The World of Numbers

WORLD OF NUMBERS TABLE OF CONTENTS

By now, veteran *CSMP* students have had a rich variety of experiences in the World of Numbers. They have met and become familiar with various kinds of numbers, and with operations and relations on them. They have encountered positive and negative integers, decimal numbers, fractions, numerical functions (such as 5x, +3, ÷10, –5, ½x), order relations (such as < and >), and the notions of multiples and divisors of a given number. They have been introduced to paper-and-pencil algorithms for addition and subtraction of whole numbers and decimal numbers, and for multiplication of whole numbers and of a fraction times a whole number. Students have had extensive experience using systematic methods for division of whole numbers and involving addition and subtraction of fractions in preparation for algorithms. Topics from combinatorics and number theory have provided many interesting problems.

In *CSMP Mathematics for the Intermediate Grades*, *Part III*, these earlier numerical experiences will be revisited, extended, and deepened through familiar games and activities, as well as in fascinating new situations. As always, *CSMP* stresses the unity and continuity of growth of mathematical ideas and concepts. The program's spiral approach does not require mastery of each lesson, but rather allows students to encounter the elements of each content strand in different situations throughout the year. It is important to recognize this approach consciously. If you strive for mastery of each single lesson, you will find yourself involved in a great deal of redundancy as the year progresses.

Further, *CSMP* presents the content in a situational framework. That is, a "pedagogy of situations" engages students in rich problem-solving activities as they construct mathematical ideas. These situations offer opportunities both to develop necessary numerical skills and to gain deeper understanding of mathematical concepts in the world of numbers. At the same time, the situations presented encourage students to develop patterns of logical thinking and strategies for attacking problems.

Perhaps the most important embodiments of the *CSMP* approach are the nonverbal languages and tools used throughout the program. These are vehicles that allow students to investigate the contexts in which the content is presented and to explore new mathematical ideas. It is hard to overstate the value of developing languages and tools that are not confined to one area of mathematical content or to one level of the development of content; that aid in attacking problems as well as in representing situations. Armed with the universally applicable languages of the *CSMP* curriculum, students grow more and more familiar with the syntax of these languages and are free to explore new content as extensions rather than think of each new mathematical idea as tied to a certain new language. This is not to say that *CSMP* students do not learn the usual descriptive language of mathematics; naturally, they do. However, in the *CSMP* approach the usual descriptive language is not a requisite for learning new concepts, but only a means for succinctly describing those ideas as they are being explored.

Standard Algorithms of Arithmetic

CSMP seeks to develop basic numerical skills as well as an understanding of the underlying mathematical ideas. We are fully in agreement with the thesis that, along with the growth of understanding of the world of numbers, there must be a concomitant growth of familiarity and facility with numbers and operations on them. But facility should not be confused with understanding; they are partners in the growth of mathematical maturity. A balanced growth of each must be maintained, neither being sacrificed for the other.

Students must eventually learn mechanical algorithms for the basic operations (addition, subtraction, multiplication, division). However, the development of these methods should occur only after students have had many experiences with prerequisite concepts. Premature presentation of these algorithms may actually inhibit a student's desire and ability to develop alternative algorithms, to do mental arithmetic, or to estimate.

CSMP believes that students should be able to solve a problem such as 672 ÷ 4 using models, pictures, or mental arithmetic before being introduced to a division algorithm. Even after students have mastered an algorithm, they should be aware that alternative methods are often more appropriate. For example, consider the problem of calculating 698 x 9. Rather than using a standard multiplication algorithm, it may be easier and more efficient to note that 700 x 9 = 6 300, so that 698 x 9 = 6 300 – 18 = 6 282. Indeed, built into this way of approaching the problem is an excellent estimate (6 300) of the product. To insist on a mechanistic response to such a problem would be to encourage inefficiency and might also inhibit the development of the flexibility necessary for problem solving. On the other hand, a rich array of situations in which students interact with numbers provides them with opportunities to gain the necessary facility with standard algorithmic procedures while retaining the openness required to respond creatively to new situations in the world of numbers.

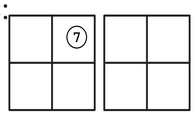
Content Overview

Multiplication

By this time your *CSMP* students are quite familiar with the concept of multiplication and with a paper-and-pencil algorithm for multiplying whole numbers. The lessons in this strand illustrate how the Minicomputer, arrows, and calculators all provide for review and extension of one concept.

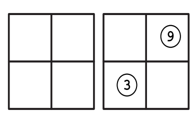
Putting numbers on the Minicomputer with weighted checkers presents challenging problems while reviewing basic multiplication facts. For example:

Put 280 on the Minicomputer with one weighted checker.



$= 7 \times 40 = 280$

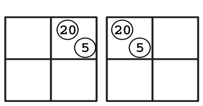
Put 42 on the Minicomputer with two weighted checkers.



$= (9 \times 4) + (3 \times 2) = 42$

The Minicomputer also provides for review of a standard paper-and-pencil algorithm for multiplication.

$25 \times 48 =$

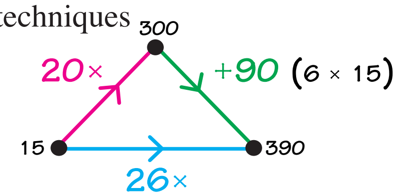


$(5 \times 48) + (20 \times 48)$

$$\begin{array}{rl} 48 & \\ \times\ 25 & \\ \hline 240 & (5 \times 48) \\ +960 & (20 \times 48) \\ \hline 1200 & \end{array}$$

Arrow pictures encourage the development of mental arithmetic techniques while providing further practice with basic facts. For example:

26 x 15 is calculated by 20 x 15 = 300 and 300 + 90 = 390.



WORLD OF NUMBERS INTRODUCTION

Calculator relations and calculator sentences make the calculator a context for posing problems that challenge students to use number and operation sense. Relations defined using the constant feature of a calculator are especially helpful to further develop multiplication concepts. A calculator allows students to immediately check their thinking, and encourages them to make other attempts when a solution is incorrect. In fact, information from what happens on the calculator may suggest a solution.

The sections on Negative Integers, Decimal Numbers, and Fractions describe extension of multiplication to these sets of numbers. Other activities apply multiplication to new situations, for example, in the analysis of combinatorics problems and of number patterns.

Lessons: N1, 3, 4, 7, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 28, 29, 30, 33, 34, and 35

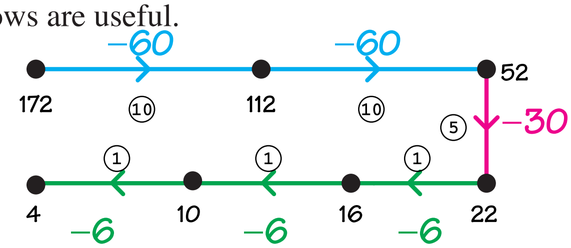
Division

CSMP students already have experience with division as a sharing process (sharing 108 books equally among three classes), as repeated subtraction (finding how many 12s are in 200), and as a multiplication inverse. The lessons in *IG-III* continue the development toward an efficient paper-and-pencil algorithm for division and extend students' experiences with division to new patterns and applications.

Arrow roads provide an ideal model for illustrating division as repeated subtraction. The problem 172 ÷ 6 can be interpreted as, "How many 6s are there in 172?" With arrows, the equivalent problem is, "How many –6 arrows are there in a road from 172 to a positive number less than 6?" But students quickly realize that not only –6 arrows are useful.

To divide 172 by 6, a student may use –60 arrows, –30 arrows, and –6 arrows to determine that there are 28 sixes in 172.

$$6\overline{)172}\quad 28\ R = 4$$



Allowing flexibility in choosing which arrows to use acknowledges that at any one time students differ dramatically in ability and confidence. Some students prefer using only –6, –60, and –600 arrows. A few students creatively use a variety of arrows, –30, –24, –120, –12, ... The choice of arrows reflects various stages of development in progressing toward a standard division algorithm.

In order to construct a broader understanding of division than the algorithm alone provides, students investigate and apply many patterns involving division. Patterns often suggest mental arithmetic techniques for solving division problems.

726 ÷ 3 = 242	5100 ÷ 15 = 340
72.6 ÷ 3 = 24.2	5115 ÷ 15 = 341
7.26 ÷ 3 = 2.42	5130 ÷ 15 = 342
	5160 ÷ 15 = 344
430 x 10 = 4 300	3.6 x 10 = 36

WORLD OF NUMBERS INTRODUCTION

Activities such as slicing apples into sections and dividing line segments into equal parts illustrate division of fractions and decimal numbers by whole numbers; for example, $1\frac{1}{3} \div 2 = \frac{2}{3}$, and $1.6 \div 4 = 0.4$. Applications such as these demonstrate when division is appropriate and provide insights into the concept of division.

Lessons: N4, 8, 9, 11, 13, 17, 19, 21, 22, 23, 24, 27, 29, 30, 32, 33, 34, and 35

Negative Integers

CSMP introduces negative integers in first grade through a story about Eli the Elephant and magic peanuts. The story leads to a model for adding integers, first by pictures, then also on the Minicomputer. By the end of fourth grade, *CSMP* students have encountered negative numbers in games, in reading outdoor temperatures, in arrow roads, and in calculator activities. These experiences extend the concept of order from whole numbers to negative numbers, and provide models for the addition of negative numbers.

The activities in this strand increase students' familiarity with negative numbers in many contexts. The goal is to portray negative integers not as a strange set of numbers, but as a natural and necessary extension of counting numbers. Therefore, few lessons focus on negative numbers but many lessons include them.

Negative numbers appear regularly on the Minicomputer in *Minicomputer Golf,* in *The String Game* with numbers, in detective stories, in *Guess My Rule* activities, in arrow roads, and in calculator activities. In these contexts, students solve problems such a $3 \times (8 - (7 + 6)) = \widehat{15}$ and $\frac{3}{5} \times \widehat{75} = \widehat{45}$. Through these experiences, student gradually learn the characteristics of negative numbers and accept them as "real" numbers.

CSMP employs a special notation for representing negative numbers. Traditional approaches to arithmetic often make no distinction on the printed page between the function "subtract 3" and the number "negative 3"; both are denoted by "–3." Only by context can a person discern the intended meaning of "–3." In *CSMP*, negative numbers are distinguished from subtraction in the following ways:

- The minus sign "–" is reserved for subtraction. Thus, for example, "–14" denotes the function "subtract 14."
- The $\frown$ symbol denotes a negative number. Thus, "$\widehat{14}$" denotes the number "negative 14." The $\frown$ symbol was introduced first in the story about Eli the Elephant.
- A raised minus sign may be used when recording a negative number, especially for results obtained from using a calculator. For example, $^{-}14$.

We recommend that you continue to use the $\frown$ notation for negative numbers and recognize alternative notations as students encounter them in other contexts (calculators, temperature, tests, and so on).

Lessons: N1, 4, 8, 10, 12, 13, 17, 20, 26, 28, and 35

Decimal Numbers

Just as students' confidence with whole numbers requires several years of growth, so must the development of decimal number concepts proceed gradually. The introductory activities in second, third, and fourth grades rely on money, on the Minicomputer, and on the number line as models for decimal numbers. These three models complement each other. Whereas all facilitate computation, the Minicomputer highlights patterns while the number line and money focus on order and relative magnitude of decimal numbers. In this strand, students perform many computations involving decimal numbers, relying on the various models to confirm their results. For example:

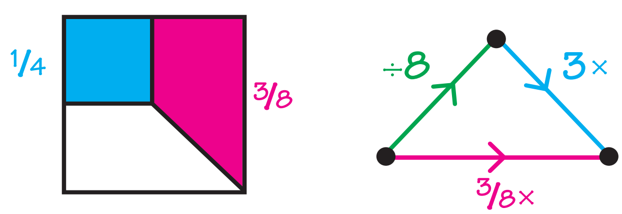
$(7 \div 2) + 10 = 13.5$	$\frac{1}{2} \times 12.25 = 6.125$
$2 - 1.7 = 0.3$	$1.5 \times 16 = 24$
$4 \div 20 = 0.2$	$30 \times 0.2 = 6$

Many decimal number patterns and computations are simply extensions of whole number patterns and computations that the students have already encountered. This semester Minicomputer, calculator, and arrow road activities illustrate these relationships. They encourage students to discover and become familiar with the subtleties of decimal numbers, without relying too early on rules and mechanical manipulation of the numbers.

Lessons: N3, 4, 8, 12, 13, 14, 17, 24, 25, 29, 32, 34, and 35

Fractions

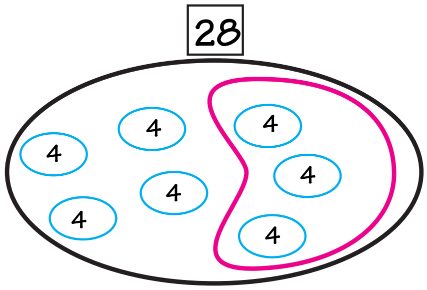
One goal this semester is to continue the parallel development of two ways of conceptualizing fractions: as parts of a whole and as functions.

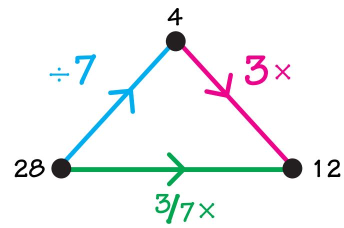


The area model emphasizes the need for equal size parts and supports calculations such as $\frac{3}{8} + \frac{1}{4}$ and $\frac{3}{8} - \frac{1}{4}$. This semester several lessons use objects such as apple slices as a variation to partitioning regions. The use of such objects precludes any confusion about the unit and provides an appropriate concrete model for calculations involving fractions greater than 1. For example, students observe the teacher slice several apples into thirds and sixths, and then they write number sentences suggested by the sliced apples. Students can invent a variety of equations involving all four operations with fractions and mixed numbers. For example:

$\frac{2}{3} + \frac{2}{3} = \frac{4}{3} = 1\frac{1}{3}$	$2 \div 6 = \frac{1}{3}$
$\frac{1}{3} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$	$3 \times \frac{1}{3} = 1$
$2 - \frac{1}{3} = \frac{5}{3} = 1\frac{2}{3}$	$4\frac{1}{3} - 1\frac{2}{3} = 2\frac{2}{3}$

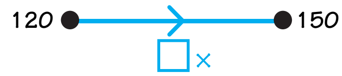
An early lesson in this semester reviews the story about a greedy monkey, Bobo, who eats more than his share of a bunch of bananas. Students use both string pictures and arrow diagrams to perform calculations.



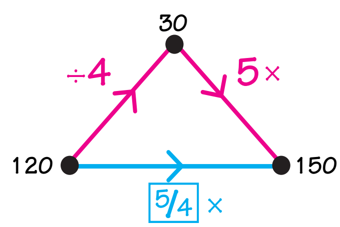


Both pictures demonstrate that $\frac{3}{7}$ x 28 = 12. The string picture clearly portrays both the fair sharing of 28 bananas among seven monkeys and Bobo's usurpation of three shares. The arrow picture succinctly records the calculations and suggests a general composition rule for multiplying an integer by a fraction.

Besides learning to perform calculations such as $\frac{3}{7}$ x 28 efficiently, students are ready to consider more challenging problems. At first glance, students may assume that this problem has no solution since 1 x 120 = 120 while 2 x 120 = 240.



Drawing a two-arrow detour and asking students to label the detour arrows leads them to discover a solution.



Students find that other pairs of arrows, for example, ÷12 and 15x, ÷8 and 10x, or ÷120 and 150x also work, reflecting that $\frac{5}{4} = \frac{15}{12} = \frac{10}{8} = \frac{150}{120}$.

Traditionally, fractions are a notoriously difficult topic to teach. The careful development over several years of two complementary but distinct models for fractions aims to provide students with a sound understanding of this peculiar representation for numbers.

Lessons: N9, 11, 13, 23, 27, 29, 30, and 35

Combinatorics

Combinatorics is a term meaning "the process of counting the number of possible outcomes in a situation." For example, here is a sample combinatorics problem:

> An ice cream store offers 31 flavors of ice cream.
> How many different double dip cones are possible?

A direct way to answer the question is to list all possible combinations. But listing presents two difficulties: inefficiency and the possibility of missing some combinations. A primary goal in the study of combinatorics is to develop efficient, systematic techniques for counting.

WORLD OF NUMBERS INTRODUCTION

The following combinatorics problems are the basis of lessons this semester in the World of Numbers strand.

- counting how many numbers can be obtained on the display of a calculator by pressing $\boxed{7}$ – $\boxed{2}$ – $\boxed{10}$ $\boxed{=}$ where one of the four operation symbols +, –, x, ÷, belongs in each blank box;
- counting how many expressions can be formed by adding two pairs of parentheses to 3 x 8 – 7 + 6;
- counting how many ways a commander could assign six spies to watch three bridges;

The lessons lead students to employ several pictorial techniques in analyzing these problems: trees, a base three abacus, and code words. Students must study each procedure to assure themselves that it accurately portrays the situation, and then they proceed to solve the problem.

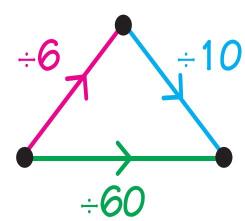
Combinatorics is a basic element of probability as well as a source of challenging recreational mathematics problems. Several lessons in the Probability and Statistics strand and the Languages of Strings and Arrows strand further apply the techniques developed in these lessons.

Lessons: N4, 12, 15, and 16

Composition of Functions

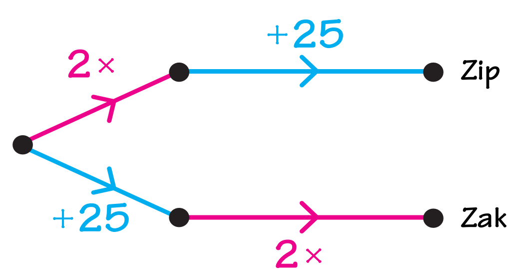
Several lessons in this strand deal with what happens when you compose a sequence of functions, that is, when you apply the functions in order, one at a time. These compositions lead to many powerful insights into the properties of numbers and operations. Arrow diagrams provide a concrete means to study this abstract but practical concept. For example, if you divide any number by 6 and then divide the result by 10, the net effect is the same as dividing the original number by 60.

Besides succinctly depicting the composition, this arrow picture also suggests that an easy way to divide any number by 60 is to divide by 6 and then divide by 10, or vice versa; the latter two operations can often be performed mentally. For example, 42 ÷ 6 = 7 and 7 ÷ 10 = 0.7, so 42 ÷ 60 = 0.7.



The section on Multiplication in this introduction includes an arrow diagram that suggests using composition to simplify a calculation such as 26 x 15 = 390.

Many pairs of functions, for example 2x and ÷3, *commute*. That is, they produce the same effect regardless of the order in which they are applied. ⅔x can be interpreted as 2x followed by ÷3 or as ÷3 followed by 2x; +98 can be interpreted as +100 followed by –2 or as –2 followed by +100. However, in this strand we explore other pairs of functions, for example, 2x and +25,, that do not commute. But there is a pattern—for example, in this illustration students find that Zak is always 25 larger than Zip, regardless of how the dots are labeled.



Calculator activities and lessons on division use arrow roads and composition to explore the relationship between repeated addition (or subtraction) and multiplication (or division). Also, students investigate several challenging problems that are solved most easily by interpreting the problem in terms of composition.

The composition of functions shows how the language of arrows is able to visually highlight rich and practical mathematical concepts and techniques. A series of lessons in the Languages of Strings and Arrows strand further supports this development with additional focus on the composition of arrows.

Lessons: N2, 3, 7, 9, 11, 13, 14, 18, 19, 25, 28, 31, 32, and 35

N1 MINICOMPUTER REVIEW N1

Capsule Lesson Summary

Review the square values and decode numbers on the Minicomputer. From a list of numbers, determine those that can be put on the Minicomputer using exactly one weighted checker. Put other numbers on the Minicomputer using two weighted checkers. Start with a configuration of checkers on the Minicomputer and then move a checker from one square to another. Determine the difference in the numbers displayed.

Materials

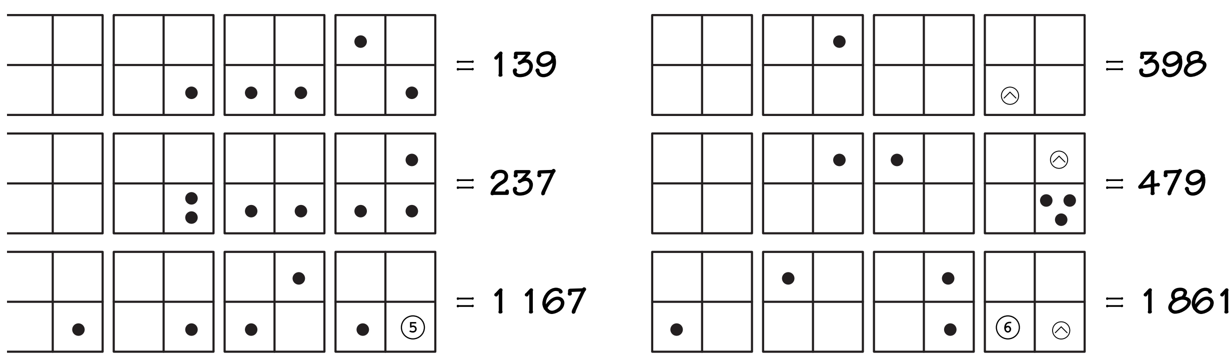
Teacher
- Minicomputer set†
- Weighted checker set

Student
- Paper
- Worksheets N1*, **, ***, and ****

Description of Lesson

Exercise 1

Display four Minicomputer boards. Move a regular checker from square to square on the Minicomputer to display the numbers 1, 2, 4, 8, 10, 100, 1 000, 4 000, 400, 40, and so on, and ask students to announce each number as you display it. Do a similar activity with a negative checker. Continue by asking the students to decode the following configurations.



You may need to remind students that a ⑤-checker is the same as five positive checkers.

Exercise 2

Display this set of weighted checkers: ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

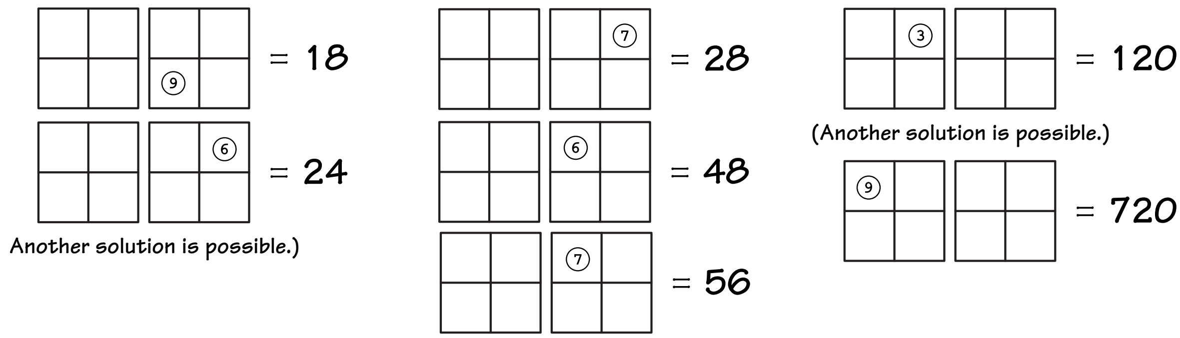
… and list these numbers on the board:

18, 24, 26, 28, 48, 54, 56, 120, 490, 720

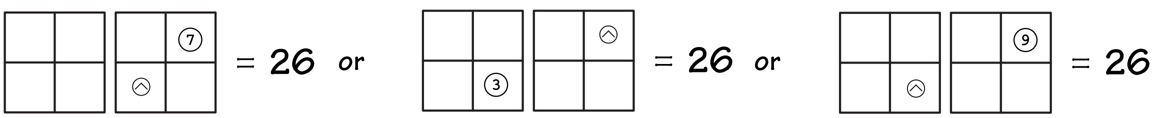
T: ***Find all of the numbers in this list that can be put on the Minicomputer using exactly one of these weighted checkers.***

†A teacher's Minicomputer set consists of four demonstration Minicomputer boards and a sufficient number of magnetic Minicomputer checkers.

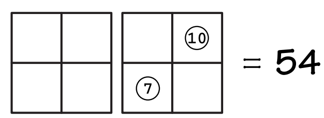
Invite students to put numbers on the Minicomputer using exactly one of the checkers. Each time, ask the rest of the class to check that the number displayed is indeed one of the numbers listed. All of the listed numbers except 26, 54, and 490 can be put on the Minicomputer using exactly one checker. The configurations below show one way to represent each of the possible numbers.



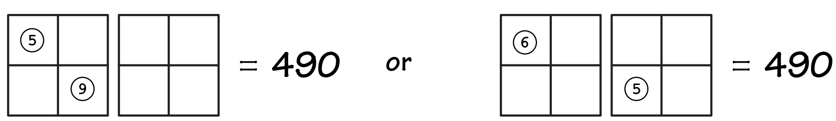
T: ***Who can put 26 on the Minicomputer using a negative checker and one of these weighted checkers?***



Who can put 54 on the Minicomputer using a ⑩-checker and one of these weighted checkers?

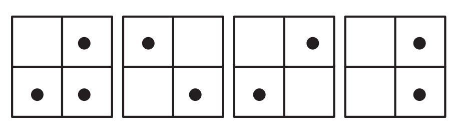


Who can put 490 on the Minicomputer using two of these weighted checkers?



Exercise 3

Put this con~~figuration on the M~~inicomputer.

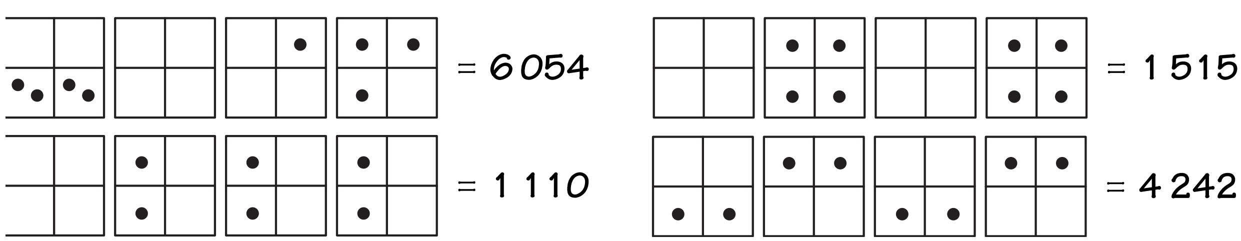


T: ***Write this number on your paper.***

Check several students' answers before asking one to answer aloud.

S: ***7 965.***

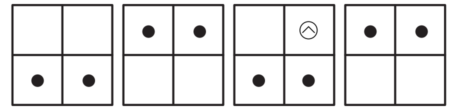
Continue the activity with these configurations.



If some students have difficulty decoding a number, invite them to make trades until the number is displayed in standard configuration.

In the above configuration for 4 242,

place a negative checker on the 40-square.

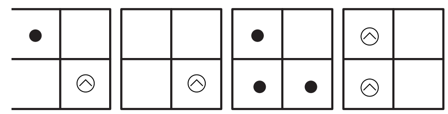


T: ***Now, what number is on the Minicomputer?***

S: ***4 202; 4 242 + $\widehat{40}$ = 4 202.***

Continue the activity by placing a negative checker on the 200-square (4 002), and then placing one negative checker on the 2-square and one on the 1-square (3 999).

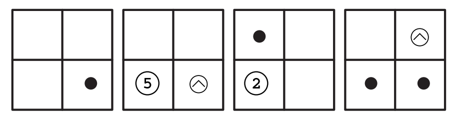
Ask students to decode this configuration. Check several papers before asking a student to announce the number.



= 7 000

Exercise 4

Put this configuration on the Minicomputer.



T: ***Let's estimate this number on the Minicomputer. Is it greater than 1 000?***

S: ***Yes, because there is a checker on the thousands board.***

T: ***Is it greater than 1 500?***

S: ***Yes. 1 000 is on the thousands board and 900 is on the hundreds board.***

T: ***Is it greater than 2 000?***

S: ***Yes, there's more than 100 on the tens boards.***

T: ***I am going to move some of the checkers on the Minicomputer. Each time I move a checker, tell me if the new number is more or less and how much.***

Move the negative checker from the 4-square to the 8-square.

S: ***Less.***

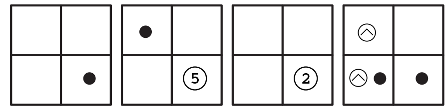
T: ***How much less?***

S: ***Four less, because a negative checker on the 4-square shows $\widehat{4}$ and a negative checker on the 8-square shows $\widehat{8}$. $\widehat{4} - 4 = \widehat{8}$.***

Continue in this manner making the following moves:

- Move the ②-checker from the 20-square to the 10-square.
  (20 less; 40 – 20 = 20)
- Move the regular checker from the 80-square to the 800-square.
  (720 more; 80 + 720 = 800)
- Move the ⑤-checker from the 200-square to the 100-square.
  (500 less; 1 000 – 500 = 500)
- Move the negative checker from the 100-square to the 2-square.
  (98 more; $\widehat{100}$ + 98 = $\widehat{2}$)

If the above moves were made, this configuration will be on the Minicomputer.



T: ***Is it possible to move a checker and make the number 18 less?***

S: ***Yes, move the ②-checker from the 10-square to the 1-square.***

Invite a student to make the move on the Minicomputer.

T: ***Can we move a checker and make the number on the Minicomputer six more?***

One of the following moves would accomplish this.

- Move a regular checker from the 2-square to the 8-square.
- Move a negative checker from the 8-square to the 2-square.
- Move the ②-checker from the 1-square to the 4-square.

Invite a student to make one of these moves. Continue in this manner, asking for the following changes:

- 450 less (Move the ⑤-checker from the 100-square to the 10-square.)
- 199 more (Move a regular checker from the 1-square to the 200-square.)

Worksheets N1*, **, ***, and **** are available for individual work.

Writing Activity

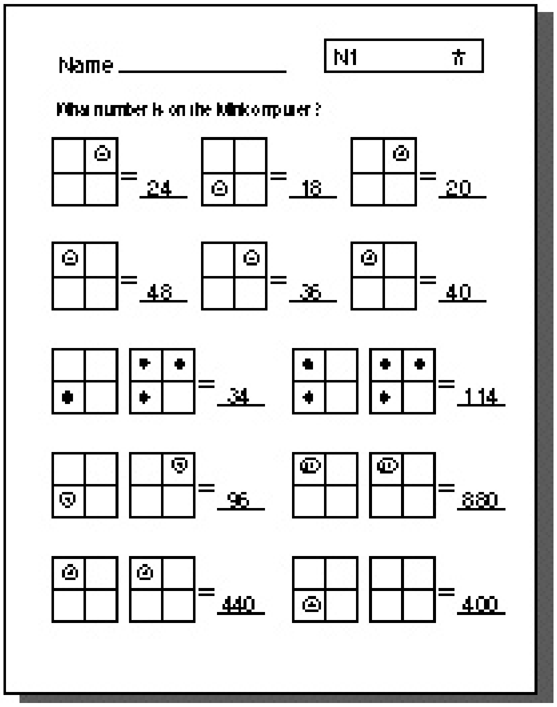
You may like students to take lesson notes on some, most, or even all their math lessons. The "Lesson Notes" section in Notes to the Teacher gives some suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students may note what they should remember about the Minicomputer review.
\

Practice/Assessment Activity

You may like to give students additional fact practice with weighted checkers on the Minicomputer. Blank Minicomputer pages are available in the Blackline section and may be used to prepare additional problems similar to the worksheets.

Home Activity

If you believe parents/guardians would benefit from a reminder about the Minicomputer, you may like to send home an introductory letter. Blacklines N1(a), (b), and (c) have a sample letter together with a home Minicomputer.


Name ____________ N1 ☆

What number is on the Minicomputer?

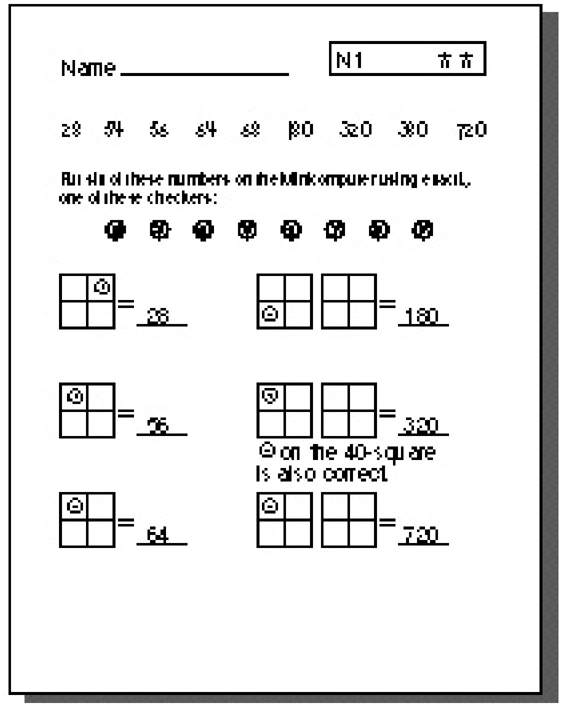
= 24 = 18 = 20

= 48 = 36 = 40

= 34 = 114

= 96 = 880

= 440 = 400


Name ____________ N1 ☆☆

28 54 56 64 68 180 320 380 720

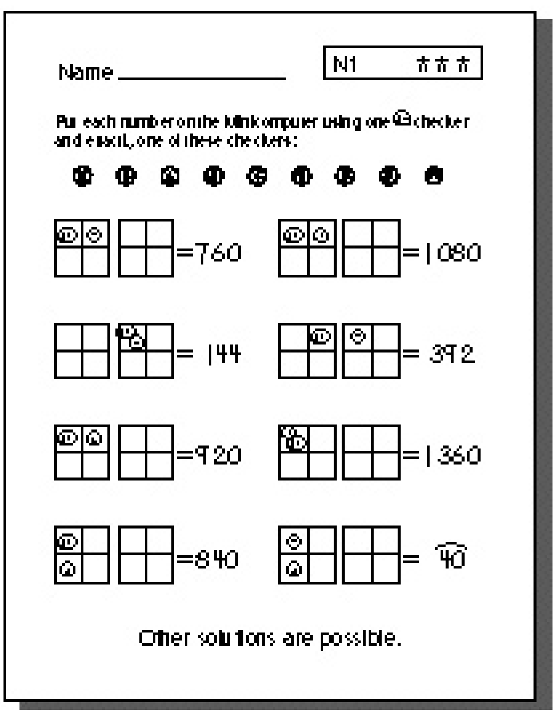
Put six of these numbers on the Minicomputer using exactly one of these checkers:

= 28 = 180

= 56 = 320

⊝ on the 40-square is also correct.

= 64 = 720


Name ____________ N1 ☆☆☆

Put each number on the Minicomputer using one ⑩ checker and exactly one of these checkers:

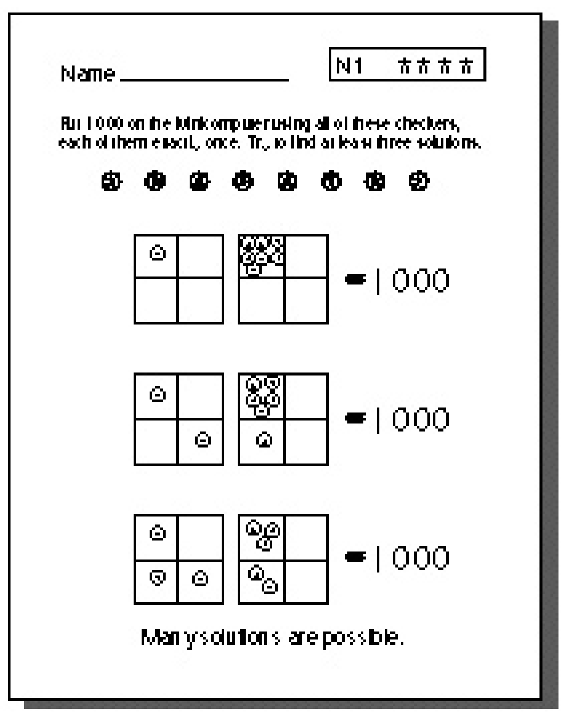
=760 =1080

= 144 = 392

=920 =1360

=840 = 40

Other solutions are possible.


Name ____________ N1 ☆☆☆☆

Put 1000 on the Minicomputer using all of these checkers, each of them exactly once. Try to find at least three solutions.

= 1000

= 1000

= 1000

Many solutions are possible.

Capsule Lesson Summary

Identify the composite of two arrows or identify an arrow knowing what the composite of it with another arrow is. Extend the activity to finding the composite of more than two arrows. Find possible locations of pairs of numbers in an arrow picture that has only the arrows identified.

Materials

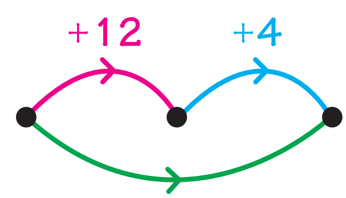
Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N2*, **, ***, and ****

Description of Lesson

Exercise 1

Draw this arrow picture on the board.



T: ***What is +12 followed by +4?***

S: ***+16.***

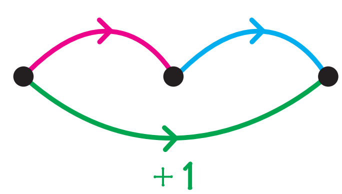
Label the green arrow +16 and relate the composition to a real situation. For example:

T: ***If we move 12 desks into a room and then move four more desks into the room, altogether we've put an additional 16 desks into the room.***

Continue with these problems, each time erasing arrow labels before beginning a new problem. (Answers are in boxes.)

+12	+4	$\boxed{+16}$	+7	$\boxed{+8}$	+15
+5	−8	$\boxed{-3}$	$\boxed{-21}$	+9	−12
−6	−6	$\boxed{-12}$	$+3\frac{3}{4}$	$\boxed{-23\frac{3}{4}}$	−20

Label only the green arrow.



T: ***What could the red and blue arrows be for?***

Accept several possibilities.

S: ***+¾ and +¼.***

S: ***+12 and –11.***

S: ***–6⅖ and + 7⅖.***

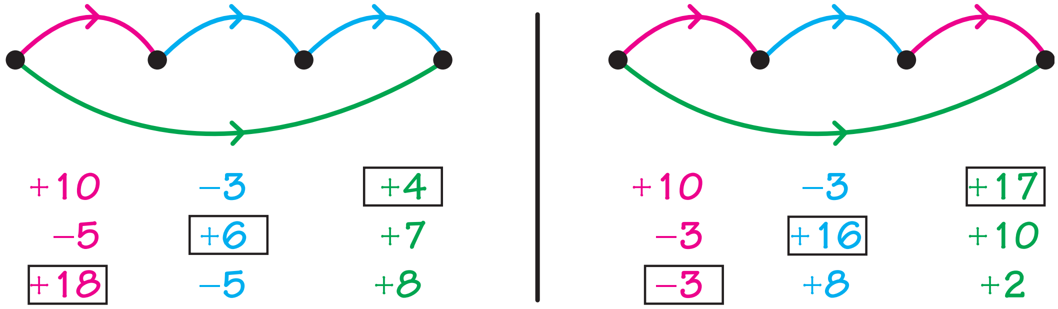
S: ***+100 and –99.***

N2

If students always give an addition function first and a subtraction function second, ask if the order is important. (No)

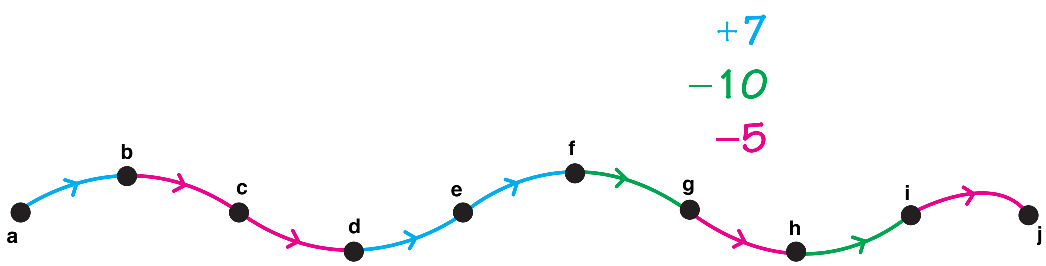
Exercise 2

Repeat Exercise 1 with a composite of more than two arrows. For example:



Exercise 3

Draw this arrow picture on the board.

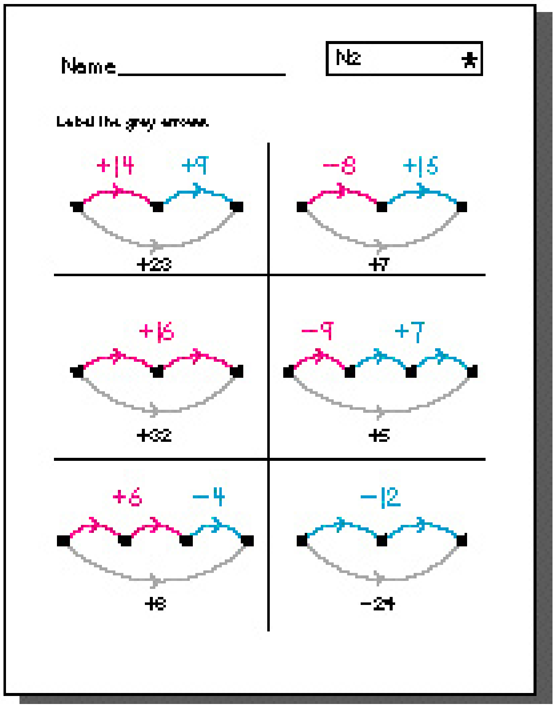


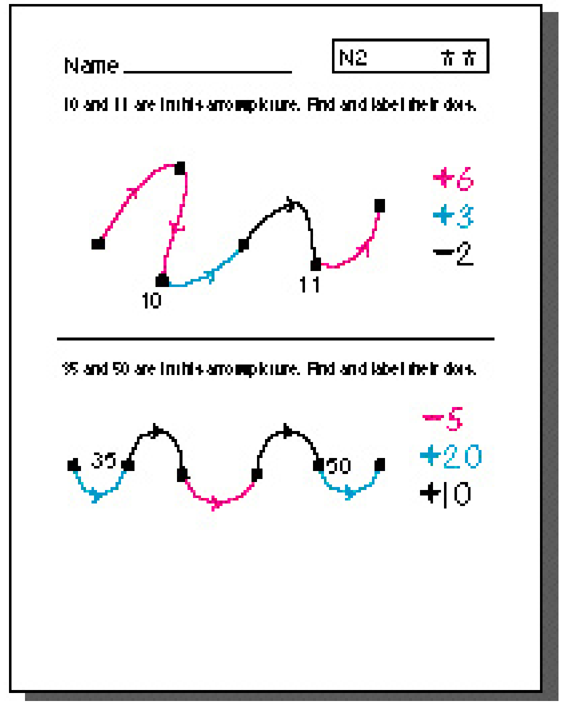
Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

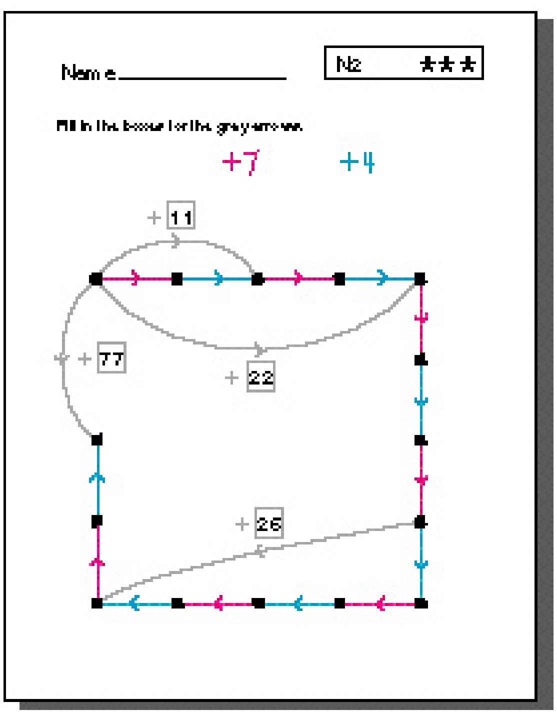
Ask the class to find where the following pairs of numbers could be in the picture. Consider each pair separately, erasing dot labels before considering more than one solution to a problem or before proceeding to the next problem. Always ask how much more (or less) one number in a pair is than the other.

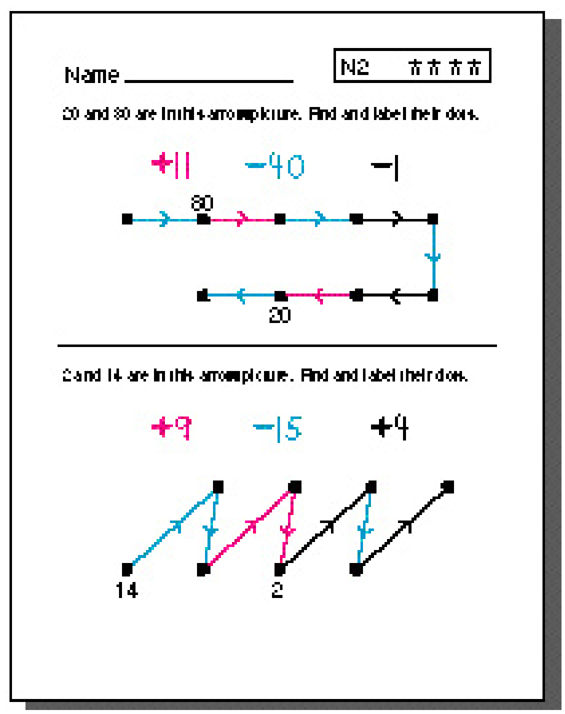
- 13 and 23 (three solutions: 13 at **d** and 23 at **b**; 13 at **g** and 23 at **f**; 13 at **i** and 23 at **h**)
- 9 and 11 (two solutions: 9 at **a** and 11 at **c**; 9 at **c** and 11 at **e**)
- 100 and 80 (100 at **g** and 80 at **j**)
- 70 and 100 (70 at **j** and 100 at **f**)
- 21 and 10 (two solutions: 21 at **f** and 10 at **a**; 21 at **d** and 10 at **i**)

If students do not find a solution for the last problem, leave it on the board for further thought. Worksheets N2*, **, ***, and **** are available for individual work.


Name
N2 ★
Label the grey arrows.
+14
+9
+23
−8
+15
+7
+16
+32
−9
+7
+5
+6
−4
+8
−12
−24


Name
N2 ☆☆
10 and 11 are in this arrow picture. Find and label their dots.
+6
+3
−2
10
11
35 and 50 are in this arrow picture. Find and label their dots.
35
50
−5
+20
+10


Name
N2 ★★★
Fill in the boxes for the grey arrows.
+7
+4
+ 11
+ 77
+ 22
+ 26


Name
N2 ☆☆☆☆
20 and 80 are in this arrow picture. Find and label their dots.
+11
−40
−1
80
20
2 and 14 are in this arrow picture. Find and label their dots.
+9
−15
+4
14
2

N3 MULTIPLICATION #1

Capsule Lesson Summary

By labeling the dots and arrows in specially designed arrow pictures, observe commutativity for addition and multiplication, for example, 27 + 33 = 33 + 27, and 30 x 48 = 48 x 30. Review some ways to multiply by two-digit numbers. For example:

14 x 12 = (10 x 12) + 48 = 120 + 48 = 168
or = (4 x 12) + 120 = 48 + 120 = 168

Materials

Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N3* and **

6 x 8 = 48	7 x 10 = 70	2 x 41 = 82	3 x 15 = 45
60 x 8 = 480	70 x 10 = 700	4 x 41 = 164	6 x 15 = 90
600 x 8 = 4 800	700 x 10 = 7 000	8 x 41 = 328	12 x 15 = 180
		16 x 41 = 656	24 x 15 = 360

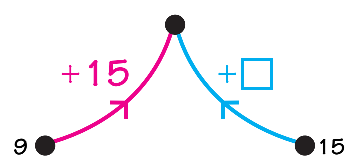
Exercise 1

Draw this arrow picture on the board and point to the unlabeled dot.

T: ***What number is here and what could the blue arrow be for?***

S: ***The dot is for 24.***

S: ***The blue arrow is for +9.***



T: ***Do you notice anything interesting about this picture?***

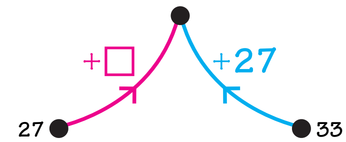
S: ***One dot is for 9, and the arrow opposite it is for +9. The other dot is for 15, and the arrow opposite it is for +15.***

Accept any reasonable observation.

T: ***The arrow picture shows 9 + 15 = 15 + 9 = 24.***

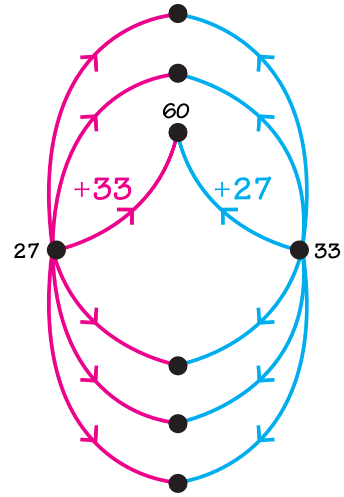
For emphasis, write the number sentence on the board next to the arrow picture.

Relabel the arrow picture as shown here.



Repeat the above activity using this arrow picture. The unlabeled dot is for 60, and the red arrow is for +33.

Add these arrows to your picture.



Ask students to copy the picture, and to label the dots and arrows. While students are working independently, draw a table on the board for possible red and blue arrows.

After a few minutes, let some students make entries into the table. For example:

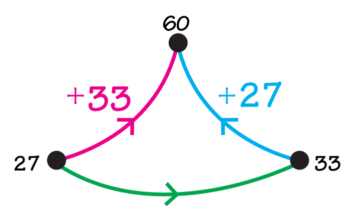
+□ (red)	33	8	10	10.5	30	37.4	25	100
+□ (blue)	27	2	4	4.5	24	31.4	19	94

T: ***Do you see any patterns in this table?***

S: ***The difference between two red arrows is the same as the difference between their corresponding blue arrows.***

S: ***Each red arrow is for plus 6 more than its blue arrow (or the difference between red and blue is always 6).***

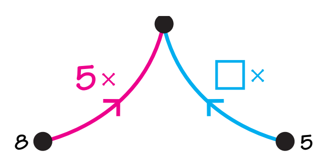
Draw an arrow from 27 to 33 in a different color.



T: ***What could this arrow be for?*** (+6)

Exercise 2

Draw this arrow picture on the board, and point to the unlabeled dot.



T: ***What number is here and what could the blue arrow be for? Write your answers on your paper.***

S: ***The dot is for 40 and the blue arrow is for 8*x.**

Label the dot and the arrow.

T: ***What do you notice about this picture?***

S: ***One dot is for 8, and the arrow opposite it is for 8*x. *The other dot is for 5, and the arrow opposite it is for 5*x.**

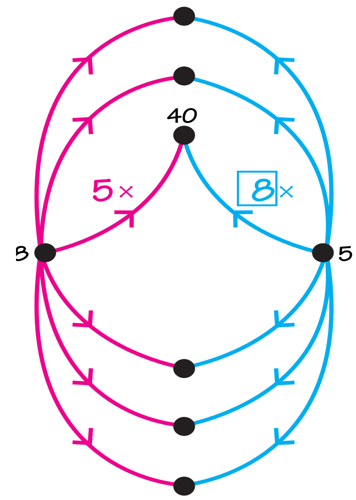
S: ***On the left is 5*** x ***8 and on the right is 8*** x ***5; they are the same.***

Accept any reasonable observation.

T: ***The arrow picture shows 5*** x ***8 = 8*** x ***5 = 40.***

For emphasis, write the number sentence on the board next to the arrow picture.

Add these arrows to the picture.



Ask students to copy the picture, and to label the dots and arrows. While students are working independently, draw a table on the board for possible red and blue arrows.

After a few minutes, let some students make entries into the table. As the table is being completed, discuss the situation. The discussion might develop along these lines.

T: ***How did you label this red arrow*** (point to some red arrow)***?***

S: ***15x.***

T: ***What number is 15 x 8?***

S: ***120.***

T: ***How did you find the answer?***

S: ***15 = 3 x 5 and 3 x 40 = 120.***

T: ***How did you label the corresponding blue arrow?***

S: ***24x, because 3 x 8 = 24.***

T: ***How did you label these two arrows?*** (Point to another pair of red and blue arrows ending at the same point).

S: ***30x and 48x; I doubled 15 and 24.***

T: ***What is 30 x 8?*** (240)
What is 48 x 5? (240)

Continue in this manner until you have a completed table. For example:

$\square$× (red)	5	10	15	30	0.5	50	100	200	45
$\square$× (blue)	8	16	24	48	0.8	80	160	320	72

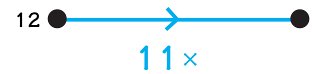
T: ***Do you see any patterns in this table?***

S: ***If a red arrow is for times some multiple of 5, then the blue arrow is for times the same multiple of 8.***

Accept any reasonable observation.

Exercise 3

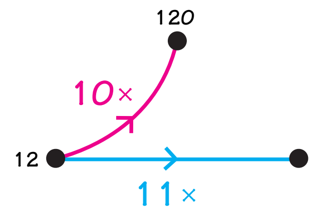
Draw this arrow picture on the board.



T: *How can we multiply 11 x 12?*

S: *Multiply 10 x 12 and then add 12.*

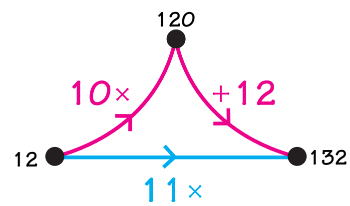
Note: There are many ways to do this problem, for example, add the results of 8 x 12 and 3 x 12. Accept many methods, but illustrate the method above.



T: *First multiply by 10. What number is 10 x 12?*

S: *120.*

Add the information to the picture.

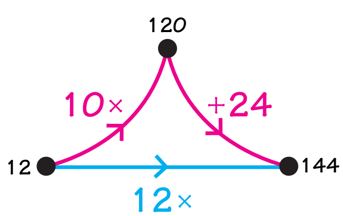


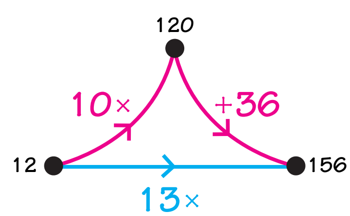
T: *Now we need to add one more 12. 1 x 12 = 12.*
What number is 120 + 12?

S: *132.*

Add the information to the picture.

Change the blue arrow to 12x and then to 13x, each time repeating the activity. Observe that you can first multiply by 10 and then add 2 x 12 or 3 x 12.





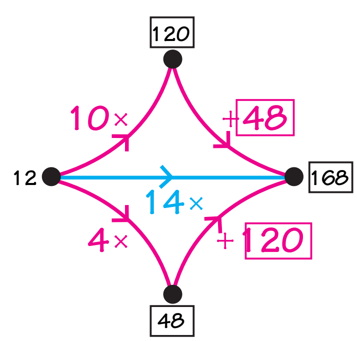
T: *Let's look at two ways to multiply by 14 in the same picture. How could we multiply 14 x 12?*

S: *Multiply 10 x 12 and then add four more 12s (4 x 12).*

S: *Multiply 4 x 12 and then add ten more 12s (10 x 12).*

Note: Again accept other methods, but tell the class that these methods are the ones most commonly used.

Draw this arrow picture on the board, and call on students to label the addition arrows as well as the dots.

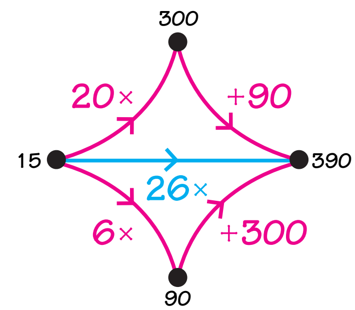


T: *What do you notice about the picture?*

S: *120 + 48 = 48 + 120 = 168.*

T: *The picture shows both that*
14 x 12 = (10 x 12) + (4 x 12)
and
14 x 12 = (4 x 12) + (10 x 12)

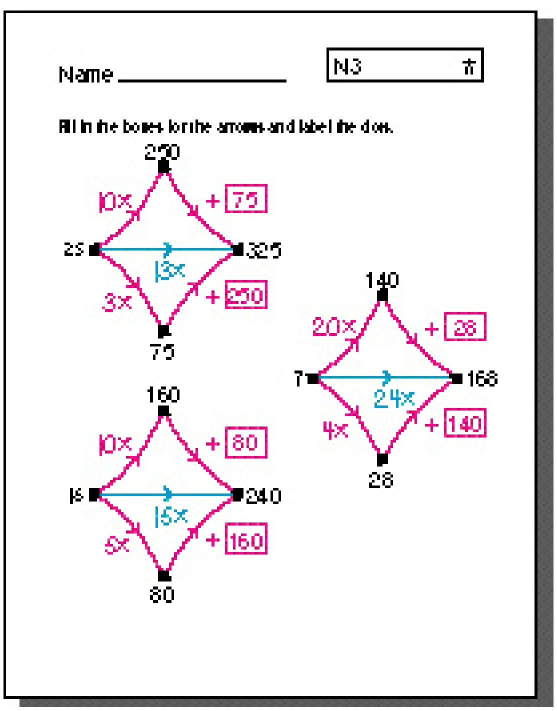
Repeat the activity using 15 as the starting dot and 26x for the blue arrow. You may like to let students complete the arrow picture individually or with a partner. A completed picture is shown here.

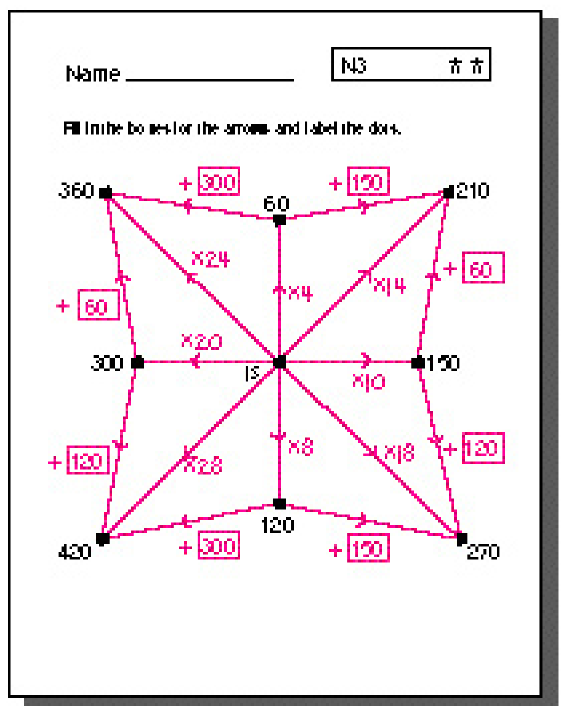


Worksheets N3* and ** are available for individual work.

Home Activity

This would be a good time to send a letter to parents/guardians about mental arithmetic. Blackline N3 has a sample letter.





N4 CALCULATOR SENTENCES

Capsule Lesson Summary

Practice multiplying and dividing whole numbers, decimal numbers, and negative numbers by 2 and by 10. Use a tree diagram and a list to count the number of ways to complete an expression, for example, 7 ☐ 2 ☐ 10, where one of the operation symbols +, –, x, ÷, is placed in each box. Also, investigate the effect of not allowing repetition of the operations in an expression.

Materials

Teacher		Student	
	• Overhead calculator • Colored chalk		• Calculator • Paper • Worksheets N4*, **, and ***

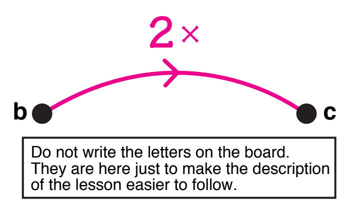
Note: In this lesson description, we assume that the calculator does chain operations from left to right. See the section on "Role and Use of Calculators" in Notes to the Teacher. If your calculator does operations in a priority order (x, ÷, +, –), you will need to adjust the lesson description accordingly.

Description of Lesson

Exercise 1

Begin the lesson with some mental arithmetic involving 2x and 10x. Select numbers appropriate for the abilities of your students, and including some negative numbers and some decimal numbers.

Draw this red arrow on the board.



T: ***If 6 is here (b), then what number is here (c)?***

S: ***12; 2 x 6 = 12.***

Select several more numbers for **b** such as 9, 27, 8, 18, 43, 4.3, 1.7, and 8.5.

Ask students to explain how they double a (non-integer) decimal number, for example, 1.7.

S: ***1.7 + 1.7 = 3.4.***

S: ***2 x 17 = 34, and so 2 x 1.7 = 3.4.***

S: ***2 x 1 = 2, and 2 x 0.7 = 1.4. Therefore, 2 x 1.7 = 2 + 1.4 = 3.4.***

Reverse the situation by giving students a number at **c** and asking for the number at **b**.

T: ***If 14 is here (c), then what number is here (b)?***

S: ***7; 2 x 7 = 14.***

S: ***7; ½ x 14 = 7 (or 14 ÷ 2 = 7).***

Trace an arrow from **c** to **b**.

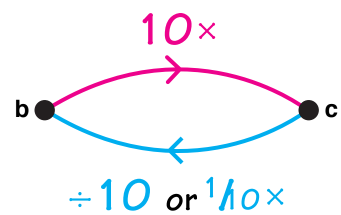
T: ***What is the opposite (return) arrow?***

S: ***½x or ÷2.***

Select several more numbers for **c** such as: 68, 30, 300, 7, 13, 4.6, and 1.8.

As necessary, use a money model to support calculations with decimal numbers. For example, \$7 ÷ 2 = \$3.50, so 7 ÷ 2 = 3.50 = 3.5.

Repeat the activity using the functions 10x and ÷10.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

Choose various numbers for **b** such as 9, 16, 43, 430, 4 300, 2.4, 16.7, 3.12, and 18.35, and ask for the corresponding numbers at **c**. Then choose various numbers for **c** such as 860, 6 000, 600, 60, 6, 18, and 32, and ask for the corresponding numbers at **b**.

Exercise 2

You may like to use an overhead calculator for this exercise. Write this calculator sentence on the board.

$\boxed{7}\ \boxed{+}\ \boxed{2}\ \boxed{\times}\ \boxed{10}\ \boxed{=}$

T: ***If we press these keys on a calculator, what number will be on the display?***

S: ***90. 7 + 2 = 9 and 9 x 10 = 90.***

Direct students to check the result on their calculators. Replace $\boxed{\times}$ with $\boxed{+}$.

$\boxed{7}\ \boxed{+}\ \boxed{2}\ \boxed{+}\ \boxed{10}\ \boxed{=}$

T: ***Now what number will appear?***

S: ***19. 7 + 2 + 10 = 19.***

Write four operation symbols and an open calculator sentence on the board.

$\boxed{7}$ _ $\boxed{2}$ _ $\boxed{10}\ \boxed{=}$ Za

× — × ÷

T: ***One of these four symbols goes in each blank box. Both blank boxes could have the same symbol. Za is a number that could appear on the display of the calculator. We found that Za could be 90 or 19. What other numbers could Za be?***

Let students work independently or with a partner to find other possibilities for Za.

S: ***Za could be 1.4. Put x in the first box and ÷ in the second box. 7 x 2 = 14 and 14 ÷ 10 = 1.4.***

S: ***15. 7 – 2 = 5 and 5 + 10 = 15.***

S: ***0.35. 7 ÷ 2 = 3.5 and 3.5 ÷ 10 = 0.35.***

After receiving several more possibilities for Za, ask,

T: ***With these four operations, how many different ways are there to complete the calculator sentence? How many possibilities for Za?***

Record students' estimates on the board, asking a few students to explain their answers. Some students are likely to suggest listing all possibilities for the two operations. Follow this suggestion, but encourage a systematic organization for the list; for example:

S: ***If we put + in the first box, then there are four possibilities for the second box: +, –,*** x***, and ÷.***

S: ***If the first operation is –, there will again be four possibilities for the second box.***

S: ***The same is true for*** x ***and ÷.***

Ordered pairs will conveniently represent the operations in the first and second boxes.

An organized list will have 16 pairs of operations.

(×,×)	(–, ×)	(×, ×)	(÷, ×)
(×,–)	(–, –)	(×, –)	(÷, –)
(×,×)	(–, ×)	(×, ×)	(÷, ×)
(×,÷)	(–, ÷)	(×, ÷)	(÷, ÷)

Compare students' estimates to 16.

Note: The above argument shows 16 ways to complete the number sentence. Due to the possibility of duplicate answers, there could be fewer than 16 numbers for Za. At this time you may like to let students check that these 16 possibilities all give different numbers for Za.

19	15	24	13.5
$^{-}1$	$^{-}5$	4	$^{-}6.5$
90	50	140	35
0.9	0.5	1.4	0.35

A student may suggest using a tree diagram, but even if not, observe that a tree in another way to show the 16 possibilities.

Draw four branches from the start as you note the following:

T: ***We have four choices for the first box: +, –,*** x***, and ÷.***
If + is in the first box, how many choices are there for the second box?

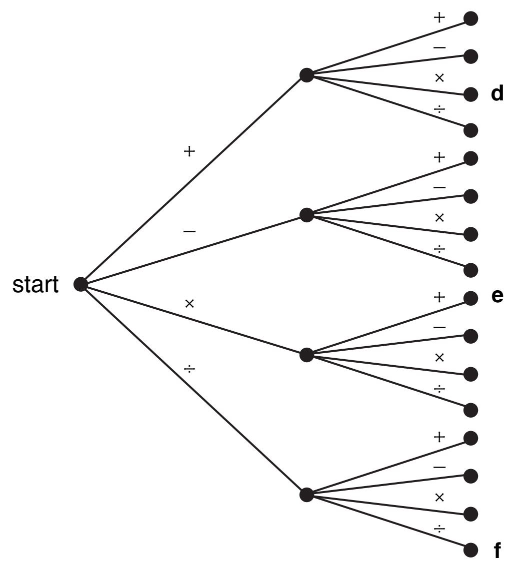
S: ***Four again: +, –,*** x***, and ÷.***

Extend the topmost branch of the tree (see the illustration).

S: ***We also have four choices for the second box if the first operation is –, x, or ÷.***

T (pointing to **d**): ***Which choice of operations does this path represent? What number would Za be for this choice?***

S: ***(+, x); Za would be 90.***



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

Similarly, let students identify Za and the two operations represented by **e** [(x, +); Za would be 24] and **f** [(÷, ÷); Za would be 0.35].

T: ***How does the tree diagram show 16 ways to complete the calculator sentence?***

S: ***There are 16 paths in the tree.***

S: ***For each possible first operation, there are four choices for the second operation. 4 x 4 = 16.***

Using the multiplication principle, students may explain that there are 16 ways to complete the calculator sentence; you can multiply 4 **x** 4 because there are four possible operations for the first box and four for the second box. Commend use of the multiplication principle and encourage students to use it to solve problems later in the lesson.

Exercise 3

Extend the open calculator sentence on the board.

