L Strand

The Languages of Strings and Arrows

LANGUAGES OF STRINGS & ARROWS TABLE OF CONTENTS

THE LANGUAGES OF STRINGS & ARROWS INTRODUCTION

Two fundamental modes of thought for understanding the world around us are the classification of objects into sets and the study of relationships among objects. In everyday life, we classify cars by brand (Ford, Chevrolet, Toyota, and so on) and we study relationships among people (Sally is Mark's sister, Nancy is Mark's cousin). Chemists classify elements by properties, and zoologists study predator-prey relationships. Similarly, mathematicians sort numbers by primeness, and they employ functions to model predicted relationships, for example, between inflation and unemployment.

Many of a child's earliest learning experiences involve attempts to classify and to discern relationships. A child classifies people by roles (the teacher, the doctor), and creates relationships between the smell and taste of foods. Part of language development depends on a child's repeated attempts to sort objects by function, and to relate words with things or events.

The role of sets and relations is so pervasive in mathematics, that perhaps the simplest definition of mathematics is "the study of sets and relations principally involving numbers and geometrical objects." Given the equally pervasive presence of these two notions in everyday life and in a child's experiences, it is natural that they should play a key role in an elementary mathematics curriculum. Yet the inclusion of classification and the study of relations require an appropriate language for representing and studying them. For that reason, *CSMP* develops the nonverbal languages of strings and of arrows.

The pictorial language of strings represents the grouping together of objects into sets. The pictorial language of arrows represents relations among objects of the same or different sets. Each of these languages permeates the different content strands of the *CSMP* curriculum, providing unity both pedagogically and mathematically. With continual use, the languages become versatile student tools for modeling situations, for posing and solving problems, and for investigating mathematical concepts.

The general aim of this strand is to present situations that are inherently interesting and thought provoking, and that involve classification or the analysis of relations. The activities emphasize the role of logical thinking in problem solving rather than the development of specific problem-solving techniques.

Classification: The Language of Strings

As the word implies, classifying means putting things into classes, or as the mathematician says, sets. The mathematics of sets can help students to understand and use the ideas of classification. The basic idea is simple: Given a set S and any object x, either x belongs to S (x is in S) or x does not belong to S (x is not in S). We represent this simple act of sorting—in or out—by using pictures to illustrate in and out in a dramatic way. Objects to be sorted are represented by dots, and the sets into which they are sorted are represented by drawing strings around dots. A dot inside the region delineated by a set's string is for an object in the set, and a dot outside a set's string is for an object not in the set.

This language of strings and dots provides a precise (and nonverbal) way of recording and communicating thoughts about classification. The ability to classify, to reason about classification, and to extract information from a classification are important skills for everyday life, for intellectual activity in general, and for the pursuit and understanding of mathematics in particular. The unique quality of the language of strings is that it provides a nonverbal language that is particularly suited to the mode of thinking involved in classification. It frees young minds to think logically and creatively about classes, and to report their thinking long before they have extensive verbal skills.

Relations: The Language of Arrows

Relations are interesting and important to us in our everyday lives, in our careers, in school, and in scientific pursuits. We are always trying to establish, explore, and understand relations. In mathematics it is the same; to study mathematics means to study the relations among mathematical objects like numbers or geometric shapes. The tools we use to understand the everyday world are useful to understand the world of mathematics. Conversely, the tools we develop to help us think about mathematical things often serve us in non-mathematical situations.

A serious study of anything requires a language for representing the things under investigation. The language of arrows provides an apt language for studying and talking about relations. Arrow diagrams are a handy graphic representation of a relation, somewhat the same way that a blueprint is a handy graphic representation of a house. By means of arrow diagrams, we can represent important facts about a given relation in a simple, suggestive, pictorial way—usually more conveniently than the same information could be presented in words. (See, for example, the storybook *A Valentine Mystery*.)

The convenience of arrow diagrams has important pedagogical consequences for introducing children to the study of relations in the early grades. A child can read—and also draw—an arrow diagram of a relation long before he or she can read or present the same information in words. The difficulty of presenting certain ideas to children lies not in their intellectual inability to grasp the ideas; rather, the limitations are often mechanical. Arrow diagrams have all the virtues of a good notation: they present information in a clear, natural way; they are attractive, colorful things to look at; they are easy and fun for children to draw. Students may use arrow pictures to study, test, and explain their thinking about concepts or situations under consideration. Discussion about an arrow picture often aids the teacher in clarifying a student's solution or misunderstanding of a problem.

Another educational bonus occurs when an arrow diagram spurs students' curiosity to investigate variations or extensions of the original problem. A minor change in an arrow picture sometimes reinforces a pattern already discussed and at other times suggests new problems to explore.

One of the purposes of this strand is to use the power of the language of arrows to help children think logically about relations. Again it must be remembered that our goals concern the thinking process and not the mechanism. The ability to draw prescribed arrows is not the objective in itself, nor is viewing an arrow diagram just another format for drill problems in arithmetic.

The general aim of the Languages of Strings and Arrows strand is to suggest situations that are inherently interesting and thought provoking, and to give children modes of thinking and appropriate languages with which they can organize, classify, and analyze. In addition to a varied assortment of lessons concerning sets and relations, this strand includes lessons involving systematic methods for solving combinatorial (counting) problems.

Content Overview

Counting and Logical Thinking

Two lessons in this strand involve students in some counting problems with sets and subsets. In all cases the language of strings provides a problem-solving strategy. Explicit attention is given to statements involving precise terms such as *exactly*, *at most*, and *at least*. Sometimes students are asked to identify statements as true, false, or can't tell. At other times they are asked to amend a string picture so that the statements are true.

Lessons: L1 and 6

The String Game

This semester *The String Game* is the main focus of four lessons. In all of these lessons, the class works collectively on a variety of analysis activities involving string game situations before actually playing the game. For example, a string picture is presented with numbers placed as clues about the string labels. After extracting all the possible information from the clues, students are asked if there is sufficient information to determine where particular numbers belong in the picture. Several variations of this activity appear in the four lessons. In collective analyses, students have an opportunity to share and further formulate playing strategies for the game.

In all four lessons, some of the analysis situations involve three-string pictures and the game is played with three strings. Also, there are hatching clues used in analysis and as starting clues in a game; knowing that one region of a string picture is empty or has a specified number of elements can powerfully eliminate many possibilities for the strings.

The String Game continues to provide rich situations that encourage logical reasoning, and further develop familiarity and ease with number properties.

Lessons: L2, 5, 7, and 12.

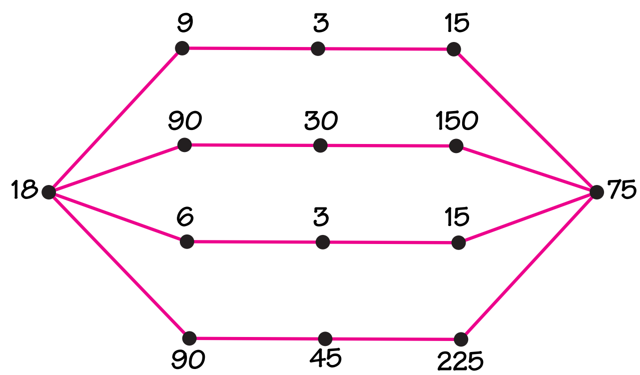
Distance

Four lessons in this strand concern the concept of distance. Traditionally, the only concept of distance in the elementary school curriculum is the well known geometric notion of Euclidean distance—distance "as the crow flies." However, there are many other distances of interest in mathematics. The general notion of distance as a special kind of function is an important mathematical concept.

In the *CSMP* curriculum, students begin thinking about distance (even in Kindergarten) in the simplified settings of taxi-geometry where distance is as a taxi-cab travels and is measured in terms of whole numbers by counting the number of city blocks traveled. Many of the properties of a distance function are experienced in the taxi-geometry activities of *CSMP Mathematics for the Upper Primary Grades*. Other kinds of distance functions are introduced this semester in the lessons on map distance and prime factor distance.

In the map distance lessons, students are asked to solve two problems based on minimizing and maximizing the distances between pairs of points. Further, they develop a method for finding the shortest route that visits all of the cities on the map and returns to the starting point. The prime factor distance lessons introduce a new kind of relation between pairs of whole numbers; that is, one number equals a positive prime number times the other. The prime factor distance is the number of cords in a shortest road between two numbers. Students explore the relation and the distance concept in a variety of problem situations.

Finding the prime factor distance between two numbers is mostly an investigation process for students since they are not yet equipped with ideas of prime factorization. However, for your information only, there is an “algorithmic” means of determining the prime factor distance between two whole numbers. First, for whole numbers b and c, divide each by their greatest common divisor. Next, look at the prime factorization of the results, $\frac{b}{b \sqcap c}$ and $\frac{c}{b \sqcap c}$. If there are n prime factors in the prime factorization of $\frac{b}{b \sqcap c}$ and m prime factors in the prime factorization of $\frac{c}{b \sqcap c}$, then the prime factor distance between b and c is n + m. For example, to find the prime factor distance between 18 and 75 (pfd (18, 75)) note that $18 \sqcap 75 = 3$ and $^{18}/_{3} = 6$ and $^{75}/_{3} = 25$. Next, look at the prime factorizations of 6 and 25; $6 = 2 \times 3$ and $25 = 5 \times 5$. Since there are two prime factors in each, pfd (18, 75) = 2 + 2 = 4. There are many shortest cord roads between 18 and 75 using the prime factor relation (a few are illustrated below), but all will have four cords.



Even though the relations in the map distance and the prime factor distance lessons are quite different, one easily notes the similar definitions of distance. The distance between two points on a map is defined as the length of a shortest road between them; the distance between two numbers is defined as the number of cords in a shortest road between them. This idea of distance underlies many road-building activities in previous parts of the *CSMP* curriculum.

Lessons: L3, 4, 10, and 11

Greatest Common Divisor and Least Common Multiple

Capsule Lesson Summary

Use string pictures to solve problems involving the number of objects in intersecting sets.

Materials

Teacher • Colored chalk

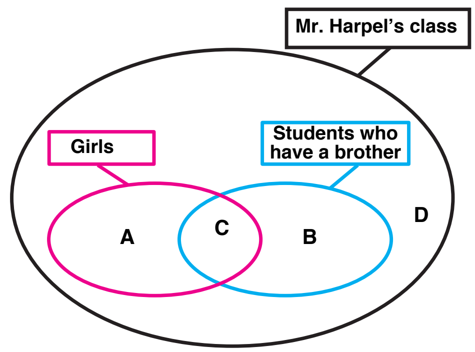
Student • Colored pencils, pens, or crayons • Paper • Worksheets L1(a) and (b)

Description of Lesson

Throughout this lesson, feel free to change the names or sets to fit the interest of your class.

Exercise 1

Draw this string picture on the board.



Write this information on the board as clues:

1) There are exactly 30 students in Mr. Harpel's class.
2) There are exactly 16 girls in the class.
3) Exactly 12 students in the class have brothers.
4) Exactly 7 girls in the class do not have brothers.

Then consider the four clues, one at a time.

T: ***What does the first clue tell us about the string picture?***

S: ***There are exactly 30 dots inside the string for Mr. Harpel's class.***

T: ***We don't know how many students are in each region (A, B, C, and D), but let's put 30 near the string for Mr. Harpel's class to remind us that there are a total of 30 dots inside this string.***

What does the second clue tell us about the string picture?

S: ***There are exactly 16 dots in the red string.***

T: ***Does this second clue tell us exactly how many dots are in region A?***

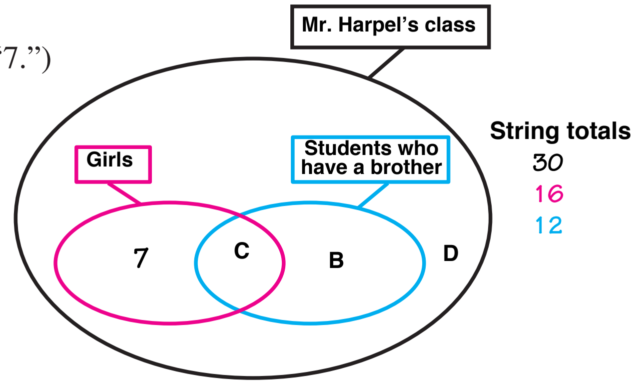
S: ***No. But there are a total of 16 dots in regions A and C together.***

T: ***Could there be three dots in region A?***

S: ***Yes. Then there would be 13 dots in region C.***

Consider the third and the fourth clues in a similar manner. The class should come to the following conclusions:

- There are 12 dots inside the blue string (in regions **B** and **C** together).
- Region **A** has 7 dots. (Replace "**A**" with "7.")

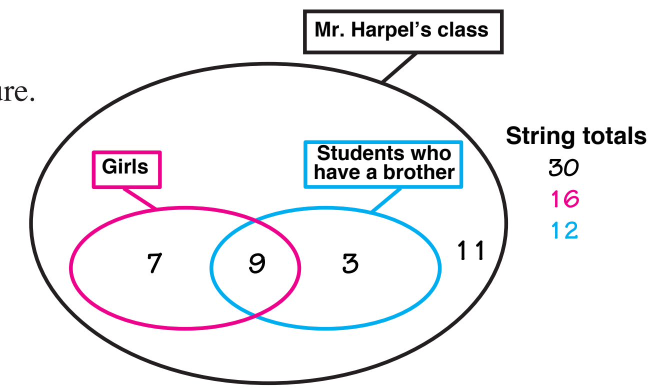


T: ***We know there are exactly 7 dots in region* A*. Can we use the clues to determine the exact number of dots in any other region?***

S: ***Since there are 16 dots total in the red string and 7 of these are in region* A*, there are 9 (16 – 7 = 9) dots in region* C*.***

Replace "**C**" with "9."

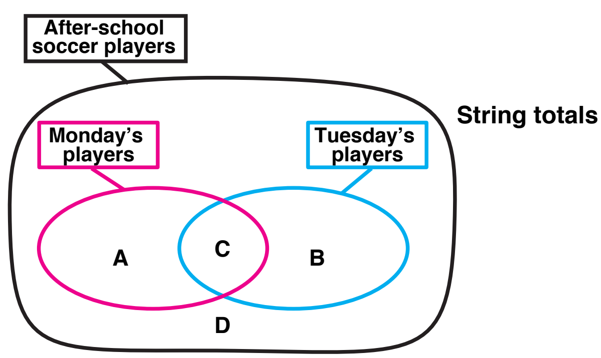
With similar arguments, let students use the first clue and the third clue to find how many dots belong in the other regions of the string picture.



Ask these questions, which require students to interpret the regions in the string diagram.

1) How many girls in the class have brothers? (9)
2) How many boys in the class do not have brothers? (11)
3) How many boys are in the class? (14 since 11 + 3 = 14 or 30 – 16 = 14)

Exercise 2

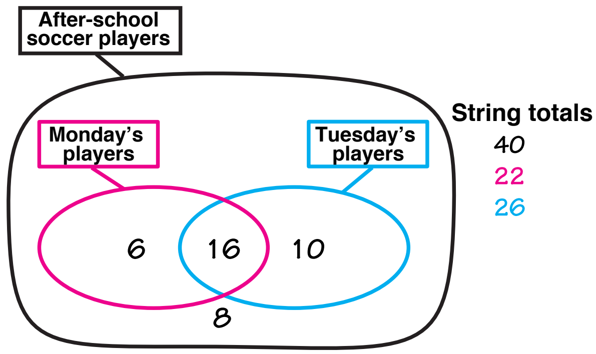


Distribute copies of Worksheet L1(a). Instruct students to work independently while you copy the string picture onto the board.

The clues from the worksheet are listed here for your convenience.

1) Exactly 40 students in the school play soccer after school hours.
2) Exactly 22 of them play soccer on Mondays. Some of these students also play soccer on other days.
3) Exactly 16 of them play soccer on both Mondays and Tuesdays.
4) Exactly 8 students play soccer after school, but neither on Mondays nor on Tuesdays.

Lead the class to a solution in a manner similar to that of Exercise 1. First, invite students to record the information obtained by considering the four clues, one at a time. Then, invite additional analysis to complete the solution.



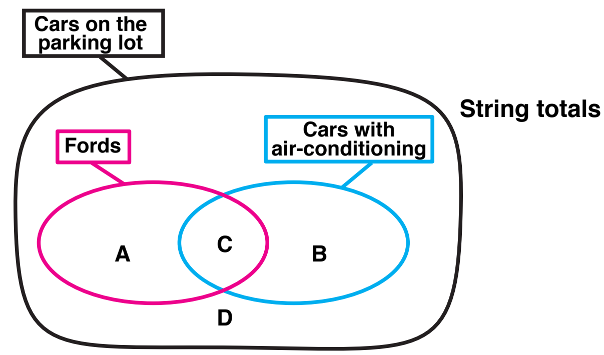
After solving the problem, collectively confirm that all four clues are now satisfied.

Ask the following questions.

1) How many students play soccer on Mondays but not on Tuesdays? (6)
2) How many students play soccer on Tuesdays? (26; 10 + 16 = 26)
3) How many students play soccer after school, but not on Tuesdays? (14; 6 + 8 = 14 or 40 – 26 = 14)

Exercise 3

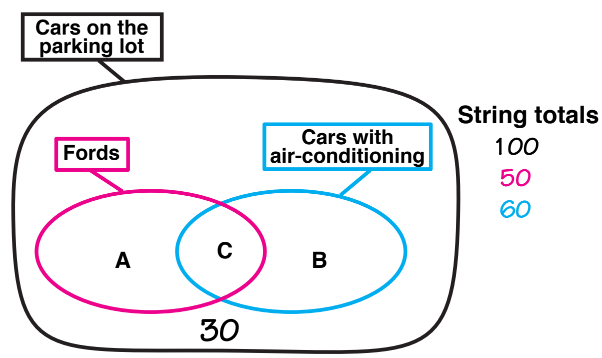
Invite students to study the problem on Worksheet L1(b) while you copy the string picture onto the board.



The clues from the worksheet are given here for your convenience.

1) There are exactly 100 cars on the parking lot.
2) Exactly 30 cars on the lot are not Fords and do not have air-conditioning.
3) Exactly 50 cars on the lot are Fords.
4) Exactly 60 cars on the lot have air-conditioning.

Invite students to record the information obtained by considering the four clues, one at a time.



T: ***We know that of the 100 cars on the parking lot, 30 are in region* D. *Where must we put the other 70 cars?***

S: ***In regions* A, B, *and* C.**

Let students discuss the problem and propose solutions. If a solution in not found in a couple of minutes, suggest the following method.

T: ***Let's use trial and error. What happens if we try 20 cars in region* A?**

S: ***If there are 20 cars in region* A, *then there are 30 cars in region* C *since there are 50 cars in the red string.***

S: ***If there are 30 cars in region* C, *there are also 30 cars in region* B *since there are 60 cars in the blue string.***

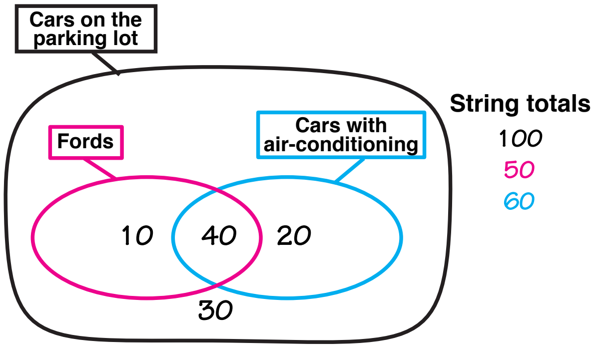
Record these (incorrect) answers in the string picture.

T: ***Is this a correct solution? Let's check all four clues.***

S: ***No, it's not correct. 20 + 30 + 30 + 30 = 110, but there are only 100 cars on the parking lot.***

S: ***Putting 20 cars in region* A *was not a correct guess.***

Erase the incorrect numbers in the string picture, and suggest that students work on their own to solve the problem. After several minutes, invite students to present a correct solution at the board.



