Geometry & Measurement

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GEOMETRY & MEASUREMENT INTRODUCTION

Geometry is useful and interesting, an integral part of mathematics. Geometric thinking permeates mathematics and relates to most of the applications of mathematics. Geometric concepts provide a vehicle for exploring the rich connections between arithmetic concepts (such as numbers and calculations) and physical concepts (such as length, area, symmetry, and shape). Even as children experience arithmetic notions in their daily lives, they encounter geometric notions. Common opinion recognizes that each child should begin developing a number sense early. There is no sound argument to postpone the development of geometric sense. These are compelling reasons why the systematic development of geometric thinking should be a substantial component of an elementary school curriculum.

A benefit of early development of geometric thinking is that it provides opportunities for all learners. Children develop concepts at individual rates. We are careful to allow students a great deal of time to explore quantitative relationships among physical objects before studying arithmetic. In the same spirit, students should be allowed to explore geometric relationships through free play and guided experiences. Before studying geometric properties of lines and shapes, they should have free play with models and with drawing instruments, using both to create new figures and patterns. Before considering parallelism and perpendicularity as relations on geometric lines, they should have free play in creating patterns with parallel and perpendicular lines. Before learning formulas for finding measurements, they should have informal experiences to establish and sharpen their intuitive notions. These early experiences are aesthetically rewarding, and just as stimulating and suggestive of underlying concepts as early experiences leading to arithmetic. Such experiences are a crucial first step in the development of a geometric sense.

A variety of constructions forms the basis for the geometry of *CSMP Mathematics for the Intermediate Grades*. Students use tools to explore geometric concepts, directly discovering their properties and interrelationships. The tools include a straightedge, a compass, mirrors, angle templates, and a translator for drawing parallel lines. The constructions foster insights into the properties of shapes, independent of the measurement of those properties. Only after students are familiar with the shapes do they begin to use rulers and protractors to measure lengths and angles. In this sequential development of geometric ideas, the measurement is viewed as the intersection of geometric concepts and arithmetic concepts.

The focus of this strand is experience. The measurement activities guide the students to refine their ability to accurately measure lengths of line segments and areas of polygons. Another sequence of lessons allows students to explore symmetry through use of a common tool, a mirror. As a natural consequence of their involvement in these activities, the students develop their knowledge and skills in geometry. The effects of this informal approach should be judged by the long-term effects on the students' knowledge, confidence, intuition, and interest in the world of geometry and measurement.

Content Overview

Measurement

Consensus calls for measurement activities in the elementary curriculum, but with no agreement on the form or scope of these activities. Rather than stress mastery of formulas for area and perimeter, or for comparison of standard units, the lessons of this strand provide open-ended experiences within rich problem-solving situations. Measurement becomes a means for investigating problems and developing concepts, rather than an end in itself. Direct experiences with the concepts and tools is central. The carefully designed problems do lead to insights concerning accurate measurement and involving area and perimeter of rectangles. The emphasis is on the development of ideas and understanding rather than on the memorization of rules. The measurement activities become the means for investigating problems and developing concepts rather than an end in themselves.

Two lessons in this strand provide experiences with constructing boxes from centimeter cubes. After an introduction to calculating volumes of boxes by multiplication of edge lengths analogous to finding areas of rectangles, students are challenged to construct boxes with specified volumes. The central role of cubes as units of volume measure warrants their special consideration. A strong connection to the World of Numbers strand occurs in the problem of decomposing positive integers as the sum of cubes of integers.

Students exercise their spatial abilities by studying maps of cubes; that is, arrangements of six squares which can be folded to construct a cube. The mapping problem posed for dice leads to a more detailed consideration of the surface of a cube. This in turn leads naturally to finding surface areas. Extending the mapping technique to boxes provides an effective means of calculating surface areas for rectangular boxes.

The lessons in this strand are supplemented regularly with pages in the *Selection of Problems* workbooks. In particular, the workbooks contain practice with linear measure and with finding area and perimeter of polygons.

Lessons: G1, 2, 5, 6, and 12

Symmetry

Suggestions of reflective symmetry are everywhere; the human body; the design of an airplane or an automobile; a soaring sea gull; a candelabra; a snowflake. The architectural embodiments of symmetry indicate its role even in the earliest geometry. A rich and fascinating study of symmetry and related ideas has developed over many centuries. Observations of symmetry provide powerful problem-solving techniques in many areas, not just those involving geometrical notions.

Two lessons in this strand extend earlier experiences with reflections and using mirrors to explore their properties. The lessons examine the relationships in terms of movement between a shape and its reflections. With a double mirror placed on a square corner, students investigate four types of reflections as composite reflections in a single mirror and describe these composites in terms of other movements. After reviewing and extending earlier experiences with composition of reflections using a double mirror, the lessons present the idea of multiple reflections with mirrors placed in parallel. Students explore relationships between component designs in a pattern formed by reflecting a design infinitely many times. These experiences complement those in the subsequent lessons on tessellations.

GEOMETRY & MEASUREMENT INTRODUCTION

Four lessons in this strand begin an exploration of tessellations. First students review the lines of symmetry of a regular polygon and, using a variety of regular polygons, create designs that tessellate the plane. They discover many interesting characteristics of tessellation patterns; for example, when one shape is used to form a tessellation, it must be a triangle, a quadrilateral, or a hexagon (and only certain hexagons will do). They learn how to code a tessellation in terms of the number of sides in the shapes that meet at a corner in the tessellation. Following the experiences with tessellation patterns made up of polygons, the lessons present a technique for creating non-polygonal shapes that will tessellate the plane. Using such shapes, students create their own Ecsher-type designs. Further they investigate using glide reflections and double glide reflections to create tessellation patterns.

In previous semesters, lessons making use of the translator developed some geometrical concepts using parallelism. They explored the motion, or transformation, geometry ideas of a translation and a parallel projection. The *IG-III* and *IV* lessons on symmetry provide experience with another motion geometry idea, that of a reflection. In keeping with the spirit of *CSMP* geometry activities, these concepts from transformation geometry build from experience.

Lessons: G3, 4, 7, 8, 9, and 10

A Grid Game

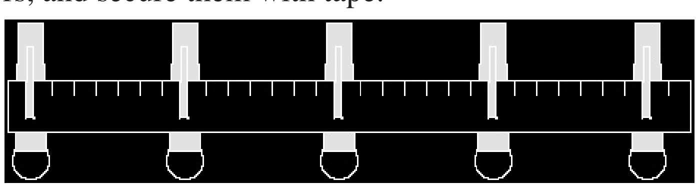
One lesson in this strand presents a game using the coordinate grid and thereby gives practice in corresponding points of the grid with ordered pairs of numbers. The game involves marking the four corners of a square with a random procedure for deciding which corners may be marked. It provides informal analytic geometry experience and develops spatial perception in still another kind of activity. Although the game is introduced in just one lesson, it could be played collectively or with a small group of students at other times.

Lesson: G11

Note on Grids

Several lesson call for demonstration on a grid board. A portable, commercially made, permanent grid chalkboard is ideal. However, if such a grid board is not available, try one of these alternatives.

- Make a grid transparency for use on an overhead projector and use erasable overhead markers. The Blackline section includes several size grids to use in making a grid transparency.
- Printed grids on large sheets of paper are provided in classroom sets of materials for *CSMP*. Tape one of these sheets to the board, use thick felt-tipped pens for drawings, and discard the sheet after use. You may prefer to laminate or mount one of the sheets on cardboard and cover it with clear contact paper to provide a protective surface. Then use erasable markers or wax crayons for drawings. These grid sheets also are useful as graphing mats.
- For each lesson, carefully draw a grid on the chalkboard using an easy-to-make tool. Either (1) use a music staff drawer with chalk in the first, third, and fifth positions; or (2) clip five pen caps, each having the diameter to hold a piece of chalk tightly, to a ruler at intervals of 4–8 centimeters, and secure them with tape.



G1 CUBES AND VOLUME #1

Capsule Lesson Summary

Introduce volume by studying rectangular boxes. Develop the technique of calculating the volume of a box by multiplying three appropriate edge lengths. Construct boxes with specified volumes.

Materials

Teacher
- A centimeter cube and a decimeter cube
- 30 cubes all the same size
- Large box (not cubical)
- 12 cardboard strips, each 1 meter long

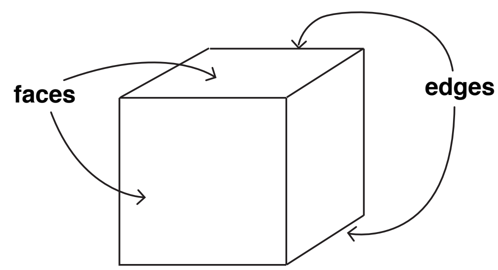
Student
- Paper

Advance Preparation: For demonstration, locate a cube with edges exactly 1 centimeter long and a second cube with edges exactly 1 decimeter (10 centimeters) long. The cubes in most sets of base-10 blocks are this size. Collect 30 cubes, all the same size, to be used for building rectangular boxes. Edge length between 2.5 cm and 5 cm, and different colors, will make the cubes easier to distinguish from a distance.

Description of Lesson

Exercise 1

Display a cube. With the class, discuss the special features of a cube, noticing its box shape, its six square faces of equal area, and its edges of equal length. A sample dialogue follows.



T: ***What is the shape of this object?***

S: ***Square.***

T (indicating one of the faces)**:** ***A side is square, but what shape is the entire object?***

S: ***It is a box.***

T: ***But this is a special box.***

S: ***It is a cube.***

T: ***Yes. What is special about a cube?***

S: ***It has sides of the same size.***

T: ***Which sides are the same size? Show us.***
We call the sides of a cube its **faces.** ***How many faces does a cube have?***

S: ***Six.***

T: ***And what shape is each face?***

S: ***Square.***

S: ***All of them are the same size.***

S: ***All of the faces are squares with the same area.***

T: ***Yes. A cube has six square faces, all of the same area.*** (Run a finger along an edge of the cube.) ***This is an edge of the cube. How many edges does a cube have?***

S: ***Twelve.***

Record the information on the board.

cube
6 faces
12 edges

Hold up a large closed box that is not a cube.

T: ***Is this box a cube?***

S: ***No, a cube has six square faces all the same size.***

S: ***That box has six faces, but they are not all squares.***

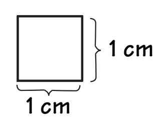
Ask students to show with their fingers how long 1 centimeter, 1 decimeter, and 1 meter are. Then ask what could be packed in a cubical box 1 cm by 1 cm by 1 cm, a box 1 dm by 1 dm by 1 dm, and a box 1 m by 1 m by 1 m. With the assistance of several students, outline a cube with edges 1 meter long, using cardboard strips. If a student suggests something too small for this size cube, ask how many of them would fit in it.

Hold up a cube with edges 1 centimeter long.

T: ***The length of one edge*** (trace with a finger) ***of this cube is 1 centimeter. What is the area of a face*** (indicate one)***?***

S: ***One square centimeter.***

Draw a square on the board to represent a face and record its area. Use the abbreviation "cm^2" for "square centimeter."



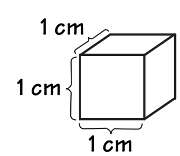
area: 1 cm^2

T: ***And what is the area of this face*** (indicate another face)***?***

S: ***Also 1 square centimeter.***

Repeat the question with the other faces to reinforce the equality of their areas.

T: ***The amount of space that an object fills is called its volume. Here is a cube that measures exactly 1 centimeter on each edge. Its volume is 1 cubic centimeter. The abbreviation that we use for cubic centimeter is "cm^3".***



volume: 1 cubic centimeter = 1 cm^3

A cube with a volume of 1 cubic centimeter is really quite small.

Pass your model around the room so that students can gain a sense of the centimeter cube.

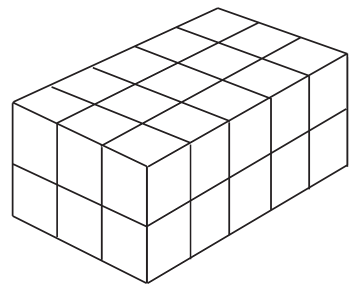
T: ***The cubic centimeter is a unit we can use to measure volume.***

Exercise 2

Display 30 (color) cubes for building boxes. Agree to treat them as if they were each 1 cubic centimeter. This convention is easily accepted by students and is similar to the convention used regularly in working with area on a grid board.

Make a pedestal on your desk with a large box. Construct a sequence of boxes (rectangular prisms) with the (color) cubes and ask students to calculate their volumes in cubic centimeters. Begin with simple boxes such as a single row of cubes, and gradually increase the complexity of the boxes. Develop the technique of calculating the volume of a box by multiplying the lengths of three appropriately chosen edges. It is not necessary to use all 30 cubes in any of the boxes you construct.

After a few simpler examples, you might construct a box like the one pictured here and lead the following discussion.



T: ***What is the volume of this box?***

S: ***There are 15 cubes in each layer. 3 x 5 = 15. With two layers there are 30 cubes in all. 2 x 15 = 30. So the volume is 30 cm³.***

S: ***I was looking at the box in a different way. There are six cubes in the front section and five such sections in all. So there are 30 cubes in all. 5 x 6 = 30.***

S: ***There is another way of looking at the box. There are ten cubes along the side. 2 x 5 = 10. There are three rows like the side, so there are 30 cubes in all. 3 x 10 = 30.***

Ask students to point to the parts of the box as they refer to them.

After a variety of boxes have been considered, show the class a cube with edges 10 cm long and ask for its volume. (1 000 cm^3)

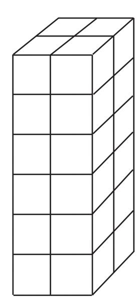
Exercise 3

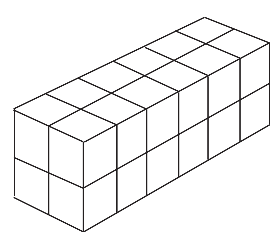
Organize the class into cooperative groups, and provide each group with 24 centimeter cubes. Direct the groups to make as many different boxes as they can with volume 24 cubic centimeters. The groups should describe the size of each box, as in this chart.

VOLUME: 24 cm^3

24 cm by 1 cm by 1 cm	6 cm by 4 cm by 1 cm
12 cm by 2 cm by 1 cm	6 cm by 2 cm by 2 cm
8 m by 3 cm by 1 cm	4 cm by 3 cm by 2 cm

Note: There are alternative descriptions of each of the boxes. For example, 6 cm by 2 cm by 2 cm describes the same box as 2 cm by 6 cm by 2 cm. The only difference is one of orientation. You may illustrate this point when such duplications are suggested by turning a box and resting it on sides of different dimensions.





Repeat the group activity using only 20 cubes to build different boxes.

VOLUME: 20 cm³	
20 cm by 1 cm by 1 cm	5 cm by 4 cm by 1 cm
10 cm by 2 cm by 1 cm	5 cm by 2 cm by 2 cm

As groups complete the above activity, suggest they list all the boxes they could construct with 120 cubes. Some groups may be slower than others in discovering the multiplication rule. At the end of the lesson, list solutions on the board.

