Probability and Statistics

PROBABILITY AND STATISTICS TABLE OF CONTENTS

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PROBABILITY AND STATISTICS INTRODUCTION

In today's world, probabilistic and statistical methods have become a part of everyday life. They are an important part of industry, business, and both the physical and social sciences. Just being an intelligent consumer requires an awareness of basic techniques of descriptive statistics, for example, the use of averages and graphs. Probability and statistics are key to the present day modeling of our world in mathematical terms. The lessons in this strand utilize both fictional stories and real data as vehicles for presenting problems and applications. The problems and questions that arise focus attention on key concepts of probability and statistics such as randomness, equally likely events, and prediction.

Probability stories fascinate most students and encourage their personal involvement in the situations. They often relate the probability activities to games they have encountered outside the classroom. This personal involvement builds students' confidence and encourages them to rely on their intuition and logical thinking to analyze the situations. In *IG-III*, students use marbles or dice to simulate a situation or to play a game. These activities help students understand the story and also form a basis for predicting the likelihood of particular outcomes. Yet simulations produce only estimates of the probabilities, leaving open the question of a true probability. Pictorial techniques make the analysis of theoretical probabilities accessible. This combination of simulation and analysis of situations demonstrates the strong interdependence between probability and statistics.

Some applications of mathematics are more easily investigated than others. The probability and statistics applications in this strand lean towards the easy end of the spectrum, both because of the small amount of theoretical knowledge required, and because the link between the situation and the mathematical model is easily perceived. In particular, the development of pictorial models for s the ease of solutions.

Content Overview

Probability

There are many methods available for determining probabilities. The simplest techniques, though usually tedious, require listing all possible outcomes. Most powerful techniques rely on formulas involving the multiplication of probabilities. The lessons in this strand review and introduce several efficient pictorial techniques that elementary students can readily apply.

Several lessons this semester deal with a classical problem in the history of probability theory. Sometimes called the problem of points, the problem is to determine each player's probability of winning a game to ten points when, for example, player A has scored nine points and player B has scored seven points. Initially analysis relies on an area model for finding the probabilities. The method makes use of graphical representation in which a square is divided into regions according to the probabilities present in the problem. This technique allows the solution of problems dealing with multi-stage random experiments in a very elegant and concrete way that avoids multiplication of fractions.

After analyzing several cases of the problem of points, students investigate some number patterns that evolve in the solutions and make use of these patterns to predict solutions to other cases. They examine the relationships between various cases and find that the solution of a specific case may be greatly simplified by applying the solutions of other cases.

Another lesson this semester involves the multiplication principal for finding probabilities of multi-stage events. In *Dangerous Roads* students explore all the possible routes between two points and find which route is the safest given probabilities of survival along component roads. Although not viewed as a probability tree, solutions involve the same multiplication principal as is used along the branches of a probability tree. The technique of sending a certain number of imaginary travelers along a route and seeing how many survive is used to justify and verify the safety probability of the route. A similar technique has been used with probability trees.

Lessons: P4, 7, 8, and 9

Statistics

Several lessons in this strand include descriptive statistics—the use of numerical and graphical techniques to summarize and compare sets of data. The activities continue to develop the students' abilities to use averages, and to read, draw, and interpret bar graphs. The goal is to increase the students' familiarity with these topics through rich experiences rather than to drill the techniques of computing an average or drawing a graph.

In *Mendel's Experiments* students analyze the numerical data that led Mendel to a probabilistic model for inherited characteristics. The lesson involves simulating Mendel's heredity experiments on pea plants and predicting future generation outcomes. Further, students discuss the applicability of the model to human traits such as eye color.

Two lessons this semester on *Shunda's Newsstand* consider an advanced operations research problem. In mathematics literature, the problem is usually called "the newsboy's problem" and is taken from inventory theory. Characteristic for this problem is that a decision about inventory is made once for the entire demand process. Every day Shunda has to decide how many newspapers to buy from her supplier. However, the demand is uncertain and the problem for Shunda is to order enough to realize full profit but not too much as to incur losses on the excess. Shunda uses a graph of daily demand to determine the most profitable inventory. The lessons provide an excellent example of how a problem, which is usually discussed on a high mathematical level, can be made understandable at the elementary school level. They show how statistics as the art of making decisions when faced with uncertainty can be treated in a meaningful way at an elementary level.

Another lesson this semester involves viewing different visual presentations of data collected from first estimating and then finding the actual count of raisins in ½-oz. boxes. The back-to-back stem-and-leaf plot and a double bar graph allow different visual perspectives and readily answer different questions.

Lessons: P1, 2, 3, 5, and 6.

Capsule Lesson Summary

Describe Gregor Mendel's experiments with pea plants. Analyze the numerical data that led Mendel to a probabilistic model for inherited characteristics. Discuss the applicability of the model to human traits such as eye color and hair color.

Materials

Teacher
- Marbles (six red and six white)
- Three cups
- Six index cards
- Blackline P1
- Colored chalk

Student
- Calculator

Advance Preparation: Use Blackline P1 to prepare six index cards with flowers on them. Color red flowers on three cards, and leave white flowers on the other three.

Description of Lesson

Exercise 1

Lead a discussion about probability situations. Students might recall various experiences with probability, such as in games with marbles, dice, and spinners, or in stories of Reynaldo walking through a maze and *The Island of Tam Tam* bugs. They might also mention uses of probabilities in weather reports, advertisements, sports, gambling, and medicine.

T: ***The first probability problems arose from games of chance in about the 16th century. We have studied some of these problems in the lessons about Bruce's games. One of the early applications of probability to a topic other than games involved heredity.***

Write the word *heredity* on the board.

T: ***What physical characteristics of humans are inherited?***

Discuss several characteristics such as eye color, height, and skin color. Mention the effect of both heredity and environment on traits such as height.

T: ***In the early 1800s very little was known about inherited characteristics for two reasons: First, scientists knew little about the structure and components of cells; and second, the theory of inherited traits seemed very complex. For example, two brown-eyed parents could have a blue-eyed child. Or sometimes a child was red-haired even though neither parents nor grandparents had red hair.***

As appropriate, let students mention traits that "unexpectedly" occurred in their families.

Write *Gregor Mendel: 1822–1884* on the board.

T: ***Gregor Mendel was a Czechoslovakian priest, scholar, and teacher. From 1854 to 1865, he carefully conducted a series of heredity experiments on pea plants. What characteristics of pea plants do you think he might have studied?***

P1

Let students make suggestions. Then begin this table on the board.

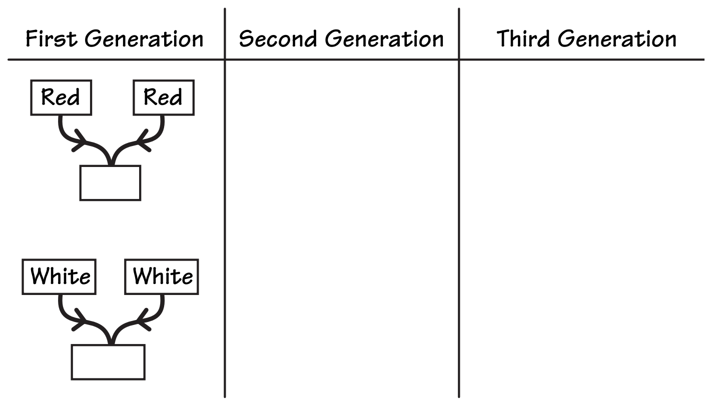
Traits	
Flower: Red or White	
Height: Tall or Short	
Seeds: Yellow or Green	
Seeds: Round or Wrinkled	
Pods: Yellow or Green	

T: ***These are five of the traits Mendel chose to study, one at a time. For example, he grew one plot of red-flowered plants and one plot of white-flowered plants.***

Write the word *pollinate* on the board.

T: ***Because Mendel's family grew fruit trees and because he studied beekeeping in school, Mendel knew about pollination. Pollen is a fine powdery substance created by many plants. For new plants to grow, pollen must be transferred from one part of a pea plant to either another part of the same plant or to another pea plant. This pollen is usually transferred by bees, wind, or gravity. But Mendel covered the pollen-carrying parts of the plant and transferred the pollen himself by hand. This way he controlled which plants pollinated which plants.***

Draw this diagram on the board.



T: ***In the first generation[†], Mendel pollinated red-flowered plants with their own pollen. What color flowers do you think the new plants had?***

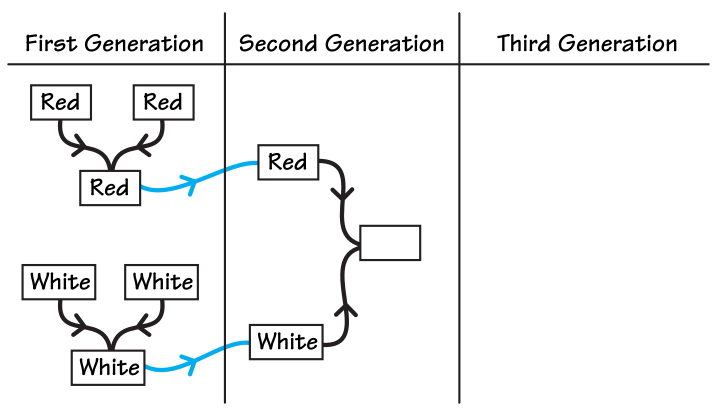
S: ***Red flowers.***

T: ***Mendel also pollinated white-flowered plants with their own pollen. What color flowers do you think resulted?***

S: ***White flowers.***

T: ***Yes, there were no surprises in the first generation.***

Add this information to the diagram on the board.



T: ***In the second generation, Mendel cross-pollinated the red-flowered plants from the first generation and the white-flowered plants from the first generation. What color flowers do you think the new pea plants had?***

[†]By growing pea plants for several years, Mendel developed pure strains of red-flowered plants and of white-flowered plants for the first generation experiment.

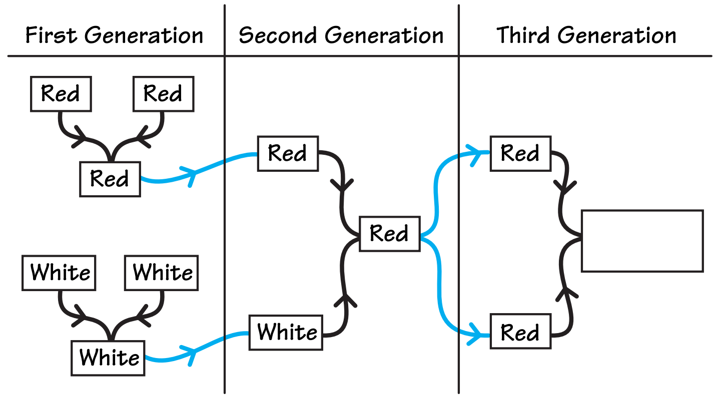
S: *Some red flowers and some white flowers.*

S: *Pink flowers.*

T: *No! The very surprising result is that all of the new plants had red flowers.*

Let students discuss this result. Emphasize that the result was not due to any experimental errors.

Add this information to the diagram on the board.

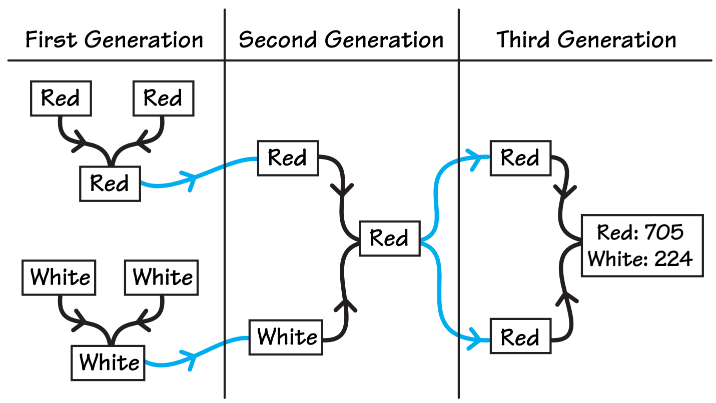


T: *Now, Mendel pollinated the red-flowered plants of the second generation with their own pollen. What color flowers do you think the third-generation plants had?*

Let the students predict the results.

T: *705 plants had red flowers, but 224 plants had white flowers.*

Add this data to the diagram on the board.



Let students comment. Emphasize that Mendel conducted his experiments very carefully and that these were indeed his results.

T: *Mendel did the same experiment for other traits of pea plants and had similar results. Each time, only one of the characteristics appeared in the second generation.*

Extend the table with traits to include the column for second generation, as in the next illustration.

T: *Do you think that all of the third generation plants were the same height or that they varied?*

S: *I think that they varied and that there were more tall plants than short plants.*

T: *Yes, in Mendel's experiment, there were 787 tall plants and 277 short plants in the third generation.*

P1

Continue in a similar manner until all of Mendel's results are recorded, as in the next table. Conclude that, for each trait, only one type appears in the second generation and that type is most common in the third generation.

Traits	Second Generation	Third Generation
Flower: Red or White	Red	705 Red; 224 White
Height: Tall or Short	Tall	787 Tall; 277 Short
Seeds: Yellow or Green	Yellow	6 022 Yellow; 2 001 Green
Seed : Round or Wrinkled	Round	5 474 Round; 1 850 Wrinkled
Pods: Yellow or Green	Green	428 Green ; 152 Yellow

T: ***Mendel now had a large amount of surprising data, but still could not explain why these results occurred. For each experiment, he decided to calculate what number times the smaller number equaled the larger number. For example, about what number times 224 equals 705?***

S: ***3 is too small, but 4 is too large.***

S: ***We should try a decimal between 3 and 4.***

Draw these red arrows in the table. Instruct students to work with two or three other students to find numbers to put in the boxes. Suggest the groups use calculators and find decimal numbers as close as they can.

Traits	Second Generation	Third Generation
Flower: Red or White	Red	$\square \times$ 705 Red; 224 White
Height: Tall or Short	Tall	$\square \times$ 787 Tall; 277 Short
Seeds: Yellow or Green	Yellow	$\square \times$ 6 022 Yellow; 2 001 Green
Seeds: Round or Wrinkled	Round	$\square \times$ 5 474 Round; 1 850 Wrinkled
Pods: Yellow or Green	Green	$\square \times$ 428 Green ; 152 Yellow

After several minutes, accept and record the best solution from each group. For example:

$$3.15 \times 224 = 705.6$$
$$2.84 \times 277 = 786.68$$
$$3.01 \times 2\,001 = 6\,023.01$$
$$2.96 \times 1850 = 5\,476$$
$$2.816 \times 152 = 428.032$$

T: ***What do you notice about all of the multipliers?***

S: ***They are all close to 3x.***

T: ***Amazingly, Mendel was able to reach the following conclusion from these experiments and data: Each trait is controlled by two genes.† In other words, each pea plant can have two genes for red flowers, two genes for white flowers, or one of each.***

T: ***Let's simulate Mendel's experiments on flower color.***

† The term

Select two students to assist you at the front of the room. Give a cup to each of them, and put two red marbles in each cup so that the class can see what you are doing.

T: ***Two plants,* A *and* B*, each have two genes for red flowers. What color flowers do you think they have?***

S: ***Red.***

Give red-flower cards to the two students.

T: ***When* A *and* B *cross-pollinate, a new plant—call it* C*—results.*
C *receives one gene from each "parent."***

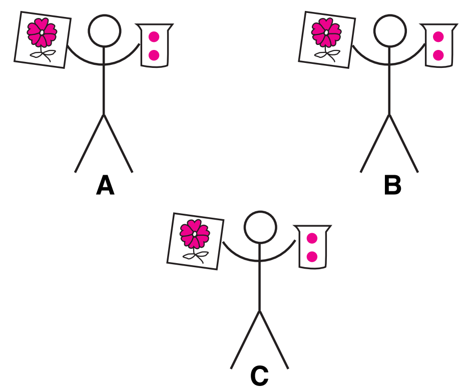
Invite a third student to select randomly one marble from each of the other two students' cups. The student should show the class the marbles before returning them to the cups, and you should give the student a cup with two marbles exactly like the ones selected.

T: ***What genes does the new plant* C *have?***

S: ***Two genes for red flowers.***

T: ***What color flowers will the new plant* C *have?***

S: ***Red flowers.***



Give the student a red-flower card.

Repeat the experiment, this time giving two students each two white marbles. The students should agree that the offspring will have white flowers.

T: ***This is like the first generation of Mendel's experiment: Red flowers yielded red flowers and white flowers yielded white flowers. But now let's look at the second generation when Mendel transferred pollen between red-flowered plants and white-flowered plants.***

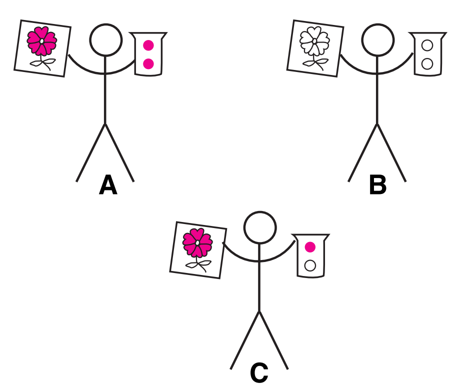
Give two red marbles and a red-flower card to one student, and two white marbles and a white-flower card to another student.

T: ***Now the new pea plant* C *receives one gene from each parent. What genes will it have?***

S: ***One gene for red flowers and one gene for white flowers.***

T: ***What color will its flowers be?***

S: ***Red! That's what Mendel discovered. All of the pea plants in the second generation had red flowers.***



Give student **C** a red-flower card.

