L Strand

The Languages of Strings and Arrows

LANGUAGES OF STRINGS & ARROWS TABLE OF CONTENTS

THE LANGUAGES OF STRINGS & ARROWS INTRODUCTION

Two fundamental modes of thought for understanding the world around us are the classification of objects into sets and the study of relationships among objects. In everyday life, we classify cars by brand (Ford, Chevrolet, Toyota, and so on) and we study relationships among people (Sally is Mark's sister, Nancy is Mark's cousin). Chemists classify elements by properties, and zoologists study predator-prey relationships. Similarly, mathematicians sort numbers by primeness, and they employ functions to model predicted relationships, for example, between inflation and unemployment.

Many of a child's earliest learning experiences involve attempts to classify and to discern relationships. A child classifies people by roles (the teacher, the doctor), and creates relationships between the smell and taste of foods. Part of language development depends on a child's repeated attempts to sort objects by function, and to relate words with things or events.

The role of sets and relations is so pervasive in mathematics, that perhaps the simplest definition of mathematics is "the study of sets and relations principally involving numbers and geometrical objects." Given the equally pervasive presence of these two notions in everyday life and in a child's experiences, it is natural that they should play a key role in an elementary mathematics curriculum. Yet the inclusion of classification and the study of relations require an appropriate language for representing and studying them. For that reason, *CSMP* develops the nonverbal languages of strings and of arrows.

The pictorial language of strings represents the grouping together of objects into sets. The pictorial language of arrows represents relations among objects of the same or different sets. Each of these languages permeates the different content strands of the *CSMP* curriculum, providing unity both pedagogically and mathematically. With continual use, the languages become versatile student tools for modeling situations, for posing and solving problems, and for investigating mathematical concepts.

The general aim of this strand is to present situations that are inherently interesting and thought provoking, and that involve classification or the analysis of relations. The activities emphasize the role of logical thinking in problem solving rather than the development of specific problem-solving techniques.

Classification: The Language of Strings

As the word implies, classifying means putting things into classes, or as the mathematician says, sets. The mathematics of sets can help students to understand and use the ideas of classification. The basic idea is simple: Given a set S and any object x, either x belongs to S (x is in S) or x does not belong to S (x is not in S). We represent this simple act of sorting—in or out—by using pictures to illustrate in and out in a dramatic way. Objects to be sorted are represented by dots, and the sets into which they are sorted are represented by drawing strings around dots. A dot inside the region delineated by a set's string is for an object in the set, and a dot outside a set's string is for an object not in the set.

This language of strings and dots provides a precise (and nonverbal) way of recording and communicating thoughts about classification. The ability to classify, to reason about classification, and to extract information from a classification are important skills for everyday life, for intellectual activity in general, and for the pursuit and understanding of mathematics in particular. The unique quality of the language of strings is that it provides a nonverbal language that is particularly suited to the mode of thinking involved in classification. It frees young minds to think logically and creatively about classes, and to report their thinking long before they have extensive verbal skills.

Relations: The Language of Arrows

Relations are interesting and important to us in our everyday lives, in our careers, in school, and in scientific pursuits. We are always trying to establish, explore, and understand relations. In mathematics it is the same; to study mathematics means to study the relations among mathematical objects like numbers or geometric shapes. The tools we use to understand the everyday world are useful to understand the world of mathematics. Conversely, the tools we develop to help us think about mathematical things often serve us in non-mathematical situations.

A serious study of anything requires a language for representing the things under investigation. The language of arrows provides an apt language for studying and talking about relations. Arrow diagrams are a handy graphic representation of a relation, somewhat the same way that a blueprint is a handy graphic representation of a house. By means of arrow diagrams, we can represent important facts about a given relation in a simple, suggestive, pictorial way—usually more conveniently than the same information could be presented in words. (See, for example, the story-workbook *Summer Camp*.)

The convenience of arrow diagrams has important pedagogical consequences for introducing children to the study of relations in the early grades. A child can read—and also draw—an arrow diagram of a relation long before he or she can read or present the same information in words. The difficulty of presenting certain ideas to children lies not in their intellectual inability to grasp the ideas; rather, the limitations are often mechanical. Arrow diagrams have all the virtues of a good notation: they present information in a clear, natural way; they are attractive, colorful things to look at; they are easy and fun for children to draw. Students may use arrow pictures to study, test, and explain their thinking about concepts or situations under consideration. Discussion about an arrow picture often aids the teacher in clarifying a student's solution or misunderstanding of a problem.

Another educational bonus occurs when an arrow diagram spurs students' curiosity to investigate variations or extensions of the original problem. A minor change in an arrow picture sometimes reinforces a pattern already discussed and at other times suggests new problems to explore.

One of the purposes of this strand is to use the power of the language of arrows to help children think logically about relations. Again it must be remembered that our goals concern the thinking process and not the mechanism. The ability to draw prescribed arrows is not the objective in itself, nor is viewing an arrow diagram just another format for drill problems in arithmetic.

The general aim of the Languages of Strings and Arrows strand is to suggest situations that are inherently interesting and thought provoking, and to give children modes of thinking and appropriate languages with which they can organize, classify, and analyze. In addition to a varied assortment of lessons concerning sets and relations, this strand includes lessons involving systematic methods for solving combinatorial (counting) problems.

Content Overview

The String Game

This semester *The String Game* is the main focus of four lessons. In all of these lessons, the class works collectively on a variety of analysis activities involving string game situations before actually playing the game. For example, a string picture is presented with numbers placed as clues about the string labels. After extracting all the possible information from the clues, students are asked if there is sufficient information to determine where particular numbers belong in the picture. Several variations of this activity appear in the four lessons. In collective analyses, students have an opportunity to share and further formulate playing strategies for the game.

In one lesson students locate numbers in the regions of a three-string picture with all the string labels given. Upon discovering an empty region, the students hatch it and then redraw the strings to show an alternative picture conveying the same information without hatching. There are also hatching clues in analysis situations where knowing that one region of a string picture is empty or has only specified numbers in it powerfully eliminates many possibilities for the strings.

Positive divisors of various numbers are used in *The String Game* as possible labels for strings. One lesson this semester further explores the positive divisor concept to investigate which numbers have a specified number of positive divisors, and to begin work on counting positive divisors.

The String Game continues to provide rich situations that encourage logical reasoning, and further develop familiarity and ease with number properties.

Lessons: L1, 3, 8, 10, and 12

Relations

One lesson this semester reviews the notion of composition. Students explore first with unspecified relations and then with specific interpretations; for example, family relations such as "you are my father" and "you are my mother," or numerical relations such as +2 or 3x. Composition of relations is a very useful idea and is used extensively in lessons in the World of Numbers strand to develop numerical concepts such as fractions and multiplication. Further, composition ideas appear in the Geometry strand with symmetry and reflections.

A series of four lessons prepares students to build an elaborate four-tailed kite arrow picture in a finite arithmetic system. The lessons review addition and multiplication modulo 10 ($+_{10}$ and x_{10}), operations first introduced in the storybook *Dancing Friends* with just the ten whole numbers 0 to 9. When ten more numbers are included, the finite system and the operations are extended to include 20 numbers (0 to 19), that is, addition and multiplication modulo 20. Students examine some interesting patterns generated by iterated composites of the relations $2\mathrm{x}_{10}$ and $2\mathrm{x}_{20}$. The four-tailed kite is a complete picture for $2\mathrm{x}_{20}$.

Another lesson this semester introduces the idea of a permutation, a special type of relation, in an exchanging names situation.

Lessons: L2, 4, 5, 6, 7, and 9

Combinatorics

Two lesson in this strand involve students in generating systematic methods of counting. This experience leads to solving combinatorics problems or problems that involve counting all possible outcomes in a situation. See the World of Numbers strand introduction for an example of such a problem.

Combinatorics is a basic element of probability as well as a source of challenging recreational mathematics problems. Several lessons in the Probability and Statistics strand further apply the techniques developed in these lessons.

Although not strictly a counting problem, the binary codes lesson uses the graphic of a decision tree and binary notation, both of which are often used for solving counting problems. Here the problem involves coding and decoding messages, and asks students to work with useful tree diagrams.

Lessons: L9, 10, and 11

L1 STRING GAME ANALYSIS #1

Capsule Lesson Summary

Put numbers into a three-string picture with the labels showing. Use hatching to illustrate empty regions, and draw a similar picture giving all of the same information without hatching. Analyze a string game situation with two starting clues, and use the resulting information to determine whether a given statement is true, false, or indeterminate.
Play *The String Game* with numbers.

Materials

Teacher
- Numerical String Game kit
- Colored chalk
- Marker or crayon

Student
- String Game analysis sheet

Description of Lesson

Exercise 1

T: ***We are going to play The String Game today, but first let's review some possible string labels.***

Draw a three-string picture on the board as in the next illustration, but label only the red string **POSITIVE DIVISORS OF 24**.

T: ***Which numbers are positive divisors of 24? Where do they go in this picture?***

All of the positive divisors of 24 (1, 2, 3, 4, 6, 8, 12, 24) should be given, and the class should note that they go inside the red string. Label the blue string **MULTIPLES OF 4**.

T: ***Name some multiples of 4.***

If students mention only positive multiples of 4, ask for multiples of 4 less than 4. If 0 is not mentioned, list the multiples of 4 on the board, omitting 0.

$$\cdots\ \widehat{16},\ \widehat{12},\ \widehat{8},\ \widehat{4},\ 4,\ 8,\ 12,\ 16,\ \cdots$$

T: ***There is a very important number missing here. Which number is it?***

S: ***0.***

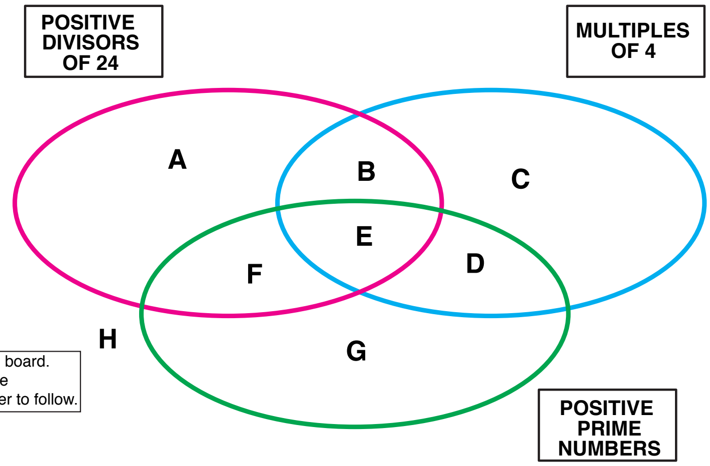
Label the green string **POSITIVE PRIME NUMBERS**.

T: ***What is a positive prime number?***

S: ***A positive number with exactly two divisors, 1 and itself.***

T: ***Name some positive prime numbers.*** (2, 3, 5, 7, 11, 13, 17, 23, …)

When an incorrect response is given (9, for example), ask for the divisors of that number (1, 3, and 9) and observe that it has more than two divisors.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: ***Where does 16 go in this picture?*** (Region **C**)

Call on a student to locate 16 inside the blue string only, getting class agreement before it is finally placed.

Continue this activity by seeking the locations of 4, 7, $\widehat{3}$, and $\widehat{12}$ (see the next illustration).

T: ***Give me a number that belongs in this region (A).***

S: ***6.*** (1 is also correct.)

T: ***Give me a number that belongs in this region (F).***

S: ***3.*** (2 is also correct.)

T: ***Is there a number that belongs in this region (D)?***

S: ***No, multiples of 4 cannot be prime because they all have at least 1, 2, and 4 as divisors.***

Do not expect such an explicit response, although students should realize that **D** is an empty region. Also, region **E** is empty for the same reason.

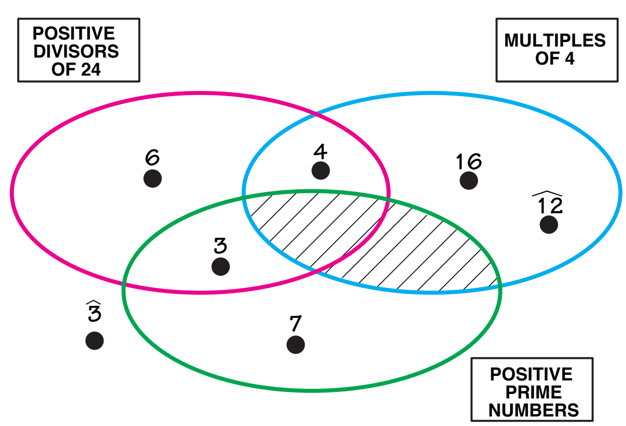
T: ***How can we show that this picture has an empty region?***

S: ***Use hatching.***

Invite a student to do the hatching.

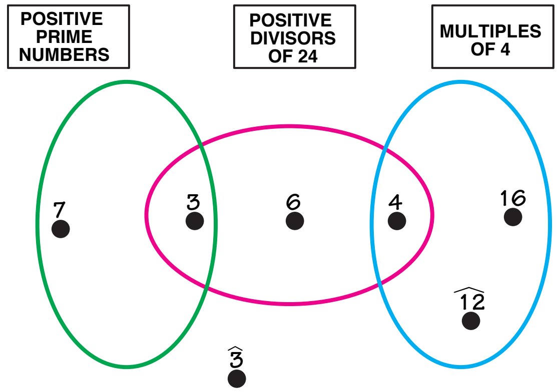
If the student hatches only region **D**, ask the students for numbers that could be in region **E** and then hatch that region also.

T: ***Can we draw a new picture with these same three strings that would give the same information without using any hatching?***



Invite a student to draw such a picture. Any picture in which the blue string and the green string do not overlap, and the red string overlaps both the blue string and the green string, would be correct.

In the new picture, label the strings and ask a student to locate the numbers from the original picture.

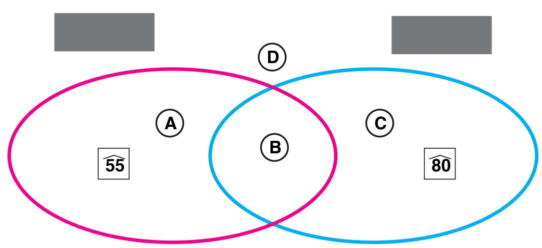


Exercise 2

Using two Numerical String Game posters, prepare your board as illustrated below. You may like to give students String Game analysis sheets, so that they can follow the class activity of eliminating impossible string labels.

Red

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN $\widehat{10}$	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN $\widehat{10}$	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27



Blue

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN $\widehat{10}$	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN $\widehat{10}$	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27

T: ***This is the starting situation for a string game. Here is the list of possibilities for the red string*** (point to the poster on the left), ***and here is the same list of possibilities for the blue string*** (point to the poster on the right)***. There are two clues. What information do these clues give us about the strings?***

Let students suggest labels to cross out on the lists. Each time, ask for a clear explanation as to why a string cannot have a label before you cross it off. For example:

S: ***The red string cannot be for*** **MULTIPLES OF 2*****, because*** $\widehat{55}$ ***is not a multiple of 2.***

S: ***The red string cannot be for*** **MULTIPLES OF 5*****, because*** $\widehat{80}$ ***is a multiple of 5 and is outside the red string.***

S: ***The blue string cannot be for*** **MULTIPLES OF 5*****, because*** $\widehat{55}$ ***is a multiple of 5 and is outside the blue string.***

Sometimes a student may suggest crossing off several labels at one time with one explanation. For example:

S: ***The red string and also the blue string cannot have any of the "positive divisor of" labels because there is a negative number inside the red and also inside the blue string.***

L1

On the two lists, cross out the labels that the strings cannot have as verified by students. When all of the information from these clues has been discussed, your class should find that the red string has been determined and that there are still three possibilities for the blue string. Circle each of the remaining possibilities.

Red

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN 10~~	ODD NUMBERS
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	~~GREATER THAN 10~~	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

Blue

MULTIPLES OF 2	~~MULTIPLES OF 3~~	MULTIPLES OF 4	~~MULTIPLES OF 5~~
MULTIPLES OF 10	~~LESS THAN 50~~	~~LESS THAN 10~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	~~GREATER THAN 10~~	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

T: ***Now I will make some statements. With this information about the strings, you must say whether each statement is true or false or you can't tell. Be ready to explain your answer. The regions are labeled* A, B, C, *and* D *to make it easier to talk about them.***

In the following dialogue, the left side has a variety of statements you can use with this situation. On the right are correct responses. Encourage students to check each other's responses and to discuss them if necessary.

T: ***105 is in the red string.***
S: ***True, because 105 is an odd number.***

T: ***105 is in the blue string.***
S: ***False. 105 cannot be in the blue string because 105 is not a multiple of 2, not a multiple of 4, and not a multiple of 10.***

T: ***So, where must we put 105?***
S: ***In region* A.**

T: ***50 is in the red string.***
S: ***False, because 50 is not an odd number.***

T: ***50 is in the blue string.***
S: ***Can't tell. It would be in the blue string if the blue string were for* MULTIPLES OF 2 *or for* MULTIPLES OF 10. *It would not be in the blue string if the blue string were for* MULTIPLES OF 4.**

T: ***So where could we put 50?***
S: ***In region* C *or region* D.**

T: ***40 is inside both strings (region* B*).***
S: ***False. 40 is not an odd number.***

T: ***40 is outside both strings (region* D*).***
S: ***False. 40 is in the blue string because it is a multiple of 2, and of 4, and of 10.***

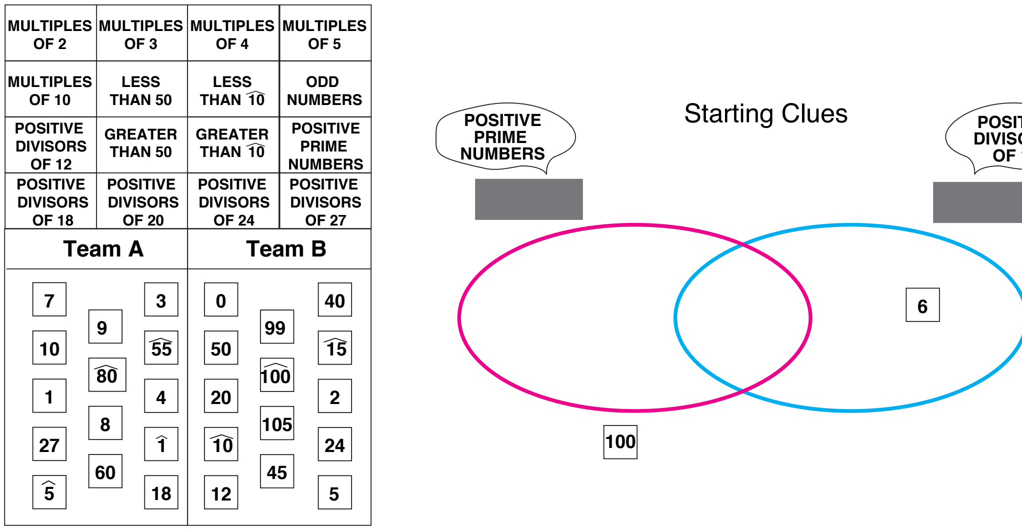
T: ***So where must we put 40?***
S: ***In region* C.**

T: ***Do you see a region that could be hatched because it must be empty?***
S: ***Region* B*, because multiples of 2, of 4, and of 10 are all even numbers.***

Hatch region **B** in the picture.

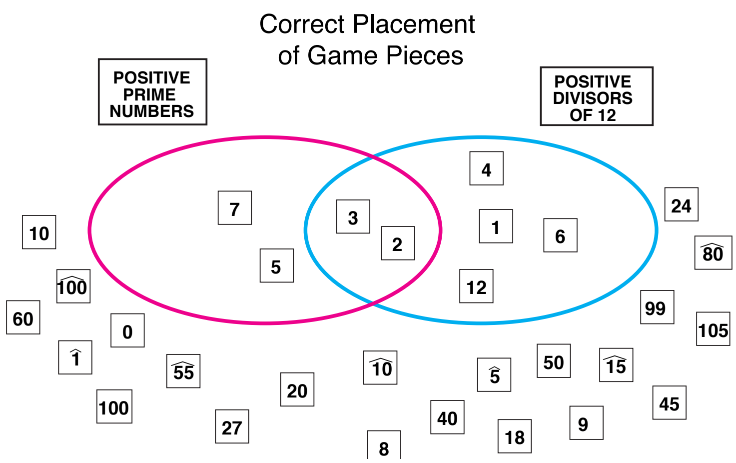
Exercise 3

Play *The String Game* in the usual way (see Appendix D). The illustration below shows a possible game with two pieces placed correctly as starting clues. Encourage students to use their lists (String Game analysis sheets) to eliminate possibilities for string labels during the game.