VOLUME: 120 cm³	
1 cm by 1 cm by 120 cm	2 cm by 2 cm by 30 cm
1 cm by 2 cm by 60 cm	2 cm by 3 cm by 20 cm
1 cm by 3 cm by 40 cm	2 cm by 4 cm by 15 cm
1 cm by 4 cm by 30 cm	2 cm by 5 cm by 12 cm
1 cm by 5 cm by 24 cm	2 cm by 6 cm by 10 cm
1 cm by 6 cm by 20 cm	3 cm by 4 cm by 10 cm
1 cm by 8 cm by 15 cm	3 cm by 5 cm by 8 cm
1 cm by 10 cm by 12 cm	4 cm by 5 cm by 6 cm

Writing Activity

The last part of Exercise 3 could include a writing experience. For example, pose the following problem:

> **At the hardware store, Mr. Ace must stack boxes of nails to make a rectangular display. He has 120 boxes of nails, all the same size and all cubes. Write a letter to Mr. Ace explaining all the ways he could do this so that the display would form a rectangular box.**

You may like students to take lesson notes on some, most, or even all their math lessons. The “Lesson Notes” section in Notes to the Teacher gives some suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students may describe all the boxes that can be made with a given number of centimeter cubes.

Capsule Lesson Summary

Find boxes with a volume of 240 cm^3. Using the idea of volume, decompose integers as sums of cubes of integers.

Materials

Teacher
- 30 cubes all the same size
- Large box

Student
- Paper
- Worksheets G2*, **, and ***

Advance Preparation: Use the same 30 cubes and the same large box that you used in Lesson G1.

Description of Lesson

Exercise 1

Place a large box on your desk to serve as a pedestal. Display 30 color cubes.

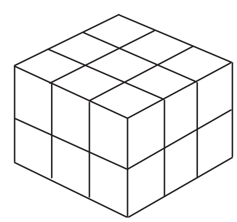
T: ***Let's pretend that an edge of each cube is 1 centimeter. What would the volume of each cube be?***

S: ***One cubic centimeter.***

T: ***I will build some boxes with these cubes. Tell me their volumes.***

Construct a variety of rectangular boxes with the cubes on the pedestal. Each time, ask a student to find the volume of the box and to indicate the technique used. Reinforce the technique of multiplying the three dimensions of a box to find its volume.

An example of a box is pictured here and a discussion follows.



T: ***What is the volume of this box?***

S: ***18 cm^3.***

T: ***How did you find it?***

S: ***I counted nine cubes on the top layer and added another nine cubes for the bottom layer.***

S: ***I multiplied 3 x 3 x 2.***

When students seem comfortable with the multiplication technique for calculating the volume of a box, continue.

T: ***If we had 240 cubes, what different sized boxes could we build using all 240 cubes?***

List solutions on the board as they are suggested, and check to see if they are correct by multiplying the dimensions. There are many solutions to this problem. It provides an opportunity for students to test their abilities to visualize constructing boxes. When a student suggests a box, ask what the box looks like. For example, a student might say "long and skinny," "almost a cube," "a little wider than the last one described," and so on.

Continue the exercise only as long as solutions are forthcoming.

VOLUME: 240 cm^3	
1 cm by 1 cm by 240 cm	2 cm by 4 cm by 30 cm
1 cm by 2 cm by 120 cm	2 cm by 5 cm by 24 cm
1 cm by 3 cm by 80 cm	2 cm by 6 cm by 20 cm
1 cm by 4 cm by 60 cm	2 cm by 8 cm by 15 cm
1 cm by 5 cm by 48 cm	2 cm by 10 cm by 12 cm
1 cm by 6 cm by 40 cm	3 cm by 4 cm by 20 cm
1 cm by 8 cm by 30 cm	3 cm by 5 cm by 16 cm
1 cm by 10 cm by 24 cm	3 cm by 8 cm by 10 cm
1 cm by 12 cm by 20 cm	4 cm by 4 cm by 15 cm
1 cm by 15 cm by 16 cm	4 cm by 5 cm by 12 cm
2 cm by 2 cm by 60 cm	4 cm by 6 cm by 10 cm
2 cm by 3 cm by 40 cm	5 cm by 6 cm by 8 cm

Exercise 2

Display 30 cubes, all the same size, on the pedestal. Reiterate that you will pretend they are all 1-centimeter cubes.

T: ***What is the largest cube that we can build with these 30 cubes?***

There may be some confusion in the beginning. Of course, the largest cube is 3 cm by 3 cm by 3 cm and has a volume of 27 cm^3. Students might start by constructing boxes using all of the blocks. Accept these solutions for discussion to observe that they are not cubes. Students may take a little time to realize that no cube can be constructed using all 30 of the cubes and that not all 30 cubes need to be used.

When the problem is solved to the satisfaction of most of the class, repeat the problem with several other numbers of centimeter cubes. For example:

- the largest cube with at most 15 centimeter cubes: 2 cm by 2 cm by 2 cm; volume 8 cm^3
- the largest cube with at most seven centimeter cubes: 1 cm by 1 cm by 1 cm; volume 1 cm^3
- the largest cube with at most 100 centimeter cubes: 4 cm by 4 cm by 4 cm; volume 64 cm^3

T: ***Now find the largest cube that we can build with 200 centimeter cubes.***

S: ***5 cm by 5 cm by 5 cm.***

T: ***What would its volume be?*** (125 cm^3)

Write the information on the board.

$$200 \longrightarrow 125$$

T: ***How many centimeter cubes are left over?*** (75)

T: ***What is the largest cube that can be built with the remaining 75 centimeter cubes?***

S: ***A cube 4 cm by 4 cm by 4 cm.***

T: ***What would its volume be?*** (64 cm^3)

Add to the information on the board.

$$200 \longrightarrow 125 + 64$$

T: ***How many centimeter cubes are left over?*** (11)
What is the largest cube that can be built with the remaining 11 centimeter cubes?

S: ***A cube 2 cm by 2 cm by 2 cm; its volume would be 8 cm^3.***

T: ***How many centimeter cubes are left over?*** (3)

S: ***We can't build any more cubes.***

S: ***But each one is a cube itself.***

$$200 \longrightarrow 125 + 64 + 8 + 1 + 1 + 1$$

T: ***Here is another way to write the volume of a cube. What are the measurements of the cube with volume 125 cm^3?***

S: ***5 cm by 5 cm by 5 cm.***

T: ***We write 5^3 for $5 \times 5 \times 5$; so $5^3 = 125$. We say "125 is the cube of 5."***
64 is also the cube of a number; which number?

S: ***4, because $4 \times 4 \times 4 = 64$.***

S: ***Then we write 4^3 for $4 \times 4 \times 4$; so $4^3 = 64$.***

T: ***What about 8?***

S: ***$2^3 = 8$, because $2 \times 2 \times 2 = 8$ and we write 2^3 for $2 \times 2 \times 2$.***

T: ***What about 1?***

S: ***$1^3 = 1$.***

In the course of this discussion, add to the information on the board.

$$200 \longrightarrow 125 + 64 + 8 + 1 + 1 + 1$$

$$200 = 5^3 + 4^3 + 2^3 + 1^3 + 1^3 + 1^3$$

Worksheets G2*, **, and *** are available for individual work. If time remains at the end of the lesson, conduct a short discussion to compare solutions.

Name ____________ G2 ☆

Find the largest cube that can be built with the given number of centimeter cubes. One problem is done for you.

Number of Centimeter Cubes	Dimensions	Volume
132 centimeter cubes	5 cm by 5 cm by 5 cm	125 cm^3
88 centimeter cubes	4 cm by 4 cm by 4 cm	64 cm^3
324 centimeter cubes	6 cm by 6 cm by 6 cm	216 cm^3
25 centimeter cubes	2 cm by 2 cm by 2 cm	8 cm^3
113 centimeter cubes	4 cm by 4 cm by 4 cm	64 cm^3
621 centimeter cubes	8 cm by 8 cm by 8 cm	512 cm^3

Name ____________ G2 ☆☆

Find the largest cube possible that can be built with 435 centimeter cubes. Then find the largest cube possible that can be built with the extra centimeter cubes. Keep doing this until no centimeter cubes are left over. The problem is started for you.

Dimensions	Volume	Number of Centimeter Cubes Available
7 cm by 7 cm by 7 cm	343 cm^3	435
4 cm by 4 cm by 4 cm	64 cm^3	92 (435 − 343 = 92)
3 cm by 3 cm by 3 cm	27 cm^3	28 (92 − 64 = 28)
1 cm by 1 cm by 1 cm	1 cm^3	1 (28 − 27 = 1)
		0 (1 − 1 = 0)

Find the largest cube possible that can be built with 664 centimeter cubes. Then find the largest cube possible that can be built with the extra centimeter cubes. Keep doing this until no centimeter cubes are left over.

Dimensions	Volume	Number of Centimeter Cubes Available
8 cm by 8 cm by 8 cm	512 cm^3	664
5 cm by 5 cm by 5 cm	125 cm^3	152 (664 − 512 = 152)
3 cm by 3 cm by 3 cm	27 cm^3	27 (152 − 125 = 27)
		0 (27 − 27 = 0)

Name ____________ G2 ☆☆☆

Find the largest cube possible that can be built with 1500 centimeter cubes. Then find the largest cube possible that can be built with the extra centimeter cubes. Keep doing this until no centimeter cubes are left over.

11 cm by 11 cm by 11 cm	1331 centimeter cubes
5 cm by 5 cm by 5 cm	125 centimeter cubes
3 cm by 3 cm by 3 cm	27 centimeter cubes
2 cm by 2 cm by 2 cm	8 centimeter cubes
2 cm by 2 cm by 2 cm	8 centimeter cubes
1 cm by 1 cm by 1 cm	1 centimeter cubes
	1500 centimeter cubes

$$1500 = 11^3 + 5^3 + 3^3 + 2^3 + 2^3 + 1^3$$

Find the largest cube possible that can be built with 2045 centimeter cubes. Then find the largest cube possible that can be built with the extra centimeter cubes. Keep doing this until no centimeter cubes are left over.

12 cm by 12 cm by 12 cm	1728 centimeter cubes
6 cm by 6 cm by 6 cm	216 centimeter cubes
4 cm by 4 cm by 4 cm	64 centimeter cubes
3 cm by 3 cm by 3 cm	27 centimeter cubes
2 cm by 2 cm by 2 cm	8 centimeter cubes
1 cm by 1 cm by 1 cm	1 centimeter cubes
1 cm by 1 cm by 1 cm	1 centimeter cubes
	2045 centimeter cubes

$$2045 = 12^3 + 6^3 + 4^3 + 3^3 + 2^3 + 1^3 + 1^3$$

Capsule Lesson Summary

Describe in terms of movements how a shape is related to its reflections. Look at a shape positioned between two mirrors forming a square corner, and determine the result of two reflections. Play *The Reflection Game*.

Materials

Teacher
- Tracing paper
- Colored markers
- Teacher triangle set
- Colored chalk

Student
- Unlined paper
- Double mirror
- Student triangle set
- Worksheets G3 (no star), *, **, and ***

Advance Preparation: Prepare a double mirror for each student. Lay two mirrors facedown, touching side-to-side. Place a piece of sturdy packing tape across them to form a hinge. With colored markers, draw on tracing paper a rectangle approximately 20 cm by 30 cm, similar to the rectangle on Worksheet G3 (no star). Then turn the tracing paper over and draw what appears through the paper. One side of the paper will have a rectangle, and the other side will have its reflection.

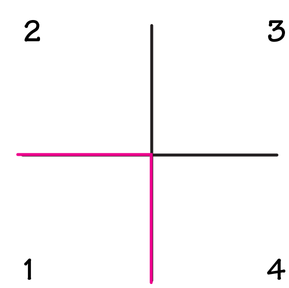
Description of Lesson

Exercise 1

Distribute double mirrors and copies of Worksheet G3 (no star) to students. Draw a pair of perpendicular line segments on the board.

T: ***Place your double mirror on the square corner in red on Worksheet G3 (no star). Where are the mirrors in the picture on the board?***

Invite a student to draw in red over the line segments that represent the double mirror.



T: ***How many rectangles do you see in the mirrors?*** (Four)

Label regions in the picture on the board.

T: ***In which of these regions is the rectangle on the worksheet?*** (Region 1)

Using the tracing paper, display the rectangle that is similar to the image that students are seeing in region 3, and orient it the same way.

T: ***In which region does the image look like this?*** (Region 3)

Flip and turn the paper several times before displaying another rectangle and asking for its location. Continue until all four images have been displayed at least once.

Using the tracing paper, display the rectangle that is similar to the image that students are seeing in region 2, and orient it the same way.

T: ***How should we move the paper to display a rectangle like the one in region 1?***

S: ***Flip the paper top to bottom.***

Ask a student to perform this action with the paper. If some students refer to this action as "turning the paper," accept this description as correct, but encourage students to use the word *flip*. Using the tracing paper, display the rectangle that is similar to the one in region 1, and orient it the same way.

T: ***How should we move the paper to show a rectangle like the image in region 3?***

S: ***Flip the paper top to bottom, and then flip it sideways.***

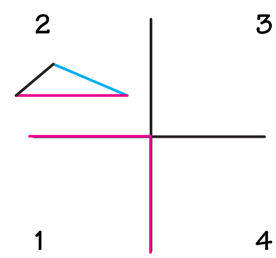
S: ***Turn the paper upside down.***

Invite students to demonstrate each of these actions. Continue this activity by asking for movements needed to display the image of the rectangle seen in:

- Region 4 (Flip the paper top to bottom.)
- Region 2 (Flip the paper top to bottom and then sideways, or turn the paper upside down.)
- Region 3 (Flip the paper sideways.)
- Region 1 (Turn the paper upside down, or flip the paper top to bottom and then sideways.)

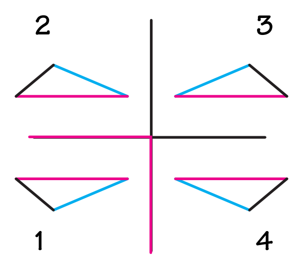
Exercise 2

Instruct students to draw perpendicular line segments on their papers, and then to place the double mirror as indicated in the picture on the board.



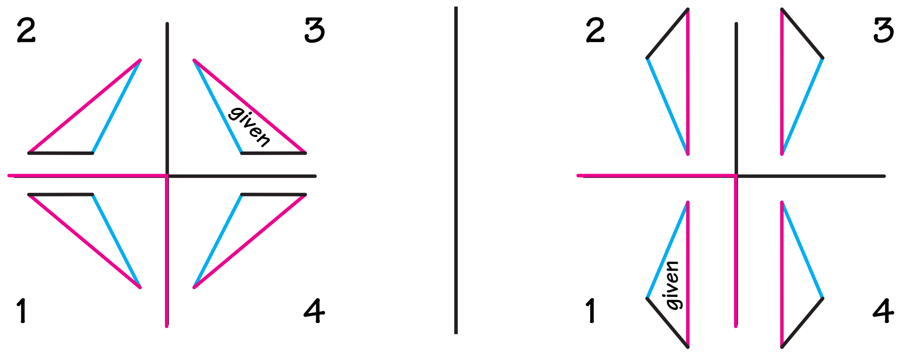
Distribute triangle sets to students and place a demonstration triangle in the picture on the board, as shown here.