$\boxed{7} - \boxed{2} - \boxed{10} - \boxed{5}\ \boxed{=}$ Clo

× – × ÷

T: ***There are four operations that could be in this new blank box. How can we extend our list or our tree to show all the possibilities for Clo? How many ways are there to complete this sentence?***

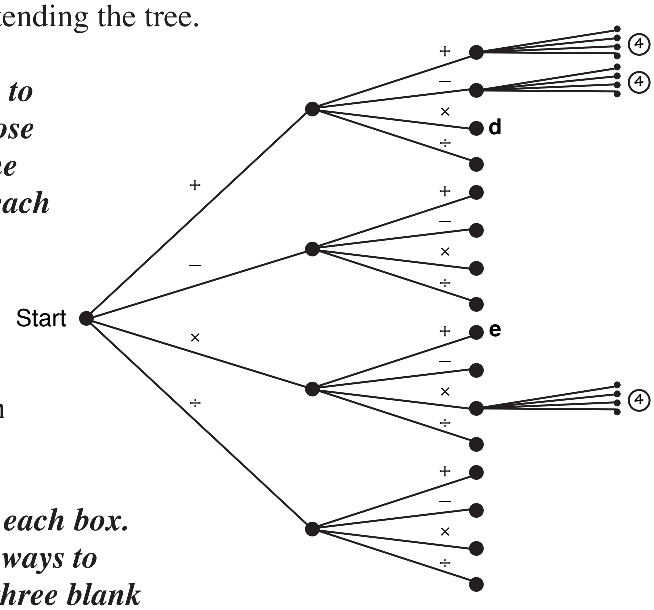
Accept students' estimates and explanations. Encourage solutions that extend the list or tree from the previous problem.

S: ***According to the list from the previous problem, there are 16 ways to fill the first two boxes. For each of those 16 pairs, there are four choices for the third box. So there are 64 ways to complete the calculator sentence since 16 x 4 = 64.***

Illustrate this on the board.

(+, +, $\begin{smallmatrix}+\\-\\\times\\\div\end{smallmatrix}$) 4	(–, +, $\begin{smallmatrix}+\\-\\\times\\\div\end{smallmatrix}$) 4	(×, +)	(÷, +)
(+, –, $\begin{smallmatrix}+\\-\\\times\\\div\end{smallmatrix}$) 4	(–, –)	(×, –)	(÷, –)
(+, ×)	(–, ×)	(×, ×)	(÷, ×)
(+, ÷)	(–, ÷)	(×, ÷)	(÷, ÷)

Another student may solve the problem by extending the tree.



S: ***The tree shows that there are 16 ways to fill the first two boxes. For each of those 16 paths, there are four choices for the third box. Draw four branches from each of the 16 end points on the tree.***

S: ***There are 64 ways to complete the calculator sentence since 16* x *4 = 64.***

Another student may extend the multiplication principle.

S: ***You have a choice of 4 operations for each box. With two blank boxes, there are 4* x *4 ways to complete the number sentence. With three blank boxes, there are 4* x *4* x *4 ways.***

Exercise 4

Write this open calculator sentence on the board.

$\boxed{100} - \boxed{5} - \boxed{2} - \quad \boxed{10}$ Kim

× − × ÷

T: ***Suppose this time we must use three different operations in the three blank boxes. Now we cannot put* x *(or + or – or ÷) in more than one box. What numbers could Kim be?***

Accept and record a few student responses, for example:

$\boxed{100} - \boxed{5} - \boxed{2} - \quad \boxed{10}$ Kim

$+ \quad - \quad \times \longrightarrow$ 1 030

$- \quad \div \quad + \longrightarrow$ 57.5

T: ***In how many ways can we complete this calculator sentence?***

Accept students' estimates and explanations.

T: ***Should the answer be more or less than 64?***

S: ***Less than 64. Both this problem and the last problem have three empty boxes. But many of the solutions to the last problem, for example, + + x, are not allowed here.***

Solve the problem either by following students' suggestions or by using a tree diagram as shown below.

S: ***Start with four branches. The first box can have +, –, x, or ÷.***

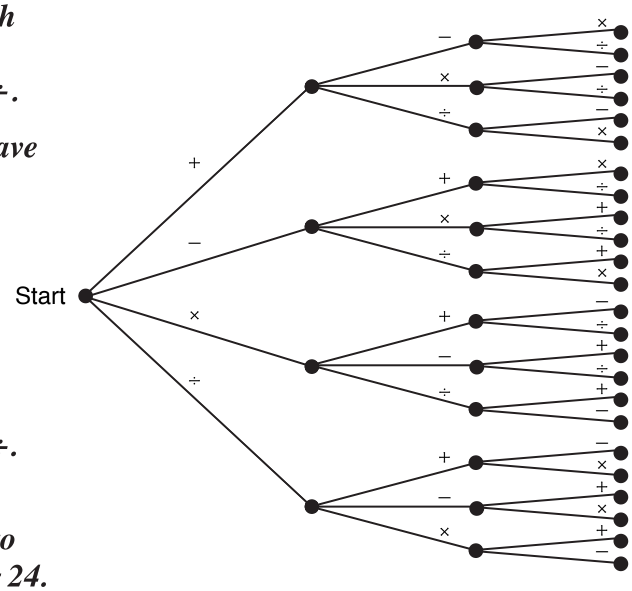
S: ***Draw only three branches from the branch for + because if the first operation is +, the second operation can only be –, x, or ÷.***

S: ***Each of the other branches also should have only three new branches. After the first operation is chosen, there are only three choices for the second operation.***

S: ***There are 12 ways to fill the first two boxes. 4 x 3 = 12.***

S: ***If the first two operations are + and –, then the third operation can only be x or ÷.***

S: ***Draw two branches from the end of each of the twelve paths. So there are 24 ways to complete the calculator sentence. 12 x 2 = 24.***



Compare students' estimates to the solution, 24 ways. Let students explain any other methods they used to solve the problem. You may wish to highlight the following method.

S: ***There are 4 choices for the first blank box. Then there are only 3 choices left for the second box and only 2 choices for the last box. So there are 24 ways to complete the calculator sentence since 4 x 3 x 2 = 24.***

Worksheets N4*, **, and *** are available for individual work. Suggest that as students work on the first clue of N4*, they record both the number they produce and the operations they use. For example, they might write:

Jo could be 12 , 8 , ...
(+, +) (–, ×)

Writing Activity

Students who like to write detective stories may like to write a **** worksheet with a detective story involving a calculator sentence.

Home Activity

This would be a good time to send a letter to parents/guardians about basic facts. Blackline N4 has a sample letter.

Name ______________ N4 ☆

Jo is a secret number.

Clue 1

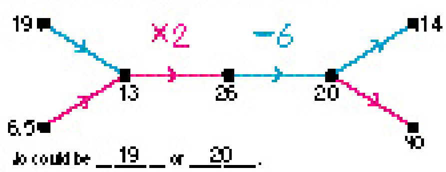
One of the symbols +, −, ×, belongs in each blank of this calculator sentence. The same symbol may be used in both blanks.

7 ☐ 3 ☐ 2 = Jo

Jo could be 12, 8, 20, 6, 2, 23, 19, or 42.

Clue 2

Jo is in this arrow picture. Label all of the dots.



Jo could be 19 or 20.

Clue 3

Jo is a prime number.

Who is Jo? 19

Name ______________ N4 ☆☆

Lou is a secret number.

Clue 1

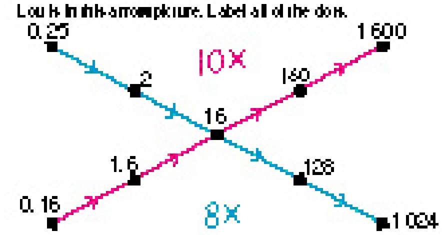
One of the symbols −, ×, ÷ belongs in each blank of this calculator sentence. Each symbol may be used only once.

6 ☐ 3 ☐ 2 ☐ = Lou

Lou could be 0.6, 15, 1.6, −1, 0, or −6.

Clue 2

Lou is in this arrow picture. Label all of the dots.



Who is Lou? 1.6

Name ______________ N4 ☆☆☆

Kir is a secret number.

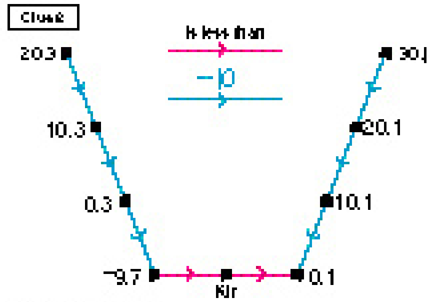
Clue 1

One of the symbols ×, ÷ belongs in each blank of this calculator sentence. The same symbol may be used in all three blanks.

9 ☐ 2 ☐ ☐ = Kir

Kir could be 31, 21, 11.1, 0.11, 24.5, 1.45, 10.45 or 0.045

Clue 2



Who is Kir? 0.045

N5 TRIANGULAR NUMBERS

Capsule Lesson Summary

By extending the model of bowling pins set up in a triangular arrangement, generate a sequence of triangular numbers and observe patterns. Use an efficient method of adding consecutive whole numbers to calculate the 49th triangular number. Solve a problem involving partitioning a rectangle whose solution unexpectedly involves triangular numbers.

Materials

Teacher
- Colored chalk
- Square tiles

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N5(a) and (b)

Description of Lesson

Exercise 1

Begin the lesson with a short introduction to the ancient Greek mathematician Pythagoras. You may want to write some highlighting words on the board as you mention dates and names.

T: ***Pythagoras was a Greek philosopher and religious leader as well as a mathematician. He lived in about 550 B.C. About how long ago is that? How do you know?***

Accept either exact answers or close estimates.

S: ***About 2 550 years; we're nearing the year 2 000 A.D. and 2 000 + 550 = 2 550.***

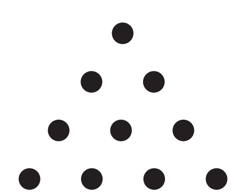
T: ***In Greece, Pythagoras founded a society of mathematicians that existed for about 200 years. The members of this society, the Pythagoreans, discovered many important ideas in arithmetic and geometry. The Pythagoreans were especially fascinated by relationships between numbers and geometric figures like squares and triangles. Today we will investigate some of the relationships that they discovered long ago.***

First, let's talk about bowling. How many of you have bowled? How many pins are used in bowling? In what shape are the pins set up?

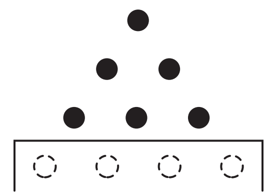
S: ***Ten pins are set up in a triangle.***

Draw a picture on the board, systematically drawing one row at a time.

T: ***This is how the ten bowling pins are set up. How many rows of pins are there?*** (Four rows)
Suppose we change the number of pins by deleting or adding rows of pins, but still placing the pins in a triangular shape. How many pins could we use? Why?



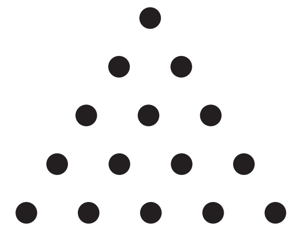
When students suggest fewer than ten pins, cover up rows of dots in the picture to confirm the number. For example:



S: ***Six pins; cover up the bottom row of dots.***

T: ***How many rows of pins do we have now?*** (Three rows)

When students suggest more than ten pins, add the appropriate rows of dots to the picture. For example:



S: ***15 pins; add a row of five dots.***

T: ***How many rows of pins now?*** (Five rows)

After several solutions have been given, draw this table on the board.

Number of rows	1	2	3	4	5	6	7	8	9	10	11
Number of dots				10							

T: ***The first triangle with ten dots had four rows. Let's record the number of dots for triangles with more or fewer rows.***

Continue this activity until at least the number of dots in triangles with one to eight rows are recorded.

T: ***What patterns do you notice?***

Let students comment on patterns and highlight ones like the following.

S: ***In the second row of numbers, 1 + 2 = 3, 3 + 3 = 6, 6 + 4 = 10, 10 + 5 = 15, 15 + 6 = 21, and so on. Each time, the number we add is one more than the previous number added.***

Number of rows	1	2	3	4	5	6	7	8	9	10	11
Number of dots	1	3	6	10	15	21	28	36			

+2 +3 +4 +5 +6 +7 +8

T: ***Does this pattern help to determine the next few numbers in the table?***

S: ***Yes. With nine rows, there are 45 dots; 36 + 9 = 45.***

S: ***With ten rows, there are 55 dots; 45 + 10 = 55.***

S: ***With 11 rows, there are 66 dots; 55 + 11 = 66. We could extend the table.***

Record this information.

Number of rows	1	2	3	4	5	6	7	8	9	10	11	12	13
Number of dots	1	3	6	10	15	21	28	36	45	55	66	78	91

+2 +3 +4 +5 +6 +7 +8 +9 +10 +11 +12 +13

T: ***Why does this addition pattern occur? For example, if a triangle with 11 rows has 66 dots, why does a triangle with 12 rows have 12 more dots?***

S: ***The new row has one more dot than the previous row.***

Write **Triangular Numbers** on the board. Point to the bottom row of numbers in the table.

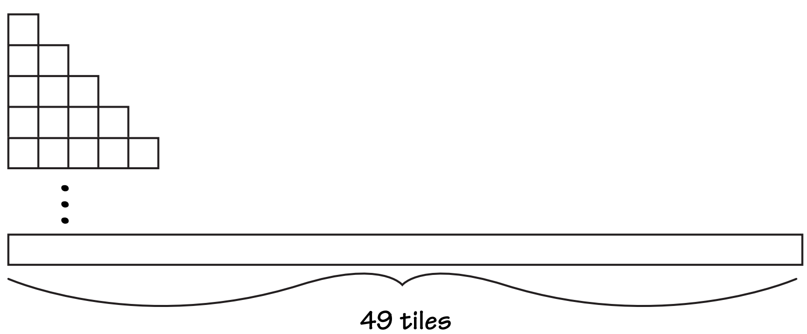
T: ***These numbers, 1, 3, 6, and so on, are called triangular numbers. With any of these number of dots, we can make a triangle of dots. The Pythagoreans were fascinated by this connection between numbers and triangles.***

Note: Students may protest calling 1 a triangular number since one dot does not form a triangle. Agree that this is a valid objection, but explain that most people still call 1 a triangular number since it fits in the pattern. You may even want to include 0 as a triangular number: 0 rows and 0 dots. Allow time for students to observe other patterns, but do not insist on any particular patterns being mentioned. Some possible patterns include the following:

- In sequence, there are two odd, two even, two odd, two even, and so on triangular numbers.
- For odd numbers of rows, the triangular number is a multiple, e.g., 1 x 1 = 1, 3 x 2 = 6, 5 x 3 = 15, 7 x 4 = 28, and so on.
- For even numbers of rows, the triangular number is a multiple plus half the number of rows, e.g., 2 x 1½ = 3, 4 x 2½ = 10, 6 x 3½ = 21, and so on.

Exercise 2

Display this picture with square tiles, or draw it on the board.



T: ***Mr. and Mrs. Laramie do tiling patterns. They have been hired to tile a very long staircase. The top row has one tile, the second row has two tiles, the third row has three tiles, and so on. The bottom row has 49 tiles. How is this pattern of squares similar to the dot patterns that we just looked at?***

S: ***The rows of squares are like the rows of dots.***

S: ***The squares form a kind of triangle pattern.***

T: ***Before they begin working, the Laramies must calculate the number of tiles they need. How could they do this?***

S: ***They need to calculate the triangular number with 49 rows.***

S: ***We know that with 11 rows there are 66 squares. So with 12 rows, there are 78 squares since 66 + 12 = 78. With 13 rows, there are 78 + 13 squares and so on. We could keep on going to 49 rows.***

Discuss any suggestions and lead the discussion toward the following method.

S: ***Add the numbers 1, 2, 3, 4, 5, 6, and so on up to 49.***

Write this sum on the board and encourage students to comment.

$$1 + 2 + 3 + 4 + 5 + \ldots + 24 + 25 + 26 + \ldots + 45 + 46 + 47 + 48 + 49$$

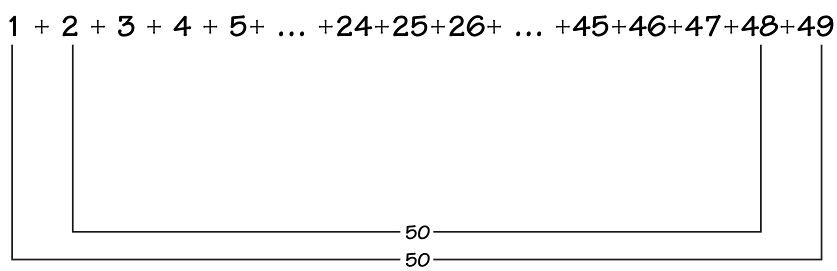
T: ***Mrs. Laramie remembered a clever way to add these numbers. She had read a story about a famous mathematician named Carl Friedrich Gauss. When Gauss was a young boy in school in about 1785, his teacher asked the students to add all of the whole numbers from 1 to 100. This took most of the students a long time, but Gauss found the correct answer in about two minutes! His method is now called the Gaussian method. It can be used for the Laramies' problem. Instead of adding the numbers in order, 1, 2, 3, …, in Gauss's method you pair numbers; first with last, second with next to last, and so on. What is 1 + 49?***

S: ***50.***

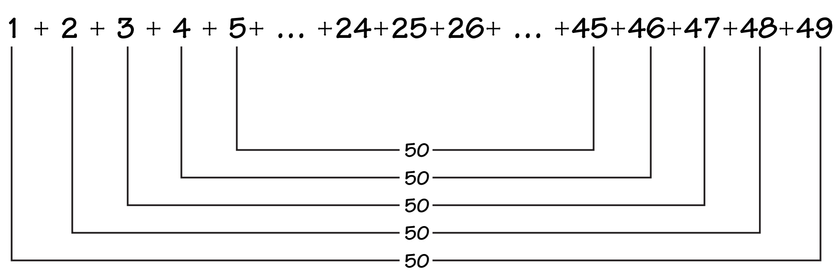
T: ***What number is 2 + 48?***

S: ***50.***

T: ***What do you pair next?***



S: ***3 and 47, 4 and 46, 5 and 45, 6 and 44, and so on. Each sum is 50.***



T: ***How does this help the Laramies?***

S: ***All they have to do is add up the 50s.***

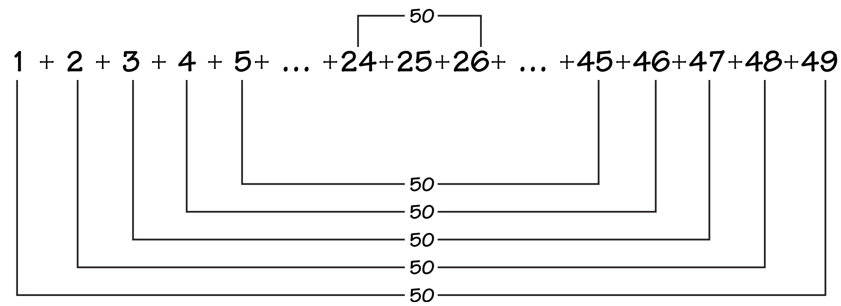
T: ***How many 50s must they add?***

This is a difficult question. Students may need the following hint.

T: ***What are the last two numbers they will pair that add to get 50?***

S: ***24 and 26. 24 + 26 = 50.***

Add this information to the picture.



T: ***So how many 50s must they add?***

S: ***24.***

T: ***What is the sum of the whole numbers 1 to 49?***

S: ***1 225. 25 was not included in any of the sums of 50, so we must add it to the twenty-four 50s. 24* x *50 = 1 200, and 1 200 + 25 = 1 225.***

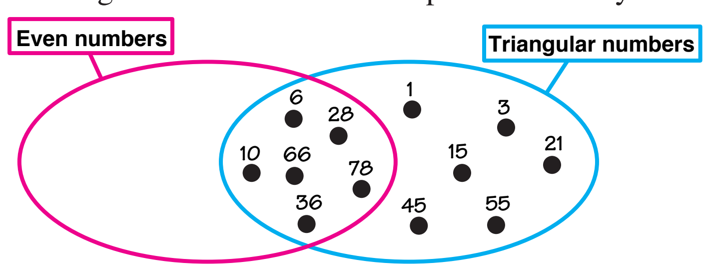
T: ***Yes, they need 1 225 tiles. Also we now know that 1 225 is a triangular number.***

Exercise 3

On the board draw two overlapping strings, one string for even numbers and the other string for triangular numbers.

T: ***Let's start with 1 and place the triangular numbers in order in the string picture. Look for patterns.***

Insist that students put the triangular numbers in the string picture in order. Continue this activity until at least the first ten triangular numbers have been placed correctly.



T: ***What patterns did you notice?***

S: ***The first two triangular numbers are odd (1, 3), the next two are even (6, 10), the next two are odd (15, 21), and so on.***

Let students comment on any other patterns they notice.

Exercise 4

Draw this picture ~~on the board~~.

T: ***How many rectangles do you see?***

S: ***Four rectangles.***

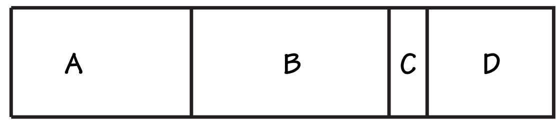
N5

If no student sees more than four or five rectangles, give a hint by tracing the following rectangle.

T: ***This is a rectangle. How many rectangles are there?***

S: ***Ten.***

Label the regions.



T: ***Let's use these letters to list the ten rectangles. The rectangle that I traced is* AB.**

Invite students to name all ten rectangles. Record the names on the board.

A B C D AB BC CD ABC BCD ABCD

Students may suggest **AC** and **BA** as additional rectangles. Point out that regions **A** and **C** do not touch, so they do not form a rectangle. Agree that **BA** is a rectangle, but that it has already been counted as **AB**.

Distribute copies of Worksheets N5(a) and (b) and refer to the table on N5(b).

Number of dividers	0	1	2	3	4	5	6	7
Number of rectangles				10				

T: ***With three dividers in the large rectangle, there is a total of ten possible rectangles. Use Worksheet 5(a) to find the number of rectangles in a picture with a rectangle and zero dividers, one divider, two dividers, and four dividers. When you finish this worksheet, check your work before you proceed to the next worksheet.***

Observe and check students' work. If many students have time to complete Worksheet N5(b), briefly summarize the lesson.

T: ***Do you recognize the numbers in the bottom row of the last table on Worksheet N5(b)?***

S: ***Yes, they are triangular numbers.***

T: ***This problem about rectangles has nothing to do with triangles, yet the triangular numbers provide the solution. Triangular numbers often appear unexpectedly as solutions for mathematics problems. That's part of the reason the Pythagoreans were so fascinated with these numbers.***

Home Activity

Send home a definition of *triangular numbers*. Suggest that students work with family members to list triangular numbers and look for patterns. You may recommend that they use a calculator, especially to check for errors.

Name ________________ N5(a)

0 Dividers

List the rectangles: A

How many rectangles? 1

A

1 Divider

List the rectangles: A, B, AB

How many rectangles? 3

A	B

2 Dividers

List the rectangles: A, B, C, AB, ABC, BC

How many rectangles? 6

A	B	C

4 Dividers

List the rectangles: A, B, C, D, E, AB, ABC, ABCD, ABCDE, BC, BCD, BCDE, CD, CDE, DE

How many rectangles? 15

A	B	C	D	E

Name ________________ N5(b)

Check your answers on N5(a). Use the answers to complete this table.

Number of Dividers	0	1	2	3	4	
Number of Rectangles	1	3	6	10	15	

3 + 3 = 6
6 + 4 = 10

What pattern do you see in the second row of numbers? 10 + 5 = 15

Each time, add one more to get to the next number.

Use your pattern to predict the number of rectangles formed when 5 dividers are used. 15+6 or 21 rectangles

5 Dividers

A	B	C	D	E	F

List the rectangles: A, B, C, D, E, F, AB, BC, CD, DE, EF, ABC, BCD, CDE, DEF, ABCD, BCDE, CDEF, ABCDE, BCDEF, ABCDEF

How many rectangles? 21 Was your prediction correct? yes

Use your previous answers and a pattern to complete this table.

Number of Dividers	0	1	2	3	4	5	6	7	8	9	
Number of Rectangles	1	3	6	10	15	21	28	36	45	55	

Do you recognize the sequence of numbers in the second row? yes

What do we call these numbers? triangular numbers

Capsule Lesson Summary

Review triangular numbers. Refer to the old game of Ninepins (from the story *Rip Van Winkle*) in which pins are a set in a square pattern to begin an investigation of square numbers. Compare triangular and square numbers.

Materials

Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons

Description of Lesson

Begin the lesson with a brief review of triangular numbers as developed in Lesson N5.

Number of rows	1	2	3	4	5	6	7	8	9	10	11	12	13	...
Number of dots	1	3	6	10	15	21	28	36	45	55	66	78	91	

Exercise 1

T: ***Does anyone know Washington Irving's story* Rip Van Winkle*?***

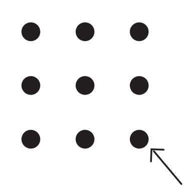
Give a brief summary of the story and then continue with the lesson as follows.

T: ***Rip Van Winkle lived in the Catskill Mountains in New York. One day he happened upon some dwarfs playing an old bowling game, "Ninepins." Instead of ten pins, they only used nine. How do you think they set up the pins?***

Note: You may wish to mention that Ninepins was a popular bowling game in the Eastern United States in the 1800s. So much gambling was associated with the game that New York and Connecticut declared Ninepins illegal. In those states, the bowlers switched to using ten pins, which was legal, and which led to the modern game of bowling.

Let students make suggestions. If no student suggests a square or diamond, do so yourself.

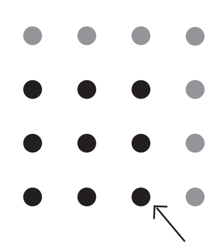
Draw this picture on the board.



T: ***They set up the nine pins like this in the shape of a square or diamond. The arrow shows the direction from which they rolled the ball.***

This square with three rows has nine pins (or dots.) Let's make a square with four rows.

Invite a student to draw additional dots in the picture.



T: ***How many new dots did we need to draw?*** (7)
How many dots are there altogether?

S: ***16 dots; 9 + 7 = 16.***

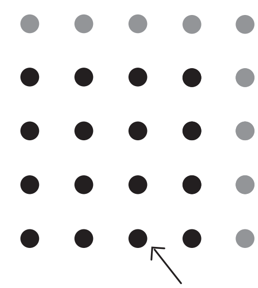
S: *16 dots; 4 x 4 = 16.*

Draw a table on the board to record the number of dots in square patterns.

Number of rows	1	2	3	4	5	6	7	8	9	10	11	12
Number of dots			9	16								

T: *If we make a square with five rows of dots, how many dots will there be?* (25)
How many new dots must we add to the dot picture? (9)

Let a student draw the new dots in the picture.



Record this information in the table and continue to find the number of dots in squares with six rows, two rows, and one row.

Number of rows	1	2	3	4	5	6	7	8	9	10	11	12
Number of dots	1	4	9	16	25	36						

T: *What patterns do you notice?*

S: *If you multiply the number of rows by itself, you get the number of dots: 1 x 1 = 1, 2 x 2 = 4,*
3 x 3 = 9, and so on.

S: *For the number of dots 1 + 3 = 4, 4 + 5 = 9, 9 + 7 = 16, 16 + 9 = 25, and so on. Each time, as we increase the number of rows, we add two more dots than we did before.*

Use arrows to show the addition pattern. Let students use patterns to extend the table.

Number of rows	1	2	3	4	5	6	7	8	9	10	11	12	13	...
Number of dots	1	4	9	16	25	36	49	64	81	100	121	144	169	

+3 +5 +7 +9 +11 +13 +15 +17 +19 +21 +23 +25

Point to the bottom row of numbers in the table.

T: *These numbers, 1, 4, 9, 16, 25, and so on, also have a special name. What do you think they are called? Why?*

S: *Square numbers, because for each number we can make a square with that number of dots.*

Refer to the table of triangular numbers and the table of square numbers, and invite students to make comparisons. The following are example comparisons, but do not expect students to notice all of these.

Triangular numbers

Number of rows	1	2	3	4	5	6	7	8	9	…
Number of dots	1	3	6	10	15	21	28	36	45	

6 + 4 = 10

Square numbers

Number of rows	1	2	3	4	5	6	7	8	9	…
Number of dots	1	4	9	16	25	36	49	64	81	

9 + 3 + 4 = 16

Number of rows	1	2	3	4	5	6	7	8	9	10	11	12	…
Number of dots	1	3	6	10	15	21	28	36	45	55	66	78	

+2 +3 +4 +5 +6 +7 +8 +9 +10 +11 +12

Number of rows	1	2	3	4	5	6	7	8	9	10	11	12	…
Number of dots	1	4	9	16	25	36	49	64	81	100	121	144	

+3 +5 +7 +9 +11 +13 +15 +17 +19 +21 +23

Number of rows	1	2	3	4	5	6	7	8	9	…
Number of dots	1	3	6	10	15	21	28	36	45	

1x 2x 3x 4x 5x

Number of rows	1	2	3	4	5	6	7	8	9	…
Number of dots	1	4	9	16	25	36	49	64	81	

1x 2x 3x 4x 5x 6x 7x 8x 9x

Number of rows	1	2	3	4	5	6	7	8	9	10	…
Number of dots	1	3	6	10	15	21	28	36	45	55	

odd even odd even odd

Number of rows	1	2	3	4	5	6	7	8	9	…
Number of dots	1	4	9	16	25	36	49	64	81	

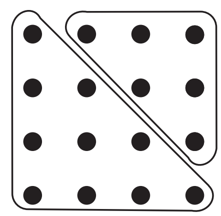
odd even odd even odd even odd even odd

Number of rows	1	2	3	4	5	6	7	8	9	…
Number of dots	1	3	6	10	15	21	28	36	45	

4 9 16 25 36 49 64 81

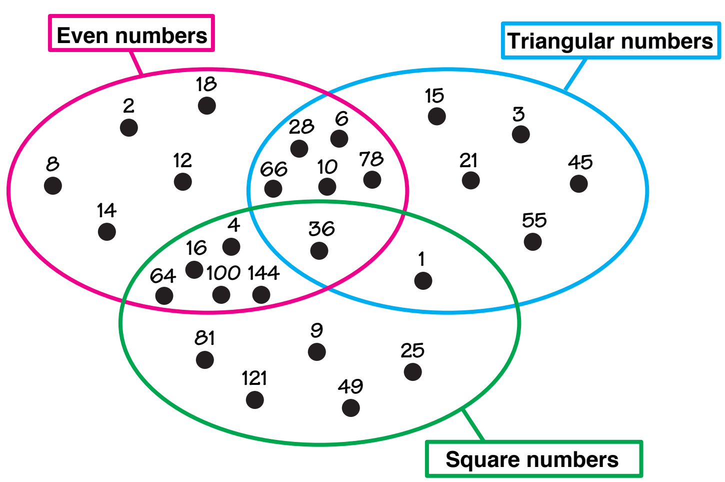
Number of rows	1	2	3	4	5	6	7	8	9	…
Number of dots	1	4	9	16	25	36	49	64	81	

The last observation suggests that a square number is the sum of two consecutive triangular numbers. This also can be seen in a pattern of dots.



Exercise 2

Draw a string picture to include even numbers, triangular numbers, and square numbers. Invite students to place numbers. During this activity students may notice the alternating patterns of odd and even in the sequences of triangular and square numbers.



T: ***What are some numbers that we know are both square and triangular?***

S: ***1 and 36.***

T: ***Numbers that are both square and triangular are difficult to find. In fact, the next such number after 36 is 1225. How do we know that 1225 is a triangular number?***

S: ***1225 was the solution to the Laramies' tile problem.***

T: ***How could we prove that 1225 is also a square number. What do we have to show?***

S: ***That some whole number times itself equals 1225.***

Write this open number sentence on the board.

$$\square \times \square = 1\,225$$

T: ***Yes, the same whole number must go in each box. Is the number more or less than 10?***

S: ***More; 10 x 10 = 100, which is much too small.***

T: ***Is the number more or less than 50?***

S: ***Less; 50 x 50 = 2 500, which is too big.***

T: ***The number in the box is between 10 and 50. Which number should we try next?***

Let students find the number in the box themselves or do so as a class.

S: ***30.***

S: ***30 x 30 = 900, so 30 is too small.***

S: ***40.***

S: ***40 x 40 = 1 600, so 40 is too big.***

S: ***35; 35 x 35 = 1225.***

T: ***So 1225 is both a square number and a triangular number. Is it odd or even?*** (Odd) ***The three smallest numbers that are both square and triangular are 1, 36, and 1225. There are infinitely many such numbers, but most are very large numbers.***

Writing Activity

Suggest that students write a report comparing square and triangular numbers.

Extension Activity

You may like to arrange for some students to work on a project to define and list a sequence of pentagonal or hexagonal numbers.

Capsule Lesson Summary

Note the use of compositions to multiply a number by 80 or by 800. Use the Minicomputer to review an algorithm for multiplying by a two-digit number.

Find the greatest perfect square less than a given number, and write some numbers as sums of squares.

$$\begin{array}{rr} & 48 \\ & \times\ 25 \\ \hline 5 \times 48 = & 240 \\ 20 \times 48 = & 960 \\ \hline & 1200 \end{array}$$

Materials

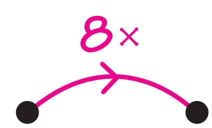
Teacher
- Colored chalk
- Minicomputer set
- Square tiles

Student
- Paper
- Colored pencils, pens, or crayons
- Calculator

Description of Lesson

Exercise 1

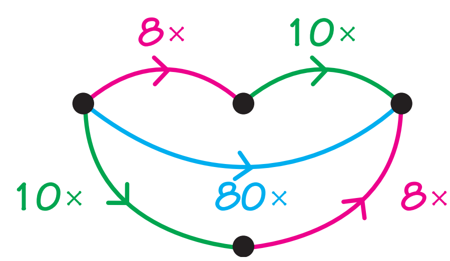
Draw an 8x arrow on the board.



T: ***If 4 is the starting dot, what number is the ending dot?*** (32)
If 6 is the starting dot, what number is the ending dot? (48)
If 72 is ending dot, what number is the starting dot? (9)
Suppose we want to multiply by 80 rather than 8. What could we do?

S: ***Multiply by 8 and then by 10.***

Extend the arrow picture.



T: ***Is there another way to multiply a number by 80?***

S: ***Multiply by 10 and then by 8.***

Note: There are many ways to multiply a number by 80, for example, 40x followed by 2x. Accept other methods, but tell the class that the two compositions shown in the arrow picture are the most commonly used.

T: ***How do we multiply a number by 50? by 70? by 90?***

Accept correct methods, but emphasize these compositions:

50x: 5x followed by 10x
10x followed by 5x

70x: 7x followed by 10x
10x followed by 7x

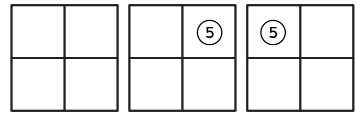
90x: 9x followed by 10x
10x followed by 9x

T: ***How can we multiply a number by 800?***

S: ***Multiply by 8 and then by 100, or vice versa.***

Exercise 2

Put 5 x 48 on the Minicomputer with ⑤-checkers.



T: ***5x what number is on the Minicomputer?*** (48)

Write the problem on the board and ask a student to solve it.

S: ***5 x 8 = 40 and 5 x 40 = 200. So the answer is 240 since 40 + 200 = 240.***

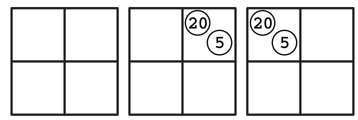
Record the answer and introduce another problem.

$$\begin{array}{r} 48 \\ \times\ 5 \\ \hline 240 \end{array} \qquad \begin{array}{r} 48 \\ \times\ 25 \\ \hline \end{array}$$

T: ***Suppose now we want to find 25 x 48. How many more 48s should we put on the Minicomputer?***

S: ***Twenty more.***

Put 20 x 48 on the Minicomputer with ⑳-checkers.



T: ***What number is 5 x 48*** (point to the ⑤-checkers)***?***

S: ***240.***

T: ***We still need to know what twenty 48s are.*** (Point to the ⑳-checkers.) ***How can we multiply a number by 20?***

S: ***Multiply by 2 and then by 10, or vice versa.***

T: ***What number is 20 x 48?***

S: ***960. 10 x 48 = 480 and 2 x 480 = 960.***

S: ***960. 2 x 48 = 96 and 10 x 96 = 960.***

$$\begin{array}{rr} & 48 \\ & \times\ 25 \\ \hline 5 \times 48 = & 240 \\ 20 \times 48 = & 960 \\ \hline & 1200 \end{array}$$

Complete the problem on the board. Note the importance of aligning digits.

Repeat the exercise with this problem.

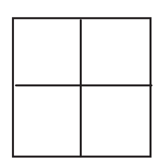
$$\begin{array}{rr} & 204 \\ & \times\ 63 \\ \hline 3 \times 204 = & 612 \\ 60 \times 204 = & 12\,240 \\ \hline & 12\,852 \end{array}$$

Let students do these problems individually, and then check the answers collectively.

$$\begin{array}{r} 53 \\ \times\ 82 \\ \hline \end{array} \qquad \begin{array}{r} 614 \\ \times\ 379 \\ \hline \end{array}$$

Exercise 3

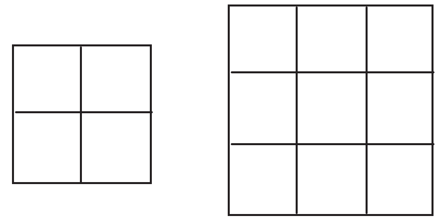
Draw this picture on the board, or make the design with square tiles.



T: ***Mr. and Mrs. Laramie make square tile designs in all different sizes. This design has two tiles on a side, using four tiles altogether. If they make a design with three tiles on a side, how many tiles would they use altogether?***

S: ***Nine tiles.***

Display the design.



T: ***Suppose a design has four tiles on a side.***
How many tiles would there be altogether?

S: ***Sixteen tiles.***

Continue discussing the situation as you generate this list.

$0 \times 0 = 0$
$1 \times 1 = 1$
$2 \times 2 = 4$
$3 \times 3 = 9$
$4 \times 4 = 16$
$5 \times 5 = 25$
$6 \times 6 = 36$
$7 \times 7 = 49$
$8 \times 8 = 64$
$9 \times 9 = 81$

T: ***What do we call numbers like 0, 1, 4, 9, 16, 25, 36, 49, 64, and 81?***

S: ***Square numbers.***

T: ***We found ten square numbers. What are some other numbers that are easy to square?***

S: ***10, 20, 30, …, 90.***

Generate this list with students' assistance.

$0 \times 0 = 0$	
$1 \times 1 = 1$	$10 \times 10 = 100$
$2 \times 2 = 4$	$20 \times 20 = 400$
$3 \times 3 = 9$	$30 \times 30 = 900$
$4 \times 4 = 16$	$40 \times 40 = 1600$
$5 \times 5 = 25$	$50 \times 50 = 2\,500$
$6 \times 6 = 36$	$60 \times 60 = 3\,600$
$7 \times 7 = 49$	$70 \times 70 = 4\,900$
$8 \times 8 = 64$	$80 \times 80 = 6\,400$
$9 \times 9 = 81$	$90 \times 90 = 8100$

T: ***One day a customer brings the Laramies 291 tiles and asks that they be used to make the largest possible square design.***

Ask for estimates as to what the dimensions of the square design will be. A sample dialogue follows.

S: ***145 tiles on a side.***

T: ***Do you think 145* x *145 will be close to 291?***

S: ***No, 2* x *145 is close to 291.***

T: ***So 145 by 145 is too large. Could it be as big as 10 tiles on a side?***

S: ***Yes, 10* x *10 = 100.***

S: ***I think about 14 tiles on a side.***

S: ***I think 16 tiles can be on each side.***

T: ***Could they put 20 tiles on a side?***

S: ***No, 20 x 20 = 400.***

T: ***So we know that they will put somewhere between 10 tiles and 20 tiles on a side of the square design.***

$$10 \times 10 < 291 < 20 \times 20$$

Record this inequality on the board.

T: ***What calculations do we need to check?***

S: ***11 x 11, 12 x 12, …, 19 x 19.***

T: ***On your paper, try to decide what size they should make.***

After a few minutes, solicit an answer.

S: ***They should make a square design that has 17 tiles on a side. The 17 by 17 square design uses 289 tiles altogether, and two tiles will be left over.***

Pose similar problems with

- 576 tiles (24 tiles by 24 tiles)
- 3 187 tiles (56 tiles by 56 tiles)
- 6 560 tiles (80 tiles by 80 tiles)

As students finish the three problems, ask them to determine if other square tile designs could be made with the leftover tiles.

576 tiles:	576 = 24 x 24
3 187 tiles:	3 187 = (56 x 56) + (7 x 7) + 2
6 560 tiles:	6 560 = (80 x 80) + (12 x 12) + (4 x 4)

As you discuss a problem collectively, let students explain which calculations they tried. Most likely they will check numbers between consecutive multiples of 10.

Additional Practice

Give students several more problems like in Exercise 2 or Exercise 3 to solve independently.

Capsule Lesson Summary

Given the location of two numbers on a number line, label the midpoint or the trisection points. Label marks on a number line that has been scaled in two different ways.

Materials

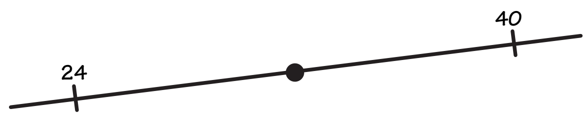
Teacher
- Colored chalk

Student
- Colored pencils, pens, or crayons
- Worksheets N8*, **, ***, and ****

Description of Lesson

Exercise 1

Draw this section of a number line on the board.



T: *The dot is halfway between the marks for 24 and 40 on this number line. What is the midpoint of 24 and 40?*

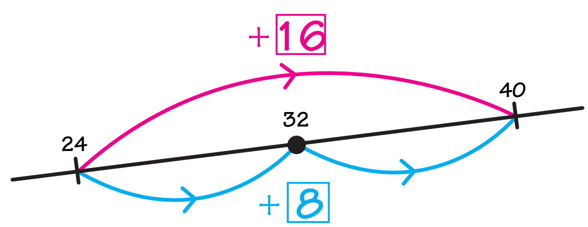
S: *32.*

T: *How can we check to see if 32 is the midpoint?*

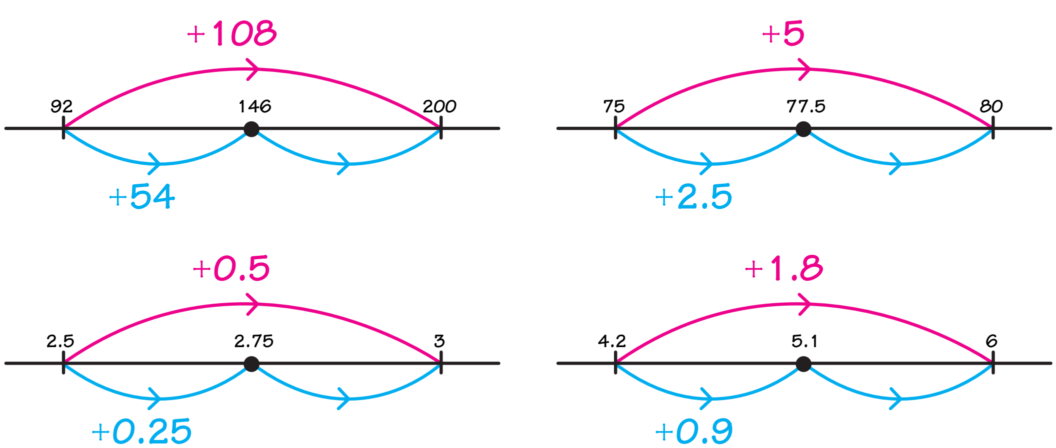
S: *32 is 8 more than 24 and 8 less than 40.*

T: *Very good. Here is a way to show this with an arrow picture.*

Draw these red and blue arrows and ask students to fill in the boxes.



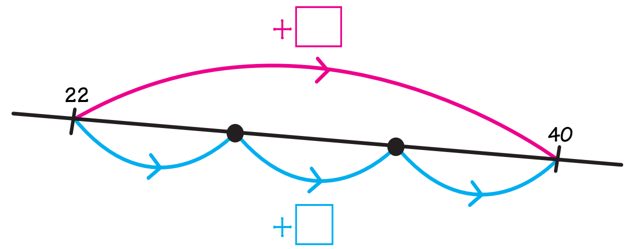
Repeat the activity with the following situations. Each time ask for the midpoint first. If students are able to answer correctly without difficulty, use the arrows to verify the answer. If students have trouble finding the midpoint, draw the arrows first. Feel free to adjust these situations, but try to include some challenging problems.



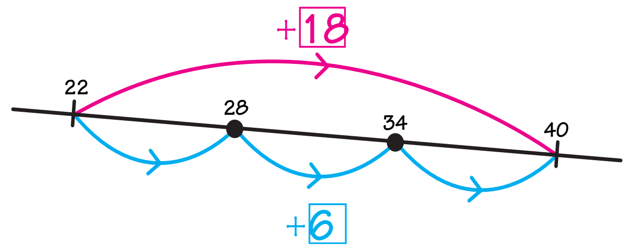
Exercise 2

Draw this picture on the board.

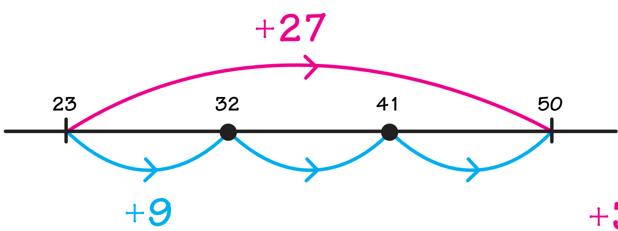
T: ***This picture shows 22 and 40 on a number line. The dots divide the segment into thirds. These arrows can help us to label the dots.***

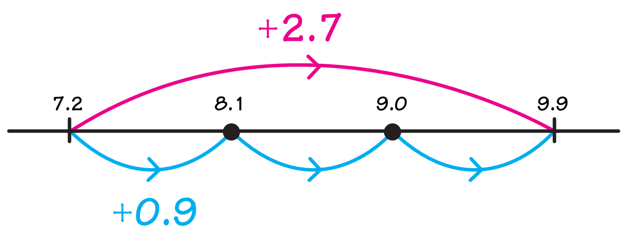


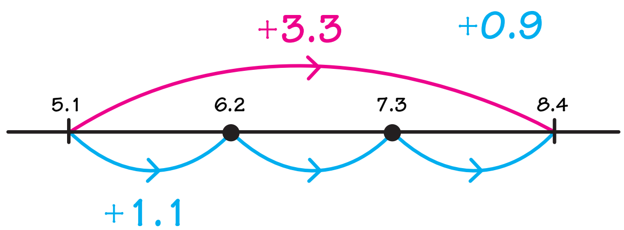
Call on students to fill in the boxes for the arrows and then to label the dots.



Repeat this activity with the following situations.

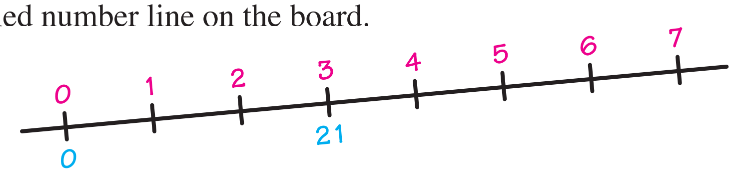






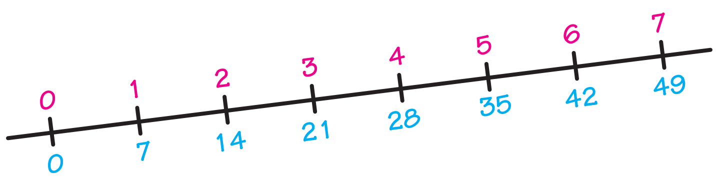
Exercise 3

Draw this double-scaled number line on the board.

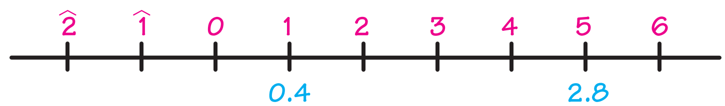


T: ***This number line has two scales. You have seen this kind of situation before with inches and centimeters on a ruler. Can you label the marks using the blue scale?***

Invite students to label the marks at the board.



Repeat this activity using the following scales.



As necessary, suggest that students determine the difference between 2.8 and 0.4.

S: ***2.8 is 2.4 more than 0.4.***

T: ***How many segments are there between 0.4 and 2.8?***

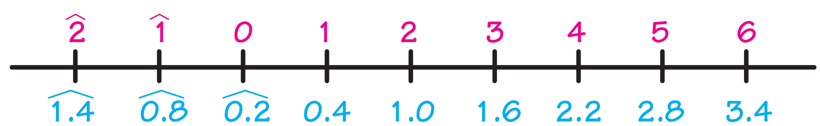
S: ***Four.***

T: ***Who can label one of the marks on the blue scale?***

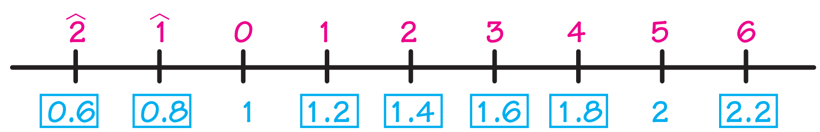
S: ***From 0.4 to 2.8 is an increase of 2.4. Since there are four segments of the same length, we can divide by 4. 2.4 ÷ 4 = 0.6.***

S: ***On the blue scale, the number to the right of 0.4 is 1 because 0.4 + 0.6 = 1.***

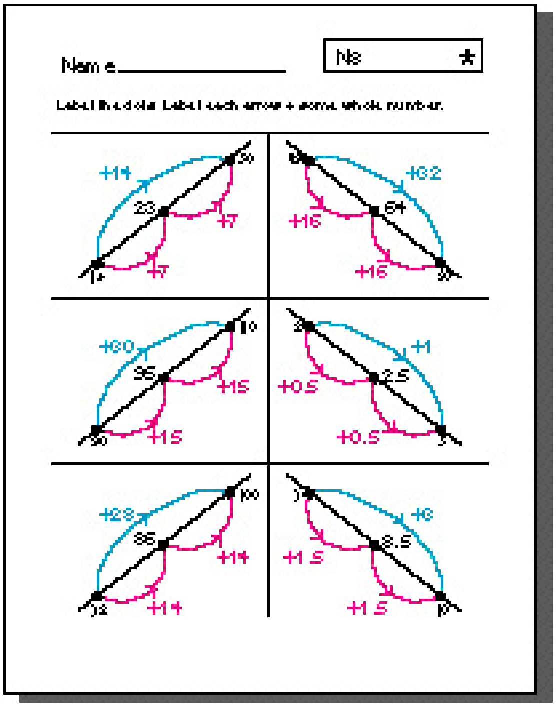
Invite students to label the marks using the blue scale.

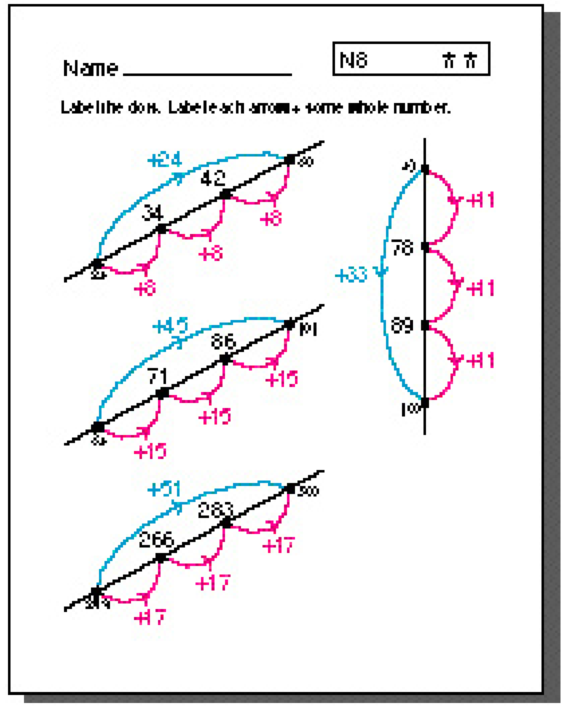


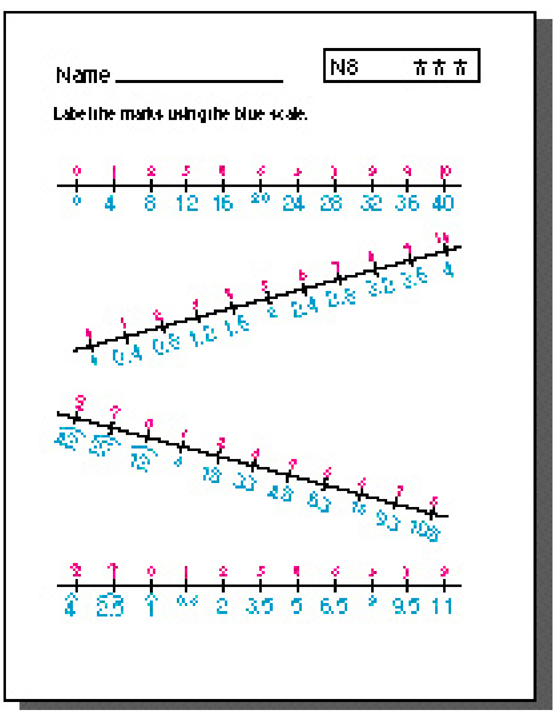
If appropriate, repeat this activity again with these scales. (Answers are in boxes.)

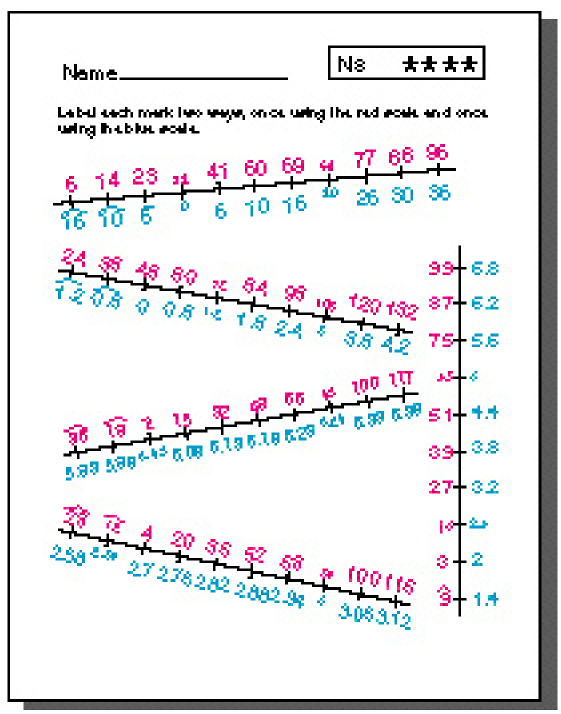


Worksheets N8*, **, ***, and **** are available for individual work.


Name
N8 ★
Label the dots. Label each arrow + some whole number.


Name
N8 ★★
Label the dots. Label each arrow + some whole number.


Name
N8 ★★★
Label the marks using the blue scale.


Name
N8 ★★★★
Label each mark two ways, once using the red scale and once using the blue scale.

Capsule Lesson Summary

In story problems, investigate common multiples of pairs of numbers. Review the story of the monkey Bobo as a model for using a string picture to multiply a fraction times a whole number, for example, $\frac{3}{5} \times 20$. Review the function $\frac{3}{5}$ x as the composition of ÷5 and 3x.

Materials

Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N9*, **, ***, and ****

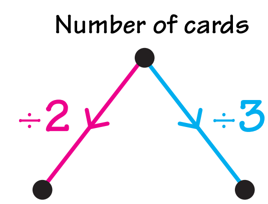
Description of Lesson

Exercise 1

Pose a problem similar to the following.

> Ms. Hernandez is buying baseball cards to share equally among the neighbor children that visit her. She doesn't know whether two or three will visit, and she wants to buy cards so she can share equally in either case. How many cards could she buy?

Allow students to find many possible solutions to the problem and make a list of the solutions. You may like to picture the problem with an arrow picture where all the dots must be for whole numbers.



Number of cards: 0, 6, 12, 18, 24, 30 …

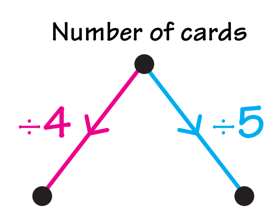
T: ***What do you notice about the number of cards Ms. Hernandez could buy?***

S: ***It must be a multiple of 6.***

Check this observation by asking students

- to try several other non-negative multiples of 6; or
- whether anyone found a solution that is not a multiple of 6.

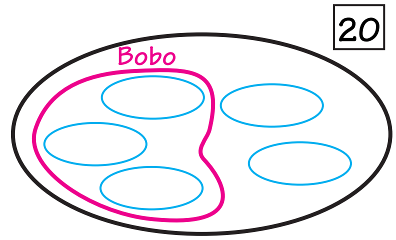
Repeat the exercise, changing the number of children that may be visiting. For example, if there may be four or five children visiting, then the number of cards that Ms. Hernandez could buy is 0 or a positive multiple of 20.



Number of cards: 0, 20, 40, 60, 80, …

Exercise 2

Erase the board and draw this string picture.



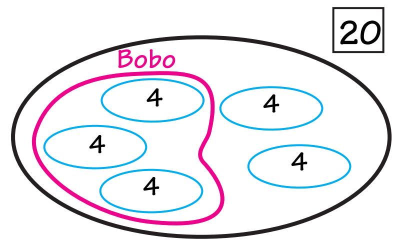
Invite students to recall from *IG-I* and *IG-II* the story about Bobo the monkey:

- Bobo lives in a zoo. Everyday the zookeeper shares bananas equally among the monkeys.
- Bobo usually eats more than his share of bananas.
- The picture and story assist in doing calculations such as $\frac{3}{5} \times 20$.

T: ***Who can tell the Bobo story with the string picture on the board?***

S: ***The zookeeper has 20 bananas to share. The five blue strings indicate that there are five monkeys. The red string shows that Bobo will eat his own share of bananas plus those of two other monkeys.***

T: ***How many bananas does Bobo eat?***



S: ***12. The zookeeper gives four bananas to each monkey since $20 \div 5 = 4$. $4 + 4 + 4 = 12$.***

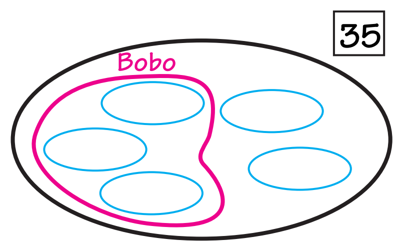
S: ***Bobo eats 12 bananas; $3 \times 4 = 12$.***

T: ***What calculation could we write about this picture?***

S: ***$\frac{3}{5} \times 20 = 12$, since Bobo eats three of the five shares.***

$$\frac{3}{5} \times 20 = 12$$

Clear the strings of numerals and put 35 in the box.



Let a student use the story and the string picture to explain that Bobo now gets 21 bananas since $35 \div 5 = 7$ and $3 \times 7 = 21$. Therefore, $\frac{3}{5} \times 35 = 21$. Similarly calculate $\frac{3}{5} \times 15 = 9$ and $\frac{3}{5} \times 45 = 27$. Emphasize that each calculation has two steps: divide the number of bananas by 5 since there are five monkeys, and multiply that answer by 3 since Bobo gets three shares.

Write this problem on the board.

$$\frac{2}{5} \times 40 =$$

T: ***What story about Bobo corresponds to this calculation?***

S: ***There are 40 bananas.***

S: ***There are still five monkeys, but Bobo only gets two shares.***

T: ***How many bananas does Bobo get?***

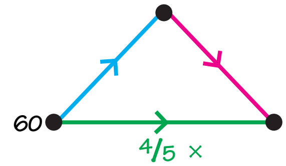
S: ***16 bananas. Each monkey gets 8 ($40 \div 5 = 8$) bananas and Bobo gets two shares; $2 \times 8 = 16$.***

In a similar manner, complete these calculations.

$\frac{2}{5} \times 40 = 16$ $\qquad$ $\frac{4}{5} \times 35 = 28$

$\frac{2}{5} \times 15 = 6$ $\qquad$ $\frac{4}{5} \times 100 = 80$

Draw this arrow picture on the board.



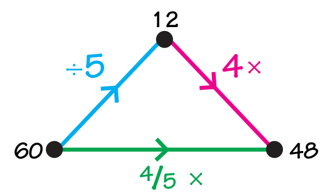
T: ***Let's record in an arrow picture how we can calculate*** $\frac{4}{5}$ x ***60. What do we do first?***

Label the arrows and dots as students explain the following:

S: ***60 ÷ 5, since there are 60 bananas and five monkeys. 60 ÷ 5 = 12.***

S: ***Bobo gets four shares, so he gets 48 bananas. 4*** x ***12 = 48.***

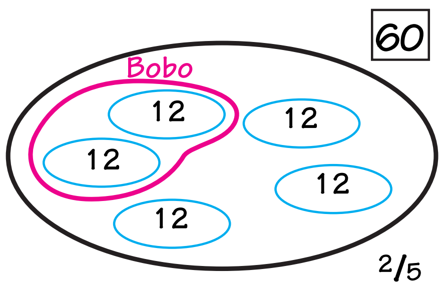
T: ***The arrow picture records that*** $\frac{4}{5}$x ***is the same as ÷5 followed by 4***x.

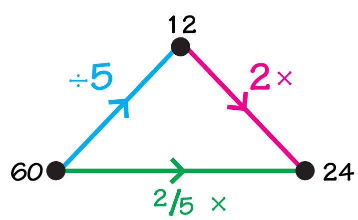


Erase the labels for the dots and use the arrow picture to do other calculations such as these:

$\frac{4}{5} \times 15 = 12$ $\qquad\qquad$ $\frac{4}{5} \times 25 = 20$

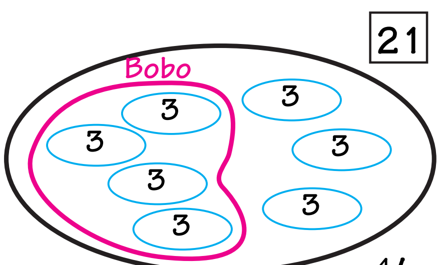
Instruct students to use either a string picture or an arrow picture to calculate $\frac{2}{5}$ x 60 on their papers. Then, following students' suggestions, draw pictures and do the calculation on the board.

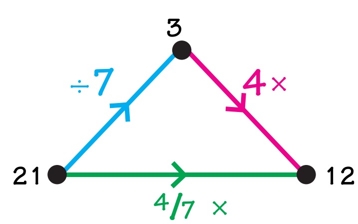




$\frac{2}{5} \times 60 = 24$

Repeat this exercise with $\frac{4}{7}$ x 21.





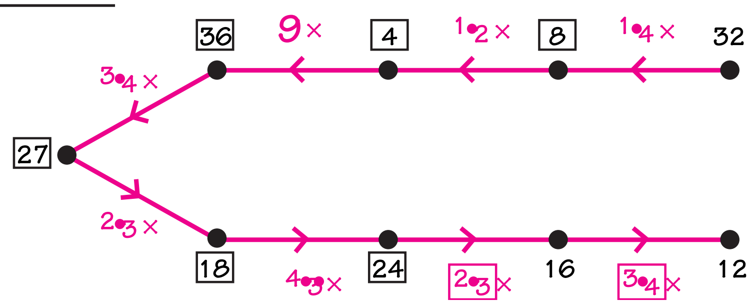
$\frac{4}{7} \times 21 = 12$

Solve the following problems in a similar manner, although you need not draw both a string picture and an arrow picture for each problem. (Answers are in boxes.)

$\frac{3}{7} \times 21 = \boxed{9}$ $\qquad$ $\frac{3}{7} \times 56 = \boxed{24}$ $\qquad$ $\frac{3}{7} \times 560 = \boxed{240}$

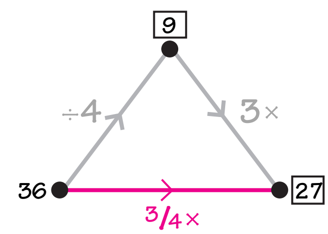
Exercise 3

Progressively draw this arrow picture on the board, asking the class to supply the boxed numbers.

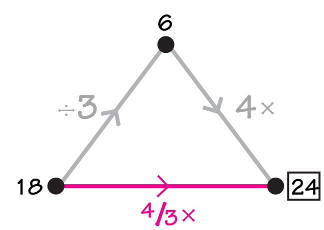


You may need to give hints for some of the problems.

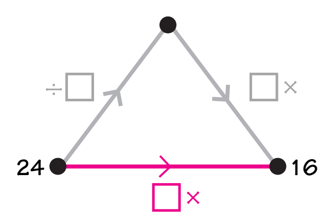
- To calculate ¾ x 36, suggest that students draw a detour:



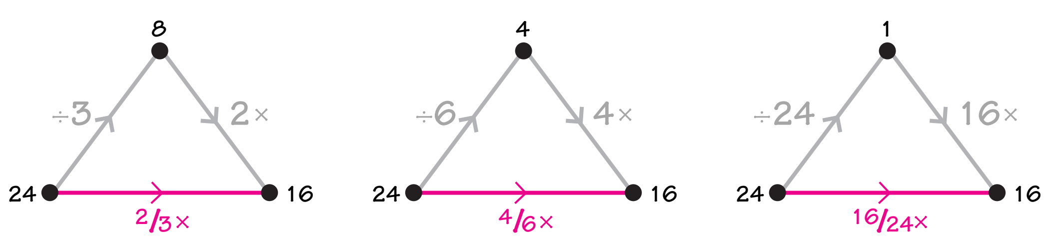
- To calculate ⁴⁄₃ x 18, use a detour with ÷3 followed by 4x.
  Note: In this case the Bobo story is inappropriate, because Bobo cannot get four shares if there are only three monkeys.



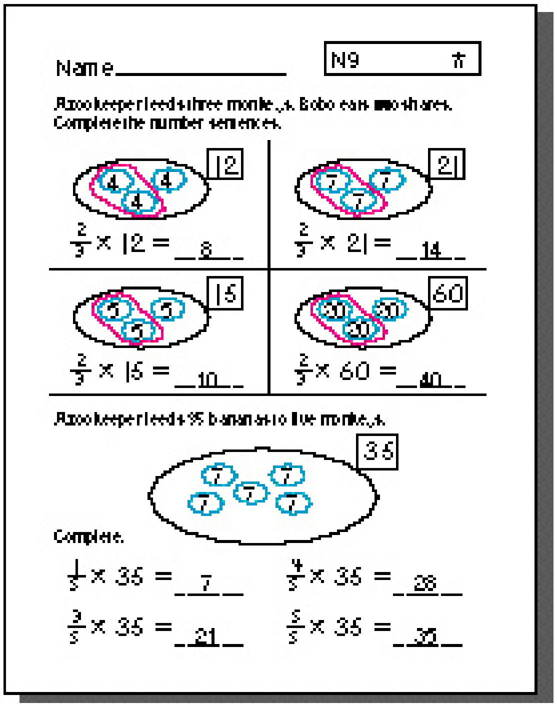
- To label the arrow from 24 to 16, draw this detour:



Invite students to label all three arrows. Any equivalent name for ⅔ suggests a different way to label the arrows. Three examples of how students might label the arrows are shown below.



Worksheets N9*, **, ***, and **** are available for individual work.



Name ____________ N9 ☆

A zookeeper feeds three monkeys. Bobo eats two shares. Complete the number sentences.

