These are some other designs that can be seen using the double mirror and these five dots. How many can you make? Do they match exactly? Circle the designs as you find them.***

Allow at least 15 minutes for students to search for the designs. When several students have found most of them, proceed with Exercise 2.

Exercise 2

Review the notion of a square corner.

T: ***What is an example of a square corner in this classroom?***

S: ***The corner of the blackboard.***

S: ***The corner of my paper.***

T: ***Place a piece of paper on your desk. Now place your double mirror on the paper and open it until it forms a square corner.***

If students have difficulty forming a square corner, suggest that they align the double mirror with a corner of their paper.

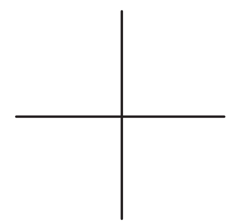
T: ***How many mirrors do you see?***

S: ***Four.***

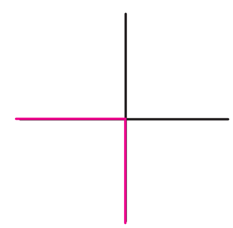
T: ***What else do you see?***

S: ***The mirrors line up and form two straight lines.***

Draw two perpendicular line segments on the board.



T: ***Is this what you mean?*** (Yes)
Keep your double mirror at a square corner, but turn it so the edge (opposite the hinge) of one mirror is pointing toward you and the same edge of the other mirror is pointing toward your left. Where are the double mirrors in this picture on the board?

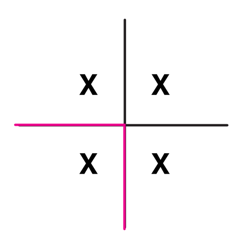


Invite a student to trace the appropriate lines and then to color them red.

T: ***Put your pencil between the mirrors.***
How many pencils do you see?

S: ***Four.***

T: ***If you write an* X *on the paper between the mirrors, how many* X*s like this one will you see?***

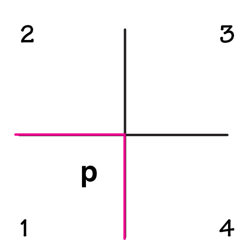


S: ***Four.***

T: ***Try it. Who can draw on the board what they see in the mirror?***

Number the four quadrants of the picture, as shown here, and erase the **X**s.

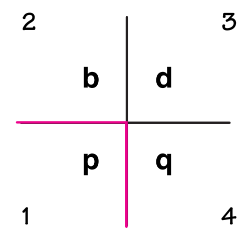
T: ***What happens if we write a* p *on our paper?***



Write **p** in region 1.

T: ***How many* p*s just like this one will you see?***

S: ***Four.***



T: ***Try it. What do you see?***

Invite students to come to the board and draw the images they see in the mirrors. Observe that the images look like **b**, **d**, and **q**.

T: ***How many* p*s*[†] *do you see in all?***

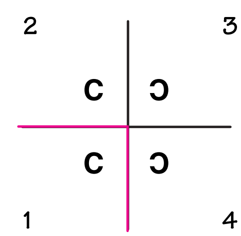
S: ***One.***

Begin the following chart on the board.

Different images in all four regions	Same images in only regions 1 and 2	Same images in only regions 1 and 3	Same images in only regions 1 and 4	Same images in all four regions
P				**X**

T: ***The image of* X *was the same in all four regions, but the image of* P *is only seen in region 1. Can you find a letter that would have the same image only in regions 1 and 2?***

Suppose a student suggests **C**.



T: ***Does it work? Try it.***

Invite a student to complete the picture on the board to show what appears in the double mirror when **C** is written on the paper.

Put **C** in the second column of the chart; then continue, asking for letters that have the same image in regions 1 and 4 only and for those that have the same image in regions 1 and 3 only. Direct students to Worksheet G8(e), asking them to classify each letter by the regions in which it appears the same. After a few minutes, collect results in the chart on the board. A completed chart is shown below.

Different images in all four regions	Same images in only regions 1 and 2	Same images in only regions 1 and 3	Same images in only regions 1 and 4	Same images in all four regions
P	**C**	**N**	**A**	**X**
F	**B**	**S**	**T**	**I**
G	**D**	**Z**	**M**	**O**
J	**E**		**U**	**H**
L	• •		**V**	
R			**W**	
Q			**Y**	

T: ***What do you notice about all of the letters with the same images in regions 1 and 2*** (point to the second column of the chart)***?***

S: ***You can flip them top to bottom and they remain the same.***

S: ***They each have a horizontal line of symmetry.***

Continue in the same manner, noticing common characteristics of the letters in each column. Students should observe the following:

- Letters in the first column have no line of symmetry.
- Letters in the third column can be turned upside down (rotated 180°) and remain the same, but they have no line of symmetry.
- Letters in the fourth column have one vertical line of symmetry.
- Letters in the fifth column have two lines of symmetry.

Challenge students to classify some words in the same way, trying to find words belonging to each of the five categories. Several examples are listed below. Record the words your class discovers on the board.

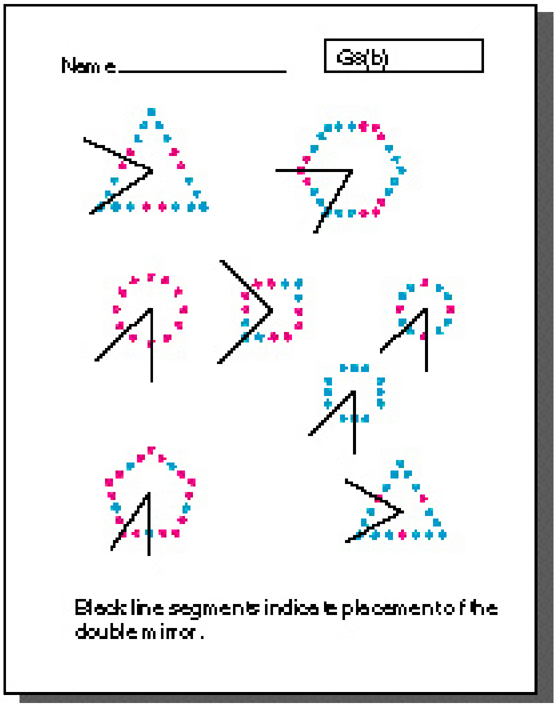
Different images in all four regions	Same images in only regions 1 and 2	Same images in only regions 1 and 3	Same images in only regions 1 and 4	Same images in all four regions
PUSH	**OHIO** **BOO**K **BOX**	**SOS** **SIS**	**MOM** **WOW** **TOT**	**I**

Extension/Writing Activity

Suggest that students consider how to use the double mirror to create a coding technique for Boris.

