Probability & Statistics

PROBABILITY AND STATISTICS TABLE OF CONTENTS

PROBABILITY AND STATISTICS INTRODUCTION

In today's world, probabilistic and statistical methods have become a part of everyday life. They are an important part of industry, business, and both the physical and social sciences. Just being an intelligent consumer requires an awareness of basic techniques of descriptive statistics, for example, the use of averages and graphs. Probability and statistics are key to the present day modeling of our world in mathematical terms. The lessons in this strand utilize both fictional stories and real data as vehicles for presenting problems and applications. The problems and questions that arise focus attention on key concepts of probability and statistics such as randomness, equally likely events, and prediction.

Probability stories fascinate most students and encourage their personal involvement in the situations. They often relate the probability activities to games they have encountered outside the classroom. This personal involvement builds students' confidence and encourages them to rely on their intuition and logical thinking to analyze the situations. In *IG-III*, students use marbles or dice to simulate a situation or to play a game. These activities help students understand the story and also form a basis for predicting the likelihood of particular outcomes. Yet simulations produce only estimates of the probabilities, leaving open the question of a true probability. Pictorial techniques make the analysis of theoretical probabilities accessible. This combination of simulation and analysis of situations demonstrates the strong interdependence between probability and statistics.

Some applications of mathematics are more easily investigated than others. The probability and statistics applications in this strand lean towards the easy end of the spectrum, both because of the small amount of theoretical knowledge required, and because the link between the situation and the mathematical model is easily perceived. In particular, the development of pictorial models for analyzing the problems facilitates the ease of solutions.

Content Overview

Probability

There are many methods available for determining probabilities. The simplest techniques, though usually tedious, require listing all possible outcomes. Most powerful techniques rely on formulas involving the multiplication of probabilities. The lessons in this strand review and introduce several efficient pictorial techniques that elementary students can readily apply.

Several lessons in this strand return to Bruce, the boy who invents games of chance to play with his friends. At first glance the games appear to be fair, but students soon begin to suspect the games favor Bruce since he wins more often that either of his two friends. In Bruce's first game, a paradox arises from the fact that when two dice are rolled, the sums of 6, 7, and 8 are each more likely than any other sum. Students determine this by using a grid to list the 36 equally likely outcomes when two dice are rolled. Besides being more compact than simply listing all possible outcomes, the grid strongly suggests that there are exactly 36 outcomes.

The analyses of Bruce's other games rely on an innovative geometrical method of solving probability problems. The method makes use of a graphical representation in which a square is divided into regions according to the probabilities present in the problem. This technique allows the solution of problems dealing with multi-stage random experiments in an elegant and concrete way that avoids multiplication and addition of fractions.

PROBABILITY AND STATISTICS INTRODUCTION

The two "random art" lessons in this strand feature Nabu as a modern artist who flips a coin to determine which colors to use in his paintings. Nabu's painting technique leads to combinatorial problems that are solved with a clever choice of an appropriate abacus. This abacus, similar to the Minicomputer which students know from lessons in the World of Numbers strand, allows students to solve quite complex counting problems.

While the major goal in these activities is the development of efficient and accessible pictorial techniques for determining probabilities, the lessons also reflect the continual development of other probabilistic themes; randomness, equally likely events, simulation, fair games, and predictions.

Lessons: P1, 2, 3, 6, 7, and 8

Statistics

Using standard relative frequencies to decipher a message presented as an unknown permutation of letters is a usual decoding technique and one that underlies even more sophisticated methods. In one *IG-III* lesson the students investigate the problem of breaking the code for a secret message using relative frequency data, and in doing so they experience its value and limitations.

Several lessons in this strand include descriptive statistics—the use of numerical and graphical techniques to summarize and compare sets of data. These activities continue to develop students' abilities to use averages and to read, draw, and interpret bar graphs. The goal is to increase the students' familiarity with these topics through rich experiences rather than simply to drill the techniques of computing an average or drawing a graph.

Lessons: P1, 4, and 5

P1 BRUCE'S GAMES #1

Capsule Lesson Summary

Roll two dice and represent all possible outcomes in a grid. Make frequency graphs for sums and for differences. Determine the probabilities of certain sums and differences appearing. Use the information to analyze the fairness of two dice games.

Materials

Teacher
- Two dice of different colors
- Colored chalk

Student
- Worksheets P1(a) and (b); P1* and **
- Colored pencils, pens, or crayons

Note: If you have access to an overhead projector, you may wish to prepare transparencies of the graphs and grids in this lesson instead of drawing them on the board.

Description of Lesson

Exercise 1

Show the class two different colored dice. Explain that *die* is singular for dice.

T: ***Each side of a die is called a face. How many faces does a die have?*** (Six) ***What is on the faces of a die?***

S: ***One dot, two dots, three dots, four dots, five dots, or six dots.***

T: ***When we roll a die, what is the probability of getting 4? … of getting 6?***

S: ***$\frac{1}{6}$; there is one chance out of six for any number 1 to 6.***

Based on their experiences in games, some people believe it is less likely to roll a 6 than other numbers. As appropriate, lead a discussion on the equal likelihood for each of the six faces.

T: ***What is the probability of rolling a number less than 3?***

S: ***$\frac{2}{6}$; there are two chances out of six of rolling a number less than 3.***

T: ***What is the probability of rolling an odd number?***

S: ***$\frac{3}{6}$ or $\frac{1}{2}$.***

T: ***Bruce, Helen, and Victor are friends. Bruce often invents games for them to play. Usually Bruce invents fair games, but sometimes he likes to make a game that favors him. Helen and Victor must always be alert for Bruce's tricks.***

In one game that Bruce invents, two dice are rolled and the sum of the numbers on the two dice determines the winner. What sums are possible?

S: ***2 through 12.***

Write this information on the board as you describe Bruce's game.

Helen	Bruce	Victor
2 3 4 5	6 7 8	9 10 11 12

P1

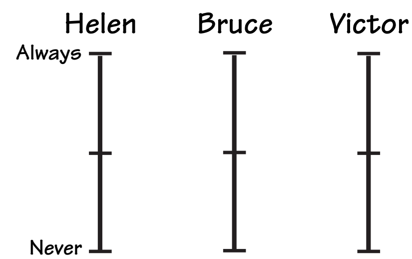
T: ***After rolling two dice, Helen wins if the sum is 2, 3, 4, or 5; Bruce wins if the sum is 6, 7, or 8; and Victor wins if the sum is 9, 10, 11, or 12. Do you think this is a fair game?***

S: ***No, I think Helen and Victor are favored. They each have four winning numbers, while Bruce has only three.***

S: ***No, I think Bruce is favored. It's easier to get the sums in the middle than to get Helen's or Victor's sums.***

S: ***It could be fair.***

On the board, draw three line segments of equal length, and label them for the three players.

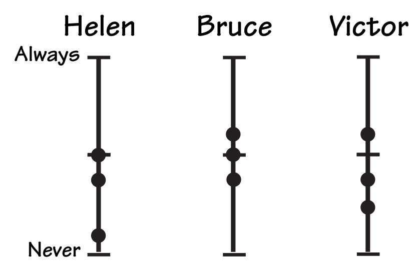


T: ***Do you remember how we use segments like these to record probabilities?***

S: ***Yes, a dot at the top of the line segment is for an event that is certain to happen. A dot at the bottom is for an event that can never happen. The probability of an event that might happen is represented by a dot somewhere along the line segment.***

T: ***Where should we draw dots for each player's probability of winning?***

Suggest that students do this on their papers, and invite several students to put their guesses on the segments on the board. For example:



Make observations appropriate to students' estimates such as the following.

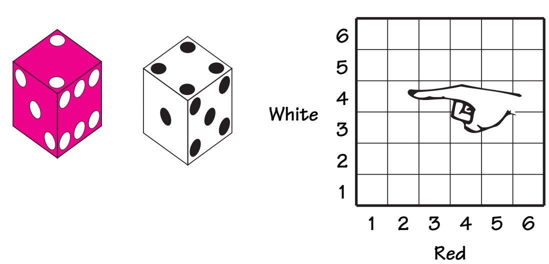
T: ***Twyla thinks that Bruce is favored. Derrick thinks that Helen and Victor have equal chances. Julie thinks that Bruce's probability of winning is exactly ½.***

Distribute copies of Worksheet P1(a), and sketch a similar six-by-six grid on the board or use an overhead transparency.

Explain that each square in the grid is for a way the dice can land when tossing two dice; it indicates what is on the red die and what is on the white die. **Note:** Use the colors of your dice.

Toss the two dice and ask,

T: ***Which square is for this way the dice land?***



Let a student locate the square while you point to 2 at the bottom edge of the grid and 4 along the side edge.

T: ***What is the outcome (sum)?***

S: ***6, because 2 + 4 = 6.***

Direct students to write 6 in the square. Call on students to suggest other possible ways for the dice to land, or allow them to experiment by tossing the dice; then ask them to locate the appropriate squares. Write outcomes (sums) in the squares. You may also point to a square in the grid and ask how the dice land. After three or four squares have been filled in, ask for a way for the dice to land, given a specific sum.

T: ***What could be on the red die and on the white die if the sum is 8.***

S: ***4 on both dice.***

S: ***3 on the red die and 5 on the white die.***

S: ***5 on the red die and 3 on the white die.***

As the ways are mentioned, locate and fill in the appropriate squares.

White						
6		8			11	
5			8			
4		6	7	8		
3					8	
2						8
1			4			
	1	2	3	4	5	6

Red

T: ***How many different ways can we get a sum of 8?***

S: ***Five; there are five 8s in the table.***

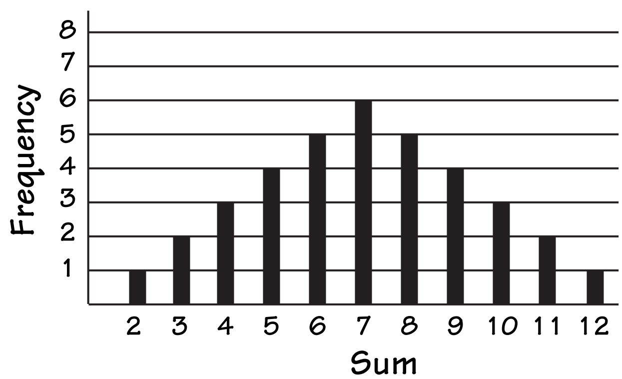
T: ***The frequency of a sum is the number of different ways to roll that sum. For example, we found that there are five ways to roll a sum of 8. A bar in this graph*** (indicate the bottom half of the worksheet) ***shows the frequency of 8 is 5.***

Direct students to fill in all 36 squares in the grid and then to complete the frequency bar graph. Encourage students to observe and use patterns.

Sum of Two Dice

White						
6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6

Red



T: ***There are three ways to get a sum of 10. What is the probability of rolling a sum of 10?***

S: ***3/36. There are three ways to get a sum of 10. There are 36 ways to roll the two dice since there are 36 (6 x 6 = 36) squares.***

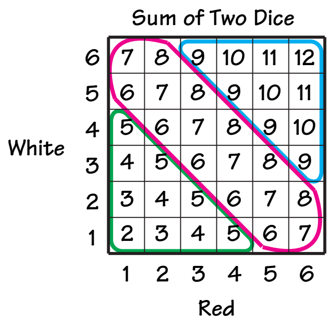
Use the grid and/or frequency graph to discuss probabilities of various outcomes. Your class may observe symmetry; that is, 2 and 12 have the same probabilities, as do 3 and 11, as do 4 and 10, and so on.

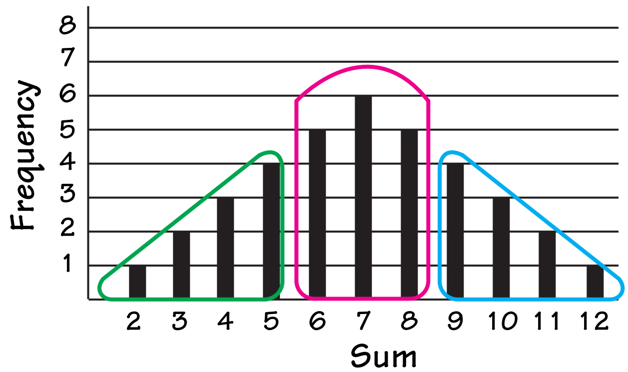
T: ***Let's check whether Bruce's game is fair. When does Helen win? In how many ways can Helen win?***

S: ***If the sum is 5 or less (sums of 2, 3, 4, and 5).***

S: ***Ten ways; ten squares have a sum of 5 or 4 or 3 or 2.***

Suggest students circle the winning sums for each player in their grid or frequency graph, and then determine the number of ways to win for each player.





Helen: 10 Bruce: 16 Victor: 10

T: ***What is each player's probability of winning?***

S: ***Helen's probability of winning is $\frac{10}{36}$, Bruce's is $\frac{16}{36}$, and Victor's is $\frac{10}{36}$.***

T: ***Is the game fair?***

S: ***No, Bruce is favored.***

T: ***Why is Bruce favored even though he wins on only three sums?***

S: ***There are more ways to roll a sum of 6, 7, or 8 than to roll any of the other sums.***

Refer back to the three line segments on which students indicated their estimates of each player's probability of winning.

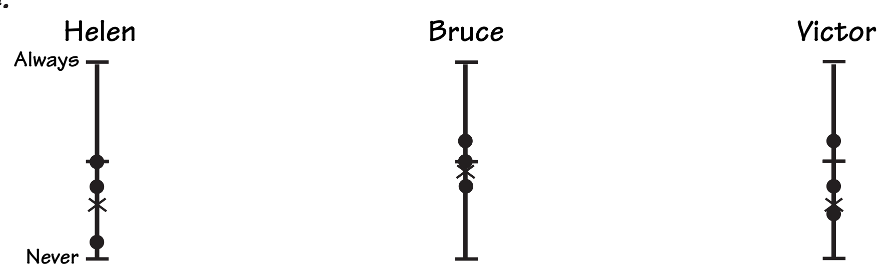
T: ***Let's put an X to show each person's actual probability of winning.***

Let the students discuss where to place Xs You may need to direct the discussion toward the following line of reasoning.

S: ***Draw Bruce's X just below the middle. His probability of winning is $\frac{16}{36}$, which is a little less than $\frac{18}{36}$ or $\frac{1}{2}$.***

S: ***Helen's and Victor's probabilities of winning are equal, so their Xs should be at similar locations. Their Xs should be lower than Bruce's since Bruce is favored.***

S: ***Helen's and Victor's probabilities of winning are both $\frac{10}{36}$. $\frac{9}{36} = \frac{1}{4}$, so draw their Xs just above $\frac{1}{4}$.***



Determine which dots indicated better estimates.

T: ***Victor likes this dice game but would like to make it fair. He thinks they should share the possible sums in a different way. How could Helen, Bruce, and Victor share the sums to make it a fair game?***

Let students explore and make suggestions. You may like to let students work with partners or in small groups to find fair games. If no one finds a fair game, give the following hints.

T: ***How many ways are there to roll two dice?***

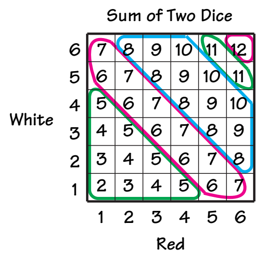
S: ***36 ways, there are 36 squares.***

T: ***To be a fair game, how many ways must each person have of winning?***

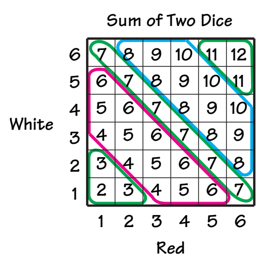
S: ***12 ways; with three players each person should have*** $\frac{1}{3} \times 36 = 12$.

