The World of Numbers

WORLD OF NUMBERS TABLE OF CONTENTS

WORLD OF NUMBERS INTRODUCTION

By now, veteran *CSMP* students have had a rich variety of experiences in the World of Numbers. They have met and become familiar with various kinds of numbers, and with operations and relations on them. They have encountered positive and negative integers, decimal numbers, fractions, numerical functions (such as $5x$, $+3$, $\div 10$, -5, $\frac{2}{3}x$), order relations (such as $<$ and $>$), and the notions of multiples and divisors of a given number. They have been introduced to paper-and-pencil algorithms for addition and subtraction of whole numbers and decimal numbers, and for multiplication of whole numbers and of a fraction times a whole number. Students have had extensive experience using systematic methods for division of whole numbers and involving addition and subtraction of fractions in preparation for algorithms. Topics from combinatorics and number theory have provided many interesting problems.

In *CSMP Mathematics for the Intermediate Grades*, *Part IV*, these earlier numerical experiences will be revisited, extended, and deepened through familiar games and activities, as well as in fascinating new situations. As always, *CSMP* stresses the unity and continuity of growth of mathematical ideas and concepts. The program's spiral approach does not require mastery of each lesson, but rather allows students to encounter the elements of each content strand in different situations throughout the year. It is important to recognize this approach consciously. If you strive for mastery of each single lesson, you will find yourself involved in a great deal of redundancy as the year progresses.

Further, *CSMP* presents the content in a situational framework. That is, a "pedagogy of situations" engages students in rich problem-solving activities as they construct mathematical ideas. These situations offer opportunities both to develop necessary numerical skills and to gain deeper understanding of mathematical concepts in the world of numbers. At the same time, the situations presented encourage students to develop patterns of logical thinking and strategies for approaching problems.

Perhaps the most important embodiments of the *CSMP* approach are the nonverbal languages and tools used throughout the program. These are vehicles that allow students to investigate the contexts in which the content is presented and to explore new mathematical ideas. It is hard to overstate the value of developing languages and tools that are not confined to one area of mathematical content or to one level of the development of content; that aid in attacking problems as well as in representing situations. Equipped with the universally applicable languages of the *CSMP* curriculum, students grow more and more familiar with the syntax of these languages and are free to explore new content as extensions rather than think of each new mathematical idea as tied to a certain new language. This is not to say that *CSMP* students do not learn the usual descriptive language of mathematics; naturally, they do. However, in the *CSMP* approach the usual descriptive language is not a requisite for learning new concepts, but only a means for succinctly describing those ideas as they are being explored.

The Minicomputer, strings, and arrows embody three fundamental concepts of mathematics: binary and decimal number systems, sets, and functions. Using these pictorial languages to highlight unifying themes counteracts the tendency to fragment mathematics instruction into a large set of independent topics. For specific examples of the situations and the ways *CSMP* uses non-verbal languages in this strand, we refer you to the brief topic summaries later in this introduction and, in particular, to the lessons themselves.

Standard Algorithms of Arithmetic

CSMP seeks to develop basic numerical skills as well as an understanding of the underlying mathematical ideas. We are fully in agreement with the thesis that, along with the growth of understanding of the world of numbers, there must be a concomitant growth of familiarity and facility with numbers and operations on them. But facility should not be confused with understanding; they are partners in the growth of mathematical maturity. A balanced growth of each must be maintained, neither being sacrificed for the other.

Students must eventually learn mechanical algorithms for the basic operations (addition, subtraction, multiplication, division). However, the development of these methods should occur only after students have had many experiences with prerequisite concepts. Premature presentation of these algorithms may actually inhibit a student's desire and ability to develop alternative algorithms, to do mental arithmetic, or to estimate.

CSMP believes that students should be able to solve a problem such as 672 ÷ 4 using models, pictures, or mental arithmetic before being introduced to a division algorithm. Even after students have mastered an algorithm, they should be aware that alternative methods are often more appropriate. For example, consider the problem of calculating 698 x 9. Rather than using a standard multiplication algorithm, it may be easier and more efficient to note that 700 x 9 = 6 300, so that 698 x 9 = 6 300 – 18 = 6 282. Indeed, built into this way of approaching the problem is an excellent estimate (6 300) of the product. To insist on a mechanistic response to such a problem would be to encourage inefficiency and might also inhibit the development of the flexibility necessary for problem solving. On the other hand, a rich array of situations in which students interact with numbers provides them with opportunities to gain the necessary facility with standard algorithmic procedures while retaining the openness required to respond creatively to new situations in the world

Content Overview

Multiplication

By this time your *CSMP* students are quite familiar with the concept of multiplication and with a paper-and-pencil algorithm for multiplying whole numbers. Here, in *IG-IV*, both familiar and new situations present many opportunities to review and apply multiplication. Arrow pictures provide an ideal vehicle for developing methods of multiplying both decimal numbers and fractions.

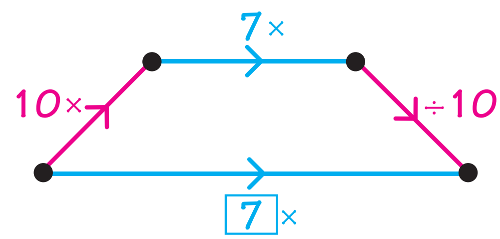
As review, students encounter multiplication in activities such as *Minicomputer Golf*, *Guess My Rule*, detective stories, and algorithm puzzles. Multiplication becomes a tool for investigating new topics, for example, a division algorithm, Cartesian graphs of various linear relations, or the least common multiple operation. It is also used to decode numbers on a new abacus, and to find ways to display numbers on a "broken" calculator. The extent and range of these activities reflect the students' growing confidence with multiplication.

WORLD OF NUMBERS INTRODUCTION

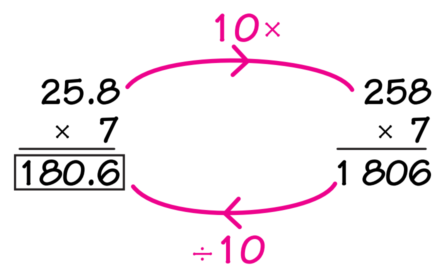
Earlier work with the Minicomputer examined sequences of calculations such as this, introducing multiplication with decimal numbers:

7 x 60 = 420   7 x 6 = 42   7 x 0.6 = 4.2   7 x 0.06 = 0.42

An arrow picture serves to generalize this process. Students first determine that a 7x arrow can be drawn from the beginning dot to the ending dot in this picture.



In other words, the result is unchanged by the combined action of the 10x and ÷10 arrows. This suggests a technique for multiplying with decimal numbers; for example, 7 x 25.8.



In other words, the problem 7 x 25.8 can be solved by performing the more familiar calculation 7 x 258, and then dividing the result by 10. While slightly less efficient than the rule "count the decimal places" (which *CSMP* students discover), this technique works well as it enhances understanding of multiplication with decimal numbers.

Refer to the sections on Fractions and Composition of Functions, for a description of the development of a standard algorithm for the multiplication of two fractions, such as illustrated here.

$$\frac{3}{4} \times \frac{2}{5} = \frac{3 \times 2}{4 \times 5} = \frac{6}{20} = \frac{3}{10}$$

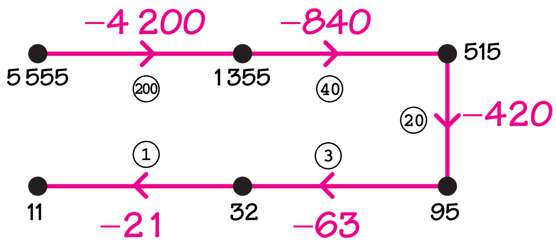
Lessons: N2, 4, 6, 7, 9, 11, 12, 16, 17, 19, 27, 28, 29, 30, 31, 34, and 36

Division

CSMP students already have experience with division as a sharing process (sharing 108 books equally among three classes), as repeated subtraction (finding how many 12s are in 200), and as a multiplication inverse. The lessons in *IG-IV* complete the development toward an efficient paper-and-pencil algorithm for division and extend students' experiences with division to new patterns and applications.

Arrow roads provide an ideal model for illustrating division as repeated subtraction. The problem "How many boxes holding 21 bottles each can be filled with 5 555 bottles?" can be interpreted as "How many 21s are there in 5 555?" In arrow terminology, an equivalent problem is "How many –21 arrows are there in a road from 5 555 to a positive number less than 21?" Students quickly realize that not only –21 arrows are useful.

To divide 5 555 by 21, this student uses several different arrows to determine that there are 264 21s in 5 555. Each arrow is labeled to indicate how many –21s it represents.



264 R = 11
21)5 555

Relying on their experience since fourth grade of using arrow roads to solve division problems by repeated subtraction, students are well prepared to make the transition to a tabular format. The table on the left in the next illustration shows an analogous solution technique of the same problem, 5 555 ÷ 21, emphasizing the story line about packing bottles in cartons.

bottles	cartons
5 555	
−4 200	200
1 355	
−840	40
515	
−420	20
95	
−63	3
32	
−21	1
11	total: 264

	264 R = 11
21) 5 555	
−4 200	200
1 355	
−840	40
515	
−420	20
95	
−63	3
32	
−21	1
11	

This table readily converts to a standard division format (above right). Note that the student chooses to fill 40 boxes and then 20 more boxes rather than 60 boxes at one time. Allowing students this flexibility acknowledges that at any one time students differ dramatically in ability and confidence. Some students prefer using only the functions –21, –210, and –2 100. Others quickly appreciate the efficiency of using functions such as –4 200, –840, –420, –63, and so on. A few students will find the most efficient subtractions. The choice of functions reflects the various stages of development in progressing toward an efficient division algorithm.

Learning a division algorithm is only one part of developing an understanding of the concept of division. Therefore, your students’ exposure to division in many contexts continues with activities involving calculators, arrow roads, number lines, fractions, and numerical patterns. In particular, old friend Nabu introduces a new abacus designed to facilitate a very specialized algorithm: the division of any number by 9.

An algorithm is a well defined procedure designed to solve a specific type of problem. While mathematics curricula emphasize algorithms for the basic operations of arithmetic (addition, subtraction, multiplication, and division), algorithms exist for many other purposes. As an extension of the work on the Minicomputer and other abaci, Nabu’s new algorithm allows students to mechanically divide a number by 9 by making a series of moves on the abacus. While fascinating in itself, this algorithm also reflects the renewed interest that computers have brought to the study of algorithms in advanced mathematics.

Lessons: N8, 11, 12, 13, 16, 17, 18, 21, 27, 28, 29, 31, 32, 33, 34, and 36

WORLD OF NUMBERS INTRODUCTION

Negative Integers

CSMP introduces negative integers in first grade through a story about Eli the Elephant and magic peanuts. The story leads to a model for adding integers, first by pictures, then also on the Minicomputer. By the end of fourth grade, *CSMP* students have encountered negative numbers in games, in reading outdoor temperatures, in arrow roads, and in calculator activities. These experiences extend the concept of order from whole numbers to negative numbers, and provide models for the addition of negative numbers.

The activities in this strand increase students' familiarity with negative numbers in many contexts. The goal is to portray negative integers not as a strange new set of numbers, but as a natural and necessary extension of counting numbers. Therefore, few lessons focus on negative numbers but many lessons include them.

Negative numbers appear regularly on the Minicomputer in *Minicomputer Golf,* in *The String Game* with numbers, in detective stories, in Cartesian graphs, in *Guess My Rule* activities, in arrow roads, and in calculator activities. In these contexts, students solve problems such as $(3 \times 10) + \widehat{5} = 25$ and $(\widehat{4} \times 0.20) + (5 \times 0.40) = 1.20$. Through these experiences, students gradually learn the characteristics of negative numbers and accept them as "real" numbers.

CSMP employs a special notation for representing negative numbers. Traditional approaches to arithmetic often make no distinction on the printed page between the function "subtract 3" and the number "negative 3"; both are denoted by "–3." Only by context can a person discern the intended meaning of "–3." In *CSMP*, negative numbers are distinguished from subtraction in the following ways:

- The minus sign "–" is reserved for subtraction. Thus, for example, "–14" denotes the function "subtract 14."
- The $\frown$ symbol denotes a negative number. Thus, "$\widehat{14}$" denotes the number "negative 14." The $\frown$ symbol was introduced first in the story about Eli the Elephant.
- A raised minus sign may be used when recording a negative number, especially for results obtained from using a calculator. For example, $^{-}14$.

We recommend that you continue to use the $\frown$ notation for negative numbers and recognize alternative notations as students encounter them in other contexts (calculators, temperature, tests, and so on).

Lessons: N2, 3, 9, 15, 24, 25, 26, and 32

Decimal Numbers

Just as students' confidence with whole numbers requires several years of growth, so must the development of decimal number concepts proceed gradually. The introductory activities in second, third, and fourth grades rely on money, on the Minicomputer, and on the number line as models for decimal numbers. These three models complement each other. Whereas all facilitate computation, the Minicomputer highlights patterns while the number line and money focus on order and relative magnitude of decimal numbers.

WORLD OF NUMBERS INTRODUCTION

Reflecting and furthering the students' growing confidence, decimal numbers appear in this semester in activities involving calculators, Cartesian coordinates, *Guess My Rule* situations, and arrow diagrams. These activities require students to perform many computations involving decimal numbers, relying on the various models to confirm their results. For example:

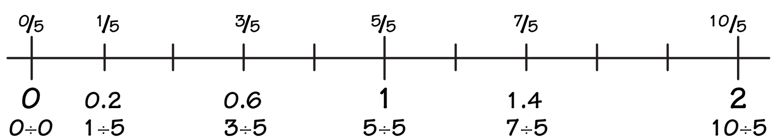
$3 \times (7 \div 2) = 10.5$ $\qquad$ $((5 + 6) \div 5) - 2 = 0.2$

$(1.5 \times 0.20) + (2.5 \times 0.40) = 1.30$ $\qquad$ $4/5 \times 2 = 1.6$

The ability to perform such calculations as well as to order decimal numbers indicate that students can discover and become familiar with the subtleties of decimal numbers without a too early reliance on rules and mechanical manipulation of numbers.

Building on the students' knowledge of various models for fractions, decimal numbers, and division, a goal of this strand is to identify the relationships among these concepts by observing equalities such as $7 \div 5 = 7/5 = 1.4$.

first students use patterns originally developed on the Minicomputer to establish relationships between decimal names for numbers and names involving division. For example, $2\,000 \div 5 = 400$, $200 \div 5 = 40$, $20 \div 5 = 4$, and $2 \div 5 = 0.4$. These equalities, along with students' ability to locate both decimal numbers and fractions on a number line, lead to the following triple labeling of a number line.



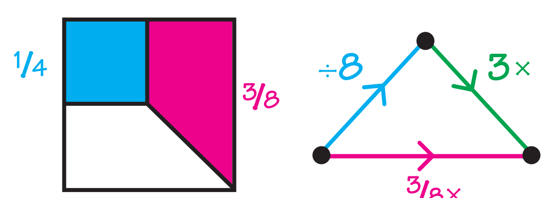
This number line clearly demonstrates equalities such as $3/5 = 0.6 = 3 \div 5$ and $10/5 = 2 = 10 \div 5$.

Students continue to build their understanding of decimal numbers by encountering them in a variety of situations. Moreover, the lessons begin to emphasize relationships among decimal numbers, fractions, and division.

Lessons: N2, 4, 8, 9, 15, 18, 19, 23, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, and 36

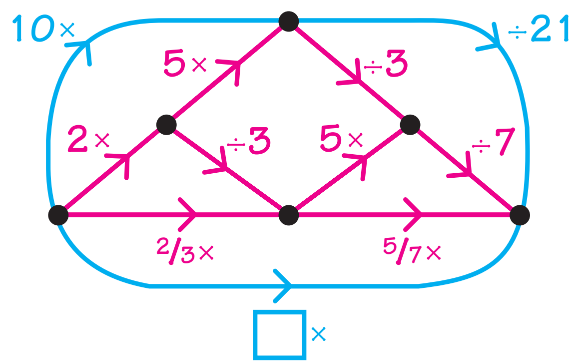
Fractions

The activities involving fractions in *IG-IV* reflect *CSMP*'s belief in the spiral approach. Early exposure to fractions began in first grade. From second through fourth grade students have gradually become familiar with two concepts involving fractions: fractional parts of a whole and certain composite functions.



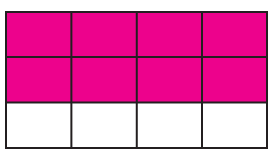
A variety of activities involving these models has increased student understanding of fractions. This background has prepared students to compute (multiply, add, and subtract) with fractions in *IG-IV*. One technique for multiplying fractions relies on the composition of multiplication functions.

Labeling the blue arrow in the diagram below requires the drawing of several detour arrows and the appropriate compositions.

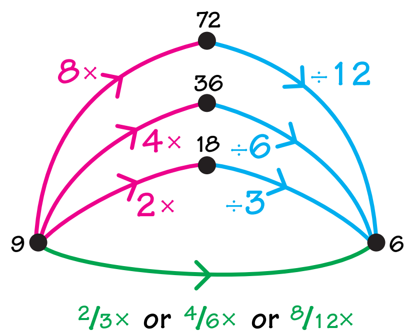


The unlabeled blue arrow is the composition of 10x followed by ÷21 or $\frac{10}{21}$x. Therefore $\frac{2}{3}$x followed by $\frac{5}{7}$x is $\frac{10}{21}$x or, analogously, $\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$. The arrow picture suggests a generalization to $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$, the standard algorithm for multiplying fractions, for example, $\frac{5}{6} \times \frac{11}{20} = \frac{55}{120} = \frac{11}{24}$. Furthermore, students learn to calculate problems such as $5\frac{3}{4} \times 8$ by considering $5 \times 8 = 40$ and $\frac{3}{4} \times 8 = 6$.

A prerequisite for adding fractions is an understanding of equivalent fractions. Both the area model and arrow pictures suggest that numbers can have different fractional names.



2/3 = 8/12



With the concept of equivalent fractions in hand, a story about Amelia and Sara cutting cakes leads to a method for adding fractions. The area model emphasizes the need for equal-sized regions, i.e., the need for a common denominator.

Sara

1/3 = 4/12

Amelia

1/4 = 3/12

$$\frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$$

In addition, in order to focus on a common error in adding fractions, students provide arguments based on area or estimation to prove that, for example, $\frac{2}{5} + \frac{3}{4} \neq \frac{2+3}{5+4}$ or $\frac{5}{9}$. In *IG-IV*, students also review calculations such as $\frac{3}{8} + \frac{5}{8} = 1$ and $9\frac{7}{8} - 1 = 8\frac{7}{8}$.

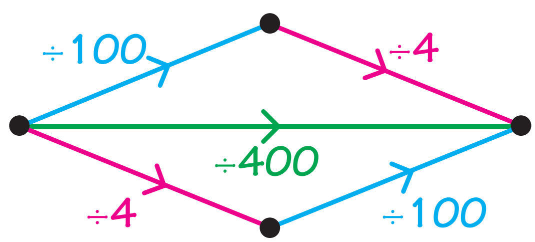
As further support of the relationship between division and fractions, students interpret $7 \div 5$ as sharing seven pizzas fairly among five families. Appropriate dividing and shading of circles reveals that first, each family receives one pizza and there are two pizzas left over; and then, by dividing the two left over pizzas in five parts each, each family receives another $\frac{2}{5}$ pizza. That is, $7 \div 5 = 1\frac{2}{5} = \frac{7}{5}$.

The section on Decimals in this introduction describes the development of methods for changing fractions to decimal numbers and vice-versa; for example, $1.4 = 1\frac{2}{5} = \frac{7}{5} = 7 \div 5$.

Lessons: N4, 6, 8, 10, 15, 18, 20, 22, 27, 28, 29, 30, 31, and 32

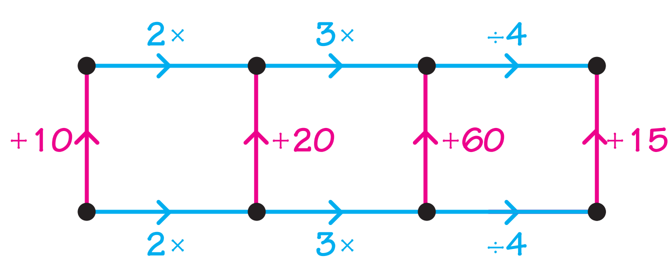
Composition of Functions and Percent

Several lessons in this strand deal with what happens when you compose a sequence of functions, that is, when you apply the functions in order, one at a time. These compositions lead to many powerful insights into the properties of numbers and operations. Arrow diagrams provide a concrete means to study this abstract but practical concept. For example, if you divide any number by 100 and then divide the result by 4, the net effect is the same as dividing the original number by 400.



Besides depicting the composition, the arrow picture also suggests that an easy way to divide any number by 400 is to divide by 100 and then divide by 4 or vice-versa; the two operations can often be performed mentally. For example, $828 \div 4 = 207$ and $207 \div 100 = 2.07$, so $828 \div 400 = 2.07$.

Many pairs of functions *commute*. That is, they produce the same effect regardless of the order in which they are applied. $\frac{2}{3}$x can be interpreted as 2x followed by $\div 3$ or as $\div 3$ followed by 2x; +98 can be interpreted as +100 followed by –2 or as –2 followed by +100. However, other pairs of functions, for example, +10 and 2x, do not commute. Yet patterns do exist—students find, for example, that +10 followed by 2x has the same effect as 2x followed by +20. The following arrow picture depicts several compositions of this kind.

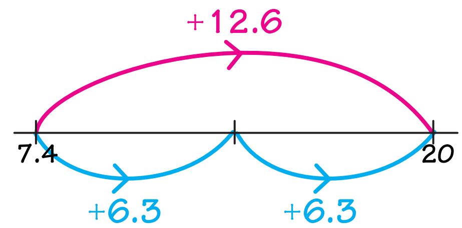


From the students' perspective, they are solving challenging problems and discovering new numerical patterns. From the mathematical viewpoint, they are investigating the commutative and distributive properties.

WORLD OF NUMBERS INTRODUCTION

The composition of functions can lead to insights in many problem-solving situations. In *Minicomputer Golf*, the use of composition aids in finding solutions to problems requiring that students move two checkers to produce a specified change. In the section on multiplication, an arrow picture involving composition indicates a method of calculating 25.8 x 7 by first calculating 258 x 7.

The idea of composition also facilitates finding the midpoint of two numbers on a number line.



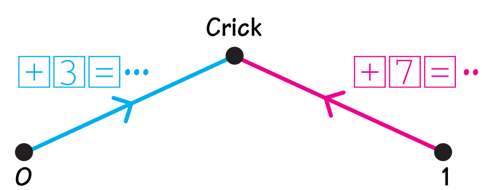
The composition of functions shows how the language of arrows is able to visually highlight rich and practical mathematical concepts and techniques. For example, three lessons this semester introduce the concept of percent as a composite function; that is, n% of is nx followed by ÷100 or ÷100 followed by nx. In this context, the exercises investigate many useful names for certain percents (for example, 20% = $\frac{2}{10}$ = $\frac{1}{5}$ = 0.2), several interesting patterns, and some helpful properties. The lessons all include some applications of percent in the solution of real life problems.

Lessons: N1, 2, 3, 4, 6, 9, 14, 15, 16, 19, 21, 25, 27, 29, 30, 31, and 34

Multiples and Divisors

The study of multiples and divisors leads to practical applications such as the addition of fractions, as well as to investigations of many fascinating properties of numbers. Your students have had many earlier experiences finding multiples and divisors of whole numbers, and through string pictures they have encountered the notions of common multiples and common divisors. In *IG-IV*, calculator activities, arrow diagrams, and string pictures provide further opportunities to explore common multiples and common divisors. Calculator relations, for example $\boxed{+}\,\boxed{7}\,\boxed{=}$ …, provide a means for posing and studying problems suggested in arrow pictures.

Students use calculators to generate a list of possibilities for Crick, namely, 15, 36, 57, 78, 99, 120, …, and they note that consecutive numbers in the list differ by 21, a common multiple of 3 and 7. Further investigation reveals that this is not a coincidence.



Guess My Rule games introduce explicitly the least common multiple and greatest common divisor operations. To gain insight into these operations, students use string pictures to study problems such as $6 \sqcup \square = 30$ and $\square \sqcap \triangle = 6$ where $\sqcup$ and $\sqcap$ are the symbols for "least common multiple" and "greatest common divisor," respectively. See the L strand for these lessons.

Common multiples also appear in the study of fractions. As preparation for the addition of fractions, arrow pictures and activities involving the fair division of rectangular cakes both lead students to generate lists of equivalent fractions. Students recognize the role of common multiples in determining the numerators and denominators of equivalent fractions.

Problems involving adding two multiples of 3 or two non-multiples of 3 and patterns generated by moving one checker in a configuration on the Minicomputer suggest interesting properties of multiples

Lessons: N1, 3, 4, 10, 11, 12, 14, 20, 22, 25, 32, and 35

N1 MINICOMPUTER DYNAMICS #1

Capsule Lesson Summary

Decide which changes in a given list can be made to a number displayed on the Minicomputer by moving exactly one checker. Moving one checker at a time, repeatedly change the number displayed on the Minicomputer by specified amounts. Play *Minicomputer Golf*.

Materials

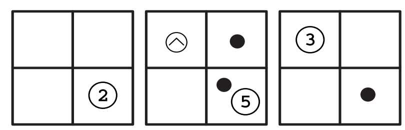
Teacher
- Minicomputer set
- Weighted checker set
- Colored chalk

Student
- Paper

Description of Lesson

Exercise 1

Put this configuration on the Minicomputer.



T: ***What number is on the Minicomputer? Write it on your paper.***

While students are decoding the number, write the following list of increases and decreases on the board. Check a few answers before asking a student to announce the number. (245)

×7	−21	−30	×9	−50
×13	−198	×99	×72	×150

T: ***This is a list of increases and decreases. Which of these changes can be made by moving exactly one checker from the square it's on to another square?***

Invite students to demonstrate the changes that are possible on the Minicomputer. Circle them in the list as they are found. Always return a checker to its original position before another change is suggested. All of the changes except –50 and +13 are possible.

×7 (Move a regular checker from the 1-square to the 8-square.)

−21 (Move the ③-checker from the 8-square to the 1-square.)

−30 (Move the regular checker from the 40-square to the 10-square, or move the ⑤-checker from the 10-square to the 4-square.)

×9 (Move the regular checker from the 1-square to the 10-square.)

−198 (Move the ②-checker from the 100-square to the 1-square.)

×99 (Move the regular checker from the 1-square to the 100-square.)

×72 (Move the negative checker from the 80-square to the 8-square.)

×150 (Move the ⑤-checker from the 10-square to the 40-square.)

N1

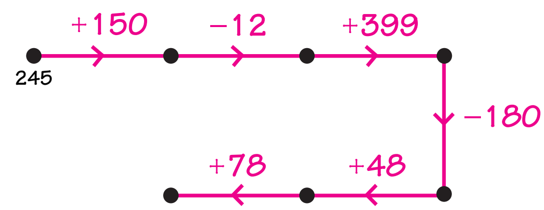
Once students determine that the above changes can be made moving one checker, you may wish to challenge the class to find solutions for +13 and –50 by moving exactly two checkers.

×13 (Move the regular checker from the 1-square to the 4-square and move the regular checker from the 10-square to the 20-square, or move the ③-checker from the 8-square to the 10-square and move the regular checker from the 1-square to the 8-square.)

–50 (Move the negative checker from the 80-square to the 100-square and move the regular checker from the 40-square to the 10-square, or move the regular checker from the 40-square to the 20-square and move the ⑤-checker from the 10-square to the 4-square.)

Exercise 2

Draw this arrow road on the board.



T: ***This arrow road starts at the number that is on the Minicomputer. For each arrow you may move exactly one checker fromone square to another. Who can move one checker and increase the number by 150?***

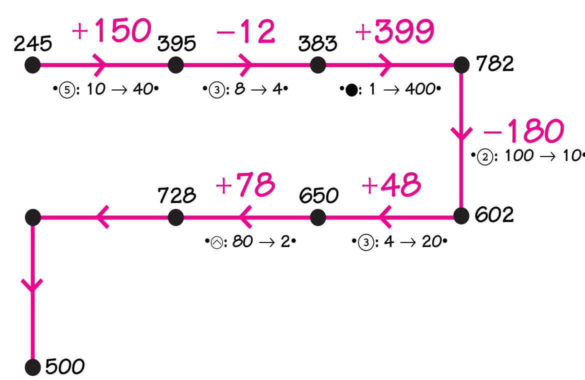
S: ***Move the ⑤-checker from the 10-square to the 40-square.***

Ask a student to make the move on the Minicomputer.

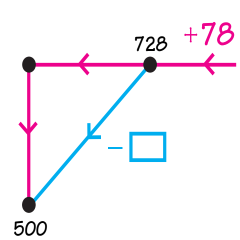
T: ***What number is on the Minicomputer now?***

S: ***395.***

Continue this activity until the dot for 728 is labeled. A completed arrow road indicatingthe corresponding moves on the Minicomputer is shown here. Extend the arrow road two arrows and put 500 at the last dot.



Challenge the class to get 500 on the Minicomputer in two moves. If students have difficulty with this problem, draw an arrow from 728 to 500.



T: ***What change do these two moves together need to make?***

S: ***–228.***

T: ***What could the red arrows be for? There are many possibilities.***

S: ***–28 and –200.***

S: ***A –28 move is not possible.***

S: ***–38 and –190.***

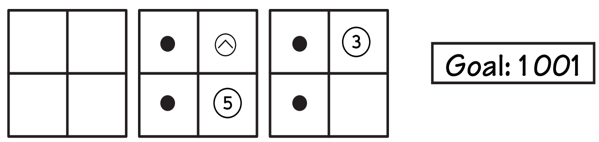
S: ***Subtract 38 by moving the negative checker from the 2-square to the 40-square.***

S: ***Move the ⑤-checker from the 40-square to the 2-square for –190.***

Make the moves and label the arrows. If a solution is not discovered, leave the problem as a challenge problem for the class.

Exercise 3

Play *Minicomputer Golf*. The following is a possible game (starting configuration) starting at 132 and with a goal of 1 001.



Writing Activity

You may like students to take lesson notes on some, most, or even all their math lessons. The "Lesson Notes" section in Notes to the Teacher gives some suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students may describe some changes and how to make them on the Minicomputer.

Home Activity

Send home a description of *Minicomputer Golf*, and suggest parents play it with their child.

Capsule Lesson Summary

Multiply numbers by 10, by 100, and by 1 000. Use a double-scaled number line to locate the answers to multiplication problems involving decimals. Relate a multiplication problem involving a non-integer decimal with one that involves only whole numbers.

Materials

Teacher
- Meter stick
- Colored chalk

Student
- Paper
- Worksheets N2* and **

Description of Lesson

Begin the lesson with some mental arithmetic involving multiplication and division by 10, 100, and 1 000.

$10 \times 4.6 = \boxed{46}$ $\quad 10 \times 1.28 = \boxed{12.8}$ $\quad 100 \times 0.05 = \boxed{5}$

$100 \times 4.6 = \boxed{460}$ $\quad 100 \times 1.28 = \boxed{128}$ $\quad 10 \times 0.05 = \boxed{0.5}$

$1\,000 \times 4.6 = \boxed{4\,600}$ $\quad 1\,000 \times 1.28 = \boxed{1280}$ $\quad 1\,000 \times 0.05 = \boxed{50}$

$860 \div 10 = \boxed{86}$ $\quad \boxed{10} \times 2.6 = 26$

$860 \div 100 = \boxed{8.6}$ $\quad \boxed{100} \times 0.81 = 81$

$860 \div 1\,000 = \boxed{0.86}$ $\quad \boxed{1000} \times 9.206 = 9\,206$

$\boxed{100} \times 32.4 = 3\,240$

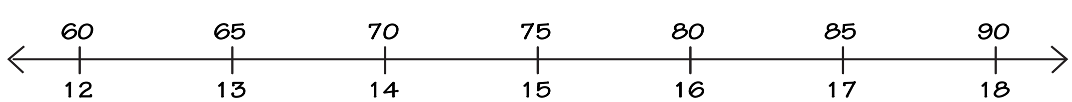
Occasionally ask the class if an answer should be more or less than another number in the problem; for example, is 860 ÷ 10 more or less than 860? Also compare answers; for example, is 10 x 0.05 more or less than 100 x 0.05? Students should notice the location of decimal points in problems and in their answers.

Exercise 1

Draw a number line with whole numbers from 12 to 18 on the board.

T: ***This is a part of a number line. I'm going to rescale the line with different numbers at the same marks above the line. See if you can figure out my rule.***

Put 75 over 15 and 90 over 18. Students should begin to suspect the 5x pattern, or they may see the upper scale is in increments of 5. Let students complete labeling the marks in the upper scale.



T: ***What number would be below 45 if we extended the number line to the left?***

S: ***9.***

T: ***What number would be above 25 if we extended the number line to the right?***

S: ***125.***

T: ***Where would 15.3 be on the number line using the bottom scale?***

Call on volunteers to point out the location, draw a mark, and label it.

T: ***What number corresponds to 15.3 on the upper scale? What two numbers is it between?***

S: ***Between 75 and 80.***

S: ***76.5. 5*** x ***15 = 75 and 5*** x ***0.3 = 1.5; 75 + 1.5 = 76.5.***

You may need to suggest this breakdown of 15.3 into 15 and 0.3. Label the mark in the upper scale.

T: ***Where would 13.24 be on the lower scale?***

A volunteer should locate 13.24 about a fourth of the distance between 13 and 14. Mark the location and label it. You may need to give a hint; for example, ask where 13.5 is located.

T: ***What number on the upper scale corresponds to 13.24?***

S: ***It is between 65 and 70.***

T: ***Which is it closer to?***

S: ***65.***

T: ***What number is halfway between 65 and 70?***

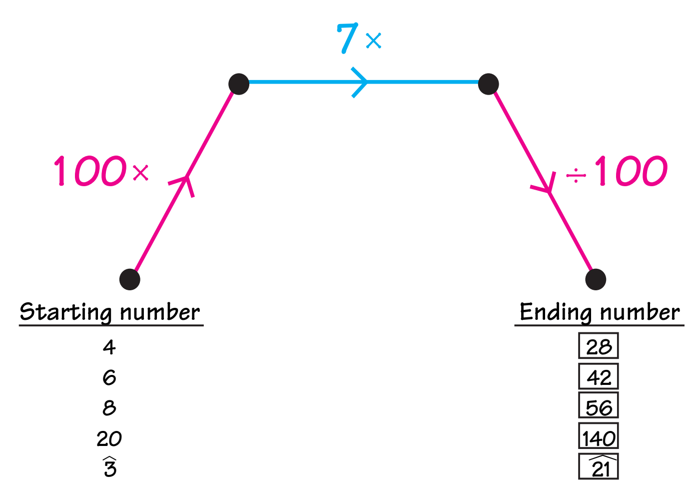
S: ***67.5.***

T: ***So the top number must be between 65 and 67.5. What number is it? What is 5*** x ***13?*** (65) ***5*** x ***0.2?*** (1) ***5*** x ***0.04?*** (0.20) ***So altogether what number is 5*** x ***13.24?*** (66.20)

Use money as a model if the students are having difficulty.

Exercise 3

Draw this arrow road on the board.



Label the starting dot with each of these numbers in turn, and ask the class for the corresponding ending numbers. (Answers are in boxes.)

Draw an arrow from the starting dot of the arrow road to its ending dot.

T: ***What is 100*x *followed by 7*x *followed by ÷100?***

S: *7*x*.*

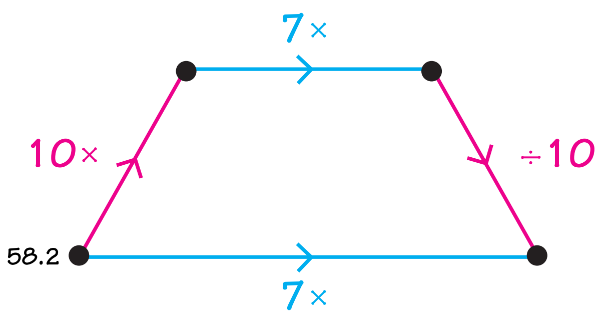
Change the 100x and ÷100 arrows to 2x and ÷2, 10x and ÷10, 5x and ÷5 in turn, in each case asking for ending numbers that correspond to various starting numbers and then for the composite of the three arrows. The composite arrow in all cases is for 7x. Students should see than *n*x and ÷*n* undo one another in these problems.

T: ***Let's go back to 10*x *followed by 7*x *followed by ÷10. The arrow picture can help us do this problem:***

An average person in Johnstown spends 58.2 minutes each day watching T.V. How many minutes does the average person watch T.V. in a week?

Write the multiplication calculation on the board and alter the arrow picture as follows.

$$\begin{array}{r} 58.2 \\ \times\ 7 \\ \hline \end{array}$$

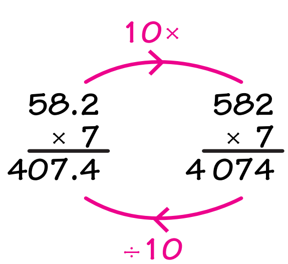


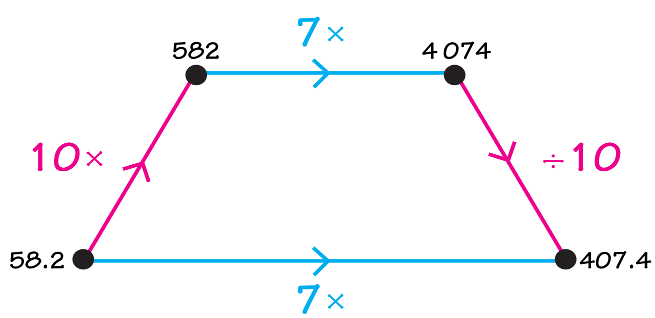
T: ***What number is 10*** x ***58.2?*** (582)
7 x ***582?*** (4 074)
4 074 ÷ 10? (407.4)

Label the dots and record the result of the multiplication problem.

T: ***We used the result of 7*** x ***582*** (trace the 7x arrow from 582 to 4 074) ***to help us calculate 7*** x ***58.2.***

Write the related whole number multiplication problem next to the decimal problem. Then draw the 10x arrow and ÷10 arrow to suggest how the arrow picture was used.





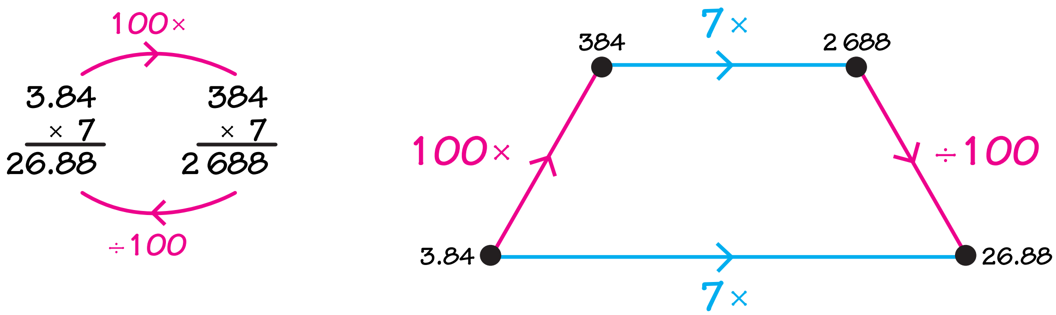
N2

For the next couple of problems you may like to ask the class to make up stories in which they would use these multiplication calculations.

T: ***Now let's try 7* x *3.84. What should we multiply 3.84 by to get 384?***

S: ***100.***

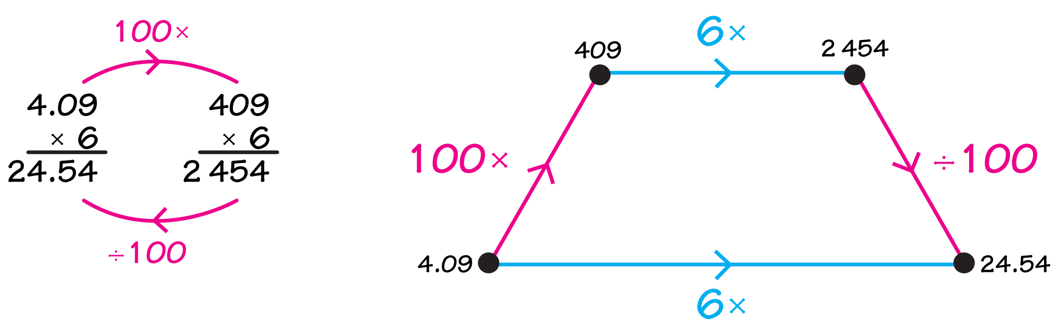
Change the arrow road to 100x followed by 7x followed by ÷100, and write the related problems to the side. Repeat the activity and arrive at this picture.



T: ***Now let's try 6* x *4.09. Instead of 7*x*, we want 6*x. (Change each of the two 7x arrows to 6x arrows.) ***What should we multiply 4.09 by to get 409?***

S: ***100.***

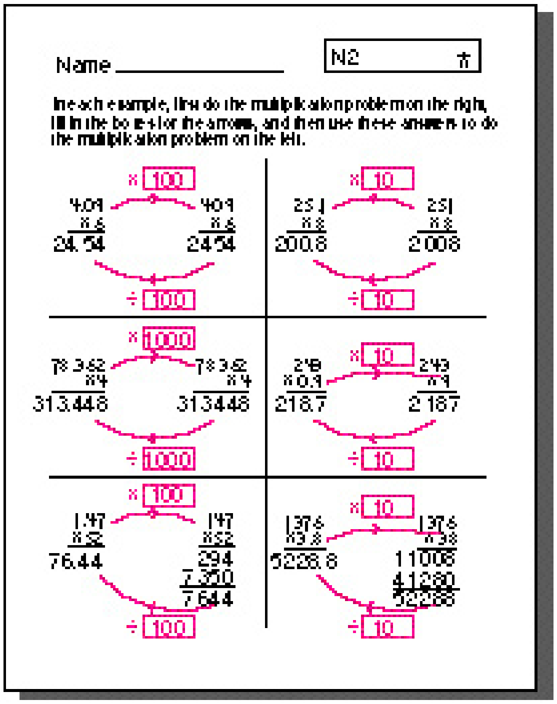
Repeat the activity and arrive at this picture.



T: ***How could we do the problem 21* x *1.032?***

S: ***Decide what times 1.032 equals 1 032 and then calculate 21* x *1 032. If we multiply by 1 000, we need to divide the answer to the problem 21* x *1 032 by 1 000.***

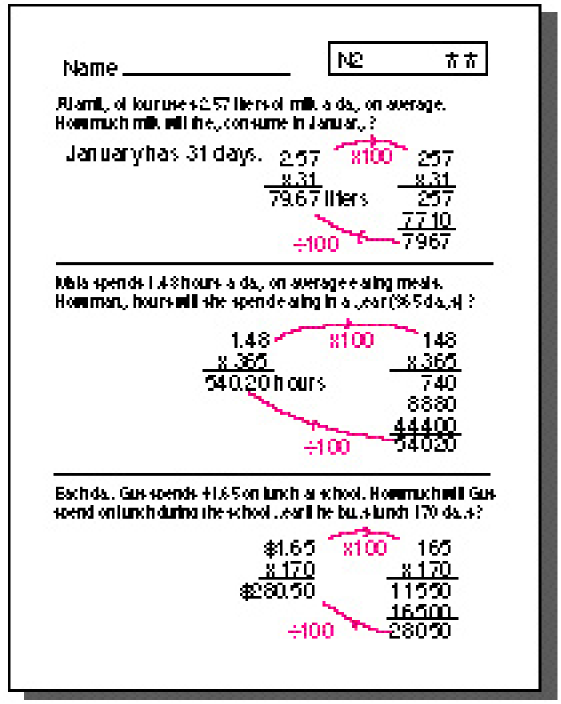
Worksheets N2 * and ** are available for individual work.



Name ____________ N2 ☆

In each example, first do the multiplication problem on the right, fill in the boxes for the arrows, and then use these answers to do the multiplication problem on the left.

4.09 × 6 = 24.54; 409 × 6 = 2454; × 100, ÷ 100	25.1 × 8 = 200.8; 251 × 8 = 2008; × 10, ÷ 10
78.362 × 4 = 313.448; 78362 × 4 = 313448; × 1000, ÷ 1000	243 × 0.9 = 218.7; 243 × 9 = 2187; × 10, ÷ 10
1.47 × 52 = 76.44; 147 × 52: 294, 7350, 7644; × 100, ÷ 100	137.6 × 3.8 = 522.88; 1376 × 38: 11008, 41280, 52288; × 10, ÷ 10



Name ____________ N2 ☆☆

A family of four uses 2.57 liters of milk a day, on average. How much milk will they consume in January?

January has 31 days. 2.57 × 31 = 79.67 liters; 257 × 31: 257, 7710, 7967; ×100, ÷100

Mala spends 1.48 hours a day, on average eating meals. How many hours will she spend eating in a year (365 days)?

1.48 × 365 = 540.20 hours; 148 × 365: 740, 8880, 44400, 54020; ×100, ÷100

Each day, Gus spends $1.65 on lunch at school. How much will Gus spend on lunch during the school year if he buys lunch 170 days?

$1.65 × 170 = $280.50; 165 × 170: 11550, 16500, 28050; ×100, ÷100

Capsule Lesson Summary

Solve a detective story in which the clues involve a composition of the calculator relations $\boxed{+}\boxed{3}\boxed{=}$ … and $\boxed{-}\boxed{10}\ \boxed{=}$ …, multiples of 8, and a pattern generated by the calculator relation $\boxed{-}\boxed{5}\boxed{=}$ … .

Materials

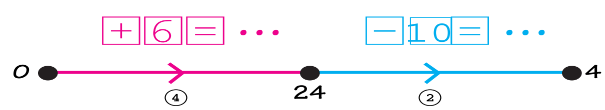
Teacher	• Colored chalk • Calculator	• Colored pencils, pens, or crayons • Worksheets N3* and **
Student	• Calculator	

Description of Lesson

Note: The first clue of this detective story may be investigated on two different levels depending on the numerical abilities of your students. Some students might consider only the number pattern, while other students might try to find an explanation for the pattern.

Clue 1

Draw this arrow picture on the board.



T: ***Paf is a secret number. Use your calculator and find some numbers Paf could be.***

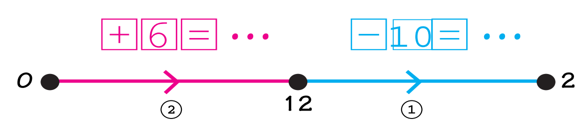
Allow a few minutes for students to explore this situation.

S: ***Paf could be 4.***

T: ***How can we get 4?***

S: ***Press $\boxed{+}$ $\boxed{6}$ and $\boxed{=}$ four times to get 24. Then press $\boxed{-}$ $\boxed{1}\boxed{0}$ and $\boxed{=}$ twice to get 4.***

As a student is giving an explanation of this kind, write the information in the arrow picture in this way.



Continue by asking for other numbers that Paf could be. It is not necessary to elicit an explanation for every suggested number; however, do list them on the board. After a few minutes you should have a list of eight or ten possibilities for Paf. Since Paf can be any even integer, we cannot predict which numbers students will suggest, but encourage them to fill in some of the gaps. Adjust the following dialogue depending on which numbers are in your list.

T: ***Could Paf be 2?***

S: ***Yes. Press $\boxed{+}$ $\boxed{6}$ and $\boxed{=}$ twice followed by $\boxed{-}$ $\boxed{1}\boxed{0}$ and $\boxed{=}$ once.***

Include 2 in your list.

T: ***What about 6 and 8?***

S: ***Paf could be 6; 6*** x ***6 = 36 and 36 – 30 = 6.***

S: ***Paf could be 8; 3*** x ***6 = 18 and 18 – 10 = 8.***

T: ***Try to find ways to put 10, 12, 14, and 16 on the display by yourself.***

Continue to find other numbers that Paf could be as long as there is interest. At some point during this discussion, ask if there are integers that Paf could not be. If no comments are forthcoming, do not worry as there will be other opportunities to explore this situation.

If appropriate, lead a discussion that explains why Paf could be any even integer.

T: ***We found that Paf could be many even integers. Do you think that Paf could be any even integer?***

Let students react to the question and listen to their arguments.

T (pointing to the middle dot)**:** ***One way of approaching this question is first to decide which numbers could be here.***

S: ***Any positive multiple of 6 could be there.***

With the class's assistance, list positive multiples of 6 up to 90. Arrange them in the order shown below.

6	12	18	24	30
36	42	48	54	60
66	72	78	84	90

T: ***Since there are infinitely many multiples of 6, we could continue to list them as long as we wished. Do you notice any patterns?***

S: ***You arranged them so the numbers in the same column end in the same digit.***

S: ***There is a column of numbers for every possible ending number.***

S: ***There is a +30 pattern in each column: 6 + 30 = 36, 36 + 30 = 66; 12 + 30 = 42, 42 + 30 = 72; and so on.***

After the last observation is made, ask what the next number would be in several columns.

T: ***Let's choose an even number and see if Paf could be that number. Could Paf be 228? First, let's consider what the middle dot could be for.***

S: ***It must be a multiple of 6 that ends in 8.***

T: ***Good. Will 18 help us?*** (No) ***48?*** (No) ***78?*** (No) ***Why?***

S: ***We cannot take away tens from 18 or 48 or 78 to get 228.***

T: ***So let's extend the 8-column.***

With the class's help, extend the 8-column using the +30 pattern until everyone agrees upon a number that will help. 258 is the first such number. (All numbers in the 8-column more than 258 will also work.) If students suggest stopping at 228, observe that in the arrow picture you need to subtract 10 at least once.

6	12	18	24	30
36	42	48	54	60
66	72	78	84	90
		108		
		138		
		168		
		198		
		228		
		258		

T: ***Now what do we do?***

S: ***Subtract three tens to get 228.***

T: ***So Paf could be 228. What's another number that you'd like to try?***

Let the class check a few other numbers in a similar way.

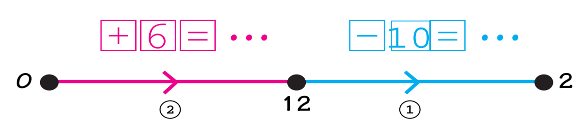
T: ***Do you think we could get any positive even integer this way?*** (Yes) ***There is another way to show that any positive even integer could be Paf. How can we get 2?***

S: ***Press*** $\boxed{+}$ $\boxed{6}$ ***and*** $\boxed{=}$ ***twice and then press*** $\boxed{-}$ $\boxed{1}\boxed{0}$ ***and*** $\boxed{=}$ ***once.***

Record the information in the arrow picture.

T: ***What number would we get if we doubled the number of times we pressed*** $\boxed{=}$ ***for each of the relations?***

S: ***4, because 4*** x ***6 = 24 and 24 – 20 = 4.***



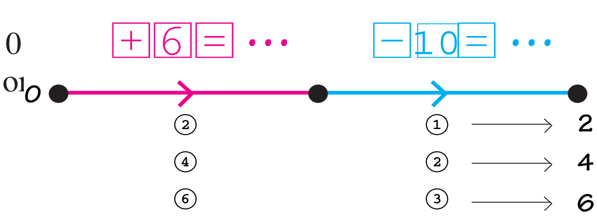
Record the information under the arrow picture.

T: ***Using this method, how could we get 6?***

S: ***Press*** $\boxed{+}$ $\boxed{6}$ ***and*** $\boxed{=}$ ***six times; then press*** $\boxed{-}$ $\boxed{1}\boxed{0}$ ***and*** $\boxed{=}$ ***three times.***

Record the information under the arrow picture. Observe that you could continue in this way to get any positive even integer.

Note: Some students may object to letting Paf be 0
0 is at the starting dot. You may agree to include 0 o



T: ***What about negative even integers?***

N3

Let students have a few minutes to find some negative even integers for Paf.

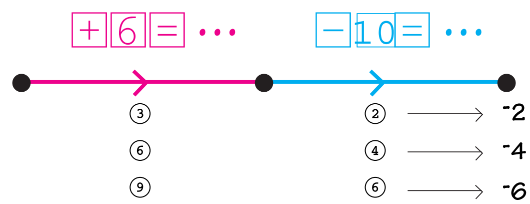
S: ***Paf could be $^{-}2$. Press $\boxed{+}$ $\boxed{6}$ and $\boxed{=}$ three times and then $\boxed{-}$ $\boxed{1}\boxed{0}$ and $\boxed{=}$ twice.***

T: ***What number do we get when we double the number of times we press $\boxed{=}$ for each of the relations?***

S: ***We get $^{-}4$, because $6 \times 6 = 36$ and $36 - 40 = {}^{-}4$.***

T: ***And if we triple the number of times we press $\boxed{=}$?*** ($^{-}6$)

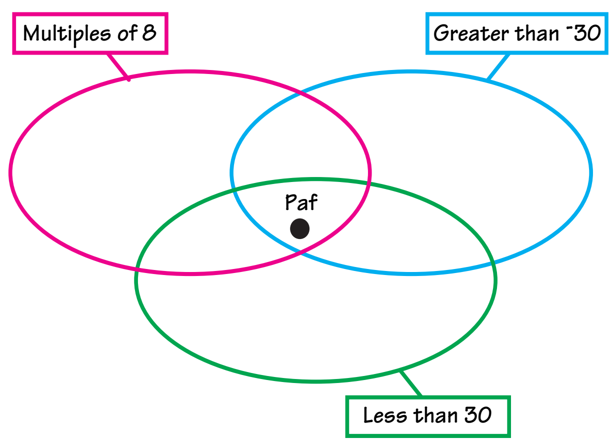
Do you think Paf could be any even negative integer?



S: ***Yes; to get the next smaller even integer increase the number of times we press $\boxed{=}$ for the red arrow by 3 and increase the number of times we press $\boxed{=}$ for the blue arrow by 2.***

Clue 2

Draw this string picture on the board.



T: ***Who could Paf be?***

S: ***8; 8 is a multiple of 8, 8 is greater than $^{-}30$, and 8 is less than 30.***

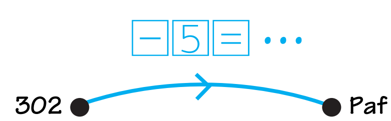
Continue this activity until the class concludes that Paf could be any of the following numbers.

$^{-}24$, $^{-}16$, $^{-}8$, 0, 8, 16, 24

Note: Your class may want to exclude 0.

Clue 3

Draw this arrow picture on the board.

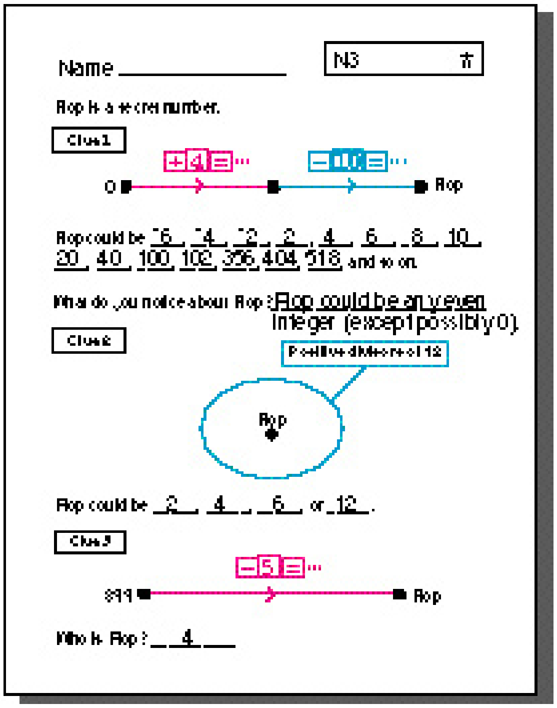


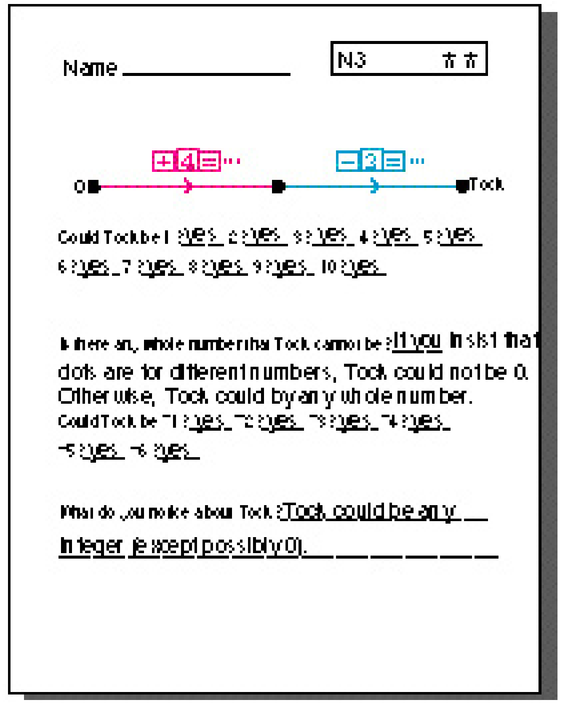
T: ***If we put 302 on the display of a calculator and press $\boxed{-}$ $\boxed{5}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ and so on, one of the numbers that will appear is Paf. Who is Paf?***

Let students have a few minutes to explore the clue.