T: ***If this triangle is the image that we see in region 2, who can place a triangle to show the image that we would see in region 4?***



Invite a student to place a triangle in region 4; then check the response by inviting other students to place triangles for the images in regions 1 and 3.

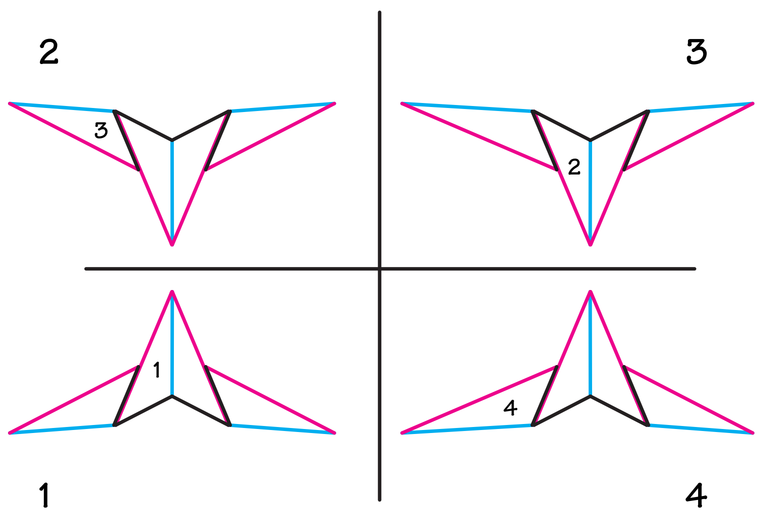
Remove the triangles from the board and repeat this activity with the following situations. Place the triangles labeled "given" yourself, and invite students to place the other triangles in any order.



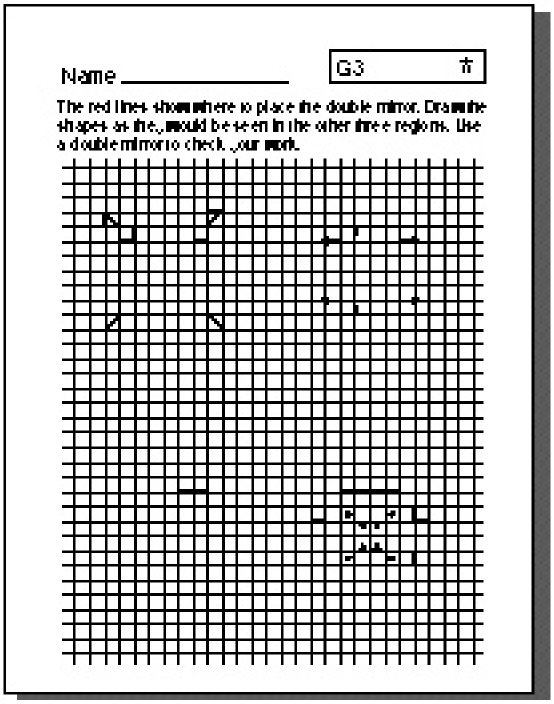
Exercise 3

Play *The Reflection Game* (see *IG-III* Lessons G5 and G9) in the usual way with double mirrors, except each time you place a triangle in the picture, ask a student to first place the image that would be seen in the non-adjacent region before asking students to place triangles in the other two regions. For example, when you place a triangle in region 1, ask a student to place the image that would be seen in region 3 next.

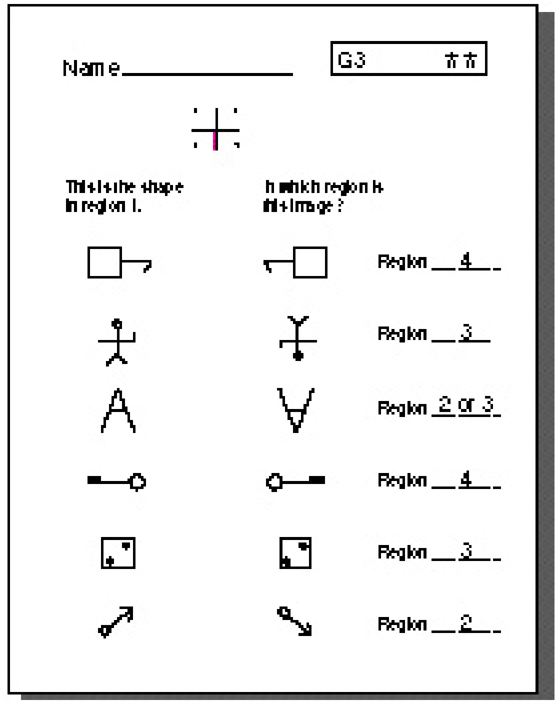
Place the triangles on the board in the order indicated by the numbered triangles in the next illustration. The reflections of those triangles are shown without numerals.



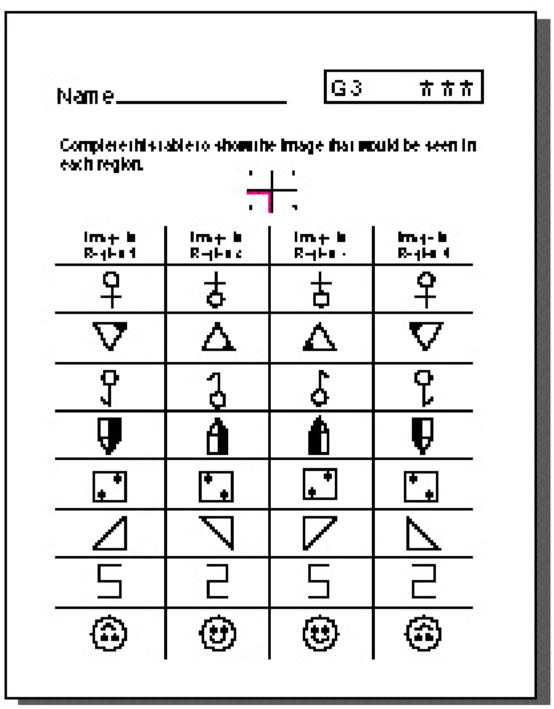
Worksheets G3*, **, and *** are available for individual work.


Name ____________ G3 ☆

The red lines show where to place the double mirror. Draw the shapes as they would be seen in the other three regions. Use a double mirror to check your work.


Name ____________ G3 ☆☆

This is the shape in region 1.	In which region is this image?	
		Region 4
		Region 3
		Region 2 or 3
		Region 4
		Region 3
		Region 2


Name ____________ G3 ☆☆☆

Complete this table to show the image that would be seen in each region.

Capsule Lesson Summary

Determine how multiple reflections appear with parallel lines of symmetry. Examine the relationships between designs in a pattern that is formed by reflecting a design infinitely many times in four directions.

Materials

Teacher
- Teacher triangle set

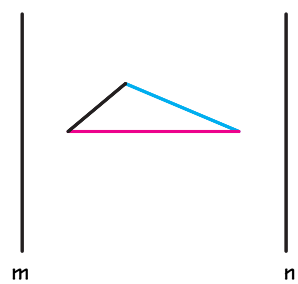
Student
- Unlined paper
- Double mirror
- Student triangle set
- Colored pencils, pens, or crayons
- Worksheets G4(a) and (b)

Description of Lesson

During this lesson, let students work with a partner. Give two double mirrors and one triangle set to each pair of students.

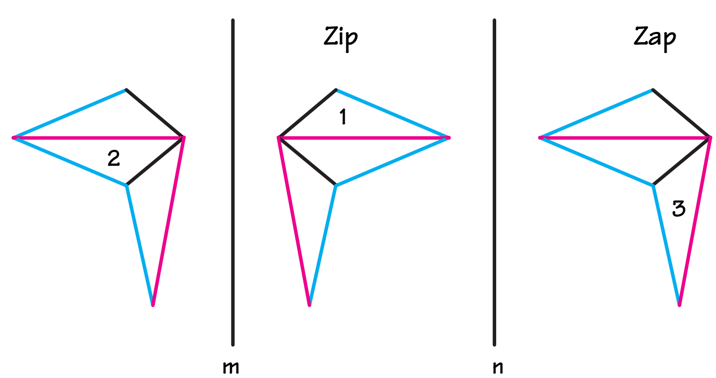
Exercise 1

Draw parallel line segments **m** and **n** on the board, and place a demonstration triangle between them.



T: ***Previously, we only looked at double reflections when the mirrors formed a square corner. Let's see what happens when two mirrors are parallel to and facing each other.***

Invite students to place triangles to show the reflections in mirror **m** and mirror **n**. Then continue this activity by placing triangles in the order indicated by the numbered triangles in the following illustration and inviting students to place triangles to indicate the reflections, which are shown without numerals. Call the middle design Zip and the design on the right Zap.



T (pointing to the design on the left)**:** ***Is this design Zip or Zap?***

S: ***Zap. It is exactly like the design on the right.***

T: ***Build a design like Zip with your partner. Then fold your double mirror back to form a single mirror. Place your mirror parallel to your partner's mirror with Zip between them as in the picture on the board. What do you see in the mirrors?***

S: ***Designs that go on forever in two directions.***

T: ***Where do the other images come from?***

S: ***They are reflections of reflections.***

T: ***Which of the images are Zap?***

S: ***Every other image.***

T: ***Which design is two reflections away from Zip; that is, the second image in the mirror?*** (Zip)
Three reflections away from Zip? (Zap)
Eight reflections? (Zip)
1 000 001 reflections? (Zap)

T: ***How can we tell if the design will be Zip or Zap by the number of reflections that it is away from Zip?***

S: ***Images that are an even number of reflections away will be Zip, and images that are an odd number of reflections away will be Zap.***

Exercise 2

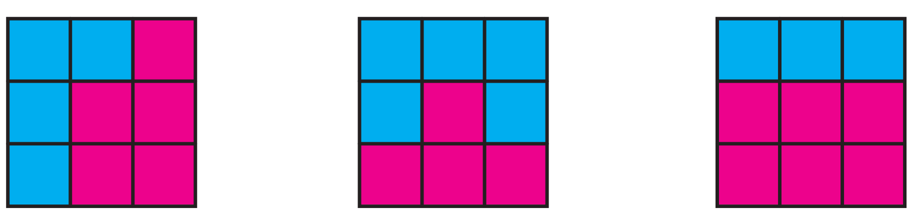
Distribute copies of Worksheets G4(a) and (b).

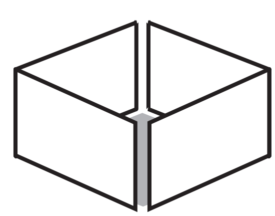
T: ***Listen carefully. Each pair of you use one copy of Worksheet G4(a). Following dotted lines, draw a path from one side of the large square to the opposite side. The path should divide the square into two parts.***

Check students' work, and ask those who do not follow instructions to start again using the partner's copy of Worksheet G4(a).

T: ***Working together, color one part of your square design red and the other part blue. Next, copy your square design from Worksheet G4(a) onto the 3-by-3 square marked with an*** **X** ***on Worksheet G4(b).***

Allow a couple minutes for students to complete these tasks. Students' square designs should resemble one of these designs.





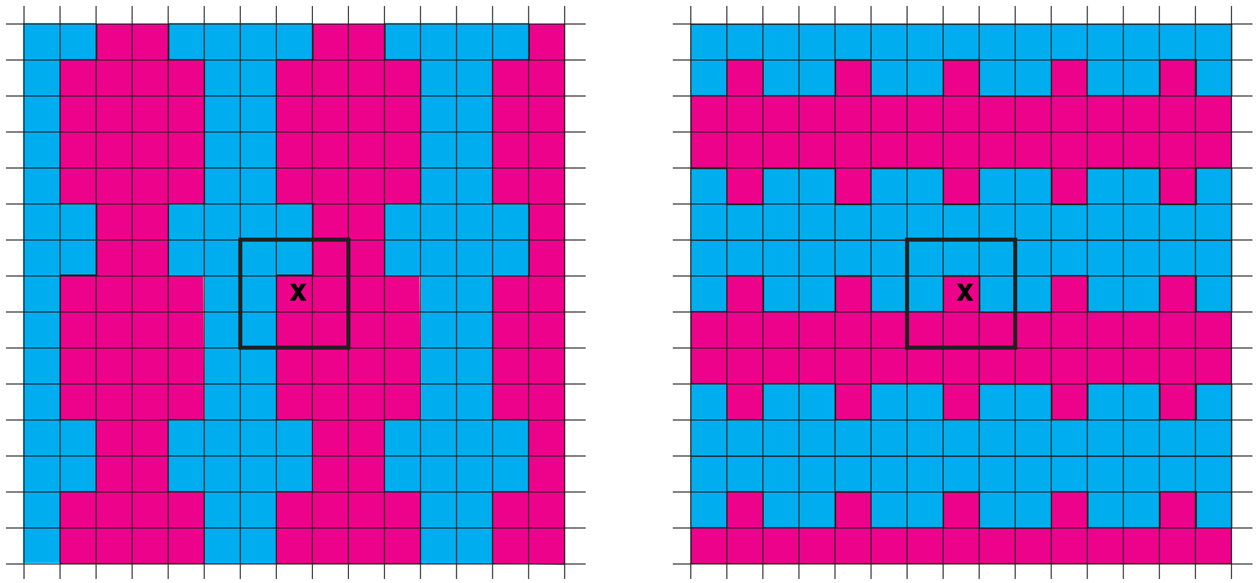
T: ***Open the double mirrors to form square corners, and place them on the sides of the large square on Worksheet G4(a). What do you see?***

S: ***A pattern that goes on forever.***

T: ***Look carefully at what you see in the mirrors and copy the pattern onto Worksheet G4(b). You have already drawn one of the square designs in your pattern. Which design is it?***

S: ***The design that is between the mirrors.***

Allow time for students to complete the pattern on their worksheets. Then hold a discussion about the results. The following are examples from the first two designs on the preceding page.



Students may observe some of the following:

- Adjacent 3-by-3 square designs are reflections of each other.
- 3-by-3 square designs two (or four) reflections apart are either the same or one turned upside down (one is a 180º rotation of the other).
- The overall pattern is repeating.

T: ***The square design can be turned upside down with one reflection left or right followed by one reflection up or down. Look for other square designs on the worksheet that are the original design upside down. Exactly how many reflections in each direction are they away from the original design?***

S: ***One reflection to the right and three reflections down.***

Record responses on the board.