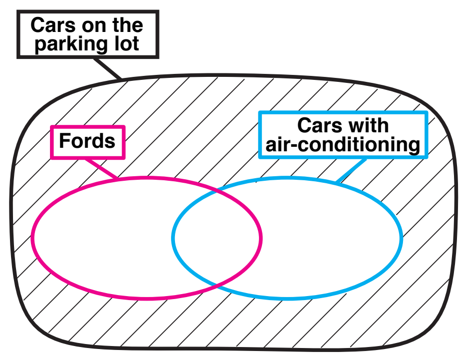
Check that this solution does satisfy all four clues. If some students believe they have found a different solution, let them present it and then let students find a clue that is not satisfied.

Erase only the numbers in the string picture.

Exercise 4

T: ***Suppose all of the cars on the parking lot are either Fords or have air-conditioning. How could we indicate this information in the picture?***

S: ***Hatch the region outside both the red and blue strings but inside the string for cars on the parking lot.***

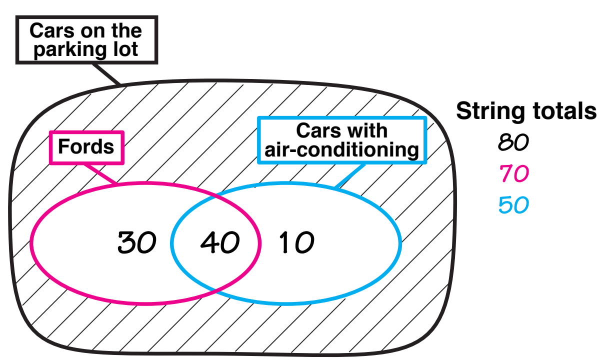


T: ***There are 80 cars on the parking lot. I'll tell you the number of Fords and the number of cars with air-conditioning. You must determine the number of Fords with air-conditioning.***

Put this information in a table on the board.

Cars on the parking lot	Fords	Cars with air-conditioning	Fords with air-conditioning
80	70	50	

Suggest that students draw the picture on their paper and determine the number of dots in each region. Encourage students who do not seem to have their own method of solution to experiment by guessing the number of cars in any one region. After a while, invite a student with a correct solution to tell you the number of dots in one region and let other students complete the solution.



Confirm that the string totals are correct. Let students disprove any other proposed solutions. Record 40 in the fourth column of the table.

Present the following problems, and let students work on their own. (Answers are in boxes.)

Cars on the parking lot	Fords	Cars with air-conditioning	Fords with air-conditioning
80	70	50	40
80	70	70	$\boxed{60}$
80	50	40	$\boxed{10}$

After a while, invite students to present their solutions and explain their methods. Do not insist on perfectly clear explanations. A student might propose the following method with reference to one of the problems; for example, the problem with string totals of 80, 70, and 70.

S: ***70 + 70 = 140 which is 60 more than 80. Therefore, 60 dots have been counted twice, so they must go in the middle region.***

Extension Activity

Add the following information to the table on the board.

Cars on the parking lot	Fords	Cars with air-conditioning	Fords with air-conditioning
80			$\square$
80			$\triangle$

Challenge your students to solve these two problems:

- Find the number of Fords and the number of air-conditioned cars that makes $\square$ as large as possible.
- Find the number of Fords and the number of air-conditioned cars that makes $\triangle$ as small as possible.

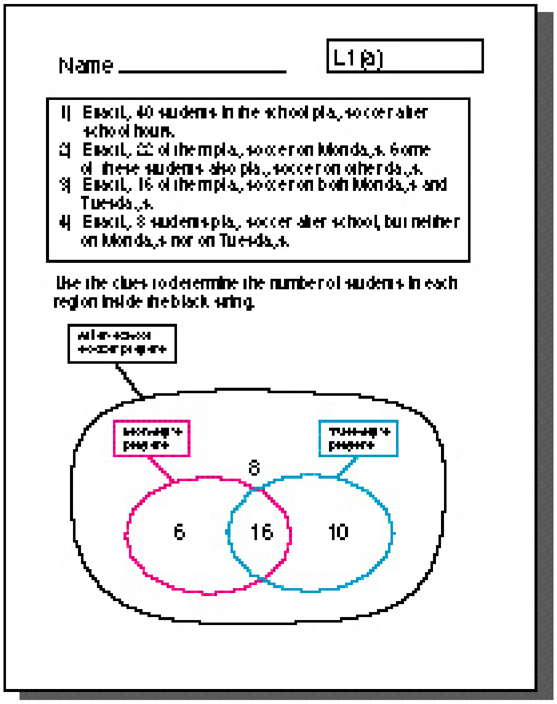
There is only one solution for the first problem, but many for the second. For example:

Cars on the parking lot	Fords	Cars with air-conditioning	Fords with air-conditioning
80	80	80	$\boxed{80}$
80	40	40	$\triangle$ 0

Any two whole numbers whose sum is 80 will yield 0 for the triangle.

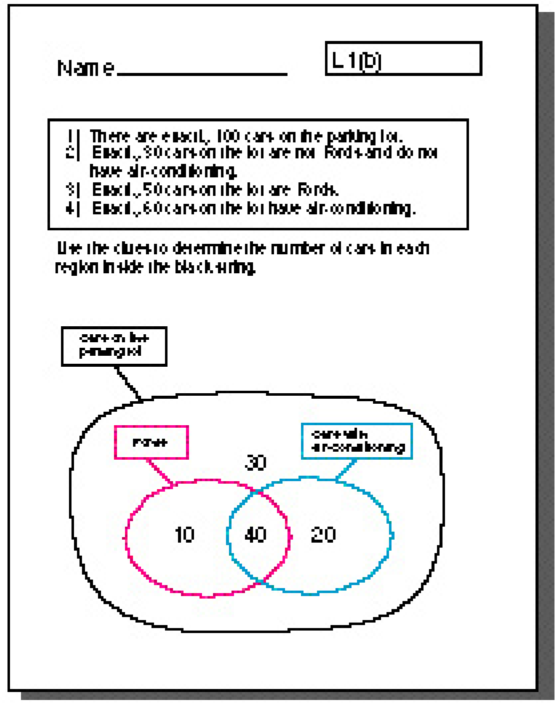
Home Activity

Send home a problem similar to that on Worksheet L1(a) for students to solve with a family member.


Name ____________ L1(a)

1) Exactly 40 students in the school play soccer after school hours.
2) Exactly 22 of them play soccer on Mondays. Some of these students also play soccer on other days.
3) Exactly 16 of them play soccer on both Mondays and Tuesdays.
4) Exactly 8 students play soccer after school, but neither on Mondays nor on Tuesdays.

Use the clues to determine the number of students in each region inside the black string.


Name ____________ L1(b)

1) There are exactly 100 cars on the parking lot.
2) Exactly 30 cars on the lot are not Fords and do not have air-conditioning.
3) Exactly 50 cars on the lot are Fords.
4) Exactly 60 cars on the lot have air-conditioning.

Use the clues to determine the number of cars in each region inside the black string.

Capsule Lesson Summary

Determine two string labels from three clues given one at a time. Analyze the information obtained from a series of three clues in a three-string picture, and then play *The String Game* to determine the string labels. Play *The String Game* with numbers.

Materials

Teacher
- Colored chalk
- Numerical String Game kit
- Numerical 3-String Game poster
- Marker or crayon

Student
- 3-String Game analysis sheet
- Colored pencils, pens, or crayons

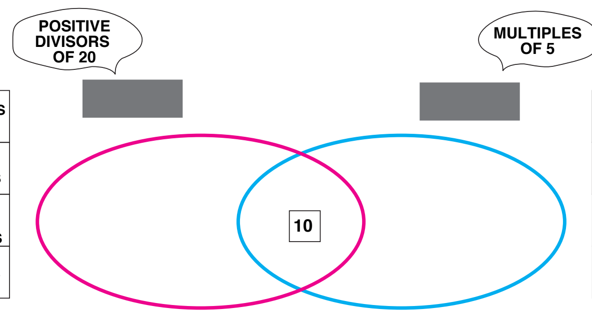
Description of Lesson

Exercise 1

Set up your board for *The String Game* with numbers as shown below. (Bubbles indicate what is on the hidden labels.)

Red

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN 10	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN 10	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27



Blue

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN 10	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN 10	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27

T: ***Suppose we are preparing to play The String Game with numbers. Here is the list of possible labels for the red string*** (point to the poster on the left)***, and here is the list for the blue string*** (point to the poster on the right)***. 10 is inside the red string and inside the blue string. What information does this clue give us about the strings?***

S: ***The red string cannot be for* MULTIPLES OF 3*, because 10 is not a multiple of 3.***

T: ***Could the blue string be for* MULTIPLES OF 3*?***

S: ***No, because 10 is also inside the blue string.***

Let students suggest labels to cross out on the lists. Each time, ask for a clear explanation as to why a string cannot have a label before you cross it off. The class should quickly notice that this clue eliminates the same labels from both lists. Continue until there are only six label remaining for each string.

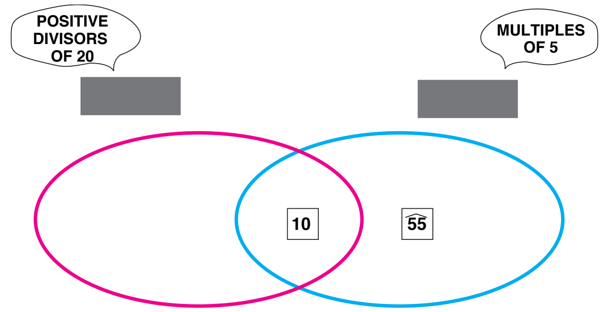
Red

MULTIPLES OF 2	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	~~LESS THAN 10~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	GREATER THAN 10	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

Blue

MULTIPLES OF 2	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	~~LESS THAN 10~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	GREATER THAN 10	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

T: ***A second clue is that*** $\widehat{55}$ ***is inside the blue string but outside the red string.*** (Place $\widehat{55}$ in the picture.) ***What new information does this give us?***



Cross off labels as students justify why they are no longer possibilities. With this clue the class should notice that when the blue string cannot have a particular label, then the red string <u>can</u> have that label, and vice versa. For example:

S: *The blue string cannot be for* **GREATER THAN** $\widehat{10}$*, because* $\widehat{55}$ *is less than* $\widehat{10}$*.*

T: *Could the red string be for* **GREATER THAN** $\widehat{10}$*?*

S: *Yes.* $\widehat{55}$ *is not in the red string, and 10, the number in the red string, is greater than* $\widehat{10}$*.*

When all of the information from this clue has been discussed, four possibilities should remain for the red string and two for the blue.

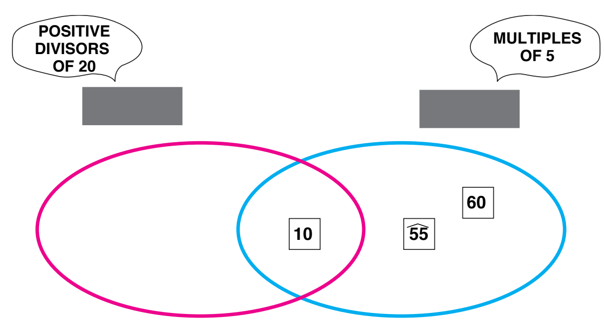
Red

MULTIPLES OF 2	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
MULTIPLES OF 10	~~LESS THAN 50~~	~~LESS THAN $\widehat{10}$~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	GREATER THAN $\widehat{10}$	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

Blue

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	MULTIPLES OF 5
~~MULTIPLES OF 10~~	LESS THAN 50	~~LESS THAN $\widehat{10}$~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	~~GREATER THAN $\widehat{10}$~~	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

T: ***A third clue is that 60 is in the blue string and not in the red string.*** (Place 60 in the picture.)



Let students eliminate labels from both lists until the red and blue strings have been determined.

Red

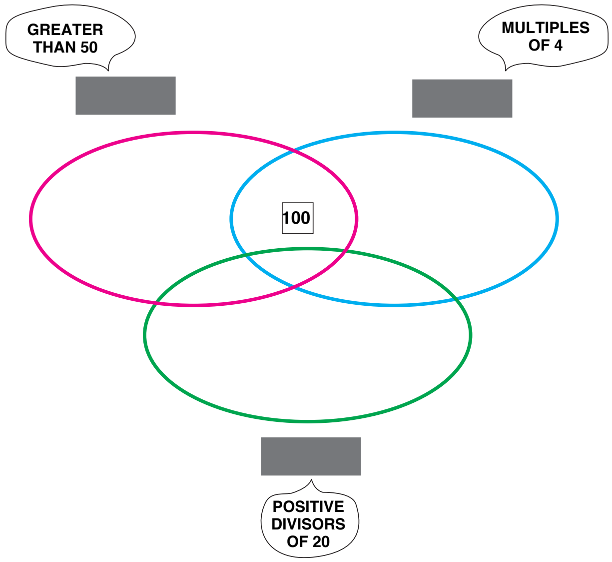
~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN $\widehat{10}$~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	~~GREATER THAN $\widehat{10}$~~	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

Blue

~~MULTIPLES OF 2~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 4~~	MULTIPLES OF 5
~~MULTIPLES OF 10~~	~~LESS THAN 50~~	~~LESS THAN $\widehat{10}$~~	~~ODD NUMBERS~~
~~POSITIVE DIVISORS OF 12~~	~~GREATER THAN 50~~	~~GREATER THAN $\widehat{10}$~~	~~POSITIVE PRIME NUMBERS~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 27~~

Exercise 2

Set up your board as illustrated below. Give students 3-String Game analysis sheets so that they can follow the class activity of eliminating impossible string labels.



RED	BLUE	GREEN
MULTIPLES OF 2	MULTIPLES OF 2	MULTIPLES OF 2
MULTIPLES OF 3	MULTIPLES OF 3	MULTIPLES OF 3
MULTIPLES OF 4	MULTIPLES OF 4	MULTIPLES OF 4
MULTIPLES OF 5	MULTIPLES OF 5	MULTIPLES OF 5
MULTIPLES OF 10	MULTIPLES OF 10	MULTIPLES OF 10
ODD NUMBERS	ODD NUMBERS	ODD NUMBERS
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS
GREATER THAN 50	GREATER THAN 50	GREATER THAN 50
LESS THAN 50	LESS THAN 50	LESS THAN 50
GREATER THAN 10	GREATER THAN 10	GREATER THAN 10
LESS THAN 10	LESS THAN 10	LESS THAN 10
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27

T: ***What information does this clue*** (point to 100 in the string picture) ***give us about the strings?***

Let students suggest labels to cross off on each list. The class should quickly notice that the same labels are eliminated for both the red string and the blue string. They should also observe that if the red and blue strings can have a particular label, then the green string cannot have that label, and vice versa. Encourage such observations and multiple cross-outs.

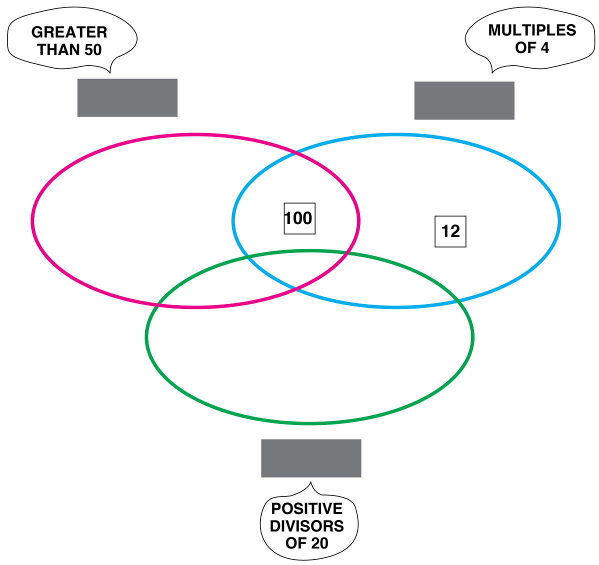
When all the information from this clue has been discussed, these labels should remain.

RED	BLUE	GREEN
MULTIPLES OF 2	MULTIPLES OF 2	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	MULTIPLES OF 3
MULTIPLES OF 4	MULTIPLES OF 4	~~MULTIPLES OF 4~~
MULTIPLES OF 5	MULTIPLES OF 5	~~MULTIPLES OF 5~~
MULTIPLES OF 10	MULTIPLES OF 10	~~MULTIPLES OF 10~~
~~ODD NUMBERS~~	~~ODD NUMBERS~~	ODD NUMBERS
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~	POSITIVE PRIME NUMBERS
GREATER THAN 50	GREATER THAN 50	~~GREATER THAN 50~~
~~LESS THAN 50~~	~~LESS THAN 50~~	LESS THAN 50
GREATER THAN 10	GREATER THAN 10	~~GREATER THAN 10~~
~~LESS THAN 10~~	~~LESS THAN 10~~	LESS THAN 10
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~	POSITIVE DIVISORS OF 12
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 18
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 24
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~	POSITIVE DIVISORS OF 27

Place 12 in the string picture as you announce,

T: ***12 is inside the blue string but outside both the red string and the green string. What information does this clue give us?***

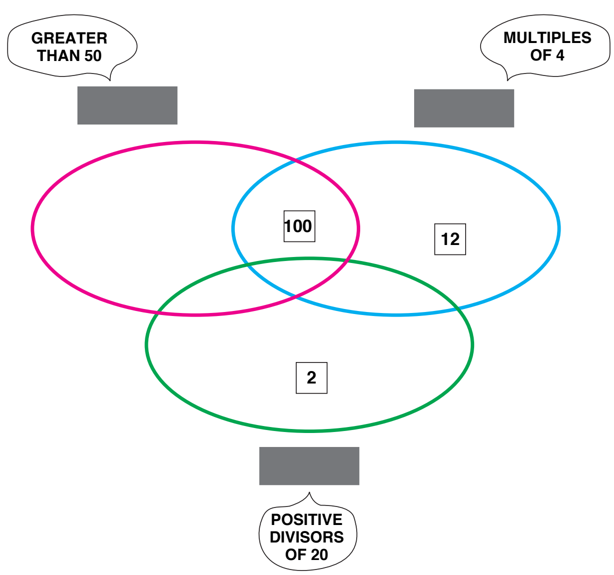
Proceed as before, collectively analyzing this clue.



RED	BLUE	GREEN
~~MULTIPLES OF 2~~	MULTIPLES OF 2	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	MULTIPLES OF 4	~~MULTIPLES OF 4~~
MULTIPLES OF 5	~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
MULTIPLES OF 10	~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
~~ODD NUMBERS~~	~~ODD NUMBERS~~	ODD NUMBERS
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~	POSITIVE PRIME NUMBERS
GREATER THAN 50	~~GREATER THAN 50~~	~~GREATER THAN 50~~
~~LESS THAN 50~~	~~LESS THAN 50~~	~~LESS THAN 50~~
~~GREATER THAN 10~~	GREATER THAN 10	~~GREATER THAN 10~~
~~LESS THAN 10~~	~~LESS THAN 10~~	LESS THAN 10
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 18
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~	POSITIVE DIVISORS OF 27

T: ***Another clue is that 2 is inside the green string but outside the red string and the blue string.*** (Place 2 correctly in the picture.)

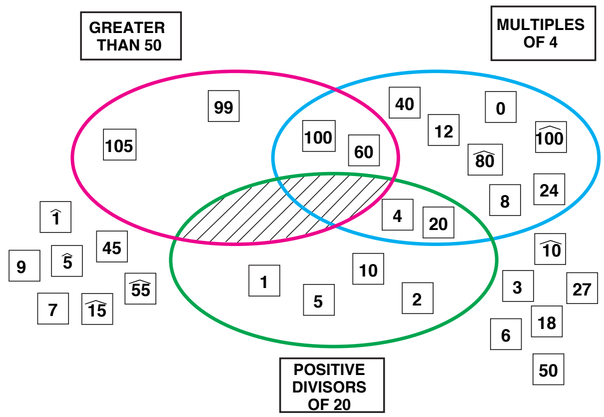
Proceed with a collective analysis of this new information.