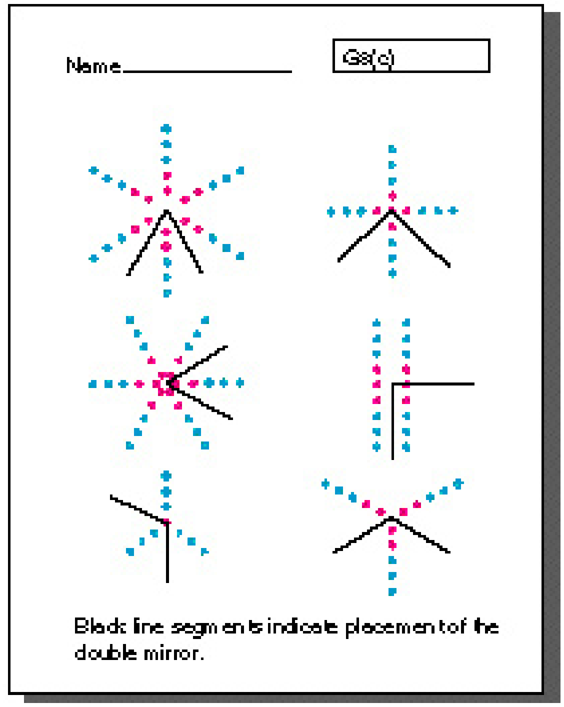
Home Activity

Give students a copy of the chart used in Exercise 2 to take home. Suggest that they work with family members to find shapes or numbers that fit in each of the five categories.



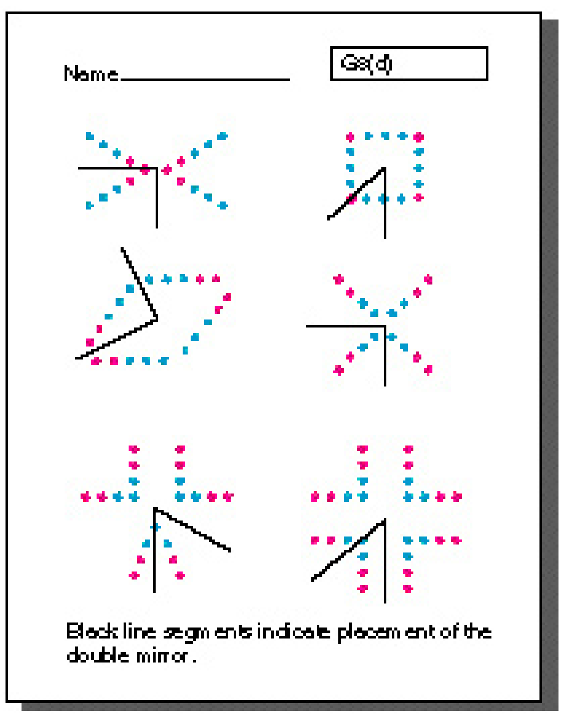
Name________________ G8(b)

Black line segments indicate placement of the double mirror.



Name________________ G8(c)

Black line segments indicate placement of the double mirror.



Name________________ G8(d)

Black line segments indicate placement of the double mirror.

Capsule Lesson Summary

Using sheets of colored paper and a double mirror, make specified designs. Determine the relationship between the four images formed when a double mirror is placed at a square corner. Play *The Reflection Game* with a double mirror.

Materials

Teacher
- Teacher triangle set
- Colored chalk

Student
- Blue and red sheets of paper
- Double mirror
- Student triangle set
- Worksheets G9(a) and (b), G9* and **

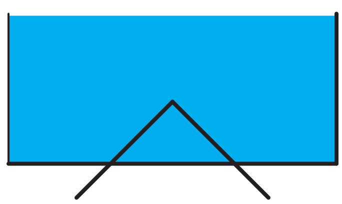
Description of Lesson

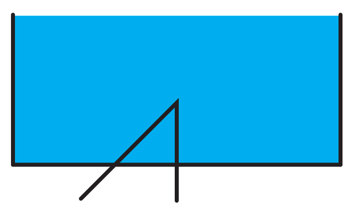
Exercise 1

For this exercise, you may like to let students work in pairs, but give each student a double mirror, one blue and one red sheet of paper, and a copy of Worksheet G9(a).

T: ***Use the blue paper and the double mirror to see a blue square.***

Assist students as necessary. Two of the possible arrangements for showing a square using a double mirror and paper are indicated below.





T: ***Can you see a shape with six sides? with three sides?***

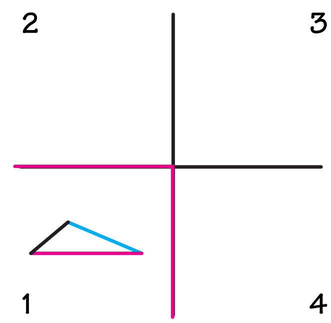
Many solutions are possible. As you observe students' work, look for a variety of solutions. You may like to suggest to some students that they find an irregular shape or a six-sided shape that is concave (has dents).

T: ***Use both the blue paper and the red paper, and see how many of the designs on Worksheet G9(a) you can see with the double mirror.***

Allow about 10–15 minutes for individual or partner work. Students who finish quickly should continue with Worksheet G9(b).

Exercise 2

Distribute student triangle sets. Put a picture on the board as you give these directions.



T: ***Make a square corner with the double mirror, and place a triangle between the mirrors as I have done at the board. How many triangles do you see?***

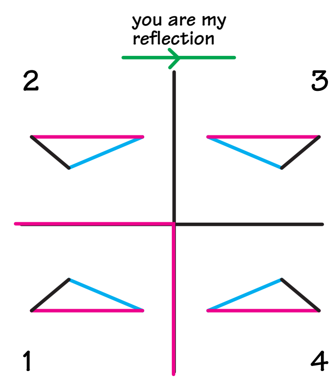
S: ***Four.***

T: ***Are they all the same?***

S: ***No, they all look different.***

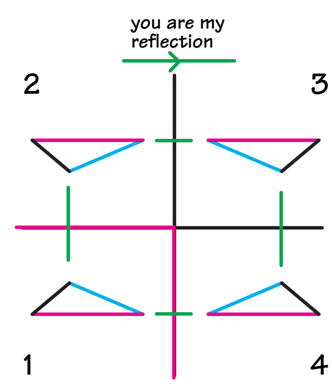
T: ***Let's see. Who can place a triangle like you see it here?*** (Point to region 2.)

Continue in this manner until all four images are displayed.