S: ***The last digit of each positive number that appears is 2 or 7. The last digit of each negative number that appears is 3 or 8. So Paf must be $^{-}8$.***

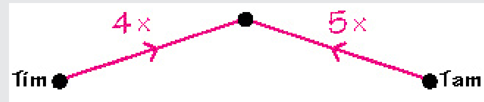
Worksheets N3* and ** are available for individual work.


Name
N3 ☆
Flop is a secret number.
Clue 1
+4= ...
−10= ...
0
Flop
Flop could be -6, -4, -2, 2, 4, 6, 8, 10, 20, 40, 100, 102, 356, 404, 518 and so on.
What do you notice about Flop? Flop could be any even integer (except possibly 0).
Clue 2
Positive divisors of 12
Flop
Flop could be 2, 4, 6, or 12.
Clue 3
−5= ...
899
Flop
Who is Flop? 4


Name
N3 ☆☆
+4= ...
−3= ...
0
Tock
Could Tock be 1? yes 2? yes 3? yes 4? yes 5? yes 6? yes 7? yes 8? yes 9? yes 10? yes
Is there any whole number that Tock cannot be? If you insist that dots are for different numbers, Tock could not be 0. Otherwise, Tock could by any whole number.
Could Tock be -1? yes -2? yes -3? yes -4? yes -5? yes -6? yes
What do you notice about Tock? Tock could be any integer (except possibly 0).

N4 MULTIPLICATION OF RATIONAL NUMBERS #1

Capsule Lesson Summary



Find pairs of numbers for Tim and Tam in this arrow picture. Observe that an arrow from Tim to Tam could be for $\frac{4}{5}$x and that its opposite arrow would be $\div\frac{4}{5}$ or $\frac{5}{4}$x. Find pairs of multiplication functions for similar arrow pictures in which the numbers Tim and Tam are given but the arrow labels are not. Find many different names for the function $\frac{2}{3}$x as the composite of two functions.

Materials

Teacher
- Colored chalk

Student
- Paper
- Worksheets N4*, **, ***, and ****

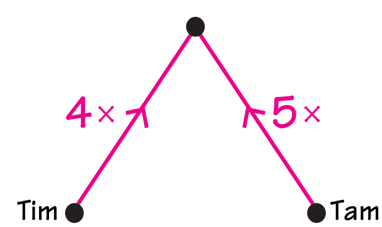
Description of Lesson

Begin the lesson with some mental arithmetic. Suggested problems are shown below. (Answers are in boxes.)

$600 \div 3 = \boxed{200}$	$810 \div 9 = \boxed{90}$	$400 \div 8 = \boxed{50}$
$60 \div 3 = \boxed{20}$	$81 \div 9 = \boxed{9}$	$40 \div 8 = \boxed{5}$
$6 \div 3 = \boxed{2}$	$8.1 \div 9 = \boxed{0.9}$	$440 \div 8 = \boxed{55}$
$0.6 \div 3 = \boxed{0.2}$	$0.81 \div 9 = \boxed{0.09}$	$44 \div 8 = \boxed{5.5}$
$0.06 \div 3 = \boxed{0.02}$	$0.081 \div 9 = \boxed{0.009}$	$4.4 \div 8 = \boxed{0.55}$

Exercise 1

Draw this arrow picture on the board.



Trace arrows as you explain,

T: ***The arrow picture shows that 4 times Tim equals 5 times Tam. Find some pairs of numbers that Tim and Tam could be.***

As students suggest them, make a list on the board of pairs of numbers that Tim and Tam could be. Examples are shown in the chart below.

Tim	5	10	15	25	20				
Tam	4	8	12	20	16				

Challenge students to consider (non-integer) decimal numbers as possibilities for Tim and Tam, as in the following dialogue.

T: ***If Tim is 0.5, which number is Tam?***

S: ***Tam is 0.4, because 4*** x ***0.5 = 2 and 5*** x ***0.4 = 2.***

T: ***If Tim is 1, which number is Tam?***

S: ***Tam is 0.8, because 4*** x ***1 = 4 and 5*** x ***0.8 = 4.***

T: ***If Tim is 0.1, which number is Tam?***

S: ***Tam is 0.08, because 4*** x ***0.1 = 0.4 and 5*** x ***0.08 = 0.40.***

T: ***Look closely at the chart. Do you notice any interesting patterns?***

If suggestions are not forthcoming, highlight any two pairs of numbers in the chart using colored chalk.

Tim	5	10	15	25	20	0.5	1	0.1	
Tam	4	8	12	20	16	0.4	0.8	0.08	

T: ***Look at these two pairs of numbers: 5 and 4, 15 and 12.***
How many times 5 is 15?

S: ***3 times.***

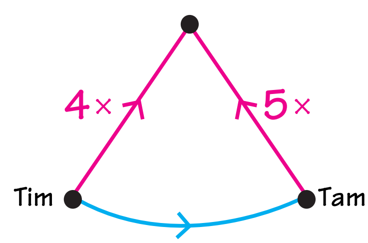
T: ***And how many times 4 is 12?***

S: ***3 times.***

Repeat this a couple times, and pick pairs of numbers to best suit the numerical abilities of your students. For example:

T: ***0.5*** x ***2 = 1 and 0.4*** x ***2 = 0.8, or 0.1*** x ***5 = 0.5 and 0.08*** x ***5 = 0.4.***

Draw an arrow from Tim to Tam in the picture.



T: ***What could this arrow be for?***

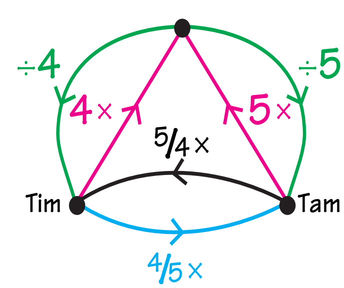
For emphasis, trace the 4x arrow followed by "against" the 5x arrow. Then trace the blue arrow.

S: $\frac{4}{5}$x.

S: ***The opposite arrow for the 5x arrow is ÷5. 4x followed by ÷5 is*** $\frac{4}{5}$**x.**

T: ***What is the opposite of a*** $\frac{4}{5}$**x *arrow?***

For emphasis, trace the 5x arrow followed by "against" the 4x arrow. Then draw an arrow from Tam to Tim.



S: ÷$\frac{4}{5}$.

S: ***The opposite arrow for the 4x arrow is ÷4.***
5x followed by ÷4 is $\frac{5}{4}$**x.**

Write the following problems on the board and ask students to copy and solve them. (Answers are in boxes.)

$\frac{4}{5} \times 15 = \boxed{12}$	$\frac{4}{5} \times 2 = \boxed{1.6}$	$\frac{5}{4} \times 100 = \boxed{125}$
$\frac{4}{5} \times 1.5 = \boxed{1.2}$	$\frac{4}{5} \times 0.2 = \boxed{0.16}$	$\frac{5}{4} \times 10 = \boxed{12.5}$
$\frac{4}{5} \times 25 = \boxed{20}$	$\frac{4}{5} \times 20 = \boxed{16}$	$\frac{5}{4} \times 2 = \boxed{2.5}$
$\frac{4}{5} \times 2.5 = \boxed{2}$		$\frac{5}{4} \times 0.2 = \boxed{0.25}$

After several minutes for independent work, check some answers collectively.

T: ***How did you calculate*** $\frac{4}{5} \times 15$***?***

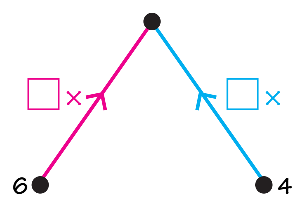
S: $4 \times 15 = 60$ ***and*** $60 \div 5 = 12$***, so*** $\frac{4}{5} \times 15 = 12$***.***

The arrow picture shows one way to multiply by $\frac{4}{5}$, but dividing by 5 first and then multiplying by 4 is also correct.

S: $15 \div 5 = 3$ ***and*** $3 \times 4 = 12$***, so*** $\frac{4}{5} \times 15 = 12$***.***

Exercise 2

Draw this arrow picture and chart on the board.



→ (red)								
→ (blue)								

T: ***What could the red arrow and the blue arrow be for? The number in the red box times 6 equals the number in the blue box times 4.***

Let the class check any set of multiplication functions suggested. Enter correct pairs in the chart. You may need to prompt students to consider (non-integer) decimal numbers.

T: ***If the red arrow is for 0.5x, what is the blue arrow for?***

S: $6 \times 0.5 = 3$ ***and*** $4 \times 0.75 = 3$***, so the blue arrow is for 0.75x.***

Your chart might look similar to this one upon completion of the activity.

→ (red)	6×	4×	10×	2×	8×	20×	2 000×	0.5×
→ (blue)	9×	6×	15×	3×	12×	30×	3 000×	0.75×

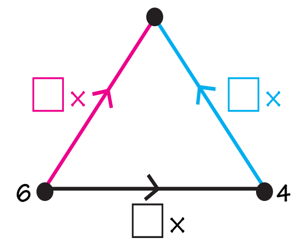
T: ***Do you notice anything interesting about this chart?***

N4

S: ***It is like the last problem. Look at the pair 4x and 6x. Half of 4 is 2 and half of 6 is 3. 2x and 3x are another pair in the table.***

S: ***Look at the pair 20x and 30x. 100 x 20 = 2 000 and 100 x 30 = 3 000. 2 000x and 3 000x are another pair in the table.***

Draw an arrow starting at 6 and ending at 4.



T: ***What could this arrow be for?***

If necessary, trace the red arrow and against the blue arrow. Then trace the new arrow.

S: ***$\frac{2}{3}$x.***

S: ***$\frac{8}{12}$x.***

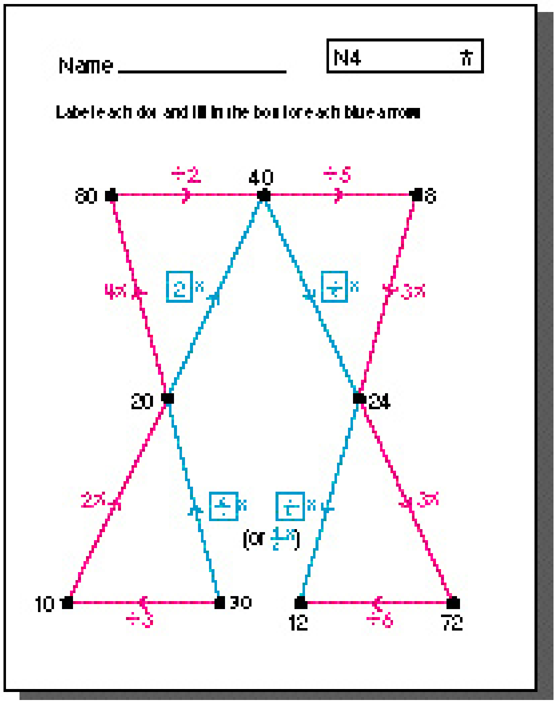
S: ***$\frac{2\,000}{3\,000}$x.***

S: ***x0.666666 … = x$0.\overset{*}{6}$.***

As students suggest names, list them on the board in a string of equalities.

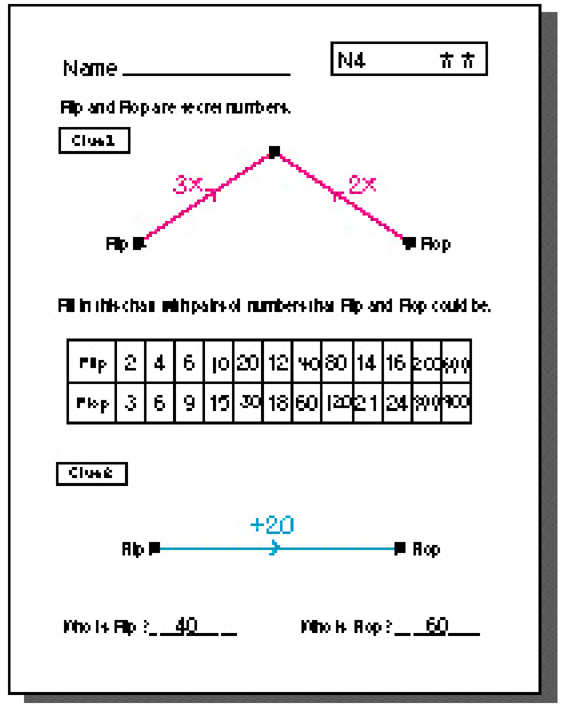
$$\times 0.6 = \frac{2}{3}\times = \frac{8}{12}\times = \frac{2\,000}{3\,000}\times = \frac{12}{18}\times = \frac{6}{9}\times = \cdots$$

Worksheets N4*, **, ***, and **** are available for individual work.



Name ______________ N4 ☆

Label each dot and fill in the box for each blue arrow.



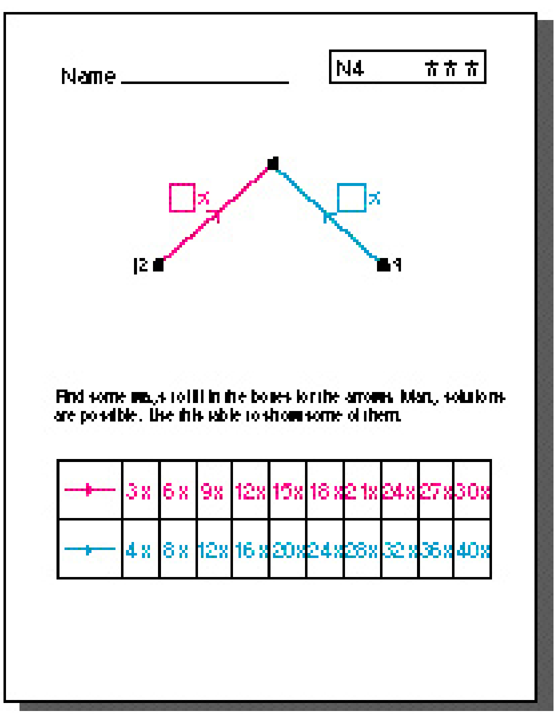
Name ______________ N4 ☆☆

Flip and Flop are secret numbers.

Fill in this chart with pairs of numbers that Flip and Flop could be.

Flip	2	4	6	10	20	12	40	80	14	16	200	400
Flop	3	6	9	15	30	18	60	120	21	24	300	600

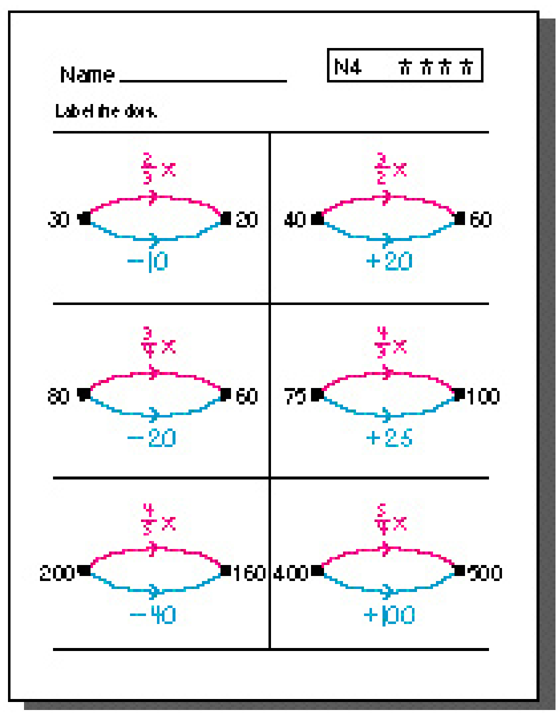
Who is Flip? 40 Who is Flop? 60



Name ______________ N4 ☆☆☆

Find some ways to fill in the boxes for the arrows. Many solutions are possible. Use this table to show some of them.

→ (red)	3x	6x	9x	12x	15x	18x	21x	24x	27x	30x
→ (blue)	4x	8x	12x	16x	20x	24x	28x	32x	36x	40x



Name ______________ N4 ☆☆☆☆

Label the dots.

Capsule Lesson Summary

Determine that the number of ancestors of a person in any one generation is a power of 2. Calculate the total number of ancestors of a person up to a certain generation. Compare the number of ancestors of a person to the number of ancestors of a bee, which differs since a male bee has a single parent. Note that the number of ancestors of a bee in any one generation is a Fibonacci number.

Materials

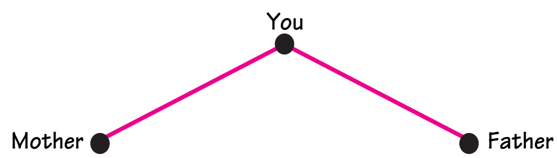
Teacher
- Colored chalk

Student
- Colored pencils, pens, or crayons
- Worksheet N5

Description of Lesson

Exercise 1

Draw this cord picture on the board.



T: ***Today we're going to talk about the number of ancestors a person has. In the first generation, a person has two ancestors: a mother and a father. In about what year were your parents born?***

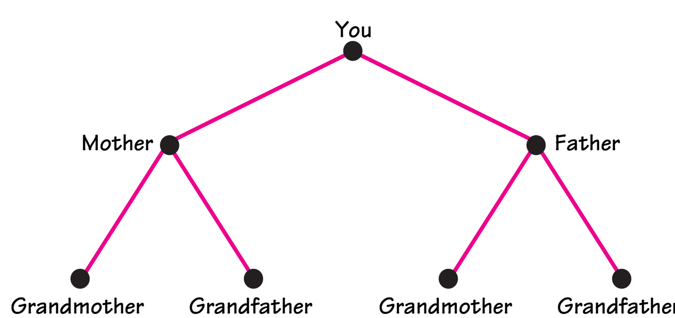
Let students estimate when their parents were born. Then select a year that is a multiple of 5, 1960 for example, as an average. Begin a table on the board.

Generation	Ancestors	Year of Birth
1st	2	1960
2nd		
3rd		

T: ***Your second generation ancestors are your grandparents. How many (blood) grandparents, both living and dead, does a person have?***

S: ***Four, each parent has two parents.***

Record this response in the table and extend the cord picture.



T: ***Let's assume that a generation is 25 years. That is, we will assume that your grandparents were 25 years old when your parents were born. If your parents were born in about 1960 about when were your grandparents born?***

S: ***1935 (1960 – 25 = 1935).***

Record this figure in the table.

T: ***What is your next set of ancestors called?***

S: ***Great-grandparents. There are eight (2 x 4 = 8) great-grandparents, and they were born in about 1910 (1935 – 25 = 1910).***

Record this data in the table. In a similar manner, complete the table for about eight generations. Continue the cord picture only until most students see the doubling pattern.

Generation	Ancestors	Year of Birth
1st	2	1960
2nd	4	1935
3rd	8	1910
4th	16	1885
5th	32	1860
6th	64	1835
7th	128	1810
8th	256	1785

T: ***What do you notice about the number of ancestors?***

S: ***It doubles each generation.***

Point out to students that the eighth generation was born shortly after the time of the American Revolution. Also note that the 256 ancestors in that generation do not include most family relations, that is, brothers, sisters, and children of all of these people.

T: ***This method of counting ancestors is accurate for a few generations but eventually leads to a paradox. A paradox is a statement that is seemingly contradictory.***

If we continued doubling, we'd find that in the 31st generation—in about the year 1210—you would have over two billion ancestors. Does that seem possible?

S: ***No, it seems too large.***

T: ***In fact, in 1210, the population of the whole earth was only about half a billion. Why doesn't our method of doubling work?***

Let students discuss this paradox. If necessary, state the following explanation yourself.

S: ***Some of the people in the family tree, especially many generations ago, might have been brothers or sisters to each other. These people would have had the same parents, grandparents, and so on, and their ancestors would therefore be listed at least twice in the family tree. As you go back further, more duplication would occur.***

Put a fourth column in the table and label it "Total (Cumulative) Ancestors."

T: ***Let's count the total number of ancestors up to each generation. After one generation there are two ancestors, your father and mother. After two generations, there are six (2 + 4 = 6) ancestors, your parents and your grandparents.***

Let students complete the fourth column of the table by adding the appropriate number of ancestors in each generation.

Point to the fourth column of numbers: 2, 6, 14, 30, … .

T: ***What patterns do you see here?***

Encourage students to describe several patterns. For example:

- (2 x 2) + 2 = 6, (2 x 6) + 2 = 14, (2 x 14) + 2 = 30, …
- 2 x (2 + 1) = 6, 2 x (6 + 1) = 14, 2 x (14 + 1) = 30, …
- 2 + 4 = 6, 6 + 8 = 14, 14 + 16 = 30, 30 + 32 = 62, …
- The total number of ancestors up to one generation is two less than the number of ancestors in the next generation.

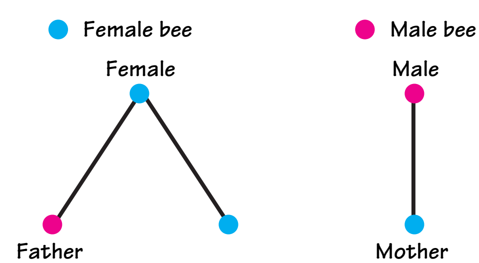
Draw arrows for +2 to highlight this last pattern.

Generation	Ancestors	Year of Birth	Total (Cumulative) Ancestors
1st	2	1945	2
2nd	4	1920	6
3rd	8	1895	14
4th	16	1870	30
5th	32	1845	62
6th	64	1820	126
7th	128	1795	254
8th	256	1770	510

Exercise 2

T: ***Most animals and insects have an ancestry pattern similar to people, but bees are different. A female bee has two parents, just as we do. However, a male bee has only one parent, a mother.†***

Draw this picture on the board.



Then draw a red dot separate from this key on the board.

T: ***Let's draw a family tree for a male bee. As we draw the tree, we must remember that all bees have a mother, but only female bees have a father. How many parents does this male bee have?***

S: ***Only one, a mother.***

Begin a tree and chart on the board.

Male Bee

Generation	Ancestors
1st	1
2nd	
3rd	
4th	

T: ***How many grandparents does a male bee have?***

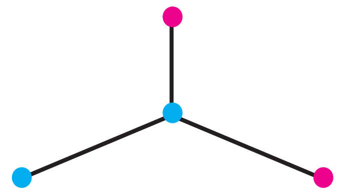
S: ***Two, its mother's two parents.***

†Give the following explanation, if you wish. Female bees lay two types of eggs: fertilized eggs which produce females, and unfertilized eggs which produce males. Therefore, a male bee has only one parent, a mother.

N5

Invite a student to extend the tree and make the next entry in the chart.

T: ***How many great-grandparents does a male bee have?***



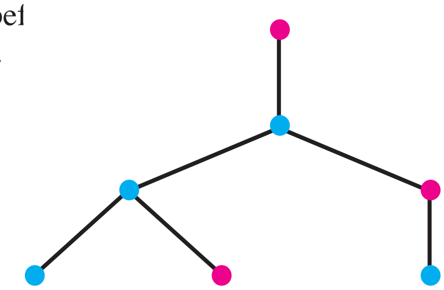
Male Bee

Generation	Ancestors
1st	1
2nd	2
3rd	
4th	

Let students explain their answers bef inviting a student to extend the tree.

Note that the male grandparent has only one parent.

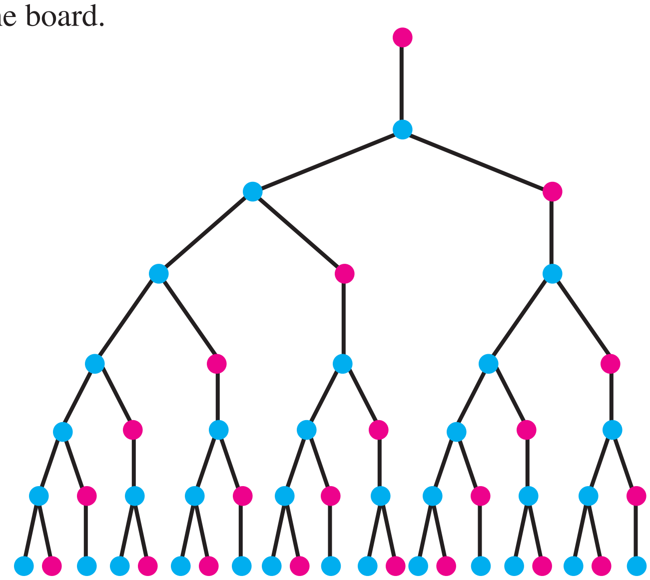
T: ***How many ancestors will there be in the fourth generation?***



Male Bee

Generation	Ancestors
1st	1
2nd	2
3rd	3
4th	

Let students make predictions before extending the tree. Distribute copies of Worksheet N5 and let students work individually or with partners to extend the tree several more generations. As you observe students' work, point out that a male bee (red dot) always has one parent and a female bee (blue dot) always has two. After several minutes, invite students to help extend the male bee's family tree on the board.



Male Bee

Generation	Ancestors
1st	1
2nd	2
3rd	3
4th	5
5th	8
6th	13
7th	21
8th	

T: ***Are there any patterns you can use to predict the number of ancestors in the eighth generation? The ninth? The tenth?***

S: ***34 in the eighth generation.***

S: ***55 in the ninth generation.***

S: ***89 in the tenth generation.***

S: ***To find the number of bees in a generation, add the number of bees in the two previous generations: 1 + 2 = 3, 2 + 3 = 5, 3 + 5 = 8, 8 + 13 = 21, 13 + 21 = 34, 21 + 34 = 55, and 34 + 55 = 89.***

Encourage students to note this rule and then to use it to extend the table several more generations.

Male Bee

Generation	Ancestors
1st	1
2nd	2
3rd	3
4th	5
5th	8
6th	13
7th	21
8th	34

Generation	Ancestors
9th	55
10th	89
11th	144
12th	233
13th	377
14th	610
15th	987
16th	

T: ***How does the size of these numbers compare to the number of ancestors for people?***

S: ***These are much smaller. For example, in the eighth generation, a bee has only 34 ancestors while a human has 256.***

Add two columns to the table, one for male ancestors and one for female ancestors.

T: ***Humans have the same number of male ancestors as female ancestors since each person in the family tree has one father and one mother. Let's check the number of male and female ancestors for bees.***

Let students use their worksheets to count the number of males and females in each generation and to record these numbers in the table.

T: ***What do you notice?***

S: ***Starting at 1 in each column, the same sequence of numbers occurs: 1, 1, 2, 3, 5, 8, 13, … .***

T: ***Does anyone remember seeing this sequence of numbers before?***

Male Bee

Generation	Ancestors	Male Ancestors	Female Ancestors
1st	1	0	1
2nd	2	1	1
3rd	3	1	2
4th	5	2	3
5th	8	3	5
6th	13	5	8
7th	21	8	13
8th	34		

Students may remember their appearance in the story-workbook *Summer Camp*.

Write *Fibonacci numbers* on the board.

T: ***This sequence of numbers is quite famous in mathematics. An Italian, Leonardo Fibonacci, was an outstanding European mathematician in the 1200s. In his famous book,* Liber Abaci*, he posed a problem about rabbits. The solution to that problem was in this sequence of numbers. The numbers in this sequence are named after him; they are called* Fibonacci numbers.**

Add a fifth column to the table for the cumulative totals.

T: ***Let's count the total number of ancestors up to each generation. In the first generation there is one ancestor. By the second generation, there are three (1 + 2 = 3); and by the third generation, there are six (3 + 3 = 6).***

Let students complete the table (see the next illustration).

T: ***What patterns do you notice among these number?***

S: ***(1 + 3) + 2 = 6, (3 + 6) + 2 = 11, (6 + 11) + 2 = 19, and so on.***

S: ***The total number of ancestors in one generation is two fewer than the number of bees two generations later.***

Add +2 arrows to highlight this last relationship.

Male Bee

Generation	Ancestors	Male Ancestors	Female Ancestors	Total (Cumulative) Ancestors
1st	1	0	1	1
2nd	2	1	1	3
3rd	3	1	2	6
4th	5	2	3	11
5th	8	3	5	19
6th	13	5	8	32
7th	21	8	13	53
8th	34			

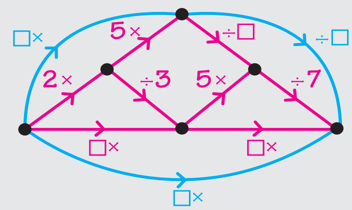
Extension Activity

Suggest that students investigate (check) some other patterns in the sequence of Fibonacci numbers. For example:

- Every third number in the sequence is even, every fourth number is a multiple of 3, and every fifth number is a multiple of 5.
- Choose four successive numbers in the sequence, for example: 5 8 13 21. Multiply the two outside numbers (5 x 21) and multiply the two inside numbers (8 x 13). How do they compare? Does this always work?
- Add the first *n* numbers (for example, the first six numbers in the sequence 1 + 1 + 2 + 3 + 5 + 8)
  and compare the sum to the *n* + 2 number in the sequence. (For example, compare the sum of the first six to the eighth.)

Capsule Lesson Summary

Use this arrow picture to motivate a method of multiplying fractions.



Materials

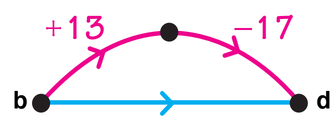
Teacher
- Colored chalk
- Calculator

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N6*, **, ***, and ****

Description of Lesson

Exercise 1

Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: ***If 15 is here*** (at **b**)***, what number is here*** (at **d**)***?***

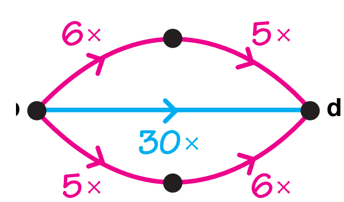
S: ***11. 15 + 13 = 28 and 28 – 17 = 11.***

Repeat the question a couple more times using, for example, 9, $\widehat{3}$, and 1.6 as the number at **b**. You may also ask what number is at **b** if the number at **d** is 10, $\widehat{2}$, or 2.8.

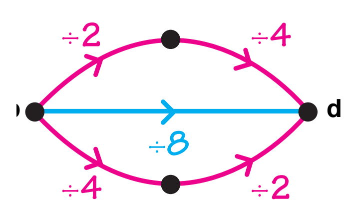
T (tracing the blue arrow)**:** ***What is +13 followed by –17?*** (–4)
If the red arrows were in the opposite order, that is, –17 followed by +13, what would the blue arrow be for?

S: ***It would still be for –4.***

Relabel the red arrows 6x and 5x. Repeat the activity using 3, 7, and $\widehat{2}$ as possible numbers at **b**. Conclude that the composition of 6x and 5x is 30x, and that 6x followed by 5x is the same as 5x followed by 6x.

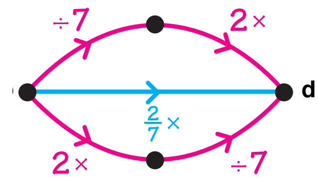


Relabel the red arrows ÷2 and ÷4. Repeat the activity using 16, 64, 7.2, and $\widehat{24}$ as possible numbers at **b**. Conclude that the composition of ÷2 and ÷4 is ÷8, and that ÷4 followed by ÷2 is the same as ÷2 followed by ÷4.



Relabel the red arrows ÷7 and 2x. Repeat the activity using 14, 21, and $\widehat{70}$ as possible numbers at **b**. If a student labels **b** with a number other than a multiple of 7, you can suggest using a calculator for the computation.

Conclude that ÷7 followed by 2x is the same as $^2\!/_7$x, and that ÷7 followed by 2x is the same as 2x followed by ÷7. Show the commutativity in the picture for emphasis.



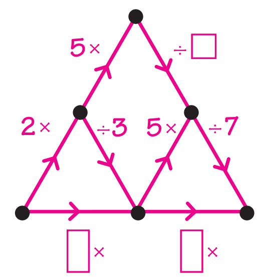
Erase the arrow labels.

T: ***Can you think of what the red arrows could be for so that the blue arrow would not be the same when we change the order?***

A multiplication or division arrow composed with an addition or subtraction arrow provide examples in which order makes a difference. Allow a couple minutes for students to consider the question and then move on to Exercise 2.

Exercise 2

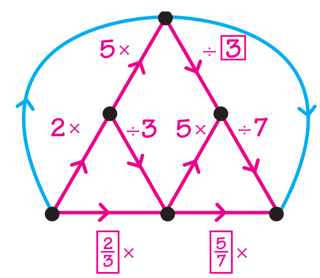
Draw this arrow picture on the board.



Invite students to fill in the boxes for the arrows. If the box for the division arrow is a problem, focus their attention on the center diamond and ask,

T: ***÷3 followed by 5x is the same as 5x followed by ÷ what number?*** (3)

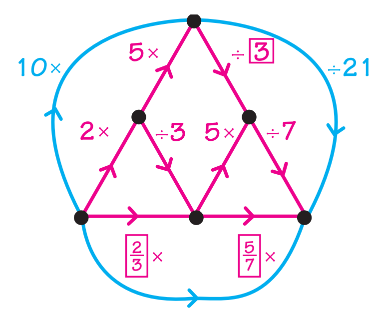
Add these two blue arrows to the picture.



T: ***What is 2x followed by 5x?*** (10x)
What is ÷3 followed by ÷7? (÷21)

If students have trouble with either of these questions, ask them to label the dots in several ways.

Label the blue arrows and then draw another blue arrow as shown here.



T: ***Look at the blue arrow along the bottom of the picture. What could it be for? Use the other two blue arrows to help you.***

Allow a few minutes for the class to consider the situation.

S: ***$^{10}\!/_{21}$x.***

T (tracing the blue arrows)**:** ***10x followed by ÷21 is $^{10}\!/_{21}$x.***

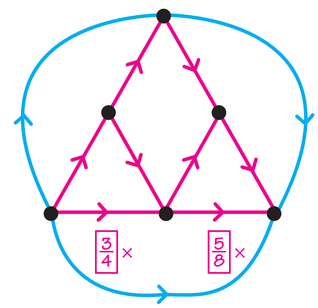
Label the blue arrow along the bottom of the picture $^{10}\!/_{21}$x and make the following observation.

T (tracing appropriate arrows)**:** ***We already know that 2x followed by 5x is 10x; 2 x 5 = 10. We also know that ÷3 followed by ÷7 is ÷21; 7 x 3 = 21. Now we see that $^2\!/_3$x followed by $^5\!/_7$x is $^{10}\!/_{21}$x, or $\frac{2 \times 5}{3 \times 7}$x. So we will conclude that $^2\!/_3$ x $^5\!/_7$ = $^{10}\!/_{21}$.***

Record the equality on the board.

$$\frac{2}{5} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$$

Erase the arrow labels in the picture. Relabel two of the red arrows as shown here.



T (tracing the blue arrow across the bottom)**:** ***What could this blue arrow be for? How do you know?***

S: ***$^{15}/_{32}$x; 3 x 5 = 15 and 4 x 8 = 32.***

T: ***$^{3}/_{4}$x followed by $^{5}/_{8}$x is $^{15}/_{32}$x.***

Write this multiplication of fractions problem on the board.

$$\frac{3}{4} \times \frac{5}{8}$$

T: ***How could we do this calculation?***

S: ***3 x 5 = 15 and 4 x 8 = 32, so $^{3}/_{4}$ x $^{5}/_{8}$ = $^{15}/_{32}$.***

T: ***To multiply two fractions, just multiply the top numbers and then multiply the bottom numbers.***

$$\frac{3}{4} \times \frac{5}{8} = \frac{3 \times 5}{4 \times 8} = \frac{15}{32}$$

As appropriate, use the terminology *numerator* and *denominator*.

Write several multiplication problems on the board, and ask students to copy and solve them. Check the problems collectively. (Answers are in boxes.)

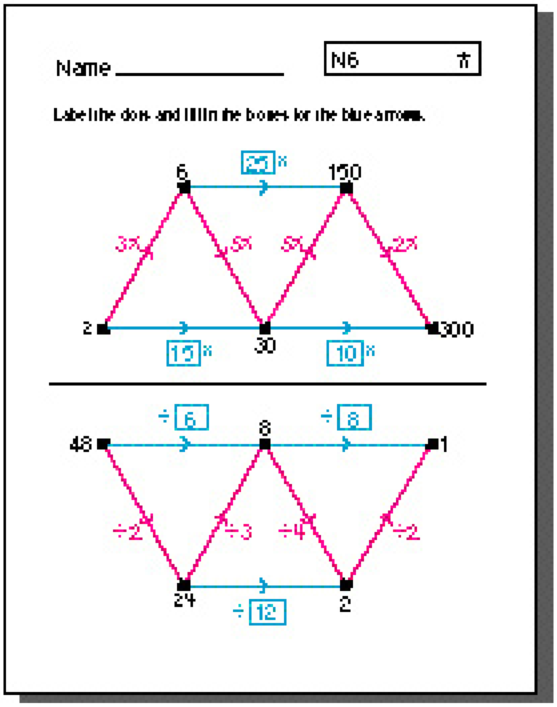
$$\frac{3}{5} \times \frac{2}{3} = \boxed{\frac{3 \times 2}{5 \times 3} = \frac{6}{15} = \frac{2}{5}}$$

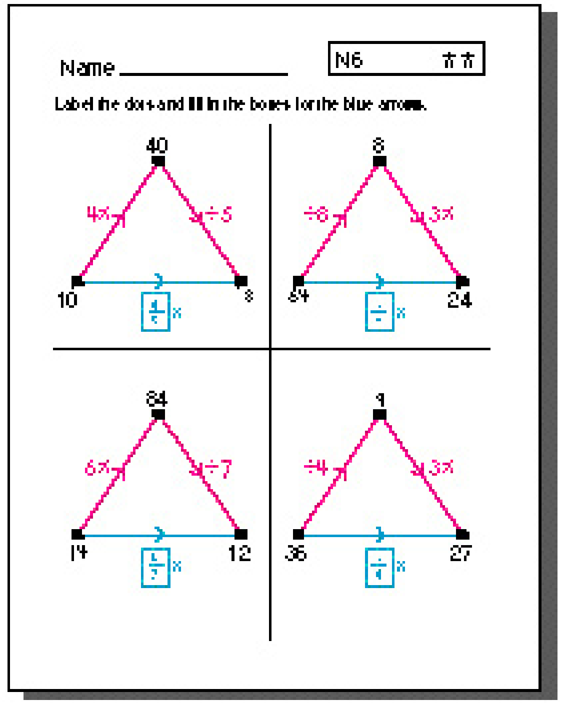
$$\frac{4}{5} \times \frac{3}{4} = \boxed{\frac{4 \times 3}{5 \times 4} = \frac{12}{20} = \frac{3}{5}}$$

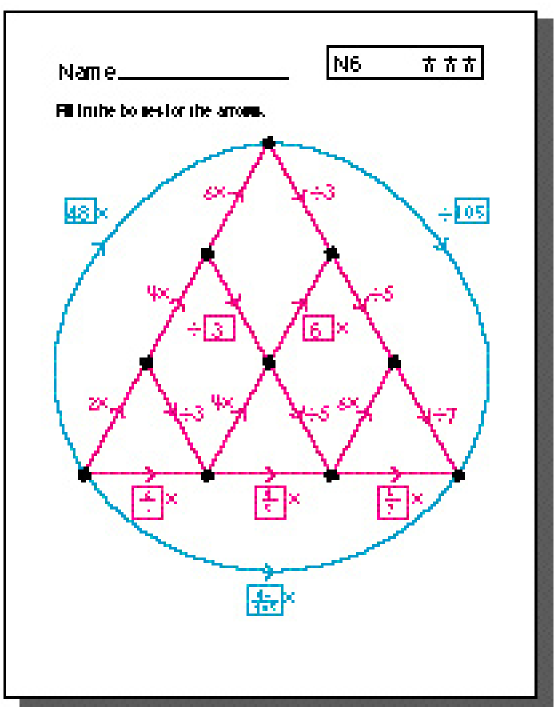
$$\frac{5}{8} \times \frac{4}{5} = \boxed{\frac{5 \times 4}{8 \times 5} = \frac{20}{40} = \frac{1}{2}}$$

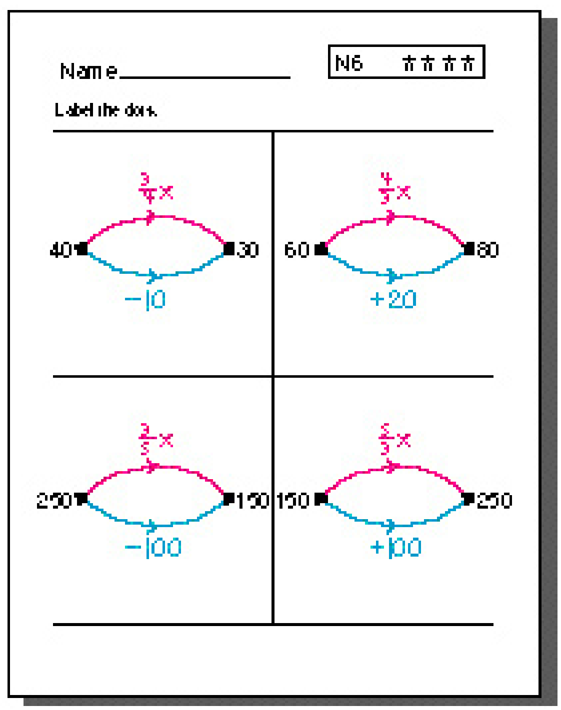
$$\frac{3}{7} \times \frac{2}{5} = \boxed{\frac{3 \times 2}{7 \times 5} = \frac{6}{35}}$$

Worksheets N6*, **, ***, and **** are available for individual work.


Name
N6
Label the dots and fill in the boxes for the blue arrows.
25x
6
150
3x
5x
5x
2x
2
30
300
15x
10x
÷6
÷8
48
8
1
÷2
÷3
÷4
÷2
24
2
÷12


Name
N6
Label the dots and fill in the boxes for the blue arrows.
40
4x
÷5
10
8
24
3x
84
6x
÷7
14
12
÷4
3x
36
27


Name
N6
Fill in the boxes for the arrows.
48x
÷105
6x
÷3
4x
÷5
÷3
6x
2x
÷3
4x
÷5
6x
÷7


Name
N6
Label the dots.
40
30
−10
60
80
+20
250
150
−100
150
250
+100

Capsule Lesson Summary

Review the standard algorithms for addition, subtraction, and multiplication. Solve some puzzle-type problems involving these algorithms. Play a multiplication or subtraction card game.

Materials

Teacher
- Chalk
- Blacklines N7(a) and (b)
- Ten index cards

Student
- Paper
- Game sheets
- Worksheets N7* and **

Advance Preparation: Use Blacklines N7(a) and (b) to make copies of the game sheets for students. Write the digits from 0 to 9 on the ten index cards, one digit on each card.

Description of Lesson

Write these three problems on the board.

$$\begin{array}{r} 7\,6\,7\,8 \\ 3\,4\,9 \\ +\,6\,2\,7\,5 \\ \hline \square\square\square\square\square \end{array} \qquad \begin{array}{r} 6\,\square\,4 \\ +\,\square\,4\,\square \\ \hline \square\,5\,7\,0 \end{array} \qquad \begin{array}{r} 4\,\square\,7\,5 \\ \square\,0\,4\,\square \\ +\,5\,3\,\square\,2 \\ \hline \square\,3\,9\,0\,3 \end{array}$$

T: ***I had three addition problems for you, but an eraser gremlin erased one answer and parts of the other calculations. I put boxes in place of the numerals the gremlin erased. Copy the problems and fill in the boxes. A single digit belongs in each box.***

When many students are finished, invite students to complete the problems at the board and to explain the calculations.

Solutions:

$$\begin{array}{r} 7\,6\,7\,8 \\ 3\,4\,9 \\ +\,6\,2\,7\,5 \\ \hline \boxed{1}\boxed{4}\boxed{3}\boxed{0}\boxed{2} \end{array} \qquad \begin{array}{r} 6\,\boxed{2}\,4 \\ +\,\boxed{9}\,4\,\boxed{6} \\ \hline \boxed{1}\,5\,7\,0 \end{array} \qquad \begin{array}{r} 4\,\boxed{4}\,7\,5 \\ \boxed{4}\,0\,4\,\boxed{6} \\ +\,5\,3\,\boxed{8}\,2 \\ \hline \boxed{1}\,3\,9\,0\,3 \end{array}$$

Present several subtraction problems in a similar manner. (Answers are in boxes.)

$$\begin{array}{r} 3\,8\,0\,2 \\ -\,2\,6\,2\,9 \\ \hline \boxed{1}\boxed{1}\boxed{7}\boxed{3} \end{array} \qquad \begin{array}{r} \boxed{8}\,9\,\boxed{8}\,6 \\ -\,6\,\boxed{3}\,5\,\boxed{2} \\ \hline 2\,6\,3\,4 \end{array} \qquad \begin{array}{r} \boxed{9}\,4\,0\,\boxed{3} \\ -\,2\,6\,\boxed{1}\,6 \\ \hline 6\,\boxed{7}\,\boxed{8}\,7 \end{array}$$

Present several multiplication problems in a similar manner. (Answers are in boxes.)

$$\begin{array}{r} 2\,3\,7\,6 \\ \times\ 4 \\ \hline \boxed{9}\boxed{5}\boxed{0}\boxed{4} \end{array} \qquad \begin{array}{r} 2\,\boxed{3}\,6\,\boxed{7} \\ \times\ 3 \\ \hline \boxed{7}\,1\,\boxed{0}\,1 \end{array} \qquad \begin{array}{r} 9\,\boxed{5}\,4 \\ \times\ \boxed{7} \\ \hline \boxed{6}\boxed{6}\,7\,8 \end{array}$$

Exercise 2

Distribute copies of the multiplication game sheet. Then explain and play the following game.

T: ***I have a deck to ten cards here with the numbers (single digits) 0 to 9 on them. I will draw four cards from the deck, one at a time, and set them aside. After each draw, write the number in one box on your game sheet. After all four boxes are filled with numbers, do the multiplication calculation. The object of the game is to make the product as large as possible, but once you put a digit in a box it must stay there. No erasing.***

Suppose, for example, that the four cards you draw are $\boxed{3}$, $\boxed{7}$, $\boxed{2}$, and $\boxed{5}$ in that order. The following are possible student responses.

$$\begin{array}{r} \boxed{5}\boxed{3} \\ \times\ \boxed{2}\boxed{7} \\ \hline 3\,7\,1 \\ 1\,0\,6\,0 \\ \hline 1\,4\,3\,1 \end{array} \qquad \begin{array}{r} \boxed{7}\boxed{2} \\ \times\ \boxed{5}\boxed{3} \\ \hline 2\,1\,6 \\ 3\,6\,0\,0 \\ \hline 3\,8\,1\,6 \end{array} \qquad \begin{array}{r} \boxed{2}\boxed{3} \\ \times\ \boxed{7}\boxed{5} \\ \hline 1\,1\,5 \\ 1\,6\,1\,0 \\ \hline 1\,7\,2\,5 \end{array} \qquad \begin{array}{r} \boxed{7}\boxed{3} \\ \times\ \boxed{5}\boxed{2} \\ \hline 1\,4\,6 \\ 3\,6\,5\,0 \\ \hline 3\,7\,9\,6 \end{array}$$

In this case, the greatest possible product is 3 816; however, it may happen that no student gets that product. Ask for a product a student believes may be close to the greatest and then check whether any student has a greater result.

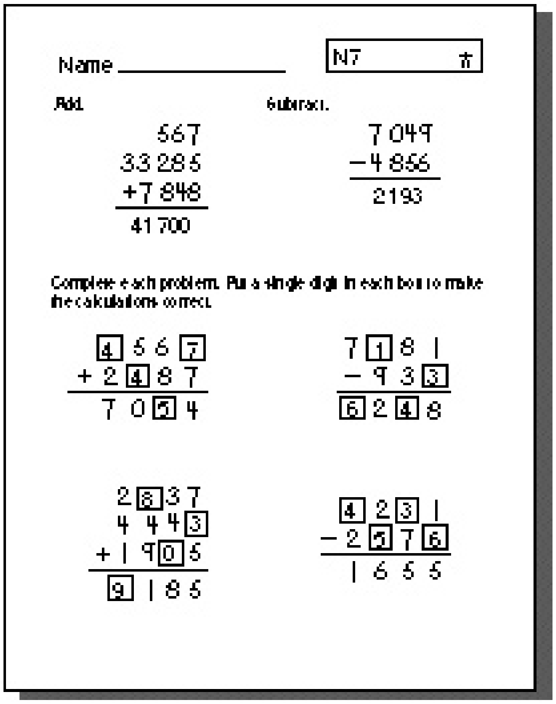
Repeat the game, drawing five cards and using the bottom of the multiplication game sheet. You can vary the game by looking for the least product.

You may like to play the game with subtraction as well, looking for either the greatest or least result (difference).

Worksheets N7* and ** are available for individual work.

Home Activity

Send home some algorithm puzzles for parents to do with their child. You may also suggest that students or parents create puzzles to challenge each other.



Name ____________ N7 ☆

Add.

$$\begin{array}{r} 567 \\ 33\,285 \\ +7\,848 \\ \hline 41\,700 \end{array}$$

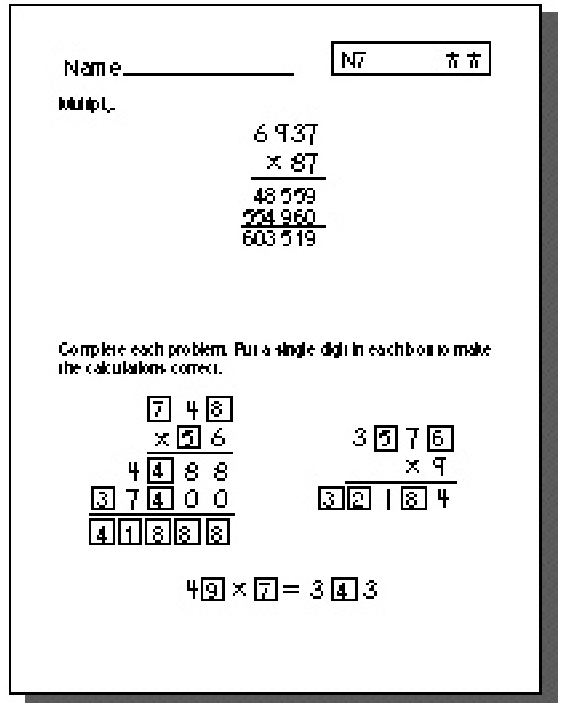
Subtract.

$$\begin{array}{r} 7\,049 \\ -4\,856 \\ \hline 2193 \end{array}$$

Complete each problem. Put a single digit in each box to make the calculations correct.

$$\begin{array}{r} \boxed{4}\,5\,6\,\boxed{7} \\ +\,2\,\boxed{4}\,8\,7 \\ \hline 7\,0\,\boxed{5}\,4 \end{array} \qquad \begin{array}{r} 7\,\boxed{1}\,8\,1 \\ -\,9\,3\,\boxed{3} \\ \hline \boxed{6}\,2\,\boxed{4}\,8 \end{array}$$

$$\begin{array}{r} 2\,\boxed{8}\,3\,7 \\ 4\,4\,4\,\boxed{3} \\ +\,1\,9\,\boxed{0}\,5 \\ \hline \boxed{9}\,1\,8\,5 \end{array} \qquad \begin{array}{r} \boxed{4}\,2\,\boxed{3}\,1 \\ -\,2\,\boxed{5}\,7\,\boxed{6} \\ \hline 1\,6\,5\,5 \end{array}$$



Name ____________ N7 ☆☆

Multiply.

$$\begin{array}{r} 6\,937 \\ \times\,87 \\ \hline 48\,559 \\ 554\,960 \\ \hline 603\,519 \end{array}$$

Complete each problem. Put a single digit in each box to make the calculations correct.

$$\begin{array}{r} \boxed{7}\,4\,\boxed{8} \\ \times\,\boxed{5}\,6 \\ \hline 4\,\boxed{4}\,8\,8 \\ \boxed{3}\,7\,\boxed{4}\,0\,0 \\ \hline \boxed{4}\,\boxed{1}\,\boxed{8}\,\boxed{8}\,\boxed{8} \end{array} \qquad \begin{array}{r} 3\,\boxed{5}\,7\,\boxed{6} \\ \times\,9 \\ \hline \boxed{3}\,\boxed{2}\,1\,\boxed{8}\,4 \end{array}$$

$$4\boxed{9} \times \boxed{7} = 3\,\boxed{4}\,3$$

Capsule Lesson Summary

Label marks on a number line with division names, fractions, and decimals. Notice the equivalence between $n \div m$ and $\frac{n}{m}$. Use this equivalence and a calculator to find the decimal names of some fractions.

Materials

Teacher	• Meter stick • Calculator	Student	• Calculator • Worksheets N8*, **, *** and ****

Description of Lesson

Exercise 1

Use an order natural to the seating arrangement in your classroom for this activity. The dialogue here assumes there are 28 students.

T: ***Let's count by sixths. If we simply say, $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$, $\frac{6}{6}$, $\frac{7}{6}$, …, $\frac{20}{6}$, …, with what fraction will we end?***

S: ***$\frac{28}{6}$, because there are 28 of us.***

T: ***Okay, but let's say the numbers a little differently. What is another name for $\frac{6}{6}$?***

S: ***1.***

T: ***Instead of $\frac{6}{6}$, we'll say 1. What would come next?***

S: ***$1\frac{1}{6}$, which is the same as $\frac{7}{6}$.***

T: ***We'll say $1\frac{1}{6}$. If we say the numbers this way, what name for $\frac{28}{6}$ will the last person give?***

If different, record students' responses on the board. Then begin the counting. Accept equivalent fractions. When the last student has responded, acknowledge correct responses given earlier.

T: ***The last number said was $4\frac{4}{6}$, or $4\frac{2}{3}$. Where is $4\frac{4}{6}$ on the number line?***

S: ***Between 4 and 5, but closer to 5.***

T: ***If we continue the count and go around again, at what number will we end?***

Let several students respond before doing the actual counting. Record these equivalences on the board.

$$\frac{28}{6} = 4\frac{4}{6}$$

$$\frac{56}{6} = 9\frac{2}{6}$$

T: ***$2 \times \frac{28}{6} = \frac{56}{6}$. $2 \times 4\frac{4}{6} = 9\frac{2}{6}$.***

Suspose we had counted by eighths.
At what number would we have ended?
What is another name for $\frac{28}{8}$?

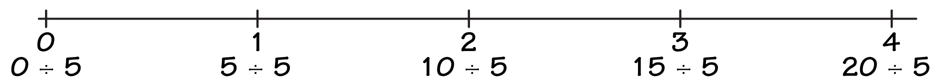
S: ***$3\frac{1}{2}$; $\frac{24}{8} = 3$; so $\frac{28}{8} = 3\frac{4}{8}$, or $3\frac{1}{2}$.***

Exercise 2

Draw a number line with five equally spaced marks on the board.

T: ***What number is 20 ÷ 5?*** (4)

Use this fact to label the mark farthest to the right. Continue with 15 ÷ 5, 10 ÷ 5, 5 ÷ 5, and 0 ÷ 5.



T: ***Where does 1 ÷ 5 belong on this number line?***

S: ***Between 0 and 1, but closer to 0.***

Record number sentences on the board as you pose the next questions.

T: ***Let's find out exactly where 1 ÷ 5 belongs. If five friends share $1 evenly, how much money does each friend get?***

S: ***20¢.***

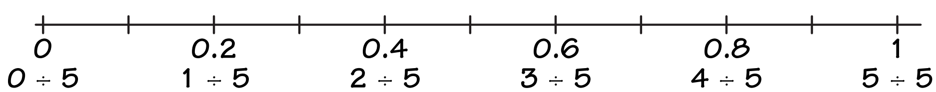
T: ***So what decimal number is 1 ÷ 5?***

S: ***0.20, or 0.2.***

T: ***How can we locate 0.2 on the number line?***

S: ***Divide the section between 0 and 1 into ten pieces of the same length. The second mark from 0 will be for 0.2.***

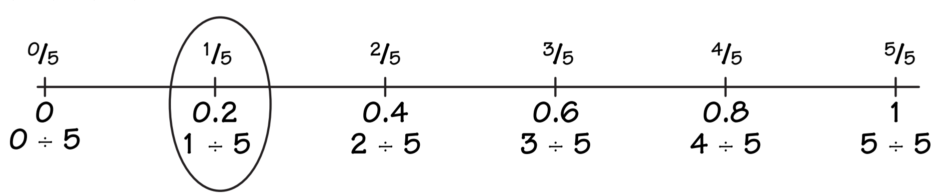
Locate 0.2 on the number line and label its mark 1 ÷ 5 as well. Continue the activity with 2 ÷ 5, 3 ÷ 5, and 4 ÷ 5.