TURNED UPSIDE DOWN
1R and 1D
1R and 3D
1R and 3U
1L and 3U
3L and 3D

T: ***Do you notice any patterns?***

S: ***Any odd number of reflections followed by an odd number of reflections will turn the square upside down.***

T: ***Look for other square designs on your worksheet that are the same as the original design. Exactly how many reflections in each direction are they away from the original design?***

Record responses on the board.

SAME
4R
4R and 4U
2R and 2D
6L and 2D

S: ***Any even number of reflections to the right or left followed by an even number of reflections up or down is the same as the original design.***

T: ***Which of the square designs that are exactly four reflections away from the original square design would be the image turned upside down of the original square design?***

S: ***The square design that is one reflection to the right and three reflections down.***

S: ***The square design that is three reflections to the left and one reflection up.***

T: ***Which of the designs that are exactly four reflections away from the original design would be the same as the original design?***

S: ***The design that is four reflections down.***

S: ***The design that is two reflections to the right followed by two reflections down.***

S: ***The design that is two reflections to the left and two reflections up.***

There are eight possibilities in each case. They are listed below.

Four reflections away

TURNED UPSIDE DOWN	SAME
1R and 3D	2R and 2D
1R and 3U	2R and 2U
1L and 3D	2L and 2D
1L and 3U	2L and 2U
3R and 1D	4R
3R and 1U	4L
3L and 1D	4D
3L and 1U	4U

Capsule Lesson Summary

Find the ways in which six squares can be positioned with adjacent sides to form a map of a cube. Cut out and fold such maps to build cubes and to study relations of parts of the cube.

Materials

Teacher
- Colored chalk
- Cardboard cube
- Blacklines G5(a)–(l)
- Scissors

Student
- 1-inch grid paper
- Scissors
- Envelope
- Colored pencils, pens, or crayons

Advance Preparation: Make copies of Blackline G5(a) so every student can have several pages of 1-inch grid paper. Use Blacklines G5(b)–(l) to make and cut out 11 maps of a cube. Also cut six squares the same size (approximately 10 cm edges) from cardboard, and tape them together to form a cube.

Description of Lesson

Exercise 1

Hold up the cardboard cube, and ask the class how many faces, edges, and corners a cube has. Record the information on the board.

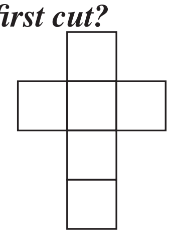
cube
6 faces
12 edges
8 corners

T: ***What is special about the edges of a cube?***

S: ***They all have the same length.***

T: ***Let's flatten this hollow cube by making cuts along some of its edges. Let's try to get just one flat piece when we are finished. Where should we make the first cut?***

Let a student trace an edge; then cut it yourself. Continue this activity until you can hold the cardboard as one flat piece, perhaps like the one illustrated here.



T: ***This is a map of the cube. It is a six-square pattern that we can fold up to make a cube.***

Fold and unfold the cardboard pattern several times. Then draw it on the board.

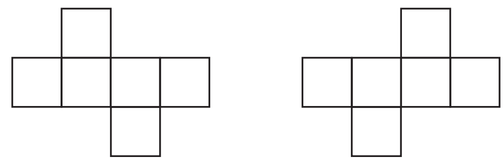
T: ***Do you think that there are other maps of a cube?***

Do not confirm or deny that there are other maps. Distribute 1-inch grid paper, scissors, and envelopes.

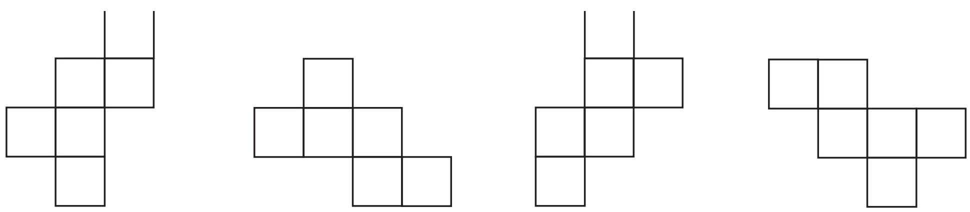
T: ***First, cut out a map for a cube exactly like the one on the board. Then, experiment to see if you can find any other maps. The maps that you make will be for a 1-inch cube, a cube with 1-inch edges, because your paper has 1-inch grid squares.***

As soon as someone discovers another map, notify the class that there are at least two maps. Allow about ten minutes for students to find different maps. When students show you maps, ask them to demonstrate that the maps fold into cubes. Each student should try to have at least three maps to put in an envelope labeled with his or her name.

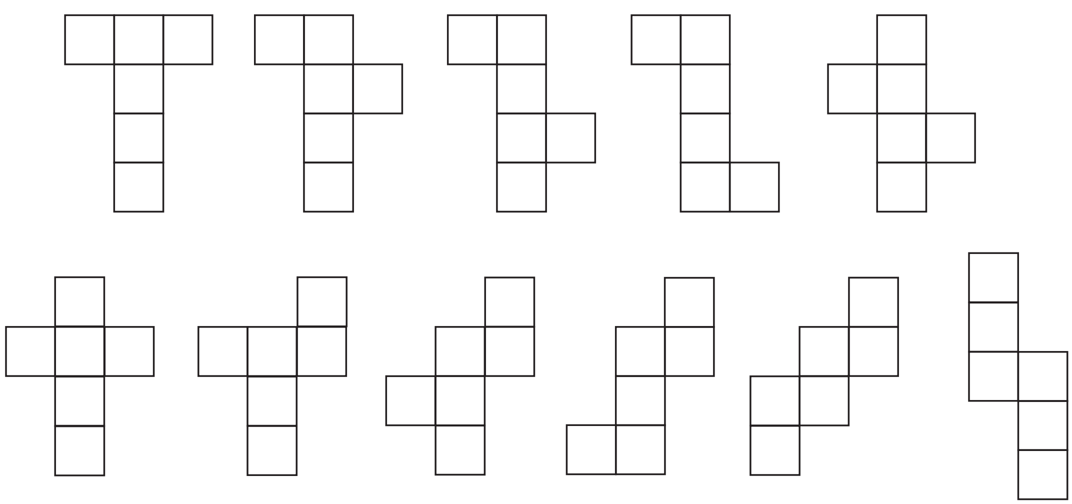
As students are working, gradually begin displaying the maps they find. Use the maps you made from Blacklines G5(b)–(l). You can poll the class to see how many students have found a particular map after you have displayed it.



Note: Students may suggest maps that differ only in orientation, as illustrated here. These two maps will be considered the same map for the purposes of this lesson, since one can be positioned exactly on top of the other by simply flipping it over. If both of these maps are suggested, demonstrate that they are identical by flipping the appropriate cut-out map. Also, make no distinctions based on rotation. For example, consider maps illustrated below to be the same.

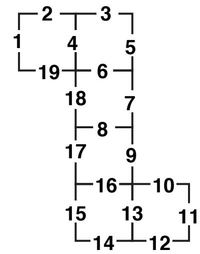


There are 11 possible maps. It is not necessary that the class finds all of them.



T (pointing to one of the maps on the board)**:** ***I see six squares for the six faces of a cube. What about the edges?***

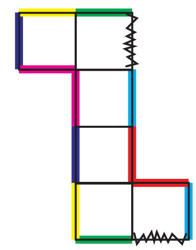
Count the line segments forming the six squares aloud. For example:



T: ***1, 2, 3, …, 19. But a cube has 12 edges.***

S: ***When you fold the map, some sides of the squares will meet to form edges.***

T: ***Come show us two segments in this map that meet in the cube.***

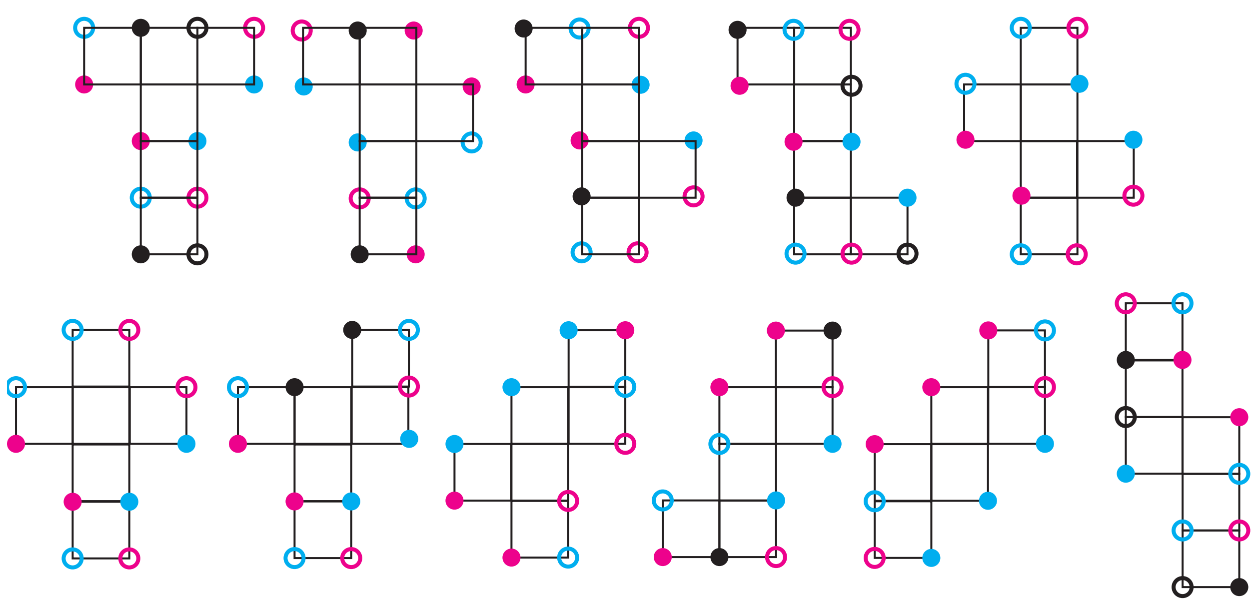


Let the student fold the display map to show two line segments that meet in the cube. Then color them alike. Continue the activity until all edges of the cube are identified.

T: ***So there are these seven edges*** (point to the colored pairs) ***and five inside edges. That makes 12 edges.***

Instruct students to use the cube maps they have in their envelopes and to mark the line segments that meet when the maps are folded. Encourage students to try to envision which segments will meet before folding the map. Students who finish quickly can make more of the possible maps and include them in their collections.

For some students, you can suggest finding the corners of the cube in a map. Just as with an edge, a corner of a cube may appear more than once in a map. Colored circles, dots, or crosses can be used to show the matching of corners. Matching corners for the 11 possible maps are shown below.



Collect students' envelopes with maps in them for use in Lesson G6.

Capsule Lesson Summary

Determine whether a configuration of connecting squares forms a map of a cube. Convince the class that there are only two ways to make a die so that opposite faces have a total of seven dots. Label maps of a cube to create maps of a die.

Materials

Teacher	• Blackline G5(a) • Four cardboard cubes • Two black markers	**Student**	• Cut-out maps of a cube • 1-inch grid paper • Scissors • Worksheets G6(a) and (b); G6* and **

Advance Preparation: Use Blackline G5(a) to make copies of 1-inch grid paper for student use. Make four cardboard cubes as you did for Lesson G5. Students will need their envelopes with maps of a cube that they made in Lesson G5.

Description of Lesson

Exercise 1

Distribute copies of Worksheets G6(a) and (b) and the students' collections of maps of a cube which were saved from Lesson G5. Have 1-inch grid paper and scissors available.

T: ***On these worksheets there are a variety of configurations of squares. Some of them are maps of a cube and some are not. Circle those that are correct maps and cross out the others. You may use grid paper to duplicate any that you wish to fold before deciding.***

Do not require that all students complete both worksheets. When many are done with G6(a), check the work collectively.

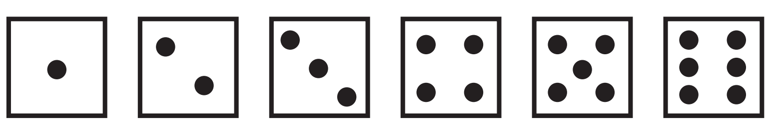
Exercise 2

T: ***Can someone describe a die for us?***

S: ***It is a cube with dots on its faces.***

S: ***There are a different number of dots on each face: one, two, three, four, five, and six.***

Draw six squares on the board, and invite students to draw the dots to illustrate the six faces.



Give two pairs of students cardboard cubes and black markers. Ask each pair to make a die by drawing dots for the faces. Hold up the dice before the class.

T: ***Do you think these two dice are exactly the same? How can we check?***

Lead a discussion, and direct it towards considering opposite faces.

Note: We are not concerned with the orientation or arrangement of the dots in determining whether two dice are the same in this lesson.

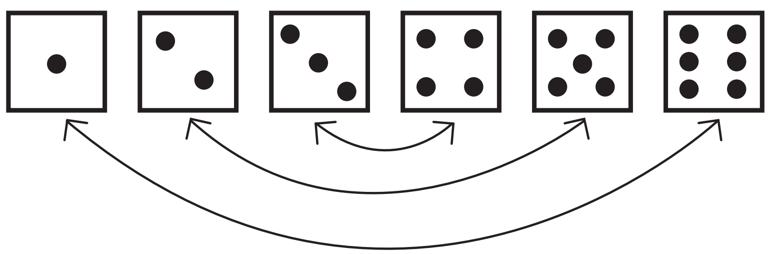
T: ***Let's first look at the faces of the cubes with one dot. I'll hold the cubes so that face of each cube is towards me. What do you see on the faces toward you?***

The class should notice whether or not the same number of dots are on those faces. Likewise, repeat the question with the other faces. It should then be clear whether the cubes are the same or not.

T: ***Is it possible to put dots on a cube so that the number of dots on opposite faces have the same sum?***

S: ***I think we could. One dot and six dots would be on opposite faces; two dots and five dots would be on opposite faces; three dots and four dots would be on opposite faces. The sum would be 7 in each case.***

Match the faces in the picture on the board.



T: ***Do you think we could put the dots on a die in more than one way and still have the number of dots on opposite faces add up to 7?***

Some students may believe there is just one way. Others may argue that there are six ways because there are six faces, or that there are three ways because there are three pairs of opposite faces. In fact, there are two ways to design a die so that opposite faces have a total of seven dots. Lead the class to this observation while constructing a die.

Display a blank cube. Hold two opposite faces between your fingers.

S: ***Put one dot on a face first.***

T: ***Does it matter which face I draw one dot on?*** (No)

Draw a dot on one of the faces and turn it toward the class.

T: ***Now what?***

S: ***Put six dots on the opposite face (toward you).***

Draw six dots on the opposite face. Show the class the face with six dots. Then with fingers holding the opposite marked faces, rotate the cube to show the class the remaining four blank faces.