$\frac{2}{3} \times 12 = \underline{8}$

$\frac{2}{3} \times 21 = \underline{14}$

$\frac{2}{3} \times 15 = \underline{10}$

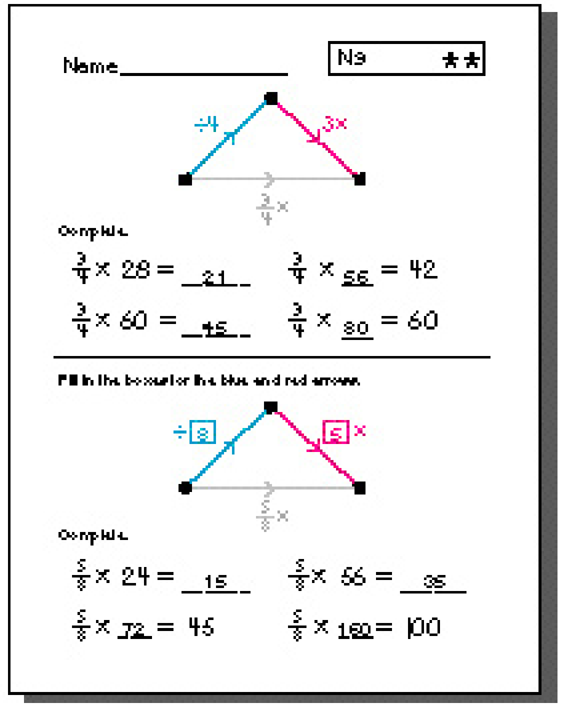
$\frac{2}{3} \times 60 = \underline{40}$

A zookeeper feeds 35 bananas to five monkeys.

Complete.

$\frac{1}{5} \times 35 = \underline{7}$ $\frac{4}{5} \times 35 = \underline{28}$

$\frac{3}{5} \times 35 = \underline{21}$ $\frac{5}{5} \times 35 = \underline{35}$



Name ____________ N9 ★★

Complete.

$\frac{3}{4} \times 28 = \underline{21}$ $\frac{3}{4} \times \underline{56} = 42$

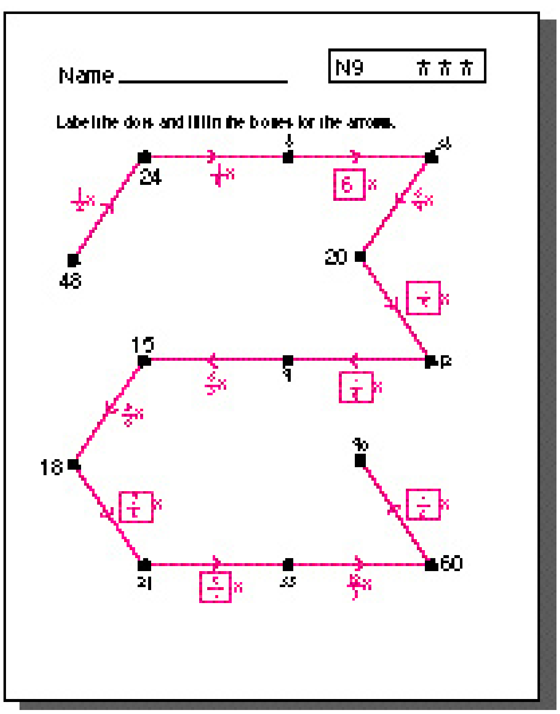
$\frac{3}{4} \times 60 = \underline{45}$ $\frac{3}{4} \times \underline{80} = 60$

Fill in the boxes for the blue and red arrows.

Complete.

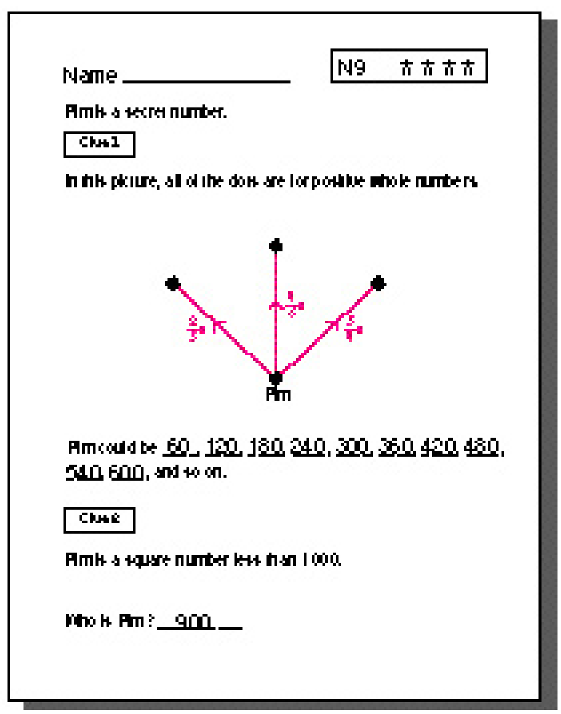
$\frac{5}{8} \times 24 = \underline{15}$ $\frac{5}{8} \times 56 = \underline{35}$

$\frac{5}{8} \times \underline{72} = 45$ $\frac{5}{8} \times \underline{160} = 100$



Name ____________ N9 ☆☆☆

Label the dots and fill in the boxes for the arrows.



Name ____________ N9 ☆☆☆☆

Pim is a secret number.

Clue 1

In this picture, all of the dots are for positive whole numbers.

Pim could be 60, 120, 180, 240, 300, 360, 420, 480, 540, 600, and so on.

Clue 2

Pim is a square number less than 1000.

Who is Pim? 900

N10 MINICOMPUTER GOLF #1

Capsule Lesson Summary

Decode numbers on the Minicomputer. By moving exactly one checker in a configuration, change the number displayed by a given amount. Play *Minicomputer Golf.*

Materials

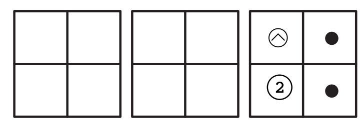
Teacher
- Minicomputer set
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons

Description of Lesson

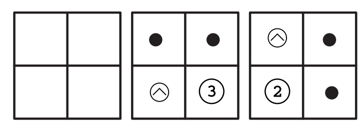
Exercise 1

Slowly put the following configuration on the Minicomputer.



T: ***What number is on the Minicomputer?*** (1)

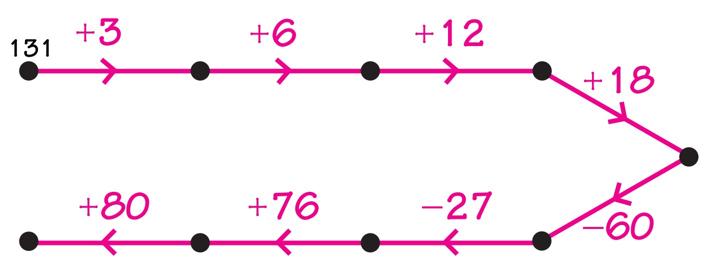
Put these checkers on the tens board.



Ask students to decode the number that is on the Minicomputer while you draw the following arrow road on the board.

T: ***What number is on the Minicomputer?*** (131)

Label the first dot of the arrow road with the number on the Minicomputer.



T: ***This arrow road starts at the number on the Minicomputer. Each arrow tells us how to change the number, but we only move one checker to make a change.*** (Point to the +3 arrow.) ***What move will make the number on the Minicomputer three greater?***

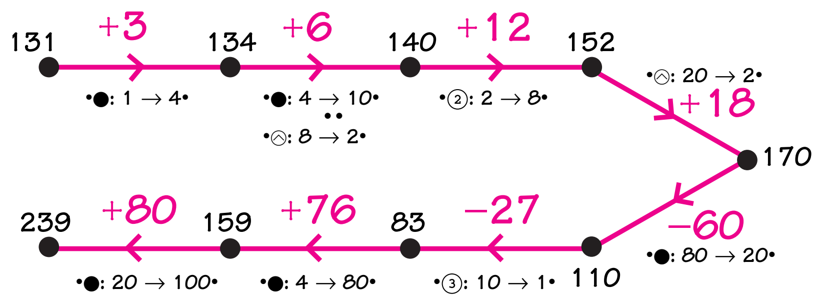
S: ***Move a regular checker from the 1-square to the 4-square.***

Invite a student to make the move on the Minicomputer.

T: ***Now, what number is on the Minicomputer?*** (134)

Label the dot for 134 and continue in the same manner until the arrow road is completed. The picture below indicates moves corresponding to the sequence of arrows in the road.

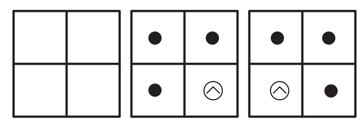
Note: (●: 1 → 4) indicates moving a regular checker from the 1-square to the 4-square. Similarly (②: 2 → 8) indicates moving a ②-checker from the 2-square to the 8-square.



Suggest that students check the configuration on the Minicomputer to verify that the ending number is 239.

Exercise 2

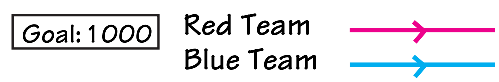
Put this configuration on the Minicomputer.



T: ***What number is on the Minicomputer?*** (141)

Let's play Minicomputer Golf using 141 as the starting number and 1 000 as the goal.

Organize the class into two groups, a Red Team and a Blue Team. Write this information on the board.



Review the rules of *Minicomputer Golf*.

- Teams alternate play and members take turns within each team.
- A player takes a turn by moving one checker from a square to any other square. As long as the number on the Minicomputer is less than the goal, the play must increase the number. If the number on the Minicomputer is more than the goal, the play must decrease the number.

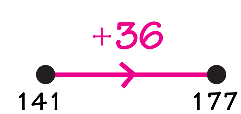
The first team to reach the goal is the winner. You can use a red-blue arrow road to record the progress of the game. A sample game is described below.

In this game the Red Team goes first and moves a regular checker from the 4-square to the 40-square.

T: ***Did you increase or decrease the number on the Minicomputer?*** (Increase)
How much greater? (36 greater)
What number is on the Minicomputer now?

S: ***177; 141 + 36 = 177.***

A red arrow records the Red Team's move.

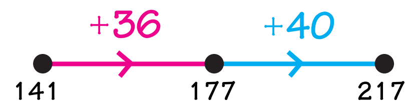


It is the Blue Team's turn and a player moves a checker from the 40-square to the 80-square.

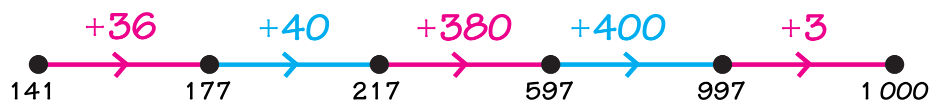
T: ***Did you increase or decrease the number on the Minicomputer?*** (Increase)
How much greater? (40 greater)
What number is on the Minicomputer now?

S: ***217; 177 + 40 = 217.***

A blue arrow records the Blue Team's move.

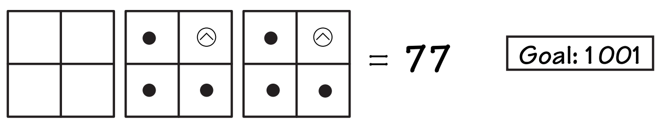


The game continues until the goal (1 000) is reached.
The record below shows a game in which the Red Team wins.



Exercise 3 (optional)

Play another game with this starting situation and goal.



Home Activity

Send home a description of *Minicomputer Golf* and suggest that students play the game with family members.

Capsule Lesson Summary

Relate division problems to dividing a line segment into parts of equal length. Use patterns to solve related division problems such as 56 ÷ 7 and 5 600 ÷ 7. Use arrow roads to solve division problems in preparation for a standard algorithm.

Materials

Teacher
- Meter stick
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Metric ruler
- Worksheets N11*, **, ***, and ****

Description of Lesson

Exercise 1

Carefully draw four 48-cm line segments on the board.

T: ***How could we divide one of these line segments into six segments of equal length?***

S: ***First, measure the length of the segment.***

Invite a student to measure the segment.

T: ***It is 48 cm long.***

S: ***48 ÷ 6 = 8; so divide the 48-cm segment into 8-cm pieces.***

T: ***If we place the 0 mark of the ruler at one end of the line segment, where should we mark the line segment?***

S: ***At 8 cm, 16 cm, 24 cm, 32 cm, 40 cm, and 48 cm.***

S: ***Those are all multiples of 8.***

Call on students to help mark one 48-cm line segment as directed by the class. Then solicit number sentences to describe the division. For example:

$48 \div 6 = 8$ $\quad$ $\frac{1}{6} \times 48 = 8$ $\quad$ $6 \times 8 = 48$ $\quad$ $48 \div 8 = 6$

Ask students to interpret the number sentences in terms of the measurements.

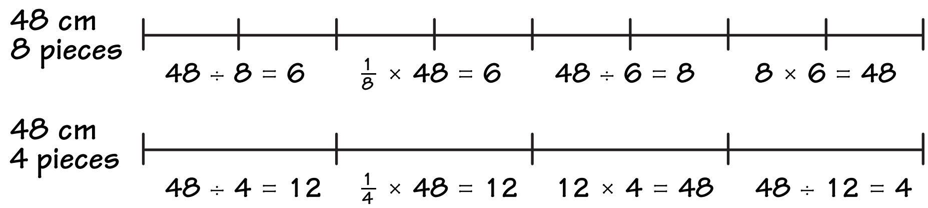
S: ***48 ÷ 6 = 8. If we divide a 48-cm line segment into six pieces of equal length, each piece is 8 cm long.***

S: ***⅙ x 48 = 8; each 8-cm piece is one-sixth of the 48-cm segment.***

S: ***6 x 8 = 48. Six 8-cm line segments form a 48-cm line segment.***

S: ***48 ÷ 8 = 6. There are six 8-cm pieces in a 48-cm line segment.***

Repeat the activity by dividing one 48-cm segment into eight pieces of equal length and another segment into four pieces of equal length.



T: ***Could we divide the last 48-cm segment into 12 pieces of equal length?***

S: ***Yes. Make each piece 4 cm long since 48 ÷ 12 = 4.***

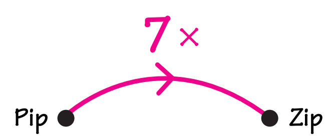
T: ***Do you see any connection between these division problems?***

S: ***There are eight 6-cm pieces and six 8-cm pieces in a 48-cm segment. 48 ÷ 6 = 8 and 48 ÷ 8 = 6.***

S: ***There are four 12-cm pieces and twelve 4-cm pieces. 4 x 12 = 48 = 12 x 4.***

Exercise 2

Draw this arrow on the board.



T: ***If Pip is 6, what number is Zip?***

S: ***42. 7 x 6 = 42.***

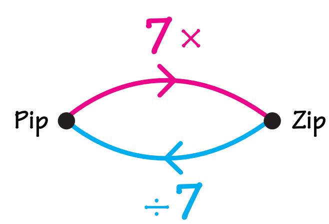
Ask students to calculate Zip when Pip is 8, 4, 40, 400, and 0.4.

T: ***If Zip is 35, what number is Pip?***

S: ***5; 7 x 5 = 35.***

S: ***5; 35 ÷ 7 = 5.***

Draw a blue arrow from Zip to Pip.



T: ***What would the return (opposite) arrow be for?***

S: ***÷7 or $\frac{1}{7}$x.***

Ask students to calculate Pip when Zip is 14, 63, 56, 560, 5 600, and 5.6.

Begin a sequence of multiplication problems on the board as you ask,

$$7 \times 60 = 420$$
$$7 \times 61 = \_\_\_\_$$

T: ***If Pip is 60, what number is Zip?***

S: ***420. 7 x 60 = 420.***

T: ***What number is 7 x 61?***

S: ***427; add one more 7 to 420.***

Continue in a similar manner to present several more problems. (Answers are in boxes.)

$7 \times 62 = \boxed{434}$

$7 \times 64 = \boxed{448}$

$7 \times 67 = \boxed{469}$

$7 \times 68 = \boxed{476}$

Exercise 3

Begin a table on the board as you present the following information.

T: ***Nabu works in a bottling factory. His job is to put soda bottles into cartons. If he has 34 bottles and he puts six bottles into each carton, how many cartons will he fill?***

Numbers of Bottles	Bottles in One Carton	Full Cartons	Extra Bottles
34	6		

S: ***Five, since 5 x 6 = 30.***

S: ***There will be four extra bottles.***

Record the information in the table.

T: ***With 68 bottles, how many cartons of six will Nabu fill?***

S: ***He will fill 11 cartons and have two bottles left over since (11 x 6) + 2 = 68.***

Record the information in the table.

T: ***68 is the double of 34, so the number of bottles doubles, but the number of full cartons more than doubles. 2 x 5 is only 10. Why does this happen?***

S: ***With 68 bottles, Nabu doubles both the number of full cartons and the number of extra bottles. So he has ten full cartons and eight bottles left over. But then he can use six of the extra bottles to fill one more carton. He fills 11 cartons and has only two bottles left over.***

Present the following problems in a similar manner. (Answers are in boxes.)

Numbers of Bottles	Bottles in One Carton	Full Cartons	Extra Bottles
34	6	5	4
68	6	11	2
50	6	$\boxed{8}$	$\boxed{2}$
100	6	$\boxed{16}$	$\boxed{4}$
200	6	$\boxed{33}$	$\boxed{2}$
420	6	$\boxed{70}$	$\boxed{0}$

T: ***How many cartons of six can Nabu fill with 426 bottles?***

S: ***71. With 420 bottles, he fills 70 cartons; with six more bottles, he can fill one more carton.***

N11

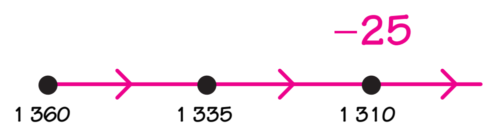
Present the following problems in a similar manner. (Answers are in boxes.)

Numbers of Bottles	Bottles in One Carton	Full Cartons	Extra Bottles
420	6	70	0
+ 6		+ 1	
426	6	71	0
+ $\boxed{6}$		+ $\boxed{1}$	
432	6	$\boxed{72}$	0
+ $\boxed{12}$		+ $\boxed{2}$	
444	6	$\boxed{74}$	$\boxed{0}$
+ $\boxed{4}$		+ $\boxed{0}$	
448	6	$\boxed{74}$	$\boxed{4}$

Exercise 4

T: ***Nabu is transferred to another part of the factory. Here he packs bottles into larger cartons. Each carton holds 25 bottles. Nabu must pack 1 360 bottles. First he must decide how many cartons he will need. About how many cartons should he order?***

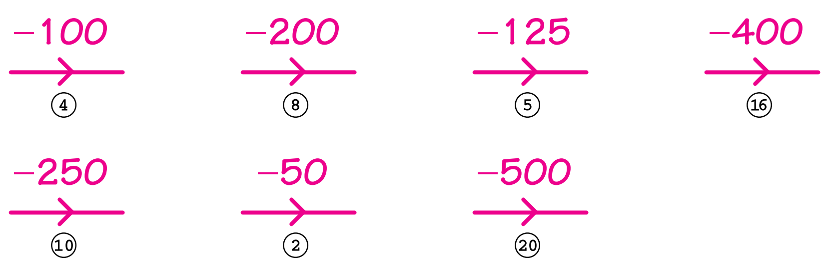
Accept and record students' estimates without comment.
Draw this picture on the board.



T: ***Nabu realizes he could build an arrow road with –25 arrows to calculate how many cartons he will need, since each carton holds 25 bottles. But he sees that he would have to build a very long road. What arrows could he use to shorten the arrow road?***

S: ***He could use –100 arrows. Each –100 arrow would represent filling four cartons.***

Accept and record several students' suggestions, for example:



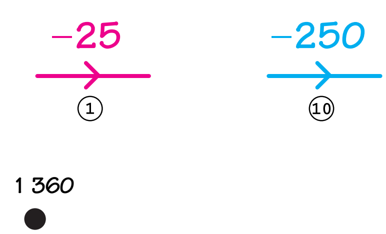
T: ***Nabu could use any of these arrows. He decides to use only –250 and –25 arrows.***

Write the information on the board.

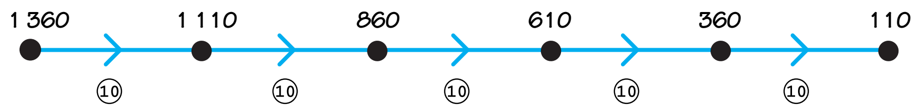
T: ***Let's help Nabu build an arrow road to solve the problem. How should we start?***

S: ***Use a –250 arrow. 1 360 – 250 = 1110.***

T: ***With ten cartons Nabu could pack only 250 bottles. He still would have 1 110 bottles to pack. What arrow should we draw next?***



Continue with blue arrows until the ending dot is less than 250. If a student suggests using a red arrow, explain that this would be correct but less efficient.



T: ***How many cartons has Nabu filled?***

S: ***50; there are five blue arrows. He still has 110 bottles to pack.***

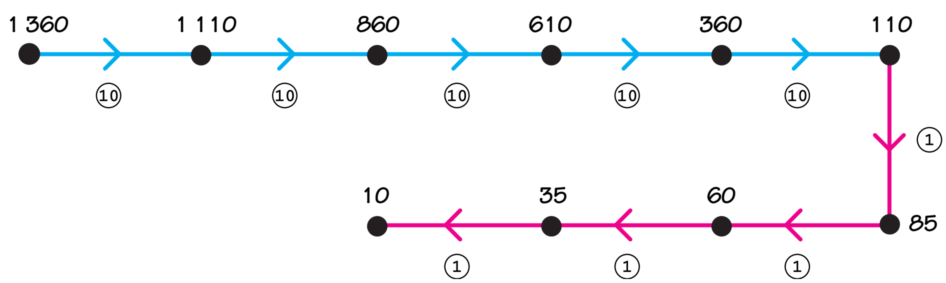
T: ***What arrows will Nabu use now?***

S: ***–25 arrows, since there are fewer than 250 bottles left to pack.***

T: ***How many red arrows will he need?***

S: ***Four, since 4 x 25 = 100.***

Let students complete the arrow road.



T: ***How many cartons does Nabu need?***

S: ***54; there are five blue arrows for ten cartons each, and four red arrows for one carton each.***

S: ***There are ten bottles left over.***

Record this calculation on the board. Compare the answer to students' estimates. If some students predicted 54, ask them to explain how they solved the problem.

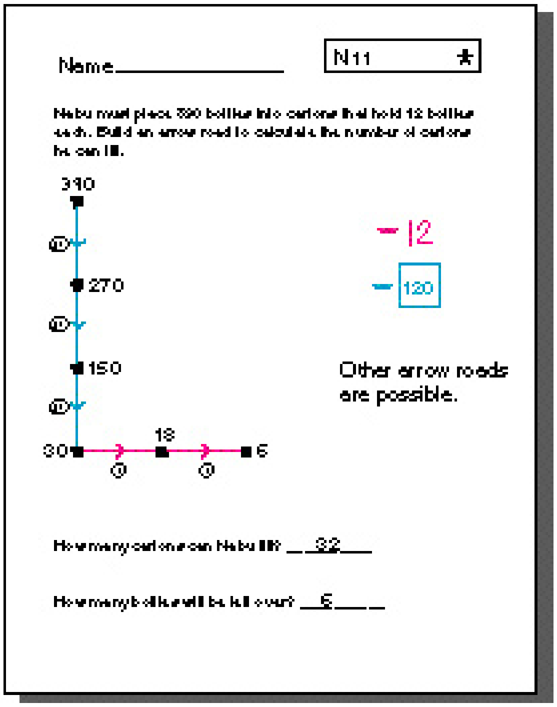
$$\begin{array}{r} 54 \text{ R} = 10 \\ 25\overline{)1\,360} \end{array}$$

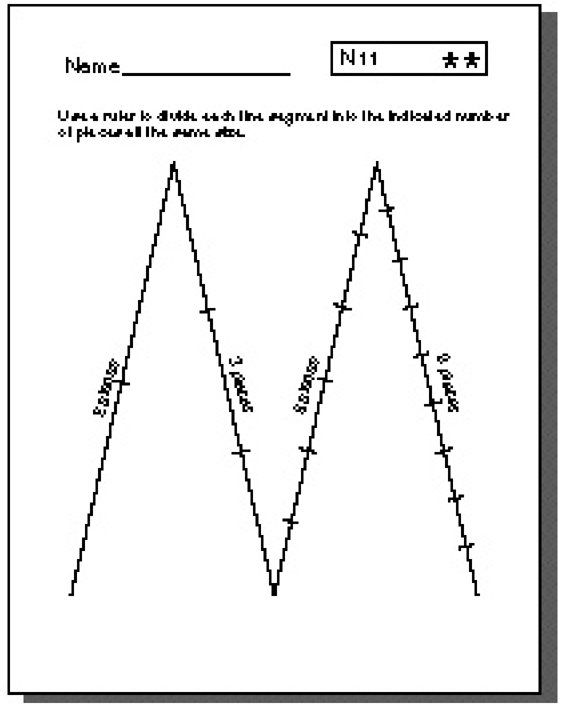
Worksheets N11*, **, ***, and **** are available for individual work. You may like to have a collective discussion of the problem on the * worksheet, letting several students with different arrow roads draw them on the board.

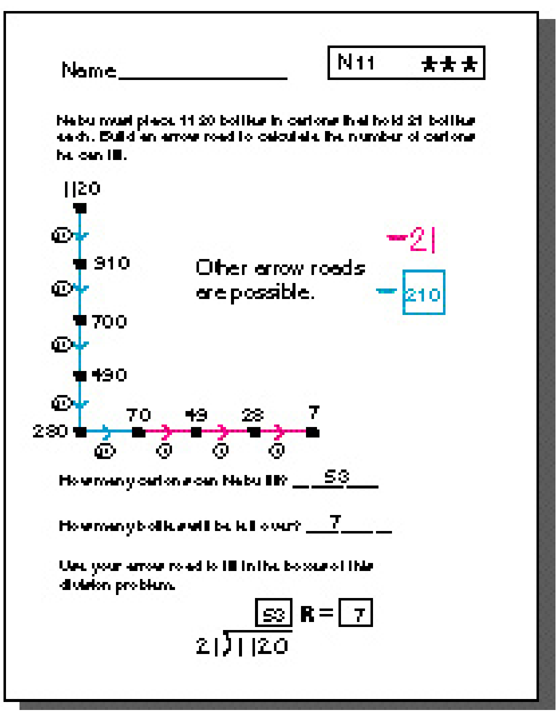
Home Activity

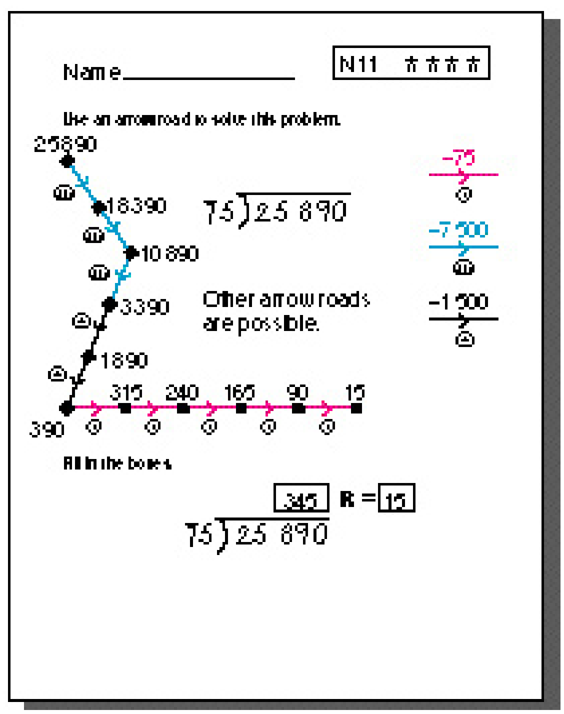
This would be a good time to send a letter to parents/guardians about division. A sample letter is available on Blackline N11.

N11


Name
N11 ★
390
270
150
30
18
6
−12
−120
Other arrow roads are possible.
32
6


Name
N11 ★★
3 pieces
6 pieces


Name
N11 ★★★
1120
910
700
490
280
70
49
28
7
−21
−210
Other arrow roads are possible.
53
7
53 R = 7
21)1120


Name
N11 ★★★★
Use an arrowroad to solve this problem.
25890
18390
10890
3390
1890
390
315
240
165
90
15
75)25 890
Other arrow roads are possible.
−75
−7 500
−1 500
Fill in the boxes.
345 R = 15
75)25 890

Capsule Lesson Summary

Find the two numbers that can be formed by adding one set of parentheses to an expression such as 7 + 3 x 6. Use a tree diagram to assist in finding all of the numbers that can be formed by adding parentheses to an expression such as 3 x 8 – 7 + 6.

Materials

Teacher
- Colored chalk

Student
- Colored pencils, pens, or crayons
- Paper
- Worksheets N12* and **

Description of Lesson

Exercise 1

Write this expression on the board.

$7 \times 3 \times 6$

T: ***Two students looked at this expression, did the calculations, and arrived at different results. They checked their work and found no errors. What could the two numbers have been?***

S: ***60; 7 + 3 = 10 and 10 x 6 = 60.***

S: ***25; 3 x 6 = 18 and 7 + 18 = 25.***

T: ***Yes, 25 and 60 were the two numbers the students came up with.***

You may at this time note that a calculator using chain operations (in which operations are done in the order they occur) would get 60, but a person (or calculator) using a priority order of operations† (in which multiplication is done before addition) would get 25. For this lesson, we will use parentheses and look at all possible orders.

$7 + 3 \times 6$

$\boxed{60,\ 25}$

Write another similar expression on the board.

$3 \times 8 - 7$

T: ***What numbers might the two students come up with for this expression?***

S: ***17. 3 x 8 = 24 and 24 – 7 = 17.***

S: ***3. 8 – 7 = 1 and 3 x 1 = 3.***

Note: In this case, both chain operations and priority order of operations give 17.

T: ***How are we getting two numbers for each of these expressions?***

S: ***It makes a difference which operation we do first.***

S: ***We did the operations in a different order.***

---

†The rule for order of operations is that all multiplications and divisions are performed before additions and subtractions. With just multiplications and divisions or just additions and subtractions, compute from left to right.

T: ***Let's look at doing the operations in different orders. In the first problem, to get 60 do we first add or multiply?***

S: ***We add first. 7 + 3 = 10 and 10*** x ***6 = 60.***

T: ***How do we get 25?***

S: ***Multiply first. 3*** x ***6 = 18 and 7 + 18 = 25.***

T: ***In the second problem, if we subtract first, is the number 17 or 3?***

S: ***3; 8 – 7 = 1 and 3*** x ***1 = 3.***

T: ***We can choose which operation to do first*** (refer to 7 + 3 x 6). ***Sometimes we might want to add first; other times we might want to multiply first. What symbols can we put to this expression to indicate which operation we want to do first?***

S: ***Parentheses.***

T: ***Who can add parentheses to make the answer to the first problem 25? Why are we now sure that this expression equals 25, not 60?***

$$7 \times (3 \times 6) = 25$$

S: ***The parentheses tell us to multiply first.***

Similarly, ask students to put parentheses in the expressions 7+ 3 x 6 and 3 x 8 – 7 to show names for 60, 17, and 3.

$$(7 \times 3) \times 6 = 60$$
$$(3 \times 8) - 7 = 17$$
$$3 \times (8 - 7) = 3$$

Continue by asking the class to add parentheses to each of the following expressions and to give the resulting numbers.

$8 - 3 + 9$	$7 \times 8 - 2$	$6 \div 2 + 10$
$(8 - 3) + 9 = 14$	$(7 \times 8) - 2 = 54$	$(6 \div 2) + 10 = 13$
$8 - (3 + 9) = \widehat{4}$	$7 \times (8 - 2) = 42$	$6 \div (2 + 10) = 0.5$

Erase the board before going on to Exercise 2.

Exercise 2

Write this expression on the board.

$$3 \times 8 - 7 \times 6$$

T: ***A group of students came up with many different numbers for this expression. What could those numbers be?***

There are five possible answers: $\widehat{15}$, 9, 11, 21, and 23. Ask students to explain how to get each possibility. For example:

S: ***9; 8 – 7 = 1, 3*** x ***1 = 3, and 3 + 6 = 9.***

T: ***You chose first to subtract, then to multiply, and then to add. Can you put parentheses in the expression to indicate that you want to subtract first?***

$$3 \times (8 - 7) \times 6$$

T: ***After subtracting, you chose to multiply next. Can you draw another set of parentheses to indicate that you want to multiply next?***

Give the students red chalk for this second set of parentheses. This may be difficult; be ready to give hints. If necessary, draw the parentheses yourself.

T: ***Let's check that this does equal 9. The parentheses tell us to first subtract. 8 – 7= 1.***

The red parentheses now tell us to multiply. 3 x 1 = 3.

Now we add. 3 + 6 = 9.

$$(3 \times (8 - 7)) + 6$$
$$(3 \times 1) + 6$$
$$3 \quad \times 6$$
$$9$$

Review other answers in a similar manner. Emphasize that the student chooses to do the three operations (x, –, +) in a particular order and that the parentheses record this order. Do not expect students to find all of the solutions, which are listed below.

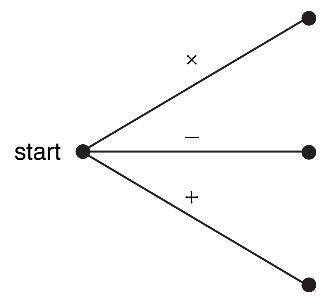
$$(3 \times (8 - 7)) + 6 = 9 \qquad 3 \times ((8 - 7) + 6) = 21$$
$$((3 \times 8) - 7) + 6 = 23 \qquad 3 \times (8 - (7 + 6)) = \widehat{15}$$
$$(3 \times 8) - (7 + 6) = 11 \qquad (3 \times 8) - (7 + 6) = 11$$

Regardless of how many solutions students have suggested, continue with the following counting procedure.

T: ***To find all of the possible numbers that this expression could equal, we'll draw a tree diagram. How many choices do we have for the first operation?***

S: ***Three:*** x***, –, or +.***

Begin drawing a tree diagram.

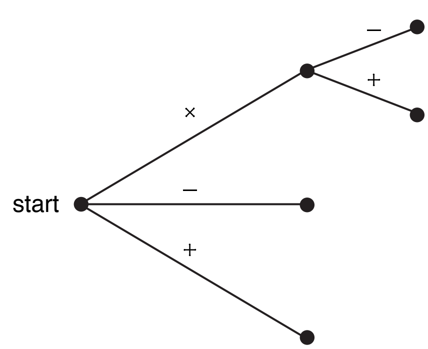


Trace the branch for x as you ask,

T: ***If we choose to multiply first, what could the second operation be?***

S: ***– or +. Draw two branches from that end point.***

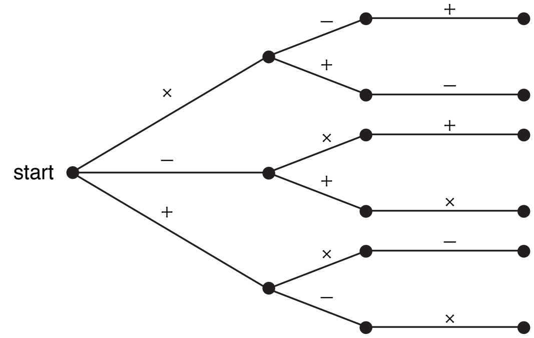
Trace the uppermost path, x followed by –, and ask,



T: ***What is the third operation here?***

S: ***+, since we already used*** x ***and –. Draw just one branch there.***

Follow student's suggestions to complete the tree diagram.



T: ***In how many different ways can we perform the three operations?***

S: ***Six ways; there are six paths in the tree diagram.***

S: ***Six ways; there are six end points on the right side of the tree diagram.***

T: ***Let's match the answers we found to paths in this tree diagram.***

Select one of the answers found by students, for example:

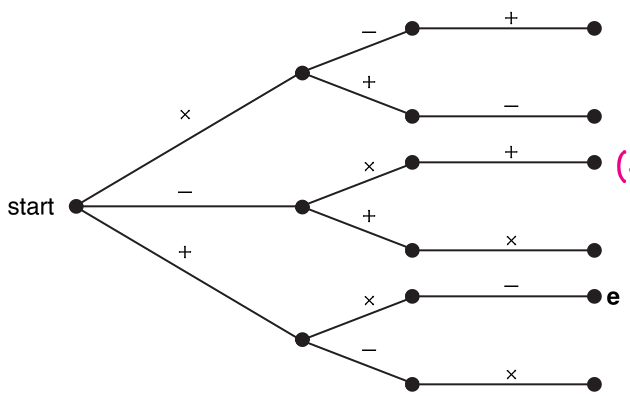
$$(3 \times (8 - 7)) + 6 = 9$$

T: ***In what order did we do the operations here?***

S: ***Subtraction first, multiplication second, and addition third.***

T: ***Which path in the tree corresponds to this order of operations?***

Let a student trace the appropriate path and write the number sentence at the end. For example:



$$(3 \times (8 - 7)) + 6 = 9$$

Do not write this letter on the board. It is here just to make the description of the lesson easier to follow.

In a similar manner, consider any other solutions students found.

It is unlikely that all six possibilities will be found. Select any unlabeled path and determine the corresponding number sentence. A sample dialogue follows.

T: ***Let's determine which number sentence corresponds to this path (e) in the tree diagram. Which operation is first?***

S: ***Addition.***

Invite a student to draw the first set of parentheses.

$$3 \times 8 - (7 \times 6)$$

T: ***According to this path, which operation is second?***

S: ***Multiplication.***

Give a student red chalk to draw the second set of parentheses. If this is difficult, give hints or add the parentheses yourself.

$$(3 \times 8) - (7 + 6)$$
$$(3 \times 8) - 13$$
$$24 - 13$$
$$11$$

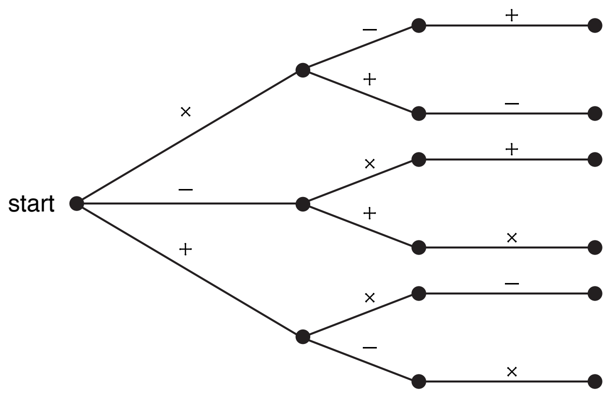
T: ***What number is this?***

S: ***11. 7 + 6 = 13 … and 3 x 8 = 24.***

24 – 13 = 11.

You may need to emphasize that the last subtraction is done left to right (24 – 13 = 11), not right to left (13 – 24 = $\widehat{11}$), even though 13 was calculated first.

In a similar manner, write a number sentence at the end of every path in the tree diagram.



$((3 \times 8) - 7) + 6 = 23$

$(3 \times 8) - (7 + 6) = 11$

$(3 \times (8 - 7)) + 6 = 9$

$3 \times ((8 - 7) + 6) = 21$

$(3 \times 8) - (7 + 6) = 11$

$3 \times (8 - (7 + 6)) = \widehat{15}$

T: ***According to the tree diagram, there are six ways to add two sets of parentheses to 3 x 8 – 7 + 6. How many different numbers are formed?***

S: ***Only five; 11 is repeated.***

T: ***Our tree diagram helped us find all of the possible numbers the expression could equal.***

Exercise 3

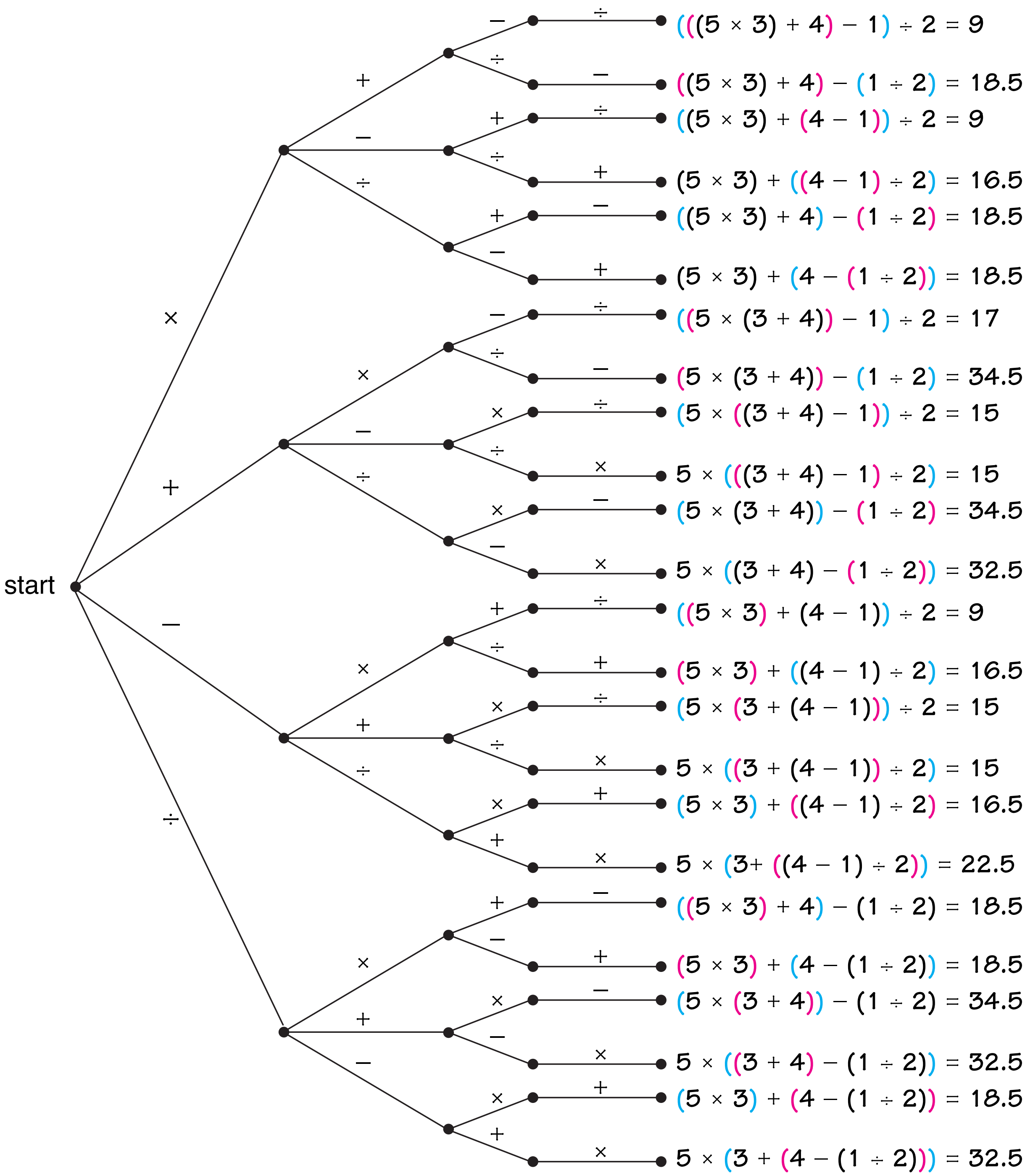
Repeat Exercise 1 for an expression in which there are four operations. First, ask students to predict how many possible numbers could be given by the expression when parentheses are put in.

In this case, you may want to start on the problem as a class and then let students work in groups to continue to find all the possibilities.

N12

An example expression and complete tree is given below.

$5 \times 3 + 4 - 1 \div 2$



Note: This would be a good time to observe how cumbersome it is to use several sets of parentheses, and to discuss how most people agree to use a rule for order of operations. See the footnote on page N-69.

Worksheets N12* and ** are available for individual work.

Name ____________ N 12 ☆

Add one pair of parentheses to make each number sentence true.

5 + (4 × 7) = 33 (5 + 4) × 7 = 63

9 − (4 + 8) = 3 (9 − 4) + 8 = 13

(11 − 8) ÷ 2 = 1.5 11 − (8 ÷ 2) = 7

Complete.

((4 × 6) − 3) + 5 = 26 4 × (6 − (3 + 5)) = 8

(4 × 6) − (3 + 5) = 16 (4 × 6) − (3 + 5) = 16

(4 × (6 − 3)) + 5 = 17 4 × ((6 − 3) + 5) = 32

Name ____________ N 12 ☆☆

Add one pair of parentheses to make each number sentence true.

((3 × 6) + 4) × 4 = 88

(3 × 6) + (4 × 4) = 34

3 × ((6 + 4) × 4) = 120

Another solution is (3 × (6 + 4)) × 4 = 120

3 × (6 + (4 × 4)) = 66

Add two pairs of parentheses to make each number sentence true.

(2 × (9 + 3)) ÷ 10 = 2.4

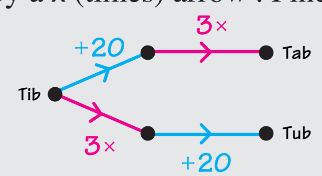
(2 × 9) + (3 ÷ 10) = 18.3

2 × (9 + (3 ÷ 10)) = 18.6

((2 × 9) + 3) ÷ 10 = 2.1

Capsule Lesson Summary

Label dots and arrows in an arrow road. When an arrow cannot be x some whole number, use a composition detour of a ÷ (divide by) arrow followed by a x (times) arrow . Find composites for some pairs of arrows in the road. In this arrow picture, establish that the difference between the numbers Tab and Tub will always be 40. Knowing the greatest number in another such picture, label the dots.



Materials

Teacher	• Colored chalk	**Student**	• Paper • Colored pencils, pens, or crayons • Worksheets N13*, **, ***, and ****

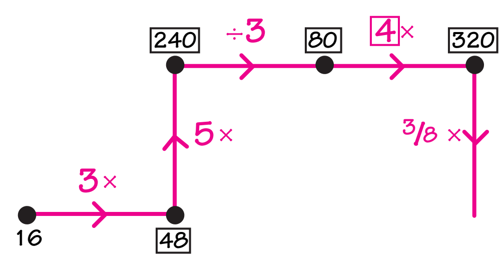
Description of Lesson

Begin the lesson with some mental arithmetic involving multiplication and division. Suggested problems are given below. (Answers are in boxes.)

4 x 8 = $\boxed{32}$	7 x 6 = $\boxed{42}$	6 x 6 = $\boxed{36}$
32 ÷ 8 = $\boxed{4}$	42 ÷ 6 = $\boxed{7}$	36 ÷ 6 = $\boxed{6}$
32 ÷ 4 = $\boxed{8}$	42 ÷ 7 = $\boxed{6}$	
4 x 9 = $\boxed{36}$	360 ÷ 4 = $\boxed{90}$	36 000 ÷ 40 = $\boxed{900}$
36 ÷ 9 = $\boxed{4}$	3 600 ÷ 4 = $\boxed{900}$	36 000 ÷ 400 = $\boxed{90}$
36 ÷ 4 = $\boxed{9}$	36 000 ÷ 4 = $\boxed{9 000}$	36 000 ÷ 4 000 = $\boxed{9}$

Exercise 1

With student input on labeling dots and arrows, draw this arrow road on the board. (Boxes indicate where students should provide labels.)



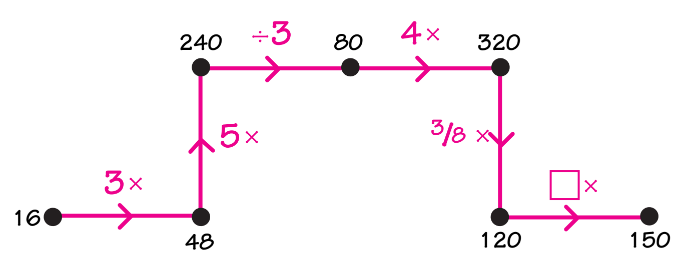
T: ***How do we calculate three-eighths of a number?***

S: ***Divide by 8 and then multiply by 3, or multiply by 3 and then divide by 8.***

T: ***What number is*** $^3/_8$ ***x 320?***

S: ***320 ÷ 8 = 40 and 3 x 40 = 120.***

Draw and label a dot for 120. Then extend the arrow road with an arrow ending at 150.

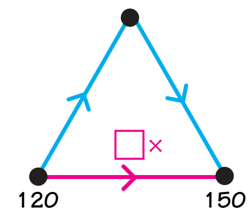


T: ***We need to label this multiplication arrow. Could the number in the box be a whole number?***

S: *No. 1 x 120 = 120 and 2 x 120 = 240, so the number in the box is between 1 and 2.*

S: *The number in the box could be a fraction or a decimal.*

T: *Let's try a detour. For a fraction times, we can draw a detour with one arrow for "divide by some number" and the other arrow for "times some number."*



There are infinitely many possibilities, so the following is only a sample dialogue.

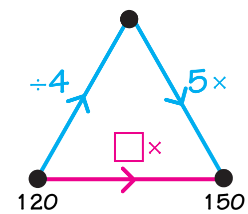
S: *We could use ÷4 for the first arrow in the detour. 120 ÷ 4 = 30.*

T: *Now we need to go from 30 to 150 with a x arrow.*

S: *5x.*

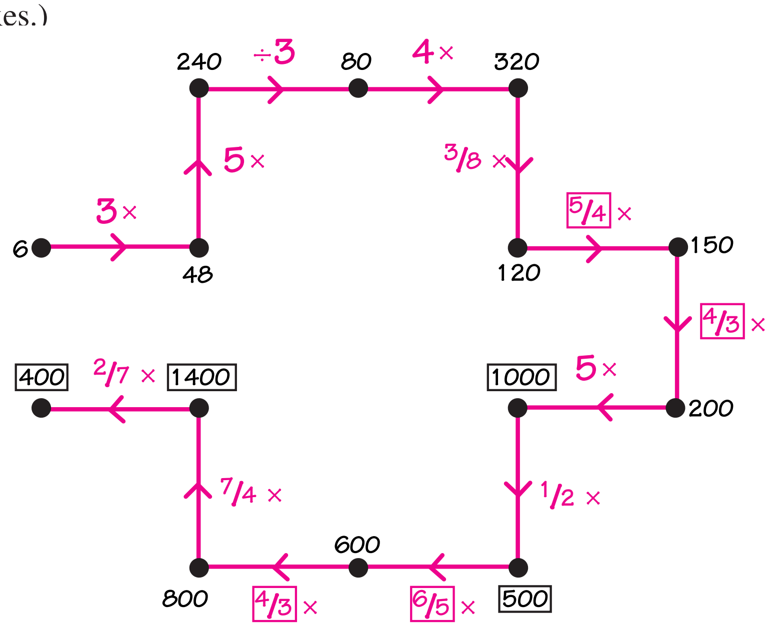
T: *What number (fraction) could go in the box? What is ÷4 followed by 5x?*

S: *$\frac{5}{4}$x.*

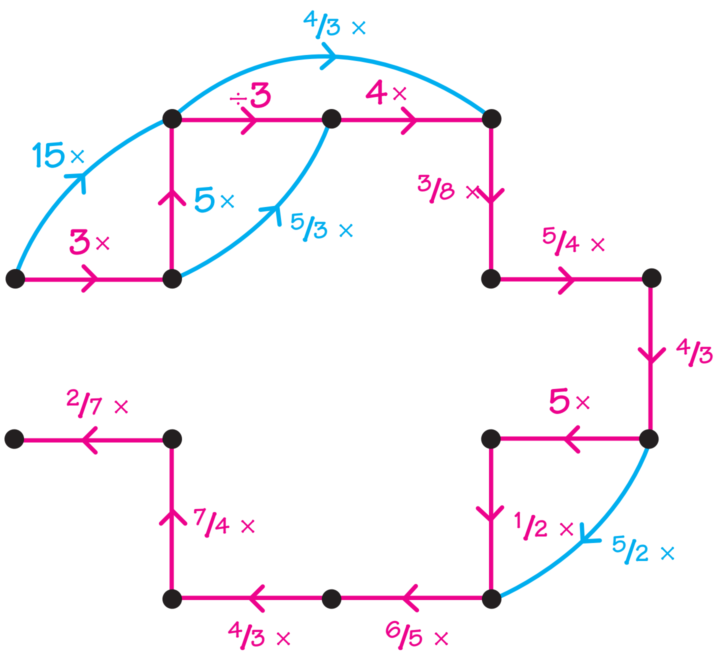


Note: Any fraction equivalent to $\frac{5}{4}$ could go in the box as well as the decimal 1.25. Keep in mind that at any time several useful detours may be suggested, and whenever this occurs note that the composite functions are equal. In this case, the following are all the same function: $\frac{5}{4}$x, $\frac{15}{12}$x, $\frac{50}{40}$x, and so on.

Continue extending the arrow road, one arrow at a time. A possible arrow picture is shown below. (Answers are in boxes.)

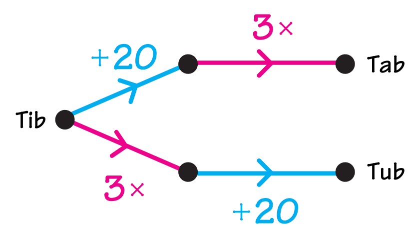


Erase the labels for all of the dots and any detour arrows in the arrow picture. Do not erase the red arrow labels. Draw several composition arrows in the picture and ask students to label them. The blue arrows shown below are examples.



Exercise 2

Draw this arrow picture on the board.



T: ***Which of these two numbers is greater, Tab or Tub? Why?***

S: ***Tab is greater. If Tib is 5, then Tab is 75 (because 5 + 20 = 25 and 3* x *25 = 75) and Tub is 35 (because 3* x *5 = 15 and 15 + 20 = 35.)***

Record this information (in a table) on the board.

Tib	Tab	Tub
5	75	35
2	66	26
11	93	53
3.2	69.6	29.6
8.5	85.5	45.5
$\widehat{40}$	$\widehat{60}$	$\widehat{100}$

Select several more starting numbers. Each time, label the dot for Tib and ask students to label the other dots. Here are some suggested starting numbers for Tib with the corresponding ending numbers (Tab and Tub). Feel free to adjust the starting numbers to the abilities of your students. Encourage students to make observations about the relative sizes of the numbers Tab and Tub.

Note: If students have difficulty calculating 3 x $\widehat{40}$, remind them that 3 x $\widehat{40} = \widehat{40} + \widehat{40} + \widehat{40}$.

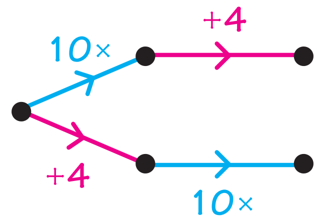
T: ***Each time, Tab is 40 more than Tub. Do you think that Tab is always 40 more than Tub?*** (Yes) ***Why?***

S: ***In the upper part of the picture, 20 is added to Tib and then the sum is tripled, so the 20 gets tripled also. In the lower part of the picture, Tib is tripled and then 20 is added to the result. So Tub is always 40 less than Tab.***

N13

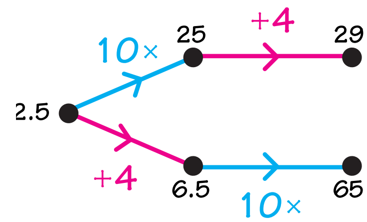
Exercise 3

Draw this arrow picture on the board.



T: ***65 is the greatest number in this picture. Copy the picture on your paper and circle the dot for 65. Then label all the dots.***

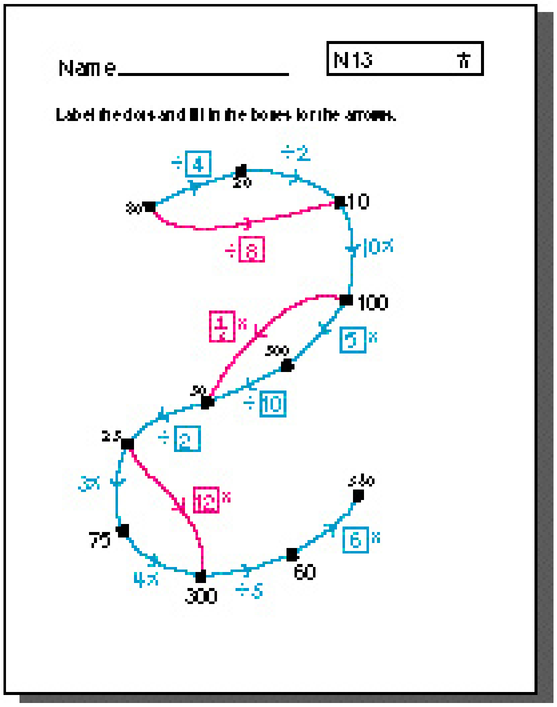
When many students have located 65 and labeled dots, invite a student to label the dot for 65 in the picture on the board. Then call on other students to complete the picture.

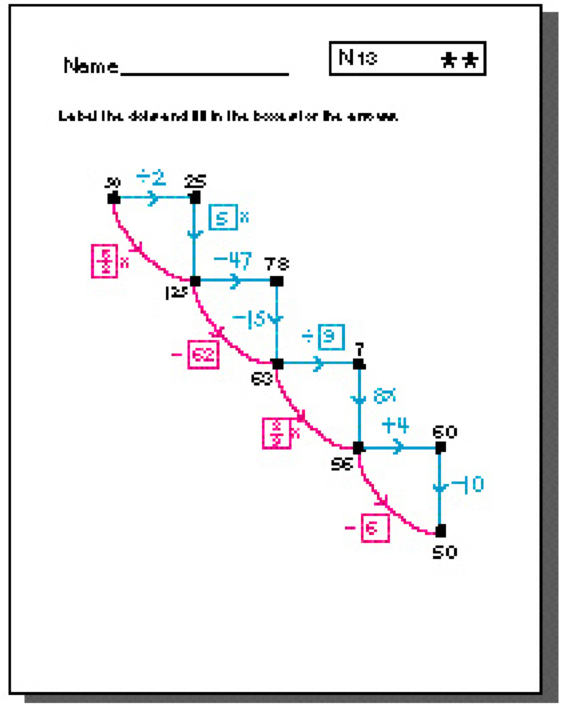


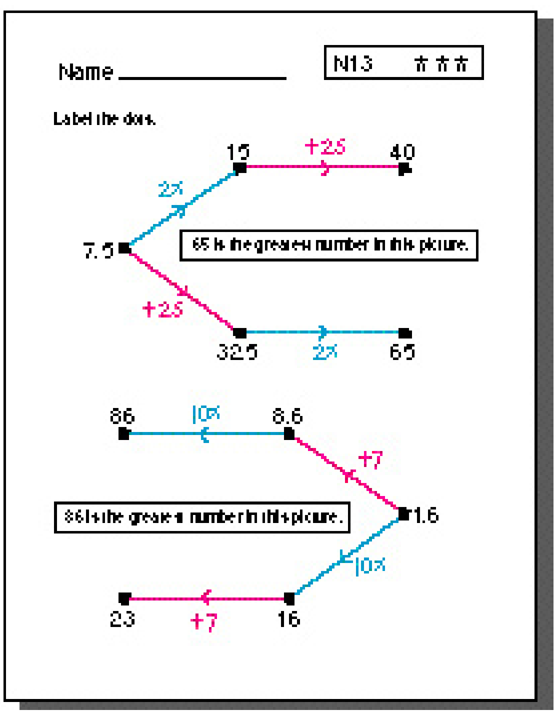
Worksheets N13*, **, ***, and **** are available for individual work.

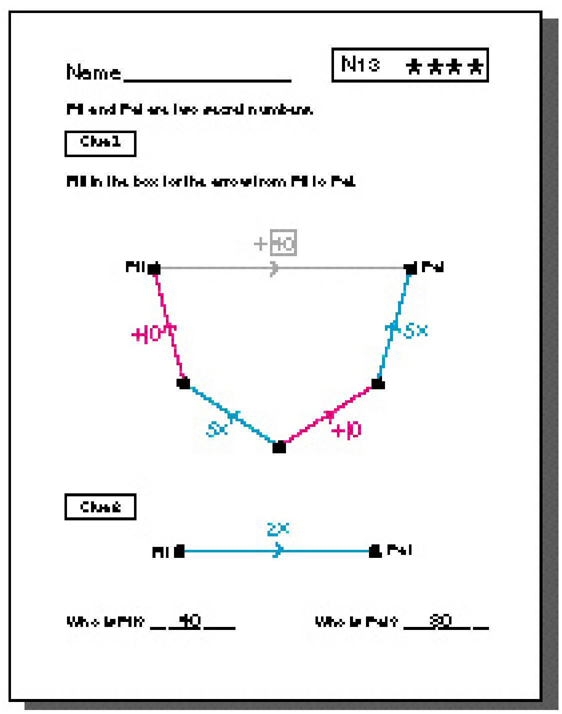
Writing Activity

Suggest that students write a problem similar to those on Worksheet N13*** to challenge a classmate or to take home and challenge a family member.


Name
N13 ☆
Label the dots and fill in the boxes for the arrows.
÷4
÷2
10
÷8
10×
100
5×
÷10
÷2
3×
12×
75
4×
300
÷5
60
6×


Name
N13 ★★
Label the dots and fill in the boxes for the arrows.
÷2
25
5×
−47
78
−15
÷9
7
−62
63
8×
+4
60
56
−10
−6
50


Name
N13 ☆☆☆
Label the dots.
15
+25
40
2×
7.5
65 is the greatest number in this picture.
+25
32.5
2×
65
86
10×
8.6
+7
1.6
86 is the greatest number in this picture.
10×
23
+7
16


Name
N13 ★★★★
Clue 1
Clue 2
2×
Who is Pil? 40
Who is Pal? 80

Capsule Lesson Summary

Extend an arrow picture with arrows for ⊞ 0.2 ⊟ … . Determine the weights (number of times ⊟ is pressed) of the arrows and their composites in the picture. Extend arrow pictures with arrows for ⊞ ⊟ … and label the arrows with equivalent multiplication functions. These functions, of course, are dependent on the choice of dot labels.

Materials

Teacher
- Colored chalk

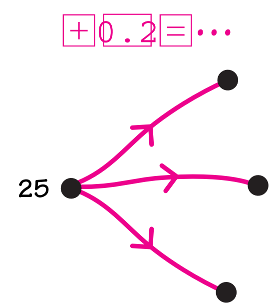
Student
- Calculator
- Paper
- Colored pencils, pens, or crayons
- Worksheets N14*, **, ***, and ****

Description of Lesson

This lesson assumes calculators have an automatic constant feature for addition. See the section on "Role and Use of Calculators" in Notes to the Teacher.

Exercise 1

Draw this arrow picture on the board.



T: ***Who can explain what the red arrows are for?***

S: ***Start with 25; then press ⊞ 0.2 ⊟*** [†] ***and keep pressing ⊟. Any number that appears can be at the ending dot of one of these red arrows.***

T: ***Put 25 on your calculator and press ⊞ 0.2 and ⊟ … as many times as you like. What is a number that could be here*** (point to an unlabeled dot)***?***

Many responses are possible. The following is a sample dialogue.

S: ***26.8.***

T: ***How many times do we press ⊟ to go from 25 to 26.8? Why?***

S: ***9 times, because 9 x 0.2 = 1.8, and 25 + 1.8 = 26.8.***

T: ***What number could be here*** (point to an unlabeled dot)***?***

S: ***31.***

T: ***How many times do we press ⊟ to go from 25 to 31.***

S: ***30 times, because pressing ⊟ five times adds 1 to the number and so pressing ⊟ 30 times (30 = 5 x 6) adds 6 to the number.***

[†] "Press ⊞ 0.2 ⊟" is an abbreviation for "press ⊞ ⊡ 2 ⊟." We will use this abbreviation in this and succeeding lessons on calculator relations with decimals.

T: ***What number could be here*** (point to the unlabeled dot)***?***

S: ***55.4***

T: ***How many times do we press*** $\boxed{=}$ ***to go from 25 to 55.4?***

S: ***152 times.***

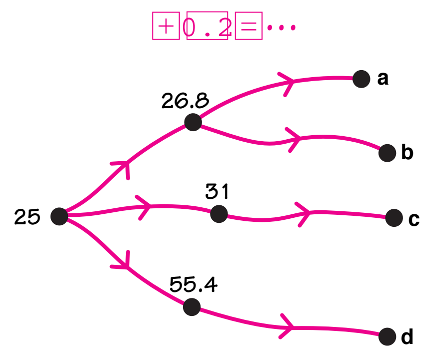
For such an ending number this question is difficult. You may ask,

T: ***How much more is 55 than 25?*** (30) ***How many times do we press*** $\boxed{=}$ ***to add 30?***

S: ***150 times; because pressing*** $\boxed{=}$ ***five times adds 1, so pressing*** $\boxed{=}$ ***150 times (150 = 5*** x ***30) adds 30 to the number.***

T: ***How many times do we press*** $\boxed{=}$ ***to go from 55 to 55.4?*** (Two times)
So how many times do we press $\boxed{=}$ ***to go from 25 to 55.4?*** (152 times)

Add these arrows to the drawing.

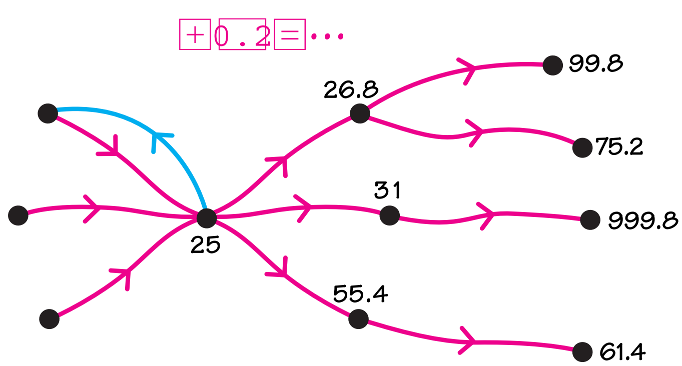


Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

Depending upon which numbers are already in the picture, make adjustments to the following questions and label dots accordingly.

T: ***What is the greatest number less than 200 that could be here*** (at **a**)***?*** (99.8)
What is the least number more than 75 that could be here (at **b**)***?*** (75.2)
What is the greatest number less than 1 000 that could be here (at **c**)***?*** (999.8)
What is the least number more than 61.3 that could be here (at **d**)***?*** (61.4)

Add these red arrows ending at 25 and a return blue arrow to the picture.



T (tracing the blue arrow)**:** ***What could this blue arrow be for?***

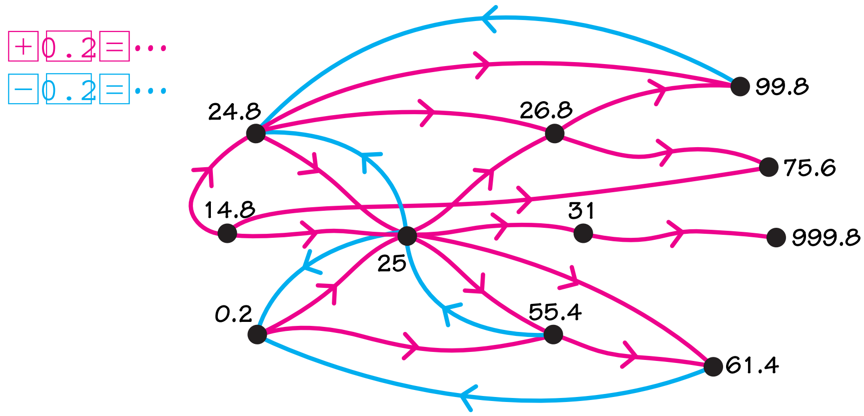
S: $\boxed{-}$ $\boxed{0.2}$ $\boxed{=}$ **. . . .**

Write a key for the blue arrow near the arrow picture.

T (pointing to the upper left dot)**:** ***What number could be here?***

Label the dot with any correct student response, for example, 24.8, 20.0, 9.6, ⁻3.8, and so on. Ask the class to label the remaining two dots with the greatest number possible less than 15 (14.8) and the least possible positive number (0.2).

Invite students to draw more red and blue arrows. Some of the many possible arrows that can be drawn in the picture are shown below.



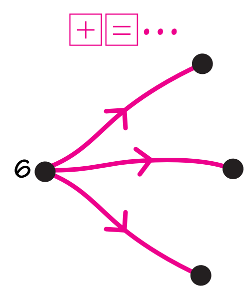
Choose a few of the newly drawn arrows and ask how many times you press $\boxed{=}$ to go from the starting number to the ending number. For example:

T: ***How many times must we press $\boxed{=}$ to go from 14.8 to 75.6?***

S: ***304 times; one time to go to 15, 300 times to go to 75, and three more times to 75.6.***

Exercise 2

Draw this picture on the board.