Erase the marks that are not labeled between 0 and 1, and trace this section of the number line as you ask,

T: ***How many sections do we have marked between 0 and 1?*** (Five)
What fraction labels could we give these marks?

S: ***0/5, 1/5, 2/5, 3/5, 4/5, and 5/5.***



Circle the labels at one mark as you observe,

T: ***1/5, 0.2, and 1 ÷ 5 are all the same number. Especially notice that 1/5 = 1 ÷ 5.***
Likewise, 0/5 = 0 ÷ 5, 2/5 = 2 ÷ 5, 3/5 = 3 ÷ 5, 4/5 = 4 ÷ 5, and 5/5 = 5 ÷ 5.

Divide the section of the number line between 1 and 2 into five pieces of equal length, and point to the first mark to the right of 1.

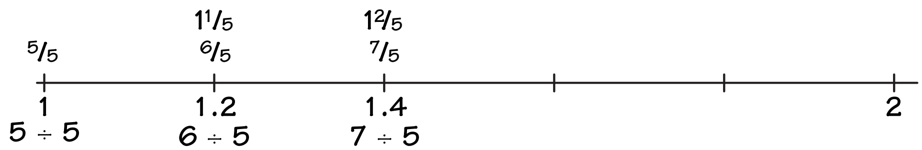
T: ***If we continue labeling, what is this mark for?***

S: ***$\frac{6}{5}$, or $1\frac{1}{5}$.***

S: ***1.2.***

S: ***$6 \div 5$.***

Continue by asking how to label the next mark.



T: ***Notice that $7 \div 5 = 1\frac{2}{5}$. Sometimes we write this division problem in another way.***

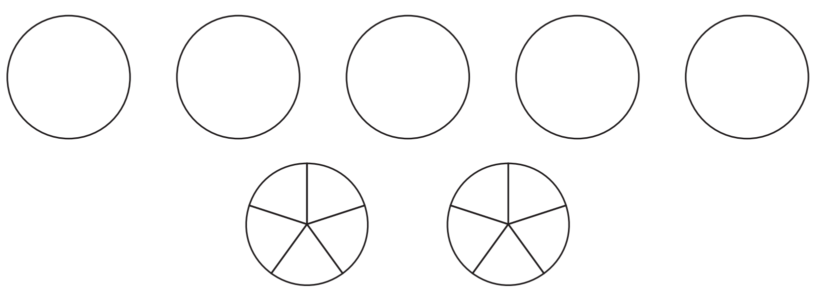
Record the division on the board in two ways.

$$7 \div 5 = 1\frac{2}{5} \qquad \begin{array}{r} 1 \text{ R} = 2 \\ 5\overline{)7} \end{array}$$

T: ***Here is another way to think about the division. Suppose five families wish to share seven pizzas. Each family gets one pizza and there are two pizzas left over*** (point to the problem on the right)***. But the families could share those two pizzas. How could they divide the two pizzas to share?***

S: ***Cut each of the two pizzas into five pieces, all the same size.***

Illustrate the pizzas on the board.



T: ***What fraction of a whole pizza would each piece be?*** ($\frac{1}{5}$)
How many $\frac{1}{5}$-pieces would each family get? (Two)
So each family would get $1\frac{2}{5}$ pizzas altogether.

As appropriate, label other marks between 1 and 2 with the class's help.

T: ***If we divide 7 by 5 on a calculator, we get the decimal 1.4. To find the decimal name for $\frac{7}{5}$, we can divide 7 by 5. How can we find the decimal name for $\frac{423}{5}$?***

S: ***Use a calculator to divide 423 by 5.***

T: ***We can use a calculator for division, but we must keep in mind when doing some problems that a calculator can display only eight digits.***

N8

Write the corresponding names for $\frac{1}{10}$ on the board as they are mentioned.

T: ***What is a division name for*** $\boldsymbol{\frac{1}{10}}$***?***

S: ***1 ÷ 10.***

$$\frac{1}{10} = 1 \div 10 = 0.1$$

Give a student a calculator to divide 1 by 10.

S: ***1 ÷ 10 = 0.1.***

Continue the activity to find the decimal names for $\frac{3}{10}$, $\frac{8}{10}$, $\frac{1}{100}$, $\frac{6}{100}$, and $\frac{24}{100}$.

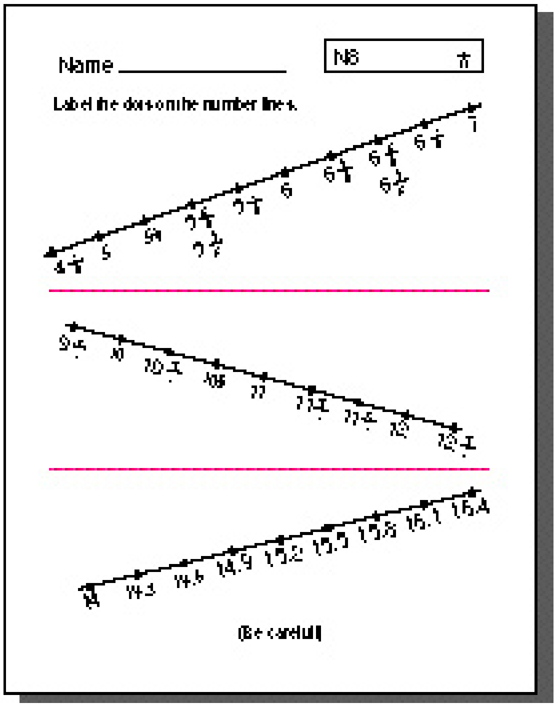
$$\frac{1}{10} = 1 \div 10 = 0.1 \qquad \frac{1}{100} = 1 \div 100 = 0.01$$

$$\frac{3}{10} = 3 \div 10 = 0.3 \qquad \frac{6}{100} = 6 \div 100 = 0.06$$

$$\frac{8}{10} = 8 \div 10 = 0.8 \qquad \frac{24}{100} = 8 \div 100 = 0.24$$

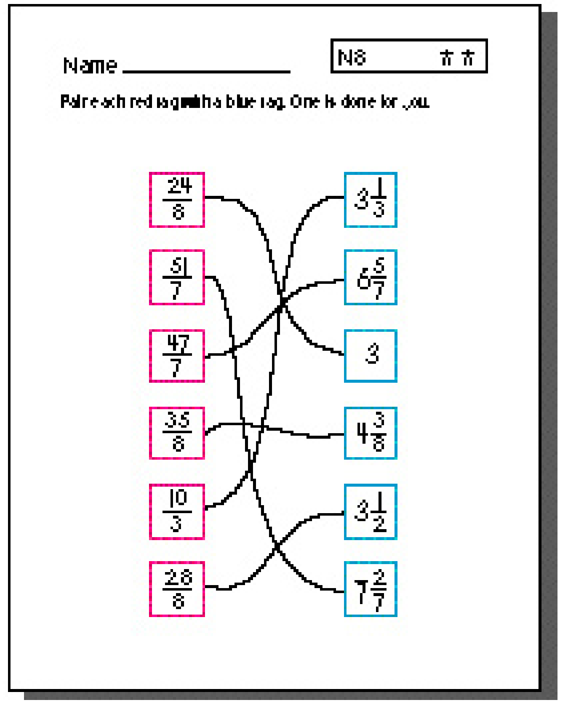
Observe that you may read 0.1 as "one tenth" and 0.01 as "one hundredth."

Worksheets N8*, **, ***, and **** are available for individual work.


Name ____________ N8 ☆

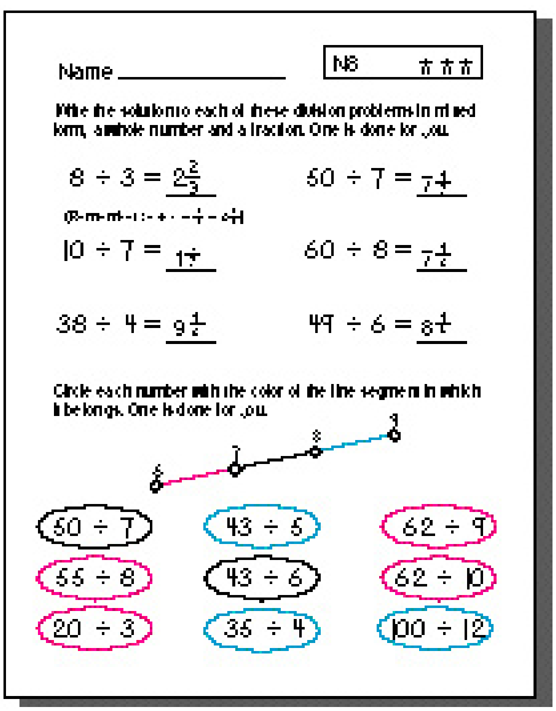
Label the dots on the number lines.

(Be careful!)


Name ____________ N8 ☆☆

Pair each red tag with a blue tag. One is done for you.

Red tags	Blue tags
$\frac{24}{8}$	$3\frac{1}{3}$
$\frac{51}{7}$	$6\frac{5}{7}$
$\frac{47}{7}$	3
$\frac{35}{8}$	$4\frac{3}{8}$
$\frac{10}{3}$	$3\frac{1}{2}$
$\frac{28}{8}$	$7\frac{2}{7}$


Name ____________ N8 ☆☆☆

Write the solution to each of these division problems in mixed form, a whole number and a fraction. One is done for you.

$8 \div 3 = 2\frac{2}{3}$

$60 \div 7 =$ ____

$10 \div 7 =$ ____

$60 \div 8 =$ ____

$38 \div 4 =$ ____

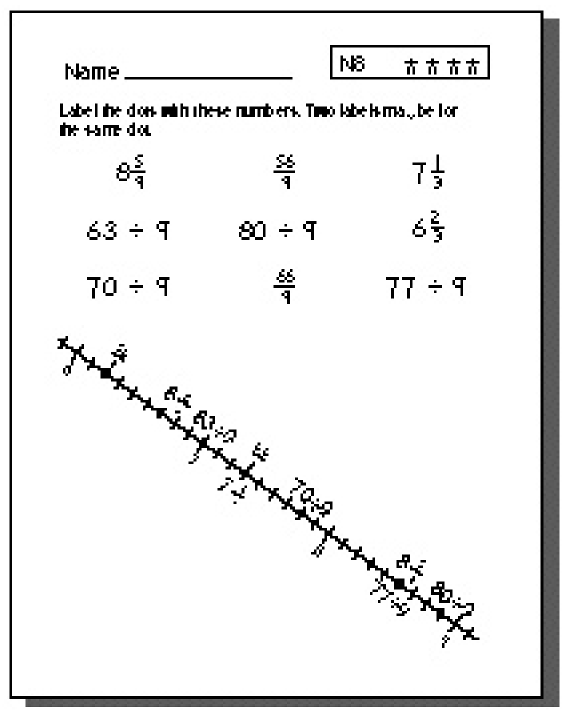
$49 \div 6 =$ ____

Circle each number with the color of the line segment in which it belongs. One is done for you.

$50 \div 7$ $43 \div 5$ $62 \div 9$

$55 \div 8$ $43 \div 6$ $62 \div 10$

$20 \div 3$ $36 \div 4$ $100 \div 12$


Name ____________ N8 ☆☆☆☆

Label the dots with these numbers. Two labels may be for the same dot.

$8\frac{5}{9}$ $\frac{56}{9}$ $7\frac{1}{3}$

$63 \div 9$ $80 \div 9$ $6\frac{2}{3}$

$70 \div 9$ $\frac{66}{9}$ $77 \div 9$

N9 GUESS MY RULE GAMES

Capsule Lesson Summary

Play a *Guess My Rule* game with the operation ∗ defined by a ∗ b = (3 x a) + b. Use this rule for * to do some calculations. Play another *Guess My Rule* game in which the rule is the function 2x followed by –5. Label the dots in some arrow roads with arrows for this function.

Materials

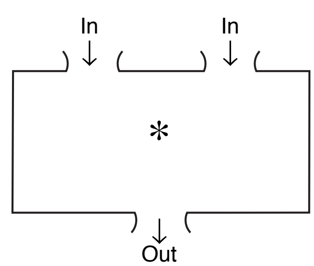
Teacher	• Colored chalk	Student	• Paper • Colored pencils, pens, or crayons • Worksheets N9*, **, and ***

Description of Lesson

When playing *Guess My Rule*, encourage students to think about what the rule could be without saying it out loud. This will allow other students a chance to discover the rule on their own. You may let students who think that they know the rule test it on numbers suggested by you or by other students. Each time, confirm or deny the result.

Exercise 1

For this exercise the rule is for an operation. You may like to use a “machine” picture to explain how an operation rule works.



T: ***I have a secret rule for ∗. This rule is like a machine. If I put two (a pair of) numbers into the machine, it sends one number out.***

I’ll give you some clues using a star (∗) for the operation. Try to figure out the secret rule for ∗.

Write several number sentences on the board as clues. Then write an open sentence and see if anyone can predict which number goes in the box.

3 ∗ 7 = 16
7 ∗ 3 = 24
1 ∗ 7 = 10
7 ∗ 1 = ☐

Note: The rule is a ∗ b = (3 x a) + b. In this case, the number in the box is 22 because 7 ∗ 1 = (3 x 7) + 1 = 22.

You might suggest that students write their guesses on paper for you to check. After a moment, let a student announce that 22 is in the box. Otherwise, put 22 in the box yourself and continue with other open sentences. For example:

7 ∗ 1 = $\boxed{22}$
4 ∗ 0 = $\boxed{12}$
0 ∗ 4 = $\boxed{4}$
10 ∗ 6 = $\boxed{36}$
6 ∗ 10 = $\boxed{28}$

When many students know the rule, let one explain it to the class. Insist on a clear explanation, and write the rule on the board as described.

$$\boxed{a * b = (3 \times a) + b}$$

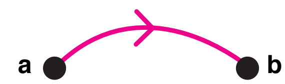
S: ***The rule for * is 3 times the first number plus the second number. For example, 10 * 6 = 36 because (3 x 10) + 6 = 36.***

Write these problems on the board. Instruct students to copy the problems and to use this rule for * to solve them. Allow several minutes for independent work before checking solutions collectively. (Answers are in boxes.)

$$\begin{aligned} 2 * 0.5 &= \boxed{6.5} \\ 0.5 * 2 &= \boxed{3.5} \\ (2 * 3) * 4 &= \boxed{31} \\ 2 * (3 * 4) &= \boxed{19} \end{aligned} \qquad \begin{aligned} 6 * \boxed{12} &= 30 \\ \boxed{8} * 6 &= 30 \\ 10 * \boxed{5} &= 25 \\ \boxed{5} * 10 &= 25 \end{aligned}$$

Exercise 2

For this exercise, the rule is for a relation (an arrow). Draw an arrow to explain how a relation rule works.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: ***This time I have a rule for an arrow. When you put a number here*** (at **a**)***, I will use my secret rule*** (trace the arrow) ***and give the ending number*** (at **b**)***.***

Draw an arrow road on the board as you explain the following.

T: ***There are clues about the rule in this arrow road. Try to figure out the rule for red arrows. What number do you think is at this last dot?***

Note: Red arrows are for the function 2x followed by –5. For example, a red arrow starting at 15 ends at 25 because (2 x 15) – 5 = 25.

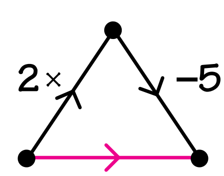
You might suggest students write the ending number on paper for you to check. After a moment, let a student announce that 25 is the ending number, or tell the class this ending number yourself. Try not to give away the rule yet.

Continue by adding a couple more arrows to the road and asking for the ending numbers.

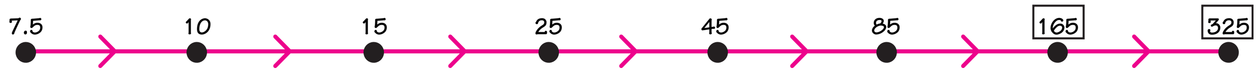
When many students know the rule, let one explain it to the class. As a student explains, record the rule on the board as a composition.

S: ***The ending number of an arrow is 2 times the starting number minus 5.***

S: ***The red arrow is for 2x followed by –5.***



Add a couple more arrows to the arrow road and ask students to label the dots, using this composition rule. (Answers are in boxes.)



Begin a new road on the board with an arrow starting at 4.5.

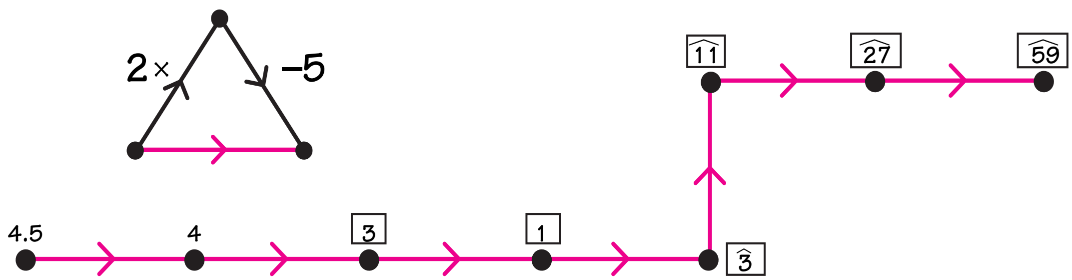
T: ***What is the ending number of this arrow?***

S: ***4, because (2 x 4.5) – 5 = 9 – 5 = 4.***

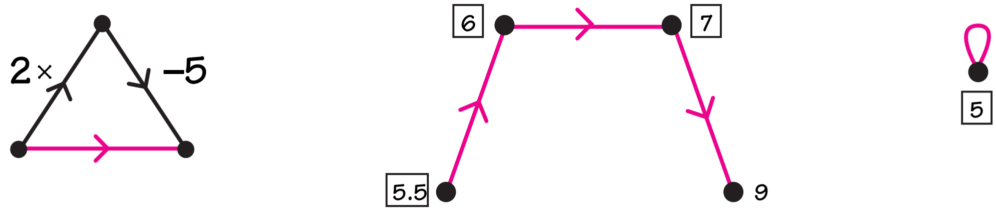
Label the ending dot and extend the arrow road with six more arrows.

T: ***Copy this picture on your paper and label the dots. Use the composition rule for red arrows.***

Allow several minutes for independent work. Then collectively complete the picture on the board. (Answers are in boxes.)



Draw this picture on the board, labeling only the dot for 9. Instruct students to copy the picture and label the dots. (Answers are in boxes.) After a while, collectively complete the picture.



Worksheets N9*, **, and *** are available for individual work.

Home Activity

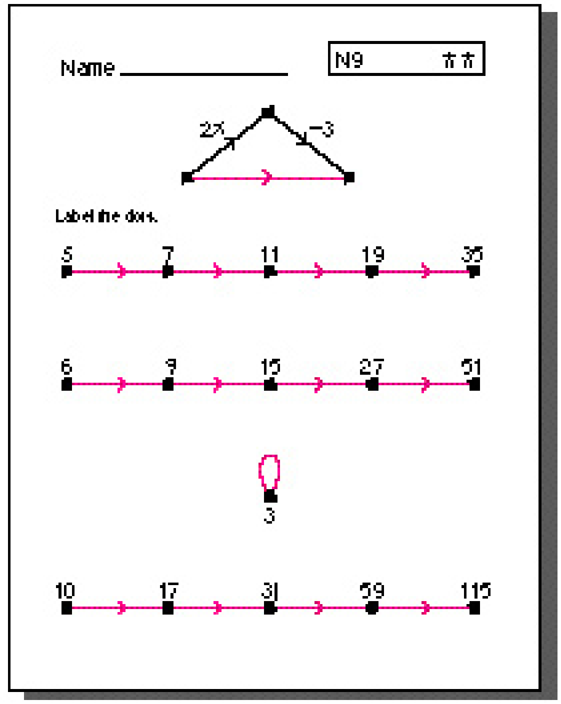
Suggest that students play *Guess My Rule* at home with family members.

Name____________ N9 ☆

$$a \ast b = (3 \times a) - b$$

Fill in the boxes.

$5 \ast 4 = 11$	$5 \ast \boxed{3} = 12$
$4 \ast 5 = \boxed{7}$	$6 \ast \boxed{4} = 14$
$10 \ast 4 = \boxed{26}$	$7 \ast \boxed{5} = 16$
$4 \ast 10 = \boxed{2}$	$8 \ast \boxed{6} = 18$
$7 \ast 6 = \boxed{15}$	$9 \ast \boxed{7} = 20$
$6 \ast 7 = \boxed{11}$	$\boxed{4} \ast 5 = 7$
$9 \ast 9 = \boxed{18}$	$\boxed{6} \ast 8 = 10$
$15 \ast 5 = \boxed{40}$	$\boxed{8} \ast 10 = 14$
$5 \ast 15 = \boxed{0}$	$\boxed{10} \ast 12 = 18$



Name____________ N9 ☆☆☆

$$a \ast b = a + (b \div 3)$$

Fill in the boxes.

$12 \ast 15 = \boxed{17}$	$2 \ast \boxed{15} = 7$
$60 \ast 27 = \boxed{69}$	$0 \ast \boxed{21} = 7$
$27 \ast 60 = \boxed{47}$	$1 \ast \boxed{27} = 10$
$0.6 \ast 1.5 = \boxed{1.1}$	$5 \ast \boxed{45} = 20$
$3.4 \ast 3.6 = \boxed{4.6}$	$8 \ast \boxed{0} = 8$
$\boxed{7} \ast 24 = 15$	$3 \ast \boxed{\hat{9}} = 0$
$\boxed{6} \ast 42 = 20$	$\hat{3} \ast \boxed{9} = 0$
$\boxed{6} \ast 72 = 30$	$\hat{12} \ast \boxed{30} = \hat{2}$

Capsule Lesson Summary

Add and subtract fractions on a number line. By dividing a rectangular region into horizontal and vertical strips, find equivalent names for $\frac{1}{3}$ and $\frac{2}{3}$. Decide what fraction of a square region is in a particular color or combination of colors.

Materials

Teacher	• Meter stick • Colored chalk • Blackline N10	Student	• Divided square • Worksheets N10*, **, ***, and ****

Advance Preparation: Use Blackline N10 to make copies of the divided square for use in Exercise 4.

Description of Lesson

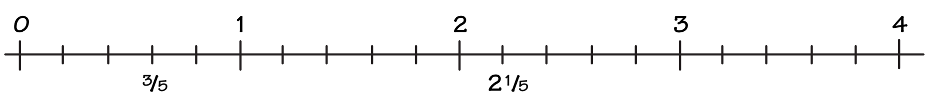
Exercise 1

On the board, draw a number line with equally spaced marks for the numbers 0, 1, 2, 3, and 4.

T: ***How can we locate $\frac{3}{5}$ or $2\frac{1}{5}$ on this number line?***

S: ***Make marks for fifths; divide each section of the number line into five parts. The third mark from 0 will be for $\frac{3}{5}$. The first mark after 2 will be for $2\frac{1}{5}$.***

Make marks on the number line for fifths, and label the marks for $\frac{3}{5}$ and $2\frac{1}{5}$.



Point to other marks and ask for their labels.

T: ***How many fifths is 1?***

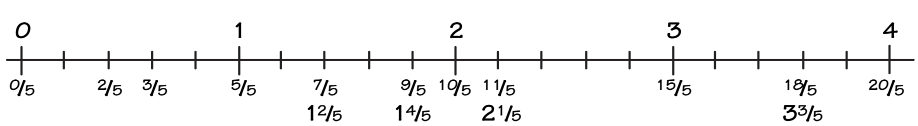
S: ***$\frac{5}{5}$.***

Give the mark for 1 this label as well.

T: ***How many fifths is 2?*** ($\frac{10}{5}$) ***3?*** ($\frac{15}{5}$) ***4*** ($\frac{20}{5}$) ***0?*** ($\frac{0}{5}$)
What is another name for $2\frac{1}{5}$?

S: ***$\frac{11}{5}$.***

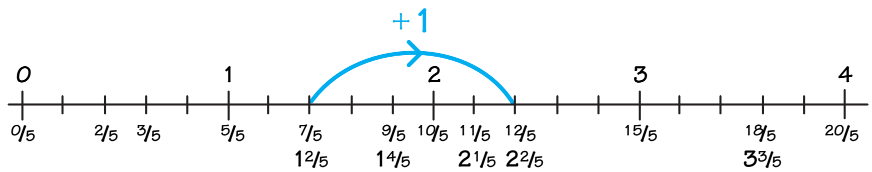
In this way, continue to label several marks with a name as fifths and a mixed fraction name.



T: ***Where should this arrow end?***

S: ***At $2\frac{2}{5}$.***

Complete the arrow.



Repeat the activity with a

- +2 arrow starting at $1\frac{4}{5}$
- –1 arrow starting at $3\frac{3}{5}$
- –2 arrow starting at $2\frac{1}{5}$

Erase the arrows.

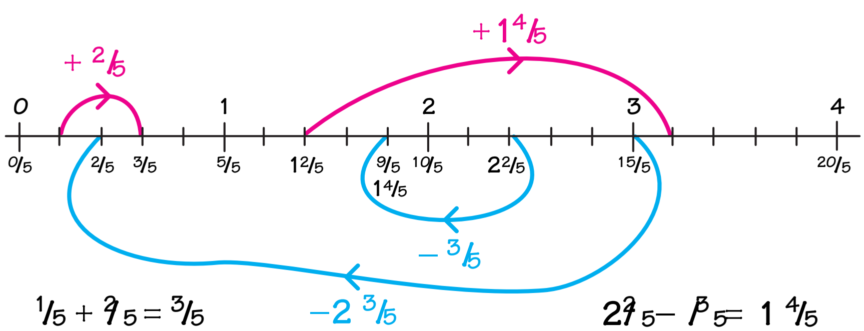
Pose the following problems as a mental arithmetic activity. (Answers are in boxes.)

$8 + 1 = \boxed{9}$	$4\frac{2}{7} - 1 = \boxed{3\frac{2}{7}}$	$1\frac{3}{16} - 1 = \boxed{\frac{3}{16}}$
$8\frac{3}{4} + 1 = \boxed{9\frac{3}{4}}$	$100\frac{21}{23} - 40 = \boxed{60\frac{21}{23}}$	$9\frac{7}{8} - 1 = \boxed{8\frac{7}{8}}$
$7\frac{3}{10} + 1 = \boxed{8\frac{3}{10}}$	$15\frac{3}{5} - 1 = \boxed{14\frac{3}{5}}$	$9 - 1 = \boxed{8}$
$22\frac{1}{11} + 1 = \boxed{23\frac{1}{11}}$	$300\frac{1}{9} - 1 = \boxed{299\frac{1}{9}}$	$9 - 1\frac{7}{8} = \boxed{7\frac{1}{8}}$

Direct students' attention back to the number line.

T: ***Where can we draw a $\frac{2}{5}$ arrow on this number line?***

Let a student draw a +$\frac{2}{5}$ arrow. Ask the class where the arrow starts and where it ends. Then write a number sentence about that arrow. For example, the student might draw an arrow from $\frac{1}{5}$ to $\frac{3}{5}$; the number sentence would be $\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$. Repeat the activity with a +$1\frac{4}{5}$ arrow, a –$\frac{3}{5}$ arrow, and a –$2\frac{3}{5}$ arrow. Your picture might look similar to this one.



$\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$ $\qquad$ $2\frac{2}{5} - \frac{3}{5} = 1\frac{4}{5}$

$1\frac{2}{5} + 1\frac{4}{5} = 3\frac{1}{5}$ $\qquad$ $3 - 2\frac{3}{5} = \frac{2}{5}$

Erase the board before going on to Exercise 2.

Exercise 2

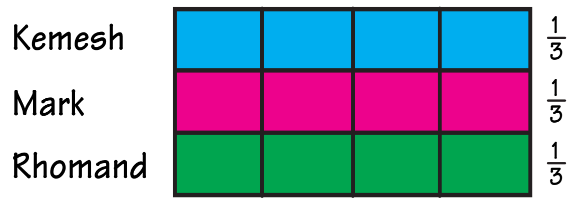
Choose three students to star in this exercise.

T: ***Kemesh, Mark, and Rhomand share a cake equally. Kemesh ices her third with blue (blueberry) frosting, Mark ices his third with red (cherry) frosting, and Rhomand ices his third with green (mint) frosting.***

Draw this picture on the board. You may just draw the rectangle and invite students to measure and make the divisions.

T: ***Rhomand wants the cake to last four days so he makes more cuts in the cake.***

Show these cuts on the cake.



T: ***Does each person have enough cake to last four days?***

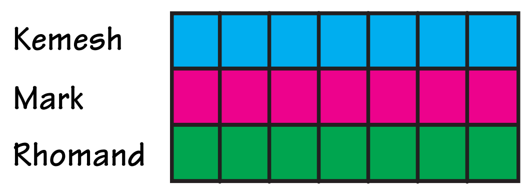
S: ***Yes, each has four pieces.***

T: ***The cake is cut into how many pieces?*** (12)
What fraction of the cake does each person get? ($\frac{4}{12}$, or $\frac{1}{3}$)
What do we know about $\frac{4}{12}$ and $\frac{1}{3}$?

$$\frac{1}{3} = \frac{4}{12}$$

S: ***They are equal. The amount of cake each person gets does not change when Rhomand makes more cuts.***

Return to the original picture and repeat the activity. This time, let Mark cut another cake to last for seven days.



$$\frac{1}{3} = \frac{7}{21}$$

T: ***What if they wanted such a cake to last ten days, how much would each of them get?***

S: ***$\frac{10}{30}$. They could cut the cake into 30 pieces and each take ten.***

T: ***What if they had cut the cake into 18 pieces? How may days would the cake last?***

S: ***Six.***

Record the equivalences on the board.

$$\frac{1}{3} = \frac{4}{12} = \frac{7}{21} = \frac{10}{30} = \frac{6}{18}$$

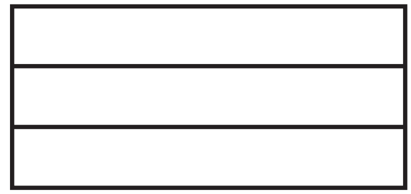
T: ***Do you notice any patterns in these equivalent fractions for $\frac{1}{3}$?***

S: ***The bottom number (denominator) is always 3 times the top number (numerator).***

S: ***If you multiply the top and bottom numbers by the same number, you can get an equivalent fraction.***

Exercise 3

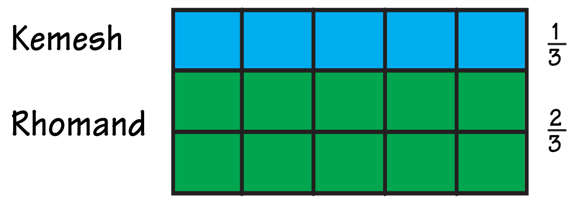
Alter the picture from Exercise 2.



T: ***One day Kemesh, Mark, and Rhomand meet to share a cake. After it is cut into thirds, Mark decides he doesn't want any and gives his share to Rhomand. How much of the cake does Rhomand get?***

S: *⅔.*

T: ***Kemesh decides she wants the cake to last five days and makes these cuts.***



How many pieces is the cake cut into? (15)
How many pieces does Kemesh get? (5)
How many pieces does Rhomand get? (10)

Record the equivalences on the board.

$$\frac{1}{3} = \frac{5}{15}$$
$$\frac{2}{3} = \frac{10}{15}$$

T: ***If Kemesh wants the cake to last four days, how many pieces would she cut the cake into?*** (12)

If Rhomand gets two-thirds of the cake, how many pieces would he get? (8)

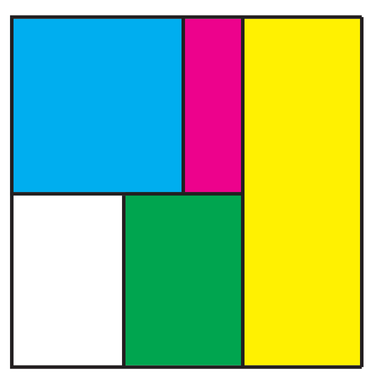
Record the equivalence ⅔ = 8⁄12 on the board. Generate other equivalent fractions for ⅔ by giving either the numerator or the denominator of such a fraction and asking the class for the other. For example:

$$\frac{1}{3} = \frac{8}{12} = \frac{\boxed{14}}{15} = \frac{20}{\boxed{30}} = \frac{\boxed{12}}{18} = \frac{18}{\boxed{27}}$$

Use the cake story as often as necessary. Ask for patterns and methods of finding these fractions. Encourage the method of finding the appropriate name for ⅓ and doubling it, and the method of multiplying both numerator and denominator by the same number. Extend the list, asking for other names for ⅔.

Exercise 4

Draw this picture on the board, and distribute copies of the divided square (Blackline N10) to students. Suggest that students color their square like yours.



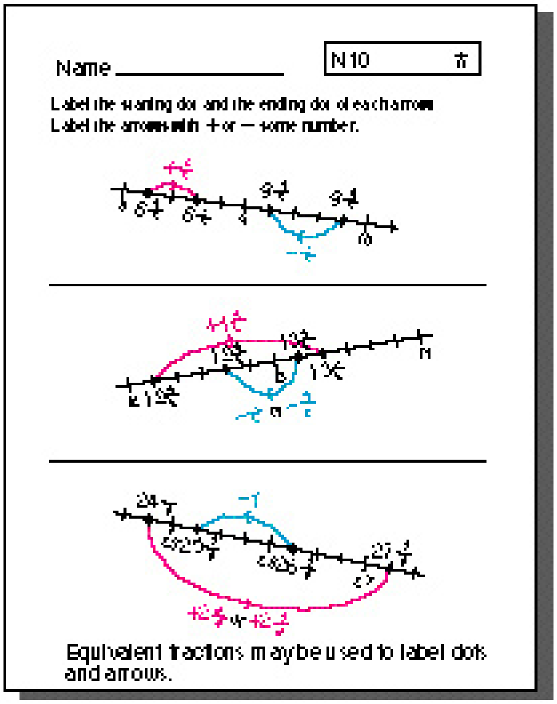
Direct students to work independently or with a partner to determine what fraction of the whole shape is in each color. After a few minutes discuss these fractions collectively, observing what fraction is not a given color or what fraction is a combination of two colors together. For example:

Yellow	Y: $\frac{1}{3}$	Not Y: $\frac{2}{3}$	Y + G: $\frac{1}{3} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$
Green	G: $\frac{1}{6}$	Not G: $\frac{5}{6}$	G + W: $\frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$
White	W: $\frac{1}{6}$	Not W: $\frac{5}{6}$	B + R: $\frac{1}{4} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$
Red	R: $\frac{1}{12}$	Not R: $\frac{11}{12}$	R + G: $\frac{1}{12} + \frac{1}{6} = \frac{3}{12} = \frac{1}{4}$
Blue	B: $\frac{1}{4}$	Not B: $\frac{3}{4}$	B + G: $\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$

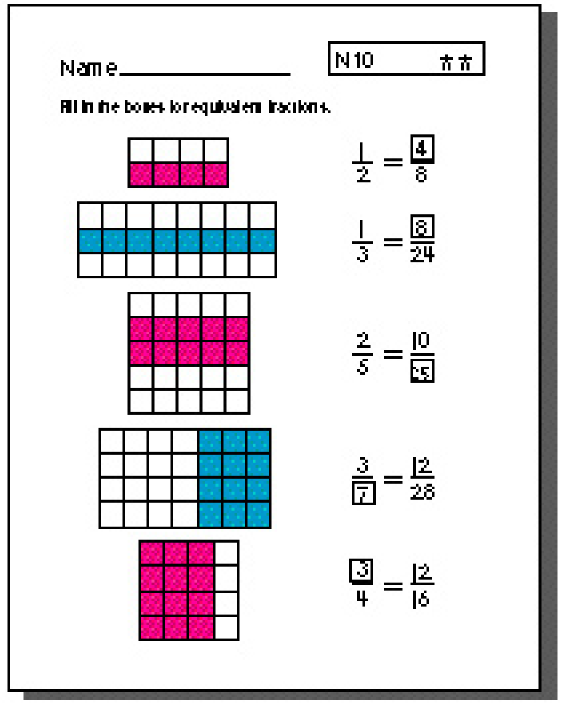
Ask students to explain the additions. For example, a student might explain that the blue and green sections together form $\frac{5}{12}$ of the shape as follows:

S: ***Two reds are the same as one green. Three reds are the same as one blue. One red is $\frac{1}{12}$ of the shape, so five reds are $\frac{5}{12}$ of the shape.***

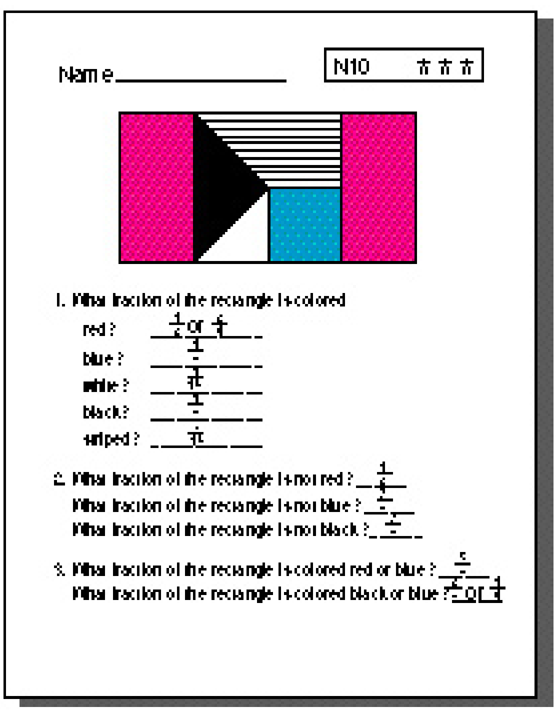
Worksheets N10*, **, ***, and **** are available for individual work.



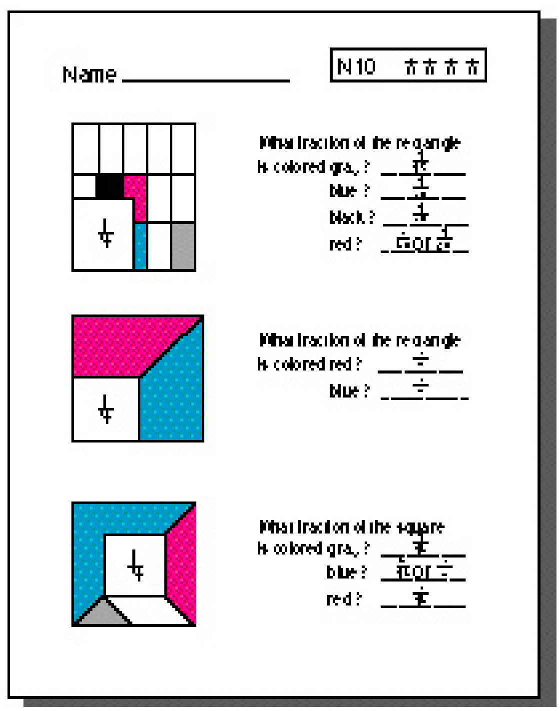
Name
N10 ☆
Label the starting dot and the ending dot of each arrow.
Label the arrows with + or − some number.
Equivalent fractions may be used to label dots and arrows.



Name
N10 ☆☆
Fill in the boxes for equivalent fractions.
1/2 = 4/8
1/3 = 8/24
2/5 = 10/25
3/7 = 12/28
3/4 = 12/16



Name
N10 ☆☆☆
1. What fraction of the rectangle is colored
red?
blue?
white?
black?
striped?
2. What fraction of the rectangle is not red?
What fraction of the rectangle is not blue?
What fraction of the rectangle is not black?
3. What fraction of the rectangle is colored red or blue?
What fraction of the rectangle is colored black or blue?



Name
N10 ☆☆☆☆
What fraction of the rectangle
is colored gray?
blue?
black?
red?
What fraction of the rectangle
is colored red?
blue?
What fraction of the square
is colored gray?
blue?
red?

N11 NABU’S ABACUS #1

Capsule Lesson Summary

Introduce a new abacus and with it determine some trades. Examine the abacus by putting numbers on it, decoding numbers, and making trades with both regular and weighted checkers.

Materials

Teacher
- Magnetic checkers
- Weighted checkers

Student
- Paper

Description of Lesson

Begin the lesson with mental arithmetic for multiplication by 9. Include these or other calculations appropriate for the numerical abilities of your students. (Answers are in boxes.)

$9 \times 6 = \boxed{54}$	$9 \times 30 = \boxed{270}$	$9 \times 15 = \boxed{135}$	$9 \times 63 = \boxed{567}$
$9 \times 7 = \boxed{63}$	$9 \times 70 = \boxed{630}$	$9 \times 16 = \boxed{144}$	$9 \times 64 = \boxed{576}$
$9 \times 8 = \boxed{72}$	$9 \times 40 = \boxed{360}$	$9 \times 17 = \boxed{153}$	$9 \times 65 = \boxed{585}$

Occasionally ask students to explain their methods, for example:

S: ***9 x 3 = 27, so 9 x 30 = 270.***

S: ***9 x 15 = 135, so 9 x 16 = 135 + 9 = 144.***

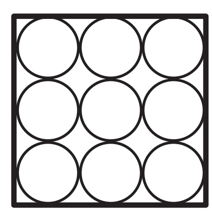
S: ***9 x 60 = 540 and 9 x 3 = 27, so 9 x 63 = 540 + 27 = 567.***

Note: To maintain the involvement of many students, you may wish to employ the following technique frequently throughout this lesson: After asking a question, ask students to write their answers on paper. Check several students’ answers, and then allow a student to give the answer.

Exercise 1

As you tell this story, draw a picture of how nine pencils could fit in a box.

T: ***I want to tell you about Nabu and his summer job at a pencil factory. Nabu is given a number of pencils; his job is to put the pencils into boxes. Each box holds nine pencils. His first assignment is to put 40 pencils into boxes. How many boxes can he fill and how many pencils will be left over?***



S: ***He can fill four boxes and will have four pencils left over. 4 x 9 = 36 and 36 + 4 = 40.***

T: ***What number sentences could we write about this situation?***

Write students’ answers on the board.
Be sure that this number sentence is included.

$$40 = (4 \times 9) \times 4$$

Continue in a similar manner, varying the number of pencils Nabu must pack into boxes that hold nine pencils each. For example:

$57 = (6 \times 9) \times 3$ (57 pencils fill 6 boxes with 3 left over.)

$72 = 8 \times 9$ (72 pencils fill 8 boxes with none left over.)

T: ***Next, Nabu must put 140 pencils into boxes. Can he fill 10 boxes? 20 boxes? 30 boxes?***

S: ***He can fill between 10 and 20 boxes, because 10 x 9 = 90 and 20 x 9 = 180.***

T: ***After Nabu uses 90 pencils to fill 10 boxes, he will have 50 pencils left. How many more boxes can he fill?***

S: ***5 boxes. He will have 5 pencils left over since (5 x 9) + 5 = 50.***

S: ***So with 140 pencils, Nabu can fill 10 + 5 = 15 boxes and will have 5 pencils left over.***

Write this number sentence on the board. $140 = (15 \times 9) \times 5$

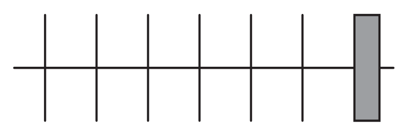
In a similar manner, continue with the following problems. Encourage students to solve the problems by using estimation techniques.

$213 = (23 \times 9) \times 6$ (213 pencils fill 23 boxes with 6 left over.)

$965 = (107 \times 9) \times 2$ (965 pencils fill 107 boxes with 2 left over.)

Exercise 2

Draw this abacus on the board.



T: ***In order to determine the number of boxes he could fill with large numbers of pencils, what could Nabu do?***

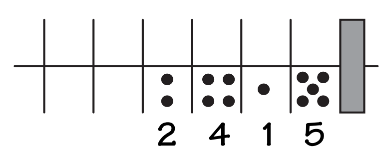
S: ***Divide the number by 9.***

T: ***To avoid doing much division, Nabu invented a new abacus to help him solve such problems.***

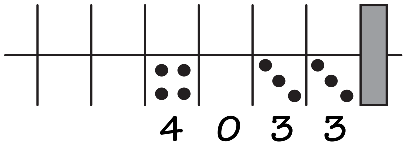
Cover the top row of the abacus as you explain the following:

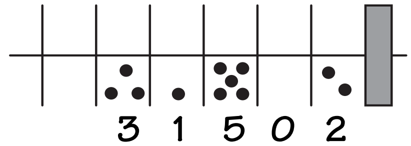
T: ***The bottom of Nabu's abacus is like a base-10 abacus. Here he can enter the number of pencils he receives.***

Demonstrate first with a single checker moved from square to square along the bottom: 1, 10, 100, 1 000, and so on. Then put on this configuration of checkers and ask students to read the number.



Give the class a couple numbers to put on the bottom of this abacus. For example:

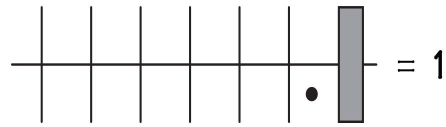




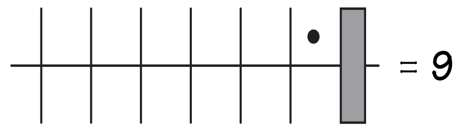
Cover the bottom row of the abacus as you continue to explain.

T: ***Nabu uses the top of his abacus to represent the number of boxes of pencils he can fill.***

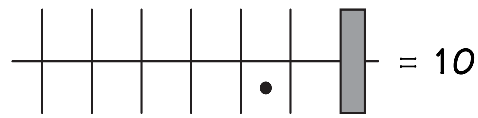
Place a checker first in the 1-square and then move it to the square directly above.



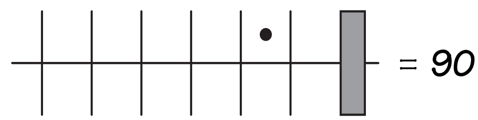
T: ***This is one pencil.***
On the top, this is one <u>box</u> of pencils.
How many pencils in one box? (9)
So this is 9.



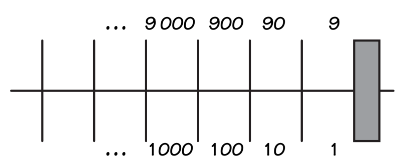
Place the checker first in the 10-square and then move it to the square directly above.



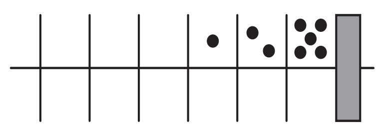
T: ***This is ten pencils.***
On the top, this is ten <u>boxes</u> of pencils.
How many pencils in ten boxes? (90)
So this is 90.



Repeat this explanation a couple times to get the square values across the top of the abacus.



Put on this configuration of checkers; ask first how many boxes, then how many pencils.



S: ***125 boxes.***

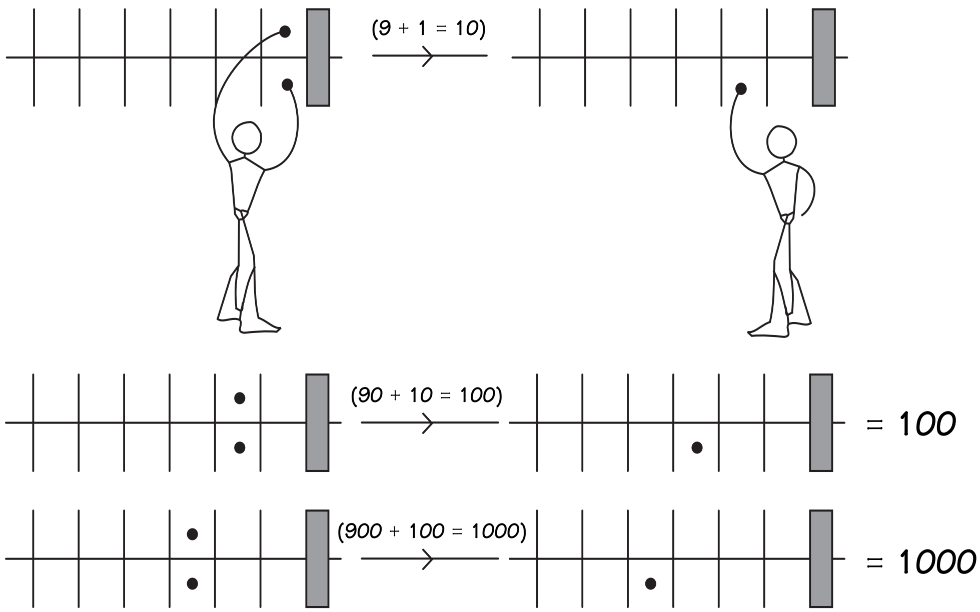
S: ***125 boxes have 9*** x ***125 = 1 125 pencils.***

Invite the class to invent trades for Nabu's abacus. You may need to remind the class of trades on other types of abaci; for example, trades on the Minicomputer or trades on a base-three or a base-five abacus. As necessary, give some hints by placing checkers that can be traded to get a simpler configuration. Eventually, students should mention or observe three kinds of trades, as explained on the following page.

N11

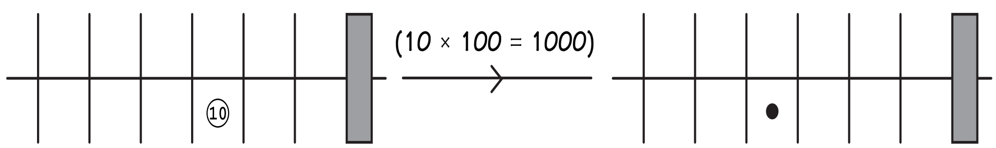
1. **Whenever there are two checkers, one in a bottom square and the other in the top square directly above, the checkers can be traded for one checker in the next bottom square to the left.**

Illustrate such trades with several examples: 9 + 1 = 10, 90 + 10 = 100, 900 + 100 = 1000, and so on.

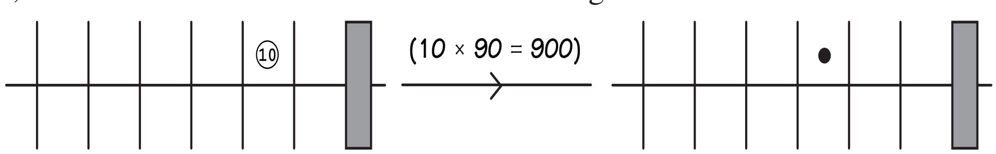


2. **Ten checkers in a square trade for one checker in the next square to the left (top or bottom).**

On the bottom, this is the notion of a base-10 abacus. Use a weighted checker to demonstrate. Ask a student to trade the ⑩-checker for a regular checker.

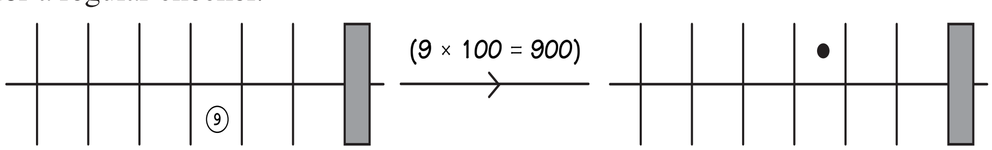


Check with the class that the same kind of trade can be done on the top row of Nabu's abacus. For example, ask a student to trade the ⑩-checker for a regular checker.



3. **Nine checkers in a bottom square trade for one checker in the top square directly above.**

Use a weighted checker to demonstrate such trades. For example, ask a student to trade the ⑨-checker for a regular checker.



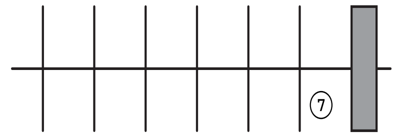
Provide several opportunities for students to practice making these three types of trades.

Exercise 3

Return the class discussion to why Nabu invented this abacus.

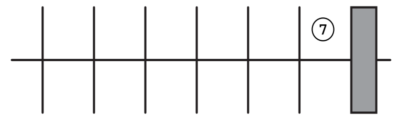
T: ***Nabu knows that it is easy to represent both the number of individual pencils and the number of boxes of pencils on this abacus.***

Put a ⑦-checker on the 1-square.



T: ***How many pencils?*** (7)

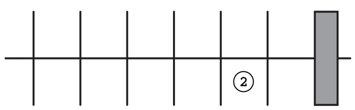
Move the ⑦-checker to the 9-square.



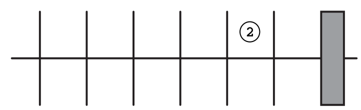
T: ***How many boxes of pencils?*** (7)
How many pencils are in 7 boxes?

S: ***7 x 9 = 63; 63 pencils in 7 boxes.***

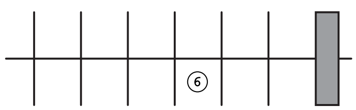
In a similar manner, use the following configurations to demonstrate the connection on Nabu's abacus between pencils and boxes of pencils.



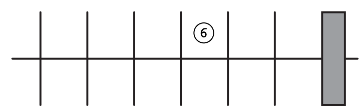
20 pencils



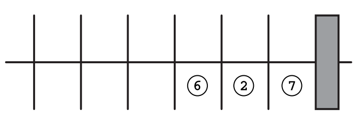
20 boxes or 180 pencils



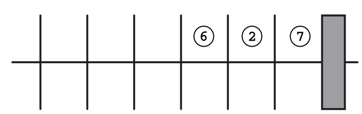
600 pencils



600 boxes
or 5 400 pencils

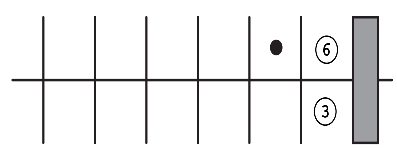


627 pencils



627 boxes
or 5 643 pencils
(5 643 = 5 400 + 180 + 63)

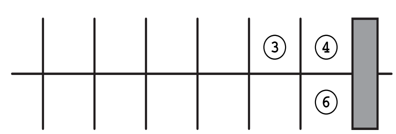
T: ***How could Nabu easily show 16 full boxes of pencils and 3 extra pencils? How many pencils is this?***



S: ***147 pencils since (16 x 9) + 3 = 147.***

S: ***147 pencils since 90 + 54 + 3 = 147.***

T: ***How could Nabu most easily show 34 full boxes of pencils and 6 extra pencils?***



S: ***This is 312 pencils since (34 x 9) + 6 = 312.***

S: ***This is 312 pencils since 270 + 36 + 6 = 312.***

T: ***Today we've learned a lot about Nabu's new abacus. Next time, we'll discover exactly how Nabu uses it to solve division problems when putting large numbers of pencils into boxes.***

N12 NABU'S ABACUS #2

Capsule Lesson Summary

Review the trades for Nabu's abacus (introduced in Lesson N11) and practice decoding numbers and making trades. Examine special configurations for numbers that allow the abacus to provide an algorithm for dividing by 9. Solve some problems involving division by 9 on the abacus.

Materials

Teacher
- Colored chalk
- Magnetic checkers
- Weighted checkers

Student
- Paper
- Worksheets N12* and **

Description of Lesson

Exercise 1

Briefly review the story context from Lesson N11.

T: ***Do you remember Nabu and his summer job?***

Let the class recall that Nabu packs pencils into boxes. Each box holds nine pencils.

T: ***Suppose Nabu has 50 pencils to put into boxes. How many boxes can he fill?***

S: ***He can fill five boxes and have five pencils left over. (5 x 9) + 5 = 50.***

Write this number sentence on the board.

$$50 = (5 \times 9) \times 5$$

Continue in a similar manner, varying the number of pencils. Encourage students to use estimation techniques. For example:

$85 = (9 \times 9) \times 4$ (85 pencils fill 9 boxes with 4 left over.)

$170 = (18 \times 9) \times 8$ (170 pencils fill 18 boxes with 8 left over.)

$340 = (37 \times 9) \times 7$ (340 pencils fill 37 boxes with 7 left over.)

$680 = (75 \times 9) \times 5$ (680 pencils fill 75 boxes with 5 left over.)

T: ***What do you notice about the numbers of pencils in these problems?***

S: ***The number of pencils doubles each time.***

T: ***Does the number of full boxes also double each time?***

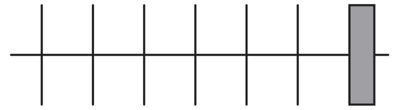
S: ***No. 18 = 2 x 9, but 37 = (2 x 18) + 1 and 75 = (2 x 37) + 1.***

T: ***When the number of pencils doubles, why does the number of boxes sometimes double and sometimes turn out to be one more than the double?***

Lead a discussion toward the observation that the number of left over pencils also doubles each time and sometimes there are enough to fill one more box.