Note: If you decide to play the game with three or more teams, create a team board with sections for more teams.

The following illustration shows correct placement of all 30 numbers and may be used by you as a crib sheet during the play of the game.



Writing Activity

You may like students to take lesson notes on some, most, or even all their math lessons. The “Lesson Notes” section in the Notes to the Teacher gives suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students can note several facts about the attributes of numbers used in *The String Game*.

Home Activity

Suggest that parents work with their child to make a list of numbers that are prime numbers, or a list of multiples of 4, or a list of positive divisors of 24.

Capsule Lesson Summary

Complete an arrow picture according to a composition rule.

Materials

Teacher
- Tape
- Colored chalk or markers

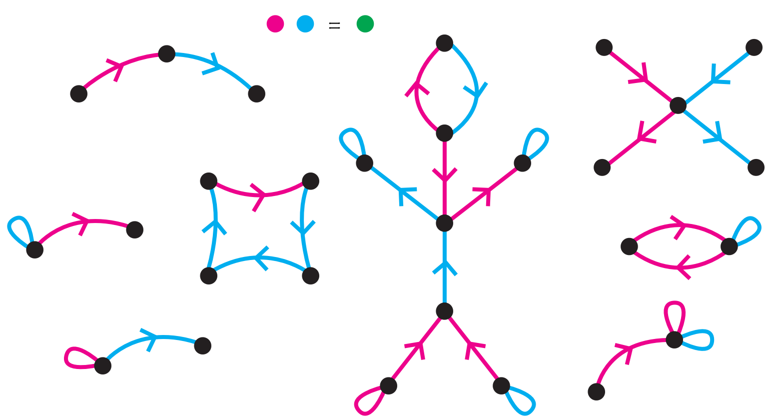
Student
- Colored pencils, pens, or crayons
- Worksheets L2*, **, and ***

Advance Preparation: Before the lesson begins, draw the arrow picture from Exercise 1 on the board or prepare it on a transparency. Give the picture plenty of space, and draw the arrows in colors that can be distinguished easily. Practice the motions described in the first activity. Use two hands, one to point at a starting dot and the other to trace arrows.

Description of Lesson

Exercise 1

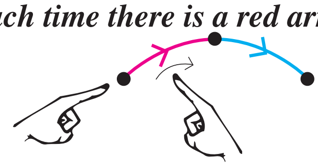
Invite students to comment on this arrow picture.



T: ***We are going to play a game with red and blue arrows. Some of you will remember the game from last year. The object of the game is to draw green arrows, but there is a rule for drawing green arrows.***

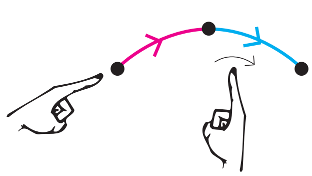
Use the upper left corner of the picture for your explanation. As you explain the rule stated in the left column below, make the motions described in the right column.

T: ***Each time there is a red arrow…***



(Point and hold your left forefinger on a dot at which a red arrow starts. Follow the red arrow with your right forefinger in the direction of the arrowhead.)

…followed by a blue arrow

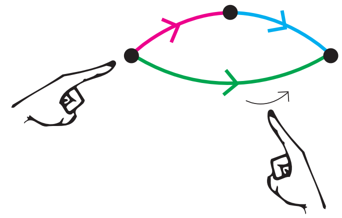


(Stop the motion of your right forefinger at the middle dot; tap the dot, and then follow the blue arrow. Hold your right forefinger at the ending dot of the blue arrow.)

L2

T: ***…then we can draw a green arrow from the dot where the red arrow starts …*** (Tap this dot several times with your left forefinger.)

…to the dot where the blue arrow ends. (Tap this dot several times with your right forefinger.)



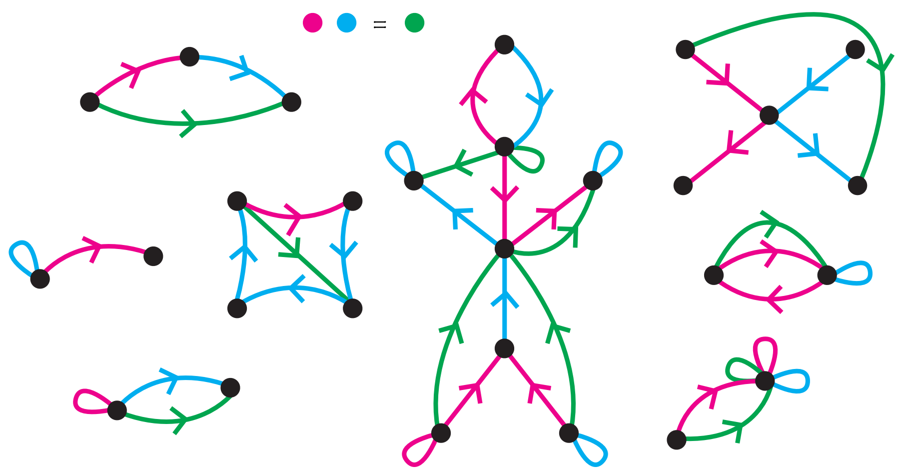
First trace and then draw the green arrow. Emphasize that your left forefinger marks the start and that your right forefinger is at the end of the green arrow.

Repeat the motions as you summarize the rule. Point out that  is a shorthand way of recording this rule:

T: ***Red followed by blue is green. Where can we draw other green arrows?***

Invite students to the board, one at a time, to show where other green arrows can be drawn (see the answer key below). Ask students to first trace a red arrow and a blue arrow following it, and then to trace how a green arrow can be drawn. Stop a student who starts to trace against the direction of an arrow, and emphasize that the direction of an arrow must be followed. Encourage the class to help you check for mistakes. Let a student draw a green arrow if it has been traced correctly. You may insist that students verbalize the "red followed by blue is green" rule each time they find a place to draw a green arrow. You may need to mention that a loop is like an arrow that starts and ends at the same dot.

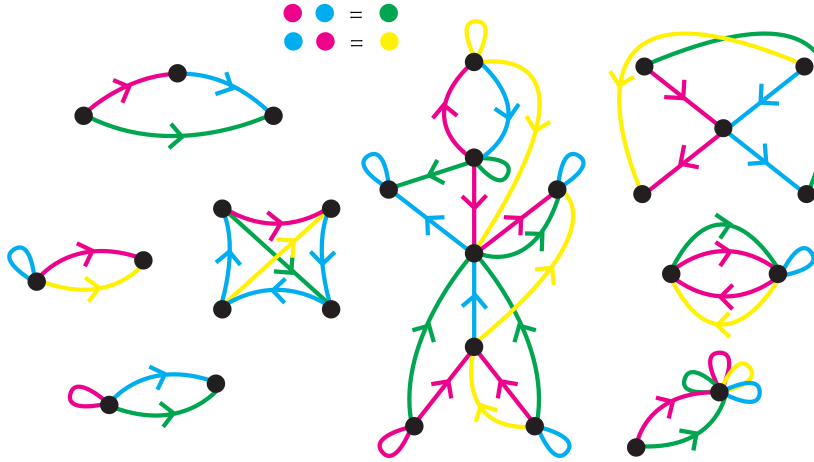
Answer Key



When many of the green arrows have been found, introduce a new rule.

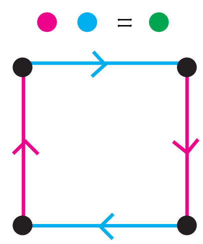
T: ***Red followed by blue is green. Let's extend the game with a rule for blue followed by red. Blue followed by red is yellow. Where can we draw yellow arrows? Why?***

Invite students to show where yellow arrows can be drawn. In the meantime, they may also find any missing green arrows. Every student should be able to participate in this activity, and it should move quickly. With a little encouragement, your class should be able to find all of the green and yellow arrows. An answer key appears below.



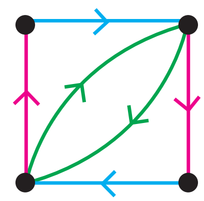
Exercise 2

Draw this arrow picture on the board.



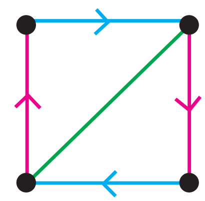
T: ***Can we draw a green arrow in this picture?***

S: ***Yes. Red followed by blue is green.***



S: ***There is also one going in the other direction.***

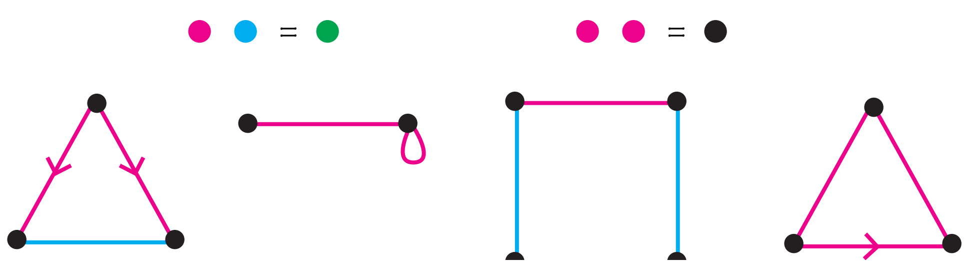
T: ***Do you remember a simpler way to show these two green arrows?***



S: ***Use a green cord in place of the two green arrows.***

T: ***That's right. When we have the same color arrow in both directions between two dots, we can use a cord to represent the two arrows.***

Draw this arrow picture on the board and point out the rules for green and black arrows.

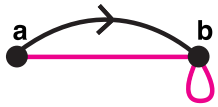


T: ***Who can draw a green arrow or a black arrow? First trace the arrow and tell the class which color arrow you are tracing; then if the class agrees, you may draw the arrow.***

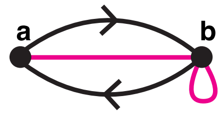
L2

Let students take turns adding green arrows or black arrows to the picture. Encourage them to use language and tracing techniques similar to those in Exercise 1. When there is one color arrow in both directions between two dots, the class should observe that it is possible to draw a cord. If necessary, mention this yourself.

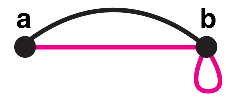
Note: For your reference, the following sequence shows some black arrows that may be difficult to discover. This is one piece of the arrow picture, with the dots labeled for easy reference.



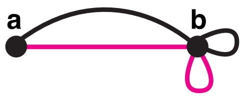
Start at **a**. Trace the red cord to **b**, and then trace the red loop to end at **b**. Draw a black arrow from **a** to **b**.



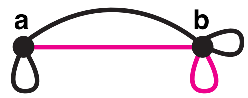
Start at **b**. Trace the red loop to **b**, and then trace the red cord to end at **a**. Draw a black arrow from **b** to **a**.



The two black arrows can be replaced by a black cord.



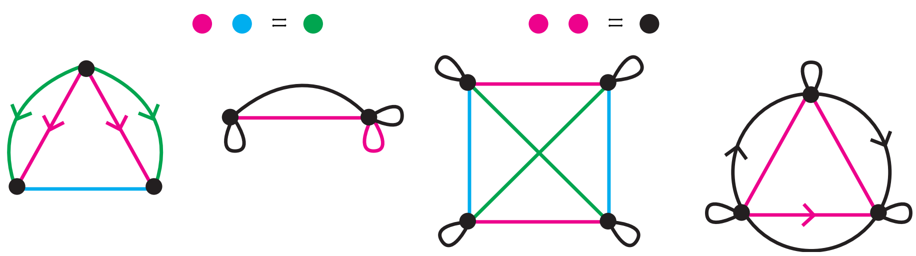
Start at **b**. Trace the red loop twice and end at **b.** Draw a black loop at **b.**



Start at **a**. Trace the red cord to **b**, and then trace the red cord to return to **a**. Draw a black loop at **a**.

The preceding sequence illustrates how two black arrows may be replaced by a cord, and how a red cord or loop can be followed twice to produce a black loop. You may want to mention that a loop or cord can be used twice.

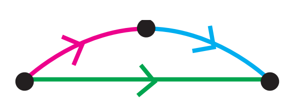
Allow the exercise to continue until the class has found all of the possible green and black arrows that can be drawn in the picture.



Erase the board before going on to Exercise 3.

Exercise 3

Draw this arrow picture and chart on the board. Trace the appropriate arrows as you say,



●	●	● ● = ●
+5	−8	

T: ***Red followed by blue is green. Suppose the dots are for numbers. If the red arrow is for +5 and the blue arrow is for –8, what could the green arrow be for?***

S: ***–3.***

Let students check –3 for the green arrow with several examples of assigning numbers to the dots. You may like to use a bag and pretend it contains a lot of something (for example, marbles or beans). Ask students to think about first putting in five objects and then taking out eight objects. The effect is a loss of three objects (–3).

Fill in the third column of the chart and then pose another problem.

●	●	● ● = ●
+5	−8	−3
you are my father	you are my mother	

T: ***Suppose the dots are for people. If the red arrow is for "you are my father" and the blue arrow is for "you are my mother," what could the green arrow be for?***

Suggest that students pretend they are at the beginning dot (start of the red arrow) and think about who could be at the other dots.

S: ***You are my grandmother.***

T: ***Does it matter which grandmother?***

S: ***On my father's side.***

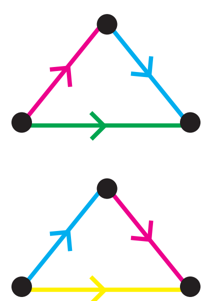
T: ***We call that person your paternal grandmother.***

Continue the activity by presenting relations as indicated in the chart, each time asking the class what the green arrow could be for.

If a numerical problem causes difficulty, put numbers at the beginning of the red arrow and ask students to label the other dots. Check several examples before deciding on the composite relation. If a non-numerical problem causes difficulty, ask that each student imagine himself or herself at the start of the red arrow and then think of who the other dots could be for.

●	●	● ● = ●
+5	−8	−3
you are my father	you are my mother	you are my paternal grandmother
−4	−7	−11
2×	3×	6×
you are in my class	you are not in my class	you are not in my class
4×	$\frac{1}{2}$×	2×
you are my father	you are my sister	you are my father's sister or you are my aunt

Extend the chart to include a fourth column for ● ● = ●. Draw another arrow picture like the one shown below. For each pair of relations, ask the class what the yellow arrow could be for. A completed chart is shown here.



●	●	● ● = ●	● ● = ●
+5	−8	−3	−3
you are my father	you are my mother	you are my paternal grandmother	you are my maternal grandmother
−4	−7	−11	−11
2×	3×	6×	6×
you are in my class	you are not in my class	you are not in my class	you are not in my class
4×	$\frac{1}{2}$×	2×	2×
you are my father	you are my sister	you are my father's sister or you are my aunt	you are my father

T: ***What do you notice about the chart?***

Lead a short discussion, comparing the third and the fourth columns of the chart.

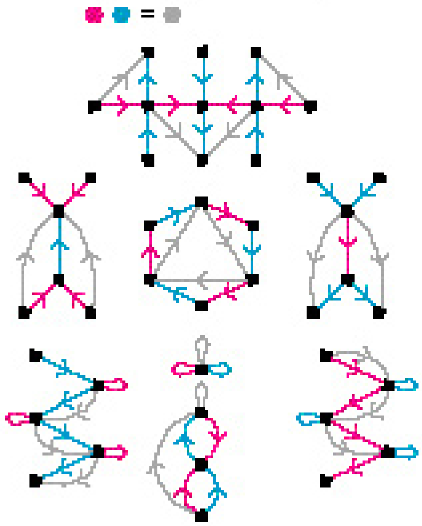
Worksheets L2*. **. and *** are available for individual work.

Home Activity

This is a good time to send home a letter to parents/guardians about the use of arrows. Blackline L2 has a sample letter.

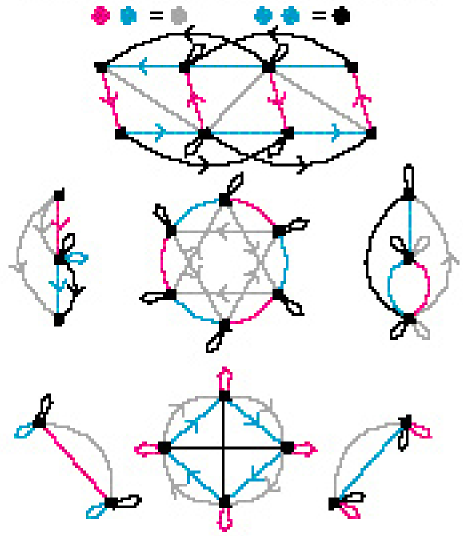
Name________________ L2 ★

Draw all of the possible gray arrows and loops.



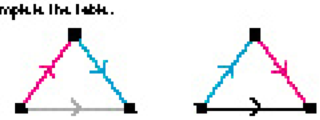
Name________________ L2 ★★

Draw all of the possible gray and black arrows, loops, and cords.



Name________________ L2 ★★★

Complete the table.



●	●	●● = ●	●● = ●
−6	−13	−19	−19
5x	4x	20x	20x
3x	2x	6x	6x
+3	−7	−4	−4
8x	½x or ÷2	4x	4x
[illegible]	[illegible]	you are my daughter	you are my nephew
you are my mother	you are my father	[illegible]	you are my paternal grandmother
you are my friend	you are my brother	[illegible]	you are my brother's friend
[illegible]	[illegible]	you are [illegible] older than I am	you are [illegible] older than I am
[illegible]	[illegible]	you are older than I am	you are older than I am

Capsule Lesson Summary

Decide where to place numbers in a string picture with one string predetermined and the other string for one of two possibilities. Use hatching as a clue in a string game analysis situation. Play *The String Game* with numbers.

Materials

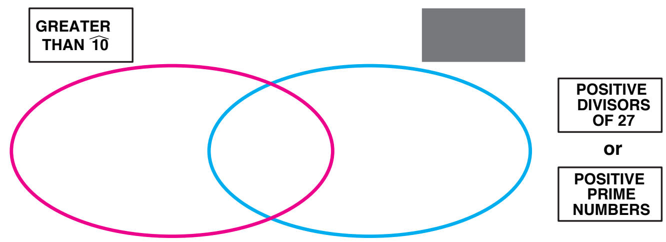
Teacher
- Colored chalk
- Numerical String Game kit

Student
- String Game analysis sheet

Description of Lesson

Exercise 1

Draw this string picture on the board.



T: ***Suppose we are playing The String Game, and know that the red string is for* GREATER THAN $\widehat{10}$ *. Suppose also that the blue string is either* POSITIVE DIVISORS OF 27 *or* POSITIVE PRIME NUMBERS*. Where would you find numbers greater than $\widehat{10}$ on our number line?***

S: ***To the right of $\widehat{10}$.***

T: ***Name some numbers greater than $\widehat{10}$.*** ($\widehat{9}$, $\widehat{8}$, $\widehat{7}$, …, 0, 1, 2, 3, …)

Write this list of numbers on the board.

3 4 0 1 $\widehat{80}$ 7

T: ***With this information given, is it possible to place any of these numbers in the string picture?***

S: ***3 goes in the middle (inside both strings).***

T: ***Are you certain? Why?***

S: ***3 is greater than $\widehat{10}$, so 3 goes inside the red string. 3 is also a positive prime number and a positive divisor of 27, so 3 goes inside the blue string.***

Place 3 correctly in the string picture, and continue the discussion with other numbers in the list.