RED	BLUE	GREEN
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	MULTIPLES OF 4	~~MULTIPLES OF 4~~
MULTIPLES OF 5	~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
MULTIPLES OF 10	~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
~~ODD NUMBERS~~	~~ODD NUMBERS~~	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~	POSITIVE PRIME NUMBERS
GREATER THAN 50	~~GREATER THAN 50~~	~~GREATER THAN 50~~
~~LESS THAN 50~~	~~LESS THAN 50~~	~~LESS THAN 50~~
~~GREATER THAN 10~~	~~GREATER THAN 10~~	~~GREATER THAN 10~~
~~LESS THAN 10~~	~~LESS THAN 10~~	~~LESS THAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 18
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

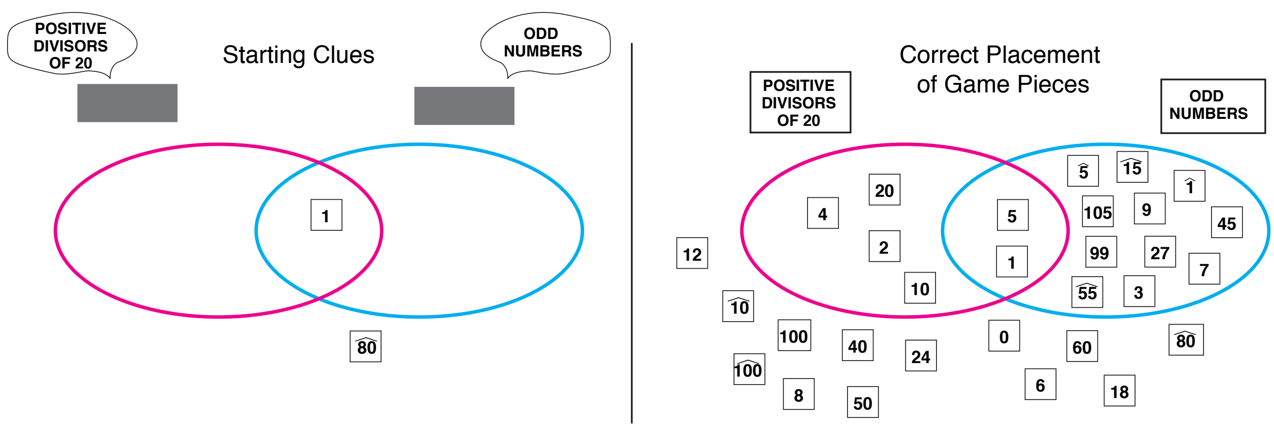
Let the class play a 3-string version of *The String Game* with numbers using the above situation as the starting point.

This illustration shows correct placement of all of the game pieces and can serve as a crib sheet for this game.



Exercise 3 (optional)

If time allows, play a 2-string version of *The String Game* with numbers in the usual way. The illustration below shows a possible game with two starting clues. Encourage students to use analysis sheets by allowing time between turns for them to cross out labels.



Writing Activity

You may like students to take lesson notes on some, most, or even all their math lessons. The "Lesson Notes" section in Notes to the Teacher gives suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students can note several facts about the attributes of numbers used in *The String Game*.

Capsule Lesson Summary

Calculate the lengths of some routes between points on a map where the lengths of the roads that make up the route are given. Define the notion of distance between points, and calculate the distance between each pair of points on the map.

Materials

Teacher
- Blackline L3
- Colored markers

Student
- Worksheets L3(a) and (b)

Advance Preparation: Make a transparency of the map on Blackline L3 for display, or carefully draw this map on the board.

Description of Lesson

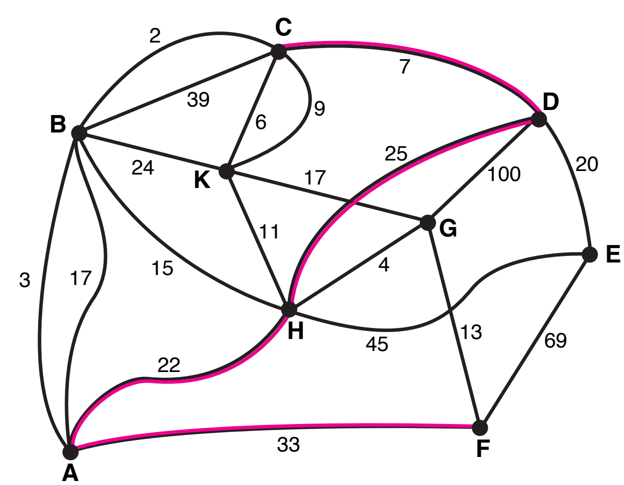
Note: In this lesson, a distinction is made between length and distance. Each route between two points has a *length*, which is calculated by adding the lengths of the roads that make up the route. The *distance* between two points is the length of a shortest route between the two points. Try to maintain this distinction as you teach the lesson.

Display the map on Blackline L3.

T: ***This is a map; the numbers are the lengths of roads between points. You might think that it is an unusual map. There are two roads between A and B; the length of one is 3*** (trace it) ***while the length of the other is 17*** (trace it)***. We aren't told, but maybe the short road is straight and the long road winds through mountains. Where else on this map do you see two different lengths for roads between two points?*** (Between **B** and **C** or between **C** and **K**).

Who can trace a route (way to go) from C to F?

Invite a student to trace and then highlight in red a route between **C** and **F**. One possibility is shown in this illustration.



T: ***What is the length of the route drawn in red?*** (87) ***How do you find the length?***

S: ***Add the lengths of the roads that make up the route: 7 + 25 + 22 + 33 = 87.***

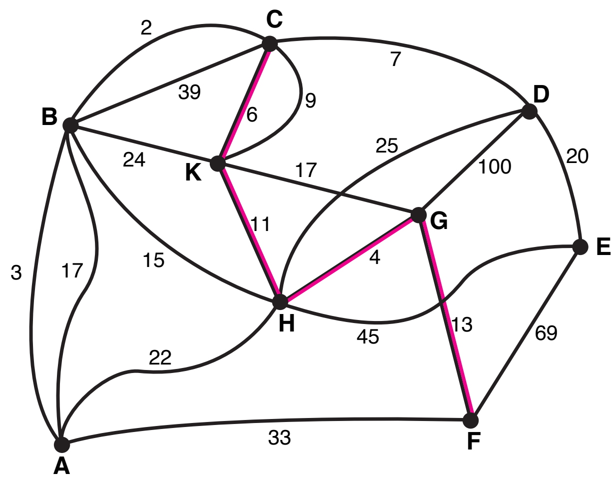
Record the student's name and the length of the route on the board, as illustrated here.

Beth: 87

Invite several students to trace routes between **C** and **F** while the class calculates their lengths. After each student traces a route, write the student's name and the length of the route on the board.

T: ***We've found several routes between C and F. Do you think that we've found a shortest route between C and F yet?***

Let students search for a shortest route until the one of length 34 shown here is suggested and until the class seems convinced that there are no shorter routes between **C** and **F**. Then confirm that 34 is the length of a shortest route.



T: ***This route with length 34 is a shortest route between* C *and* F*. Therefore, we say that the distance between* C *and* F *is 34.***

Write this definition on the board.

> The distance between two points on our map is the length of a shortest route between the two points.

T: ***This is how we write that the distance between* C *and* F *is 34.***

$d\,(C, F) = 34$

Distribute copies of Worksheets L3(a) and (b).

T: ***One of the worksheets has a copy of the map. The other has a table that we'll use to record the distances we find. Why is 34 already in the table twice?***

S: ***We found that the distance between* C *and* F *is 34. But the distance between* F *and* C *is also 34, so we record the answer twice.***

If necessary, explain how to read the table. In a number sentence on the board, record that the distances between two points taking either point first are the same.

$d\,(C, F) = 34 = d\,(F, C)$

T: ***Let's find the distance between two more points. Which two points would you like to use?***

Let the class find distances between other pairs of points that they suggest. (There is a worksheet answer key at the end of this lesson description.) On the board, record each distance the students find in number sentences, as shown above. Be sure to include the possibility that both points in a pair can be the same and that the distance from a point to itself is defined to be 0, for example, **d (F, F) = 0**. When a distance is determined, tell students to record that distance on their worksheets.

When you feel students can continue on their own, organize the class into groups of two or three to complete the table. In groups, students may like to assign different parts of the table to different members to complete. As you observe students' work, tell them when a number they have recorded does not represent the distance, that is, the length of a shortest route, between the two points. You may need to emphasize that sometimes, even when a direct road exists between two points, for example **B** and **K**, the road might not be a shortest route between those two points.

Note: Although there are a lot of entries in the table, at least half are duplicates and many are very quick to fill in. For example, **d (A, B) = 3** and **d (B, C) = 2**.

When most groups of students have completed the worksheet, or when there are about 10 minutes left in the class period, call the class back together.

T: ***For which pairs of points was it difficult to find a shortest route between them?***

When a student suggests a pair of points, for example, **E** and **G**, write the distance between that pair of points on the board.

d (E, G) = 48

Then challenge the class to find a route of that length between the points. Continue in this manner with other pairs of points. Check as many answers as time allows.

2

	A	B	C	D	E	F	G
A	0	3	5	12	32	33	22
B	3	0	2	9	29	32	19
C	5	2	0	7	27	34	21
D	12	9	7	0	20	41	28
E	32	29	27	20	0	61	48
F	33	32	34	41	61	0	13
G	22	19	21	28	48	13	0
H	18	15	17	24	44	17	4

L4 MAP DISTANCE #2

Capsule Lesson Summary

Review the notion of distance on the map from Lesson L3. Within a story about Snoopy, Woodstock, and a cat, solve two problems based on minimizing and maximizing the distances between pairs of points. Develop a method for finding the shortest route for a salesman to visit all of the cities on a map and to return home.

Materials

Teacher	• Colored marker • Calculator (optional) • Blackline L3	**Student**	• Worksheets L4(a), (b), and (c) • Colored pencils, pens, or crayons • Calculator (optional)

Advance Preparation: Make a transparency of the map on Blackline L3 for display, or carefully draw this map on the board.

Description of Lesson

Note: The same distinction between length and distance that was mentioned at the beginning of Lesson L3 applies to this lesson.

Exercise 1

Display the map on Blackline L3, and distribute copies of Worksheets L4(a) and (b) to students.

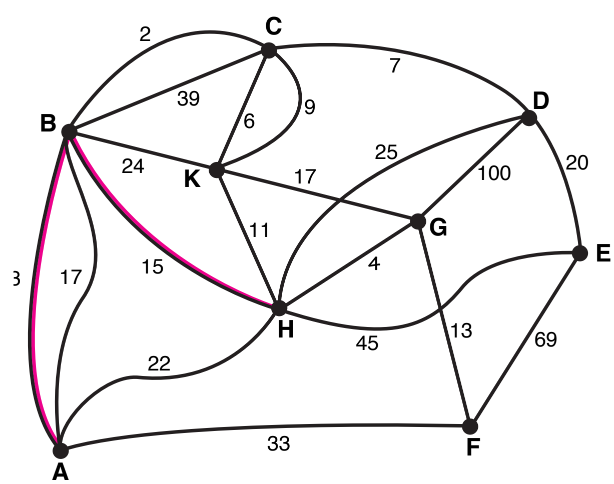
T: ***Do you remember this map that we used last week? How do we define the distance between two points on the map?***

S: ***The distance is the length of a shortest route between the two points.***

T: ***This map is on one side of your worksheet; on the other side there is a table of the distances between all pairs of points on the map. What is the distance between* A *and* H*?***

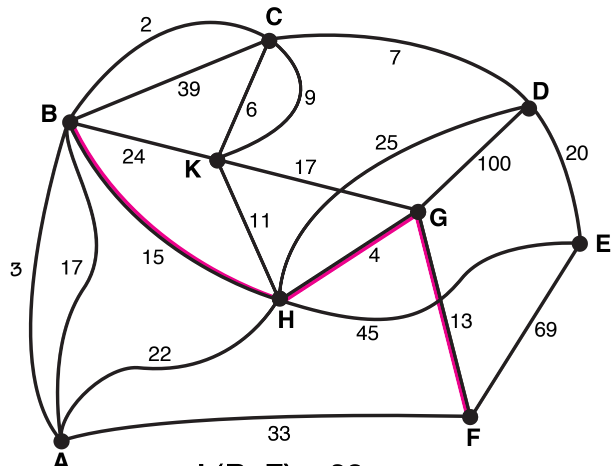
S: ***18.***

T: ***On the map, find a route from* A *to* H *of length 18.***

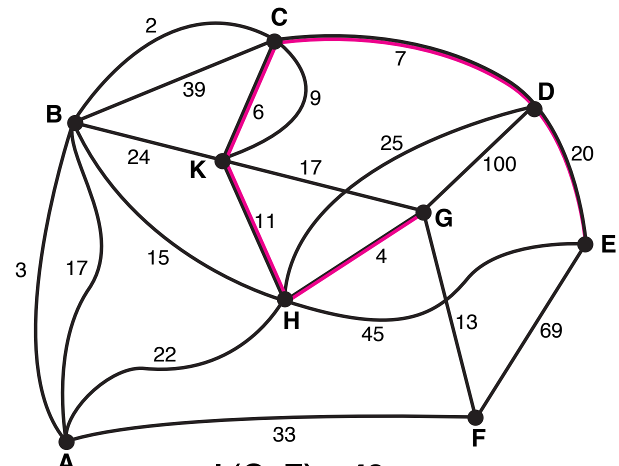


d (A, H) = 18

In a similar manner, invite students to find a shortest route between **B** and **F** (length: 32) and a shortest route between **G** and **E** (length: 48). Encourage students to read the table to find each distance, but insist that they then find a route on the map of that length between the pair of points. After several students have found a shortest route between a pair of points, let a student come to the board to trace a route.



d (B, F) = 32



d (G, E) = 48

T: ***We're going to use the map to solve two problems about some famous comic strip characters: Snoopy and Woodstock. What do you know about Snoopy and Woodstock?***

Let students briefly talk about the dog Snoopy and the bird Woodstock from the comic strip *Peanuts*.

T: ***Snoopy fears a ferocious cat, and Woodstock likes to be near Snoopy. Let's place the three animals at three different points on the map in order to meet these two concerns. First, find locations for Snoopy and the cat so that they are as far apart as possible. Then place Woodstock as close to Snoopy as possible.***

Write this information on the board.

First: Place Snoopy and the cat as far apart as possible.
Second: Place Snoopy and Woodstock as close as possible.

Make sure that the students understand the problem. Let students work individually or with a partner for a while. When many students have found the locations (Snoopy at **F**, Woodstock at **G**, and the cat at **E**), solve the problem collectively.

T: ***What should we do first?***

S: ***Place Snoopy and the cat as far apart as possible. I looked for the greatest distance in the table, which is 61, the distance between* E *and* F.**

Some students may claim to have found greater distances on the map. For example, someone may look at the route between **D** and **G** of length 100 and claim that **D** and **G** would be good locations for Snoopy and the cat. If this occurs, ask for the distance between **D** and **G**, and allow other students to point out that shorter routes exist between **D** and **G**. So the distance is not 100. In fact, the distance between **D** and **G** is recorded in the table as 28.

T: ***What can we conclude about Snoopy and the cat?***

S: ***Snoopy is at* E *and cat is at* F*, or the cat is at* E *and Snoopy is at* F*.***

T: ***We don't know yet whether Snoopy should be at* E *or at* F*. The second condition tells us about Woodstock. How does this help us find where Snoopy should be?***

S: ***Woodstock must be as close to Snoopy as possible.***

T: ***How do we use this information?***

S: ***Snoopy must be at* E *or* F*. To place Woodstock, find the point that is the shortest distance from* E *or* F*. Look at the two rows (or columns) of the table for* E *and* F*, and find the smallest number. It is 13, the distance from* F *to* G*. Therefore, Snoopy should be at* F*, Woodstock at* G*, and the cat at* E*.***

On the map, use red to label **G** Woodstock, **F** Snoopy, and **E** cat.

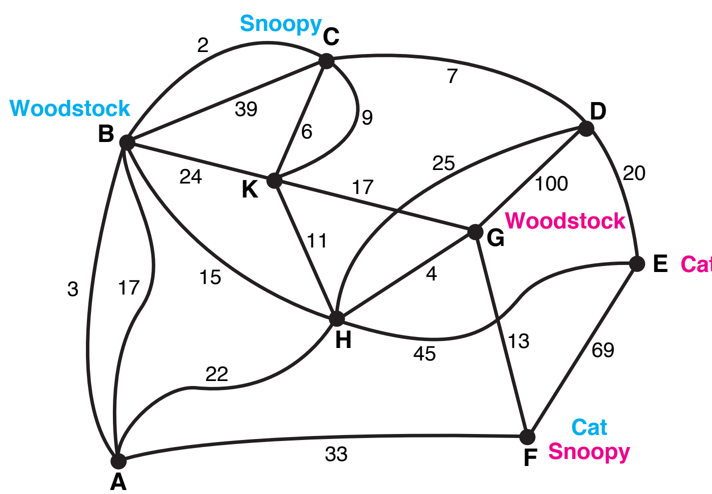
Switch "First" and "Second" in the statement of the problem.

Second: Place Snoopy and the cat as far apart as possible.
First: Place Snoopy and Woodstock as close as possible.

T: ***Let's switch our priorities. First, we'll put Snoopy and Woodstock close to each other and then place the cat far away. Do you think that this switch will affect the solution?***

Let students express their opinions.

Solve the problem in a way similar to the previous problem, letting students work independently or with a partner for a few minutes. Then discuss and solve the problem as a class. Determine that Woodstock is at **B**, Snoopy is at **C**, and the cat is at **F**. Write the solution in blue on your display map. Encourage students to compare the two problems and solutions.



Exercise 2 ______________

Display a clean copy of the map on Blackline L3.

T: ***Pretend the dots are for cities and a salesman lives at* A*. He must plan a route that starts at* A*, visits each city at least once, and returns home to* A*. Try to find his shortest possible route.***

Let students work individually or with partners. As you reiterate the problem, note that the salesman may visit a city more than once and may use a road as often as he wishes or not at all. As students work, occasionally compare their findings and record the length of the shortest route found thus far in the class. You may then challenge students to find an even shorter route.

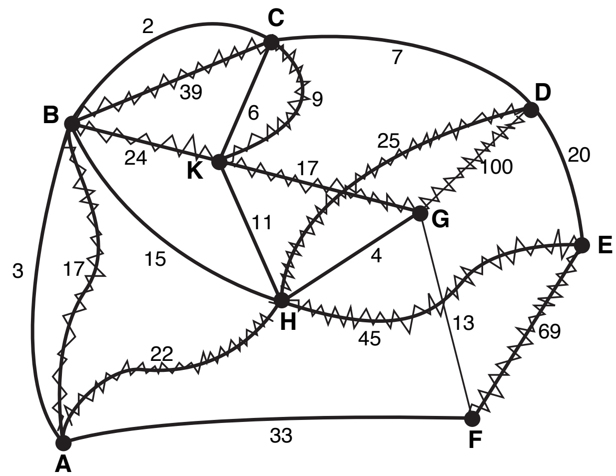
After a while, invite students to trace the shortest routes they discovered on the display map. Confirm that a route starts and ends at **A** and visits each city. Students may use a calculator to check the accuracy of the total length.

T: ***Do you think that this is the shortest route for the salesman? How could we check it?***

Let students give their opinions.

T: ***To help find a shortest route, let's simplify the map. Are there any roads the salesman would never take because a shorter route exists between the same two cities?***

Cross out roads as students justify why they would not be used.



S: ***He would never take the road of length 17 from* A *to* B *because there is another road of length 3 from* A *to* B*.***

S: ***Eliminate the road of length 24 from* B *to* K *because there is a route of length 8 (2 + 6 = 8) from* B *to* C *to* K*.***

S: ***Eliminate the road of length 69 from* F *to* E *because according to our table, the distance from* F *to* E *is 61.***

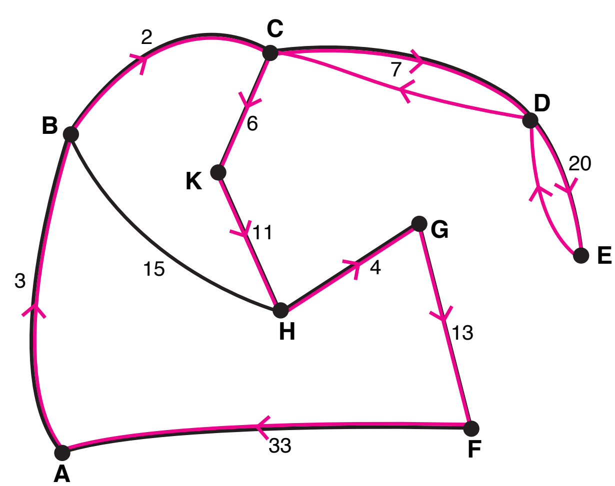
S: ***Yes, the route* F → G → H → K → C → D → E *has length 61.***

Continue until the class eliminates all of the roads crossed out here. Give hints as necessary.