T: ***The red arrows are for $\boxed{+}$ $\boxed{=}$ …, $\boxed{=}$ as many times as you like. In this picture we start with 6 on our calculators and press $\boxed{+}$ $\boxed{=}$ $\boxed{=}$ and so on. What are some numbers that could be at these ending dots?***

Encourage students to explore, using their calculators. The discussion might develop along these lines. Label the dots with numbers suggested by your students.

S: ***48.***

T: ***Could we label one of the dots with a number less than 48?***

S: ***12.***

T: ***Could we label a dot with a number greater than 48?***

S: ***1 038.***

T: ***What do you notice about these numbers?***

S: ***They are all even.***

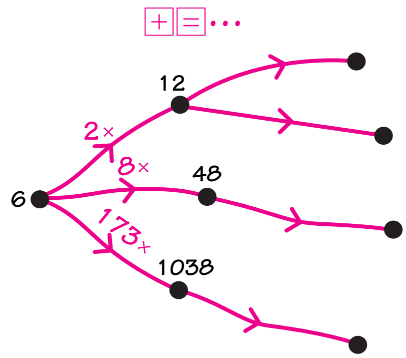
S: ***They are all multiples of 6.***

T: ***If they are all multiples of 6, then we can label each arrow some number times. How should we label this arrow?***

N14

Label each arrow x some number. Allow students to use calculators.

Add these arrows to the picture.



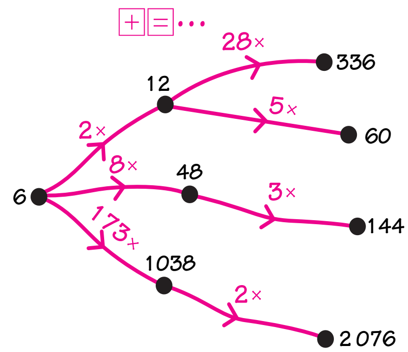
T: ***Put 12 on your calculator and press* ⊞ ⊟ … . *What number could go here*** (point to the ending dot of an arrow starting at 12)***?***

S: ***60.***

T: ***Could we label this arrow*** x ***some number?***

S: ***5*x*, because 5*** x ***12 = 60.***

Continue in this manner, labeling dots and the corresponding arrows. After a while, your picture might look like this.



T: ***Can we draw some more red arrows in the picture?***

Invite several students to draw arrows.

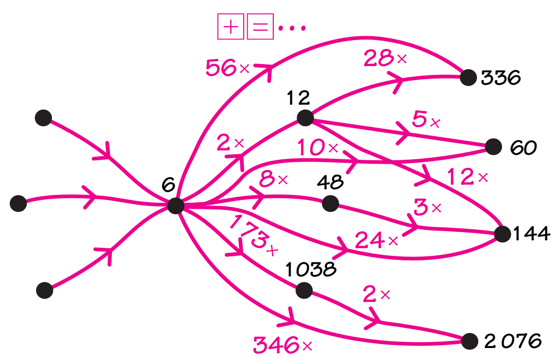
For each additional arrow, ask the class to label it some number times. For example:

T (tracing an arrow starting at 6 and ending at 336)**:** ***Could you label this arrow*** x ***some number?***

S: ***2*x *followed by 28*x *is 56*x.**

S: ***56*x*; 336 ÷ 6 = 56.***

Add these arrows ending at 6 to the picture.



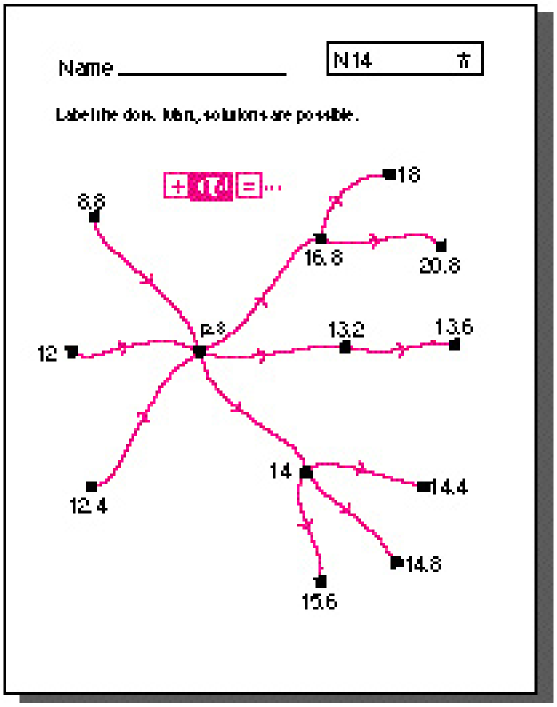
T (pointing to the new unlabeled dots)**:** ***What numbers could be here?***

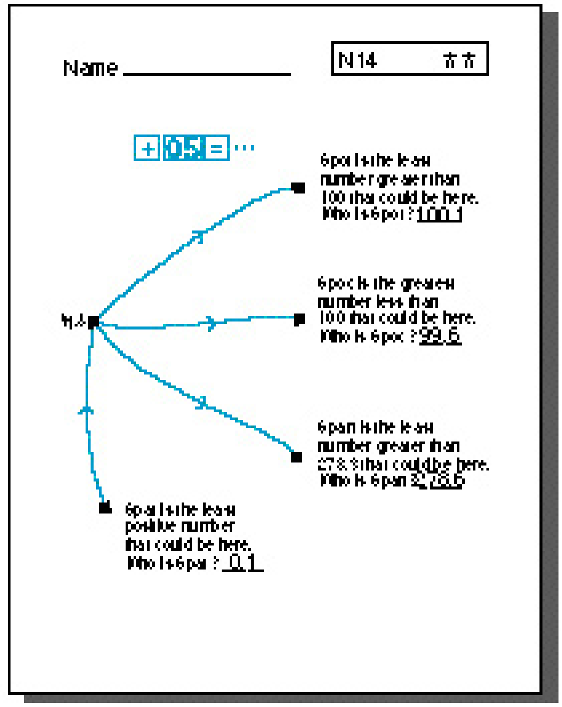
The first suggestions are likely to be 1, 2, and 3. These numbers are correct, but encourage students to think of some (non-integer) decimal numbers, for example, 1.5, 0.6, 0.01, and 0.2. Label the dots with students' suggestions and label the arrows with the class's help.

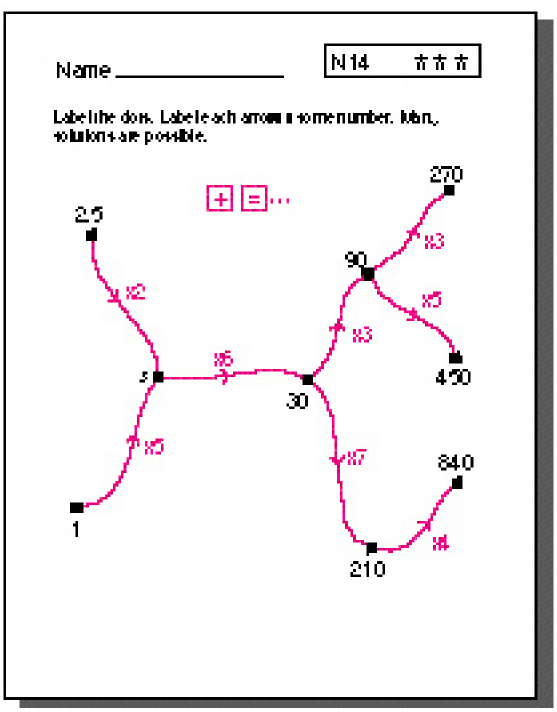
Worksheets N14*, **, ***, and **** are available for individual work.

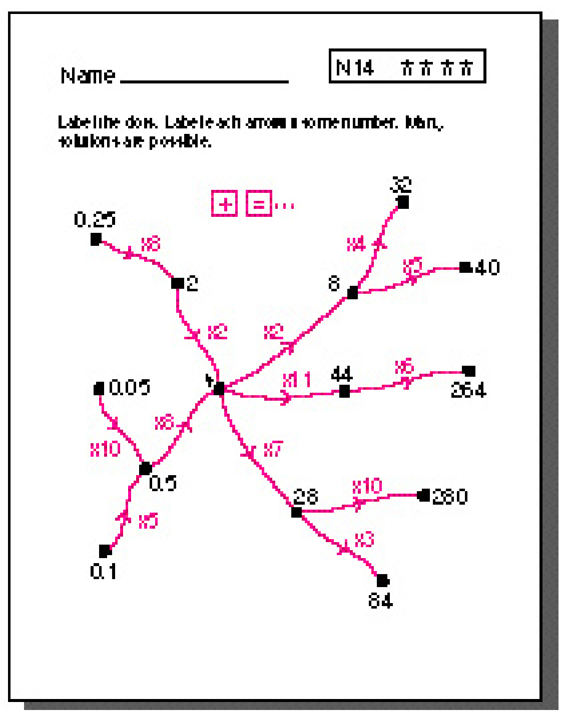
Home Activity

This would be a good time to send a letter to parents/guardians about the use of calculators. Blackline N14 has a sample letter.


Name
N14 ☆
Label the dots. Many solutions are possible.
+ 1.6 = ...
8.8
18
16.8
20.8
12
13.2
13.6
12.4
14
14.4
14.8
15.6


Name
N14 ☆☆
+ 0.5 = ...
Spot is the least number greater than 100 that could be here. Who is Spot? 100.1
Spoc is the greatest number less than 100 that could be here. Who is Spoc? 99.6
Span is the least number greater than 278.3 that could be here. Who is Span? 278.6
Spat is the least positive number that could be here. Who is Spat? 0.1


Name
N14 ☆☆☆
Label the dots. Label each arrow a some number. Many solutions are possible.
+ = ...
2.5
270
x2
90
x3
x5
x6
x3
30
450
x5
x7
840
1
210
x4


Name
N14 ☆☆☆☆
Label the dots. Label each arrow a some number. Many solutions are possible.
+ = ...
0.25
32
x8
x4
2
8
x5
40
x2
x2
0.05
x11
44
x6
264
x6
x10
x7
0.5
28
x10
280
x5
x3
0.1
84

N15 SPIES AND BRIDGES #1

Capsule Lesson Summary

Put numbers on an abacus with the rule that three checkers on a board represent the same number as one checker on the next board to the left. Relate configurations of checkers on this abacus to secret numerical messages in a story about six spies watching three bridges. Using this base three abacus, determine how many messages can be written in code 3.

Materials

Teacher
- Magnetic checkers
- Colored chalk
- Colored pencils, pens, or crayons
- Worksheets N15 (no star), *, **, and ***

Student
- Paper

Description of Lesson

Exercise 1

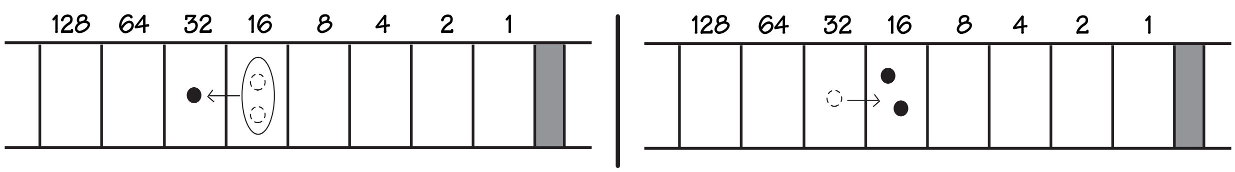
Draw part of a binary abacus on the chalkboard. Call on students to tell you how to label the boards.

T: ***Do you remember the rule for trading checkers on the binary (or base two) abacus?***

S: ***Two checkers on a board represent the same number as one checker on the next board to the left.***

Demonstrate a trade on the abacus, and then reverse the trade.

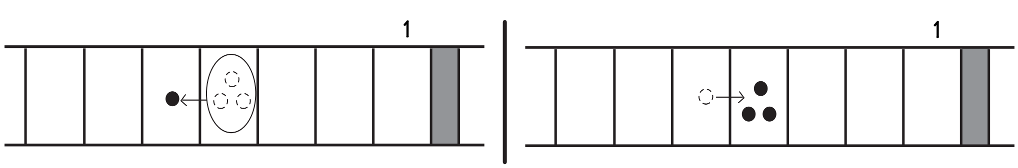
Note: If your chalkboard is not magnetic, draw and erase the checkers on the abacus.



Erase all of the board labels. Place three checkers on any board to the left of the bar.

T: ***Today we are going to use a new abacus with a new rule for making trades. A checker on this board*** (point to the ones board) ***is still for one*** (label the board)***; but three checkers on a board trade for one checker on the next board to the left.***

Demonstrate a trade on this abacus, and reverse the trade.

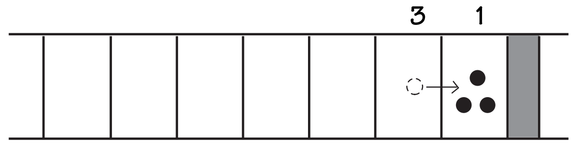


Place one checker on the board to the left on the ones board.

T: ***What number is this?***

S: ***3. We can trade that checker for three checkers on the ones board, which is 3.***

Demonstrate the trade and label the board.

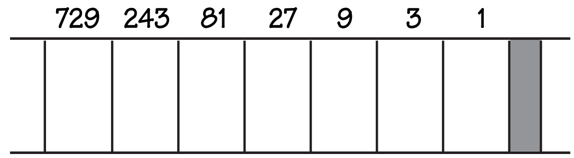


Place a checker on the board to the left of the threes board.

T: ***What number is this?***

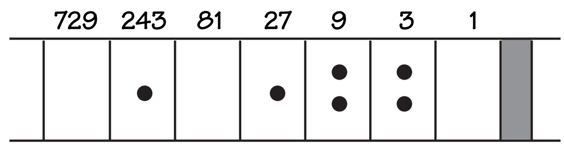
S: ***9. It is the same as three checkers on the threes board. 3*** x ***3 = 9.***

Label the board and continue until six or seven boards to the left of the bar are labeled.



T: ***Because the rule is trade three checkers for one checker, we call this a*** **base three** ***abacus.***

Put this configuration on the abacus.

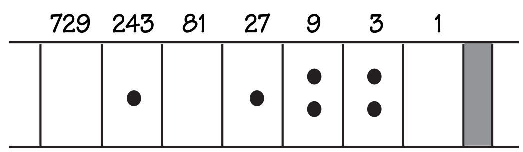


T: ***What number is this? Write the answer on your paper.***

After checking several answers, call on a student for the number.

S: ***34. 27 + 3 + 3 + 1 = 34.***

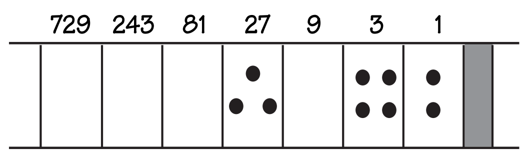
Similarly, let students decode the following configurations. (Answers are in boxes.)



= $\boxed{294}$

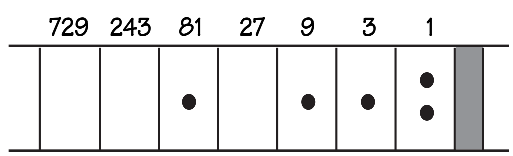
T: ***How did you find that this number is 95?***

S: ***I added the numbers shown by the checkers.***



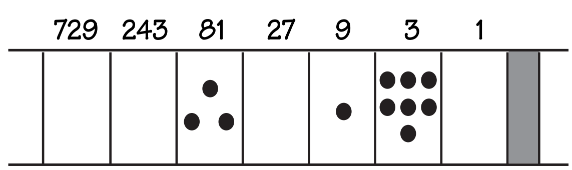
= $\boxed{95}$

Invite students to make trades, and check that the simplified configuration is 95.

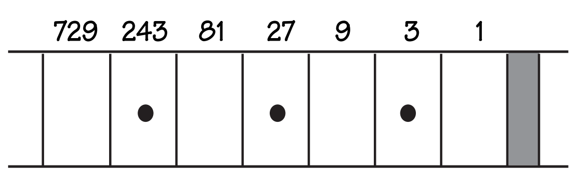


= 95

Repeat the activity with this configuration.



Encourage students to calculate the number on the abacus by first making trades. After several trades, this configuration results.

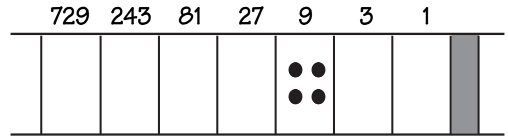


T: ***What number is this?*** (273)
To put a number on this abacus, do we ever need more than two checkers on a board?

S: ***No. If there are more than two checkers on a board, we can make trades.***

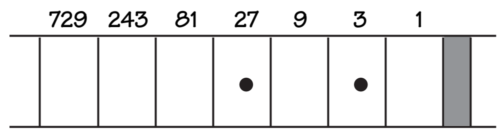
To emphasize this idea, put four checkers on a board and observe the following:

T: ***We could show 36 this way since 4*** x ***9 = 36. But do we need four checkers on one board?***

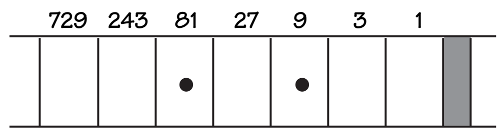


S: ***No, trade three of the checkers for one checker on the next board to the left. 27 + 9 = 36.***

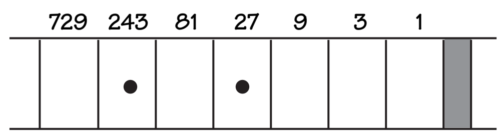
Invite students, one at a time, to put 30, 90, and 270 on the abacus. If a number is put on with three or more checkers on a board, ask for trades to get a simpler configuration.



= 30



= 90



= 270

T: ***Do you notice patterns when we put 30, 90, and 270 on the abacus?***

S: ***Each time there is one blank board between the two checkers.***

S: ***The checkers move one board to the left each time.***

T: ***Why does this happen? What is special about these numbers?***

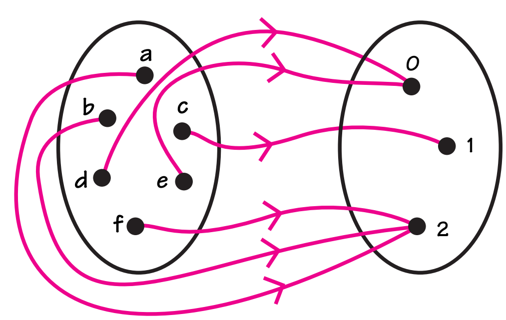
S: ***We multiply by 3 each time. 3*** x ***30 = 90 and 3*** x ***90 = 270. On a base three abacus, moving the checkers one board to the left multiplies by 3.***

Exercise 2

Draw the strings from the following picture on the board as you tell this story.

T: ***This is a story about a spy, Boris, and his six assistants whose code names are* a*,* b*,* c*,* d*,* e*, and* f*. Each day Boris assigns his six assistants to guard three bridges. He calls the bridges 0, 1, and 2. Depending on enemy activity, Boris might assign his six spies to observe all three bridges, or maybe only two bridges, or he might even assign all the spies to observe one bridge. That's his choice each day.***

Boris uses an arrow picture to show an assignment. One day Boris makes this assignment.



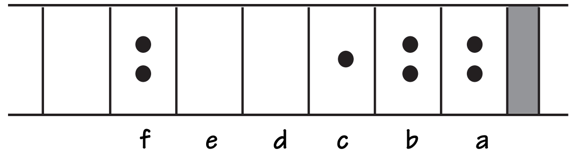
Add arrows to the picture.

T: ***Which bridge does spy* d *observe?*** (0)
Who watches bridge 1? (Only **c**)
Who watches bridge 2?

S: ***Spies* a*,* b*, and* f*.***

T: ***Each day Boris must transmit his assignment to headquarters. He could send this arrow picture, but it is certainly not very secret and there is the danger that the enemy might intercept the message. Boris convinces headquarters to use the base three abacus. How could he use the abacus?***

Allow a few minutes to discuss students' suggestions. If no one suggests the following method, do so yourself. Label the boards of an abacus **a** through **f**, and put this configuration on the abacus.



T (pointing to the arrow picture and then to the abacus)**:** ***This is Boris's method of coding the assignment. Who can explain his method?***

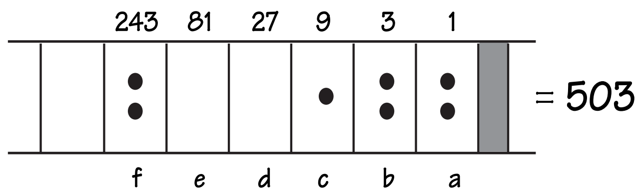
S: ***Spies*** **a**, **b**, ***and*** **f** ***observe bridge 2, so he puts two checkers on each of their boards.***

S: ***Spy*** **c** ***observes bridge 1, and there is one checker on the board for*** **c**.

S: ***Spies*** **d** ***and*** **e** ***observe bridge 0, so there are no checkers on those boards.***

T: ***Boris could send a picture of the abacus, but thinks that this is still too easy for the enemy to decode. He decides to send a single number for each assignment. What number do you think he sends for this assignment? Why?***

S: ***503. I added the numbers shown by the checkers on a base three abacus: 243 + 243 + 9 + 3 + 3 + 1 + 1 = 503.***



Label the boards of the base three abacus.

Write this message on the board.

503
code 3

T: ***Boris sends this message to headquarters. Code 3 reminds headquarters that Boris is in charge of three bridges.***

Boris and headquarters must agree that this is a good system for sending codes. First, is it secretive?

S: ***Yes, I doubt that the enemy could decode it if they receive only the number.***

T: ***Second, can Boris code each assignment?***

S: ***Yes, he draws the arrow picture, puts 0, 1, or 2 checkers on each board, and calculates the number on the abacus.***

T: ***Finally, we must check whether headquarters can decode a message.***

Distribute copies of Worksheet N15 and write the message on the board.

150
code 3

T: ***How will headquarters decode this message?***

S: ***Put 150 on a base three abacus, and then draw the corresponding arrow picture.***

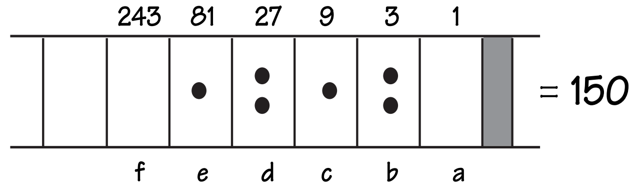
T: ***Solve this problem on your worksheet.***

Let students work individually or with a partner for a few minutes. After a while, invite a student to present the solution to the class. (See the answer key at the end of this lesson.)

The following exercise is optional. If you choose to skip it, ask students to continue working individually on Worksheets N15*, **, and ***.

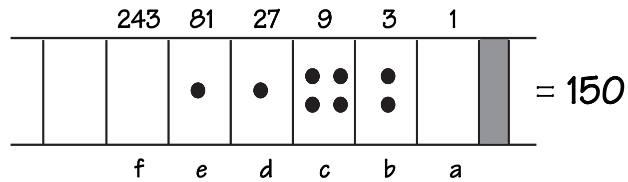
Exercise 3 (optional)

Refer to the configuration for 150 used to solve the problem on Worksheet N15.



T: ***Can we be sure that headquarters will represent 150 in this way to find Boris's assignment? Aren't there other ways to show 150 on this abacus?***

Invite a student to show 150 in another way. For example:



T: ***Why wouldn't headquarters represent 150 in this way and then draw a different arrow picture?***

S: ***This representation doesn't work. Here spy*** **c** ***must observe bridge 4, but headquarters knows there is no bridge 4.***

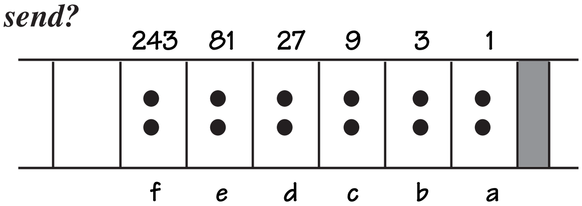
Let students suggest one or two other ways to put 150 on the abacus. Note that every other representation results in the type of difficulty discussed above. Conclude that there is only one way to represent 150 with fewer than three checkers on any board. Therefore, headquarters will always be able to determine Boris's assignment.

T: ***Boris decides to make a different assignment every day, seven days a week. How many weeks, months, or years do you think Boris could do this before he would have to repeat an assignment?***

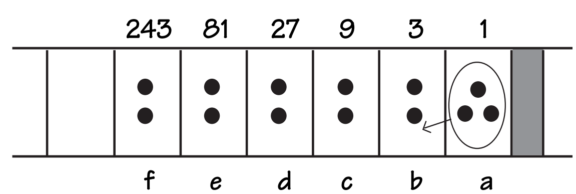
List students' estimates on the board.

T: ***What is the greatest code number Boris could send?***

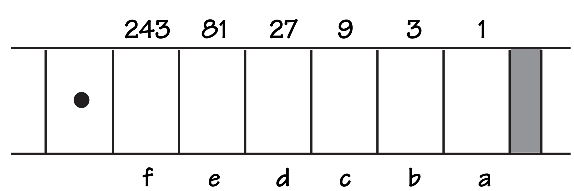
S: ***Put two checkers on each board. All of the spies watch bridge 2.***



T: ***One way to find this number is to add the numbers represented by the checkers, but here's an easier way. Consider what we get when we add 1.***



Place a third checker on the ones board and invite students to make trades until only one checker remains. Comment on the cascade effect.



T: ***What number is this?***

S: ***729; 3 x 243 = 729.***

Return to the original configuration with two checkers on each board.

T: ***What number is this?***

S: ***728, since we added 1 to get 729.***

T: ***What is the least code number Boris could send?***

S: ***0. Do not put any checkers on the abacus; all spies watch bridge 0.***

T: ***Does every number between 0 and 728 correspond to an assignment? In other words, can every number between 0 and 728 be put on this abacus with two or fewer checkers on any board?***

Let students express their opinions. The following is a possible explanation:

> To put any number between 0 and 728 on the abacus, put that number of checkers on the ones board. Then make trades (many!) until there are at most two checkers on each board. Each representation corresponds to a unique spy assignment.

T: ***How many different messages can Boris send?***

S: ***729. There are 728 whole numbers from 1 to 728, and 729 whole numbers from 0 to 728.***

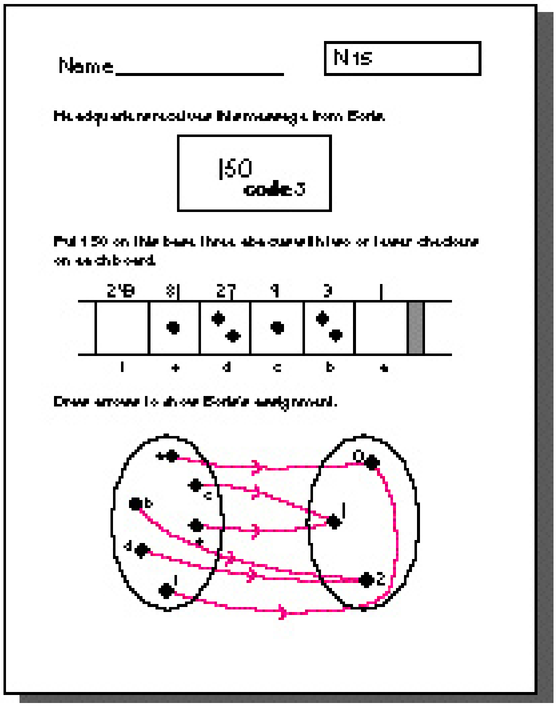
T: ***If Boris makes one assignment per day, for how many months or years can he make different assignments?***

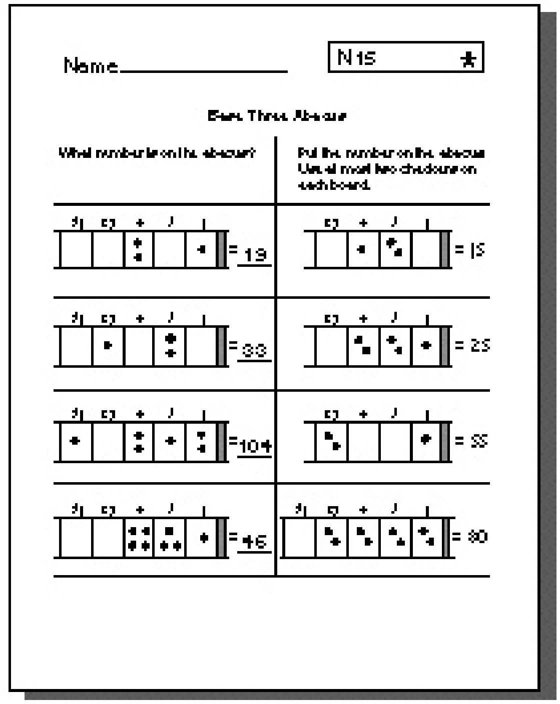
S: ***Almost two years. There are 365 days in a year. 2 x 365 = 730.***

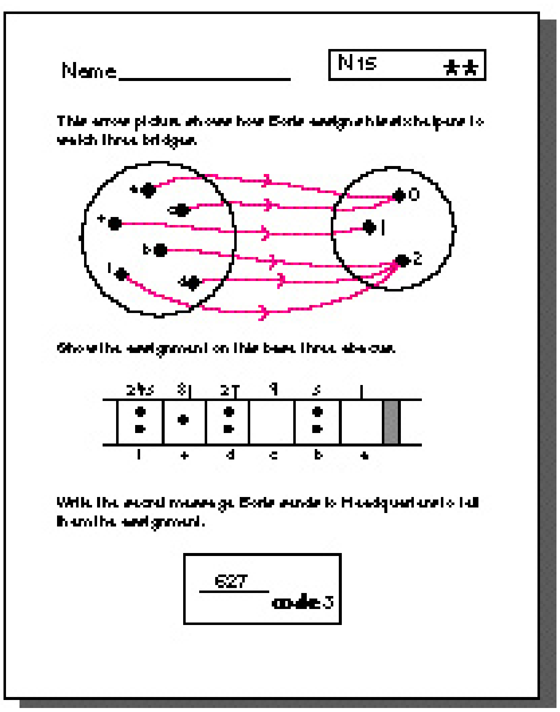
Compare the solution with students' estimates. Worksheets N15*, **, and *** are available for

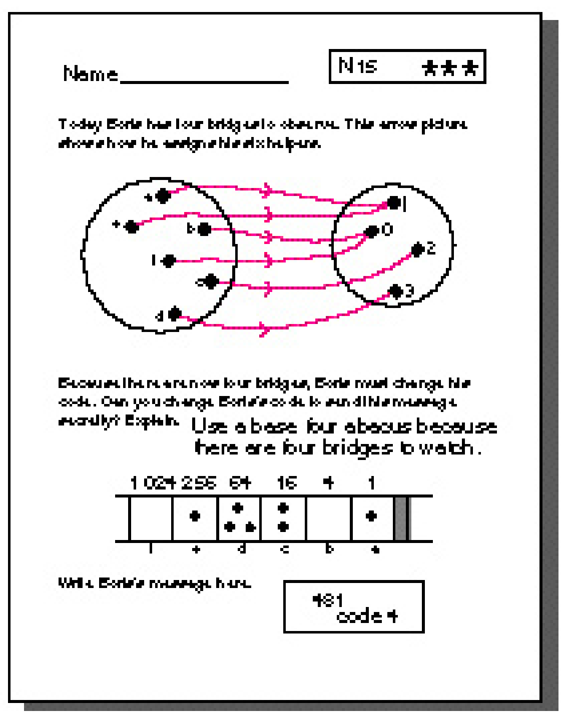
Writing/Home Activity

Suggest that students write a letter to a family member or a friend explaining how to decipher one of Boris's code 3 messages to headquarters.


Name
N 15
150
code 3
243 81 27 9 3 1


Name
N 15
= 19
= 15
= 33
= 25
= 104
= 55
= 46
= 80


Name
N 15
243 81 27 9 3 1
627
code 3


Name
N 15
Use a base four abacus because there are four bridges to watch.
1 024 256 64 16 4 1
481
code 4

Capsule Lesson Summary

Compare a base three abacus with a base five abacus. Relate the base five abacus to a story about six spies watching five bridges. Determine how many different secret messages can be written in code 5.

Materials

Teacher
- Magnetic checkers
- Colored chalk

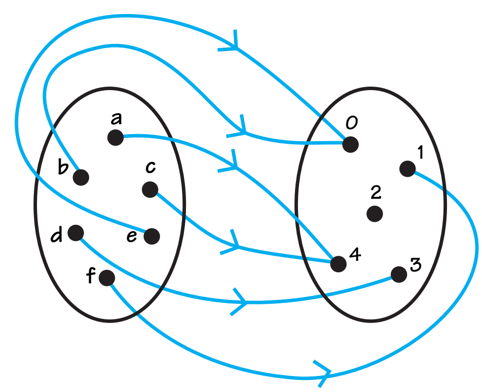
Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N16 (no star), *, and **

Description of Lesson

Begin the lesson with a quick review of the story about six spies and three bridges. Ask students to recall the story, and be sure they mention the use of a base three abacus.

Exercise 1

Draw this picture on the board as you tell the following story.

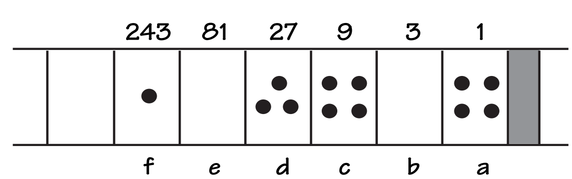


T: ***Today I'll tell you about another spy, Natasha. She also has six assistants, but they have five bridges to watch. The spies are* a, b, c, d, e *and* f*; the bridges are numbered 0, 1, 2, 3, and 4. This arrow picture shows one assignment.***
Which spies are watching bridge 0? (**b** and **e**)
Which bridge is* c *watching? (Bridge 4)
Is anyone else watching bridge 4? (Yes, **a**)
Are all the bridges being watched?

S: ***No. No one is watching bridge 2.***

T: ***Natasha must secretly transmit the assignment each day to headquarters. At first she thinks she might be able to use the same code that Boris used. What do you think?***

Let students comment. Students might point out that "code 3" in a message from Boris reminds headquarters that his assistants watch three bridges. They might even suggest that Natasha use a base five abacus because her assistants watch five bridges. If appropriate, let the class see the confusion that would result at headquarters if a number were represented on a base three abacus with from zero to four checkers allowed on a board. The following is an example.



Invite students to place checkers on a base three abacus to show the assignment indicated by the arrow picture on the board.

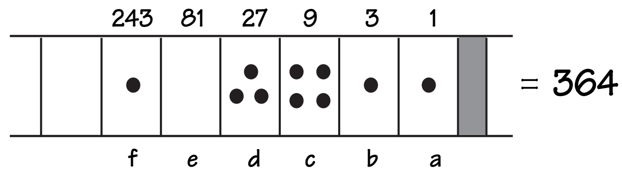
T: ***What message would Natasha send to headquarters?***

S: ***364, code 3.***

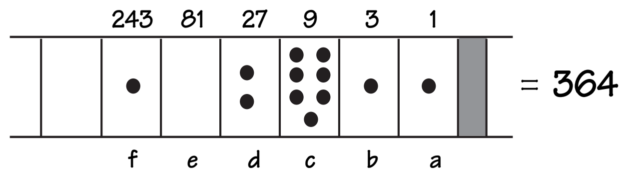
T: ***Natasha was going to send this message, but fortunately she realized that it could cause confusion at headquarters. Why might there be confusion?***

S: ***We can make trades and show 364 in other ways on the base three abacus. Headquarters wouldn't know which way Natasha used; so headquarters might reach the wrong spy assignment.***

Encourage students to make trades to show several other ways to display 364, for example:



Some students might show 364 using more than four checkers on a board, for example:



If this occurs, ask students which bridge spy **c** is watching. According to the configuration on the abacus, **c** is watching bridge 7, but there is no bridge 7. Conclude that headquarters would not believe 364 was shown in this way since they know Natasha is responsible for watching only five bridges.

T: ***There are many ways for headquarters to put 364 on the base three abacus with fewer than five checkers on a board. Therefore, headquarters wouldn't know Natasha's assignment. What could Natasha do to avoid this problem?***

Accept student suggestions, but lead to the following solution.

S: ***Use a base five abacus.***

T: ***Why?***

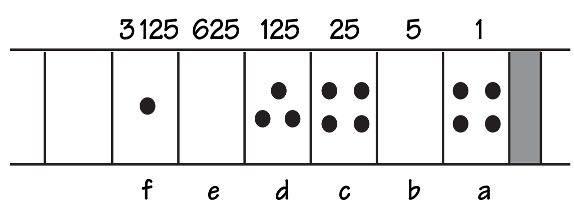
S: ***This time there are five bridges instead of three bridges.***

S: ***There are five bridges: 0, 1, 2, 3, and 4. There would be no more than four checkers on each board of the abacus, and, on a base five abacus, you need at least five checkers on a board to make a trade for a checker on the next board to the left.***

T: ***Natasha decides to use a base five abacus. Would she still send the number 364 to headquarters?***

S: ***No. Checkers go on the same position boards of the base five abacus, but the labels of the boards are different.***

With student help, label the boards of a base five abacus and place checkers according to the arrow picture.



T: ***Can we make any forward trades on this base five abacus?***

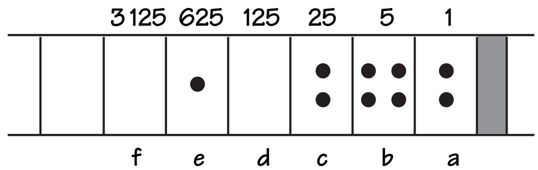
S: ***No, we would need at least five checkers on a board.***

T: ***What message would Natasha send for this assignment?***

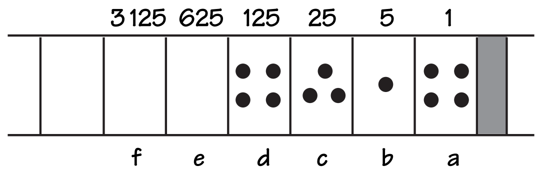
S: ***3 604, code 5.***

3 604
code 5

Present the following configurations and ask students to write code numbers on their papers. Collectively check answers, as shown here in boxes.



697 code 5



584 code 5

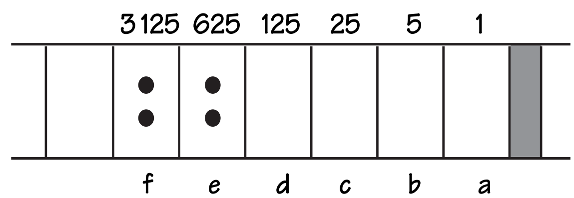
Distribute copies of Worksheet N16 (no star) and ask students to solve the problems individually or with a partner. After a short while, invite students to present their solutions at the board. (Refer to the answer key following the lesson description.)

T: ***What do you notice about the first four problems?***

S: ***All of the checkers move one board to the left each time.***

S: ***Each number is five times the previous number: 5 x 12 = 60; 5 x 60 = 300; 5 x 300 = 1 500.***

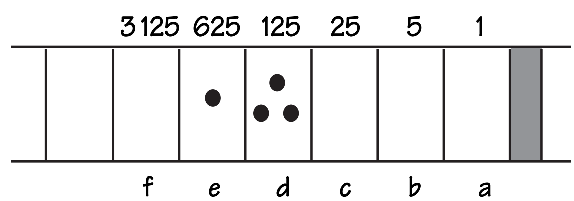
Put this configuration on the abacus.



T: ***What number is this?***

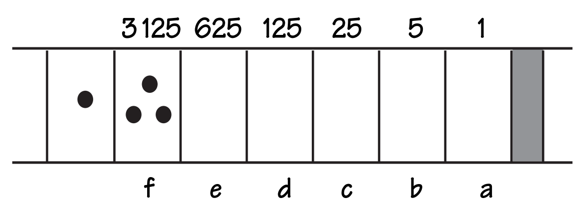
S: ***7 500. You moved the checkers one more board to the left; 5 x 1500 = 7 500.***

Put this configuration on the abacus.



T: ***This is the last problem on the worksheet. What number is this?*** (1000)

Let's move all of the checkers two boards to the left.

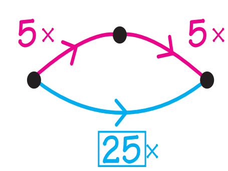


Now what number is this? Why?

S: ***25 000. By moving the checkers two boards to the left, we multiply by 5 twice. 5 x 1000 = 5 000 and 5 x 5 000 = 25 000.***

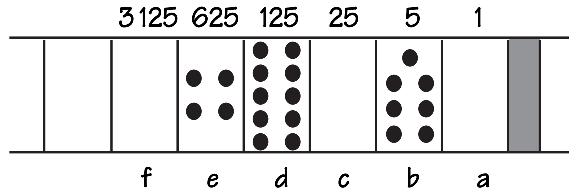
S: ***5x followed by 5x is the same as 25x. 25 x 1000 = 25 000.***

You may like to draw an arrow picture to illustrate this composition.



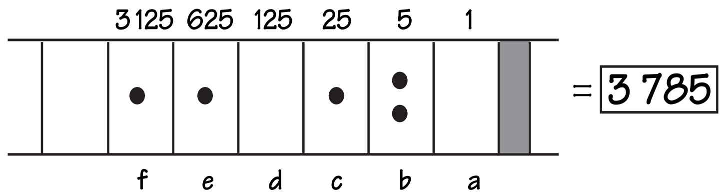
If some students believe that the number on the abacus is 10 000 (10 x 1 000) or that the blue arrow is for 10x, check the number on the abacus or label dots in the arrow picture.

Put this configuration on the abacus.

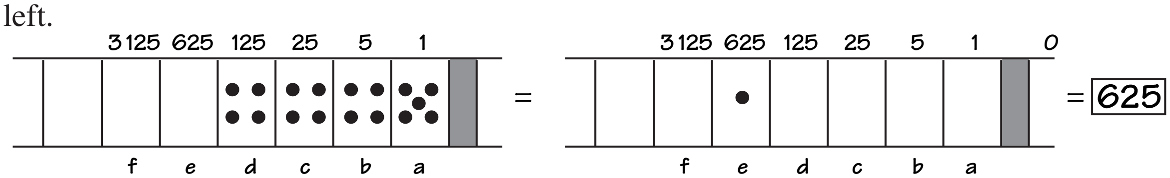


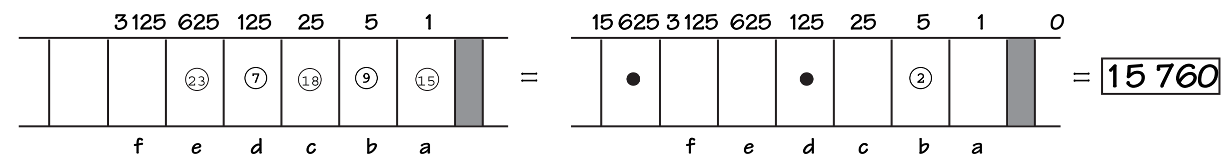
T: ***To calculate this number, we can first make trades. Do this on your papers.***

Invite students to make trades until there are fewer than five checkers on a board, and then calculate the number.



In a similar manner, invite students to make trades and decode the numbers in the configurations on the left.





Exercise 2

Note: If you presented Exercise 3 in Lesson N15, allow students to solve the following problem with minimal guidance.

T: ***Natasha decides to transmit her spy assignments via code 5. She plans to make a different assignment every day, seven days a week. How many weeks, months, or years do you think she could continue to make different assignments?***

List students' estimates on the board.

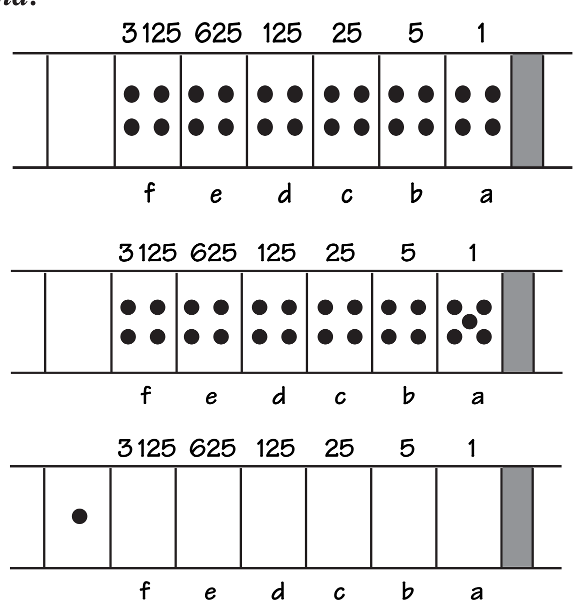
T: ***What is the greatest number Natasha could send?***

S: ***Put four checkers on each board and add up the numbers. All six spies would watch bridge 4.***

T: ***Adding all those numbers is tiresome. Can anyone suggest an easier way?***

If no student suggests placing another checker on the ones board, do so yourself.

Invite students to make trades until only one checker remains. Comment on the cascade effect.



T: ***What number is this?***

S: ***15 625; 5 x 3 125 = 15 625.***

T: ***What number was on the abacus before we added the extra checker?***

S: ***15 624, one less than 15 625.***

T: ***Yes, the greatest number Natasha could send is 15 624. What is the least number Natasha could send?***

S: ***0. Assign all six spies to watch bridge 0, and put no checkers on the abacus.***

T: ***Could Natasha use all numbers between 0 and 15 624 for different assignments?***

Let students discuss this question. After a while, offer this explanation:

T: ***To put any whole number between 0 and 15 624 on the abacus, put that number of regular checkers on the ones board. For example, to put 1 736 on the abacus, Natasha could put 1 736 checkers on the ones board. Then she could make many trades until there are at most four checkers on each board. This method is not very practical, but it does show that any whole number between 0 and 15 624 does correspond to a spy assignment.***

How many different assignments could Natasha make?

S: ***15 625. There are 15 624 whole numbers from 1 to 15 624, so there are 15 625 whole numbers from 0 to 15 624.***

T: ***If she makes one assignment every day, for how many months or years could she make different assignments?***

Encourage students to estimate the number of years.

S: ***More than ten years since there are about 3 650 days in ten years.***

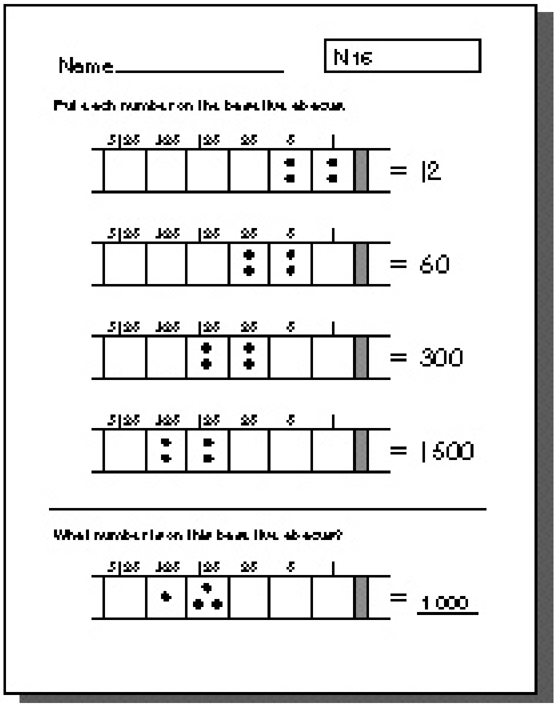
S: ***Less than 100 years since 100*** x ***365 = 36 500.***

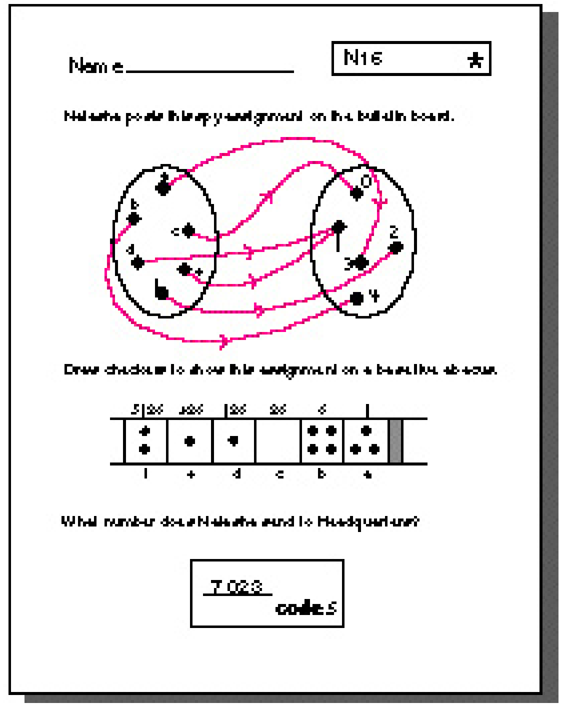
S: ***Less than 50 years since ½ of 36 500 = 18 250.***

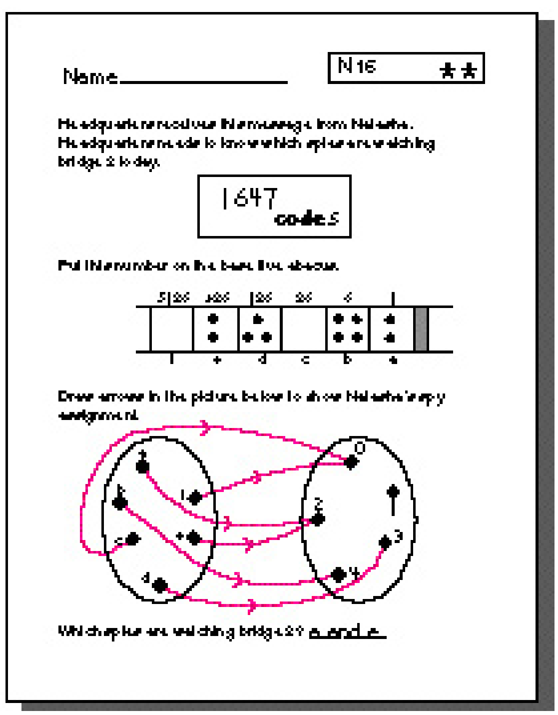
S: ***Between 40 and 50 years.***

Continue until the class concludes it would take between 42 and 43 years. Compare this answer to students' estimates.

Worksheets N16* and ** are available for individual work.


Name
N16
= 12
= 60
= 300
= 1500
= 1000


Name
N16 ★
7 023
codes


Name
N16 ★★
1647
codes

N17 MULTIPLICATION AND DIVISION WITH ___

Capsule Lesson Summary

Given the number at either the starting dot or the ending dot of a 6x arrow, label the other dot. Use estimation to properly place a decimal point in a product. Use patterns to solve division problems with decimals. Identify pairs of numbers whose product is 24.

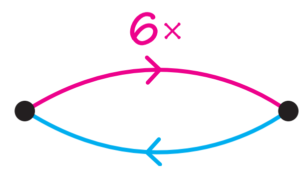
Materials

Teacher	• Colored chalk	Student	• Paper • Colored pencils, pens, or crayons • Worksheets N17* and **

Description of Lesson

Exercise 1

Draw this arrow picture on the board. Trace the blue arrow.

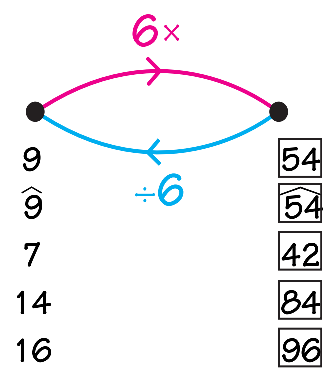


T: ***What is the return or opposite of a 6*x *arrow?***

S: ***÷6 or ⅙x.***

Label the blue arrow.

T: ***I will label one of the dots in this picture. Tell me what the other dot is for.***



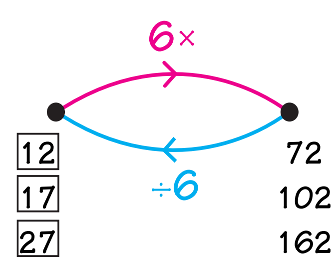
Label the dot on the left and ask students for the corresponding number at the right. Here are some possible choices with student responses shown in boxes.

You may like to ask students how they calculate 6 x 14.

S: ***6* x *4 = 24 and 6* x *10 = 60, so 6* x *14 = 60 + 24 = 84.***

S: ***6* x *7 = 42, so 6* x *14 is 2* x *42, or 84.***

Continue this activity, giving the number on the right and asking for the corresponding number on the left.



Erase all the dot labels, and then label the dot on the left 27.8.

T: ***Let's estimate the number here*** (point to the dot on the right)***. What number would be close to 6* x *27.8? Why?***

S: ***162 because 6* x *27 = 162.***

S: ***168 because 6* x *28 = 168.***

N17

Note that 6 x 27.8 is between 162 and 168 and record the inequality on the board.

$$162 < 6 \times 27.8 < 168$$

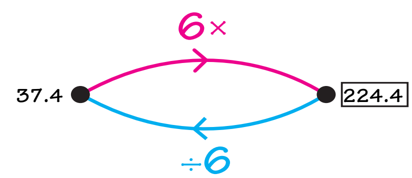
Then ask a student to calculate 6 x 278.

$$\begin{array}{r} 278 \\ \times\ 6 \\ \hline 1668 \end{array}$$

T: *6 x 278 is 1 668, so what number is 6 x 27.8?*

S: *166.8, because it is between 162 and 168.*

Repeat this activity to first estimate 6 x 37.4 and then to use the estimate to correctly place the decimal point in the answer.



Exercise 2

Write this problem on the board.

$$726 \div 3 = \boxed{}$$

T: *If we share $726 among three people, how much will each person receive?*

S: *$242.*

Record the answer and present another problem.

$$726 \div 3 = \boxed{242}$$
$$72.6 \div 3 = \boxed{}$$

T: *If we share $72.60 among three people, how much will each person receive?*

S: *$24.20.*

Record the answer and present another problem.

$$726 \div 3 = \boxed{242}$$
$$72.6 \div 3 = \boxed{24.2}$$
$$7.26 \div 3 = \boxed{}$$

T: *What if we share $7.26 among three people?*

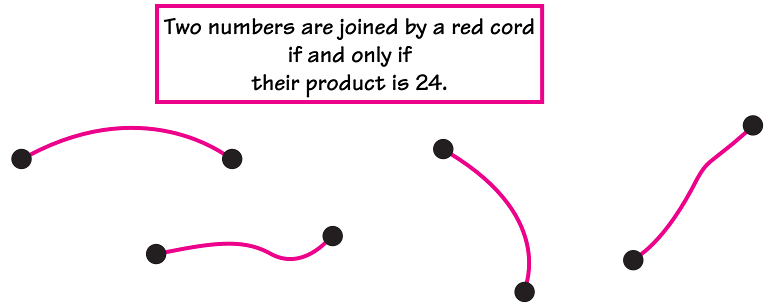
S: *Each person will get $2.42.*

In a similar manner, complete the following sequences of calculations. If students have difficulty when a decimal is introduced, use money as a model. (Answers are in boxes.)

$$800 \div 4 = \boxed{200} \qquad 21 \div 7 = \boxed{3}$$
$$812 \div 4 = \boxed{203} \qquad 42 \div 7 = \boxed{6}$$
$$8120 \div 4 = \boxed{2\,030} \qquad 420 \div 7 = \boxed{60}$$
$$8124 \div 4 = \boxed{2\,031} \qquad 427 \div 7 = \boxed{61}$$
$$812.4 \div 4 = \boxed{203.1} \qquad 434 \div 7 = \boxed{62}$$
$$81.24 \div 4 = \boxed{20.31} \qquad 43.4 \div 7 = \boxed{6.2}$$
$$8.124 \div 4 = \boxed{2.031} \qquad 4.34 \div 7 = \boxed{0.62}$$

Exercise 3

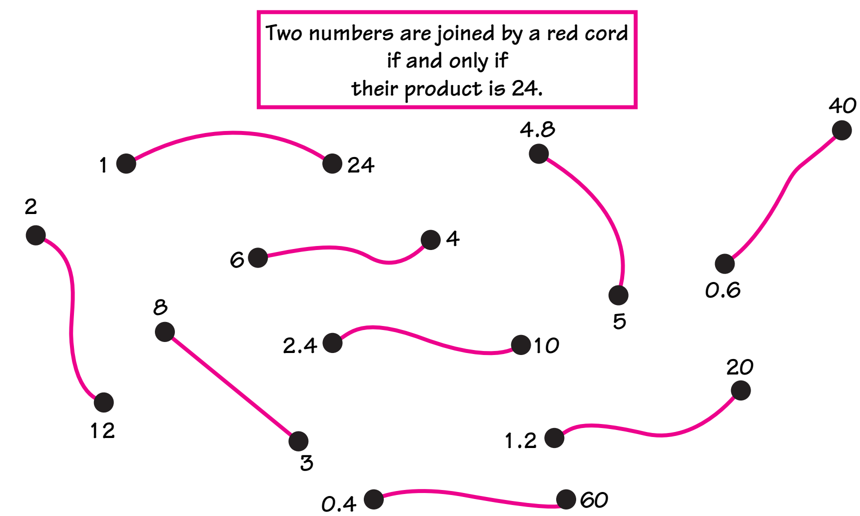
Draw this cord picture on the board and direct students to draw several red cords on their papers.



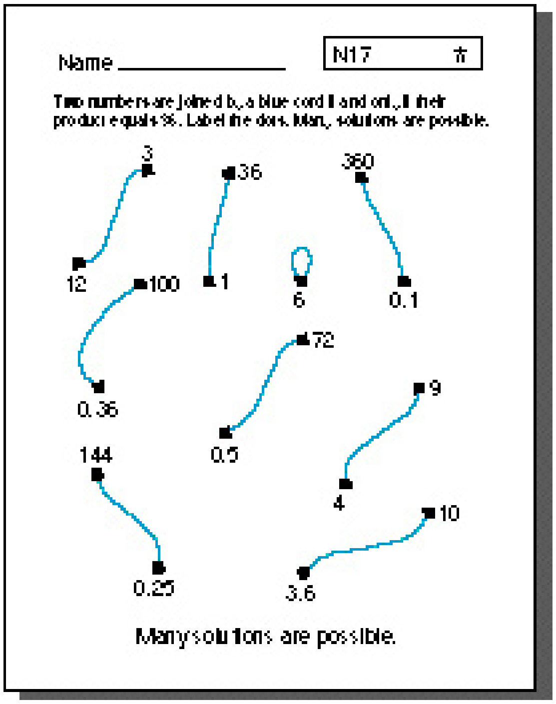
T: ***In this picture, two numbers are joined by a cord if and only if their product is 24. What are some numbers we could join with a red cord?***

Invite students to label dots in the picture on the board. Add more dots and cords to your picture as students suggest more pairs of numbers. If only whole numbers are suggested, label a dot 10 (or 20 or 5 or 0.4) and ask what number to join with it.

Examples of pairs of numbers that can be joined by red cords are shown below.



Worksheets N17* and ** are available for individual work.



Name ____________ N17 ☆☆

Complete. Watch for patterns to help you.

8 × 8 = 64	72 ÷ 8 = 9
8 × 16 = 128	720 ÷ 8 = 90
8 × 1.6 = 12.8	7 200 ÷ 8 = 900
8 × 32 = 256	7 256 ÷ 8 = 907
8 × 320 = 2 560	7.2 ÷ 8 = 0.9
8 × 3.2 = 25.6	736 ÷ 8 = 92
8 × 0.32 = 2.56	73.6 ÷ 8 = 9.2
	72.56 ÷ 8 = 9.07

27.5 ÷ 5 = 5.5	15.6 × 4 = 62.4
94.5 ÷ 9 = 10.5	21.3 × 7 = 149.1
8.16 ÷ 3 = 2.72	5.62 × 6 = 33.72

Capsule Lesson Summary

Investigate patterns that emerge when multiplying a number by 101; by 1 001; and by 10 001. Multiply a three-digit number by 13, the answer by 7, and that answer by 11. Discover that the composition of 13x followed by 7x followed by 11x is 1 001x. Review the standard algorithms for addition, subtraction, and multiplication by solving some puzzle-type problems.

Materials

Teacher
- Colored chalk

Student
- Calculator
- Colored pencils, pens, or crayons
- Paper

Description of Lesson

Exercise 1

Write this number sentence on the board.

$\boxed{} \times 26 = 2\,626$

T: ***What number should we multiply 26 by to get 2 626?***

Allow a few minutes for students to think about this problem.

T: ***What number is 100*** x ***26?*** (2 600)
Is the number in the box more or less than 100? (More)
How much greater is 2 626 than 2 600? (26 more)
What number (point to the box) ***should we multiply by?***

S: *101.*

$\boxed{101} \times 26 = 2\,626$
$\boxed{101} \times 81 = 8181$
$\boxed{101} \times 90 = 9\,090$

Continue with these problems. Ask students to explain answers and observe patterns.

Begin another list nearby.

$\boxed{} \times 452 = 452\,452$
$\boxed{} \times 306 = 306\,306$
$\boxed{} \times 500 = 500\,500$

T: ***What number do we multiply 452 by to get 452 452?***

Suppose a student incorrectly answers 101.

T: ***What number is 100*** x ***452?*** (45 200) ***And what is one more 452?***

S: *45 200 + 452 = 45 652.*

T: *So 101* x *452 is less than 452 452.*

S: *I think we multiply by 1 001.*

T: *Convince us.*

S: *1 000* x *452 = 452 000, and one more 452 is 452 452.*

Continue until students find that 1 001 is in all three boxes.

$\boxed{101} \times 26 = 2\,626$ $\boxed{1001} \times 452 = 452\,452$

$\boxed{101} \times 81 = 8181$ $\boxed{1001} \times 306 = 306\,306$

$\boxed{101} \times 90 = 9\,090$ $\boxed{1001} \times 500 = 500\,500$

T: ***What patterns do you notice?***

S: ***The numbers in the boxes all end in 1 and all start with 1. There are zeroes in between.***

S: ***The digits of the numbers here (middle column) repeat in the answers. For example, 101* x *26 = 2 626.***

T: ***Let's look at that pattern. Suppose we start with 1 232 and then write the digits again. Commas placed in such big numbers make them easier to read.***

Write this number sentence on the board.

$\boxed{} \times 1232 = 12{,}321{,}232$

T: ***What number belongs in the box?***

S: ***10 001.***

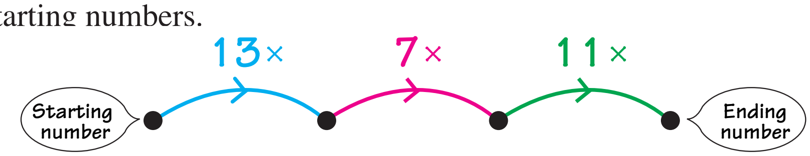
Ask the class to check this answer. If a student suggests 1 001 is the number in the box, ask for 1 000 x 1 232, and then ask for that number plus another 1 232. (1,233,232) Observe that this number is less than 12,321,232.

Exercise 2

Suggest that students use calculators for this exercise.

T: ***Choose any three-digit number and put it on the display of your calculator. Multiply your number by 13, then multiply the answer by 7, and then multiply that answer by 11. Write your starting and ending numbers on paper.***

Draw this arrow picture on the board as you repeat the instructions. Some students may like to try several different starting numbers.



When most students have at least one pair of starting and ending numbers, collect some results in a table on the board, as illustrated here.

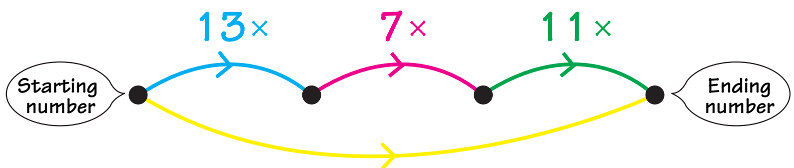
Starting Number	Ending Number
111	111111
100	100100
802	802 802
325	325 325

When your table has several entries, ask a student to announce only a starting number. Then ask the class to determine what the corresponding ending number is without using calculators. The student can then confirm the ending number. Record the pair in the chart. Repeat this activity by asking another student to reveal only an ending number.

T: ***Why is it so easy to determine the starting or ending number when we know the other one?***

S: ***You get the ending number by writing the digits of the starting number again. If you start at 341, you end at 341 341.***

Draw this composition arrow in your picture.



T: ***What could the arrow from start to end be for? Times what number?***

S: ***1 001.***

T: ***Why is 13*x *followed by 7*x *followed by 11*x *the same as 1 001*x*?***

S: ***13* x *7* x *11 = 1 001.***

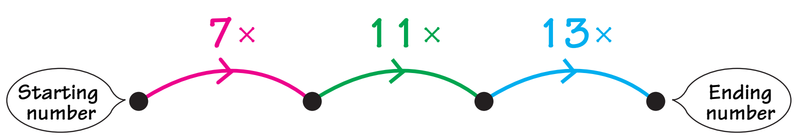
Direct students to calculate 13 x 7 x 11.

$$\begin{array}{r} \overset{2}{1}3 \\ \times\ 7 \\ \hline 91 \end{array} \qquad \begin{array}{r} 91 \\ \times\ 11 \\ \hline 91 \\ 910 \\ \hline 1001 \end{array}$$

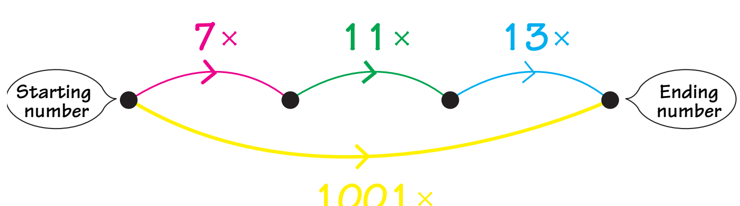
Label the composition arrow and keep the arrow picture on the board.

T: ***Now choose a three-digit number and multiply it by 7, multiply the answer by 11, and then multiply that answer by 13.***

Draw this arrow picture on the board.



If the class suspects that 1 001 times the starting number equals the ending number, ask why (the order of the arrows does not matter), collectively try a starting number, and then draw and label the composite arrow. Otherwise, let your students try starting numbers individually and discover that the same pattern emerges.



You might wish to challenge students to see what happens if the starting number (in either picture) is a two-digit or a four-digit number.

Erase the board before going on to Exercise 3.

Exercise 3

T: ***I was adding some numbers when an eraser gremlin came along and erased part of my work. I'll put boxes where digits are missing. Try to figure out what digit belongs in each box.***

Write these problems on the board and allow several minutes for students to work independently. (Answers are in boxes.) When many students are done, invite three students to complete the problems at the board and to explain their solutions.

$$\begin{array}{r} 4\boxed{4}8 \\ +\,23\boxed{1} \\ \hline 679 \end{array} \qquad \begin{array}{r} \boxed{3}\boxed{6}\boxed{7} \\ +\,8\boxed{2}5 \\ \hline 1192 \end{array} \qquad \begin{array}{r} 1\boxed{4}42 \\ \boxed{7}95\boxed{8} \\ +\,36\boxed{8}1 \\ \hline 13081 \end{array}$$

Erase the board and then write these two subtraction problems on the board.

$$\begin{array}{r} 103 \\ -\,47 \\ \hline \end{array} \qquad \begin{array}{r} 5168 \\ -\,2173 \\ \hline \end{array}$$

Invite students to demonstrate various algorithms they use for solving subtraction problems. The two most common methods are shown below.