T: ***So we must find three sets of sums with each set giving 12 ways to win. How could we do that?***

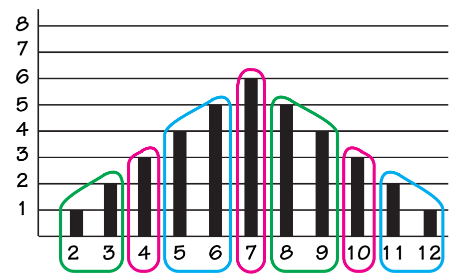
There are many ways to share the sums among three people to produce a fair game, for example:



Helen {2, 3, 4, 5, 11}
Bruce {6, 7, 12}
Victor {8, 9, 10}



Helen {2, 3, 7, 11, 12}
Bruce {4, 5, 6}
Victor {8, 9, 10}



Helen {2, 3, 8, 9}
Bruce {4, 7, 10}
Victor {5, 6, 11, 12}

As a homework assignment, you may wish to challenge students to find other solutions to this problem.

Erase the board before going on to Exercise 2.

Exercise 2

Explain that the difference between two numbers is the result after subtracting the smaller from the larger. Roll the two dice a few times and ask students to announce the difference between the two numbers rolled.

T: ***What is the greatest possible difference when we roll two dice?***

S: ***5; roll a 6 and a 1.***

T: ***What is the least possible difference?***

S: ***0; roll the same number on each die.***

T: ***Are all differences between 0 and 5 possible?*** (Yes)

Write this information on the board.

Helen	Bruce	Victor
0 1	2 3	4 5

T: ***Helen suggests a new dice game. One player rolls the two dice. The difference between the two numbers determines the winner. Do you think Helen's game is fair?***

Let students express their opinions about the fairness of this game.

Direct students to complete Worksheet P1(b) to determine if Helen's game is fair. You may like to let students work with partners or in small groups.

After a while, hold a collective discussion of the findings. Decide that this game is unfair since Helen's probability of winning is $\frac{16}{36}$, Bruce's is $\frac{14}{36}$, and Victor's is $\frac{6}{36}$. Similar to Exercise 1, let the class determine how to make a different game fair for the three players. One possibility is to give each player two differences: {0, 3}, {1, 5}, and {2, 4}.

Worksheet Pl* and ** are available for individual work.

Writing Activity

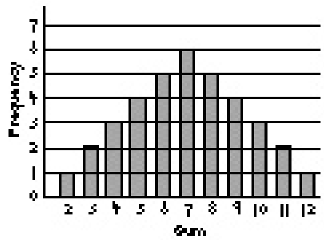
You may like students to take lesson notes on some, most, or even all their math lessons. The "Lesson Notes" section in Notes to the Teacher gives some suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students can note how to use probabilities to determine when a game is fair. They may also review how to calculate probabilities of various sums and differences when tossing two dice.

Name_______________ P1(a)

Sum of Two Dice

6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6

Draw a bar graph to show the frequency of each sum. One is done for you.



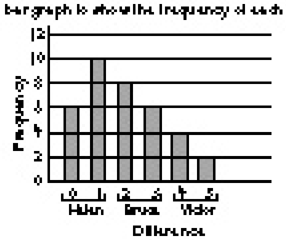
Name_______________ P1(b)

Complete. Two squares are filled in for you.

Difference of Two Dice

6	5	4	3	2	1	0
5	4	3	2	1	0	1
4	3	2	1	0	1	2
3	2	1	0	1	2	3
2	1	0	1	2	3	4
1	0	1	2	3	4	5
	1	2	3	4	5	6

Draw a bar graph to show the frequency of each difference.



How many ways does Helen have to win? 16

How many ways does Bruce have to win? 14

How many ways does Victor have to win? 6

Name_______________ P1 ★

Use the information on Worksheet P1(a) to answer these questions.

What is the probability that the sum is 6? $\frac{5}{36}$

What is the probability that the sum is not 6? $\frac{31}{36}$

What is the probability that the sum is more than 6? $\frac{21}{36}$ or $\frac{7}{12}$

What is the probability that the sum is less than 6? $\frac{10}{36}$ or $\frac{5}{18}$

When Bruce goes home, Helen and Victor decide to continue playing the sum game. They wish to play a fair game. List the sums each person could take to make the game fair.

Helen 2, 4, 6, 8, 10, 12

Victor 3, 5, 7, 9, 11

Many solutions are possible.

Explain why your solution produces a fair game.

There are 18 ways to get an odd sum and 18 ways to get an even sum.

Many explanations are possible.

Name_______________ P1 ★★

Use the information on Worksheet P1(b) to answer these questions.

What is the probability that the difference is 1? $\frac{10}{36}$ or $\frac{5}{18}$

What is the probability that the difference is not 1? $\frac{26}{36}$ or $\frac{13}{18}$

What is the probability that the difference is 0? $\frac{6}{36}$ or $\frac{1}{6}$

What is the probability that the difference is not 0? $\frac{30}{36}$ or $\frac{5}{6}$

When Helen goes home, Bruce and Victor decide to continue playing the difference game. They want to play a fair game. List the differences each person could take to make a game fair.

Bruce 0, 1, 5

Victor 2, 3, 4

Many solutions are possible.

Explain why your solution produces a fair game.

Bruce has 18 ways to win and Victor has 18 ways to win

Many explanations are possible.

Capsule Lesson Summary

Tell a story about a princess who can help the one she loves in a situation that could mean his death or their marriage. Use marbles to simulate a walk through a maze. Use an area technique to determine the probabilities involved in the choice that confronts the princess.

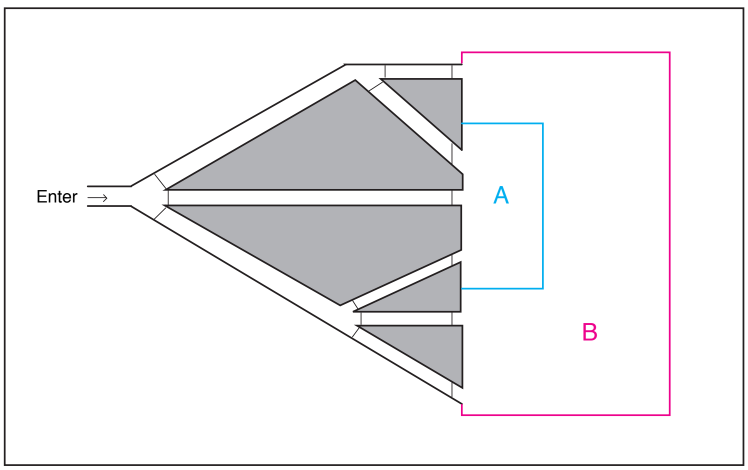
Materials

Teacher
- Meter stick
- Colored chalk
- Three marbles of different colors
- *IG-III Probability Poster* #1
- Crayon or marker

Student
- Colored pencils, pens, or crayons
- Metric ruler
- Worksheets P2* and **

Description of Lesson

Display *IG-III Probability Poster* #1.



Tell the following story in a manner appealing to your class. Stop periodically to let students comment or contribute to the story. (This story is inspired by a popular short story, "The Lady or the Tiger?" by Frank R. Stockton in *A Storyteller's Pack: A Frank R. Stockton Reader*.)

T: ***This is a love story about a beautiful princess. Her parents, the king and queen, had arranged for her to marry Prince Cuthbert from a nearby kingdom. The princess did not consider Cuthbert very appealing, but in the royal tradition she had to accept the marriage planned by her parents.***

As the wedding day approached everything was fine until the princess met Reynaldo, a poor farmer from the village. Reynaldo was handsome, clever, and romantic. It was love at first sight. But being a poor farmer, Reynaldo was not even allowed to talk to the princess. Despite the laws, they began to meet secretly. Reynaldo and the princess were careful, but one day the king caught them together. The king was extremely angry that Reynaldo, a commoner, would dare to fall in love with his daughter.

What do you think was the punishment for this?

Let students suggest possible punishments.

T: ***In this country, the punishment was to put the person in a room full of tigers. So the king ordered Reynaldo sent to the tigers. But when the princess heard this, she wept and protested that she now loved Reynaldo and would never marry Cuthbert. The king was very upset and confused. According to tradition, Reynaldo needed to be punished. If the king allowed the princess to marry Reynaldo instead of Cuthbert he would lose face. Still, the king loved his daughter and did not wish to hurt her. Because of his daughter's plea, the king agreed to discuss the problem with the queen. For the night, he sent Reynaldo to the dungeon instead of giving him to the tigers.***

What should the king do?

Encourage the students to comment on the king's dilemma.

T: ***The next morning the king called for Reynaldo and the princess to tell them of his plan. "The queen and I cannot decide whether to let you, Reynaldo, marry our daughter or to send you to the tigers. So we will let the two of you determine your own fate." The king continued, "Tonight Reynaldo will walk through a maze. At the end of the maze he will arrive at a door that leads to one of two rooms. He must open the door and enter the room."***

Tell your class that the poster is a picture of the maze, but that Reynaldo, of course, did not get to see this picture. Ask your class what they think is waiting in each room.

T: ***The king told Reynaldo that the princess would be waiting in one room and hungry tigers would be waiting in the other. He then sent Reynaldo back to the dungeon to wait until it was time for him to walk the maze.***

The king turned to his daughter and said, "If you love Reynaldo and are wise, you can help him. Here is a map of the maze." The king showed the princess a picture like this one (point to the poster). ***He then explained to her, "Reynaldo has not seen this maze. He must enter, choose a door, and walk down one of the three paths. Each door is made so that it springs open even if he opens it only slightly. And each door closes behind him so he cannot turn back. He must continue until he enters one of the rooms and finds either you or the tigers. The rooms are soundproof, so Reynaldo will not be able to hear you or the tigers. If Reynaldo finds you, I will allow him to marry you. Otherwise, he will go to the tigers and you will marry Cuthbert."***

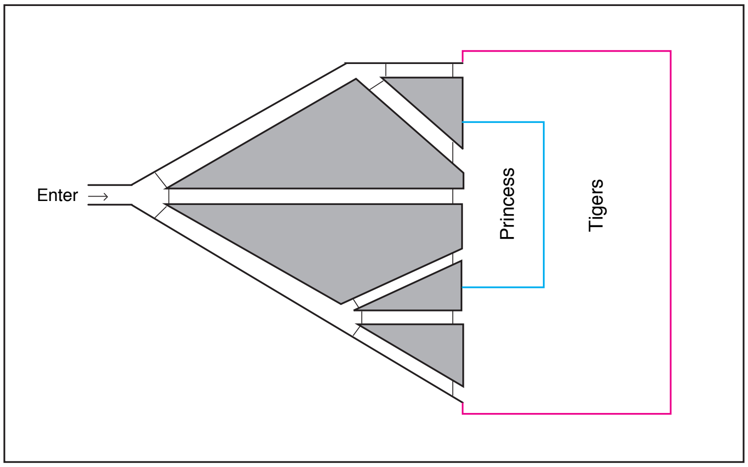
The princess agreed to this plan, understanding that she could choose which room to wait in, A or B, and that the tigers would then be in the other room.

Discuss the princess's choice with the class. You might ask the following questions.

T: ***Having no map, Reynaldo can only guess which paths to follow. But do you think that he is more likely to enter room A or room B? Or are the chances for each the same? If you were the princess, which room would you choose to stay in?***

Encourage discussion. Some students might suggest that Reynaldo's probability for entering each room is $\frac{3}{6}$ or $\frac{1}{2}$ because there are three doors into each room. If no one questions this response, ask whether Reynaldo is equally likely to arrive at each of the six doors. The class should realize that Reynaldo is more likely to arrive at some doors (for example, the third door from the top) than at other doors (for example, the fifth door from the top) due to the layout of the maze. Therefore, we cannot easily tell if Reynaldo's probability of entering each room is $\frac{1}{2}$.

After a class discussion, take a vote on which room the princess should choose. Then mark the rooms as decided by the vote. The dialogue in the remainder of this lesson is based on placing the princess in room **A**. If your class decides to put the princess in room **B**, use the same dialogue and pictures, but reverse all references to the princess and to the tigers.

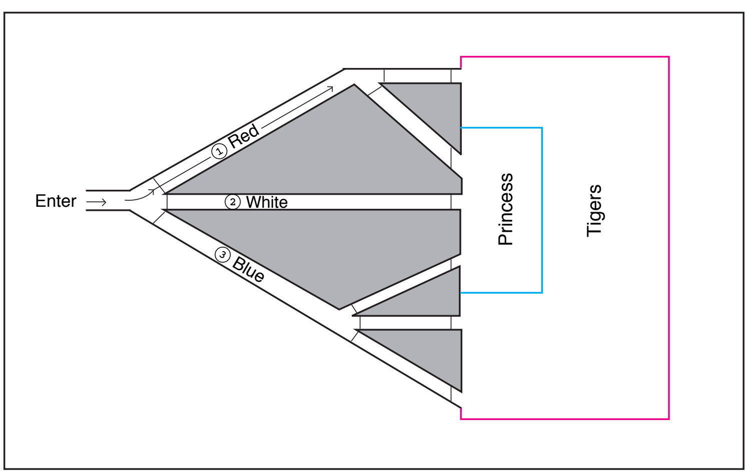


T: ***Let's pretend that we are Reynaldo and must walk through the maze. Reynaldo hates to make life and death decisions at each door; he prefers to randomly decide which door to open. Reynaldo happens to have some marbles in his pocket. How could he use marbles to decide which doors to open?***

After discussing students' suggestions, lead to the following technique.

S: ***Reynaldo could use three marbles of different colors. At the first junction, just beyond the entrance, he could assign one marble color to each of the three doors. Then he could randomly select a marble and open the corresponding door.***

Show the class your three marbles. (This discussion assumes you have one red marble, one white marble, and one blue marble.) On the poster, label each of the three doors near the entrance with a color. Mix the marbles behind your back and, without looking, select one marble. Draw the appropriate path on the poster. Suppose you select the red marble.



T: ***Reynaldo proceeds to the next set of doors. How could he use the marbles to determine which door to open?***

S: ***Use only two of the three marbles since Reynaldo is facing only two doors. Assign one color to each door and randomly select a marble.***

P2

Complete this simulation to determine whether Reynaldo meets the princess or the tigers. Repeat the simulation several times, asking students how Reynaldo should use the marbles at each choice of doors.

After completing several trials, ask if anyone has changed his or her opinion of the room in which the princess should wait. Then adapt the following dialogue to the results in your class.

T: ***In our simulations, Reynaldo found the princess three times and the tigers twice. If we did this five more times, would we get the same results?***

S: ***Maybe, but they could be different.***

T: ***Five simulations are not enough to tell us if the princess made the better choice of rooms. Perhaps Reynaldo just hit a lucky (or unlucky) streak in these five tries. We could do many more simulations. But instead, let's look at a new way to calculate the probability that Reynaldo will find the princess.***

Draw a 60-cm square on the board.