Distribute copies of Worksheet L4(c).

T: ***On this worksheet the map has all of the roads erased that we know the salesman would not use. Use the new map to try to find the shortest route that starts at* A*, visits all of the cities, and returns to* A*.***

Again, let students work individually or with partners. At the end of the period, confirm that a shortest route for the salesman has length 126, and invite a student to draw such a route on the poster.



Capsule Lesson Summary

In a two-string picture, determine the string labels from three starting clues given all at one time. Analyze the information obtained from a series of four clues in a three-string picture, and then discuss how playing some of the remaining pieces could determine the string labels. Play *The String Game* with numbers.

Materials

Teacher
- Numerical String Game kit
- Colored chalk
- Numerical 3-String Game poster
- Marker or crayon

Student
- 3-String Game analysis sheets
- Colored pencils, pens, or crayons

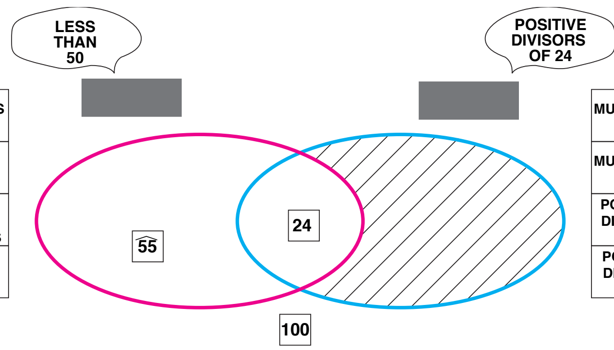
Description of Lesson

Exercise 1

Prepare your board for *The String Game* as shown below. You may like to give students String Game analysis sheets, so that they can follow the class activity of eliminating impossible string labels.

Red

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN $\widehat{10}$	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN $\widehat{10}$	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27



Blue

MULTIPLES OF 2	MULTIPLES OF 3	MULTIPLES OF 4	MULTIPLES OF 5
MULTIPLES OF 10	LESS THAN 50	LESS THAN $\widehat{10}$	ODD NUMBERS
POSITIVE DIVISORS OF 12	GREATER THAN 50	GREATER THAN $\widehat{10}$	POSITIVE PRIME NUMBERS
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 27

Remind the class that hatching indicates the region is empty.

T: ***The strings are determined by these three number clues and the hatching. What can we cross off the lists of possible labels?***

When possible, encourage students to eliminate several possibilities at once. Some sample responses follow.

S: ***The red string cannot be for the positive divisors of any number because there is a negative number ($\widehat{55}$) inside.***

S: ***Neither string can be for multiples of 10, 5, 4, or 2 because 100 is a multiple of each of those numbers and is outside of both strings.***

S: ***Neither string is for* GREATER THAN 50 *because 24 is inside both strings.***

Continue the analysis to determine the strings. Most likely your class will have no difficulty in determining that the red string is for **LESS THAN 50**, but they may hesitate as to whether the blue string is for **MULTIPLES OF 3** or for **POSITIVE DIVISORS OF 24**. A sample dialogue follows.

T: *We found that the blue string could be for* **MULTIPLES OF 3** *or for* **POSITIVE DIVISORS OF 24***. How can we decide which it is?*

S: *Suppose the blue string is for* **MULTIPLES OF 3***. There are multiples of 3 that are greater than 50, but that region in the picture is hatched. So the blue string cannot be for* **MULTIPLES OF 3***.*

T: *Name a multiple of 3 that is greater than 50.*

S: *60.*

T: *Is 60 less than 50?* (No) *So 60 is outside the red string.*

Is 60 a multiple of 3? (Yes) *So, if the blue string were for* **MULTIPLES OF 3***, 60 would be inside the blue string.*

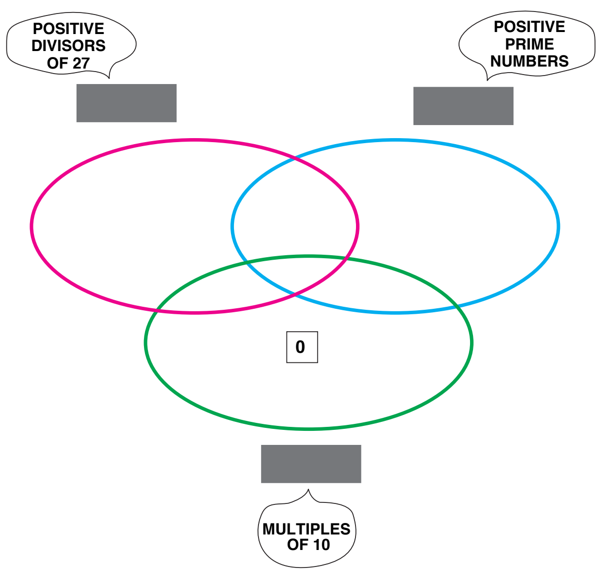
Where would 60 be in the picture? (In the hatched region) *But 60 cannot be there.*

What are some other numbers that would have no place in the picture if the blue string were for **MULTIPLES OF 3***?* (105, 99, any multiple of 3 greater than 50)

Conclude that the blue string is for **POSITIVE DIVISORS OF 24**.

Exercise 2

Set up your board as illustrated below. Give students 3-String Game analysis sheets so that they can follow the class activity of eliminating impossible string labels.



RED	BLUE	GREEN
MULTIPLES OF 2	MULTIPLES OF 2	MULTIPLES OF 2
MULTIPLES OF 3	MULTIPLES OF 3	MULTIPLES OF 3
MULTIPLES OF 4	MULTIPLES OF 4	MULTIPLES OF 4
MULTIPLES OF 5	MULTIPLES OF 5	MULTIPLES OF 5
MULTIPLES OF 10	MULTIPLES OF 10	MULTIPLES OF 10
ODD NUMBERS	ODD NUMBERS	ODD NUMBERS
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS
GREATER THAN 50	GREATER THAN 50	GREATER THAN 50
LESS THAN 50	LESS THAN 50	LESS THAN 50
GREATER THAN $\widehat{10}$	GREATER THAN $\widehat{10}$	GREATER THAN $\widehat{10}$
LESS THAN $\widehat{10}$	LESSTHAN $\widehat{10}$	LESS THAN $\widehat{10}$
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27

T: *What information does this clue* (point to 0 in the picture) *give us about the strings? Use the information from this clue to cross out some of the labels from your lists.*

After several minutes of individual work, let the class discuss why certain labels can be crossed out. Cross out these labels on the demonstration analysis sheets and ask students to cross them off their analysis sheets if they haven't already done so. Encourage the class to observe that, with this clue, the same labels can be crossed out for the red string and the blue string. For example:

S: *Cross out* **GREATER THAN $\widehat{10}$** *for the red string.*

T: ***Can we cross out this label for the blue string also?***

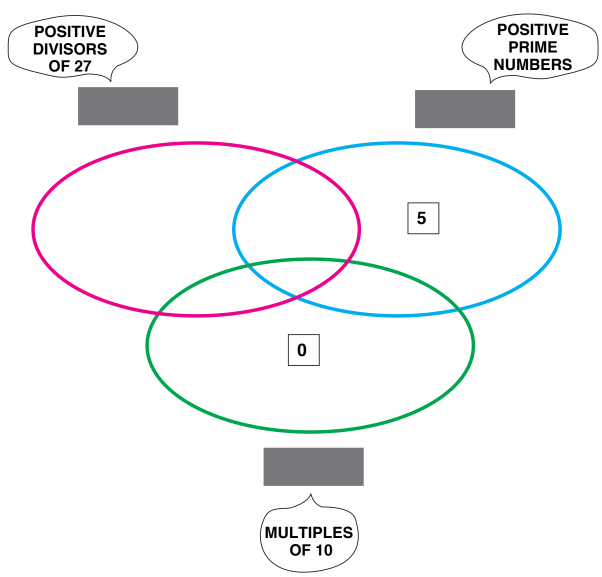
S: ***Yes, anything we cross out for the red string can also be crossed out for the blue string; 0 is outside both the red and blue strings, and there are no number clues inside these strings. It is the same situation for both strings.***

After all of the information from this clue has been discussed, these possibilities will remain for each string.

RED	BLUE	GREEN
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~	MULTIPLES OF 2
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	MULTIPLES OF 3
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~	MULTIPLES OF 4
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~	MULTIPLES OF 5
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~	MULTIPLES OF 10
ODD NUMBERS	ODD NUMBERS	~~ODD NUMBERS~~
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
GREATER THAN 50	GREATER THAN 50	~~GREATER THAN 50~~
~~LESS THAN 50~~	~~LESS THAN 50~~	LESS THAN 50
~~GREATER THAN 10~~	~~GREATER THAN 10~~	GREATER THAN 10
LESS THAN 10	LESS THAN 10	~~LESS THAN 10~~
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 18~~
POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~

T: ***The second clue is that the number 5 is in the blue string, but not in the red string or in the green string.*** (Put 5 in the picture.)

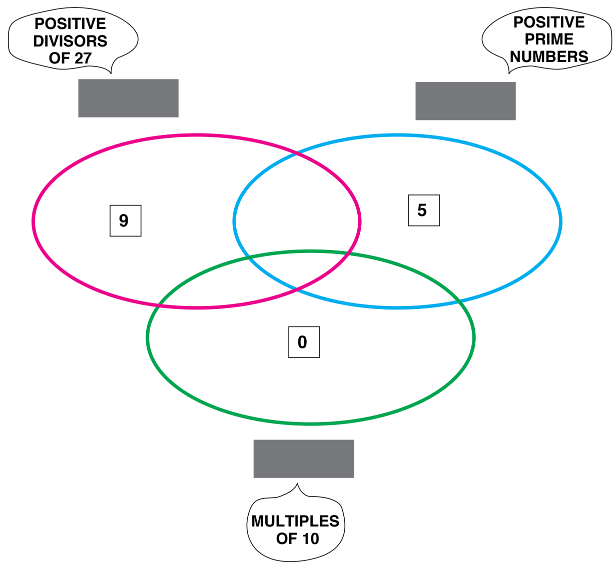
Consider the strings, one at a time, and ask students which additional labels can be crossed out on the lists. A student might suggest that the same labels be crossed out for both the red and the green strings. Such a suggestion is good, however, you should observe that some labels from one of the two strings may already be crossed out as a result of the previous clue, so they do not need to be crossed out again. These will be the remaining labels:



RED	BLUE	GREEN
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~	MULTIPLES OF 2
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	MULTIPLES OF 3
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~	MULTIPLES OF 4
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~	MULTIPLES OF 10
~~ODD NUMBERS~~	ODD NUMBERS	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
GREATER THAN 50	~~GREATER THAN 50~~	~~GREATER THAN 50~~
~~LESS THAN 50~~	~~LESS THAN 50~~	~~LESS THAN 50~~
~~GREATER THAN 10~~	~~GREATER THAN 10~~	~~GREATER THAN 10~~
LESS THAN 10	~~LESS THAN 10~~	~~LESS THAN 10~~
POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

T: ***The third clue is that the number 9 is in the red string, but not in the blue string nor in the green string.*** (Put 9 in the picture.) ***Which additional labels can we cross out?***

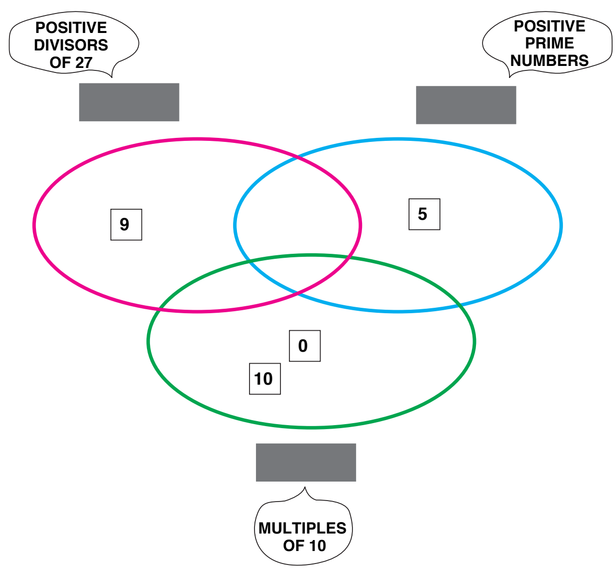
These will be the remaining labels.



RED	BLUE	GREEN
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~	MULTIPLES OF 2
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~	MULTIPLES OF 4
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~	MULTIPLES OF 10
~~ODD NUMBERS~~	~~ODD NUMBERS~~	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
~~GREATER THAN 50~~	~~GREATER THAN 50~~	~~GREATER THAN 50~~
~~LESS THAN 50~~	~~LESS THAN 50~~	~~LESS THAN 50~~
~~GREATER THAN 10~~	~~GREATER THAN 10~~	~~GREATER THAN 10~~
~~LESS THAN 10~~	~~LESS THAN 10~~	~~LESS THAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

T: ***The fourth clue is that the number 10 is in the green string, but not in the red string nor in the blue string,*** (Put 10 in the picture.) ***Which additional labels can we cross out?***

These will be the remaining labels:



RED	BLUE	GREEN
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~	MULTIPLES OF 2
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~	MULTIPLES OF 10
~~ODD NUMBERS~~	~~ODD NUMBERS~~	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
~~GREATER THAN 50~~	~~GREATER THAN 50~~	~~GREATER THAN 50~~
~~SMALLER THAN 50~~	~~SMALLER THAN 50~~	~~SMALLER THAN 50~~
~~GREATER THAN 10~~	~~GREATER THAN 10~~	~~GREATER THAN 10~~
~~SMALLER THAN 10~~	~~SMALLER THAN 10~~	~~SMALLER THAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITI VE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 1~~	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

At this point, the blue string has been determined; there are two possibilities remaining for the red string and two possibilities for the green string. Depending upon how well your students did with the first part of this exercise, you can proceed in either of two ways before continuing with Exercise 3.

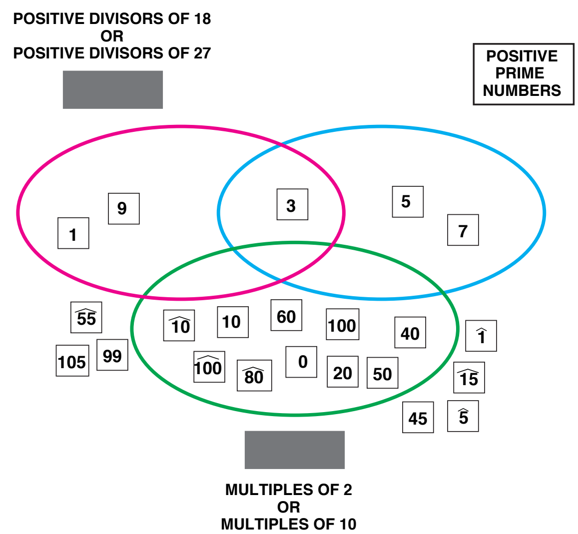
- Ask students which game pieces they would like to know the position of in the picture. Place the pieces, one at a time, analyzing the information from each correctly placed piece until the two remaining strings have been determined. (This is the easier of the two options.)
- Continue the analysis as in the following dialogue.

T: *We have two possibilities left for the red string:* **POSITIVE DIVISORS OF 18** *and* **POSITIVE DIVISORS OF 27***. We also have two possibilities for the green string:* **MULTIPLES OF 2** *and* **MULTIPLES OF 10***. The blue string is for* **POSITIVE PRIME NUMBERS***.*

Point to the game pieces (numbers) that are not in the string picture.

T: ***Even though we have not determined the labels for the red string and the green string, some of these pieces can be placed correctly in the string picture. Which ones?***

Let students give numbers, explain where they go in the picture, and place the pieces. There are 18 pieces that can be placed with the available information. This picture shows all the numbers whose placements are determined.



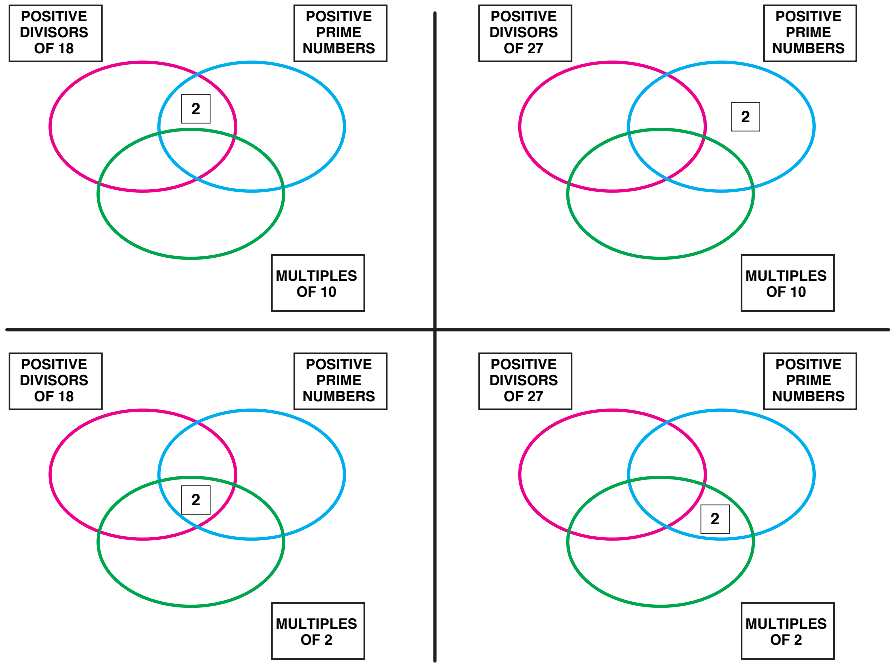
These numbers cannot be put in the picture without further information.

8	4	24	6
18	12	27	2

Move 2, 6, and 18 to a prominent location on the board.

T: ***Knowing the location of any of these three numbers would determine the red string and the green string. Where could 2 belong in the string picture?***

S: ***2 belongs in the blue string because 2 is a prime number, but we don't know where in the blue string.***

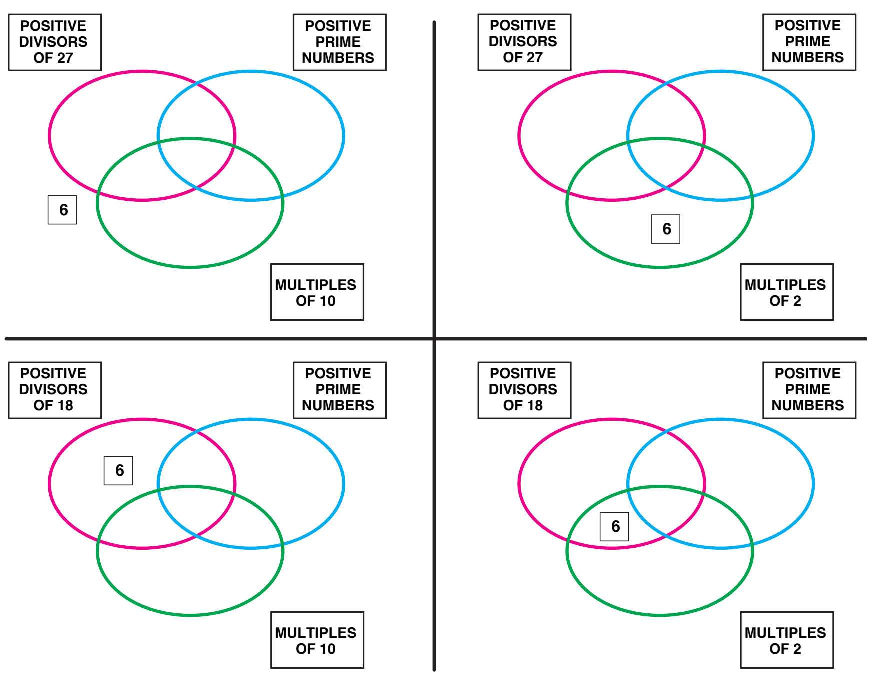


T: *So no matter where (in the blue string) we play 2, if we get a "yes" answer, we will know what the other two strings are for.*

Both 6 and 18 are divisors of 18 and multiples of 2, but neither is a divisor of 27 or a multiple of 10. So whatever information we get from playing 6 will be the same as we would get from playing 18. So let's consider only 6.