Compensation
(Nick's Method)

$$\begin{array}{r} 1\,{}^{10}0\,{}^{10}3 \\ -\,{}^{1}4\,7 \\ \hline 56 \end{array} \qquad \begin{array}{r} 5\,{}^{10}1\,{}^{10}6\,8 \\ -\,{}^{1}2\,{}^{1}1\,7\,3 \\ \hline 2995 \end{array}$$

Regrouping
(Borrowing)

$$\begin{array}{r} \overset{0}{\cancel{1}}\,\overset{\overset{9}{\cancel{10}}}{\cancel{0}}\,\overset{13}{\cancel{3}} \\ -\,47 \\ \hline 56 \end{array} \qquad \begin{array}{r} \overset{4}{\cancel{5}}\,\overset{\overset{10}{\cancel{0}}}{\cancel{1}}\,\overset{16}{\cancel{6}}\,8 \\ -\,2173 \\ \hline 2995 \end{array}$$

T: ***Again the eraser gremlin attacked some of my subtraction work. See if you can figure out what digit belongs in each box.***

Write these problems on the board. (Answers are in boxes.) After several minutes of individual work, check the answers collectively.

$$\begin{array}{r} \boxed{3}49 \\ -\,2\boxed{1}5 \\ \hline 134 \end{array} \qquad \begin{array}{r} 62\boxed{1} \\ -\,4\boxed{3}5 \\ \hline \boxed{1}86 \end{array} \qquad \begin{array}{r} 75\boxed{9}1 \\ -\,3\boxed{8}6\boxed{2} \\ \hline 3729 \end{array}$$

Erase the board and continue the activity with multiplication problems. (Answers are in boxes.)

$$\begin{array}{r} 2\boxed{6} \\ \times\,3 \\ \hline 78 \end{array} \qquad \begin{array}{r} 3\boxed{5}9 \\ \times\,\boxed{4} \\ \hline 1436 \end{array} \qquad \begin{array}{r} \boxed{6}2\boxed{8} \\ \times\,5 \\ \hline 3140 \end{array}$$

Home Activity

This would be a good time to send a letter to parents/guardians about addition, subtraction, and multiplication practice. Blackline N18 has a sample letter.

N19 DIVISION #2

Capsule Lesson Summary

Determine the number of students in a class through clues that focus on the students lining up in rows. Extend this model to solve a sequence of related division problems; for example, 5100 ÷ 15; 5115 ÷ 15; and 5145 ÷ 15. Build an arrow road to calculate 7946 ÷ 24.

Materials

Teacher
- Colored chalk

Student
- Colored pencils, pens, or crayons
- Worksheets N19* and **

Description of Lesson

This lesson is presented as a story situation. You may, of course, adjust the story to your liking.

Exercise 1

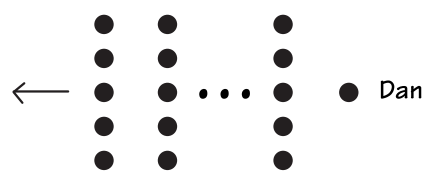
T: ***Tall Dan Zydrich is a fifth-grade student. Dan usually finds himself at the back when his class lines up to go somewhere. Do you know why?***

S: ***His family name starts with Z. So if the class lines up alphabetically, he'll most likely be at the very end of the line.***

S: ***He's tall; so if they line up by height, he'll be at the end.***

T: ***You're right. As a matter of fact, not only is Dan at the end, but when his class lines up in rows of fives, Dan is all alone at the end.***

Draw this picture on the board.



Explain that the picture shows Dan's class lined up in rows of five with Dan alone at the end. Point out that the three dots between the second row and the last row indicate the possibility of more rows.

T: ***We don't know how many students are in Dan's class. My picture has dots for 16 students , but there might not be even that many or there might be more. All we know is that when the class lines up in rows of five, Dan is alone at the end. How many students could there be in Dan's class?***

S: ***There could be 36 students.***

T: ***How many rows of five students would there be?***

S: ***Seven rows, since 7 x 5 = 35 and Dan makes 36.***

S: ***There could be 31 students. 30 ÷ 5 = 6. There would be six full rows plus Dan.***

Record students' suggestions on the board until this list of possibilities is generated.

1, 6, 11, 16, 21, 26, 31, 36, 41, 46, 51, 56, ⋯

T: ***Can you describe all of the numbers that would work?***

S: ***Any whole number whose ones digit is 1 or 6.***

S: ***Any whole number that is one more than a multiple of 5.***

S: ***Start at 1 and count by fives.***

T: ***Suppose we know that there are fewer than 50 students in Dan's class, and that there are other students besides Dan in the class.***

Erase 1 and all numbers greater than 50 from the list.

T: ***On another day, Dan's class lines up in rows of two. Can you guess what happens?***

S: ***Dan is alone at the end.***

T: ***Yes. How many students could be in Dan's class?***

S: ***21 students. 21 is in our list and 10 x 2 = 20. 20 students plus Dan makes 21 students.***

S: ***There could be 11 students. 10 ÷ 2 = 5. There would be five rows of two plus Dan.***

S: ***Any odd number in the list: 11, 21, 31, or 41. With an even number of students, the whole class would be in rows of two and Dan would not be alone.***

Indicate the possibilities in the list.

~~6~~, 11, ~~16~~, 21, ~~26~~, 31, ~~36~~, 41, ~~46~~

T: ***The teacher doesn't want Dan to always be alone at the end, so she asks the class to line up in rows of three. But alas! Tall Dan is alone at the end again! How many students could be in Dan's class?***

S: ***31 students. There would be ten rows of three with Dan at the end. 10 x 3 = 30 and 30 + 1 = 31.***

S: ***31 students. 30 ÷ 3 = 10. There would be ten full rows and Dan.***

Circle 31 in the list on the board and check that 11, 21, and 41 are no longer possibilities. For example:

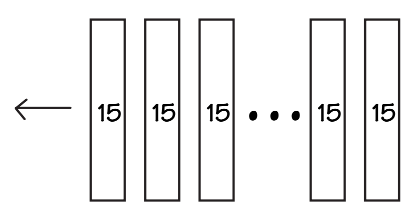
T: ***Could there be 11 students?***

S: ***No, 3 x 3 = 9 and 11 – 9 = 2, thus leaving two students in the last row.***

Exercise 2

Erase the board and draw this picture.

T: ***Later in life, tall Dan Zydrich joins the army and still is always in the last row. At least in the army, he is always in a full row of 15 soldiers. This picture represents soldiers marching in rows of 15.***



Write 4 500 on the board as you ask,

T: *If Dan's division has 4 500 soldiers and they march in rows of 15, how many rows are there?* (300 rows) *Why?*

S: *3* x *15 = 45, 30* x *15 = 450, and 300* x *15 = 4 500.*

S: *45 ÷ 15 = 3; so 4 500 ÷ 15 = 300.*

During the explanations, write number sentences on the board to describe the situation. Be sure to include multiplication and division number sentences.

$$300 \times 15 = 4\,500 \qquad 4\,500 \div 15 = 300 \qquad 15\overline{)4\,500}^{\,300}$$

T: *A battalion of 600 soldiers is also marching in rows of 15.*
How many rows of 15 do they have?

S: *40 rows. 60 ÷ 15 = 4, so 600 ÷ 15 = 40.*

S: *40 rows. 4* x *15 = 60, so 40* x *15 = 600.*

T: *Suppose this battalion and Dan's division join forces and march in rows of 15.*
How many soldiers are there?

S: *5 100 soldiers. 4 500 + 600 = 5 100.*

T: *How many rows of soldiers are there?*

S: *340 rows. 300 + 40 = 340.*

Write this information on the board. The blue arrow represents the number of rows added; the red arrow represents the number of soldiers added.

$$\begin{array}{rrcl} & 5\,100 \div 15 = 340 & & \\ +\square & \square \div 15 = 341 & & +1 \end{array}$$

T: *With 5 100 soldiers, there are 340 rows of 15. New recruits join to add one new row.*
How many new soldiers are there?

S: *15 soldiers, since there are 15 soldiers in each row.*

T: *What is the total number of soldiers now?*

S: *There are 5 115 soldiers. 5 100 + 15 = 5 115.*

Add the following information to the picture.

$$\begin{array}{rlr} & 5\,100 \div 15 = 340 & \\ +\boxed{15} & \boxed{5\,115} \div 15 = 341 & +1 \\ \square & \square \div 15 = 343 & +2 \end{array}$$

T: *Now two more rows of soldiers are added.*
How many new soldiers are there?

S: *30 soldiers, since 2* x *15 = 30.*

S: *There are now 5 145 soldiers. 5 115 + 30 = 5 14*

Continue in a similar manner with the problems suggested here. (Answers are in boxes.)

$$\begin{array}{rlr} & 5\,145 \div 15 = 343 & \\ +\boxed{30} & \boxed{5\,175} \div 15 = 345 & +2 \\ +\boxed{15} & \boxed{5\,190} \div 15 = 346 & +1 \\ +\boxed{60} & \boxed{5\,250} \div 15 = 350 & +4 \end{array}$$

Erase the board before going on to Exercise 3.

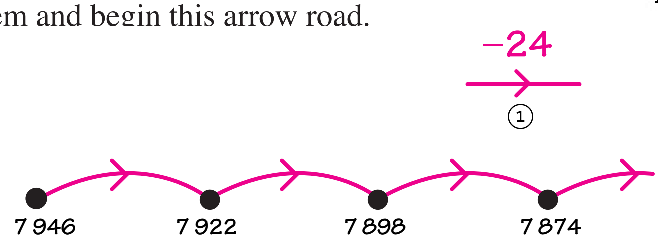
Exercise 3

T: ***Dan's division now has 7 946 soldiers and has been told to march in lines of 24. Dan's sergeant asks him to calculate the number in rows of soldiers. What division problem must Dan solve?***

S: ***7 946 ÷ 24.***

Write the division problem and begin this arrow road.

$24\overline{)7\,946}$



T: ***At first, Dan starts drawing this arrow road. Each red arrow represents one row of 24 soldiers. Is this a good method to solve the problem?***

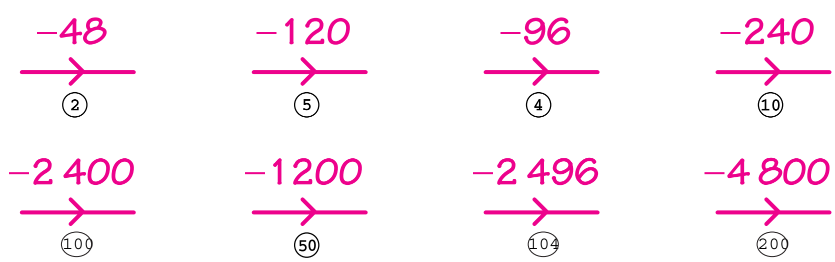
S: ***Yes, but it requires drawing a lot of arrows.***

T: ***What other kind of arrows would make the calculation quicker?***

S: ***–48 arrows. Each –48 arrow represents two rows of soldiers since 2 x 24 = 48.***

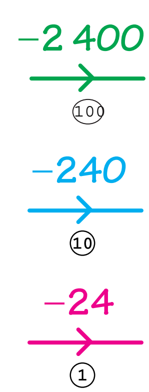
S: ***–240 arrows. Each –240 arrow represents ten rows.***

Record and label arrows as suggested. For example:



T: ***Dan decides to use only –2 400 arrows, –240 arrows, and –24 arrows.***

Near the edge of the board, draw keys for these arrows.
Ask students for the number of rows represented by each arrow.



T: ***Why do you think Dan chooses these three arrows?***

S: ***It's easier to subtract 240 or 2 400 than it is to subtract 96 or 2 496.***

S: ***It's obvious how many rows of 24 there are with each of those arrows.***

T: ***Let's build an arrow road that starts at 7 946 and uses these three kinds of arrows. How could we start?***

S: ***With a –2400 arrow; 7 946 – 2 400 = 5 546.***

Note: Students could first use a –24 arrow or –240 arrow, but this choice might later reduce the efficiency of the solution.

Draw the arrow road on the board with student direction. After three –2 400 arrows have been drawn, ask,

T: ***We've used three –2 400 arrows. How many soldiers does this represent?***

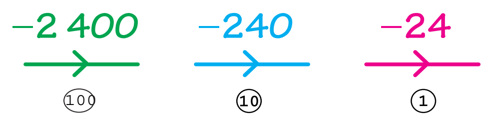
S: ***7 200 soldiers. 3 x 2 400 = 7 200.***

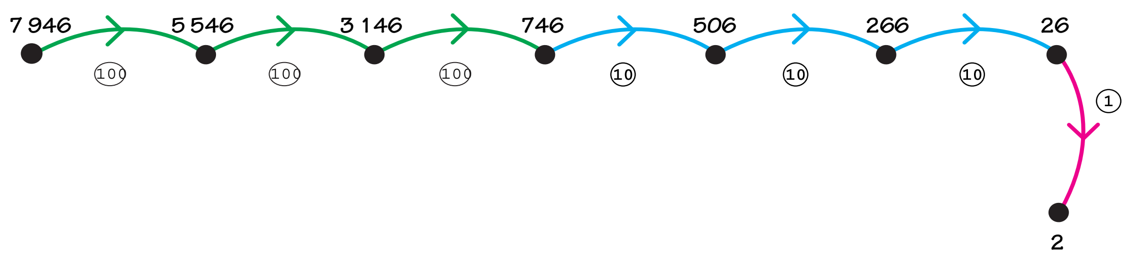
T: ***How many rows of soldiers do we have so far?***

S: ***300 rows; each arrow represents 100 rows.***

T: ***We must still consider 746 soldiers. What arrow should we use now?***

Let students direct the completion of this arrow road.





T: ***Only two soldiers are left, so the arrow road is finished.***
How many rows of 24 soldiers are there?

S: ***331 rows.***

Refer to the division problem written on the board.

$$\begin{array}{r} 331 \text{ R} = 2 \\ 24\overline{)7\,946} \end{array}$$

T: ***What is the answer to this problem?***

S: ***331 with a remainder of 2.***

T: ***I wonder if Dan will be one of those soldiers.***

Briefly review the method just used to do the division problem.

Exercise 4

Distribute copies of Worksheets N19* and **. Call students' attention to the problem on N19*.

T: ***797 soldiers march in rows of 15. To calculate the number of rows, we could build an arrow road using –150 and –15 arrows. What other arrows might be useful?***

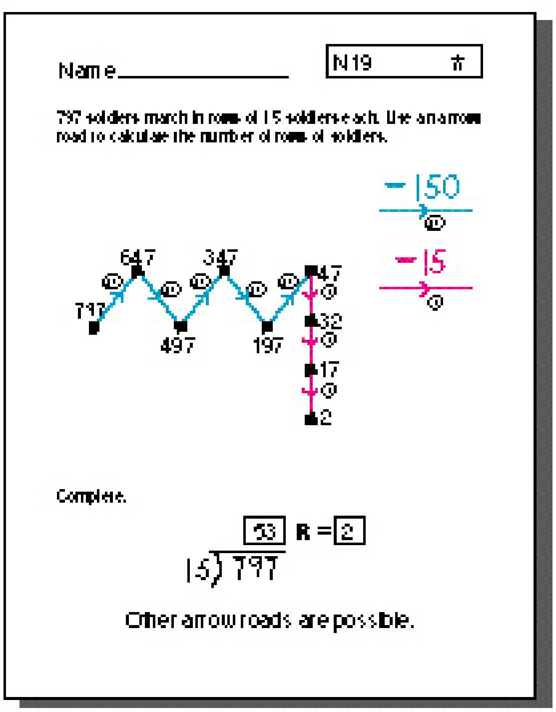
For each student response, record the label of the arrow and the number of rows it represents on the board, for example:

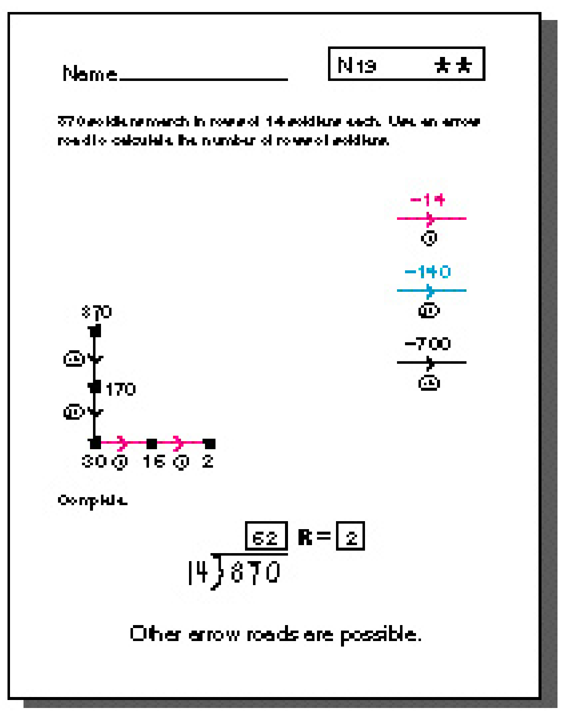
T: ***The worksheet suggests that you use –150 arrows and –15 arrows. Actually you may use any arrows you wish. Be sure to clearly label the arrows you use.***

Worksheets N19* and ** are available for individual work. Assist any students needing help. Write these two problems on the board for students who finish quickly.

$45\overline{)5\,600}$ $\qquad\qquad$ $23\overline{)13\,930}$

Near the end of class, you may wish to discuss and solve the problems on the worksheets.





Capsule Lesson Summary

From a list of numbers, select those that can be put on the Minicomputer using exactly one of a given set of checkers. Put other numbers on the Minicomputer using exactly two of a given set of checkers. By moving exactly one checker in a configuration, increase or decrease the number displayed by a given amount. Play *Minicomputer Golf*.

Materials

Teacher
- Minicomputer set
- Weighted checker set
- Colored chalk

Student
- Paper
- Worksheet N20*, **, ***, and ****

Description of Lesson

Exercise 1

Display two Minicomputer boards and these weighted checkers: ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨.

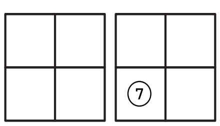
List these numbers on the board.

14 34 36 42 52 64 72 160 400 600

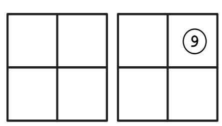
T: ***Find all of the numbers in this list that can be put on the Minicomputer using exactly one of these weighted checkers.***

Invite students to put numbers on the Minicomputer using exactly one of the checkers. Each time, ask the class to identify the number and to confirm that it is in the list. Continue this activity, circling numbers as they are shown on the Minicomputer.

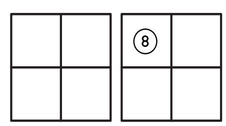
Six of the numbers can be put on the Minicomputer with exactly one of the checkers displayed.



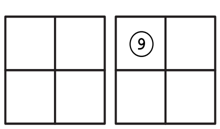
= 14



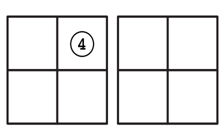
= 36



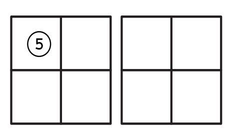
= 64



= 72



= 160

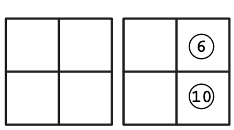


= 400

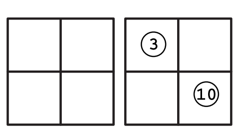
Other solutions are possible.

14 34 36 42 52 64 72 160 400 600

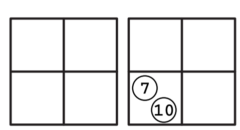
T: ***Who can put 34 on the Minicomputer using a ⑩-checker and one of these other weighted checkers?***



= 34



= 34

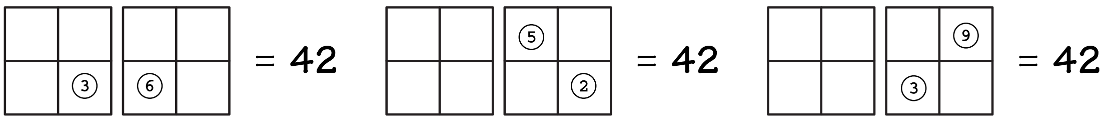


= 34

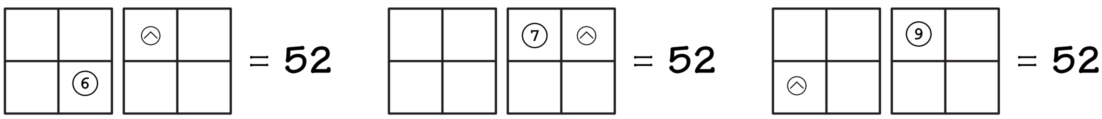
Note: You may like to call on two students to solve this problem. Let one student place just the ⑩-checker first, and then ask the other student to place another weighted checker, or vice-versa.

Continue in a similar manner with these problems.

T: ***Who can put 42 on the Minicomputer using any two of these weighted checkers ② through ⑨?***

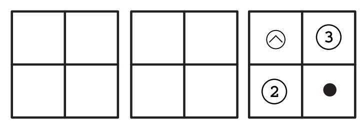


T: ***Who can put 52 on the Minicomputer using a negative checker and one of the other checkers?***



Exercise 2

Put this configuration on the Minicomputer.

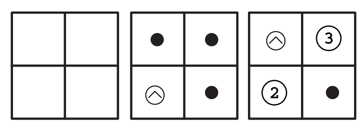


T: ***Write this number on your paper.***

Check several students' answers before asking one to answer aloud.

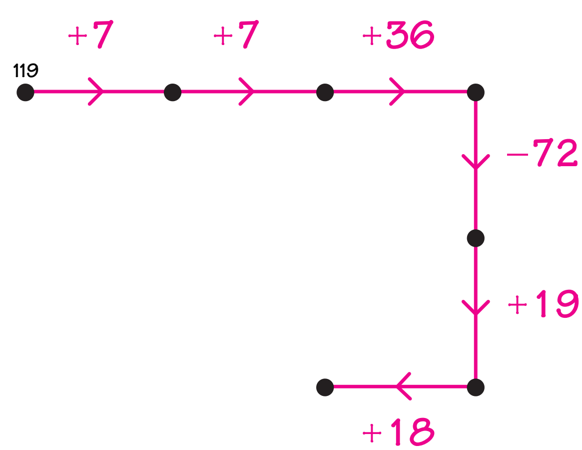
S: ***9.***

Then add these checkers to the tens board.



T: ***Write this number on your paper.***

While students are decoding the number on the Minicomputer, draw this arrow road on the board. Again check several papers before asking a student to announce that 119 is on the Minicomputer. Label the first dot of the arrow road as you explain,

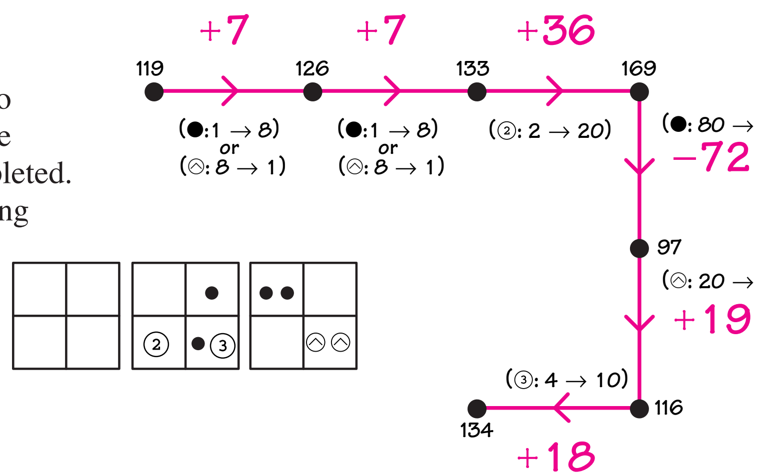


T: ***This arrow road starts at the number on the Minicomputer. Each arrow tells us how to change the number, but we may only move one checker to make a change.*** (Point to the first +7 arrow.) ***What move will make the number on the Minicomputer seven greater?***

S: ***Move the regular checker from the 1-square to the 8-square.***

S: ***Move the negative checker from the 8-square to the 1-square.***

Invite a student to make one of the moves on the Minicomputer; then ask a student to announce the new number. Continue in the same manner until the arrow road is completed. A completed arrow road with corresponding moves on the Minicomputer indicated is shown here.

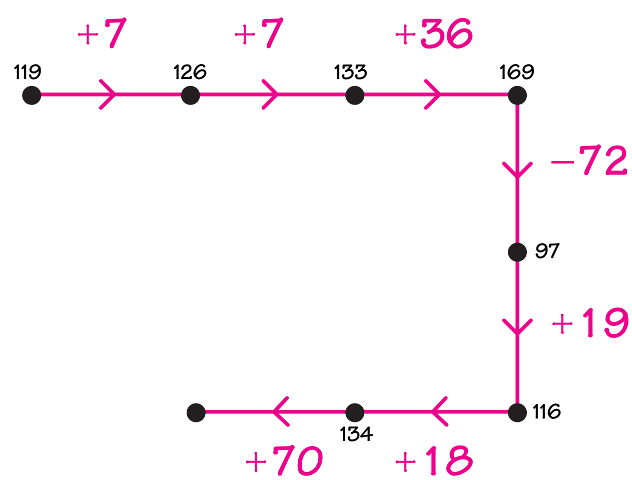


At this point you should have this configuration on the Minicomputer.

Ask students to verify that the number on the Minicomputer is 134.

T: ***Let's try to get 200 on the Minicomputer by moving checkers but with as few moves as possible. How much greater is 200 than 134?***

S: ***66; 134 + 66 = 200.***



Allow students a few minutes to work on the problem of increasing the number by 66. Students may find other longer solutions, but lead to the following. First extend the arrow road with a +70 arrow.

T: ***Is there a single move that would make th***

S: ***Move the regular checker from the 10-squ***

Let a student make the move.

T: ***What number is on the Minicomputer now?***

S: ***204.***

T: ***How can we get 200?***

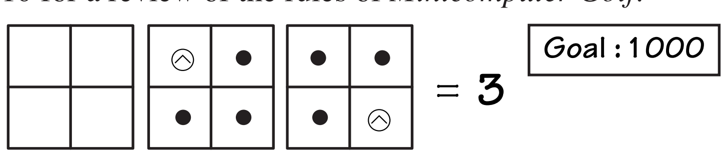
S: ***Move a regular checker from the 8-square to the 4-square, a decrease of 4.***

Extend the arrow road with a –4 arrow ending at 200. Make the final move and ask students to verify that the number on the Minicomputer is 200.

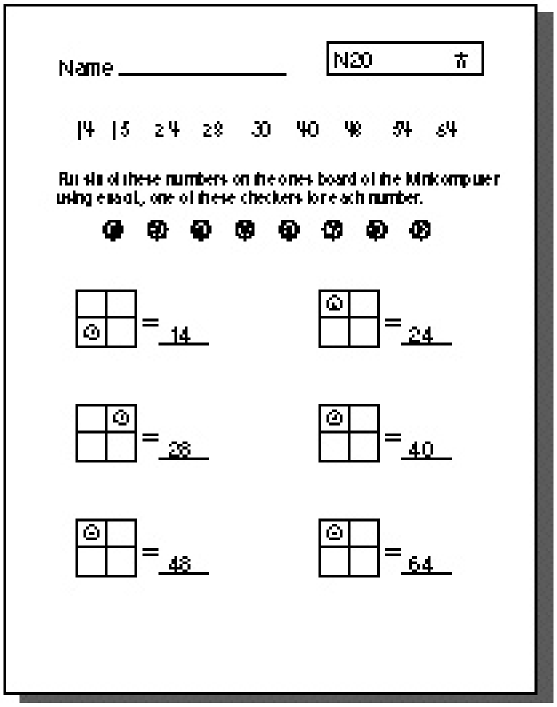
Exercise 3

Play *Minicomputer Golf*. A possible starting configuration and goal are given below.

Note: See Lesson N10 for a review of the rules of *Minicomputer Golf*.



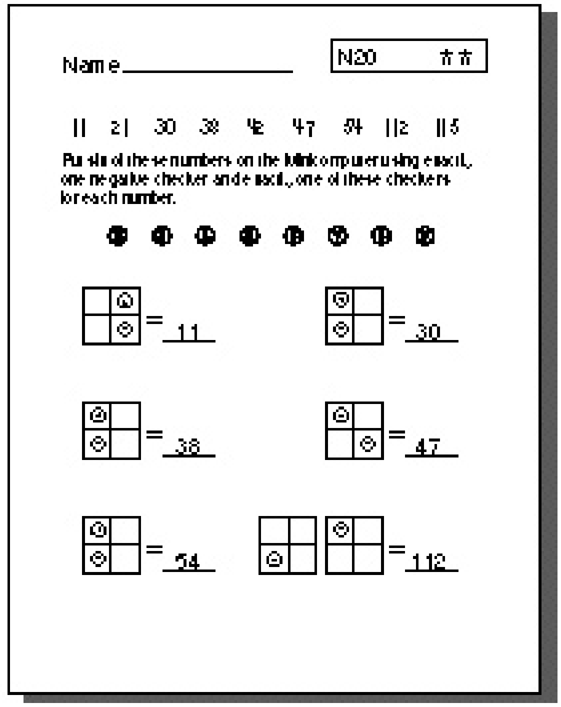
Worksheets N20*, **, ***, and **** are available for individual work if sufficient time is available.


Name ______________ N20 ☆

14 15 24 28 30 40 48 54 64

Put six of these numbers on the ones board of the Minicomputer using exactly one of these checkers for each number.

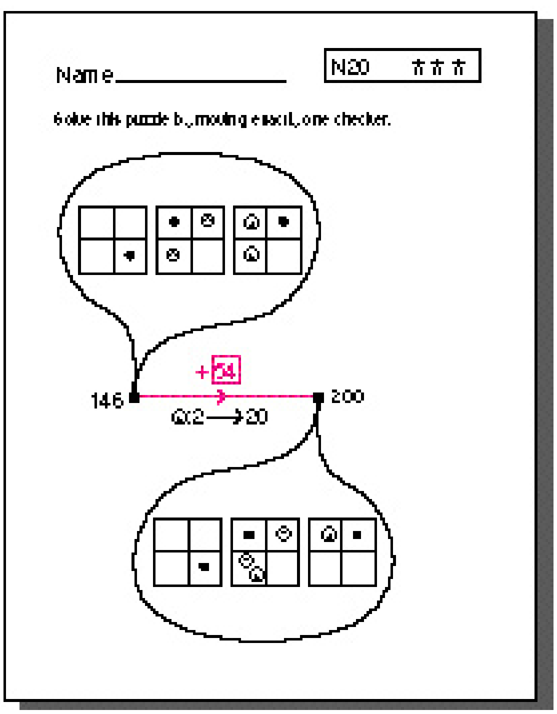
= 14
= 24
= 28
= 40
= 48
= 64


Name ______________ N20 ☆☆

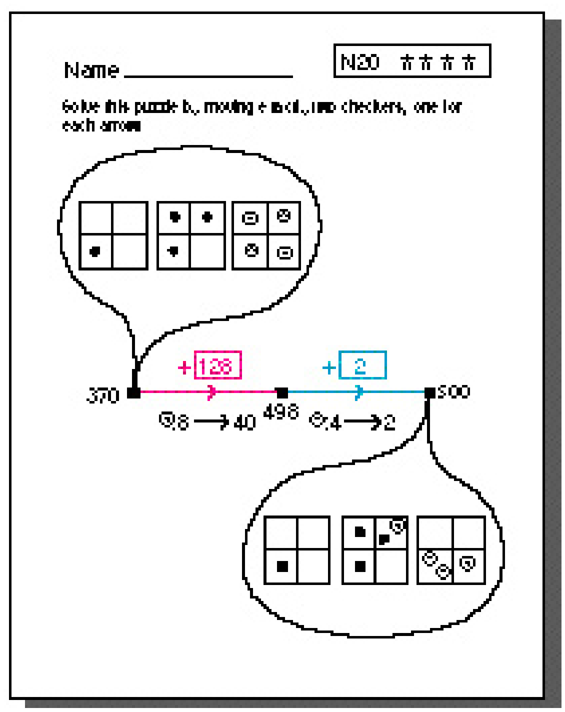
11 21 30 38 42 47 54 112 115

Put six of these numbers on the Minicomputer using exactly one negative checker and exactly one of these checkers for each number.

= 11
= 30
= 38
= 47
= 54
= 112


Name ______________ N20 ☆☆☆

Solve this puzzle by moving exactly one checker.


Name ______________ N20 ☆☆☆☆

Solve this puzzle by moving exactly two checkers, one for each arrow.

Capsule Lesson Summary

Find a shortest connection between two given whole numbers in which two whole numbers are joined by a red cord if and only if one of them is one more than the other, and the numbers are joined by a blue cord if and only if one of them is ten times the other. Solve a detective story involving these relations.

Materials

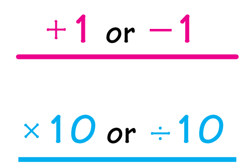
Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N21(a) and (b)

Description of Lesson

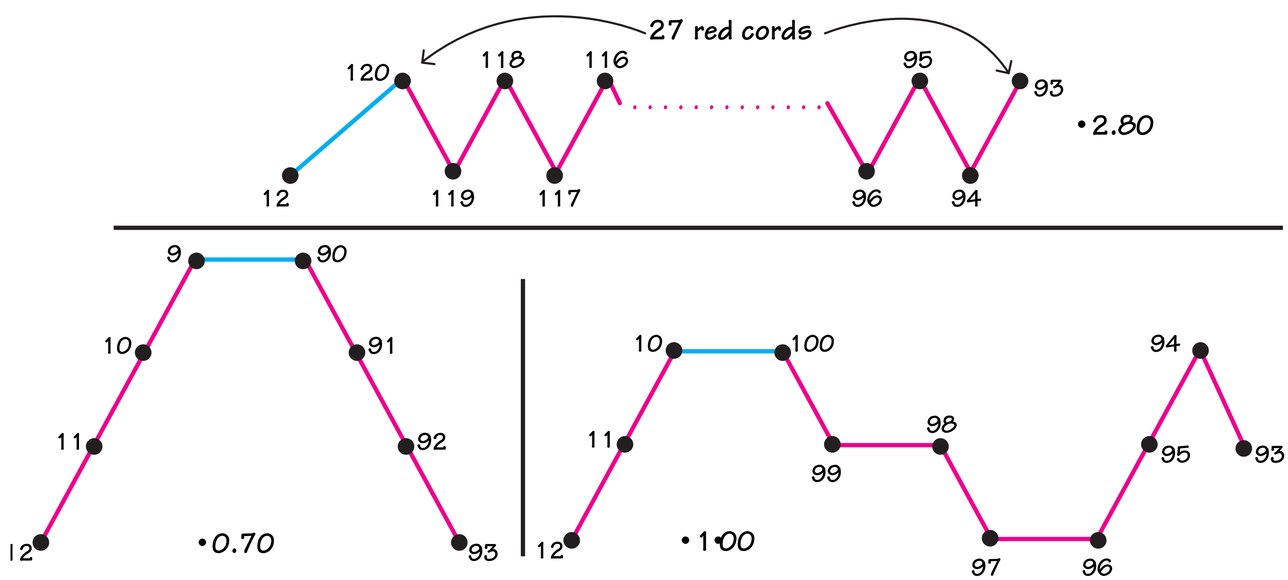
Exercise 1

Put cord keys on the board as you explain this situation.



T: ***The whole numbers live in a town with two bus lines—red and blue. Buses travel between houses as shown by the red and blue cords. Passengers pay 10¢ for each section of a trip (each time the bus stops). Let's find several ways to travel from 12 to 93 and calculate the cost of each trip.***

Draw dots for 12 and 93 and encourage students to suggest several solutions. Draw and calculate the cost of each road. Try to get at least two roads, including the 70¢ road shown below. You need not draw all of the red cords for especially long solutions.



T: ***What is the cost of the shortest road?***

S: ***70¢, since there are seven cords.***

T: ***This is the shortest road. For this road, even thought 93 is more than 12, we subtracted 1s to go from 12 to 9. Why was that useful?***

S: ***Because we wanted to get a number close to 93. 10 x 9 = 90 and 90 is close to 93.***

Erase the roads and draw two dots on the board, one for 42 and one for 69.

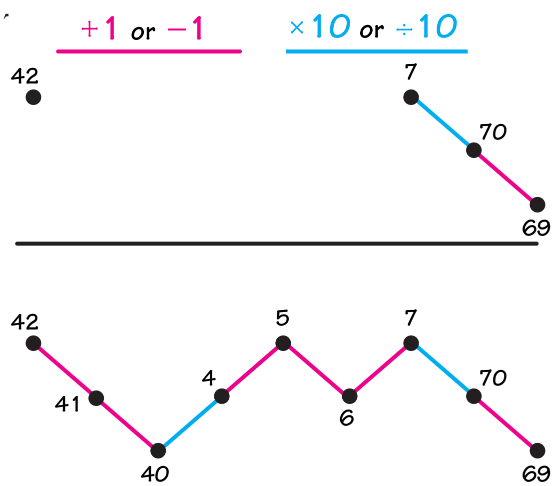
T: ***Using the same rules (red and blue cords), let's find roads between 42 and 69. Remember that we are in the world of whole numbers, so all of the dots are for whole numbers.***

Draw and calculate the cost of roads students suggest. This problem is harder than the previous one, and the class may have difficulty finding a shortest road. If necessary, give hints for finding a shortest road as in the following dialogue.

T: ***Starting at 42, our goal is 69. Can we multiply by 10 to reach a number close to 69? What number might we aim for?***

S: ***70, since 70 is a multiple of 10. Then we could connect 70 to 7 because 10 x 7 = 70.***

Draw a red cord from 69 to 70 and a blue cord from 70 to 7.



T: ***To finish the road, we must go from 42 to 7.***

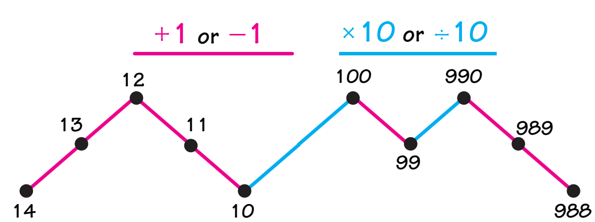
S: ***Subtract two 1s to get 40 and then divide by 10.***

S: ***Add three 1s to go from 4 to 7.***

T: ***What is the cost of this road?*** (80¢) ***This is the shortest road. It's interesting that to build a shortest road from 42 to 69, we went through 4, a number much less than either 42 or 69.***

Draw two dots on the board, one for 14 and one for 988.

Again ask for roads between 14 and 988, and attempt to find a shortest road. If after several attempts no one suggests the 90¢ solution shown below, give hints as in the following dialogue. Emphasize the role multiples of 10 play in finding a shortest road.



T: ***What multiple of 10 is closest to 988?*** (990)

S: ***Draw red cords from 990 to 989 to 988.***

S: ***Now draw a blue cord from 99 to 990.***

T: ***Let's try to build a short road from 14 to 99.***

S: ***100 is a multiple of 10 close to 99.***

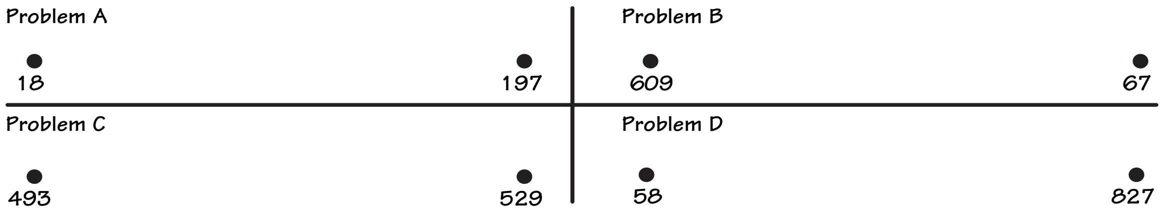
S: ***We could draw a blue cord from 10 to 100.***

S: ***Use red cords to go from 10 to 14.***

T: ***How much does a bus trip along this shortest road cost?*** (90¢)

Erase the board and pose these four problems.

+1 or −1 ×10 or ÷10



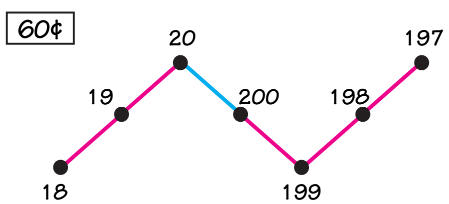
T: ***Build roads between the numbers in each problem. When you finish a problem, I'll tell you if you found a shortest road.***

Note: Each of these problems has only one shortest solution; however, some problems of this type do have more than one shortest solution. In Lesson N22 there is an example.

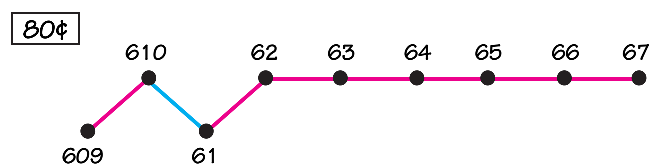
Allow students to work individually for 10 to 15 minutes. You may wish to give hints to students having difficulty. These are some suggestions.

+1 or −1 ×10 or ÷10

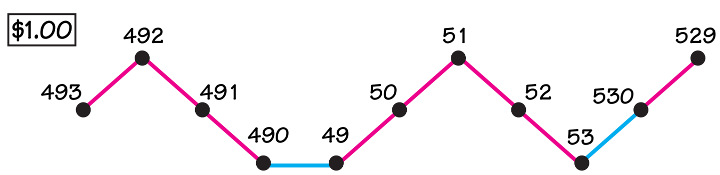
Problem A
What multiple of 10 is closest to 197? (200)



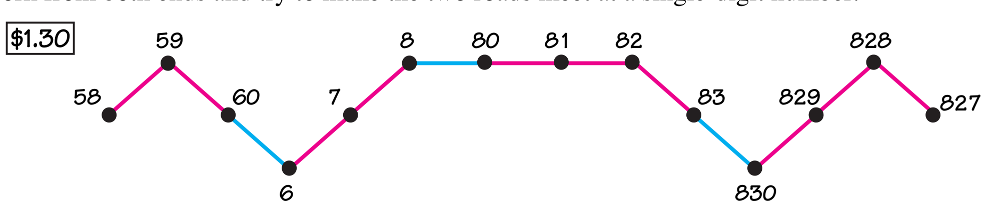
Problem B
- 607÷ 10 = 60.7; 607 ÷ 10 ≠ 67.
- What multiple of 10 is close to 609? (610)



Problem C
The least number in the shortest road is 49.



Problem D
Work from both ends and try to make the two roads meet at a single-digit number.



Exercise 2

Distribute copies of Worksheets N21(a) and (b). Prepare to solve this detective story collectively. You may like to let students work on the clues with partners before collecting information from the class .

T: ***This detective story uses the same red cords and blue cords we've been using for roads. We are going to try to find the address number of Julia's friend Theresa.***

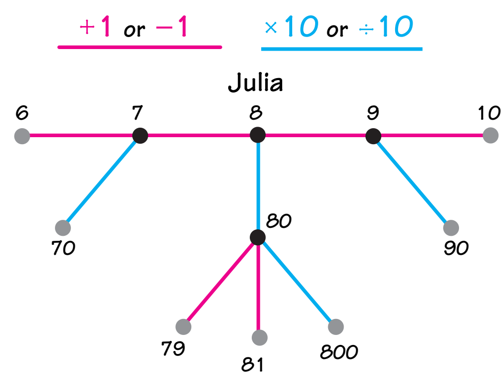
Clue 1

T: ***Julia lives at 8. A cheapest bus ride from Theresa's house to Julia's house costs 20¢. Theresa's address number is a whole number. Where could Theresa live? Why?***

S: ***At 81. We can draw a blue cord from 8 to 80 and a red cord from 80 to 81.***

Accept two or three more possibilities for Theresa's address number and put these in a picture on the board. Then ask students to use Worksheet N21(a) to find all of the possibilities for Theresa's address number.

As they work, invite students to the board to show these numbers. Continue until your picture looks similar to this and includes all seven possibilities: 6, 10, 70, 79, 81, 90, or 800.



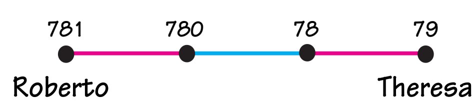
Clue 2

Put the information from Worksheet N21(b) on the board.

T: ***Theresa's friend Roberto lives at 781. A cheapest bus ride from Theresa's house to Roberto's house costs 30¢. To find where Theresa lives, build a road with three cords from 781 to one of the possible address numbers for Theresa.***

Let students work independently or with a partner until many students have determined that Theresa's address number is 79.

Invite a student to draw a road with three cords between 781 and 79 on the board.



T: ***Did anyone find another possibility for Theresa's address?***

S: ***No. There are many other numbers that are a 30¢ bus ride from Roberto's house, but the ones that I found are not in the list of possibilities from the first clue.***

Conclude that Theresa does live at 79.

Note: Of the possible addresses listed in the first clue, only 79 is three cords from 781. This fact need not be proven in class, but you may wish to discuss the following method of proof: Draw a cord picture that shows all of the numbers three cords from 781. Check that of the possible addresses listed in the first clue, only 79 is three cords from 781.

N22 DECIMAL DISTANCE #2

Capsule Lesson Summary

Using red and blue cords as defined in Lesson N21, find a shortest road between two numbers. Find the greatest number and the least number at a specified decimal distance from a given whole number, the decimal distance being the number of cords in a shortest road between two numbers. Find numbers that are at the same decimal distance from two given whole numbers.

Materials

Teacher	• Colored chalk	**Student**	• Paper • Colored pencils, pen, or crayons • Worksheets N22* and **

Description of Lesson

Exercise 1

Put cord keys on the board as you recall the situation.

+1 or −1

×10 or ÷10

T: ***Do you remember the town of whole numbers with red and blue bus lines?***

Let students recall the story. These four points should be mentioned:

- Red and blue cords indicate the routes that the two buses may take.
- Passengers pay 10¢ for each section of the road (stop) the bus takes.
- Passengers try to find the shortest routes to their destinations.
- All address numbers are whole numbers.

T: ***How could we travel between 6 and 87 for the least amount of money?***

Consider any routes that students suggest. Except for especially long solutions, draw and calculate the cost of each route. You might let several students contribute to a solution.

S: ***Go from 6 to 9 with three red cords.***

S: ***Draw a blue cord from 9 to 90.***

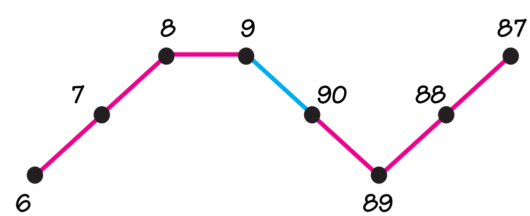
S: ***Draw three red cords from 90 to 87.***

T: ***How much does this route cost?***

S: ***70¢, since there are seven cords.***

+1 or −1

×10 or ÷10

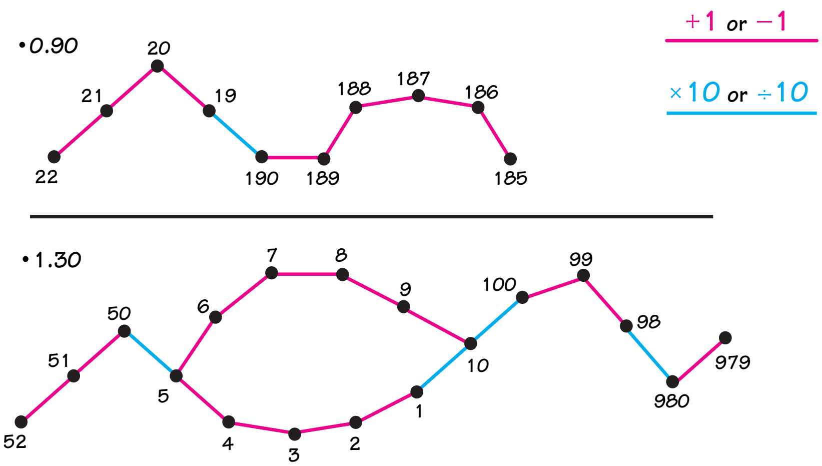


This is the shortest road between 6 and 87. Encourage the class to find this road.

In a similar manner, present and solve the following two problems, giving hints when necessary. You may like to let students work in pairs before solving the problems collectively.

- Build a shortest road between 22 and 185. (Hint: Aim for a multiple of 10 near 185.)
- Build a shortest road between 52 and 979. (Hint: Work from both ends. Try to meet at a single-digit number.)

If the class only discovers one solution for the second problem, show the other solution yourself. Emphasize that there are two distinct paths of the same length between 5 and 10.



Erase the board before going on to Exercise 2.

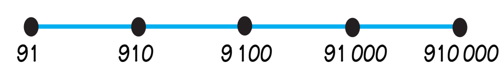
Exercise 2

Display a picture with the given information as you pose this problem.

T: ***Suppose we start at 91. What is the greatest number we can reach using exactly four cords? Write your answer on a piece of paper.***

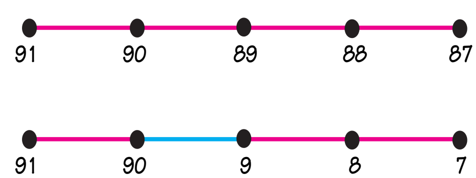
Check several answers before asking students to illustrate a solution on the board.

S: ***910 000. Use four blue arrows for*** x***10.***

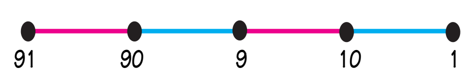


T: ***Suppose we again start at 91. What is the least number we can reach with exactly four cords? Draw a cord road to show your solution.***

Allow several minutes for students to work individually or with a partner on this problem. For those who draw one of these two roads, challenge them to find a number less than 87 (or 7). You might give a hint such as, "The cords from 91 to 90 and then to 9 are a good start."



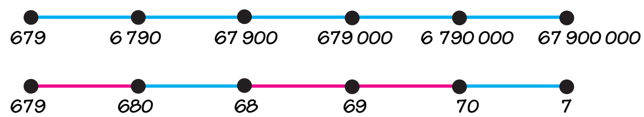
When several students find a road to 1, invite them to draw it on the board.



Present these problems in a similar manner.

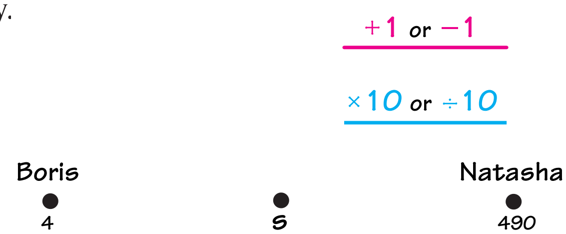
- Starting at 679, what is the greatest number we can reach with five cords?
- Starting at 679, what is the least number we can reach with five cords?

Solutions:



Exercise 3

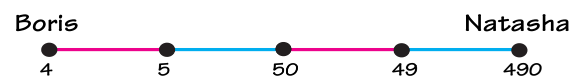
Put this information on the board as you tell this story.



T: ***Boris and Natasha are two spies. Their archenemy is Medussa. Medussa bugged Boris's phone and overheard part of a telephone conversation between Boris and Natasha. Medussa heard that Boris and Natasha were planning a secret meeting. Medussa did not hear where they would meet, but she did hear that Boris was at 4 and Natasha was at 490. They each would take a bus and spend the same amount of money going to the meeting place.***

I've marked the secret meeting place* s*. The number of cords from 4 to* s *must equal the number of cords from 490 to* s*. Of course, each spy takes a direct route. Medussa doesn't know how long each bus ride is. If you were Medussa, at which house would you wait?

Direct students to work on the problem individually or with a partner. After a few minutes, select a student who gives 50 as a solution to draw the appropriate cord road on the board.



T: ***Could Boris and Natasha meet at 50?***

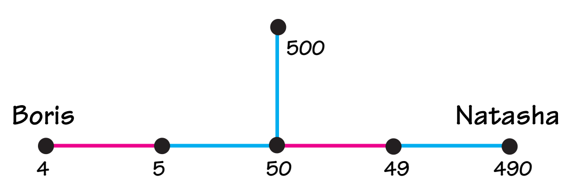
S: ***Yes, 50 is two cords from both 4 and 490.***

T: ***Well, Medussa also thought they would meet at 50, but she missed them! Where else could they have met?***

If necessary, tell students that Boris and Natasha each spent more than 20¢.

Note: All possible solutions pass through 50. Let students discover this fact in the picture that evolves on the board.

Invite students to describe their solutions, and record them in one picture as shown. For example, a student may suggest 500.



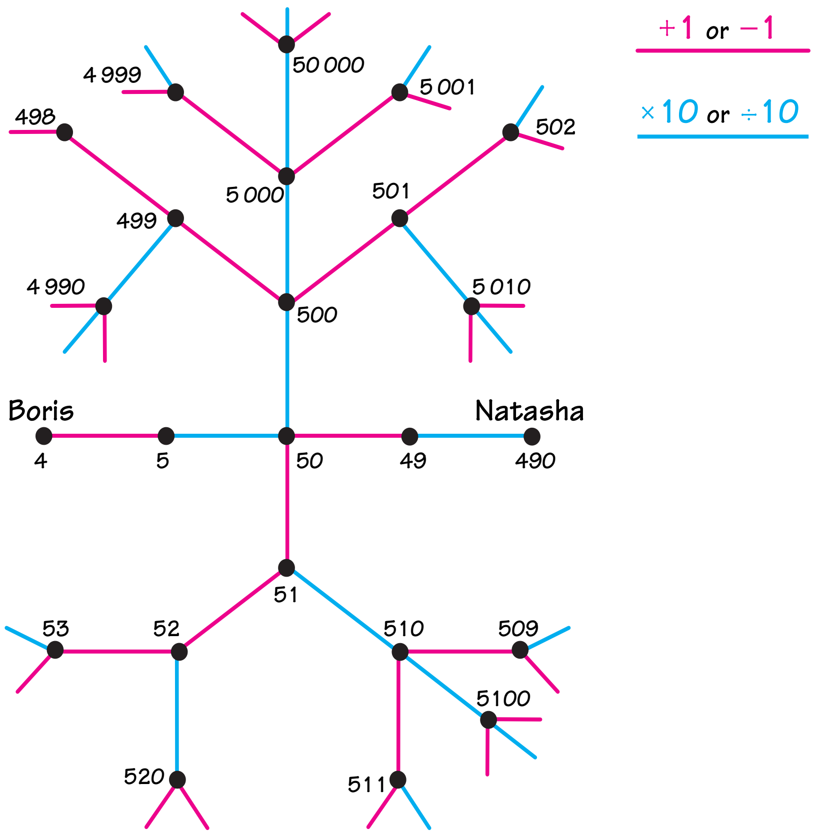
T: ***Why is 500 a possible meeting place?***

S: ***500 is 30¢ away from 4 and from 490.***

Students may protest that 500 is not a solution since they could have met at 50, which is closer to each spy. Explain that 500, as well as 50, is a possible solution because the shortest bus ride from 4 to 500 is 30¢ and the shortest bus ride from 490 to 500 is also 30¢, so each spy took the shortest route to 500 and each spent the same amount of money.

T: ***Where else could Boris and Natasha have met?***

All of the numbers (except 4, 5, 49, and 490) in the following picture are possible solutions, as well as many others. Students certainly will not find all of the solutions. Continue until students realize that many meeting places are possible and that all of the roads from Boris and Natasha to the possible meeting places go through 50.



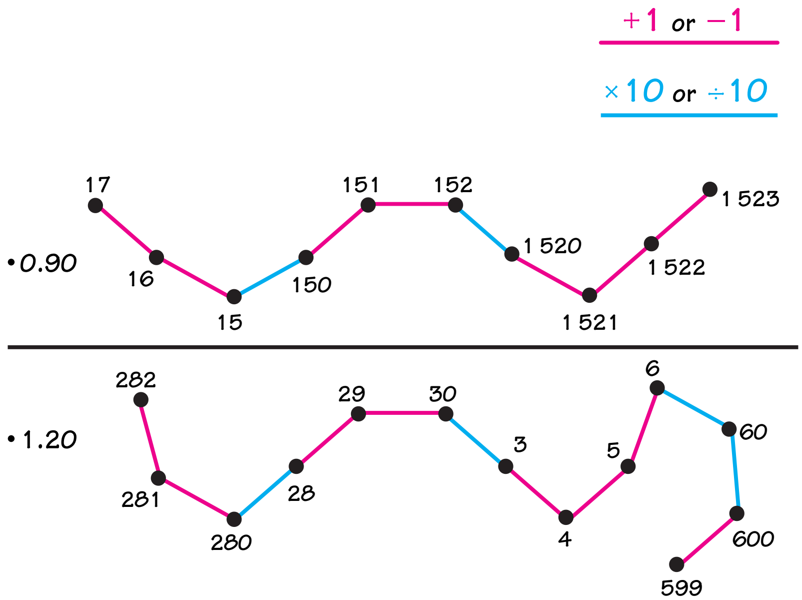
Conclude that no one knows for sure where Boris and Natasha met that day, but we do know that Medussa was waiting at 50 and missed them. Unfortunately for her, she did not spot them when their buses went by.

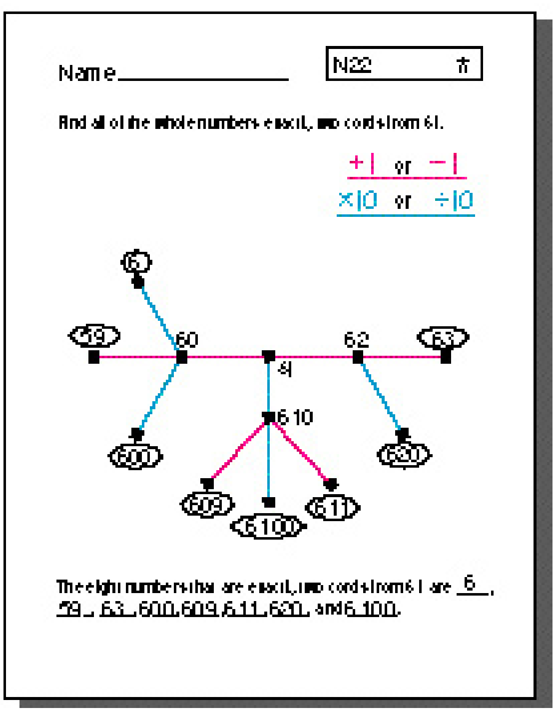
Note: Not all branches of the tree continue indefinitely. For example, 50, 51, 52, 53, 54, and 55 are solutions, but 56 is not a solution. The shortest road from 490 to 56 goes through 50 and has eight cords, while the shortest road from 4 to 56 goes through 60 and has only seven cords.

Worksheets N22* and ** are available for individual work. You may wish to begin the * problem collectively, find two or three solutions, and then ask students to complete the problem on their own. Note that 61 is not a solution.

For students who finish early, write these two problems on the board.

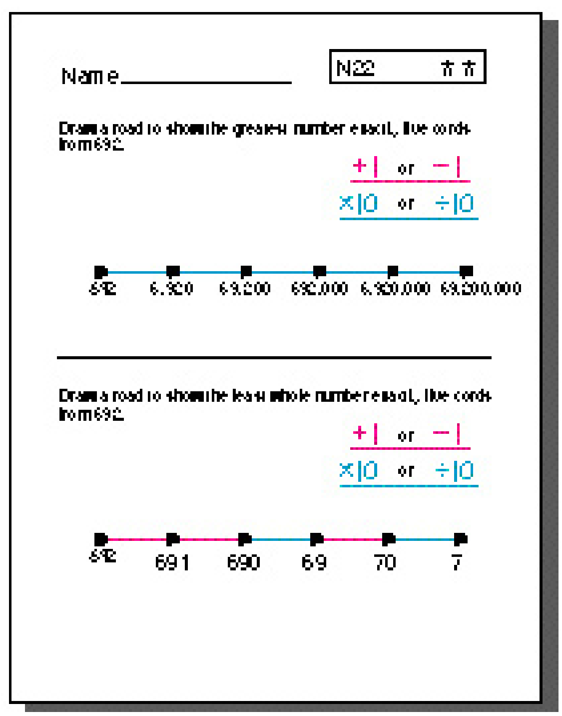
- Build a shortest road between 17 and 1 523.
- Build a shortest road between 282 and 599.





Find all of the whole numbers exactly two cords from 61.

The eight numbers that are exactly two cords from 61 are 6, 59, 63, 600, 609, 611, 620, and 6 100.



Draw a road to show the greatest number exactly five cords from 692.

Draw a road to show the least whole number exactly five cords from 692.

Capsule Lesson Summary

Associate a fraction with each of the regions in a shape by comparing areas. Cut apples into halves, fourths, and eighths as a model for doing calculations such as $\frac{1}{4} + \frac{1}{2}$, $6 \times \frac{1}{4}$, $1 \div 2$, and $3\frac{1}{2} - 1\frac{3}{4}$. Count forward and backward by fourths. Build an arrow road using $+\frac{3}{4}$ and $+\frac{1}{2}$ arrows.

Materials

Teacher
- Five to eight apples (same size)
- Knife
- Paper towels
- Bowl of water
- Colored chalk

Student
- Paper
- Worksheets N23*, **, and ***

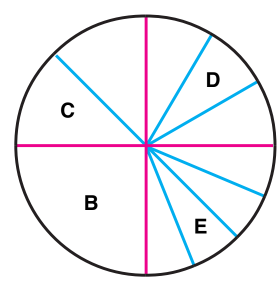
Advance Preparation: Before starting the lesson, prepare the circle and square pictures for Exercise 1 on the board. You may prefer to make overhead materials with Blacklines N23(a) and (b).

Description of Lesson

Note: You may prefer to do Exercise 1 with a fraction manipulative. Blackline N23(a) may be used to prepare your own fraction manipulatives on cardstock. With manipulatives, let students demonstrate how many pieces of a particular size fill the circle.

Exercise 1

Draw this picture on the board.



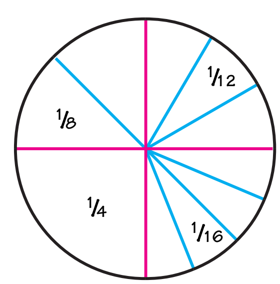
Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: ***The inside of this circle is divided into sections. What fraction of the whole shape is this section*** (trace the border of **B**)***?***

S: ***$\frac{1}{4}$. Four pieces of that size would fill the circle exactly.***

Label section **B**.

Similarly, invite students to label sections **C**, **D**, and **E**.



T: ***What other fraction names for $\frac{1}{4}$ does the picture suggest?***

S: ***$\frac{2}{8}$. It takes two of the $\frac{1}{8}$ pieces to fill $\frac{1}{4}$ of the circle.***

Use students' suggestions to make a list of equivalent names for $\frac{1}{4}$.

$$\frac{1}{4} = \frac{2}{8} = \frac{3}{12} = \frac{4}{16}$$

T: ***What patterns do you notice in this list of names for $\frac{1}{4}$?***

S: ***The top numbers increase by one while the bottom numbers increase by four.***

S: ***The bottom numbers are multiples of 4.***

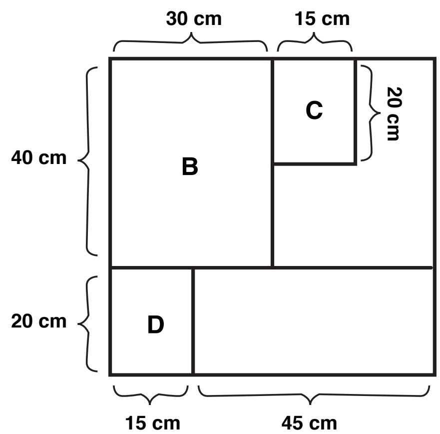
N23

S: ***To get another name for ¼, multiply both the numerator and the denominator by the same number. For example, ¼ = 3⁄12 since 3 x 1 = 3 and 3 x 4 = 12.***

T: ***The picture on the board suggests several names for ¼. Can you use the patterns we've noticed to find more names for ¼?***

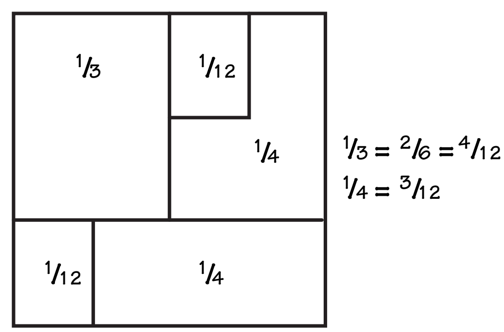
Extend your list with additional names for ¼; for example, $\frac{1}{4} = \frac{2}{8} = \frac{3}{12} = \frac{4}{16} = \frac{5}{20} = \frac{6}{24} = \frac{8}{32} = \frac{25}{100}$.

Accurately draw the following picture on the chalkboard. A 60-cm square is convenient. (Measurements are given for your information; do not write them on the board.)



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

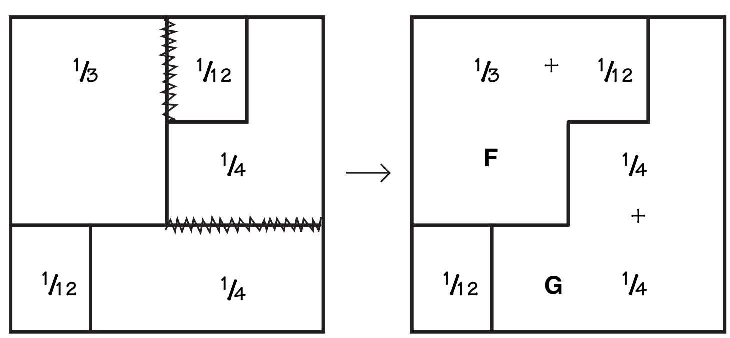
Starting with the smaller regions, invite students to tell what fraction of the whole square is covered and to explain their answers. Equivalent fractions arise naturally in the situation. List equivalent fractions on the board, but write the simplest names in the regions.



S: ***This region (B) is 4⁄12 of the square. One of the small pieces (C or D) is 1⁄12 of the square, and it takes four of those pieces to cover the region (B).***

S: ***That region (B) is ⅓ of the square. Three of those pieces would cover the whole square.***

Erase two line segments from the picture as indicated by the zigzags in the illustration on the left below.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: ***What fraction of the square is this region (F)?***

S: ***5⁄12. Five 1⁄12 pieces would cover it.***

T: ***If we join regions covering ⅓ of and 1⁄12 of the square, we get a region covering 5⁄12 of the square. ⅓ + 1⁄12 = 5⁄12.***

Similarly conclude that region **G** is ½ of the square and that ¼ + ¼ = 2⁄4 = ½.

Exercise 2

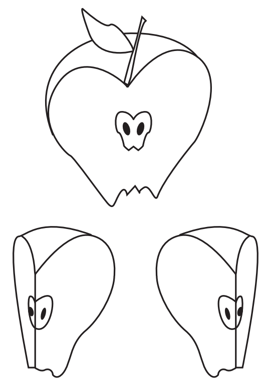
On a table, arrange five to eight apples, a knife, paper towels, and a bowl of water. During this exercise, place the apple pieces in the bowl of water to retard their browning and to insure their palatability at the end of the lesson.

T: ***I am going to cut the apples, and your job is give us number sentences about the cuts.***

Cut an apple in half. Encourage students to mention several number sentences involving various operations, but accept only ones suggested by the cut. List students' sentences on the board, for example:

$$\tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{2}{2} = 1 \qquad 2 \times \tfrac{1}{2} = 1 \qquad 1 \div 2 = \tfrac{1}{2}$$
$$\tfrac{1}{2} \times 1 = \tfrac{1}{2} \qquad 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

Cut $\frac{1}{2}$ of the apple in half to produce two $\frac{1}{4}$ pieces.
Show the half and two quarters of the apple to the class.