S: ***Now put two dots on one of the blank faces.***

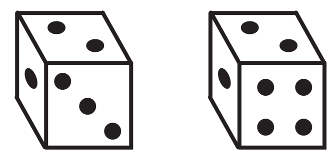
As before, hold the cube with the 1-face and the 6-face between your fingers. Rotate the cube, asking the class whether the cube would be any different if you put the two dots on a different face. (No) Draw two dots on one of the four blank faces.

S: ***Now draw four dots on the opposite face.***

Draw four dots on the face opposite the 2-face. Hold the cube with one of the blank faces toward the class.

T: ***Where should I draw three dots? Does it matter? We have two choices.***

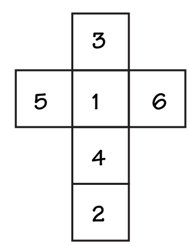
Make a replica of the cube with dots arranged the same way as on the one you have been using. Hold each with a blank face toward the class and with the same orientation of the other faces. Complete the dice in the two possible ways. Hold them with the 3-face of one toward you and the 3-face of the other toward the class.



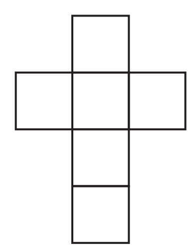
Pass the cubes around so that students can see that the cubes are different.

T: ***There are only two ways to design a die so that opposite faces have a total of seven dots. Some people refer to them as a right-handed die and a left-handed die.***

Note: Dice were not always marked in this way. The development of carefully marked and balanced dice is relatively recent in the history of such devices. One result of the modern labeling is that no cycle of four faces is favored for low or high scores in rolling. A die marked as shown here could be carefully rolled along the 3-1-4-2 faces to produce a consistently low score, or it could be rolled along the 3-6-4-5 faces to produce a consistently high score. Such manipulation of the die is prevented by the presently used labeling. You might like to label a cube as shown here to demonstrate and discuss how it is unfair with the class.

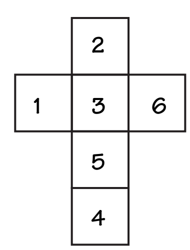


Draw this map of a cube on the board.

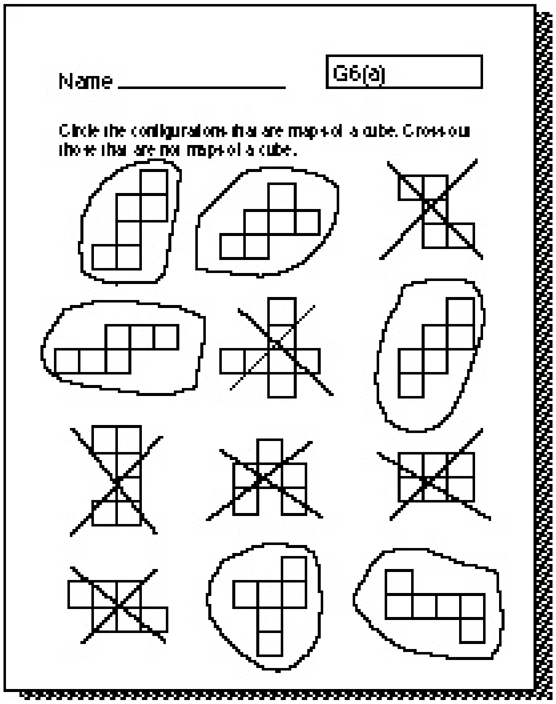


T: ***Let's mark its faces to make a map of a die.***

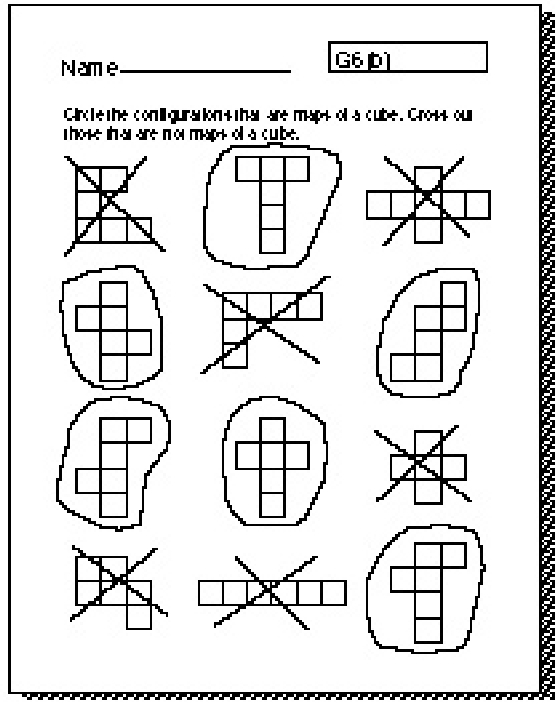
Invite students to draw the dots on the faces. Be sure that opposite sides are recognized and marked with a total of seven dots. You might prefer to write numerals rather than draw dots to speed up the process, as illustrated here.



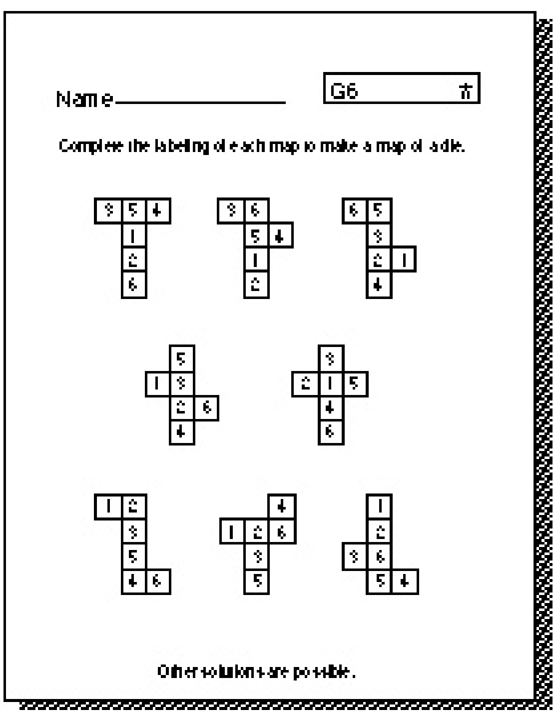
Worksheets G6* and ** are available for individual work.


Name ____________ G6(a)

Circle the configurations that are maps of a cube. Cross out those that are not maps of a cube.

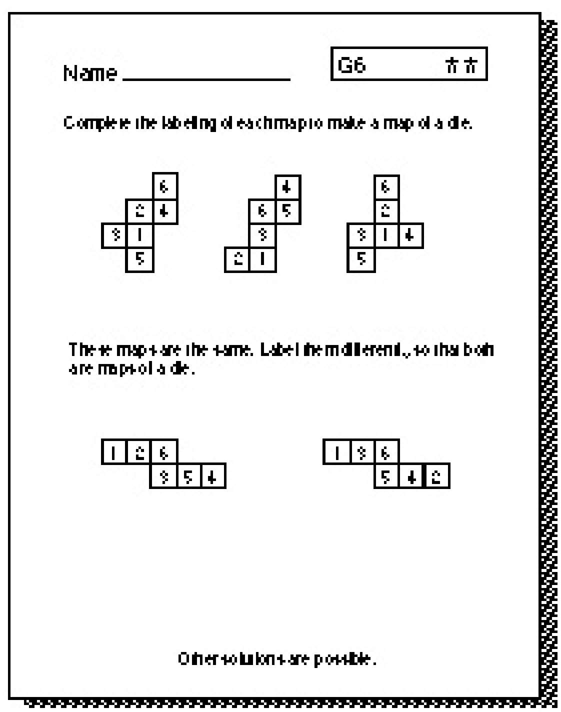

Name ____________ G6(b)

Circle the configurations that are maps of a cube. Cross out those that are not maps of a cube.


Name ____________ G6 ☆

Complete the labeling of each map to make a map of a die.

Other solutions are possible.


Name ____________ G6 ☆☆

Complete the labeling of each map to make a map of a die.

These maps are the same. Label them differently, so that both are maps of a die.

Other solutions are possible.

Capsule Lesson Summary

Determine the lines of symmetry of regular polygons. Use regular polygons to create designs that are tessellations of the plane.

Materials

Teacher	• Overhead projector • Tessellation set • Straightedge	Student	• Tessellation set • Unlined paper • Straightedge • Colored pencils, pens, or crayons

Description of Lesson

Arrange for students to work in pairs during this lesson. Each pair of students needs unlined paper and two tessellation sets, one red and one blue. Ask students in each pair to share the tessellation pieces so that each has some red and some blue pieces. Allow five to ten minutes for the students to make designs with the tessellation pieces. Challenge students to construct designs with exactly one line of symmetry, designs with exactly two lines of symmetry, and designs with exactly three lines of symmetry.

Exercise 1

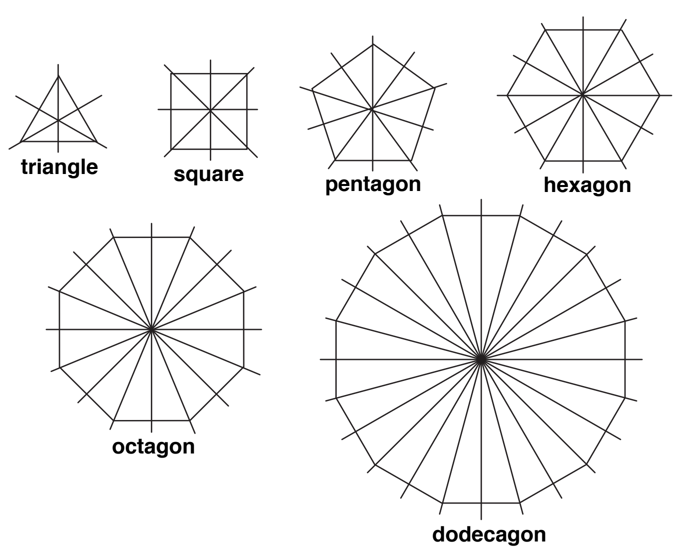
Solicit names for all the kinds of shapes in a tessellation set. (See the next illustration.) Also, solicit observations such as the following:

- All the shapes are regular (equal length sides).
- The side length of each shape is the same as any other shape.

T: ***You have several kinds of shapes. Take one of each kind and draw around it on your paper. For each shape, draw all of its lines of symmetry.***

While students are working, place one shape of each kind on an overhead and project them onto the board. After a few minutes, invite students to the board to draw the lines of symmetry for each shape. Ask the class to confirm that the lines of symmetry are correct.

Use the standard name for each shape when you refer to it. The tessellation shapes are shown here with their standard names and lines of symmetry.



T: ***What do you notice?***

S: ***Each shape has as many lines of symmetry as it has sides (or corners).***

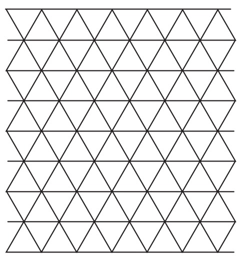
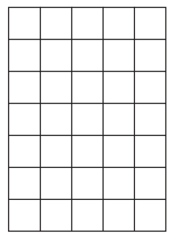
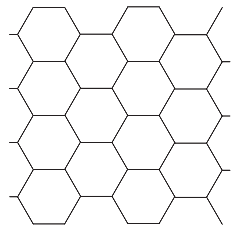
G7

Exercise 2

T: ***Imagine that these pieces are tiles with which we might like to tile a floor. If we could only use one kind of shape, with which kinds could we tile a floor? Remember, a tiling pattern covers the floor without any gaps or holes between the pieces and without overlapping pieces.***

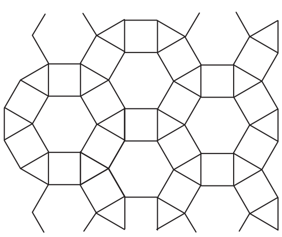
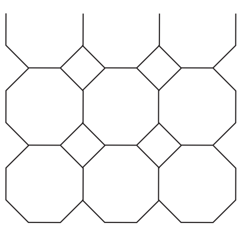
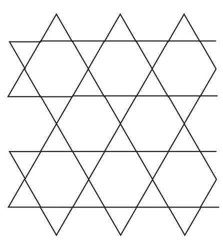
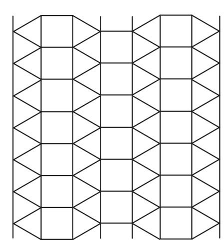
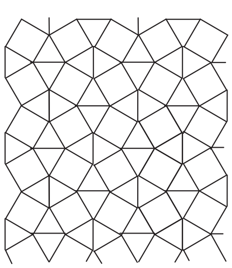
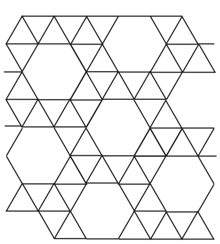
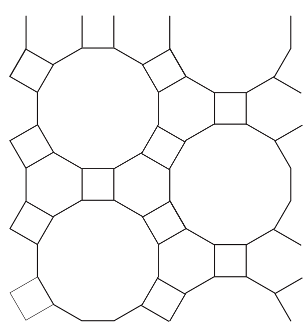
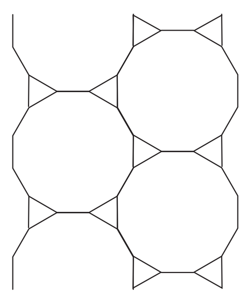
Note: You may want to suggest that the floor goes on and on, and that students should not worry about where the floor ends or meets the wall. In this exercise, students are generating tessellation patterns that potentially cover a plane.

Students should discover that only three regular polygons will tessellate: the equilateral triangle, the square, and the regular hexagon.

T: ***Now suppose that we could use more than just one kind of shape in the tiling. Try to find patterns that cover the floor without any gaps between pieces and without overlapping pieces. Use your pieces to create tiling patterns we could use.***

As students discover patterns that tessellate, ask them to record their patterns by carefully outlining parts of them on paper. Students should outline enough of a pattern to allow them to reconstruct it if necessary. Encourage students to find several different patterns. As you monitor students' work, be sure patterns are clear; that is, be sure students can explain how the patterns repeat. Save copies of the patterns drawn by students for use in Lesson G8. The following are some of the patterns students are likely to find.

Note: You may like to use the word *tessellation* in talking about the patterns. Also, students may like to color their patterns, but color does not describe the tessellations.

Home Activity

Suggest that students look for tessellation patterns in their homes or in buildings. They may notice tessellations being used for floor and wall patterns, or on rugs, quilts, clothing, furniture, wire netting, and so on. You may instruct students to sketch tessellation patterns they find to share with the class.

Capsule Lesson Summary

Determine and record a code for each tessellation created in Lesson G7. Discover that all triangles, all quadrilaterals, and some hexagons will tessellate.