S: *6 is not a prime number so it cannot be in the blue string.*

T: *In the picture, there are four possible regions into which 6 can go. As I place 6 in each region, tell me if 6 could be there, and if so, what the red and green strings would be for.*



T: ***So again, if we place 6 in any of these regions, a "yes" answer will tell us what the red and green strings are for.***

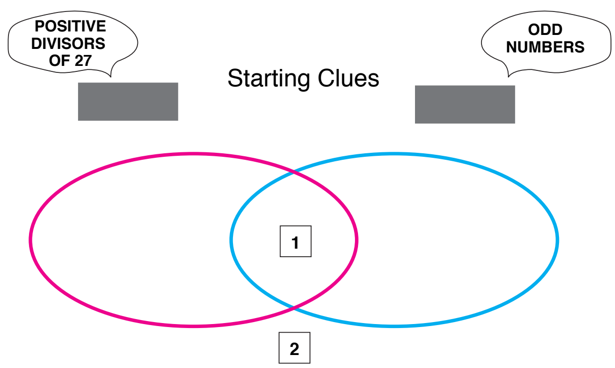
We still don't know what the two strings are so I'll give you the final clue.

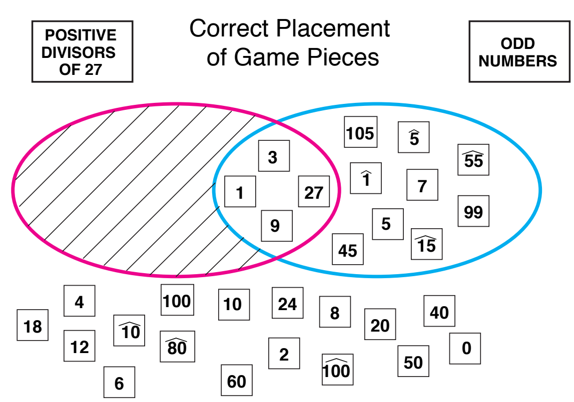
Place 6 outside all of the strings.

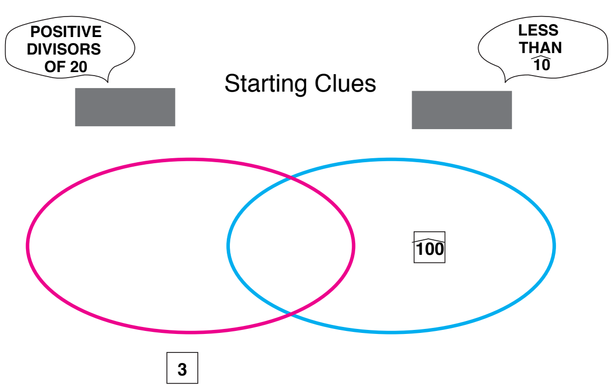
S: ***The red string is for* POSITIVE DIVISORS OF 27 *and the green string is for* MULTIPLES OF 10.**

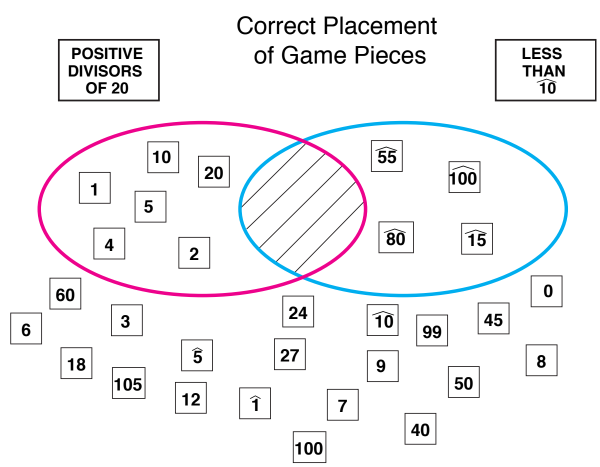
Exercise 3

Play *The String Game* with numbers in the usual way. The illustrations below show two possible games, each with two starting clues. Encourage students to use analysis sheets by allowing time between turns for them to cross out labels.









Capsule Lesson Summary

Given a string picture, determine whether statements about the picture are true, false, or indeterminate. Include statements involving *exactly*, *at most*, and *at least*. Given a statement, make additions to a string picture so that the statement is true. Find the number of dots in each of the regions of a three-string picture, given a series of clues.

Materials

Teacher
- Colored chalk

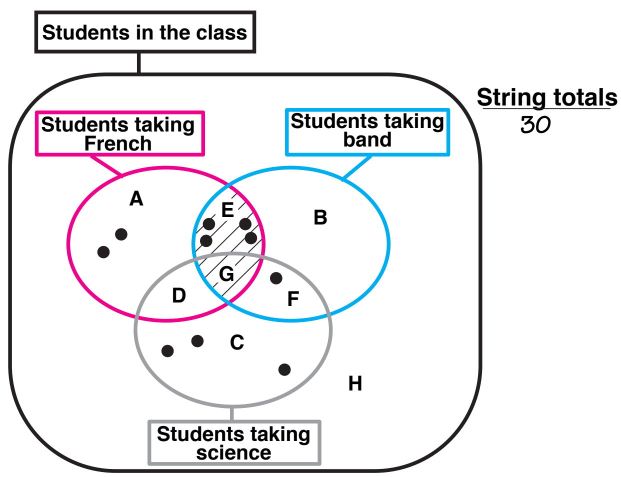
Student
- Paper
- Colored pencils
- Worksheet L6

Description of Lesson

Throughout this lesson, feel free to change the names or sets to fit the interest of your class.

Exercise 1

Draw this string picture on the board.



T: *This string picture is for a class of 30 students. Some students take French, some take band, and some take science. Is there a dot drawn in the picture for every student in the class?*

S: *No, there are fewer than 30 dots drawn.*

T: *That's right—I have not shown you where all of the 30 dots belong. Where would the dot be for a student who takes French and band, but not science?*

S: *In region* **E.**

Continue with these problems:

	Region
• A student who takes band, but neither French nor science.	**B**
• A student who does not take French, band, or science.	**H**
• A student who takes French and science, but not band.	**D**

T: *Kirk is taking French but not band. Where could his dot be?*

S: *Either in region* **A** *or* **D.** *It depends on whether he is taking science.*

Continue with these problems:

	Region
• A student does not take French or science.	**B or H**
• A student takes band and science.	**F (G is empty)**
• A student does not take French.	**B, C, F, or H**

For the following dialogue, we assume the string picture looks as it did in the beginning of the exercise.

L6

T: ***I am going to make some statements. After each statement, tell me if it is true, false, or if you can't tell.***

Repeat each statement slowly two or three times before accepting students' responses.

T: ***There are exactly four students in the class taking band and French.***

T: ***There are exactly six students in the class taking French.***

T: ***There are at least three students in the class taking science.***

T: ***There are at most four band students in the class.***

T: ***At least one student in the class takes French, band, and science.***

T: ***At least two students in the class do not take science.***

T: ***At least one student in the class takes none of these three subjects.***

T: ***Every band student in the class also takes French.***

T: ***There are no students in the class who take both French and science.***

S: ***True. Students taking band and French are in regions*** **E** ***and*** **G*****. There are four dots, and the hatching tells us there are no more.***

S: ***We can't tell. There are six dots that we know for sure are in the French string, but we do not know if there are more dots in regions*** **A** ***and*** **D*****.***

S: ***True. "At least three" means "three or more than three." There are more than three dots in the science string because there are already four dots in it, three in region*** **C** ***and one in region*** **F*****.***

S: ***False. "At most four" means "four or less than four." There are five dots shown in the band string, four in region*** **E** ***and one in region*** **F*****, and there could be more dots in regions*** **B** ***and*** **F*****.***

S: ***False. Region*** **G** ***is empty and hatched.***

S: ***True. There are six dots in the picture that are not in the science string.***

S: ***Can't tell. Region*** **H** ***is not hatched and it has no dots in it. Either some missing dots belong there*** (statement is true) ***or it is empty*** (statement is false.)

S: ***False. There is a dot in the band string that is not in the French string. It is in region*** **F*****.***

S: ***Can't tell. Region*** **D** ***could be empty or not. Since it is not hatched we do not know if any missing dots belong in it.***

Erase the dots and the hatching from the string picture.

Exercise 2 ____________

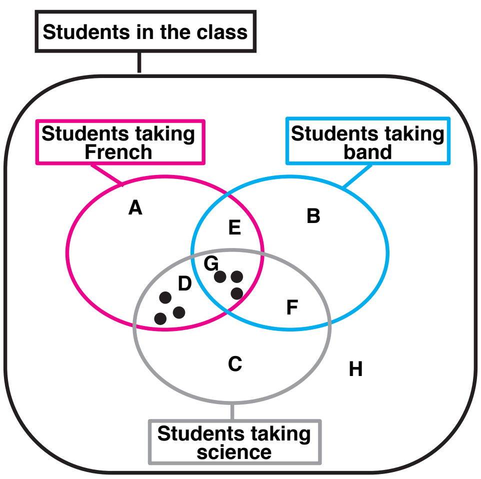
Distribute copies of Worksheet L6.

T: ***I will make a statement about the class in our string picture. Each time, draw dots and hatch regions in one of the string pictures on the worksheet to make the statement true. There are many solutions for each statement. Assume there are 30 students in the class.***

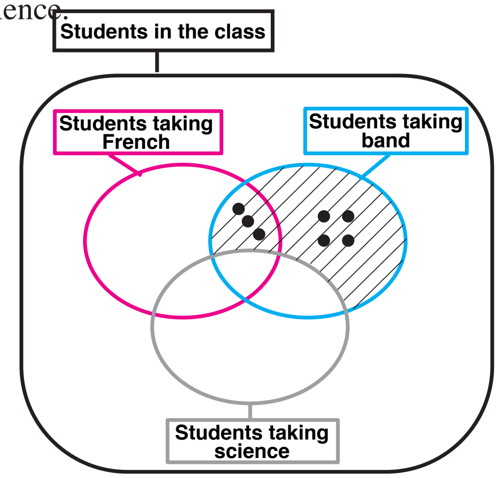
After each statement, allow time for students to draw a solution on their worksheets. There are copies of the string picture on both sides of the worksheet. After a short time, invite a student to draw a solution on the board. Let the class discuss whether or not it is correct. Encourage other students to come to the board to show alternate solutions.

There are many correct responses for each statement. A "minimal" solution is shown next to each statement. Alternate solutions involve moving dots, adding dots, or adding hatching. Be sure that students understand that solutions with extra dots or hatching are correct and are allowed, as long as the statement is still true for that picture. For example, in the first solution, students may add dots to any region, including regions **D** and **G**. Before each new statement, erase any dots and hatching in the string picture on the board.

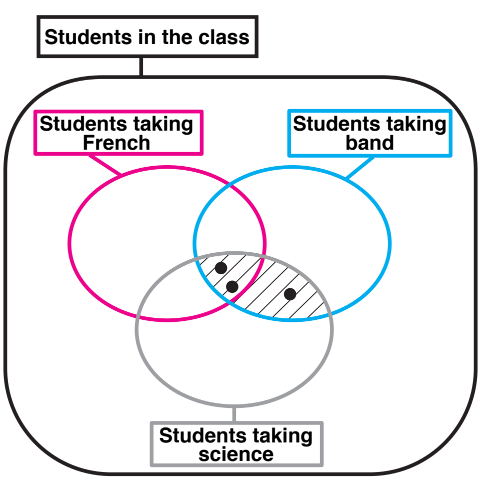
At least six students take French and science.



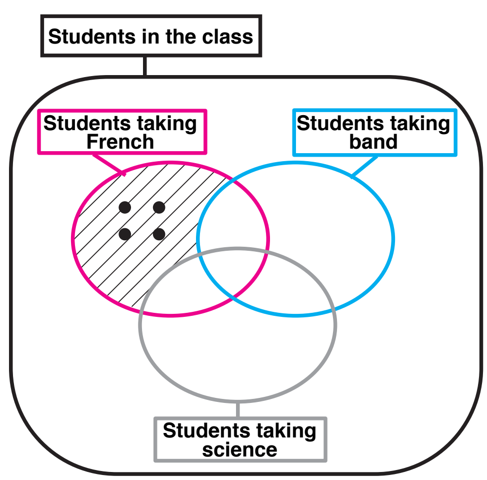
Exactly seven students take band but not science.



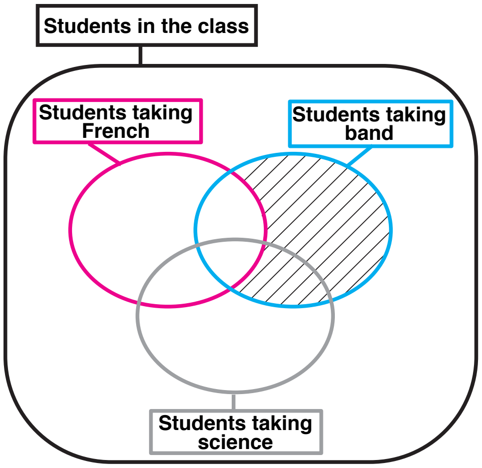
At most three students take both band and science.



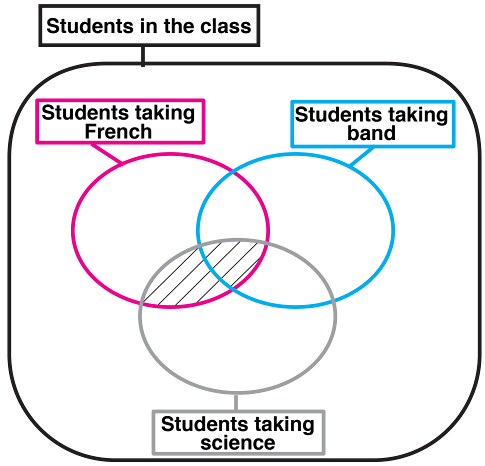
Exactly four students take French, but neither band nor science.



Every band student also takes French.



No student takes both French and science.



Erase any dots and hatching from the string picture on the board.

Exercise 3

Each student should have an unaltered copy of the string picture on Worksheet L6 for use in this exercise. Direct them to label the regions with letters as in the first picture of this lesson.

T: ***There are 30 students in the class. I will give several clues about how many students take various courses. My first clue is this: Exactly eight students take French and science. With this clue, can we record the number of dots in any one region?***

S: ***No. We only know that there are a total of eight dots in regions* D *and* G*.***

T: ***If, after a clue, you know the number of students in a region, record it in your string picture. If not, record the regions and the total number of dots.***
My second clue is this: Exactly five students take all three subjects.

D and G: 8

Repeat the clue. Let students briefly consider the clue and use their worksheets. Then invite students to comment on the clue.

S: ***There are five dots in region* G*.***

S: ***We also know that there are three dots in region* D*, since there are a total of eight dots in regions* D *and* G*; 8 – 5 = 3.***

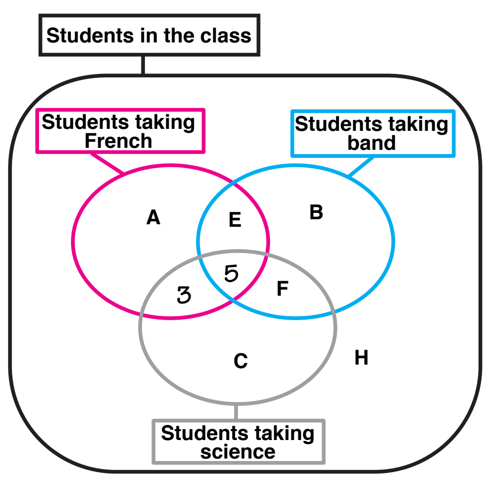
Record the information in the string picture, and ask students to record it on their worksheets.

Continue in a similar manner for the following clues.

T: ***Exactly 15 students take French.***

S: ***There must be 15 dots in the French string. Already eight dots are recorded there. So there are seven dots in regions* A *and* E*; 15 – 8 = 7.***

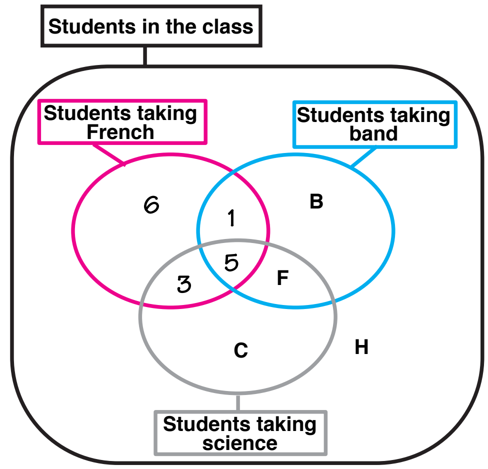
A and E: 7



T: *Exactly six students take French and band.*

S: *Since five students take all three subjects, one other student must take French and band, but not science.*

S: *From the previous clue, we know there are seven students in regions* **A** *and* **E**, *so there must be six students in region* **A**; *7 – 1 = 6.*



T: *Twelve students take science.*

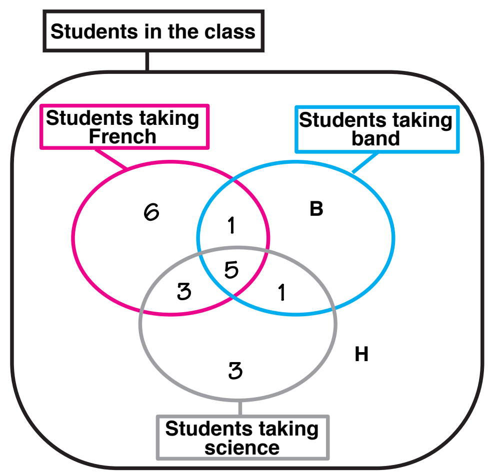
S: *Since eight students take both science and French, four students must take science but not take French. 12 – 8 = 4.*

S: *There must be a total of four dots in regions* **C** *and* **F**.

C and F: 4

T: *Exactly three students take science, but neither French nor band.*

S: *There are three dots in region* **C**.

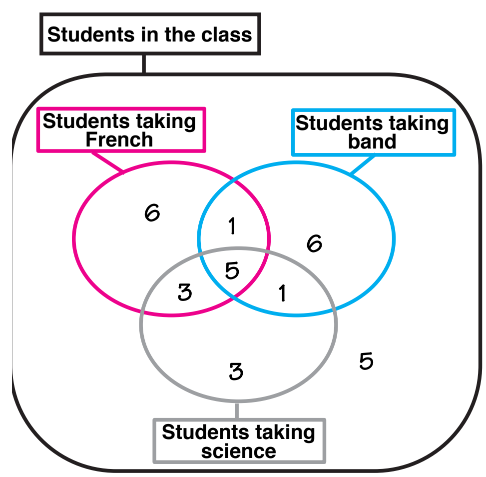


S: *Then there is one dot in region* **F**, *since we know that there must be a total of four dots in regions* **C** *and* **F**. *4 – 3 = 1.*

T: *Thirteen students take band.*

S: *There are six dots in region* **B**, *since seven of the 13 dots in the band string are already correctly placed. 13 – 7 = 6.*

S: *Then there are five dots in region* **H**, *since there are 30 students and 25 dots are already placed. 6 + 1 + 3 + 5 + 6 + 1 + 3 = 25.*



Capsule Lesson Summary

Play *The String Game* with numbers using three strings. Present a three-string picture in which one string is for the positive prime numbers and the other two strings are each for the multiples of an unknown number. Request the locations of various numbers in the picture, and discover what the strings are for.