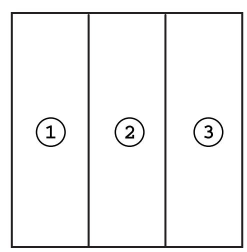
T: ***We will use this square to show what could happen to Reynaldo. When he enters the maze, he must choose one of three paths. On the square, we show this choice by dividing the square into three parts of equal size. How could we do this?***

S: ***Measure the length of one side of the square.***

T: ***It is 60 cm.***

S: ***Draw dividing lines at 20 cm and 40 cm since 60 ÷ 3 = 20.***

Using the meter stick, carefully divide the square into thirds and label the pieces.

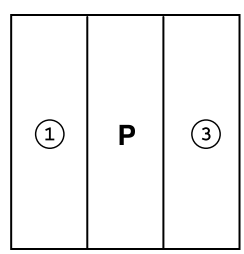


T: ***Reynaldo is equally likely to choose each of the three doors (①, ②, and ③), so we've divided the square into three parts of equal size.***

Suppose Reynaldo chooses path ②. What happens?

S: ***He goes straight to the room with the princess.***

T: ***Let's mark the region for path ② with a* P*.***
Now, suppose he chooses path ①. What happens?



S: ***His chances of going to the princess are the same as his chances of going to the tigers.***

T: ***Why?***

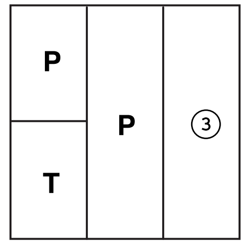
S: ***After following path ①, he comes to a junction and must select one of two paths. One path leads to the princess and one path leads to the tigers.***

T: ***How can we show that on the square?***

S: ***Divide the region for path ① into two parts of the same size. Mark one part* P *for princess and one part* T *for tigers.***

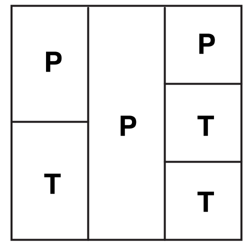
Use a meter stick to divide the region for path ③ in half.

T: ***Let's see what happens if Reynaldo chooses path ③.***



S: ***He comes to a junction with three doors. He has two chances to go to the tigers and one chance to go to the princess. On the square, divide the region for path ③ into three parts of the same size. Mark two parts with a* T *and one part with a* P*.***

If no student gives the above suggestion, do so yourself. Accurately divide the region for path ③ into thirds and mark the parts appropriately.



T: ***Is Reynaldo more likely to find the room with the princess or the room with the tigers?***

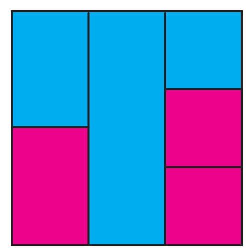
The students are likely to observe that more of the square is marked **P** than is marked **T**.

T: ***Look at the paths in the maze. Can anyone explain why Reynaldo is more likely to find the room with the princess than the room with the tigers?***

S: ***Reynaldo may be very lucky and choose path ② which leads directly to the princess. If he chooses path ① or path ③, he could go to the princess or to the tigers.***

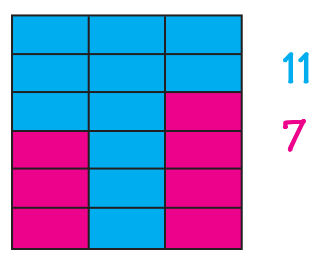
Color the regions of the square marked **P** one color (blue) and those marked **T** another color (red).

T: ***I colored the regions for the princess blue and the regions for the tigers red. Let's calculate exactly Reynaldo's chances of finding the princess and his chances of finding the tigers. Can anyone divide the square into small pieces all the same size so we can count the red pieces and the blue pieces?***



Invite students to the board to explain methods of dividing the square. Several methods are possible. The most natural division is shown here.

T: ***How many blue pieces?*** (11)
How many red pieces? (7)



T: ***Reynaldo has 11 chances to find the princess and 7 chances to find the tigers. What is Reynaldo's probability of finding the princess? The tigers?***

S: ***His probability of finding the princess is $^{11}/_{18}$; he has 11 chances out of 18.***
His probability of finding the tigers is $^{7}/_{18}$; he has 7 chances out of 18.

Write the probabilities as fractions near the square.

T: ***What would the results have been if you had placed the princess in the other room?***

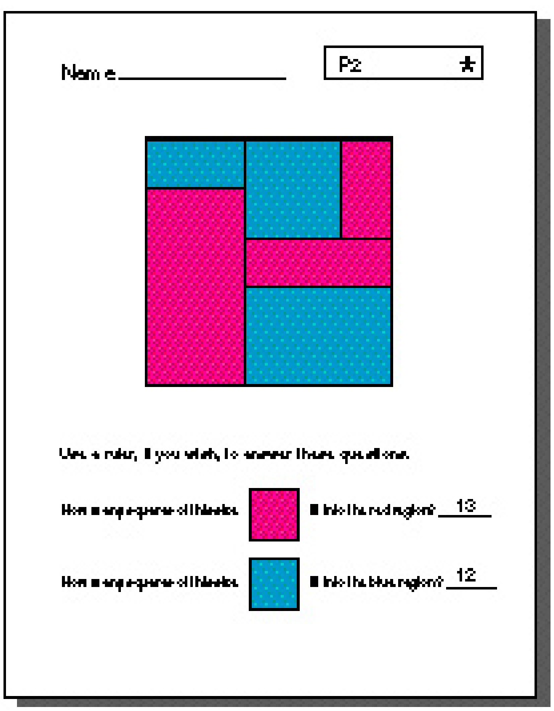
S: ***Just the opposite, Reynaldo would have had 11 chances to find the tigers and 7 chances to find the princess.***

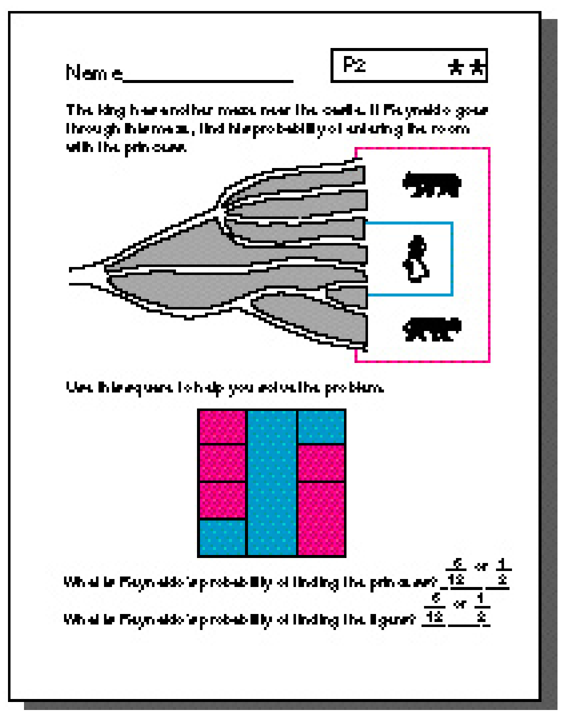
Many students are likely to be curious how the story ended, that is, what really happened to Reynaldo when he walked the maze. We suggest that you either invent an appropriate ending to the story or assign writing an appropriate ending as homework.

Worksheets P2* and ** are available for individual work.

Extension/Writing Activity

Suggest that students create their own mazes for Reynaldo to walk through, and then calculate probabilities of finding the princess (or tigers). You may give them specific probabilities to attain. For example, create a maze where Reynaldo's probability of reaching the princess if $\frac{9}{12}$ or $\frac{3}{4}$.





Capsule Lesson Summary

Use a probability model to generate equivalent fractions. Use a cord diagram and an area method to determine the fairness of several two-stage probability games. Investigate the effect that changing an aspect of the game has on the probabilities of winning for each player.

Materials

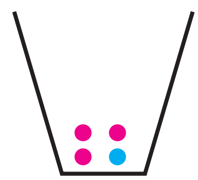
Teacher
- Colored chalk
- Meter stick

Student
- Paper
- Metric ruler
- Worksheet P3

Description of Lesson

Exercise 1

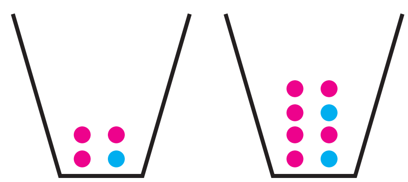
Draw a corresponding picture on the board as you ask,



T: ***If we randomly select one marble from a cup with three red marbles and one blue marble, what is the probability of our selecting a red marble?***

S: ***$\frac{3}{4}$; we'd have three chances out of four of selecting a red marble.***

Record the probability and draw another picture as you ask,



P (Red) = $^3/_4$

T: ***If we add another three red marbles and one blue marble, what is the probability of selecting a red marble?***

S: ***$\frac{6}{8}$; we'd have six chances out of eight of selecting a red marble.***

Record the probability on the board.

T: ***With which cup are we more likely to randomly select a red marble?***

S: ***Neither; the probability of selecting a red marble is the same for both.***

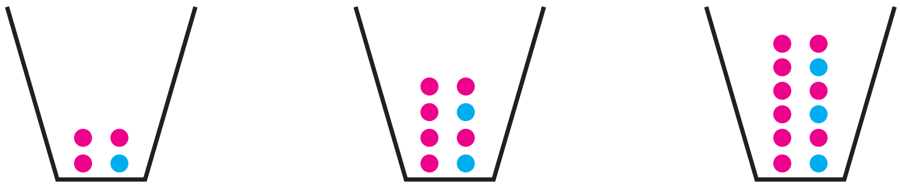
T: ***Why?***

S: ***You just doubled the number of marbles of each color.***

S: ***The probabilities of selecting a red marble, $\frac{3}{4}$ and $\frac{6}{8}$, are equal.***

P3

Draw a third picture on the board as you ask,



P (Red) = $^3/_4$ P (Red) = $^6/_8$

T: ***If we add three more red and one more blue marble to the cup, what is the probability of randomly selecting a red marble?***

S: ***$^9/_{12}$ or $^3/_4$; since you just added three red marbles and one blue marble, you did not change the probability of selecting a red marble.***

S: ***There are three red marbles for each blue marble.***

T: ***Can you suggest other probabilities equal to $^3/_4$?***

Let students generate fractions equivalent to $^3/_4$. For example:

$$\frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{12}{16} = \frac{24}{32} = \frac{48}{64} = \frac{96}{128} = \frac{15}{20} = \frac{21}{28}$$

Ask students to describe the patterns they used to generate equivalent fractions. For example:

- Add 3 to the numerator and 4 to the denominator:
  $\frac{9}{12} = \frac{9+3}{12+4} = \frac{12}{16}$ and $\frac{12}{16} = \frac{12+3}{16+4} = \frac{15}{20}$.

- Add numerators and add denominators:
  $\frac{9}{12} = \frac{12}{16} = \frac{9+12}{12+16} = \frac{21}{28}$ and $\frac{6}{8} = \frac{24}{32} = \frac{6+24}{8+32} = \frac{30}{40}$.

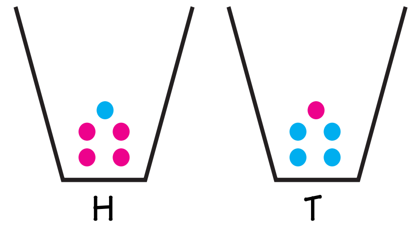
- Multiply both numerator and denominator by the same number:
  $\frac{3}{4} = \frac{5 \text{ x } 3}{5 \text{ x } 4} = \frac{15}{20}$ and $\frac{3}{4} = \frac{7 \text{ x } 3}{7 \text{ x } 4} = \frac{21}{28}$.

Do not expect students to mention all of the above methods. However, if the whole class tends to use only one method, for example, doubling, suggest problems that force another method by providing either the numerator or denominator of a new fraction.

$$\frac{3}{4} = \frac{\square}{20} \quad \text{or} \quad \frac{3}{4} = \frac{21}{\square}$$

Exercise 2

Draw this picture on the board.



Alice: 2 Red
Bruce: 1 Blue, 1 Red
Carl: 2 Blue

T: *Do you remember any stories about Bruce?*

S: *Bruce likes to invent games to play with his friends. The games often look fair but actually favor Bruce.*

T: *Here is Bruce's new game for three people. He has two cups. An* **H** *cup has four red marbles and one blue marble. A* **T** *cup has one red marble and four blue marbles. One player flips a coin to determine which cup to use:* **H** *for heads,* **T** *for tails. Then a player selects two marbles from that cup. Alice wins if both marbles are red, Bruce wins if one marble is red and one marble is blue, and Carl wins if both marbles are blue. Do you think this game is fair or that one player is favored?*

Let students express their opinions.

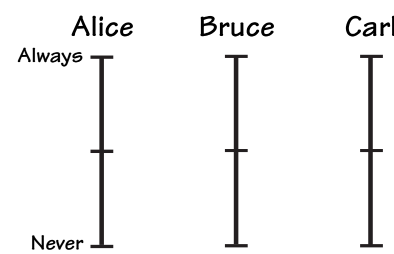
S: *I think Bruce is favored because he can win with either cup.*

S: *Carl is favored with tails.*

S: *Alice is favored with heads.*

S: *It could be a fair game.*

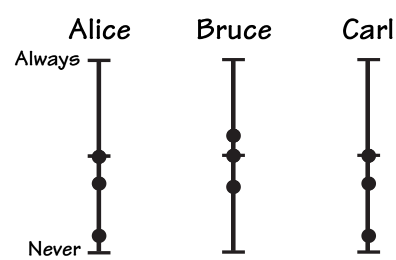
Draw three line segments of the same length on the board, and label them for the three players.



T: *Do you remember how to use segments like these to record probability estimates?*

S: *Draw dots for estimates of probabilities on a scale from* **Never** *to* **Always**. *A dot at the bottom is for an event that can never happen. A dot at the top is for an event that always occurs. Dots between* **Never** *and* **Always** *are for events that might occur.*

Suggest that students copy these segments on their papers and draw dots to indicate their estimates of each player's probability of winning. Invite several students to put their guesses on the board, as illustrated here.



Encourage students to explain their estimates.

Draw a 60-cm square on the board.

T: *Let's use this square to help calculate each player's probability of winning. Do you remember another time when we used a square to help calculate probabilities?*

S: *In the story about the princess and the tigers; we calculated Reynaldo's probability of finding the princess when he went through a maze.*

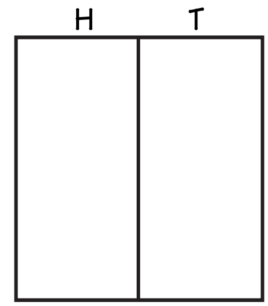
T: ***In this game, a player first flips a coin. How should we divide the square to show the possible outcomes of the coin flip?***

S: ***Divide the square in two equal parts: half for heads and half for tails.***

T: ***The square is 60 cm on each side. How could we divide it in half?***

S: *½* x ***60 = 30, so draw a line (segment) at 30 cm.***

T (pointing to the **H** region): ***Suppose that heads is flipped. Who could win the game?***



S: ***Only Alice or Bruce.***

T: ***How can we calculate Bruce's probability of winning when heads is flipped? How can we calculate Alice's probability of winning?***

You may need to remind students of the following method used in some *IG-II* lessons.

S: ***Draw colored dots for the marbles. Draw cords to show how each person could win.***

Draw one blue dot and four red dots on the board.

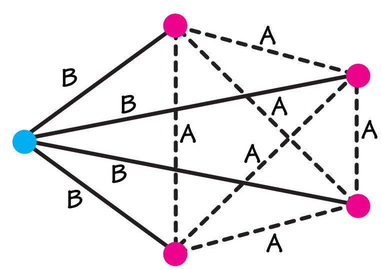
T: ***In how many ways can Bruce win?***

S: ***In four ways.***

Invite a student to draw the appropriate cords (solid).