S: ***4 (or 0) is inside the red string because it is greater than $\widehat{10}$, but 4 (or 0) is neither a positive divisor of 27 nor a positive prime number.***

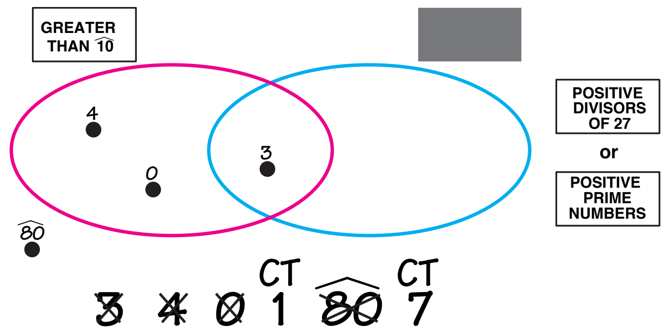
S: ***1 is greater than $\widehat{10}$ and is a positive divisor of 27.***

S: ***Yes, but 1 is not a positive prime number.***

T: ***Then we cannot tell for sure where 1 goes in this picture, so let's label it* CT *for "can't tell."***

S: ***$\widehat{80}$ goes outside of both strings because $\widehat{80}$ is not greater than $\widehat{10}$ and is not positive.***

S: ***You can't tell about 7 because it would be in the middle if the blue string is for* POSITIVE PRIME NUMBERS *but it would be only in the red string if the blue string is for* POSITIVE DIVISORS OF 27.**



At this point, allow students to suggest some other numbers that can be placed in the string picture. In doing so, they should notice that the region inside the blue string but outside the red string is empty. If not, point to the region and ask,

T: ***What numbers could we place here?***

S: ***None, because every number that is a positive divisor of 27 or a positive prime number is greater than $\widehat{10}$ (being positive) and belongs in the middle.***

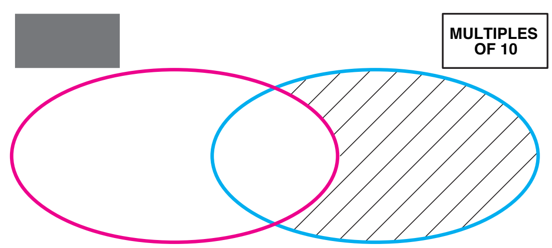
T: ***How can we show that this region is empty?***

S: ***Hatch it.***

Invite a student to hatch this region.

Exercise 2

For this exercise, arrange that each student has a String Game analysis sheet. Draw this string picture on the board.



T: ***Pretend we are playing The String Game and we determine that the blue string is for* MULTIPLES OF 10. *Suppose also we know that this region*** (point to the hatched region) ***is empty. What could the red string be for?***

Let students make suggestions. Encourage the class to check each suggestion.

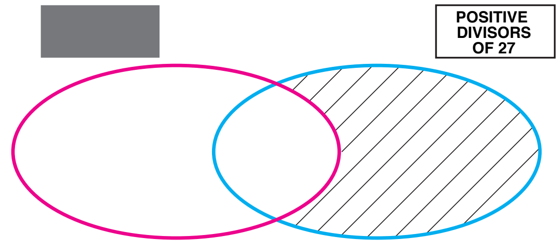
S: **MULTIPLES OF 5*, because every multiple of 10 is a multiple of 5.***

S: **MULTIPLES OF 2*, because every multiple of 10 is a multiple of 2.***

When an incorrect label for the red string is suggested, find some numbers that would be in the hatched region. For example, the red string cannot be for **MULTIPLES OF 4** because 10 or 30 or 50 would be in the hatched region.

Note: You may accept that both strings could have the same label, **MULTIPLES OF 4**.

Relabel the blue string **POSITIVE DIVISORS OF 27**.



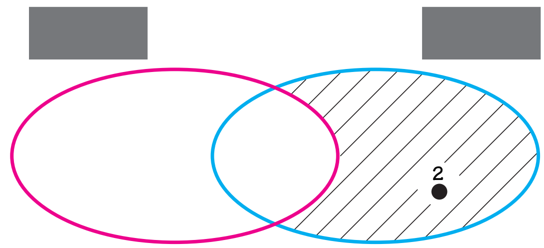
T: ***Suppose the blue string is for* POSITIVE DIVISORS OF 27*. What could the red string be for?***

S: **ODD NUMBERS*, because every positive divisor of 27 is an odd number.***

S: **LESS THAN 50.**

S: **GREATER THAN $\widehat{10}$.**

Erase the labels from the string picture and put 2 in the hatched region.



T: ***Instead of this region*** (point to the hatched region) ***being empty, suppose there is exactly one number in that region, the number 2.***

What could the red string and the blue string be for? Remember, you can use your analysis sheets to help you.

Let students make suggestions and ask the class to check each suggestion. For example:

S: ***The blue string could be for* MULTIPLES OF 2 *and the red string for* MULTIPLES OF 4.**

S: ***No. 6 is a multiple of 2, but not a multiple of 4, so 6 would be in the hatched region.***

S: ***The blue string could be for* POSITIVE DIVISORS OF 18 *and the red string for* POSITIVE DIVISORS OF 27.**

S: ***No. 18 is a positive divisor of 18, but not a positive divisor of 27, so 18 would be in the hatched region.***

After some trial and error, students most likely will discover the identity of the strings.

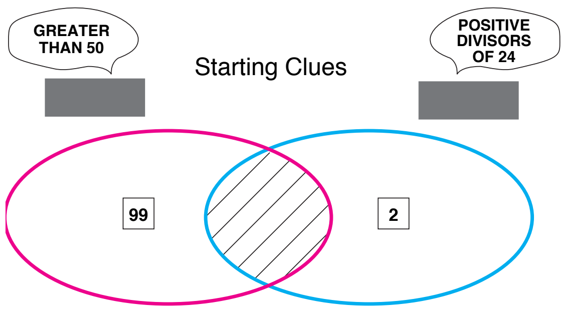
S: ***The blue string is for* POSITIVE PRIME NUMBERS *and the red string is for* ODD NUMBERS.**

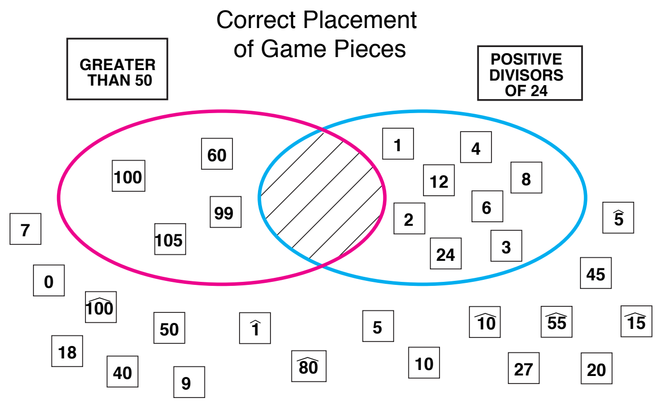
S: ***Yes. The only positive prime number that is not an odd number is 2.***

If the analysis of this situation becomes too difficult and students appear frustrated, leave the exercise as a challenge puzzle.

Exercise 3

Play *The String Game* with numbers in the usual way. The illustration below shows an interesting starting situation you can use for the game. One clue is that the middle region is empty. Remind that class that the hatching should give some information about the strings.





Capsule Lesson Summary

Review addition and multiplication modulo 10 using the games with ten number friends first introduced in the storybook *Dancing Friends*. Practice calculating with these operations. Observe a pattern generated by multiplying increasing numbers of 2s modulo 10.

Materials

Teacher	• None	**Student**	• Paper • Worksheets L4* and **

Description of Lesson

Begin the lesson with some mental arithmetic involving multiplication patterns. (Answers are in boxes.)

$6 \times 8 = \boxed{48}$	$7 \times 9 = \boxed{63}$	$4 \times 9 = \boxed{36}$
$6 \times 80 = \boxed{480}$	$7 \times 90 = \boxed{630}$	$4 \times 90 = \boxed{360}$
$60 \times 8 = \boxed{480}$	$70 \times 9 = \boxed{630}$	$40 \times 9 = \boxed{360}$
$60 \times 80 = \boxed{4\,800}$	$70 \times 90 = \boxed{6\,300}$	$40 \times 90 = \boxed{3\,600}$

Exercise 1

T: ***Does anyone remember the storybook Dancing Friends?*** [†] ***In the story, the boy invites the ten whole numbers 0 through 9 over to his house to play games.***

Write these numerals on the board.

0 1 2 3 4 5 6 7 8 9

T: ***The ten number friends are upset at first because they don't know any games that just the ten of them can play. They can't do multiplication. Do you know why?***

S: ***8* x *9 = 72, and 72 is not one of the ten friends.***

A student may resist and say that 7 and 2 are present and therefore that 72 is present. If so, point out that 7, 2, and 72 are three different numbers.

T: ***Are there other operations they cannot do?***

The class should rule out addition, subtraction, and division. Consider any other operations or relations that your students suggest, and decide with the class whether or not the ten numbers could play them. For example, a student could suggest that the numbers play GCD (greatest common divisor) in which the greatest common divisor of any two of these numbers is again one of the ten numbers. Note that any whole number 1 to 9 is a divisor of 0.

T: ***Well, 0 invents a game that the ten friends can play. You might not remember the game, so I'll give you some clues. I'll use * (read as "star") to signify the operation.***

[†] It is not necessary that students have read this storybook prior to this lesson.

Note: To perform the $*$ operation on two numbers, add the two numbers and keep only the ones digit; or add the two numbers and then subtract 10 when the sum is greater than 10. Do not reveal the operation at this time to the class.

Write these number sentences on the board as clues.

$$2 * 3 = 5$$
$$4 * 8 = 2$$
$$5 * 9 = 4$$
$$8 * 7 =$$

T: ***If you think you know the rule, what number is 8 * 7?*** (5)

At this point do not let students explain the rule. Record $8 * 7 = 5$ even if no one gives the correct answer. Then repeat the activity with the following problems. (Answers are in boxes.)

$$2 * 7 = \boxed{9}$$
$$3 * 9 = \boxed{2}$$
$$9 * 3 = \boxed{2}$$

T: ***Who can explain the rule?***

S: ***Add the two numbers and keep the ones digit.***

S: ***Add the two numbers. If the sum is less than 10, keep it. Otherwise, subtract 10.***

Demonstrate these descriptions of the rule using some of the examples on the board.

T: ***Since this operation involves addition, but differs from regular addition by using just ten numbers, we'll call it "circle-plus."***

Change the $*$s to $\oplus$s in the list on the board.

T: ***I'll put some problems on the board. Copy them, and then decide which of the numbers 0 through 9 to put in each box to make true number sentences.***

Write the following problems on the board. As you observe students' work, you may want to remind them that only the numbers 0 through 9 are playing in this game. (Answers are in boxes.)

$$6 \oplus 9 = \boxed{5}$$
$$4 \oplus 6 = \boxed{0}$$
$$9 \oplus 8 = \boxed{7}$$
$$\boxed{4} \oplus 3 = 7$$
$$\boxed{7} \oplus 5 = 2$$

$$7 \oplus 7 \oplus 7 = \boxed{1}$$
$$8 \oplus 7 \oplus 6 = \boxed{1}$$
$$9 \oplus 9 \oplus 9 \oplus 9 = \boxed{6}$$
$$5 \oplus \boxed{8} \oplus 4 = 7$$

$$1 \oplus 2 \oplus 3 \oplus 4 \oplus 5 \oplus 6 \oplus 7 \oplus 8 \oplus 9 = \boxed{5}$$

When most students have finished, go over answers collectively asking students to explain. Observe that when operating with more than two numbers, students may add all of the numbers and then take the ones digit (or subtract tens from the sum until the result is a number from 0 through 9), or they may use the $\oplus$ operation on two numbers at a time. For example, you might get any of the following explanations for $8 \oplus 7 \oplus 6 = 1$.

S: ***8 + 7 + 6 = 21 and I kept only the 1.***

S: ***8 + 7 + 6 = 21. 21 – 10 = 11. 11 isn't one of the ten friends, so I subtracted 10 again. 11 – 10 = 1.***

S: ***$8 \oplus 7 = 5$ and $5 \oplus 6 = 1$.***

The last problem in the list is of particular interest. If no one suggests the following way of doing the problem, suggest it yourself.

T: ***We could add all of the numbers in the usual way to get 45 and then just keep the ones digit (5), or we could circle-plus two numbers at a time until we get the answer. But there is a clever way of doing the problem.***

Indicate the pairing of numbers as you ask,

T: ***What is $1 \oplus 9$?***

S: ***0.***

T: ***Do you see other numbers we could pair to get 0's in this calculation?***

S: ***2 and 8; $2 \oplus 8 = 0$. 3 and 7; $3 \oplus 7 = 0$. 4 and 6; $4 \oplus 6 = 0$.***

$$1 \oplus 2 \oplus 3 \oplus 4 \oplus 5 \oplus 6 \oplus 7 \oplus 8 \oplus 9 = \square$$

T: ***What is the result of this calculation?*** (5)

Erase the board before going on to Exercise 2.

Exercise 2

Write this problem on the board. $3 \otimes 4 =$

T: ***These ten numbers also play a multiplication game.***
What do you think $3 \otimes 4$ (read as "three circle-times four") is?

S: ***2, because $3 \times 4 = 12$ and we get 2 if we keep only the ones digit in 12.***

Write the following problems on the board and ask students to solve them on their papers.

$4 \otimes 6 = \boxed{4}$	$\boxed{6} \otimes 3 = 8$
$3 \otimes 9 = \boxed{7}$	$\boxed{2 \text{ or } 7} \otimes 2 = 4$
$2 \otimes 3 = \boxed{6}$	$4 \otimes \boxed{1 \text{ or } 6} = 4$
$5 \otimes 4 = \boxed{0}$	$8 \otimes \boxed{4 \text{ or } 9} = 2$

When most students have finished, go over the answers collectively asking students to explain. Accept explanations of the rule in terms of keeping the ones digit or in terms of subtracting tens to get a number from 0 to 9. The class should discover that there is more than one solution to each of the last three problems. When discussing these problems, ask students if they are sure that they have found all of the solutions.

Erase the board before going on to Exercise 3.

Exercise 3

Pose these problems to the class, one at a time.
(Answers are in boxes.)

$$2 \otimes 2 = \boxed{4}$$
$$2 \otimes 2 \otimes 2 = \boxed{8}$$
$$2 \otimes 2 \otimes 2 \otimes 2 = \boxed{6}$$

If the response "16" is given for the last problem, remind the class that only the numbers 0 through 9 are playing the game.

T: ***It gets tiring to write all those 2s and all those ⊗s. So let's use a shorthand way of writing. Let's replace $2 \otimes 2 \otimes 2 \otimes 2$ with 2^4*** (read as "two to the fourth power")***, $2 \otimes 2 \otimes 2$ with 2^3*** (read as "two to the third power") ***and $2 \otimes 2$ with 2^2*** (read as "two to the second power")***. What number is 2^1*** (read as "two to the first power")***?*** (2)

$$2^1 = 2$$
$$2^2 = 4$$
$$2^3 = 8$$
$$2^4 = 6$$
$$2^5 =$$

What number is 2^5?

S: ***2. I multiplied $2 \times 2 \times 2 \times 2 \times 2$ and kept the ones digit of the answer.***

S: ***$2^4 = 6$ and $6 \otimes 2 = 2$.***

Record this result and then continue the sequence of problems to 2^{12}. Students may notice a repeating pattern.

S: ***There's a pattern; they repeat in order 2, 4, 8, 6; 2, 4, 8, 6; and so on.***

Highlight this pattern by organizing the calculations on the board in groups of four.

$2^1 = 2$	$2^5 = 2$	$2^9 = 2$
$2^2 = 4$	$2^6 = 4$	$2^{10} = 4$
$2^3 = 8$	$2^7 = 8$	$2^{11} = 8$
$2^4 = 6$	$2^8 = 6$	$2^{12} = 6$

T: ***Be careful, because now I'm going to skip around.***
What number is 2^{14}?

S: ***4.***

Write the result on the board and continue with other problems. For each problem, allow a minute so that students have time to use the pattern or calculate. (Answers are in boxes.)

$2^{14} = 4$	$2^{21} = \boxed{2}$	$2^{28} = \boxed{6}$
$2^{18} = \boxed{4}$	$2^{23} = \boxed{8}$	$2^{40} = \boxed{6}$

Some students may make guesses. For example, with the jump to 2^{40}, you might get several incorrect answers before the correct one.

T: ***Was it necessary to guess or was there a way to be sure it was 6 from the beginning?***

S: ***I was sure it was 6, because 40 is a multiple of 4, and here 2 to a power that is a (positive) multiple of 4 is always 6.***

Do not force students to be so explicit if an explanation is not forthcoming. The explanation is rather sophisticated, and you should not necessarily expect it.

If students seem to be catching on to the pattern, continue with the following problems. (Answers are in boxes.)

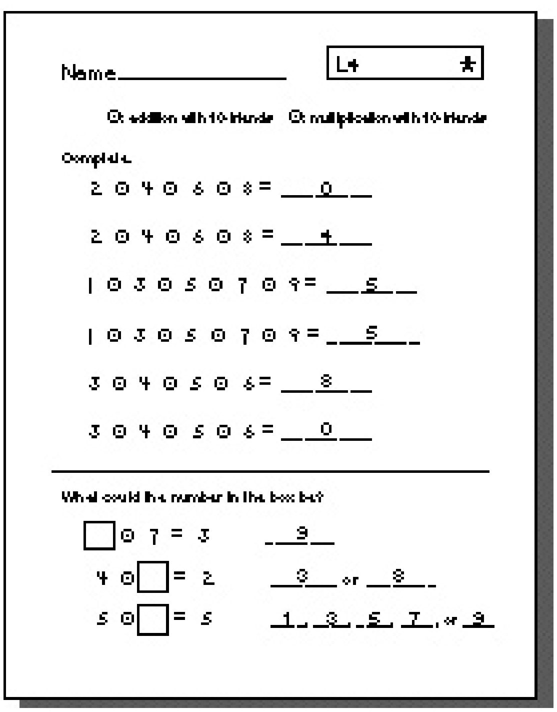
$2^{41} = 2$	$2^{53} = \boxed{2}$	$2^{103} = \boxed{8}$
$2^{50} = \boxed{4}$	$2^{100} = \boxed{6}$	$2^{1001} = \boxed{2}$

Note: For your information, a quick way to find 2^n is to divide n by 4 and consider the remainder.

1) If the remainder is 0, then $2^n = 6$.
2) If the remainder is 1, then $2^n = 2$.
3) If the remainder is 2, then $2^n = 4$.
4) If the remainder is 3, then $2^n = 8$.

Again, you should not expect the class to discover this exact pattern, but encourage them to continue searching. If a student does suggest this pattern, commend the insight and let others continue to think about it.

Worksheets L4* and ** are available for individual work.



Name______________ L4 *

$\odot$: addition with 10-hands $\odot$: multiplication with 10-hands

Complete.

$2 \odot 4 \odot 6 \odot 8 =$ __0__

$2 \odot 4 \odot 6 \odot 8 =$ __4__

$1 \odot 3 \odot 5 \odot 7 \odot 9 =$ __5__

$1 \odot 3 \odot 5 \odot 7 \odot 9 =$ __5__

$3 \odot 4 \odot 5 \odot 6 =$ __8__

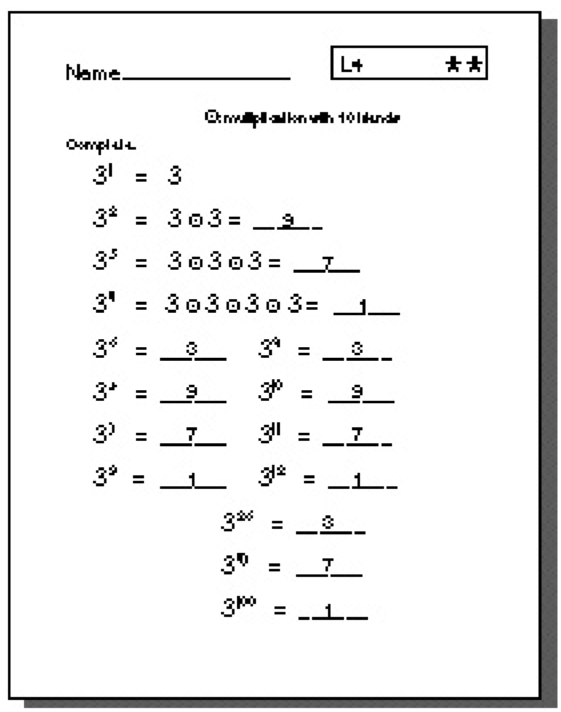
$3 \odot 4 \odot 5 \odot 6 =$ __0__

What could the number in the box be?

$\square \odot 7 = 3$ __9__

$4 \odot \square = 2$ __3__ or __8__

$5 \odot \square = 5$ _1_, _3_, _5_, _7_, or _9_



Name______________ L4 **

$\odot$: multiplication with 10-hands

Complete.

$3^1 = 3$

$3^2 = 3 \odot 3 =$ __9__

$3^3 = 3 \odot 3 \odot 3 =$ __7__

$3^4 = 3 \odot 3 \odot 3 \odot 3 =$ __1__

$3^5 =$ __3__ $3^9 =$ __3__

$3^6 =$ __9__ $3^{10} =$ __9__

$3^7 =$ __7__ $3^{11} =$ __7__

$3^8 =$ __1__ $3^{12} =$ __1__

3[illegible] = __3__

3[illegible] = __7__

$3^{100} =$ __1__

Capsule Lesson Summary

Review addition and multiplication modulo 10. Draw an arrow picture for the relation 2⊗ (multiplication by 2 modulo 10) and examine iterated composites of this relation.