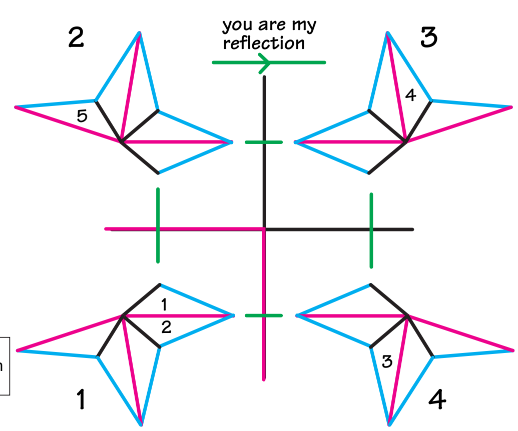
Put a key for green arrows ("you are my reflection") in the picture, and ask students where to draw green arrows. Allow students to draw several arrows in the picture on the board. Students should realize quickly that the arrows come in pairs and cords can be drawn appropriately.

Continue until this picture is on the board.



Some students may suggest drawing a cord between images 1 and 3 or between images 2 and 4. Compare the two triangles in these locations by placing one on top of the other. Students should see the triangles are not reflections of each other, rather they are rotations of each other.

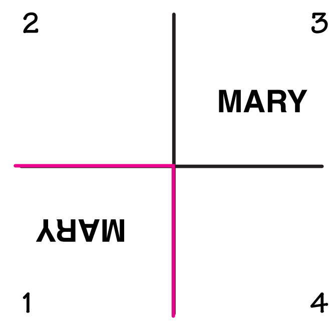
Continue by playing *The Reflection Game* with the double mirror. Place triangles on the board in the order indicated by the numbered triangles in this illustration. Reflections of these triangles are shown without numerals.

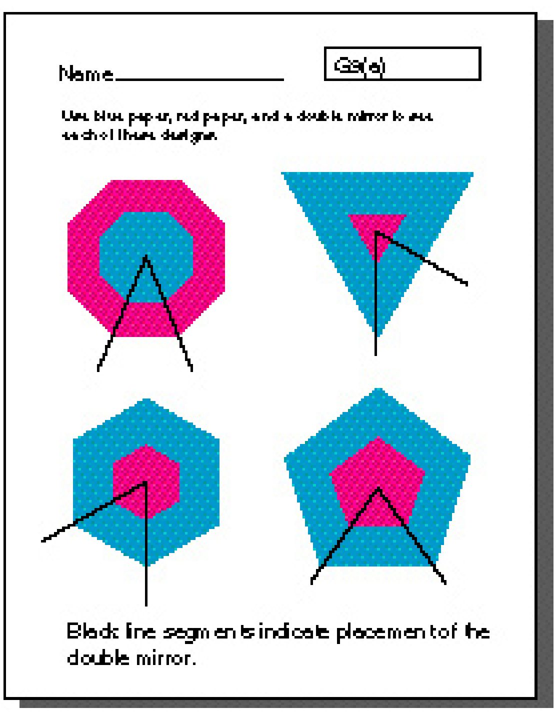


Do not write the numerals on the demonstration triangles. They are shown here to make the lesson description easier to follow.

Conclude the lesson by asking each student to write his or her name on a piece of paper in such a way that it appears correctly in region 3, such as illustrated here.

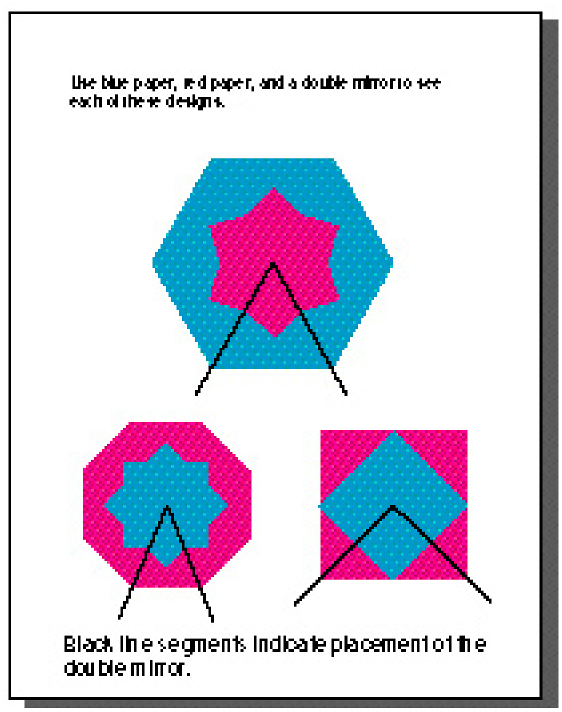
Worksheets G9* and ** are available for individual work.



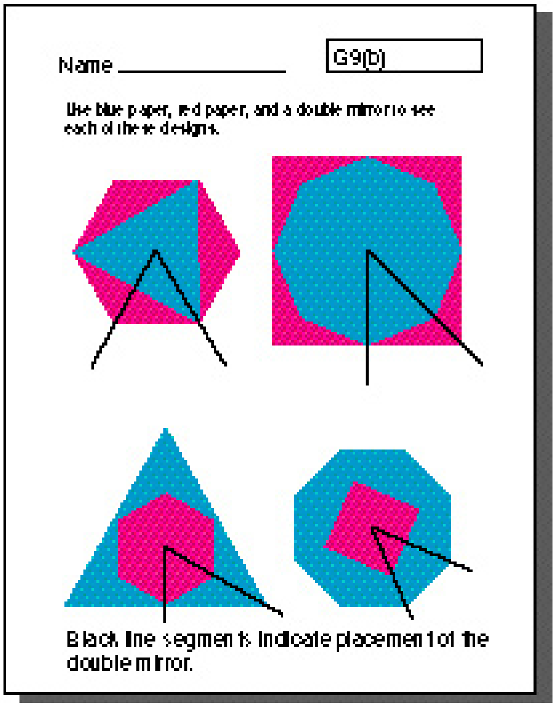

Name____________ G9(a)

Use blue paper, red paper, and a double mirror to see each of these designs.

Black line segments indicate placement of the double mirror.


Use blue paper, red paper, and a double mirror to see each of these designs.

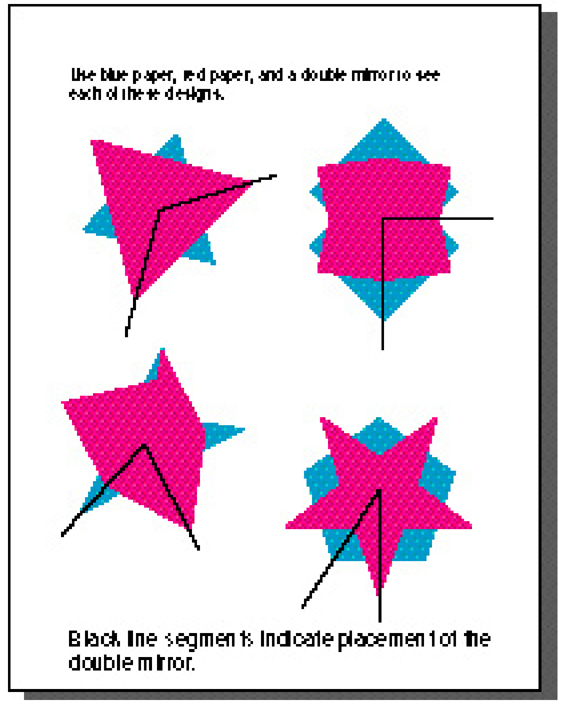
Black line segments indicate placement of the double mirror.