T: ***In how many ways can Alice win?***

S: ***In six ways.***



Invite a student to draw the appropriate cords (dotted).

T: ***Who is favored if heads is flipped?***

S: ***Alice; she has six ways to win and Bruce has only four ways to win.***

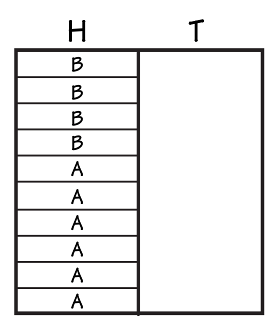
T: ***How can we show this on the square?***

S: ***Divide the*** **H** ***region in ten equal parts since 4 + 6 = 10; four parts for Bruce and six parts for Alice.***

T: ***The*** **H** ***region is 60 cm by 30 cm. How could we divide it into ten equal parts?***

S: ***Make ten strips each 6 cm wide since 10*** x ***6 = 60.***

Accept any correct method, as there are many.



T (pointing to the **T** region)**:** ***Suppose that tails is flipped. Who could win the game?***

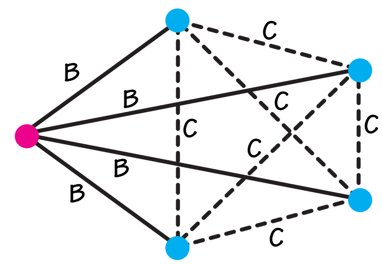
S: ***Bruce or Carl.***

T: ***In how many ways can each person win?***

S: *Bruce can win in four ways, and Carl can win in six ways.*

T: *Why?*

S: *In the picture with colored dots and cords, just switch the colors of the marbles. Then change* **A** *for Alice to* **C** *for Carl.*



Make changes to the earlier picture as indicated, or redraw it.

T: *How can I show these results on the square?*

S: *Divide the* **T** *region into ten equal parts; four parts for Bruce and six parts for Carl.*

H	T
B	B
B	B
B	B
B	B
A	C
A	C
A	C
A	C
A	C
A	C

T: *Who is favored in this game? Why?*

S: *Bruce. In the square, we see that Bruce has eight chances of winning while Carl and Alice each have only six chances of winning.*

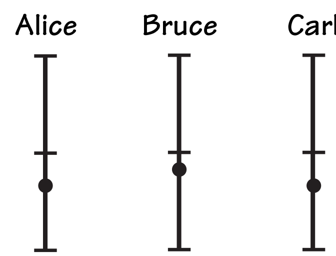
T: *What is each player's probability of winning?*

S: *Since there are 20 equal parts in the square, 8/20 for Bruce and 6/20 each for Alice and Carl.*

Record these results and check where to locate dots for the actual probabilities on the line segments.

$$P\ (\text{Alice}) = 6/20 \qquad P\ (\text{Bruce}) = 8/20 \qquad P\ (\text{Carl}) = 6/20$$

Encourage students to use good estimation techniques in locating the dots.

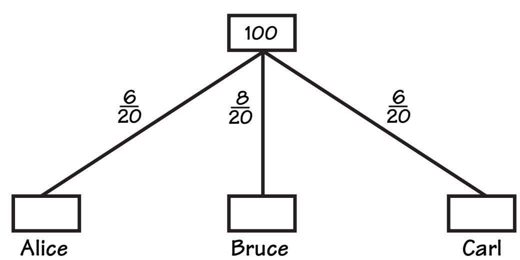


S: *8/20 is just less than 10/20 or ½, so Bruce's dot is just below the midpoint.*

S: *Place Alice's dot and Carl's dot lower than Bruce's dot since 6/20 is less than 8/20.*

You may want to make the line segments a length divisible by 20 to make more accurate locations. Compare these results to the students' earlier estimates. Highlight the fact that Bruce is favored even though his probability of winning is less than ½.

Draw this probability tree on the board.



T: *Pretend that Alice, Bruce, and Carl play this game 100 times. Using our results, what is the best prediction for the number of games each player will win?*

S: *Alice should win about 30 games since 6/20 x 100 = 30.*

T: *How did you calculate 6/20 x 100?*

S: *100 ÷ 20 = 5 (or 1/20 x 100 = 5) and 6 x 5 = 30.*

S: *6* x *100 = 600 and 600 ÷ 20 = 30.*

S: *Carl should also win about 30 games since his probability of winning is the same as Alice's, $\frac{6}{20}$.*

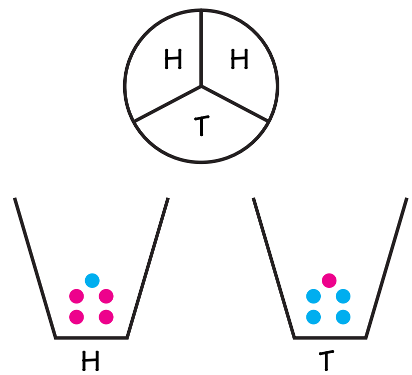
T: *About how many games should Bruce win?*

S: *40, since 100 – 30 – 30 = 40*

S: *40, since $\frac{8}{20}$* x *100 = 40.*

Exercise 3

Draw this picture on the board.



T: *Bruce and Carl agree to let Alice change the game since the original game favors Bruce. Alice decides to use the same marble distribution but to use a spinner to decide which cup to select. Alice's spinner is divided into three equal parts: two parts are for cup* H*, one part for cup* T*. How does this spinner change the game? Who does it help? Who does it hurt?*

S: *It helps Alice because the probability that cup* H *will be selected is $\frac{2}{3}$ and Alice is more likely to win when cup* H *is used.*

S: *It hurts Carl because the probability that cup* T *will be selected is less.*

As in the previous activity, ask students to predict each player's probability of winning by drawing dots on segments. Then direct the class to calculate probabilities using the method of dividing up a square. You may like to suggest that students first try to do this calculation themselves on papers, or at least draw a squares on their papers and follow along.

T: *How should we first divide the square now that we are using this spinner?*

S: *Divide the square into three equal-sized parts; two parts for* H *and one part for* T*.*

Let students measure and divide the square into three parts all the same size.

T: *Now what should we do?*

S: *Divide each region into ten parts just like last time since the distribution of marbles has not changed.*

H	H	T
A	A	C
A	A	C
A	A	C
A	A	C
A	A	C
A	A	C
B	B	B
B	B	B
B	B	B
B	B	B

S: *In each* H *region, six parts are for Alice and four parts are for Bruce.*

S: *In the* T *region, six parts are for Carl and four parts are for Bruce.*

T: *What is each player's probability of winning?*

S: *$\frac{12}{30}$ for both Alice and Bruce and $\frac{6}{30}$ for Carl.*

Record the probabilities on the board.

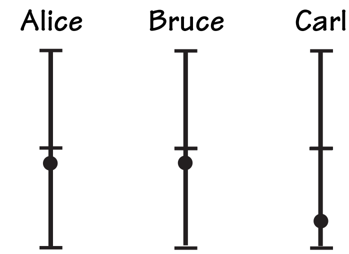
T: *What has changed from last time?*

S: ***Alice and Bruce have equal probabilities of winning. Both are favored over Carl.***

S: ***Carl's chances have decreased from 6 out of 20 to 6 out of 30.***

S: ***Bruce's probability of winning has stayed the same since he still has four out of ten chances to win in each third of the square.***

Locate the winning probabilities for the three players on their respective segments, and let students check their predictions.



Exercise 4

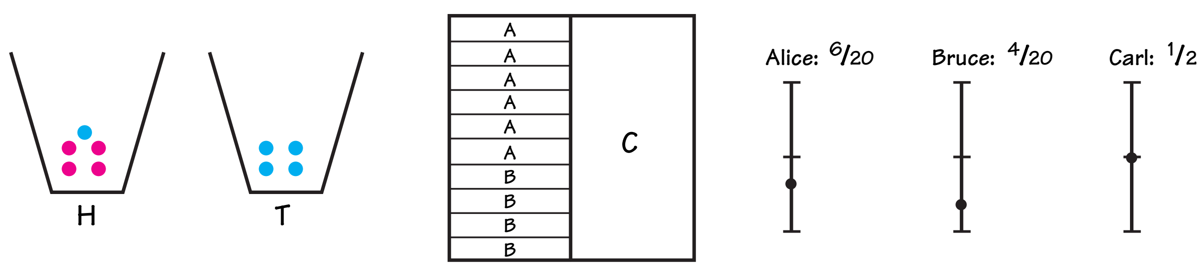
T: ***Bruce and Alice now let Carl change the game. How could Carl change the game so that it is more likely that he will win?***

Encourage students to suggest several changes that would favor Carl. For example:

- Switch the **H** and **T** regions on the spinner, or switch the cups.
- Use a die to determine from which cup to select. Select from cup **H** if 6 is rolled, otherwise select from cup **T**.
- Put four blue marbles and one red marble in each cup.

If some of the games suggested by students can be quickly analyzed using a square, you may wish to stop to do the analysis. For example:

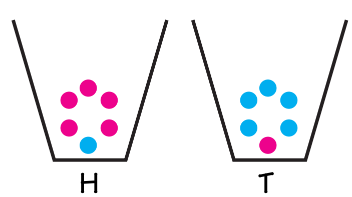
S: ***Use the coin again, not the spinner, and remove the red marble from the second cup.***



Exercise 5 (optional)

T: ***Instead of trying to favor himself, Carl decides to try to make the game fair. He chooses to use a coin again, not a spinner. And he decides to add a red marble to the first cup and a blue marble to the second cup.***

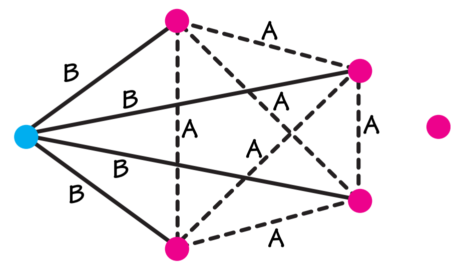
Compared to the first game, who should this change help?



S: ***Both Alice and Carl; each is more likely to win if their cup is selected.***

P3

Redraw the cord picture from Exercise 2 and add another red dot (marble). Divide a square in half as in Exercise 2.



T: *We've solved the problem with four red marbles and one blue marble in the cup. Now I've added one extra red marble. Who will this help more, Alice or Bruce?*

S: *Alice; we will have only one new cord for Bruce (from the new red dot to the blue dot) but we will have four new cords for Alice.*

S: *There will be ten cords (6 + 4 = 10) for Alice and five cords (4 + 1 = 5) for Bruce.*

S: *Divide the* H *region into 15 equal-sized parts: Ten for Alice and five for Bruce.*

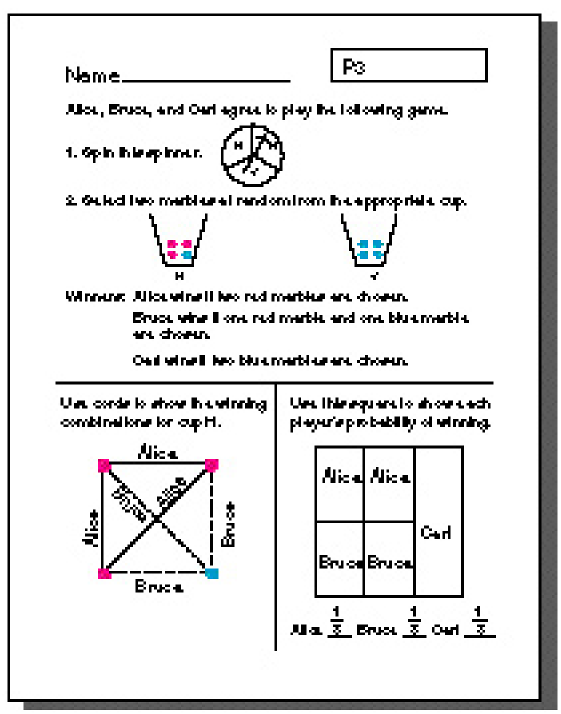
T: *I won't actually divide the region; I will just record the results.*

S: *Similarly, if tails is flipped, Carl has ten ways to win and Bruce has only five ways to win.*

S: *The game is fair! Each player has ten ways to win. Each player's probability of winning is $^{10}/_{30}$ or $^{1}/_{3}$.*

H	T
A: 10 B: 5	C: 10 B: 5

Worksheet P3 is available for individual work.



P4 SECRET MESSAGES

Capsule Lesson Summary

Decode a message written in a known code. Decode another message written in an unknown code by comparing letter frequencies to already determined standards.

Materials

Teacher
- Colored chalk
- Tape
- *IG-III Probability Poster #2*
- Colored markers
- Blacklines P4(a) and (b)
- Colored pencils, pens, or crayons

Student
- Worksheets P4(a) and (b)

Advance Preparation: Use Blacklines P4(a) and (b) to prepare transparencies for display or to make multiple copies to distribute.

Description of Lesson

With the class, recall Boris, the spy who specializes in codes.

T: ***Sometimes Boris receives coded messages, other times he sends coded messages, and sometimes he even intercepts coded messages from the enemy. Here is a message that Boris received.***

Write the message on the board. Leave enough space below each letter to write the decoded message.

GVZFZ UB KP ZPZOJ
OZBBZPWZF QP GVZ
PZIG OUYPUWVG GFKUP.

T: ***Why did headquarters write the message this way?***

S: ***So no one except Boris could read it.***

Display or distribute the code from Blackline P4(a). You may make a transparency or give students copies of the code. To give more visual distinction, you may like to color over the gray letters on the blackline to make them red.

CODE

A	B	C	D	E	F	G	H	I
Q	S	U	W	Z	R	T	V	X
J	K	L	M	N	O	P	Q	R
Y	A	B	C	L	M	N	O	P
S	T	U	V	W	X	Y	Z	
K	J	I	H	G	F	D	E	

T: ***This is Boris's secret code. What does the letter G represent in the secret message?***

S: ***T.***

Ask a student to write **T** in red below each **G** in the message.

Invite students to continue in this way until the message is fully decoded.

G V Z F Z   U B   K P   Z P Z O J
T H E R E   I S   A N   E N E M Y

O Z B B Z P W Z F   Q P   G V Z
M E S S E N G E R   O N   T H E

P Z I G   O U Y P U W V G   G F K U P
N E X T   M I D N I G H T   T R A I N

T: ***Boris meets the midnight train, and lo and behold who is on the train? Medussa, his archenemy. Medussa is the most clever and most important of all of the enemy spies. Boris knows immediately that she is carrying an important message. A great chase begins. First Boris chases Medussa inside the train, then on top of the train, then in and out of the train. Finally he leaps up and grabs her just in time, before she can swallow a secret message. But unfortunately, she has already swallowed the secret code. This is what the message looks like.***

Display *IG-III Probability Poster* #2 and distribute copies of Worksheet P4(a).

W	E		J	E		J	X	U		S	H	U	U	A		D	U	Q	H			
J	X	U		B	Q	I	J		J	H	Q	Y	D		I	J	Q	J	Y	E	D	
M	X	U	H	U		Q		J	H	U	U		X	Q	I		V	Q	B	B	U	D
E	L	U	H		J	X	U		M	Q	J	U	H		Y	D	I	Y	T	U		
J	X	U		J	H	U	U		O	E	K		M	Y	B	B		V	Y	D	T	
Q		V	E	H	C	K	B	Q		V	E	H		Q		I	U	S	H	U	J	
F	E	J	Y	E	D		M	Y	J	X		J	X	Y	I		F	E	J	Y	E	D
M	U		M	Y	B	B		H	K	B	U		J	X	U		M	E	H	B	T	

T: ***What does the message say?***

Let the class comment on how they could decipher this message.