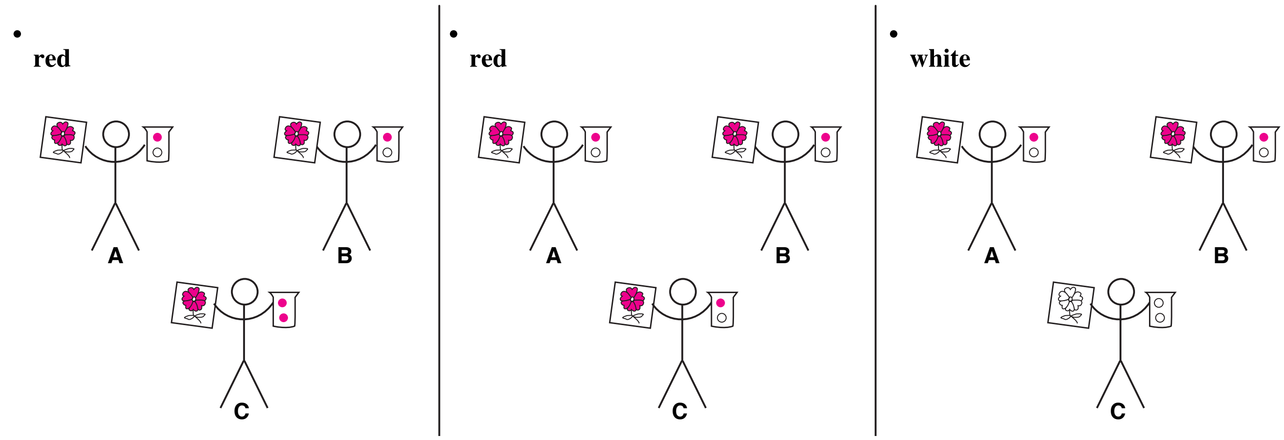
T: ***Yes, Mendel concluded correctly, that a pea plant with one red-flower gene and one white-flower gene always has red flowers. The red-flower gene* dominates.**

Let students discuss this surprising conclusion, which is crucial to the rest of the lesson.

T: ***Let's look at what happens in the third generation.***

Give one red marble, one white marble, and a red-flower card to each of two students. Let a third student choose and replace a marble from each cup, letting the class see the colors of the marbles.

Based on the selection, ask what color flowers the new plant will have.

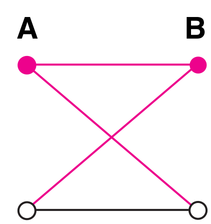


Repeat this simulation several times until all three possibilities have at least been mentioned. Then draw a picture with red and white marbles for **A** and **B**, as illustrated below.

T: ***These dots represent* A*'s and* B*'s genes. Let's draw red cords to show the possible combinations of genes that would result in* C *having red flowers.***

Let a student draw the red cords in the picture.

T: ***Draw white cords to show how* C *could have white flowers.***



Let a student draw the white cord in the picture.

Note: On the board, the black cord in this illustration will be white.

T: ***What is the probability that* C *will have red flowers? White flowers?***

S: ***$\frac{3}{4}$ for red and $\frac{1}{4}$ for white, since three of the four cords are red and one cord is white.***

Record this information in a probability tree on the board (see the next illustration).

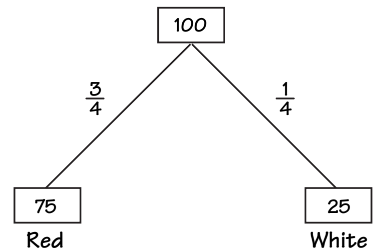
T: ***Pretend that we repeat this experiment 100 times. About how often would* C *have red flowers? White flowers?***

S: ***C would have red flowers about 75 times, since $\frac{3}{4} \times 100 = 75$.***
C would have white flowers about 25 times, since $\frac{1}{4} \times 100 = 25$.

Record the results.

T: ***Does this agree with Mendel's results?***

S: ***Yes, 3 x 25 = 75 and Mendel found out that in the third generation there were about three times as many red-flowered pea plants as white-flowered pea plants (3 x 224 ≈ 705).***



T: ***By assuming that there are two genes for each trait and by assuming that one dominates, Mendel was able to explain his experimental results. Only much later were scientists able to find direct evidence for genes. Mendel used careful experiments and probability to reach an amazing conclusion.***

Let students comment on Mendel's experiments and conclusions.

T: ***Mendel conducted his experiments for 11 years and then wrote a magazine article about them in 1865. But scientists ignored his conclusions. Mendel died in 1884 without recognition of his work.***

In about 1900, Dutch, German, and English scientists concurrently did experiments to rediscover Mendel's rules. Upon finding his 1865 article, they all credited Mendel with the results for which he is now famous.

Exercise 2

T: ***Not all plants are as genetically simple as pea plants. There is a Japanese plant for which cross-pollination of red-flowered plants and white-flowered plants yields pink-flowered plants. When these pink-flowered plants are pollinated by themselves, they yield red-flowered plants, pink-flowered plants, and white-flowered plants. Also, for humans, a trait like height is controlled by many pairs of genes, and by one's health and diet. At one time scientists believed eye color of people was as simple to understand as the flower color of pea plants. What are two major eye colors?***

S: ***Brown and blue.***

T: ***Brown eyes play the same role as red flowers. Blue eyes play the same role as white flowers. That is, brown eyes dominate blue eyes. Lisa (select a student in the class) has blue eyes. What two genes for eye color could she have?***

S: ***She must have two genes for blue eyes.***

T: ***Ali (select a student in the class) has brown eyes. What two genes for eye color could he have?***

S: ***Two genes for brown eyes, or one gene for brown eyes and one gene for blue eyes since brown dominates blue.***

T: ***If both parents have brown eyes, can their child have blue eyes?***

S: ***Yes, if each parent has one gene for blue eyes and one gene for brown eyes, they would each have brown eyes. Their child could receive one gene for blue eyes from each parent and thereby have blue eyes.***

Let students talk about eye colors in their own families. Encourage them to use genes to explain the eye colors of parents and children. You might mention that some green eyes are actually blue eyes with yellow pigment. Gray eyes are blue eyes with a thicker coating than normal that allows less light.

Note: More recent discoveries in genetics suggest that there may be other genes for eye color or distinguishing features such as color streaks. Further, another theory about eye color says that a group of genes acting together control eye color. Still, parents with pure blue eyes will have blue-eyed children, but parents with other eye colors could have children with a number of different eye colors as a result of polygene inheritance.

An inherited trait that might be fun to investigate with your class is the ability to roll the tongue (sides rolled up). Those who can have a dominant gene, and those who cannot have inherited two recessive genes.

Writing Activity

Ask students to find out the eye color of their parents and siblings and then write a possible explanation on how each individual got his or her eye color.

You may like students to take lesson notes on some, most, or even all their math lessons. The "Lesson Notes" section in Notes to the Teacher gives some suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students can describe the probability activity used to simulate Mendel's experiments.

P2 SHUNDA'S NEWSSTAND (LESSON ONE)

P2

Capsule Lesson Summary

Present the story of *Shunda's Newsstand* and discuss the general situation. Calculate profit from various sales records, and investigate different ways of recording data.

Materials

Teacher		Student	
	• Colored chalk • *Shunda's Newsstand* Story-Workbook		• Paper • Colored pencils, pens, or crayons • *Shunda's Newsstand* Story-workbook

Description of Lesson

Begin the lesson by asking students whether any of them has ever sold newspapers. Encourage the class to discuss the arrangements that a newspaper seller makes for such a business. Raise questions such as, How much does a newspaper seller pay for papers? How much does a newspaper seller charge customers for papers? What does a newspaper seller do with unsold papers? and so on.

Distribute copies of the story-workbook *Shunda's Newsstand*. You may like to arrange for students to work in pairs during the lesson, but each student will need a copy of the workbook in which to record his or her work. Ask that they not go ahead of the class in the workbook.

Pages 2–5

Read pages 2 to 5 together and write the following information on the board as it occurs in the reading.

Buys: 10¢
Sells: 20¢
Returns: 5¢

Sold +10
Returned −5

balances

Let students explain how a gain of 10 cents is made when one newspaper is sold, how a loss of 5 cents is had when a newspaper is returned, and how one newspaper sold balances two newspapers returned. Encourage students to associate *balances* with *break-even profit* or *gain and loss are the same* or *profit of $0*.

Page 6

Let the class discuss the page. They should observe how the picture illustrates a supply of 22 newspapers and the final sales for the day.

T: ***How many newspapers did Shunda sell this day?***

S: ***10; there are 10 blue squares.***

T: ***How many newspapers did Shunda return to the dealer?***

S: ***12; there are 12 red squares.***

T: ***Could you find how many newspapers were returned without counting the red squares?***

S: ***10 were sold. 22 – 10 = 12; so 12 were returned.***

T: ***Now, let's check the calculations. How did she get a gain of $1.00?***

S: ***10 newspapers sold and a gain of 10 cents for each newspaper sold makes a gain of 100 cents (10 x 10). That's $1.00.***

T: ***How did she get a loss of $.60?***

S: ***12 newspapers returned and a loss of 5 cents for each newspaper returned makes a loss of 60 cents (12 x 5).***

T: ***How does she get a profit of $.40?***

S: ***$1.00 – $.60 = $.40.***

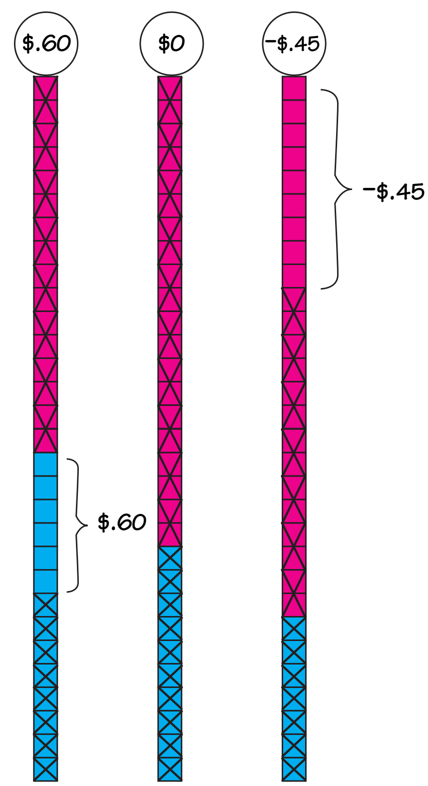
Note: Throughout this story the word *profit* is used to mean the difference between what Shunda takes in for newspapers (either by selling or returning) and what she paid for the newspapers. On a balance sheet, the profit is found simply as gain minus loss. In some cases the profit will be negative; for example, when the gain is $.60 and the loss is $1.00, the profit is ⁻$.40. This use of the word *profit* may initially bother some students because often *profit* connotes *positive*. However, if you use the word consistently, and perhaps note that for Shunda's records the profit could be positive or negative, students should easily adapt. Do not encourage a lengthy discussion of various meanings of the word as that will only be distracting.

Page 7

Collectively check the calculations that resulted in the three profit-o-meters on this page. One way to do this is to make a balance sheet for each day.

Supply	Gain	Loss	Profit
30	$1.40	$ .80	$ .60
30	$1.00	$1.00	$ 0
30	$ .70	$1.15	⁻$ .45

A student might suggest matching two red squares with one blue square and, because they balance, crossing them off. Then only squares that were not crossed off would need to be considered. For example:



This method should be encouraged if it is suggested. If both the balance sheet method and this method are suggested, use one to check the other.

Page 8

Let students work independently or with a partner to do the calculations necessary to complete the balance sheet and profit-o-meter on this page. Observe students' work, giving help as necessary. When most students are finished, check the work collectively. An answer key for this story-workbook follows Lesson P3.

Page 9

Collectively fill in the blanks at the top of the page.

Ask students to complete the balance sheet and profit-o-meter. While they are working, put some similar problems on the board, such as those listed below. As students finish the work on page 9, ask them to solve these problems, making a balance sheet and finding the profit.

1) Supply: 27
   Sold: 20

2) Supply: 27
   Sold: 10

3) Supply: 27
   Sold: 5

4) Shunda has a supply of 27 newspapers. How many must she sell to break-even (that is, have a profit of $0)?

5) Shunda has a supply of 27 newspapers. How many could she sell and have a negative profit?

Collectively check page 9 and the problems on the board. Ask students to check their work.

1) Supply: 27
   Sold: 20

Gain	Loss	Profit
$2.00	$ .35	$1.65

2) Supply: 27
   Sold: 10

Gain	Loss	Profit
$1.00	$ .85	$ .15

3) Supply: 27
   Sold: 5

Gain	Loss	Profit
$ .50	$1.10	−$ .60

4) Supply: 27
   Sold: $\boxed{9}$

Gain	Loss	Profit
$ .90	$ .90	$ 0

5) Supply: 27
   Sold: $\boxed{\text{0 to 8}}$

The solution to the fourth problem is 9, because 9 newspapers sold balance 18 newspapers returned. This observation may not be obvious; many students will use trial and error until they get a balance sheet with $0 profit. For the fifth problem, any number sold from 0 to 8 would give a negative profit.

Pages 10 and 11

Briefly discuss the meaning of these zigzag pictures as "stylized" profit-o-meters.

T: ***Do you know what these pictures are?***

S: ***They are like profit-o-meters but with line segments rather than squares.***

T: ***Why is $.20 in this circle?***

S: ***That is the profit when Shunda sells three newspapers (three blue segments) and returns ten newspapers (ten red segments).***

Record the result in a balance sheet.

Gain	Loss	Profit
$ .30	$ .50	−$ .20

Direct students to work independently or with a partner to complete pages 10 and 11. After a while, check the work on these pages collectively. The following dialogue concerns page 11.

T: ***This is a very long profit-o-meter. How should we find the profit?***

S: ***Count the blue squares and the red squares, and then make a balance sheet.***

T: ***Yes, we could do that. Has anyone already counted them?***

S: ***There are 136 blue squares and 129 red squares.***

It is likely there will be some counting errors, but eventually the class should agree on these numbers.

T: ***Did anyone solve this problem another way?***

Gain	Loss	Profit
$13.60	$6.45	$7.15

S: ***I crossed off one blue square and two red squares, then one blue and two red, then one blue and two red, then one blue and two red, and so on. There were 72 blue and one red left over. $7.20 – $.05 = $7.15. So the profit is $7.15.***

Encourage students to explore different approaches to this problem to make it easier, but do not spend too much time doing so.

Pages 12 and 13

Discuss the pictures on these pages until the class understands that they are, again, stylized profit-o-meters.

T: ***Do you understand the faces?***

S: ***A happy (smiling) face means the profit is positive, a sad (frowning) face means the profit is negative, and a "so-so" (neutral) face means $0 profit or break-even.***

Allow a few minutes for students to work on these pages before checking them collectively.

T (pointing to the upper left picture)**:** ***Why is this first face a so-so face?***

S: ***Because there was $0 profit; Shunda had a break-even day.***

T: ***How can we be sure from the picture?***

S: ***Because there is twice as much red as blue, and one blue (newspapers sold) balances two red (newspapers returned).***

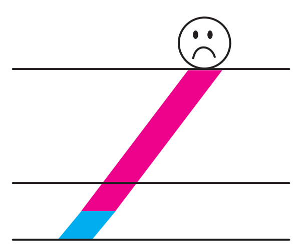
T: ***Yes; the profit-o-meter is one-third blue and two-thirds red.***

Continue, if necessary, to discuss why certain faces are smiling or frowning. In checking what kind of face should be in a circle, sometimes ask a student to explain. For example:

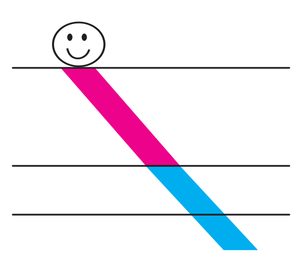
S: ***It's a sad face because red is below the in-between lines.***

S: ***To break-even, blue has to be up to the in-between line, and it is not; so the profit is negative.***

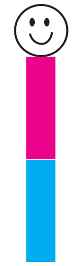
S: ***Red is more than two-thirds, and blue is less than one-third.***



S: ***It's a happy face because blue is below the bottom line. To break-even, blue would be from the in-between line just to the bottom line. But here there is more blue (sold), so the profit is positive.***



S: ***It's a happy face because half of the newspapers were sold and half returned. To break-even or have a negative profit, Shunda would have returned more than she sold. So this must be a positive profit.***



Note: An example might provide a good explanation in this case.

Supply: 40
Sold: 20
Returned: 20

Gain	Loss	Profit
$2.00	$1.00	$1.00

Collect the story-workbooks and have them ready for use in Lesson P3.

Capsule Lesson Summary

Review the story and sales records in *Shunda’s Newsstand*. Present the problem of having to determine a constant daily supply of newspapers. Make a record of the data for a 20-day experimental period; determine the total profit for different choices of daily supply, and thus arrive at the most lucrative one for Shunda.

Materials

Teacher
- Colored chalk
- *Shunda’s Newsstand* Story-Workbook
- Calculator

Student
- Colored pencils, pens, or crayons
- *Shunda’s Newsstand* Story-Workbook

Description of Lesson

Distribute students’ copies of the story-workbook and recall the story of *Shunda’s Newsstand*. Write this information on the board as it is mentioned.