Name ________________ G9(b)

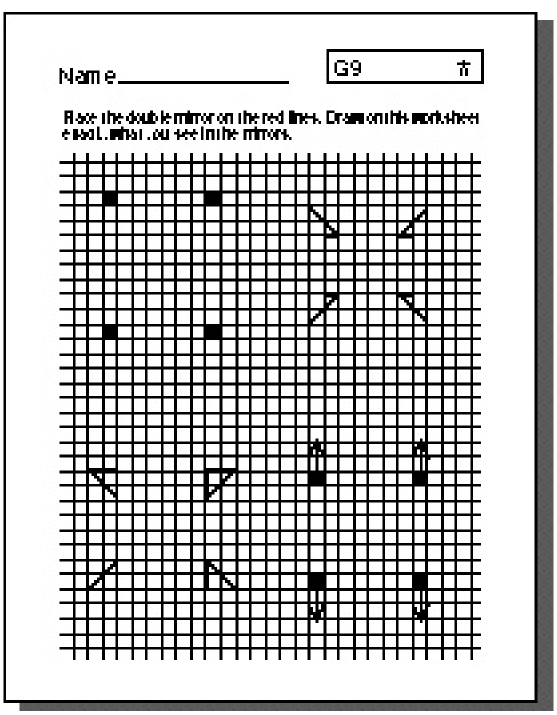
Use blue paper, red paper, and a double mirror to see each of these designs.

Black line segments indicate placement of the double mirror.



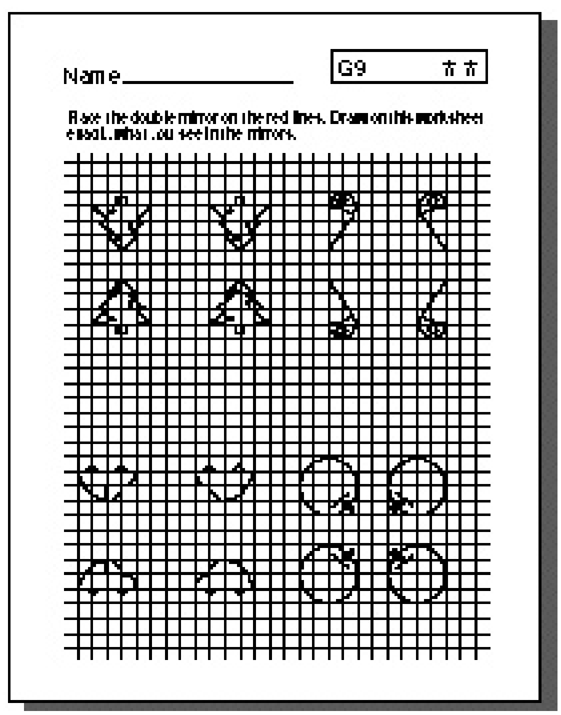
Use blue paper, red paper, and a double mirror to see each of these designs.

Black line segments indicate placement of the double mirror.



Name ________________ G9 ☆

Place the double mirror on the red lines. Draw on this worksheet exactly what you see in the mirrors.



Name ________________ G9 ☆☆

Place the double mirror on the red lines. Draw on this worksheet exactly what you see in the mirrors.

G10 AREA AND PERIMETER #1

G10

Capsule Lesson Summary

Find areas of rectangles by multiplication. Find the rectangle of largest area among those of perimeter 100 m.

Materials

Teacher
- Colored chalk
- Grid board
- Meter stick

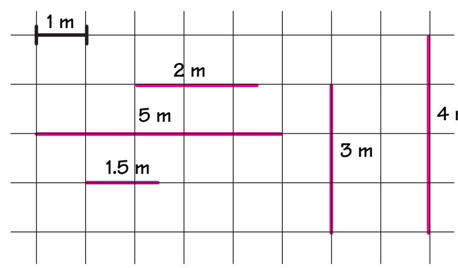
Student
- Worksheet G10
- Calculator

Description of Lesson

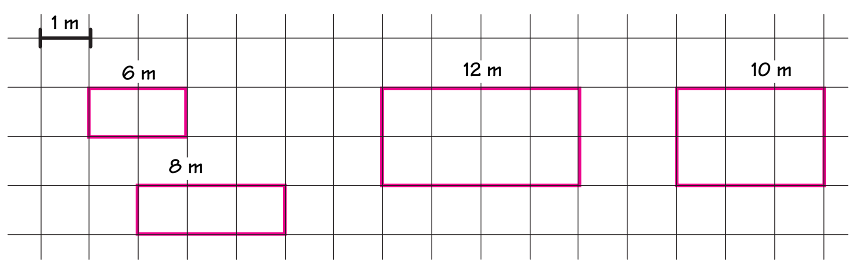
Exercise 1

Display a grid board, and tell the class that they should pretend that the grid squares are 1 meter on a side.

Draw a few line segments on the grid board following grid lines. As you draw each segment, ask a student to state the length of the segment.



Erase the line segments from the grid and draw rectangles, one at a time. For each rectangle, ask what is the perimeter.



T: ***So far we have only been measuring the perimeters of the rectangles. Let's measure their areas as well. If the length of one side of a square of the grid is 1 meter, as I have written here, what is the area of one square?***

S: ***1 square meter.***

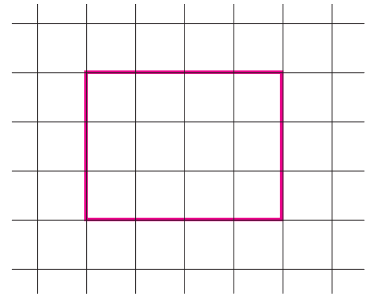
Write this abbreviation on the board.

1 square meter = 1 m^2

Ask for the area of each rectangle on the grid board. Continue this activity, determining the areas of a few more rectangles whose sides are aligned with grid lines. Some students will simply count the squares of the grid that comprise the rectangles, one by one; some may use other counting methods.   Encourage a multiplication method for calculating areas or other methods that suggest multiplication.

T: ***Who has a quick way of finding the area of the rectangles?***

S: ***Count the number of rows of squares in the rectangle and the number of squares in each row. Then multiply the two numbers.***



T: ***What is the area of this rectangle?***

S: ***There are three rows and four squares in each row. The area is 12 square meters.***

Note: Insist that students give a unit when giving the area or any measurement.