Materials

Teacher

- Colored chalk
- Numerical String Game kit
- Numerical 3-String Game poster

Student

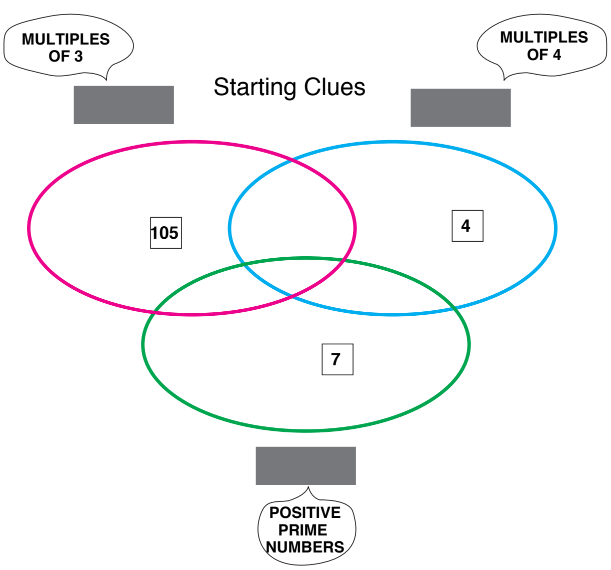
- 3-String Game analysis sheet
- Worksheets L7* and **

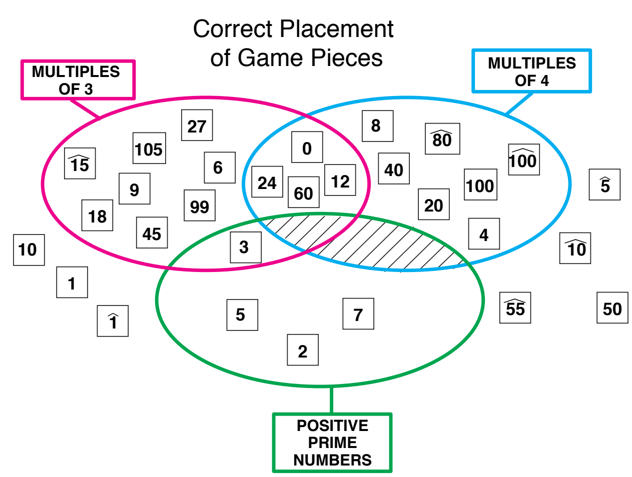
Description of Lesson

Exercise 1

Play a three-string version of *The String Game* with numbers in the usual way. The illustration below shows a possible game with three starting clues. Encourage students to use the 3-String Game analysis sheets during the play of the game.

When the game is finished, ask if there are any regions that should be hatched. The intersection on the blue and green strings should be hatched because no multiple of 4 is a prime number (a multiple of 4 has at least three positive divisors: 1, 2, and 4).

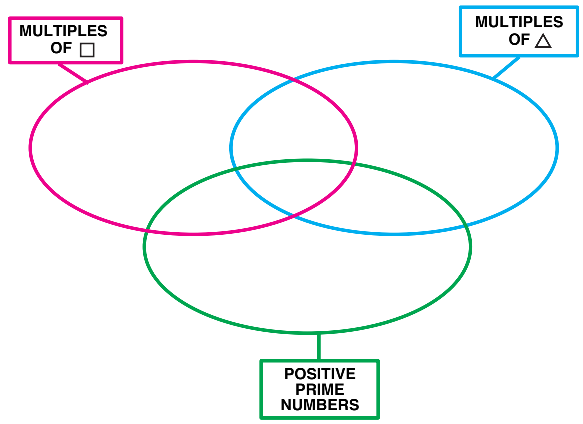




Exercise 2

Put away the string cards and game pieces from *The String Game*. Announce that you are no longer only interested in string labels and numbers that occur in the game. Leave only the three strings on the board.

Label the strings in the picture on the board as you say,



T: ***Suppose the green string is for* POSITIVE PRIME NUMBERS*, the red string is for the multiples of some number—I put a box for this number—and the blue string is for the multiples of some number—I put a triangle for this number.***

T: ***What numbers did we have for the box and the triangle in the game we just played?***

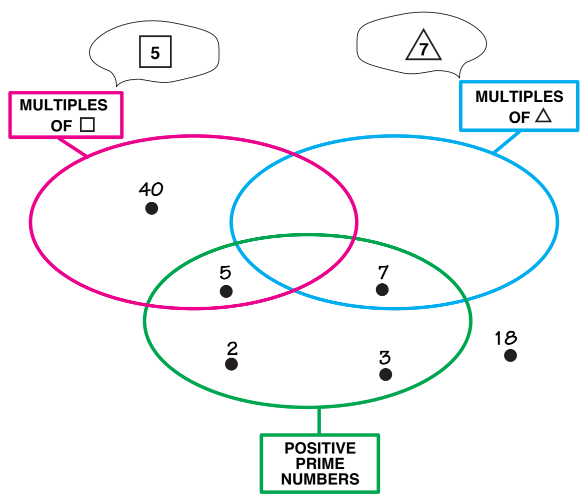
S: ***3 and 4.***

T: ***I've chosen some new numbers for the box and the triangle, but you have to figure out what they are.***

Note: Choose 5 for the box and 7 for the triangle, but do not tell the class.

T: ***Suggest some numbers and I'll locate them in the picture. See how quickly you can determine what the strings are for.***

Let students request the locations of numbers of their choosing. Continue until several students indicate that they know what the numbers in the box and the triangle are. Perhaps your picture will look similar to this one.



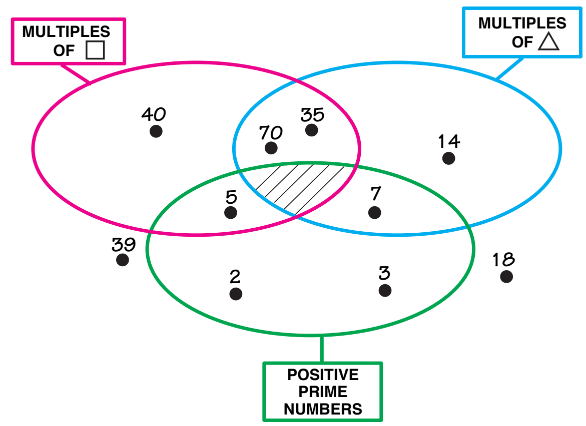
Ask students who think they know the numbers in the box and the triangle to tell the location of numbers suggested by other members of the class. After several more numbers have been located in the picture, ask a student to reveal the identities of the numbers in the box and the triangles. Put 5 in the box and 7 in the triangle.

Point to any empty regions in the string picture and ask if any numbers could go there. Continue this activity until the students conclude that only the center region is empty.

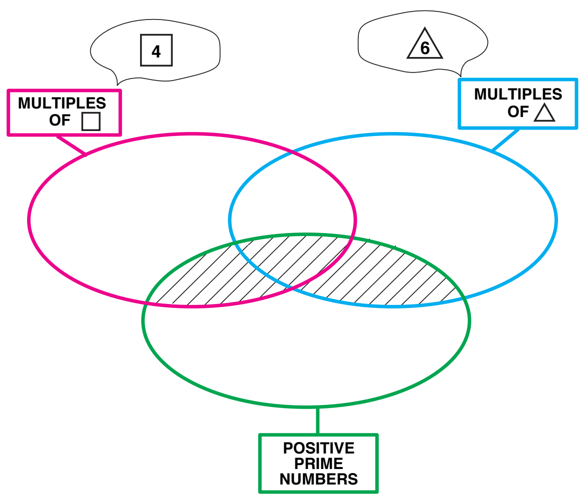
T: ***Why is the center region empty?***

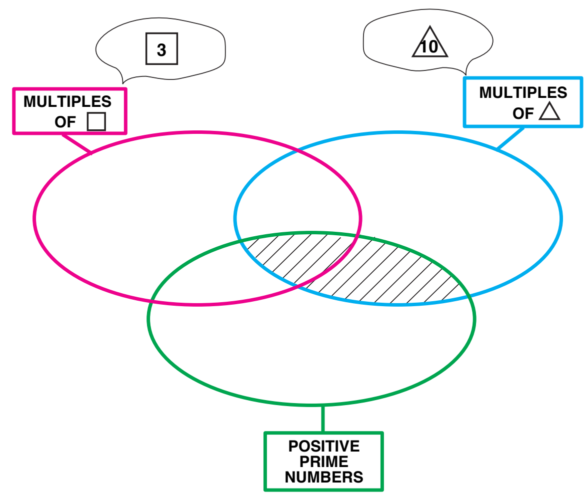
S: ***Because a number that is a multiple of 5 and a multiple of 7 cannot be a prime number. It has at least four positive divisors: 1, 5, 7, and 35.***

Hatch this region. Perhaps the string picture will look like this.



Repeat this activity several times. Two situations are given below. The pictures show which regions are empty and hatched. They should be hatched only after the strings have been identified.

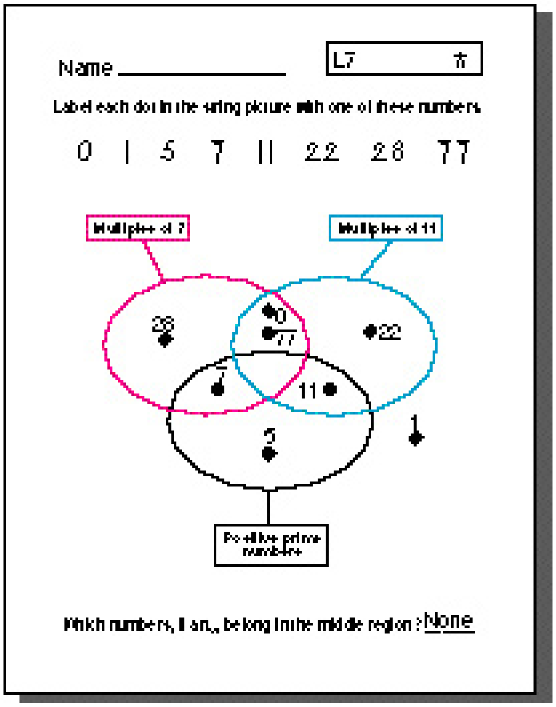


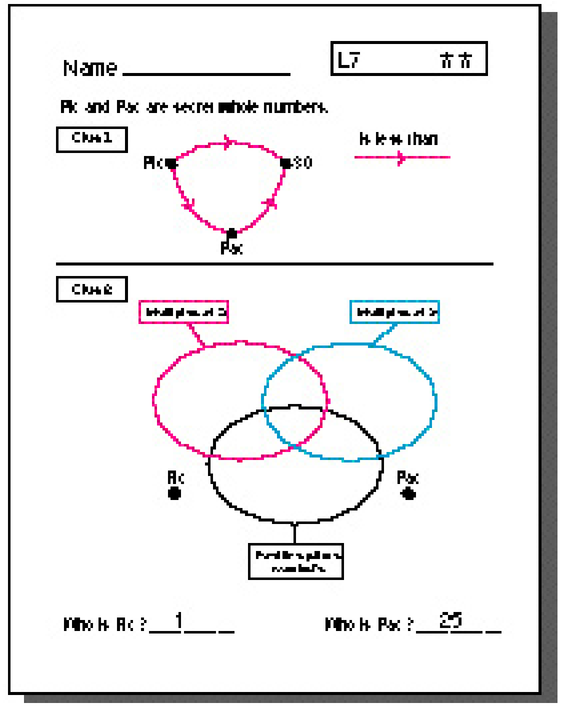


Worksheets L7* and ** are available for individual work. If some students have difficulty with the ** worksheet, suggest they put whole numbers in the picture in succession: 0, 1, 2, 3, … .

Home Activity

Send home a problem similar to that on Worksheet L7* for students to solve with a family member.


Name
L7
Label each dot in the string picture with one of these numbers.
0 1 5 7 11 22 28 77
Multiples of 7
Multiples of 11
28
0
77
22
7
11
5
1
Positive prime numbers
Which numbers, if any, belong in the middle region? None


Name
L7
Pic and Pac are secret whole numbers.
Clue 1
is less than
Pic
30
Pac
Clue 2
Pic
Pac
Who is Pic? 1
Who is Pac? 25

Capsule Lesson Summary

Play *Guess My Rule* with the operations *greatest common divisor* and *least common multiple*. Find numbers whose greatest common divisor with 6 is 2, and pairs of numbers whose greatest common divisor is 6.

Materials

Teacher	• Colored chalk	Student	• Paper

Description of Lesson

In this lesson, operations are introduced in a *Guess My Rule* activity. As necessary, review the idea of a secret rule for an operation * using a "machine" picture (see Lesson N9).

Write several number sentences on the board as clues. Then write an open sentence and see if anyone can predict which number goes in the box.

$$8 * 12 = 4 = 12 * 8$$
$$1 * 12 = 1 = 12 * 1$$
$$12 * 18 = \boxed{} = 18 * 12$$
$$10 * 15 = 5 = 15 * 10$$

T: ***These number sentences give you some information about my secret rule. Can you guess what it is?***

Note: The rule is "a ∗ b is the greatest common divisor of a and b." For example, 8 ∗ 12 = 4 because 4 is the greatest common divisor of 8 and 12. In this case, the number in the box is 6 because 6 is the greatest common divisor of 12 and 18.

You might suggest that students write their guesses on paper for you to check. After a moment, let a student announce that 6 is in the box. Do not let the student explain why 6 is in the box at this time.

Continue the activity with several more sentences involving ∗, letting students who think they know the rule provide the numbers in the boxes.

$$12 * 20 = \boxed{4} = 20 * 12$$
$$12 * 21 = \boxed{3} = 21 * 12$$
$$14 * 21 = \boxed{7} = 21 * 14$$
$$80 * 100 = \boxed{20} = 100 * 80$$
$$8 * 8 = \boxed{8}$$
$$24 * 30 = \boxed{6} = 30 * 24$$
$$6 * 7 = \boxed{1} = 7 * 6$$

T: ***What is my rule?***

S: ***The answer is a positive divisor of both of the numbers, and it is the greatest divisor of both.***

Find a counter example on the board to any incorrect rule suggested. When a good statement of the rule is given, discuss one of the examples.

T: ***What are the positive divisors of 12?***
What are the positive divisors of 18?

L8

As students name positive divisors of 12 and 18, list them on the board.

$12 * 18 = 6$

12	18
12	18
6	9
4	6
3	3
2	2
1	1

T: ***Which of these numbers are divisors of both 12 and 18 (common divisors)?***

S: ***6, 3, 2, and 1.***

T: ***Which of the common divisors is the greatest?***

S: ***6.***

Ask students to check with a partner that $12 * 20 = 4$ by listing the positive divisors of 12 and 20 on their papers. While they are doing so, write these problems on the board. Then direct students to solve these problems on their papers. (Answers are in boxes.)

$9 * 24 = \boxed{3}$ $\qquad (12 * 30) * 8 = \boxed{2}$

$12 * 40 = \boxed{4}$ $\qquad 12 * (30 * 8) = \boxed{2}$

$75 * 100 = \boxed{25}$ $\qquad (12 * 8) * 30 = \boxed{2}$

Encourage students who have difficulty to list all of the positive divisors of each number as in the previous dialogue. Check the answers collectively.

T (pointing to $(12 * 30) * 8 = \square$)**:** ***What do the parentheses tell us?***

S: ***Do 12 * 30 first.***

T: ***What is 12 * 30?*** (6) ***What should we do now?***

$$(12 \overset{6}{*} 30) * 8 = \square$$

S: ***Find 6 * 8.***

S: ***6 * 8 = 2.***

Discuss $12 * (30 * 8)$ and $(12 * 8) * 30$ in a similar manner.

T: ***What do you notice about these last three problems?***

S: ***2 is the greatest common divisor of all three numbers.***

S: ***Moving the parentheses doesn't change the answer.***

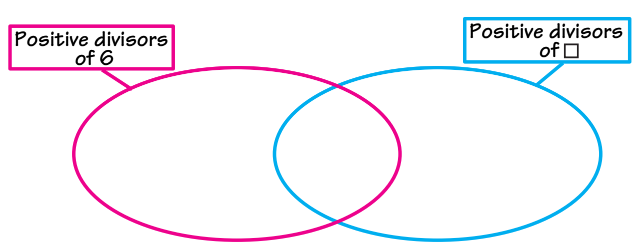
Exercise 2

Write this open sentence on the board.

$$6 * \square = 2$$

T: **** is for the greatest common divisor. What numbers could we put in the box?***

Let students work independently or with a partner while you draw this string picture on the board.



T: ***What are the positive divisors of 6?***

S: ***1, 2, 3, and 6.***

T: ***Where do these numbers belong in this string picture?***

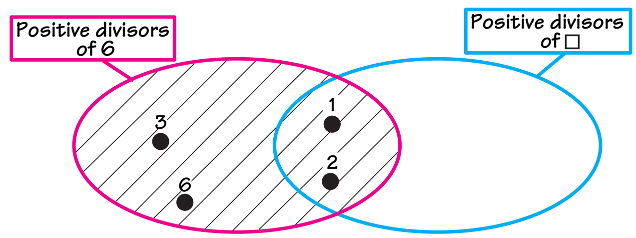
There may be a few false starts, but students should catch mistakes and correct themselves.

S: ***1 is in the middle because 1 is a divisor of every number.***

S: ***2 is the only other number that can be in the middle because 2 is a positive divisor of 6 and of the number in the box; it is their greatest common divisor.***

S: ***3 and 6 must be inside the red string but outside the blue string.***

Ask students to put the numbers in the string picture as they discuss them. Someone might suggest that the inside of the red string be hatched, as shown here.



T: ***What can you tell me about a number that could go in the box?***

S: ***It is even because 2 is a divisor of it.***

S: ***It is neither a multiple of 6 nor of 3 because 3 and 6 are not divisors of it.***

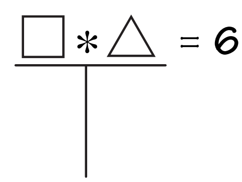
S: ***It is any even number that is not a multiple of 3.***

T: ***Which numbers could be in the box?***

Let students give several examples such as 2, 4, 8, 10, 14, 16, 20, 22, and so on.

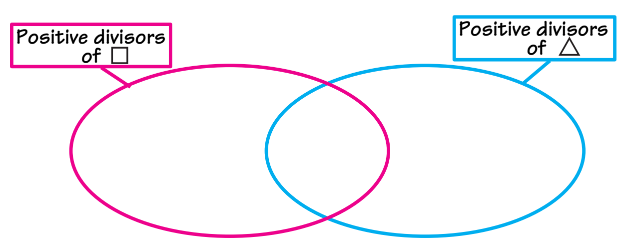
Exercise 3

Draw this chart on the board.



T: ***Copy this chart on your paper and find many pairs of numbers that could be in the □ and the △ .***

While the students are working, draw this string picture near the table.



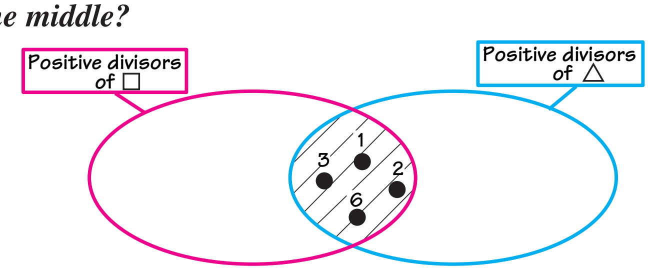
T: ***What numbers can we put in this string picture for sure?***

S: ***1 goes in the middle because 1 is a positive divisor of every number.***

S: ***If 6 is a divisor of both the number in the box and the number in the triangle, then so are 3 and 2. So 6, 3, and 2 go in the middle.***

T: ***Are there any other numbers that go in the middle?***

S: ***No, that region should be hatched.***



Fill in the chart with pairs of numbers that the students have found for the box and the triangle. Check a few of the pairs by putting the numbers in the string picture.

$\square * \triangle = 6$	
6	12
12	18
36	42
6	6
24	30
48	54
12	66

Exercise 4

Note: Because of time constraints, you may prefer to omit this exercise now. However, present it to your class (on an Adjustment Day, for example) prior to Lesson L9.

T: ***I am thinking of a new secret rule. These are some hints. Try to guess my rule.***

Put several number sentences on the board as clues. Then write an open sentence, and see if anyone can predict which number goes in the box.

$$4 * 6 = 12 = 6 * 4$$
$$3 * 6 = 6 = 6 * 3$$
$$7 * 3 = 21 = 21 * 3$$
$$8 * 12 = \square = 12 * 8$$

Note: The rule is "a ∗ b is the least common (positive) multiple of a and b." For example, 4 ∗ 6 = 12, because 12 is the least common positive multiple of 4 and 6.