Buys: 10¢
Sells: 20¢
Returns: 5¢

Sold +10
Returned −5

balances

Page 14

Read page 14 collectively. Ask the class how Shunda could decide what would be the best constant supply. You might get comments such as these:

S: ***She should buy only as many newspapers as she is sure to sell.***

S: ***She should get 35 newspapers, because then she will always have enough for customers.***

S: ***She should buy about 17 newspapers because she most often sells 17 newspapers.***

S: ***She should buy about 22 newspapers, because half the time she sells more than 22 newspapers and half the time she sells 22 or less than 22 newspapers.***

S: ***She should buy 20 newspapers because 20 is halfway between 5 (the fewest number sold) and 35 (the largest number sold).***

S: ***She should buy the average number sold.***

Note: In this case the mean average is 22, because the total numbers sold in the 20 days is 440 and 440 ÷ 20 = 22.

You may want to introduce some of the following terms for various kinds of statistical averages as students' comments suggest them:

- *mode* – the most frequent number sold
- *median* – the number in the middle with half the time more sold and half the time less sold
- *mean* – the average found by totaling the number sold and dividing by 20
- *mid-point of range* – the number halfway between the fewest sold and the most sold on any one day

Discuss whether any of these numbers (various statistical averages) is really the best choice for Shunda. You will want to mention that Shunda wants to make a choice that will help her make as much money as possible. Also, none of these averages takes into account how she gains or loses money.

Page 15

Let students complete this page independently or with a partner. They should notice that the bar graph is another way to record the demand for the 20-day experimental period.

Pages 16 and 17

Collectively read and discuss these pages. The class should observe that these pages are just a reorganization of page 15.

Pages 18 and 19

Read page 18 collectively and discuss how this 20-day balance sheet should be completed. Students might suggest counting all of the red squares and all of the blue squares on page 19 and then using these numbers to find the total gain and total loss. This method is correct, but rather time consuming.

With some prodding, the class should discover that to find the number of papers returned, you can add the numbers of newspapers sold on all 20 days (a calculator would be useful to do this addition rapidly) and then subtract the sum (440) from 700 (20 x 35 = 700), the total supply for the 20 days.

With this information, ask students to complete the balance sheet by themselves; then check their work collectively.

Total Supply: 700
Sold: 440
Returned: 260

Daily Supply	Total Gain	Total Loss	Total Profit
35	$44.00	$13.00	$31.00

Before going on to page 20, ask the class whether they think Shunda could make a better choice than 35 for the constant daily supply. Accept opinions without comment.

Pages 20 and 21

Read page 20 collectively, and discuss how the 20-day balance sheet should be completed. Students might suggest counting the red squares because there are not too many, and then subtracting this number from 400 (the total supply) to find the number of blue squares. Accept any reasonable method that leads to the needed information.

Let students complete the balance sheet on their own; then check their work collectively.

Total Supply: 400
Sold: 351
Returned: 49

Daily Supply	Total Gain	Total Loss	Total Profit
20	\$35.10	\$2.45	\$32.65

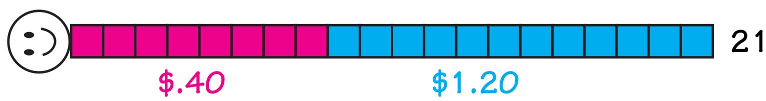
T: ***Is a constant daily supply of 20 a better choice than 35?***

S: ***Yes, the profit is greater. Here it is \$32.65; with 35, it is only \$31.00.***

Let the class again discuss whether there might be a still better choice for the constant daily supply. Again, accept opinions without comment.

Pages 22 and 23

Read page 22 collectively and, with class discussion, check that going from a supply of 20 to 21 does increase the profit by \$.80.

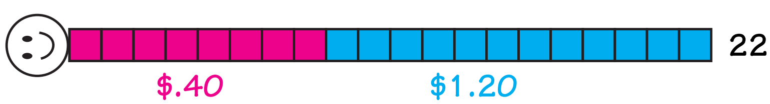


Note: Adding this row to the profit-o-meter increases the total gain by \$1.20 and increases the total loss by \$.40. Therefore, the increase of profit is \$.80.

T: ***Do you see why Shunda sees immediately that changing the daily supply from 21 to 22 would again increase her profit by \$.80?***

S: ***The row for 22 is the same as the row for 21.***

Ask students to check this on the graph on page 23, coloring blue squares for newspapers sold and red squares for newspapers returned.



Increase in Profit
\$.80

Pages 24 and 25

Direct students to complete pages 24 and 25 independently or with a partner before checking collectively.

T: ***Does this work help us to decide which is the best choice for a constant daily supply?***

S: ***Shunda's profit increased until the daily supply was 26. Then when we looked at a daily supply of 27, the profit decreased (there is a sad face). So the best choice is 26.***

During a class discussion, comments will likely suggest the idea indicated above but may not be clearly stated. Some students will benefit from several explanations of the same idea.

P3

Pages 26 and 27

Read page 26 collectively and ask students to complete the balance sheet independently. Then check the calculations collectively.

T: ***There are 408 blue squares. How much money is 408 dimes? ($40.80)***
There are 112 red squares. How much money is 112 nickels?

S: ***$5.60; 112 dimes is $11.20, so 112 nickels is half as much.***

T: ***And the profit?***

S: ***$35.20; 40.80 – 5.60 = 35.20.***

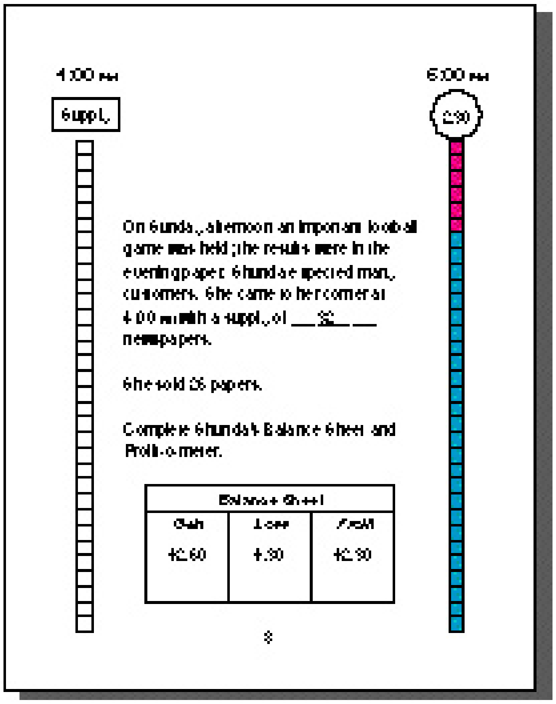
Daily Supply	Total Gain	Total Loss	Total Profit
26	$40.80	$5.60	$35.20

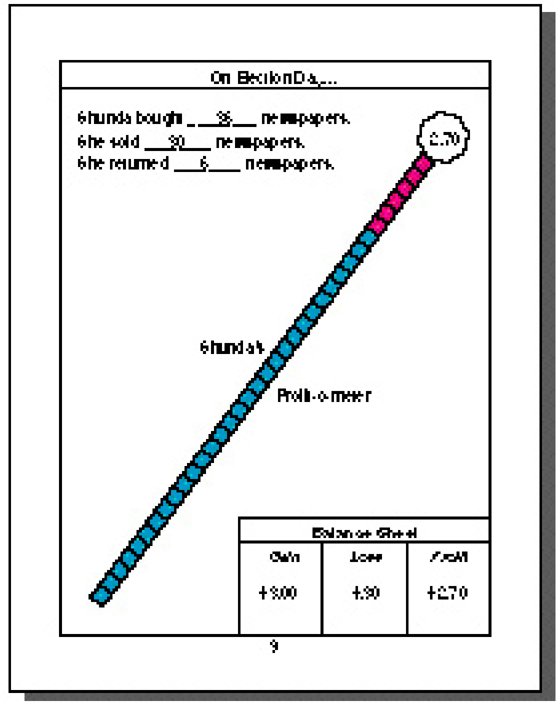
Pages 28 and 29

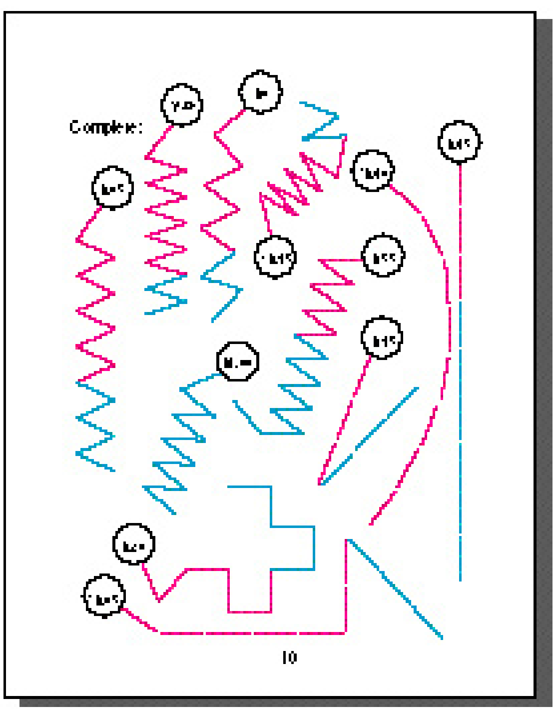
Read and discuss these pages collectively. Lead the discussion so that the class examines page 29 and relates the break-even point (one newspaper sold balances two newspapers returned) with the determination of 26 as the best choice for a constant daily supply.

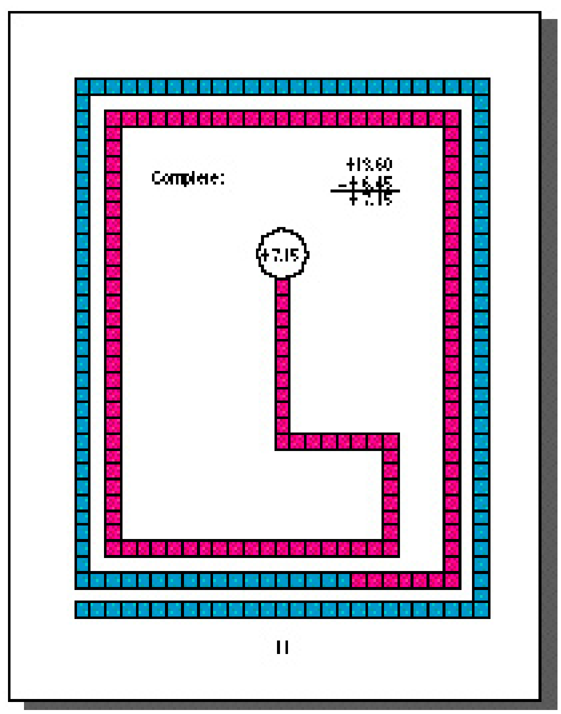
Page 30

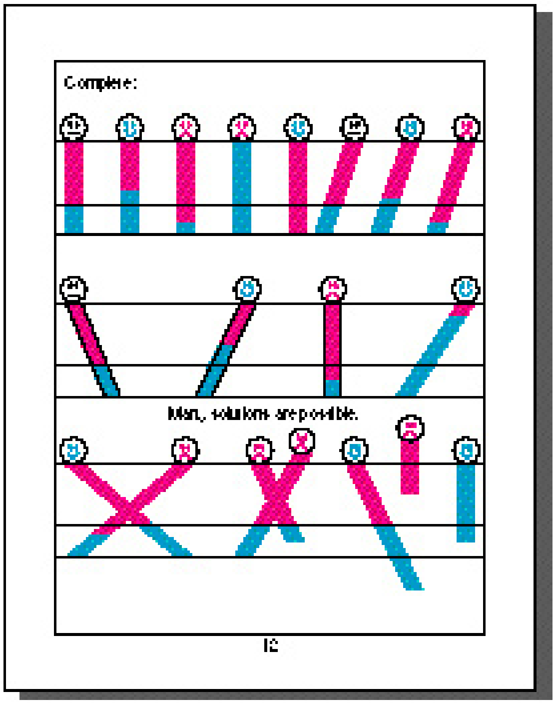
Read this page collectively and let students comment further, if they wish, about what other factors could affect Shunda's business.

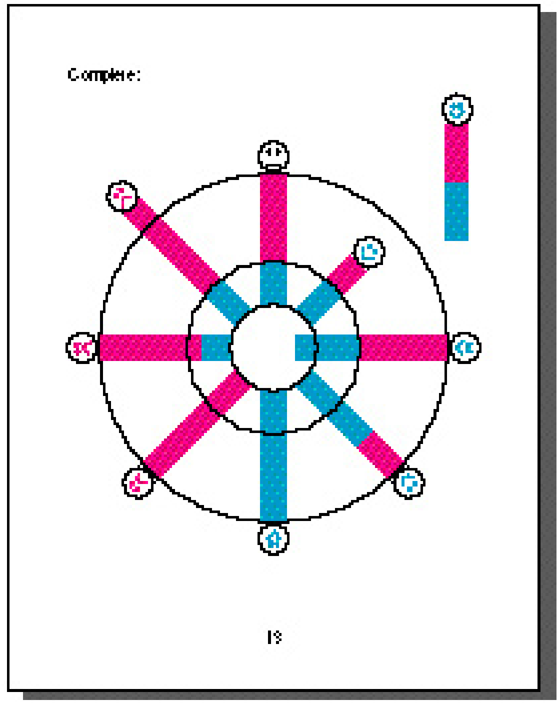

4:00 PM
6:00 PM
Supply
2.90
On Sunday afternoon an important football game was held; the results were in the evening paper. Shunda expected many customers. She came to her corner at 4:00 PM with a supply of ___32___ newspapers.
She sold 26 papers.
Complete Shunda's Balance Sheet and Profit-o-meter.
Balance Sheet
Gain
Loss
Profit
+2.60
+.30
+2.30
8

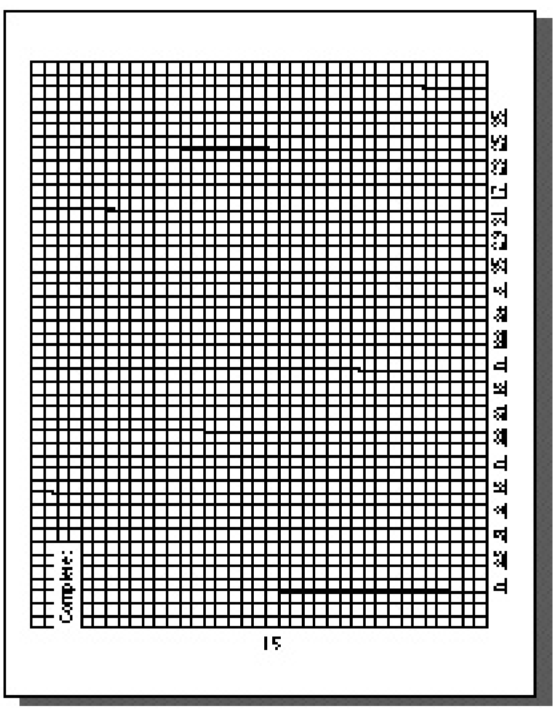

On Election Day...
Shunda bought ___36___ newspapers.
She sold ___30___ newspapers.
She returned ___6___ newspapers.
2.70
Shunda's
Profit-o-meter
Balance Sheet
Gain
Loss
Profit
+3.00
+.30
+2.70
9


Complete:
10


Complete:
+13.60
− +6.45
+7.15
+7.15
11







The largest demand during the 20 day experimental period was 35. Suppose Shunda would decide on a constant daily supply of 35 newspapers.

Page 19 shows the record (profit-o-meter) for the 20 day experimental period

IF

the constant daily supply is 35 newspapers.

Complete:

Balance Sheet For 20 day experimental period			
Daily Supply	*Total Cash*	*Total Loss*	*Total Profit*
35	$44.00	$13.00	$31.00

With such a supply, Shunda could satisfy her customers every day, but is it her most profitable choice?

Don't you think there are too many red squares in this record?

18

Suppose Shunda would decide on a constant daily supply of 20 newspapers.

Page 21 shows the record (profit-o-meter) for the 20 day experimental period

IF

the constant daily supply is 20 newspapers.

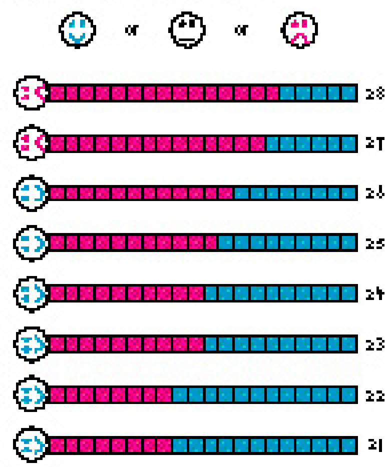
Complete:

Balance Sheet for 20 day experimental period			
Daily Supply	*Total Gain*	*Total Loss*	*Total Profit*
20	$95.10	$2.45	$92.65

Shunda found that 20 was a better choice than 25 for the constant daily supply. But is it the best possible choice?

19

Color the strips and complete the faces:



25

Shunda's Conclusion

During the 20 day experimental period, the largest profit would have been obtained with a daily supply of 26 newspapers.

Complete:

Balance Sheet for 20 day experimental period			
Daily Supply	*Total Gain*	*Total Loss*	*Total Profit*
26	$100.80	$5.60	$95.20

26

Capsule Lesson Summary

When multiplying a number by a fraction, determine conditions needed for the result to be greater than the starting number. Given the probability of surviving a trip along each road in a dangerous country, determine the safest route between two towns.

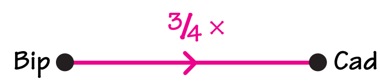
Materials

Teacher		Student	
	• Colored chalk		• Worksheets P4* and **

Description of Lesson

Exercise 1

Draw this arrow on the board.