Materials

Teacher	• Colored chalk	**Student**	• Colored pencils, pens, or crayons • Paper • Worksheets L5(a) and (b)

Description of Lesson

Briefly review the story about ten number friends with ⊕ and ⊗ operations.

T: ***Ten number friends play the games ⊕ and ⊗*** (read as "circle-plus" and "circle-times")***. Do you remember which numbers they are?***

S: ***0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.***

Record the numbers on the board. 0 1 2 3 4 5 6 7 8 9

Pose these problems, one at a time, to the class. Let students explain how to do the calculations. (Answers are in boxes.)

$5 \oplus 8 = \boxed{3}$	$2 \otimes 4 = \boxed{8}$
$3 \oplus 2 = \boxed{5}$	$7 \otimes 6 = \boxed{2}$
$4 \oplus 6 = \boxed{0}$	$4 \otimes 9 = \boxed{6}$
$6 \oplus 4 \oplus 7 = \boxed{7}$	$3 \otimes 2 \otimes 4 = \boxed{4}$

T: ***Today we are going to look at the relation 2⊗. We've already used arrows to look at many kinds of relations: +5, –6, 2x, ÷6, ⅓x, "you are taller than me," and "you are my sister." Let's try to draw an arrow picture for 2⊗ with the ten numbers 0 to 9.***

Indicate a color key for 2⊗ on the board and draw a dot. Let the class choose any of the ten numbers for that dot. This example assumes the class chooses to start with 1. Draw a 2⊗ arrow starting at 1.

T: ***What number is 2 ⊗ 1?***

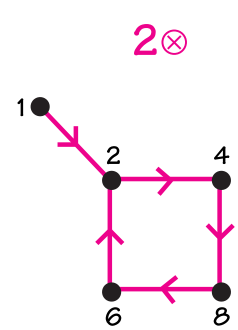
S: ***2.***

Put 2 at the ending dot of the arrow, and draw an arrow starting at 2.

T: ***What number is 2 ⊗ 2?***

S: ***4.***

Continue in this manner until your picture looks like this one. Try to arrange your picture in the same way.



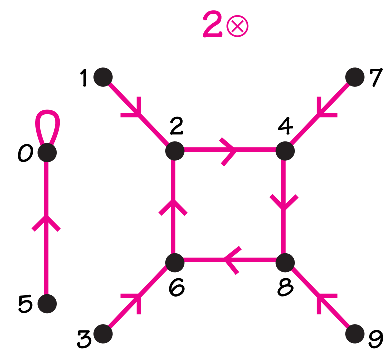
T: ***So far we have five of the ten number friends in our 2⊗ picture.***

Let the class choose a number not yet in the picture, for example, 7.

T: ***What number is 2 ⊗ 7.*** (4)

Extend the picture by drawing a dot for 7 and an arrow from 7 to 4.

Continue in this way until all ten numbers 0 to 9 are in the picture. You may need to prompt students to see that there is a loop at 0.



T: ***What do you notice about the picture?***

Let students comment freely.

T: ***How many arrows start at each of the dots in the picture?***

S: ***Exactly one.***

Point to the various dots, one at a time, and ask the class to check that there is exactly one arrow starting at each of these dots.

Announce that you want to look at compositions with this relation.

T: ***Suppose we call our 2x relation R. Let's draw blue arrows for the relation R followed by R; that is, the rule for blue arrows is red followed by red.***

Add this information to the key.

$R = 2\otimes$

$R \circ R = R^2$

T: ***We can read this as R circle R, R followed by R, R to the second power, or just R-two.***

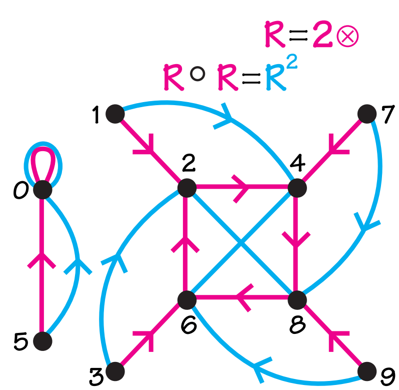
Invite students to locate places to draw blue arrows in the picture. Remind the class, as necessary, of the rule "red followed by red is blue" to help them find places to draw blue arrows. For example, put your left forefinger on the dot for 1, and trace the red arrows from 1 to 2 and from 2 to 4 with your right forefinger. As you do this, say,

T: ***Red followed by red …***

Emphasize that you started at 1 (your left forefinger is there) and that you ended at 4 (your right forefinger is there). Then trace an arrow starting at 1 and ending at 4 as you say,

T: ***… is blue.***

Continue by asking students to trace and then draw more blue arrows. A student may suggest replacing a round trip with blue arrows by a blue cord. If not, suggest this yourself in order to simplify the drawing. When all of the blue arrows have been drawn, your picture should look like this:



T: ***Earlier we noticed that exactly one red arrow starts at each dot. Is that true of R^2? Does exactly one blue arrow start at each dot?***

S: ***Yes.***

Point to several dots in the picture and ask the class to check that this is true.

Write this statement on the board.

$$R \circ R = R^2 = \underline{\quad}\otimes$$

T: ***Can we put one of the ten number friends in the box? Is R^2 for some number ⊗? Write your answer on a piece of paper.***

Allow a few minutes, and then check some of the students' responses. Discuss the problem collectively.

S: ***The blue arrows could be for 4⊗.***

T: ***How can we be sure?***

S: ***Check each arrow.***

S: ***A blue arrow starts at 7 and ends at 8. $4 \otimes 7 = 8$.***

Quickly check that the other blue arrows could be for 4⊗.

$4 \otimes 0 = 0$	$4 \otimes 5 = 0$
$4 \otimes 1 = 4$	$4 \otimes 6 = 4$
$4 \otimes 2 = 8$	$4 \otimes 7 = 8$
$4 \otimes 3 = 2$	$4 \otimes 8 = 2$
$4 \otimes 4 = 6$	$4 \otimes 9 = 6$

T: ***All of the blue arrows agree with these calculations, so 4⊗ is correct.***

Record 4 in the box. Distribute copies of Worksheets L5(a) and (b).

You may like students to work in small groups or pairs on these worksheets.

T: ***On this worksheet (front and back), there are eight copies of the relation R (2⊗). First, color the R^2 blue and in that picture copy the blue arrows that we just drew on the board. Also, put 4 in the box since we found that $R^2 = 4\otimes$. Then draw arrows for R^3 on the copy in the upper right-hand corner. Use any color you wish except red and blue. Be sure to fill in the R^3 in the color you choose. To find an R^3 arrow, start at some dot and follow three R (red) arrows in order to find the ending dot. Then draw the arrow in the color you chose for R^3. When you finish drawing all of the R^3 arrows, try to find a number to fill in the box for ⊗. Then continue to draw R^4, R^5, R^6, R^7, R^8, and R^9 arrows using a variety of colors.***

As students are working, you may need to help those who are having difficulty getting started. Emphasize that to find R^3 arrows, you follow three red arrows in order.

About five minutes before the end of the period, discuss the worksheets collectively. Students should tell you which numbers belong in the boxes, and most likely they will spontaneously tell you about how some pictures look.

S: ***R^2 is the same as R^6.***

S: ***They are both for 4⊗.***

T: ***So we can use the same color of arrows for R^6 as we used for R^2.***

L5

Begin a list of equivalences on the board.

$R^2 = R^6$

S: ***R^8 is the same as R^4; they are both for 6⊗.***

S: ***The arrows for R^7 are the same as the arrows for R^3.***

S: ***R^9 is the same as R^5.***

T: ***R^9 and R^5 are for ⊗ what number?***

S: ***2⊗.***

T: ***If you look at those pictures, you'll see that the arrows are just duplicates of the red arrows. R, R^5, and R^9 are all the same relation.***

As students name equivalences, list them on the board. Observe that students may have many different colors for the same relation, but you will simply choose four colors for the list on the board.

$2⊗ = R^1 = R^5 = R^9$
$4⊗ = R^2 = R^6$
$8⊗ = R^3 = R^7$
$6⊗ = R^4 = R^8$

T: ***Do you notice any patterns in the list?***

S: ***It looks like the even powers of R are either 4⊗ or 6⊗, and the odd powers of R are either 2⊗ or 8⊗.***

S: ***They repeat in groups of 4.***

T: ***What about R^{10}?***

S: ***It's the same as R^2 and R^6.***

S: ***$R^{10} = 4⊗$.***

T: ***So I'll write R^{10} in blue.***

Continue with R^{11} through R^{16}.
The list should look like this.

$2⊗ = R^1 = R^5 = R^9 = R^{13}$
$4⊗ = R^2 = R^6 = R^{10} = R^{14}$
$8⊗ = R^3 = R^7 = R^{11} = R^{15}$
$6⊗ = R^4 = R^8 = R^{12} = R^{16}$

Also, ask which colors the following relations would be.

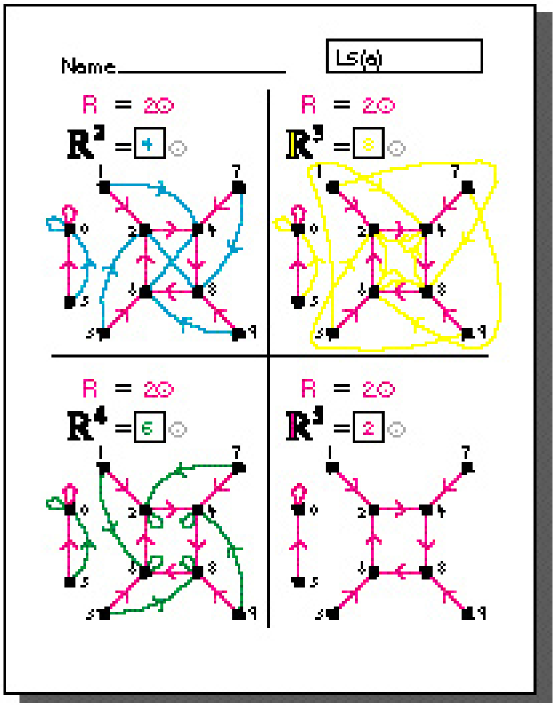
R^{18} (blue)	R^{40} (green)
R^{20} (green)	R^{39} (yellow)

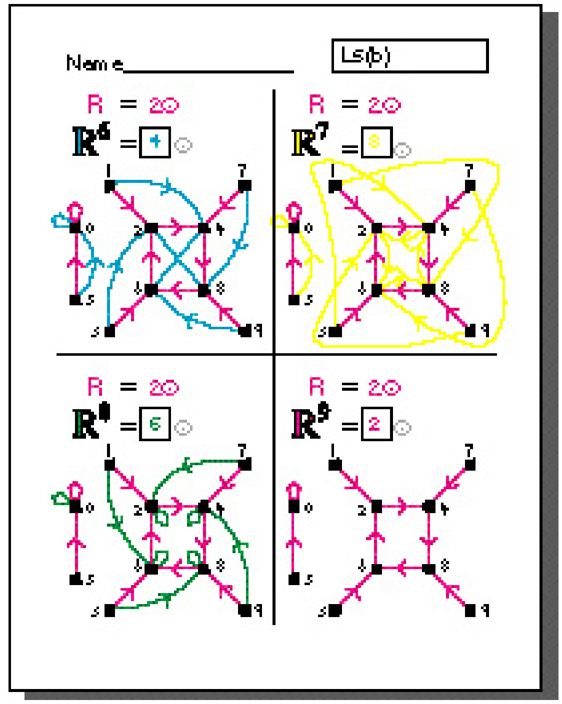
T: ***Now here's a hard one. Think about it and tell me the answer tomorrow.***

Write this expression on the board.

$R^{2\,042} = \_⊗$

Note: Since 2 042 = (4 x 510) + 2, then $R^{2\,042} = 4⊗$.


Name
L5(a)
R = 20
R^2 = 4
R = 20
R^3 = 8
R = 20
R^4 = 6
R = 20
R^5 = 2


Name
L5(b)
R = 20
R^6 = 4
R = 20
R^7 = 8
R = 20
R^8 = 6
R = 20
R^9 = 2

L6 OPERATIONS WITH 20 FRIENDS

Capsule Lesson Summary

Review addition and multiplication modulo 10, and introduce addition and multiplication modulo 20. Practice various calculations with these operations.

Materials

Teacher	• None	Student	• Paper • Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

Ask the class to recall the story about a boy who invited ten number friends to play games together. Solicit the ten numbers from the class, and record them in an upper corner of the board.

0 1 2 3 4 5 6 7 8 9

T: ***When these ten numbers were trying to decide which games to play, what problems did they have?***

S: ***Sometimes answers were more than 9 or less than 0, and those numbers were not invited.***

S: ***6 + 9 = 15 and 15 is not one of the ten friends.***

S: ***5 x 8 = 40 and 40 is not one of the ten numbers.***

T: ***What games did they decide upon?***

S: $\oplus$ ***and*** $\otimes$.

Review these operations by posing several problems and asking students for explanations as to how to do the calculations. (Answers are in boxes.)

$7 \oplus 9 = \boxed{6}$ $\quad$ $4 \oplus 8 = \boxed{2}$ $\quad$ $9 \oplus \boxed{} = 4$

$4 \otimes 2 = \boxed{8}$ $\quad$ $7 \otimes 7 = \boxed{9}$ $\quad$ $\boxed{} \otimes 9 = 7$

Whenever a sum or product is 9 or less, the operation $\oplus$ or $\otimes$ gives the same answer as + or x. But when a sum or product is 10 or more, students should explain the rule in terms of keeping the ones digit, or in terms of subtracting tens from the sum or product to get one of the numbers 0 through 9.

Exercise 2

T: ***One day the boy decides to invite not only these ten number friends, but ten more numbers to play. How many number friends gather on that day?***

S: ***Twenty numbers.***

T: ***Can you guess which numbers they are?***

S: ***The numbers 0 through 19.***

0 1 2 3 4 5 6 7 8 9
10 11 12 13 14 15 16 17 18 19

Extend the list of numbers on the board.

T: ***How do you know there are 20 numbers here*** (point to the list)***?***
Ten numbers 0 to 9 and ten numbers 10 to 19; altogether that's 20.

S: ***There are 19 numbers from 1 to 19, and with 0 there are 20 numbers altogether.***

T: ***The 20 friends decide upon new rules for the games so that all 20 numbers can play. See if you can discover the rules.***

Note: In this case, to perform the * operation on two numbers, add the two numbers. If the sum is 19 or less, keep it. If it is 20 or more, subtract 20 from the sum. Do not reveal the rule at this time.

Write these number sentences on the board.

$$5 * 3 = 8 \qquad 7 * 5 = 12$$
$$14 * 11 = 5 \qquad 19 * 10 =$$

T: ***If you think you know what the rule is, what number is 19 * 10?*** (9)

Let several students answer, but simply record the answer without yet describing the rule. Repeat the activity with these problems. (Answers are in boxes.)

$$18 * 12 = \boxed{10} \qquad 17 * 16 = \boxed{13}$$
$$13 * 3 = \boxed{16} \qquad 4 * 16 = \boxed{0}$$

If many students are still having trouble guessing the rule, give a hint that the operation has something to do with addition. Then pose another problem such as 10 * 15.

S: ***10 * 15 = 5.***

T: ***10 + 15 = 25 and 10 * 15 = 5. Who can explain the rule?***

$$10 * 15 = \boxed{5}$$

S: ***Add the two numbers. If the sum is less than 20, keep it. If it is 20 or more, subtract 20 to get one of the 20 number friends.***

Ask students to check that this rule gives the correct answers to the problems on the board. Also check other rules suggested by students in this way.

T: ***To help us remember that this is addition with 20 friends, let's use this symbol:*** $+_{20}$ ***.***

Replace * by $+_{20}$ in all of the number sentences on the board. (Read "$+_{20}$" as "plus with twenty friends.") Continue by posing a few more problems. (Answers are in boxes.)

$$8 +_{20} 13 = \boxed{1}$$
$$2 +_{20} 15 = \boxed{7}$$
$$9 +_{20} 19 = \boxed{18}$$

Instruct students to do these problems on their papers. Remind them that only the numbers 0 through 19 are playing the game. Then check the answers collectively, asking for explanations.

You might get different explanations for the last two problems. For example:

$$13 +_{20} \boxed{16} = 9$$
$$\boxed{17} +_{20} 6 = 3$$
$$15 +_{20} 15 +_{20} 15 = \boxed{5}$$
$$[illegible] +_{20} 8 +_{20} 5 +_{20} 10 = \boxed{12}$$

S: ***I added 15 + 15 + 15 = 45. 45 – 20 = 25. 25 is not one of the 20 friends, so I subtracted 20 again. 25 – 20 = 5.***

S: ***15 + 15 = 30 and 30 – 20 = 10. 10 + 15 = 25 and 25 – 20 = 5.***

You may like to put in parentheses to help a student explain this second approach.

$$\underbrace{15 +_{20} 15)}_{10} +_{20} 15 = \boxed{5}$$

T: ***For this next problem, put the same number in both boxes.***

Write this problem on the board.

$$\square +_{20} \square = 6$$

S: ***We could put 3 in each box, because 3 + 3 = 6.***

S: ***We could put 13 in each box, because 13 + 13 = 26 and 26 – 20 = 6.***

If students suggest 23, remind them that only the 20 numbers 0 to 19 are playing the game. 3 and 13 are the only solutions to the problem.

Continue in the same way with this problem.

$$\square +_{20} \square = 16$$

(Answers: 8 and 18)

T: ***For the next problem, you may put the same number in the triangle and in the box, but you may also put in different numbers.***

Write this problem on the board.

$$\triangle +_{20} \square = 12$$

Let students make suggestions, and list correct solutions on the board. They should quickly notice that reversing the order of two numbers in one solution gives another solution. Because of this, you need only consider 11 solutions even though there are actually 20.

0	12
1	11
2	10
3	9
4	8
5	7
6	6
3	19
4	18
5	17
6	16

T: ***The 20 whole numbers 0 to 19 can also play a multiplication game called "Times with 20 Friends."*** (Write the symbol $\times_{20}$ on the board.) ***Let's see if you can figure out the rule.***

Present several calculations involving $\times_{20}$ until it appears that many students understand the rule.

$5 \times_{20} 3 = \boxed{15}$ $\qquad$ $10 \times_{20} 2 = \boxed{0}$

$5 \times_{20} 5 = \boxed{5}$ $\qquad$ $4 \times_{20} 12 = \boxed{8}$

$6 \times_{20} 6 = 16$ $\qquad$ $7 \times_{20} 9 = \boxed{3}$

$5 \times_{20} 11 = \boxed{15}$

T: ***Who can explain the rule?***

A student may be able to explain the rule in general, but more likely a student will use one of the calculations recorded on the board to explain it. A sample dialogue follows.

S: ***5 x 11 = 55, 55 – 20 = 35, and 35 – 20 = 15. So 5 $\times_{20}$ 11 = 15.***

T: ***How did you know how many 20s to subtract?***

S: ***I subtracted 20s until I got one of the 20 friends, a number less than 20.***

T: ***What about 5 $\times_{20}$ 3? Who can explain how the rule works with those numbers?***

S: ***5 x 3 = 15. 15 is one of the 20 friends, so we keep it.***

T: ***Therefore the rule is this: Multiply the numbers. If the product is less than 20, keep it. If it is 20 or more, subtract 20s (or a multiple of 20) until you get one of the numbers 0 to 19.***

Instruct students to do the following problems on their papers, and then check the answers collectively.

$9 \times_{20} 8 = \boxed{12}$

$10 \times_{20} 15 = \boxed{10}$

$\boxed{4 \text{ or } 14} \times_{20} 6 = 4$

$10 \times_{20} \boxed{} = 0$ (Answers: even numbers from 0 to 18)

Invite students to write problems involving $+_{20}$ and $\times_{20}$ to challenge the class.

Writing Activity

Suggest that students write and illustrate a story about the 20 friends and the operations $+_{20}$ and $\times_{20}$ to explain the lesson.

Extension Activity

Some students may like to consider what the operations would be like if there were 30 or 40 or 50 friends. Perhaps you could suggest they extend their stories to include such cases.

Capsule Lesson Summary

Review addition and multiplication modulo 10 and modulo 20. Explore a pattern generated by repeatedly multiplying by 2 modulo 20. Construct roads starting at 1, 3, 7, and 9 using arrows for 2 x80. Find that these roads connect to one another, resulting in an arrow picture that resembles a kite with four tails.