T: ***What fraction of the apple is each of the small pieces?***

S: ***$\frac{1}{4}$.***

T: ***What number sentences do these pieces suggest? You may use addition, subtraction, multiplication, or division.***

You may need to give a few examples yourself to encourage creativity.
Let students explain each example in terms of the apples.

$$4 \times \tfrac{1}{4} = \tfrac{1}{4} \times \tfrac{1}{4} \times \tfrac{1}{4} \times \tfrac{1}{4} = 1 \qquad \tfrac{1}{4} \times \tfrac{1}{2} = \tfrac{3}{4}$$
$$\tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4} \qquad 1 - \tfrac{1}{4} = \tfrac{3}{4} \qquad 1 - \tfrac{3}{4} = \tfrac{1}{4}$$
$$\left(\tfrac{1}{2} \times \tfrac{1}{2}\right) \times \left(\tfrac{1}{2} \times \tfrac{1}{2}\right) = \tfrac{1}{2} \qquad \tfrac{1}{2} \div 2 = \tfrac{1}{4}$$

Cut one quarter of the apple in half to produce two $\frac{1}{8}$ pieces. Ask for more number sentences.

$$\tfrac{1}{2} \times \tfrac{1}{4} = \tfrac{1}{8} \qquad \tfrac{1}{8} \times \tfrac{1}{8} = \tfrac{1}{4} \qquad \tfrac{1}{8} \times \tfrac{7}{8} = 1$$
$$1 - \tfrac{1}{8} = \tfrac{7}{8} \qquad 1 - \tfrac{1}{4} - \tfrac{1}{8} = \tfrac{5}{8} \qquad \tfrac{1}{4} \div 2 = \tfrac{1}{8}$$
$$\tfrac{1}{8} \times \tfrac{1}{2} = \tfrac{5}{8} \qquad \tfrac{1}{8} \times \tfrac{1}{4} = \tfrac{3}{8} \qquad \tfrac{1}{2} - \tfrac{1}{8} = \tfrac{3}{8}$$
$$\tfrac{5}{8} = \tfrac{1}{8} \times \tfrac{1}{8} \times \tfrac{1}{8} \times \tfrac{1}{8} \times \tfrac{1}{8} = \tfrac{1}{2} \times \tfrac{1}{8} = \tfrac{1}{4} \times \tfrac{1}{4} \times \tfrac{1}{8} = \tfrac{1}{4} \times \tfrac{3}{8}$$

Set aside the cut apple. Cut one apple into halves, two apples into fourths, and one apple into eighths. Write this problem on the board.

$$\tfrac{3}{4} \times \tfrac{3}{4} =$$

T: ***What number is $\frac{3}{4} + \frac{3}{4}$?***

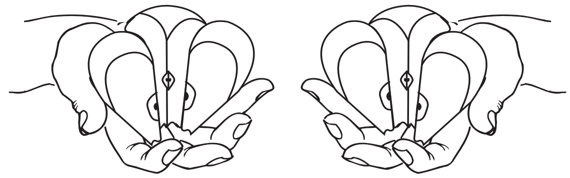
Suppose that some students incorrectly add the fractions by adding the numerators and adding the denominators, thus arriving at $\frac{6}{8}$. Use the apple cut in eighths as you ask,

T: ***Is $\frac{6}{8}$ apple (six $\frac{1}{8}$ pieces) more or less than 1 apple?***

S: ***Less.***

N23

Hold three $\frac{1}{4}$ pieces of an apple in each hand.



T: ***I have $\frac{3}{4}$ apple in one hand and $\frac{3}{4}$ apple in the other. Do I have more or less than 1 apple?***

S: ***More.***

T: ***$\frac{3}{4} + \frac{3}{4}$ is more than 1, but $\frac{6}{8}$ is less than 1; so $\frac{3}{4} + \frac{3}{4}$ does not equal $\frac{6}{8}$. To add $\frac{3}{4} + \frac{3}{4}$, think of the apples. How much am I holding?***

S: ***$\frac{6}{4}$ apples.***

S: ***$1\frac{2}{4}$ or $1\frac{1}{2}$ apples.***

Show that the six $\frac{1}{4}$ pieces form $1\frac{1}{2}$ apples.
Complete this number sentence on the board.

$$\frac{3}{4} \times \frac{3}{4} = \frac{6}{4} = 1\frac{2}{4} = 1\frac{1}{2}$$

Present some or all of the following problems, each in a similar manner. (Answers are in boxes.)

- Write the problem on the board.
- Ask for the answer and explanations.
- Show the class the appropriate apple pieces.
- Invite a student to demonstrate the answer with pieces of apple.
- Complete the appropriate number sentences.

$\frac{1}{2} + \frac{3}{4} = \boxed{1\frac{1}{4}}$ $\quad 2\frac{1}{2} + 1\frac{1}{4} = \boxed{3\frac{3}{4}}$

$1\frac{1}{4} + 2\frac{3}{4} = \boxed{3\frac{4}{4} = 4}$ $\quad 3\frac{3}{4} + 2\frac{3}{4} = \boxed{5\frac{6}{4} = 6\frac{2}{4} = 6\frac{1}{2}}$

$3\frac{1}{2} - 2 = \boxed{1\frac{1}{2}}$ $\quad 3\frac{1}{2} - \frac{1}{4} = \boxed{3\frac{1}{4}}$

$3\frac{1}{2} - 1\frac{1}{2} = \boxed{2}$ $\quad 3\frac{1}{2} - 3\frac{1}{4} = \boxed{\frac{1}{4}}$

$3\frac{1}{2} - \frac{3}{4} = \boxed{2\frac{3}{4}}$ $\quad 3\frac{1}{2} - 1\frac{3}{4} = \boxed{1\frac{3}{4}}$

Exercise 3

T: ***If I gave $\frac{1}{4}$ of an apple to each of you, how many apples would I distribute?***

Ask students to explain their answers. Do not indicate correct answers yet.

T: ***Let's find the answer by counting by fourths. I'll start: 0.***

Let students count by fourths in an order natural to their seating arrangement. As appropriate, use apple pieces to demonstrate an equivalence such as $\frac{5}{4} = 1\frac{1}{4}$.

S: ***$\frac{1}{4}$.***

S: ***$\frac{2}{4}$, or $\frac{1}{2}$.***

S: ***$\frac{3}{4}$.***

S: ***$\frac{4}{4}$, or 1.***

S: ***$\frac{5}{4}$, or $1\frac{1}{4}$.***

S: *$1^2/_4$, or $1^1/_2$.*

S: *$1^3/_4$.*

S: *2.*

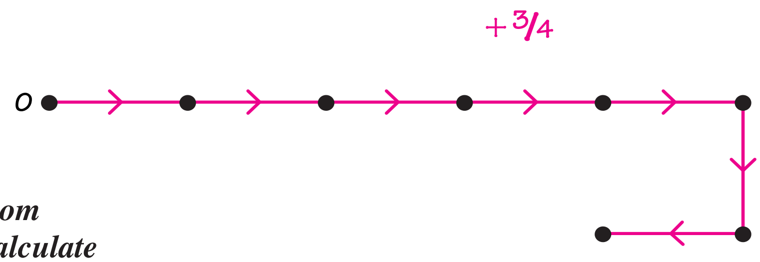
Continue until each student has responded exactly once. Check whether any student predicted the exact number of apples needed.

T: ***Let's count backward by fourths. I'll start at 10.***

Let the students count backward: $9^3/_4$, $9^2/_4$ or $9^1/_2$, $9^1/_4$, 9, $8^3/_4$, …. Continue until each student has responded exactly once. Use apples to demonstrate any calculations that students find difficult, for example, $9 - {}^1/_4$.

Exercise 4

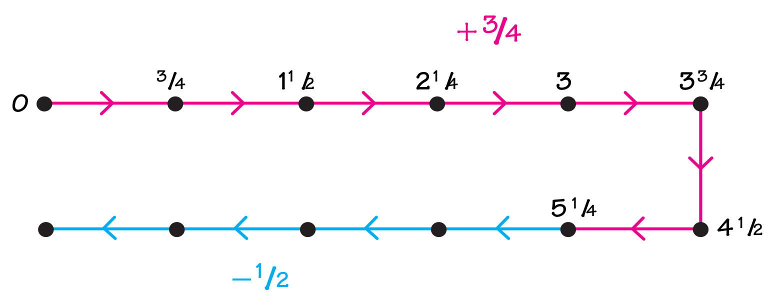
Draw this arrow road on the board.



T: ***Traditionally teachers receive apples from students. I have a friend who received $^3/_4$ apple from each of seven students. Let's calculate the number of apples that she received.***

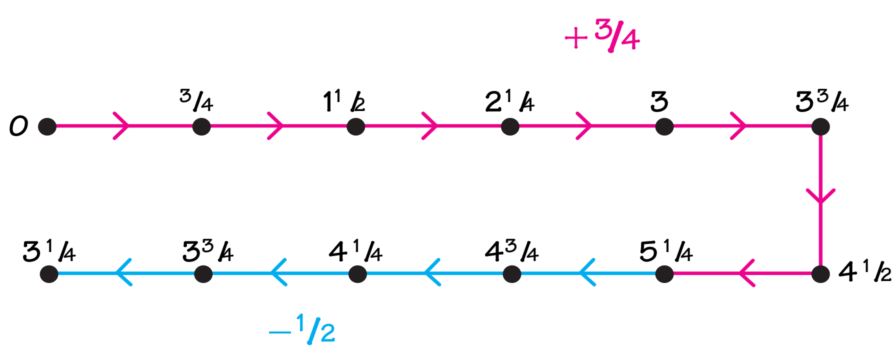
Invite students to label the dots for the red arrows. As students suggest answers, use the apples to demonstrate a calculation. For example, show that $1^1/_2$ apples plus $^3/_4$ apple equals $2^1/_4$ apples.

Add these blue arrows to the road.



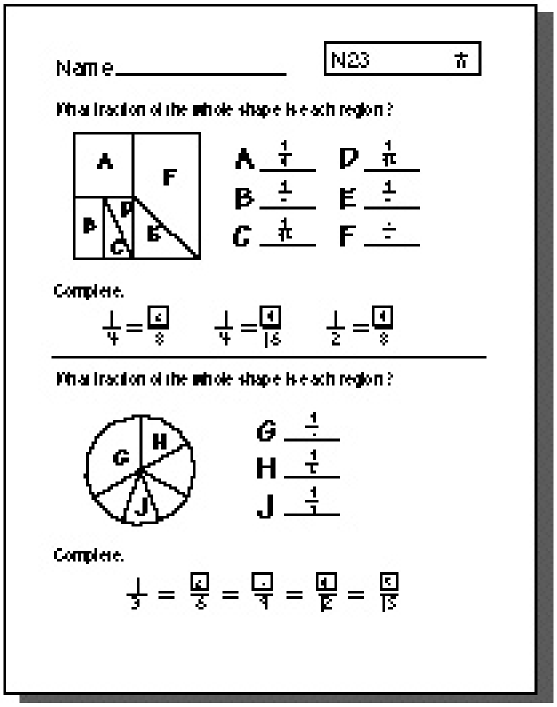
T: ***The teacher began eating a $^1/_2$ apple at a time.***

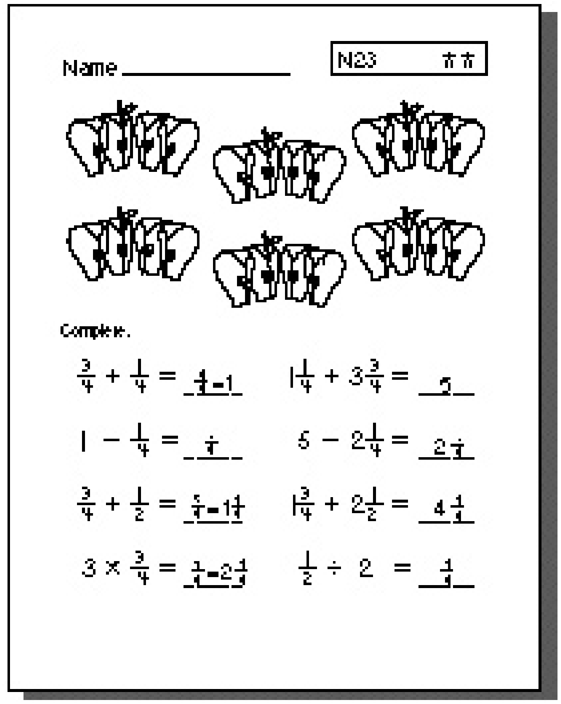
Invite students to label the dots. Show the appropriate number of apples for each calculation.

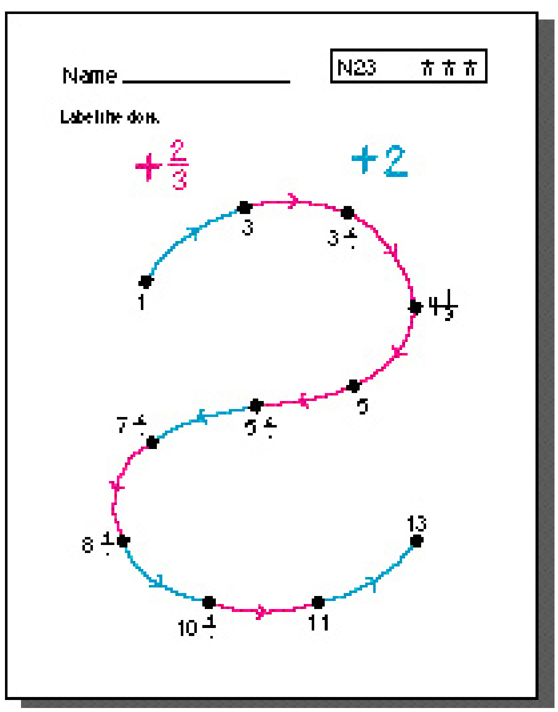


Worksheets N23*, **, and *** are available for individual work.

If you wish, give the pieces of apples to students.


Name
N23 ☆
What fraction of the whole shape is each region?
A B C D E F
Complete.
What fraction of the whole shape is each region?
G H J
Complete.


Name
N23 ☆☆
Complete.
$\frac{3}{4} + \frac{1}{4} =$
$1\frac{1}{4} + 3\frac{3}{4} =$
$1 - \frac{1}{4} =$
$5 - 2\frac{1}{4} =$
$\frac{3}{4} + \frac{1}{2} =$
$1\frac{3}{4} + 2\frac{1}{2} =$
$3 \times \frac{3}{4} =$
$\frac{1}{2} \div 2 =$


Name
N23 ☆☆☆
Label the dots.
$+\frac{2}{3}$
$+2$
1
3
5
11
13

Capsule Lesson Summary

Label dots and arrows in an arrow road. Build the shortest possible road between two numbers using only 2x and +0.3 arrows.

Materials

Teacher
- Colored chalk

Student
- Calculator
- Paper
- Colored pencils, pens, or crayons
- Worksheets N24*, **, ***, and ****

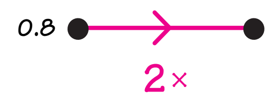
Description of Lesson

Exercise 1

Ask students to copy the arrow road as you draw it on the board one arrow at a time.

T: ***Today we are going to build an arrow road, and you will label some of the dots and arrows.***

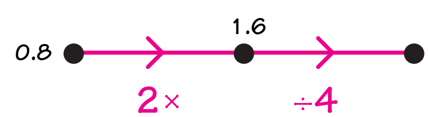
Start the road with a 2x arrow beginning at 0.8.



T: ***What number is here*** (point to the ending dot)***?***

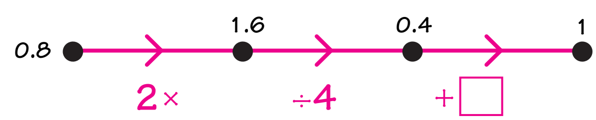
S: ***1.6.***

Label the dot and extend the arrow road.



T: ***What number is here*** (point to the ending dot of the ÷4 arrow)***?*** (0.4)

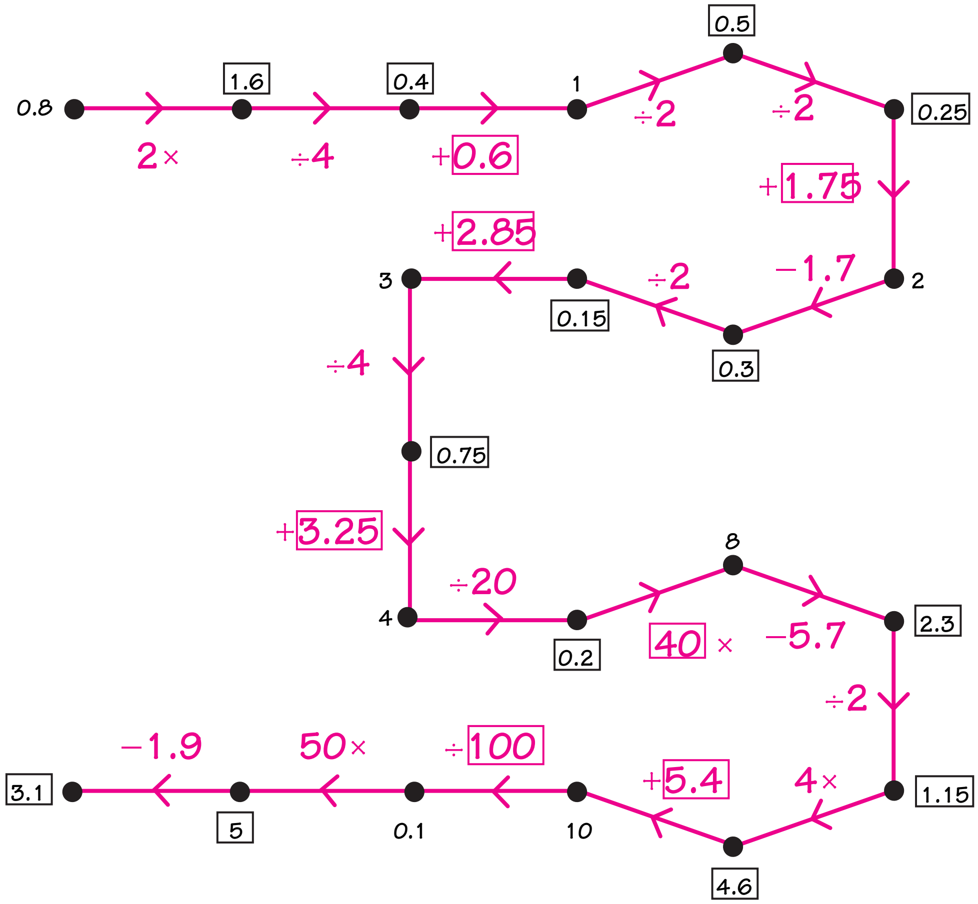
Label the dot and extend the arrow road.



T: ***It may help us to think about money in order to label this arrow.***
How much money would this be (point to 0.4)***?*** (40¢)
How much must we add to 40¢ to get $1? (60¢)
How do we write that as a decimal? (0.6)

N24

Continue extending the arrow road shown below as long as there is interest. (Answers are in boxes.)



Exercise 2

Put keys for red and blue arrows on the board.

Draw dots for the numbers 1 and 10.1 as you explain this problem.

T: ***Build an arrow road from 1 to 10.1. Try to use as few of these red and blue arrows as possible.***

Give students time to work independently or with partners. As students begin finding solutions, record and continually update the smallest number of arrows used. Encourage students to beat the number on the board. The shortest possible road has six arrows. When many students have found at least one solution, continue with the collective discussion.

T: ***Many of you have drawn at least one arrow road from 1 to 10.1. Let's try to find a way to build the shortest road. Rather than start at 1, suppose we look first at an arrow ending at 10.1. Which arrow—red or blue—should we use?***

S: ***Blue.***

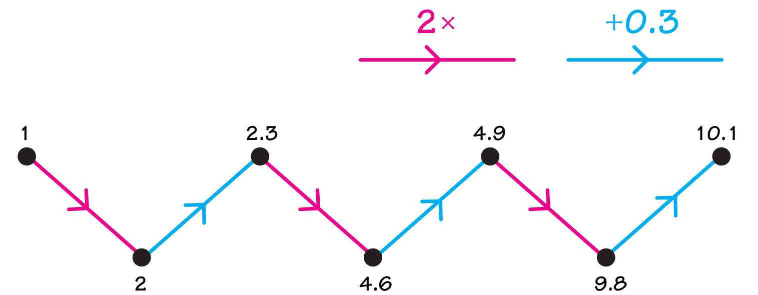
Perhaps a student will be able to explain that 2 x 5.05 = 10.1 and that it is impossible to build a road from 1 to 5.05 with these red and blue arrows.

Draw a blue arrow ending at 10.1.

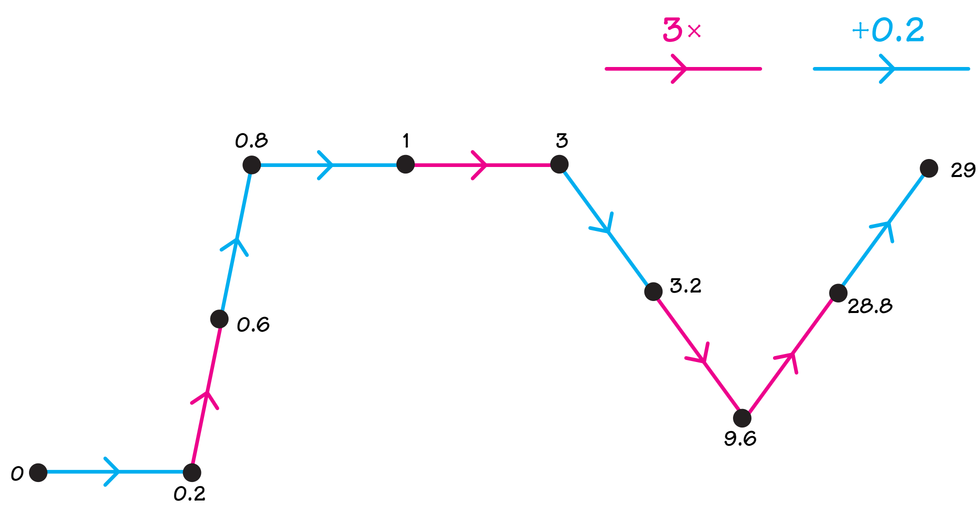
T (pointing to the starting point of the blue arrow)**:** ***What number is here?***

S: ***9.8.***

Continue following the class's directions until the road is complete. It is possible that your class will decide at some point to change the color of a previously agreed upon arrow. The shortest such road is shown below.



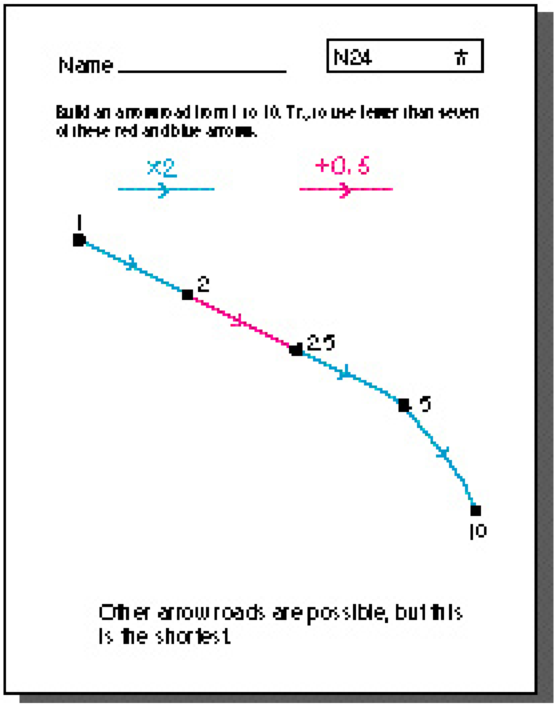
Repeat this activity, asking students to build an arrow road from 0 to 29 using only 3x and +0.2 arrows. The shortest such road is shown below.

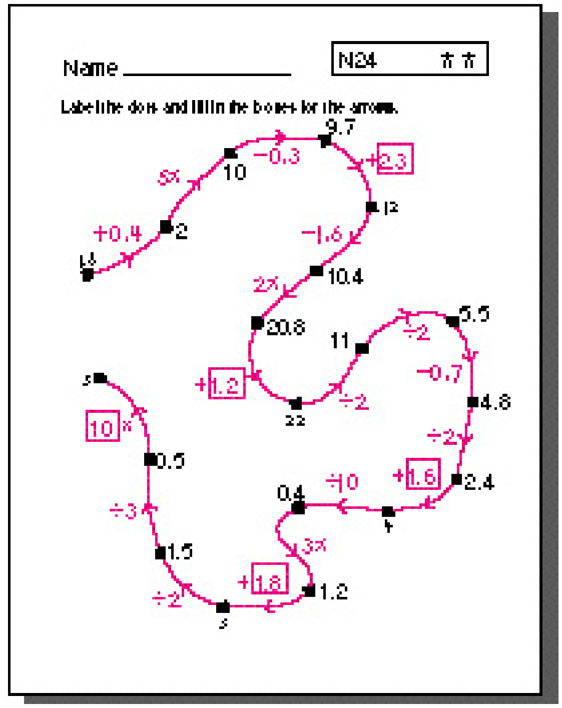


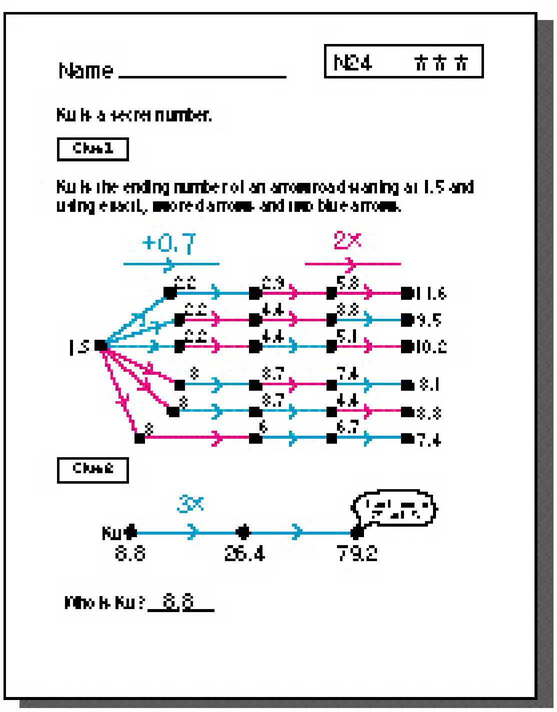
Worksheets N24*, **, ***, and **** are available for individual work.

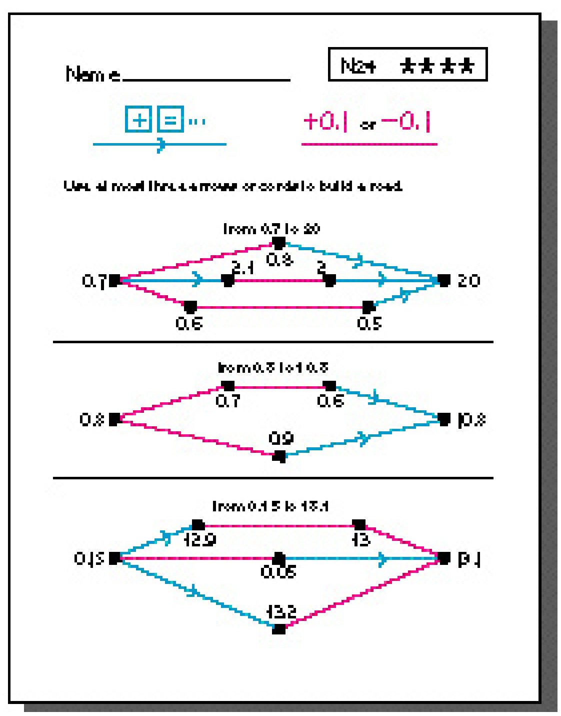
Home Activity

Send home a problem similar to that on Worksheet N24* for students to solve with a family member.


Name
N24 ☆
Build an arrow road from 1 to 10. Try to use fewer than seven of these red and blue arrows.
×2
+0.5
1
2
2.5
5
10
Other arrow roads are possible, but this is the shortest.


Name
N24 ☆☆
Label the dots and fill in the boxes for the arrows.
9.7
10
−0.3
+2.3
5×
12
+0.4
2
−1.6
10.4
2×
20.8
11
5.5
÷2
−0.7
+1.2
22
÷2
4.8
10×
0.5
÷2
÷10
+1.6
2.4
0.4
÷3
3×
1.5
+1.8
1.2
÷2


Name
N24 ☆☆☆
Ku is a secret number.
Clue 1
Ku is the ending number of an arrow road starting at 1.5 and using exactly three red arrows and two blue arrows.
+0.7
2×
1.5
11.6
9.5
10.2
8.1
8.8
7.4
Clue 2
3×
Ku
8.8
26.4
79.2
Who is Ku? 8.8


Name
N24 ☆☆☆☆
+0.1 or −0.1
Use at most three arrows or cords to build a road.
from 0.7 to 20
0.7
0.8
2.1
2
20
0.6
0.5
from 0.8 to 10.8
0.8
0.7
0.6
10.8
0.9
from 0.15 to 13.1
0.15
12.9
13
13.1
0.05
13.2

Capsule Lesson Summary

Label dots and arrows in pictures with arrows for “times a whole number.” Identify composites of relations in the arrow pictures.

Materials

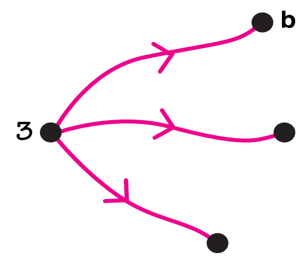
Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N25*, **, ***, and ****

Description of Lesson

Exercise 1

Draw this arrow picture on the board.



Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

T: ***Each red arrow is for multiplication by some whole number.***

Trace the red arrow from 3 to **b**.

T: ***Some whole number times 3 equals this number (at* b*). Could this number be 6?***

S: ***Yes, if the red arrow is for 2*x.**

T: ***Could the number (at* b*) be 15?***

S: ***Yes, if the red arrow is for 5*x.**

T: ***Could the number be 1.5?***

S: ***No, the red arrow would be for ½*x *and ½ is not a whole number.***

T: ***So this number (at* b*) cannot be 1.5. What other number could be here (at* b*)?***

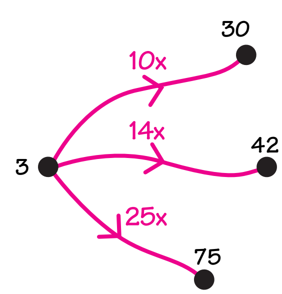
The class discussion might develop along these lines. Label dots with students’ suggestions.

S: ***30.***

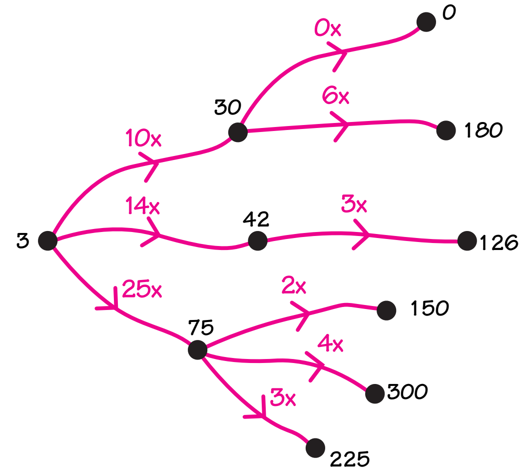
T: ***In order for 30 to go here, we have to find a whole number so that the red arrow is for times that whole number. What whole number will work?***

S: ***10.***

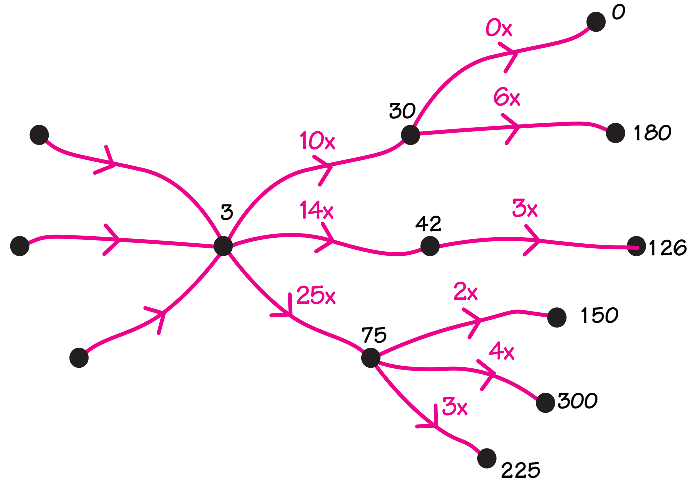
Continue in this manner, labeling each of the unlabeled dots and the corresponding arrows. You may like to add more red arrows starting at 3.



Continue this activity by extending the arrow picture as shown here, and labeling each dot and its corresponding arrow. Upon completion of the activity, the arrow picture might look like this.

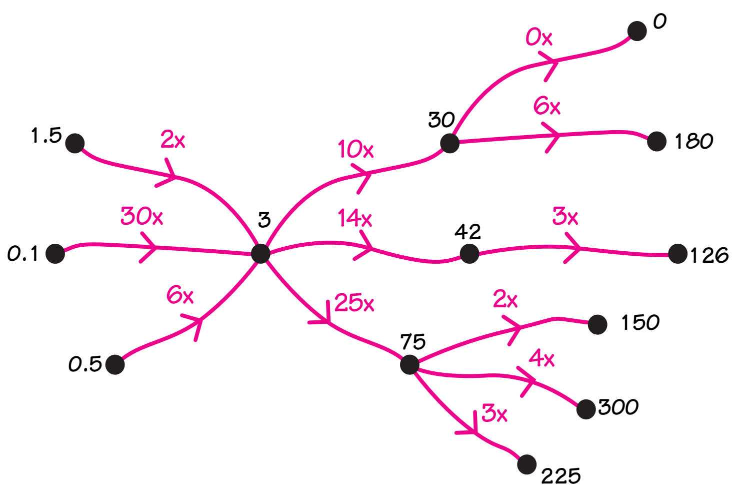


Add three arrows ending at 3 to the picture.



T: ***Copy only the three arrows ending at 3 on your paper. Label the dots as you wish, but be sure each red arrow can be for multiplication by a whole number. The dots can be for (non-integer) decimal numbers.***

Let students work independently on this problem for a few minutes. Ask faster workers to draw and label more arrows ending at 3 or at other numbers in their pictures. Invite students to label the three dots and the corresponding red arrows in the picture on the board. For example:

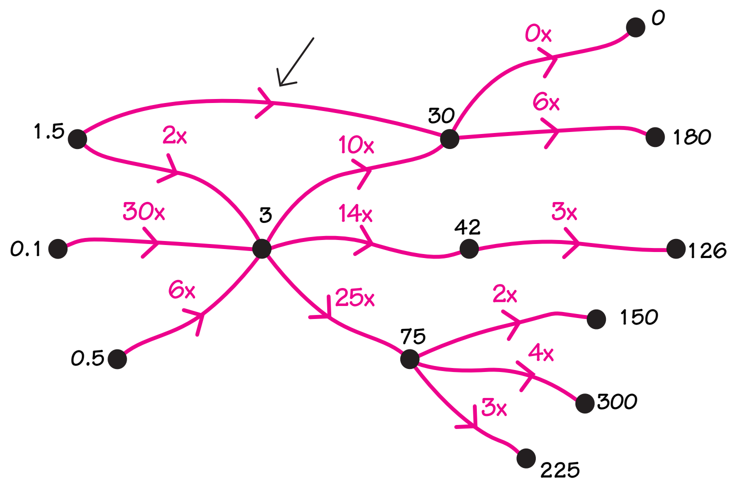


T (pointing to the ending dots of the three arrows starting at 3)**:** ***What do we call numbers like these?***

S: ***Multiples of 3.***

T: ***Do you see any other red arrows that we could draw?***

Ask a student to trace an arrow; then draw it. Suppose an arrow starting at 1.5 and ending at 30 is traced.



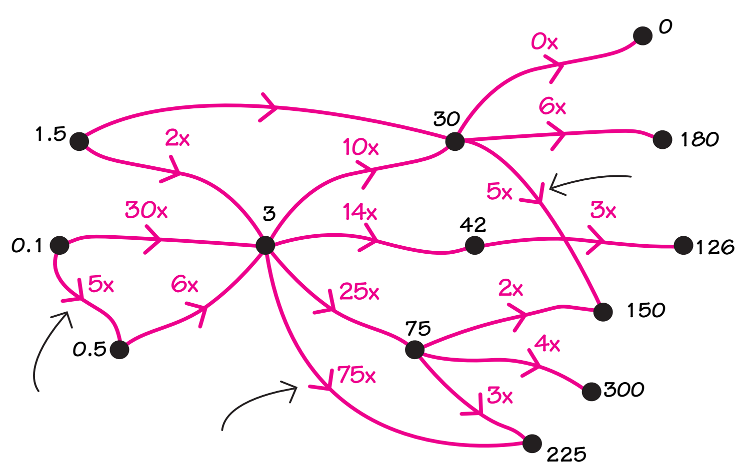
T: ***A red arrow must be for times some whole number. Let's use the picture to see if this is a good red arrow.***

Trace the 2x arrow followed by the 10x arrow from 1.5 to 30.

T: ***What is 2*x *followed by 10*x*?***

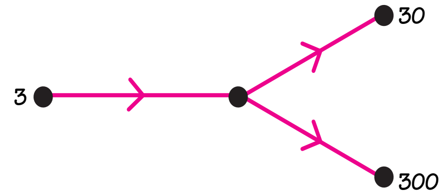
S: ***20*x.**

Ask for a few more red arrows and label them times some whole number. Point out when a red arrow is the composite of two other red arrows. There are many possibilities; a few are shown below.



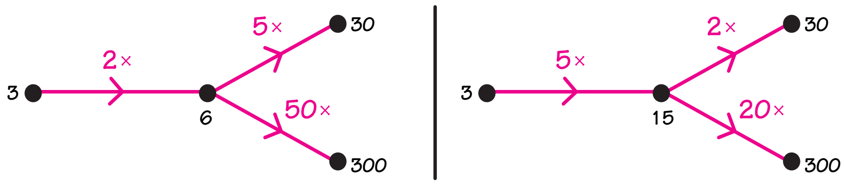
Exercise 2

Draw this arrow picture on the board.



T: ***Copy this picture on your paper. Label the dot and the arrows. Remember, each red arrow must be labeled times some whole number.***

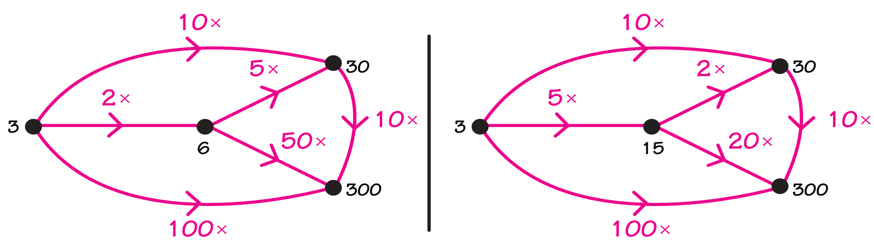
Two solutions to this problem are shown below. After giving students a few minutes to find a solution, ask them to put both solutions on the board.



Point to either of the arrow pictures and ask,

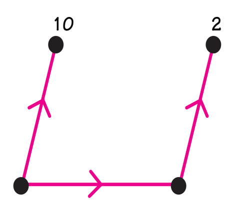
T: ***Where can more red arrows be drawn in this picture?***

Ask students to draw and label red arrows at the board. For example:



Erase the dot labels but not the arrow labels. Then ask if the dots can be labeled another way. After students have successfully relabeled the dots, ask for another way. The class should suspect there are infinitely many ways, arguing that the composites always hold.

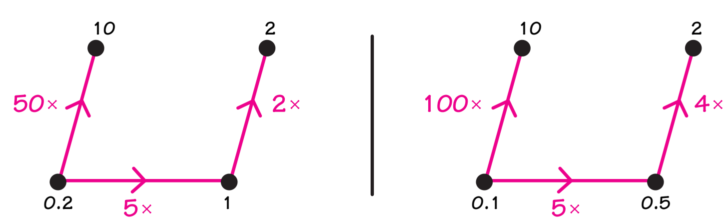
Draw this arrow picture on the board.



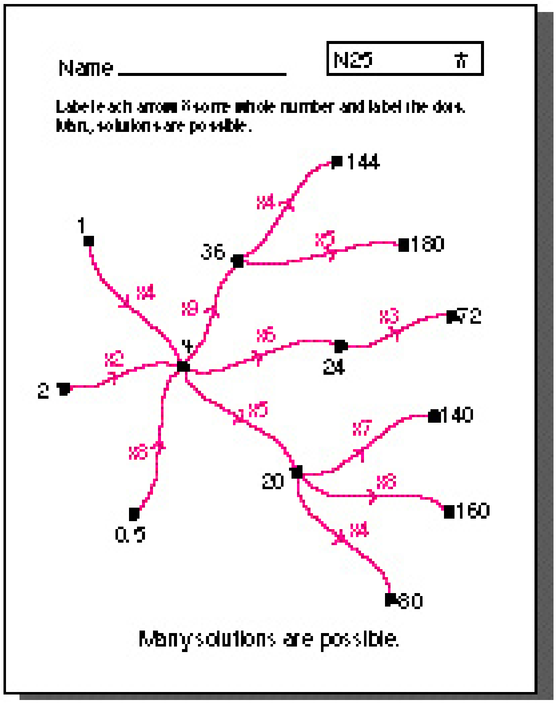
T: ***Copy this picture, and label the dots and arrows. Remember, each red arrow must be for times a whole number. The dots may be for decimal numbers. There are many possible solutions.***

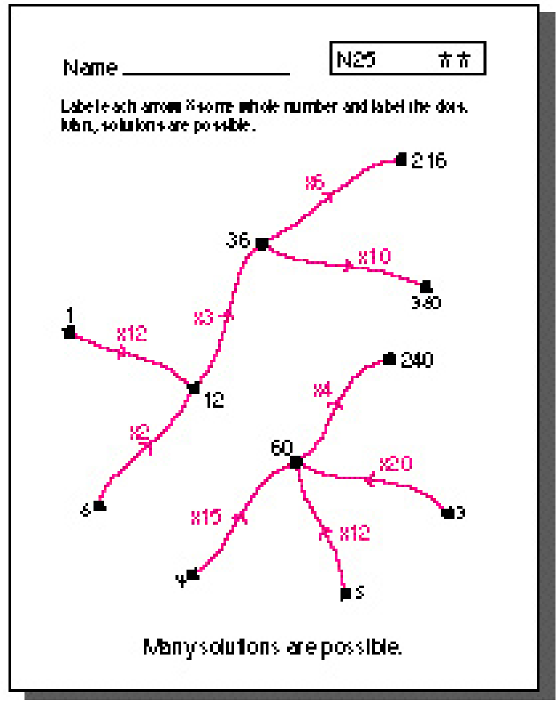
Let students work on this problem independently for a few minutes. Then put several solutions on the board.

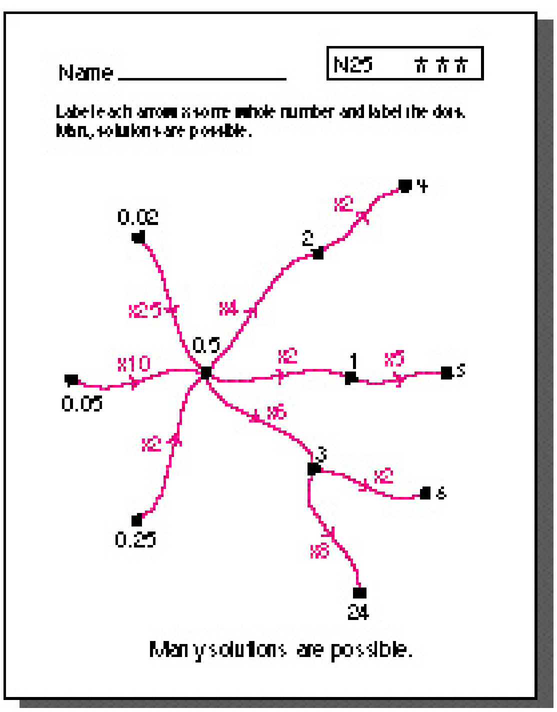
Two possible solutions are shown here.

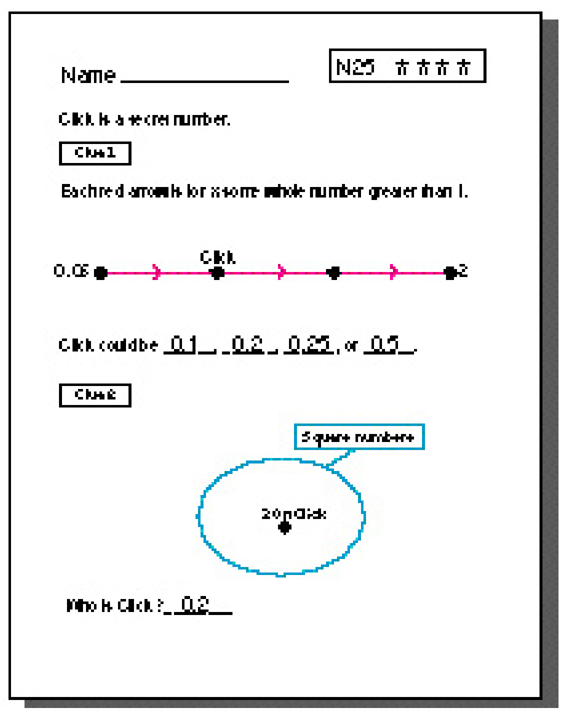


Worksheets N25*, **, ***, and **** are available for individual work.


Name
N25 ☆
Many solutions are possible.
144
180
72
24
140
160
80
36
20
4
1
2
0.5
Many solutions are possible.


Name
N25 ☆☆
Many solutions are possible.
216
36
340
1
12
240
60
Many solutions are possible.


Name
N25 ☆☆☆
Many solutions are possible.
0.02
4
2
0.5
0.05
1
0.25
3
24
Many solutions are possible.


Name
N25 ☆☆☆☆
Click is a secret number.
Clue 1
Each red arrow is for x some whole number greater than 1.
0.05
Click
2
Click could be 0.1, 0.2, 0.25, or 0.5.
Clue 2
Square numbers
20 x Click
Who is Click? 0.2

N26 MINICOMPUTER GOLF #3

Capsule Lesson Summary

Estimate a number on the Minicomputer displayed with positive, negative, and weighted checkers. Choose an appropriate move to increase or decrease the number by a specified amount. Play *Minicomputer Golf*.

Materials

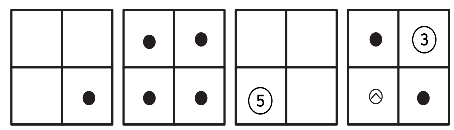
Teacher
- Minicomputer set
- Weighted checker set
- Colored chalk

Student
- Paper
- Minicomputer set (optional)
- Calculator (optional)

Description of Lesson

Exercise 1

Put this configuration on the Minicomputer.



T: ***Let's estimate this number. Is this number more or less than 2 000?***

S: ***More. There is 1 000 on the thousands board and more than 1000 on the hundreds board.***

T: ***Is this number more or less than 3 000?***

S: ***Less. There is 2 500 on the thousands and hundreds boards together, and the number on the other boards is less than 500.***

Attempt to get a more accurate estimate, or agree that the number is between 2 500 and 3 000.

T: ***By moving exactly one checker, can you make the number nine greater?***

S: ***Move the regular checker from the 1-square to the 10-square.***

Make the move as a student suggests; then return the checker to its original position and continue the activity by asking similar questions. Feel free to adjust the level of difficulty of the questions to the numerical abilities of your students.

T: ***Who can make the number 12 greater?***
(Move the regular checker from the 8-square to the 20-square, or move the ③-checker from the 4-square to the 8-square.)

6 less?
(Move a regular checker from the 8-square to the 2-square, or move the negative checker from the 2-square to the 8-square.)

98 less?
(Move a regular checker from the 100-square to the 2-square, or move the negative checker from the 2-square to the 100-square.)

T: ***300 greater?***
(Move the regular checker from the 100-square to the 400-square, or move a 5-checker from the 20-square to the 80-square.)

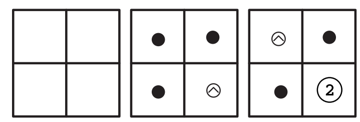
48 greater?
(Move the 3-checker from the 4-square to the 20-square.)

992 less?
(Move a regular checker from the 1 000-square to the 8-square.)

Clear the Minicomputer before going on to Exercise 2.

Exercise 2

Slowly put this configuration on the Minicomputer.

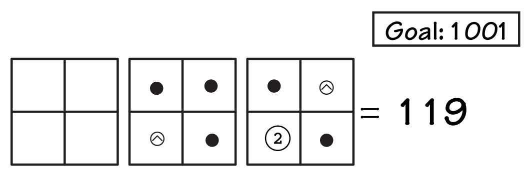


T: ***What number is this?***

S: ***130.***

Play *Minicomputer Golf* starting with the configuration above and a goal of 1 000.

If time allows, play a second game with this starting configuration and a goal of 1001.



Exercise 3 (optional)

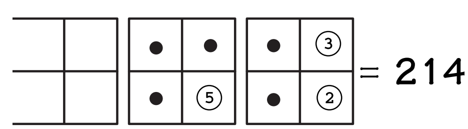
Note: For this activity, each pair or foursome of students will need a desk Minicomputer set and a calculator. These materials are not supplied with the classroom set of materials for *CSMP Mathematics for the Intermediate Grades, Part III*. Desk Minicomputer sets can be borrowed from an earlier level *CSMP* class; or individual Minicomputer sheets and sheets of punch-out checkers can be purchased.

For the remainder of the lesson, ask students or student partners to pair up and play *Minicomputer Golf*—one against the other or two against two.

Weighted checkers can be made by the students as needed. Just turn a cardboard checker over and write a numeral on the back.

A starting configuration is suggested here.

Goals can be 700 or 1000 or any number the students choose.



Extension Activity

Perhaps your class would enjoy organizing and playing a *Minicomputer Golf* tournament.

N27 FRACTIONS #3

Capsule Lesson Summary

Associate a fraction with each of the subdivisions of a rectangular region by comparing areas. Use an area model to generate equivalent fractions. Cut apples into thirds and sixths to suggest a variety of number sentences involving fractions. Count forward and backward by sixths.

Materials

Teacher
- Six apples (same size)
- Knife
- Paper towels
- Bowl of water
- Meter stick
- Colored chalk

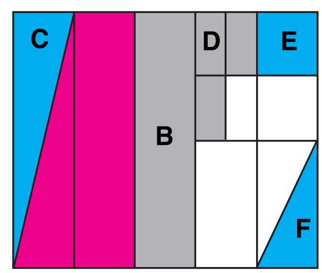
Student
- Metric ruler
- Worksheets N27 (no star), *, **, and ***

Description of Lesson

Exercise 1

Distribute copies of Worksheets N27 (no star).

On the board, carefully copy and enlarge the picture on this worksheet. A convenient size is 80 cm by 100 cm.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: ***The shape on the board is like the one on your worksheet. The inside of the large rectangle is divided into regions; some are red, some are blue, some are gray, and some are white. What fraction of the whole shape is this region*** (trace the border of **B**)***?***

S: ***$\frac{1}{5}$; the shape is divided into five vertical strips all the same size.***

Likewise continue with regions **C**, **D**, **E**, and **F**. Encourage a variety of explanations. Let students come to the board if they wish. The following list provides answers and sample explanations.

C: $\frac{1}{10}$. Two of these triangles fill one vertical strip, so ten of these triangles would fill the large rectangle.
D: $\frac{1}{40}$. Eight small rectangles this size fill one vertical strip, so forty of them would fill the large rectangle.
E: $\frac{1}{20}$. This region is twice the area of region **D**.
F: $\frac{1}{20}$. This region has the same area as region **E**.

T: ***What fraction of the whole shape is red?***

S: ***$\frac{1}{5} + \frac{1}{10}$. The red region is a rectangle ($\frac{1}{5}$) and a triangle ($\frac{1}{10}$).***

S: ***$\frac{3}{10}$. The red region is the same size as three of these triangles ($\frac{1}{10}$).***

Write this number sentence on the board.

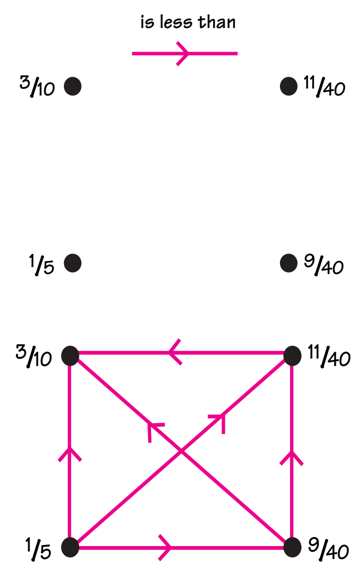
Red: $\frac{1}{5} \times \frac{1}{10} = \frac{3}{10}$

As you discuss this number sentence, point out that $\frac{1}{5} = \frac{2}{10}$.

In a similar manner, ask students what fraction of the shape is gray, blue, or white. Lead to the following number sentences:

$$\text{Gray: } \tfrac{1}{5} \times \tfrac{3}{40} = \tfrac{11}{40}$$
$$\text{Blue: } \tfrac{1}{10} \times \tfrac{1}{20} \times \tfrac{1}{20} = \tfrac{2}{10} = \tfrac{1}{5}$$
$$\text{White: } \tfrac{1}{10} \times \tfrac{1}{20} \times \tfrac{1}{20} \times \tfrac{1}{40} = \tfrac{9}{40}$$

Draw this picture on the board, and ask students to indicate where you can draw red arrows.



S: ***From $\frac{1}{5}$ to $\frac{3}{10}$.***

T: ***Why?***

S: ***In the picture on the board, $\frac{3}{10}$ of the rectangle is colored red, while one vertical rectangle is $\frac{1}{5}$ of the whole rectangle. Clearly, one vertical rectangle is less than the red region.***

S: ***$\frac{1}{5} = \frac{2}{10}$. Therefore $\frac{1}{5} < \frac{3}{10}$.***

Invite students to draw all of the possible red arrows in the picture. Encourage a variety of explanations.

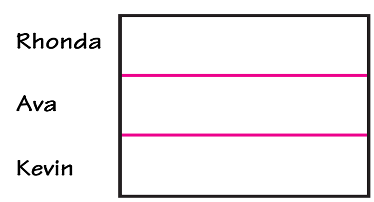
Erase the board before going on to Exercise 2.

Exercise 2

Choose three of your students to star in this exercise. In the dialogue, the students' names are Rhonda, Ava, and Kevin. Draw a rectangle 60 cm by 100 cm on the board.

T: ***This is a rectangular cake. The cake is for Rhonda, Ava, and Kevin. I'd like to divide the cake lengthwise into three pieces of the same size. How can I do that?***

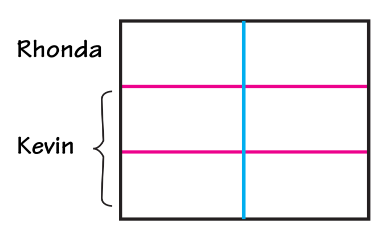
Using a meter stick, follow students' suggestions to divide the cake.



T: ***What fraction of the cake does Rhonda receive?*** ($\frac{1}{3}$)

Ava doesn't like this cake and gives her piece to Kevin. What fraction of the cake does Kevin have? ($\frac{2}{3}$)

The children decide to make the cake last for two days, so they cut their pieces in halves. What fraction of the cake does Rhonda have?



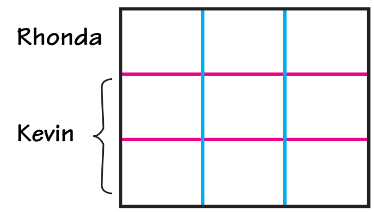
S: ***Still $\frac{1}{3}$ or $\frac{2}{6}$.*** $\quad \frac{1}{3} = \frac{2}{6}$

T: ***How much of the cake does Kevin have?***

S: ***$\frac{4}{6}$, or $\frac{2}{3}$.*** $\quad \frac{2}{3} = \frac{4}{6}$

T: ***What if, instead of two days, the children decide to make the cake last three days?***

Erase the blue segment and let students instruct you on how to accurately use a meter stick to divide the rectangle.



As above, conclude that $\frac{1}{3} = \frac{3}{9}$ and $\frac{2}{3} = \frac{6}{9}$.

Repeat the above activity asking, “What if the children decide to make the cake last for four days?”
Record the equivalent fractions.

$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$$
$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12}$$

T: ***What patterns do you notice among these equivalent fractions?***

Let students describe any patterns they notice. Encourage students to use their patterns to generate more fractions equivalent to $\frac{1}{3}$ and $\frac{2}{3}$, and list them on the board.

S: ***For $\frac{1}{3}$, the top numbers increase by one and the bottom numbers increase by three. So $\frac{5}{15}$, $\frac{6}{18}$, $\frac{7}{21}$, $\frac{8}{24}$, … are also equivalent to $\frac{1}{3}$.***

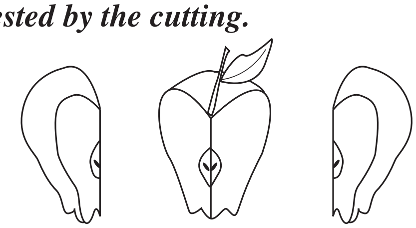
S: ***You can double both numerators and denominators. So $\frac{2}{3}$, $\frac{4}{6}$, $\frac{8}{12}$, $\frac{16}{24}$, $\frac{32}{48}$, … are equivalent to each other.***

Exercise 3

On a table, arrange six apples, a knife, paper towels, and a bowl of water. During this exercise, place the pieces of apples in the bowl of water to retard their browning and to insure their palatability at the end of the lesson.

T: ***As I cut the apples, tell me number sentences suggested by the cutting.***

Cut an apple into thirds as accurately as you can.



Write number sentences given by students on the board, but accept only ones suggested by the cutting. For example:

$$\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = 1 \qquad 3 \times \frac{1}{3} = 1 \qquad \frac{1}{3} \times \frac{1}{3} = \frac{2}{3}$$
$$1 \div 3 = \frac{1}{3} \qquad 1 - \frac{1}{3} = \frac{2}{3} \qquad \frac{1}{3} \times 1 = \frac{1}{3}$$

Cut another apple into thirds. Ask for new number sentences involving the cut apples, for example:

$$\frac{2}{3} \times \frac{2}{3} = \frac{4}{3} = 1\frac{1}{3} \qquad \frac{3}{3} \times \frac{3}{3} = \frac{6}{3} = 2 \qquad 6 \times \frac{1}{3} = 2$$
$$2 - \frac{1}{3} = 1\frac{2}{3} = \frac{5}{3} \qquad 2 \div 6 = \frac{1}{3}$$

If students create very few number sentences, use the above equations to produce problems. For example, ask students to calculate $2 - \frac{1}{3}$ and use the apples to show that $2 - \frac{1}{3} = \frac{5}{3} = 1\frac{2}{3}$.

T: ***Let’s use our cut-up apples to help solve some problems.***

Write the following or similar problems on the board. (Answers are in boxes.)

$$3 - \frac{1}{3} = \boxed{2\frac{2}{3}} \qquad 2\frac{1}{3} + 3\frac{1}{3} = \boxed{5\frac{2}{3}}$$
$$4\frac{2}{3} - 2\frac{1}{3} = \boxed{2\frac{1}{3}} \qquad 2\frac{1}{3} + \frac{2}{3} = \boxed{3}$$
$$1\frac{2}{3} + 1\frac{2}{3} = \boxed{3\frac{1}{3}} \qquad 4\frac{1}{3} - 1\frac{2}{3} = \boxed{2\frac{2}{3}}$$
$$3 \times 2\frac{2}{3} = \boxed{8} \qquad 1\frac{1}{3} \div 2 = \boxed{\frac{2}{3}}$$

Invite students to solve these problems and to explain their answers. Use the apples, as necessary, to model calculations. For example, to calculate $4\frac{1}{3} - 1\frac{2}{3}$ display four whole apples (three uncut, one cut) and another $\frac{1}{3}$ piece. Then let a student remove $1\frac{2}{3}$ apples. Show that $2\frac{2}{3}$ apples remain.

Take three $\frac{1}{3}$ pieces of apple and cut each in half to form six $\frac{1}{6}$ pieces of apple. Ask for number sentences involving sixths, for example:

$$\frac{1}{3} = \frac{2}{6} \qquad \frac{1}{3} \div 2 = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$
$$5/6 \times 1/3 = \frac{7}{6} = 1\frac{1}{6} \qquad \frac{1}{3} \times \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$
$$\frac{2}{3} - \frac{1}{6} = \frac{3}{6} = \frac{1}{2} \qquad 1\frac{1}{6} - \frac{1}{3} = \frac{5}{6}$$

T: ***Let's count by sixths. If I start at 0 and each of you responds exactly once, what number will the last student say?***

Let students explain their answers, but do not indicate a correct answer yet.

T: ***If we have problems, we'll use the apples. I'll start: 0.***

S: ***$\frac{1}{6}$.***

S: ***$\frac{2}{6}$, or $\frac{1}{3}$.***

S: ***$\frac{3}{6}$, or $\frac{1}{2}$.***

S: ***$\frac{4}{6}$, or $\frac{2}{3}$.***

S: ***$\frac{5}{6}$.***

S: ***$\frac{6}{6}$, or 1.***

S: ***$\frac{7}{6}$, or $1\frac{1}{6}$.***

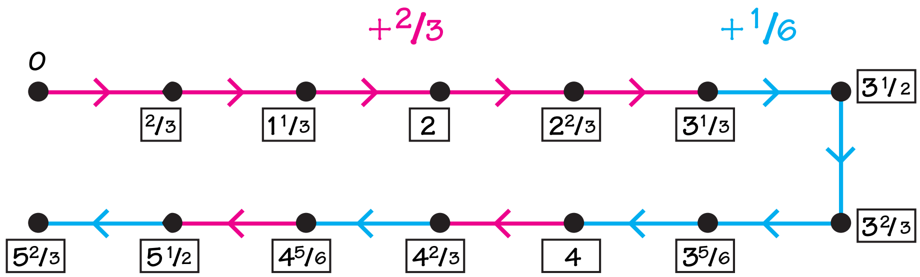
Accept $\frac{7}{6}$ but also ask for the common name: $1\frac{1}{6}$. Continue until all students have contributed exactly once.

Compare the final student's response with students' predictions.

T: ***Let's count backward by sixths. I'll start at 5.***

Let students count backward: $4\frac{5}{6}$, $4\frac{4}{6}$ or $4\frac{2}{3}$, $4\frac{3}{6}$ or $4\frac{1}{2}$, $4\frac{2}{6}$ or $4\frac{1}{3}$, $4\frac{1}{6}$, 4, $3\frac{5}{6}$, … until each student has responded exactly one. Use apples to help explain any difficult problems.

Draw this arrow road on the board. Invite students to label the dots. Use apples to demonstrate any calculations that the students find difficult. (Answers are in boxes.)

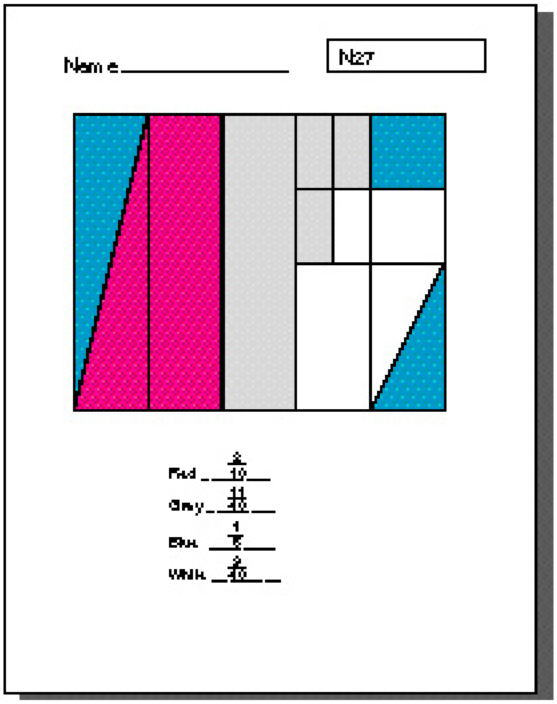


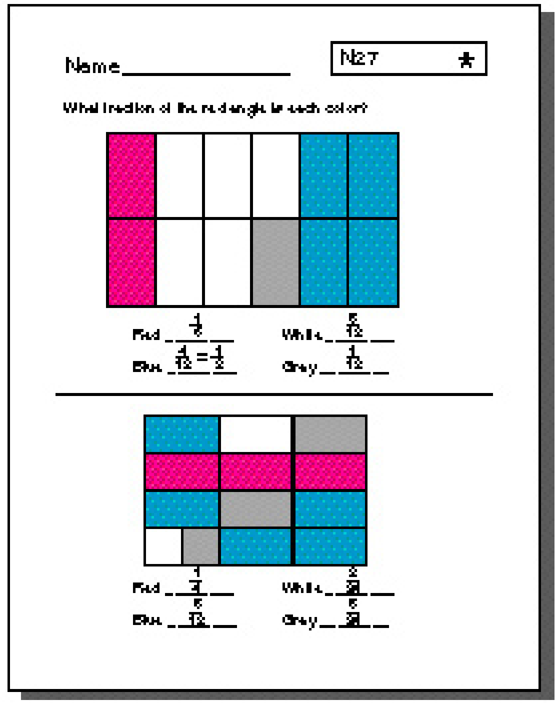
Worksheets N27*, **, and *** are available for individual work. Be prepared to assist students with the second problem on N27**.

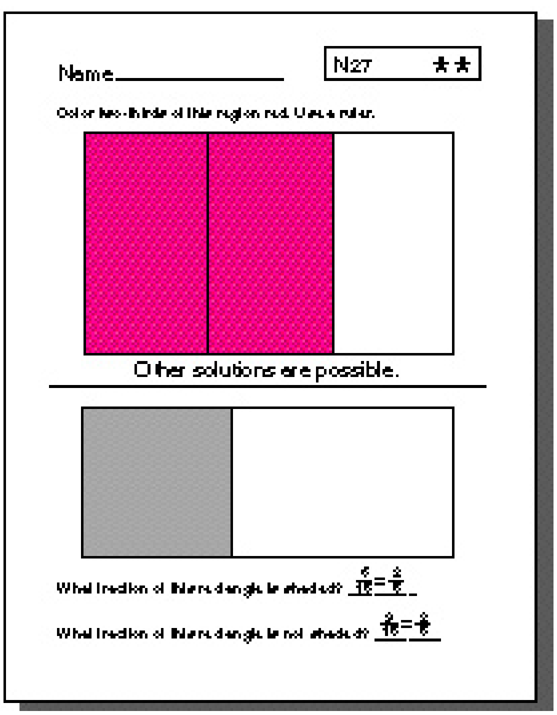
If you wish, give the pieces of apples to students.