T: ***Bip is an integer. Will Bip always be greater than Cad, less than Cad, or sometimes greater and sometimes less than Cad?***

Let students express their opinions.

T: ***Perhaps we should try some numbers. What number do you want to try for Bip?***

S: ***8.***

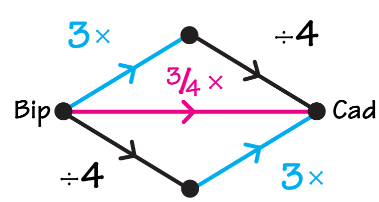
T: ***How can we calculate three-fourths of a number?***

S: ***Multiply by 3 and then divide by 4.***

S: ***Or divide by 4 and then multiply by 3.***

S: $\frac{3}{4} \times 8 = 6$.

Show the compositions in the arrow picture. Draw a table on the board for recording possibilities for Bip and Cad.



Let students suggest several other numbers for Bip and calculate three-fourths of each number. You may need to help students calculate three-fourths of a non-multiple of 4. For example:

Bip	Cad
8	6
12	9
20	15

S: ***Let Bip be 15.***

S: ***If Bip is 15, Cad is*** $11\frac{1}{4}$. $3 \times 15 = 45$ ***and*** $45 \div 4 = 11\frac{1}{4}$ ***because if 45 objects are shared among four people, each person receives*** $11\frac{1}{4}$ ***objects.***

S: ***Cad is*** $11\frac{1}{4}$. $15 \div 4 = \frac{15}{4} = 3\frac{3}{4}$. $3 \times 3\frac{3}{4} = 11\frac{1}{4}$ ***since*** $3 \times 3 = 9$, $3 \times \frac{3}{4} = \frac{9}{4} = 2\frac{1}{4}$, ***and*** $9 + 2\frac{1}{4} = 11\frac{1}{4}$.

P4

Continue until the class has tried about six possibilities for Bip. Then return to the original question about the relative size of Bip and Cad. If only positive numbers were suggested for Bip, continue with the following questions. Otherwise, proceed directly to the conclusions that follow.

T: ***$\frac{3}{4}$ x Bip = Cad. Which is greater, Bip or Cad?***

S: ***For the numbers we tried, Bip is always greater.***

S: ***I think that if you divide any number by 4 and multiply it by 3, the answer will be less than the original number.***

T: ***But, in fact, Bip is not always greater than Cad. Let's try to find numbers for which Bip is less than Cad.***

S: ***Let Bip be 0. $\frac{3}{4}$ x 0 = 0. Then Bip and Cad are both 0.***

S: ***Let Bip be $\widehat{12}$. $\frac{3}{4}$ x $\widehat{12} = \widehat{9}$, and $\widehat{12} < \widehat{9}$. So, if Bip is $\widehat{12}$, Bip is less than Cad.***

After a few more examples, lead to the following conclusions:

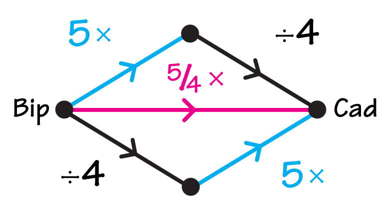
- If Bip is positive, Bip is greater than three-fourths of Bip.
- If Bip is 0, Bip equals three-fourths of Bip.
- If Bip is negative, Bip is less than three-fourths of Bip.

Note: Students may want to extend the conclusions, allowing Bip to be a non-integer as well. The conclusions, of course, still hold.

Change the arrow picture so that the red arrow is $\frac{5}{4}$x, and erase the table.

T: ***Let's check which is greater, Bip or Cad, when the red arrow is for $\frac{5}{4}$x. Will Bip or Cad be greater?***

Let students express their opinions, label the other arrows, and suggest several possibilities for Bip.



Bip	Cad
16	20
4	5
10	$12\frac{1}{2}$
100	125
$\widehat{16}$	$\widehat{20}$
0	0

Conclude the following about $\frac{5}{4}$x:

- If Bip is positive, Bip is less than Cad.
- If Bip is 0, Bip equals Cad.
- If Bip is negative, Bip is greater than Cad.

T: ***If Bip is positive, why is three-fourths of Bip less than Bip whereas five-fourths of Bip is greater than Bip?***

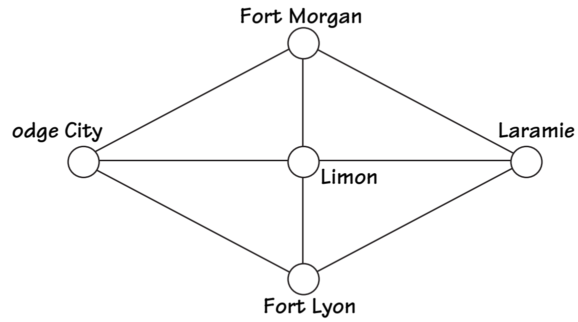
Let students compare the two functions. They may comment that $\frac{3}{4}$ is less than 1 while $\frac{5}{4}$ is greater than 1.

Exercise 2

Feel free to choose an alternate adventure for this exercise.

Draw this picture on the board.

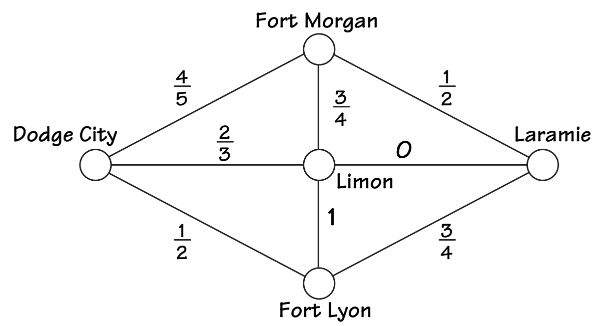
T: ***This is a map of a portion of the wild, dangerous West more than 100 years ago. The map shows routes between two towns—Dodge City and Laramie. The cords indicate roads, and the small circles represent towns or forts. Let's go back in time and trace a route a cowboy might take from Dodge City to Laramie.***



Let several students trace routes from Dodge City to Laramie. If a student suggests a route that includes returning to Dodge City or visiting a town or a fort twice, tell the class that the cowboy would not take such a route due to the dangers involved in travel.

Add these numbers to the picture.

T: ***It's 100 years ago, and travel in the West is dangerous due to bandits, Indians, and dangerous animals. The numbers on each road indicate the probability that a lone traveler will survive one passage along the road. Which road is most dangerous? Safest?***



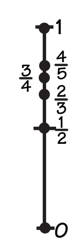
S: ***The road from Limon to Laramie is the most dangerous. The probability of surviving is 0 meaning that no one survives.***

S: ***The road from Limon to Fort Lyon is safest. The probability of surviving is 1 meaning that everyone survives on that road.***

On the board, draw a vertical line segment approximately 60 cm long. Indicate its midpoint.

T: ***The top of this probability stick is 1 and the bottom is 0. The top is for the safest and the bottom is for the most dangerous of the six roads. Of the other roads, which is safest?***

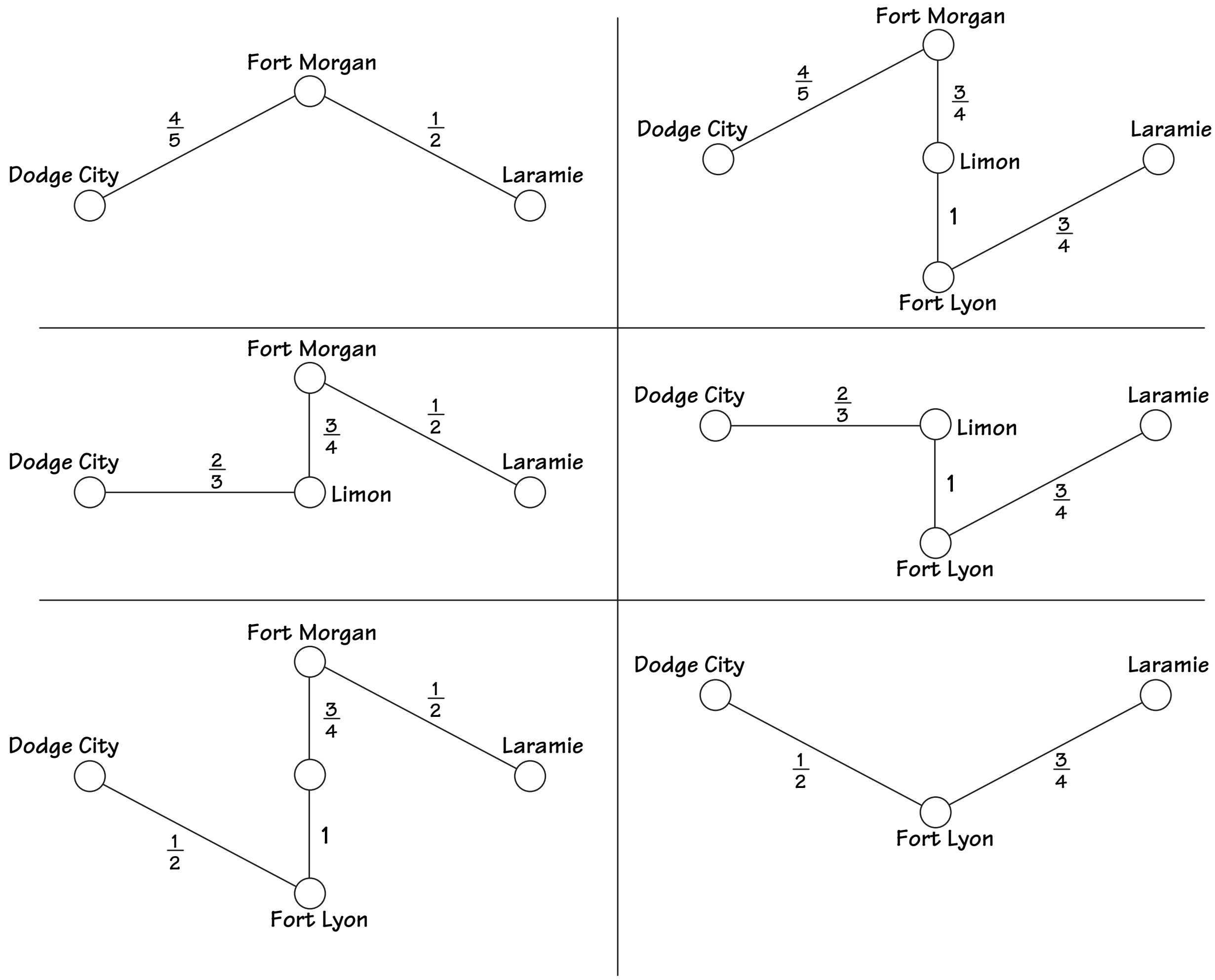
Let a student select a road and, right or wrong, ask someone to use a meter stick to locate accurately a dot on the probability stick for the corresponding probability. Continue in a similar manner until all of the probabilities are recorded.



T: ***Which route from Dodge City to Laramie do you think is safest?***

For each route a student suggests, draw it on the board, as in the next illustration.

T: ***Let's draw all of the routes from Dodge City to Laramie that do not include repeated towns and do not include the fatal road from Limon to Laramie. Then we can compare their safety.***



T: ***How can we determine which of these six possible routes is the safest route?***

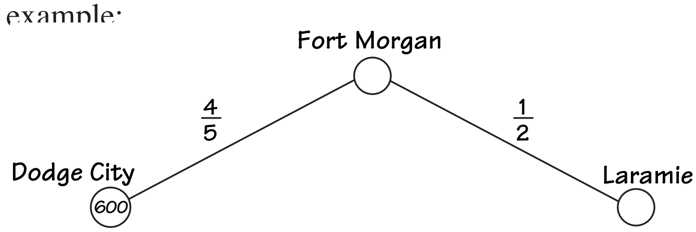
Suggest the following idea yourself if no student does.

S: ***Pretend that a large number of cowboys tries each route. Check which route results in the most survivors.***

S: ***How many cowboys should we use?***

Any number is acceptable. However, you may prefer to choose multiples of 120 since then all fraction calculations will result in whole numbers. Suppose 600 is chosen.

Check the path that the class feels is safest, for example:



T: ***Starting with 600 cowboys from Dodge City, about how many of them should arrive at Fort Morgan safely?***

S: ***About 480, since*** $\frac{4}{5} \times 600 = 480$.

Put 480 in the small circle for Fort Morgan.

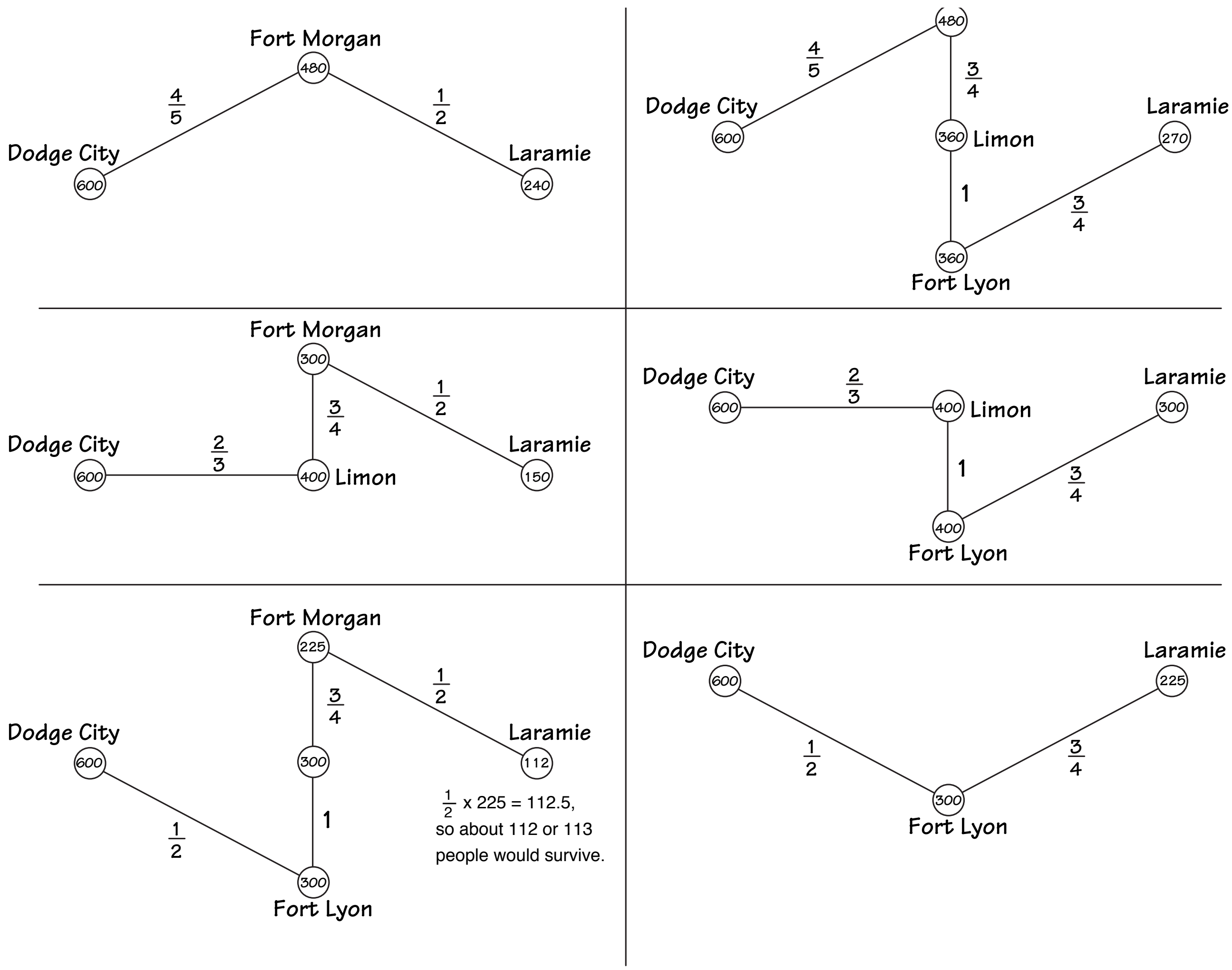
T: ***Of those 480, about how many should reach Laramie?***

S: ***About 240; ½ x 480 = 240.***

Put 240 in the small circle for Laramie.

T: ***I'll let you check whether this is the safest route.***

Divide the class into five groups. Assign one of the other five routes to each group. Tell each group to start with 600 cowboys at Dodge City and to calculate how many would reach Laramie along the route assigned to their group. When most groups are finished, ask one student in each group to announce the group's findings. Record all of the solutions on the board.