Materials

Teacher
- Colored chalk

Student
- Paper
- Blue pencil
- Colored pencils, pens, or crayons

Description of Lesson

Give several sheets of paper and a blue pencil to each student.

Exercise 1

With the class, review the addition and multiplication games with 10 friends and with 20 friends. Do this by doing some calculations. Instruct students first to do the calculations on their papers and then check answers collectively.

$8 +_{10} 9 = \boxed{7}$ $\quad 7 \times_{10} 9 = \boxed{3}$ $\quad \boxed{6} +_{10} 6 = 2$

$3 \times_{10} \boxed{7} = 1$ $\quad (3 \times_{10} \boxed{9}) +_{10} 5 = 2$

T: ***How many friends play these games (point to $+_{10}$ and $\times_{10}$)? Who are they?***

S: ***Ten; the whole numbers from 0 to 9.***

Continue with these problems. For the last problem, remind the class that the same number goes in each box.

$8 +_{20} 15 = \boxed{13}$ $\quad 7 \times_{20} 11 = \boxed{17}$ $\quad \boxed{7} +_{20} 16 = 3$

$3 \times_{20} \boxed{10} = 10$ $\quad (3 \times_{20} \boxed{5}) +_{20} 7 = 2$

$\boxed{} +_{20} \boxed{} = 8$ (Answers: 4 and 14)

T: ***How many friends play these games*** (point to $+_{20}$ and $\times_{20}$)***? Who are they?***

S: ***Twenty; the whole numbers from 0 to 19.***

Exercise 2

One at a time, ask the class to do these calculations as you list them on the board. (Answers are in boxes.)

$$2 \times_{20} 2 = \boxed{4}$$
$$2 \times_{20} 2 \times_{20} 2 = \boxed{8}$$
$$2 \times_{20} 2 \times_{20} 2 \times_{20} 2 = \boxed{16}$$
$$2 \times_{20} 2 \times_{20} 2 \times_{20} 2 \times_{20} 2 = \boxed{12}$$

T: ***It's tiring to write $2\times_{20}$ so many times, so let's use a shorthand notation.***

Write the appropriate power of 2 next to each calculation on the board, and continue with more calculations in the same manner. Read 2^2 as "two to the second power," 2^3 as "two to the third power," 2^4 as "two to the fourth power," and so on.

$$2 \times_{20} 2 = 2^2 = 4$$
$$2 \times_{20} 2 \times_{20} 2 = 2^3 = 8$$
$$2 \times_{20} 2 \times_{20} 2 \times_{20} 2 = 2^4 = 16$$
$$2 \times_{20} 2 \times_{20} 2 \times_{20} 2 \times_{20} 2 = 2^5 = 12$$
$$2^6 = 4$$
$$2^7 = 8$$

T: ***Can you explain why $2^7 = 8$?***

S: ***$2^6 = 4$, so I calculated $2 \times_{20} 4$.***

Continue the calculations and arrange your list to suggest the emerging pattern.

$2^2 = 4$	$2^6 = 4$	$2^{10} = \boxed{4}$	$2^{14} = \boxed{4}$
$2^3 = 8$	$2^7 = 8$	$2^{11} = \boxed{8}$	$2^{15} = \boxed{8}$
$2^4 = 16$	$2^8 = \boxed{16}$	$2^{12} = \boxed{16}$	
$2^5 = 12$	$2^9 = \boxed{12}$	$2^{13} = \boxed{12}$	

T: ***Do you see a pattern in these calculations?***

S: ***If we have an even power of 2, the answer is either 4 or 16. If we have an odd power of 2, the answer is either 8 or 12.***

S: ***The solutions are repeating 4, 8, 16, 12; 4, 8, 16, 12; and so on. These four numbers are always the answers.***

T: ***Can you predict what 2^{25} will be?***

S: ***12, because $2^{16} = 16$, $2^{20} = 16$, $2^{24} = 16$, and $2^{25} = 12$.***

S: ***$2^5 = 2^9 = 2^{13} = 2^{17} = 2^{21} = 2^{25} = 12$.***

T: ***Can you predict what 2^{44} will be?***

S: ***16, because 44 is a multiple of 4; whenever the power is a multiple of 4 we get 16.***

You might not receive such clearly stated explanations of the pattern; however, your students will probably make use of this pattern.

T: ***Yes. Let's see if you can use this pattern to solve other problems.***

Write the problems on the board and instruct students to solve them on their papers. Then check answers collectively.

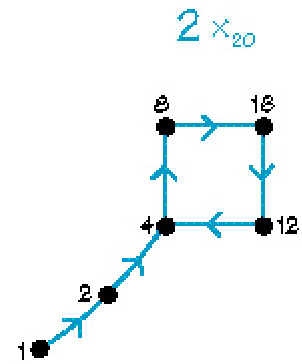
$2^{20} = \boxed{16}$	$2^{41} = \boxed{12}$	$2^{50} = \boxed{4}$	$2^{110} = \boxed{4}$
$2^{23} = \boxed{8}$	$2^{45} = \boxed{12}$	$2^{108} = \boxed{16}$	$2^{403} = \boxed{8}$

Exercise 3

With the class, recall how you previously drew an arrow picture for the ten friends and $2x_{10}$ (or $2\otimes$). Tell them that today they are going to see what a picture looks like when there are 20 friends and the relation is $2x_{20}$.

T: ***Let's start at 1 and use blue arrows for $2x_{20}$***
(read as "two times with twenty friends").

Draw this picture with the help and active participation of the class.

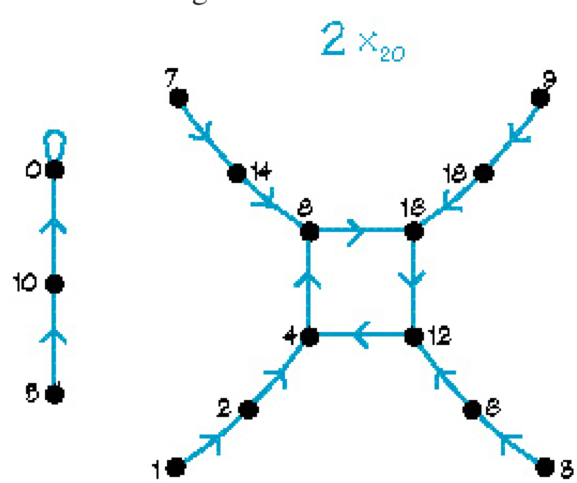


S: ***It looks like a kite.***

Divide the class into four groups. Ask that students in each group draw a road with blue arrows for $2x_{20}$. The students in the first group should start their roads at 3, the second group should start at 5, the third group should start at 7, and the fourth group should start at 9. Allow several minutes for this individual work.

Choose one member from the first group to draw the road starting at 3 on the board and to show how it is connected to the road starting at 1 that is already there. Then choose a member from the third group to do the same for the road starting at 7; then a member from the fourth group to do the same for the road starting at 9. You can influence how the resulting picture will appear by putting the dots for 3, 7, and 9 on the board yourself. Each of the new roads is a "kite" that connects to the first kite at the center; the resulting picture looks like a four-tailed kite as in the next illustration.

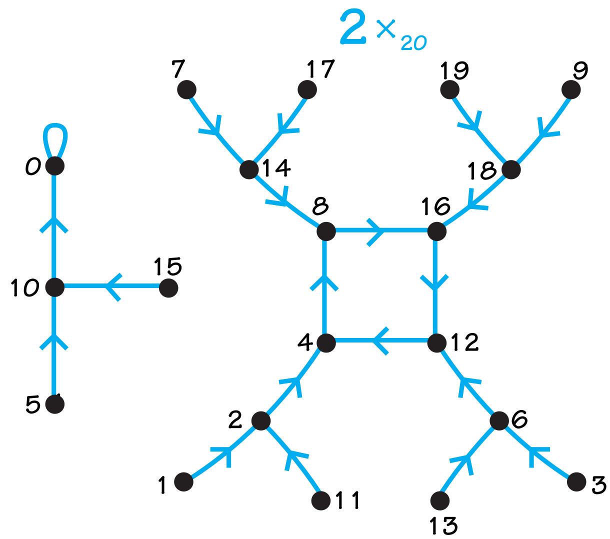
The second group will complain that their road does not connect. Let a student draw their road separate from the big kite picture. Encourage the class to comment on the picture.



Perhaps some students will recognize that the repeating numbers from the calculations in Exercise 2 appear in the center of the picture.

T: ***Are any of the 20 numbers missing from the picture?***

Let students discover the five missing numbers (11, 13, 15, 17, and 19) and suggest how to fit them in the picture.



Extension Activity

Students who investigated games with 30, 40, or 50 friends can extend those investigations to consider what the corresponding $2x_{30}$ or $2x_{40}$ or $2x_{50}$ pictures would be like.

L8 STRING GAME ANALYSIS #3

Capsule Lesson Summary

Decide where to place numbers in a string picture with one string determined and the other string for one of two possibilities. Given a string picture with two possibilities for each string, decide whether statements are true, false, or indeterminate. Play *The String Game* with numbers.

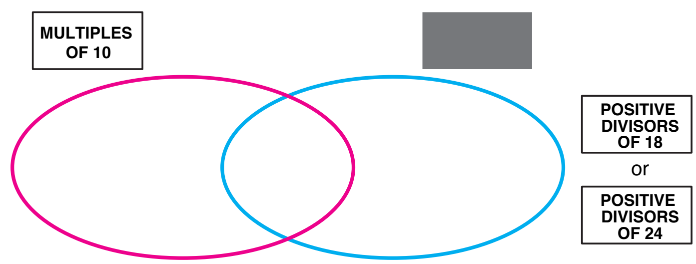
Materials

Teacher	• Numerical String Game kit • Colored chalk	Student	• String Game analysis sheet • Worksheets L8*, **, ***, and ****

Description of Lesson

Exercise 1

Draw this string picture on the board.



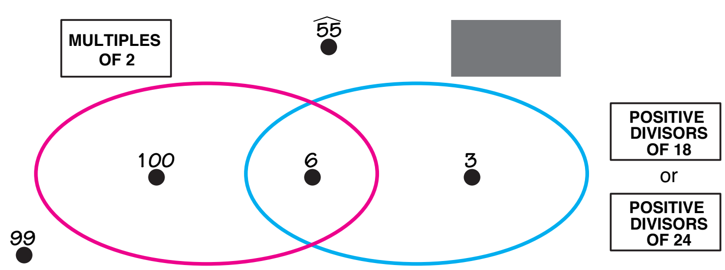
T: ***Pretend that we are playing The String Game. We know that the red string is for* MULTIPLES 2 *and that the blue string is for either* POSITIVE DIVISORS OF 18 *or* POSITIVE DIVISORS OF 24.**

Write this list of numbers on the board.

6 4 3 9 100 18 55 99

T: ***Even though the blue string still has two possible labels, some of these numbers can be placed correctly in the string picture. Who can show us the correct place for one of these numbers?***

Let students respond by discussing the numbers in any order they wish. They should say that a number is in a certain region or that they cannot be sure which of two regions a number is in. Encourage the class to check each response before continuing. Put the numbers whose placement can be determined into the picture. Mark **CT** (can't tell) above those numbers whose locations are uncertain.



CT CT CT

~~6~~ 4 ~~3~~ 9 ~~100~~ 18 ~~55~~ ~~99~~

T: ***Suppose it is your turn in the game. Where would you try to place 4?***

S: ***In one of the two regions inside the red string.***

If a student indicates a region outside the red string, point out that then they are sure to get a "no" answer.

Let a student attempt to place 4 in the string picture. For example, suppose 4 is placed inside the red string but outside the blue string.

T: ***If I say yes, what would that tell you about the blue string?***

S: ***That the blue string is for* POSITIVE DIVISORS OF 18*, because 4 is a positive divisor of 24 and is not in the blue string.***

T: ***Suppose, instead, I say no. What would you know about the blue string?***

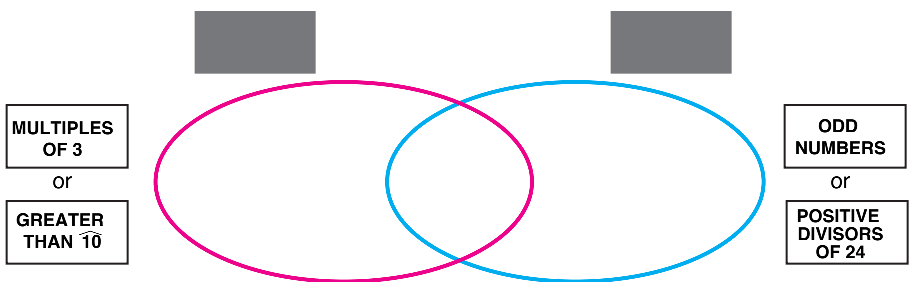
S: ***Then 4 would have to be in the middle (inside both the red and the blue string). The blue string would have to be for* POSITIVE DIVISORS OF 24 *because 4 is not a positive divisor of 18.***

T: ***So no matter how I answer, you will know what the blue string is for.***

Repeat this activity, asking where students would try 9 and 18.

Exercise 2

Draw this string picture on the board. Near each string, write two possible labels for that string, as shown here.

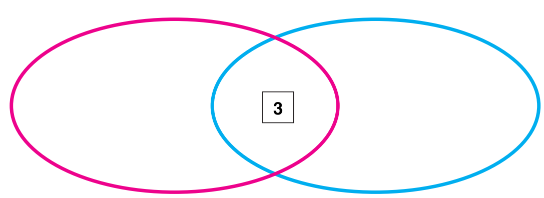


T: ***Let's pretend that we are playing The String Game and we determine that each string must have one of two possible labels. The red string could be for* MULTIPLES OF 3 *or* GREATER THAN $\widehat{10}$*. The blue string could be for* ODD NUMBERS *or* POSITIVE DIVISORS OF 24*. Now I will give you some statements. You say whether the statement is true or false, or that you can't tell.***

As you make each statement, put the number referred to in the indicated region. If the statement is true, leave the number in the picture. If the statement is false or you can't tell, remove it from the picture. Encourage class discussion about each statement, and request an explanation about why the statement is true or false, or why you can't tell.

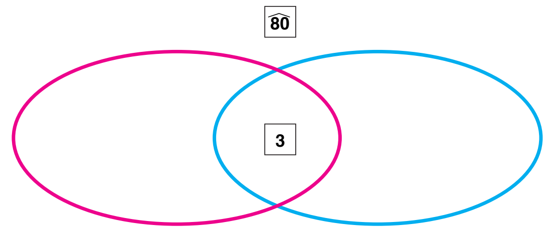
T: ***3 is in the red string and the blue string.***

S: ***True. 3 is a multiple of 3 and greater than $\widehat{10}$; therefore, 3 is inside the red string. 3 is an odd number and a positive divisor of 24; therefore, 3 is inside the blue string. So 3 must be in both the red string and the blue string.***



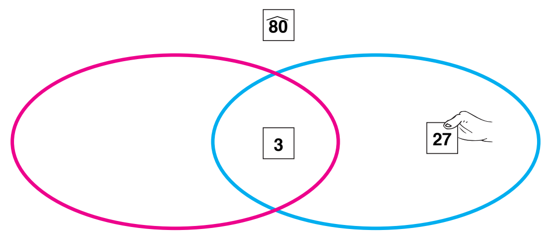
T: *$\widehat{80}$ is outside both the blue string and the red string.*

S: *True. $\widehat{80}$ is not a multiple of 3, is not greater than $\widehat{10}$, is not odd, and is not a positive divisor of 24; therefore, $\widehat{80}$ is outside of both strings.*



T: *27 is inside the blue string but outside the red string.*

S: *False. 27 is a multiple of 3 and is greater than $\widehat{10}$; therefore, 27 must be inside the red string.*

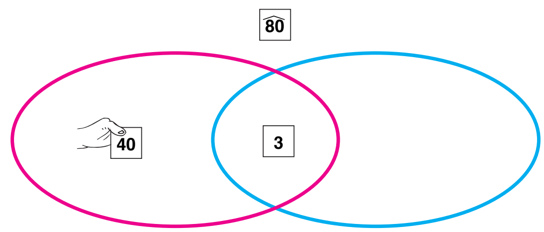


T: *Can we tell where 27 belongs?*

S: *No. 27 is in the middle if the blue string is for* **ODD NUMBERS**, *and 27 is just in the red string if the blue string is for* **POSITIVE DIVISORS OF 24.**

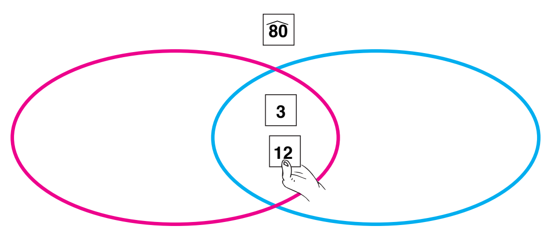
T: *40 is inside the red string but outside the blue string.*

S: *Can't tell. 40 is outside the blue string because 40 is not odd and not a positive divisor of 24. But 40 would be outside both strings if the red string is for* **MULTIPLES OF 3.**



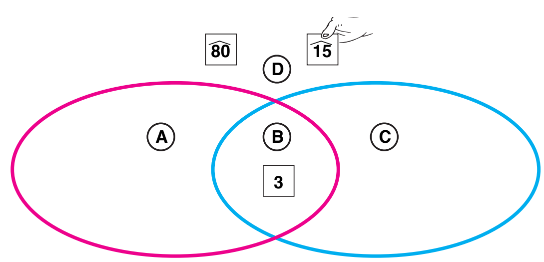
T: *12 is in both the blue string and the red string.*

S: *Can't tell. If the blue string is for* **ODD NUMBERS**, *then 12 is outside the blue string; if the blue string is for* **POSITIVE DIVISORS 24**, *then 12 is in both the red string and the blue string.*



T: *$\widehat{15}$ is outside both strings.*

S: *Can't tell.*



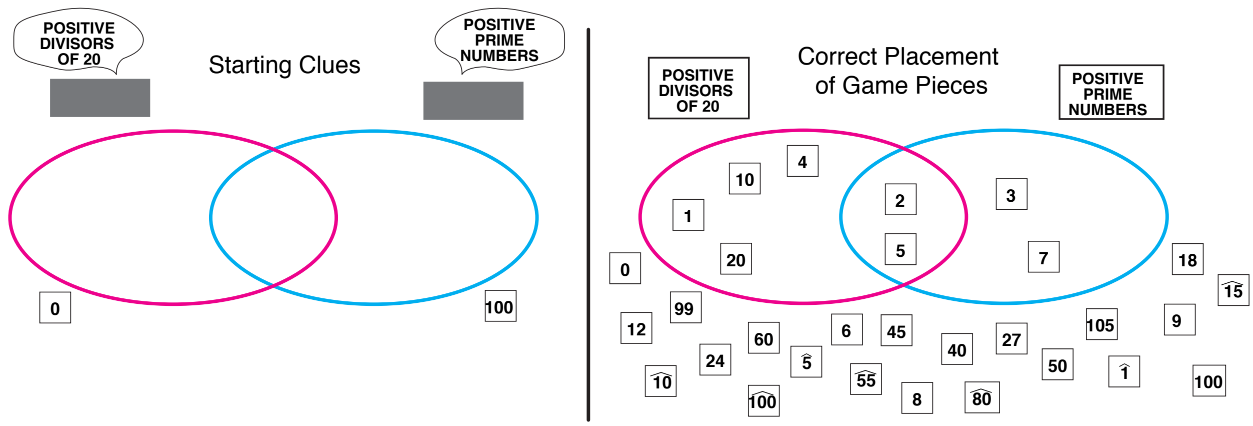
Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

Note: $\widehat{15}$ could be in any of the four regions, as indicated by this table.