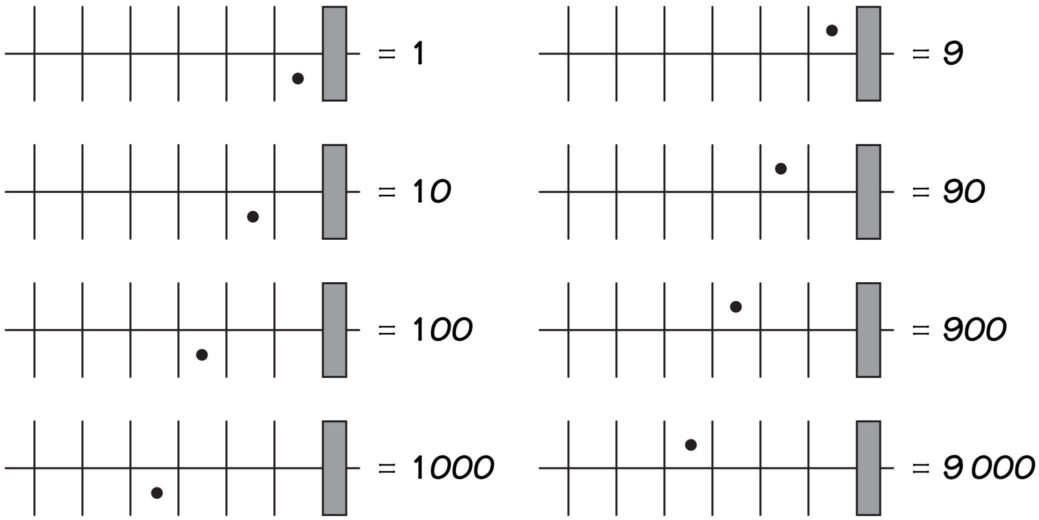
Exercise 2

Draw this abacus on the board.

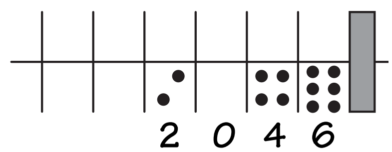


T: ***Do you remember the abacus Nabu invented to help him with the division problems?***

Review the values of the squares by moving a checker from square to square and asking for the number.



Put this configuration of checkers on the abacus and instruct students to write the number on their papers.

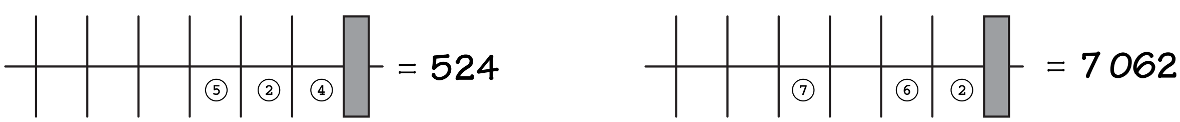


T: ***What number is this?***

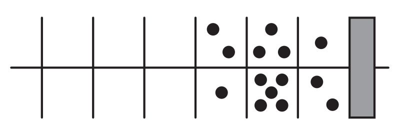
S: ***2 046. 2 000 + 40 + 6 = 2 046.***

S: ***2 046. The bottom row of squares is base-10, so you can read the number directly from the number of checkers on each square.***

Similarly, ask students to decode or put on the following numbers.

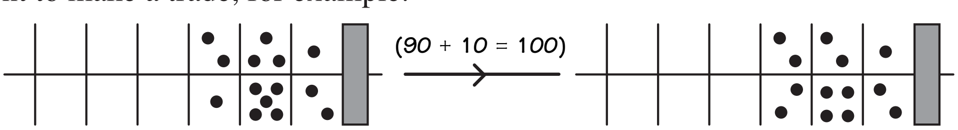


Display this configuration of checkers.

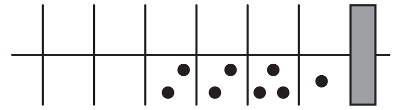


T: ***Let's make trades before decoding this number. Who can show the same number, but with one less checker?***

Invite a student to make a trade, for example:



Invite students to make additional trades until the number is easy to read with all the checkers on the bottom row of the abacus.

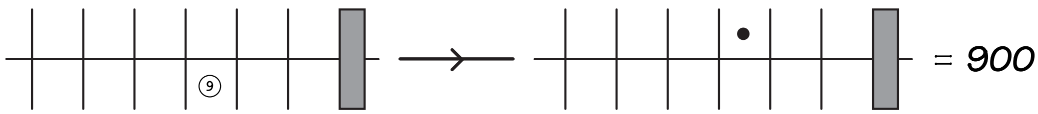


T: ***What number is on the abacus?***

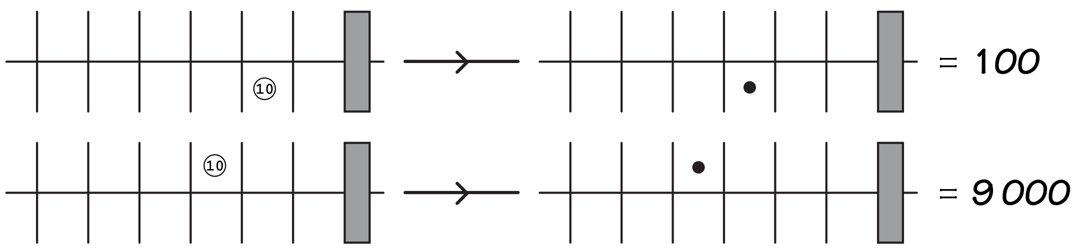
S: ***2 231.***

T: ***What other kinds of trades did we discover last time we looked at this abacus?***

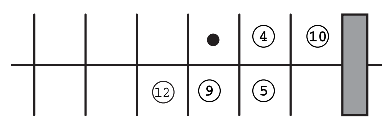
S: ***A ⑨-checker on a bottom square can be traded for one regular checker on the square directly above.***



S: ***A ⑩-checker on any square can be traded for one regular checker on the square directly to the left.***

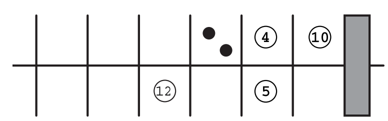


Display this configuration on the abacus.

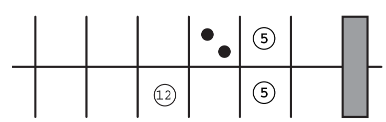


T: ***Who can trade the ⑨-checker for a regular checker and show the same number?***

Who can make a trade with the ⑩-checker?

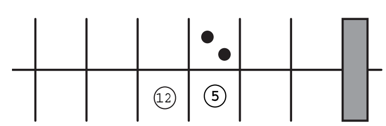


Who can make a trade with both ⑤-checkers?

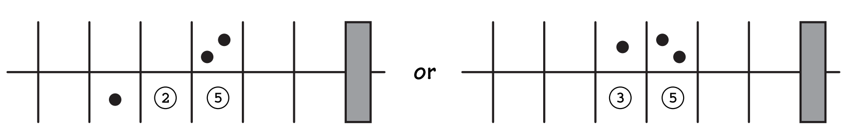


Mention that the last trade is equivalent to making five 90 + 10 = 100 trades with regular checkers.

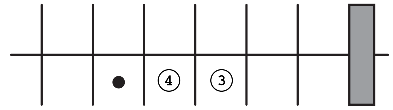
T: ***Who can make a trade with the ⑫-checker?***



Depending upon which of two trades students make, one of these two configurations will be on the abacus.



In either case, invite students to make additional trades until there are only checkers on the bottom part of the abacus.



T: ***What number is on the abacus?***

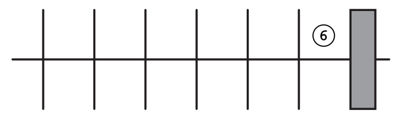
S: ***14 300.***

Exercise 3

Review Nabu's reason for inventing this abacus.

T: ***How does Nabu represent 6 boxes of pencils on the abacus?***

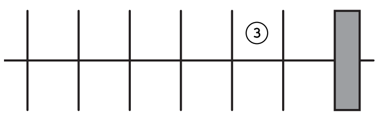
Let a student put a ⑥-checker on the abacus.



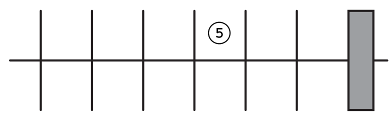
T: ***How many pencils does this represent?***

S: ***54; 6 x 9 = 54.***

Similarly, ask students to put on checkers for 30 boxes and 500 boxes of pencils.

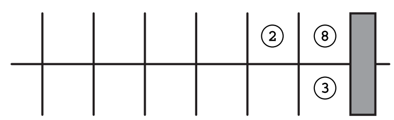


30 boxes, or
270 pencils



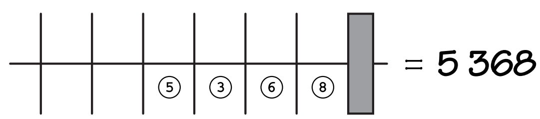
500 boxes, or
4 500 pencils

T: ***On the abacus, how could Nabu represent 28 boxes of pencils and 3 extra pencils? How many pencils is this?***



S: ***255 pencils; (28 x 9) + 3 = 255.***

Return to one of Nabu's original problems.



T: ***One day Nabu gets 5 368 pencils to pack into boxes. Who can put 5 368 on the abacus?***

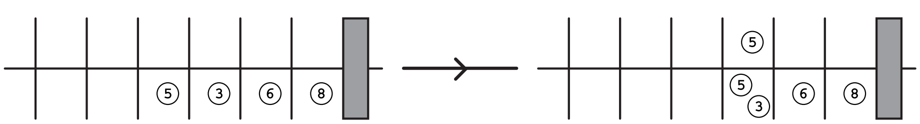
Nabu wants to use the abacus to determine how many boxes he can fill. Can anyone suggest how he can do this?

Let students suggest methods Nabu could use. Lead the discussion to describe the following method.

T: ***Nabu needs to get 5 368 represented on the abacus with checkers mostly on the top squares to see how many boxes.***

S: ***Nabu could make backward trades to move as many checkers as possible to the upper squares of the abacus. That would tell him how many boxes he could fill.***

T: ***Who can make a backward trade with the ⑤-checker?***



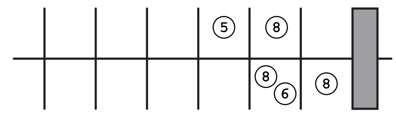
Exchange the ③-checker and the⑤-checker on the same square for an ⑧-checker on that square.

T: ***So far how many boxes does the abacus show that Nabu can fill?*** (500)
And how many pencils are left over? (868)
Can more boxes be filled? (Yes)

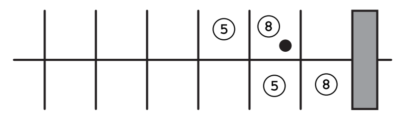
Point to the ⑧-checker on the 100-square.

T: ***Who can make a backward trade with this ⑧-checker?***

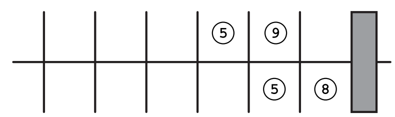
We now have an ⑧-checker and a ⑥-checker on the same square, which is the same as having 14 checkers on that square. We could make backward trades with a ⑭ -checker, but what else could we do?



S: ***Trade the ⑧-checker and the ⑥-checker for a ⑨-checker and a ⑤-checker. Then trade the ⑨-checker for one checker on the square above it.***

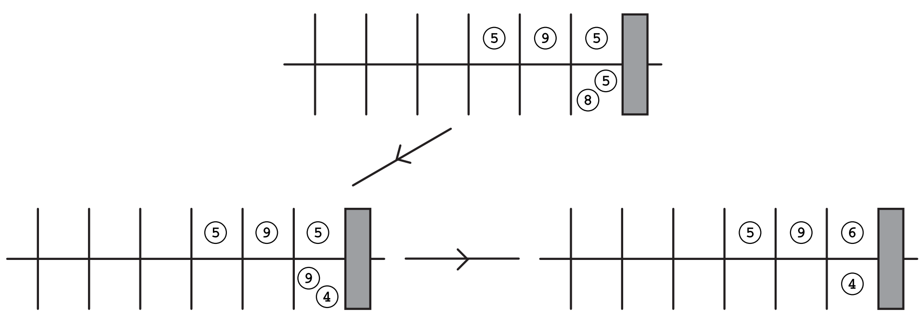


Exchange the ⑧-checker and the regular checker on the same square for a ⑨-checker.



T: ***So far how many boxes does the abacus show that Nabu can fill?*** (590)
And how many pencils are left over? (58)
Can more boxes be filled? (Yes)

Continue to direct students to make trades.



T: ***Can we get anymore checkers on the upper squares of the abacus?*** (No)

How many boxes can Nabu fill?

S: ***596. He has 4 pencils left over.***

T: ***This is Nabu's way to determine the number of boxes he can fill. What division problem is related to this situation?***

S: ***5 368 ÷ 9.***

N12

Write this information on the board.

$$5\,368 = (596 \times 9) + 4 \qquad \begin{array}{r} 596 \text{ R} = 4 \\ 9\overline{)5\,368} \end{array}$$

T: ***With this abacus, Nabu has a mechanical way to divide any whole number by 9. It's a little bit slow, but he only has to know how to make trades. He doesn't need to know any multiplication or division facts.***

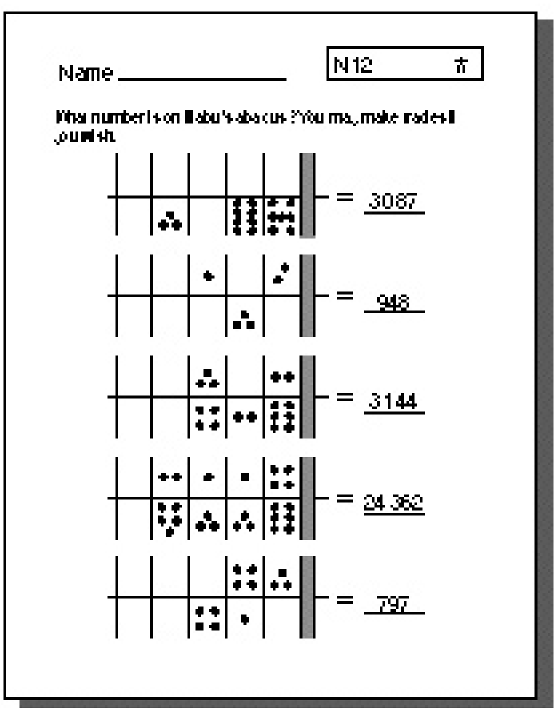
However, this abacus only works for dividing by 9. What if Nabu wanted to divide by another number, for example, by 7 or 12?

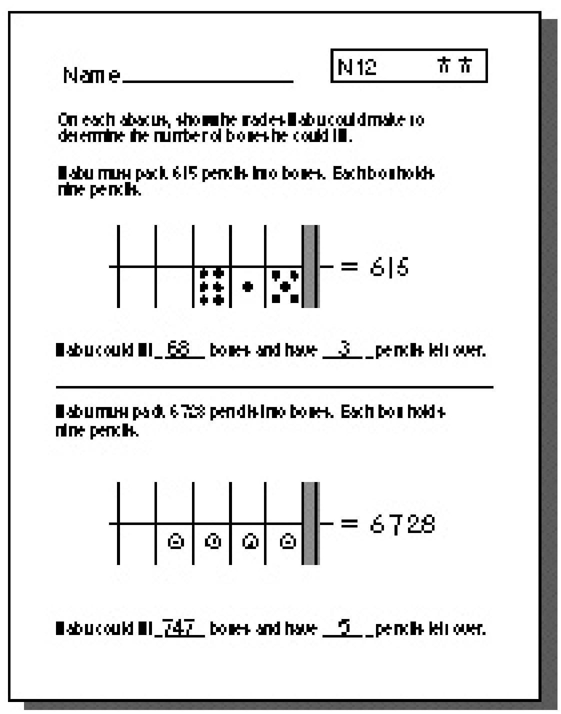
S: ***He could devise new trading rules for a similar abacus.***

Worksheets N12* and ** are available for individual work.

Extension/Writing Activity

Suggest that students devise an abacus Nabu could use to solve his problems if his job required him to put 7 pencils per box or 12 pencils per box. Make this a writing activity by asking students to write a report about the new abacus.





N13 DIVISION ALGORITHM #1

Capsule Lesson Summary

Practice mental arithmetic patterns involving multiplying by multiples of 10. Solve division problems suggested by situations in which a given number of bottles are packed into cartons, each carton holding a specified number of bottles. Present a form of the standard division algorithm.

Materials

Teacher		Student	
	• Chalk		• Paper • *Division Problems* Booklet

Description of Lesson

Begin the lesson with mental arithmetic. The following are possible sequences of problems. Stop occasionally to ask students to describe the patterns they are using. If necessary, write a few problems from a sequence on the board.

100 x 9	10 x 7	100 x 16	10 x 42
200 x 9	20 x 7	200 x 16	20 x 42
300 x 9	30 x 7	300 x 16	30 x 42
⋮	⋮	⋮	⋮
1000 x 9	100 x 7	1 000 x 16	100 x 42

Exercise 1

Begin this table on the board.

Point to the appropriate places in the table as you describe the problem.

Numbers of Bottles	Bottles in One Carton	Number of Filled Cartons	Left Over Bottles
70	6		

T: ***Nabu's new job is at a bottle recycling plant. His job is to fill cartons with bottles. One day he must put 70 bottles in cartons that hold six bottles each. How many cartons can he fill completely and how many bottles will be left over?***

S: ***11 cartons can be filled because 11* x *6 = 66.***

S: ***Four bottles will be left over.***

Record the answer in the table and in a division format.

$$6\overline{)70}\quad 11\ R = 4$$

T: ***What number sentence does this problem suggest?***

Write the students' answers on the board. Be sure to include this number sentence.

$$(6 \times 11) + 4 = 70$$

Add the following problems to the table and invite students to present solutions. (Answers are in boxes.)

Numbers of Bottles	Bottles in One Carton	Number of Filled Cartons	Left Over Bottles
70	6	11	4
136	10	$\boxed{13}$	$\boxed{6}$
2 543	10	$\boxed{254}$	$\boxed{3}$

Begin a line in the table with 181 bottles and 8 bottles per carton.

T: ***Nabu must pack 181 bottles into cartons. Each carton holds 8 bottles. Can Nabu fill as many 10 cartons? 100 cartons?***

S: ***He can fill at least 10 cartons since 10 x 8 = 80.***

S: ***He cannot fill 100 cartons since 100 x 8 = 800.***

T: ***Can he fill 20 cartons? 30? 50?***

S: ***He can fill between 20 and 30 cartons because 20 x 8 = 160 and 30 x 8 = 240.***

T: ***Nabu fills 20 cartons with 160 bottles. How many bottles must he still pack?***

S: ***21; 181 – 160 = 21.***

S: ***Nabu can use those 21 bottles to fill 2 more cartons, since 2 x 8 = 16. There are 5 bottles left over.***

S: ***So with 181 bottles, Nabu can fill 22 (20 + 2 = 22) cartons. He has 5 bottles left over.***

Record the results in the table. Present the following problems in a similar manner, encouraging estimation when appropriate. (Answers are in boxes.)

Numbers of Bottles	Bottles in One Carton	Number of Filled Cartons	Left Over Bottles
181	8	22	5
440	6	$\boxed{73}$	$\boxed{2}$
500	20	$\boxed{25}$	$\boxed{0}$
3 628	100	$\boxed{36}$	$\boxed{28}$

Exercise 2

T: ***One day Nabu has to pack 5 555 bottles into large cartons (cases) that hold 24 bottles each. He must tell his supervisor how many cases he will need. Let's calculate the number of cartons he will fill. How have we solved division problems in other lessons?***

S: ***We built an arrow road. We could start an arrow road at 5 555 and use –24 arrows.***

T: ***What other kinds of arrows would also be useful?***

S: ***–48. Each arrow would represent filling two cases.***

S: ***–240. Each arrow would represent filling 10 cases.***

T: ***Let's solve this problem in a similar way but without drawing an arrow road. Can Nabu fill 10 cases? 100? 1 000?***

S: ***Nabu can fill more than 100 cases since 100*** x ***24 = 2 400, but he needs fewer than 1 000 cases since 1 000*** x ***24 = 24 000.***

T: ***Can Nabu fill 200 cases? 300? 400?***

Let students discuss this question. Accept a response of either 100 or 200 cases.

S: ***Try 200 cases. 4 800 bottles fill 200 cases; 200*** x ***24 = 4 800.***

Show the filling of 200 cases by writing this calculation on the board.

Bottles	Cases
5 555	
−4 800	200
755	

T: ***Nabu packed 4 800 bottles into 200 cases.***
How many bottles must he still pack?

S: ***755; 5 555 – 4 800 = 755.***

T: ***Now Nabu must put 755 bottles into cases of 24.***
Can he fill another 100 cases? (No)
Can Nabu fill 10 more cases? 20? 30? 80?

Let students briefly discuss this question. Accept 10, 20, or 30.

S: ***Let's try 20 cases. 20*** x ***24 = 480.***

Show the filling of 20 cases

Bottles	Cases
5 555	
−4 800	200
755	
−480	20
275	

Note: 30 cases is a better estimate than 20 at this stage. Accept a student's suggestion even if it is not the best estimate. This freedom allows students to adapt the procedure to their own abilities. Gradually they will begin making more efficient choices.

T: ***How many cases has Nabu filled so far?*** (220)
How many bottles must he still pack? (275)

Continue in a similar manner until fewer than 24 bottles remain. Encourage students to use multiples of 10 as estimates until the number of bottles still to pack is less than 10 x 24. A sample solution is shown here.

Bottles	Cases
5 555	
−4 800	200
755	
−480	20
275	
−240	10
35	
−24	1
11	

T: ***Can Nabu fill any more cases?***

S: ***No, only 11 bottles remain.***

T: ***How many cases can Nabu fill?***

S: ***231; 200 + 20 + 10 + 1 = 231.***

T: ***What division problem have we been solving?***

S: ***5 555 ÷ 24.***

Convert the calculation on the board to a standard division format.

$$
\begin{array}{r|l}
\multicolumn{2}{r}{231 \text{ R} = 11} \\
24\overline{)5\,555} & \\
-4\,800 & 200 \\ \hline
755 & \\
-480 & 20 \\ \hline
275 & \\
-240 & 10 \\ \hline
35 & \\
-24 & 1 \\ \hline
11 &
\end{array}
$$

T: ***What number sentence involving multiplication does this problem suggest?***

Write the students' answers on the board.
Be sure to include this number sentence.

$$(231 \times 24) \times 11 = 5\,555$$

Repeat the above activity, solving this problem collectively.

$$32\overline{)39\,815}$$

$$
\begin{array}{r|r}
\multicolumn{2}{r}{1\,244 \text{ R} = 7} \\
32\overline{)39\,815} & \\
-32\,000 & 1\,000 \\ \hline
7\,815 & \\
-6\,400 & 200 \\ \hline
1\,415 & \\
-640 & 20 \\ \hline
775 & \\
-640 & 20 \\ \hline
135 & \\
-96 & 3 \\ \hline
39 & \\
-32 & 1 \\ \hline
7 & 1\,244
\end{array}
$$

In solving the problem, encourage students

- to first use powers of 10, 100, or 1 000 as a first estimate of the solution; and
- to next use multiples of those numbers; for example, 20, 50, 200, 300, … for more accurate estimates and simpler multiplications.

A sample solution is shown here.

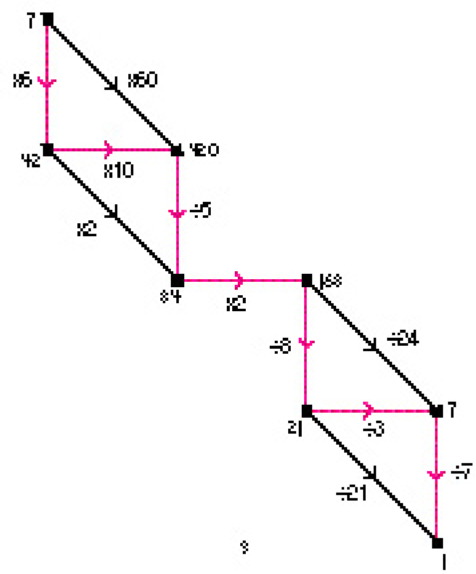
A 16-page booklet, *Division Problems*, contains worksheets to use with a sequence of lessons involving the division algorithm (N13, N21, and N33). Allow about 10–15 minutes for independent work in this booklet. At the end of the lesson, collect the booklets, check them, and store them for use in the remaining lessons on the division algorithm.

Divide.

72 ÷ 9 = 8	54 ÷ 6 = 9	36 ÷ 3 = 12
$5\overline{)110}$ = 22	$7\overline{)56}$ = 8	$8\overline{)96}$ = 12
108 ÷ 12 = 9	56 ÷ 4 = 14	78 ÷ 6 = 13

2

Label each arrow with × or ÷ some whole number.



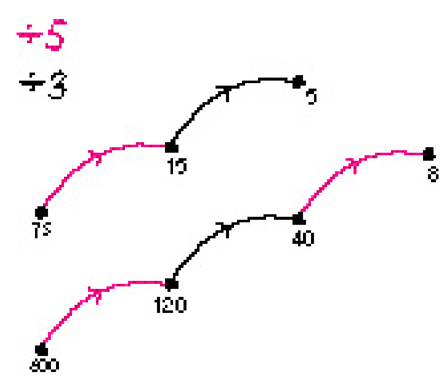
3

Complete.

6 × 8 = 48	7 × 4 = 28
48 ÷ 6 = 8	28 ÷ 7 = 4
48 ÷ 8 = 6	28 ÷ 4 = 7
480 ÷ 8 = 60	280 ÷ 4 = 70
90 ÷ 9 = 10	36 ÷ 3 = 12
99 ÷ 9 = 11	42 ÷ 3 = 14
108 ÷ 9 = 12	48 ÷ 3 = 16
117 ÷ 9 = 13	54 ÷ 3 = 18
60 ÷ 6 = 10	100 ÷ 4 = 25
24 ÷ 6 = 4	36 ÷ 4 = 9
84 ÷ 6 = 14	136 ÷ 4 = 34

4

Label the dots.



Complete.

75 ÷ 15 = 5   600 ÷ 15 = 40   120 ÷ 15 = 8

5

$$\boxed{728 \div 14 = 52}$$

The boxed division fact above is correct. Use this fact to help complete the following division problems.

$714 \div 14 =$ 51	$756 \div 14 =$ 54
$700 \div 14 =$ 50	$770 \div 14 =$ 55
$672 \div 14 =$ 48	$798 \div 14 =$ 57
$532 \div 14 =$ 38	$938 \div 14 =$ 67
$1064 \div 14 =$ 76	$1008 \div 14 =$ 72
$1596 \div 14 =$ 114	$2016 \div 14 =$ 144

6

Divide.

31 R=8

14)442	
−280	20
162	
−140	10
22	
−14	1
8	

323 R=2

12)3878	
−3600	300
278	
−240	20
38	
−36	3
2	

409 R=5

18)7367	
−3600	200
3767	
−3600	200
167	
− 90	5
77	
−72	4
5	

647

15)9705	
−7500	500
2205	
−1500	100
705	
−600	40
105	
−75	5
30	
−30	2

7

How many 32¢ stamps can you buy with $2.00? 6

6 R=8

32)200

3 x 32¢ = 96¢
6 x 32¢ = 192¢ = $1.92
six stamps with 8¢ left over

The area of a rectangular room is 294 square feet. If the length of the room is 21 feet, what is the width of the room? 14 feet

14

21)294	
−210	10
84	
−84	4

The largest watermelon ever grown weighed 279 pounds. If an average watermelon weighs about 23 pounds, how many times the average was the record-holding watermelon? about 12 times

12 R=3

23)279	
−230	10
49	
−46	2
3	

The volume of a box (rectangular prism) is 240 cm³. If the bottom of the box is 12 cm by 5 cm, how tall is the box? 4 cm

240 ÷ 12 = 20
and
20 ÷ 5 = 4

Bottom of the box is
5 cm x 12 cm = 60 cm²
240 ÷ 60 = 4

8

Complete.

$120 \div 20 =$ 6	$140 \div 7 =$ 20
$120 \div 40 =$ 3	154 $\div 7 = 22$
$120 \div 30 =$ 4	161 $\div 7 = 23$
$120 \div 15 =$ 8	$175 \div 7 =$ 25
$1600 \div 10 =$ 160	$4200 \div 50 =$ 84
$1600 \div 100 =$ 16	$4200 \div 25 =$ 168
$1600 \div 200 =$ 8	$4200 \div 60 =$ 70
$1600 \div 1000 =$ 1.6	$4200 \div 70 =$ 60
$1600 \div 500 =$ 3.2	$4200 \div 35 =$ 120

13)265 = 20 R=5   13)267 = 20 R=7   13)270 = 20 R=10   13)274 = 21 R=1

9

Write a story problem in which you would use the division calculation. Then complete the division.

$316 \div 4 = 79$

Billy collected 316 golf balls from the golf course. If he shares them equally with three other friends (four ways), how many golf balls will each child have?

Many stories are possible.

$905 \div 17 = 53$ R 4

Erin is bundling 905 pounds of newspapers to take to the recycling center. If each bundle must weigh 17 pounds, how many bundles will she be able to make?

Many stories are possible.

10

$24 \div 6 = 4$	$72 \div 9 = 8$	$54 \div 6 = 9$	$28 \div 4 = 7$
$48 \div 12 = 4$	$216 \div 27 = 8$	$108 \div 12 = 9$	$112 \div 16 = 7$

Do you see a pattern? Can you explain why you think the pattern works?

It works because you are multiplying both numbers by the same amount so the answers stay the same.

$24 \div 6 = \frac{24}{6}$
$48 \div 12 = \frac{48}{12}$

If you multiply numerator and denominator by the same number, the resulting fraction is equivalent.

Make up your own examples using this pattern.

$32 \div 8 = 4$ $\quad$ $50 \div 5 = 10$ $\quad$ $48 \div 6 = 8$
$64 \div 16 = 4$ $\quad$ $250 \div 25 = 10$ $\quad$ $144 \div 18 = 8$

Use the pattern to do the calculations below.

$144 \div 12 = 12$ $\quad$ $416 \div 8 = 52$
$288 \div 24 =$ 12 $\quad$ $832 \div 16 =$ 52
$72 \div 6 =$ 12 $\quad$ $2496 \div 48 =$ 52
$216 \div 18 =$ 12 $\quad$ $4992 \div 96 =$ 52

11

Complete. Put a single digit in each box to make the calculations correct.

```
      [1][5]  R-[2]
 4 ) 6  2
   -[4] 0 | [1]0
   [2][2]
   -2 [0] |  [5]
      [2]
```

```
      [2][3]  R-[4]
 7 ) 1  6 [5]
  -[1][4][0] | [2]0
      [2][5]
     -[2][1] |  3
          4
```

12

Hector has a 270 page novel to read. How long will it take to read the book if he reads

... 5 pages per day? 54 days
... 10 pages per day? 27 days
... 15 pages per day? 18 days
... 20 pages per day? $13\frac{1}{2}$ days

Suppose Hector wants to finish the book in one week. How many pages should he plan to read each day? 39 pages

```
    38 R 4
7 ) 270
   -266
      4
```

(He'll only have to read 36 pages on the last day)

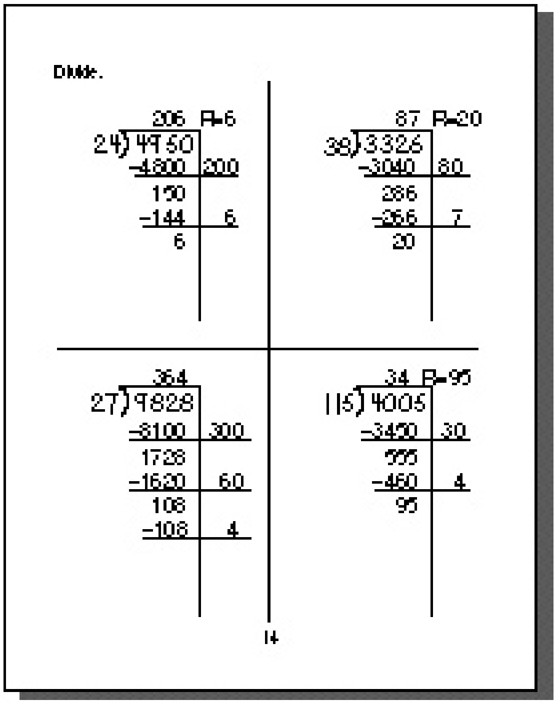
or 38 pages for six days and 42 pages on the last day

or 38 pages for three days and 39 pages for four days

Collene had six math tests this semester. She had an average score of 88 and her lowest score was 73. If the teacher ignores the lowest score, what would Collene's average test score be? 91

Six math tests – average score 88
total score $6 \times 88 = 528$
Ignore lowest score 73: $528 - 73 = 455$
Then total score for five tests is 455
average score $455 \div 5 = 91$

13



Divide.

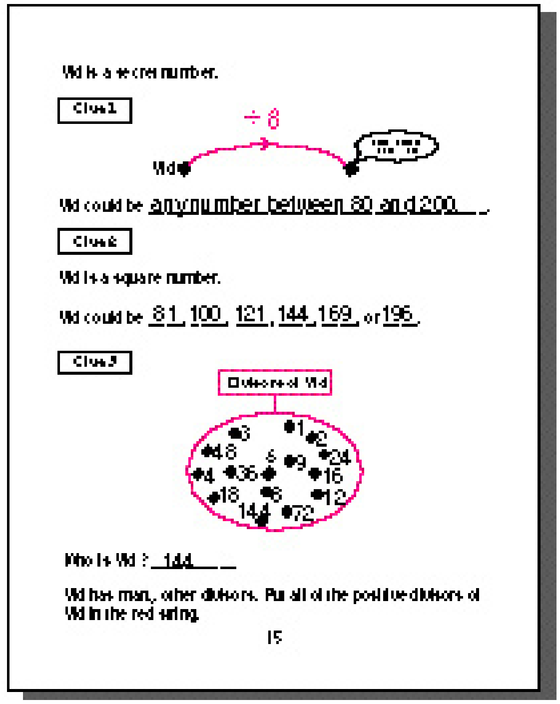
206 R=6
24)4950
−4800 | 200
150
−144 | 6
6

87 R=20
38)3326
−3040 | 80
286
−266 | 7
20

364
27)9828
−8100 | 300
1728
−1620 | 60
108
−108 | 4

34 R=95
115)4005
−3450 | 30
555
−460 | 4
95

14



Vid is a secret number.

Clue 1

Vid could be any number between 80 and 200.

Clue 2

Vid is a square number.

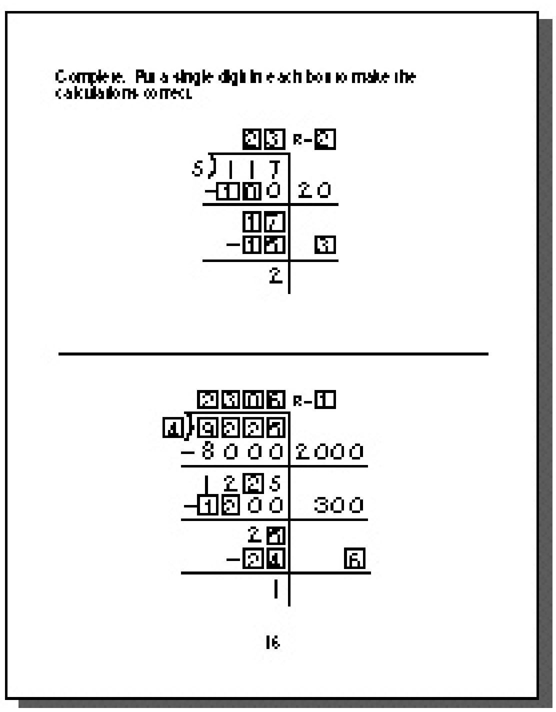
Vid could be 81, 100, 121, 144, 169, or 196.

Clue 3

Who is Vid ? 144

Vid has many other divisors. Put all of the positive divisors of Vid in the red string.

15



Complete. Put a single digit in each box to make the calculations correct.

23 R-2
5)117
−100 | 20
17
−15 | 3
2

2306 R-1
4)9225
−8000 | 2000
1225
−1200 | 300
25
−24 | 6
1

16

Capsule Lesson Summary

Solve a detective story in which clues involve patterns generated by the combined use of the calculator relations + 3 = ···, + 7 = ···, + 2 = ..., and + 5 = ···, and a three-string picture.

Materials

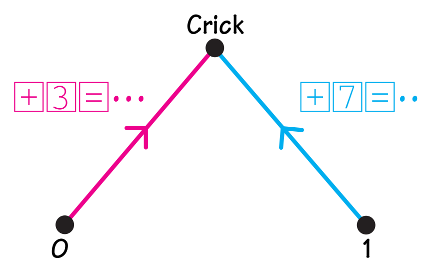
Teacher		Student	
	• Colored chalk • Calculator		• Calculator • Worksheets N14* and **

Description of Lesson

Present this detective story about a secret number named Crick.

Clue 1

Draw this arrow picture with calculator relations on the board.



T: ***Today's detective story is about a secret number called Crick. Crick is in this arrow picture. Which numbers could Crick be?***

Allow several minutes for students to explore this situation independently or with a partner.

S: ***Crick could be 36.***

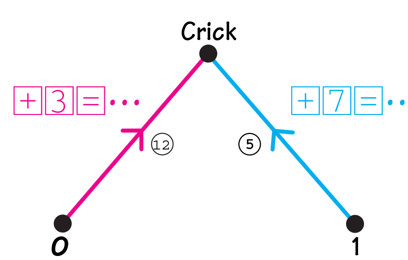
S: ***Crick could be 15.***

As students find numbers that Crick could be, list them in order on the board (even though students may not suggest them in order). Occasionally ask the class to verify that Crick could be a particular number. For example:

T: ***Let's check that Crick could be 36. If we start from 0 and press + 3, how many times do we press = to get 36?***

S: ***Twelve times, because 12 x 3 = 36.***

T: ***And if we start from 1 and press + 7, how many times do we press = to get 36?***



S: ***Five times, because 1 + (5 x 7) = 36.***

T: ***What is the least number that Crick could be?***

S: ***15. 5 x 3 = 15, and 1 + (2 x 7) = 15.***

T: ***What are some other numbers Crick could be?***

S: ***78.***

S: *246.*

S: *57.*

Continue until six to ten possibilities have been suggested.

15, 36, 57, 78, 99, 120, ,162, ,246

T: ***Do you notice anything interesting about this list of numbers?***

Let students study the list for a couple minutes.

S: ***There is a +21 pattern. 15 + 21 = 36, 36 + 21 = 57, and so on.***

T: ***Why do you think there is a +21 pattern?***

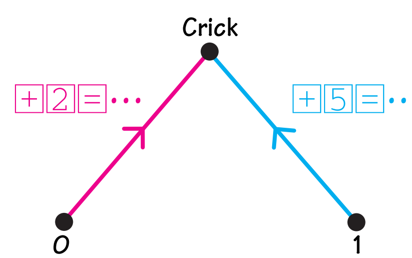
S: ***21 is a multiple of 3 and a multiple of 7.***

Accept any reasonable explanation.

At this point you might ask students to find several more numbers that Crick could be, especially to fill in some gaps in your list. The class should conclude that Crick could be one of these numbers: 15, 36, 57, 78, 99, 120, 141, 162, 183, 204, 225, 246, 267, and so on. Leave this list of numbers on the board.

Clue 2

Draw this arrow picture on the board.



T: ***Crick is also in this arrow picture. What does this picture tell us about Crick?***

Allow several minutes for students to explore the new clue.

S: ***Crick must end with 6 (have 6 in the ones place).***

T: ***Use both the first clue and this clue. Which numbers could Crick be?***

S: ***36.***

S: ***246.***

T: ***What is the next number that Crick could be? How do you know?***

S: ***456 is the next number because 246 + 210 = 456.***

T: ***That's correct, but why did you add 210?***

S: ***In the first clue there is a +21 pattern. In the second clue there is a +10 pattern. So I started from 36 and picked every tenth number in the list (36, 246, 456, …). Picking every tenth number in the list is the same as adding 210 (10 x 21) = 210.***

T: ***What are some other numbers that Crick could be?***

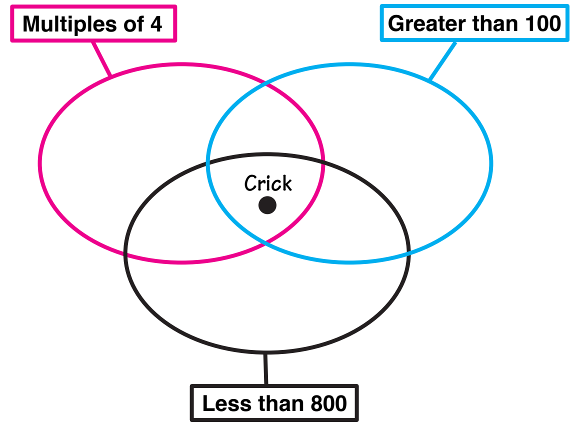
S: ***666, because 456 + 210 = 666.***

S: ***876, because 666 + 210 = 876.***

The class should conclude that Crick could be one of these numbers: 36, 246, 456, 666, 876, and so on. Erase the previous list of numbers for Crick and put this list on the board.

Clue 3

Draw this string picture on the board.



T: ***The third clue is given by this string picture. What does this picture tell us about Crick?***

S: ***Crick is a multiple of 4 and between 100 and 800.***

T: ***Who is Crick?***

Allow a few minutes for students to study the clue.

S: ***Crick could be 246 or 456 or 666 because these numbers are between 100 and 800.***

S: ***Yes, but Crick is also a multiple of 4.***

S: ***Crick cannot be 246; 240 is a multiple of 4 (60 x 4 = 240). Next comes 244 followed by 248 and 246 is skipped.***

S: ***Crick cannot be 666. 600 is a multiple of 4 and so is 60, so 660 must be a multiple of 4. Next comes 664 followed by 668; 666 is skipped.***

S: ***Crick is 456.***

T: ***How many times 4 is 456?***

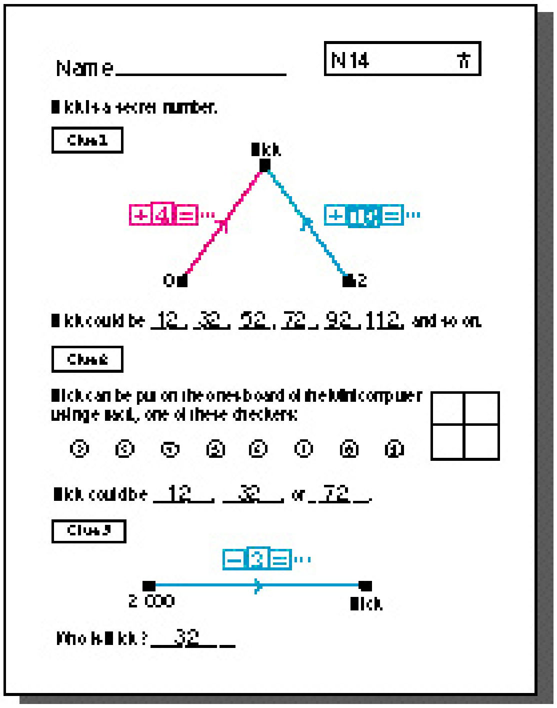
S: ***114 x 4 = 456.***

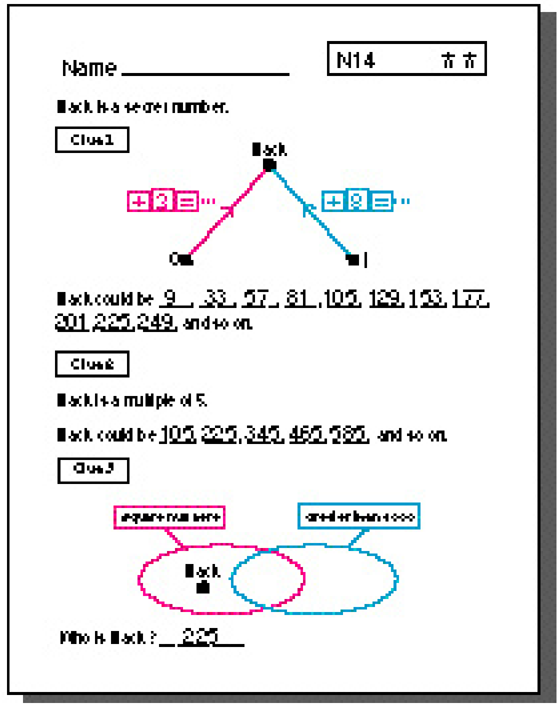
The class should conclude that Crick is 456. Worksheets N14* and ** are available for individual work.

Writing Activity

Invite students to write detective stories similar to those on the worksheets to challenge you or their classmates.

N14


Name
N14
is a secret number.
Clue 1
+4=
2
could be 12, 32, 52, 72, 92, 112, and so on.
Clue 2
one of these checkers.
could be 12, 32, or 72.
Clue 3
−3=
2 000
32


Name
N14
Clue 1
+3=
+8=
could be 9, 33, 57, 81, 105, 129, 153, 177, 201, 225, 249, and so on.
Clue 2
is a multiple of 5.
could be 105, 225, 345, 465, 585, and so on.
Clue 3
225

Capsule Lesson Summary

Label arrows in situations involving the addition of decimals. Locate decimals on a number line. Label graduation marks on a partially labeled, double-scaled number line.

Materials

Teacher
- Colored chalk

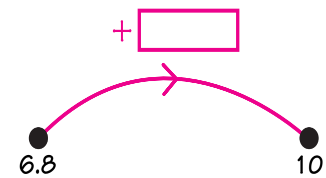
Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N15*, **, and ***

Description of Lesson

Exercise 1

Draw this arrow picture for a problem such as the following:

> Luna is taking a 10 mile bike ride. Her odometer says she has gone 6.8 miles. How much further does she have to go?

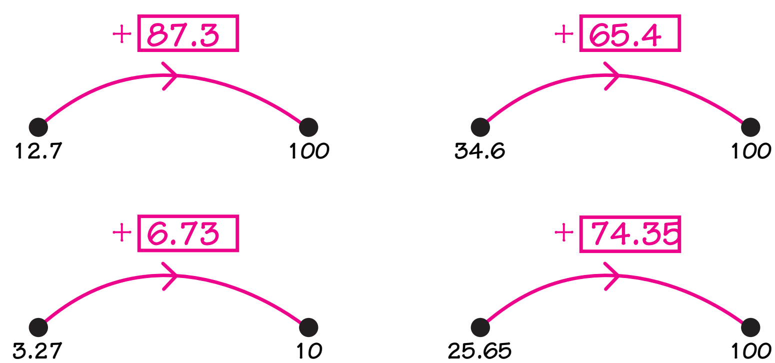


Ask students to copy the picture and to label the arrow.

T: ***What is the arrow for?***

S: ***+3.2. Luna has 3.2 miles more to ride.***

Repeat this activity several more times using problems appropriate for the numerical abilities of your students. Some examples with correct answers in boxes are given below. Ask students to tell story problems for each calculation. Often a money problem will help students having difficulty with decimals.



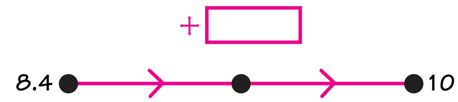
T: ***Here is another way of looking at the last two problems.***

Write the addition problems with empty boxes as shown here. Then call on students to explain how to fill in the boxes.

$$\begin{array}{r} 3.27 \\ +\ \boxed{6}.\boxed{7}\boxed{3} \\ \hline 10.00 \end{array} \qquad \begin{array}{r} 25.65 \\ +\ \boxed{7}\boxed{4}.\boxed{3}\boxed{5} \\ \hline 100.00 \end{array}$$

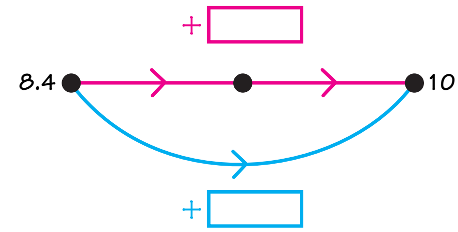
Exercise 2

Draw this arrow picture on the board.



Ask students to copy the picture and to fill in the box for the arrows.

After a few minutes, draw a single blue arrow from 8.4 to 10.



T: ***What could the blue arrow be for?***

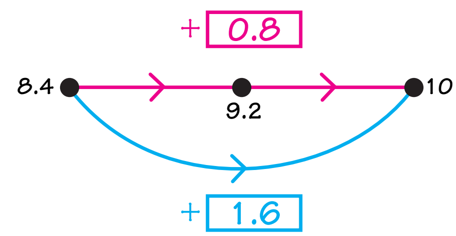
S: ***+1.6.***

T: ***Now what could the red arrows be for?***

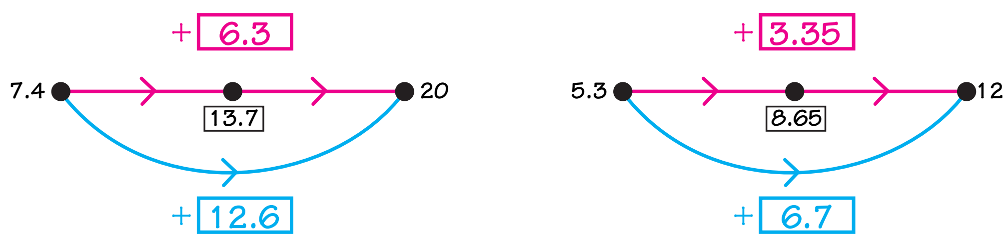
S: ***+0.8.***

T: ***What number is at the middle dot?***

S: ***9.2.***



Repeat this activity with the following arrow pictures. (Answers are in boxes.)



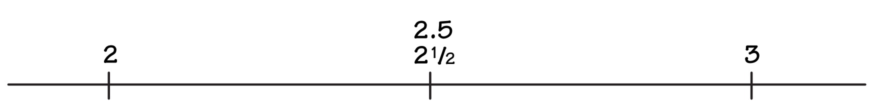
Exercise 3

Draw a number line on the board with marks for 2 and 3 approximately one meter apart.

Invite a student to locate 2½ on the number line.

T: ***What is another name for 2½?***

S: ***2.5.***



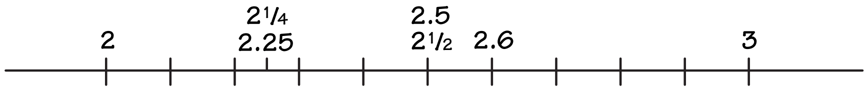
T: ***Who can locate 2.6?***

It is likely students will suggest dividing the segment between 2 and 3 into ten segments of equal length. Perform the division and ask a student to locate 2.6.

T: ***Where is 2¼ on the number line?***
What is a decimal name for 2¼?

S: ***2.25.***

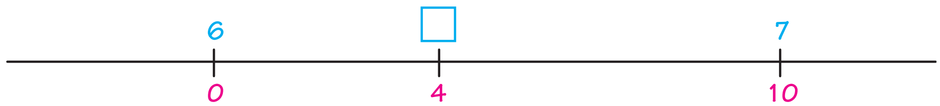
S: ***$2\frac{1}{4}$ is halfway between 2.2 and 2.3.***



Continue the activity by asking for the locations of 2.35, 2.75, and 2.05 on the number line.

Exercise 4

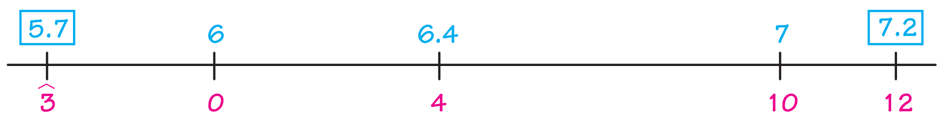
Draw this number line on the board.



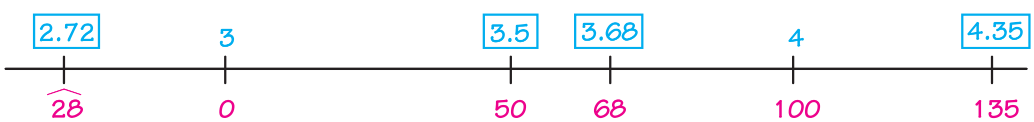
T: ***This number line has two scales, a red scale and a blue scale. Which number should be in the blue box?***

S: ***Divide the segment between 6 and 7 (blue scale) into ten segments of equal length. The red 4 will be at the fourth mark, so the number in the box is 6.4.***

Extend the number line as shown below, and ask students for the numbers on the blue scale corresponding to $\widehat{3}$ and 12 on the red scale. (Answers are in boxes.)



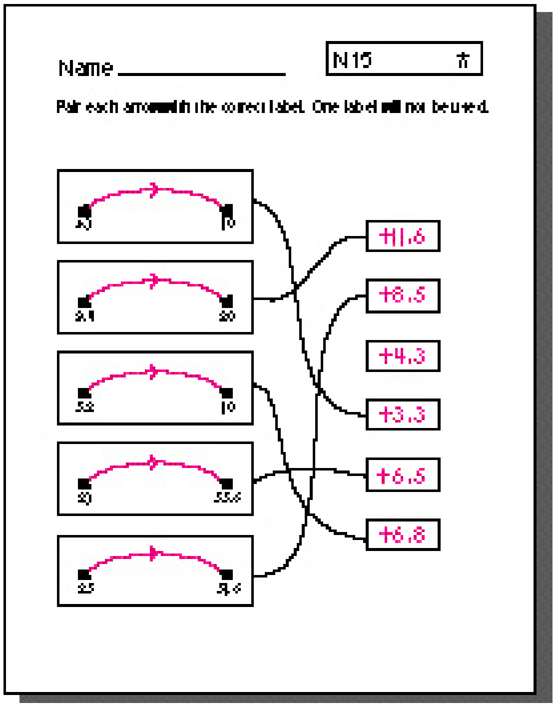
Repeat this activity using the picture below. (Answers are in boxes.)

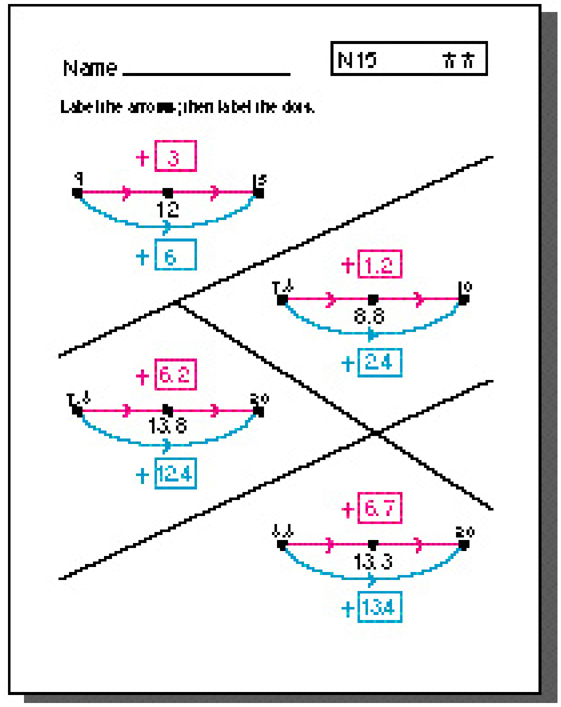


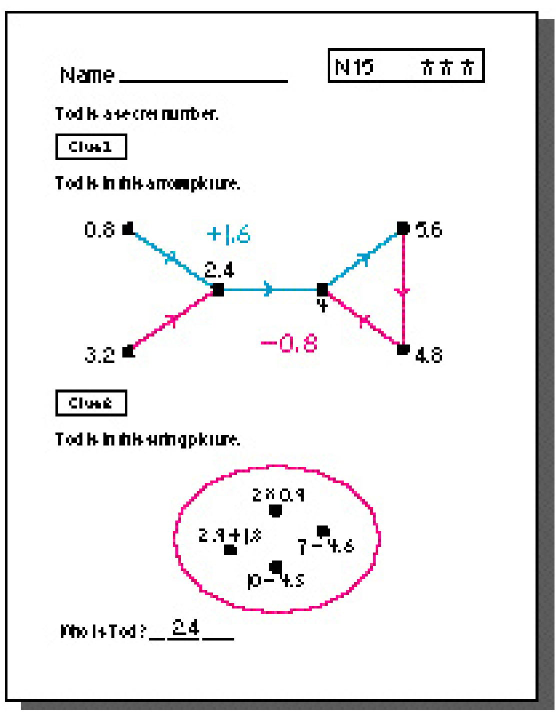
Worksheets N15*, **, and *** are available for individual work.