S: ***So the route from Dodge City to Laramie via Limon and Fort Lyon is safest. About 300 people out of 600 should survive.***

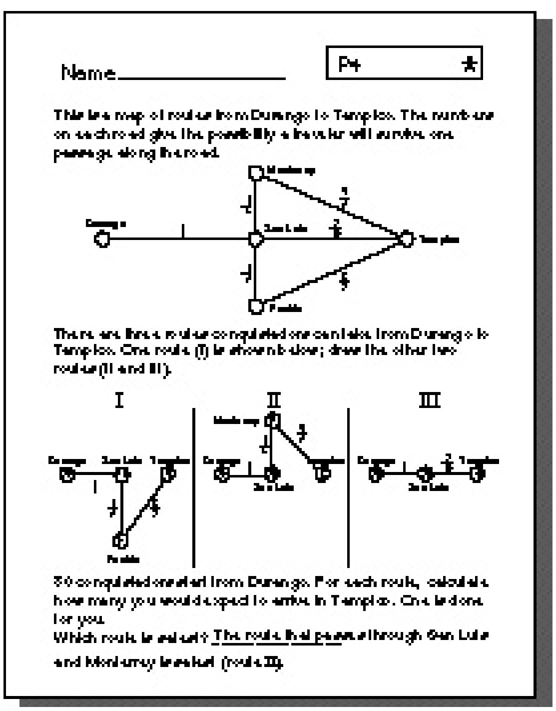
T: ***If you were to take this route, what would be your chances of survival?***

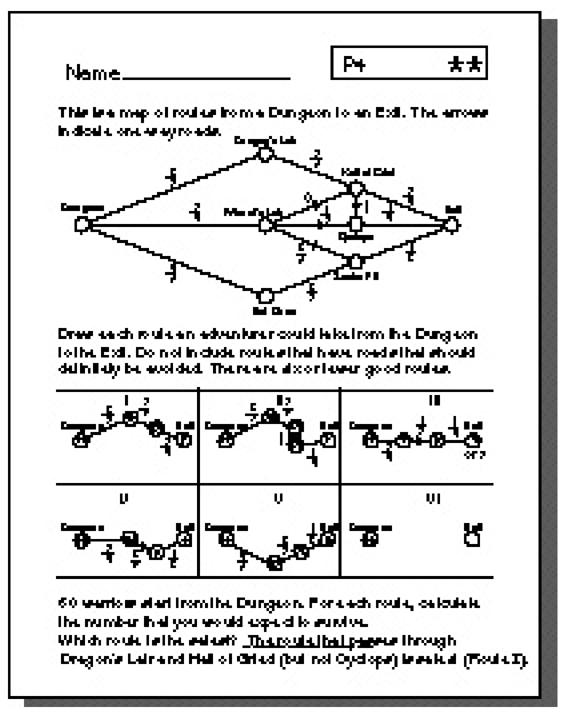
S: ***One out of two or ½, since ½ x 600 = 300.***

T: ***So one chance out of two is the best we can do.***

Worksheets P4* and ** are available for individual work.

You may want to reiterate that, on the worksheets, students need only consider routes that do not involve any repeated paths or locations, and that do not include paths whose probability of survival is 0.





P5 AVERAGES

Capsule Lesson Summary

Identify three numbers using their mean as a clue. Draw a cross-section of a lake when only the mean depth is given, and then when the mean, mode, and range of depth are given.

Materials

Teacher
- Grid board
- Colored chalk
- Meter stick

Student
- Worksheets P5(a) and (b)
- Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

Note: Mean, median, and mode are three types of averages. In common usage, the word *average* often refers specifically to the mean.

T: ***I am thinking of three numbers whose mean (average) is 100. What numbers could they be?***

S: ***80, 70, and 150.***

Record a response on the board.

Mean (Average) is 100		
80	70	150

T: ***How could we check to see that the mean (average) of these numbers is 100?***

S: ***Add the three numbers and then divide by 3.***

S: ***Take 50 from 150, and give 20 to 80 and 30 to 70.***

Ask for additional sets of numbers whose mean is 100. List correct responses. Then put 90 in one column.

Mean (Average) is 100		
80	70	150
100	100	100
99	101	100
0	200	100
40	130	130
100	130	100
	90	

T: ***If one of the numbers is 90, what could the other two numbers be?***

S: ***110 and 100.***

S: ***80 and 130.***

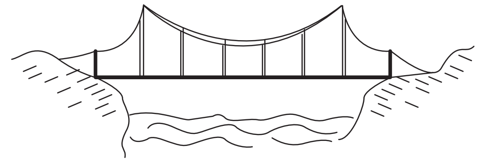
S: ***10 and 200.***

Record these responses in the list, and repeat the activity using 65 as one of the three numbers.

T: ***Here is another clue about my three numbers: One of the numbers is 100 and the other two differ by 60. Can you find my three numbers?***

Allow a few minutes for students to work on the problem. Then ask a student to announce the numbers.

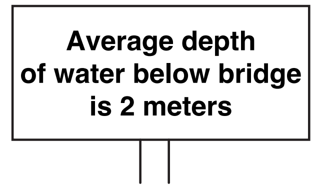
S: ***70, 100, and 130.***



Exercise 2

Draw a picture of a bridge on the board.

T: ***Giambrone Park has a lake where people swim. In the lake there is an island connected to the shore by a bridge. This sign is on the bridge.***



Write the sign information on the board.

Solicit comments on the meaning of the sign. Use a meter stick to show how deep 2 meters is. Take a vote to determine how many students would be willing to dive into the water below the bridge.

Lead the discussion to consider how the average depth could have been computed. Students may suggest measuring the wet part of a stick or of a weighted rope that has been lowered to the bottom of the lake.

T: ***If the average depth was found by computing the m these measurements have been?***

Use student responses to record possible measurements on the board. Several possibilities are listed here.

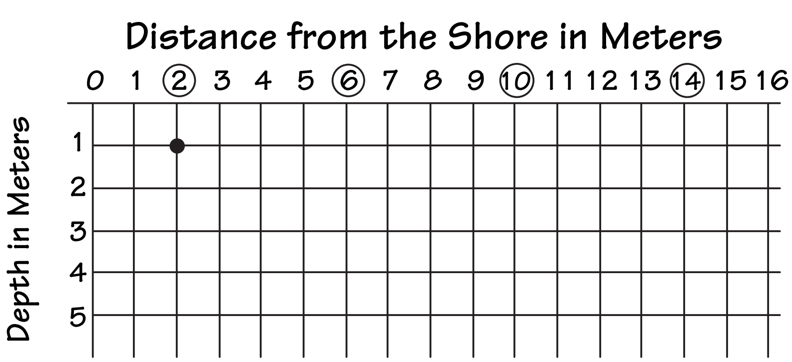
Average is 2 Meters

2 m	2 m	2 m	2 m
1 m	3 m	2 m	2 m
4 m	2 m	1 m	1 m
5 m	1 m	1 m	1 m
6 m	1 m	0.5m	0.5m
2 m	3 m	1 m	2 m

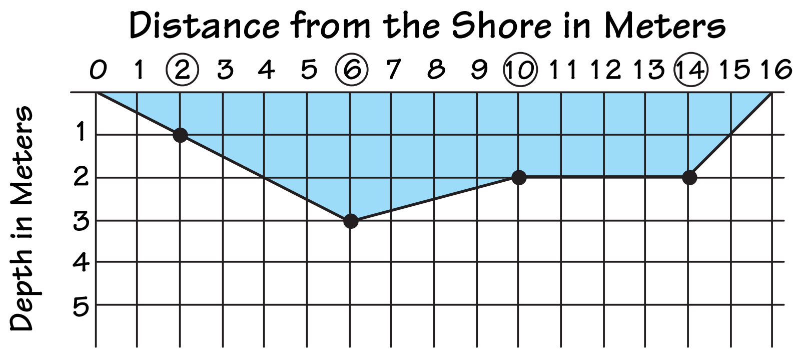
T: ***Let's use one of these sets of measurements to draw a picture of what the profile of the lake under the bridge might look like.***

On a grid, draw the graph in the next illustration as you explain,

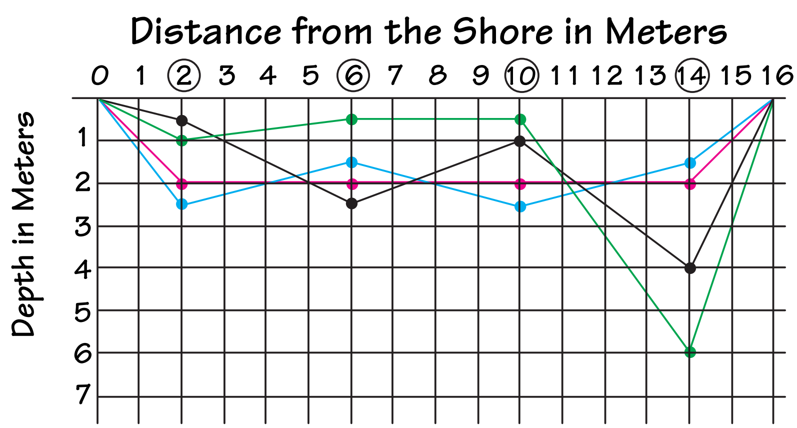
T: ***The bridge is 16 meters long. Let's assume the four measurements were taken at the circled locations. If the measurements taken were 1 m, 3 m, 2 m, and 2 m, who can place a dot to show the depth of the lake 2 meters from shore?***



Continue the activity until all four points have been plotted. Mention to the class that the depth of the lake at 0 meters and 16 meters from the shore is surely 0 meters, and draw a zigzag from (0,0) to (16, 0) that connects the four points. Ask a student to shade where the water is in the picture.



Distribute copies of Worksheet P5(a). Ask students to choose one of the other sets of data and to draw on their worksheets a picture of what the profile of the lake under the bridge might be with that data. Challenge students who finish quickly to find four possible measurements with an average of 2 m that are not listed on the board and to draw a picture of that data. Invite several students to draw their pictures of the profile of the lake under the bridge on the grid board. For example:



T: ***Does the average depth give enough information for us to know the profile of the lake under the bridge?***

S: ***No, there are many possibilities.***

T: ***What other information would be helpful?***

Accept several comments before announcing,

T: ***Here is some more information I received from the park ranger.***

Record this information on the board.

T: ***Out of eight measurements, the depth that occurred most frequently was 1 meter; that is, the mode was 1 meter. Also, the eight measurements ranged from 1 meter to 8 meters.***

8 measurements

Mean: 2 meters

Mode: 1 meter

Range: 1 to 8 meters

P5

Before continuing, you may wish to return to the data already on the board and determine the range and mode for several sets of four measurements.

T: ***Use this new information to determine what the eight measurements could have been.***

Allow a short while for students to work (perhaps with a partner) on the problem. Record several responses and check them against the information given. There are many possible solutions, but students are likely to discover a variation of these measurements.

1, 1, 1, 2, 8, 1, 1, 1

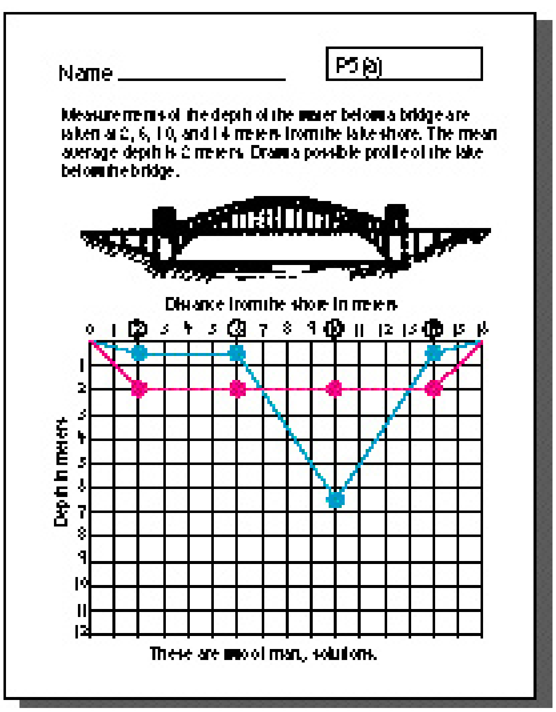
Ask students to draw a picture of the profile of the lake under the bridge on Worksheet P5(b). Briefly discuss the situation at the end of class.

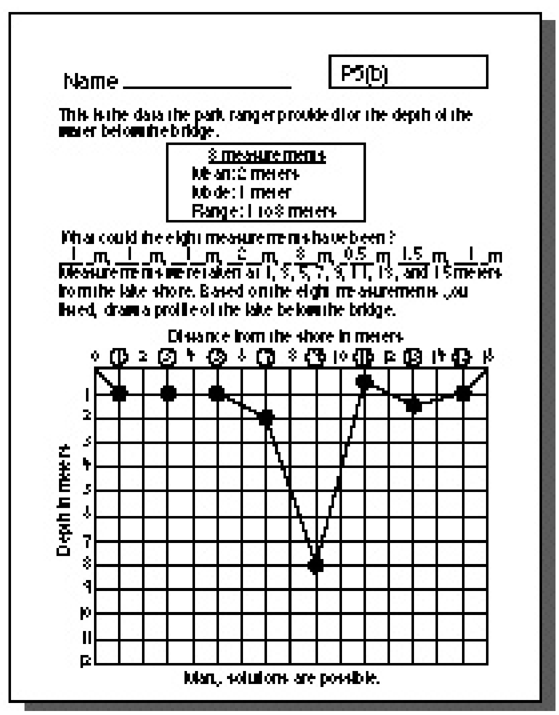
T: ***Do we know exactly what the profile of the lake looks like below the bridge?***

S: ***No, the measurements could be in any order.***

T: ***Would you dive in?***

S: ***No, the water is mostly shallow.***





Capsule Lesson Summary

Estimate the number of raisins in ½-oz. boxes and then find actual counts. Display the data from both estimates and actual counts in a back-to-back stem-and-leaf plot. Make a double bar graph of individual student data (estimates and actual counts.) Compare and contrast what information is communicated by the two visual representations.

Materials

Teacher
- Blackline P6
- Paper

Student
- ½-oz. box of raisins
- Colored pencils, pens, or crayons

Advance Preparation: Obtain ½-oz. boxes of raisins, enough for every student. As an alternative, use fun size packages of M&Ms or small containers with 1 tablespoon of dried beans in each. For the beans, you can use film canisters as containers and, at the appropriate time, show students the 1 tablespoon measure you used to measure the beans. Make several copies of Blackline P6 for group use in Exercise 2.

Description of Lesson

In this lesson, students should observe that different visual presentations of data allow different visual perspectives and readily answer different questions. This lesson is adapted and simplified from an activity in *Developing Graph Comprehension: Elementary and Middle School Activities* by Frances R. Curio.

Exercise 1

Give each student a ½-oz. box of raisins (or an alternative). Announce that they may not open the boxes yet.

T: ***Have you ever eaten raisins from a box this size?***
How many raisins do you think you get in one of these boxes?

Let students comment on the box size and how raisins are packed into the box. Then direct everyone to write on a piece of paper an estimate of the number of raisins they think will be in the box.

Call on each student in turn to give an estimate, and record the responses in a stem-and-leaf plot on the board. Explain to the class how you are recording their estimates.

For example:

S: *25.*

S: *32.*

S: *19.*

S: *20.*

S: *25.*

Tens	Ones
0	
1	9
2	5 0 5
3	2
4	
5	

Here the stem is the tens column and the leaves are the ones. Each estimate is represented by a leaf; the estimate of 25 is represented by 5 (ones) in a row next to 2 (tens) in the stem.

P6

After you record a few estimates, you may let students come to the board to record their own estimates in the stem-and-leaf plot. When estimates for all students are recorded, your stem-and-leaf plot may look similar to the one on the left below. At this point, suggest to the class that you order or rearrange the leaf data as in the plot on the right below.

Tens	Ones
0	
1	9 6 8
2	5 0 5 2 0 2 5 3 0 7
3	2 0 0 5 0 1 5 0
4	0 5 0 2
5	0 0

Tens	Ones
0	
1	6 8 9
2	0 0 0 2 2 3 5 5 5 7
3	0 0 0 0 1 2 5 5
4	0 0 2 5
5	0 0

Tell students that now they can open their boxes and count the raisins. Insist that everyone checks the count, and writes it on the same paper with his or her estimate. Suggest they label the numbers on the paper "estimate" and "actual."

Call on several students to announce their actual counts, and record these results on the left side of the stem-and-leaf plot. Explain to the class that you are recording actual numbers back-to-back with the estimates. In this way you will construct a back-to-back stem-and-leaf plot, as illustrated here.

S: *38.*

S: *35.*

S: *40.*

S: *38.*

S: *39.*

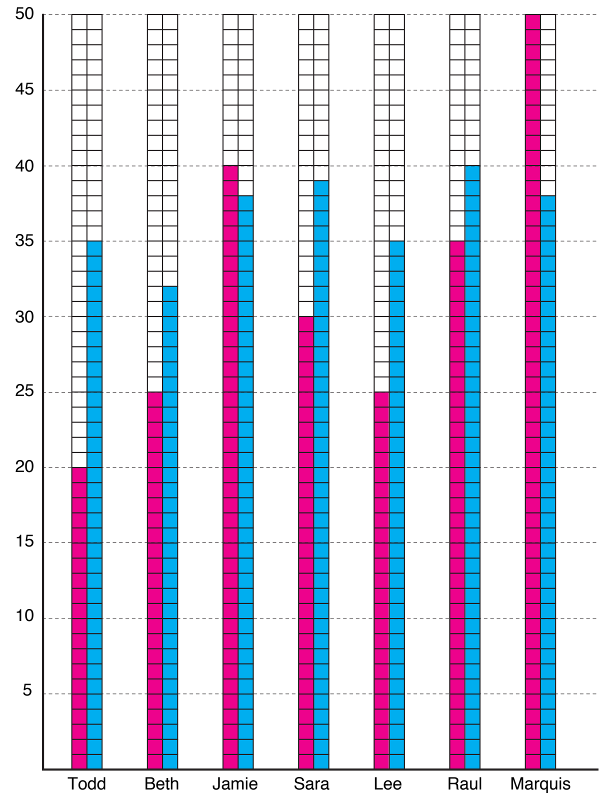
Actual Ones	Tens	Estimate Ones
	0	
	1	6 8 9
	2	0 0 0 2 2 3 5 5 5 7
9 8 5 8	3	0 0 0 0 1 2 5 5
0	4	0 0 2 5
	5	0 0

After you record a few actual counts, you may let students come to the board to record their own actual numbers in the stem-and-leaf plot. When actual counts for all students are recorded, rearrange the leaves on the actual side of the stem-and-leaf plot so it looks similar to one below.