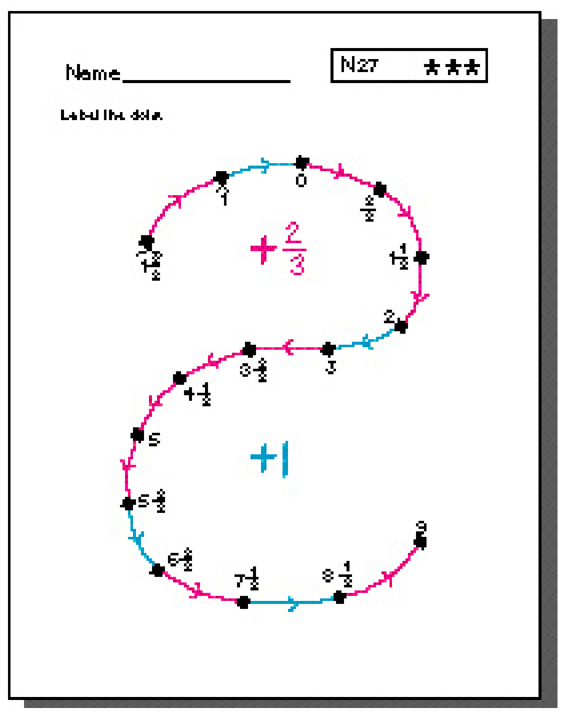
Home Activity

Suggest that students practice counting by halves (or thirds or fourths) at home with a family member.


Name
N27

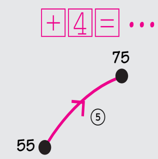

Name
N27 ★


Name
N27 ★★
Other solutions are possible.


Name
N27 ★★★
$+\frac{2}{3}$
$+1$

Capsule Lesson Summary

With the relation $\boxed{+}$ $\boxed{4}$ $\boxed{=}$ … , introduce weighted arrows; for example, ⑤ in the picture here indicates that $\boxed{=}$ is pressed five times in the relation $\boxed{+}$ $\boxed{4}$ $\boxed{=}$ … . Draw arrow roads using these weighted arrows and determine the weights of composites.



Materials

Teacher
- Colored chalk
- Calculator

Student
- Paper
- Colored pens, pencils, or crayons
- Calculator
- Worksheets N28*, **, ***, and ****

Description of Lesson

Exercise 1

Involve students in mental arithmetic as you introduce the idea of weighted arrows with a calculator relation.

T: ***Display 55 on your calculator and follow my instructions carefully. Hide the display and press*** $\boxed{+}$ $\boxed{4}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$***. What number is on the display? Why?***

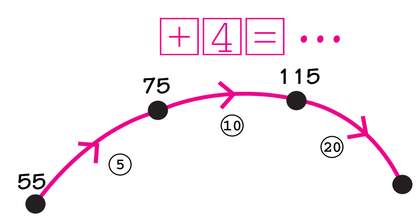
S: ***75, because we pressed*** $\boxed{=}$ ***five times: 5 x 4 = 20 and 55 + 20 = 75.***

Draw the first arrow in the picture below on the board.

T: ***The arrow shows what we just did on the calculator. ⑤ indicates that we pressed*** $\boxed{=}$ ***five times. Hide the display again and press*** $\boxed{=}$ ***ten more times. What number is on the display? Why?***

S: ***115, because 10 x 4 = 40 and 75 + 40 = 115.***

Extend the arrow picture.



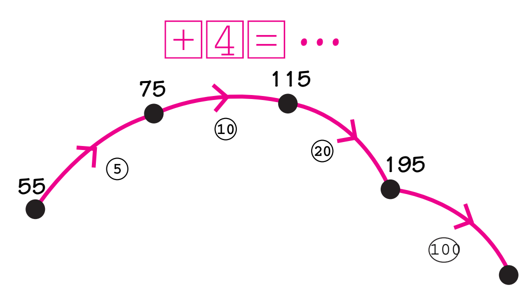
T: ***What does this ⑳ tell us to do now?***

S: ***Press*** $\boxed{=}$ ***twenty more times.***

T: ***Hide the display and press*** $\boxed{=}$ ***twenty more times. What number should be on the display? Why?***

S: ***195, because 20 x 4 = 80 and 115 + 80 = 195.***

Add an arrow starting at 195 to the picture.

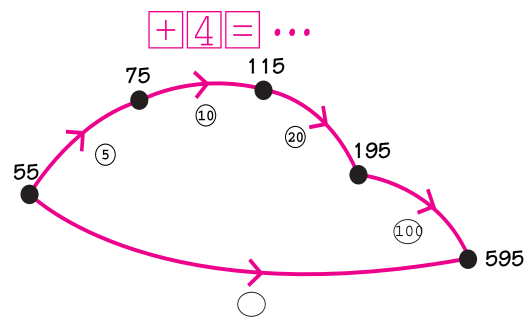


T: ***Don't press*** $\boxed{=}$***; just write the ending number on your paper.***

Let students check their answers with a partner before asking one student to announce the number and the calculations that arrived at it.

S: ***595, because 100 x 4 = 400 and 195 + 400 = 595.***

Label the dot for 595 and draw an arrow from 55 to 595.



T: ***What number should we write in the circle beside this arrow?*** (135)
How can we check it?

S: ***Put 55 on the calculator and press + 4 = …, pressing = 135 times.***

S: ***Calculate 55 + (135 x 4).***

T: ***What number is 135 x 4?*** (540)
What number is 55 + 540? (595)

Draw another arrow starting at 55 and point to its ending dot as you ask,

T: ***If we start at 55 and press + 4 = …, what is the greatest possible number less than 1000 that will appear? Write the answer on your paper.***

Allow a couple minutes for individual work.

S: ***999.***

T: ***How did you find 999?***

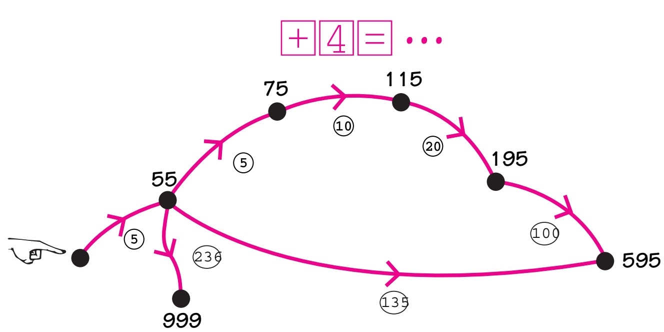
S: ***We can get 595, and from there we can get 995 (595 + 400 = 995); then 995 + 4 = 999.***

T: ***How many times must = be pressed to get 999? Why?***

S: ***236, because 236 x 4 = 944, and 55 + 944 = 999.***

S: ***We get 595 after pressing = 135 times. To get 995 we must press = 100 more times; then we get 999 by pressing = one more time. Altogether we press = 236 times.***

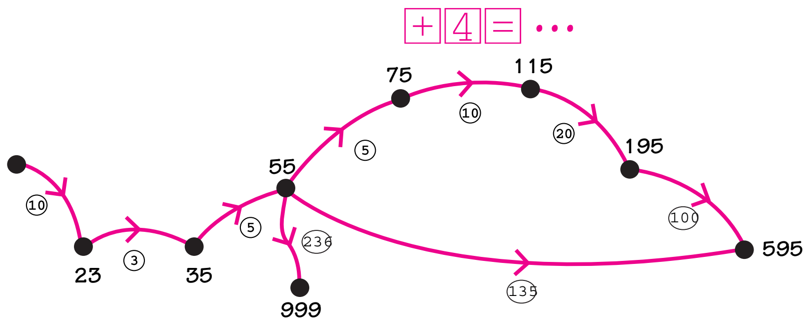
Draw an arrow ending at 55, and point to its starting dot as you ask,



T: ***From this number, we can get 55 by pressing + 4 and then = five times.***
What number is it?

S: ***35; 5 x 4 = 20, and 35 + 20 = 55 (or 55 – 20 = 35).***

Repeat this activity by first drawing an arrow ending at 35 (ask for its starting dot) and then an arrow ending at 23.



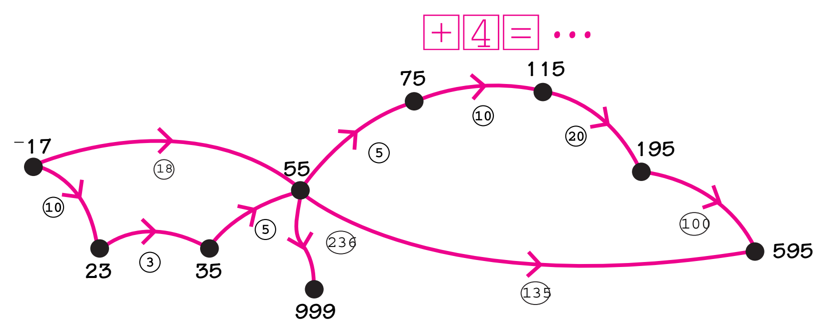
T (pointing to the unlabeled dot)**:** ***What number is here?***

S: ***$^-17$; 10 x 4 = 40 and 23 – 40 = $^-17$.***

Draw an arrow from $^-17$ to 55.

T: ***What number should we put in a circle beside this arrow?***

S: ***18, because 10 + 3 + 5 = 18.***



T: ***Can we have a red arrow from 0 to 55? If not, what is the closest number to 0 that could be a starting number for a red arrow ending at 55?***

Allow a couple minutes for students to consider the problem.

S: ***$^-1$.***

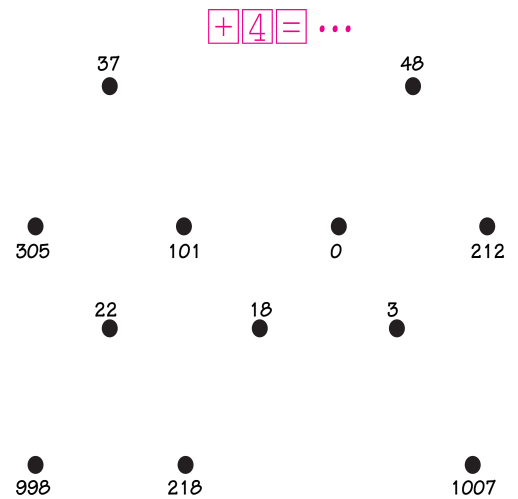
T: ***How many times would we press $\boxed{=}$ to go from $^-1$ to 55?*** (14 times)
Who can tell us how to check this?

S: ***14 x 4 = 56 and $^-1$ + 56 = 55.***

N28

Exercise 2

Carefully draw this picture on the board and ask students to copy it on their papers.

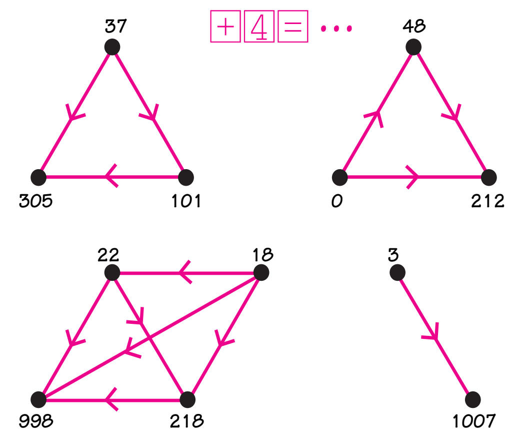


T: ***Try to draw all of the red arrows that are possible in this picture. Use a calculator, if you like, to help.***

To be certain everyone understands the problem, you may ask a student to draw one arrow at the board. Allow several minutes for independent work before continuing.

T: ***Who can draw another arrow in the picture?***

Call on several students to assist in completing the arrow picture on the board.

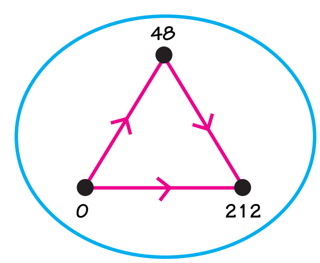


T: ***Do you notice anything interesting about this arrow picture?***

S: ***There are four separate pieces.***

T: ***Let's look at this piece.***

Draw a string around this part of the arrow picture.

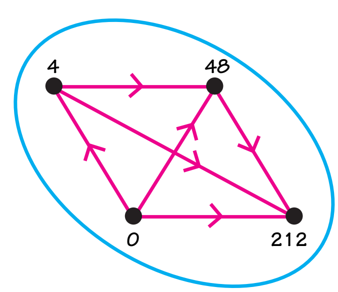


T: ***What do you notice about the numbers in this piece of the picture?***

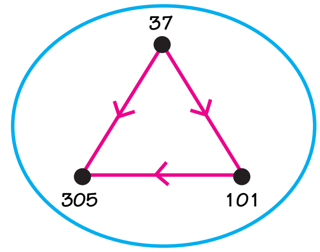
S: ***They are all multiples of 4.***

T: ***What is the least positive number that can be added to this piece of the arrow picture?*** (4)

Extend the picture to include 4.



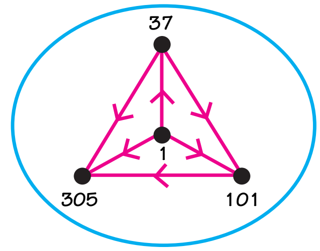
Draw a string around this part of the arrow picture.



T: ***What is the least positive number that could be joined to this piece of the arrow picture?***

S: ***1.***

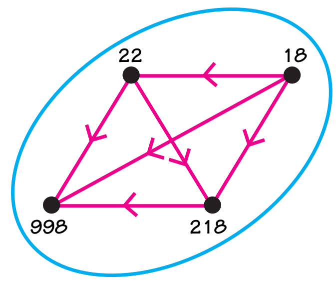
Extend the picture to include 1.



T: ***Do you notice anything interesting about these numbers?***

S: ***Each number is one more than a multiple of 4. 37 = 36 + 1 and 36 is a multiple of 4; 101 = 100 + 1 and 100 is a multiple of 4; 305 = 304 + 1 and 304 is a multiple of 4.***

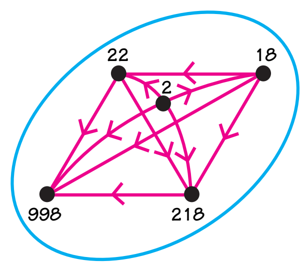
Direct students' attention to this group of numbers.



T: ***What is the least positive number that could be joined to this piece of the arrow picture?***

S: ***2.***

Extend the picture to include 2.

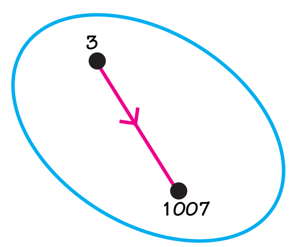


T: ***What do you notice about these numbers?***

S: ***Each is a multiple of 4, plus 2.***

S: ***Each is 2 less than a multiple of 4.***

Direct students' attention to this group of numbers.



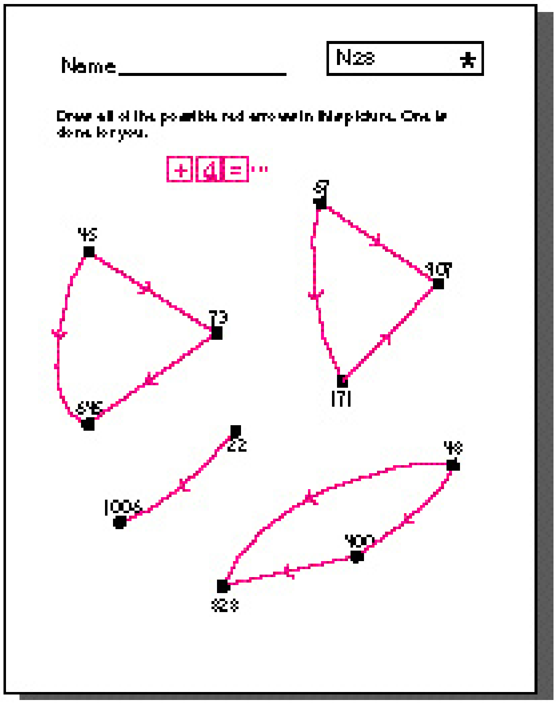
T: ***What are some numbers that can be joined to this piece of the arrow picture?*** (…, $^-1$, 7, 23, 47, 93, …)

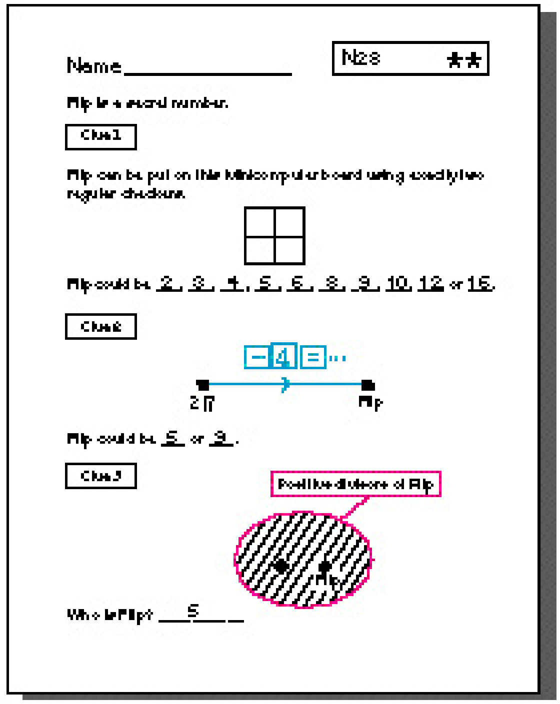
S: ***Any number that is three more than a multiple of 4.***

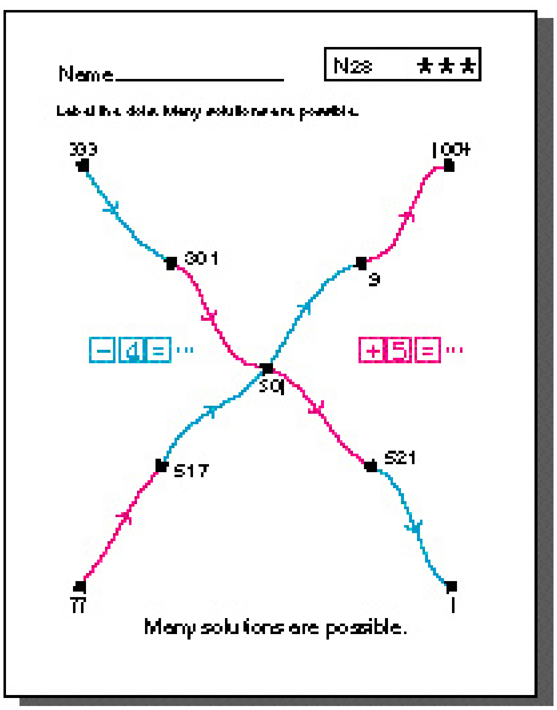
S: ***Any number that is one less than a multiple of 4.***

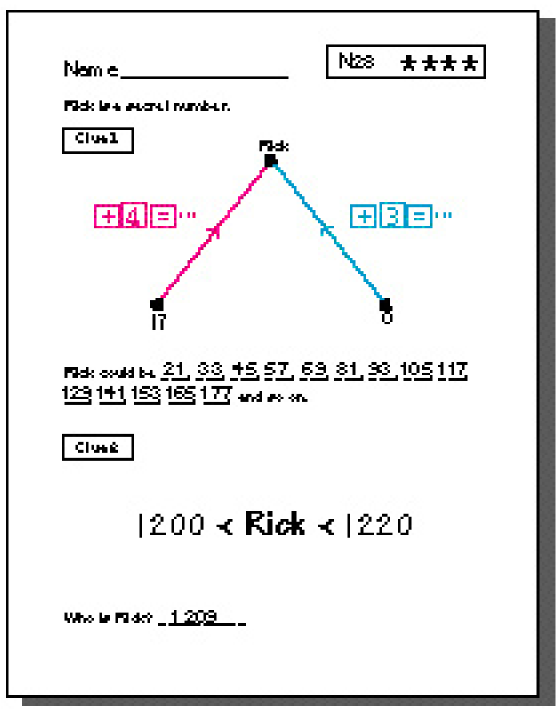
Worksheets N28*, **, ***, and **** are available for individual work.

N28


Name
N28 ★
Draw all of the possible red arrows in this picture. One is done for you.
+ 4 = ...
45
73
645
57
407
171
22
1006
48
400
828


Name
N28 ★★
Flip is a secret number.
Clue 1
Flip can be put on this Minicomputer board using exactly two regular checkers.
Flip could be 2, 3, 4, 5, 6, 8, 9, 10, 12 or 16.
Clue 2
− 4 = ...
21
Flip
Flip could be 5 or 9.
Clue 3
Positive divisors of Flip
Flip
Who is Flip? 5


Name
N28 ★★★
Label the dots. Many solutions are possible.
333
1004
301
9
− 4 = ...
+ 5 = ...
301
517
521
77
1
Many solutions are possible.


Name
N28 ★★★★
Rick is a secret number.
Clue 1
Rick
+ 4 = ...
+ 3 = ...
17
0
Rick could be 21, 33, 45, 57, 69, 81, 93, 105, 117, 129, 141, 153, 165, 177 and so on.
Clue 2
1200 < Rick < 1220
Who is Rick? 1209

N29 A NEW NAME FOR ONE-THIRD

Capsule Lesson Summary

Review calculating one-half or one-third of a number on the Minicomputer. Extend the binary abacus to include boards to the right of the bar, and use its rule for trading checkers to label the boards. Put numbers on this abacus, decode numbers, and note patterns involving doubling and halving. Use both the binary abacus and a story about sharing a cake among friends to determine that $\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$.

Materials

Teacher
- Meter stick
- Colored chalk

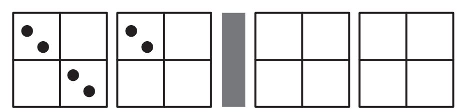
Student
- Worksheet N29
- Paper
- Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

This exercise is a review and should proceed quickly.

Display this configuration of checkers on the Minicomputer.



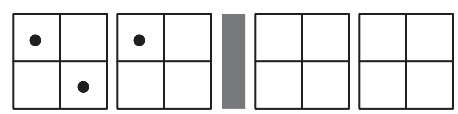
T: ***What number is on the Minicomputer?***

S: ***196.***

Let students explain what calculations they did.

T: ***What number is $\frac{1}{2}$ x 196?*** (98)

Call on a student to remove one checker from each pair and then record the fact on the board.

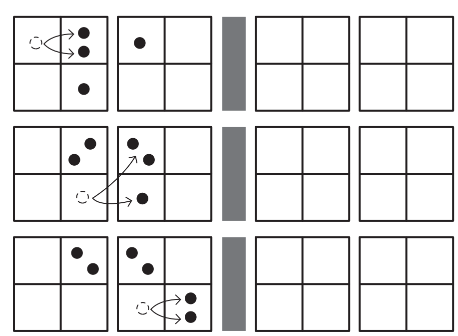


$\frac{1}{2} \times 196 = 98$

T: ***Let's calculate $\frac{1}{2}$ x 98 on the Minicomputer. What do we do?***

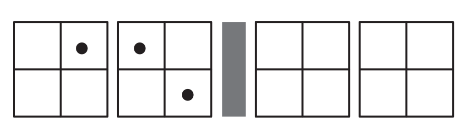
S: ***Make trades until all of the checkers are in pairs.***

Invite students to make the trades. Here is a sample sequence, but other correct sequences of trades and final configurations are possible.



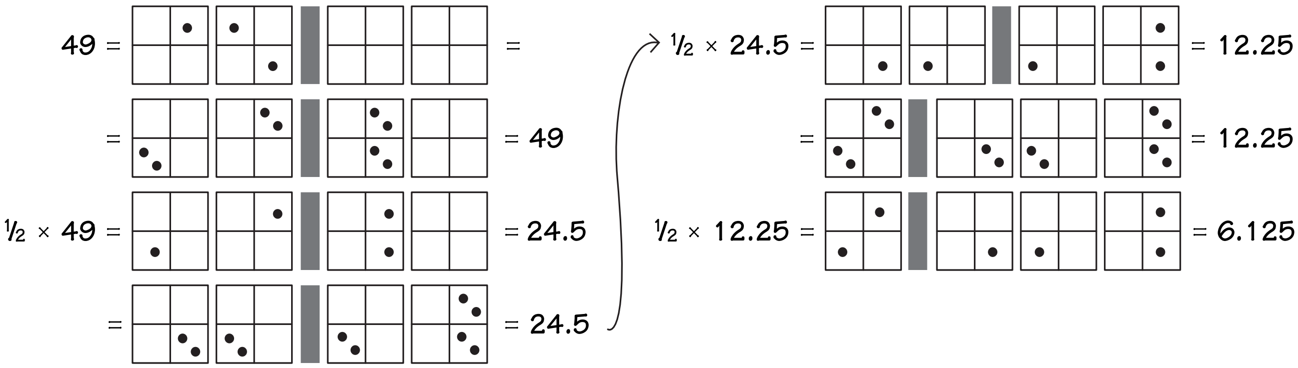
T: ***What number is $\frac{1}{2}$ x 98?***

S: ***49.***



$\frac{1}{2} \times 98 = 49$

In a similar manner, invite students to make trades on the Minicomputer to calculate ½ x 49, ½ x 24.5, and ½ x 12.25.

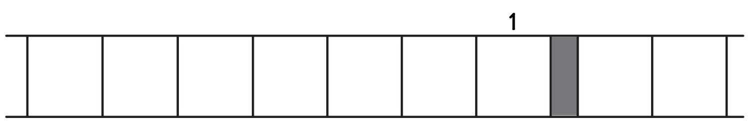


T: ***How would we use the Minicomputer to calculate one-third of a number?***

S: ***Put the number on the Minicomputer. Make trades until all checkers are in groups of three. Then remove two checkers from each group of three.***

Exercise 2

Put away the Minicomputer boards and draw this part of an abacus on the board.

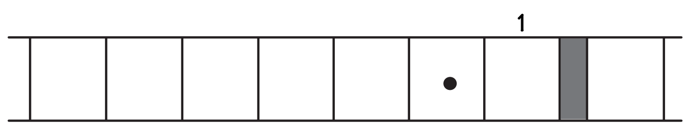


T: ***Do you remember the binary abacus and its rule for trading checkers?***

Invite a student to explain the binary abacus rule to the class. If no student recalls the rule, remind the class that the rule is this:

> Two checkers on a board
> trade for
> one checker on the next board to the left.

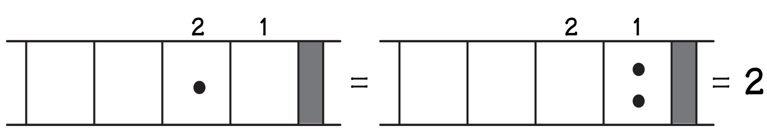
Place a checker on the board to the left of the ones board.



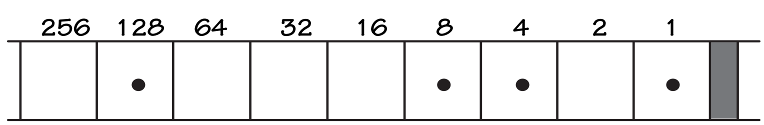
T: ***What number is this? Why?***

S: ***2. One checker on that board is the same as two checkers on the ones board.***

Show the trade and label the twos board.



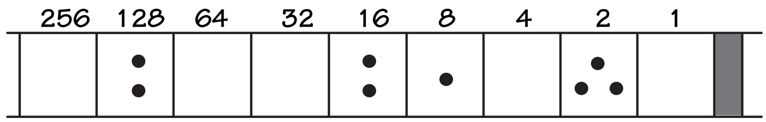
In a similar manner, continue labeling about nine binary places to the left of the bar. Then put this configuration of checkers on the abacus.



T: ***What number is this?***

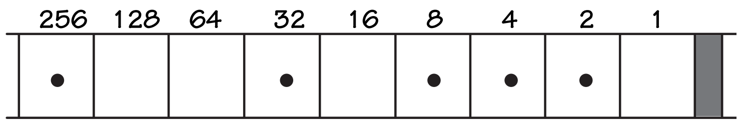
S: ***141; 128 + 8 + 4 + 1 = 141.***

Put this configuration of checkers on the abacus.



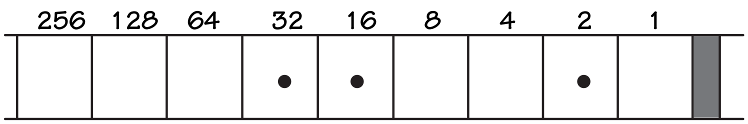
T: ***What number is this?***

As students wish, let them make trades before calculating the number on the abacus. This configuration may be obtained.

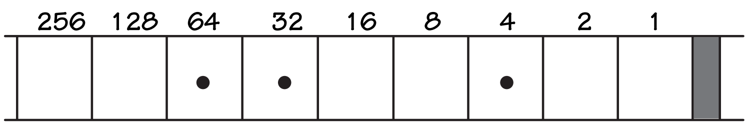


S: ***The number on the abacus is 302.***

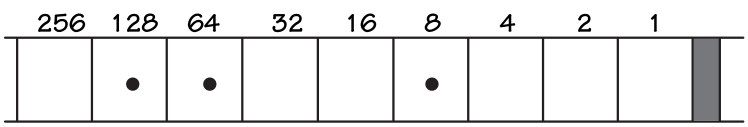
Distribute copies of Worksheet N29 and let students work independently. After a few minutes, check answers to the first three problems collectively and invite students to show 50, 100, and 200 on the abacus on the board.



= 50



= 100

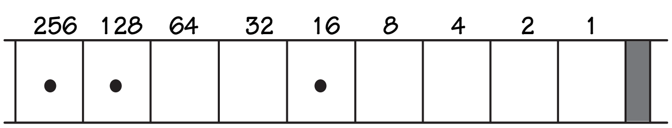


= 200

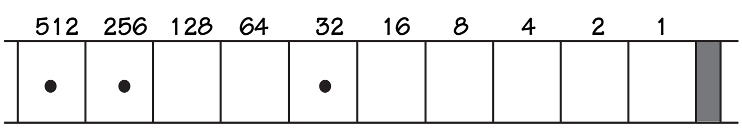
T: ***What patterns do you notice?***

S: ***The numbers double each time: 2*** x ***50 = 100 and 2*** x ***100 = 200.***

T: ***To double a number on the binary abacus, just move all the checkers one board to the left. Can we use this doubling pattern on the abacus to show quickly 400? 800?***

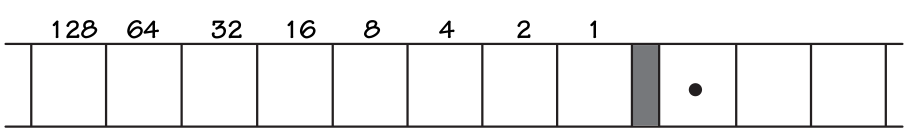


= 400



= 800

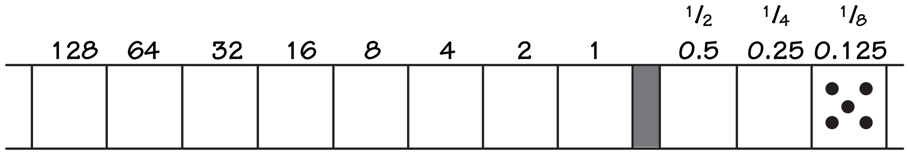
Extend the abacus a few places to the right of the bar, and put on this configuration.



T: ***What number is this? Why?***

S: ***½, or 0.5. Two checkers on that board show 1 since we could trade them for one checker on the ones board. So one checker on that board shows ½.***

Ask students to label several boards to the right of the bar. Then, place five checkers on the ⅛s board.

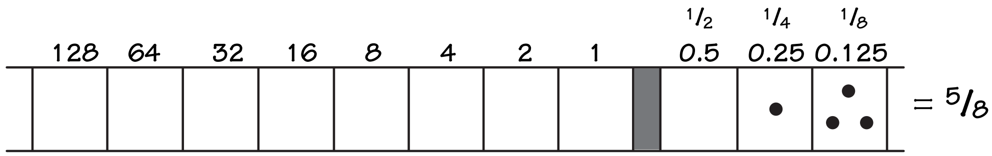


T: ***What number is this?***

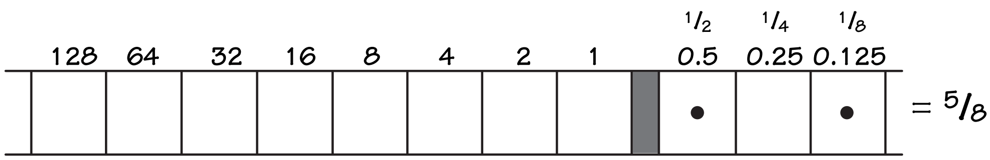
S: *⅝.*

T: ***Yes, this is ⅝ since ⅛ + ⅛ + ⅛ + ⅛ + ⅛ = ⅝. Who can make a trade and show ⅝ with one less checker?***

Let a student make the trade.



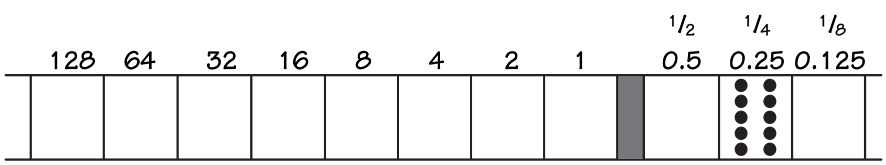
Observe that ⅝ = ¼ + ⅛ + ⅛ + ⅛. Invite students to make trades until there is at most one checker on a board. After each trade, observe a new representation for ⅝.



T: ***What addition fact for ⅝ does this suggest?***

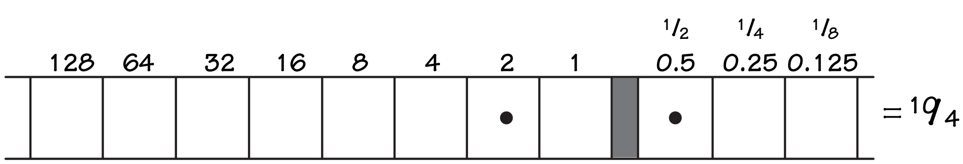
S: ***⅝ = ½ + ⅛.***

Place ten checkers on the ¼s board.



T: ***What number is this?*** (10/4)

As above, invite students to make trades and write appropriate representations for 10/4 until this configuration of checkers is reached.



T: ***What is another name for 10/4?*** (2½)

Exercise 3

Extend the binary abacus to include seven or eight places to the right of the bar. Invite students to label the boards with fraction names.

T: ***It's quite easy to put fractions such as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{3}{8}$, and $\frac{11}{16}$ on the binary abacus. I'd like to put $\frac{1}{3}$ on this abacus. How can we do it?***

Listen to students' suggestions. After a while, lead students to use the following method.

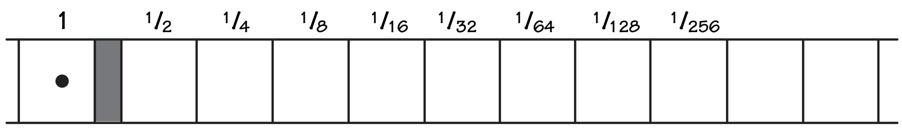
T: ***What is $\frac{1}{3}$ of 1?*** ($\frac{1}{3}$)
How do we use the Minicomputer to calculate one-third of a number?

$$\frac{1}{3} \times 1 = \frac{1}{3}$$

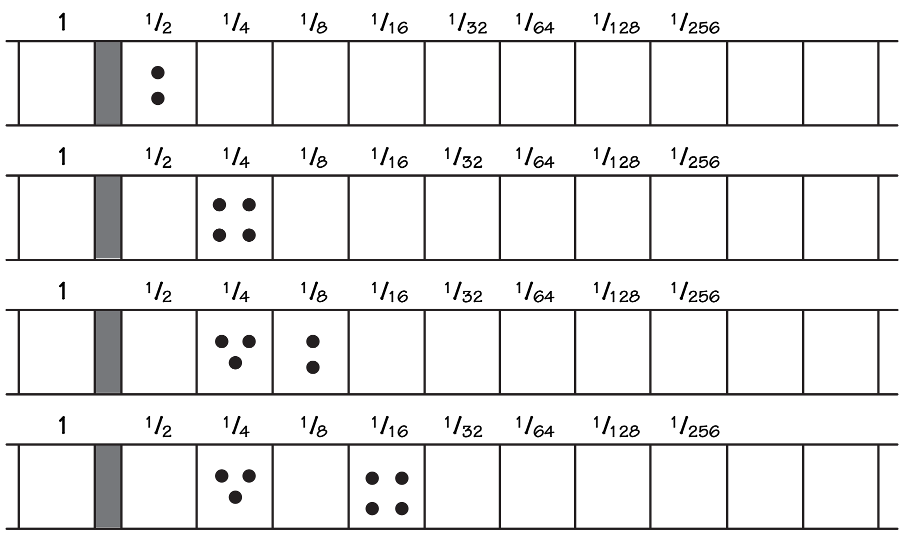
S: ***Put the number on the Minicomputer and make backward trades until the checkers are in groups of three.***

T: ***Let's use a similar method on the binary abacus to calculate $\frac{1}{3}$ x 1.***

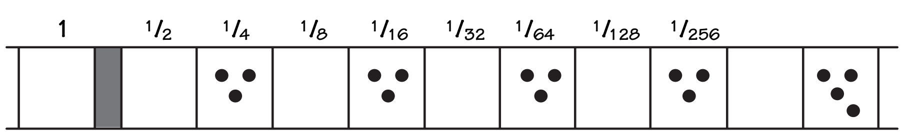
Put 1 on the abacus.



Invite students to make backward trades. Whenever there are at least three checkers on a board, push three checkers into a small group. Students should notice the repetitive pattern after several trades.



Continue until this configuration is on the abacus.

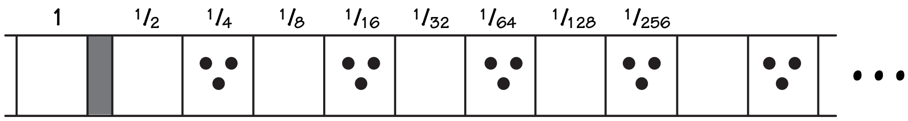


T: ***Will this ever end?***

S: ***No, there's always one extra checker. We can go on and on making backward trades.***

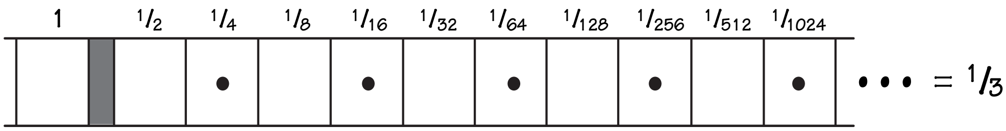
N29

Indicate the repeating pattern by removing the extra checker and writing three dots.



T: ***Now, how can we show ⅓?***

Let a student remove two checkers from each group of three checkers.



T: ***This suggests a new name for ⅓.***
The addition (pattern) goes on and on.

Write this number sentence on the board.

$$\frac{1}{3} = \frac{1}{4} \times \frac{1}{16} \times \frac{1}{64} \times \frac{1}{256} \times \frac{1}{1024} \times \cdots$$

Exercise 4

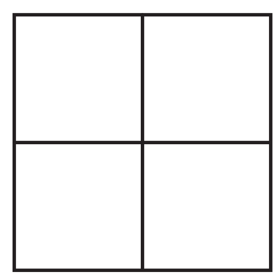
You may like to ask the class to give you four names for children to use in this story.

T: ***Four children—Brenda, Julio, Suzanne, and Artin—want to divide a cake equally among themselves.***

Draw a large square on the board. The region will be subdivided several times as the story unfolds, so make the square as large as possible. Then it will be easier to view the subdivisions. You may also like to suggest that students draw a 16-cm square on their papers and follow along with the divisions and colorings on their papers.

T: ***Show us how to divide the cake so it is easy to see that each child gets an equal amount.***

There are many ways to divide the cake. Let several students respond, and then divide the cake into four squares.



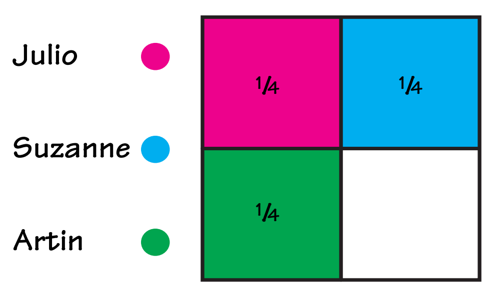
T: ***What fraction of the entire cake does each child receive?***

S: ***One-fourth.***

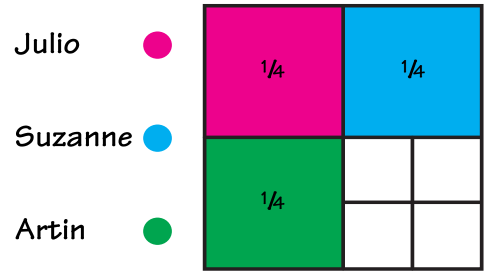
Label three regions as you announce,

T: ***Julio, Suzanne, and Artin each eat their share.***

But Brenda says, "I am really not very hungry. I'll share my portion equally among the four of us."



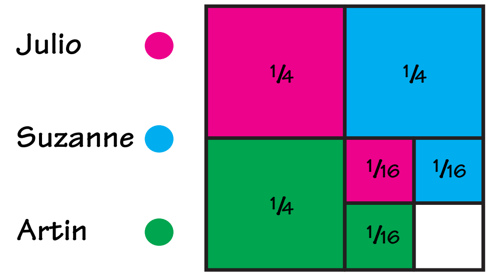
Let a student divide Brenda's portion, again making square subdivisions.



T: ***What fraction of the whole cake are each of these pieces?***

S: ***One-sixteenth. Sixteen small squares would fill the whole square.***

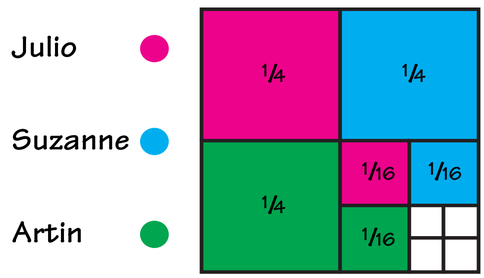
Label three regions as you announce,



T: ***Again, Brenda's three friends eat their shares.***

But Brenda says "My piece of cake is still too big! I will share my portion equally among the four of us."

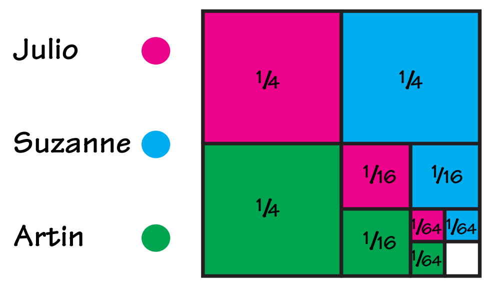
Let a student make the division.



T: ***What additional fraction of the whole cake does each child receive?***

S: ***One sixty-fourth.***

T: ***I will shade three of the portions as before.***



Continue the story with Brenda protesting that her portion is too large. Each time, let a student make the division, name the fraction represented ($\frac{1}{256}$, $\frac{1}{1024}$, …), and shade the appropriate regions. Repeat the procedure until it is too difficult to show the divisions.

T: ***The story goes on and on since Brenda always refuses her portion of the cake. If we could continue forever, what fraction of the cake would Julio receive?***

S: $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256}$ ***and so on.***

S: $\frac{1}{3}$***, since only three children would receive equal amounts of cake and Brenda would eat none of the cake.***

Write this equality on the board.

$$\frac{1}{3} = \frac{1}{4} \times \frac{1}{16} \times \frac{1}{64} \times \frac{1}{256} \times \cdots$$

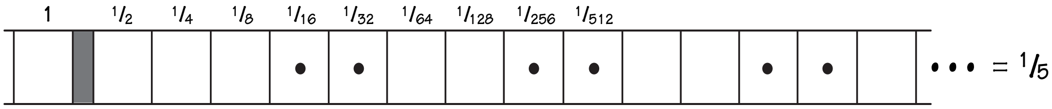
T: ***Do you see any connection between this story and the problem on the binary abacus?***

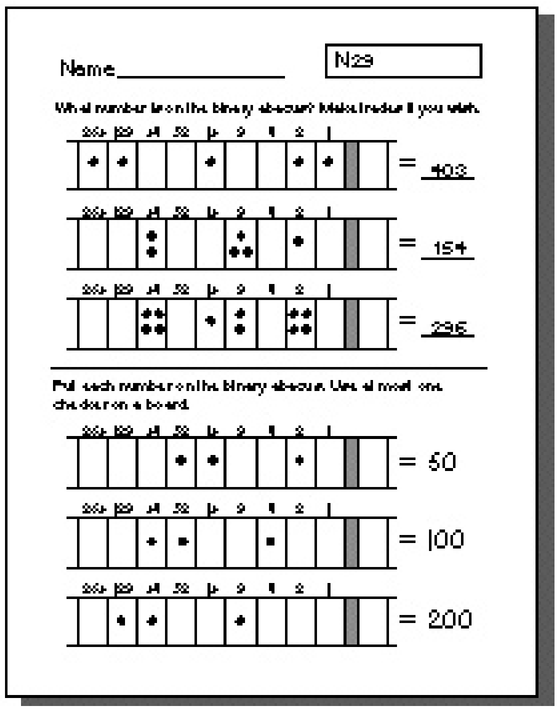
S: ***Both the story about Brenda and the problem on the abacus result in the same new name for*** $\frac{1}{3}$.

Encourage students to draw analogies between the two situations. For example, the fourth checker on a square on the binary abacus plays the same role as Brenda in the story about the cake.

Extension Activity

You may like to challenge some students to represent ⅕ on the binary abacus.





Capsule Lesson Summary

Decode numbers and put numbers on a base three abacus with regular, weighted, and negative checkers. Extend the base three abacus to include boards to the right of the bar, and use its rule for trading checkers to label the boards. Demonstrate a method of adding fractions on the abacus. Use both the base three abacus and a story about sharing a cake among friends to determine that $\frac{1}{2} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$.

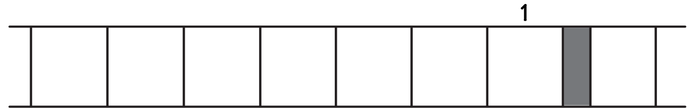
Materials

Teacher	• Meter stick • Colored chalk	**Student**	• Paper • Colored pencils, pens, or crayons • Worksheets N30 (no star), *, and **

Description of Lesson

Exercise 1

Draw this part of an abacus on the board.

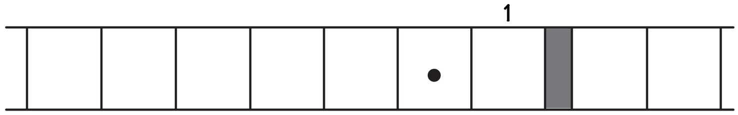


T: ***Do you remember the base three abacus and its rule for trading checkers?***

Invite a student to explain the rule to the class. If no student recalls the rule, remind the class:

> Three checkers on a board
> trade for
> one checker on the next board to the left.

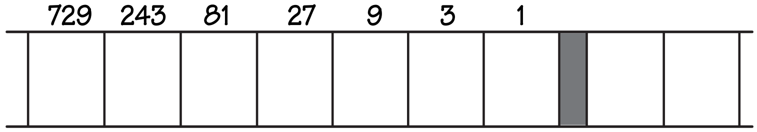
Put one checker on the board to the left of the ones board.



T: ***What number is this?***

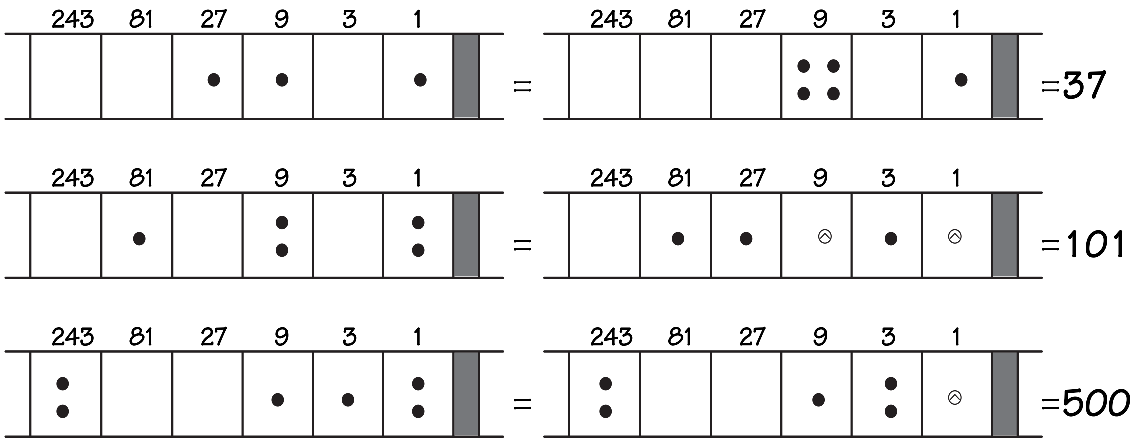
S: ***3. One checker on that board is the same as three checkers on the ones board.***

Demonstrate the trade and label the threes board. Continue by letting students label another five or six places to the left of the threes board. Use the trading rule to justify each place value: 3 x 3 = 9;
3 x 9 = 27; 3 x 27 = 81; and so on.

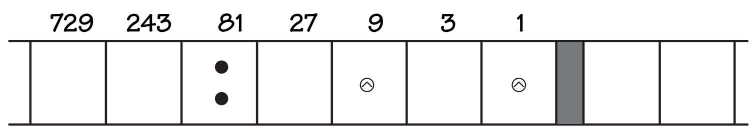


N30

Invite students to put numbers such as 37, 101, and 500 on the abacus. Allow students to find several solutions and use negative checkers if they wish. For example:



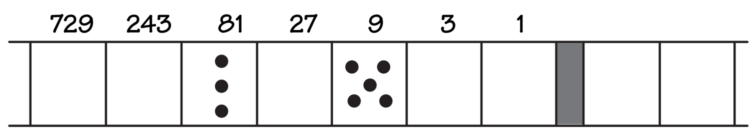
Put this configuration on the abacus.



T: ***What number is this?***

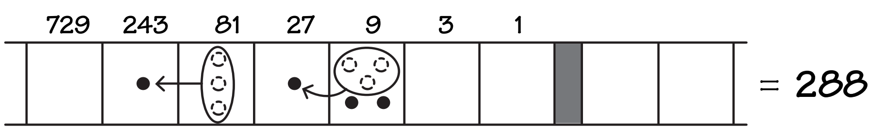
S: ***152. 2 x 81 = 162, $\widehat{9} + \widehat{1} = \widehat{10}$, and 162 + $\widehat{10}$ = 152.***

Display this configuration.



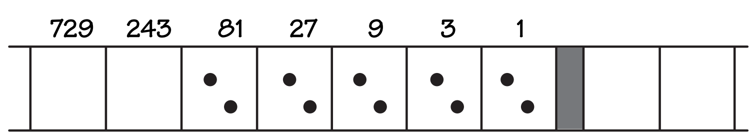
T: ***What number is this?*** (288)

As students wish, let them make trades before calculating the number on the abacus.



Distribute copies of Worksheet N30 (no star) and let students work independently for about five minutes. Then check answers collectively, discussing only those problems that seemed difficult for many students.

Display this configuration on the abacus.

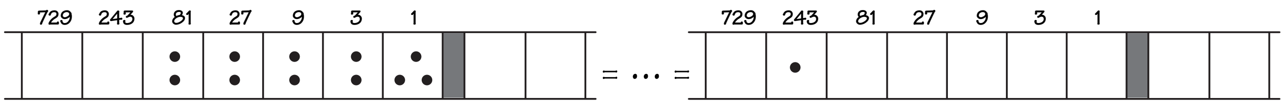


T: ***What number is this? Try to calculate it without doing many additions.***

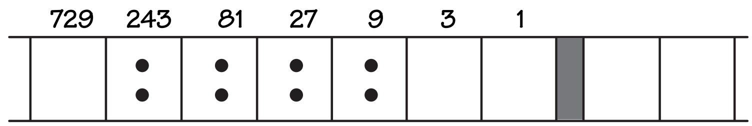
S: ***242.***

S: ***If we add one to the number on the abacus by putting one more checker on the ones board, we can make trades until there is just one checker on the 243s board. So the number on the abacus now is 242 since 242 + 1 = 243 (or 243 – 1 = 242).***

Suggest this method if no one mentions it. Then demonstrate the sequence of trades that putting another checker on the ones board triggers.



Display this configuration.

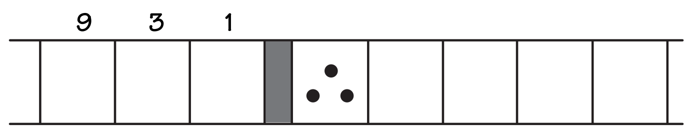


T: ***What number is this?***

S: ***720; 729 – 9 = 720. If we put another checker on the nines board, we could make trades until there was just one checker on the 729s board.***

Exercise 2

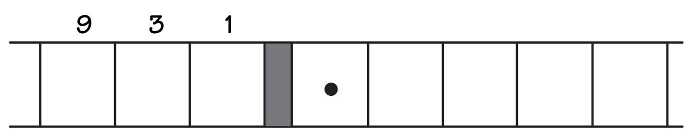
Extend the abacus several places to the right of the bar, and put on this configuration.



T: ***What number is this?***

S: ***1. We can make a trade.***

Remove two checkers.

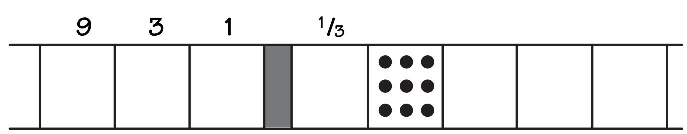


T: ***What number is this?***

S: ***$\frac{1}{3}$; $\frac{1}{3}$*** x ***1 = $\frac{1}{3}$.***

S: ***$\frac{1}{3}$; $\frac{1}{3}$ + $\frac{1}{3}$ + $\frac{1}{3}$ = 1.***

Label the $\frac{1}{3}$s board and then display this configuration.



T: ***What number is this?***

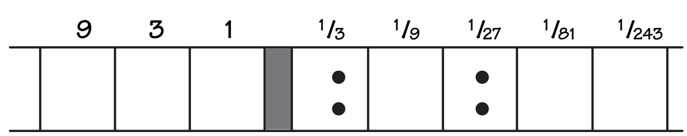
S: ***1. We could make several trades.***

T: ***How should we label this board?***

S: ***$\frac{1}{9}$.***

Label the $\frac{1}{9}$s board. Invite students to label a few more boards to the right.

Display this configuration.



T: ***What number is on this board*** (point to the $\frac{1}{3}$s board)***?***

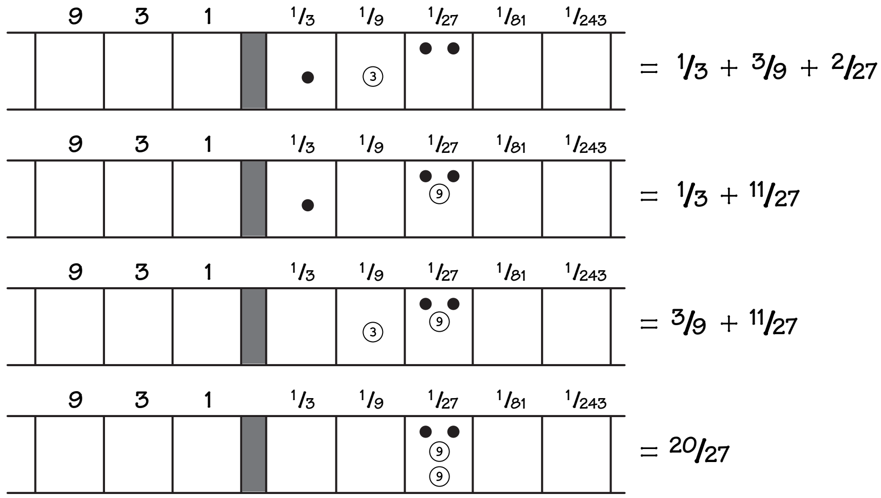
S: ***$\frac{2}{3}$.***

T: ***What number is on this board*** (point to the $\frac{1}{27}$s board)***?***

S: ***$\frac{2}{27}$.***

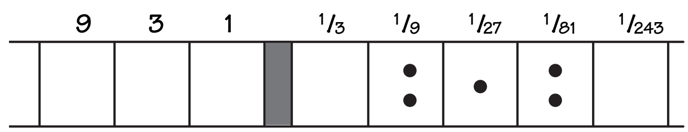
T: ***The number on the abacus is $\frac{2}{3}$ + $\frac{2}{27}$. One way to find this sum is to make trades until all of the checkers are on the same board of the abacus.***

Invite students to make backward trades. After each trade, write an equivalent name for $\frac{2}{3} + \frac{2}{27}$. Allow students to comment on the various equivalences. For example:

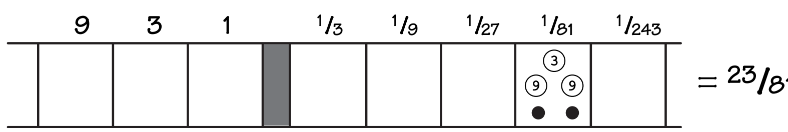


T: ***We found many names for $\frac{2}{3} + \frac{2}{27}$, including $\frac{20}{27}$.***

Display this configuration on the abacus.



Invite students to make trades to show that $\frac{2}{9} + \frac{1}{27} + \frac{2}{81} = \frac{23}{81}$.



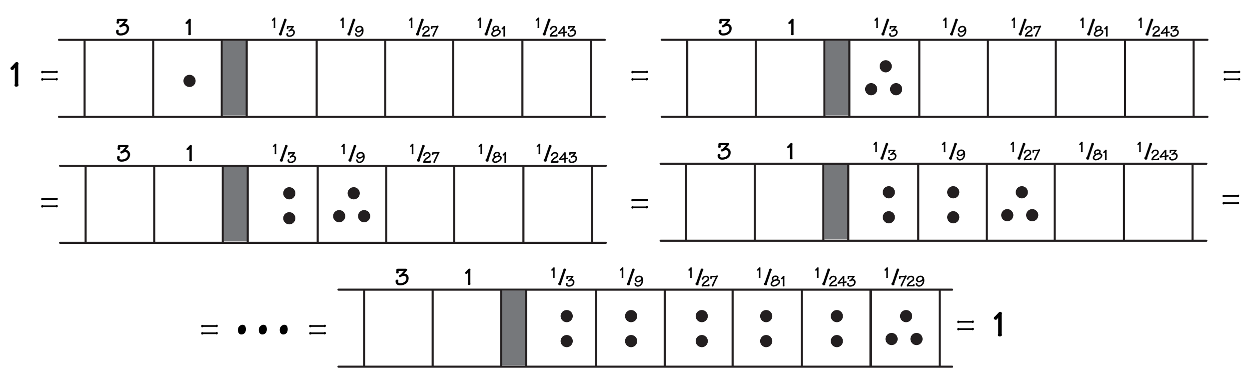
Exercise 3

Refer to Exercise 3 of Lesson N29 as you observe the following.

T: ***When we work with the binary abacus, it is easy to show fractions such as $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{8}$, and $\frac{5}{16}$, but it is harder to show $\frac{1}{3}$. On the base three abacus, it is easy to put on $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, and so on; but how could we put on $\frac{1}{2}$?***

Encourage students to adapt the method they used to show $\frac{1}{3}$ on the binary abacus.

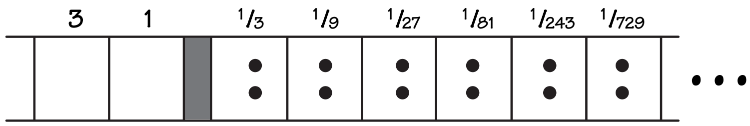
S: ***Put 1 on the abacus and make backward trades to get the checkers in groups of two.***



T: ***Will this ever end?***

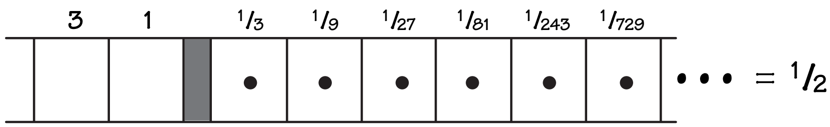
S: ***No, there is always one extra checker that we can use to make another backward trade.***

Indicate the repeating pattern by removing the extra checker and writing three dots.



T: ***This is 1. How do we show ½?***

S: ***Remove one checker from each pair.***



Write this number sentence on the board.

$$\frac{1}{2} = \frac{1}{3} \times \frac{1}{9} \times \frac{1}{27} \times \frac{1}{81} \times \frac{1}{243} \times \cdots$$

Exercise 4

Draw a large rectangle on the board. You may like to suggest that students draw an 18-cm by 27-cm rectangle on their papers, and follow along with the divisions and colorings.

T: ***Do you remember the story about four friends who shared a cake?***

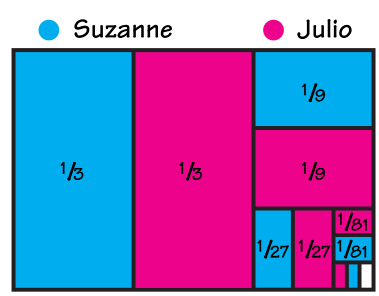
S: ***They divided the cake into four pieces all the same size. Three of the friends took their shares, but Brenda wasn't hungry so she divided her share into four pieces of the same size. Again Brenda didn't eat her share and divided it into four pieces of the same size. This just kept on going. Eventually each of Brenda's friends would receive ⅓ of the cake, and we discovered a new name for ⅓.***

T: ***How might we change the story to consider ½ instead of ⅓?***

S: ***Let three friends share a cake, and make one keep dividing her share.***

S: ***First divide a cake into three pieces all the same size.***

With the class, choose names for three friends and divide the rectangle. In the picture that follows, Brenda is still the one who is not hungry and continually divides her share further. Let students explain what happens at each step.



Repeat several times until it is difficult to make more divisions and the class sees the pattern.

T: ***We will stop the drawing, but the story goes on and on. What fraction of the whole cake would be red?***

S: ***½.***

S: ***⅓ + 1/9 + 1/27 + 1/81 + 1/243 and so on.***

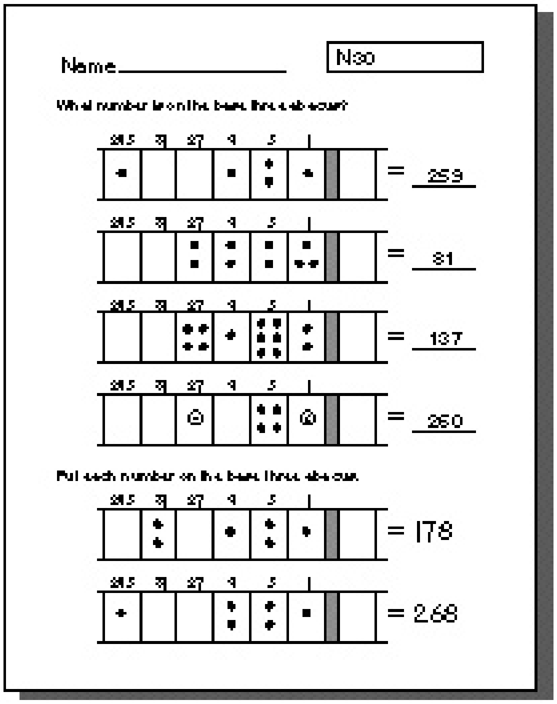
If the first answer is not suggested, ask about what portion of the cake Suzanne (or Julio) would receive.

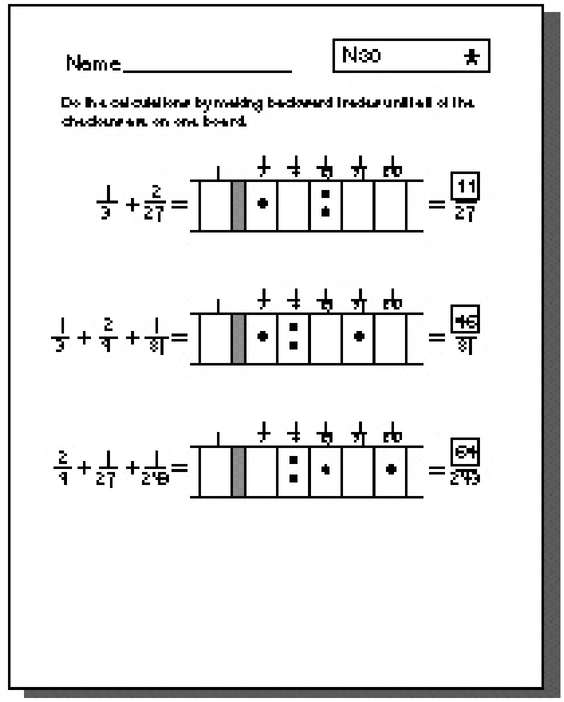
Write this equality on the board.

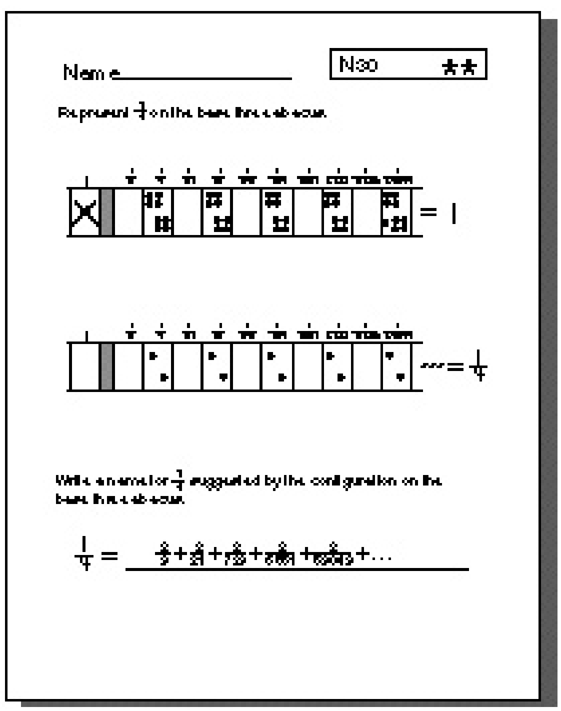
$$\tfrac{1}{2} = \tfrac{1}{3} \times \tfrac{1}{9} \times \tfrac{1}{27} \times \tfrac{1}{81} \times \tfrac{1}{243} \times \cdots$$

Encourage students to discuss and make comparisons between the two problems in this lesson and also between the names for ½ and ⅓.

Worksheets N30* and ** are available for individual work.







N31 WHO ARE ZIG AND ZAG?

Capsule Lesson Summary

Review the concepts of prime and square numbers. Solve a detective story with clues involving these concepts and the relations: is at least 20 less than, +100, and +80. Utilize a number line to examine the clues.

Materials

Teacher • Colored chalk

Student • Worksheets N31(a) and (b)

Description of Lesson

Exercise 1

Initiate a discussion of prime numbers.

T: ***What are some positive prime numbers?***

S: ***23, 17, and 31.***

T: ***What are some numbers that are not prime?***

S: ***9, 1 000, 16, and 10.***

T: ***What is a positive prime number?***

S: ***A number with exactly two positive divisors.***

T: ***What is the greatest prime number less than 50?*** (47)
What is the least positive prime? (2)
What is the next prime after 2? (3) ***After 3?*** (5)

Continue until the class names all of the positive prime numbers less than 50. (2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47)

T: ***Is there a square number between 5 and 50?***

S: ***9 (or 16 or 25 or 36 or 49).***

T: ***Why is 9 a square number?***

S: ***Because 3 x 3 = 9.***

T: ***What is a square number?***

S: ***A number that is the product of a number and itself.***

T: ***Are there other square numbers between 5 and 50?***

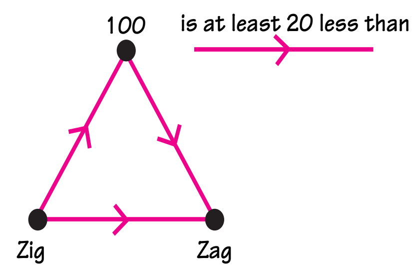
S: ***Yes; 16, 25, 36, and 49.***

Exercise 2

Tell the class that you have a detective story about two secret numbers, Zig and Zag, for them to solve.