Display the chart from Blackline P4(b) on letter frequency in English. You may put this on a transparency, or write the list on the board.

T: ***Boris knows that letters normally occur with these frequencies in books written in English. How can Boris use this information to help him break the code?***

S: ***Find out how often letters occur in the message, and list them in order from the most frequent to the least frequent.***

T: ***Good! Let's do that.***

Letter	Frequency in English
E	12.3
T	9.6
A	8.1
O	7.9
N , I	7.2
S	6.6
R	6.0
H	5.1
L	4.0
D	3.7
C, U	3.2
F, P, M	2.3
W, Y	1.9
B, G	1.6
V	0.9
K	0.5
Q, X	0.2
J, Z	0.1
	per 100

You may like to let students work with in pairs or in small groups on this task. Instruct them to find the frequency of letters in the message and to record the results on Worksheet P4(b). When most groups are done, collect the results as a class on the right side of Blackline P4(b). You may want groups to have their own copies of the chart on Blackline P4(b) on which to record the results.

T: ***Which letters appear most often in the message?***

S: ***U appears 22 times.***

S: ***Next, J appears 18 times.***

Continue until all of the letters are recorded. List letters with the same frequency together. In this activity, groups may check each other's counts.

Letter	Frequency in English	Letter	Frequency in Message
E	12.3	U	22
T	9.6	J	18
A	8.1	H	13
O	7.9	E	12
N , I	7.2	Y, Q	11
S	6.6	B	10
R	6.0	X	9
H	5.1	D	8
L	4.0	M	7
D	3.7	I	6
C, U	3.2	V	4
F, P, M	2.3	T, K	3
W, Y	1.9	F, S	2
B, G	1.6	A, C, L, O, W	1
V	0.9	G, N, P, R, Z	0
K	0.5		
Q, X	0.2		
J, Z	0.1		
	per 100		per 146

T: ***How can this information help Boris decode the message?***

S: ***U most likely corresponds to E because it appears the most often.***

S: ***J most likely corresponds to T because it appears the second most often.***

Ask students to write **E** below each **U**, and **T** below each **J** in the message.

At this point your students may have some difficulty determining how to continue decoding the message. Encourage suggestions, and note that a perfect frequency match should not be expected.

S: ***I think X corresponds to H because a common three-letter word that starts with T and ends with E is THE.***

S: ***I think E in the message corresponds to O because TO is a two letter word and E has about the right frequency for O.***

Let other students comment on these suggestions before entering them in decoding the message.

W	E		J	E		J	X	U		S	H	U	U	A		D	U	Q	H			
	O		T	O		T	H	E				E	E				E					
J	X	U		B	Q	I	J		J	H	Q	Y	D		I	J	Q	J	Y	E	D	
T	H	E					T		T							T		T		O		
M	X	U	H	U		Q		J	H	U	U		X	Q	I		V	Q	B	B	U	D
	H	E		E				T		E	E		H								E	
E	L	U	H		J	X	U		M	Q	J	U	H		Y	D	I	Y	T	U		
O		E			T	H	E				T	E								E		
J	X	U		J	H	U	U		O	E	K		M	Y	B	B		V	Y	D	T	
T	H	E		T		E	E			O												
Q		V	E	H	C	K	B	Q		V	E	H		Q		I	U	S	H	U	J	
			O								O						E			E	T	
F	E	J	Y	E	D		M	Y	J	X		J	X	Y	I		F	E	J	Y	E	D
		T		O					T	H		T	H						T		O	
M	U		M	Y	B	B		H	K	B	U		J	X	U		M	E	H	B	T	
	E										E		T	H	E			O				

Note: It is helpful to note in the frequency chart the number of times a letter occurs in the message, and to compare it to the number of times you decode that letter in the message to check that you find them all.

Let students work with in pairs or in small groups to continue decoding the message by trial and error.

W	E		J	E		J	X	U		S	H	U	U	A		D	U	Q	H			
G	O		T	O		T	H	E		C	R	E	E	K		N	E	A	R			
J	X	U		B	Q	I	J		J	H	Q	Y	D		I	J	Q	J	Y	E	D	
T	H	E		L	A	S	T		T	R	A	I	N		S	T	A	T	I	O	N	
M	X	U	H	U		Q		J	H	U	U		X	Q	I		V	Q	B	B	U	D
W	H	E	R	E		A		T	R	E	E		H	A	S		F	A	L	L	E	N
E	L	U	H		J	X	U		M	Q	J	U	H		Y	D	I	Y	T	U		
O	V	E	R		T	H	E		W	A	T	E	R		I	N	S	I	D	E		
J	X	U		J	H	U	U		O	E	K		M	Y	B	B		V	Y	D	T	
T	H	E		T	R	E	E		Y	O	U		W	I	L	L		F	I	N	D	
Q		V	E	H	C	K	B	Q		V	E	H		Q		I	U	S	H	U	J	
A		F	O	R	M	U	L	A		F	O	R		A		S	E	C	R	E	T	
F	E	J	Y	E	D		M	Y	J	X		J	X	Y	I		F	E	J	Y	E	D
P	O	T	I	O	N		W	I	T	H		T	H	I	S		P	O	T	I	O	N
M	U		M	Y	B	B		H	K	B	U		J	X	U		M	E	H	B	T	
W	E		W	I	L	L		R	U	L	E		T	H	E		W	O	R	L	D	

Extension/Writing Activity

Suggest that students write a coded message similar to the ones in this lesson and give it to another student to decode.

Home Activity

Suggest that parents/guardians try to solve a cryptogram puzzle with their child.

Capsule Lesson Summary

Use bowling scores to define the concept of *mean average*. Use averages to determine the winner of a game and to interpret the results of a sampling experiment.

Materials

Teacher
- Colored chalk
- A large bag
- 30 red marbles
- 20 blue marbles

Student
- Paper
- Calculator

Advance Preparation: Before class put 30 red marbles and 20 blue marbles into a large bag. You may substitute beads, bottle caps, dried beans, or any other items that have uniform shape and two distinct colors.

Description of Lesson

Note: Mean, median, and mode are three types of statistical averages. In common usage, the word *average* often refers to the mean. For example, when one gives a bowling average, one is giving the mean of some bowling scores. In this lesson, since only one type of average is referred to, use the words *average* and *mean* interchangeably.

Exercise 1

Students will need calculators and paper.

T: ***How many of you have gone bowling? What is the highest score a bowler can get in one game?*** (300)
Zach is a bowler. He bowled 300 one day. Is that good? (Yes)
Only Zach took two games to do it.

Begin a chart on the board.

T: ***What scores could he have had?***

S: ***150 for both games.***

Encourage students to suggest several possible pairs of scores and record them in the chart. If no one suggests scores of 150 for both games, ask what Zach's scores would have been if he had the same score in both games.

Zach

1st Game	2nd Game	Total
150	150	300
1	299	
200	100	
125	175	
160	140	

S: ***The scores of 150 and 150 are special because Zach would have bowled the same score in each game.***

T: ***150 is the* mean *of Zach's bowling scores; 150 is Zach's bowling* average.**

P5

Write *mean average* on the board.

T: ***A person's bowling average may be calculated by pretending his or her score was the same in each game. Zach has a friend named Zelda. She bowled a total of 444 in three games. What was Zelda's bowling average for those games?***

S: ***148.***

T: ***What calculation did you do?***

S: ***444 ÷ 3.***

Record the information on the board.

T: ***What are some other scores Zelda could have bowled?***

Record several suggestions in a chart. For example:

Zelda

1st Game	2nd Game	3rd Game	Total
148	148	148	444
222	111	111	
0	222	222	
300	100	44	

T: ***If Zelda bowled 150 in the first game and 148 in the second game*** (record 150 and 148 in the chart)***, what was her score in the third game? Why?***

S: ***146, because 150 + 148 = 298, and 444 – 298 = 146.***

S: ***150 is two more than her average, and 148 is the same as her average; so the last game is two less than her average, or 146.***

T: ***If Zelda bowled 140 in the first game, what could her scores have been for the last two games?***

S: ***148 and 156.***

Many answers are possible.

T: ***Zach and Zelda have a friend named Zeno. This is what Zeno bowled in three games.***

Record the information on the board.

Zeno

1st Game	2nd Game	3rd Game
120	86	130

T: ***What was Zeno's bowling average for the three games?***

S: ***112; I added the scores and divided by 3.***

S: ***112; I balanced the scores for three games.***

T: ***Who is the best bowler?***

S: ***Zach, because he has the highest average.***

S: ***Zelda, because she has almost the same average as Zach but for three games, not two.***

T: ***One day Zeno, Zelda, and Zach decide to have a contest to see who is the best bowler. Zeno is late, as usual, and only bowls in the last two games. These are their scores.***

Record this information on the board.

	1st Game	2nd Game	3rd Game	Total
Zach	140	162	130	
Zelda	160	170	60	
Zeno	X	140	170	

T: ***When they finish bowling, they all think they've won. Can you figure out why?***

Allow several minutes for students to consider this problem.

T: ***Why does Zach think he is the winner?***

S: ***Because he has the highest total.***

Invite students to enter each person's total in the chart on the board.

	1st Game	2nd Game	3rd Game	Total
Zach	140	162	130	432
Zelda	160	170	60	390
Zeno	X	140	168	308

T: ***Why do you think Zelda considers herself the winner?***

S: ***Because she won two of the three games, and she had the highest score.***

T: ***What about Zeno?***

S: ***Zeno has the best average.***

T: ***Let's see. What is Zach's average?***

S: ***144; add his scores and divide by 3.***

T: ***What is Zelda's average?***

S: ***130; divide her total 390 by 3.***

T: ***What is Zeno's average?***

S: ***154; divide Zeno's total by 2 because he only bowled two games.***

Exercise 2

Students will need paper and pencil.

T: ***Today we are going to play a tally game. The object of the game is to see which team can make the most tally marks (||||) in ten seconds.***

Divide the class into four teams with different numbers of students in each team, for example, with twelve, seven, three, and eight members.

T: ***Raise your hands in the air. When I say, "Go," write as many tally marks as you can. When I say, "Stop," put your hands up again. Go!***

Stop the students after ten seconds and ask two students from each team to count the number of tally marks for their team. Record the results on the board. For example:

	Teams			
	A	B	C	D
Number of Tally Marks	342	184	88	245

T: ***Who won?***

S: ***Team A.***

T: ***Was this a fair game?***

S: ***No! Team A had more people.***

T: ***How could we make the game fair using these scores?***

S: ***Find the (mean) average for each team and let the team with the highest average be the winner.***

T: ***What information will that give us?***

S: ***About how many marks each person on the team made.***

Invite students to use calculators to find the (mean) average for each team. Record each team's average in the chart on the board.

	Team			
	A	B	C	D
Number of Tally Marks	342	184	88	245
Number of People	12	7	3	8
Average	28.5	26.285714	$29.\dot{3}$*	30.625

Compare the teams' averages and order them from highest to lowest.

Exercise 3

You will ne~~ed the bag of marb~~les or other objects for this exercise.

T: ***A farmer stocked a pond with equal numbers of catfish and trout. A year later he had the pond tested to see which of the two types of fish was doing better. What information would he know about his pond if the test showed that there were more catfish then trout?***

Let the class discuss the situation briefly.

S: ***The pond is polluted.***

S: ***The temperature of the water is high.***

S: ***There is more food for the catfish.***

S: ***The water is muddy.***

T: ***How might the farmer discover which of the two types of fish is doing better?***

Let students offer suggestions. It is unlikely that they will suggest the method that follows, but it is important to discuss other methods in order to realize the significance of this sampling method.

After students have made several suggestions, hold up the bag with marbles in it.

T: ***The farmer's method was to catch five fish with a net each day for two weeks; record how many of each type of fish were caught; and then return all five of the fish to the pond at once. Let's see how this would work. In this bag, there are red and blue marbles. Red marbles will represent trout, and blue marbles will represent catfish. Who would like to go fishing?***

Let a student reach into the bag with both hands and pick out five marbles without looking. Announce the results of the pick and return the marbles to the bag. Shake the bag and repeat the experiment a total of 14 times to simulate two weeks. You may ask students to record the results of fishing on the board. Perhaps your results will be similar to those listed here.

T: ***How do the numbers of trout and catfish compare?***

S: ***There are probably more trout.***

T: ***What is the total number of trout caught during the two weeks? What is the total number of catfish?***

S: ***40 trout and 30 catfish.***

T: ***What is the (mean) average number of trout caught each day?***

S: ***2.8571428; I divided 40 by 14 using the calculator.***

T: ***Is 2.8571428 closer to 2.8 or to 2.9?***

S: ***2.9.***

T: ***What is the average number of catfish caught each day?***

S: ***2.1428571.***

T: ***Is the (mean) average closer to 2.1 or 2.2?***

S: ***2.1.***

Trout	Catfish
3	2
2	3
2	3
3	2
3	2
3	2
3	2
2	3
1	4
4	1
4	1
3	2
4	1
3	2

Record the size of the sample and the (approximate) average caught per day in a chart.

Number of Fish in the Sample	Number of Trout	Number of Catfish
5	2.9	2.1
10		

T: ***The (mean) average tells us that if there were five fish caught, about 2.9 would be trout and 2.1 would be catfish. What if there were ten fish caught?***

S: ***There would be about 5.8 trout and 4.2 catfish.***

Put these numbers in appropriate columns in the chart on the board. Continue in the same manner for samples of 100, 500, and 50 fish in the pond.

Number of Fish in the Sample	Number of Trout	Number of Catfish
5	2.9	2.1
10	5.8	4.2
100	58	42
500	290	210
50	29	21

T: ***There are 50 marbles in this bag for 50 fish. Let's see how close our prediction of 29 trout (red marbles) and 21 catfish (blue marbles) is to the actual number.***

Give the bag of marbles to some students to count. Ask one of the students to announce the results.

S: ***30 red marbles and 20 blue marbles.***

T: ***In this case our prediction was only off by one marble. Will our results always be this good?***

S: ***No, but most of the time they should be close.***

Note: There is a 15% chance that your class's predictions will differ by five or more than the actual number of marbles.

T: ***What could the farmer do to make his results more accurate?***

S: ***Catch more fish each day.***

S: ***Continue sampling the fish for more than two weeks.***

Capsule Lesson Summary

Determine how Bruce can increase his probability of winning over an opponent in a game he invents for a school fair.

Materials

Teacher
- Six marbles (three red and three blue)
- Two cups
- Meter stick
- Colored chalk

Student
- Colored pencils, pens, or crayons
- Worksheet P6

Description of Lesson

Exercise 1

T: ***Do you remember the stories about Bruce?***

Let students briefly recall that Bruce invents games that appear to be fair but usually favor Bruce.

T: ***Bruce's fifth-grade class is planning a weekend trip to the state capitol. They must earn most of the money to pay for the trip. The class decides to have a school fair and to invite the families of all the students in the school. What are some ways the class could make money from a school fair?***

S: ***Sell food.***

S: ***Raffle off a prize.***

S: ***Have carnival rides and play games.***

T: ***Bruce suggests a game for the school fair. It is played with three red marbles, three blue marbles, and two cups. Bruce chooses how to put the marbles into the two cups. Then, without looking, the player selects a cup and a marble from that cup. The player wins 25¢ if the marble is red and loses 25¢ if the marble is blue.***

Record this information on the board.