Actual Ones	Tens	Estimate Ones
	0	
	1	6 8 9
9	2	0 0 0 2 2 3 5 5 5 7
9 9 9 9 8 8 8 8 5 5 5 4 4 2 2 2 2 1	3	0 0 0 0 1 2 5 5
1 1 1 1 0 0 0 0	4	0 0 2 5
	5	0 0

Hold a brief discussion about the results and, in particular, ask what students notice about the estimates compared to the actual counts. For example, the class may observe that the estimates are more spread out than the actual counts. Hold off from too much discussion of the data at this time so that you can have the discussion with two different visual displays after Exercise 2.

Exercise 2



Organize the class in groups of six or seven. Provide each group with one copy of Blackline P6, and explain that they are to present their individual estimates and actual counts in a double bar graph. Suggest that each student in a group write his or her name under a double column of little squares. On the left side of a column, the student colors a bar red to represent his or her estimate. Then, on the right side, the student colors a bar blue to represent his or her actual count. Students should have these numbers labeled on their papers. Here is a sample double bar graph from a group.

When a group completes and checks their double bar graph, ask them to work together to develop one other way to visually display their data or the class data.

Exercise 3

Collect the double bar graphs from all the groups. Make a display with the graphs next to each other on the board, separate but close to the back-to-back stem-and-leaf plot.

Begin a discussion with the class to interpret the two visual representations of the same data. This discussion should elicit comments about what information is better communicated by which type of graph. For example:

Back-to-Back Stem-and-Leaf Plot Graphs
- cannot see individual's estimate and actual count
- easy to find the frequency of a given estimate or actual count
- easy to see range and distribution of estimates and actual counts

Double Bar Graph
- readily compare an individual's estimate with actual count
- more difficult to find the frequency of a given estimate or actual count
- cannot easily see range and distribution of estimates and actual counts

You may ask specific questions to compare the two data displays. For each question, check both displays to see if one is easier to use in answering the question. For example:

- Which estimate was the highest? … lowest?
- What was the *range* of estimates (actual counts)?
- Which student(s) had the best estimate?
- Did most students estimate too high or too low?
- Which estimate (actual count) occurred most often (mode)?
- Which estimate (actual count) was in the middle (median)?
- How can we calculate the mean average of the actual counts?

If the groups developed other visual representations of the class data, you may at this point like to compare their displays to the double bar graph or the back-to-back stem-and-leaf plot.

Capsule Lesson Summary

Tell the history of a classic probability problem, the problem of points. Introduce the problem as a game played by two people. Calculate the chances of each player winning when the score is not tied.

Materials

Teacher
- *IG-IV Probability Poster* #1
- Meter stick
- Colored chalk
- One red and one blue marble
- One coin
- Tape

Student
- Paper
- Colored pencils, pens, or crayons
- Metric ruler
- Worksheet P7

Description of Lesson

Exercise 1

The following dialogue concerning the historical background of the central problem in this lesson is optional.

T: ***Most mathematics deals with certain events. Problems, whether easy or hard, usually have one or more definite answers. However, probability focuses on uncertain events. No one can consistently predict the roll of a die, the flip of a coin, or tomorrow's weather. We can only state probabilities for such events.***

Because of its uncertainty, mathematicians did not study probability for a long time. Only in the 1500s did interest in probability begin to grow. What activities do you think were the source of the first problems in probability?

S: ***Games and gambling that involved dice and cards.***

T: ***Today we will study a probability problem that arose in several of these two-person games. The game could be as simple as flipping a coin in which one player gets points for heads and the other player gets points for tails. The first player to get ten points wins.***

Select two students to play the game, one student for heads and the other for tails. Flip a coin and record one point for the winner. Continue until it is clear that most students understand the game. For example, here are the results of a three-flip sequence.

1. **Heads** - Player **A** gets a point; score: **A**-1, **B**-0
2. **Tails** - Player **B** gets a point; score: **A**-1, **B**-1
3. **Tails** - Player **B** gets a point; score: **A**-1, **B**-2

Write this information on the board.

A: 9
B: 8

$100

T: ***Here is the probability problem: Each player puts $50 into a pot and agrees that the first player to get ten points wins the $100 pot. However, the players have to stop the game when the score is A-9; B-8. How should they split the $100?***

Let students discuss the problem and offer possible solutions. For example:

- Divide the pot \$50-\$50 since the game is not finished.
- Give all of the money to **A** since **A** is leading.
- Give $\frac{9}{17}$ of the pot to **A** and $\frac{8}{17}$ to **B**.

For each solution, ask which player is most likely to object to the solution. For example, player **A** needs only one more point to win and might object to the pot being divided evenly or to getting $\frac{9}{17}$ of it, which is not much more than half. But player **B** is only one point behind **A** and might object to giving all of the pot to **A**.

T: ***An Italian mathematician, Tartaglia, claimed to have solved this problem in the 1550s. However, there was no record of his solution, and mathematicians continued to discuss the problem for about 100 years. In about 1650, the Chevalier de Mére, a member of the court of King Louis XIV of France and also a gambler, asked his friend Blaise Pascal for a solution. Pascal, a young mathematician, wrote about the problem to his older and more famous friend Pierre de Fermat. After about a year of correspondence, they solved the problem in three different ways. Their methods provided a beginning to the mathematical study of probability.***

The method that we will use to solve the problem is similar to one of the methods Pascal and Fermat used about 350 years ago.

Exercise 2

T: ***Rita and Bruce are playing a game. Instead of a coin, Rita has one red marble and one blue marble. Keeping her hands behind her back, she mixes the marbles and then puts one marble in each hand. Bruce chooses a hand. If he chooses the hand with the blue marble, he scores one point. Otherwise, Rita scores a point. The first player to get ten points wins.***

Select two students to play the roles of Rita and Bruce. Let them play the game in front of the class, recording the score of the game on the board. Let them continue without interruption until one player reaches six points. Here we assume that the score is 4–6.

● Rita: 4
● Bruce: 6

T: ***The score is now 4-6; let's always read Rita's score first. What could the next score be?***

S: ***5-6, if the red marble is chosen.***

S: ***4-7, if the blue marble is chosen.***

Let the game continue. After each point, ask students for the next two possible scores. Continue until a player wins by reaching ten points.

T: ***What is the shortest game Rita and Bruce could play?***

S: ***A game that ends 10-0 or 0-10.***

T: ***What is the longest game they could play?***

S: ***A game that ends 10-9 or 9-10.***

Exercise 3

Write the following information on the board. A convenient length for the line segment is 32 cm.

● Rita: 7
● Bruce: 8

T: ***One afternoon Rita and Bruce play the game until they have to stop. But the game is not over; they stop when the score is 7-8 and agree to continue the next day. When they continue the game the next day, who is favored to win?***

S: ***Bruce, because he is ahead.***

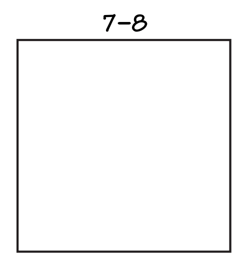
T: ***What do you think is Bruce's probability of winning the game? Draw a dot on the probability stick for your estimate.***

If necessary, remind students that a dot at 1 means that Bruce would always win and that a dot at 0 means that Bruce would never win. Let students estimate Bruce's probability of winning by drawing dots on the probability stick, as illustrated here.

Any dots placed at or below the midpoint should be challenged by other students. Since Bruce is leading, his probability of winning is definitely greater than ½.

Draw a large square on the board, as illustrated here.

● Rita
● Bruce



T: ***Let's use this square to calculate Bruce's chances of winning when the score is 7-8. What could the next score be?***

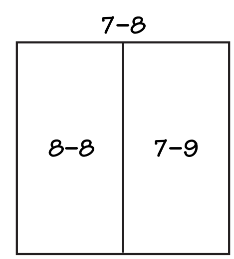
S: ***8-8 if the red marble is chosen, or 7-9 if the blue marble is chosen.***

T: ***Which next score is more likely, 8-8 or 7-9?***

S: ***They are equally likely, since there is one red marble and one blue marble.***

Divide the square in half; indicate half for 8-8 and half for 7-9.

● Rita
● Bruce



T: ***Now we must consider these two games, one with a score of 8-8, the other with a score of 7-9. First, let's pretend that the score is 7-9. If the score is 7-9, what could the next score be?***

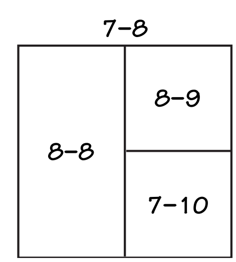
S: ***8-9 if the red marble is chosen, or 7-10 if the blue marble is chosen.***

T: ***These two scores, 8-9 and 7-10, are equally likely; therefore, let's divide the region for 7-9 in half with one half for 8-9 and the other half for 7-10.***

What do we know if the score in a game is 7-10?

S: ***Bruce has won. Color that region blue for Bruce.***

● Rita
● Bruce

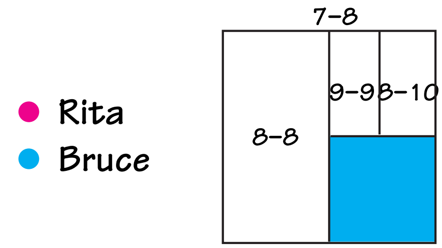


T: *If the score is 8-9, what could the next score be?*

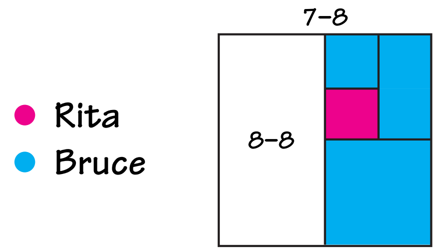
S: *9-9 or 8-10. Divide the region for 8-9 in half; one half 9-9 and the other half for 8-10.*

T: *What happens if the score is 8-10?*

S: *Bruce wins. Color that region blue.*



S: *If the score is 9-9, the next score could be 9-10 or 10-9. Divide the region for 9-9 in half and color one half red for Rita and one half blue for Bruce.*



T: *We still must consider a game when the score is 8-8.*

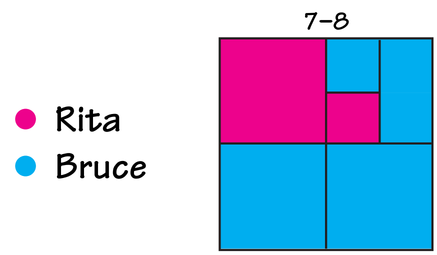
S: *The next score could be 9-8 or 8-9. Divide the region for 8-8 in half, one half 9-8 and the other half 8-9.*

T: *We could do that. When the score is 8-8, who is favored, Rita or Bruce?*

S: *Neither. Each player has a probability of ½ of winning.*

T: *Therefore, we can divide the region for 8-8 in half and color one half red and the other half blue.*

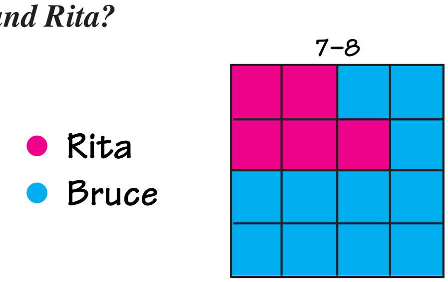
When the score in a game is 7-8, the blue regions represent Bruce's chances for winning and the red regions represent Rita's chances for winning. From the picture, can we tell who is more likely to win?



S: *Bruce, because more of the square is blue than red.*

T: *How can we calculate the chances of winning for Bruce and Rita?*

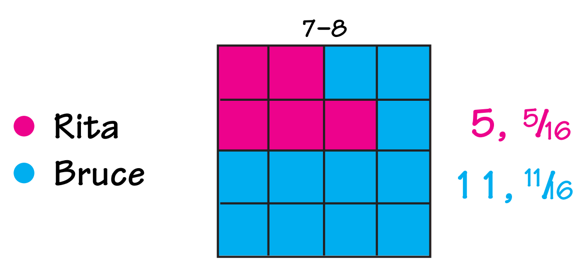
Invite class discussion. Encourage any suggestions to compare the areas shaded red and blue by choosing a small piece and counting how many of those pieces fit in the blue region and how many fit in the red region. It may be useful to subdivide the larger regions.



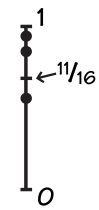
T: *The square shows that Rita has 5 chances of winning while Bruce has 11 chances of winning. Bruce is heavily favored to win.*

S: *Since 5 + 11 = 16, Rita's probability of winning is $\frac{5}{16}$ and Bruce's probability of winning is $\frac{11}{16}$.*

Record the chances and probabilities for winning near the square.



Refer back to the probability stick on the board on which students drew dots for their estimates of Bruce's probability of winning earlier in this lesson.



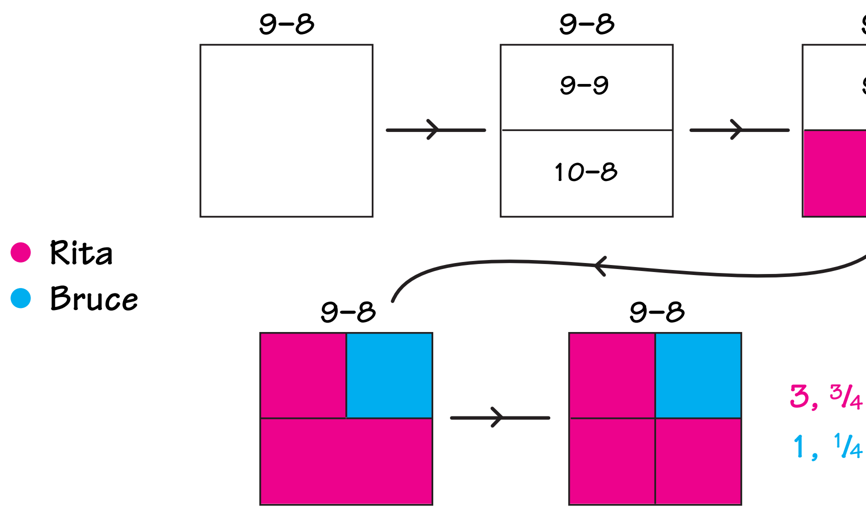
T: ***Whose dot is closest to $^{11}/_{16}$, Bruce's probability of winning?***

Let students discuss where $^{11}/_{16}$ is located on the line segment Invite students to use a meter stick to divide the segment into 16 pieces of the same length and to locate $^{11}/_{16}$. Determine whose estimate is closest.

Distribute copies of Worksheet P7.

T: ***Now let's solve the actual problem Pascal and Fermat worked on. Pretend Rita and Bruce each bet 50¢ on a game. The game stops when Rita leads 9-8. Use the square to calculate each player's probability of winning. Use these probabilities to determine how they should split the $1.00 pot.***

Allow several minutes for students to work independently. Then, using student suggestions, proceed gradually to solve the problem collectively at the board. Allow time between steps for students to complete the problem on their own. The squares drawn below show one procedure to calculate the chances. Your class might divide and color the square in a different way, but they should arrive at the same result.



T: ***How should Rita and Bruce divide the $1.00 pot when the game stops at 9-8?***

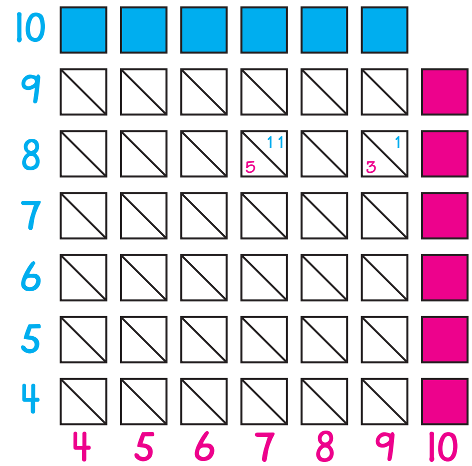
S: ***$0.75 for Rita and $0.25 for Bruce because ¾ x 1.00 = 0.75 and ¼ x 1.00 = 0.25.***

Display *IG-IV Probability Poster* #1. Record the results for the score 9-8 in the square that is in the 9-column (for Rita's score) and the 8-row (for Bruce's score). (See the next illustration.)

T: ***On this poster we can record the chances in our problems. In addition to the score 9-8, we also calculated Rita's and Bruce's chances of winning when Bruce leads 7-8. Who remembers the chances?***

S: ***When Bruce leads 7-8, he has 11 chances of winning and Rita has 5 chances of winning.***

Invite a student to record this result for 7-8 in the chart. Explain to the class that the result for 7-8 gets recorded in the square in the 7-column and the 8-row.



T: ***In later lessons, we will complete this chart by finding Bruce's and Rita's winning chances for many other starting scores. Hopefully, we can find some shortcuts and patterns so that we do not need to divide and color a square for every entry in the chart.***

Save the poster for use again in Lessons P8 and P9.

Capsule Lesson Summary

Review the story and results from Lesson P7 concerning a 16th century probability problem. Develop efficient techniques for determining the winning probabilities of the two players given different intermediate scores.