Home Activity

Suggest that parents/guardians find opportunities to solve decimal addition problems like those in Exercise 1 with their child.

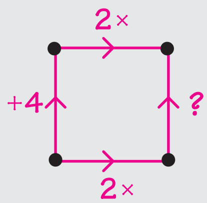

Name
N15 ☆
Pair each arrow with the correct label. One label will not be used.
+1.6
+8.5
+4.3
+3.3
+6.5
+6.8

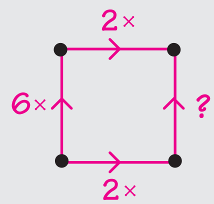

Name
N15 ☆☆
Label the arrows; then label the dots.
+ 3
9
12
15
+ 6
+ 1.2
7.6
8.8
10
+ 2.4
+ 6.2
7.6
13.8
20
+ 12.4
+ 6.7
6.6
13.3
20
+ 13.4


Name
N15 ☆☆☆
Tod is a secret number.
Clue 1
Tod is in this arrow picture.
0.8
+1.6
2.4
5.6
4
−0.8
3.2
4.8
Clue 2
Tod is in this string picture.
Who is Tod? 2.4

Capsule Lesson Summary

Label an arrow to complete a cycle with addition and subtraction arrows, and then do the same to complete a cycle with multiplication and division arrows. Using composition of addition and multiplication functions, label arrows in pictures such as these:





Materials

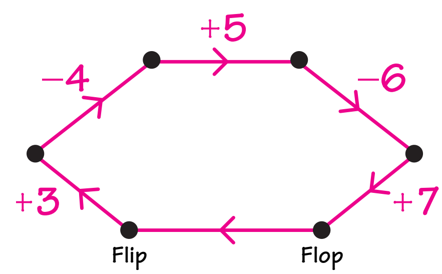
Teacher
- Colored chalk
- Bag of counters (optional)

Student
- Calculator
- Worksheets N16*, **, ***, and ****

Description of Lesson

Exercise 1

Draw this arrow picture on the board.



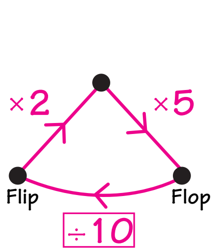
T: ***Choose any number you wish for Flip and display the number on your calculator. Hide the display and then push the appropriate keys to follow the arrow road until you come to Flop. Next, try to get back to Flip with just one operation.*** (Trace the unlabeled arrow.) ***Check and see if the number you started with is back on the display.***

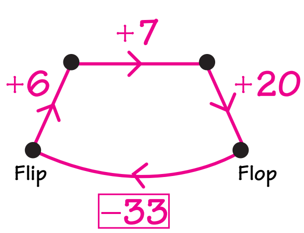
Allow several minutes for students to experiment.

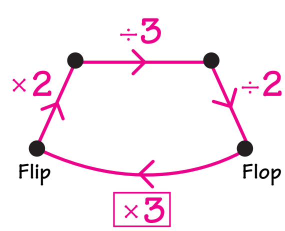
T: ***What could this arrow (the unlabeled arrow) be for? How do you know?***

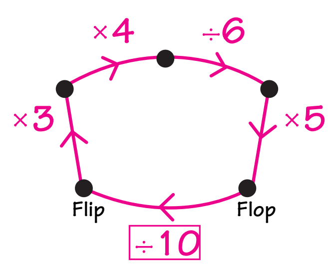
S: ***–5. I followed the arrow road: +3, –4, +5, –6, +7. I knew that Flop was five more than Flip, so the arrow from Flop to Flip could be –5.***

Repeat this activity using some of the arrow roads shown below, or choose problems more appropriate for your students. (Answers are in boxes.)









For the third arrow road above, it is possible that some students will not return to their original numbers even if they correctly press ☒ ③ ⊟. If this situation occurs, you may have the following collective discussion.

T: ***Let Flip be 5. Press ☒ ② ⊟. What number is on the display?***

S: ***10.***

T: ***Now press ⊡ ③ ⊟.***

S: ***3.3333333.***

T: ***Is it correct?***

S: ***No, $10 \div 3 = 3\frac{1}{3} = 3.333$ and so on with 3s.***

S: ***The 3s must go on and on, but the calculator can only show eight places.***

T: ***Press ⊡ ② ⊟. Now what number is on the display?***

S: ***1.6666666.***

T: ***Is it correct?***

S: ***No, $3\frac{1}{3} \div 2 = 10 \div 6 = 1\frac{2}{3} = 1.666$ and so on with 6s.***

S: ***The calculator only shows the first seven 6s.***

T: ***So Flop is 1.6666 …, but the calculator cuts off the 6s after seven. Now press ☒ ③ ⊟. What number is on the display?***

S: ***4.9999998.***

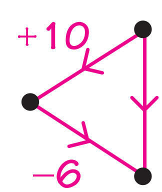
S: ***But $3 \times 1\frac{2}{3} = 5$.***

T: ***Yes. The calculator has made a very small error in the calculations (a* truncation *error) because it could not show that Flop is $1.\overset{*}{6}$; the calculator cannot show that the 6s go on and on. We should keep this in mind whenever we use a calculator.***

It may be mentioned either by you or a student that this difficulty will not occur if a multiple of 3 is chosen for Flip.

Exercise 2

Draw this picture on the board, and trace the unlabeled arrow.

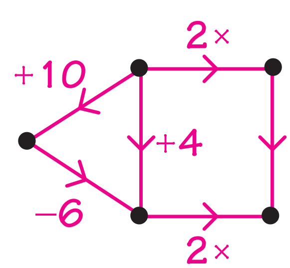


T: ***What could this arrow be for?***

S: ***+4; +10 followed by –6 is +4.***

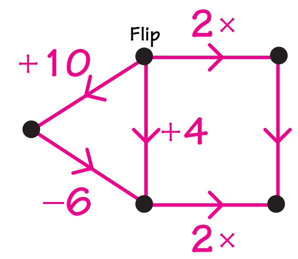
Label the arrow and extend your picture.

Again, trace the unlabeled arrow.



T: ***What could this arrow be for?***

Allow a few minutes for students to consider this situation; then proceed with the following discussion to show that the unlabeled arrow could be for +8. Label the upper left dot Flip.†



T: ***Suppose some number Flip is here*** (point to the starting dot of the +4 arrow)***. What can we say about the number here*** (trace the +4 arrow to its ending dot)***?***

S: ***It is four more than Flip.***

Label the dot Flip +4. Point to the ending dot of the upper 2x arrow.

T: ***What do we know about the number here?***

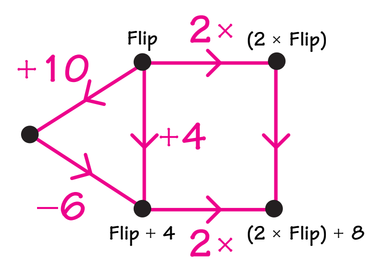
S: ***It is two times Flip.***

Label the dot 2 x Flip. Point to the ending dot of the lower 2x arrow.

S: ***It is for 2*** x ***(Flip + 4).***

S: ***2*** x ***(Flip + 4) = (Flip + 4) + (Flip + 4) = (2*** x ***Flip) + 8.***

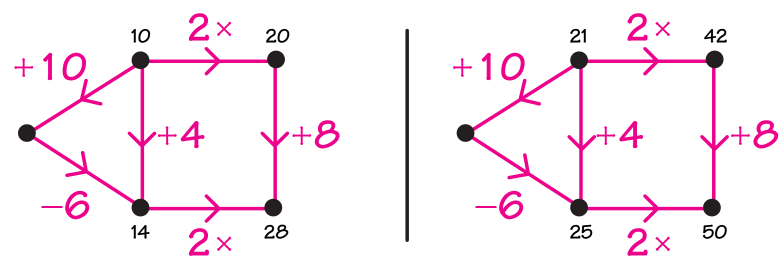
You may need to help students see that 2 x (Flip + 4) = (2 x Flip) + 8. Label the dot accordingly.



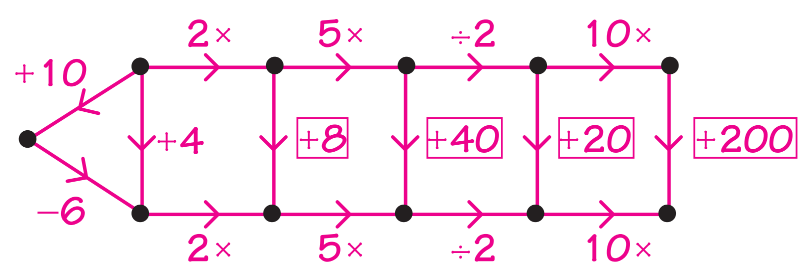
T (tracing the unlabeled arrow)**: *What could this arrow be for?***

S: ***+8.***

Label the arrow and illustrate the +8 relationship by choosing several different numbers for Flip and labeling the other dots. For example:



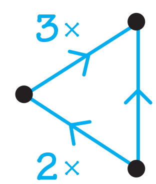
Continue this activity by extending the arrow picture and asking students to provide the labels shown in boxes below.



†You may like to act out the discussion, using a bag of counters for Flip. Then when you follow the +4 arrow, add four more counters to the bag. When you follow a 2x arrow, put out another bag of counters.

Exercise 3

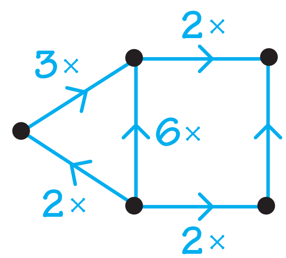
Draw this picture on the board, and trace the unlabeled arrow.



T: ***What could this arrow be for?***

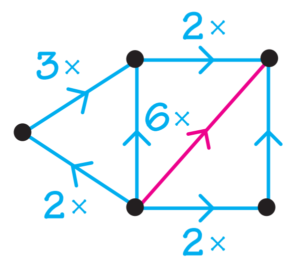
S: *6*x.

Label the arrow and extend your picture.



Again trace the unlabeled arrow.

T: ***What could this arrow be for?***



Allow a moment for students to consider the situation. If students have difficulty, add this supplementary arrow to the picture. First trace the 6x arrow followed by a 2x arrow. Then trace the red arrow.

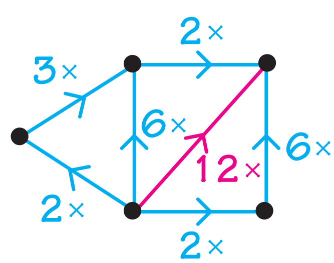
T: ***This red arrow is 6*x *followed by 2*x*.***
What could it be for?

S: *12*x.

First trace the 12x arrow; then trace the lower 2x arrow followed by the unlabeled arrow.

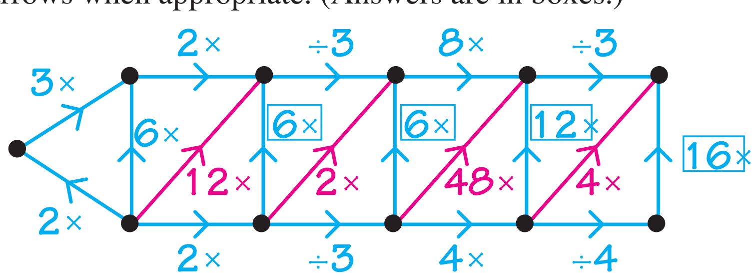
T: ***The 12*x *arrow is 2*x *followed by what kind of arrow?***

S: *6*x.



Continue this activity by extending the arrow picture, each time adding three arrows to the picture and asking students to provide labels for them.

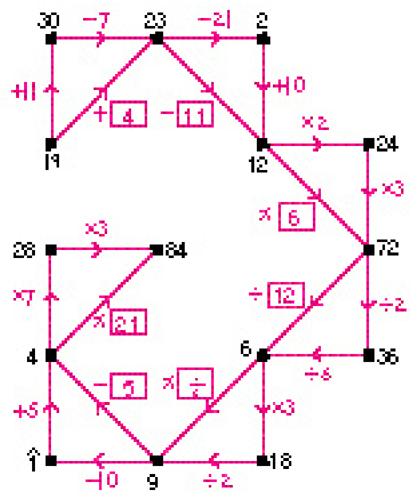
Use supplementary arrows when appropriate. (Answers are in boxes.)



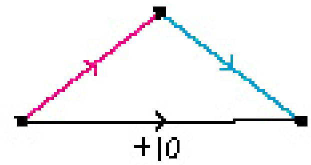
Worksheets N16*, **, ***, and **** are available for individual work.

Name ____________ N16 ☆

Label the dots and fill in the boxes for the arrows.



Name ____________ N16 ☆☆



Complete the charts.

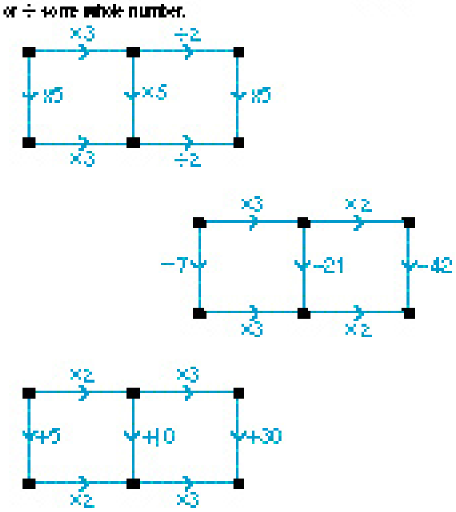
→	→
+3	+7
−3	+13
+25	−15
+4	+6
−11	+21
−100	+110
−5	+15
+50	−40

→	→
+3.6	+6.4
+8.1	+1.9
−2.5	+12.5
−3.7	+13.7
+2.65	+7.35
+11.25	−1.25
+0.05	+9.95
−7.43	+17.43

Many solutions are possible.

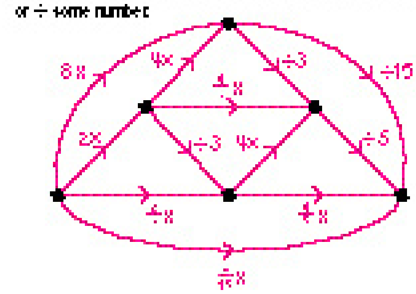
Name ____________ N16 ☆☆☆

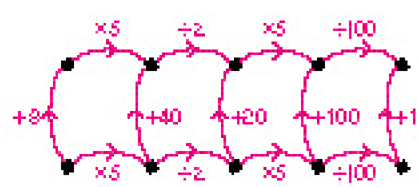
Label the unlabeled arrows. Each arrow should be +, −, ×, or ÷ some whole number.



Name ____________ N16 ☆☆☆☆

Label the unlabeled arrows. Each arrow should be +, −, ×, or ÷ some number.





N17 CALCULATOR PUZZLES #1

Capsule Lesson Summary

Put selected numbers on the display of a calculator using a restricted set of keys. Find solutions that require pressing keys fewer than ten times. Between two given numbers, build an arrow road in which each arrow is for +, –, x, or ÷ some number with digits limited to 5, 6, 8, and 9.

Materials

Teacher	• Calculator • Colored chalk	Student	• Calculator • Paper • Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

List these calculator keys on the board, and refer to them as you give the following directions.

$\boxed{5}\ \boxed{6}\ \boxed{8}\ \boxed{9}$

$\boxed{+}\ \boxed{\times}\ \boxed{-}\ \boxed{\div}\ \boxed{=}$

T: ***Today we are going to solve some calculator puzzles. The puzzles require that you use only certain keys on the calculator. You may use these keys in any way that you like, but these are the only keys you may use. Try to put 400 on the display of your calculator.***

Allow several minutes for students to experiment. You may like to talk to students individually about their attempts, and encourage some students to find several different solutions to the puzzle. After a few minutes, call for the class's attention and ask for solutions. When a student offers a solution, invite another student to check it. Make a list of solutions on the board. There are many solutions; here is a sampling.

400: $\boxed{5}\boxed{\times}\boxed{8}\boxed{5}\boxed{-}\boxed{5}\boxed{=}\boxed{=}\boxed{=}\boxed{=}\boxed{=}$

$\boxed{9}\boxed{9}\boxed{9}\boxed{-}\boxed{5}\boxed{9}\boxed{9}\boxed{=}$

$\boxed{8}\boxed{8}\boxed{-}\boxed{8}\boxed{\times}\boxed{5}\boxed{=}$

$\boxed{9}\boxed{6}\boxed{\times}\boxed{5}\boxed{-}\boxed{8}\boxed{5}\boxed{+}\boxed{5}\boxed{=}$

$\boxed{+}\boxed{5}\boxed{=}\boxed{=}\boxed{=}\boxed{=}\boxed{\times}\boxed{=}$

$\boxed{9}\boxed{\times}\boxed{5}\boxed{=}\boxed{=}\boxed{-}\boxed{5}\boxed{=}$

$\boxed{8}\boxed{5}\boxed{-}\boxed{6}\boxed{5}\boxed{\times}\boxed{=}$

This activity offers an excellent opportunity for students to use calculators in a very open and creative way. Some students will find solutions that make use of interesting features of the calculator. In such cases, discuss how the calculator operates to get the solution. Try to get a variety of solutions. Sometimes one student's solution will result in several similar solutions from other students.

Note: The list of solutions here assumes the calculator does chain operations and has an automatic constant feature (see "Role and Use of Calculators" in Section One: Notes to the Teacher). As necessary, make adjustments for the calculators in use by your students.

N17

Exercise 2

Refer to solutions on the board for 400 as you say,

T: ***Suppose that it costs 1¢ (or $1.00) every time we press a key. Which of these solutions for 400 costs the least?***

With the sample solutions here, there are two 7¢ solutions but none cheaper. If your class has only more expensive solutions, you may like to challenge them to find a less expensive one.

Put several numbers on the board; for example, 28, 33, 44, 49, and 125. Instruct students to try to find solutions costing less than 10¢ for each of these numbers. Below are several possible solutions for each number.

28:	$\boxed{5}\ \boxed{\times}\ \boxed{5}\ \boxed{+}\ \boxed{9}\ \boxed{-}\ \boxed{6}\ \boxed{=}$	**8¢**
	$\boxed{6}\ \boxed{\times}\ \boxed{6}\ \boxed{-}\ \boxed{8}\ \boxed{=}$	**6¢**
	$\boxed{5}\ \boxed{\times}\ \boxed{5}\ \boxed{+}\ \boxed{8}\ \boxed{-}\ \boxed{5}\ \boxed{=}$	**8¢**
	$\boxed{8}\ \boxed{8}\ \boxed{-}\ \boxed{6}\ \boxed{5}\ \boxed{+}\ \boxed{5}\ \boxed{=}$	**8¢**
	$\boxed{9}\ \boxed{5}\ \boxed{\div}\ \boxed{5}\ \boxed{+}\ \boxed{9}\ \boxed{=}$	**7¢**
33:	$\boxed{6}\ \boxed{\times}\ \boxed{6}\ \boxed{-}\ \boxed{9}\ \boxed{+}\ \boxed{6}\ \boxed{=}$	**8¢**
	$\boxed{9}\ \boxed{\times}\ \boxed{5}\ \boxed{-}\ \boxed{6}\ \boxed{=}\ \boxed{=}$	**7¢**
	$\boxed{8}\ \boxed{\times}\ \boxed{6}\ \boxed{-}\ \boxed{9}\ \boxed{-}\ \boxed{6}\ \boxed{=}$	**8¢**
	$\boxed{9}\ \boxed{9}\ \boxed{-}\ \boxed{6}\ \boxed{6}\ \boxed{=}$	**6¢**
	$\boxed{8}\ \boxed{5}\ \boxed{\div}\ \boxed{5}\ \boxed{+}\ \boxed{8}\ \boxed{=}\ \boxed{=}$	**8¢**
44:	$\boxed{6}\ \boxed{\times}\ \boxed{6}\ \boxed{+}\ \boxed{8}\ \boxed{=}$	**6¢**
	$\boxed{5}\ \boxed{9}\ \boxed{-}\ \boxed{9}\ \boxed{-}\ \boxed{6}\ \boxed{=}$	**7¢**
	$\boxed{5}\ \boxed{\times}\ \boxed{9}\ \boxed{-}\ \boxed{6}\ \boxed{+}\ \boxed{5}\ \boxed{=}$	**8¢**
	$\boxed{9}\ \boxed{9}\ \boxed{-}\ \boxed{5}\ \boxed{5}\ \boxed{=}$	**6¢**
49:	$\boxed{5}\ \boxed{8}\ \boxed{-}\ \boxed{9}\ \boxed{=}$	**5¢**
	$\boxed{5}\ \boxed{\times}\ \boxed{8}\ \boxed{+}\ \boxed{9}\ \boxed{=}$	**6¢**
	$\boxed{9}\ \boxed{-}\ \boxed{8}\ \boxed{+}\ \boxed{6}\ \boxed{\times}\ \boxed{=}$	**7¢**
	$\boxed{8}\ \boxed{\times}\ \boxed{8}\ \boxed{-}\ \boxed{5}\ \boxed{=}\ \boxed{=}\ \boxed{=}$	**8¢**
125:	$\boxed{6}\ \boxed{9}\ \boxed{+}\ \boxed{5}\ \boxed{6}\ \boxed{=}$	**6¢**
	$\boxed{5}\ \boxed{\times}\ \boxed{=}\ \boxed{=}$	**4¢**
	$\boxed{+}\ \boxed{6}\ \boxed{5}\ \boxed{=}\ \boxed{=}\ \boxed{-}\ \boxed{5}\ \boxed{=}$	**8¢**

Exercise 3

Pose a different type of calculator puzzle, still using just the keys $\boxed{5}$, $\boxed{6}$, $\boxed{8}$, $\boxed{9}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, and $\boxed{=}$.

T: ***In our next problem, an arrow can be for add, subtract, multiply, or divide by some number with digits limited to 5, 6, 8, and 9. Give us an example of such an arrow.***

S: ***÷8.***

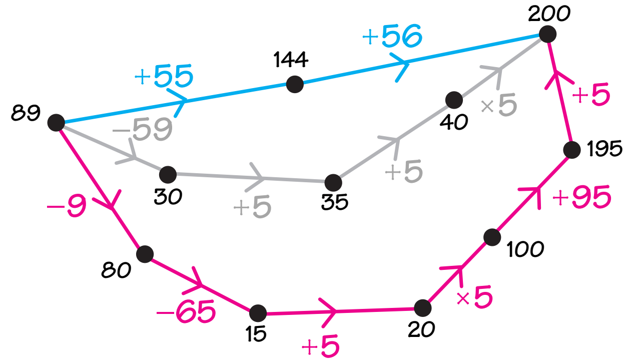
S: ***+59.***

S: ***–888.***

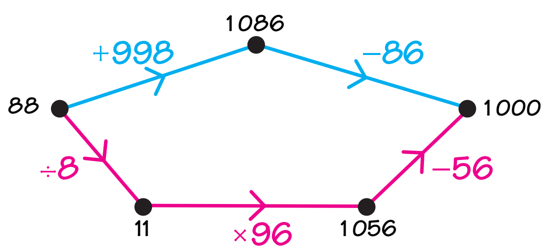
If students do not suggest immediately examples such as +59 or –888, do so yourself.

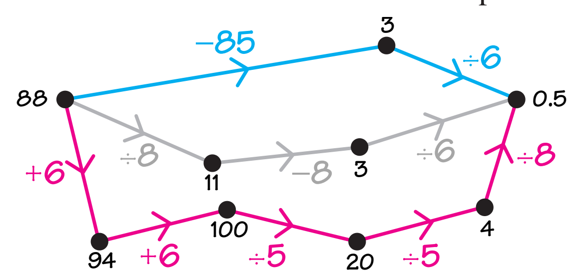
T: ***Now the problem is to build an arrow road from 89 to 200 using only these kinds of arrows.***

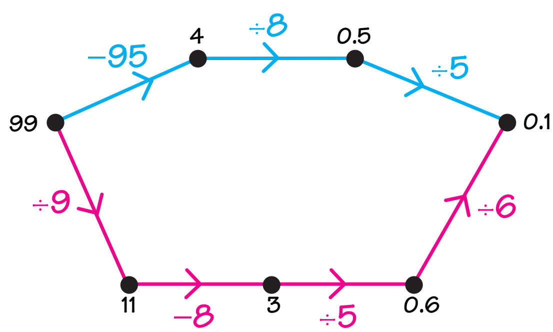
Allow several minutes for students to work independently on this problem. Encourage each student to find at least one solution. Challenge some students to find solutions with fewer arrows. When most students have found at least one solution, invite some students to draw their roads on the board. There are many possible solutions, three of which are given below. With the class, observe which solution has the fewest arrows, and challenge the class to find a shorter (fewer arrows) solution, if appropriate.



Continue this activity by asking students to draw other roads, for example, from 88 to 1000, from 88 to 0.5, or from 99 to 0.1. You may suggest they try to find solutions with as few arrows as possible.







Home Activity

Create calculator puzzles for students to work on at home with family members. For example:

> The only keys you may use are $\boxed{4}$, $\boxed{7}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, and $\boxed{=}$. You may use the keys in any way you like. Start at 0 and try to put 100 (or 150 or 0.5) on the display.

Capsule Lesson Summary

Label marks on a number line with division names, fractions, and decimals. Notice the equivalence of n ÷ m and $\frac{n}{m}$. Use this equivalence to find the decimal name of a fraction with a calculator. Introduce memory keys on a calculator, and then use these keys to add fractions with a calculator.

Materials

Teacher
- Meter stick
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Calculator

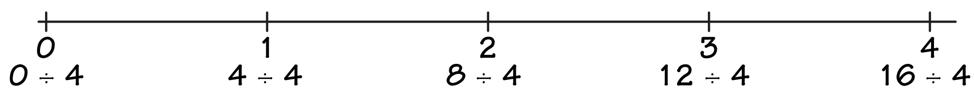
Description of Lesson

Exercise 1

Draw a number line with five equally spaced marks on the board.

T: ***What number is 16 ÷ 4?*** (4)

Use this fact to label the mark farthest to the right. Continue with 12 ÷ 4, 8 ÷ 4, 4 ÷ 4, and 0 ÷ 4 to label the other four marks.



T: ***Where does 6 ÷ 4 belong on this number line?***

S: ***Halfway between 1 and 2.***

T: ***Let's check. How much does each person get if $6 is shared equally among four people?***

S: ***$1.50.***

T: ***So what decimal number is 6 ÷ 4?***

S: ***1.50, or 1.5.***

T: ***Where is 1.5 on this number line?***

S: ***If we divide the section between 1 and 2 into ten pieces of the same length, the fifth mark from 1 will be for 1.5. That is halfway between 1 and 2.***

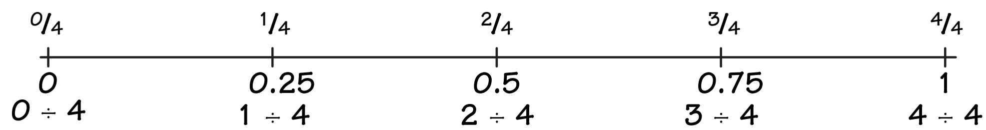
Locate 6 ÷ 4 or 1.5 on the number line and label its mark. Continue the activity, labeling marks for 2 ÷ 4, 1 ÷ 4, and 3 ÷ 4. Sharing $2, $1, or $3 among four people will help motivate giving decimal numbers as answers. Trace the section of the number line from 0 to 1 as you ask,

T: ***How many sections do we have marked between 0 and 1?*** (Four)
What fractions could we put at these marks?

S: $\frac{0}{4}$, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, ***and*** $\frac{4}{4}$.

Label the marks with these fractions.

N18



Observe that $\frac{1}{4} = 1 \div 4$, $\frac{2}{4} = 2 \div 4$, $\frac{3}{4} = 3 \div 4$, and $\frac{4}{4} = 4 \div 4$.

T: ***How many fourths is 2?*** ($\frac{8}{4}$) ***3?*** ($\frac{12}{4}$) ***4?*** ($\frac{16}{4}$) ***1.5?*** ($\frac{6}{4}$)

Divide the rest of the number line into fourths and give these fraction names to 2, 3, 4, and 1.5. Point to various marks and ask the class for a fraction, a decimal, and a division name for each.

Suppose one mark the class labels is for $7 \div 4$.

$$7 \div 4 = \frac{7}{4} = 1\frac{3}{4}$$

T: ***$7 \div 4 = \frac{7}{4}$ or $1\frac{3}{4}$.***

You may wish to look at the division problem in two ways. Point to the appropriate division expression as you observe the following:

$$4\overline{)7} = 1 \text{ R} = 3$$

T: ***If we divide seven watermelons among four families, we could give each family one watermelon and have three watermelons left over, or we could cut the three extra watermelons into fourths to give each family $1\frac{3}{4}$ watermelons in all.***

You may wish to illustrate the situation by cutting up circles drawn on the board. Note that $7 \div 4$ is the number $1\frac{3}{4}$, and that is how we locate it on the number line.

Note: The following discussion about calculators refers to simple four-function calculators. If your students have calculators that display fractions, you might instead use them to go back and forth between fraction, decimal, and division names.

T: ***We cannot display a fraction like $\frac{3}{4}$ on the calculator, but we can display its decimal name. How?***

S: ***$\frac{3}{4} = 0.75$, so just press $\boxed{0}\ \boxed{.}\ \boxed{7}\ \boxed{5}$.***

S: ***$\frac{3}{4} = 3 \div 4$, so press $\boxed{3}\ \boxed{\div}\ \boxed{4}\ \boxed{=}$.***

Ask the class to press $\boxed{3}\ \boxed{\div}\ \boxed{4}\ \boxed{=}$.

T: ***What are some other fractions equivalent to $\frac{3}{4}$?***

S: ***$\frac{9}{12}$.***

T: ***If $\frac{3}{4} = \frac{9}{12}$, then $\frac{9}{12}$ has the same decimal name as $\frac{3}{4}$. Let's check. How can we display the decimal name for $\frac{9}{12}$ on the calculator?***

S: ***Press $\boxed{9}\ \boxed{\div}\ \boxed{1}\boxed{2}\ \boxed{=}$.***

Let the class confirm that $\frac{9}{12} = 0.75$ on the calculator. Repeat the activity using a few other fractions equivalent to $\frac{3}{4}$.

T: ***Using the calculator, find the decimal name for $^3/_{10}$, but do not use the $\boxed{\cdot}$ key. What keys do you press?***

S: $\boxed{3}\ \boxed{\div}\ \boxed{1}\boxed{0}\ \boxed{=}$.

T: ***Some people read 0.3 as "three tenths" the same as they read $^3/_{10}$. 0.3 and $^3/_{10}$ name the same number. How would they read 0.5?***

S: ***"Five tenths," the same as $^5/_{10}$.***

Repeat the activity with $^4/_{100}$, $^{35}/_{10}$, $^{495}/_{100}$, and $^3/_{1000}$. Ask students for fractional names for 0.07, 0.008, and 0.012.

Exercise 2

Instruct students to clear the display of their calculators (get 0), and then to hide the display.

Record these keys on the board as you tell students to press them on their calculators.

$$\boxed{4}\ \boxed{\times}\ \boxed{5}\ \boxed{+}\ \boxed{1}\boxed{0}\ \boxed{\times}\ \boxed{8}\ \boxed{=}$$

Ask several student to predict what number they think is on their calculator display before letting all of the students uncover their displays. The number is 240. Record it next to the calculator expression.

T: ***What number sentence could we write? Remember to use parentheses to show the order in which the operations are performed.***

Suggest students write the number sentence on their papers while you invite a student to write the number sentence on the board. Let students explain the use of parentheses.

$$\boxed{4}\ \boxed{\times}\ \boxed{5}\ \boxed{+}\ \boxed{1}\boxed{0}\ \boxed{\times}\ \boxed{8}\ \boxed{=}\ \ 240$$

$$((4 \times 5) + 10) \times 8 = 240$$

T: ***First we calculate 4 x 5, so we use the red parentheses. Then we add 10 to the answer, so we use the blue parentheses. Then we multiply that answer by 8 to get 240.***

Write this expression on the board.

$$(4 \times 5) + (10 \times 8)$$

T: ***What number is this?***

S: ***100. 4 x 5 = 20, 10 x 8 = 80, and 20 + 80 = 100.***

T: ***How can we get the calculator to do the calculations in this way?***

Allow a couple minutes for students to experiment. Perhaps someone will make a comment similar to this one.

S: ***Press $\boxed{4}\ \boxed{\times}\ \boxed{5}\ \boxed{=}$. You will get 20. Write it down and clear the display. Press $\boxed{1}\boxed{0}\ \boxed{\times}\ \boxed{8}\ \boxed{=}$ and then press $\boxed{+}\ \boxed{2}\boxed{0}\ \boxed{=}$. You will get 100.***

T: ***There is a way to do the same thing but without clearing the display. You said to record a partial result and then to add it in later. The calculator can do a calculation and then remember the result while you do another calculation. We say the calculator has "memory."***

Show the class which keys are for memory on their calculators. This lesson assumes calculators have a memory plus key [M+] and a memory recall key [MRC].

T: ***I'll show you a way to use memory keys to do this problem.. Press [4] [×] [5] [=]. We now have 20 on the display. Let's ask the calculator to remember that number—you put 20 in memory by pressing [M+]. Now press [1][0] [×] [8] [+], then recall the number in memory by pressing [MRC], and then press [=]. What number is on the display?***

S: ***100.***

Note: There are other ways to do this problem by pressing a sequence of keys; for example, [4] [×] [5] [M+] [1][0] [×] [8] [M+] [MRC] or [1][0] [×] [8] [M+] [4] [×] [5] [+] [MRC] [=]. You may wish to mention other ways. Record one or more sequences of keys for this problem and the resulting number.

Suggest that students do these two problems on their calculators. They should also do them on paper or in their heads to check the results.

$$((6 \times 5) - 4) + 3 \qquad\qquad (6 \times 5) - (4 + 3)$$

A common error with the second problem is to put 30 (6 x 5) in memory and then to subtract that from 7 (4 + 3): [4] [+] [3] [−] [MRC] [=]. In the previous collective work, 20 (4 x 5) or 80 (10 x 8) could be stored in memory and then memory added to the other. Order is not important because addition is commutative. However, if your students are using a [M−] key, they can press [6] [×] [5] [M+] [4] [+] [3] [M−] [MRC]. Remind students making such an error that they want to subtract 4 + 3 from 6 x 5, not 6 x 5 from 4 + 3.

Go over solutions collectively.

$$((6 \times 5) - 4) + 3 = 29 \qquad\qquad (6 \times 5) - (4 + 3) = 23$$

[6] [×] [5] [−] [4] [+] [3] [=] **29**

[6] [×] [5] [M+] [4] [+] [3] [M−] [MRC] **23**

or

[4] [+] [3] [M+] [6] [×] [5] [−] [MRC] [=] **23**

Exercise 3

T: ***We can use a calculator to add fractions, although the calculator shows the answer in decimal form and may be limited by the number of digits it can display. What is ½ + ½?***

S: ***1.***

T: ***How can we use the calculator to do the calculation?***

S: ***Use a decimal name for ½.***

S: ***Use division: 1 ÷ 2 for ½.***

Record this calculator expression on the board.

$$1/2 + 1/2 = 1$$

[1] [÷] [2] [+] [1] [÷] [2]

T: ***On your calculator, press* [1] [÷] [2] [+] [1] [÷] [2] [=].**
What number is on the display?

S: ***0.75 not 1.***

T: ***What did the calculator do?***

S: ***It divided 1 by 2 to get 0.5, then added 1 to get 1.5, and then divided that number by 2 to get 0.75.***

T: ***What do we want the calculator to do?***

S: ***To divide 1 by 2 and to add that number to itself.***

S: ***Press* [1] [÷] [2] [M+] [1] [÷] [2] [+] [MRC] [=] .**

Accept any correct method.

Repeat the activity with this problem.

$$5/2 - 4/10$$

$$(5 \div 2) - (4 \div 10)$$

[5] [÷] [2] [M+] [4] [÷] [1][0] [M-] [MRC]   **2.1**

[4] [÷] [1][0] [M+] [5] [÷] [2] [−] [MRC] [=]   **2.1**

Capsule Lesson Summary

Progressively build an arrow picture with various multiplication and division functions. Draw arrows for other multiplication or division functions in the picture, recognizing them as composites of other arrows already there. Perform some multiplication calculations using addition and subtraction and results of other related calculations. Observe some patterns in a sequence of multiplication calculations.

Materials

Teacher
- Colored chalk
- Grid board†

Student
- Paper
- Worksheets N19*, **, and ***

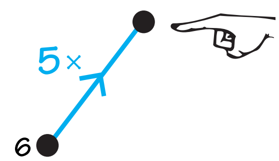
Description of Lesson

Exercise 1

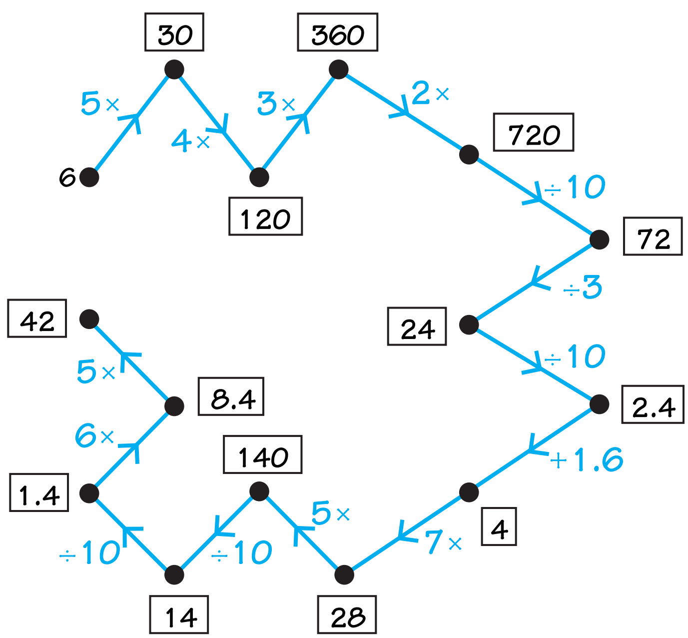
Progressively draw an arrow road on the board. Call on students to give the ending number of an arrow before drawing the next arrow. Begin the arrow picture at 6 with a 5x arrow.

T: ***Which number is here?***

S: ***30; 5 x 6 = 30.***



Continue extending the arrow picture and asking students to calculate ending numbers. A suggested arrow picture is shown below with correct responses in boxes. Feel free to adjust the calculations according to the abilities of your students. Stop the activity if many students lose interest.



T: ***What kind of arrow could we use to go from 42 back to 6 (the beginning number)?***

†See the "Note on Grids" section in the introduction to the *Geometry and Measurement* strand.

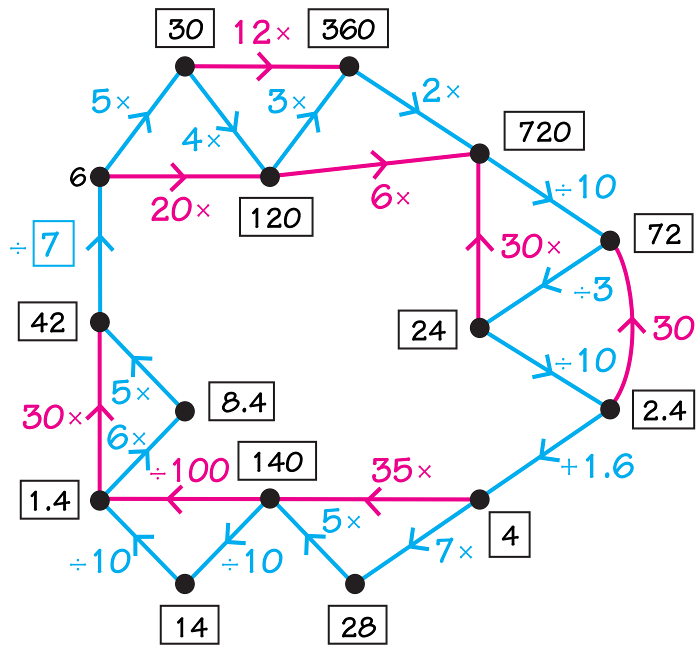
S: *÷7.*

S: *–36.*

T: ***Do you see where we could draw another 6x arrow in this picture?***

Invite a student to trace and then draw a 6x arrow. Encourage the class to note that 3x followed by 2x is 6x.

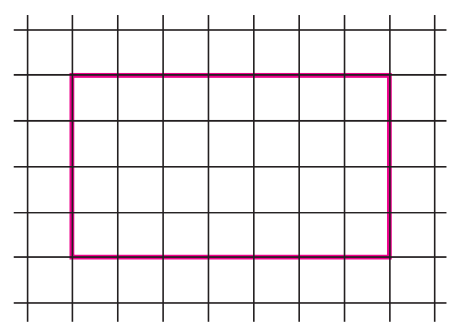
Continue this activity by asking for 12x, 20x, 30x, 35x, and ÷100 arrows. These arrows (in some cases more than one) are included in the following illustration.



Exercise 2

Display a grid board and draw a rectangle 7 grid lengths by 4 grid lengths.

T: ***The area of a shape is how much is needed to cover the shape, or the amount of surface. Suppose each little square in this grid has area 1 cm². What is the area of the rectangle?***



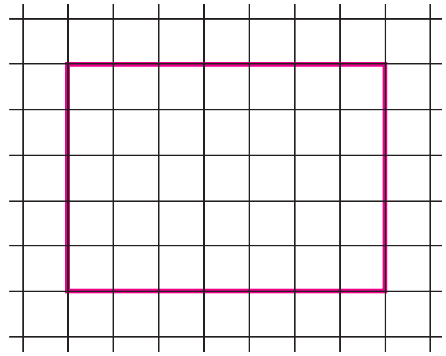
S: ***28 cm². I counted all of the small squares inside the rectangle.***

S: ***28 cm². I multiplied length times width; 7 x 4 = 28.***

T: ***Let's make the rectangle wider and let it enclose another row of small squares. What is the area of the new rectangle?***

Drop the lower edge of the rectangle one grid length.

S: ***35 cm²**. I added 28 + 7 since there are 7 more (little) squares.*



S: ***I counted all of the small squares inside the rectangle.***

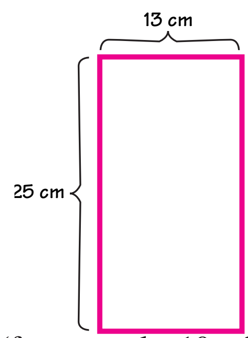
S: ***I multiplied 5 x 7 = 35.***

Continue modifying the picture. Ask for the area of each new rectangle formed by changing the size of the rectangle so that it encloses

- two more rows of grid squares (49 cm^2);
- one more column of grid squares (56 cm^2);
- one more row and one more column of grid squares (72 cm^2).

Each time, ask how much more area the new rectangle has.

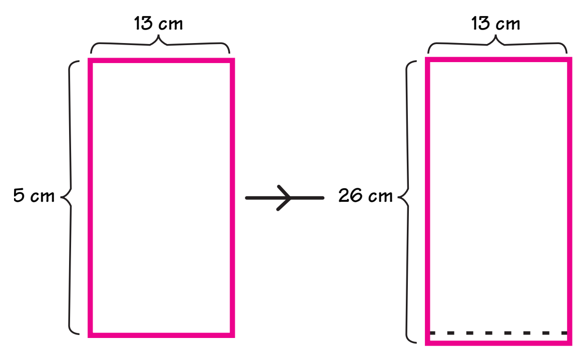
Put the grid board aside and draw a rectangle approximately twice as long on one side as the other. Label two adjoining sides as shown here.



T: ***Let's try working with a larger rectangle. What is the area of a rectangle 13 cm by 25 cm? On a grid board, it would enclose 25 rows of 13 little squares. You could also look at it as enclosing 13 columns of 25 little squares.***

Allow a couple minutes for students to do the calculation mentally (for example, 10 x 25 = 250; 3 x 25 = 75; 250 + 75 = 325) or on paper. Then record the multiplication fact 13 x 25 = 325 on the board.

T: ***If we make the rectangle wider by letting it enclose another row of squares, what would the dimensions of the new rectangle be?***



S: ***13 cm by 26 cm.***

T: ***What is its area?***

S: ***338 cm^2. I multiplied 13 and 26.***

S: ***I added 325 + 13.***

Record 13 x 26 = 338 on the board. Repeat the activity, changing the size of the rectangle to let it enclose

- another column of 1 cm^2 squares (364 cm^2);
- another column of 1 cm^2 squares (390 cm^2).

N19

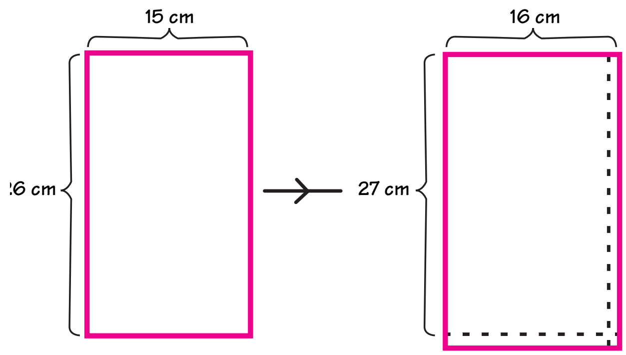
You should now have this list of multiplication facts on the board.

$$13 \times 25 = 325$$
$$13 \times 26 = 338$$
$$14 \times 26 = 364$$
$$15 \times 26 = 390$$

T: ***The rectangle is now 15 cm by 26 cm. What if this time we add a row and a column—what would the dimensions be?***

S: ***16 cm by 27 cm.***

T: ***What is the area of this new rectangle?***



Allow a few minutes for students to do the calculation.

S: ***432 cm². I multiplied 16 x 27.***

T: ***Is there any way we can use the picture to find the answer?***

S: ***A column adds 26 to the area, a row adds 15, and the corner adds 1. 26 + 15 + 1 = 42. We just need to add 42 cm² to the area of the previous (15 cm by 26 cm) rectangle.***

You may need to help students see the increase of 42 cm². Some students may eventually generalize to $n + m + 1$ in which n and m are the numbers of rows and columns.

T: ***What number is 390 + 42?*** (432)

Record the number story and show the increase in area with a blue arrow.

$$\begin{aligned} 15 \times 26 &= 390 \\ 16 \times 27 &= 432 \end{aligned}\Big\} +42$$

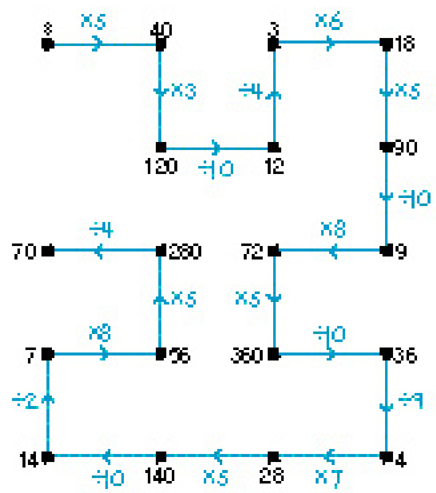
Repeat the activity, adding one row and one column several times. Students should begin to see a pattern in the blue arrows; some may still resort to multiplying, and others may use the picture. Accept all correct methods, but always confirm the answers.

$$\begin{array}{ll} 15 \times 26 = 390 & \\ & +42 \\ 16 \times 27 = 432 & \\ & +44 \\ 17 \times 28 = 476 & \\ & +46 \\ 18 \times 29 = 522 & \\ & +48 \\ 19 \times 30 = 570 & \end{array}$$

Worksheets N19*, **, and *** are available for individual work.

Name ______________ N19 ☆

Label the dots.



Name ______________ N19 ☆☆

Multiply.

68	137
×42	×59
136	1233
2720	6850
2856	8083

Complete each problem. Put a single digit in each box to make the calculations correct.

$\boxed{3}\boxed{2}7$	98
$\times\ \boxed{2}6$	$\times\ 2\boxed{4}$
$1\,9\,6\boxed{2}$	$3\boxed{9}\boxed{2}$
$\boxed{6}\boxed{5}\boxed{4}0$	$\boxed{1}\boxed{9}\boxed{6}\boxed{0}$
$\boxed{8}\boxed{5}0\boxed{2}$	$\boxed{2}\boxed{3}\boxed{5}\boxed{2}$

Name ______________ N19 ☆☆☆

$$16 \times 24 = 384$$

Complete.

$17 \times 24 = 384 + \boxed{24} =$ 408

$15 \times 24 = 384 - \boxed{24} =$ 360

$16 \times 25 = 384 + \boxed{16} =$ 400

$16 \times 23 = 384 - \boxed{16} =$ 368

$26 \times 24 = 384 + \boxed{240} =$ 624

$16 \times 34 = 384 + \boxed{160} =$ 544

$116 \times 24 = 384 + \boxed{2400} =$ 2 784

$16 \times 124 = 384 + \boxed{1600} =$ 1 984

N20 ADDITION OF RATIONAL NUMBERS

Capsule Lesson Summary

Make divisions in a region to show equivalent fractions. Introduce a method for adding two fractions with unlike denominators.

Materials

Teacher
- Meter stick
- Colored chalk
- Blackline N20
- Metric ruler
- Colored pencils, pens, or crayons
- Paper
- Worksheets N20*, **, ***, and ****

Students
- Rectangle

Advance Preparation: Use Blackline N20 to make copies of a rectangle for use in Exercise 2.

Description of Lesson

Exercise 1

Solicit examples of two fractions whose sum is 1.

S: ***$\frac{1}{2}$ and $\frac{1}{2}$.***

S: ***$\frac{1}{4}$ and $\frac{3}{4}$.***

S: ***$\frac{9}{10}$ and $\frac{1}{10}$.***

T: ***Can you find two fractions whose sum is 1 but with neither fraction having 1 as numerator?***

S: ***$\frac{2}{5}$ and $\frac{3}{5}$.***

S: ***$\frac{3}{8}$ and $\frac{5}{8}$.***

S: ***$\frac{40}{100}$ and $\frac{60}{100}$.***

Write this open sentence on the board, and ask for a fraction to put in the box.

$$\frac{3}{4} + \boxed{} = 1$$

Any fraction equivalent to $\frac{1}{4}$ can be put in the box. Try to include a fraction such as $\frac{2}{8}$.

$$\frac{3}{4} + \boxed{\frac{2}{8}} = 1$$

T: ***We agree that $\frac{3}{4} + \frac{2}{8} = 1$. Notice that when we add fractions we do not add numerators and denominators.***

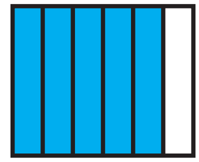
Exercise 2

Draw a large rectangle on the board and give students a similar rectangle (Blackline N20).

T: ***Let's color $\frac{5}{6}$ of the rectangle. How can we do it?***

S: ***Divide it into six pieces all the same size. Then color five of the pieces.***

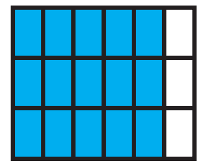
Let students do the division and coloring. They may want to use a ruler (or meter stick) to accurately divide the rectangle.



$5/6$

T: ***Now let's cut each of the strips into three pieces all the same size.***

Use a ruler or meter stick to do the division accurately.



$5/6 = 15/18$

T: ***How many pieces are there now?***

S: ***18; 3 x 6 = 18.***

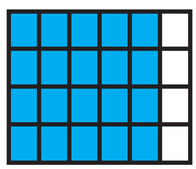
T: ***How many of them are colored?***

S: ***15; 3 x 5 = 15.***

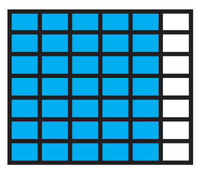
T: ***15 of the 18 pieces are colored. What do we know about ⅚ and $^{15}/_{18}$?***

S: ***⅚ = $^{15}/_{18}$. The same amount of the square is colored as before.***

Repeat this exercise to generate the equivalences ⅚ = $^{20}/_{24}$ and ⅚ = $^{35}/_{42}$. You may give students more copies of the rectangle on Blackline N20, or ask them to draw a rectangle on their papers.



$5/6 = 20/24$



$5/6 = 35/42$

Exercise 3

T: ***Sara and Amelia are friends who like to treat their families with cake for dessert. The bakery they go to sells square cakes all the same size but is willing to sell pieces of cakes. One day Sara wants ½ of a fudge cake and Amelia wants ¼ of a fudge cake. Will one cake be enough for both orders?***

S: ***Yes, ½ + ½ = 1 and ¼ < ½.***

S: ***½ + ¼ = ¾ since ½ = $^2/_4$.***

Record the number sentence on the board.

$$1/2 + 1/4 = 2/4 + 1/4 = 3/4$$

T: ***If the girls take ¾ of a fudge cake altogether, how much of the cake will be left to sell?***

S: ***¼.***

T: ***The next day Sara places the same order, but because Amelia's cousins are visiting, Amelia wants ¾ of a fudge cake. Will one cake be enough for both orders?***

S: ***No; if Amelia takes ¾ of one cake, there will be only ¼ of a cake left and Sara wants more than that.***

T: ***How many cakes do they need?***

S: ***Two, but there will be cake left over.***

S: ***$1\frac{1}{4}$ cakes; Sara and Amelia can each take $\frac{1}{2}$ of one cake and Amelia can take $\frac{1}{4}$ of another cake.***

S: ***$1\frac{1}{4}$ cakes; Amelia takes $\frac{3}{4}$ of a cake and Sara takes the other $\frac{1}{4}$. Then Sara takes $\frac{1}{4}$ of another cake. $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.***

Illustrate the cakes on the board if it would be helpful. Record the number sentence on the board.

$$\tfrac{1}{2} \times \tfrac{3}{4} = \tfrac{2}{4} \times \tfrac{3}{4} = \tfrac{5}{4} = 1\tfrac{1}{4}$$

T: ***The next week Sara orders $\frac{1}{3}$ of a coconut cake and Amelia orders $\frac{1}{4}$ of a coconut cake. But the bakery has only one coconut cake left. Will one cake be enough for both orders?***

S: ***Yes, $\frac{1}{3} < \frac{1}{2}$ and $\frac{1}{4} < \frac{1}{2}$.***

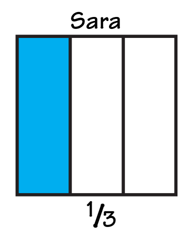
T: ***The baker would like to know what fraction of the cake they want altogether so that he knows how much cake will be left over to sell. Do they want more than $\frac{1}{2}$ of the cake?***

S: ***Yes, $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ and $\frac{1}{3} > \frac{1}{4}$.***

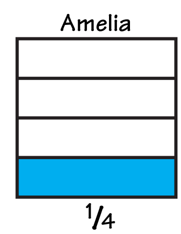
T: ***How can Sara and Amelia figure out much of the cake they want altogether?***

Listen to any suggestions from students as to how to solve the problem. Then tell the class about Sara and Amelia's method.

T: ***Sara and Amelia decide to draw pictures to help. Sara draws a square for a cake and divides it into three columnsall the same size. She colors one of them because she wants $\frac{1}{3}$ of the coconut cake. Her picture looks like this.***

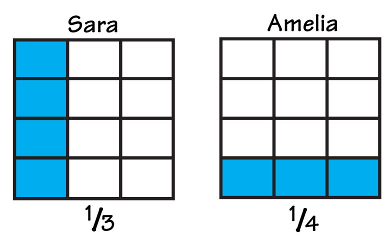


Amelia draws this same size square for a cake and divides it into four rows all the same size. She colors one of them because she wants $\frac{1}{4}$ of the coconut cake. Her picture looks like this.



It is still difficult to tell how much cake they want altogether. After much thought, they come up with this method.

Sara makes cuts like Amelia's, and Amelia makes cuts like Sara's.



How does that help?

S: ***All of the pieces are the same size.***

T: ***How many pieces is each square divided into?*** (12)
How many twelfths is $\frac{1}{3}$ of a cake? ($\frac{4}{12}$)
How many twelfths is $\frac{1}{4}$ of a cake? ($\frac{3}{12}$)

Conclude that $\frac{1}{3} = \frac{4}{12}$ and $\frac{1}{4} = \frac{3}{12}$.