T: ***The length of one side of this rectangle is 3 meters and the length of the other side is 4 meters. The area is 12 square meters; 3 x 4 = 12. Let's check that method with the other rectangles.***

For each of the other rectangles, ask students to find the length and width and then to multiply to find the area.

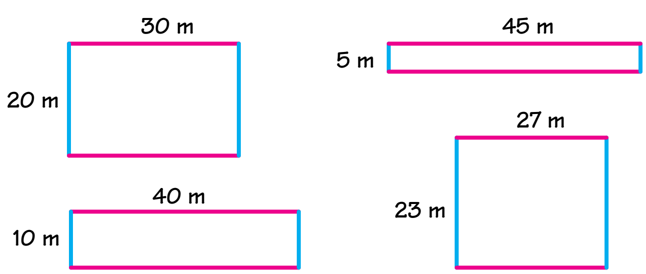
Exercise 2

T: ***A farmer has 100 meters of fencing material. The farmer wants to build a rectangular pen for cows that will use all of the fencing material.*** (Draw a rectangle on the board to represent the pen.) ***What will the perimeter of the pen be?***

S: ***A pen that uses all of the 100 meters of fencing will have a perimeter of 100 meters.***

T: ***What size could such a pen be?***

Of course, there are many correct answers. Draw a small-scale rectangle on the chalkboard for each answer as it is suggested. Draw one pair of opposite sides in red and the other in blue, as illustrated below. Be sure to check that the perimeter of each suggested rectangle is 100 m. Begin a chart to record answers. After a few solutions are recorded, change to asking only for the length of one side of such a rectangle from one student and then for the length of an adjacent side from another student. A sample chart is shown below.



● ●	● ●
20 m	30 m
10 m	40 m
5 m	45 m
23 m	27 m
35 m	15 m
32.5 m	17.5 m

Eventually most students will recognize that the sum of the lengths of two adjacent sides of a correct solution is 50 m.

T: ***Now let's find the areas of these rectangles.***

Pick one of your listed solutions that has dimensions that are multiples of 10, for example, 20 m by 30 m.

T: ***What is the area of this rectangle?***

S: ***20 x 30 = 600, so the area is 600 square meters.***

Extend the chart to list the areas of the rectangles. As appropriate, demonstrate some of the multiplications at the board. If an example involves multiplying two decimals, suggest that students do the calculations on a calculator.

● ●	● ●	Area
20 m	30 m	600 m^2
10 m	40 m	400 m^2
5 m	45 m	225 m^2
23 m	27 m	621 m^2
35 m	15 m	525 m^2
32.5 m	17.5 m	568.75 m^2

Distribute copies of Worksheet G10 and provide calculators.

T: ***Our farmer would like to build a pen with the largest possible area. Which of these rectangles has the largest area? On your worksheet, list more solutions to the farmer's problem and try to find a solution with the largest possible area.***

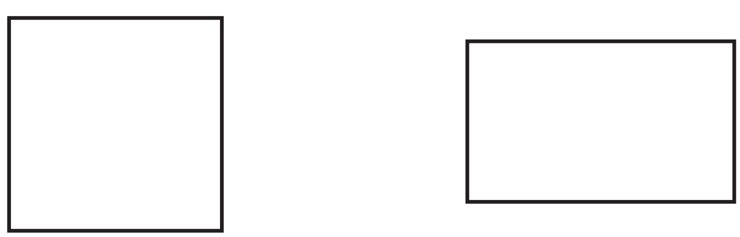
As you monitor students' work, look for different solutions (for example, involving non-integer decimals or having a large area close to the maximum) to add to the chart on the board. The maximum area is 625 square meters for the rectangle with dimensions 25 m by 25 m. Be sure that this solution is recorded in the chart at some point.

Note: This is an excellent opportunity for students to use calculators. Use of this tool allows students to consider many solutions and to concentrate on the problem (to find a rectangle with largest area) rather than on calculation. Moreover, students using calculators may be more willing to explore solutions involving decimals.

Exercise 3 (optional)

Depending on the interest and sophistication of your students, present the following optional justification for the best possible solution being a square.

Draw a square 25 cm by 25 cm and a non-square rectangle with perimeter 100 cm on the board.

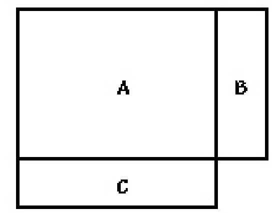


T: ***On the left is how the farmer's pen would look if built as a square. On the right is another way the pen could look. Both shapes (rectangles) have the same perimeter. (A square is a particular kind of rectangle.) How can we compare their areas?***

Let students make suggestions, and then present the following method.

T: ***We could put one on top of the other and compare them.***

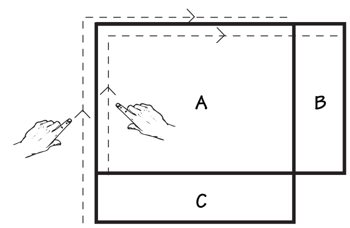
Redraw the shapes, one imposed on the other.



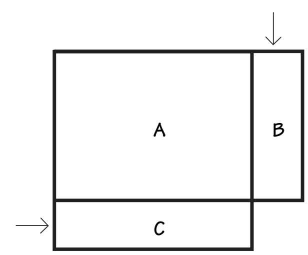
T: ***A large part of the square is covered by the other rectangle. But part of the square is not covered, and part of the other rectangle does not cover any of the square. Let's call the three parts* A, B*, and* C.**

Now the square has two parts,* A *and* C*, and the rectangle has two parts,* A *and* B*. Which do you think has larger area,* B *or* C*?

Trace two adjacent sides of the square, and then two adjacent sides of the other rectangle as you note that the total lengths are the same (50 cm, or half of the perimeter) in both cases.



Conclude that the short dimension of **C** is the same as the short dimension of **B**.

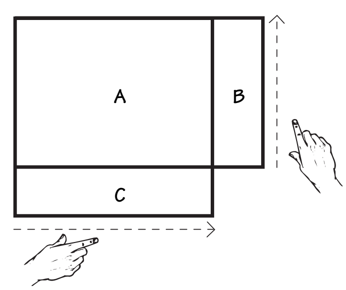


T: ***We are concerned with which rectangle has more area,* B *or* C*. How can we find the area of any rectangle?***

S: ***Multiply its length times its width.***

T: ***Rectangles* B *and* C *have the same width. But do they have the same length?***

Trace the length of rectangle **C** and then the length of rectangle **B**.



S: ***No, rectangle* B *is not as long as rectangle* C.**

Observe that the length of **B** is shorter than the side of the square, but the length of **C** is the side of the square.