Materials

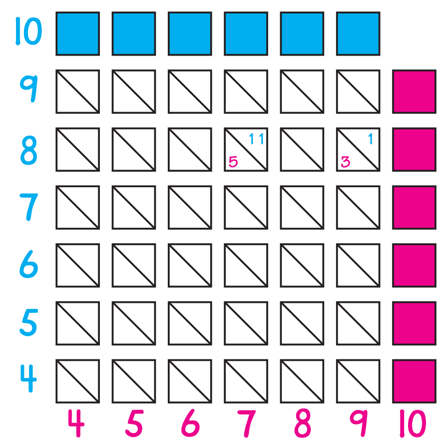
Teacher
- *IG-IV Probability Poster* #1
- One red and one blue marble
- Calculator
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets P8* and **

Description of Lesson

Exercise 1

Display *IG-IV Probability Poster* #1 with the entries made in Lesson P7. If necessary, make the following entries on a clean copy of the poster.



T: ***Who remembers the story about the famous probability problem Pascal and Fermat solved in the 1650s? What game did Rita and Bruce play that was similar to that problem?***

Let students recall details of the game and the history of the problem.

Students should mention the following aspects:

- The game for Rita and Bruce uses one red and one blue marble.
- Rita scores one point each time a red marble is chosen; Bruce scores one point each time a blue marble is chosen.
- The first player to get ten points wins.

Write this information on the board.

- Rita: 7
- Bruce: 6

T: ***Pretend that the score in a game is 7-6. What could the next score be?***

S: ***8-6 if a red marble is chosen, or 7-7 if a blue marble is chosen.***

Mix one red and one blue marble behind your back. Put one marble in each hand, and let a student select a hand, look at that marble, and give either Rita or Bruce a point. Record the new score on the board.

Continue in a similar manner until either Bruce or Rita wins by reaching ten points. At each turn, ask students for the next two possible scores.

P8

Direct the class's attention to the poster.

T: ***Last week we solved some problems and recorded the results on this poster.***

Point to the square for the score 7-8 (in the 7-column and 8-row).

T: ***What do red 5 and blue 11 in this square mean?***

S: ***If the score in a game is 7-8, Rita has 5 chances of winning as compared to Bruce's 11 chances of winning.***

T: ***What is each player's probability of winning?***

S: ***$\frac{5}{16}$ for Rita and $\frac{11}{16}$ for Bruce, since 11 + 5 = 16.***

T: ***How should Rita and Bruce split a prize pot if they stop the game at 7-8?***

S: ***Divide the amount in the pot into 16 equal parts. Give Rita 5 parts and Bruce 11 parts.***

S: ***Rita receives $\frac{5}{16}$ of the pot, and Bruce receives $\frac{11}{16}$ of the pot.***

At this point, you may like to choose an amount for the pot (say $1.00) and ask students to calculate Rita's share and Bruce's share. Depending on the amount chosen, students can use a calculator to divide the pot into 16 equal parts and then give 5 parts to Rita and 11 to Bruce. For example, with a $1.00 pot Rita's share would be 31¢ and Bruce's share 69¢.

Direct students' attention to the square for 9-8 on the poster.

T: ***If the score is 9-8, what are Bruce's and Rita's chances of winning?***

S: ***Bruce has 1 chance of winning and Rita has 3.***

T: ***If we know Rita's and Bruce's chances of winning when the score is 9-8, do we also know the results for another score?***

S: ***Yes, for 8-9. Rita has 1 chance and Bruce has 3 chances to win.***

T: ***Why?***

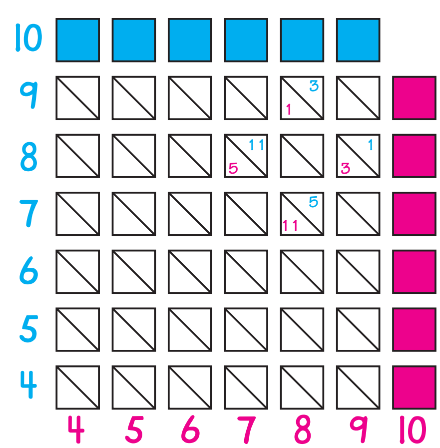
S: ***Since there is one red marble and one blue marble, Rita's advantage when the score is 9-8 is the same as Bruce's advantage when the score is 8-9.***

Invite a student to record the result for 8-9 on the poster.

T: ***Can we determine the result for another score?***

S: ***Yes, 8-7, because we already know the result for 7-8. When the score is 8-7, Rita has 11 chances of winning and Bruce has 5 chances.***

Invite a student to record this result for 8-7 on the poster.



Refer to the results for the scores 7-8 and 8-9 on the poster. You may rewrite them on the board.

7-8:	5	11
8-9:	1	3

T: ***Bruce is favored for each of these scores. Which of these two scores is better for Bruce?***

Encourage both intuitive answers based on the game situation and quantitative answers based on the chances.

T: ***8-9 is better for Bruce because in both situations Rita is one point behind Bruce. But when the score is 8-9, Rita has less time to catch up and win since Bruce is only one point from victory.***

S: ***For 8-9, Bruce has 3 chances out of 4 to win so his probability of winning is $\frac{3}{4}$. For 7-8, Bruce has 11 chances out of 16 to win, so his probability of winning is $\frac{11}{16}$. The 8-9 score is better for Bruce since $\frac{3}{4} = \frac{12}{16}$ and $\frac{12}{16}$ is greater than $\frac{11}{16}$.***

S: ***We calculated that for 8-9 Bruce would receive \$0.75 of a \$1.00 pot, while for 7-8 Bruce would receive about \$0.69. So 8-9 is better for Bruce.***

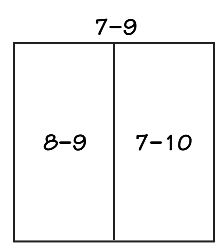
S: ***8-9 is the better score for Bruce, because he is three times more likely to win than Rita. When the score is 7-8, Bruce is only about two times as likely to win as Rita since 11 is just a little more than 2 x 5.***

Exercise 2

Draw a square on the board in preparation for calculating chances when the score is 7-9.

T: ***Let's compute Rita's and Bruce's probabilities of winning for some other scores at which they might stop playing. We will try to find some shortcuts as we solve these problems.***

One day Rita and Bruce stop playing when the score is 7-9. Let's calculate each player's chances of winning if they continue the game the next day. If the score is 7-9, what could the next score be?



S: ***8-9 if the red marble is chosen, or 7-10 if the blue marble is chosen. Divide the square in half, one half for 8-9 and the other half for 7-10.***

S: ***Bruce wins if the score is 7-10. Color that half blue.***

T: ***What do we know if the score is 8-9?***

S: ***The next score could be 9-9 or 8-10. We should divide the region for 8-9 in half.***

T: ***Yes, we could calculate Rita's and Bruce's winning chances when the score is 8-9 that way. But do we have to do all that work? What do we already know when the score is 8-9*** (refer to the poster square for 8-9)***?***

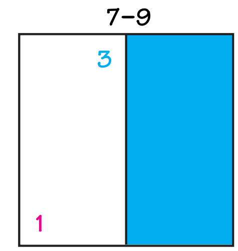
S: ***Bruce has 3 chances to win and Rita has only 1 chance.***

Invite a student to point to where this result is recorded on the poster.

T: ***So if we divided the region for 8-9 into smaller pieces, we know that we would get three blue pieces and one red piece.***

P8

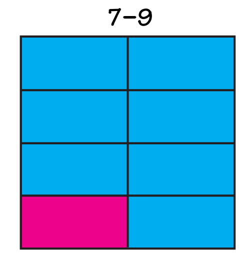
Record the information as shown here.



T: ***If we divide this uncolored region into three blue parts and one red part of the same size, what fraction of the whole square would be blue? Red?***

S: ***$\frac{7}{8}$ blue and $\frac{1}{8}$ red.***

Invite a student to show this by dividing the uncolored region into four parts and coloring them appropriately. Divide the large blue region in the same way the first half was divided. For example:

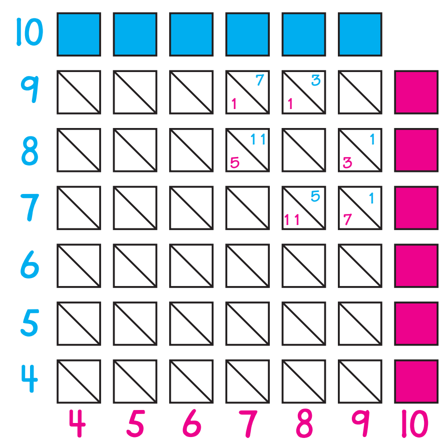


T: ***What are Rita's and Bruce's chances of winning when the score is 7-9?***

S: ***Bruce has 7 chances of winning and Rita has 1 chance.***

S: ***Bruce's probability of winning is $\frac{7}{8}$; Rita's is $\frac{1}{8}$.***

Ask a student to record this result for 7-9 on the poster.



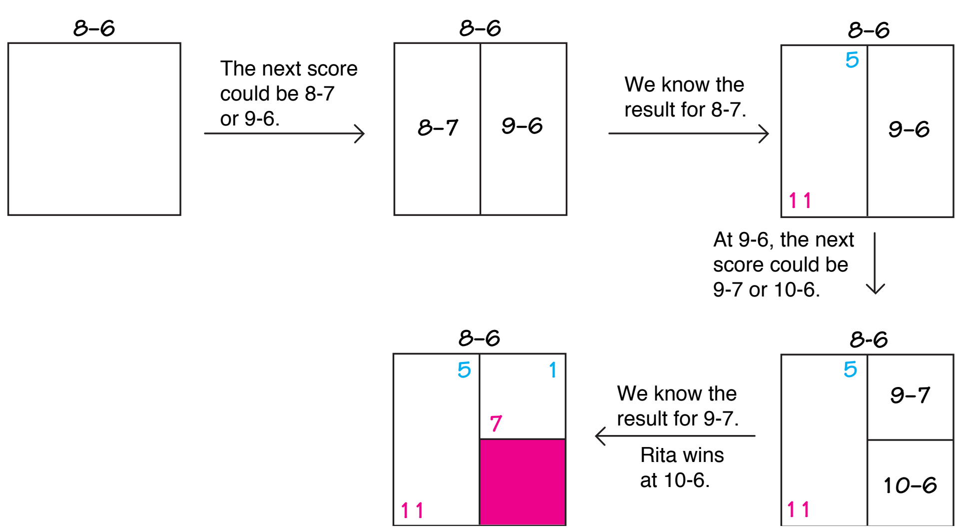
T: ***What other result can we record since we know the result for 7-9?***

S: ***9-7; Rita has 7 chances of winning and Bruce has 1.***

Invite a student to record the result for 9-7 on the poster.

Erase the board. Then draw a square on the board for a score of 8-6.

If your students are ready, let them solve this problem independently. Otherwise, let students lead you through the solution at the board. Insist that they use previous results when possible. The following illustration shows one procedure to calculate the probabilities. Key comments are included.



T: ***It should not be necessary to actually divide each region into small pieces, but let's imagine that we do it. How would we divide and color each region?***

S: ***Divide the region with blue 5 and red 11 into 16 equal-sized pieces; color five blue and 11 red.***

S: ***Divide the region with blue 1 and red 7 into eight equal-sized pieces; color one blue and seven red.***

S: ***Divide the red region into eight pieces, because that region is the same size as the region above it.***

Put 8 in red near the red region.

T: ***If we divide the regions in this way, would all of the small pieces be the same size?***

Let the class discuss this question until they agree that all of the pieces would be the same size.

T: ***In the whole square, how many small pieces would be blue?***

S: ***6 (5 + 1 = 6).***

T: ***How many would be red?***

S: ***26 (11 + 7 + 8 = 26).***

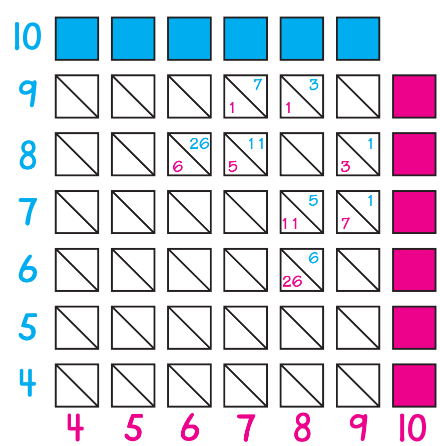
T: ***Therefore, when the score is 8-6, Rita has 26 chances of winning and Bruce has 6 chances. What is each player's probability of winning?***

S: ***$^{26}/_{32}$ for Rita and $^{6}/_{32}$ for Bruce since 6 + 26 = 32.***

T: ***Do we also know the chances of winning when the score is 6-8?***

S: ***For 6-8, Bruce has 26 chances of winning and Rita has 6 chances.***

Invite a student to record these results on the poster.



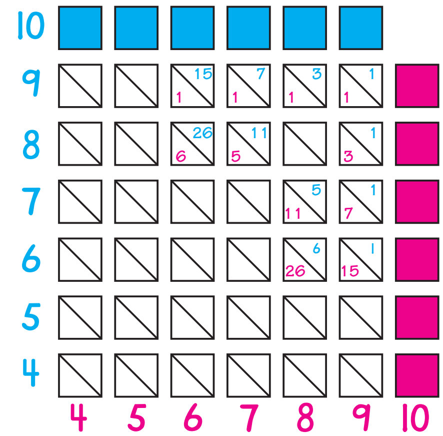
Distribute copies of Worksheets P8* and ** for individual work.

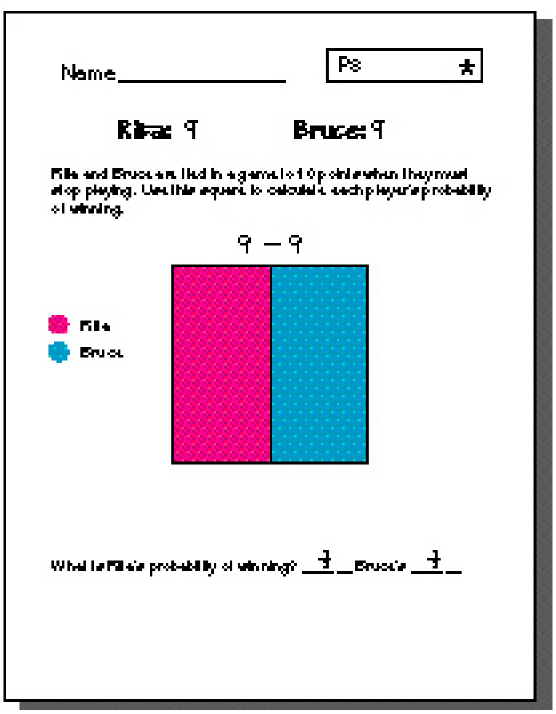
P8

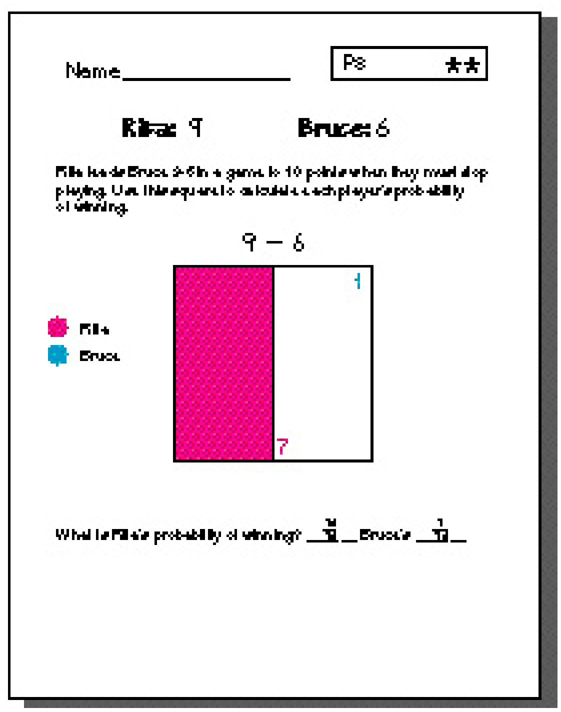
At the end of the lesson, invite students to record the results for 9-9, 9-6, and 6-9 on the poster.

T: ***Today we found a few shortcuts for filling in this chart. We learned that if we know the chances for one score, for example, 8-6, we also know the chances for 6-8. We also found out how to use previous results to help solve new problems. Next time we will complete the chart by finding some number patterns in this chart.***

Save the poster for use in Lesson P9.







Capsule Lesson Summary

Review the probability game from Lessons P7 and P8, and recall how to read the chart on the poster. Investigate some patterns in the chart and use them to complete the chart. Discuss and solve some problems about related probability games using the completed chart.

Materials

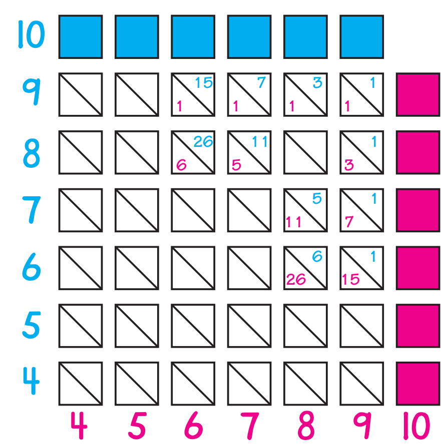
Teacher
- *IG-IV Probability Poster #1*

Student
- Colored chalk
- Paper
- Colored pencils, pens, or crayons
- Worksheet P9

Description of Lesson

Exercise 1

Distribute copies of Worksheet P9 and display *IG-IV Probability Poster #1*. The results from Lessons P7 and P8 should be recorded on the poster as they are on the worksheet.



T: ***Do you remember the game that Rita and Bruce were playing?***

Make certain that at least the following aspects of the game are mentioned.

- One red marble and one blue marble are used.
- Rita scores a point if the red marble is chosen, and Bruce scores a point if the blue marble is chosen.
- The first player to score ten points wins.