- Bruce wins
- Player wins

T: ***Let's try Bruce's game.***

Invite one student to be Bruce and another student to play the game. Demonstrate the game by following these steps:

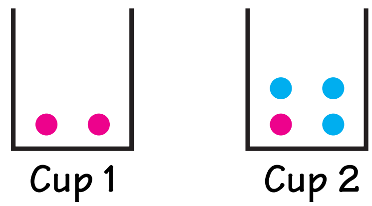
- Bruce puts the six marbles into the two cups in any way he wishes, as long as there is at least one marble in each cup.
- Bruce shuffles the two cups behind his back.
- The player chooses one of the cups (one of Bruce's hands).
- The player selects, without looking, one marble from the cup.
- The player wins 25¢ by choosing a red marble or loses 25¢ by choosing a blue marble.

Repeat this procedure several times with different pairs of students. Emphasize that, each time, Bruce decides how to distribute the marbles into the two cups. Continue this activity until everyone understands Bruce's game.

T: ***Each game is played for 25¢. If Bruce wins, the class gets 25¢ from the player. If Bruce loses, the class gives 25¢ to the player. Do you think Bruce's game is fair? Why or why not?***

Do you think Bruce's distribution of the six marbles is important? If Bruce wants to get as much money for the class as possible, could he distribute the marbles into the cups in a certain way to maximize his probability of winning?

Lead a discussion of these questions. It is not necessary that your class reach definite conclusions at this time. On the board, draw this picture of one way of distributing the marbles into two cups.



T: ***If Bruce distributes the marbles like this, do you think the probability that Bruce will win is ½, more than ½, or less than ½?***

Let students express their opinions, but do not indicate the correct answer yet. Draw a 60-cm square on the board.

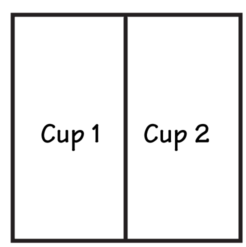
T: ***Let's use this square to calculate Bruce's and the player's probabilities of winning in this case. We'll analyze the player's choices, one at a time, and use the square to record the results of those choices. After Bruce distributes the marbles into the cups and mixes the cups behind his back, what does the player do?***

S: ***Selects one of the two cups.***

T: ***How can we show the choice on the square?***

S: ***Divide the square in half since each cup is equally likely to be chosen.***

Use measurement to accurately divide the square in half.



T: ***What happens if the player selects Cup 1?***

S: ***The player gets a red marble for sure and wins. Color the region for Cup 1 red.***

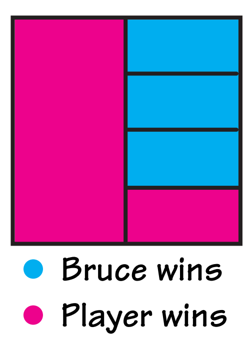
T: ***What happens if the player chooses Cup 2?***

S: ***The chances are three out of four that Bruce will win.***

T: ***How can we show that on the square?***

S: ***Divide the region for Cup 2 into four parts of the same size. Color one part red and three parts blue.***

Invite a student to divide the region and to do the coloring, such as illustrated here.



Encourage students to discuss the probabilities, and then ask some questions about this game.

T: ***If Bruce distributes the marbles in this way, is he more likely to win or lose?***

S: ***Lose, because less than half of the square is blue.***

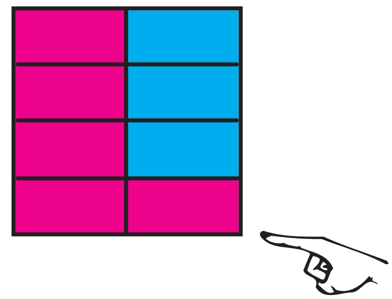
T: ***What are Bruce's chances of winning? of losing?***

S: ***Bruce has three chances to win and five chances to lose.***

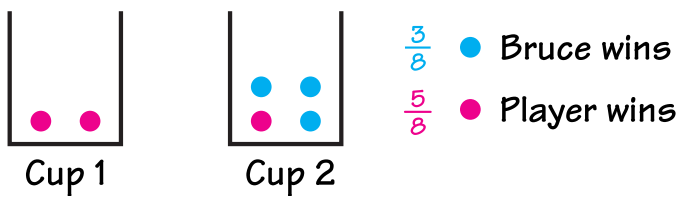
The class may further divide the square to see three blue pieces and five red pieces, all the same size.

T: ***For this distribution, what is the probability of Bruce winning? of Bruce losing?***

S: ***His probability of winning is 3/8; he has three chances out of eight. His probability of losing is 5/8; he has five chances out of eight.***

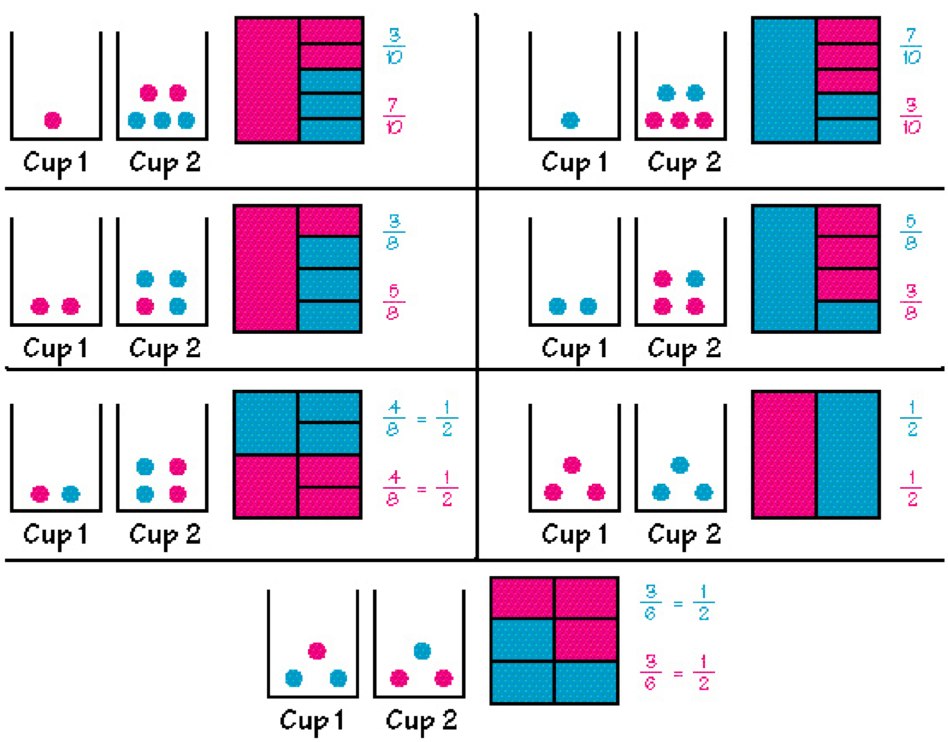


Record the probabilities near the marble distribution at one end of the board.



Invite a student to suggest a different distribution of the six marbles into two cups and to draw a picture for the distribution. Draw a square for the class to use to determine Bruce's probabilities of winning and losing with the new distribution of marbles.

For your reference, the probabilities for the seven distinct distributions are given below.



Distribute copies of Worksheet P6.

T: ***Select a distribution of marbles that we haven't tried yet, and draw the six marbles in the two cups. Use the square to calculate Bruce's probability of winning. The same thing is on both sides of the worksheet so you can analyze two different distributions. Try to find one distribution that favors Bruce and another that makes a fair game.***

After about ten minutes, call on students to tell which distributions they chose and the results for those distributions. Record each distribution by drawing a picture of two cups on the board to represent it and writing the probabilities nearby. If there is disagreement over probabilities, draw a square and determine them.

T: ***How should Bruce distribute the marbles so that he will have the greatest probability of winning?***

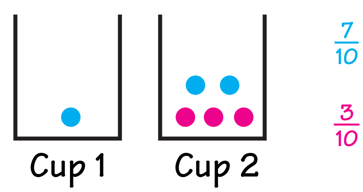
S: ***Put one blue marble in one cup and put the other five marbles in the other cup. For this distribution, Bruce's probability of winning is 7/10. He has seven chances out of ten of winning.***

T: ***Did anyone find a distribution of the six marbles that resulted in more of the square being colored blue?*** (No)

Invite the class to compare and discuss results from many different distributions of marbles. Perhaps they will notice symmetry involved when red marbles and blue marbles are switched. See the side-by-side cases in the previous illustration.

Also, they may notice that it is easy to predict when the probabilities are both ½ simply by studying the distribution without dividing a square.

Refer to the distribution that most favors Bruce.

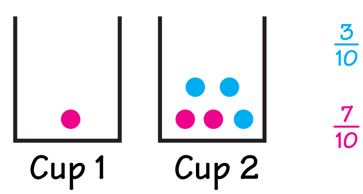


T: ***Can you explain why this distribution gives the greatest probability that Bruce will win?***

S: ***It only takes one blue marble to insure complete success with one of the cups. Then, if we put the other two blue marbles in with the red marble in the other cup, the probability of getting a blue marble from that cup will be close to ½.***

T: ***If you were the player, how would you want Bruce to distribute the marbles?***

S: ***Put one red marble in one cup and the other five marbles in the other cup. Then the probability of Bruce winning is only 3/10; he has three chances out of ten of winning.***



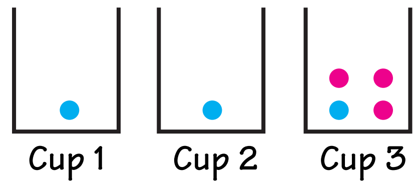
Exercise 2

T: ***Suppose Bruce has three cups instead of two, but he still has three red and three blue marbles. How should he distribute the marbles to maximize his probability of winning?***

You may want to allow a few minutes for students to consider this problem.

S: ***Put one blue marble in each of two cups, and put the other four marbles in the third cup.***

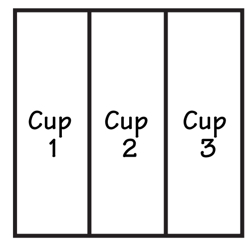
Illustrate the distribution on the board.



T: ***Do you think using three cups instead of two cups increases or decreases the probability that Bruce will win?***

S: ***Increases.***

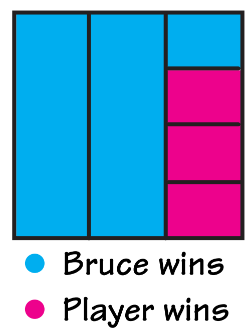
Draw a 60-cm square on the board.



T: ***How should we divide this square to calculate the probability that Bruce will win?***

S: ***First, divide the square into thirds because the player can choose one of three cups.***

S: ***Then color the regions for Cup 1 and Cup 2 blue. Divide the region for Cup 3 into fourths and color one piece blue and three pieces red.***



T: ***What is the probability that Bruce will win? that the player will win?***

S: ***Bruce's probability of winning is $9/12$, or $3/4$; he has nine chances out of twelve of winning. The player's probability of winning is $3/12$ or $1/4$.***

Record the probabilities on the board.

T: ***Can Bruce have a greater probability of winning with two cups or three cups? He can have a $9/12$ probability of winning with three cups and a $7/10$ probability of winning with two cups. Which is greater, $9/12$ or $7/10$?***

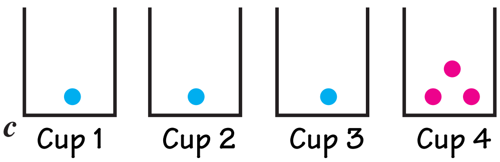
S: ***Three cups. $9/12 = 3/4 = 0.75$ and $7/10 = 0.7$; $0.75 > 0.7$. So $9/12 > 7/10$.***

S: ***Three cups. $9/12 = 3/4 = 15/20$ and $7/10 = 14/20$. Therefore $3/4 > 7/10$.***

T: ***If there are four cups and the same six marbles, how should Bruce distribute the marbles to maximize his probabilities of winning?***

S: ***Put a blue marble in each of three cups, and put the three red marbles in the other cup.***

Illustrate the distribution on the board.



T: *What is the probability that Bruce will win?*

S: *¾.*

T: *Is Bruce's probability of winning greater with three c*

S: *The probabilities are the same.*

If they wish, let students discuss why using three cups instead of two cups increases the probability that Bruce will win, but using four cups does not further increase the probability.

If there is time, present the same situation using five cups and then using six cups, and notice that the probabilities decrease with the addition of more cups.

Exercise 3 (optional)

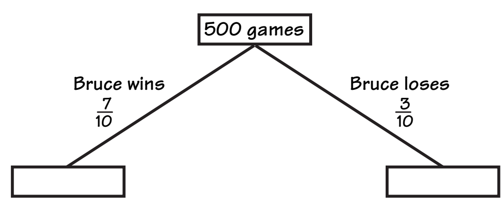
T: *At the school fair, Bruce plans to play the game about 500 times. Suppose he puts one blue marble in one ~~cup and the other five~~ marbles in the second cup every time. Let's predict about how much money Bruce should earn for the class.*

Draw a probability tree on the board.

T: *With his best distribution of marbles, what is Bruce's probability of winning? of losing?*

S: *Bruce has $\frac{7}{10}$ probability of winning and $\frac{3}{10}$ probability of losing.*

Label the picture appropriately.

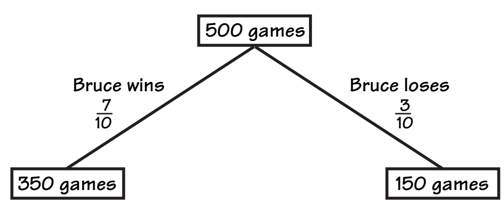


T: *Of 500 games, how many is Bruce likely to win? How many is he likely to lose?*

S: *He should win about 350 games since $\frac{7}{10} \times 500 = 350$.*

S: *He should lose about 150 games since $\frac{3}{10} \times 500 = 150$. Also $500 - 350 = 150$.*

Record the answers in the boxes.



T: *In each game, Bruce either wins 25¢ for the class or loses 25¢. About how much money should the class get overall?*

S: *If Bruce wins 350 games, he wins \$87.50; $350 \times 0.25 = 87.50$. If he loses 150 games, Bruce loses \$37.50; $150 \times 0.25 = 37.50$. Overall, the class's profit would be \$50 ($87.50 - 37.50 = 50$).*

S: *If Bruce loses 150 games, his losses in those 150 games are balanced by 150 of his wins. So the profit is what Bruce gets in the extra 200 ($350 - 150$) games he wins. The class's profit would be \$50 ($200 \times 0.25 = 50$).*

P7 RANDOM ART #1

Capsule Lesson Summary

Determine how many ways a 2-by-2 picture of four squares can be colored with red or blue squares. Decide which pictures are essentially alike if you can rotate them.

Materials

Teacher

- Two marbles (one red and one blue)
- Colored chalk
- *IG-III Probability Poster* #3
- Blackline P7

Student

- Colored pencils, pens, or crayons
- 2 x 2 picture sheet
- Worksheets P7* and **

Advance Preparation: Make copies of the 2 x 2 picture sheet using Blackline P7 for use in Exercise 1. For Exercise 2, cut out the six demonstration pictures with four squares that are provided on the *IG-III Probability Poster* #3. You will need the other pictures in Lesson P8.

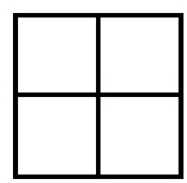
Description of Lesson

Exercise 1

If you have previously used the character Nabu, tell the class that today you have another story about Nabu.