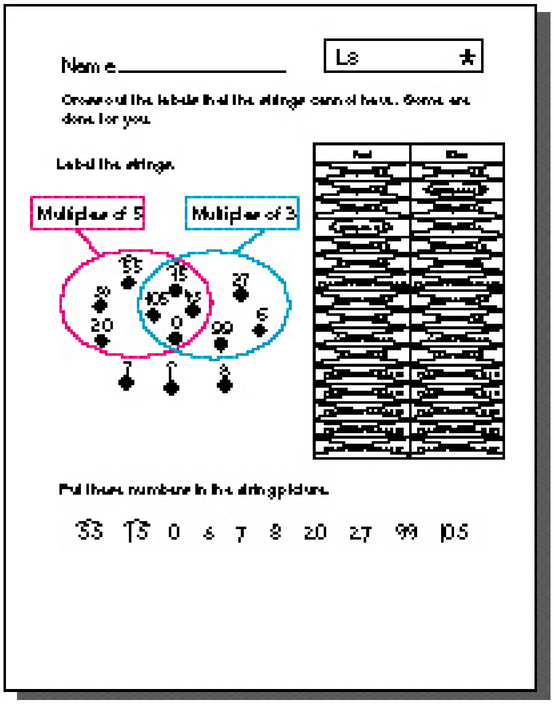
RED	BLUE	$\widehat{15}$ is in region
MULTIPLES OF 3	ODD NUMBERS	B
MULTIPLES OF 3	POSITIVE DIVISORS OF 24	A
GREATER THAN $\widehat{10}$	ODD NUMBERS	C
GREATER THAN $\widehat{10}$	POSITIVE DIVISORS OF 24	D

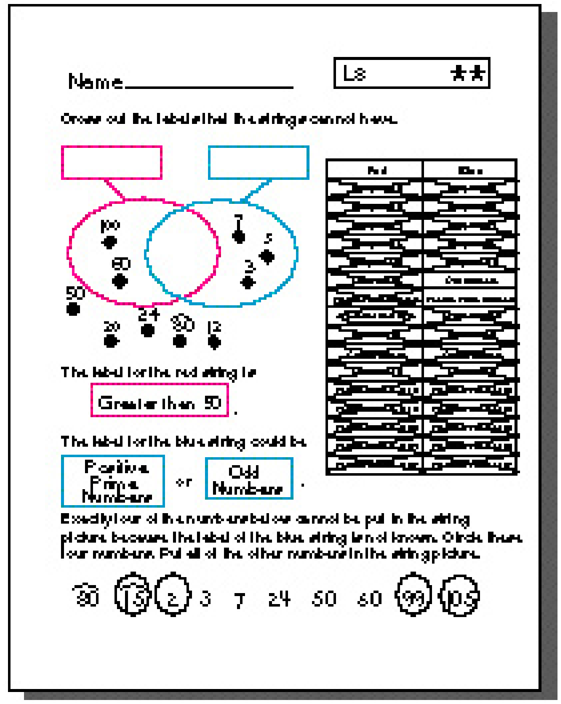
Exercise 3

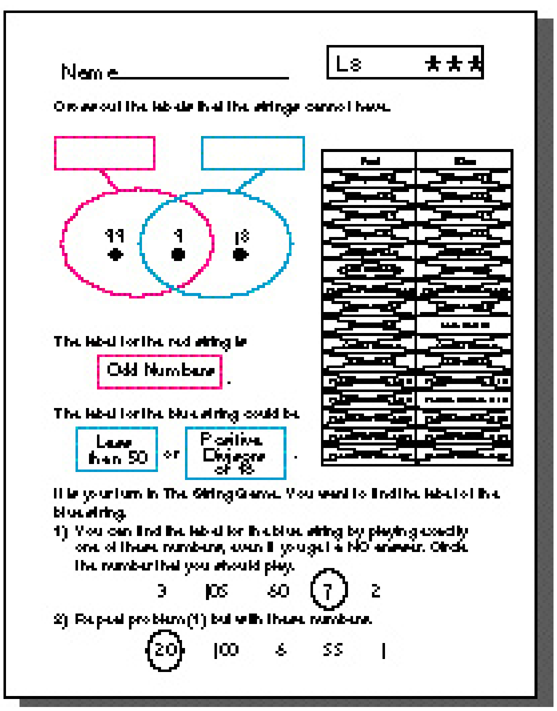
Play *The String Game* with numbers in the usual way. The illustration below shows a possible game with starting clues.

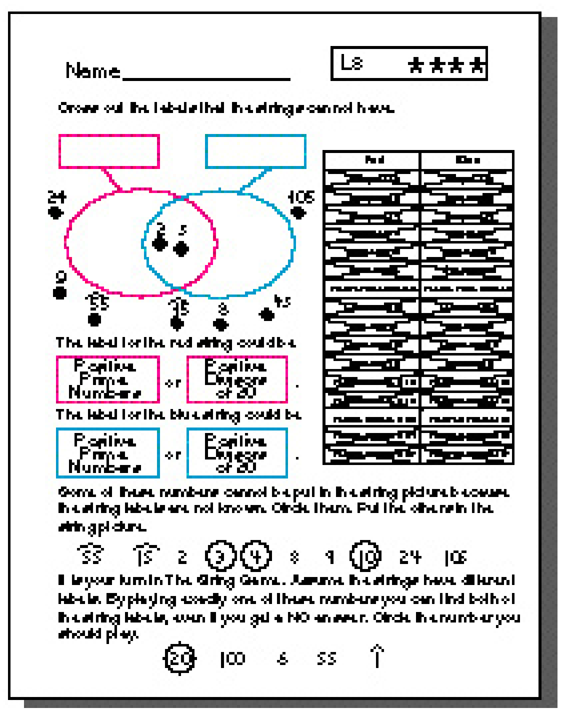


Worksheets L8*, **, ***, and **** are available for individual work.


Name
L8 ★
Cross out the labels that the strings cannot have. Some are done for you.
Label the strings.
Multiples of 5
Multiples of 3
55 30 20 15 105 45 0 27 6 99 7 1 8
Put these numbers in the string picture.
35 15 0 6 7 8 20 27 99 105


Name
L8 ★★
Cross out the labels that the strings cannot have.
100 60 50 7 5 3 20 24 80 12
The label for the red string is
Greater than 50
The label for the blue string could be
Positive Prime Numbers or Odd Numbers
Exactly four of the numbers below cannot be put in the string picture because the label of the blue string is not known. Circle these four numbers. Put all of the other numbers in the string picture.
80 15 2 3 7 24 50 60 99 105


Name
L8 ★★★
Cross out the labels that the strings cannot have.
99 9 18
The label for the red string is
Odd Numbers
The label for the blue string could be
Less than 50 or Positive Divisors of 18
It is your turn in The String Game. You want to find the label of the blue string.
1) You can find the label for the blue string by playing exactly one of these numbers, even if you get a NO answer. Circle the number that you should play.
3 105 60 7 2
2) Repeat problem (1) but with these numbers.
20 100 6 55 1


Name
L8 ★★★★
Cross out the labels that the strings cannot have.
24 105 2 5 9 55 15 8 45
The label for the red string could be
Positive Prime Numbers or Positive Divisors of 20
The label for the blue string could be
Positive Prime Numbers or Positive Divisors of 20
Some of these numbers cannot be put in the string picture because the string labels are not known. Circle them. Put the others in the string picture.
55 15 2 3 4 8 9 10 24 105
It is your turn in The String Game. Assume the strings have different labels. By playing exactly one of these numbers you can find both of the string labels, even if you get a NO answer. Circle the number you should play.
20 100 6 55 1

L9 HOW MANY PERMUTATIONS?

Capsule Lesson Summary

Find the number of arrow pictures that can be drawn with eight dots for people and using arrows for a name exchange (a permutation). Determine the solution inductively by first solving the equivalent problem for two people, three people, four people, and so on.

Materials

Teacher
- Eight slips of paper with student names
- Colored chalk

Student
- Colored pencils, pens, or crayons
- Unlined paper
- Worksheets L9(a) and (b)

Advance Preparation: Choose eight students in your class whose first names start with different letters. Write each name on a separate piece of paper and fold it so the name is not visible.

Description of Lesson

To make the description of the lesson easier to follow, these eight names will be used: Angela, Dirk, Charles, Edward, James, Beth, Martha, and Kim.

Ask eight students (the ones whose names are on your papers) to stand at the front of the classroom. Draw a dot for each of them on the board. Label the dots with the first letters of the students' names. Tell the class that they are to imagine that these eight students are in a family (or club) that is exchanging names for a gift exchange.

T: ***I have eight pieces of paper with one of your names written on each piece. I will mix up the pieces of paper and let each of you take one piece. Do not open it yet. When I say, "Go," unfold the paper and read the name. Then, with your right hand, point to the person whose name is on your piece of paper.***

Distribute the papers and say, "Go!" Watch the students to make sure they follow instructions.

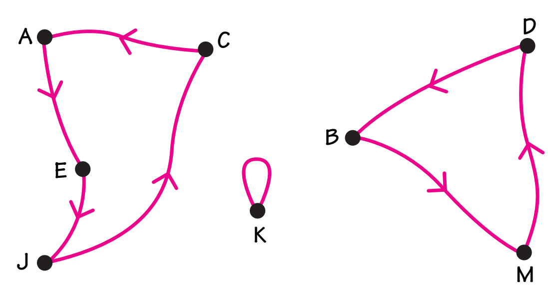
T: ***Let's draw red arrows to show the relation "I give you a gift."***

Invite students not participating in the game to show where to draw arrows on the board.

S: ***Charles has Angela's name, so Charles gives a gift to Angela. We can draw an arrow from*** **C** ***to*** **A.**

S: ***Kim got her own name—there should be a loop at*** **K.**

Complete the arrow picture, such as illustrated here.



Note: Each dot in the picture must have exactly one arrow starting at it and exactly one arrow ending at it.

T: ***Each dot has exactly one arrow starting at it or has a loop. Why did that happen?***

S: ***Each person in the game got one other person's name.***

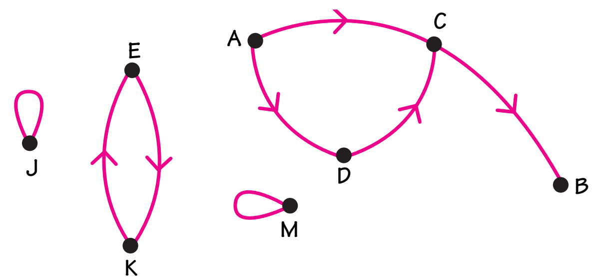
T: ***Each dot has exactly one arrow ending at it or has a loop. Why?***

S: ***The name of each person in the game is on exactly one piece of paper.***

Instruct students to draw eight dots on a clean piece of unlined paper. Then tell them to label the eight dots with the same letters that were used on the board.

T: ***Imagine that the students have to exchange names a second time. Draw an arrow picture for this imaginary exchange. Make your picture different than the one on the board. If we did another exchange, the result would most likely be different.***

Let students work independently on their arrow pictures for a few minutes. As you observe students' work, give help to those having difficulty getting started, and correct obvious errors. After many students have completed their pictures, call the class's attention to an arrow picture you have drawn on the board. Use the initials of your students.



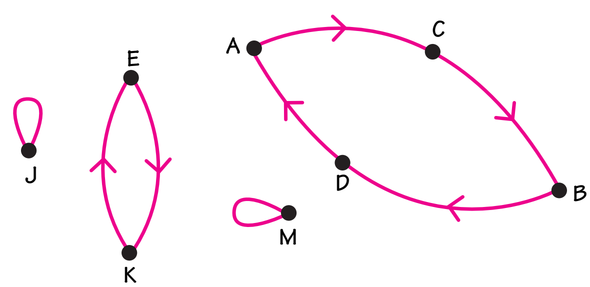
T: ***Could this arrow picture describe what could happen in the name exchange?*** (No) ***What's wrong with it?***

There are several things wrong with the picture that students should mention.

- Two arrows end at **C**, which would mean that two people received Charles's name.
- Two arrows start at **A**, which would mean that Angela received two names.
- No arrow ends at **A**, which would mean that no one received Angela's name.
- No arrow starts at **B**, which would mean that Beth received no name.

T: ***Can you help me correct this arrow picture?***

Accept suggestions until the arrow picture is correct. There are many ways to correct the picture, for example, like this:



Suggest that students study their arrow pictures to see if they made any similar errors.

T: ***Look at our arrow picture on the board. Into many groups have the eight people been divided?***

S: ***Four.***

T: ***On your papers, into how many groups have you divided the eight people? Does anyone have an arrow picture showing just one group?***

S: ***Yes; they are all in one large circle.***

T: ***What is the greatest number of groups that you could have?***

S: ***Eight. There could be eight loops.***

T: ***Do you think that any two of you drew exactly the same arrow picture?***

Let students briefly discuss the possibility.

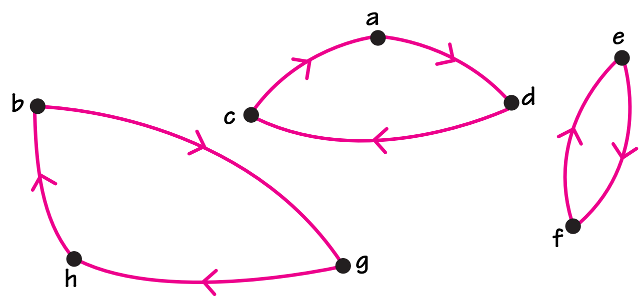
T: ***Can you estimate the number of different arrow pictures we could draw for eight people exchanging names?***

Record students' estimates and save them for future reference.

T: ***We could try to draw all of the pictures, but that might take a long time, and it would be difficult to check for duplicates. Instead, I will show you a special code that will make the problem easier.***

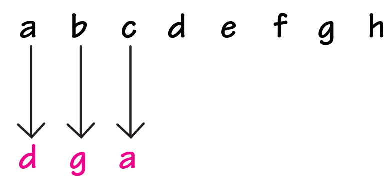
Draw this picture on the board.

T: ***Suppose eight people whose first names start with the letters* a *through* h *exchange names. This arrow picture shows one possible outcome.***



Explain that to code this situation, you first list the letters from **a** to **h** on the board.

T: ***An arrow goes from* a *to* d*, from* b *to* g*, and from* c *to* a.**

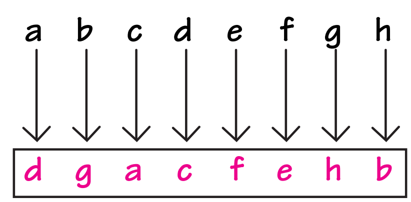


Write the corresponding letters in red.

T: ***What letter would go under* d*? Why?***

S: **c.** ***There's an arrow from* d *to* c.**

Let students complete the correspondence to obtain the code word (in red) for this picture. Highlight the code word for emphasis.



T: ***This is the code word for this arrow picture.***

Erase the arrows from the picture on the board, and then write a new code word.

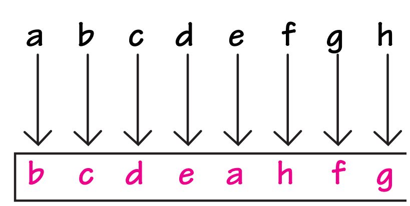
T: ***This is a code word for another arrow picture. Let's draw the arrow picture. How should we start?***

$\boxed{\text{b c d e a h f g}}$

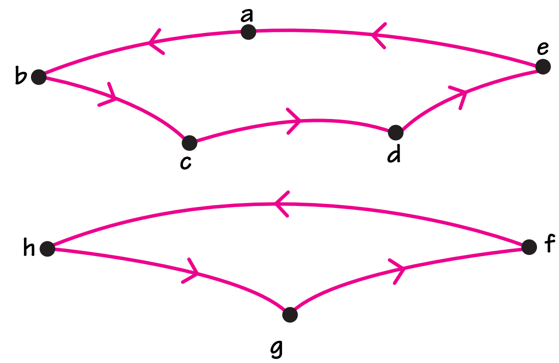
S: ***Draw an arrow from* a *to* b.**

S: ***Draw an arrow from* b *to* c.**

If necessary, show the correspondence.



Let students complete the corresponding arrow picture.



Distribute copies of Worksheets L9(a) and (b). Instruct students to write the code word for the picture on the L9(a) worksheet and to draw the arrow picture for the given code word on L9(b).

When many students complete the worksheets, check them collectively or let students check each other's. (An answer key follows the lesson description.)

Write this code word on the board.

f	g	b	a	b	c	h

T: ***An arrow picture describing an exchange has exactly one arrow starting at each dot and exactly one arrow ending at each dot. Could this be the code word for such an arrow picture?*** (No) ***Why not?***

S: ***The letter* b *appears twice.***

S: ***The letters* d *and* e *are not used in the code word.***

S: ***The code word has seven instead of eight letters.***

Invite students to change the code word so that it has eight letters and each of the letters **a** through **h** appears exactly once. For example:

f	g	b	a	e	c	d	h

T: ***For each arrow picture there is one code word, and for each code word there is one arrow picture. How many eight-letter code words, using each letter exactly once, do you think there are?***

Let students have an opportunity to make estimates or to change estimates made earlier. Some students may want to explain how to count the number of eight-letter code words.

T: ***The problem of finding how many code words there are for eight letters is not an easy one. Let us first look at a similar problem with only two letters. On your paper, write all of the code words with two letters,* a *and* b*, using each letter once.***

In a few minutes, invite a student to give the code words $\boxed{\text{a, b}}$ and $\boxed{\text{b, a}}$.

Begin a table on the board.

Number of Letters	Number of Code Words
2	2

T: ***Now write all of the code words with exactly three letters,* a*,* b*, and* c*.***

After a few minutes, solve the problem collectively, writing the code words on the board as they are mentioned.

S: ***There are six code words with three letters.***

T: ***Which of those six code words start with* a*?***

S: $\boxed{\text{abc}}$ ***and*** $\boxed{\text{acb}}$.

$\boxed{\text{abc}}$
$\boxed{\text{acb}}$

T: ***Which three-letter code words start with* b*?***

S: bac ***and*** bca.

T: ***Which code words start with* c*?***

S: cab ***and*** cba.

T: ***Are there any three-letter code words that we missed?*** (No)

S: ***There are two code words that start with each of the three letters. So there are six code words; 3* x *2 = 6.***

Three-Letter Code Words

abc	bac	cab
acb	bca	cba

Add this information to the table.

Number of Letters	Number of Code Words
2	2
3	6 = 3 × 2

T: ***Now try to find how many four-letter code words there are. You don't have to write all of them, but try to find a way of counting them without missing any.***

Encourage students to try to find a method for counting the four-letter code words. If some students finish quickly, ask them to go on finding the number of code words for five letters, six letters, seven letters, and eight letters.

After a few minutes, solve the four-letter problem collectively.

T: ***How many four-letter code words did you find?*** (24)
Let's check. How many four-letter code words start with* a*?

S: ***Six.***

T: ***Let's list them.***

abcd
abdc
acbd
acdb
adbc
adcb

Let students give the code words as you list them at the board.

T: ***How many start with* b*?***

S: ***Six.***

Continue in this way for the code words that start with **c** and **d**, listing them as they are given.

You might wish to put only one of each kind in the list.

Four-Letter Code Words

abcd	bacd	cabd	dabc
abdc	badc	cadb	d - - -
acbd	bcad	cbad	d - - -
acdb	bcda	cbda	d - - -
adbc	bdac	cdab	d - - -
adcb	bdca	cdba	d - - -
6	6	6	6

T: ***How many four-letter code words are there?***

S: ***24. There are four groups of six code words. 4* x *6 = 24.***

Add this information to the table.

Number of Letters	Number of Code Words
2	2
3	$6 = 3 \times 2$
4	$24 = 4 \times 3 \times 2$

T: ***Now try to find the number of code words with five letters.***

After a couple minutes, solve the problem collectively.

T: ***How many five-letter code words are there?*** (120) ***Let's check this answer. How many five-letter code words begin with* a*?***

S: ***24.***

T: ***Why?***

S: ***After* a *is fixed as the first letter, there are four more letters to consider. But we know from the previous problem that there are 24 code words with four letters.***

Check that everyone understands this explanation, and write the information on the board. (See the next illustration.)

T: ***How many five-letter code words begin with* b*?***

S: ***24 for the same reason. Also, there are 24 code words that begin with* c*, and* d*, and* e.

Five-Letter Code Words

a – – – –	b – – – –	c – – – –	d – – – –	e – – – –
24	24	24	24	24

T: ***How many five-letter code words altogether?***

S: ***120; 5* x *24 = 120.***

Add this information to the table.