Point to the square for 6-8 on the poster.

T: ***What does this entry in the chart tell us?***

S: ***When Bruce leads 6-8 in the game, he has 26 chances of winning while Rita has only 6 chances.***

S: ***Bruce's probability of winning is $^{26}/_{32}$ and Rita's is $^{6}/_{32}$.***

T: ***How did we calculate these probabilities?***

Let students briefly describe how they used a square to calculate the probabilities.

T: ***When Rita leads 8-7, what are Rita's and Bruce's chances of winning?***

Let a student point to the square for 8-7 on the poster.

S: ***Rita has 11 chances to win and Bruce has 5 chances.***

S: ***Rita's probability of winning is $^{11}/_{16}$ and Bruce's is $^{5}/_{16}$.***

Point to the row of blue squares.

T: ***Why are these squares colored blue?***

S: ***Those squares are for scores of 4-10, 5-10, 6-10, and so on, all games that Bruce wins.***

Similarly, conclude that the column of red squares are for scores of games that Rita wins.

Exercise 2

Point to the square for 5-9 in the chart as you ask,

T: ***Let's try to find number patterns to predict what some other entries on the chart might be. For example, can you predict Rita's chances when the score is 5-9?***

S: ***Rita most likely has 1 chance of winning. There are only red 1s in that row.***

T: ***What do you think are Bruce's chances when the score is 5-9?***

S: ***31.***

T: ***What pattern did you use?***

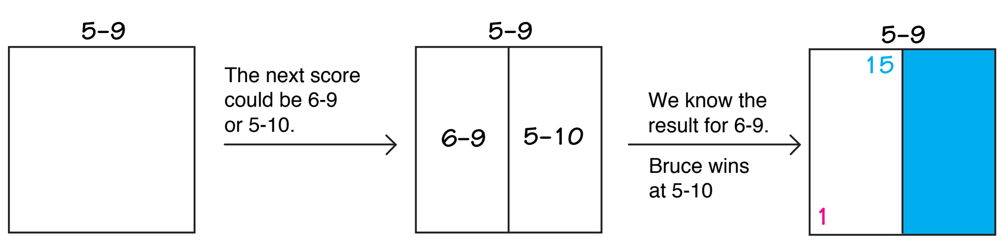
Elicit explanations from several students if they noticed different patterns.

S: ***The blue numbers in that row are 1, 3, 7, and 15. To get the next blue number, multiply the previous number by 2 and add 1. For example, (2 x 1) + 1 = 3, (2 x 3) + 1 = 7, and (2 x 7) + 1 = 15. Therefore, the next blue number is 31 since (2 x 15) + 1 = 31.***

S: ***I noticed that 1 + 2 = 3, 3 + 4 = 7, and 7 + 8 = 15. The number you add each time—2, 4, 8—is doubled. The next number should be 31, since 15 + 16 = 31 and 16 is the double of 8.***

S: ***I also noticed that 1 + 2 = 3, 3 + 4 = 7, and 7 + 8 = 15. The number you add each time is one more than the previous number. The next number should be 31, since 15 + 16 = 31 and 16 is one more than 15.***

Draw a square on the board for 5-9, and invite students to tell you how to use the square to calculate the chances and check the predictions. For example:



T: ***What are Rita's and Bruce's chances of winning?***

S: ***1 for Rita and 31 for Bruce.***

T: ***Why?***

S: ***The left half of the square could be divided into 16 pieces: 1 red and 15 blue. To use the same size pieces, the right half must also be divided into 16 pieces, all blue. Then there would be 1 red piece and 31 (15 + 16 = 31) blue pieces.***

Invite students to record the results for 5-9 and 9-5 on the poster and on their worksheets.

Ask students to use patterns to predict the entry for 9-4.

S: ***Bruce has just 1 chance. All of the blue numbers in the 9-column are 1s.***

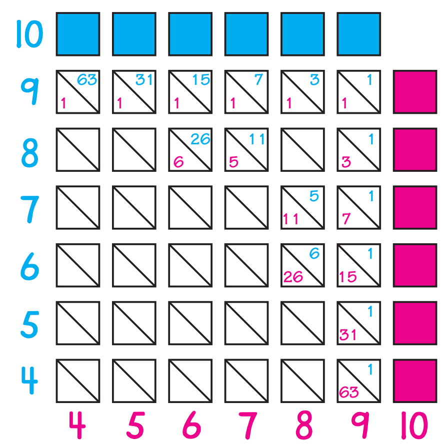
If students are using different patterns, encourage several explanations for Rita's chances.

S: ***63, since (2 x 31) + 1 = 63.***

S: ***63, since 31 + 32 = 63 and 32 is the double of 16.***

S: ***63, since 31 + 32 = 63 and 32 is one more than 31.***

Invite students to record the entries for 9-4 and 4-9 both on the poster and on their worksheets.



T: ***Can you predict Rita's and Bruce's chances when the score is 8-8?***

S: ***1 for Rita and 1 for Bruce because 8-8 is a fair game.***

T: ***That's correct. We could record the result for 8-8 as 1 chance for Rita and 1 for Bruce. But are there other ways to record chances for a fair game?***

S: ***We could record 2 chances for each player or 3 chances for each, or any number of chances as long as it is the same for each player.***

T: ***Do you see any patterns in our chart that might suggest a way to record the results for the score 8-8?***

S: ***If we enter a red 4, the red numbers in the 8-row increase by one: 3, 4, 5, 6.***

S: ***If we enter a blue 4, the blue numbers in the 8-column also increase by one: 3, 4, 5, 6.***

T: ***To maintain these patterns, we'll record the results for the score 8-8 as 4 chances for Rita and 4 chances for Bruce.***

Record the result on the poster.

T: ***Can you predict Rita's chances when the score is 5-8?***

S: ***7 chances, because the red numbers in the 8-row increase by one: 3, 4, 5, 6, 7.***

T: ***What are Bruce's chances when the score is 5-8?***

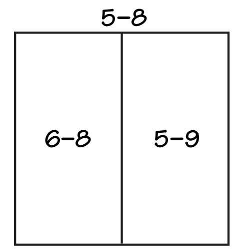
Record students' predictions and let them explain their answers.

T: ***On your own, draw a square on a piece of paper and calculate Bruce's and Rita's chances when the score is 5-8. Use previous results when possible. If you finish quickly, do the same for the score 8-4.***

You may need to assist some students and give hints to the class. After a while, collectively work with the class as follows.

T: ***Let's calculate the winning chances when the score is 5-8. How do we start?***

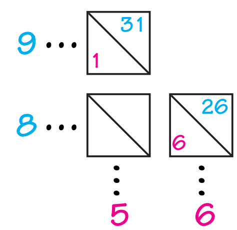
S: ***Divide the square for 5-8 in half; one half for 6-8 and the other half for 5-9.***



T: ***Do we know the results for the scores 6-8 and 5-9?***

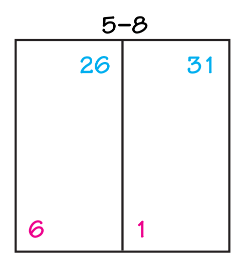
S: ***Yes, if the score is 6-8, Bruce has 26 chances to win Rita has 6 chances. If the score is 5-9, Bruce has 31 chances and Rita has 1 chance.***

Refer to this part of the chart as you point out the relationships among the squares for 5-8, 6-8, and 5-9.



T: ***To calculate the chances for 5-8, we can use the results recorded in this square for 6-8 and the results recorded in this square for 5-9.***

Record these results for 6-8 and for 5-9 in their respective regions of the large square on the board.



T: ***If we divided each half of the square into small pieces as indicated, would the pieces on both halves be the same size?***

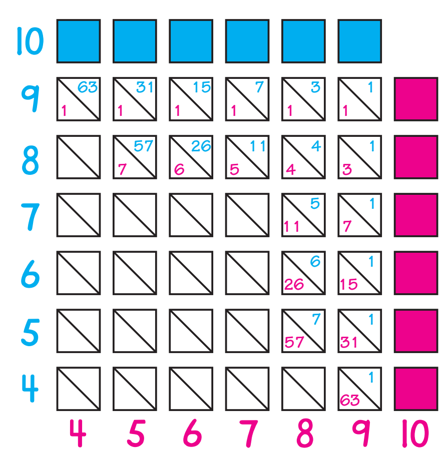
S: ***Yes. There would be 32 (26 + 6 = 32 and 31 + 1 = 32) pieces on each side.***

T: ***What are the winning chances for Bruce and for Rita when the score is 5-8?***

S: ***57 (26 + 31 = 57) for Bruce and 7 (6 + 1 = 7) for Rita.***

T: ***Which other result do we know immediately?***

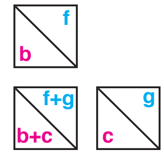
S: ***When the score is 8-5, Rita has 57 chances to win and Bruce has 7.***



Instruct students to record these results for 5-8 and 8-5 on their worksheets. Invite a student to fill in the same squares on the poster.

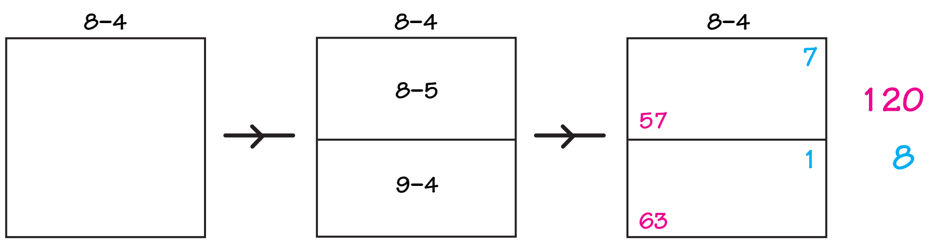
Check whether any students predicted 57 chances for Bruce. If so, ask them to explain how their predictions were made.

Note: The goal in the next part of the lesson is to lead students to observe a pattern that will allow them to quickly complete the rest of the chart. For your reference only, this illustration uses symbols to explain this pattern.

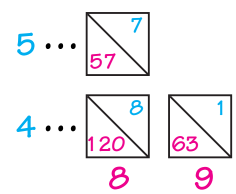


To find the results for a square when the results for the squares directly above and to the right are already known, simply add the numbers as indicated. The discussion of the score 5-8 has already suggested this technique. As soon as one or more students discover the rule, ask them to explain it to the class.

Allow a few more minutes for students to solve the problem for 8-4. Then solve the problem collectively at the board, and record the results on the poster. The following illustration indicates steps that can be followed.



Refer to this part of the poster, and emphasize that the solution was reached quickly because we knew the results for 8-5 and for 9-4.



Lead the class to observe that 57 + 63 is 120 (red numbers), 7 + 1 is 8 (blue numbers). That is, you can add the red numbers above and to the right, and the same for the blue numbers.

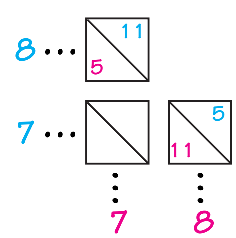
Encourage students to observe that this pattern works for these scores that have already been recorded: 5-8, 6-8, 8-4, 8-5, and 8-6.

T: ***Let's use this pattern to calculate the results for some more scores. For example, what are Rita's and Bruce's winning chances when the score is 7-7?***

S: ***Rita and Bruce both have 16 chances to win.***

T: ***Why?***

Direct a student to point to the appropriate part of the poster.

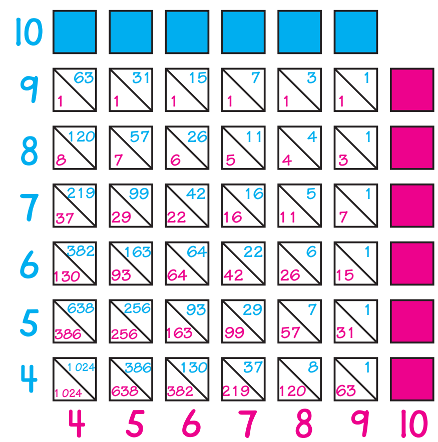


S: ***Add blue 11 and 5 to get Bruce's winning chances for 7-7, and add red 5 and 11 to get Rita's winning chances for 7-7.***

A student might observe that when the score is tied at 7-7, Rita and Bruce have the same chances to win and we could put 1 for Rita and 1 for Bruce. If a student suggests this, agree that it is a correct answer, but note that this formulation of the answer does not follow the pattern just discovered and will not help us complete other squares in the chart.

T: ***Why do Rita and Bruce have the same number of winning chances, namely 16, when the score is 7-7?***

S: ***The game is tied at 7-7. Rita and Bruce must each have the same number of chances to win.***



Invite the class to use the pattern to quickly calculate the results for a couple other scores such as 6-7 and 5-7. Then direct students to use the pattern to complete their worksheets. As students are working, invite students with correct answers to complete the chart on the poster.

Exercise 3 (optional)

T: ***Now that we have completed the chart, let's use it to solve some related problems concerning this game.***

Write this information on the board.

9–7: 7 1
8–6: 26 6

T: ***Rita is favored for both of the scores 9-7 and 8-6. If you were Bruce, would you prefer the score to be 9-7 or 8-6?***

S: ***8-6, because when the score is 8-6, Bruce's probability of winning is $\frac{6}{32}$. At 9-7, his probability of winning is $\frac{1}{8}$. $\frac{6}{32}$ is greater than $\frac{1}{8}$ since $\frac{1}{8} = \frac{4}{32}$.***

S: ***8-6, because then Bruce would have more time to catch up. For both scores, Bruce is two points behind Rita.***

S: ***8-6, because if the score is 9-7, Rita is seven times more likely to win than Bruce. But if the score is 8-6, Rita is only about four times more likely to win, since 26 is about 4 x 6.***

Erase the board and write this information.

Game to 100
97–96

T: ***Rita and Bruce are now playing the same game, but the first to score 100 points wins. If the score is 97-96, can you use the same chart to find Rita's and Bruce's chances to win?***

S: ***When the score is 97-96 in a game to 100, Rita needs three points to win and Bruce needs four points to win. This situation is the same as when the score is 7-6 in the game to 10; Rita needs three points and Bruce needs four points. Therefore, when the score is 97-96, Rita has 42 chances to win and Bruce has 22.***

You may need to lead to this observation by asking,

T: ***If the score is 97-96 and the goal is 100, how many points does Rita need to win?*** (3) ***Bruce?*** (4)

In their old game to ten points, is there ever a situation when Rita needs three points to win and Bruce needs four points to win?

S: ***Yes, when the score is 7-6. Therefore, at both a score of 97-96 in a game to 100 and at a score of 7-6 in a game to 10, Rita has 42 winning chances and Bruce has 22 winning chances.***

Write this information on the board.

<u>Game to 25</u>
22–24

T: ***On another day, Rita and Bruce set a goal of 25. The score in their game is 22-24. What are Rita's and Bruce's chances to win?***

S: ***7 for Bruce and 1 for Rita. When the score is 22-24, Rita needs three points to win and Bruce needs one point. That is the same as when the score is 7-9 in the game to 10.***

Write this information on the board.

<u>Game to Rita 15, Bruce 12</u>
10–10

T: ***One day Rita was winning more often than Bruce. They decided that in their next game Rita would have to score 15 to win and Bruce would only need to score 12. If the score is 10-10, what are their chances of winning?***

S: ***57 for Bruce and 7 for Rita. When the score is 10-10 in this game, Rita needs five points to win and Bruce needs two points to win. That is the same as when the score is 5-8 in the game to 10.***

Exercise 4 (optional)

T: ***Let's look for some other number patterns in this chart.***

Write this series of numbers on the board.

$\square$
7
11
16
22
29
37
$\square$

Point to the blue numbers in the 7-column on the poster.

T: ***These are the blue numbers from the 7-column. Try to find a pattern to this sequence of numbers. According to your pattern, what number would come before 7 and what number after 37? Write your answers on a piece of paper.***

Check several answers before asking students to explain their answers.

S: ***46 is after 37. I saw that 7 + 4 = 11, 11 + 5 = 16, 16 + 6 = 22, 22 + 7 = 29, and 29 + 8 = 37. The number you add is one more each time. The next number should be 46 because 37 + 9 = 46.***

P9

Highlight this pattern on the board.

S: ***4 is before 7. You must add 3 to some number to get 7. 4 + 3 = 7.***

$$\boxed{}, 7 \xrightarrow{+4} 11 \xrightarrow{+5} 16 \xrightarrow{+6} 22 \xrightarrow{+7} 29 \xrightarrow{+8} 37 \xrightarrow{+9} \boxed{46}$$

Erase the board and write this sequence of numbers.

$$\boxed{},\quad \boxed{},\quad 120,\quad 57,\quad 26,\quad 11,\quad 4,\quad 1$$

T: ***These are the blue numbers from the 8-row. Can you find a pattern?***

S: ***Starting from 1 at the right, you double 1 and add 2 to get 4. Then double 4 and add 3 to get 11. Then double 11 and add 4 to get 26. This continues; each time, you double the number and add one more than you did the last time.***

Following students' suggestions, highlight a pattern on the board.

$$\boxed{},\quad \boxed{},\quad \underset{(2\times57)+6}{120},\quad \overset{(2\times26)+5}{57},\quad \underset{(2\times11)+4}{26},\quad \overset{(2\times4)+3}{11},\quad \underset{(2\times1)+2}{4},\quad 1$$

T: ***On your paper, write what the next two numbers should be.***

S: ***247 comes next because (2 x 120) + 7 = 247.***

S: ***502 comes after 247 because (2 x 247) + 8 = 502.***