T: ***So do the rectangles* B *and* C *have the same area?***

S: ***No,* C *has larger area.***

T: ***So which pen would have more area,* A *with* B *or* A *with* C*?***

S: **A *with* C.**

T: ***So the square pen is the best solution for the farmer.***

Name________________

G10

What is the largest possible area? 625 m^2

perimeter 100 m

		Area
20 m	30 m	600 m^2
5 m	45 m	225 m^2
16 m	34 m	544 m^2
1 m	49 m	49 m^2
18.3 m	31.7 m	580.11 m^2
25 m	25 m	625 m^2
40 m	10 m	400 m^2

Use the other side to record other solutions.

Many solutions are possible.

G11 AREA AND PERIMETER #2

Capsule Lesson Summary

Find the largest possible area for a rectangular pen that a farmer can construct using a stream for one side and 100 m of fencing for the other three sides.

Materials

Teacher
- Colored chalk
- Grid board
- Meter stick

Student
- Worksheet G11
- Calculator

Description of Lesson

Exercise 1

Display a grid board. Begin by asking students to show approximately how big a centimeter, a decimeter, and a meter are. Review these equivalences: 10 dm = 1 m; 10 cm = 1 dm; 100 cm = 1m. Start with a unit (side of a small square of the grid) of 1 dm and periodically change it among 1 m, 1 dm, and 1 cm. Draw a variety of rectangles, and ask students to calculate their areas and perimeters, using the appropriate units. Encourage the technique of multiplying the lengths of two adjacent sides to find the area of a rectangle, or adding the lengths of two adjacent sides and doubling to find the perimeter.

Recall the problem of the farmer with 100 meters of fence who wanted to build a pen with the largest possible area. On the board, draw a rectangle and record the desired perimeter.


perimeter = 100 m

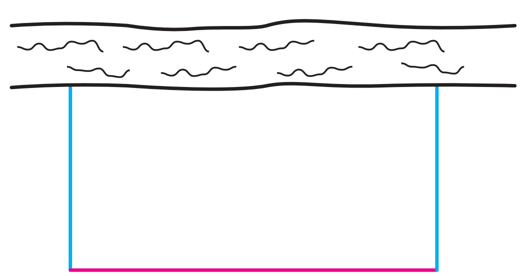
Ask for the possible dimensions of the rectangle, and record them along with the area in a chart, as in Lesson G10.

		Area
20 m	30 m	600 m^2
5 m	45 m	225 m^2
25 m	25 m	625 m^2
24.5 m	25.5 m	624.75 m^2
26 m	24 m	624 m^2
20.25 m	29.75 m	602.4375 m^2

Continue until there are six to eight entries in the chart, including the solution with largest area, 25 m by 25 m. The class should remember that a square pen is best for the farmer.

T: ***There is another farmer in the area who has a stream flowing through the farm. This farmer also has 100 meters of fencing material and wants to build a rectangular pen so that the stream forms one side.***

Illustrate the situation on the board.



T: ***What are some solutions to this farmer's problem?***

Begin a chart to record answers. For each suggestion, check that the total length of the three fence sides of the rectangles is 100 m. In order to engage more students in the mental calculations, occasionally request only the length of one side of such a rectangle from one student and then ask another student for the length of an adjacent side.

● ●	●
20 m	60 m
15 m	70 m
40 m	20 m
5 m	90 m
32.5 m	35 m
27.25 m	45.5 m

After several solutions have been given, ask students to calculate the areas as well. If an example involves multiplying decimals, let students do the calculation with a calculator. Occasionally ask what the pen would look like; for example, a 5 m by 90 m pen would be long and narrow.

● ●	●	Area
20 m	60 m	1200 m^2
15 m	70 m	1050 m^2
40 m	20 m	800 m^2
5 m	90 m	450 m^2
32.5 m	35 m	1137.5 m^2
27.25 m	45.5 m	1239.875 m^2

Distribute copies of Worksheet G11.

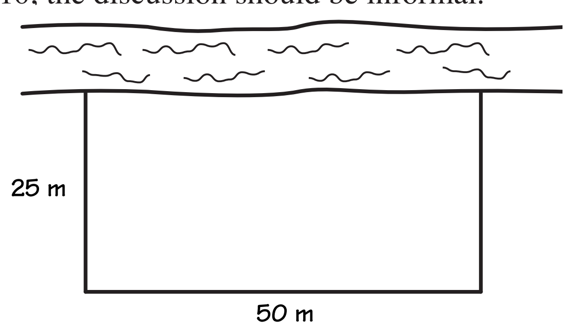
T: ***This farmer also wants to build a pen with the largest possible area. Investigate the problem and record solutions on your worksheet. Remember, you are looking for the largest possible area.***

This is another excellent opportunity to use calculators in order to focus attention on the problem of maximizing the area of the pen rather than on the calculations. With calculators, you can encourage students to find solutions such as 26.37 (blue) by 47.26 m (red), giving an area of 1 246.2462 m^2. Students should notice that when they increase the blue dimension, they must decrease the red dimension by twice as much to keep the perimeter 100 m. Students eventually should find that the maximum area to be had is 1 250 m^2, with a rectangle of 25 m by 50 m.

Complete the lesson with one or both of these options:

1) Ask students to repeat the activity with other lengths of fencing, for example 200 m, 300 m, or 500 m. The maximum areas in these cases are 5 000 m^2, 11 250 m^2, and 31 250 m^2, respectively. In each case, the best solution is a rectangular pen, using half of the fencing for the red side and distributing the other half equally between the two blue sides.
2) Informally justify why the maximum area in the problem is 1 250 m^2.

If you opt to do (2), continue as follows. As in Lesson G10, the discussion should be informal. Begin by drawing a rectangle in the proportions of 25 m by 50 m, using a stream to form one side. (For example, your rectangle on the board might be 25 cm by 50 cm.) Do not record the actual dimensions on the board.



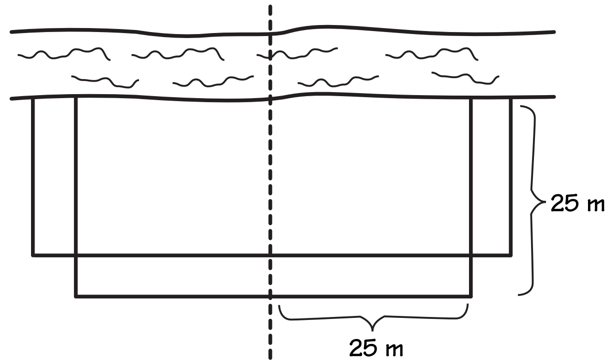
T: ***The pen of largest area that we have found is 25 m by 50 m. What do you notice about the dimensions of this pen?***

S: ***The length is half of the width.***

Divide the rectangle with a dotted vertical line into two pieces of equal size.