Materials

Teacher
- Overhead projector
- Tessellation set
- Colored chalk

Student
- Tessellation set
- Cardboard or construction paper
- Scissors
- Colored pencils, pens, or crayons
- Tessellation patterns from Lesson G7

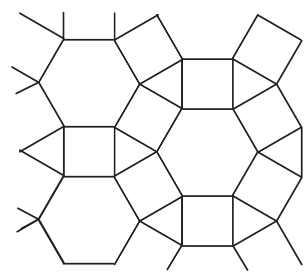
Description of Lesson

Exercise 1

Arrange for students to work in the same pairs as in Lesson G7, and return students' work from that lesson. Distribute tessellation sets and remind students that they were using pieces from these sets to make tiling patterns.

T: ***Find a tiling pattern that is different from the ones you found before. These patterns have a special name; they are called* tessellations.**

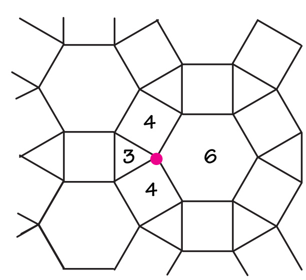
Write *tessellations* on the board, and invite a student with a tessellation involving three shapes to construct it on the overhead. For example:



T: ***We have created many different tessellations that involve only these shapes. To draw all of them on the board would take too long, but we can record them using a unique code that mathematicians often use.***

Note: You may like to put this coding problem in a story context. For example, describe a tile store that has tiles in shapes like those in the tessellation set. Explain that when the store sells tiles for a tiling pattern, rather than draw a picture, they give the buyer a code for recreating the pattern.

On the board, draw a red dot at any corner in the projected tessellation. Point to each of the shapes that has a corner at the red dot, ask the class to count how many sides it has, and record the number of sides. For example:



6-4-3-4

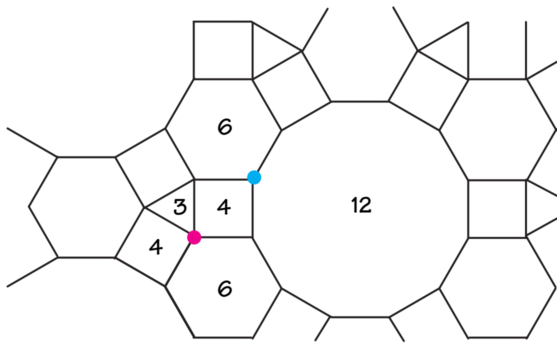
T: ***Mathematicians (the tile store) would describe this point as "6-4-3-4" (read as "six four three four").***

Note: The sequence of the numbers is important but the starting number is not. Thus, this point could also be described as 4-3-4-6, 3-4-6-4, or 4-6-4-3.

Choose several other points on the pattern and ask students to describe those points. The class should observe that 6-4-3-4 describes all of the points of this tessellation.

T: ***The code for this tessellation is 6-4-3-4 because it describes all of the points of this tessellation. Use this method to code each of the tessellations you found. You may need to use several points in some tessellations, because all of the points may not have the same description. A code must describe all the different kinds of points.***

An example of a tessellation with points having differing point descriptions is shown below. The blue point would be described as 12-6-4 and the red point as 6-4-3-4. There are only two types of points, so this tessellation would be coded 6-4-3-4/12-6-4 or 12-6-4/6-4-3-4.



After a short while, call on a pair of students to tell the class the code of a tessellation they found. Record the code on the board, and then ask other students to construct the corresponding tessellation. Repeat this by letting student pairs trade codes and try to construct each other's tessellations.

Below is a complete list of the codes for tessellations whose points all have the same description. Do not insist that students find all of them.

3-3-3-3-3-3	3-3-3-4-4	3-3-4-3-4
4-4-4-4	3-4-6-4	3-12-12
6-6-6	4-8-8	4-6-12
3-3-3-3-6	3-6-3-6	

The following is an incomplete list of codes for tessellations that have points with different descriptions. Probably only a few such tessellations will be discovered by your students.

3-12-12/3-4-3-12	3-3-4-3-4/ 3-3-4-12/3-4-3-12
3-6-3-6/3-3-6-6	3-3-3-3-3-3/3-3-4-12/3-3-4-3-4
3-3-3-3-3-3/3-3-4-12	3-3-3-4-4/3-3-4-3-4/3-4-6-4
3-4-6-4/4-6-12	3-3-3-3-3-3/3-3-3-4-4/3-3-4-3-4
3-3-4-3-4 / 3-4-6-4	3-4-6-4/3-4-4-6

Others are possible.

Exercise 2

Review which shapes by themselves can be used in a tessellation.

Place one triangle, one square, and one hexagon on the overhead.

T: ***When one shape can be used to form a tessellation, we say that the shape will tessellate. Is there anything special about a shape that tessellates?***

S: ***It has 3, 4, or 6 sides all the same length and all of its corners are the same.***

T: ***These are special cases of 3-sided, 4-sided, and 6-sided shapes. Do you think any other 3-sided, 4-sided, or 6-sided shapes tessellate? How could we answer this question?***

S: ***Make other 3-sided, 4-sided, and 6-sided shapes and see if they tessellate.***

Distribute scissors and cardboard or construction paper. Divide the class into three groups (keeping partners together). Each pair of students in the first group should cut out an irregular triangle and carefully make about 12 duplicates of it. Student pairs in the second group should do the same with irregular quadrilaterals, and student pairs in the third group should do the same with irregular hexagons. Ask that each student pair use their set of shapes to try to form a tessellation. After a few minutes, let students trade their sets of shapes with students from another group. Continue this activity until everyone has attempted to make a tessellation with an irregular triangle, with an irregular quadrilateral, and with an irregular hexagon.

After a while, ask the class if there were any sets of triangles that did not tessellate. If a set is offered, check to make sure all of the triangles are the same size and shape, and then collectively use them to form a tessellation on the overhead. Continue this activity with quadrilaterals and with hexagons.

The class should find that every triangle and every quadrilateral will tessellate, but not every hexagon. A hexagon will tessellate only if it has a pair of non-adjacent sides that are parallel and equal in length.

Extension Activity

Post a list of tessellation codes recorded during this lesson and invite students to add to this list at another time.

Capsule Lesson Summary

By deforming a square, construct a non-polygonal shape that will tessellate the plane. Use this shape to create an Escher-type print.

Materials

Teacher
- Cardstock or construction paper
- Scissors
- Tape
- Overhead projector
- Blacklines G9(a), (b), and (c)

Student
- 5 cm square
- Scissors
- Tape
- Unlined paper
- Colored pencils, pens, or crayons

Advance Preparation: Use Blacklines G9(a), (b), and (c) to make transparencies for display during this lesson. Cut 5 cm squares from cardstock, index cards, or heavy construction paper, enough for each student plus one for demonstration. You may like to trace cut lines on the demonstration square so that your tessellation piece in Exercise 1 looks like the one in the illustrations. This is the shape used to create the tessellations on Blacklines G9(b) and (c).

Description of Lesson

Exercise 1

Project the pictures on Blackline G9(a), and ask students if they have seen pictures like these before. Perhaps someone will know that these are M.C. Escher pictures.

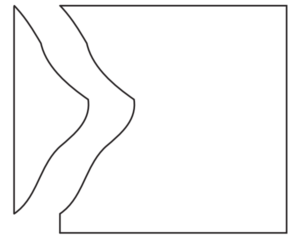
T: ***Who has heard of M.C. Escher? M.C. Escher was a Dutch artist who lived from 1898 to 1981. Many of his drawings are tessellations. But instead of using ordinary shapes like triangles or squares, he liked to create pictures of horses, fish, salamanders, and birds that would tessellate.***

Trace several examples of the shape that is repeated in each picture.

T: ***Watch carefully as I show you how you can make tessellations like Escher's.***

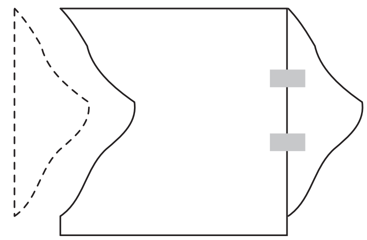
Place a demonstration 5 cm square on the overhead and ask a student to indicate how a square tessellates. Have scissors and tape readily available.

T: ***Step 1 is to cut a piece from one side of a square. The piece can be any shape.***



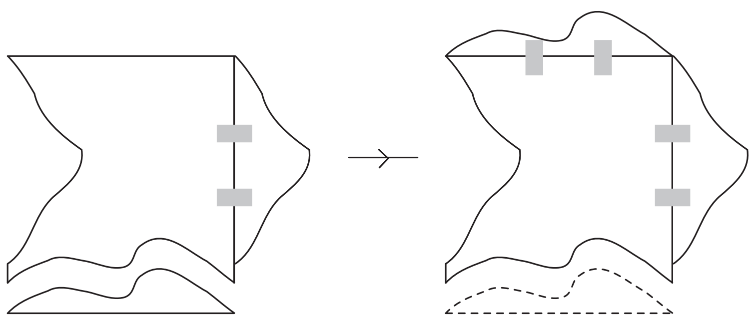
Step 2 is to slide the piece directly across the square and tape it to the other side. Do not move the piece up or down.

How will this shape tessellate?



Let a student demonstrate how the shape can be repeated right and left, and up and down.

T: ***Step 3 is to do the same thing to another side of the square. Cut out a piece, slide it to the opposite side, and tape it. Be careful not to move the piece to the right or left.***



How will this shape tessellate?

Let a student demonstrate how the shape can be repeated up and down, and right and left.

T: ***How can we actually show that this shape will tessellate?***

S: ***Make several copies of the shape and make a design.***

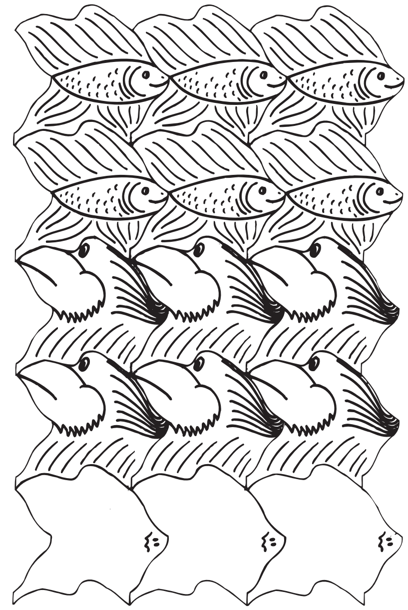
S: ***Trace around the shape on a piece of paper.***

T: ***Step 4 is to use the shape to make a tessellation.***

Project the tessellation on Blackline G9(b). If your tessellation shape matches, move it around on the transparency to show that the shape tessellates.

T: ***In Escher's tessellations, the shapes are often animals. Can you think of something this shape could be?***

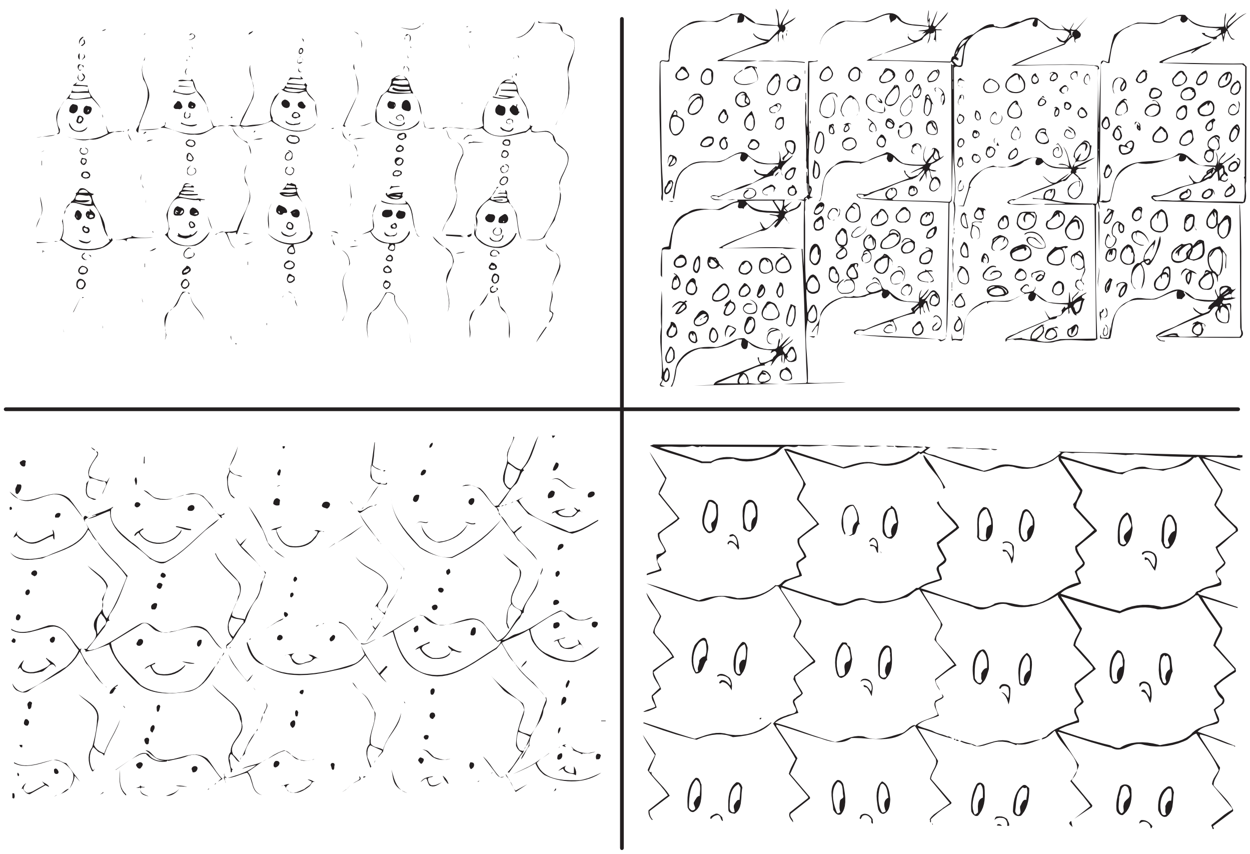
Let the students make several suggestions before placing a transparency of Blackline G9(c) on top of the one from Blackline G9(b). Here the picture shows the shape could suggest a fish, a gorilla, or a ghost.



Distribute 5 cm squares, tape, scissors, and unlined paper to students.

T: ***Use your imagination to create your own Escher-type print using the steps that I've just shown you.***

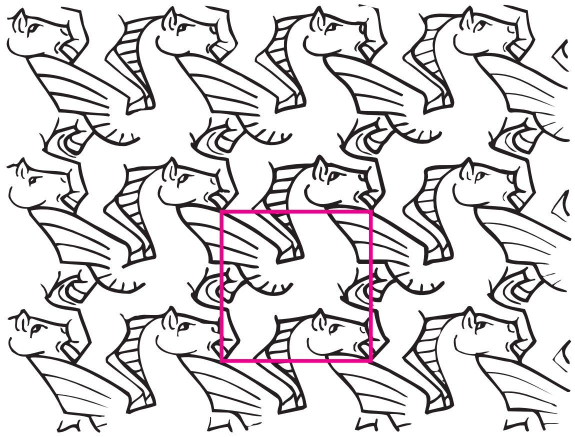
These are a few of the pictures created by students in other *CSMP* classes.