Record this number sentence on the board.

$$\tfrac{1}{3} \times \tfrac{1}{4} = \tfrac{4}{12} \times \tfrac{3}{12}$$

N20

T: ***How could the one coconut cake that the bakery has be cut to fill Sara's and Amelia's orders?***

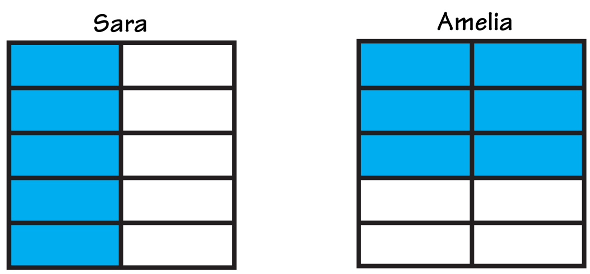
S: ***Cut the cake into 12 pieces all the same size. Sara takes four, Amelia takes three, and the baker has five pieces left to sell.***

T: ***What number is $^1/_3 + {^1/_4}$?***

S: ***$^7/_{12}$.***

Complete the number sentence. $^1/_3 \times {^1/_4} = {^4/_{12}} \times {^3/_{12}} = {^7/_{12}}$

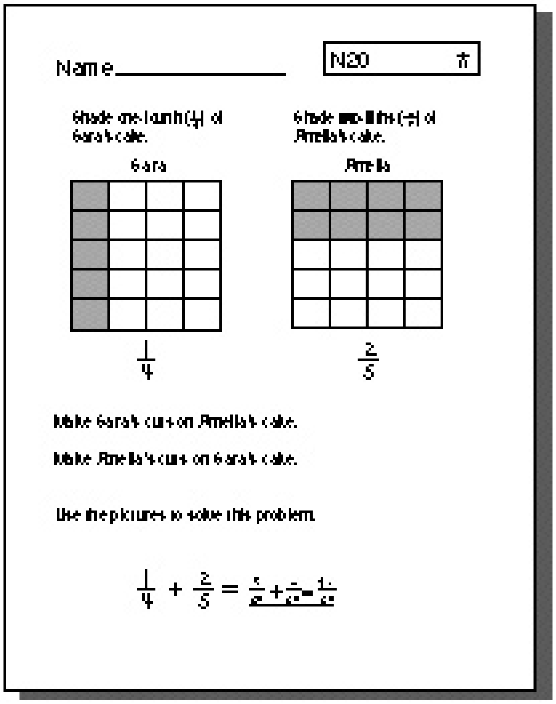
Pose the problem $^1/_2 + {^3/_5}$, and ask students to use Sara's and Amelia's method to solve it. As the class solves the problem collectively, establish that Sara always divides squares for cakes into columns and Amelia always divides squares for cakes into rows.



$^1/_2 = {^5/_{10}}$ $^3/_5 = {^6/_{10}}$

$^1/_2 + {^3/_5} = {^5/_{10}} + {^6/_{10}} = {^{11}/_{10}} = 1\,{^1/_{10}}$

Worksheets N20*, **, ***, and **** are available for individual work.


Name ______________ N20 ☆

Shade one-fourth ($\frac{1}{4}$) of Sara's cake.

Shade two-fifths ($\frac{2}{5}$) of Amelia's cake.

Sara $\frac{1}{4}$

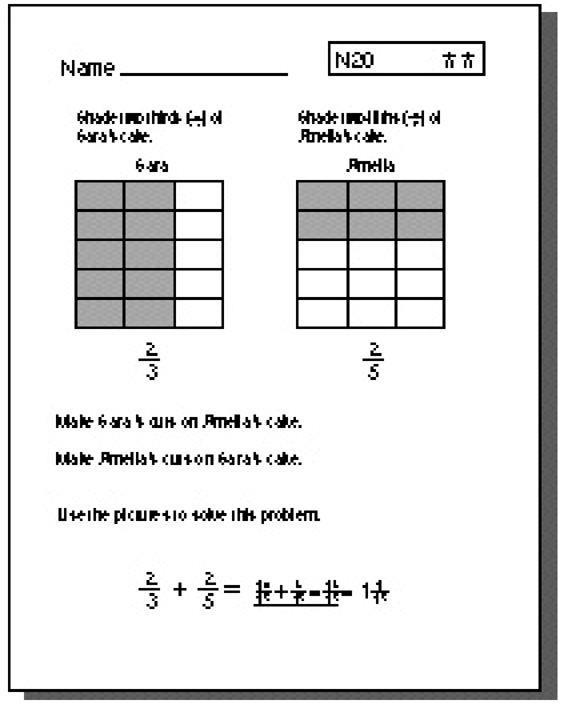
Amelia $\frac{2}{5}$

Mark Sara's cuts on Amelia's cake.

Mark Amelia's cuts on Sara's cake.

Use the pictures to solve this problem.

$\frac{1}{4} + \frac{2}{5} = \frac{5}{20} + \frac{8}{20} = \frac{13}{20}$


Name ______________ N20 ☆☆

Shade two-thirds ($\frac{2}{3}$) of Sara's cake.

Shade two-fifths ($\frac{2}{5}$) of Amelia's cake.

Sara $\frac{2}{3}$

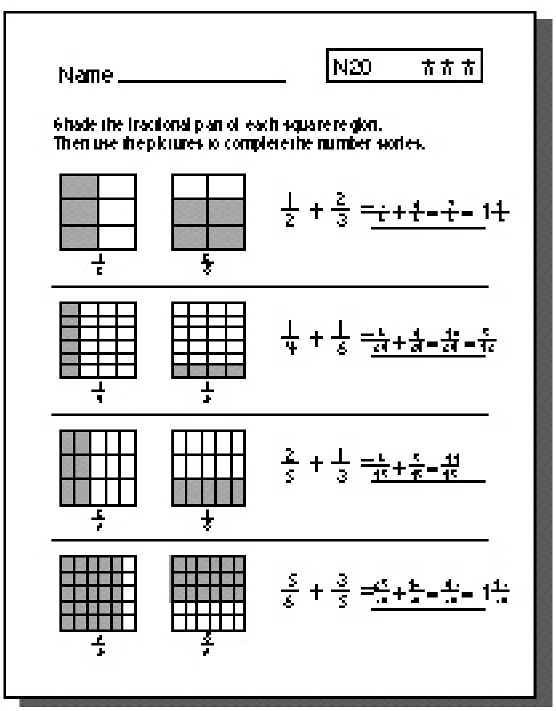
Amelia $\frac{2}{5}$

Mark Sara's cuts on Amelia's cake.

Mark Amelia's cuts on Sara's cake.

Use the pictures to solve this problem.

$\frac{2}{3} + \frac{2}{5} = \frac{10}{15} + \frac{6}{15} = \frac{16}{15} = 1\frac{1}{15}$


Name ______________ N20 ☆☆☆

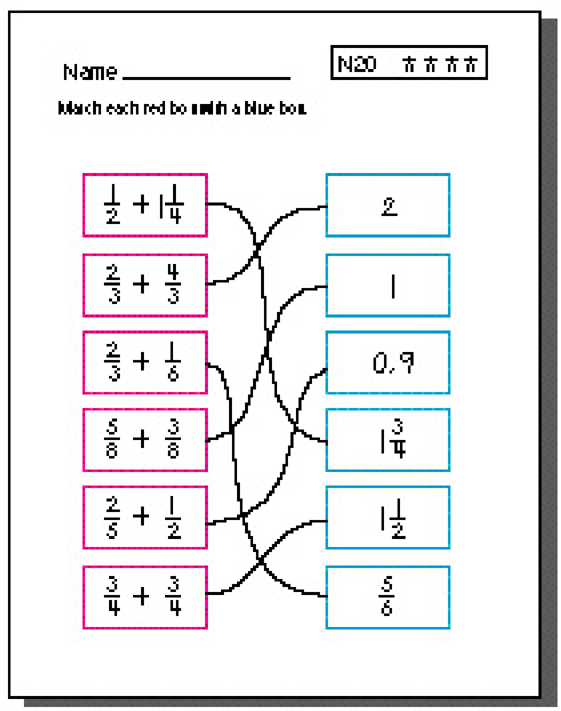
Shade the fractional part of each square region.
Then use the pictures to complete the number sentence.

$\frac{1}{2} + \frac{2}{3} = \frac{3}{6} + \frac{4}{6} = \frac{7}{6} = 1\frac{1}{6}$

$\frac{1}{4} + \frac{1}{6} = \frac{6}{24} + \frac{4}{24} = \frac{10}{24} = \frac{5}{12}$

$\frac{2}{5} + \frac{1}{3} = \frac{6}{15} + \frac{5}{15} = \frac{11}{15}$

$\frac{5}{6} + \frac{3}{5} = \frac{25}{30} + \frac{18}{30} = \frac{43}{30} = 1\frac{13}{30}$


Name ______________ N20 ☆☆☆☆

Match each red box with a blue box.

$\frac{1}{2} + 1\frac{1}{4}$	2
$\frac{2}{3} + \frac{4}{3}$	1
$\frac{2}{3} + \frac{1}{6}$	0.9
$\frac{5}{8} + \frac{3}{8}$	$1\frac{3}{4}$
$\frac{2}{5} + \frac{1}{2}$	$1\frac{1}{2}$
$\frac{3}{4} + \frac{3}{4}$	$\frac{5}{6}$

N21 DIVISION ALGORITHM #2

Capsule Lesson Summary

Solve a sequence of related division problems. Solve division problems suggested by situations in which a given number of bottles are packed into cartons, each carton holding a specified number of bottles. Present a form of the standard division algorithm. Individually solve some puzzles involving the division algorithm.

Materials

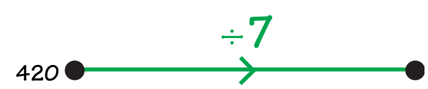
Teacher
- Colored chalk

Student
- Paper
- *Division Problems* Booklet

Description of Lesson

Exercise 1

Draw this arrow on the board.



T: ***I don't want an answer yet. Instead, tell me a story or situation in which the problem 420 ÷ 7 might arise.***

Call on several students to tell the stories. Encourage different types of stories, including some stories that involve dividing 420 objects into 7 equal groups and some that involve dividing 420 objects into groups of 7 objects each (or finding how many groups of 7 are in 420).

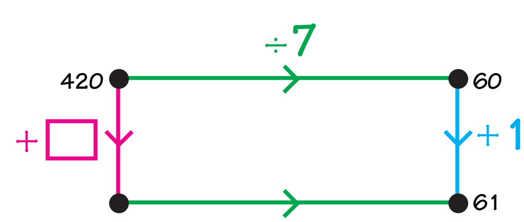
Select one of the students' stories, and refer to it as you solve the problems in this exercise. Here is a sample:

S: ***Our class must sell 420 raffle tickets in 7 days. If we sell the same number of tickets every day, how many tickets must we sell each day?***

T: ***What is 420 ÷ 7?***

S: ***60. 42 ÷ 7 = 6, so 420 ÷ 7 = 60.***

Add this information to the arrow picture

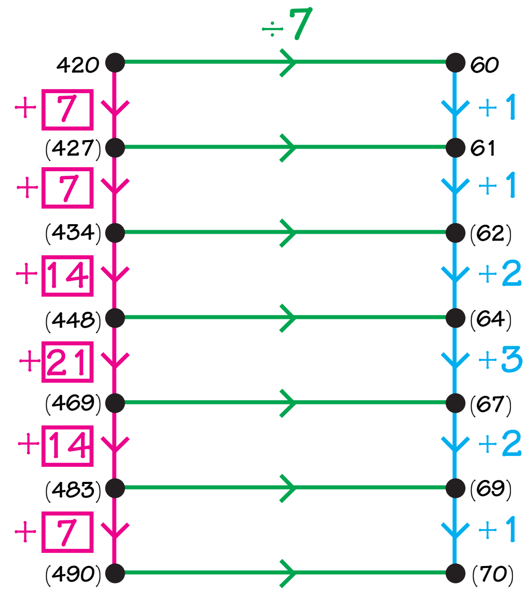


T: ***What if instead our class sells 61 tickets per day. How many tickets would we sell in 7 days?***

S: ***427. If we sell one more ticket per day, we would sell a total of 7 more tickets. 420 + 7 = 427.***

S: ***The red arrow is for +7.***

Extend the arrow picture and solve the suggested problems in a similar manner. Refer to the students' stories as often as useful. Point out that the labels for the blue arrows and the red arrows vary. (Answers are in parentheses and in boxes.)



T: ***We've used patterns to determine that 490 ÷ 7 = 70? Is that true?***

S: ***Yes. 7 x 70 = 490, so 490 ÷ 7 = 70.***

S: ***Yes, 49 ÷ 7 = 7, so 490 ÷ 7 = 70.***

Exercise 2

Begin this table on the board.

Numbers of Bottles	Bottles in One Carton	Number of Filled Cartons	Extra Bottles
98	6		

Recall the story about Nabu working in a bottle recycling plant.

T: ***Nabu must pack 98 bottles into cartons. Each carton holds 6 bottles. Can he fill 10 boxes? 20? 30?***

S: ***He can fill between 10 and 20 cartons, since 10 x 6 = 60 and 20 x 6 = 120.***

T: ***If Nabu fills 10 cartons, how many bottles will be left over?***

S: ***38; 98 – 60 = 38.***

S: ***He can fill 6 more cartons with those 38 bottles, since 6 x 6 = 36. There will be 2 extra bottles.***

Add this information to the table.

Numbers of Bottles	Bottles in One Carton	Number of Filled Cartons	Extra Bottles
98	6	16	2
103	8	$\boxed{12}$	$\boxed{7}$
165	7	$\boxed{23}$	$\boxed{4}$
578	60	$\boxed{9}$	$\boxed{38}$
627	10	$\boxed{62}$	$\boxed{7}$
3 548	100	$\boxed{35}$	$\boxed{48}$

Continue in a similar manner with the problems suggested in this table. Feel free to adjust the level of difficulty. When appropriate, encourage students to use estimation as in above example. (Answers are in boxes.)

Exercise 3

Write this problem on the board.

$26\overline{)6\,700}$

T: ***Nabu must put 6 700 bottles into cartons. Each carton holds 26 bottles. Let's determine how many cartons Nabu will fill.***

Solve the problem in a manner similar to that in Lesson N13. As the class solves the problem, encourage students to follow these steps.

1. Try multiples of powers of 10 (10; 20; …100; 200; …; 1 000; …) for an estimate of the number of cartons.
2. Make several estimates as necessary.
3. Continue until the number of bottles left is less than 26.

At each step, refer to the story about Nabu packing cartons. Do not insist that students always find best estimates. For example, in this sample solution, the teacher allows students to consider 200 cartons, 20 cartons, 20 cartons, then 10 cartons, instead of insisting on first using 200 and then 50 cartons immediately.

26) 6 700	
−5 200	200
1 500	
−520	20
980	
−520	20
460	
−260	10
200	
−130	5
70	
−52	2
18	

T: ***How many cartons does Nabu fill?***

S: ***257; 200 + 20 + 20 + 10 + 5 + 2 = 257.***

T: ***How many bottles are left over?***

S: ***18.***

Write the answer on the board above the problem.

```
     257 R =18
26)6 700
```

Distribute students' copies of the *Division Problems* Booklet and allow 10–15 minutes for individual work. Encourage students to correct errors and to complete unfinished pages before starting new pages. At the end of the lesson, collect the booklets, check them, and have them ready for use in Lesson N33. An answer key for the *Division Problems* Booklet follows Lesson N13.

Home Activity

This is a good time to send a letter to parents/guardians about a division algorithm. Blackline N21 has a sample letter.

Capsule Lesson Summary

Add fractions using the Sara-and-Amelia method of cutting cakes. Find combinations of fractions with a given sum when the numbers involved are limited by the circumstances of the situation in which they are called for.

Materials

Teacher
- Overhead projector (optional)
- Meter stick
- Colored markers
- Blacklines N22(a), (b), and (c)

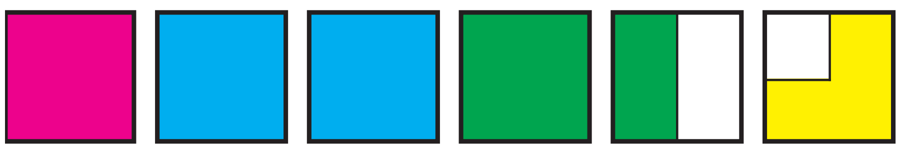
Student
- Metric ruler
- Colored pencils, pens, or crayons
- Worksheets N22*, **, and ***

Advance Preparation: Use Blacklines N22(a), (b), and (c) to make the transparencies for display during Exercise 3, or otherwise prepare the squares for display.

Description of Lesson

Exercise 1

Draw 6 squares on the board. Invite students to color 1 square red, 2 squares blue, 1½ squares green, and ¾ of a square yellow. Ask students to use white to complete the shading of the 6 squares. One possible way of doing the colorings is shown below.



As you ask the following questions, record the appropriate number sentences on the board.

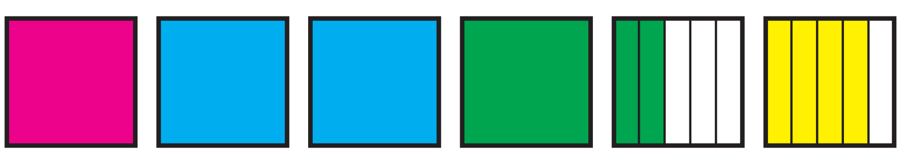
T: ***In terms of squares this size*** (trace the red square)***, how many squares are colored red or blue?*** (3 squares) ***Blue or green?*** (3½ squares) ***Red or yellow?*** (1¾ squares) ***White?*** (¾ square)

How do the white pieces compare to the yellow piece(s)?

S: ***They are the same (in area).***

$$1 + 2 = 3$$
$$2 + 1\tfrac{1}{2} = 3\tfrac{1}{2}$$
$$1 + \tfrac{3}{4} = 1\tfrac{3}{4}$$
$$\tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}$$

Repeat the activity, coloring 1 square red, 2 squares blue, 1⅖ squares green, and ⅘ of a square yellow.



$2 + 1\tfrac{2}{5} = 3\tfrac{2}{5}$ (blue or green)

$1 + \tfrac{4}{5} = 1\tfrac{4}{5}$ (red or yellow)

$\tfrac{3}{5} + \tfrac{1}{5} = \tfrac{4}{5}$ (white)

N22

Exercise 2

Pose the problem ⅖ + ¾. Let students suggest how to do the calculation. You may find that someone suggests adding numerators and denominators. Point out that it doesn't work that way, perhaps with examples such as these:

$\frac{1}{2} \times \frac{1}{2} \neq \frac{2}{4} = \frac{1}{2}$ or $\frac{1}{2} \times \frac{1}{4} \neq \frac{2}{6} = \frac{1}{3} < \frac{1}{2}$

$(\frac{1}{2} \times \frac{1}{2} = 1)$ $(\frac{1}{2} \times \frac{1}{4} = \frac{3}{4})$

Draw two squares of the same size on the board.

T: ***Do you remember Sara and Amelia?***

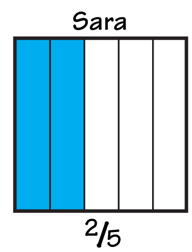
S: ***Yes; they buy cake to treat their families.***

S: ***The bakery that they go to sells cakes all the same size and will sell parts of a cake.***

S: ***They use squares to figure out how much cake they want altogether.***

T: ***How would Sara show that she wants ⅖ of a cake?***

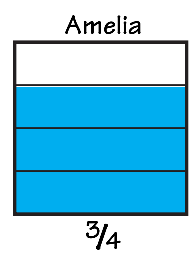
S: ***She would draw a square, divide it into five columns all the same size, and then color two of them.***



Illustrate on the board, using a meter stick to measure.

T: ***How would Amelia show that she wants ¾ of a cake?***

S: ***She would draw a square, divide it into four rows all the same size, and then color three of them.***



Illustrate on the board, using a meter stick to measure.

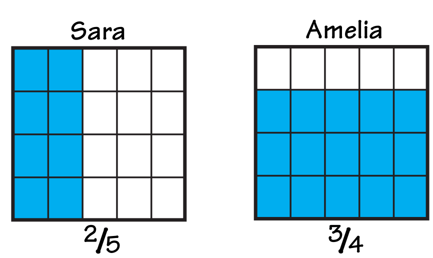
T: ***If we combine Sara's amount of cake and Amelia's amount, will there be more or less than one cake?***

S: ***It looks like a little more.***

T: ***Let's find out for sure. How do we use the squares?***

S: ***Make Amelia's cuts on Sara's square, and make Sara's cuts on Amelia's square.***

Indicate the cuts in the pictures on the board.



T: ***Why does this help?***

S: ***All of the pieces are the same size.***

T: ***How many pieces does each cake have now?***

S: ***20.***

Solicit and record equivalences for ⅖ and ¾ in twentieths.

$\frac{2}{5} = \frac{8}{20}$ $\frac{3}{4} = \frac{15}{20}$

T: ***Will one cake be enough to fill both orders?***

S: ***No,*** $\frac{8}{20} + \frac{15}{20} = \frac{23}{20}$. $\frac{20}{20} = 1$, ***so*** $\frac{23}{20} = 1\frac{3}{20}$.

Record the calculation on the board. $\frac{2}{5} \times \frac{3}{4} = \frac{8}{20} \times \frac{15}{20} = \frac{23}{20} = 1\frac{3}{20}$

Worksheets N22*, **, and *** are available for individual practice, using this method to add fractions with unlike denominators. Notice that the second problem on each worksheet can be done by looking at the regions in the squares left unshaded.

Exercise 3

T: ***One day Amelia and Sara draw squares for cakes, make cuts in their own ways, add shading to show how much cake they want, and then make each other's cuts. Each square ends up in 30 small pieces. How could they have divided the squares?***

Record possibilities in a table on the board.

Columns	Rows
6	5
10	3
2	15
1	30
5	6
3	10
15	2
30	1

S: ***Into 6 columns and 5 rows. 6 x 5 = 30.***

S: ***Into 10 columns and 3 rows. 10 x 3 = 30.***

S: ***Into 2 columns and 15 rows. 2 x 15 = 30.***

S: ***Into 1 column and 30 rows. 1 x 30 = 30.***

T: ***What would it mean to have 1 column or 1 row?***

S: ***One of the girls wants a whole cake.***

S: ***We could switch the number of columns and the number of rows in each case.***

T: ***When they finish, Sara and Amelia have shaded 17 pieces altogether. What fraction of a cake could each of them have wanted originally?***

Accept different answers and let students comment.

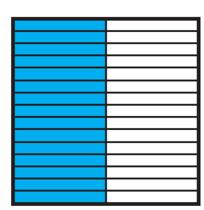
T: ***Let's try one combination—let's try 2 columns and 15 rows.***

Project a transparency made from Blackline N22, or otherwise display square regions divided as shown here.

T: ***If Sara divides her square into two columns, how much of a cake does she want?***

S: ***One-half; because if she wants all or none of a cake, she doesn't cut it.***

Shade one column in the picture of Sara's cake.



$\frac{1}{2} = \frac{15}{30}$

T: ***One-half is how many thirtieths?***

S: ***15.***

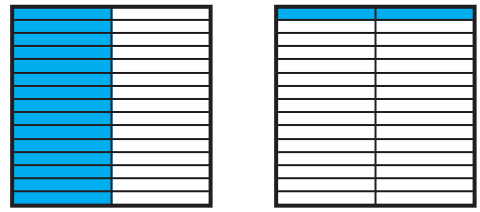
N22

T: ***Amelia divides her square into 15 rows. How many rows (fifteenths) should we shade on Amelia's cake to get a total of 17 pieces (thirtieths) shaded altogether?***

Write this number sentence on the board.

$$\frac{1}{2} \times \frac{?}{15} = \frac{17}{30}$$

S: ***One row ($\frac{1}{15}$), because $\frac{1}{2} = \frac{15}{30}$ and $\frac{1}{15} = \frac{2}{30}$; $\frac{15}{30} + \frac{2}{30} = \frac{17}{30}$.***

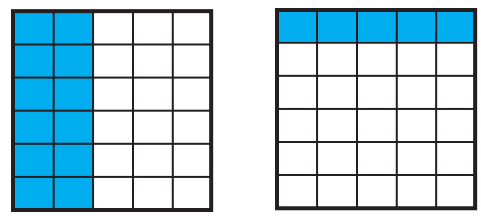


Record the solution on the board.

$$\frac{1}{2} \times \frac{1}{15} = \frac{15}{30} \times \frac{2}{30} = \frac{17}{20}$$

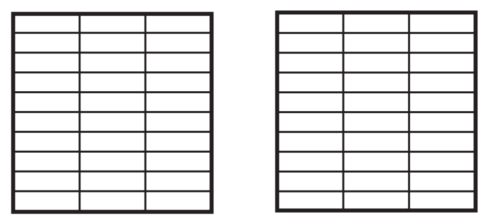
Try other combinations of rows and columns. Use the square regions pictured on Blacklines N22(b) and (c) and let the class experiment until they find a right combination. Often an incorrect attempt leads quickly to a correct solution.

The 5-by-6 case involves finding the combination of groups of five and groups of six that make 17. (2 groups of six plus 1 group of five make 17; $\frac{2}{5} + \frac{1}{6} = \frac{17}{30}$.)

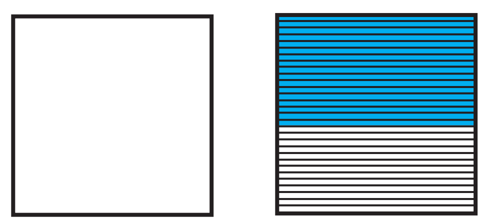


$\frac{2}{5}$ $\frac{1}{6}$

The 3-by-10 case has no solution. There is no combination of groups of three and groups of ten that makes 17.



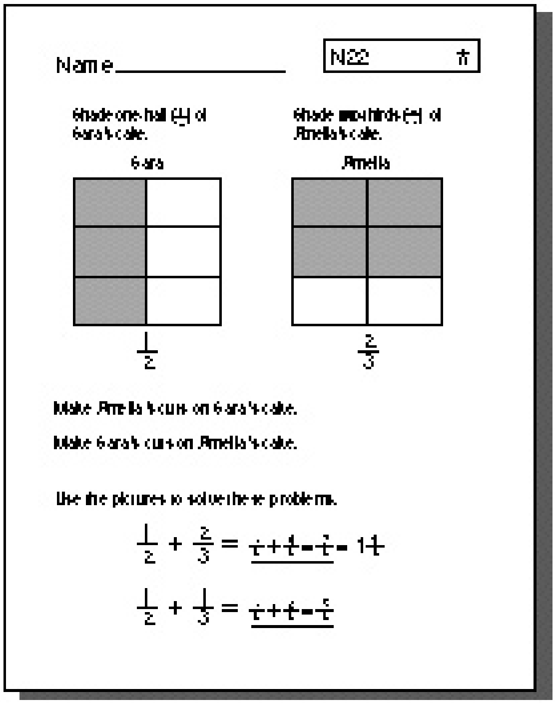
The 1-by-30 case results in one girl wanting no cake ($\frac{17}{30} < 1$) and the other girl wanting $\frac{17}{30}$.



0 $\frac{17}{30}$

Home Activity

Let students take home the worksheet to explain Sara's and Amelia's method for adding fractions to parents or other family members. Suggest that parents (guardians) create another problem involving addition of fractions to do with their child using this method.



Name ________________

N22 ☆

Shade one-half ($\frac{1}{2}$) of Sara's cake.

Shade two-thirds ($\frac{2}{3}$) of Amelia's cake.

Sara

$\frac{1}{2}$

Amelia

$\frac{2}{3}$

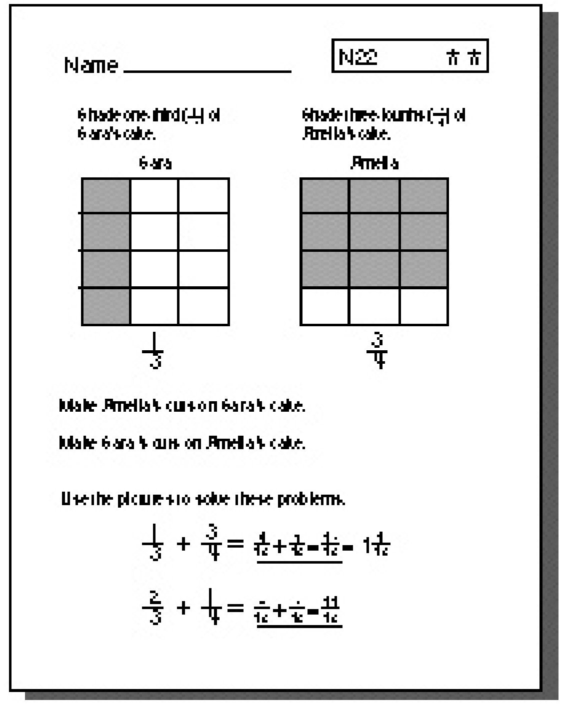
Make Amelia's cuts on Sara's cake.

Make Sara's cuts on Amelia's cake.

Use the pictures to solve these problems.

$\frac{1}{2} + \frac{2}{3} =$ [illegible]

$\frac{1}{2} + \frac{1}{3} =$ [illegible]



Name ________________

N22 ☆☆

Shade one-third ($\frac{1}{3}$) of Sara's cake.

Shade three-fourths ($\frac{3}{4}$) of Amelia's cake.

Sara

$\frac{1}{3}$

Amelia

$\frac{3}{4}$

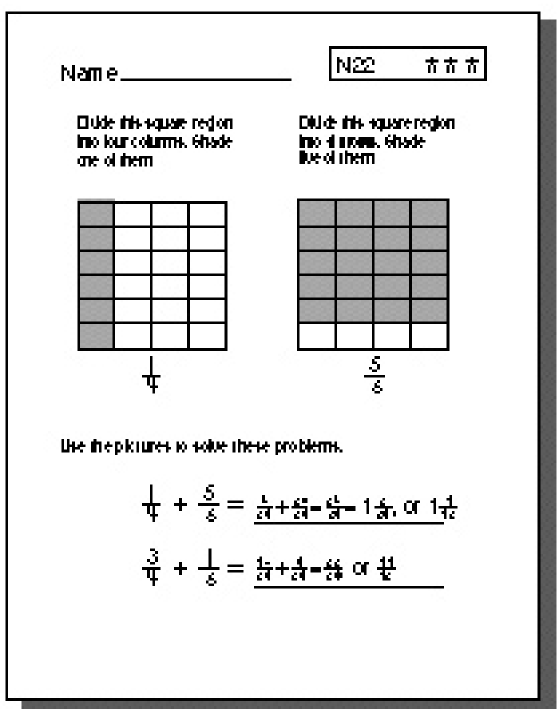
Make Amelia's cuts on Sara's cake.

Make Sara's cuts on Amelia's cake.

Use the pictures to solve these problems.

$\frac{1}{3} + \frac{3}{4} = \frac{4}{12} +$ [illegible] $= 1\frac{1}{12}$

$\frac{2}{3} + \frac{1}{4} =$ [illegible] $= \frac{11}{12}$



Name ________________

N22 ☆☆☆

Divide this square region into four columns. Shade one of them.

Divide this square region into 6 rows. Shade five of them.

$\frac{1}{4}$

$\frac{5}{6}$

Use the pictures to solve these problems.

$\frac{1}{4} + \frac{5}{6} =$ [illegible] or $1\frac{1}{12}$

$\frac{3}{4} + \frac{1}{6} =$ [illegible] or $\frac{11}{12}$

N23 PURCHASE VECTOR SPACE #1

Capsule Lesson Summary

Let students estimate their weights† in kilograms. Represent purchases of apples and peaches on a Cartesian grid. Plot points for possible $4.00 purchases and observe that they lie on a straight line.

Materials

Teacher

- *IG-IV World of Numbers Poster #1*
- Kilogram weight
- Meter stick
- Translator (optional)
- Colored markers
- Metric scale (optional)
- Tape

Student

- Worksheets N23(a) and (b)

Advance Preparation: Either locate a kilogram weight or fill a box with enough material that it weighs one kilogram (approximately 2.2 pounds). If available, borrow a translator from a fourth grade *CSMP* classroom for use in Exercise 2.

Description of Lesson

Exercise 1

Note: You may prefer to omit this exercise and begin with Exercise 2 using pounds rather than kilograms. In this case change kilograms to pounds on the poster.

Show the kilogram weight to the class, and pass it around so that students can get a feeling for the weight of one kilogram.

T: ***A newborn infant weighs on the average about 3.4 kilograms. How many kilograms do you weigh?***

S: ***About 30 kilograms.***

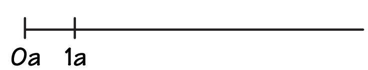
Let several students respond. You may wish to let students weigh themselves on a metric scale.

Exercise 2

Introduce the fruit market as a context for this lesson.

T: ***The fruit market that I go to sells apples by the kilogram. Each time I go, I record my purchase of apples on a line.***

Draw this line segment on the board.



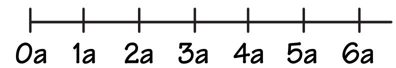
†In the metric system, a *kilogram* is a unit of mass and a *newton* is a unit of force or weight. To avoid the need in these lessons to distinguis□
to refer to mass.

N23

T: ***If I buy no apples, I draw a dot here*** (point to the mark for 0a)***. If I buy one kilogram of apples, I draw a dot here*** (point to the mark for 1a)***. Where should I draw a dot for a purchase of five kilograms of apples?***

S: ***Make a mark for 5a to the right of 1a so that the distance from 0a to 5a is five times the distance from 0a to 1a.***

Use a meter stick to accurately locate the mark for 5a. Similarly, draw marks for 2a, 3a, 4a, and 6a.



Invite students to estimate the location of dots for these purchases: 2.5a, 1.2a, and 4.7a.

T: ***I also buy peaches by the kilogram at this market. How could I record purchases of peaches?***

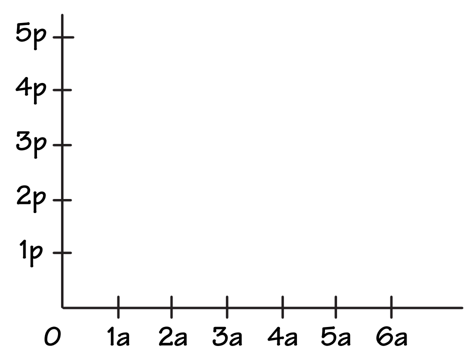
Listen to students' suggestions. Perhaps someone will suggest drawing another line for purchases of peaches.

T: ***I would like a method that would allow me to draw just one dot to represent a purchase of both apples and peaches.***

Let the class anticipate the method and perhaps conjecture what it might be.

T: ***Let's draw two lines perpendicular to one another, one for apples and one for peaches.***

Invite students to help you make a square corner, and then to mark the peaches line.



T: ***Where could we draw a dot to represent a purchase of five kilograms of apples and three kilograms of peaches?***

S: ***Draw a dot above 5a and to the right of 3p.***

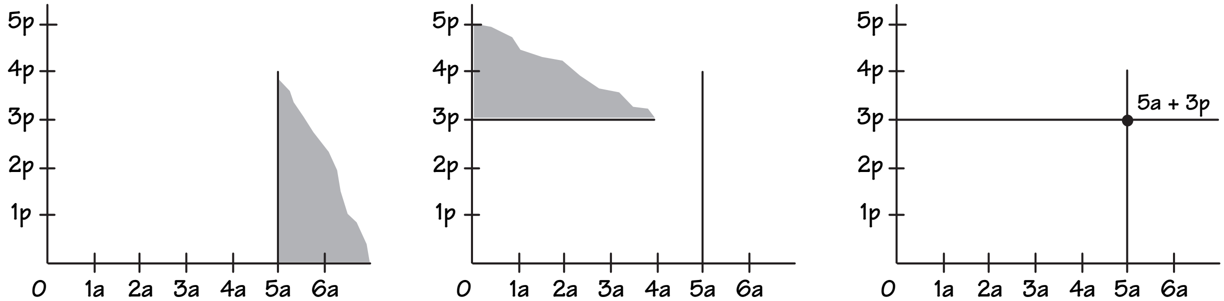
Let a student locate the dot (approximately).

Invite students to locate (approximately) dots for the purchases 4a + 2p and 2a + 4p.

T: ***You've estimated the location of these dots quite well. But how could we locate the dots more accurately?***

Let students explain their methods. Encourage the use of a variety of tools, for example, a meter stick, a square corner, a translator, and a compass. If students suggest a method using a meter stick, insist that they address the problem of making sure that the meter stick is precisely horizontal or vertical. Suggest or lead to the following methods involving a square corner and a translator to locate, for example, the dot for 5a + 3p.

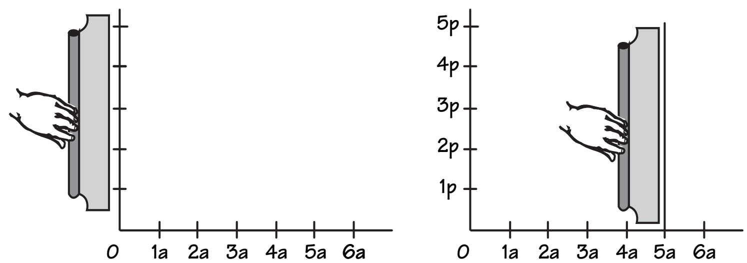
Square Corner



1. Draw a vertical line at 5a.
2. Draw a horizontal line at 3p.
3. Extend the lines so that they intersect, and draw a dot for 5a + 3p.

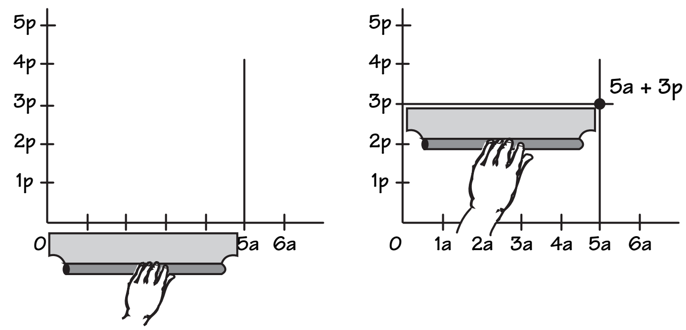
Translator

1. Draw a vertical line at 5a, parallel to the peach line.



2. Draw a horizontal line at 3p, parallel to the apple line.

3. Extend the lines so that they intersect and draw a dot for 5a + 3p.



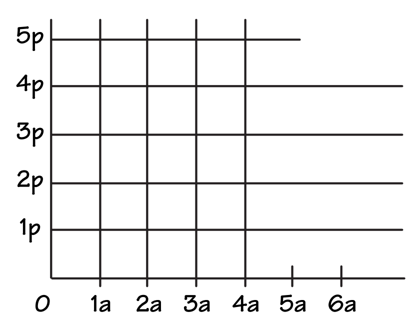
Let students use either a square corner or the translator to

- check the location of the dots for 4a + 2p and 2a + 4p;
- locate dots for 1a + 3p, 6a + 5p, and 3.5a + 4p; and
- identify purchases for several dots you draw on the graph.

T: ***Instead of using the square corner or the translator to locate each dot, we could use one of these tools to draw grid lines. Then we could use the grid lines to locate dots or to identify purchases.***

Use either the square corner or the translator to demonstrate how to draw a few grid lines in each direction.

Distribute copies of Worksheets N23(a).



T: ***Use the grid lines to label the dots on the graph. Then draw and label a dot for each purchase listed at the bottom of the worksheet.***

While students work, display *IG-IV World of Numbers Poster* #1. After about five minutes, discuss any difficult problems on the worksheet. In this discussion, you may refer to the poster but do not write on it.

T: ***My last several purchases were all for two kilograms of apples and varying amounts of peaches. Where could the dots for these purchases be?***

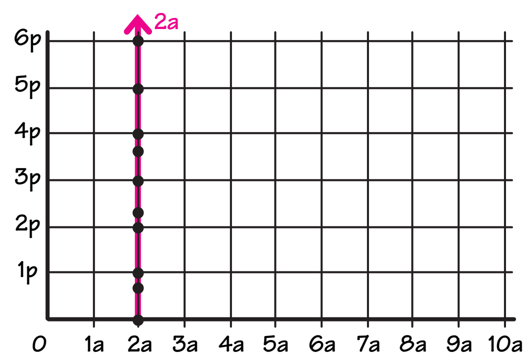
Invite students to draw and identify dots for several such purchases, for example, 2a + 3.5p or 2a + 0.75p. Include exactly two kilograms of apples, 2a + 0p, as a possible purchase.

T: ***Where will the dots for all such purchases lie?***

S: ***On the vertical line through 2a.***

Draw a red vertical line at 2a.

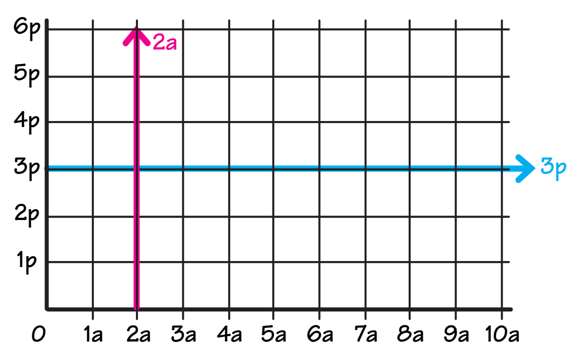
T: ***We'll call this the "two-apple" or "2a" line.***



Similarly, ask students to draw dots for purchases involving exactly three kilograms of peaches, and draw a 3p line in blue.

T: ***In what part of this grid would dots for purchases of many kilograms of peaches with few kilograms of apples lie?***

S: ***In the upper left.***



Similarly, ask students to identify the region of the grid representing purchases of many kilograms of both peaches and apples (upper right), and of many kilograms of apples but few kilograms of peaches (lower right).

Exercise 3

Write this information on the board.

$p(1a) = 40¢ \qquad p(1p) = 80¢$

T: ***At this market, one kilogram of apples costs 40¢ and one kilogram of peaches costs 80¢. How much do three kilograms of apples cost?***

S: ***\$1.20; 3 x 0.40 = 1.20.***

$p\,(3a) = \boxed{\$1.20}$

Ask for the cost of several more single item purchases. (Answers are in boxes.)

$(6p) = \boxed{\$4.80} \qquad p\,(3.5a) = \boxed{\$1.40} \qquad p\,(4.5p) = \boxed{\$3.60}$

T: ***What is the price of the purchase of three kilograms of apples and one kilogram of peaches?***

S: ***\$2.00; 1.20 (3 x 0.40) + 0.80 = 2.00.***

$p\,(3a + 1p) = \boxed{\$2.00}$

Ask for the cost of several more two item purchases. (Answers are in boxes.)

$(1a + 3p) = \boxed{\$2.80} \qquad p\,(2a + 0.5p) = \boxed{\$1.20}$

$(3a + 3p) = \boxed{\$3.60} \qquad p\,(1.5a + 2.5p) = \boxed{\$2.60}$

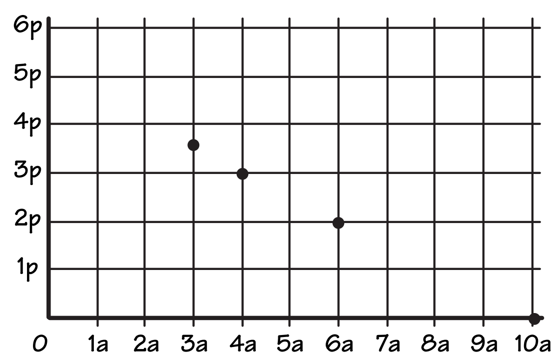
T: ***One day my friend goes to the market and spends exactly \$4.00 on apples and peaches. What combination could he have bought?***

S: ***Four kilograms of apples and three kilograms of peaches. (4 x 0.40) + (3 x 0.80) = 4.00.***

S: ***Six kilograms of apples and two kilograms of peaches; the double of the \$2.00 purchase, p (3a + 1p) = \$2.00.***

Begin a list of \$4.00 purchases. Draw a dot on the poster for each \$4.00 purchase. Accept three or four answers from the students. For example:

\$4.00
4a + 3p
6a + 2p
10a + 0p
3a + 3.5p



Direct students to turn to Worksheet N23(b).

T: ***Copy the \$4.00 purchases from the board and poster onto your worksheet. Find at least three more \$4.00 purchases. List them and draw dots on the grid.***

N23

Let students work independently or with partners for awhile. Encourage some students to find solutions without whole numbers of apples or peaches, for example, 1.5a or 3.5p. Occasionally give hints, for example, write an open sentence on the board and say,

T: ***Lisa spent $4.00 on peaches and apples. She bought 2 kilograms of peaches. How many kilograms of apples did she buy?***

p (2a × ___p) = $4.00

You may like to write this problem on the board for students who finish quickly.

List at least six $2.80 purchases.

On Worksheet N23(b), draw dots for each $2.80 purchase.

With approximately five minutes remaining in the class period, ask students for additional $4.00 purchases to list on the board and to draw dots for on the poster.

T: ***What do you notice about these dots?***

S: ***They all lie on a line.***

Draw a line in red through these dots.

$4.00
4a + 3p
6a + 2p
10a + 0p
3a + 3.5p
2a + 4p
8a + 1p
0a + 5p
6.5a + 1.75p

T: ***Does the red line segment suggest other $4.00 purchases?***

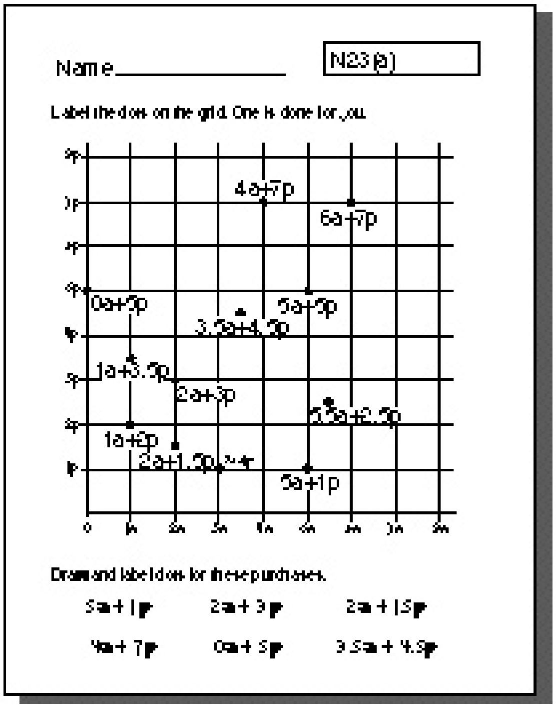
S: ***Any dot on the red line is a $4.00 purchase.***

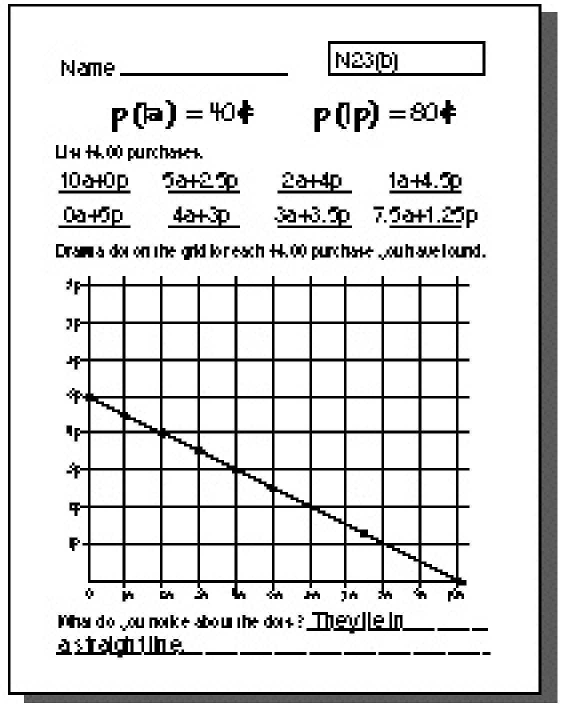
Check one or two examples with the class by

- drawing a dot on the red line segment;
- identifying the purchase it represents, for example, 7a + 1.5p; and
- confirming that it is a $4.00 purchase, for example, (7 x 0.40) + (1.5 x 0.80) = 4.00.

Conclude that all of the dots for $4.00 purchases lie on the red line segment, and all of the dots on the red line segment represent $4.00 purchases.

If time allows and if some students did the corresponding problem for $2.80 purchases, let them announce their results to the class. They should notice that the dots for $2.80 purchases not only lie on a line segment, but that the $2.80 line segment is parallel to the $4.00 line segment.


Name
N23(a)
Label the dots on the grid. One is done for you.
4a+7p
6a+7p
0a+5p
5a+5p
3.5a+4.5p
1a+3.5p
2a+3p
5.5a+2.5p
1a+2p
2a+1.5p
5a+1p
Draw and label dots for these purchases.
5a+1p
2a+3p
2a+1.5p
4a+7p
0a+5p
3.5a+4.5p


Name
N23(b)
p(1a) = 40¢
p(1p) = 80¢
List $4.00 purchases.
10a+0p
5a+2.5p
2a+4p
1a+4.5p
0a+5p
4a+3p
3a+3.5p
7.5a+1.25p
Draw a dot on the grid for each $4.00 purchase you have found.
What do you notice about the dots? They lie in a straight line.

N24 PURCHASE VECTOR SPACE #2

Capsule Lesson Summary

Review the representation of purchases of apples and peaches on a Cartesian grid. Plot points for purchases satisfying certain conditions and discover that they lie on a line. Extend the Cartesian grid to include negatives by allowing the return of apples and peaches.

Materials

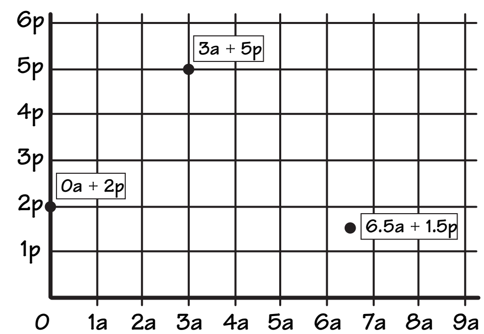
Teacher
- *IG-IV World of Numbers Posters* #1 and #2
- Meter stick
- Colored markers
- Tape

Student
- Colored pencils
- Worksheets N24(a), (b), and (c)

Description of Lesson

Exercise 1

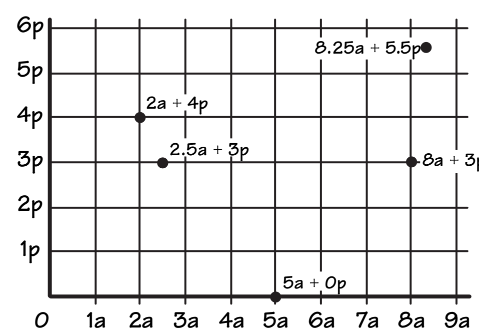
Display a clean copy of *IG-IV World of Numbers Poster* #1.



Invite students to recall the story about purchasing apples and peaches at a fruit market. Draw several dots on the poster, and ask students to identify the purchases and to label the dots. (Example answers are in boxes.)

Distribute copies of Worksheets N24(a), and instruct students to locate and label dots for the purchases listed on the worksheet.

After a few minutes, invite students to draw and label the dot for each purchase on the poster.



Next, invite students to draw dots on the poster for purchases such as:.

T: ***Come and draw dots for purchases that include exactly five kilograms of peaches.***

Continue until they suggest purchases involving decimals, for example, 3.5a + 5p.

T: ***Where do all of these dots seem to lie?***

S: ***On a horizontal line crossing the peach line at 5p.***

Invite a student to draw the horizontal line. (See the next illustration.)

T: ***A friend of mine always buys the same quantity (weight) of apples as she buys of peaches. What could her purchase be?***

S: ***Three kilograms of apples and three kilograms of peaches.***

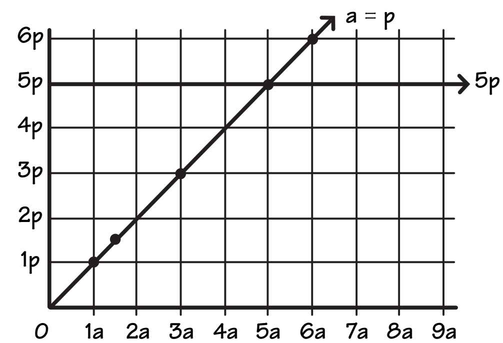
T: ***Yes. Draw a dot on your worksheet for 3a + 3p. Also draw dots for many other purchases that she might make.***

As students are working, invite several to draw dots on the poster for these purchases.

T: ***What do you notice about these dots?***

S: ***They all lie on a diagonal line.***

Invite a student to draw the line.



Write this information on the board.

$p(1a) = 40¢$ $\quad$ $p(1p) = 80¢$

T: ***How many kilograms of apples and peaches could you buy for $2.40?***

S: ***Four kilograms of apples and one kilogram of peaches. (4*** x ***0.40) + 0.80 = 2.40.***

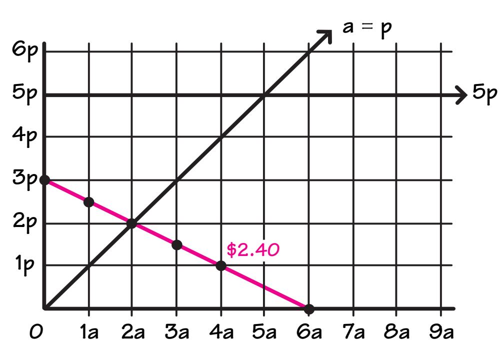
Begin a list of $2.40 purchases on the board, and invite a student to draw a dot for this purchase on the poster.

T: ***On your worksheet, list a few more $2.40 purchases and draw a dot on the graph for each one. What do you think will happen?***

S: ***I think that the dots for $2.40 purchases lie on a line, since last time the dots for $4.00 purchases were on a line.***

After a while, invite students to list their $2.40 purchases on the board and to draw dots on the poster for them. Once there are a sufficient number of dots in the picture to observe that they lie on a line, draw the appropriate line segment. For example:

$2.40
4a + 1p
2a + 2p
0a + 3p
6a + 0p
3a + 1.5p
1a + 2.5p



T: ***How can we use this line segment to find more $2.40 purchases?***

Call on a student to draw a new dot on the segment, for example, at 5a + 0.5p.

T: ***Let's check that this dot is for a \$2.40 purchase. What purchase is it for?***

S: ***5a + 0.5p. 5a costs \$2.00 and 0.5p costs 40¢, so it is a \$2.40 purchase.***

Select an amount of apples not yet considered, for example, 3.5 kilograms. Write this open sentence on the board.

p (3.5a × ___p) = \$2.40

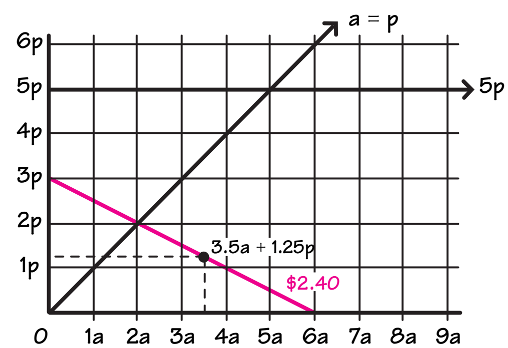
T: ***If we bought 3.5 kilograms of apples and some quantity of peaches and spent a total of \$2.40, how many kilograms of peaches would we have bought?***

S: ***1.25 kilograms of peaches.***

Encourage both a numerical and a graphical solution.

S: ***3.5 kilograms of apples cost \$1.40. For the other \$1.00, you can buy 1¼ or 1.25 kilograms of peaches.***

S: ***On the grid, 3.5a + 1.25p is on the \$2.40-line.***



Exercise 2

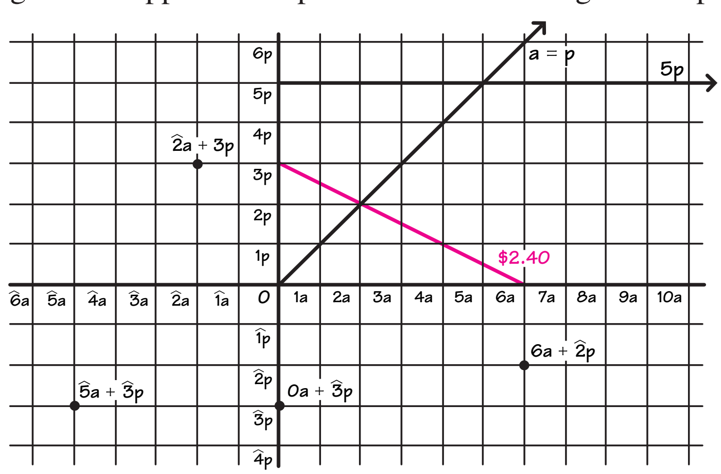
Extend the story context as follows.

T: ***The fruit market allows people to bring good fruit either to return or to trade. How could we change the picture to show a return of two kilograms of apples?***

Let students discuss the question. Lead to the use of negative numbers and the extension of the apple line to the left. Indicate that you would read "$\widehat{2}$a" as "a return of two kilograms of apples" rather than "negative two apples."

T: ***How could we show a return of three kilograms of peaches?***

Once students suggest extending the peach line downward, replace the poster on the board with *IG-IV World of Numbers Poster #2*. Invite students to label the dots on the poster. Read "$\widehat{2}$a + 3p" as "a return of two kilograms of apples and a purchase of three kilograms of peaches."