T: ***One of Nabu's interests is painting. He does not paint portraits or landscapes; he paints pictures of random red and blue squares. Let me show you how he paints a picture.***

Draw this outline on the board.



T: ***Nabu draws this outline of a picture. Then he takes a red marble and a blue marble in his hands, puts them behind his back, and shakes them. Then, without looking, he brings one marble forward.***

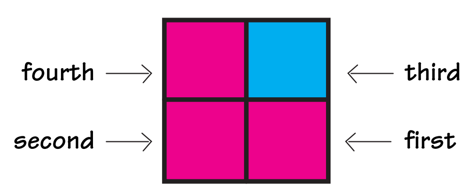
Act out this part of the story with marbles as you tell it. Let a student take Nabu's role. Suppose the red marble is chosen first.

T: ***The red marble was selected so Nabu paints the first square (in the lower right corner) red.***

Color your picture according to the color of the marble that was selected first.

T: ***Nabu puts both marbles back in his hands and shakes them behind his back. He again brings one marble forward. He colors the next square with the color of the marble he selects.***

Repeat the shaking and coloring until all four squares are colored. Suppose the completed picture looks like this one.



P7

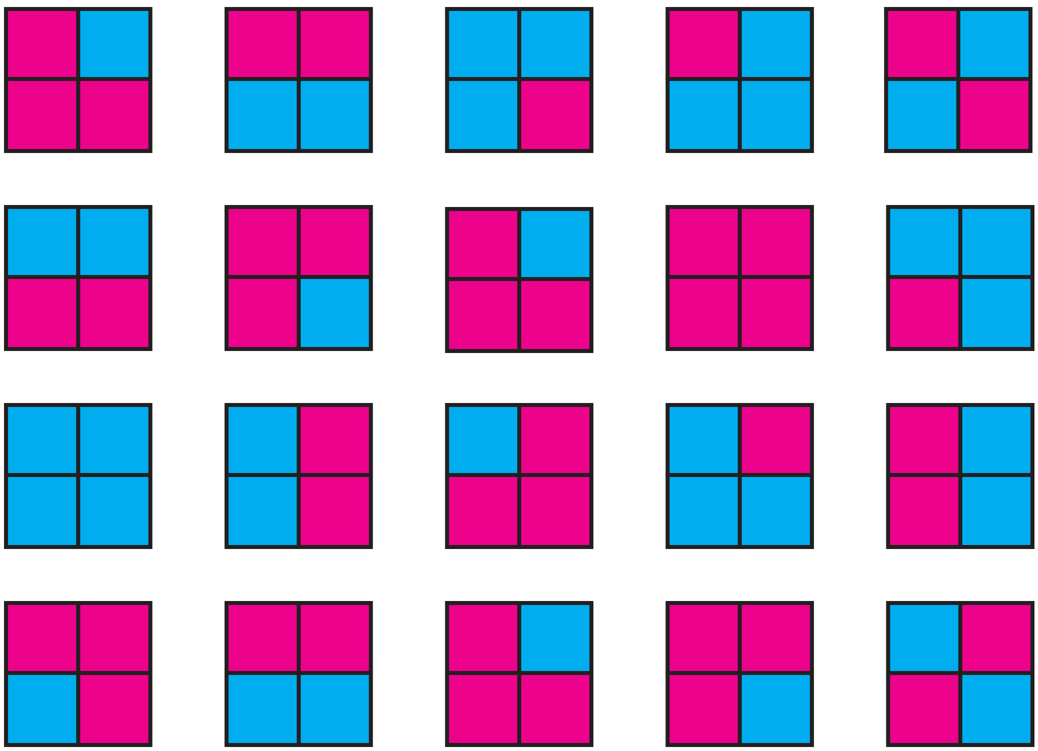
T: ***How many different pictures could Nabu paint?***

Record students' estimates in one corner of the board for future reference. Then distribute copies of the 2 x 2 picture sheet.

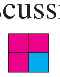
T: ***Color all of the different pictures that Nabu could paint.***

While students are working on this problem independently or with a partner, draw 19 more picture outlines on the board. One at a time, ask students to go to the board and copy one of the colored pictures from their worksheets. Ask them to try to color a different picture than the ones already on the board, but for the moment do not worry if there are some duplications.

The following lesson description supposes that these pictures are on the board. Adjust it to your class results.



T: ***How many different pictures do you think there are now?***

Encourage discussion. Students should see that there are some duplicates on the board.
(For example,  appears twice in the illustration above.)

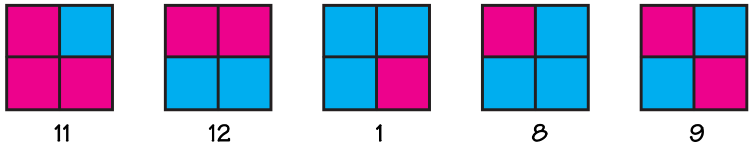
They might decide that estimates in the 15 to 20 range are close to the actual number of pictures. In any case, lead the discussion to the suggestion that what is needed is a systematic way of checking that all of the different pictures are shown and there are no duplicates.

T: ***I have a code that will help us determine how many different pictures Nabu could paint; it will also reveal duplicates if we have any. Watch as I number the pictures, and see if you can discover my code.***

Slowly and progressively number the pictures according to this code:

Treat the picture like a Minicomputer board ones board.
Red can represent squares with one regular checker on them.
Blue can represent empty squares.

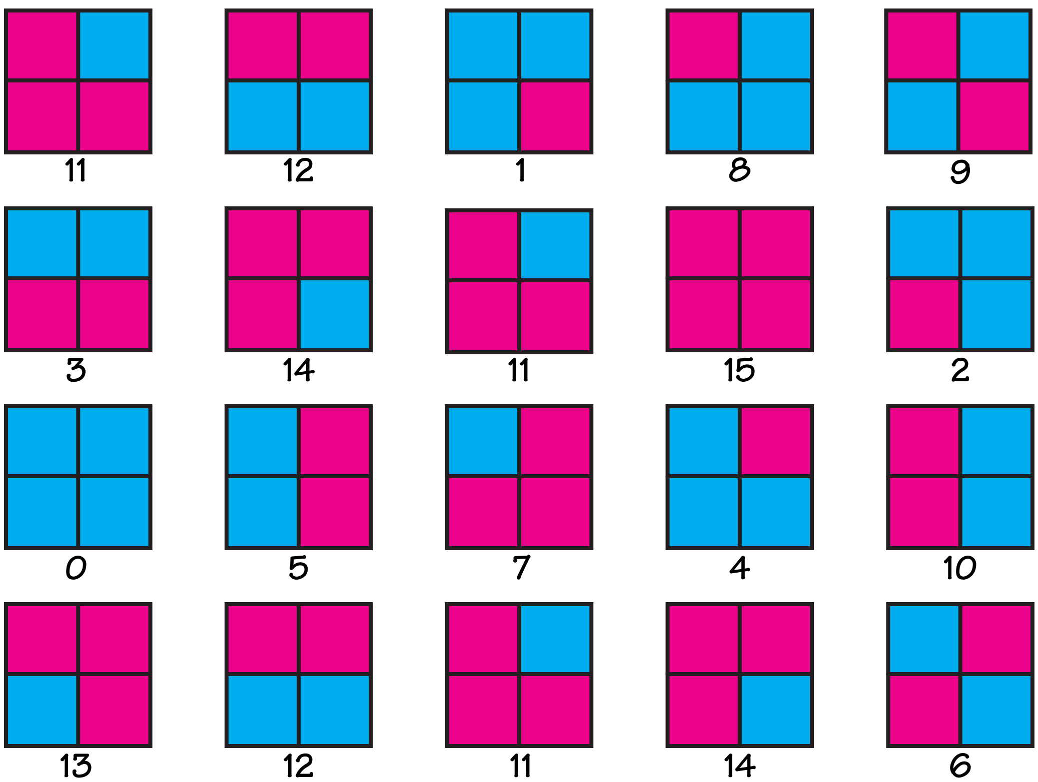
For example:



After you have numbered several pictures, let students try to guess what the next picture will be numbered, but do not let them give away your code. If after nine or ten pictures have been numbered some students have not discovered the code, suggest that they think about the Minicomputer.

Continue numbering the pictures and letting students guess the code number. When only a few pictures remain to be numbered, ask a student to explain the code. Then all students can try out the code on the last few pictures.

When all the pictures are numbered, students may more easily notice duplicates.



T: ***What is the greatest possible code number we can give to one of Nabu's pictures?***

S: ***15; the picture with all four squares painted red has code number 15.***

T: ***What is the least code number?***

S: ***0; the picture with all four squares painted blue has code number 0.***

T: ***Do you think there is a picture for each of the numbers between 0 and 15?*** (Yes)

T: ***How does this code help us to check for duplicates?*** (Pictures with the same code number are duplicates.)
How does this code help us find how many pictures Nabu can paint?

S: ***There should be a picture for each number from 0 to 15, and for each number from 0 to 15 there should be just one picture.***

T: ***How many different pictures are there?***

S: ***16 pictures, because there are 16 numbers from 0 to 15.***

Check to see if any pictures are missing, and erase any duplicates.

Exercise 2

T: ***Nabu enjoys painting, but he also likes to sell his pictures. Nabu knows that he should get more money for rare pictures and less for those that are more common. What do you think a rare painting would be?***

Let students discuss the word *rare* and how Nabu could decide which of his pictures are rare. Students might suggest that the pictures numbered 0 to 15 are rare because they have the least and greatest code numbers. Other students might suggest that code number 8 is not rare because it is a lot like code numbers 1, 2, and 4. Accept comments, and then proceed with the following discussion.

T: ***Suppose Nabu is going to paint a picture. He uses the red and blue marbles to decide how to color each square. What is the probability that Nabu will paint the picture with code number 6?***

S: ***$\frac{1}{16}$ (or 1 out of 16).***

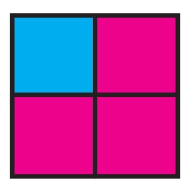
T: ***What is the probability that he will paint the picture with code number 12?***

S: ***$\frac{1}{16}$.***

T: ***What is the probability that he will paint any one of these pictures?***

S: ***$\frac{1}{16}$.***

Display the picture with code number 7 from *IG-III Probability Poster* #3.



T: ***One day Nabu meets a man who wants to buy the picture with code number 7. The man thinks this is a one-of-a-kind painting. What do you think?***

Let students react. A student might suggest that, by turning the painting, it could also be the picture with code number 11 or the picture with code number 13 or the picture with code number 14. If this does not happen, turn the painting yourself and let students note the different possibilities.

S: ***The picture with code number 7 can be turned to make the picture with code number 11 or the picture with code number 13 or the picture with code number 14. They are really all the same painting.***

List these four numbers in a box on the board.

$\boxed{7, 11, 13, 14}$

T: ***What is the probability that the picture Nabu paints has one of these code numbers?***

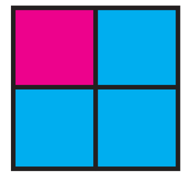
S: ***$\frac{4}{16}$ (or 4 out of 16), or $\frac{1}{4}$.***

Write $\frac{1}{4}$ next to the box containing this list of numbers.

T: ***So is the man who wants the picture with code number 7 going to buy a one-of-a-kind picture?***

S: ***No.***

Display the painting with code number 8 from *IG-III Probability Poster #3*.



T: ***Suppose the man was interested in the picture with code number 8. Do you think that it is a one-of-a-kind painting?***

S: ***No, it is the same as the picture with code number 2 and the picture with code number 1 and the picture with code number 4. Just turn it.***

Invite a student to turn the picture with code number 8 to show that it is like the others. List these four numbers in another box on the board.

T: ***What is the probability that the picture Nabu paints has one of these four code numbers?***

S: ***$\frac{1}{4}$.***

$\boxed{7, 11, 13, 14}$ $\frac{1}{4}$

$\boxed{1, 2, 4, 8}$ $\frac{1}{4}$

Write $\frac{1}{4}$ next to the box containing this list of numbers.

T: ***What do you think about the picture with code number 0?***

S: ***It's a one-of-a-kind.***

Continue in this way, letting students classify the pictures and giving the probabilities for each classification. The lists on the board should be as follows:

$\boxed{7, 11, 13, 14}$ $\frac{1}{4}$ $\boxed{0}$ $\frac{1}{16}$

$\boxed{1, 2, 4, 8}$ $\frac{1}{4}$ $\boxed{15}$ $\frac{1}{16}$

$\boxed{3, 5, 10, 12}$ $\frac{1}{4}$ $\boxed{6, 9}$ $\frac{1}{8}$

T: ***If Nabu painted 160 pictures using his red and blue marbles to decide the colors of the squares, about how many would be like the picture with code number 7?***

S: ***40.***

Repeat this question for each classification; answers are given below in parentheses.

7, 11, 13, 14	$\frac{1}{4}$	(40)	0	$\frac{1}{16}$	(10)
1, 2, 4, 8	$\frac{1}{4}$	(40)	15	$\frac{1}{16}$	(10)
3, 5, 10, 12	$\frac{1}{4}$	(40)	6, 9	$\frac{1}{8}$	(20)

T: ***Which pictures are the rarest?*** (Those with code numbers 0 and 15)
So these pictures are least likely to occur.
Which are the most common? (Those with code numbers 1, 3, 7, and their likes)

Nabu decides to price his pictures according to how rare they are. He will charge $20 for the rarest pictures, 0 and 15. How much should he charge for the pictures with code numbers 7, 11, 13, and 14?

S: ***$5 each. There are four alike and ¼ x 20 = 5.***

T: ***So the man wanting picture number 7 would be charged $5.***

How much should Nabu charge for the pictures with code numbers 3, 5, 10, and 12, and for the pictures with code numbers 1, 2, 4, and 8?

S: ***$5 each.***

T: ***How much should he charge for the pictures with code numbers 6 and 9?***

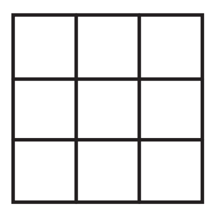
S: ***$10 each. There are two alike and ½ x 20 = 10.***

Erase the board before going on to Exercise 3.

Exercise 3

T: ***Nabu's paintings soon saturate the market, so he thinks he should create some new designs. He decides to start with a larger square divided into nine smaller squares.***

Draw this outline on the board.



T: ***Again, Nabu uses his red and blue marbles to decide how to color each square. How many different pictures of this type could Nabu paint?***

Record estimates somewhere on the board for later reference.

T: ***What code might we use?***

S: ***Give each square a number: 1, 2, 4, 8, 16, 32, 64, 128, 256.***

This suggestion may not be forthcoming, and you may have to start the process of assigning numbers to the squares. Begin in the lower right-hand corner and label the squares step-by-step, asking students to make suggestions as you proceed.

256	128	64
32	16	8
4	2	1

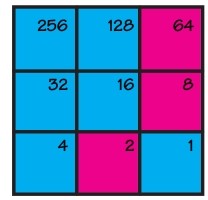
S: ***It's like the binary abacus only in a square rather than in a line.***

T: ***How can we use these numbers to make a code?***

S: ***A square colored red will be a square with a checker on it. A square colored blue will be an empty square.***

Emphasize the binary relationship: two possibilities—red or blue.

Color your picture as shown here.