Collect students' tessellation shapes and Escher-type prints to save for use in Lesson G10. Allow at least five minutes for Exercise 2.

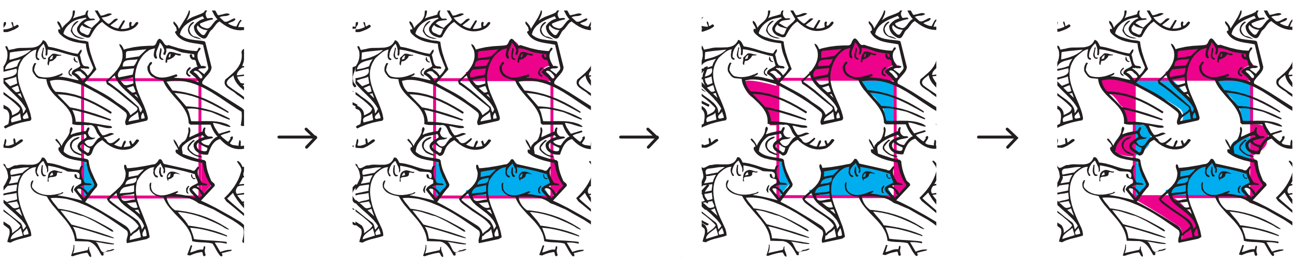
Exercise 2

Project the picture on Blackline G9(a), and draw the square indicated below. Notice that each corner of the square is at the tip of the jaw of one of the horses.



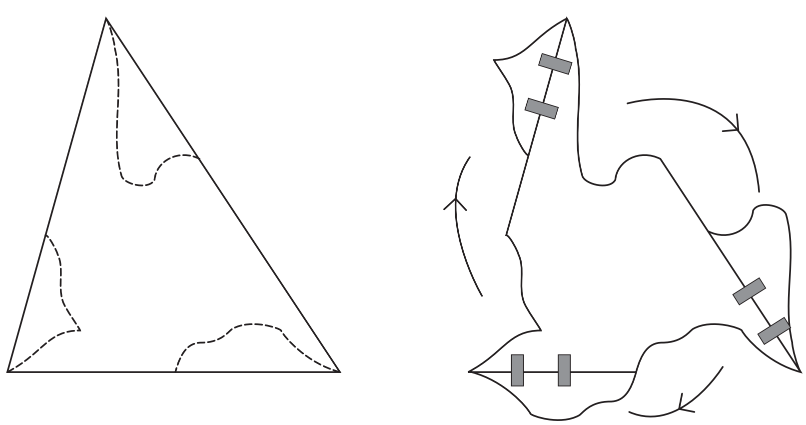
T: ***Let's start with this square and determine where pieces could be cut out and slid across the square to create this horse design.***

Invite students to suggest how the horse could be formed from the square. Shade pieces that would be moved in blue, and shade where they would be relocated in red. The following pictures show how the horse could be formed from a square.



Extension Activity

Some students may like to create Escher-type prints starting with a triangle or an irregular quadrilateral rather than a square. Deciding how to move and place pieces cut out of a side is a little more difficult in these cases. See the following illustration with a triangle.



G10 ESCHER DESIGNS #2

Capsule Lesson Summary

Use a glide reflection and then a double-glide reflection to create shapes that can be used to tessellate the plane. Use these tessellations to create Escher-type prints.

Materials

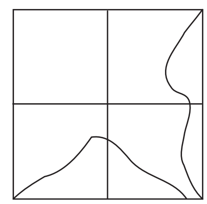
Teacher

- Overhead projector
- Blank transparency
- Erasable overhead pen
- Meter stick
- Tessellation shape from Lesson G9
- Tape

Student

- Tessellation shape and Escher-type print from Lesson G9
- Unlined paper
- Tape
- Colored pencils, pens, or crayons
- Metric ruler

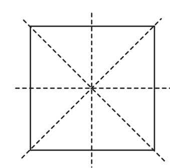
Advance Preparation: Before teaching this lesson, reassemble the pieces of the demonstration tessellation shape into a square, and accurately draw these two lines of symmetry on both sides of it.



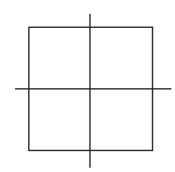
Also, draw two line segments 9 cm long that are perpendicular bisectors of each other on a blank transparency.

Description of Lesson

Return the students' tessellation shapes and Escher-type prints from Lesson G9. Students will also need paper, tape, pencils, and metric rulers.



Carefully draw a square on the board, and ask students to trace its lines of symmetry.



Invite students to the board to accurately draw the square's non-diagonal lines of symmetry.

T: ***How could we check that these are lines of symmetry?***

S: ***Place a mirror on the line and see if a square appears in the mirror.***

S: ***Measure the sides of the square to see if a line of symmetry goes through the midpoints.***

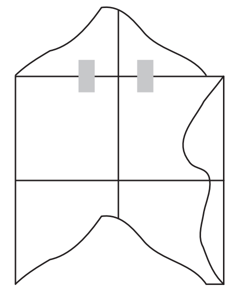
Ask students to check the lines of symmetry by measuring.

Place the demonstration tessellation square on the transparency that you prepared, aligning the line segments (lines of symmetry) on the square with the line segments on the transparency, and project them.

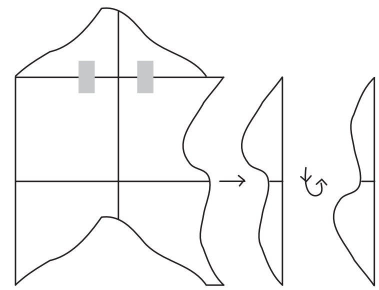
T: ***Today we are going to use these lines of symmetry to create tessellations different than the ones you made before. Using tape, rearrange the pieces of your tessellation shape into the original square. Then draw these two lines of symmetry on both sides of the square. Use a ruler to draw the lines as accurately as possible.***

Allow a few minutes for students to finish this task. Then use the tessellation shape on the overhead to demonstrate the following procedure.

T: ***Step 1 is to take the cut-out piece from one side of the square, slide it across the square, and tape it to the opposite side. Make sure the two pieces of the line of symmetry are lined up.***

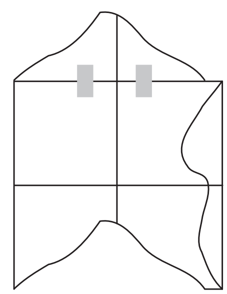


Step 2 is to take the other cut-out piece and flip it end-to-end.

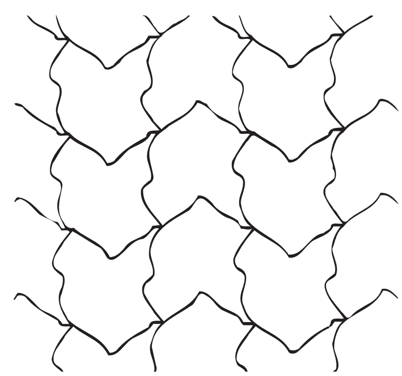


Step 3 is to slide that piece across the square and attach it to the opposite side with tape. Make sure the two pieces of the line of symmetry are lined up.

Could we use this shape to create a tessellation?



Allow students to experiment. Wipe the transparency clean and place it on the overhead. Collectively begin to create a tessellation on the transparency with the shape. Before the shape is moved to a new location on the transparency, ask a student to describe how the shape will be moved. Encourage the students to describe such a movement as "a slide" or as "a flip and a slide." Conclude that the shape can be used to create a tessellation.



T: ***Step 4 is to create a tessellation on paper by outlining the shape like we just did on the transparency. Create your own tessellation using these steps.***

Encourage students who finish quickly to create an Escher-type print by decorating their tessellations. When most everyone has drawn several copies of their shape in a tessellation, continue with the collective discussion.

T: ***Compare this new tessellation with the tessellation you made last time. How are they different?***

S: ***The design is different.***

S: ***In the first tessellation all of the shapes are the same, but in the second tessellation there are two different shapes.***

T: ***How are the two shapes related?***

S: ***They are reflections of each other.***

Ask a student to show how the two shapes are organized in the tessellation on the board. The student should indicate that the shapes in the same column (or row depending on orientation) are the same and that the shapes in adjacent columns (or rows) are different.

T: ***Why does this tessellation have reflections in it?***

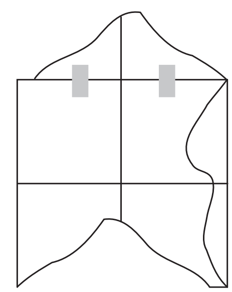
S: ***Because we flipped one of the pieces that was cut out of the square, and when you flip a shape, you get its reflection.***

T: ***What would happen if we flipped both pieces that were cut out of the square? Could that shape be used to create a tessellation?***

Let students comment; then reassemble the pieces of the tessellation pattern, and demonstrate the following procedure to the students on the overhead.

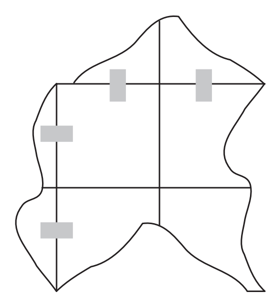
T: ***Step 1 is to take the cut-out piece from one side of the square and flip it end-to-end.***

Step 2 is to slide that piece across the square and attach it to the opposite side with tape. Make sure the two pieces of the line of symmetry are lined up.

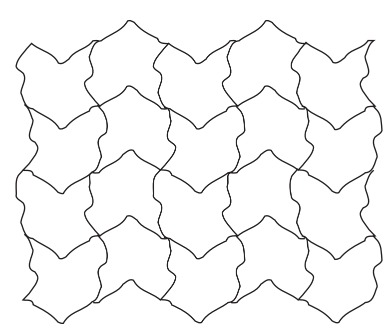


Step 3 is to take the other cut-out piece and flip it end-to-end.

Step 4 is to slide that piece across the square and attach it to the opposite side with tape. Check to see that the two pieces of the line of symmetry are lined up.



Step 5 is to create a tessellation using this shape. Create your own tessellation by following these steps.



Encourage students who finish quickly to create an Escher-type print by decorating their tessellations

Home Activity

Suggest that students take home one or more of their Escher-type prints to show family members. They should explain to the viewer how they created the Escher-type prints.

G11 A GRID GAME

Capsule Lesson Summary

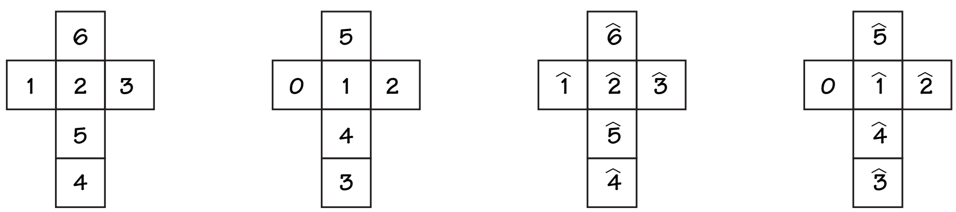
Correspond the intersection points of a grid with ordered pairs of integers. Mark four points on the grid so that they are the corners of a square. Using the results of rolling specially labeled cubes to designate ordered pairs, locate points on the grid until the four corners of a square have been marked. Play a game using four cubes in which points are marked on a grid, and the points are chosen from those that can be named from the results of rolling the cubes. The first person to mark the four corners of a square is the winner.

Materials

Teacher
- Grid
- Four specially marked number cubes
- Magnetic checkers
- Blacklines G11(a) and (b)

Student
- Cube game grid sheet
- Colored pencils, pens, or crayons

Advance Preparation: Prepare four cubes (dice or other cubes which roll well) by putting labels on their faces as indicated in these cube cutouts.



Prepare a grid board with horizontal and vertical axes as pictured in Exercise 1, or use Blackline G11(a) to make a transparency of this grid. Use Blackline G11(b) to make cube game grid sheets for students.

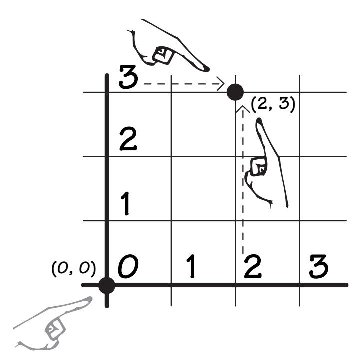
Description of Lesson

Exercise 1

Display a grid with horizontal and vertical axes drawn, as shown in the second illustration below.

Demonstrate and point to the various points on the grid as you mention them.

T: ***Each point on the grid corresponds to a couple (or ordered pair) of numbers. The intersection of the two axes, this point, is (0,0). (2,3) is another couple. The first number in a couple tells us how far to move to the right or to the left of (0,0) along the horizontal axis. For (2,3), we move two spaces to the right. The second number in a couple tells us how far to move up or down from (0,0) along the vertical axis. For (2,3) we move three spaces up. Then we bring our fingers together to locate the couple.***

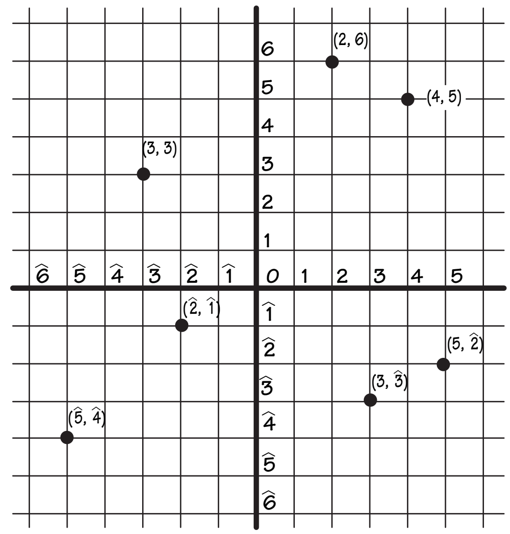


Mention several other pairs of numbers, and let students locate the corresponding points on the grid. Reinforce the rules:

- Start at (0,0), the intersection of the axes.
- The first number indicates how far to move right or left.
- The second number indicates how far to move up or down.
- Positive numbers indicate moving right or up.
- Negative numbers indicate moving left or down.

Continue until the class is secure in the technique.

Next, mark a few points on the grid and ask students to name the corresponding ordered pairs of numbers. For example, you could use the points indicated here.

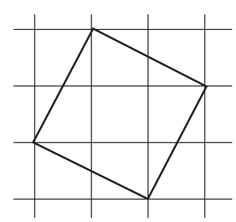


Exercise 2

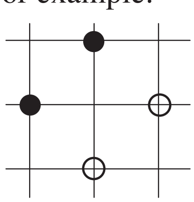
Clear the graph. If your grid board is magnetic, use magnetic checkers for convenience to mark the points; otherwise use colored chalk or markers.