Clue 1

Draw this picture on the board.



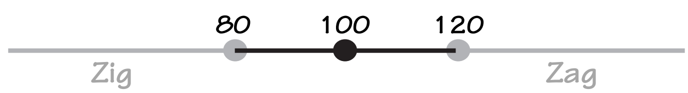
T: ***The first clue about Zig and Zag is in this arrow picture. What numbers could Zig be? Why? What numbers could Zag be? Why?***

S: ***Zig could be 78, because 78 is 22 less than 100.***

S: ***Zag could be 125, because 100 is 25 less than 125.***

Accept several more numbers for both Zig and Zag. Then distribute copies of Worksheet N31(a) and let students work independently for a few minutes. In a collective discussion of the worksheet, ask students to explain their answers. An answer key for this worksheet can be found at the end of this lesson description.

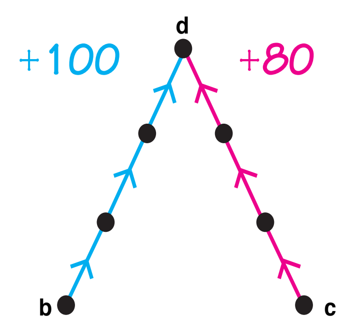
Draw a part of the number line on the board with 100 centrally located. Ask the class to locate Zig and Zag. That is, students should find that Zig could be 80 or any number less than 80 and Zag could be 120 or any number greater than 120.



Do not erase the number line as it will be needed later in the lesson.

Clue 2

Direct students to look at Worksheet N31(b) and to locate Zig and Zag in the arrow picture. While students are working, draw the arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

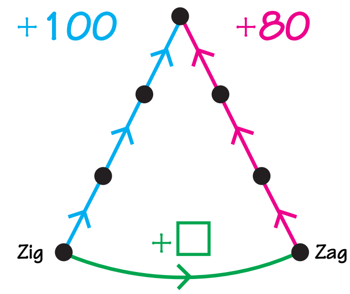
T: ***Where is the greatest number in this picture?*** (at **d**) ***Circle it on your worksheet. Where is the least number in this picture?*** (at **b**)

Zig and Zag are the two least numbers in this picture. Where did you put Zig and Zag?

It is likely that most students will agree that **b** and **c** are the least two numbers in the arrow picture. However, there may be some disagreement over which of the two is Zig and which is Zag. In this case, refer back to clue 1 to note that Zig is less than Zag.

S: ***Zig is less than Zag, so Zig goes here (at b) and Zag goes here (at c).***

Locate Zig and Zag in the picture, and connect them with a green arrow.

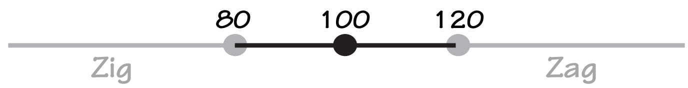


T: ***This arrow picture gives more information about Zig and Zag; it shows how much less Zig is than Zag. What could the green arrow be for?***

S: ***+60, because 300 – 240 = 60.***

S: ***Zig + 300 = Zag + 240, and 300 is 60 more than 240. So Zig is 60 less than Zag.***

Refer back to the number line as you ask,



T: ***What are some numbers that Zig and Zag could be?***

S: ***Zig could be 65 and Zag could be 125.***

S: ***Zig could be 70 and Zag could be 130.***

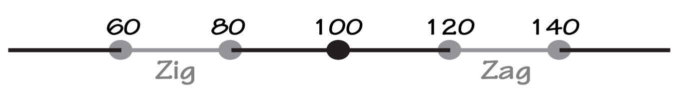
You may like to suggest a possibility for Zag and then check for Zig, or vice versa. In this way, students should begin to notice that there is a least possibility for Zig and a greatest for Zag.

S: ***Zig could be 90.***

T: ***What is the least number Zig could be? Why? What is the greatest number Zag could be? Why?***

S: ***80 is the greatest number for Zig, and Zag is 60 more, so 140 is the greatest number that Zag could be.***

S: ***120 is the least number for Zag, and Zig is 60 less, so 60 is the least number Zig could be.***



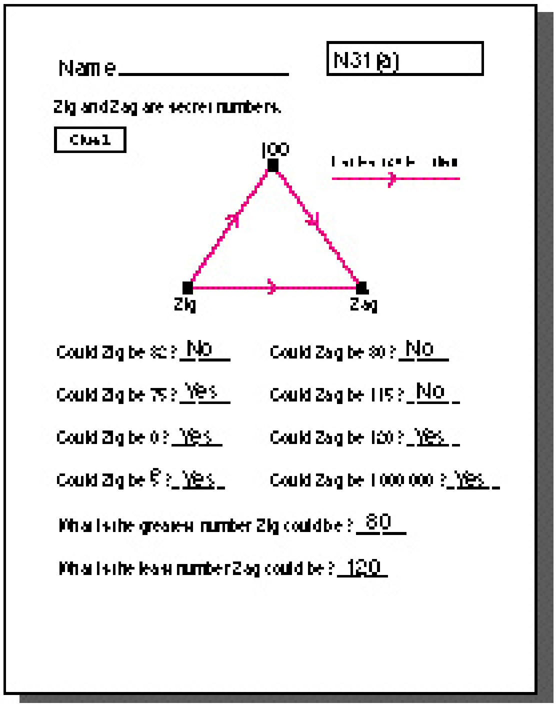
Clue 3

T: ***The last clue is that one of the secret numbers is prime and the other is a square number. Who are Zig and Zag?***

Allow students to work with a partner on this clue. The class should conclude that Zig is 61 and Zag is 121.

Writing Activity

Invite students to write a detective story of their own that uses prime and square numbers in clues.


Name ______ N31(a)

Zig and Zag are secret numbers.

Clue 1

Could Zig be 82? No

Could Zag be 80? No

Could Zig be 75? Yes

Could Zag be 115? No

Could Zig be 0? Yes

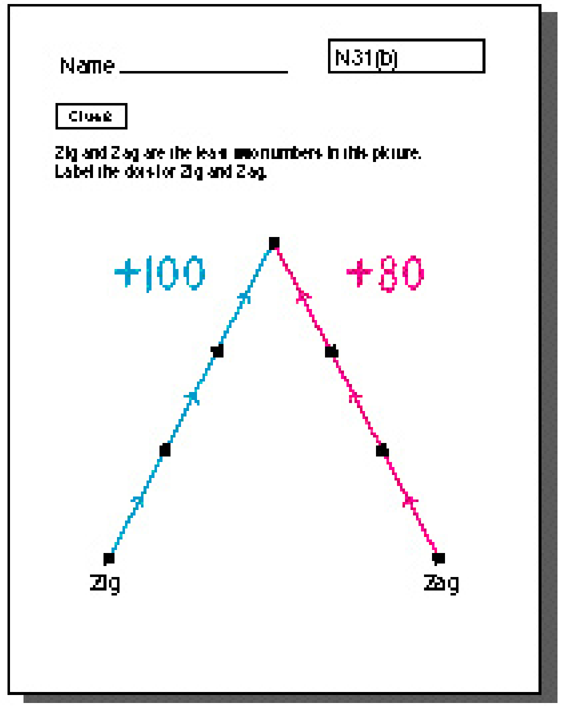
Could Zag be 120? Yes

Could Zig be [illegible]? Yes

Could Zag be 1 000 000? Yes

What is the greatest number Zig could be? 80

What is the least number Zag could be? 120


Name ______ N31(b)

Clue 2

Zig and Zag are the least two numbers in this picture.
Label the dots for Zig and Zag.

Capsule Lesson Summary

Practice multiplying and dividing by multiples of 10. Label the dots in an arrow picture with arrows for "divided by a whole number." Identify composites of relations in the arrow pictures.

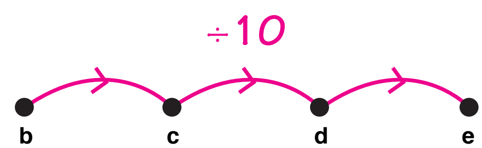
Materials

Teacher	• Colored chalk	Student	• Colored pencils, pens, or crayons • Worksheets N32*, **, ***, and ****

Description of Lesson

Exercise 1

Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

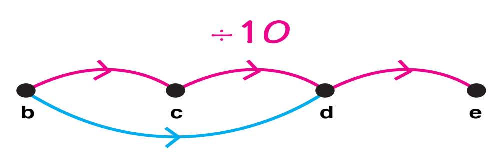
Label the appropriate dots as you ask,

T: ***If 860 is here*** (at **b**)***, what numbers are these*** (at **c, d,** and **e.**)

S: ***86, 8.6, and 0.86.***

Repeat the activity several times. Label one of the dots and ask students to label the other three. Vary which dot's label is given to start. For example, you could start with 42 at **b**, 3.7 at **c**, 56 at **d**, and 0.081 at **e**. When appropriate, introduce return arrows to the picture. Encourage students to comment on ÷10 and 10x patterns.

Draw a blue arrow from **b** to **d**.



T: ***What could this blue arrow be for?***

S: ***÷100.***

Label the blue arrow.

T: ***If 17 000 is here*** (at **b**)***, what number is here*** (at **d**)***?***

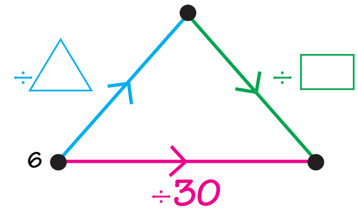
S: ***170 since 17 000 ÷ 100 = 170.***

S: ***170 because 17 000 ÷ 10 = 1 700, and 1700 ÷ 10 = 170.***

Present additional problems to practice dividing and multiplying by 100. For example, start with 263 at **b**; 479 at **d**; or 0.816 at **d**. Let students comment on how to multiply and divide by 100.

T: ***What number is 6 ÷ 30?***

Let students comment on the problem. Emphasize that 30 ÷ 6 does not equal 6 ÷ 30. Draw this arrow picture on the board.



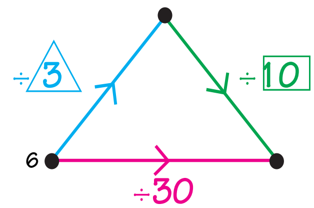
T: ***What could the blue arrow and the green arrow be for?***

S: ***÷10 and ÷3.***

S: ***÷6 and ÷5.***

S: ***÷15 and ÷2.***

Check some of the responses by labeling dots. Then label the arrows as shown here.



T: ***One easy way to divide by 30 is first to divide by 3 and then to divide by 10. What number is 6 ÷ 30?***

S: ***0.2, since 6 ÷ 3 = 2, and 2 ÷ 10 = 0.2.***

Solve the following problems in a similar manner. Draw corresponding arrow pictures when necessary. For example, decompose ÷700 into ÷7 and ÷100. (Answers are in boxes.)

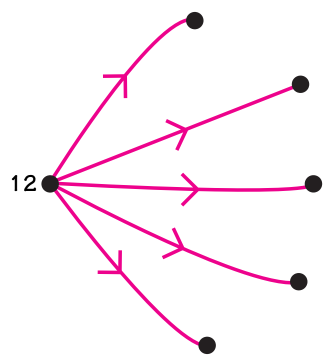
$42 \div 60 = \boxed{0.7}$ $\qquad$ $56 \div 700 = \boxed{0.08}$

$30 \div 20 = \boxed{1.5}$ $\qquad$ $540 \div 900 = \boxed{0.6}$

$1.2 \div 40 = \boxed{0.03}$ $\qquad$ $3.6 \div 400 = \boxed{0.009}$

Exercise 2

Draw this arrow picture on the board.



T: ***Each red arrow is for "divide by some whole number." The dots may be for any decimal numbers. Label one of the red arrows and give us the corresponding ending number.***

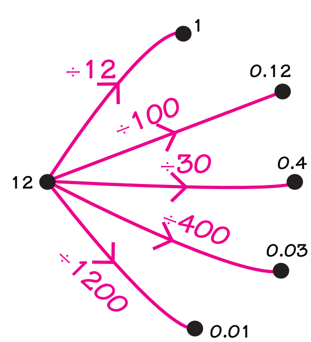
S: ***12 ÷ 4 = 3.***

Label a red arrow ÷ 4. The ending dot is 3.

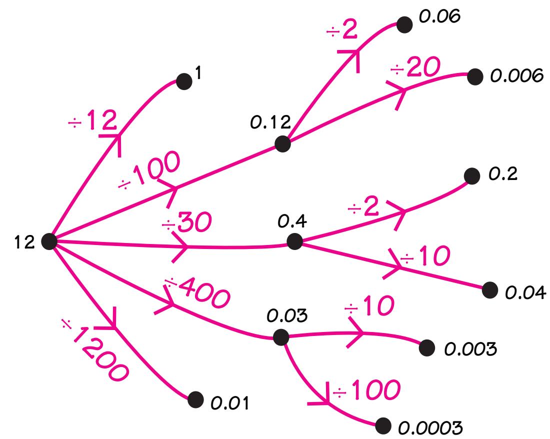
S: ***12 ÷ 10 = 1.2.***

Label a red arrow ÷10. The ending dot is 1.2.

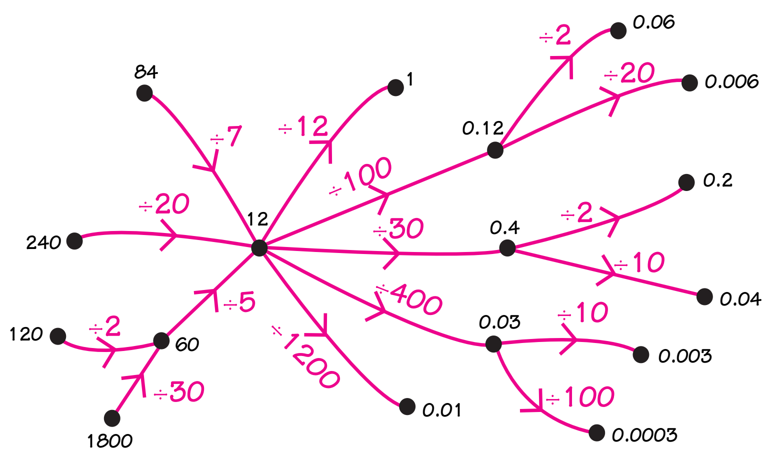
Label all of the dots and arrows as suggested by students. When the picture is complete, erase the students' labels for dots and arrows, and relabel the arrows in ways not yet suggested by students. Invite students to label the ending dots. For example:



Extend the arrow picture to the right as in the following picture. Invite students to label the new arrows and corresponding ending dots. For example:



Draw several arrows ending at 12 in the picture; invite students to label these arrows and corresponding starting dots. If necessary, remind students that all red arrows are for “divide by some whole number.” You might extend the picture even further to the left, as shown below, and continue the activity.



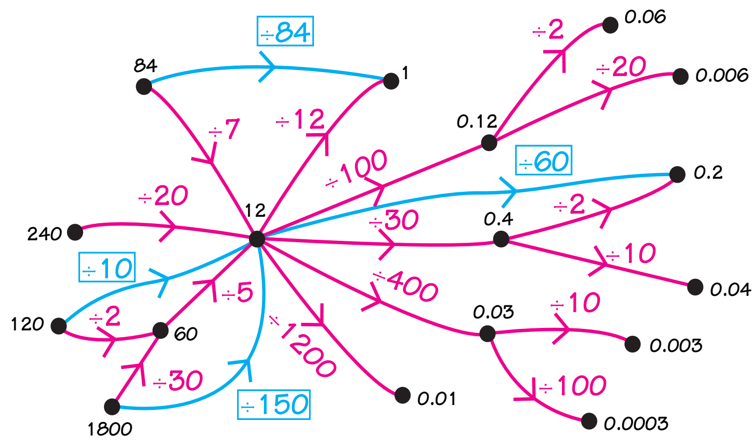
Point to the numbers at the start of red arrows ending at 12 or leading to 12.

T: ***Starting at all of these numbers there are red arrow roads leading to 12. What kind of numbers are they?***

S: ***Multiples of 12.***

Other answers such as “even numbers” or “multiples of 6” are correct. Use return arrows to direct a discussion so that students see these numbers as multiples of 12.

In the arrow picture, draw some blue arrows that are composites of red arrows. Invite students to label the blue arrows. Encourage them to use composition. For example:

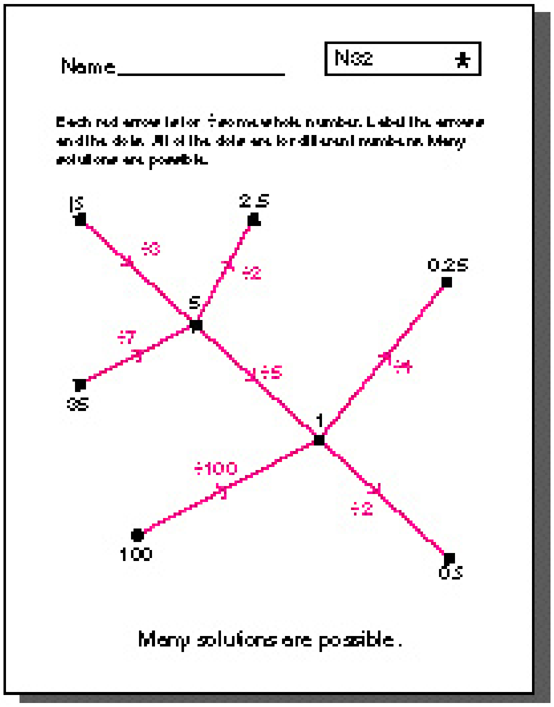


Exercise 3

Distribute copies of Worksheet N32*.

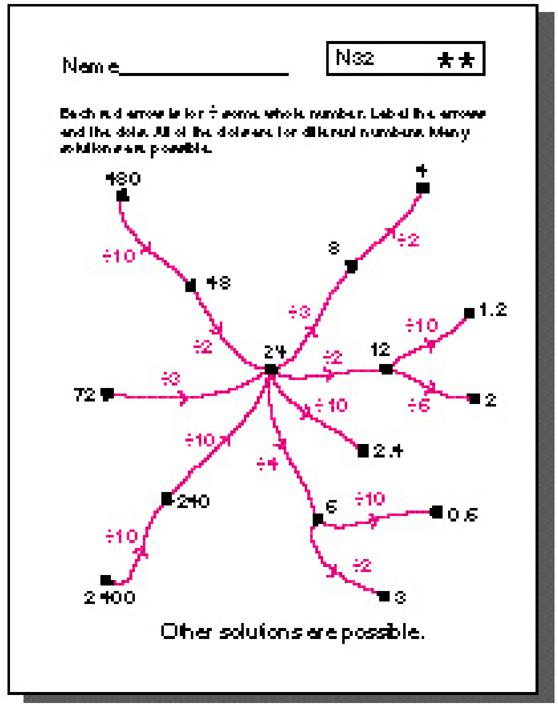
T: ***Each red arrow is for "divide by some whole number." Label the arrows and the dots. Try to make all of the dots for different numbers.***

As students work, draw the arrow picture from Worksheet N32* on the board. Allow students, when ready, to proceed to Worksheets N32**, ***, and ****. After a few minutes, collectively solve the problem on Worksheet N32* and observe that any solution builds a road from 15 to 0.5. Discuss several students' solutions. One solution is shown in the answer key.


Name________________ N32 ★

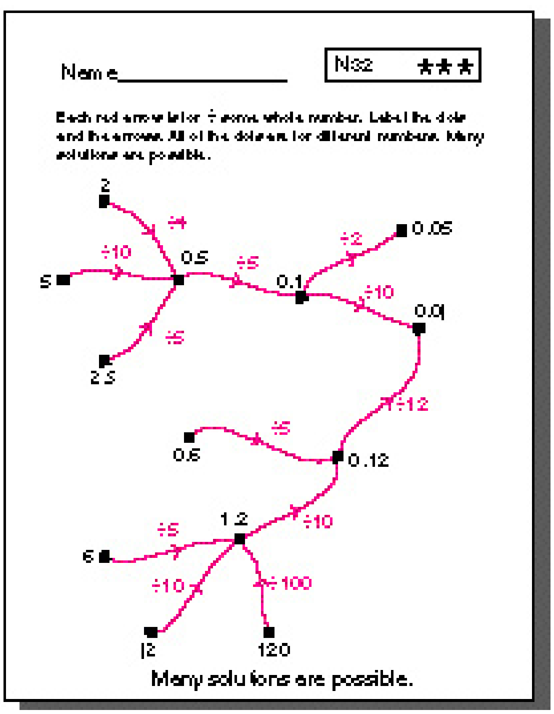
Each red arrow is for ÷ some whole number. Label the arrows and the dots. All of the dots are for different numbers. Many solutions are possible.

Many solutions are possible.


Name________________ N32 ★★

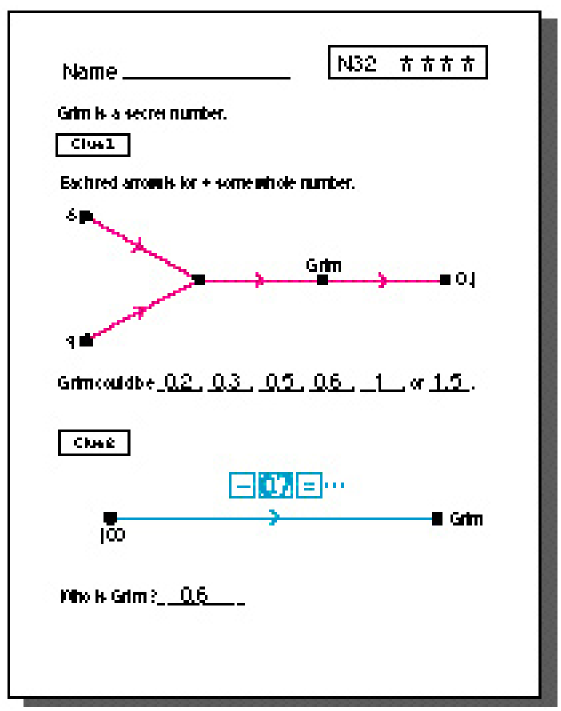
Each red arrow is for ÷ some whole number. Label the arrows and the dots. All of the dots are for different numbers. Many solutions are possible.

Other solutions are possible.


Name________________ N32 ★★★

Each red arrow is for ÷ some whole number. Label the dots and the arrows. All of the dots are for different numbers. Many solutions are possible.

Many solutions are possible.


Name ________________ N32 ★★★★

Grim is a secret number.

Clue 1

Each red arrow is for ÷ some whole number.

Grim could be 0.2, 0.3, 0.5, 0.6, 1, or 1.5.

Clue 2

Who is Grim? 0.6

Capsule Lesson Summary

In cooperative groups, solve two comparison problems, and present or write explanations for methods of solution.

Materials

Teacher
- Colored Chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Calculator
- Counters or other props

Description of Lesson

Organize the class into small cooperative groups for problem solving. Each group should have supplies such as paper, colored pencils, calculator, and props or manipulatives. Distribute copies of Worksheets N33(a) and (b).

Direct the groups to work on the two problems on the worksheets. Encourage students to find methods of solving the problems that everyone in their group can explain. You may want students to answer questions and write explanations on their worksheets individually, but suggest that each group prepare to present their solutions to the class.

As you observe group work, look for different solution techniques so that you can arrange that these different approaches are presented to the whole class.

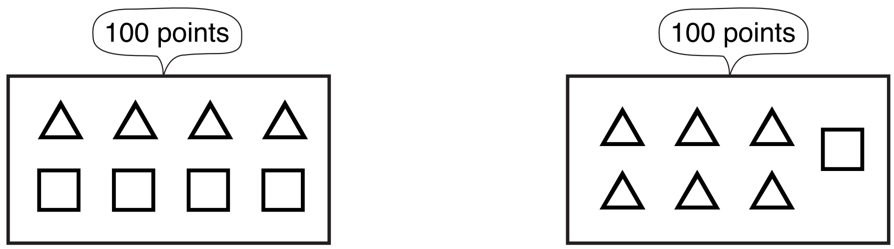
For your information, some different ways to solve each problem are shown on the following two pages (solutions are shaded in gray). Your students are likely to find their own methods and explanations.

Writing Activity

Suggest that students or groups write comparison problems similar to the two in this lesson. You may like to let groups or individual students exchange problems, or you may suggest students take home one of their problems to do with a family member.

Name ____________________ N33(a)

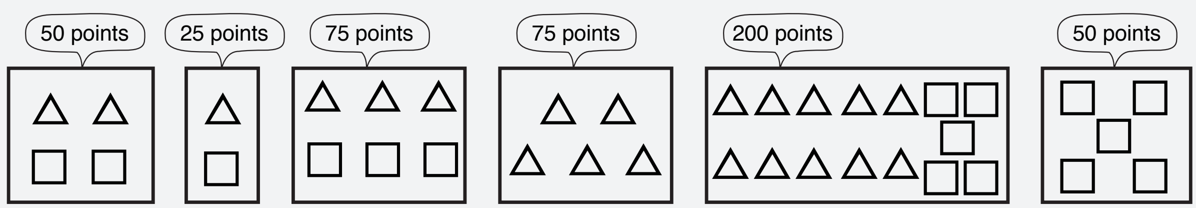
Imagine a game in which a player gets points for triangles (△) and squares (□). Four △s and four □s give 100 points. Also, six △s and one □ give 100 points.



Which shape gives more points in this game? △

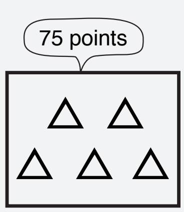
- Consider the following: 4 △s, 4 □s $\overset{\text{balance}}{=}$ 6 △s, 1 □
  If you take away points for 4 △s … 4 □s = 2 △s, 1 □
  And then take away points for 1 □ … 3 □s = 2 △s
  Since the points from 3 □ s equal the points from 2 △s, △s must give more points.
- On the left, eight things give 100 points; on the right, seven things give 100 points.
  Since there are more △s on the right, △s must give more points.

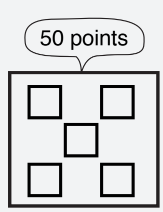
Find the number of points for some different combinations of △s and □s.



In this game, a player gets
how many points for a △? 15
how many points for a □? 10

- Trial and error methods may be used, especially along with one of the different point combinations above.
- Since 2 △s = 3 □s, then replacing △s in one of the 100 point sets with □s gives 10 □s = 100 or 10 points for a □ and 15 points for a △.
- From combinations



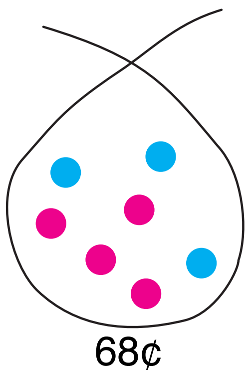


△ = 15 points □ = 15 points

Name ____________________

N33(b)

Jon bought a bag with 3 blue and 4 red marbles for 68¢.
Jan bought a bag with 4 blue and 3 red marbles for 72¢.

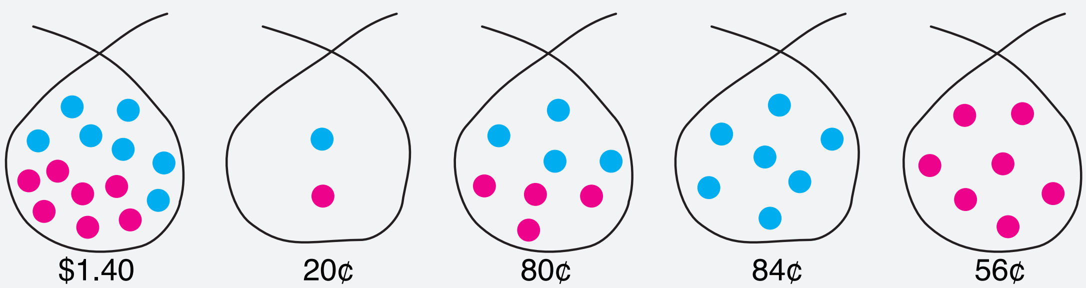


Which costs less, a blue or a red marble? Red

- Since both bags have seven marbles and the one with more red marbles is cheaper, red marbles must cost less.
- Take an equal amount from both bags, say three blue and three red marbles. That will reduce the cost of the bags by the same amount. The bag on the left with one red marble remaining will still be 4¢ less than the bag on the right with one blue marble remaining. A red marble costs 4¢ less than a blue marble.

Find the cost for some other combinations of marbles.

(Many solutions are possible.)



Find the cost for one blue marble. 12¢

one red marble. 8¢

- Trial and error methods may be used, especially along with one of the different cost combinations above.
- Since a red marble costs 4¢ less than a blue marble, you can replace the three blue marbles in the 68¢ bag with three red marbles, reducing the cost by 3 x 4 or 12¢. Then seven red marbles cost 56¢, and one red marble costs 8¢. One blue marble costs 4¢ more, or 12¢.
- When you put both bags together, you get a bag with seven red and seven blue marbles costing $1.40. There are seven red-blue combinations, so one red and one blue marble together cost $1.40 ÷ 7, or 20¢. The red marble is 4¢ less than the blue marble, so the costs are 12¢ for blue and 8¢ for red.

Capsule Lesson Summary

Calculate ½ or ⅓ of a number. Use the Minicomputer to show that ⅓ x 19 = 6.333 …. Complete calculator sentences, such as $\boxed{7}$ – 2.5 – $\boxed{4}$ $\boxed{=}$ 18 , in which one of the symbols +, –, x, ÷ goes in each blank box to make the sentence true.

Materials

Teacher
- Mini computer set
- Calculator

Student
- Paper
- Worksheets N34*, **, ***, and ****

Advance Preparation: Borrow extra Minicomputer boards from another classroom, or be prepared to draw extra boards on the chalkboard.

Description of Lesson

Exercise 1

Write these problems on the board and ask students to solve them independently.

$2 \times \boxed{} = 100$ $\quad 16 \times \boxed{} = 100$

$4 \times \boxed{} = 100$ $\quad 32 \times \boxed{} = 100$

$8 \times \boxed{} = 100$ $\quad 64 \times \boxed{} = 100$

After a few minutes of individual work, invite students to announce the numbers in the boxes for the first three problems.

$2 \times \boxed{50} = 100$

$4 \times \boxed{25} = 100$

$8 \times \boxed{12.5} = 100$

T: ***Do you see any patterns?***

S: ***Each time the first number (the number we multiply by) doubles, but the result is still 100.***

T: ***What do you notice about the numbers in the boxes?***

S: ***Each time the number is one-half of the number above it.***

Point to the next problem as you ask,

$16 \times \boxed{6.25} = 100$

$32 \times \boxed{3.125} = 100$

$64 \times \boxed{} = 100$

T: ***16 times what number equals 100?***

S: ***6.25; ½ x 12.5 = 6.25.***

T: ***32 times what number equals 100?***

You may indicate 6.25 = 6.250 as a hint.

S: ***3.125; ½ x 6 = 3 and ½ x 0.250 = 0.125, so ½ x 6.250 = 3.125.***

T: ***What do we have to do for the last calculation?***

S: ***Find ½ x 3.125.***

N34

Allow a few minutes for students to attempt the calculation themselves. Then write the problem in two or three parts on the board.

$$\begin{array}{l} \tfrac{1}{2} \times 3 \\ \tfrac{1}{2} \times 0.12 \\ \underline{\tfrac{1}{2} \times 0.0050} \\ \tfrac{1}{2} \times 3.1250 \end{array}$$

T: ***What number is ½ x 3?***

S: ***1.5.***

T: ***What is ½ x 0.12?***

S: ***0.06.***

T: ***What is ½ x 0.0050?***

S: ***0.0025.***

T: ***What is ½ x 3.1250?***

S: ***Just add; 1.5 + 0.06 + 0.0025 = 1.5625.***

S: ***So 64 x 1.5625 = 100.***

$$\begin{array}{lcl} \tfrac{1}{2} \times 3 & = & 1.5 \\ \tfrac{1}{2} \times 0.12 & = & 0.06 \\ \tfrac{1}{2} \times 0.0050 & = & 0.0025 \\ \hline \tfrac{1}{2} \times 3.1250 & = & 1.5625 \end{array}$$

Do a related activity with the following problems. Answers to the first three problems are given in the boxes. After solving the first three problems, lead the class to recognize the 3x relationship of the multipliers and the ⅓x relationship of numbers in the boxes, which is necessary to insure the same product (513).

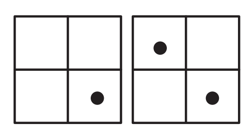
$$3 \times \boxed{171} = 513 \qquad 81 \times \boxed{} = 513$$
$$9 \times \boxed{57} = 513 \qquad 243 \times \boxed{} = 513$$
$$27 \times \boxed{19} = 513$$

T: ***81 times what number is 513?***

S: ***We need to calculate ⅓ x 19.***

T: ***Let's use the Minicomputer to help us calculate ⅓ x 19.***

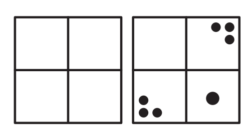
Display two Minicomputer boards and ask a volunteer to put on 19.



T: ***How do we calculate ⅓ of a number on the Minicomputer?***

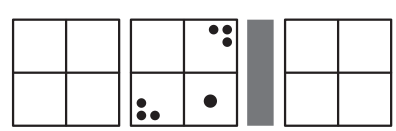
S: ***Make backward trades until all of the checkers are in groups of three. Then take two checkers away from each group of three checkers.***

Invite students to make trades, working towards getting groups of three checkers, until this configuration is on the Minicomputer.



T: ***The checkers are in groups of three except for this extra checker on the 1-square.***

Draw a bar and place a Minicomputer board to the right.

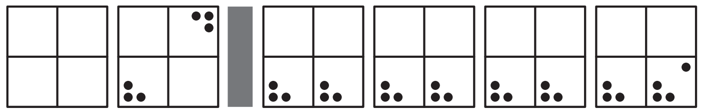


T: ***Did the number on the Minicomputer change?***

S: ***No, 19.0 is the same as 19.***

T: ***Who can make a backward trade with the checker on the 1-square?***

Invite students to continue making trades until all of the checkers are in groups of three except for a checker on the white square. Continue placing boards on the right and inviting students to make trades until you have the following configuration.



T: ***What would happen if we had more Minicomputer boards?***

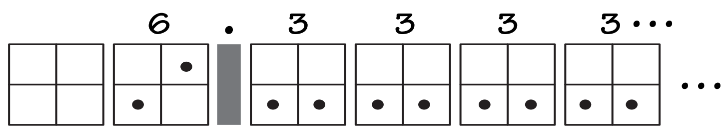
S: ***There would still be groups of three on the red squares and the white squares.***

T: ***Would the pattern ever stop?***

S: ***No, it would go on forever.***

T: ***Let's remove the extra checker and put three dots to indicate that this pattern goes on forever.***

Invite a student to remove two from every group of three checkers, getting ⅓ of 19 on the Minicomputer.



S: ***6.3333 and so on with 3s.***

T: ***So 81 times what number is 513?***

S: ***6.3333….***

T: ***Writing a star above the 3 is another way to show that the 3 repeats.***

$$6.3333\ldots = 6.3\overset{*}{3}$$

T: ***Who can do the last calculation?***

S: ***2.111 and so on with 1s.***

Complete the number sentences on the board.

$$81 \times \boxed{6.3\overset{*}{3}} = 513$$
$$243 \times \boxed{2.1\overset{*}{1}} = 513$$

Exercise 2

Write this calculator sentence† on the board.

$\boxed{7}\ \boxed{}\ \boxed{2.5}\ \boxed{}\ \boxed{4}\ \boxed{=}\ 18$

T: ***Make this calculator sentence true by filling each blank box with one of the symbols: +, x, –, ÷. You may use a symbol more than once. Solve this problem on your paper.***

After a couple minutes, check the work collectively.

$\boxed{7}\ \boxed{-}\ \boxed{2.5}\ \boxed{\times}\ \boxed{4}\ \boxed{=}\ 18$

Worksheets N34*, **, ***, and **** are available for individual work. Students should have access to calculators as they wish.

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†A ca☐
correspond to the keys pressed. You may want to modify this use of calculator sentences to fit your class calculators. Note that we abbreviate $\boxed{2}\ \boxed{\cdot}\ \boxed{5}$ with $\boxed{2.5}$ to make the sentences easier to read.

Name______________ N34 ★

Put one of the symbols +, ×, −, ÷ in each blank box to make the calculator sentence true. A symbol may be used twice in the same sentence.

3 × 7 + 19 = 40

18 ÷ 3 × 7 = 42

4 × 7 +/− 0 = 28

750 ÷ 10 ×/÷ 1 = 75

Name______________ N34 ★★

Complete.

2 × 120 = 240	3 × 40 = 120
4 × 60 = 240	6 × 20 = 120
8 × 30 = 240	12 × 10 = 120
16 × 15 = 240	24 × 5 = 120
32 × 7.5 = 240	48 × 2.5 = 120
64 × 3.75 = 240	96 × 1.25 = 120
7 × 26 = 182	3 × 135 = 405
14 × 13 = 182	9 × 45 = 405
28 × 6.5 = 182	27 × 15 = 405
56 × 3.25 = 182	81 × 5 = 405

Name______________ N34 ★★★

Put one of the symbols +, ×, −, ÷ in each blank box to make the calculator sentence true. A symbol may be used twice in the same sentence.

1.8 × 10 ÷ 9 = 2

10 ÷ 4 − 0.7 [ ] 1.8

4.4 × 5 × 5 = 110

5 × 6 ÷ 100 = 0.3

0.7 + 0.8 × 5 = 7.5

Name______________ N34 ★★★★

Put one of the symbols +, ×, −, ÷ in each blank box to make the calculator sentence true. A symbol may be used more than once in the same sentence.

5 × 8 ÷ 20 ÷ 5 [ ] 0.4

1.7 + 4.3 × 7 ÷ 4 = 10.5

3 × 6 − 2 ÷ 8 = 2

0.4 + 0.6 ÷ 10 − 0.1 [ ] 0

0 ×/÷ 387 + 25 ÷ 50 = 0.5

2 − 1.2 × 50 ×/÷ 0 [ ] 40

8 × 8 − 8 ÷ 8 = 7

5 ÷ 5 + 5 ÷ 5 = 1.2

N35 GUESS MY RULE

Capsule Lesson Summary

Play *Guess My Rule* with arrows. First use the function ${}^2/_3$x; then use 2x followed by –10. Once a rule is discovered and explained, label dots on an arrow road according to the rule.

Materials

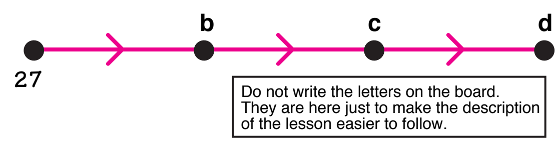
Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N35*, **, ***, and ****

Description of Lesson

Exercise 1

Draw an arrow road on the board.

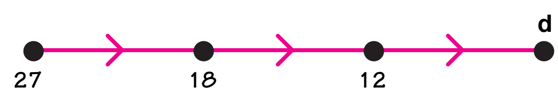


T: ***I have a secret rule for red arrows. You are going to try to guess what the rule is. All of the red arrows are for the same rule.***

Note: The secret rule for this exercise is ${}^2/_3$x. Do not announce the rule; instead, let students discover it from your clues. Label the corresponding dots as you give clues.

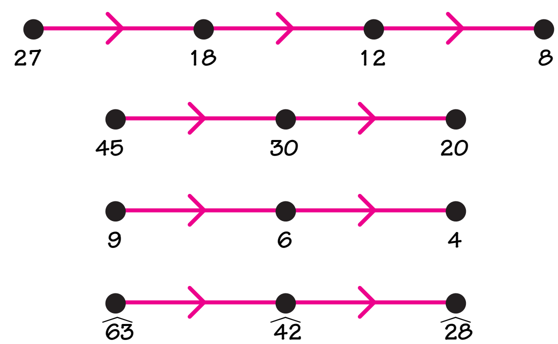
T: ***If the first dot is for 27, then using my rule, this dot*** (point to **b**) ***is for 18 and the next dot*** (point to **c**) ***is for 12.***

If you now think you know the rule, don't explain it yet. Instead, tell me what number this dot (d) is for.



Let many students give their guesses to you individually (whispering or writing them on paper). If a guess is incorrect, say out loud, for example, "No, 6 is not here." Acknowledge students who know the correct number at **d** saying only, "Yes, when you use my rule, 8 is here (at **d**)." Then let a student label the dot.

In a similar manner, let students label the dots on other arrow roads without yet stating the rule. Remind the class that all of the red arrows are for the same secret rule.



T: ***Who can explain the rule for red arrows?***

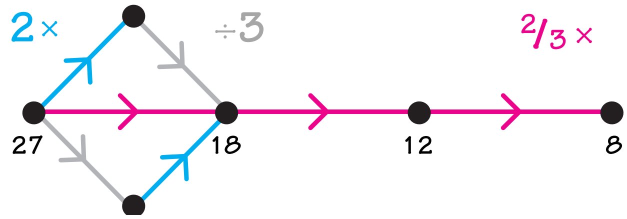
S: ***It's a two-step rule: ÷3 followed by 2x.***

S: ***I used 2x followed by ÷3.***

S: ***The rule is ${}^2/_3$x.***

N35

Write a key for red arrows as $\frac{2}{3}$x, and draw arrows for suggested compositions as well.



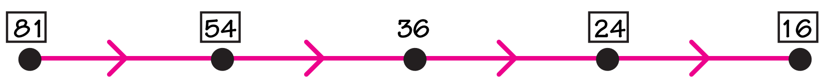
T: ***I was indeed thinking of the function*** $\frac{2}{3}$x***, but the other ways you described the rule are also correct since*** $\frac{2}{3}$x ***is the same as 2x followed by ÷3, or ÷3 followed by 2x. Let's check some of the answers. Does*** $\frac{2}{3}$ x ***45 = 30?***

S: ***Yes, 2*** x ***45 = 90 and 90 ÷ 3 = 30.***

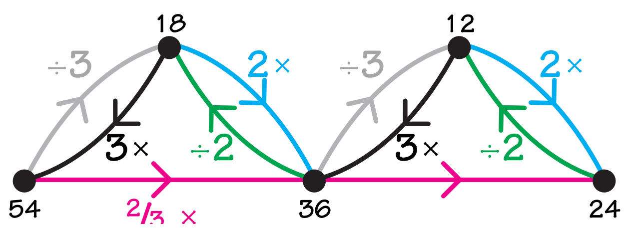
S: ***Yes, 45 ÷ 3 = 15 and 2*** x ***15 = 30.***

Similarly, let students check that $\frac{2}{3}$ x 18 = 12, $\frac{2}{3}$ x 30 = 20, and $\frac{2}{3}$ x $\widehat{63}$ = $\widehat{42}$. You may like to write these number sentences on the board to reinforce the arithmetic.

Draw the following arrow road on the board with 36 at the middle. Invite students to label the other dots according to the same rule, $\frac{2}{3}$x. (Answers are in boxes.)

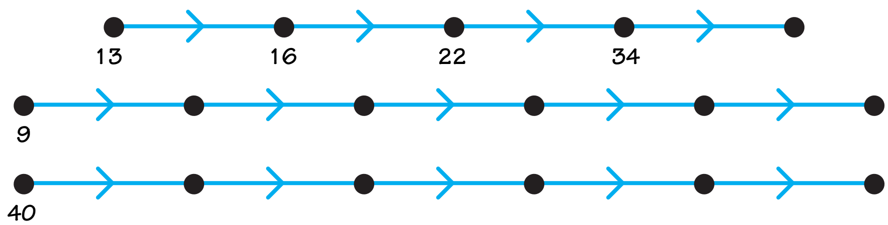


Students may use detours and return arrows as necessary. Emphasize that a red arrow can be represented as a composite of two operations: 2x followed by ÷3 or vice versa. For example,



Exercise 2

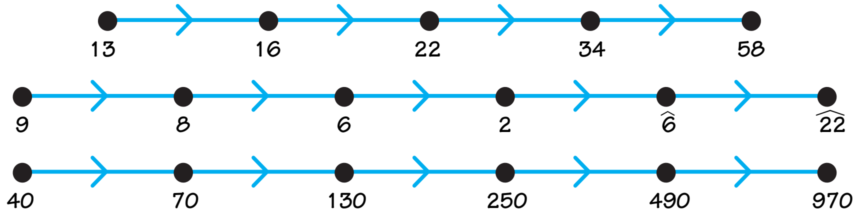
Erase the board and announce that you have a different secret rule, and this time you'll picture it with blue arrows. Then draw these arrow roads.



Note: Blue arrows are for 2x followed by –10. Do not announce this rule, but let students discover it from your clues. There are other correct ways of defining the blue arrow.

T: ***I am thinking of a secret rule for the blue arrows.*** (Point to the first arrow road.) ***Following my rule, there is a blue arrow from 13 to 16, from 16 to 22, and from 22 to 34.***

In a manner similar to Exercise 1, invite students to label other dots but without yet stating the rule.



Once all of the dots are labeled correctly, let students explain the rule.

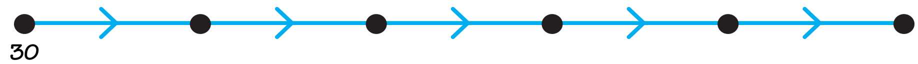
S: ***The blue arrow is for 2x followed by –10. For example, 2 x 13 = 26 and 26 – 10 = 16. Also 2 x 2 = 4 and 4 – 10 =*** $\widehat{6}$***.***

S: ***I used –5 followed by 2x. For example, 16 – 5 = 11 and 2 x 11 = 22. Also, 40 – 5 = 35 and 2 x 35 = 70.***

S: ***On each arrow road, the difference between consecutive dots doubles each time. For example, 13 + 3 = 16, 16 + 6 = 22, 22 + 12 = 34, and 34 + 24 = 58. The differences, 3, 6, 12, and 24, are doubling.***

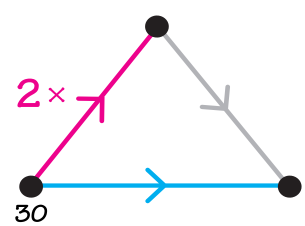
Encourage several explanations, although it is not necessary that all of the above methods be explained.

Draw this arrow road on the board.



T: ***Let's use the rule for blue arrows to label these dots.***

Students who use only the "double consecutive difference" pattern will be unable to get started. If appropriate, use this opportunity to discuss the advantages of the other statements of the rule and invite a student to label the next dot. If no student explains the rule as "2x followed by –10" or as "–5 followed by 2x," give a hint by drawing a composition arrow picture.



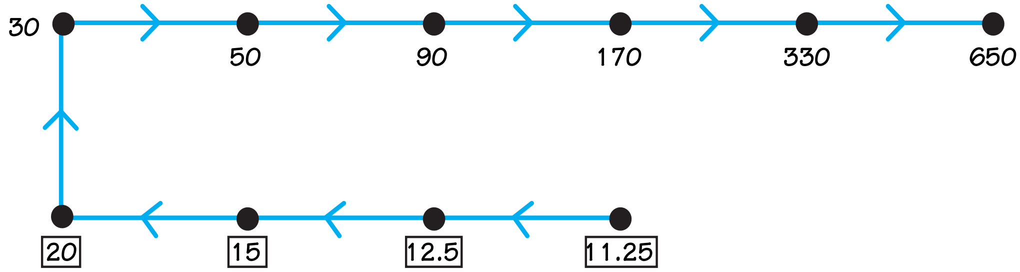
T: ***My secret rule for the blue arrow is a two-step rule; the first step is to multiply by 2. What is the second step (gray arrow)?***

Let students discover that the gray arrow is for –10.

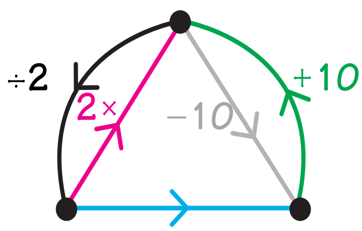
With students, continue labeling the dots on the blue arrow road. Note when students' other explanations of the rule all give the same answers for the dots.

N35

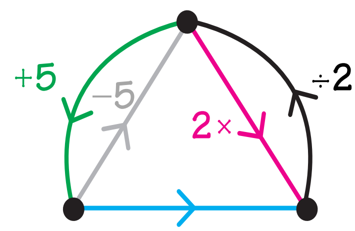
Extend the arrow road on the board by adding four more blue arrows at the beginning (leading to 30). Invite students to label the dots and to explain their methods. (Answers are in boxes.)



Some students may halve consecutive differences; for example, since 50 – 30 = 20, then the next differences are 10, 5, 2.5, and 1.25. Other students may use detours and return arrows.



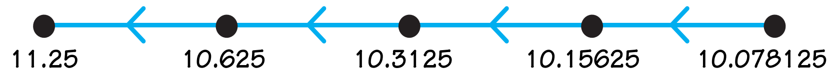
or



Worksheets N35*, **, ***, and **** are available for individual work. Emphasize that the rules for the red or blue arrows are different than the rules used in class and that each arrow is a composite; for example, ÷5 followed by 3x, or 2x followed by –7.

Extension Activity

1. In Exercise 2, further extend the blue arrow road backward from 11.25. Let students use calculators to label the dots.



   The class should notice that the numbers keep getting closer to 10, which is the limit of this sequence.

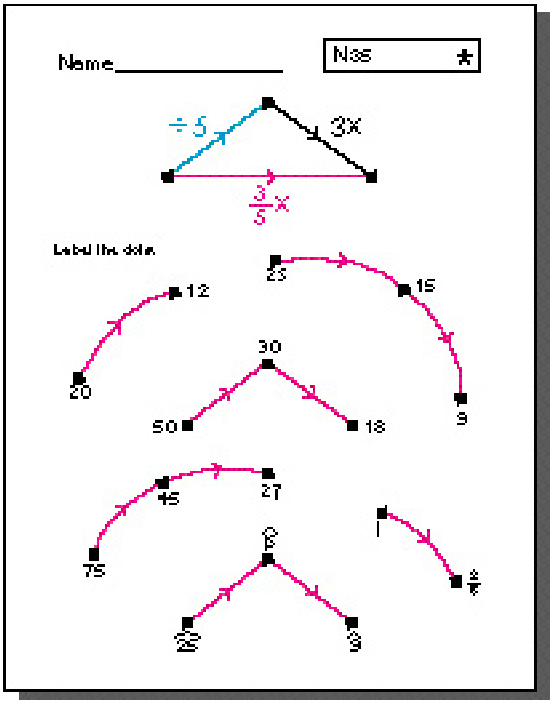
2. Draw a loop on the board.

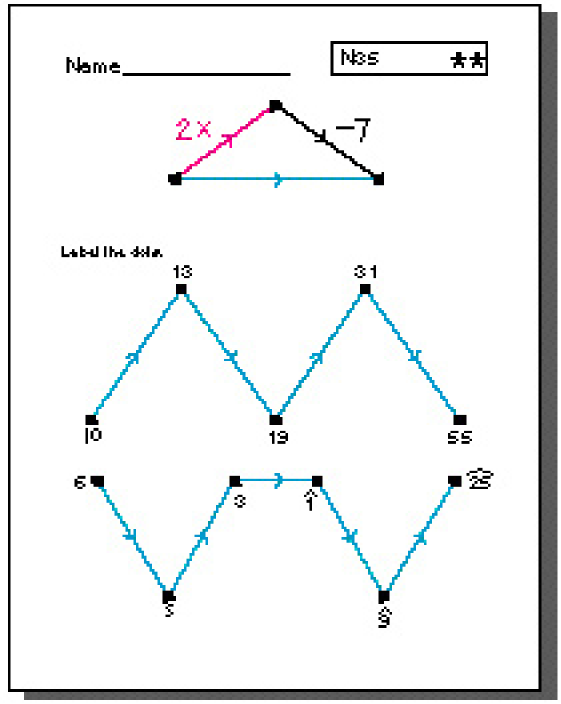
   Using the rule in Exercise 2, invite students to label the dot.
   (Answer: 10, since 2 x 10 = 20 and 20 – 10 = 10.)

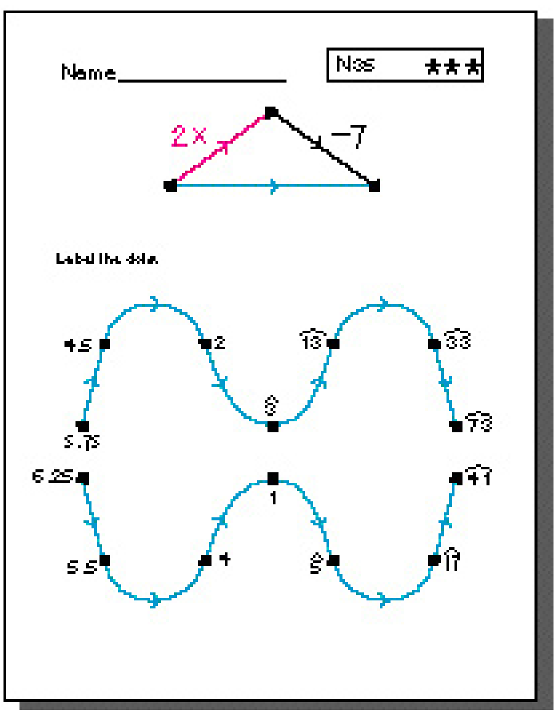
3. Ask a student to invent a new rule for some green arrows, and let that student conduct the *Guess My Rule* activity as you did in Exercises 1 and 2.

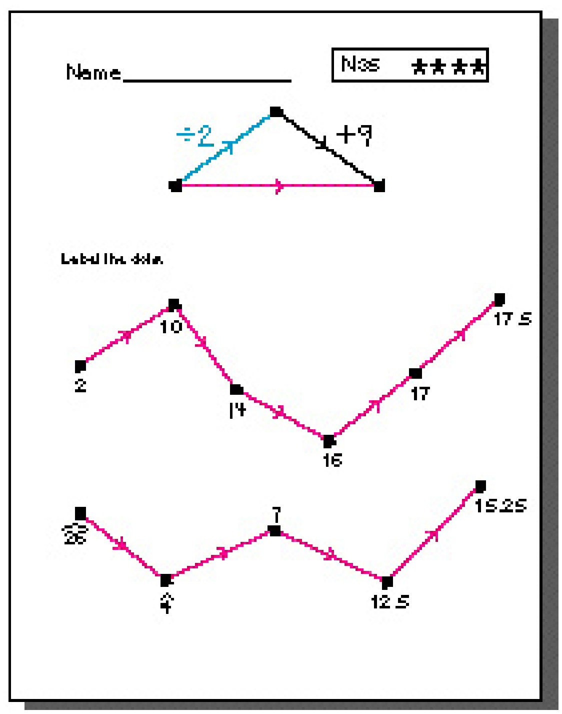
Home Activity

Suggest that students play *Guess My Rule* at home with family members.


Name
N35 ★
÷5
3×
$\frac{3}{5}$×
Label the dots.
12
20
15
9
30
50
18
45
27
75
1


Name
N35 ★★
2×
−7
Label the dots.
13
31
10
19
55
6
3


Name
N35 ★★★
2×
−7
Label the dots.
4.5
2
5.75
6.25
1
5.5
4


Name
N35 ★★★★
÷2
+9
Label the dots.
10
17.5
2
14
17
16
7
15.25
12.5

Capsule Lesson Summary

Solve a detective story with clues that use the relations $\boxed{+}\ \boxed{5}\ \boxed{=}$ …, $\boxed{+}\ \boxed{4}\ \boxed{=}$ …, and $\boxed{-}\ \boxed{3}\ \boxed{=}$ …, and a clue using weighted checkers on the Minicomputer.

Materials

Teacher
- Minicomputer set
- Weighted checker set
- Colored chalk

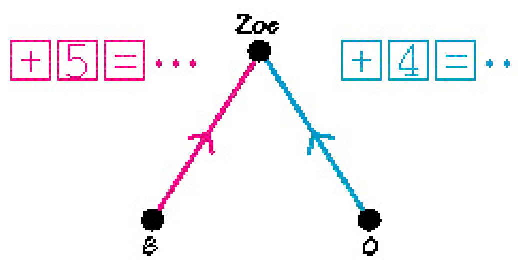
Student
- Calculator
- Paper
- Colored pencils, pens, or crayons
- Worksheets N36*, **, ***, and ****

Description of Lesson

Tell the class that today they must be detectives. First they will collectively try to discover your secret number, and then they can work individually (or with a partner) to discover some other secret numbers.

Clue 1

Draw this picture on the board.



T: ***Zoe is a secret number. What does this arrow picture tell us about Zoe?***

S: ***Zoe is a multiple of 4.***

S: ***Zoe ends in 1 or 6 (has 1 or 6 as its ones digit).***

T: ***What are some numbers that Zoe could be?***

Let students explore independently for a few minutes before letting them suggest numbers for Zoe.

T: ***What is the least number that Zoe could be?*** (16)
The next greater number? (36)

List numbers on the board in order; continue until students recognize a +20 pattern.

16, 36, 56, 76, 96, 116, 136, 156, …

T: ***What patterns do you see?***

S: ***Each number is 20 more than the number before it.***

S: ***The tens digit is always odd, and the ones digit is always 6.***

T: ***What is the greatest number less than 1 000 that Zoe could be?***

S: ***996.***

T: ***What is the least number more than 1 500 that Zoe could be?***

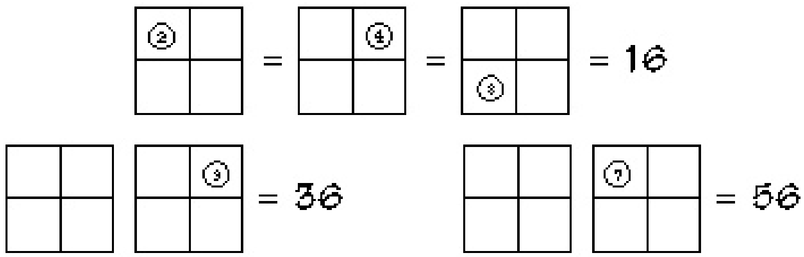
S: ***1 516.***

Clue 2

Display two Minicomputer boards and these weighted checkers: ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨.

T: ***Zoe can be put on this Minicomputer using exactly one of these checkers. Who can put one of the numbers in our list on the Minicomputer using exactly one of these checkers?***

The class should find that Zoe could be 16, 36 or 56.



T: ***Do you think these numbers are the only three numbers that Zoe could be? Why?***

Let students comment on this question.

S: ***72 is the greatest number that we can put on the ones board of the Minicomputer using exactly one of these checkers. Any greater number would have to be put on the tens board.***

T: ***What do you know about the number when we put a checker on the tens board?***

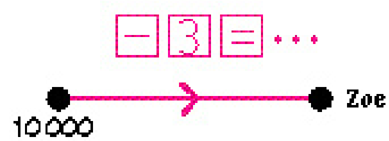
S: ***It ends in 0 (0 in the ones place).***

T: ***What about the numbers Zoe could be?***

S: ***They end in 6; so 16, 36, and 56 are the only numbers that Zoe could be.***

Clue 3

Draw this arrow picture on the board and ask students to copy it on their papers. Ask students to work independently or with a partner, using this clue to fine Zoe.



S: ***Zoe is 16.***

T: ***How do you know?***

S: ***If we start with 10 000 and press $\boxed{-}$ $\boxed{3}$ $\boxed{=}$ …, we can get 1 000 because 10 000 – 9 000 = 1000, and we can get 100 because 1000 – 900 = 100. So Zoe must be 16, because we cannot get 56 or 36 when we start with 100.***

S: ***I started with 10 000 and did not get 9 956 or 9 936, but I did get 9 916. So Zoe must be 16—just subtract 9 900.***

Another way of using this clue is described below. If this idea is not mentioned, suggest it yourself.

T: ***Think about 10 000. Is it a multiple of 3?***

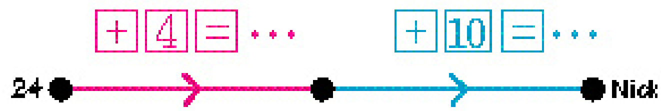
S: ***No, it is a multiple of 3 plus 1 (10 000 = 9 999 + 1).***

T: ***Which of the three numbers that Zoe could be is also a multiple of 3 plus 1?***

S: ***16 = 15 + 1. 10 000 and 16 are both multiples of 3, plus 1.***

Worksheets N36*, **, ***, and **** are available for individual work. You may like to discuss the first clue of Worksheet N36* collectively as in the following dialogue.

Draw this arrow picture on the board.



T: ***What are some numbers that Nick could be?***

Encourage students to find numbers for Nick in order—38, 42, 46, 48, 50, 52, 54, and so on.

T: ***These are all even numbers. Do you think Nick could be an odd number?***

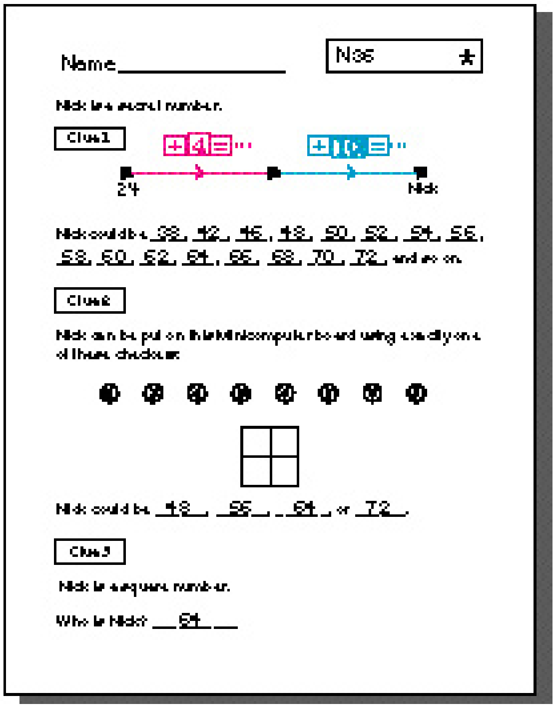
S: ***No. We start with an even number and always add an even number.***

T: ***Do you think we will miss any even numbers?***

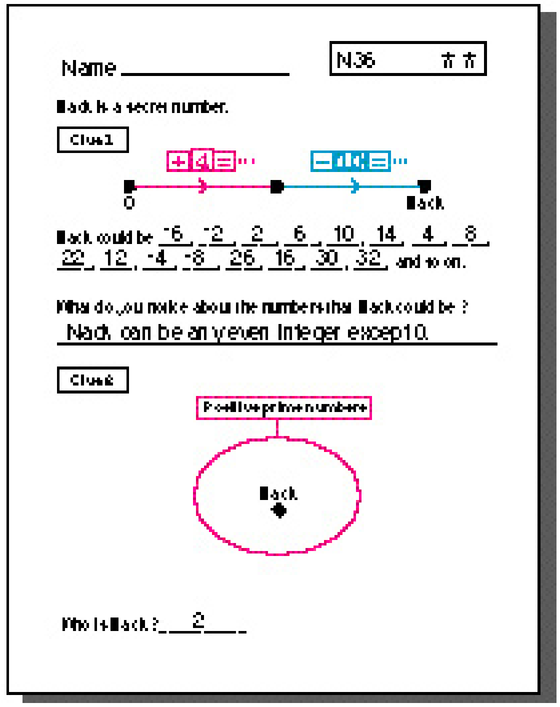
S: ***We miss 40 and 44.***

S: ***There are five even numbers in a row: 46, 48, 50, 52, and 54. If 10 is added to each, we get the next five (56, 58, 60, 62, 64) and so on. So, we can display every even number greater than 46.***

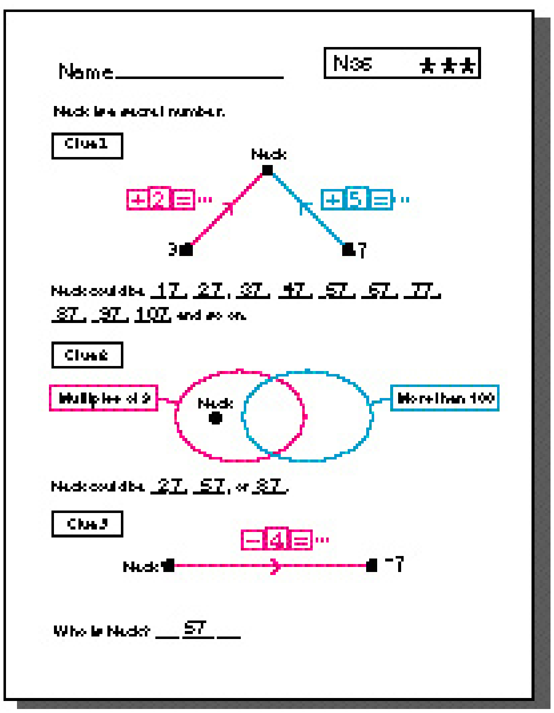
Accept any reasonable explanation, even if it is not as complete as the above.



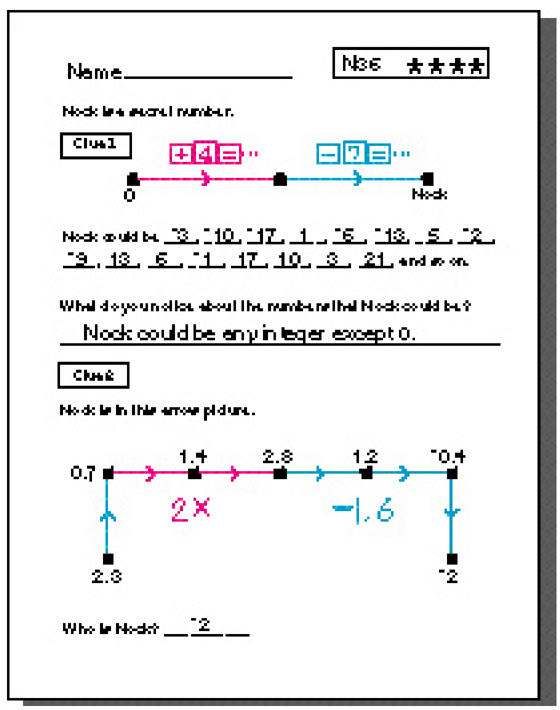
Name ______ N36 ★
Nick is a secret number.
Clue 1
+4= ... +10= ...
24
Nick
Nick could be 38, 42, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, and so on.
Clue 2
Nick can be put on this Minicomputer board using exactly one of these checkers.
Nick could be 48, 56, 64, or 72.
Clue 3
Nick is a square number.
Who is Nick? 64



Name ______ N36 ★★
Nack is a secret number.
Clue 1
+4= ... −10= ...
0
Nack
Nack could be ⁻6, ⁻2, 2, 6, 10, 14, 4, 8, 22, 12, ⁻4, ⁻8, 26, 16, 30, 32, and so on.
What do you notice about the numbers that Nack could be?
Nack can be any even integer except 0.
Clue 2
Positive prime numbers
Nack
Who is Nack? 2



Name ______ N36 ★★★
Neck is a secret number.
Clue 1
Neck
+2= ... +5= ...
3
7
Neck could be 17, 27, 37, 47, 57, 67, 77, 87, 97, 107, and so on.
Clue 2
Multiples of 3
More than 100
Neck
Neck could be 27, 57, or 87.
Clue 3
−4= ...
Neck
⁻7
Who is Neck? 57



Name ______ N36 ★★★★
Nock is a secret number.
Clue 1
+4= ... −7= ...
0
Nock
Nock could be ⁻3, ⁻10, ⁻17, 1, ⁻6, ⁻13, 5, ⁻2, ⁻9, 13, 6, ⁻1, 17, 10, 3, 21, and so on.
What do you notice about the numbers that Nock could be?
Nock could be any integer except 0.
Clue 2
Nock is in this arrow picture.
0.7
1.4
2.8
1.2
⁻0.4
2x
−1.6
2.3
⁻2
Who is Nock? ⁻2