Direct students to turn to Worksheet N24(b) and to locate dots for the given purchases: $3a + 4p$, $3a + \widehat{4}p$, $\widehat{2}a + 1p$, $\widehat{4}a + \widehat{4}p$, and $\widehat{3}a + 0p$. After a couple of minutes, invite students to locate the dots for these purchases on the poster.

T: ***Many people come to this fruit market and make even exchanges of apples for peaches or peaches for apples. For example, how many kilograms of peaches could a person trade for six kilograms of apples?***

S: ***Three kilograms of peaches. Six kilograms of apples are worth \$2.40 (6 x 0.40 = 2.40) and three kilograms of peaches are worth \$2.40 (3 x 0.80 = 2.40).***

Record the information on the board. Then invite students to find several more even exchanges and list them on the board, as illustrated here.

Even Exchanges	
$\widehat{6}a + 3p$	$6a + \widehat{3}p$
$\widehat{10}a + 5p$	$5a + \widehat{2.5}p$
$8a + \widehat{4}p$	$12a + \widehat{6}p$

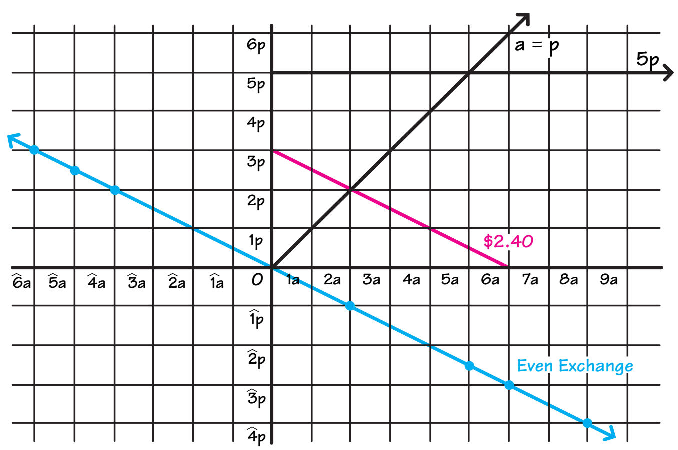
T: ***What patterns do you notice?***

S: ***The number of kilograms of apples is always twice the number of kilograms of peaches.***

S: ***If we double the weight of apples, we must double the weight of peaches to have an even exchange.***

Suggest that students draw dots in blue on the grid on Worksheet N24(b) for the even exchanges listed on the board. Some students may find other even exchanges. During the independent work, invite some students to draw dots for even exchanges on the class poster.

Once students observe that the dots for even exchanges lie on a line, draw a blue line through these dots. For example:



T: ***What do you notice about the line for even exchanges?***

S: ***It's parallel to the red line segment for \$2.40 purchases.***

T: ***What purchases does this line*** (point to the 5p line) ***represent?***

S: ***All purchases that involve exactly five kilograms of peaches.***

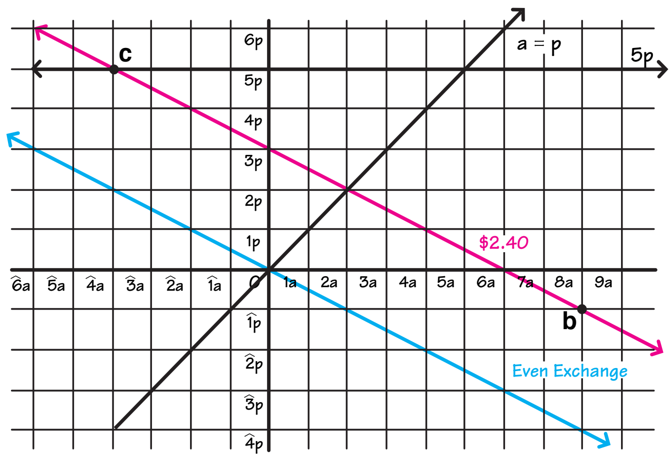
T: ***If we extend this line, should we still label it as the 5p line?***

S: ***Yes, the dots on the new part of the line still represent purchases of five kilograms of peaches. The only difference is that apples are now being returned, not purchased.***

To confirm this observation, ask students to identify a couple of dots on an extension of the 5p line, for example, $\widehat{4}a + 5p$ and $\widehat{2.5}a + 5p$.

Similarly, review that the a = p line represents all purchases of equal weights of apples and peaches. Extend the a = p line and conclude that a point on the extended part of the line represents a return of the same weight of apples and peaches.

Extend the $2.40 line segment in both directions.



Do not label the dots for $8a + \widehat{1}p$ (b) and $\widehat{4}a + 5p$ (c) on the graph. They are labeled here just to make the description of the lesson easier to follow.

T: ***Let's check a few dots on the extended $2.40 line to be sure that they, too, represent $2.40 purchases. What purchase does this dot*** (point to **b**) ***represent?***

S: ***$8a + \widehat{1}p$.***

T: ***Should that purchase cost $2.40?***

S: ***Yes. Eight kilograms of apples cost $3.20, but one kilogram of peaches is worth $0.80. So the customer would have to pay $2.40 (3.20 – 0.80 = 2.40).***

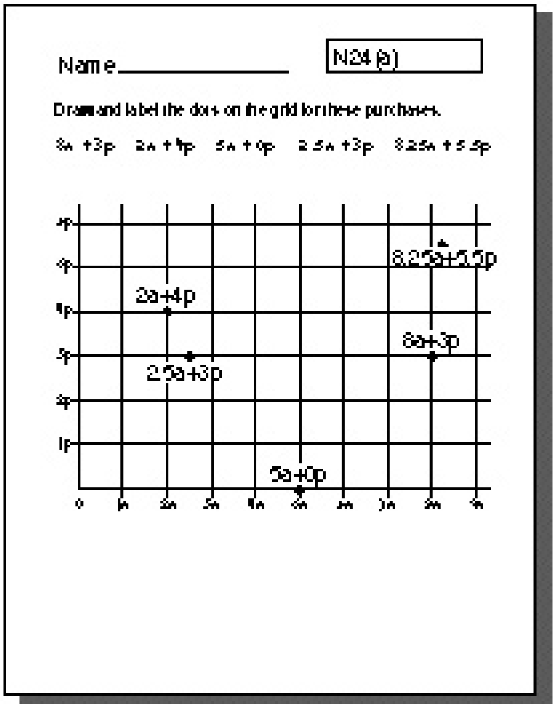
Similarly, confirm that **c** ($\widehat{4}a + 5p$) represents a $2.40 purchase:

$$(5 \times 0.80) - (4 \times 0.40) = 4.00 - 1.60 = 2.40$$

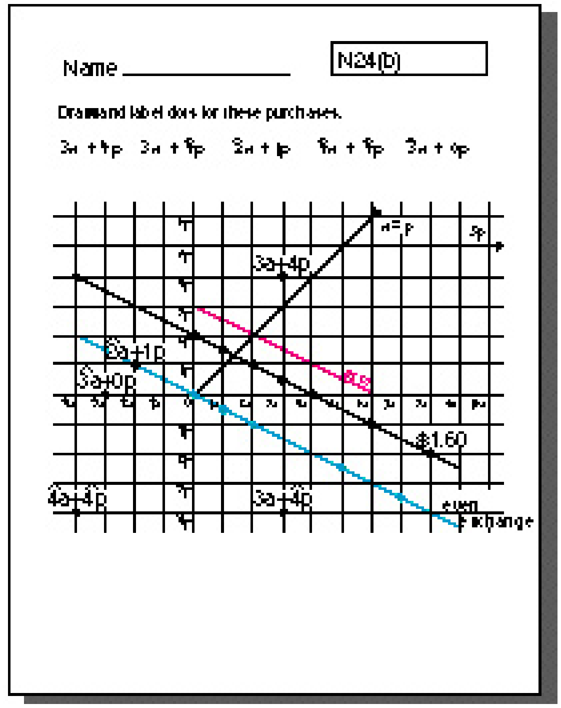
T: ***Use the red line to find some more $2.40 purchases.***

Let students draw and identify dots on the red line, for example, $\widehat{2}a + 4p$, $10a + \widehat{2}p$, and $\widehat{3}a + 4.5p$.

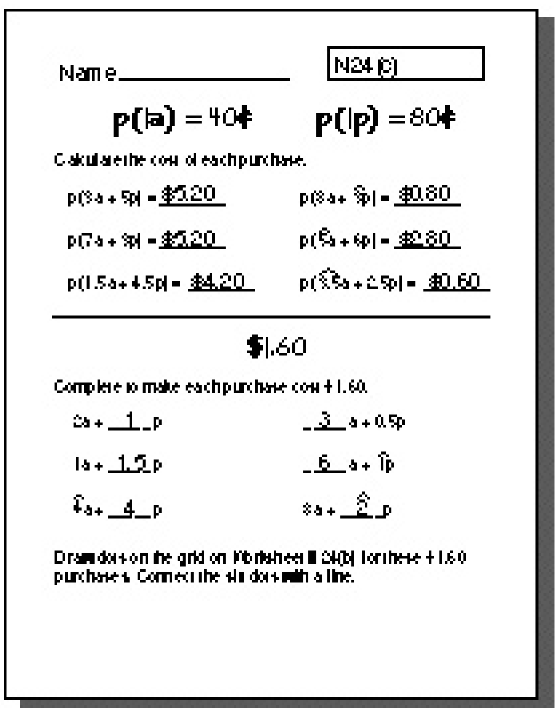
Worksheet N24(c) is available for individual work.


Name ________________ N24(a)

Draw and label the dots on the grid for these purchases.


Name ________________ N24(b)

Draw and label dots for these purchases.


Name ________________ N24(c)

$p(1a) = 40¢$ $p(1p) = 80¢$

Calculate the cost of each purchase.

p(8a + 5p) = $5.20 p(8a + 5p) = $0.80

p(7a + 3p) = $5.20 p(5a + 6p) = $2.80

p(1.5a + 4.5p) = $4.20 p(5.5a + 2.5p) = $0.60

$1.60

Complete to make each purchase cost $1.60.

2a + 1 p 3 a + 0.5p

1a + 1.5 p 6 a + 1p

4a + 4 p 8a + 2 p

Draw dots on the grid on Worksheet N24(b) for these $1.60 purchases. Connect the six dots with a line.

Capsule Lesson Summary

Decide which changes in a given list can be made to a number displayed on the Minicomputer by moving exactly one checker. When there is no one move that will increase or decrease the number on the Minicomputer by a specified amount, find a two move solution.

Materials

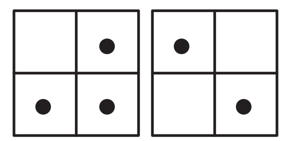
Teacher
- Minicomputer set
- Weighted checker set
- Colored chalk

Student
- Paper

Description of Lesson

Exercise 1

Put this configuration on the Minicomputer.

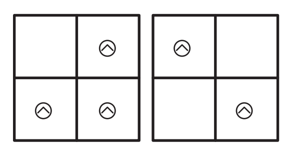


T: ***What number is on the Minicomputer?*** (79)
If we move one checker to another square, by how much can we change the number on the Minicomputer?

Invite a student to move a checker from one square to another on the Minicomputer, and ask another student to announce and record the change. Then return the checker to its original position. Continue in this way until several possible changes are recorded. For example:

+3	−4	−38	−19
−6	+79	+60	+1

Do not erase the list of increases and decreases. Replace each positive checker with a negative checker.



T: ***What number is on the Minicomputer?*** ($\widehat{79}$)
If we move one of these negative checkers to another square, by how much can we change the number on the Minicomputer?

Let students suggest changes until they notice that the possible changes are opposites of those previously found.

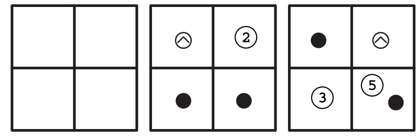
T: ***Earlier we were able to increase the number by 3 (by moving a regular checker from the 1-square to the 4-square). Is there a move now that will decrease the number by 3?***

S: ***Move a negative checker from the 1-square to the 4-square.***

Do a few more examples to suggest this generalization.

Exercise 2

Put this configuration on the Minicomputer.



T: ***What number is on the Minicomputer? Write it on your paper.***

While students are decoding the number, write the following list of increases and decreases on the board.

+30	–1	+15	–3	+40
+5	+24	–64	+17	–15

Check a few answers before asking a student to announce the number. (46)

T: ***Which of these changes can be made by moving exactly one checker from the square it's on to another square?***

Invite students to demonstrate the changes that are possible on the Minicomputer. Circle them in the list as they are found. Always return a checker to its original position before another change is suggested. All of the changes except –1, +17, and –15 are possible.

+30 (Move the regular checker from the 10-square to the 40-square.)

+5 (Move the ⑤-checker from the 1-square to the 2-square.)

+24 (Move the ③-checker from the 2-square to the 10-square.)

+15 (Move the ⑤-checker from the 1-square to the 4-square.)

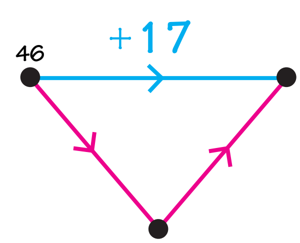
–64 (Move the ②-checker from the 40-square to the 8-square.)

–3 (Move the ③-checker from the 2-square to the 1-square.)

+40 (Move the negative checker from the 80-square to the 40-square.)

T: ***It is not possible to increase this number by 17 by moving just one checker, but can we make the change by moving two checkers?***

Draw this arrow picture on the board.



T: ***What two changes*** (trace the red arrows) ***could we make to add 17?***

S: ***+10 and +7.***

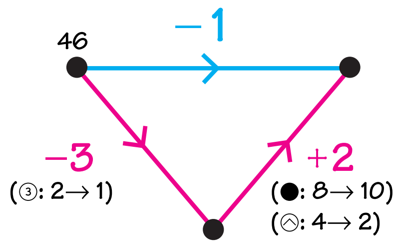
S: ***+19 and –2.***

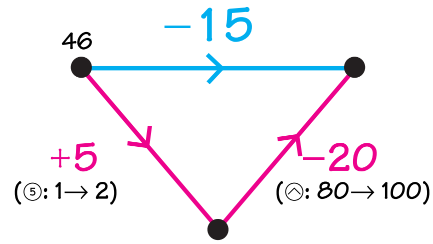
For each suggestion, let students describe how to move the checkers. For example:

S: ***+10 and +7; move the regular checker from the 10-square to the 20-square, and move the regular checker from the 1-square to the 8-square.***

S: ***+19 and –2; move the regular checker from the 1-square to the 20-square, and move the regular checker from the 10-square to the 8-square.***

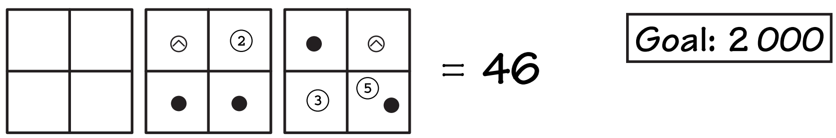
Many solutions are possible. Repeat this activity to decrease by 1 (–1) or decrease by 15 (–15) by moving two checkers. Possible solutions are pictured below.





Exercise 3

Play *Minicomputer Golf*. The following is a possible game (starting configuration for 46) with a goal of 2 000.



Home Activity

If you did not send home a description of *Minicomputer Golf* earlier, you may choose to do so now. See the home activity for Lesson N1.

Capsule Lesson Summary

Recall the graphic representation of purchases of apples and peaches at a fruit market. Determine the type of purchases that lie on a given line. Consider the effect of using a non-rectangular grid.

Materials

Teacher
- Meter stick
- Translator
- *IG-IV World of Numbers Posters* #3 and #4
- Colored markers
- Tape

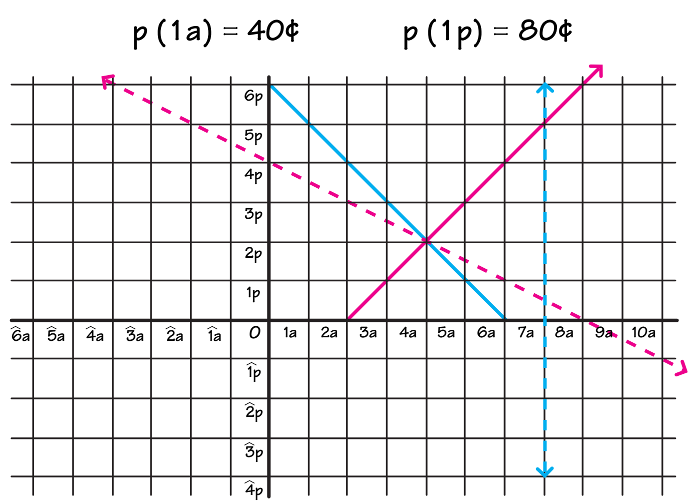
Student
- Colored pencils, pens, or crayons
- Worksheets N26*, **, and ***

Advance Preparation: Borrow a translator from a *CSMP* fourth grade classroom. If this is not possible, skip the translator construction in Exercise 2 and use *IG-IV World of Numbers Poster* #4.

Description of Lesson

Exercise 1

Display *IG-IV World of Numbers Poster* #3. Write the cost information above the poster.



Call the class's attention to the dotted blue line on the right side of the poster.

T: ***Name some purchases whose dots are on this line.***

List the students' responses on the board. Insist that they include purchases that involve negative numbers and decimal numbers, as illustrated below.

$7a \times 4p$ $\qquad$ $7a \times 0p$ $\qquad$ $7a \times \hat{3}p$ $\qquad$ $7a \times \widehat{2.5}p$

T: ***What's common to all of the purchases on the dotted blue line?***

S: ***Each purchase includes exactly seven kilograms of apples.***

T: ***Let's check whether other purchases involving exactly seven kilograms of apples are on the same line. Are these purchases on that line: 7a + 4.8p; 7a + $\hat{2}$p; and 7a + 16p?***

Let students locate dots for these purchases on the dotted blue line or convince you that the dot would lie on a extension of the line.

N26

Consider the solid red line, the solid blue segment, and the dotted red line in a similar manner. A discussion summary is presented below.

Line	Some points on the line		Pattern
Solid Red	2a + 0p 4.5a + 2.5p	7a + 5p 6.25a + 4.25p	For each purchase, there are two more kilograms of apples than peaches.
Solid Blue	0a + 6p 2a + 4p	6a + 0p 4.5a + 1.5p	For each purchase, the total amount of fruit weights six kilograms.
Dotted Red	8a + 0p 10a + 1p 2.5a + 2.75p	2a + 3p 4a + 6p 1a + 4.5p	The cost of each purchase is $3.20

Conclude that many different types of conditions on the purchases result in dots that lie on lines. As a challenge problem, you may suggest that students find other conditions that have the same effect.

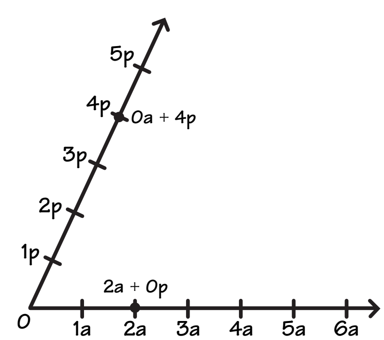
T: ***There is one dot on the graph that is on the red line, on the dotted red line, and on the blue line segment. What purchase does it represent?***

S: ***4a + 2p.***

Exercise 2

Draw an apple line and a peach line at about a 65° angle to one another.

T: ***In another class I asked a student to draw the apple line and peach line for a grid. She didn't have a square corner, so she drew the lines like this*** (point to your drawing)***. I was a bit upset when I saw what she had done, but she insisted that her lines would work. In this picture, where would she draw a dot for the purchase 2a + 0p? 0a + 4p?***



Let students draw dots on the board for these two purchases.

T: ***Where would she draw a dot for the purchase 2a + 4p?***

Let students offer suggestions.

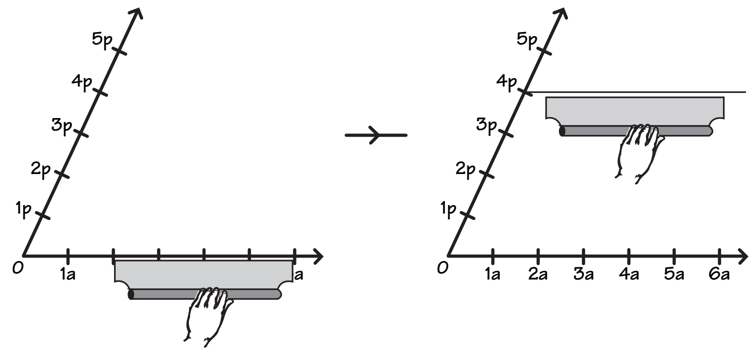
T: ***What tools did we use to locate dots and to draw grid lines when the apple line and the peach line met at a square corner?***

S: ***A translator or a square corner, and a meter stick.***

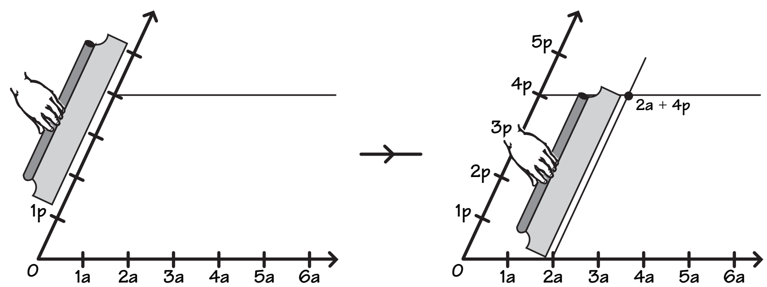
T: ***Let's use a translator to locate the dot for 2a + 4p.***

Let students suggest how to locate the dot. Lead to the following method.

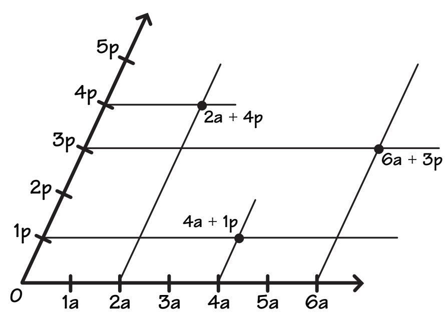
- Draw a horizontal line at 4p, parallel to the apple line.



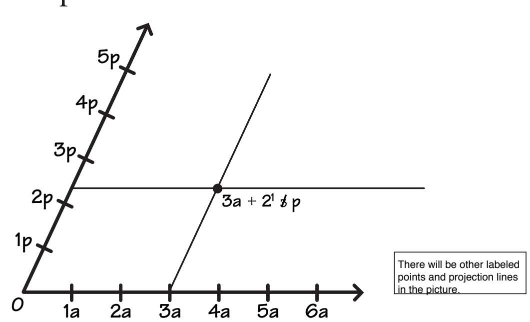
- Draw a line at 2a, parallel to the peach line. Extend the lines, if necessary, until they meet. Draw a dot for 2a + 4p.



Also invite students to use the translator to locate dots for 4a + 1p and for 6a + 3p.



Draw a dot on the grid and ask for the purchase it represents. A student should use a translator, rolling from the apple line to the dot and rolling from the peach line to the dot. Ask the class to estimate the location if the projection onto the apple line or onto the peach line does not fall on one of the graduation marks. For example:



Draw a few more dots and repeat the activity, asking students for the corresponding purchases.

N26

T: ***Again, grid lines are useful for locating and identifying dots. We've used the translator to draw a few grid lines, and we could draw the rest. The pictures on your worksheets will have the grid lines drawn.***

Display *IG-IV World of Numbers Poster* #4 next to #3. Direct students' attention to poster #3.

T: ***On this grid, many conditions on the purchases resulted in dots on a straight line. Do you think that under the same conditions, the dots will also lie on a straight line on this new grid*** (poster #4)***?***

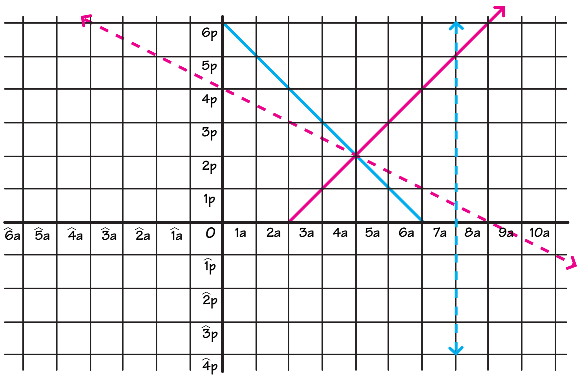
Let students express their opinions.

T: ***This line*** (point to the dotted blue line on poster #3) ***represents all purchases that involve seven kilograms of apples. On this new grid*** (poster #4)***, let's check whether the dots for all purchases that involve seven kilograms of apples lie on a line.***

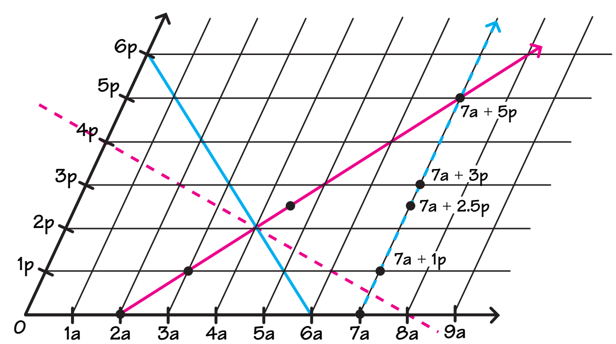
Invite students to draw and identify dots for several such purchases. Conclude that they do lie on a line.

Check the corresponding conditions for one or more of the other lines on poster #3 to find they also describe a line on this new grid.

Poster #3

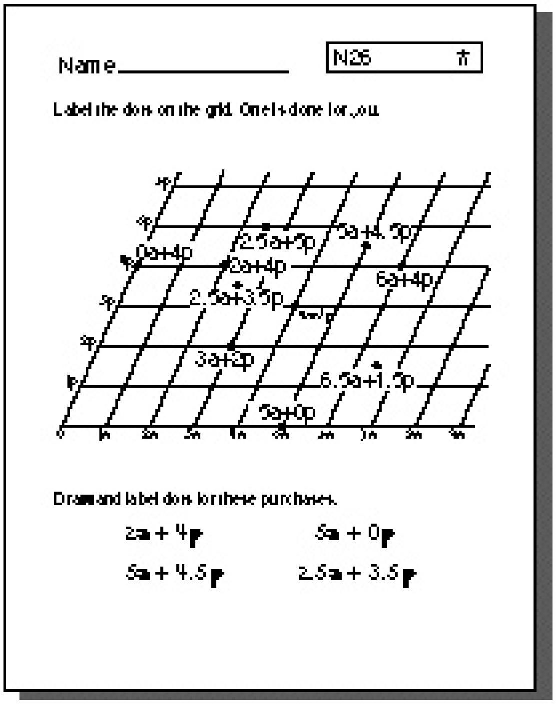


Poster #4



T: ***On the worksheets, check whether other conditions on the purchases result in straight lines on this grid.***

Worksheets N26*, **, and *** are available for individual work.

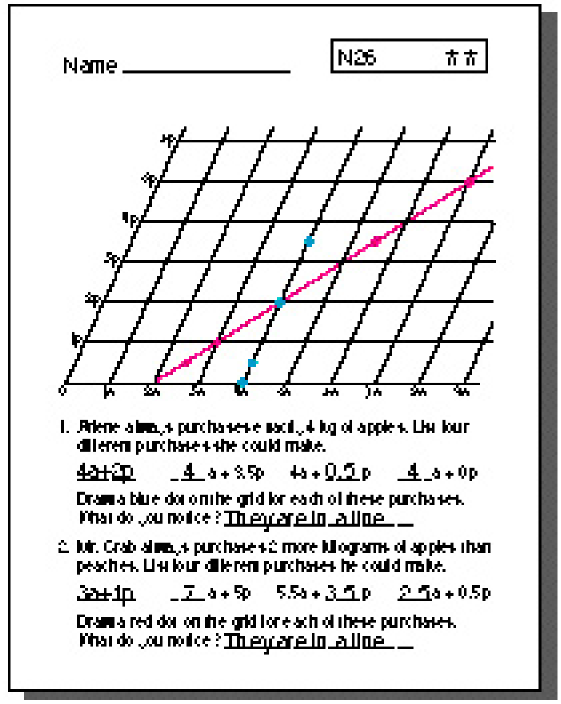

Name ____________ N26 ☆

Label the dots on the grid. One is done for you.

Draw and label dots for these purchases.

2a + 4p   5a + 0p

5a + 4.5p   2.5a + 3.5p


Name ____________ N26 ☆☆

1. Arlene always purchases exactly 4 kg of apples. List four different purchases she could make.

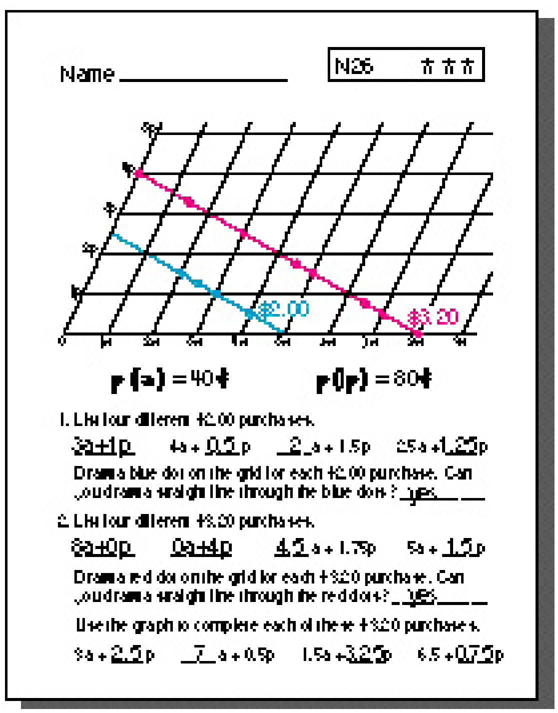
   4a+2p   4 a + 3.5p   4a + 0.5 p   4 a + 0p

   Draw a blue dot on the grid for each of these purchases. What do you notice? They are in a line.

2. Mr. Crab always purchases 2 more kilograms of apples than peaches. List four different purchases he could make.

   3a+1p   7 a + 5p   5.5a + 3.5p   2.5a + 0.5p

   Draw a red dot on the grid for each of these purchases. What do you notice? They are in a line.


Name ____________ N26 ☆☆☆

p(a) = 40¢   p(p) = 80¢

1. List four different $2.00 purchases.

   3a+1p   4a + 0.5 p   2 a + 1.5p   2.5a + 1.25p

   Draw a blue dot on the grid for each $2.00 purchase. Can you draw a straight line through the blue dots? yes

2. List four different $3.20 purchases.

   8a+0p   0a+4p   4.5 a + 1.75p   5a + 1.5p

   Draw a red dot on the grid for each $3.20 purchase. Can you draw a straight line through the red dots? yes

   Use the graph to complete each of these $3.20 purchases.

   3a + 2.5 p   7 a + 0.5p   1.5a + 3.25p   6.5 + 0.75p

Capsule Lesson Summary

Using an arrow picture, solve some problems involving ¼x and ⅙x. Find labels for several pairs of arrows, one ÷□ and one □x, from 48 to 36. Label the composite arrow in different ways, and introduce *percent* as a new name. Use patterns to do percent calculations.

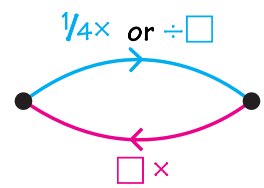
Materials

Teacher		Student	
	• Colored chalk		• Paper • Worksheets N27* and **

Description of Lesson

Exercise 1

Draw this arrow picture on the board.



T: ***What division name does ¼x have?*** (÷4)
What is the opposite of ¼x or ÷4? (4x)

Fill in the boxes for the arrows. Refer to the arrow picture when completing the following problems. (Answers are in boxes.)

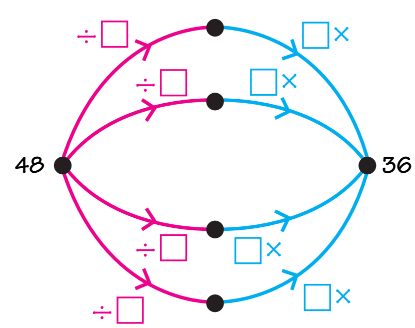
$\frac{1}{4} \times 20 = \boxed{5}$	$\frac{1}{4} \times \boxed{60} = 15$	$\frac{1}{4} \times 2\,400 = \boxed{600}$
$\frac{1}{4} \times \boxed{28} = 7$	$\frac{1}{4} \times 360 = \boxed{90}$	$\frac{1}{4} \times \boxed{120} = 30$

Change the arrows in the picture to ⅙x or ÷6 and 6x. Then solve these problems. (Answers are in boxes.)

$\frac{1}{6} \times 54 = \boxed{9}$	$\frac{1}{6} \times \boxed{72} = 12$	$\frac{1}{6} \times 3\,600 = \boxed{600}$
$\frac{1}{6} \times 42 = \boxed{7}$	$\frac{1}{6} \times \boxed{180} = 30$	$\frac{1}{6} \times \boxed{2\,400} = 400$

Exercise 2

Draw this arrow picture on the board.

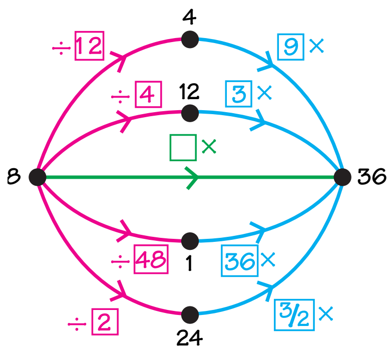


Invite students to find labels for pairs of arrows, red followed by blue. Record correct solutions in a table such as the one shown on the next page.

S: ***If a red arrow is for ÷12, then the following blue arrow is for 9x; 48 ÷ 12 = 4, and 9 x 4 = 36.***

N27

After several solutions have been found, draw a green arrow from 48 to 36.



→ (red)	→ (blue)
÷12	9×
÷4	3×
÷48	36×
÷8	6×
÷24	18×
÷2	$\frac{3}{2}$×

T: ***What could this green arrow be for?***

S: ***$\frac{9}{12}$x, because it is ÷12 followed by 9x.***

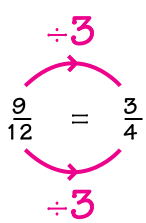
Make a list of possible names for the green arrow. Most will be suggested by the picture and the table, but students may give others. The table can be extended at the same time with more entries for the red and blue arrows.

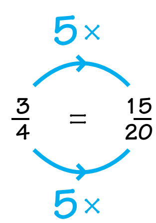
$\frac{9}{12}$× $\frac{3}{4}$× $\frac{36}{48}$× $\frac{6}{8}$× $\frac{15}{20}$× $\frac{18}{24}$× $\frac{3/2}{2}$×

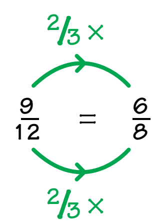
T: ***All of the fractions in these names for the green arrow are equivalent.***

$$\frac{9}{12} = \frac{3}{4} = \frac{36}{48} = \frac{6}{8} = \frac{15}{20} = \frac{18}{24} = \frac{3/2}{2}$$

With the class, discuss how to check that fractions are equivalent. You may draw arrows between some pairs of equivalent fractions to illustrate a technique for checking equality. For example:





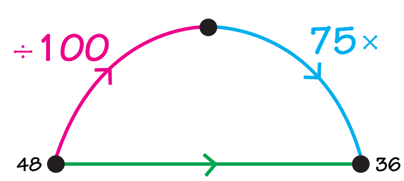


Present each of the following problems, asking students to provide the missing numerator or denominator. (Answers are in boxes.)

$\frac{3}{4} = \frac{\boxed{30}}{40}$ $\frac{3}{4} = \frac{\boxed{75}}{100}$ $\frac{3}{4} = \frac{300}{\boxed{400}}$ $\frac{3}{4} = \frac{1}{\boxed{4/3}}$

Return to the table and the arrow picture to include ÷100 followed by 7x, if it is not already listed. Erase all of the other arrows leaving this picture.

T: ***We already know many names for the green arrow. What fraction times do these arrows suggest for the green arrow?*** ($\frac{75}{100}$x) ***Another name for the green arrow is 75% of*** (read as "seventy-five percent of")***. 75% is a percent name for ÷100 followed by 75x, or for 75x followed by ÷100.***



Include $\frac{75}{100}$x and 75% in the list of equivalent names for the green arrows.

Exercise 3 ______________

Begin this exercise by asking the class where they sometimes hear percents being used. The discussion might include the following:

- weather (percent chance of rain)
- sports (percent wins or success)
- sales (percent discounts or percent off original price)
- test results (percent correct)

Use these examples to give context to some of the percents mentioned in this exercise.

Draw this table on the board, and invite students to complete it. Some students might envision arrow pictures similar to the one at the end of Exercise 2; for example:

- 50% of a number is the same as ½x the number, because 50x followed by ÷100 is the same as ÷2, or ½x.
- 20% = $\frac{20}{100}$ = $\frac{2}{10}$ = $\frac{1}{5}$, because "20% of" is ÷100 followed by 20x.

Fraction	Percent
	50%
	25%
$\frac{3}{4}$	
	20%

Other students may rely on their intuitive knowledge of percent and on the use of patterns; for example:

- 50% attendance means that one-half on the people are present.
- ¾ = 75%, since ¼ = 25% and 3 x 25 = 75.

Extend the table, asking students for the equivalent percents. (Answers are in boxes.)

Fraction	Percent
$\frac{2}{5}$	$\boxed{40\%}$
$\frac{3}{5}$	$\boxed{60\%}$
$\frac{4}{5}$	$\boxed{80\%}$
$\frac{5}{5}$	$\boxed{100\%}$

Highlight this information from the table.

$$\frac{1}{2} = 50\% \qquad \frac{1}{4} = 25\% \qquad \frac{1}{5} = 20\%$$

T: ***What do you notice?***

S: ***The numerator in each fraction is 1.***

S: ***2* x *50, 4* x *25 and 5* x *20 all equal 100.***

Write the following problems on the board, and ask students to do the calculations using fraction equivalences for the percents or using patterns. (Answers are in boxes.)

$$50\% \text{ of } 60 = \boxed{30} \qquad 100\% \text{ of } 60 = \boxed{60}$$
$$25\% \text{ of } 60 = \boxed{15} \qquad 10\% \text{ of } 60 = \boxed{6}$$
$$75\% \text{ of } 60 = \boxed{45} \qquad 5\% \text{ of } 60 = \boxed{3}$$

Encourage students to explain each calculation in several ways. For example:

S: ***25% of 60 = 15, because 50% of 60 = 30 and $\frac{1}{2}$ x 30 = 15.***

S: ***25% = $\frac{1}{4}$, so 25% of 60 = $\frac{1}{4}$ x 60 = 15.***

S: ***75% of 60 = 45, because 25% of 60 = 15 and 3 x 15 = 45.***

S: ***50% of 60 = 30 and 25% of 60 = 15, so 75% of 60 = 30 + 15 = 45.***

S: ***5% of 60 = 3, because 10% of 60 = 6 and $\frac{1}{2}$ x 6 = 3.***

S: ***50% of 60 = 30 and 30 ÷ 10 = 3, so 5% of 60 = 3.***

S: ***"5% of" is ÷100 followed by 5x or $\frac{5}{100}$x.***
$\frac{5}{100}$x = $\frac{1}{20}$x = ÷20, so 5% of 60 = 60 ÷ 20 = 3.

Suggest that students use results already on the board to do these percent calculations. (Answers are in boxes.)

55% of 60 = $\boxed{33}$ 95% of 60 = $\boxed{57}$
80% of 60 = $\boxed{48}$ 1% of 60 = $\boxed{0.6}$
40 % of 60 = $\boxed{24}$ 125 % of 60 = $\boxed{75}$
35 % of 60 = $\boxed{21}$ 8 % of 60 = $\boxed{4.8}$

Accept several explanations for each calculation; for example:

S: ***95% of 60 = 57 because 100% of 60 = 60 and 5% of 60 = 3, so 95% of 60 = 60 – 3 = 57.***

S: ***75% of 60 = 45 and 10% of 60 = 6, so 95% of 60 = 45 + (2 x 6) = 57.***

Worksheets N27* and ** are available for individual work.

Name ____________ N27 ☆

Complete.

50% of 40 = 20 10% of 40 = 4

25% of 40 = 10 20% of 40 = 8

75% of 40 = 30 100% of 40 = 40

Use the above results to help solve these problems.

40% of 40 = 16 30 % of 40 = 12

85% of 40 = 34 60 % of 40 = 24

35% of 40 = 14 80 % of 40 = 32

Name ____________ N27 ☆☆

Complete this table of test results for an 80 question true-false test.

Student	Number Correct	% Correct
Willy	64	80%
Nancy	40	50%
George	44	55%
Ki Jong	60	75%
Maria	72	90%
Alphonso	48	60%

If 70% or better is a passing grade on this test, who passes? Willy, Ki Jong, and Maria

How many questions must a person get correct to have a passing grade? 56 or more

N28 CALCULATOR PUZZLES #2

Capsule Lesson Summary

Put selected numbers on the display of a calculator using a restricted set of keys. Find solutions that require pressing fewer than ten keys. Between two given numbers, build an arrow road in which each arrow is for +, –, x, or ÷ some number with digits limited to 2, 3, 5, and 6.

Materials

Teacher
- Calculator
- Colored chalk

Student
- Calculator
- Paper
- Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

List these calculator keys on the board, and refer to them as you explain the following.

$\boxed{2}\ \boxed{3}\ \boxed{5}\ \boxed{6}$

$\boxed{\div}\ \boxed{\times}\ \boxed{+}\ \boxed{-}\ \boxed{=}$

T: ***Today we are going to solve some calculator puzzles. The puzzles require that we use only these keys on our calculators. Try to put 31 on the display of your calculator using only these keys.***

Allow several minutes for students to experiment. You may talk with individual students about what they did and encourage some students to find several different solutions. After a few minutes, call for the class's attention and ask for solutions. When a student offers a solution, invite another student to check it. Make a list of solutions on the board. There are many solutions, the following is only a sampling.

Note: Many solutions here assume the calculator has an automatic constant feature. See "Role and Use of Calculators" in Section One: Notes to the Teacher. As necessary, make adjustments for the calculators in use by your students.

31: $\boxed{5}\ \boxed{\times}\ \boxed{6}\ \boxed{+}\ \boxed{6}\ \boxed{-}\ \boxed{5}\ \boxed{=}$

$\boxed{6}\ \boxed{\times}\ \boxed{6}\ \boxed{-}\ \boxed{5}\ \boxed{=}$

$\boxed{2}\ \boxed{2}\ \boxed{+}\ \boxed{2}\ \boxed{3}\ \boxed{-}\ \boxed{6}\ \boxed{=}\ \boxed{=}\ \boxed{-}\ \boxed{2}\ \boxed{=}$

$\boxed{3}\ \boxed{2}\ \boxed{+}\ \boxed{2}\ \boxed{-}\ \boxed{3}\ \boxed{=}$

$\boxed{2}\ \boxed{\times}\ \boxed{3}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{+}\ \boxed{5}\ \boxed{+}\ \boxed{2}\ \boxed{=}$

T: ***Now, suppose it costs 1¢ (or $1.00) for each key that we press.***
Which of these solutions cost less than 10¢?
Which of these solutions for 31 costs the least? Could we find a cheaper solution?

With the sample solutions here, only the third and last solutions do not cost less than 10¢. Here the cheapest solution is 6¢. A cheaper 5¢-solution can be found; for example, $\boxed{3}\ \boxed{6}\ \boxed{-}\ \boxed{5}\ \boxed{=}$ or $\boxed{2}\ \boxed{6}\ \boxed{+}\ \boxed{5}\ \boxed{=}$.

N28

Put several numbers on the board such as 77, 97, 11, and 111. Instruct students to try to find solutions costing less than 10¢ for each of these numbers.

77:	$\boxed{5}\,\boxed{5}\,\boxed{+}\,\boxed{2}\,\boxed{2}\,\boxed{=}$	6¢	**11:**	$\boxed{5}\,\boxed{+}\,\boxed{6}\,\boxed{=}$	4¢
	$\boxed{3}\,\boxed{6}\,\boxed{+}\,\boxed{=}\,\boxed{+}\,\boxed{5}\,\boxed{=}$	7¢		$\boxed{2}\,\boxed{2}\,\boxed{\div}\,\boxed{2}\,\boxed{=}$	5¢
	$\boxed{6}\,\boxed{6}\,\boxed{+}\,\boxed{5}\,\boxed{+}\,\boxed{6}\,\boxed{=}$	7¢		$\boxed{6}\,\boxed{6}\,\boxed{-}\,\boxed{5}\,\boxed{5}\,\boxed{=}$	6¢
	$\boxed{5}\,\boxed{5}\,\boxed{+}\,\boxed{=}\,\boxed{-}\,\boxed{3}\,\boxed{3}\,\boxed{=}$	8¢		$\boxed{2}\,\boxed{\times}\,\boxed{5}\,\boxed{+}\,\boxed{3}\,\boxed{-}\,\boxed{2}\,\boxed{=}$	8¢
97:	$\boxed{6}\,\boxed{2}\,\boxed{+}\,\boxed{3}\,\boxed{5}\,\boxed{=}$	6¢	**111:**	$\boxed{5}\,\boxed{5}\,\boxed{\times}\,\boxed{2}\,\boxed{+}\,\boxed{3}\,\boxed{-}\,\boxed{2}\,\boxed{=}$	9¢
	$\boxed{2}\,\boxed{\times}\,\boxed{5}\,\boxed{2}\,\boxed{-}\,\boxed{5}\,\boxed{-}\,\boxed{2}\,\boxed{=}$	9¢		$\boxed{5}\,\boxed{6}\,\boxed{+}\,\boxed{5}\,\boxed{5}\,\boxed{=}$	6¢
	$\boxed{3}\,\boxed{\times}\,\boxed{3}\,\boxed{2}\,\boxed{+}\,\boxed{3}\,\boxed{-}\,\boxed{2}\,\boxed{=}$	9¢		$\boxed{5}\,\boxed{3}\,\boxed{\times}\,\boxed{2}\,\boxed{+}\,\boxed{5}\,\boxed{=}$	7¢
	$\boxed{3}\,\boxed{6}\,\boxed{\times}\,\boxed{2}\,\boxed{+}\,\boxed{2}\,\boxed{5}\,\boxed{=}$	8¢		$\boxed{6}\,\boxed{6}\,\boxed{6}\,\boxed{-}\,\boxed{5}\,\boxed{5}\,\boxed{5}\,\boxed{=}$	8¢

Exercise 2

Tell the class that for the next numbers they do not have to spend less than 10¢.

T: ***Try to put 0.5 on your calculator using only these keys.***

Allow a few minutes for students to work on this problem before offering guidance.

T: ***What are some fractional names for 0.5?***

As names are suggested, write them on the board.

T: ***What are the corresponding division names for 0.5 suggested by these fractions?***

Write the division names underneath the fractional names.

$\frac{1}{2}$	$\frac{2}{4}$	$\frac{3}{6}$	$\frac{4}{8}$	$\frac{5}{10}$	$\frac{6}{12}$ ···
$1 \div 2$	$2 \div 4$	$3 \div 6$	$4 \div 8$	$5 \div 10$	$6 \div 12$···

T: ***Perhaps these names for 0.5 will help you put 0.5 on the calculator.***

S: ***Press*** $\boxed{3}\,\boxed{\div}\,\boxed{6}\,\boxed{=}$.

Accept and record several solutions for 0.5 on the board.

0.5: $\boxed{3}\,\boxed{\div}\,\boxed{6}\,\boxed{=}$

$\boxed{5}\,\boxed{\div}\,\boxed{2}\,\boxed{-}\,\boxed{2}\,\boxed{=}$

$\boxed{3}\,\boxed{-}\,\boxed{2}\,\boxed{\div}\,\boxed{2}\,\boxed{=}$

$\boxed{2}\,\boxed{\div}\,\boxed{2}\,\boxed{\div}\,\boxed{2}\,\boxed{=}$

$\boxed{3}\,\boxed{3}\,\boxed{\div}\,\boxed{6}\,\boxed{6}\,\boxed{=}$

$\boxed{2}\,\boxed{2}\,\boxed{\div}\,\boxed{2}\,\boxed{\div}\,\boxed{2}\,\boxed{2}\,\boxed{=}$

Continue this activity to find ways to display 0.1, 0.2, and 0.3.

0.1:

$\boxed{3}\ \boxed{\div}\ \boxed{6}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$

$\boxed{3}\ \boxed{-}\ \boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$

$\boxed{3}\ \boxed{\div}\ \boxed{5}\ \boxed{\div}\ \boxed{6}\ \boxed{=}$

$\boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{=}\ \boxed{=}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$

0.3:

$\boxed{3}\ \boxed{\div}\ \boxed{2}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$

$\boxed{6}\ \boxed{\div}\ \boxed{2}\ \boxed{=}\ \boxed{=}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$

$\boxed{6}\ \boxed{\div}\ \boxed{5}\ \boxed{\div}\ \boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$

$\boxed{3}\ \boxed{3}\ \boxed{\div}\ \boxed{5}\ \boxed{-}\ \boxed{6}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$

0.2:

$\boxed{5}\ \boxed{+}\ \boxed{6}\ \boxed{\div}\ \boxed{5}\ \boxed{-}\ \boxed{2}\ \boxed{=}$

$\boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$

$\boxed{5}\ \boxed{\div}\ \boxed{5}\ \boxed{=}\ \boxed{=}$

$\boxed{5}\ \boxed{+}\ \boxed{5}\ \boxed{\div}\ \boxed{=}\ \boxed{=}\ \boxed{\times}\ \boxed{2}\ \boxed{=}$

$\boxed{6}\ \boxed{\div}\ \boxed{5}\ \boxed{-}\ \boxed{3}\ \boxed{+}\ \boxed{2}\ \boxed{=}$

$\boxed{3}\ \boxed{\div}\ \boxed{5}\ \boxed{\div}\ \boxed{3}\ \boxed{=}$

Exercise 3

Pose a different type of calculator puzzle, still using just the keys $\boxed{2}$, $\boxed{3}$, $\boxed{5}$, $\boxed{6}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, and $\boxed{=}$.

T: ***In this next problem, an arrow can be for add, subtract, multiply, or divide by some number with digits limited to 2, 3, 5, and 6. Give us an example of such an arrow.***

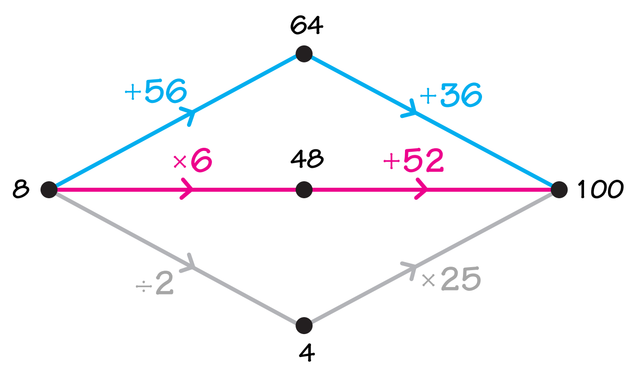
S: ***–6.***

S: ***÷356.***

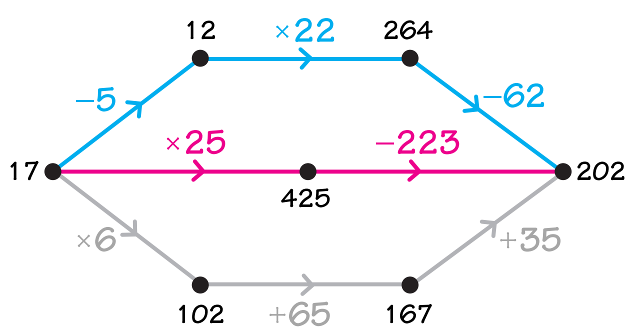
S: ***+22.***

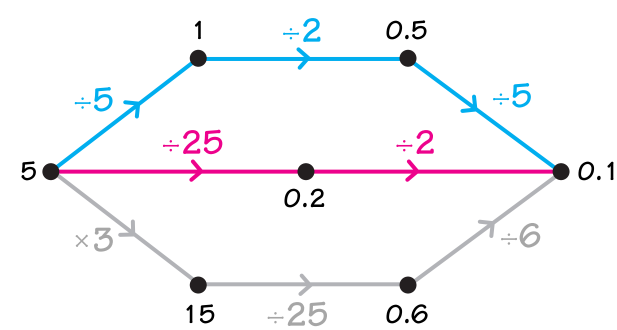
T: ***Now, the problem is to build an arrow road from 8 to 100 using only these kinds of arrows. Try to use as few arrows as possible.***

Allow several minutes for students to work independently and to draw their solutions. Encourage each student to find at least one solution whether or not it is a shortest road. When most students have found at least one solution, invite some students to draw their roads on the board. There are many solutions, three of which are given here.



Continue this activity by asking students to draw a road from 17 to 202 and from 5 to 0.1. Some solutions are given below.





N29 PERCENT #2

Capsule Lesson Summary

Color part of a 10 x 10 grid, and observe what fractional part is colored and then what percent is colored. Color a given percent of a 10 x 10 grid. Find what percent of students in a class of 25 are boys, wear sneakers, walk to school, and so on. Determine several equivalent names for 60% and use these to do calculations involving 60%.

Materials

Teacher		Student	
	• Blackline N29 • 10 x 10 grid transparency		• 10 x 10 grids • Colored pencils, pens, or crayons

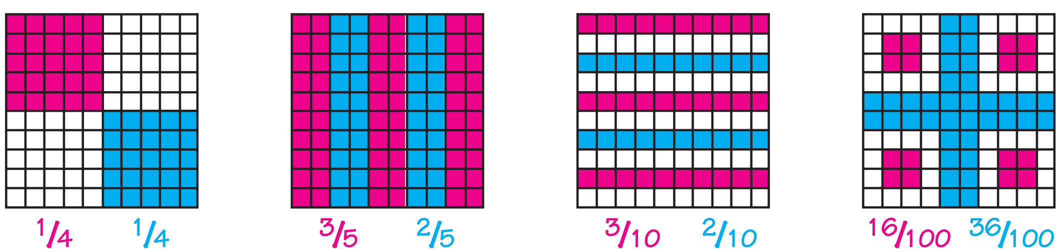
Advance Preparation: Use Blackline N29 to make several copies of a 10 x 10 grid for each student. Use the same blackline to make a 10 x 10 grid transparency.

Description of Lesson

Exercise 1

Tell students to use one of their 10 x 10 grids, and to color some little squares red and some blue. Let students decide for themselves how many little squares to make each color.

After a few minutes, instruct students to write next to the grid what fraction of the big square (the 10 x 10 grid) they colored red and what fraction blue. Choose several students to display their grids and explain how they determined what fraction was each color. For example:



During the discussion of fractional parts, students may observe the total number of little squares (100) and the number colored red or blue. Perhaps this will generate equivalent fractions such as:

$$\tfrac{1}{4} = \tfrac{25}{100} \qquad \text{or} \qquad \tfrac{3}{5} = \tfrac{6}{10} = \tfrac{60}{100}$$

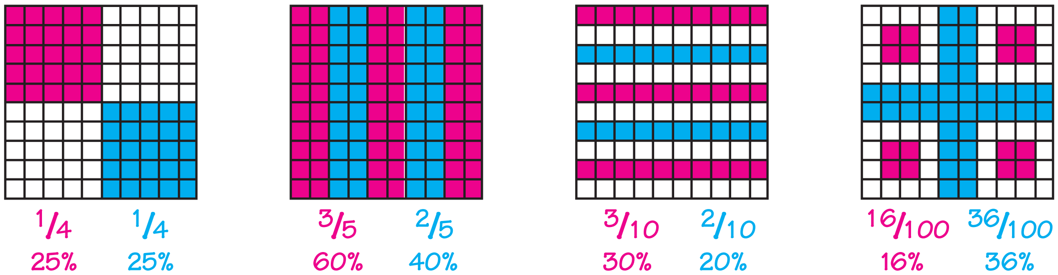
T: ***Now, decide what percent of your big square (10 x 10 grid) is red and what percent is blue.***

Instruct students to write the percents next to their grids. With the class, observe that "percent" means "per hundred" or "out of a hundred."

T: ***In a previous lesson we found percent as a composition: ÷100 followed by x. Here the big square is already divided into 100 little squares. So what can we do to determine what percent is colored red?***

S: ***Count how many little squares are colored red.***

Direct students to write on their papers what percent of the big square (10 x 10 grid) is colored red and what percent blue.