Number of Letters	Number of Code Words
2	2
3	$6 = 3 \times 2$
4	$24 = 4 \times 3 \times 2$
5	$120 = 5 \times 4 \times 3 \times 2$

T: ***Let's calculate the number of code words for six letters, seven letters, and eight letters. We could use the same method, but do you notice any patterns in the table?***

S: ***Look at the products. For four letters, it's 4* x *3* x *2; for five letters, it's 5* x *4* x *3* x *2; for six letters, it should be 6 times the product for five letters (i.e., 6* x *120 or 6* x *5* x *4* x *3* x *2).***

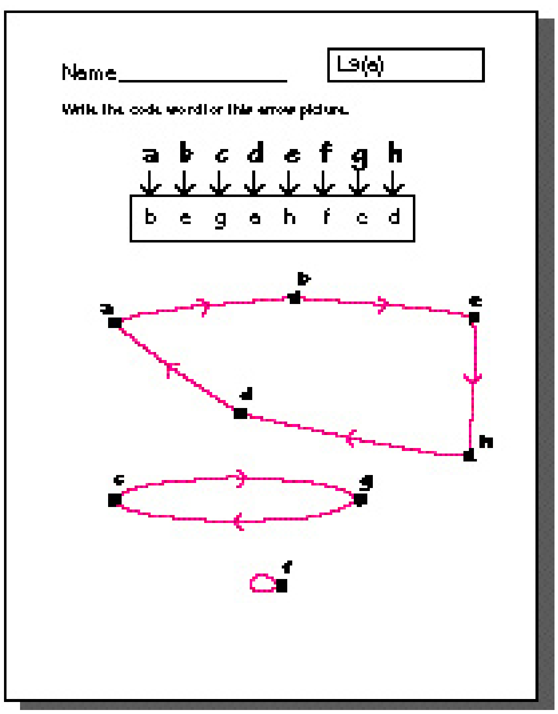
Accept any explanation along these lines. Once a pattern is established, ask students to complete the table. A completed table is shown below.

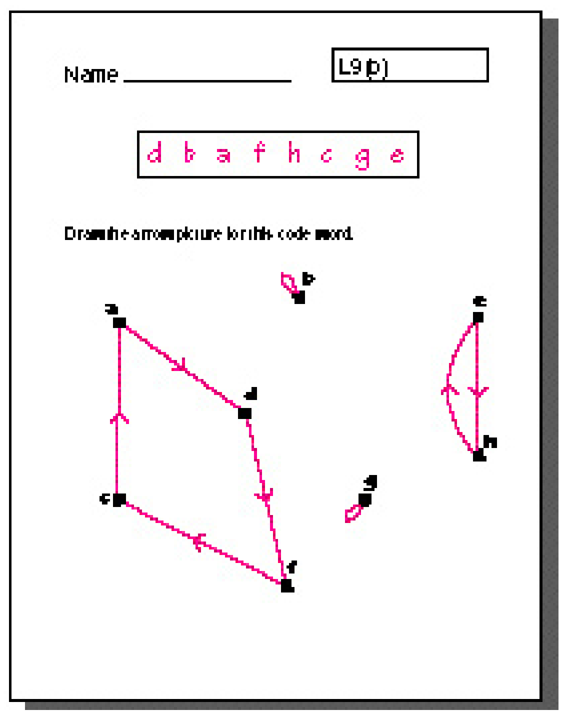
Number of Letters	Number of Code Words
2	2
3	$6 = 3 \times 2$
4	$24 = 4 \times 3 \times 2$
5	$120 = 5 \times 4 \times 3 \times 2$
6	$720 = 6 \times 5 \times 4 \times 3 \times 2$
7	$5\,042 = 7 \times 6 \times 5 \times 4 \times 3 \times 2$
8	$40\,320 = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2$

T: ***There are 40 320 code words with eight letters. How many different arrow pictures are there?***

S: ***40 320.***

Be sure to compare 40 320 to students' estimates that were made earlier. Also comment that with 40 320 different possible arrow pictures, it's unlikely that two students would have drawn the same arrow picture at the beginning of the lesson unless they did so on purpose.





Capsule Lesson Summary

Make a list of positive numbers with exactly four positive divisors. Determine which of the listed numbers are products of two distinct primes and which are cubes of primes. Observe that all numbers with exactly three positive divisors are squares of prime numbers.

Materials

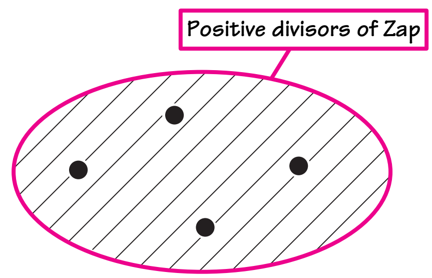
Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

Draw this string picture on the board.



T: ***Zap is a secret whole number. What does this string picture tell us about Zap?***

S: ***Zap has exactly four positive divisors.***

T: ***What is a number that Zap could be?***

S: ***6.***

T: ***What are the positive divisors of 6?*** (1, 2, 3, and 6)

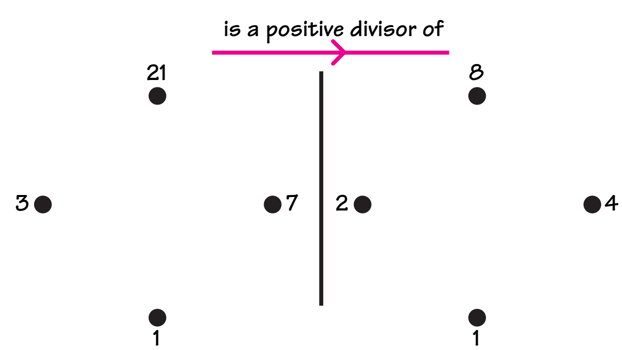
Zap could be 6 because 6 has exactly four positive divisors. On your paper, list some other numbers that Zap could be.

Let students work independently or with a partner to find several numbers that Zap could be. Then collect their findings in a list on the board. Students should verify suggestions by naming the divisors of those numbers. Suppose students have suggested these numbers:

6, 8, 10, 14, 15, 21, 22, 27, 22, 33, 35

Note: The following discussion assumes that both 27 and 8 are listed. If either of these numbers has been omitted, solicit numbers with exactly four positive divisors between 20 and 30, or between 0 and 10.

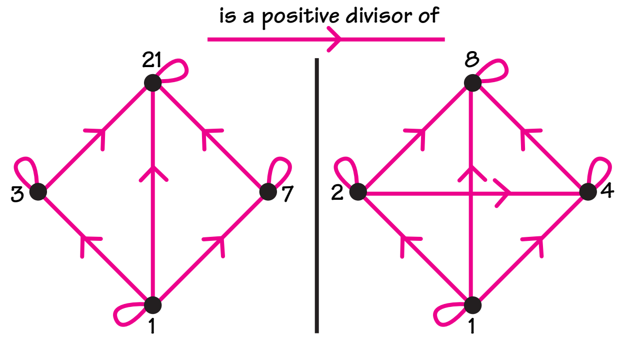
Draw these pictures on the board.



T: ***The picture on the left has the positive divisors of 21, and the picture on the right has the positive divisors of 8. Copy each of these pictures on your paper and draw as many red arrows as possible in each picture.***

L10

Allow a few minutes for students to work independently, and then invite two students to complete the pictures at the board.



T: ***Compare the two pictures.***
How are they alike?
How are they different?

S: ***In the arrow picture for 8, 2 is a positive divisor or 4. In the arrow picture for 21, 3 is not a positive divisor of 7.***

Accept any reasonable ideas, pointing out that the pictures have mostly the same arrows, their difference being that the one on the right has one additional arrow.

T: ***Now choose any two other numbers from the list and draw arrow pictures for them. Try to find another number with an arrow picture like the one on the left, and another with an arrow picture like the one on the right.***

Allow several minutes for independent work. If appropriate, encourage some students to draw several more arrow pictures.

T: ***What do you notice about the numbers with arrow pictures like 21.***
What about those with pictures like 8?

Accept any reasonable comments. If necessary, encourage students to express the numbers in each color group as products of positive prime numbers. Record these products on the board.

$6 = 2 \times 3$	$21 = 3 \times 7$	$8 = 2 \times 2 \times 2$
$0 = 2 \times 5$	$22 = 2 \times 11$	$27 = 3 \times 3 \times 3$
$4 = 2 \times 7$	$33 = 3 \times 11$	
$5 = 3 \times 5$	$35 = 5 \times 7$	

S: ***Each number with a picture like 21 can be written as the product of two different prime numbers.***

S: ***Each number with a picture like 8 is a prime number times itself and then times itself again.***

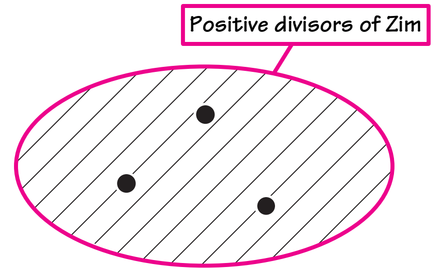
In other words, each of these numbers is the cube, or third power, of a prime number.

T (pointing to each of the two lists in turn)**:** ***Can you find at least one more number like 8 and one more number like 21?***

Spend a few minutes soliciting several more numbers in each of the two categories.

Exercise 2

This exercise develops in the same spirit as Exercise 1.



Draw this picture on the board.

T: ***Zim is a secret whole number. What does this string picture tell us about Zim?***

S: ***Zim has exactly three positive divisors.***

T: ***What is a number that Zim could be?***

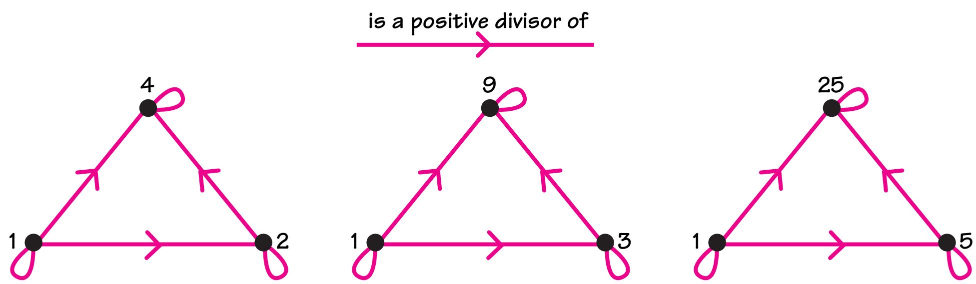
S: ***4, because the positive divisors of 4 are 1, 2, and 4.***

T: ***On your paper, list some other numbers that Zim could be.***

Encourage students to find several numbers that Zim could be. List the numbers on the board as they are found. For example:

4 9 25 49 121

Direct students to draw arrow pictures (with red arrows for "is a positive divisor of") with the three divisors of these numbers. You may give students different numbers to use. For example:



Invite several students, using different numbers, to draw their arrow pictures on the board. Observe the following:

S: ***All of the arrow pictures are the same.***

T: ***Let's try to find a pattern for these numbers.***

Encourage students to express each number as a product of primes.

$4 = 2 \times 2$ $25 = 5 \times 5$ $121 = 11 \times 11$
$9 = 3 \times 3$ $49 = 7 \times 7$

T: ***What do you notice?***

S: ***Each number is a square number.***

T: ***Do you think that Zim could be any square number?***

S: ***No; 36 is a square number, but 36 has more than three divisors.***

S: ***Zim is the square of a prime number.***

T: ***What other numbers could Zim be?***

S: ***169; 13*** x ***13 = 169.***

Repeat this exercise, looking for numbers with exactly two positive divisors. The class should quickly notice that these are the prime numbers—in fact, we define prime numbers as those numbers with exactly two positive divisors.

T: ***Is there a number that has exactly one positive divisor?*** (Yes, 1)
What are some numbers with exactly five positive divisors? (16, 81, 625, …)

You may wish to leave this last question as a challenge for students: to find such numbers and describe a pattern for them. Some students may discover that such a number is a prime number raised to the fourth power.

Extension Activity

Assign several pairs of multiplication problems with the same product. Ask students to try to find a pattern or to explain why, in each case, the two products are the same.

48 x 6	36 x 8		24 x 9	27 x 8		35 x 9	45 x 7		156 x 12	144 x 13

L11 BINARY CODES WITH TREES

Capsule Lesson Summary

Use a decision tree to decipher a binary code. Find the frequency of letters in a message, and create a decision tree and binary code for the message.

Materials

Teacher
- Blackline L11
- Colored chalk

Student
- Coded message
- Worksheets L11* and **

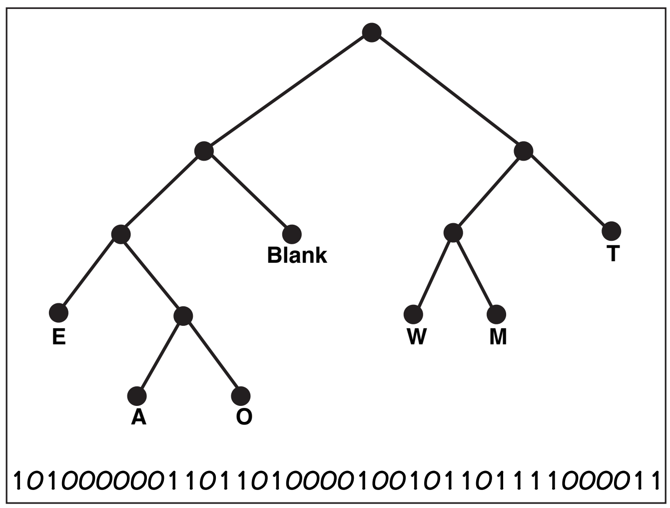
Advance Preparation: Use Blackline L11 to make copies of a coded message for use in Exercise 1.

Description of Lesson

With the class, recall Boris the spy who specializes in codes. Sometimes Boris receives messages that he must decode, sometimes he sends messages, and sometimes he intercepts messages.

Exercise 1

Display a copy of the coded message on Blackline L11 or put this information on the board. You may prefer to give copies of the coded message to students.

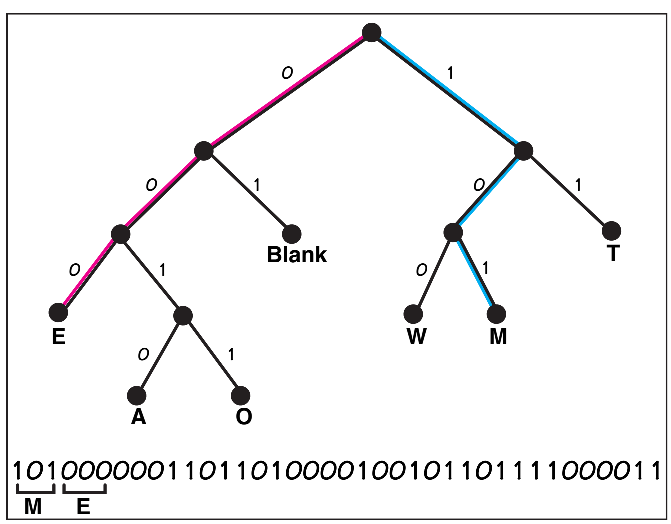


T: ***Boris receives this message from his assistant, Mr. Huffman. The message is a string of 0s and 1s like binary code; the tree tells Boris how to read the message. Mr. Huffman reminds Boris that in a decision tree 0 is left and 1 is right.***

Label the branches of the tree, and then demonstrate how to use the tree to read the message.

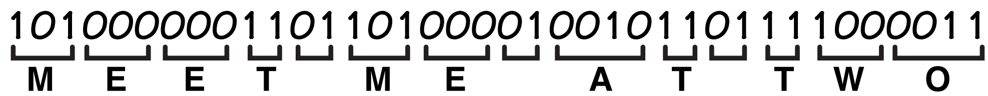
Start at the beginning of the message reading 1s and 0s. As you read, follow 0 or 1 branches in the tree. If you come to a letter (end point in the tree), record that letter. Then start again at the top of the tree.

Find the first couple letters of the message collectively, and then invite students to continue on their own. Call on students in pairs, one to read the next digits of the message and one to follow the appropriate branches in the tree, saying "stop" when he or she comes to a letter.



L11

The class should decode the message on Blackline L11 as follows:



Let the class discuss the message and the method of decoding. They may be concerned that it is not very secret. Perhaps some will suggest ways to maintain secrecy such as sending the tree and the binary code (string of 0s and 1s) separately.

Exercise 2

T: ***Boris wants to send a message back to Mr. Huffman. His message is this:***

Write the message on the board. **Change time to eight**

T: ***Boris uses Mr. Huffman's directions to make a tree.***
First, he lists all the letters used in the message, and then he determines the number of times each letter is used.

Call on students to help make a list of the letters in the message and to find their frequency of use.

Here is what students should find for this message:

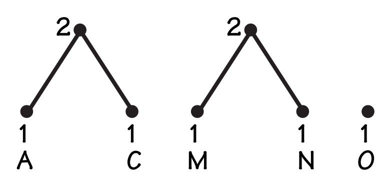
Letter	A	C	E	G	H	I	M	N	O	T	Blank
Frequency	1	1	3	2	2	2	1	1	1	3	3

T: ***Mr. Huffman tells Boris that the tree should have shorter paths for letters that occur more often and longer paths for letters that occur less often. Why do you suppose this is useful?***

S: ***Then the list of 0s and 1s won't be so long.***

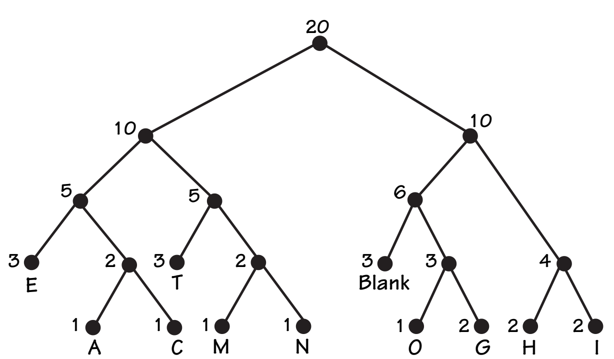
Begin a tree on the board as you explain the next step.

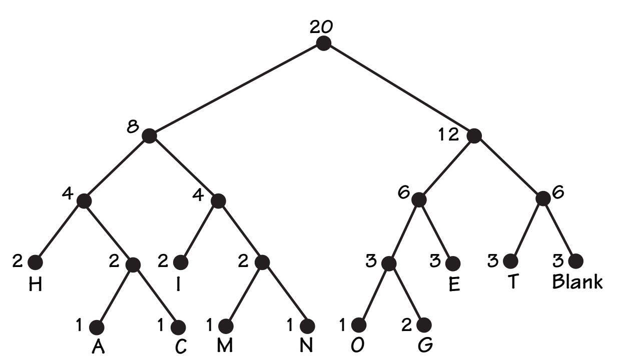
T: ***Second, Mr. Huffman tells Boris to start with the letters of frequency 1 and pair them. Build the tree from the bottom, making connections and adding frequencies.***



Next, continue to connect the two smallest numbers remaining (letter frequencies or nodes in the emerging tree).

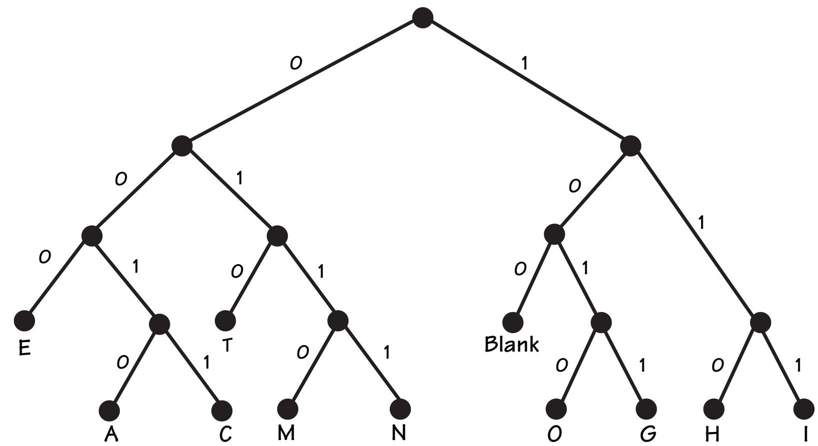
Note: The tree can be completed in many ways. The 2s can be combined in any order, and the same is true for the 3s. Two possible completed trees are shown below.



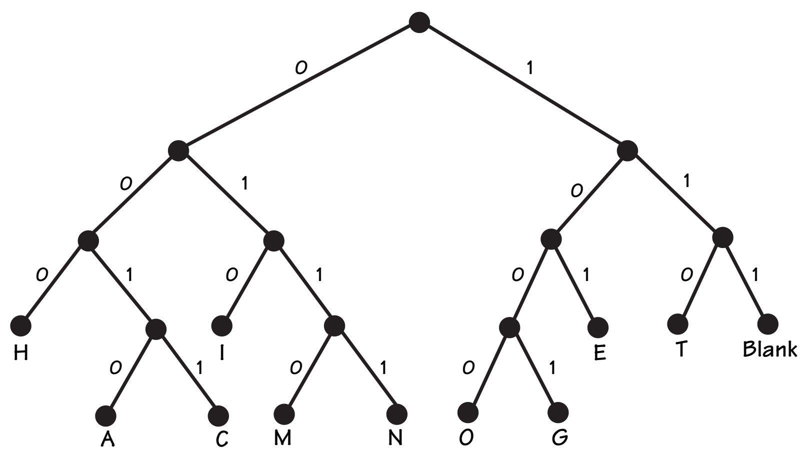


T: ***Finally, Boris writes the 0-1 message from the tree.***

Invite students to help you write the string of 0s and 1s from the tree you have developed. The resulting messages from the two preceding examples are given below.