T: ***If we divide the rectangle down the middle vertically, what shape will each piece be?***

S: ***A square.***

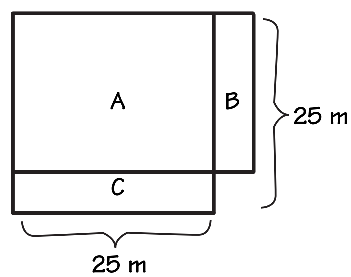


Now draw another solution rectangle oriented on top of the first. The dotted dividing line should also divide the second rectangle into two pieces of the same size. If the first rectangle is 25 cm by 50 cm, you could draw a second one 20 cm by 60 cm. All that is important is to keep the sum of the lengths of the three sides in each rectangle the same.

T: ***This picture now shows two ways the farmer could use the 100 m of fence, one imposed on top of the other.***

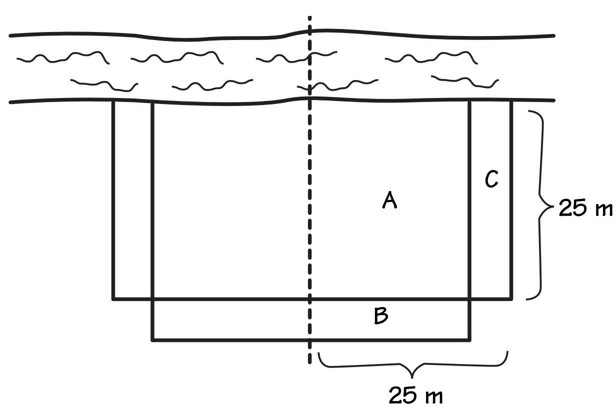
Let the class discuss the picture. Make two observations in the discussion:

1) The dotted line divides both rectangles in half. The pen with the larger half is the larger pen. So the problem is to compare the areas of the two half-pens.
2) The total length of fencing in each half-rectangle is 50 m. Each is one-half of the total length of fencing, or one-half of 100 m.



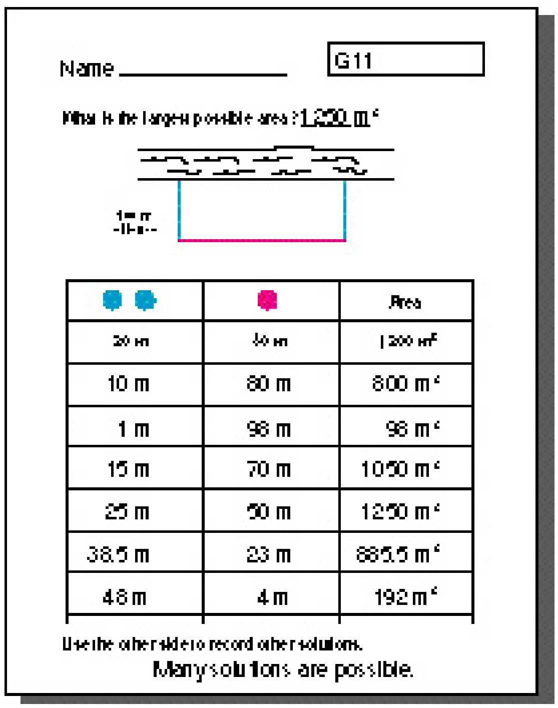
Cover the left half of the picture with a large piece of paper, and label the visible rectangles **A**, **B**, and **C**.

Students may see that this is the same situation as in Lesson G10 except the square is smaller. The argument given there showed that **A** and **C** together had a greater area than **A** and **B** together, because **C** was larger than **B**. Review as much of that argument as you think necessary. Uncover the entire drawing, and conclude that the 25 m by 50 m rectangle has a larger area than the other rectangle.



Note: In order to complete the argument, you may consider another case; the rectangles could be oriented as in the following drawing.

When you cover the left-hand side, the drawing is similar to the previous case. It is not necessary to check it again.



Name ____________ G 11

What is the largest possible area? 1,250 m²

		Area
20 m	60 m	1200 m²
10 m	80 m	800 m²
1 m	98 m	98 m²
15 m	70 m	1050 m²
25 m	50 m	1250 m²
38.5 m	23 m	885.5 m²
48 m	4 m	192 m²

Use the other side to record other solutions.

Many solutions are possible.

G12 AREA AND PERIMETER #3

Capsule Lesson Summary

Find the largest possible total area for a rectangular pen that has three sections, and that can be constructed using a stream for one side and 100 m of fencing for the other sides. Similarly, consider such a pen with four sections and with two sections.

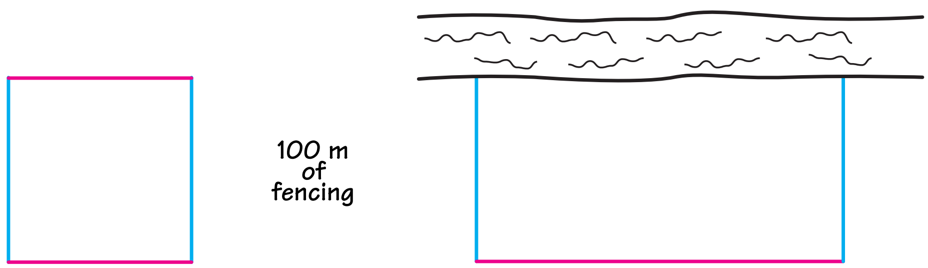
Materials

Teacher
- Colored chalk

Student
- Worksheets G12(a), (b), (c), and (d)
- Calculator

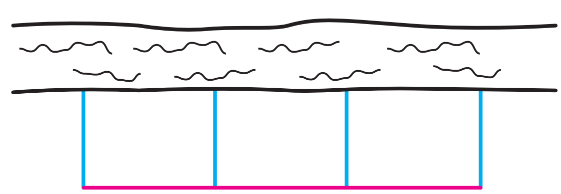
Description of Lesson

Recall the problems of the two farmers from Lessons G10 and G11. Each farmer had 100 m of fencing with which to construct a rectangular pen, but the second farmer had a stream to use for one side of the pen. Each wanted to build a pen of largest possible area. Illustrate the situations as you recall them.



T: ***Downstream from the second farmer is another farmer who has cows, sheep, and goats. This farmer also has 100 m of fencing but needs three pens.***

Illustrate the situation on the board.



T: ***How many blue segments are there?***

S: ***Four.***

T: ***We can look at the pen as having one long red side*** (trace it) ***and four blue segments. The blue segments have the same length. Who can give us a possible length of a blue segment and the length of the corresponding red segment? Remember, the total length of the one red and four blue segments must be 100 m.***

Accept and check solutions with the class. Record correct solutions in a chart. Occasionally ask one student for a possible length of a blue segment (or red) and ask another student for the length of the corresponding red (or blue) segment.

● ● ● ●	●
10 m	60 m
$6\,^1/_4$ m	75 m
5 m	80 m
12 m	52 m
7.5 m	70 m

After recording four or five solutions, ask the class to find the areas of the rectangles. Trace the outer border of the entire pen.