Check several answers before asking a student to answer aloud. (The number in the box is 24.) Continue this activity with the problems given below. (Answers are in boxes.)

$8 * 8 = \boxed{8}$ $\quad 40 * 30 = \boxed{120}$ $\quad 10 * 25 = \boxed{50}$

$5 * 20 = \boxed{60}$ $\quad 24 * 36 = \boxed{72}$ $\quad 50 * 60 = \boxed{300}$

By now several students should have discovered the rule. Discuss it collectively in a manner similar to that in Exercise 1. Some students may benefit by choosing one example and listing multiples of each of the numbers to emphasize the notion of common multiple. For example:

$8 * 12 = 24$	
8	12
16	24
24	36
32	48
40	60
48	72
⋮	⋮
72	

After several common multiples have been listed, pick out the least common positive multiple.

Ask students to copy and solve these problems involving least common multiple (∗). (Answers are in boxes.)

$10 * 12 = \boxed{60}$ $\quad 12 * 36 = \boxed{72}$ $\quad 60 * 70 = \boxed{420}$

$10 * 15 = \boxed{30}$ $\quad 24 * 30 = \boxed{120}$ $\quad 18 * 15 = \boxed{90}$

L9 GREATEST COMMON DIVISORS AND LEAST COMMON L9

Capsule Lesson Summary

Review what is meant by the greatest common divisor (gcd) and the least common multiple (lcm) of two numbers. Introduce the notation $\sqcap$ for gcd and $\sqcup$ for lcm. Explore relationships between statements about $\sqcap$ ($\sqcup$)and pictures with strings for positive divisors (multiples). Present a detective story with clues involving $\sqcup$, prime divisors, and a string picture.

Materials

Teacher	• Colored chalk	Student	• Paper • Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

Write this information and these problems on the board.

$\sqcap$: greatest common divisor	$\sqcup$: least common multiple
$12 \sqcap 18 = \boxed{}$	$12 \sqcup 18 = \boxed{}$

T: ***In a previous lesson, we played some Guess My Rule games with the operations* greatest common divisor *and* least common (positive) multiple*. Each time, we used ∗ to denote the rule. For convenience and to avoid confusion, we will use the two symbols on the board instead of the ∗.*** (Point to $12 \sqcap 18$.) ***How can we find the greatest common divisor of 12 and 18?***

S: ***List the divisors of each number, and find the greatest number that is in both lists.***

Ask students to name positive divisors of 12 and 18, and list them on the board.

T: ***What is the greatest common divisor of 12 and 18? Why?***

S: ***6, because it is the greatest number that divides both 12 and 18.***

Continue in a similar manner to find $12 \sqcup 18 = 36$. Solve several more problems collectively, reviewing the meanings of greatest common divisor and least common multiple. (Answers are in boxes.)

$\sqcap$: greatest common divisor	$\sqcup$: least common multiple
$12 \sqcap 18 = \boxed{6}$	$12 \sqcup 18 = \boxed{36}$
$15 \sqcap 20 = \boxed{5}$	$15 \sqcup 20 = \boxed{60}$
$1 \sqcap 9 = \boxed{1}$	$1 \sqcup 9 = \boxed{9}$
$8 \sqcap 9 = \boxed{1}$	$8 \sqcup 9 = \boxed{72}$
$14 \sqcap 21 = \boxed{7}$	$14 \sqcup 21 = \boxed{42}$

Continue this activity by writing these problems on the board. Instruct students to copy and solve them on their own. (Answers are in boxes.)

$12 \sqcap 20 = \boxed{4}$	$12 \sqcup 20 = \boxed{60}$
$15 \sqcap 25 = \boxed{5}$	$15 \sqcup 25 = \boxed{75}$
$75 \sqcap 100 = \boxed{25}$	$75 \sqcup 100 = \boxed{300}$
$(12 \sqcap 18) \sqcap 24 = \boxed{6}$	$(12 \sqcup 18) \sqcup 24 = \boxed{72}$
$12 \sqcap (18 \sqcap 24) = \boxed{6}$	$12 \sqcup (18 \sqcup 24) = \boxed{72}$
$(12 \sqcap 24) \sqcap 18 = \boxed{6}$	$(12 \sqcup 24) \sqcup 18 = \boxed{72}$

Exercise 2

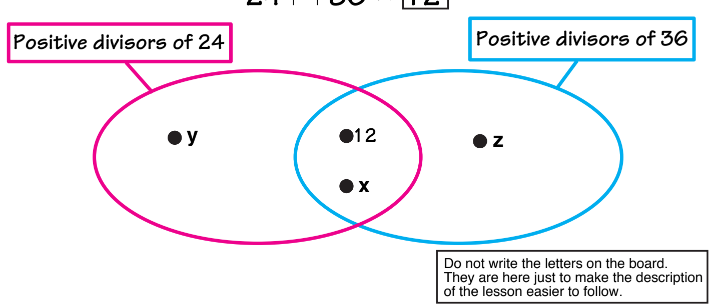
On the board, write this open number sentence, and draw overlapping strings for positive divisors of 24 and positive divisors of 36.

$$24 \sqcap 36 = \boxed{}$$

T: ***What number is 24 $\sqcap$ 36?***

S: ***12.***

$24 \sqcap 36 = \boxed{12}$



Invite a student to put 12 in the string picture. Then put three unlabeled dots in the string picture, as shown here.

T: ***What numbers could be here*** (at **x**)***?***

The class should find that **x** could be for 1, 2, 3, 4, or 6.

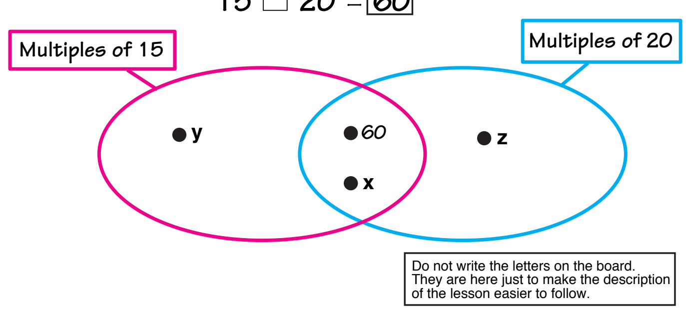
Proceed in a similar manner with dots **y** and **z**. The class should find that **y** could be for 8 or 24, and **z** could be for 9, 18, or 36.

T (pointing to the middle region)**:** ***What do you notice about the numbers in this region?***

S: ***Each number in the middle region is a positive divisor of 12.***

Repeat this activity for 15 $\sqcup$ 20.

$15 \sqcup 20 = \boxed{60}$



x could be for 0, 120, 180, 240, …
or for $\widehat{60}$, $\widehat{120}$, $\widehat{180}$, …
y could be for 15, 30, 45, 75, 90, …
or for $\widehat{15}$, $\widehat{30}$, $\widehat{45}$, $\widehat{75}$, …
z could be for 20, 40, 80, 100, 140, …
or for $\widehat{20}$, $\widehat{40}$, $\widehat{80}$, …

Each number in the middle region is a multiple of 60.

Exercise 3

Present a detective story in which Bim is a secret whole number.

Clue 1

Write this information on the board.

$\sqcup$ Bim = 30

T: ***What information does this clue give us about Bim?***

S: ***30 is the least common multiple of 6 and Bim.***

S: ***Bim is a positive divisor of 30.***

T: ***Yes. Write the positive divisors of 30 on your paper.***

List the positive divisors of 30 on the board, as students name them.

1, 2, 3, 5, 6, 10, 15, 30

T: ***Which of these numbers cannot be Bim?***

S: ***Bim cannot be 1, because 6 $\sqcup$ 1 = 6.***

Cross off 1 from your list.

S: ***Bim cannot be 2, 3, or 6. The least common multiple of each of these numbers and 6 is 6, not 30.***

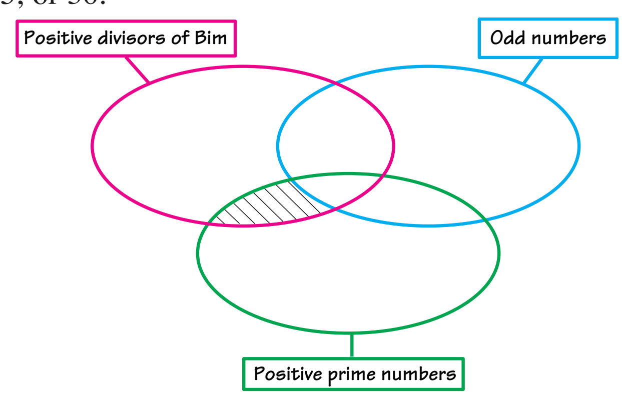
T: ***Do you notice anything interesting about the numbers we crossed off?***

S: ***Each number is a positive divisor of 6.***

The class should conclude that Bim could be 5, 10, 15, or 30.

Clue 2

Draw this string picture on the board.



T: ***What information does this picture give us about Bim?***

S: ***Bim has only odd prime divisors.***

T: ***Which numbers could be Bim?***

S: ***Bim cannot be 10, because 2 is a prime divisor of 10.***

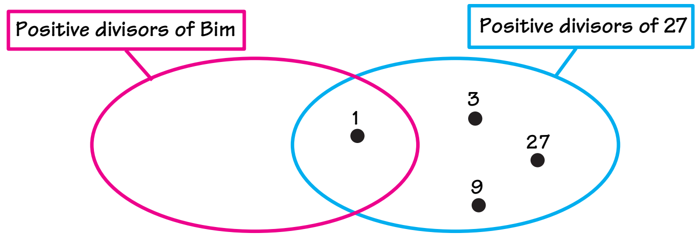
S: ***Cross of 30, because 30 has 2 for a prime divisor.***

S: ***Bim could be 5 or 15. Their only prime divisors are 5 and 3.***

L9

Clue 3

Draw overlapping red and blue strings with four dots in the blue string, as illustrated here. Invite students to label the dots. Perhaps a student will suggest hatching the blue string.



T: ***What information does this picture give us about Bim?***

S: ***1 is a positive divisor of Bim; 3, 9, and 27 are not positive divisors of Bim.***

T: ***Who is Bim?***

S: ***Bim is 5, because 3 is a divisor of 15 but 3 is not a divisor of Bim.***

L10 PRIME FACTOR DISTANCE #1

Capsule Lesson Summary

Find pairs of whole numbers that are related by the *prime factor relation*; i.e., one number equals a prime number times the other. Label the dots in some pictures involving the prime factor relation. Define *prime factor distance* and determine the prime factor distance between pairs of whole numbers.

Materials

Teacher	• Colored chalk	**Student**	• Paper • Colored pencils, pens, or crayons • Worksheets L10* and **

Description of Lesson

Exercise 1

Review that a positive prime number is any positive number with exactly two divisors. Ask students to name some positive prime numbers; then ask for the least positive prime number (2) or a prime number between 25 and 30. (29)

With the class's help, make a list on the board of the positive prime numbers less than 30. This list will be used throughout the lesson.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29

Write **prime factor relation** on the board above a red cord.

T: ***Today, we will use a new relation called the* prime factor relation. *Dots will be only for whole numbers. Two whole numbers are joined by a red cord if and only if one of the numbers equals the other number times a prime number. Can you suggest two numbers that we could join with a red cord?***

S: ***4 and 12. 3 x 4 = 12 and 3 is a prime number.***

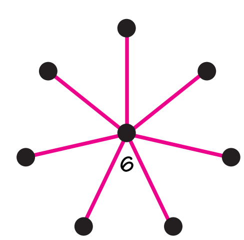
Join 4 and 12 with a red cord.

Ask for several more pairs of numbers that can be joined by a red cord, for example, 3 and 15 since 5 x 3 = 15 and 5 is a prime number, or 6 and 42 since 7 x 6 = 42 and 7 is prime. Encourage students to find examples using different prime numbers as multipliers. Note that the dots can be, but need not be, prime numbers.

T: ***Can 5 and 20 be joined by a red cord?***

S: ***No, 4 x 5 = 20, but 4 is not a prime number.***

Draw this picture on the board.



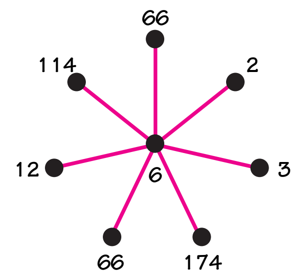
T: ***Which whole numbers can be joined to 6 with a red cord?***

S: ***2, because 3 x 2 = 6 and 3 is a prime number.***

S: ***3, because 2 x 3 = 6 and 2 is a prime number.***

S: ***66, because 11 x 6 = 66 and 11 is prime.***

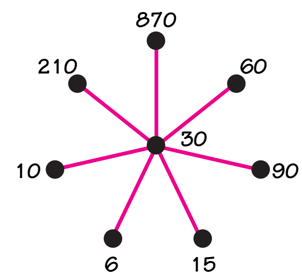
Invite students to label all the dots. Many answers are possible, but encourage students to suggest numbers both less than 6 and more than 6 that could be joined to 6. Observe that because it is a cord picture, students can either multiply or divide 6 by a prime number. This is a (possible) completed picture.



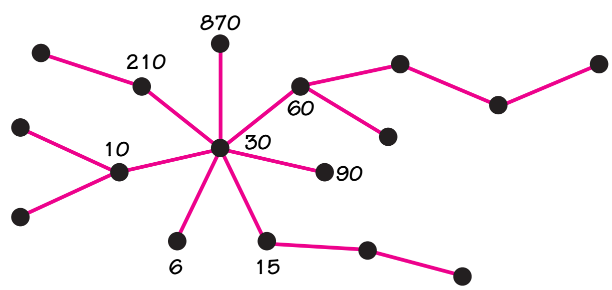
Erase the numbers from the picture on the board, and put 30 at the center dot.

T: ***Copy this picture on your paper, and label the dots with whole numbers. Try to use some numbers less than 30 and some numbers greater than 30.***

After a while, invite students to label the dots in the picture on the board. The illustration here includes some of the numbers that could be joined to 30.

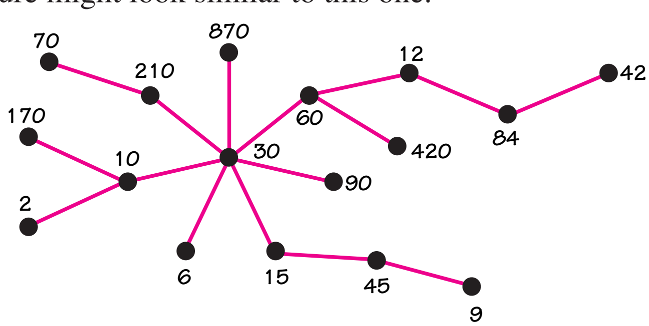


Extend your picture similar to one below. The numerical abilities of your students should guide your choice of how to extend the picture.



Invite students to label the dots. Insist that each dot be for a different number. For some dots, specify that you would like a number greater (or less) than the number at the other end of the cord. Ask the class to check each answer.

A completely labeled picture might look similar to this one.



T: ***This picture is beginning to look like a map. Do you remember some other maps and distances we studied?***

S: ***There was a funny map with towns and roads. The distance between two points was the length of a shortest route. We solved a problem about Snoopy and Woodstock on that map.***

T: ***Using the prime factor relation, we have drawn another kind of map. How do you suppose we could define distance between two numbers on this new map?***

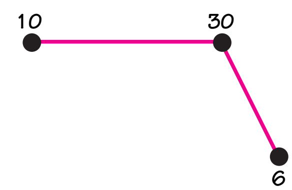
S: ***Count the number of cords in a road between the two numbers.***

T: ***Can we use any road between the two numbers?***

S: ***No, we must find a shortest road.***

T: ***Yes, the prime factor distance[†] between two numbers is the number of cords in a shortest road between the two numbers.***

Draw this cord picture on the board.



T: ***What is the prime factor distance between 10 and 6?***

S: ***2, because there are two red cords in a shortest road from 10 to 6.***

T: ***How do we know that there is no shorter road from 10 to 6?***

S: ***Because 10 and 6 cannot be joined by one red cord.***

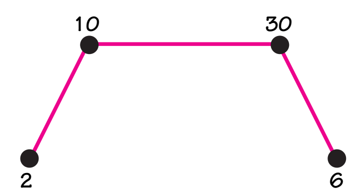
Record the prime factor distance on the board.

prime factor distance
pfd (10, 6) = 2

T: ***Can 10 be joined to 2 by a red cord?***

S: ***Yes. 5*** x ***2 = 10 and 5 is a prime number.***

Draw a red cord from 2 to 10.



T: ***What is the prime factor distance between 2 and 6?***

S: ***1. On the board, there is a three-cord road from 2 to 6. But we can join 2 and 6 with a red cord since 3*** x ***2 = 6 and 3 is a prime number. Therefore, the prime factor distance between 2 and 6 is 1, not 3.***

Draw a red cord from 2 to 6 and write pfd (2, 6) = 1 near the picture.

T: ***To find a prime factor distance, we must be sure to find the length of a shortest road between two numbers, not just any road.***

Refer to the large cord picture on the board.

T: ***In our picture, which numbers are at a prime factor distance of 2 from 30?***

The answer depends on the picture drawn by your class. In the picture on the previous page, several numbers (2, 12, 45, 70, 170, and 420) are at a distance of 2 from 30.

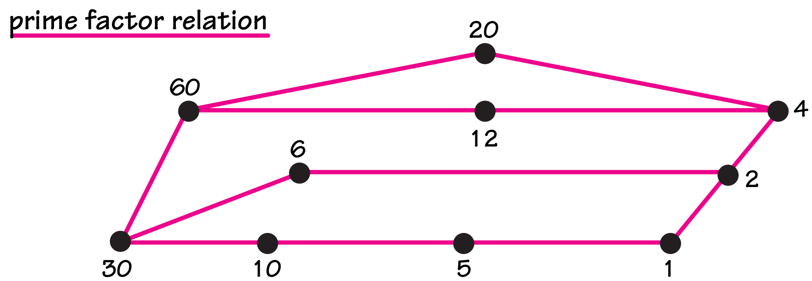
[†]See the introduction to this strand for an algorithm you can use to find the prime factor distance between two whole numbers.

Exercise 2

Erase the board except for the key indicating that a red cord represents the prime factor relation. Draw dots for 30 and 4 as you give the following instructions.

T: ***On your paper, build a road with red cords between 4 and 30. Try to build a shortest road. Remember, you can either multiply by or divide by a positive prime number. Build more than one road if you have time.***

Let students work independently or with partners for a few minutes. While others are still working, invite some students to draw their solutions on the board. Attempt to find students who offer differing solutions, not only the shortest roads. For example, your picture might look similar to this one.



T: ***Did anyone find a shorter road between 4 and 30?***

S: ***No.***

T: ***What is the prime factor distance between 4 and 30?***

S: ***3, because there are three cords in a shortest road between 4 and 30.***

If none of the roads suggested has length 3, challenge students to find such a road. Then add one of the roads found to the picture and conclude that the prime factor distance between 4 and 30 is 3. Write pfd (4, 30) = 3 on the board.

Write these problems on the board.