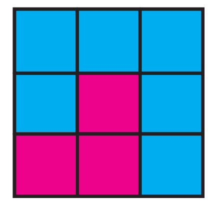


T: ***Which code number would this picture have?***

S: ***74.***

T: ***Who can draw the picture for code number 22?***

Invite a student to draw the picture on the board.



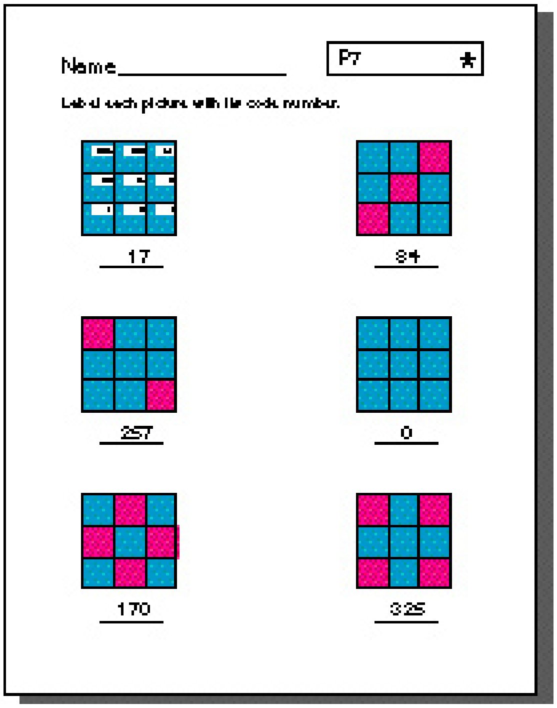
T: ***How many different pictures of nine red and blue squares could Nabu paint?***

Give students an opportunity to change their estimates. You will probably find that most students will increase their estimates. Do not expect students to determine that there are 512 such pictures, and do not give the answer away at this time. In Lesson P8, this problem will be solved completely. Record students' estimates on a piece of paper so that they can be reviewed in Lesson P8.

Worksheets P7* and ** are available for individual work.

Check the worksheets collectively, if time permits. Tell the class that your story of Nabu is not finished. Nabu is interested in knowing how many pictures are rare and how many two-of-a-kind pictures there are. These problems will be investigated in a later lesson.

P7


Name________________ P7 ★

Label each picture with the code number.

17

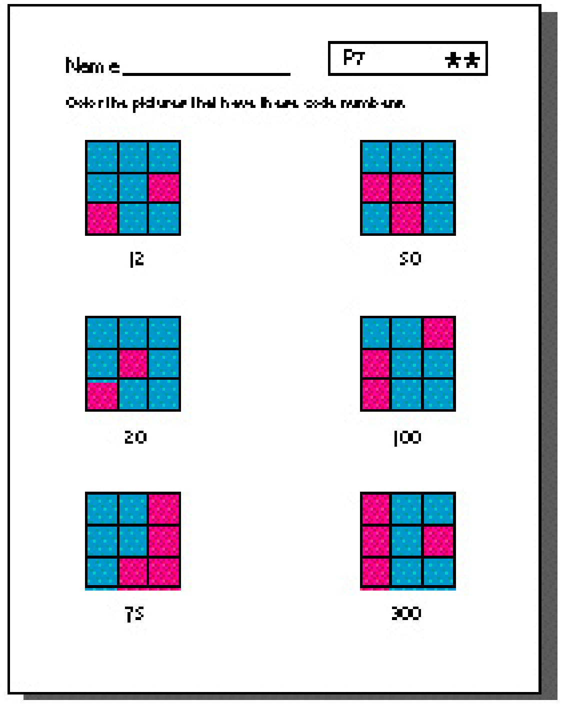
84

257

0

170

325


Name________________ P7 ★★

Color the pictures that have these code numbers.

12

50

20

100

75

300

P8 RANDOM ART #2

Capsule Lesson Summary

Find out how many ways a 3-by-3 picture of nine squares can be colored with red or blue squares. Determine which pictures are one-of-a-kind, two-of-a-kind, and so on.

Materials

Teacher
- *IG-III Probability Poster* #3
- Colored chalk
- List of estimates from Lesson P7
- Blackline P8

Student
- Colored pencils, pens, or crayons
- 3 x 3 picture sheet

Advance Preparation: For Exercise 2, make copies of the 3 x 3 picture sheet using Blackline P8. Also, cut out the ten demonstration pictures with nine squares that are provided on the *IG-III Probability Poster* #3, which you used parts of in Lesson P7.

Description of Lesson

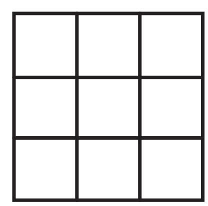
Exercise 1

Briefly review the story of Nabu and his paintings from Lesson P7.

Let students tell as much of the story as they can recall. Be sure they mention that there are 16 possible red and blue paintings with four squares; however, if you consider that some pictures are the same as other pictures when they are turned, there are only six different pictures.

T: ***After Nabu saturates the market with red and blue pictures with four squares, he decides to create some red and blue pictures with nine squares in exactly the same way.***

Draw this outline on the board.

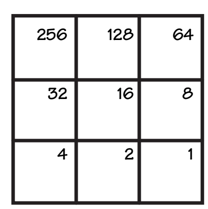


T: ***During the last lesson, we recorded estimates for how many ways Nabu could paint this picture.***

Write the estimates in a corner of the board.

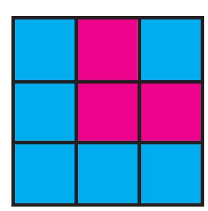
T: ***Do you remember the code we used to number the pictures Nabu could paint?***

S: ***The squares were numbered like a binary abacus: 1, 2, 4, 8, 16, 32, 64, 128, and 256.***



Label the squares of your picture.

Draw this colored picture nearby.

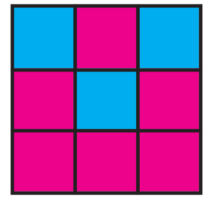


T: ***What is the code number of this picture?***

S: ***152; 128 + 16 + 8 = 152.***

P8

Invite a student to draw the colored picture with code number 175.



T: ***Let's find the greatest code number one of Nabu's paintings could have.***

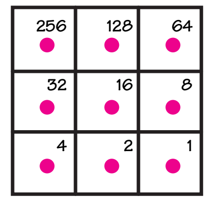
Allow several minutes for students to consider the problem.

T: ***What does the picture with the greatest code number look like?***

S: ***All of the squares are red.***

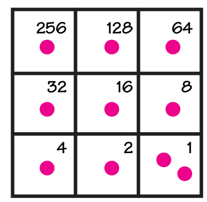
T: ***Let's find its code number. I'm going to put red checkers on squares just like we put checkers on the binary abacus.***

How can we calculate this number quickly?

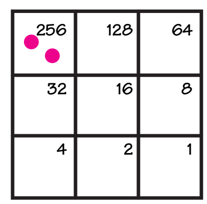


S: ***Put another checker on the 1-square.***

S: ***Now make trades.***



Invite students to make trades until there are two checkers on the 256-square.



S: ***2*** x ***256 = 512.***

T: ***Is 512 the greatest possible code number?***

S: ***No. We added 1 with an extra checker on the 1-square, so 512 – 1 = 511 is the greatest possible code number.***

T: ***Which is the least possible code number?***

S: ***0; all blue squares.***

T: ***Could Nabu paint a picture for every number from 0 to 511?***

S: ***Yes.***

If your class is not convinced of this, start with 0 on the binary abacus and keep adding 1 to get successively 1, 2, 3, 4, 5, 6, …. You will not need to continue this activity very long before most students realize that every number between 0 and 511 is the code number for a picture.

T: ***How many pictures in all?***

S: ***From 0 to 511 there are 512 numbers. So there are 512 pictures.***

Exercise 2

This exercise involves sorting the 512 possible pictures into four groups; first rotating and then counting the pictures. The groups are as follows:

1. One-of-a-kind pictures (8)
2. Two-of-a-kind pictures (24)
   - None with 0, 1, 8, or 9 red squares
   - Four each with 2, 3, 4, 5, 6, or 7 red squares
3. Three-of-a-kind pictures (0)
4. Four-of-a-kind pictures [512 – (8 + 24) = 480]

Counting pictures in groups 3 and 4 is an extension activity.

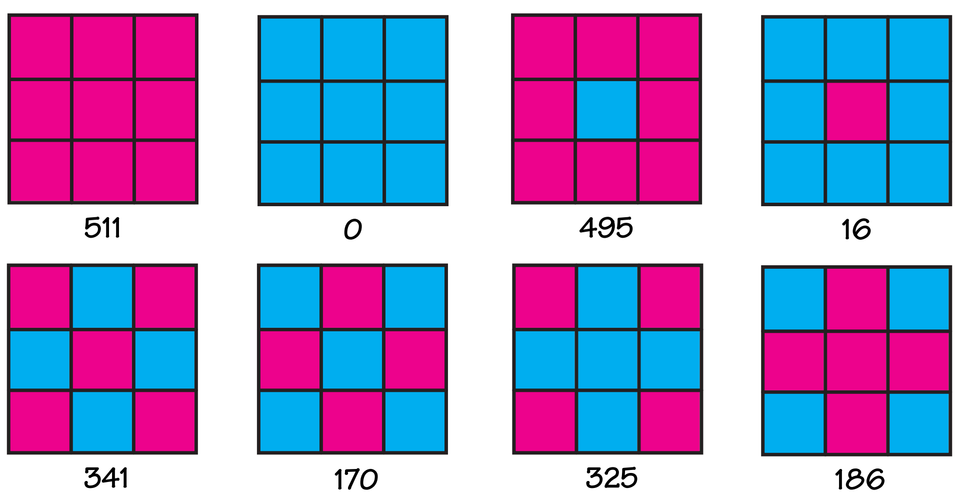
T: ***Nabu's customers are still interested in purchasing the rare pictures. Many would like to have one-of-a-kind pictures. What would a one-of-a-kind picture look like?***

S: ***A picture that is the same no matter how you turn it; any turn has the same code number.***

T: ***Can you draw some of these one-of-a-kind pictures?***

Distribute copies of the 3 x 3 picture sheet. Let students work independently or with a partner on the problem of finding one-of-a-kind pictures. Invite students to draw them on the board. The eight one-of-a-kind pictures are shown below. Ask for the corresponding code number of each picture. Code numbers should enable students to refer to pictures by number and to check for duplicates.

One-of-a-Kind Pictures



T: ***We found eight one-of-a-kind pictures. What do you notice about these pictures?***

S: ***The pictures come in pairs. If you find one picture and reverse the red and blue, you will get another one-of-a-kind picture; for example, 495 and 16.***

S: ***Also, one picture in the pair will have an even number and the other will have an odd number.***

Perhaps a student will notice that the sum of the numbers of a complementary pair of pictures is 511; for example, 495 + 16 = 511.

T: ***Remember that these pictures come in pairs. This will be useful to us later.***

P8

Begin a list in a corner of the board.

8 one-of-a-kind pictures

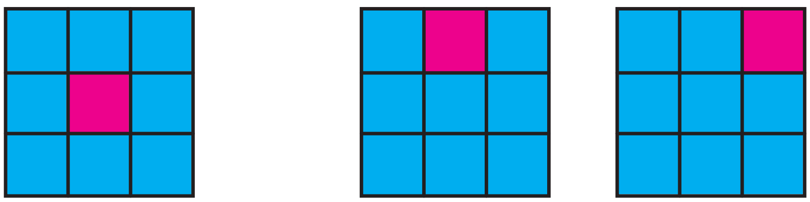
T: ***If Nabu charges $60 for a one-of-a-kind picture,***
how much would he charge for a two-of-a-kind picture?

S: ***$30. ½ x 60 = 30.***

T: ***Let's try to find two-of-a-kind pictures. Are there any two-of-a-kind pictures with no red squares?*** (No; the all blue picture is a one-of-a-kind)
Are there any two-of-a-kind pictures with one red square?

S: ***No. Pictures with one red square are either one-of-a kind or four-of-a-kind.***

Hold up these three pictures with one red square from *IG-III Probability Poster* #4. Rotate the four-of-a-kind pictures to show the four possibilities for each.



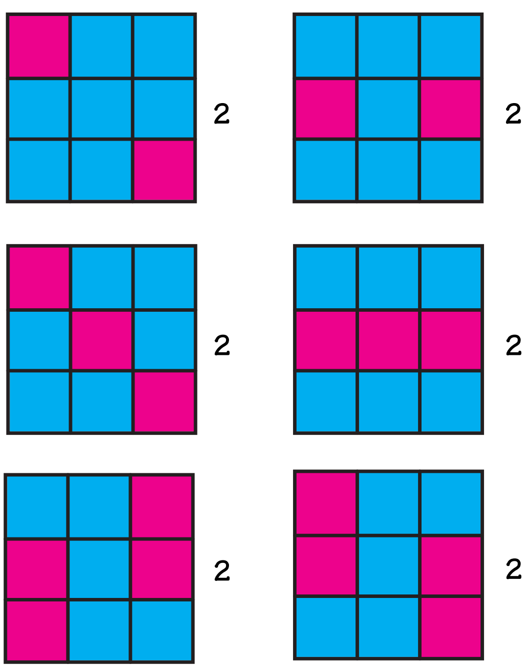
One-of-a Kind   Four-of-a Kind

T: ***Are there any two-of-a-kind pictures with exactly two red squares?***

Suggest that students try to color such pictures (on Blackline P8). They should find two basic types. Hold up two pictures from *IG-III Probability Poster* #4 (see the next illustration).

Rotate each of the pictures before the class to show the other two-of-a-kind pictures. Tape the cut-outs of these two pictures to the board and write 2 by each. Next, consider the two-of-a-kind pictures with three red squares and then with four red squares. Tape the cut-outs of these pictures to the board.

Two-of-a-Kind



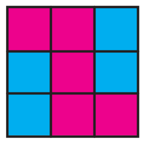
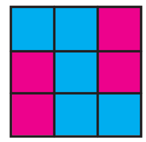
4 with exactly two red squares

4 with exactly three red squares

4 with exactly four red squares

T: ***How many two-of-a-kind pictures with exactly five red squares are there? Do we need to draw them or do we already know?***

S: ***A picture with five red squares has four blue squares. We know how many two-of-a-kind pictures there are with four red squares, so we know automatically how many there are with four blue squares—they just have the colors reversed.***

Hold this cut-out  next to this one  which is already on the board.

T: ***How many two-of-a-kind pictures do we have with exactly four red squares?***

S: ***Four.***

T: ***So there are four two-of-a-kind pictures with exactly five red squares.***

Record the information near the list of two-of-a-kind pictures. Likewise, conclude there are four two-of-a-kind pictures with exactly six red squares (the complements of those with exactly three red squares) and four with exactly seven red squares (the complements of those with exactly two red squares). Also conclude that there are none with exactly eight or nine red squares.

4 with exactly five red squares
4 with exactly six red squares
4 with exactly seven red squares

T: ***So how many two-of-a-kind pictures are there?***

S: ***24.***

Record this number on the board in the list started earlier.

8 one-of-a-kind pictures
24 two-of-a-kind pictures

Extension Activity

Some students may wish to consider three-of-a-kind and four-of-a-kind pictures. They should discover that there are no three-of-a-kind pictures. Since there are 512 pictures altogether, 8 one-of-a-kind pictures, and 24 two-of-a-kind pictures, students should conclude there are 480 four-of-a-kind pictures [512 – (8 + 24) = 480]. There are two examples of four-of-a-kind pictures on the poster.