T: ***How can we find the area of the entire pen?***

S: ***Multiply the length of a blue segment times the length of the corresponding red segment.***

G12

Ask students to calculate the areas of each rectangle. Examples involving decimals can be done on a calculator. You may want to suggest rewriting a length involving fractions; for example, $6\frac{1}{4} = 6.25$.

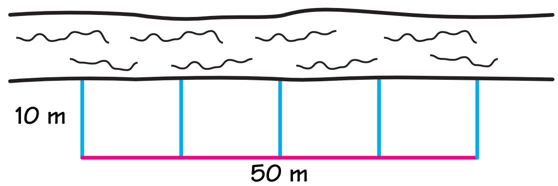
● ● ● ●	●	Total Area
10 m	60 m	600 m^2
$6\ ^1/_4$ m	75 m	468.75 m^2
5 m	80 m	400 m^2
12 m	52 m	624 m^2
7.5 m	70 m	525 m^2

Distribute copies of Worksheet G12(a).

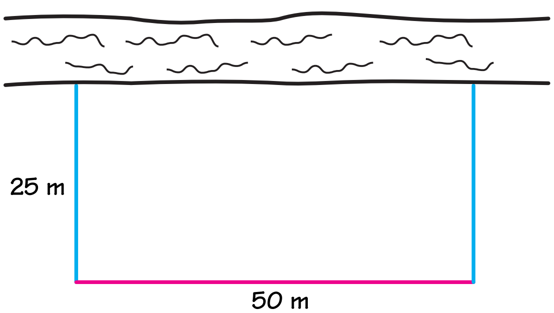
T: ***This farmer also wants to build pens that have the largest possible area. Investigate the problem and record solutions on your worksheets.***

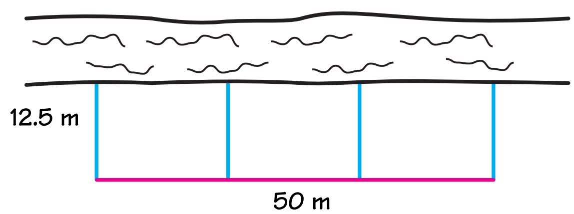
If many students believe the largest area is 624 m^2 for rectangles of dimensions 12 m by 52 m and 13 m by 48 m, suggest trying a width between 12 m and 13 m. Students may notice that when they increase blue segments by 1 m they must correspondingly decrease the red segment by 4 m. Also, students should begin observing in these farmer problems that you can increase the blue segments only so much before you begin decreasing area.

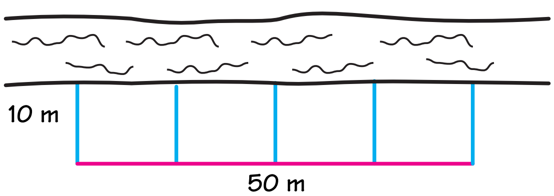
When the class concludes that a pen 12.5 m by 50 m is the best size for the farmer, pose a similar problem with four pens. Distribute copies of Worksheet G12(b) for students to record solutions. The class should find that the best solution in this case is a pen 10 m by 50 m.

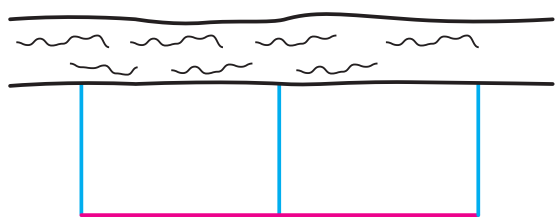


Illustrate these four situations on the board.









T: ***What patterns do you notice*** (point to the illustration of the pen with one section, with three sections, and with four sections)***?***

S: ***The red length is always 50 m.***

T: ***So the farmer uses half of the fencing for the length of the pen and must use the other half to make the sections. If a farmer only wanted two sections*** (point to the appropriate illustration)***, what might you expect the length of the pen to be?***

S: ***50 m.***

T: ***How wide do you think the pen should be?***

S: ***We need to calculate 50 ÷ 3.***

S: ***The calculator gives 16.666666.***

T: ***And in fact the 6s go on forever, but the calculator has a limited display.***

S: ***$\frac{1}{3}$ of 48 = 16 and $\frac{1}{3}$ of 2 = $\frac{2}{3}$, so $\frac{1}{3}$ of 50 = $16\frac{2}{3}$.***

If there is time remaining, let students convince themselves that a rectangle $16\frac{2}{3}$ (or 16.6666 …) m by 50 m is the best solution for the farmer. Worksheet G12(c) has a chart for recording solutions. Also, Worksheet G12(d) is available for students who would like to check a similar problem with five sections.

Writing Activity

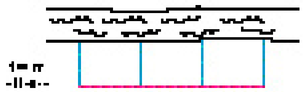
Suggest that students write a letter to the farmers explaining how to decide what dimensions to make their pens in order to maximize the area.

Home Activity

You may like to send one of the worksheets home for students to explain to family members how to find the largest area.

Name ______________ G 12 (a)

What is the largest possible area? 625 m²



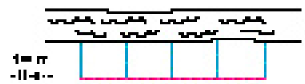
		Total Area
10 m	40 m	400 m²
20 m	20 m	400 m²
15 m	40 m	600 m²
12 m	52 m	624 m²
12.5 m	50 m	625 m²
13 m	48 m	624 m²
13.5 m	46 m	621 m²

Use the other side to record other solutions.

Many solutions are possible.

Name ______________ G 12 (b)

What is the largest possible area? 500 m²



		Total Area
5 m	75 m	375 m²
15 m	25 m	375 m²
10 m	50 m	500 m²
11 m	45 m	495 m²
12 m	40 m	480 m²
11.5 m	42.5 m	488.75 m²
10.5 m	47.5 m	498.75 m²

Use the other side to record other solutions.

Many solutions are possible.

Name ______________ G12(c)

What is the largest possible area? 833 1/3 m²

		Total Area
15 m	55 m	825 m²
20 m	40 m	800 m²
1 m	97 m	97 m²
18 m	46 m	828 m²
17 m	49 m	833 m²
16 m	52 m	832 m²
16.5 m	50.5 m	833.25 m²
16.25 m	51.25 m	832.8125 m²

Many solutions are possible, but a rectangle 16 2/3 m by 50m has maximum area, which is 833 1/3 m².

Name ______________ G12(d)

What is the largest possible area? 416 2/3 m²

		Total Area
10 m	40 m	400 m²
15 m	10 m	150 m²
5 m	70 m	350 m²
9 m	46 m	414 m²
8 m	52 m	416 m²
8.5 m	49 m	416.5 m²
8.75 m	47.5 m	415.625 m²
8 1/3 m	50 m	416 2/3 m²

Many solutions are possible, but a rectangle 8 1/3 m by 50m has maximum area.