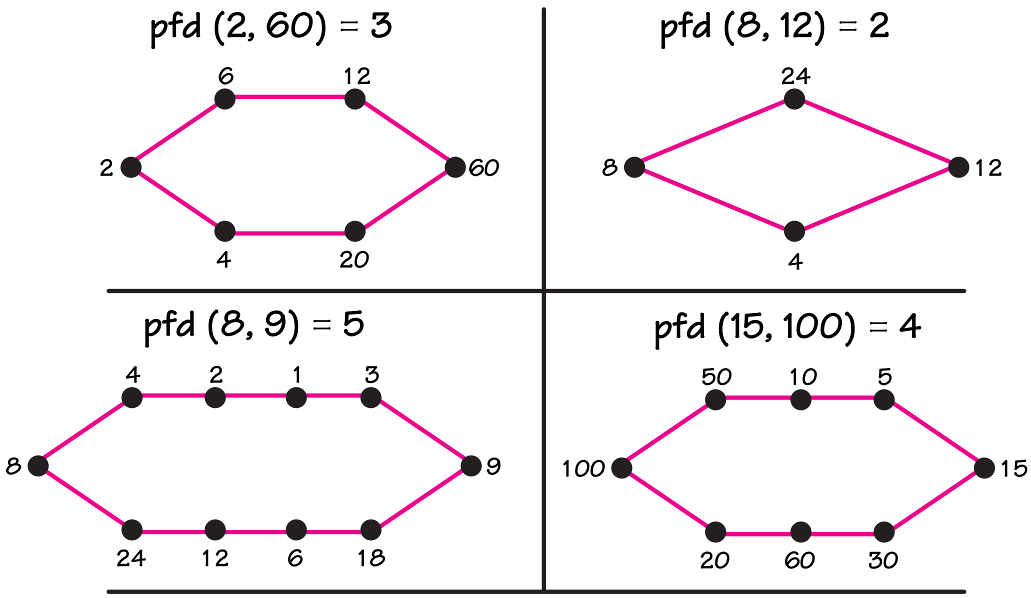
pfd (2, 60) = ___ pfd (8, 12) = ___

pfd (8, 9) = ___ pfd (15, 100) = ___

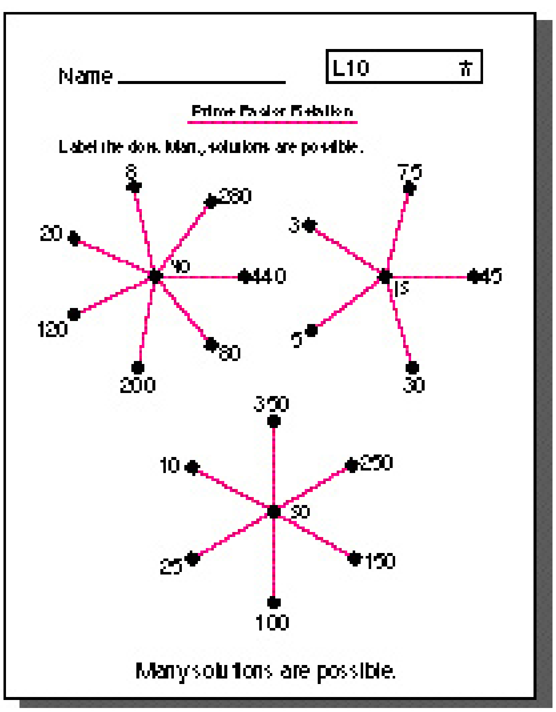
T: ***We found that the prime factor distance between 4 and 30 is 3. Find these prime factor distances in a similar manner. For each problem, try to build a road between the two numbers, and then check to see if it is a shortest road.***

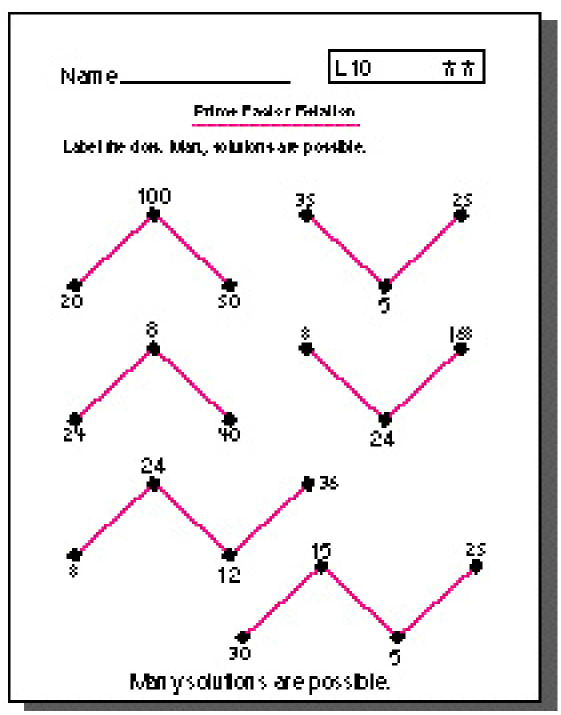
Let students work independently or in pairs on these problems. As students find roads, invite them to draw several roads for each problem on the board. If some students find these problems too difficult, you may give them Worksheet L10* to do first. If many students cannot find any roads, solve one or more problems collectively at the board.

Near the end of the period, direct the class's attention to the roads drawn on the board. Invite any student who has found a shorter road for one of the problems to add that road to the picture. Then ask for the prime factor distance between each pair of numbers. The prime factor distance and two shortest roads for each problem are shown below.



Worksheets L10* and ** are available for individual work.





Capsule Lesson Summary

Review the definition of the prime factor relation, and label dots in a picture involving the prime factor relation. Build roads using the prime factor relation, and find the prime factor distance between some pairs of whole numbers. Find whole numbers less than 50 that are at a prime factor distance of 2 from 10.

Materials

Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons

Advance Preparation: You may want to draw the picture for Exercise 1 on the board before the lesson begins.

Description of Lesson

Exercise 1

Ask the class to name all of the positive primes less than 50. List them on the board.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

Put a key on the board indicating that a red cord is for the prime factor relation.

T: ***Red cords will be for the prime factor relation. This relation works with whole numbers only. Do you remember the rule for this relation?***

S: ***Two whole numbers are joined by a red cord if and only if one of the numbers equals a prime number times the other number.***

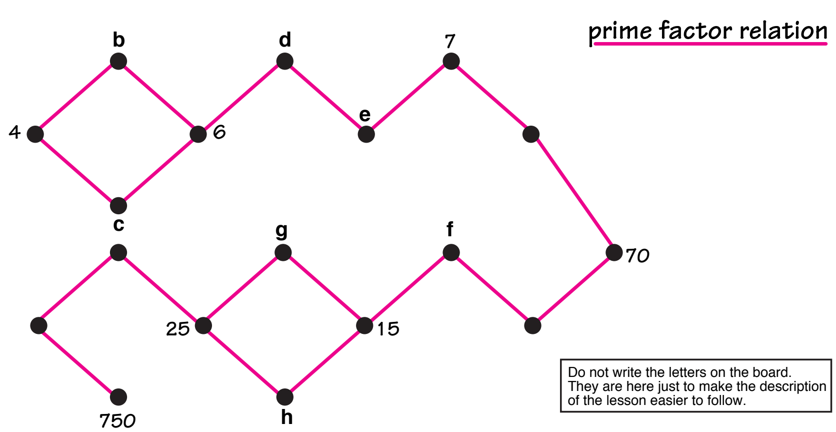
Draw this red cord as you ask,

T: ***Which numbers could be joined to 12 with a red cord?***

S: ***4, because 3 x 4 = 12 and 3 is a prime number.***

S: ***60, because 5 x 12 = 60 and 5 is prime.***

Solicit several more numbers that could be joined to 12. Then refer to this picture on the board.



T: ***The dots are for whole numbers. We will try to get different numbers for all of the dots. The red cords are for the prime factor relation. Which number could be here*** (point to **b**)***? Why?***

S: ***2. 2 x 2 = 4, 3 x 2 = 6, and both 2 and 3 are prime numbers.***

S: ***12. 3 x 4 = 12 and 2 x 6 = 12, and both 3 and 2 are prime numbers.***

Put 2 at **b** and 12 at **c**. Then refer to the dots for 6 and 7.

T: ***Let's try to build a road between 6 and 7 with three cords. Remember, we want each dot in the picture to be for a different number. How could we start?***

Accept students' suggestions until a solution is found. The class discussion might develop along these lines.

S (pointing to **d**)**:** ***This could be 18, because 3 x 6 = 18 and 3 is prime.***

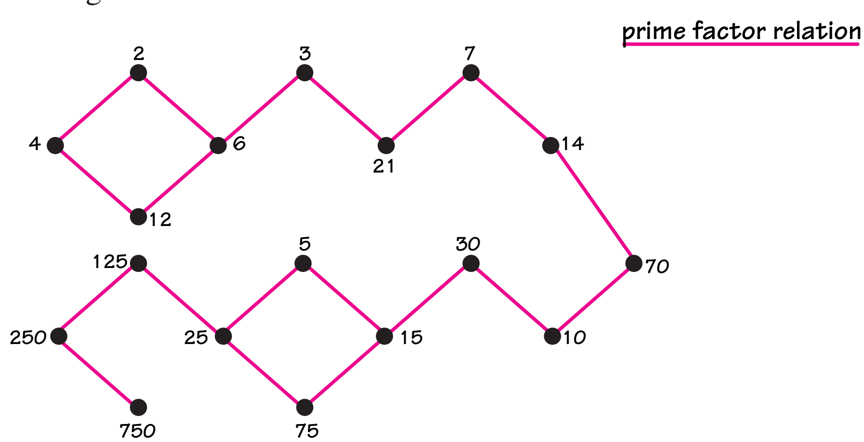
T: ***If 18 were here (***at **d*****) could we build a road from 18 to 7 with just two cords?***

When the class realizes that this is impossible, look for other numbers to try at **d**.

S: ***This number (***at **d*****) could be 3 and the next number (***at **e*****) could be 21.***

Continue in this manner until all of the dots are labeled. Insist each dot be for a different number. While labeling the dots, it may become necessary to change a previous label for one of the dots. For example, 5, 30, or 105 can be at **f**. Also, 5 must be at **g** or at **h**. Therefore, if the students first choose 5 for **f**, they will be unable to label dots **g** and **h**. If this occurs, suggest they use 5 for **g** or **h** and attempt to change the label for **f**.

A completed picture might look similar to the one below.



T: ***What is the prime factor distance between 15 and 25?***

pfd (15, 25) = 2

S: ***2. In the picture, there is a road with two cords from 15 to 25. No shorter road exists since 15 and 25 cannot be joined with a red cord.***

T: ***What is the prime factor distance between 6 and 7?***

pfd (6, 7) = 3

S: ***3. There is a road with three cords between 6 and 7.***

T: ***Do you think we could build a shorter road between 6 and 7?***

S: ***No.***

The picture shows a road from 6 to 15 with eight cords. Challenge students to find a shorter road.

pfd (6, 15) = 2

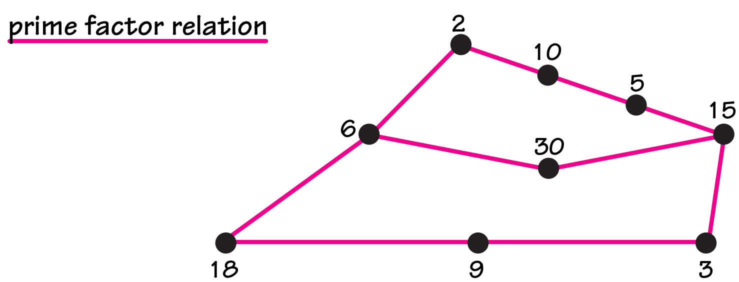
S: ***The prime factor distance between 6 and 15 is 2.***
We can build a road from 6 to 3 to 15 or from 6 to 30 to 15.

Exercise 2

Draw dots for 15 and 18 on the board as you pose this problem.

T: ***Let's find the prime factor distance between 15 and 18 by building a shortest road between 15 and 18.***

Accept student suggestions until at least one road with three cords is drawn. Allow students to draw more than one cord road, if they wish. Many roads are possible. For example, your picture might look similar to this one.



T: ***What is the length of our shortest road between 15 and 18?*** (3)
Can anyone find a shorter road? (No)

pfd (15, 18) = 3

Write these problems on the board, and ask students to try to find the prime factor distances. Instruct students to draw roads between the numbers and to look for a shortest road.

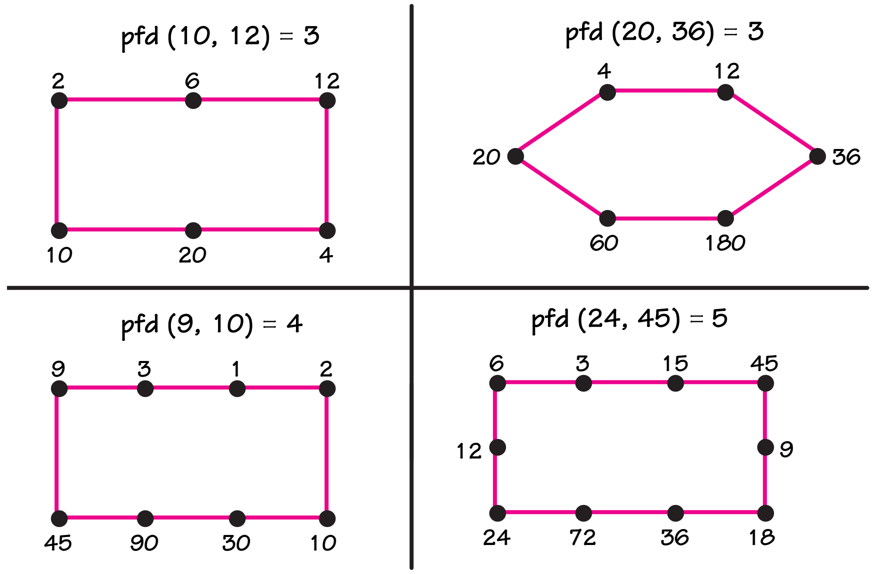
pfd (10, 12) = ___

pfd (20, 36) = ___

pfd (9, 10) = ___

pfd (24, 45) = ___

If many students have difficulty building any road, solve one or two of the problems collectively. Otherwise, after most students have solved at least one problem, invite students to draw their roads on the board. Encourage students to draw more than one road for each problem. Do not insist that only shortest roads be drawn, but let students who find shorter roads add them to the picture. After a while, ask for the prime factor distance between each pair of numbers. The prime factor distance and two of the shortest roads for each problem are shown below. Many other roads are possible.



Exercise 3

Write this information on the board.

T: ***Flip is a whole number less than 50. The prime factor distance between 10 and Flip is 2. Which numbers could Flip be?***

Flip < 50
pfd (10, Flip) = 2

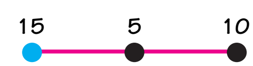
Adapt the following discussion to your students' suggestions.

S: ***Flip could be 15.***

T: ***Why?***

S: ***Draw a red cord from 10 to 5, because 2* x *5 = 10 and 2 is prime. Then draw a red cord from 5 to 15, because 3* x *5 = 15 and 3 is prime.***

Draw this picture on the board.



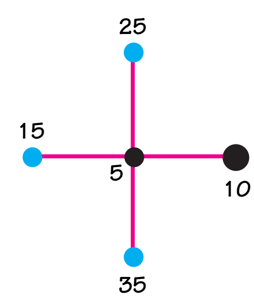
Note: Blue dots are used to emphasize the numbers that Flip could be.

T: ***What other numbers could Flip be?***

S: ***25, because 5* x *5 = 25 and 5 is a prime number.***

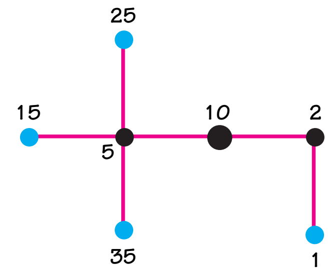
S: ***35, because 7* x *5 = 35 and 7 is a prime number.***

Extend the picture on the board.



T: ***Are there other numbers that Flip could be?***

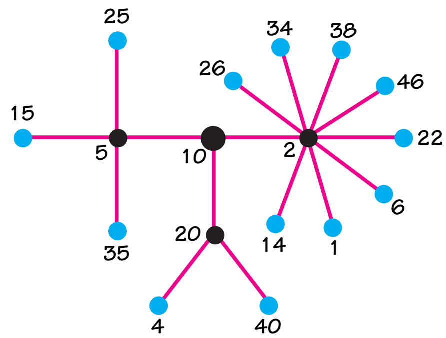
S: ***Flip could be 1, because you can draw a red cord from 10 to 2 (5 x 2 = 10 and 5 is prime), and then draw a cord from 2 to 1 since 2 x 1 = 2.***



Add this information to your picture.

After the class finds four or five numbers that Flip could be, instruct students to copy the picture from the board and to find all of the whole numbers less than 50 at a prime factor distance of 2 from 10. Let students work independently or with partners for a while.

Invite students to complete the picture on the board. There are many ways to draw the paths to the 13 numbers that Flip could be. A completed picture might look like the one below.



T: ***Which numbers could Flip be?***

S: ***1, 4, 6, 14, 15, 22, 25, 26, 34, 35, 38, 40, and 46.***

Capsule Lesson Summary

In a three-string picture, determine the string labels from five starting clues given all at once. Play *The String Game* with numbers.

Materials

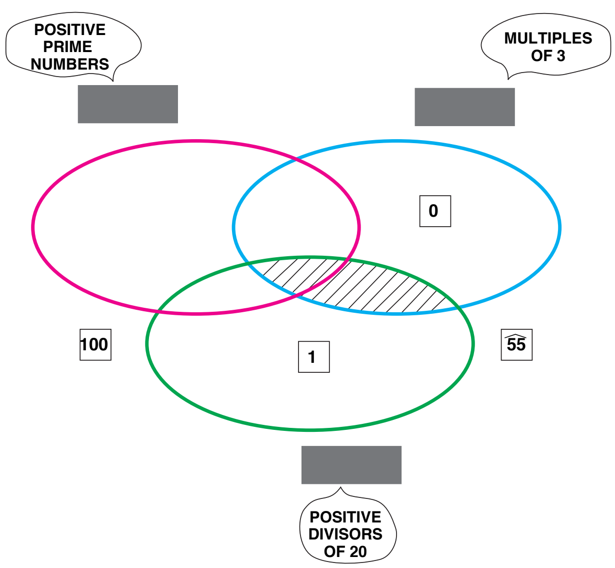
Teacher
- Colored chalk
- Numerical String Game kit
- Numerical 3-String Game posters

Student
- 3-String Game analysis sheet

Description of Lesson

Exercise 1

Prepare your board for *The String Game* as shown below. Distribute 3-String Game analysis sheets to students.



RED	BLUE	GREEN
MULTIPLES OF 2	MULTIPLES OF 2	MULTIPLES OF 2
MULTIPLES OF 3	MULTIPLES OF 3	MULTIPLES OF 3
MULTIPLES OF 4	MULTIPLES OF 4	MULTIPLES OF 4
MULTIPLES OF 5	MULTIPLES OF 5	MULTIPLES OF 5
MULTIPLES OF 10	MULTIPLES OF 10	MULTIPLES OF 10
ODD NUMBERS	ODD NUMBERS	ODD NUMBERS
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS
GREATER THAN 50	GREATER THAN 50	GREATER THAN 50
LESS THAN 50	LESS THAN 50	LESS THAN 50
GREATER THAN $\widehat{10}$	GREATER THAN $\widehat{10}$	GREATER THAN $\widehat{10}$
LESS THAN $\widehat{10}$	LESS THAN $\widehat{10}$	LESS THAN $\widehat{10}$
POSITI VE DIVISORS OF 12	POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27

T: ***The strings are determined by these number clues and the hatching. Take a few minutes to cross labels off on your sheets that the strings could not have.***

Allow time for the students to analyze the situation individually before doing so collectively.

T: ***Let's combine our efforts to determine the string labels. What can we cross off the list of possibilities?***

S: *1 is not a prime number, so the green string cannot be for* **POSITIVE PRIME NUMBERS.**

S: *0 is a multiple of every integer. Since 0 is not in the red string or in the green string, cross off all of the "multiple" labels for those two strings.*

S: *$\widehat{55}$ is less than 50 and is outside all three strings, so we can cross off* **LESS THAN 50** *for all three strings.*

Continue the analysis to determine the strings. Most likely your class will have little difficulty in determining that the red string is for **POSITIVE PRIME NUMBERS** and the blue string is for **MULTIPLES OF 3**, but they may have trouble using the hatching to determine which of the "positive divisor" labels the green string has. In this case, ask where various multiples of 3 belong in the picture. 12's location would lead to eliminating **POSITIVE DIVISORS OF 12** and **POSITIVE DIVISORS OF 24**; 3's location would lead to eliminating four of the five possibilities. Conclude that the green string is for **POSITIVE DIVISORS OF 20**.

Note: A natural variation of this activity is to present the clues, one at a time, as in Exercise 2 of Lesson L5 *String Game Analysis #2*. You may wish to use that format if you think it more appropriate for your class at this time.

Exercise 2

Play a three-string version of *The String Game* with numbers in the usual way. The illustration below shows a possible game with two numbers and hatching as starting clues. Encourage students to use the 3-String Game analysis sheets during the play of the game. You may do this by allowing time before turns begin and between turns for students to cross out labels on their lists.