0011110001001111011000100010111011000010001010101000001111011110010

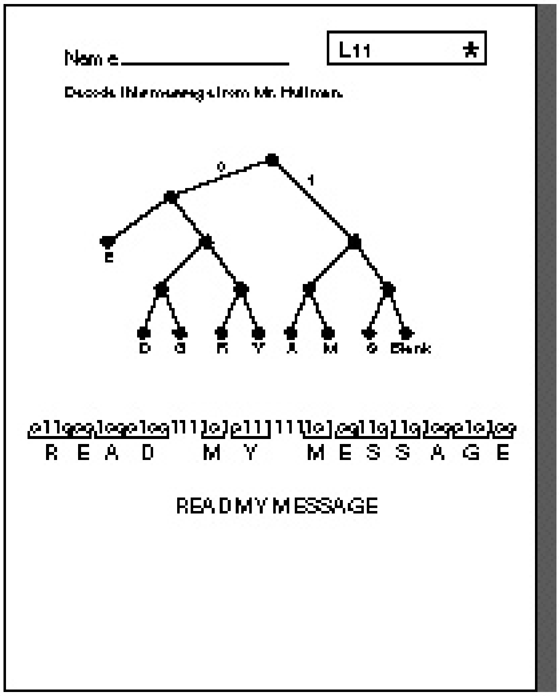


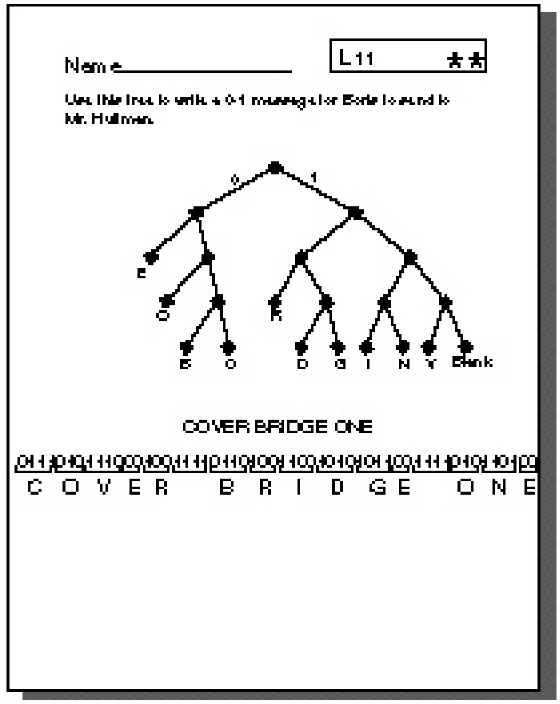
0011000001001111001101111110010011010111111010001111010101001000110

for individual work.

Writing/Home Activity

Suggest students create a three- or four-word message to encode and then take home to solve with a family member.


Name
L11 ★
Decode this message from Mr. Hullman.
0
1
E
D G R Y A M S Blank
R E A D M Y M E S S A G E
READ MY MESSAGE


Name
L11 ★★
Use this tree to write a 0-1 message for Boris to send to Mr. Hullman.
0
1
E
O
R
B C D G I N Y Blank
COVER BRIDGE ONE
C O V E R B R I D G E O N E

L12 STRING GAME ANALYSIS #4

Capsule Lesson Summary

In a string game situation, present a sequence of clues that give sufficient information for the class to determine the strings. During the analysis, pose questions concerning the location of specific numbers in the picture. Play *The String Game* with numbers.

Materials

Teacher
- Numerical String Game kit
- Colored chalk

Student
- String Game analysis sheet
- Worksheets N12*, **, ***, and ****

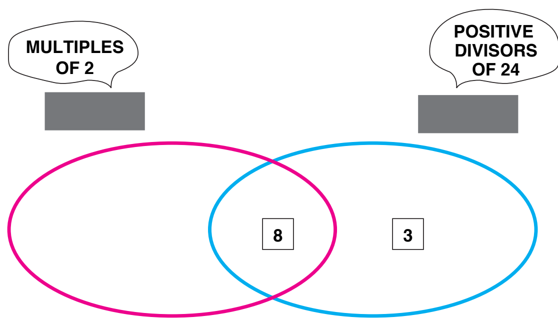
Description of Lesson

Exercise 1

Using two numerical string game posters, prepare your board for a string game as illustrated below. Bubbles show what is on the hidden tags. Distribute String Game analysis sheets.

Red

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN $\hat{10}$	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN $\hat{10}$	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27



Blue

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN $\hat{10}$	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN $\hat{10}$	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27

T: ***This is the starting situation for a string game. The posters list the possible labels for the strings.*** $\boxed{8}$ ***and*** $\boxed{3}$ ***are located in the picture as clues. What information do these clues give us?***

Let students suggest labels to cross out on the lists. Each time ask for a clear explanation as to why a string cannot have a label before you cross it off. Encourage students to cross out the appropriate labels on their analysis sheets as they are crossed out at the board. The class should eliminate all except two labels on the Red list and all except three labels on the Blue list.

Red

MULTIPLES OF 2	~~MULTIPLES OF 3~~	MULTIPLES OF 4	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN $\hat{10}$~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	~~GREATER THAN $\hat{10}$~~	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

Blue

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	LESS THAN 50	~~LESS THAN $\hat{10}$~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	GREATER THAN $\hat{10}$	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 27~~

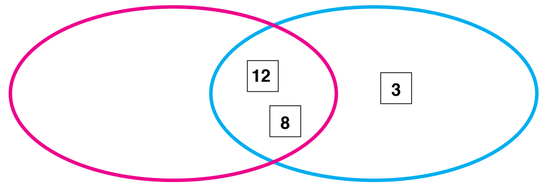
T: ***Now I will make some statements. You decide whether my statement is true or false, or that you can't tell.***

L12

As you make a statement, put the number referred to in the indicated region. If the statement is true, leave the number in the picture. If the statement is false or indeterminate, remove it from the picture. Encourage class discussion, especially when there is some disagreement.

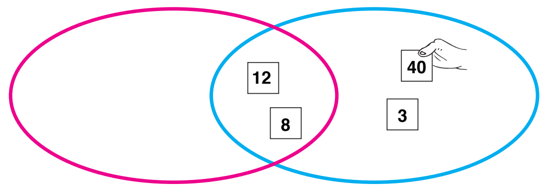
T: ***12 is inside the blue string and the red string.***

S: ***True. 12 is a multiple of 2 and of 4; also 12 is a positive divisor of 24, is less than 50, and is greater than $\widehat{10}$.***



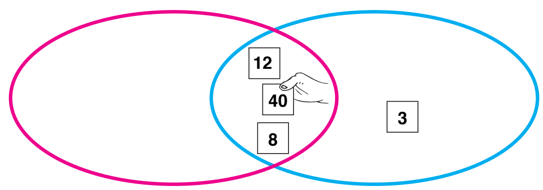
T: ***40 is inside the blue string but not inside the red string.***

S: ***False. 40 is a multiple of 2 and of 4, so 40 must be inside the red string.***



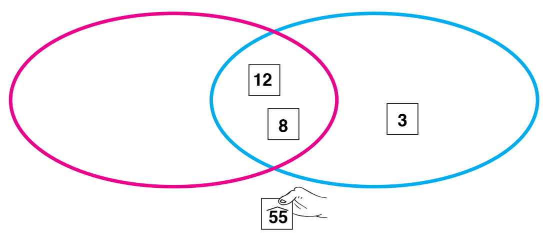
T: ***40 is inside both the red string and the blue string.***

S: ***Can't tell. 40 belongs inside both strings if the blue string is for*** **LESS THAN 50** ***or*** **GREATER THAN $\widehat{10}$*****, but 40 belongs outside the blue string if the blue string is for*** **POSITIVE DIVISORS OF 24.**



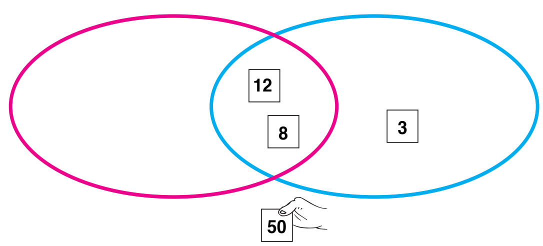
T: ***$\widehat{55}$ is outside of both strings.***

S: ***Can't tell. $\widehat{55}$ is outside the red string, but $\widehat{55}$ would be in the blue string if the blue string were for*** **LESS THAN 50.** ***Otherwise it would be outside both strings.***



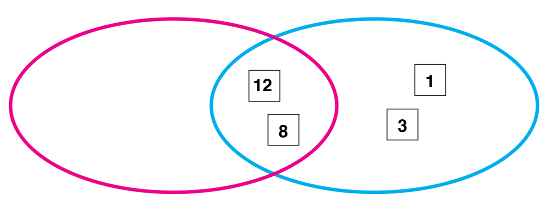
T: ***50 is outside of both strings.***

S: ***Can't tell. 50 could be in any one of the four regions.***



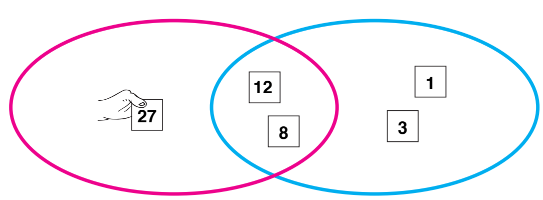
T: ***1 is inside the blue string but not inside the red string.***

S: ***True. 1 is not a multiple of 2 or of 4; 1 is a positive divisor of 24, is less than 50, and is greater than $\widehat{10}$.***



T: ***27 is inside the red string but not inside the blue string.***

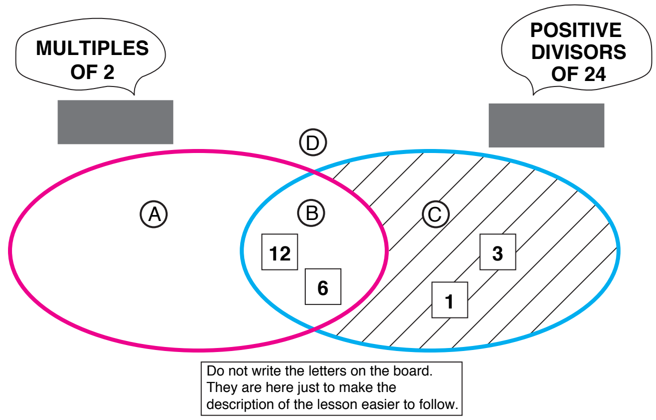
S: ***False. 27 is not a multiple of 2 or of 4.***



T: *We know that the red string could be for* **MULTIPLES OF 2** *or for* **MULTIPLES OF 4.** *We know the blue string could be for* **POSITIVE DIVISORS OF 24** *or for* **GREATER THAN $\widehat{10}$** *or for* **LESS THAN 50.** *Here is another clue: 1 and 3 are the only numbers inside the blue string and outside the red string.*

Indicate this clue by hatching region **C** as shown here.

T: *With the new clue, you can determine what the strings are for.*



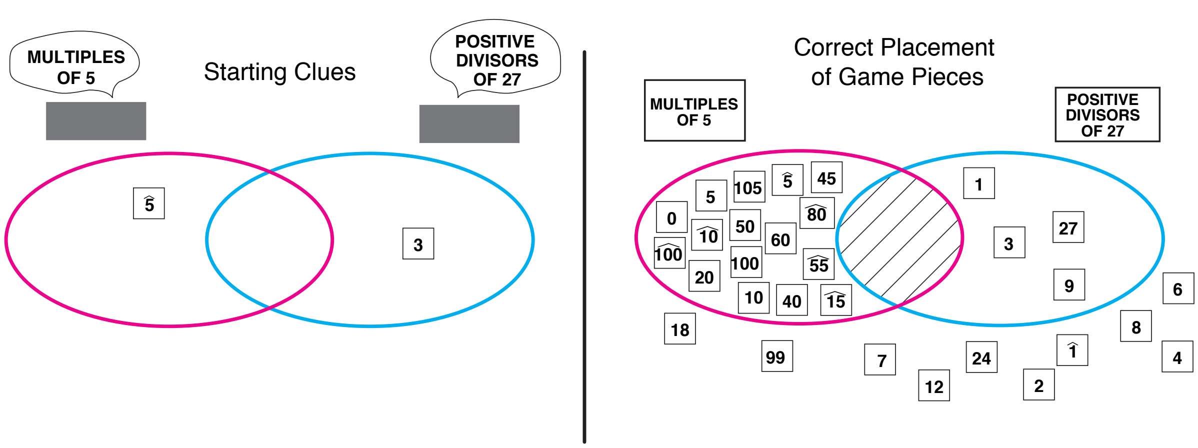
Give the class several minutes to think about this new clue. Then ask what the strings could be for. Check each suggestion. After a while, the class should resolve that the red string is for **MULTIPLES OF 2** and the blue string is for **POSITIVE DIVISORS OF 24** The table below shows why each of the other possibilities can be eliminated.

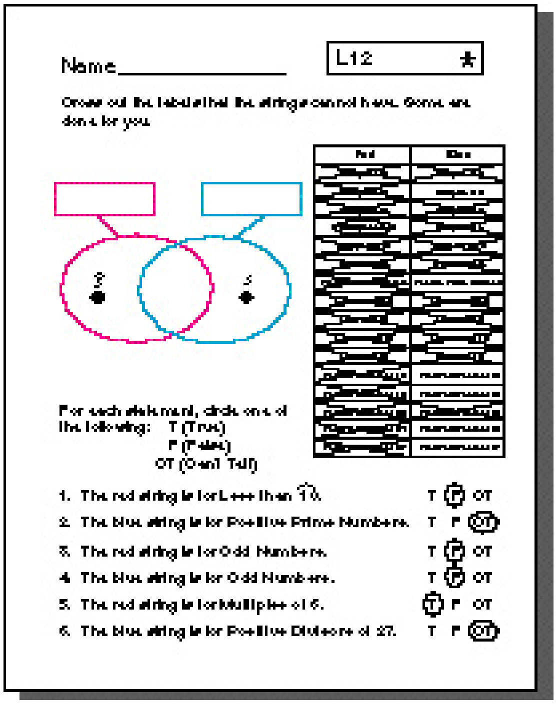
RED	BLUE	
MULTIPLES OF 2	LESS THAN 50	27 (for example) would be in region C.
MULTIPLES OF 2	GREATER THAN $\widehat{10}$	27 (for example) would be in region C.
MULTIPLES OF 4	POSITIVE DIVISORS OF 24	6 would be in region C.
MULTIPLES OF 4	LESS THAN 50	27 (for example) would be in region C.
MULTIPLES OF 4	GREATER THAN $\widehat{10}$	27 (for example) would be in region C.

Distribute copies of Worksheets L12* and **. Let students work independently for about 15 minutes. Students who enjoy these string game problems may wish to continue working on L12*** and **** in their free time.

Exercise 2

Play *The String Game* with numbers in the usual way. The illustration below shows a possible game with starting clues.



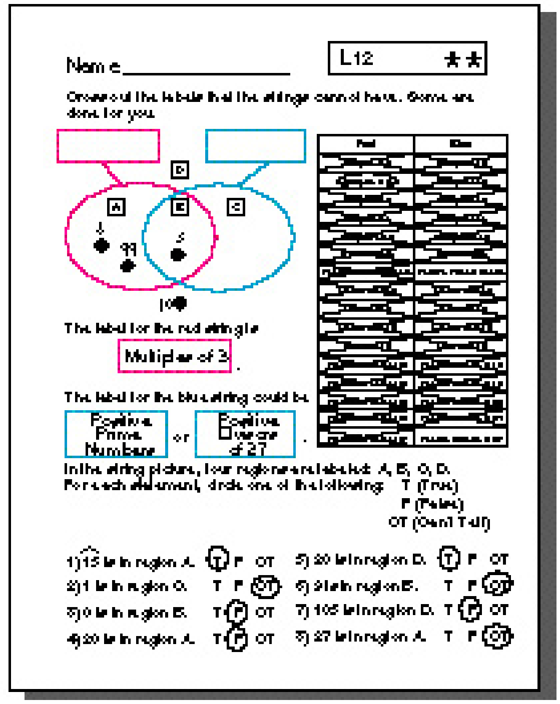


Name____________ L12 ★

Cross out the labels that the strings cannot have. Some are done for you.

For each statement, circle one of the following: T (True) F (False) OT (Can't Tell)

1. The red string is for Less than 10. T F OT
2. The blue string is for Positive Prime Numbers. T F OT
3. The red string is for Odd Numbers. T F OT
4. The blue string is for Odd Numbers. T F OT
5. The red string is for Multiples of 6. T F OT
6. The blue string is for Positive Divisors of 27. T F OT



Name____________ L12 ★★

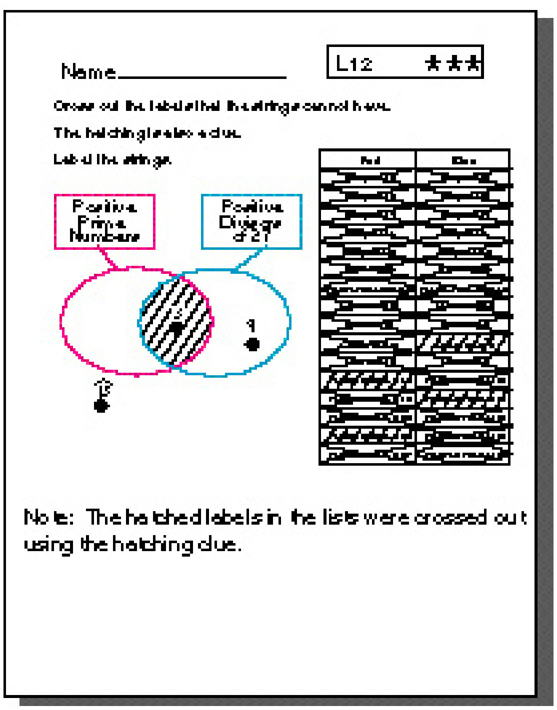
Cross out the labels that the strings cannot have. Some are done for you.

The label for the red string is Multiples of 3.

The label for the blue string could be Positive Prime Numbers or Positive Divisors of 27.

In the string picture, four regions are labeled: A, B, C, D. For each statement, circle one of the following: T (True) F (False) OT (Can't Tell)

1) 15 is in region A. T F OT
2) 1 is in region C. T F OT
3) 0 is in region B. T F OT
4) 20 is in region A. T F OT
5) 20 is in region D. T F OT
6) 2 is in region B. T F OT
7) 105 is in region D. T F OT
8) 27 is in region A. T F OT



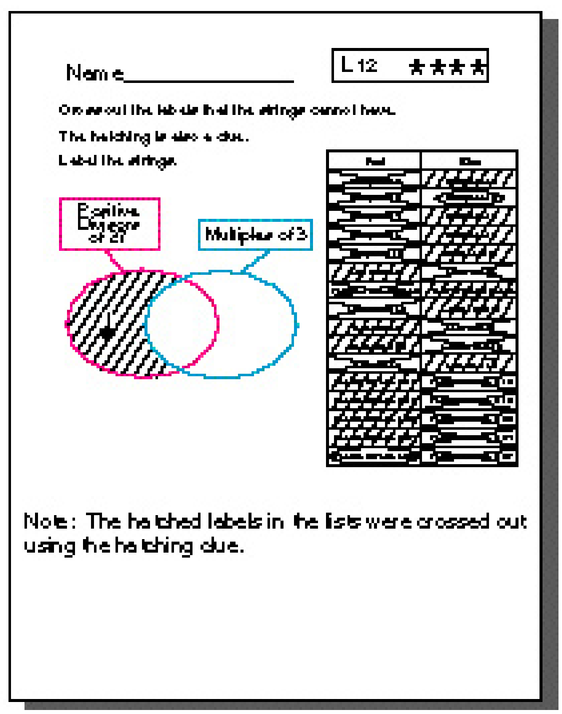
Name____________ L12 ★★★

Cross out the labels that the strings cannot have.

The hatching is also a clue.

Label the strings.

Note: The hatched labels in the lists were crossed out using the hatching clue.



Name____________ L12 ★★★★

Cross out the labels that the strings cannot have.

The hatching is also a clue.

Label the strings.

Note: The hatched labels in the lists were crossed out using the hatching clue.