T: ***Here are four checkers. Who can put them at intersection points on the grid and mark the four corners of a square?***

Ask several students to do this. Expect some error; someone will probably suggest the corners of a rectangle that is not a square. Encourage class discussion to correct any errors. There are many, many squares that can be shown on the grid using only points at the intersections of grid lines. In most classes, the last squares to be discovered are those skew to the grid lines, as illustrated here.



To exercise students' perception of squares on the grid and to introduce skewed squares if necessary, devote a few minutes to a game in which you mark three points for three corners of a square and ask a student to find the fourth corner. Then change the game by only locating two points and requiring students to find two others to complete a square. Note that there will be more than one solution to the latter problem. For example:



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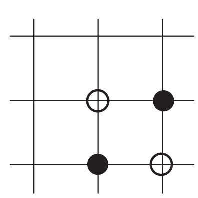

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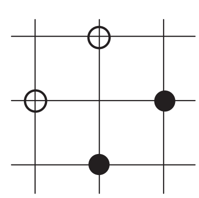
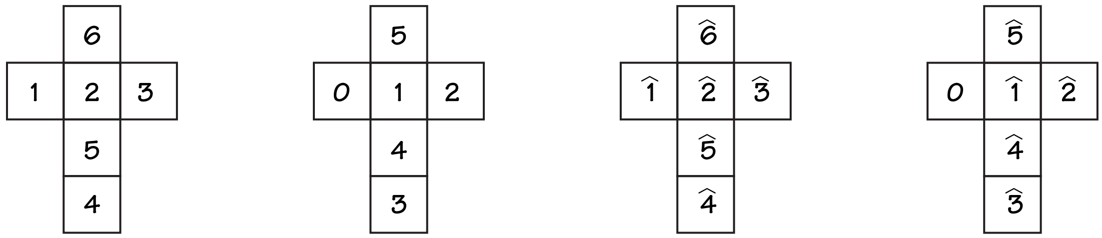


Exercise 3

Remove any marking of corners from the grid and display the four specially marked cubes. Let students describe the cubes as you picture them in maps on the board.



T: ***When we roll all four cubes, we will read the four numbers that come up on the top faces.***

Roll the cubes on a desk at the front of the room. Let a student read the numbers as you write them on the board. For example:

$\widehat{4}$ 3 $\widehat{2}$ 1

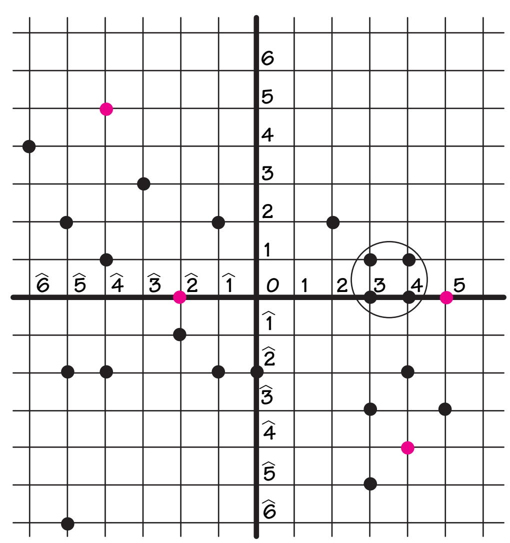
S: ***$\widehat{4}$, 3, $\widehat{2}$, and 1.***

T: ***We can use these cubes to choose points on the grid. Each time we roll the cubes, we choose two of the numbers to find a point and then mark it on the grid. Which points could we mark with this roll?***

There are 12 that could be marked with the roll given in this example.

($\widehat{4}$, 3)	(3, $\widehat{4}$)	($\widehat{2}$, $\widehat{4}$)	(1, $\widehat{4}$)
($\widehat{4}$, $\widehat{2}$)	(3, $\widehat{2}$)	($\widehat{2}$, 3)	(1, 3)
($\widehat{4}$, 1)	(3, 1)	($\widehat{2}$, 1)	(1 ,$\widehat{2}$)

T: ***After picking a point, we roll again and mark another point. The object is to mark the four corners of a square.***



Continue rolling, choosing, and marking points until the corners of a square are marked. Each time, the class should check carefully that the point chosen is correct in that the two numbers of its ordered pair show on the cubes. The result of a game with 20 cube tosses (points chosen) is shown here with black dots. The game ends when the class observes that the four corners of a square are marked. (See the circled section of the picture.) In fact, in this game the corners of another square [($\widehat{5}$, $\widehat{2}$); ($\widehat{5}$, 2); ($\widehat{1}$, 2); ($\widehat{1}$, $\widehat{2}$)] had been marked earlier but went unnoticed. Also, there were several possibilities for completing squares. Legally marking any of the three red points might also have finished the game.

If the class is confident in the procedure of the game, proceed to Exercise 4. Otherwise, play another collective game.

Exercise 4

G11

Distribute copies of the cube game grid sheet on Blackline G11(b).

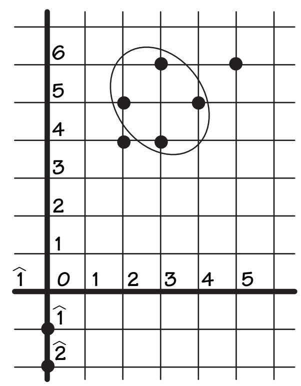
T: ***Now let's play the game individually. I will roll the cubes and write the results of each roll on the board. Each time, you may choose two of the numbers to locate a point on your grid. You may not change a point once you have marked it. Let's see who will mark the corners of a square first.***

Play the game. List the results of each roll in a column on the board. Continue until a student announces having completed a square. Ask the student to mark the corners of the square on the grid board for checking by the class. Verify the legality of each of the suggested corners by comparing the ordered pairs for the corners against the list of rolls.

Repeat the game as long as there is interest. You may prefer to let students play the game in groups of four or five.

Here is the tabulation of rolls and the winning plays of a possible game.

Roll	1	2	3	4	5	6
	4	5	2	3	2	3
	5	6	5	4	4	6
	$\hat{1}$	$\hat{2}$	$\hat{1}$	0	$\hat{1}$	$\hat{2}$
	$\hat{3}$	$\hat{6}$	$\hat{3}$	$\hat{5}$	$\hat{3}$	$\hat{5}$
points chosen	(4,5)	(5,6)	(2,5)	(3,4)	(2,4)	(3,6)



Capsule Lesson Summary

Extend the idea of drawing maps of a cube to rectangular boxes. Use maps of a box to find its surface area. Calculate the volume of a box. Find and make maps of other boxes with the same volume, and calculate the surface area of each one.

Materials

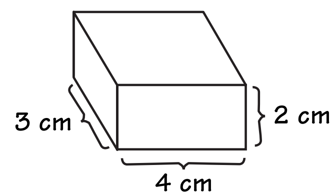
Teacher
- Colored chalk
- Grid board (optional)

Student
- Colored pencils, pens, or crayons
- Centimeter grid paper
- Scissors

Description of Lesson

Exercise 1

I am thinking about a box. I'll draw a picture of it.



Draw a picture of a box on the board and indicate its dimensions.

T: ***My box is 3 cm wide, 4 cm long, and 2 cm high. My box is painted with three colors: red, blue, and yellow. How many faces does my box have?***

S: ***Six.***

T: ***What shapes are the faces?***

S: ***Rectangles.***

T: ***What sizes are they?***

S: ***There are two faces that are 2 cm by 3 cm.***

S: ***There are two faces that are 2 cm by 4 cm.***

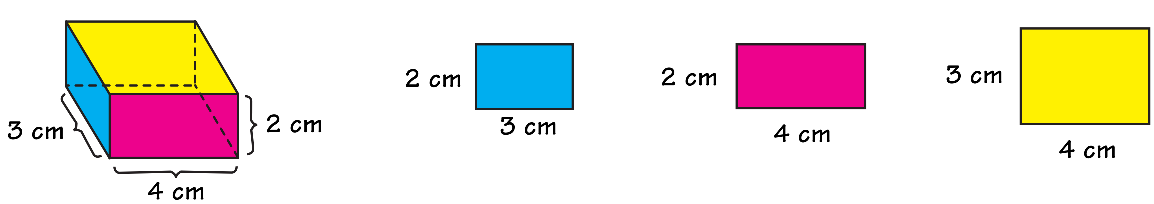
S: ***There are two faces that are 3 cm by 4 cm.***

Draw enlargements of the three different-sized rectangles on the board, and record their dimensions.

Note: You may want to tell students that such a box is also called a *rectangular prism.*

T: ***Suppose we paint each face with one color and the opposite face the same color. Two faces are blue, two faces are red, and two faces are yellow.***

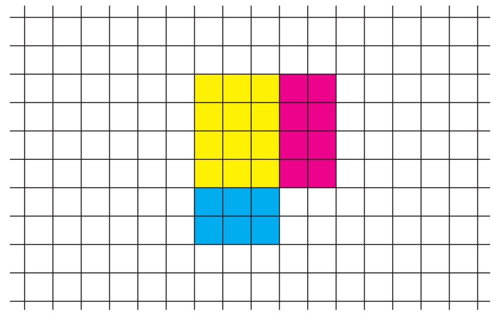
Ask students how to color three faces of the box in the picture, and then color the three rectangles to correspond.



Refer students to their centimeter grid paper, scissors, and colored pencils.

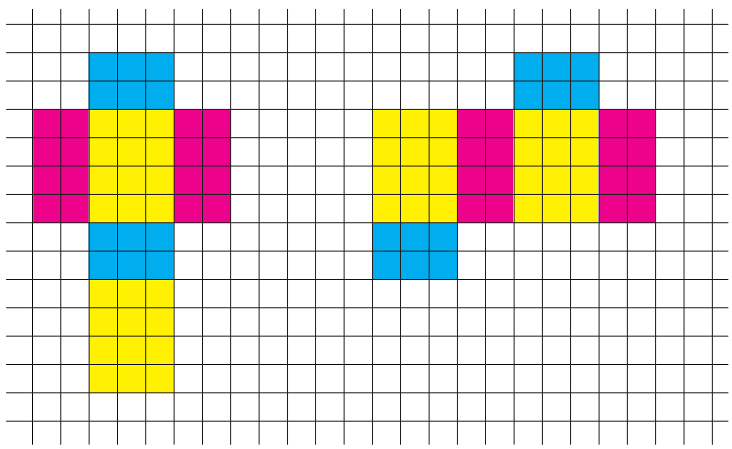
T: ***Do you remember how we made maps of cubes? Today we are going to make maps, or patterns, for the box. On your grid paper, try to make such a map. Color rectangles for the six faces, and remember, opposite faces have the same color. Then cut out the map and see if you can fold it to make this box.***

Let students experiment. This will not be easy for many. You will want to point out how the edges meet in the box; that is, edges of two rectangles can meet in the map only if they are the same length. If necessary, you may get some students started with one of each of the three rectangles connected, as shown here.



T: ***There is another yellow face, another red face, and another blue face. Can you figure out how to connect them in your picture so the result can be folded up into a box? Experiment. There are many ways.***

Students should cut out their maps and fold them. Several attempts may be necessary. As students complete maps of the box, hold them up for the class to see. Two possibilities are shown below.



Encourage students to draw and cut out several different maps of this same box.

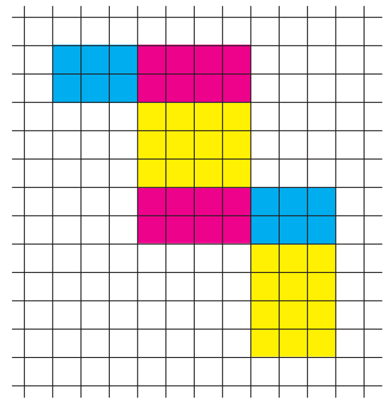
You may like to draw several of the students' solutions on the grid board, or tape the cut-outs on the board. Students can check maps easily by cutting them out and attempting to fold them into boxes.

T: ***What is the area covered by a map of my box?***

Pick one of the maps. Discuss with the class how to calculate its area by considering the six rectangles that comprise the map. Points to explore include the following:

- The area of the map is the sum of the areas of the six rectangles.
- Rectangles of the same color have the same dimensions and, hence, the same area.
- The area of the map can be found by calculating the areas of the different colored rectangles and then doubling their sum.

Based on the following map, the discussion might go like this:



T: ***This map has six pieces, the six colored rectangles. How can we find its area?***

S: ***Find the areas of all the rectangles.***

T: ***What is the area of a blue rectangle?***

S: ***6 cm² (2 x 3 = 6).***

T: ***What is the area of a red rectangle?***

S: ***8 cm² (2 x 4 = 8).***

T: ***And what is the area of a yellow rectangle?***

S: ***12 cm² (3 x 4 = 12).***

T: ***What is the sum of these three areas?***

S: ***26 cm²; 12 + 8 + 6 = 26.***

S: ***But there are two blue rectangles, two red rectangles, and two yellow rectangles in the map.***

S: ***Two blue rectangles together have an area of 12 cm². Two red have an area of 16 cm². Two yellow have an area of 24 cm².***

S: ***The area of the map is 52 cm²; 24 + 16 + 12 = 52.***

S: ***I doubled the sum of the three different areas: 2 x 26 = 52.***

Write the area next to the map.

T: ***The area of this map is 52 cm². That is the total area of the six faces of my box. The faces together are called the*** **surface** ***of the box. The total area of the faces is called the*** **surface area** ***of the box. What is the surface area of my box?***

S: ***52 cm².***

Check the calculation by using another of the maps drawn on the board. The point to make is that the surface area of the box does not depend on the map, as long as the map is correct.

Exercise 2

Equip students with more centimeter grid paper.

T: ***The surface area of my box is 52 cm². What is its volume?***

S: ***24 cm³; 2 x 3 x 4 = 24.***

T: ***Suppose this is a candy box just the right volume for a given weight of candy. Do you think that another size box would do?***

Let students react to the applied problem.

T: ***Can anyone build any other boxes with the same volume?***

S: ***6 cm by 2 cm by 2 cm.***

S: ***1 cm by 1cm by 24 cm.***

T: ***Each of you think of a box with volume 24 cm^3. Draw a map of the box and calculate its surface area.***

There are six boxes with edges of whole number length.

Dimensions	Volume	Surface Area
1 cm by 1 cm by 24 cm	24 cm^3	98 cm^2
1 cm by 2 cm by 12 cm	24 cm^3	76 cm^2
1 cm by 3 cm by 8 cm	24 cm^3	70 cm^2
1 cm by 4 cm by 6 cm	24 cm^3	68 cm^2
2 cm by 2 cm by 6 cm	24 cm^3	56 cm^2
2 cm by 3 cm by 4 cm	24 cm^3	52 cm^2

Encourage those who work quickly to find several boxes with volume 24 cm^3. End the lesson with a discussion comparing the surface area of the boxes. Which is largest? Which is smallest? What might be the advantages of any one size?

Extension Activity

Suggest that students bring boxes from home to find surface area and volume. Then let them design other boxes that would have the same volume or the same surface area.