Refer students to the other big square (10 x 10 grid) on their papers. Give each student a number between 0 and 100, and ask the student to color that percent of the big square. Let students exchange papers to check the colorings. You may like to invite a few students to explain how they colored for their percent.

Exercise 2

For this problem, use your own class or describe another class with either 20 or 25 students.

T: ***How many students are in class today?***

S: ***25.***

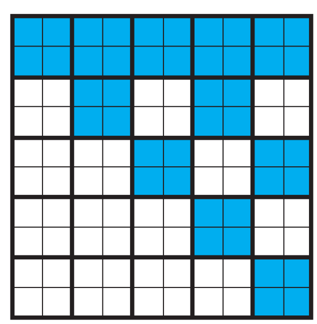
T: ***How many boys are there? How many girls?***

S: ***11 boys and 14 girls.***

T: ***If someone were to ask, "What percent of the class is boys?" how could we answer their question?***

Let students discuss the problem. Eventually lead the discussion toward considering the fractional part for boys, and recall that percent means "out of 100."

11 boys out of 25 students $^{11}/_{25} = {}^{?}/_{100}$ boys out of 100 students



You may like to display a 10 x 10 grid and darken every other line to "overlay" a 5 x 5 grid. On the 5 x 5 grid, the 25 squares could represent the 25 students in class. Let a student color 11 squares blue for boys.

T: ***What percent of the class is boys?***

S: ***44%; $^{11}/_{25} = {}^{44}/_{100}$.***

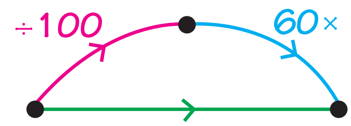
Pose some other similar percent questions for the same class of 25 students.

T: ***I see that 19 out of 25 students in class today are wearing sneakers. What percent of the class are wearing sneakers?*** (76%)

6 out of 25 students in class today walked to school. What percent of the class walked to school? (24%)

Exercise 3

Draw this arrow picture on the board.

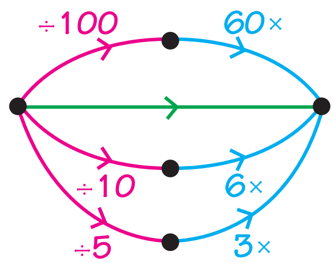


T: ***The green arrow is a composite, ÷100 followed by 60x. What could the green arrow be for?***

Solicit many equivalent names for the green arrow, including a percent name. For example:

$^{60}/_{100}\times$ $\quad$ $^{30}/_{50}\times$ $\quad$ $^{3}/_{5}\times$ $\quad$ $^{6}/_{10}\times$ $\quad$ $0.6\times$ $\quad$ 60% of

You may like to illustrate names like $^3/_5$x or $^6/_{10}$x with other pairs of arrows ÷☐ followed by ☐x.



Label the green arrow "60% of," and observe that the various names for the green arrow all are ways to calculate 60% of a number.

Put several calculations involving 60% on the board, and call on students to explain how they would do the calculations. (Answers are in boxes.)

60% of 20 = $\boxed{12}$ $\quad$ 60% of 35 = $\boxed{21}$ $\quad$ 60% of 150 = $\boxed{90}$

S: ***60% of 20 = 12. I did 20 ÷ 10 = 2 and 6 x 2 = 12.***

S: ***60% of 35 = 21. I used 35 ÷ 5 = 7 and 3 x 7 = 21.***

S: ***60% of 150 = 90. I did 150 ÷ 5 = 30 and 3 x 30 = 90.***

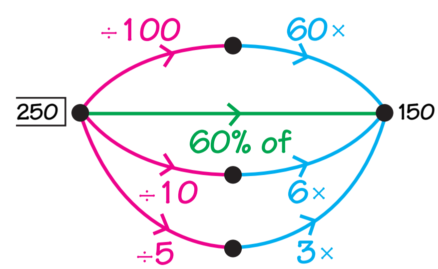
Write this calculation on the board.

60% of ☐ = 150

T: ***How could we use the arrow picture to solve this problem?***

S: ***Put 150 at the ending dot of the 60% arrow and go backward.***

S: ***Use opposites. For example, 150 ÷ 3 = 50 (the opposite of 3x is ÷3) and 5 x 50 = 250 (the opposite of ÷5 is 5x). So, 60% of 250 = 150.***



Similarly, ask students to calculate the following:

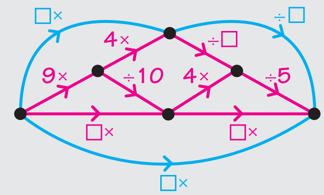
60% of $\boxed{60}$ = 36 $\quad$ and $\quad$ 60% of $\boxed{45}$ = 27

Writing Activity

Suggest that students write a problem similar to those in Exercise 2 in which they ask a question about percent.

Capsule Lesson Summary

Use this arrow picture to motivate a method of multiplying fractions. Multiply decimals by finding fractions equal to the decimals and multiplying the fractions.



Materials

Teacher		Student	
	• Colored chalk		• Calculator • Paper • Colored pencils, pens, or crayons • Worksheets N30*, **, ***, and ****

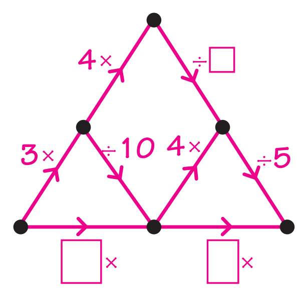
Description of Lesson

Begin the lesson with mental arithmetic. Suggested problems are given below. (Answers are in boxes.)

$2 \times 3 = \boxed{6}$	$4 \times 6 = \boxed{24}$	$3 \times 7 = \boxed{21}$	$5 \times 8 = \boxed{40}$
$\frac{1}{2} \times 3 = \boxed{1.5 \text{ or } 1\frac{1}{2}}$	$\frac{2}{3} \times 6 = \boxed{4}$	$\frac{1}{7} \times 7 = \boxed{1}$	$\frac{3}{4} \times 8 = \boxed{6}$
$2\frac{1}{2} \times 3 = \boxed{7.5 \text{ or } 7\frac{1}{2}}$	$4\frac{2}{3} \times 6 = \boxed{28}$	$3\frac{1}{7} \times 7 = \boxed{22}$	$5\frac{3}{4} \times 8 = \boxed{46}$

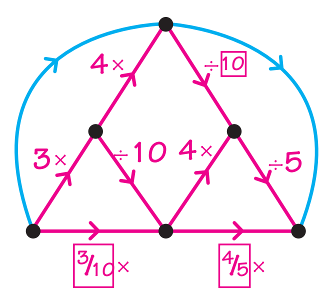
Exercise 1

Draw this arrow picture on the board.



Invite students to fill in the boxes for the arrows. If the class has trouble filling in the box for the division arrow, focus attention on the center diamond and ask, "÷10 followed by 4x is the same as 4x followed by ÷ what number?" (10)

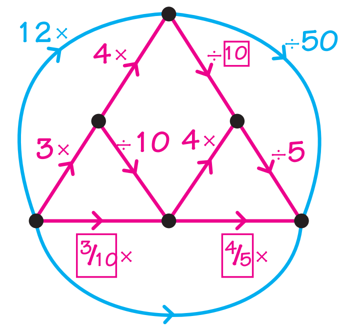
Add these two blue arrows to the picture.



T: ***What is 3x followed by 4x?*** (12x)
What is ÷10 followed by ÷5? (÷50)

You may like to check the compositions by labeling dots in several ways. Label the blue arrows and then draw another blue arrow as the composition of $\frac{3}{10}$x and $\frac{4}{5}$x.

T: ***Look at the blue arrow along the bottom of the picture. What could it be for? Use the other two blue arrows to help you.***



Allow a few minutes for the class to examine the situation.

S: ***$\frac{12}{50}$x.***

T (tracing blue arrows)**:** ***12x followed by ÷50 is $\frac{12}{50}$x.***

Label the blue arrow along the bottom.

T (tracing appropriate arrows)**:** ***3x followed by 4x is 12x; 3 x 4 = 12. ÷10 followed by ÷5 is ÷50; 10 x 5 = 50. $\frac{3}{10}$x followed by $\frac{4}{5}$x is $\frac{12}{50}$x, or $\frac{3 \times 4}{10 \times 5}$ x.***

Record this equality on the board

$$\frac{12}{50}\times \;=\; \frac{3 \times 4}{10 \times 5}\times$$

T: ***So we conclude that $\frac{3}{10} \times \frac{4}{5} = \frac{12}{50}$. Can someone give another name for $\frac{12}{50}$?***

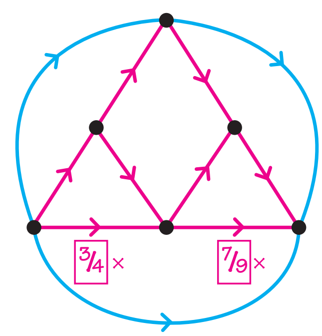
S: ***$\frac{6}{25}$.***

Record the equalities on the board.

$$\frac{3}{10} \times \frac{4}{5} = \frac{12}{50} = \frac{6}{25}$$

Erase the arrow labels in the picture. Relabel two of the red arrows as shown here.

T (tracing the lower blue arrow)**:** ***What could this blue arrow be for?***



S: ***$\frac{21}{36}$x, 3 x 7 = 21 and 4 x 9 = 36.***

S: ***$\frac{21}{36}$x = $\frac{7}{12}$x.***

T: ***What is $\frac{3}{4} \times \frac{7}{9}$?***

S: ***$\frac{21}{36}$, or $\frac{7}{12}$.***

T: ***To multiply two fractions, multiply numerators (top numbers) and multiply denominators (bottom numbers).***

Exercise 2

Refer students to their calculators.†

T: ***We cannot put a fraction like $\frac{2}{5}$ on the display of the calculator, but how can we put on its decimal name?***

S: ***Press $\boxed{2}$ $\boxed{\div}$ $\boxed{5}$ $\boxed{=}$.***

S: ***Press $\boxed{0}$ $\boxed{\cdot}$ $\boxed{4}$.***

Mention that $\frac{2}{5}$ means 2 ÷ 5, and ask students to press $\boxed{2}$ $\boxed{\div}$ $\boxed{5}$ $\boxed{=}$ to see 0.4.

T: ***Try $\frac{1}{3}$.***

†If students have calculators that display fractions, adjust this exercise for use with such calculators.

S: ***0.3333333.***

T: ***Is this the full decimal name for ⅓?***

S: ***No, the calculator only displays eight digits of ⅓'s decimal name.***

You may want to note that sometimes when we write a decimal name for ⅓, we use a * or bar over the 3 to indicate it repeats forever: ⅓ = $0.\overset{*}{3}$.

Continue by asking students to put decimal names for several other fractions on their calculators while you write them on the board. For example:

$\frac{2}{5}$ = $\boxed{2 \div 5 = 0.4}$	$\frac{7}{10}$ = $\boxed{7 \div 10 = 0.7}$
$\frac{15}{10}$ = $\boxed{15 \div 10 = 1.5}$	$\frac{39}{100}$ = $\boxed{39 \div 100 = 0.39}$
$\frac{40}{10}$ = $\boxed{40 \div 10 = 4}$	$\frac{325}{100}$ = $\boxed{325 \div 100 = 3.25}$
$\frac{6}{100}$ = $\boxed{6 \div 100 = 0.06}$	$\frac{500}{100}$ = $\boxed{500 \div 100 = 5}$

T: ***Now I'll give you a decimal and you tell me a fraction equal to it. There are always many answers to problems like this: 0.4.***

S: ***4⁄10.***

T: ***Let's check on the calculator. 4⁄10 = 4 ÷ 10. Press*** $\boxed{4}$ $\boxed{\div}$ $\boxed{1}\boxed{0}$ $\boxed{=}$***. Do you get 0.4?*** (Yes)

S: ***2⁄5.***

T: ***Check. Press*** $\boxed{2}$ $\boxed{\div}$ $\boxed{5}$ $\boxed{=}$***.***

S: ***We get 0.4 again.***

Check any other fractions given. Any fraction equivalent to 4⁄10 will be correct.

Continue with these decimals. One possible answer for each is shown below.

0.5 = $\boxed{\frac{5}{10}}$	0.03 = $\boxed{\frac{3}{100}}$
0.9 = $\boxed{\frac{9}{10}}$	0.71 = $\boxed{\frac{71}{100}}$
2.9 = $\boxed{\frac{29}{10}}$	2.83 = $\boxed{\frac{283}{100}}$

If a student suggests 2 9⁄10 for 2.9, accept it as perfectly correct. Tell the class that 2 9⁄10 = 9⁄10 + 2, so they can put 9⁄10 on by pressing $\boxed{9}$ $\boxed{\div}$ $\boxed{1}\boxed{0}$ $\boxed{=}$, and then add 2 by pressing $\boxed{+}$ $\boxed{2}$ $\boxed{=}$. The same is true for 2.83 = 2 83⁄100.

Exercise 3

Direct students to put away their calculators for this exercise.

Write this problem on the board.

$$0.2 \times 0.3$$

Note: You may want to give some context to this problem. For example:

> Jake got 0.3 kilograms of fudge for his birthday. On that day he ate 0.2 (two tenths) of the fudge. Did he eat very much fudge? What part of a kilogram of fudge did he eat?

Solicit some estimates such as more or less than ½ kilogram, or more or less than ¼ kilogram.

T: ***Calculate 0.2 x 0.3 on your papers.***

Write some of the results (correct or incorrect) on the board. A common mistake is to answer 0.6.

T: ***Several suggestions have been made. How did you do this calculation?***

S: ***I used fractions.***

If this method is not suggested, suggest it yourself.

T: ***What is a fraction for 0.2?*** ($\frac{2}{10}$) ***For 0.3?*** ($\frac{3}{10}$)

Begin the calculation this way.

$$0.2 \times 0.3 = \frac{2}{10} \times \frac{3}{10}$$

T: ***What number is $\frac{2}{10} \times \frac{3}{10}$?***

S: ***$\frac{6}{100}$; 2 x 3 = 6 and 10 x 10 = 100.***

T: ***What is the decimal name for***

S: ***0.06.***

$$0.2 \times 0.3 = \frac{2}{10} \times \frac{3}{10} \times \frac{6}{100} = 0.06$$

In contrast to the above product, pose the following two problems. Ask students to explain the calculations. (Answers are in boxes.)

$$2 \times 0.3 = \boxed{0.6}$$

$$0.2 \times 3 = \boxed{0.6}$$

Continue the activity with problems such as the ones below. Write problems on the board and direct students to copy and solve them. After a few minutes check the work collectively, asking students to explain the calculations. (Answers are in boxes.)

$$3 \times 7 = \boxed{21}$$

$$3 \times 0.7 = \boxed{3 \times \frac{7}{10} = \frac{21}{10} = 2.1}$$

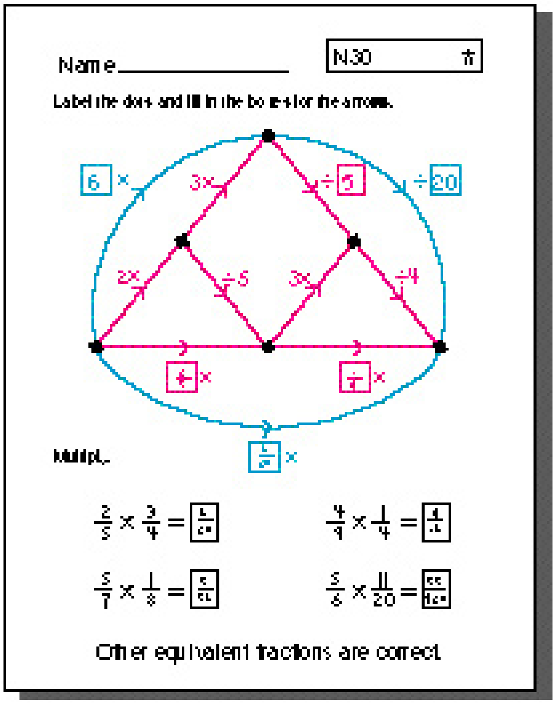
$$0.3 \times 0.7 = \boxed{\frac{3}{10} \times \frac{7}{10} = \frac{21}{100} = 0.21}$$

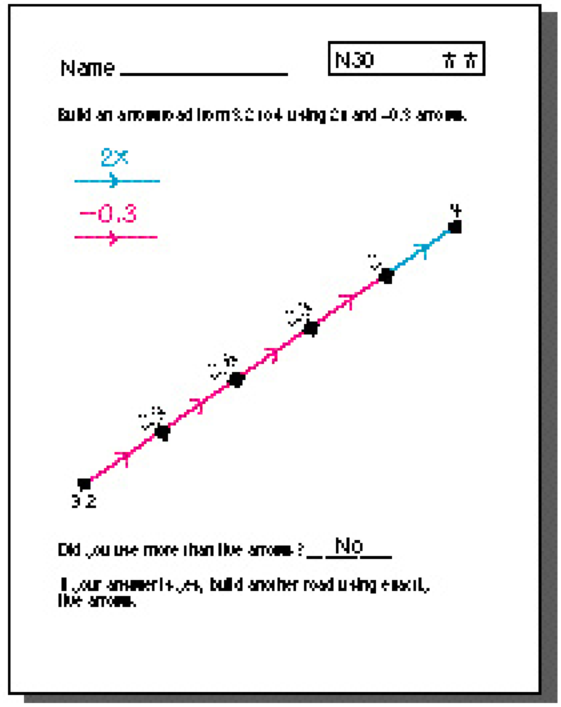
$$30 \times 0.7 = \boxed{30 \times \frac{7}{10} = \frac{210}{10} = 21}$$

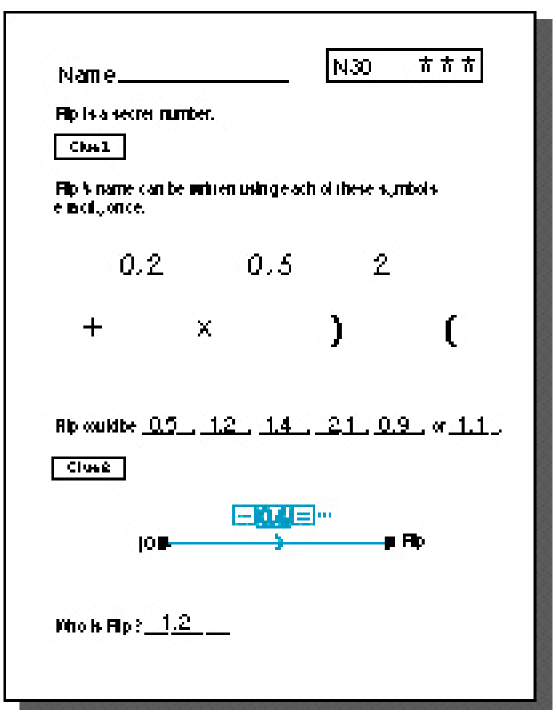
$$0.3 \times 70 = \boxed{\frac{3}{10} \times 70 = \frac{210}{10} = 21}$$

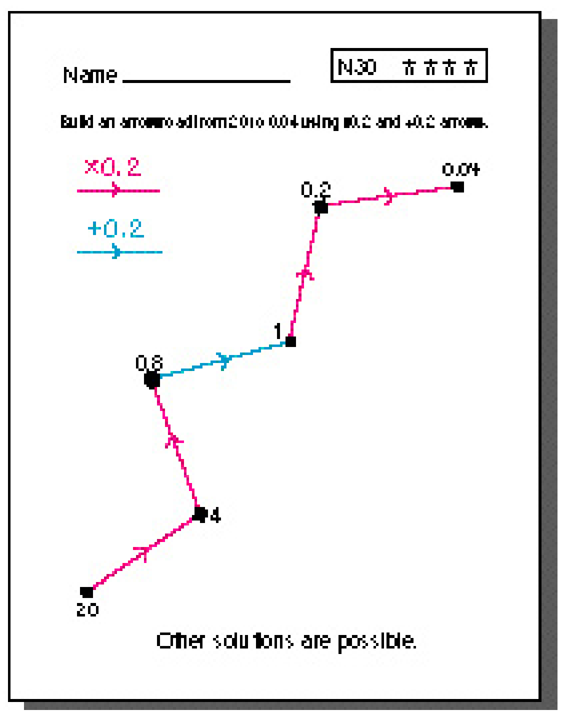
$$30 \times 0.07 = \boxed{30 \times \frac{7}{100} = \frac{210}{100} = 2.1}$$

Worksheets N30*, **, ***, and **** available for individual work.


Name
N30 ☆
Label the dots and fill in the boxes for the arrows.
Multiply.
Other equivalent fractions are correct.


Name
N30 ☆☆
Build an arrow road from 3.2 to 4 using 2x and −0.3 arrows.
2x
−0.3
4
3.2
Did you use more than five arrows? No
If your answer is yes, build another road using exactly five arrows.


Name
N30 ☆☆☆
Flip is a secret number.
Clue 1
Flip's name can be written using each of these symbols exactly once.
0.2
0.5
2
+
×
)
(
Flip could be 0.5, 1.2, 1.4, 2.1, 0.9, or 1.1.
Clue 2
Flip
Who is Flip? 1.2


Name
N30 ☆☆☆☆
Build an arrow road from 20 to 0.04 using ×0.2 and +0.2 arrows.
×0.2
+0.2
0.04
0.2
1
0.8
4
20
Other solutions are possible.

Capsule Lesson Summary

Discuss everyday uses of percent and percents that are easy to use in calculations. Combine easy percents to do calculations such as 35% of 80. Use composition of ÷100 and ☐x to define ☐% of, and find equivalent names for ☐% using equivalent compositions. Observe patterns for doing percent calculations, and apply this to calculating a 15% tip on a meal in a restaurant.

Materials

Teacher
- Colored chalk

Student
- Paper
- Worksheets 31*, **, and ***

Description of Lesson

Exercise 1

Begin the lesson by asking students about some of the ways they hear percents used in everyday situations. Elicit a variety of responses and discuss these uses of percent. For example:

S: ***The weather report might say there is a 10% chance of rain.***

T: ***What does that mean to you? Would you plan to have an umbrella or raincoat with you? Do you think the weather is likely to force a sporting event to be cancelled? Is it likely to rain?***

S: ***A test score was 75%.***

T: ***What does 75% on a test indicate to you?***
If the test had 20 items, how many were correct? (75% of 20 = 15)

S: ***A store might advertise a sale with 40% off.***

T: ***Is that a good sale? Is it better than a half-price sale?***
Approximately what would you expect to pay for an item that originally sold for $50?

After discussing a few examples of the use of percents in everyday contexts, ask,

T: ***What percents are quite easy to use in calculations?***

Accept suggestions such as 50%, 25%, 100%, 10%, and 1%. For example:

S: ***50%, because it is the same as ½ or ÷2.***

T: ***What is 50% of 14?*** (7) ***40?*** (20) ***3,000?*** (1500)

Continue until it is clear that most students can readily find 50% of a number.

In a similar manner, ask students to do calculations involving 25%, 10%, and 1%.

T: ***What number is 200% of 30?***

S: ***60. 100% of 30 = 30, so 200% of 30 = 2*** x ***30 = 60.***

S: ***200% is the same as 2*****x.**

Ask students to calculate 200% of other numbers. Then invite students to calculate 300% and 400% of various numbers. Students should realize that 300% and 400% are the same as 3x and 4x, respectively.

Suggest students solve the following problems by combining easy percent calculations. (Answers are in boxes.)

125% of $48 = \boxed{60}$ $\qquad$ 35% of $80 = \boxed{28}$

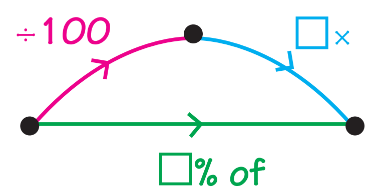
60% of $20 = \boxed{12}$ $\qquad$ 270% of $200 = \boxed{540}$

S: ***100% of 48 = 48 and 25% of 48 = ¼*** x ***48 = 12, so 125% of 48 = 48 + 12 = 60.***

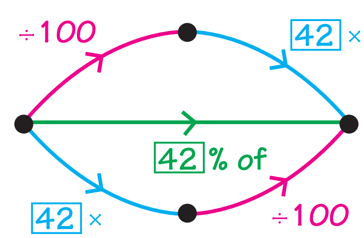
S: ***50% of 20 = 10 and 10% of 20 = 20 ÷ 10 = 2, so 60% of 20 = 10 + 2 = 12.***

Exercise 2

Draw this arrow picture on the board.



T: ***Percent can be defined as a composition of two arrows, as in the picture. With the red arrow for ÷100, the same number goes in each box. For example, if the green arrow is for 42%, then the blue arrow is for 42*****x.**

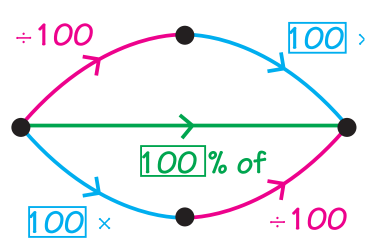


Label the arrows, and point out that the red and blue arrows could be reversed. So, 42% is ÷100 followed by 42x as well as 42x followed by ÷100.

Erase the numbers in the boxes.

T: ***If the blue arrow is for 7*****x*****, then what is the green arrow for?*** (7%)
If the green arrow is for 100%, then what is the blue arrow for? (100x)

Label the arrows.



T: ***What is 100% of a number?***

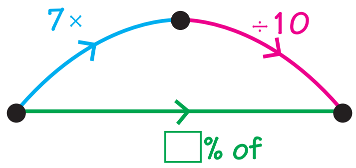
S: ***The same number.***

T: ***How does the arrow picture show that 100% of a number is the same number?***

S: ***100*****x** ***followed by ÷100 is the same as ${}^{100}/_{100}$*****x** ***or 1*****x*****. 100% is the same as 1 and so an arrow for 100% should be a loop like 1*****x*****.***

Repeat this line of questioning to observe that 300% of a number is 3x the number.

Draw this arrow picture on the board.

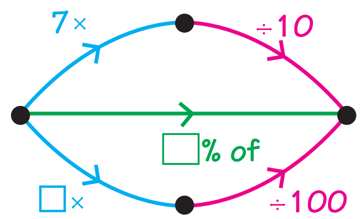


T: ***How should we fill in the box for the green arrow?***

If some students respond, "7%," observe,

T: ***If the red arrow were for ÷100, then the green arrow would be for 7%. But the red arrow is for ÷10.***

Add these arrows to the picture.



T: ***Let's consider another composition for the green arrow where the red arrow is for ÷100. How should we fill in the box for this blue arrow?***

S: ***70x. 70x followed by ÷100 is the same as 7x followed by ÷10 since $\frac{70}{100}$x = $\frac{7}{10}$x.***

T: ***What is the green arrow for?***

S: ***70% of. 70x followed by ÷100 is 70% of.***

Label the green arrow.

Exercise 3

Write the following information on the board.

□% of 18 < 18 □% of 18 = 18 □% of 18 > 18

T: ***How could we fill in the boxes to make these statements true?***

S: ***□% of 18 = 18 has only one solution, 100, since 100% of 18 = 18.***

S: ***Any number less than 100 makes □% of 18 < 18 true.***

S: ***Any number more than 100 makes □% of 18 > 18 true.***

T: ***What number is 150% of 18?***

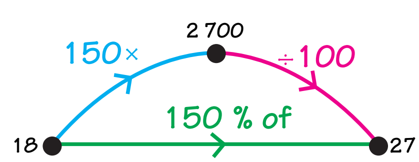
S: ***27. 100% of 18 = 18 and 50% of 18 = 9, so 150% of 18 = 18 + 9 = 27.***

Indicate this method of calculating 150% of 18 on the board.

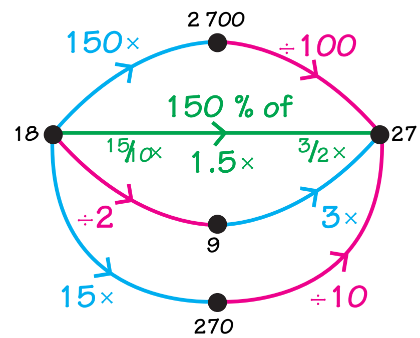
100% of 18 = 18
50% of 18 = 9
150% of 18 = 27

S: ***We could use arrows. 150% of is the same as 150x followed by ÷100.***

Draw this arrow picture on the board, and label the dots with student direction.



You may also find several equivalent compositions and equivalent names for 150%, as illustrated here.



Leave the calculation for 150% of 18 on the board when you proceed to Exercise 4.

Exercise 4

Lead a brief discussion of the practice of tipping waitresses, waiters, and other service workers. Explain that tipping is a major part of the income of many such people. Ask whether any student knows how much is an appropriate tip for a waiter or a waitress.

T: ***For good service, one might tip 15% of the total cost of the meal. If a meal costs $20, how much would a 15% tip be?***

Lead the class to use the following method of calculating 15% of $20.

10% of $20 = $2 …since 10% = $\frac{1}{10}$, $\frac{1}{10}$x = ÷10, and 20 ÷ 10 = 2

5% of $20 = $1 …since 5% is half of 10% and ½ x 2 = 1

15% of $20 = $3

Ask students to calculate 15% tips for meals costing $18 and $46. Encourage similar methods of solution.

15% of $18 = $1.80 × $0.90 = $2.70

15% of $46 = $4.60 × $2.30 = $6.90

Note that 150% of 18 = 27 and 15% of 18 = 2.7.

Worksheets N31*, **, and *** are available for individual work.

Home Activity

Send home percent problems like those on Worksheets N31* and ** for students to do with family help. Especially note for parents/guardians the use of patterns in solving many percent calculations.

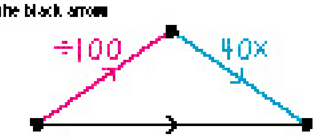
Name ______________ N31 ☆

Complete.

100% of 16 = 16	50% of 80 = 40
50% of 16 = 8	25% of 80 = 20
150% of 16 = 24	10% of 80 = 8
75% of 16 = 12	5% of 80 = 4
25% of 16 = 4	15% of 80 = 12
125% of 16 = 20	35% of 80 = 28
10% of 60 = 6	50% of 72 = 36
5% of 60 = 3	25% of 72 = 18
15% of 60 = 9	75% of 72 = 54
20% of 60 = 12	10% of 72 = 7.2
40% of 60 = 24	35% of 72 = 25.2
45% of 60 = 27	85% of 72 = 61.2

Name ______________ N31 ☆☆

Label the black arrow.



$\boxed{\frac{40}{100}}$ x or $\boxed{40}$ % of

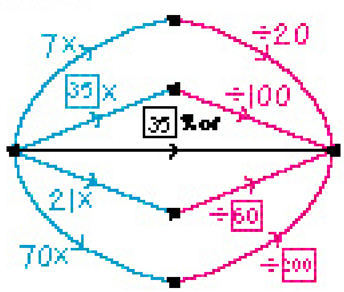
Give two other names for the black arrow: $\boxed{.4}$ x or $\boxed{\frac{2}{5}}$ x

Complete.

40% of 15 = $\boxed{6}$	40% of 34 = $\boxed{13.6}$
40% of 80 = $\boxed{32}$	40% of 12 = $\boxed{4.8}$
40% of $\boxed{200}$ = 80	40% of $\boxed{30}$ = 12

Name ______________ N31 ☆☆☆

Label the arrows.



Complete.

$\frac{7}{20}$ x 300 = $\boxed{105}$	35% of 40 = $\boxed{14}$
35% of 300 = $\boxed{105}$	35% of 14 = $\boxed{4.9}$
35% of $\boxed{800}$ = 280	35% of $\boxed{40}$ = 14

Capsule Lesson Summary

Give fractional names for 1, 2, and 3. Find decimal names for fractions with denominators of 10 and 100. Insert parentheses in the expression 7 – 5 x 3 – 2, and decide how to use calculators to perform the calculations in the order called for by the parentheses. Do simple fraction calculations using a calculator.

Materials

Teacher
- Colored chalk

Student
- Paper
- Calculator

Description of Lesson

Exercise 1

Ask the class to name fractions for 1, fractions for 2, and fractions for 3. List responses in a chart. Examples of fractions in each category are given below.

Fractions for 1	Fractions for 2	Fractions for 3
$3/3$, $5/5$, $\hat{5}/\hat{5}$, $18/18$, $10/10$, $1\,000/1\,000$, $1/1$	$2/1$, $6/3$, $4/2$, $\hat{4}/\hat{2}$, $750/375$, $20/10$	$9/3$, $3/1$, $12/4$, $24/8$, $\hat{3}/\hat{1}$, $30/10$

Ask the class to include a fraction with denominator 50 in each section of the chart, a fraction with denominator 100, and a fraction with denominator 72.

Exercise 2

Refer students to their calculators.†

T: ***We cannot display a fraction on our calculator, but what can we do?***

S: ***Display its decimal name up to eight digits.***

T: ***What can we do to get a decimal for $11/15$?***

S: ***Press $\boxed{1}\boxed{1}$ $\boxed{\div}$ $\boxed{1}\boxed{5}$ $\boxed{=}$.***

S: ***0.7333333 is on the display.***

T: ***$11/15$ = 0.733 and so on with 3s. The calculator displays only eight digits. Now find a decimal name for $3/4$.***

S: ***$3/4 = 3 \div 4 = 0.75$.***

†If students have calculators that display fractions, adjust this exercise for use with such calculators.

Repeat this activity, getting decimal names for several more fractions. Then give the class a decimal and ask them for a fraction. The following gives a possible sequence of problems with likely responses in boxes.

$3/4 = \boxed{3 \div 4 \ = 0.75}$ $\qquad$ $0.8 = \boxed{8 \div 10 \ = 8/10}$

$3/10 = \boxed{3 \div 10 \ = 0.3}$ $\qquad$ $0.02 = \boxed{2 \div 100 \ = 2/100}$

$9/100 = \boxed{9 \div 100 = 0.09}$ $\qquad$ $0.86 = \boxed{86 \div 100 = 86/100}$

$23/100 = \boxed{23 \div 100 = 0.23}$

Direct students to turn off their calculators and to give decimal names for these fractions. (Answers are in boxes.)

$2/10 = \boxed{0.2}$ $\qquad$ $9/10 = \boxed{0.9}$ $\qquad$ $30/100 = \boxed{0.30}$

$30/10 = \boxed{3}$ $\qquad$ $500/10 = \boxed{50}$ $\qquad$ $400/100 = \boxed{4}$

$32/10 = \boxed{3.2}$ $\qquad$ $507/10 = \boxed{50.7}$ $\qquad$ $430/100 = \boxed{4.3}$

Exercise 3

Write this expression on the board. $\qquad 7 - 5 \times 3 - 2$

T: ***Zef signed his name this way but forgot to put in the parentheses. Find some numbers Zef could be.***

Let students work independently on this problem for a few minutes. Collect possibilities and record them on the board. A complete list is provided here.

$(7 - 5) \times (3 - 2) = 2$ $\qquad$ $7 - ((5 \times 3) - 2) = \widehat{6}$

$(7 - (5 \times 3)) - 2 = \widehat{10}$ $\qquad$ $7 - (5 \times (3 - 2)) = 2$

$((7 - 5) \times 3) - 2 = 4$

T: ***If we press*** $\boxed{7}\ \boxed{-}\ \boxed{5}\ \boxed{\times}\ \boxed{3}\ \boxed{-}\ \boxed{2}\ \boxed{=}$***, which number will be on the display?***

S: *4.*†

Record this calculator expression on the board next to the appropriate number sentence.

Point to this number sentence and ask the class how to get the calculator to do the calculations in the order called for by the parentheses. $\qquad (7 - 5) \times (3 - 2) = 2$

†This res□
subtraction (using the algebraic order of operations), the result will be ⁻10. Adjust the exercise to fit your calculators.

Go over the problem carefully to review the use of memory keys. Because the order in multiplication does not affect the answer (multiplication is commutative), either of the following sequences of key presses can be used.

$\boxed{3}\ \boxed{-}\ \boxed{2}\ \boxed{\text{M+}}\ \boxed{7}\ \boxed{-}\ \boxed{5}\ \boxed{\times}\ \boxed{\text{MRC}}\ \boxed{=}\quad 2$

$\boxed{7}\ \boxed{-}\ \boxed{5}\ \boxed{\text{M+}}\ \boxed{3}\ \boxed{-}\ \boxed{2}\ \boxed{\times}\ \boxed{\text{MRC}}\ \boxed{=}\quad 2$

Record one of these expressions on the board next to the appropriate number sentence.

Allow time for students to try to find ways of using the calculator to do any of the other four strings of calculations. Check solutions collectively. The class may only find one or two solutions. A list showing one possibility for each is provided here.

$(7 - 5) \times (3 - 2) = 2$ $\qquad \boxed{3}\ \boxed{-}\ \boxed{2}\ \boxed{\text{M+}}\ \boxed{7}\ \boxed{-}\ \boxed{5}\ \boxed{\times}\ \boxed{\text{MRC}}\ \boxed{=}\quad 2$

$(7 - (5 \times 3)) - 2 = \widehat{10}$ $\qquad \boxed{5}\ \boxed{\times}\ \boxed{3}\ \boxed{\text{M+}}\ \boxed{7}\ \boxed{-}\ \boxed{\text{MRC}}\ \boxed{-}\ \boxed{2}\ \boxed{=}\quad {}^{-}10$

$((7 - 5) \times 3) - 2 = 4$ $\qquad \boxed{7}\ \boxed{-}\ \boxed{5}\ \boxed{\times}\ \boxed{3}\ \boxed{-}\ \boxed{2}\ \boxed{=}\quad 4$

$7 - ((5 \times 3) - 2) = \widehat{6}$ $\qquad \boxed{5}\ \boxed{\times}\ \boxed{3}\ \boxed{-}\ \boxed{2}\ \boxed{\text{M+}}\ \boxed{7}\ \boxed{-}\ \boxed{\text{MRC}}\ \boxed{=}\quad {}^{-}6$

$7 - (5 \times (3 - 2)) = 2$ $\qquad \boxed{3}\ \boxed{-}\ \boxed{2}\ \boxed{\times}\ \boxed{5}\ \boxed{\text{M+}}\ \boxed{7}\ \boxed{-}\ \boxed{\text{MRC}}\ \boxed{=}\quad 2$

(uses commutativity of ×)

$\boxed{3}\ \boxed{-}\ \boxed{2}\ \boxed{\text{M+}}\ \boxed{5}\ \boxed{\times}\ \boxed{\text{MRC}}\ \boxed{\text{MC}}\ \boxed{\text{M+}}\ \boxed{7}\ \boxed{-}\ \boxed{\text{MRC}}\ \boxed{=}\quad 2$

(uses a memory clear key $\boxed{\text{MC}}$; using $\boxed{\text{MRC}}$ again may also clear the memory)

Exercise 4

Note: This exercise may be omitted if your students have calculators that display fractions.

T: ***Using a calculator, we can get a decimal close to or equal to the sum, difference, product, or quotient of two fractions. The calculator is limited by the number of digits it can display. Let's first look at a simple fraction problem that we can do easily in our heads. What is $\frac{1}{4} + \frac{3}{4}$?***

S: ***$\frac{4}{4}$ or 1. $\frac{1}{4} + \frac{3}{4} = 1$.***

$\frac{1}{4} \times \frac{3}{4} = 1$

T: ***How could we have done the calculation on a calculator?***

S: ***Use the division names for the fractions.***

$1 \div 4 \times 3 \div 4$

T: ***Where do parentheses belong?***

Invite a student to put in parentheses.

$(1 \div 4) \times (3 \div 4)$

T: ***What keys can we press to do these calculations?***

S: $\boxed{1}\ \boxed{\div}\ \boxed{4}\ \boxed{\text{M+}}\ \boxed{3}\ \boxed{\div}\ \boxed{4}\ \boxed{+}\ \boxed{\text{MRC}}\ \boxed{=}$.

N32

Let students try the sequence of keys suggested, and check that the result is 1 on the display. Repeat the activity to find $\frac{2}{5} + \frac{1}{4}$, $\frac{3}{2} \times \frac{1}{5}$, and $\frac{3}{8} \div \frac{3}{4}$ on the calculator.

$$\frac{2}{5} + \frac{1}{4} = (2 \div 5) + (1 \div 4)$$

$\boxed{2}\ \boxed{\div}\ \boxed{5}\ \boxed{\text{M+}}\ \boxed{1}\ \boxed{\div}\ \boxed{4}\ \boxed{+}\ \boxed{\text{MRC}}\ \boxed{=}$ 0.65

$$\frac{3}{2} \times \frac{1}{5} = (3 \div 2) \times (1 \div 5)$$

$\boxed{3}\ \boxed{\div}\ \boxed{2}\ \boxed{\text{M+}}\ \boxed{1}\ \boxed{\div}\ \boxed{5}\ \boxed{\times}\ \boxed{\text{MRC}}\ \boxed{=}$ 0.3

$$\frac{3}{8} \div \frac{3}{4} = (3 \div 8) \div (3 \div 4)$$

$\boxed{3}\ \boxed{\div}\ \boxed{4}\ \boxed{\text{M+}}\ \boxed{3}\ \boxed{\div}\ \boxed{8}\ \boxed{\div}\ \boxed{\text{MRC}}\ \boxed{=}$ 0.5

Note that the order is very important with division: $4 \div 2 \neq 2 \div 4$. In this last problem, we do not have a choice as to whether to put $3 \div 4$ in memory or $3 \div 8$ in memory.

Writing/Home Activity

Suggest that students write a letter to a parent or sibling explaining how to use a calculator to add fractions.

Capsule Lesson Summary

Solve some division problems in the context of planning a trip. Decide how to use a calculator to find a whole number quotient and remainder for a division calculation.

Materials

Teacher
- Colored chalk

Student
- Paper
- Calculator

Description of Lesson

For this lesson, students should not have calculators for Exercise 1, but they should have calculators for Exercise 2.

Exercise 1

Begin the exercise with a discussion about what might be part of planning a trip. In particular, include questions about what problems might need to be solved in the planning. For example, students may suggest a variety of cost related problems, questions about routes (where to go) and distances, or decisions about where to stay or what to see.

If possible, pose a division problem related to something mentioned in the opening discussion. For example:

T: ***The Rathburns plan a two week (14 day) trip, mapping out a route that covers a total of 3 820 miles. Ms. Rathburn asks, "How many miles will we need to average a day to cover this distance in the two weeks?"***

Can you help answer Ms. Rathburn's question? What calculation should she do?

S: ***Divide 3 820 miles by 14 days to find a number of miles to cover in one day.***

S: ***Estimate and check. Then try to get a closer estimate.***

Instruct students to do the division calculation on their papers in any way they like. Then check whether results are reasonable. For example:

Algorithm

272	R = 12
14)3 820	
−2 800	200
1020	
−700	50
320	
−280	20
40	
−28	2
12	

Estimation

200 × 14 = 2 800	low
250 × 14 = 3 500	low
300 × 14 = 4 200	high
275 × 14 = 3 850	high
270 × 14 = 3 780	low
272 × 14 = 3 808	low
273 × 14 = 3 822	close

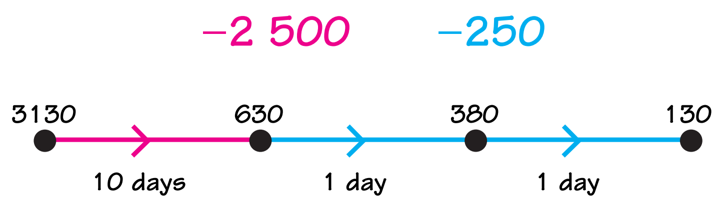
The class should decide that the Rathburns should average very close to 273 miles per day.

Pose a similar problem, but this time one in which the average is given and the number of days is in question. For example:

T: ***The Andersons plan a trip with a route that covers 3 130 miles. Mr. Anderson thinks they can average 250 miles per day. He wants to know how many days they should plan for their trip.***

What calculation could we do to solve Mr. Anderson's problem?

Students should again decide that this is a division problem that they could answer using one of their division methods. Here, you may get a suggestion to draw an arrow picture.



12 days at 250 miles/day
and 1 day at 130 miles

13 days

Invite students to pose other trip planning problems that involve division calculations.

Exercise 2

Refer students to their calculators. Explain that very often when people have division problems like those involved in planning a trip, they use a calculator.

Write this division problem on the board and ask,

$$48\overline{)13\,900}$$

T: ***Suppose you need to find a whole number quotient and a remainder for this problem. How would you use the calculator to do this?***

Let students comment. Perhaps someone will say,

S: ***Press*** $\boxed{1}\boxed{3}\boxed{9}\boxed{0}\boxed{0}\ \boxed{\div}\ \boxed{4}\boxed{8}\ \boxed{=}$.

T: ***Do it. What do you get on the calculator display?***

S: ***289.58333.***

Discuss this result. Students should observe that the calculator most likely truncated the answer because it could only display eight digits.

T: ***Remember, we want to find the whole number quotient and the remainder. Can we do that with this result?***

S: ***289 is the quotient, but the calculator does not display the remainder.***

T: ***Since the calculator does not display the remainder, how can we find it?***

Let students work on this problem for a moment. Then solicit some possible explanations. There are several methods that might be suggested.

- Multiply the divisor (48) times the quotient (289). Then subtract this result from the dividend (13 900).

$$48 \times 289 = 13\,872 \text{ and } 13\,900 - 13\,872 = 28$$

$$\begin{array}{r} 289 \text{ R} = 28 \\ 48\overline{)13\,900} \end{array}$$

- Multiply the divisor (48) times the decimal part of the result (0.58333). The result should be close to the remainder; the remainder is the closest whole number.

$$48 \times 0.58333 = 27.99984$$

$$\text{remainder is } 28$$

- Estimate the remainder; then compare the fraction $\frac{\text{divisor}}{\text{remainder}}$ to the decimal part of the result.

  For example, with the result 289.58333 on the calculator, the decimal part is greater than ½. So an estimate of the remainder should be greater than 24.

  Try 26: $26/48 = 26 \div 48 \approx 0.5416666 < 0.58333$.
  Try 30: $30/48 = 30 \div 48 = 0.625 > 0.58333$.
  Try 28: $28/48 = 28 \div 48 \approx 0.5833333$.

Put several division problems on the board, and instruct students to use their calculators to find a whole number quotient and remainder for each problem.

$79\overline{)2\,694}$ $\quad$ $65\overline{)8\,923}$ $\quad$ $128\overline{)7\,556}$ $\quad$ $311\overline{)19\,762}$

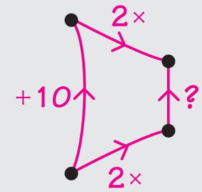
Writing Activity

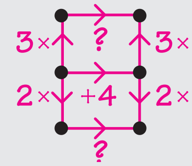
Suggest students write directions on how to use a calculator to find a whole number quotient and remainder for a division problem.

N34 COMPOSITION #2

Capsule Lesson Summary

Using composition of addition and multiplication functions, label arrows in pictures such as the ones shown here. With calculator support, label addition, subtraction, multiplication, and division arrows that complete a cycle.





Materials

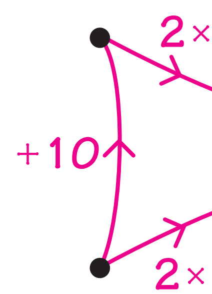
Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Calculator
- Worksheets N34*, **, and ***

Description of Lesson

Exercise 1

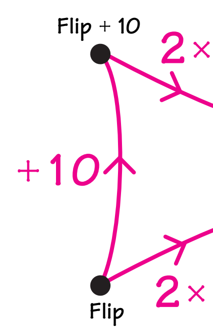
Draw this picture on the board and trace the unlabeled arrow as you ask,



T: ***What could this arrow be for?***

Allow a few minutes for students to consider the situation; then proceed with the following discussion to show that the unlabeled arrow could be for +20.

Label the lower left dot *Flip*. Progressively label the other dots as students announce how.



T: ***Suppose some number Flip is here*** (point to the starting dot of the +10 arrow)***. What can we say about the number here*** (trace the +10 arrow to its ending dot)***?***

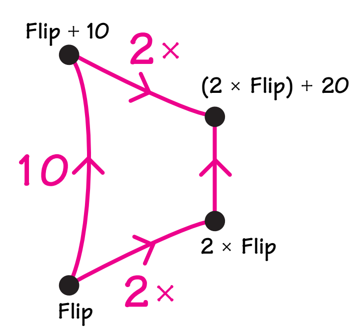
S: ***It is ten more than Flip.***

Trace the lower 2x arrow to its ending dot.

T: ***What do we know about the number here?***

S: ***It is 2*** x ***Flip.***

Trace the upper 2x arrow to its ending dot.



T: ***What do we know about the number here?***

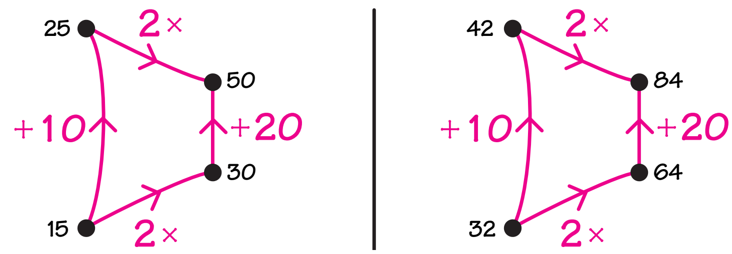
S: ***It is*** 2x ***(Flip + 10).***

S: ***2*x *(Flip + 10) = (Flip + 10) + (Flip + 10) = (2*** x ***Flip) + 20.***

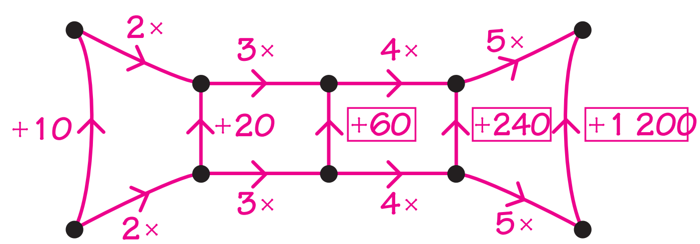
T (pointing to the unlabeled arrow)**:** ***What could this arrow be for?***

S: ***+20.***

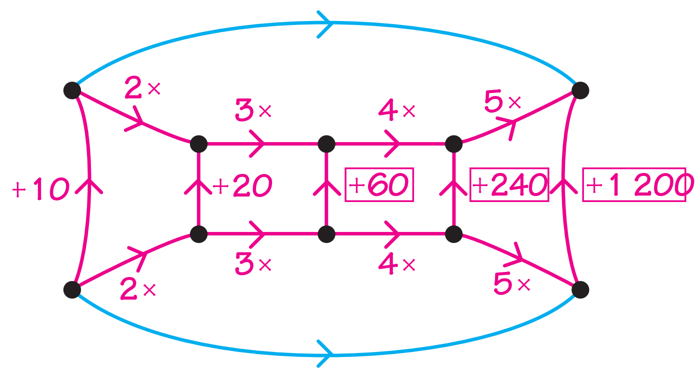
Illustrate the +20 relationship by choosing several different numbers for Flip and then labeling the other dots. For example:



Continue this activity by extending the picture, each time adding three arrows and asking students to provide a label for one of them. (Answers are in boxes.)



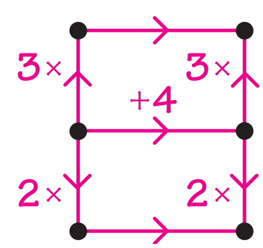
At this point, add these two blue arrows and ask for their labels.



S: ***Both blue arrows are for 120x. 2x followed by 3x followed by 4x followed by 5x is 120x.***

Exercise 2

Draw this arrow picture on the board.



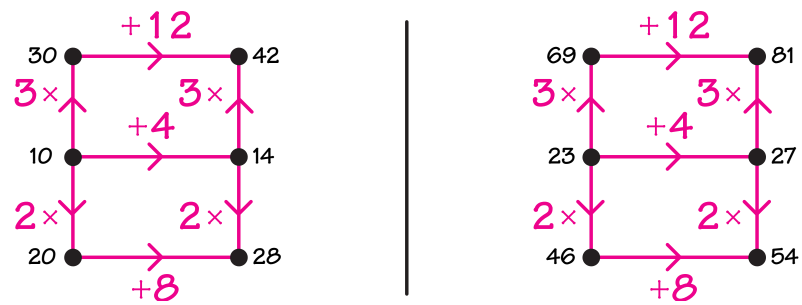
T: ***How could we label the top and bottom arrows?***

Allow a few minutes for students to study the situation.

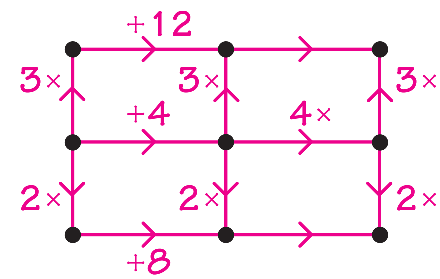
S: ***The top arrow is for +12.***

S: ***The bottom arrow is for +8.***

Illustrate the +12 and +8 relationships by choosing several numbers for the dot at the start of the +4 arrow and labeling the remaining dots. For example:

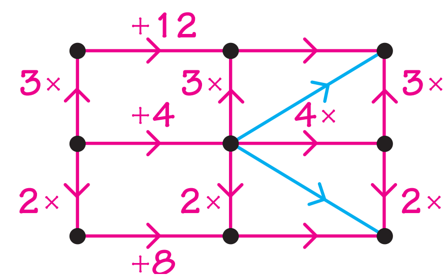


Extend the arrow picture as shown.



T: ***What could these arrows be for*** (trace the unlabeled arrows)***?***

A good hint to give if students are having difficulty is to draw blue arrows as in the next picture.

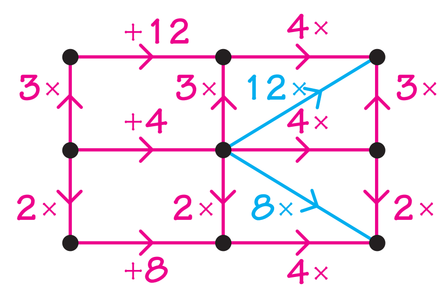


Trace the 4x arrow, then the 3x arrow following it, and finally the upper blue arrow as you say,

T: ***What is 4x followed by 3x?***

S: ***12x.***

Label the upper blue arrow 12x. Trace the middle 3x arrow, then the arrow following it, and finally the 12x arrow as you ask,

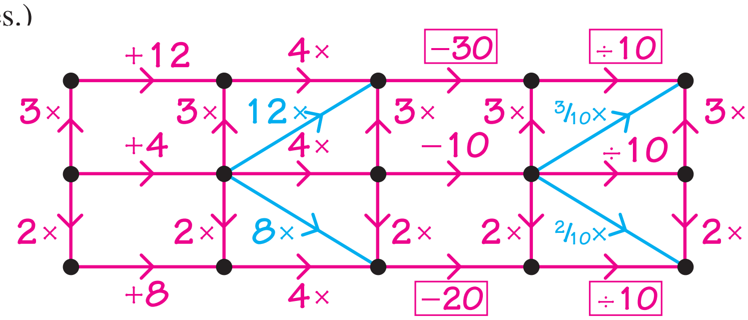


T: ***3x followed by what kind of arrow is 12x?***

S: ***4x.***

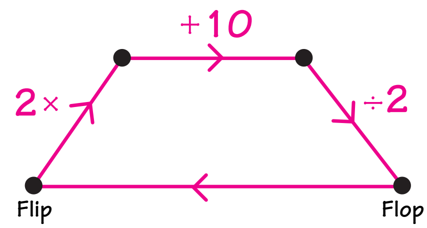
Use a similar line of questioning to find that the lower blue arrow could be for 8x and that the other unlabeled arrow could be for 4x.

Continue this activity by extending the picture, each time adding five arrows to the picture and asking students to provide the labels for two of them. Use supplementary arrows when appropriate. (Answers are in boxes.)



Exercise 3

Draw this arrow picture on the board.



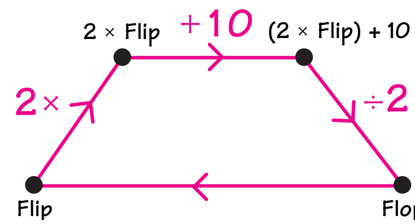
T: ***Choose any number you wish for Flip and display it on your calculator. Hide the display, and then push the appropriate keys to follow the arrow road until you come to Flop. Next, come back to Flip with just one operation. Check to see if the number you started with is back on the display.***

Allow a few minutes for students to experiment.

T: ***What could this arrow be for*** (trace the unlabeled arrow)***?***

S: ***–5.***

Trace the 2x arrow and point to its ending dot as you ask,



T: ***What can you tell me about the number here?***

S: ***It is 2 x Flip.***

Trace the +10 arrow and point to its ending dot as you ask,

T: ***And what about the number here?***

S: ***It is (2 x Flip) + 10.***

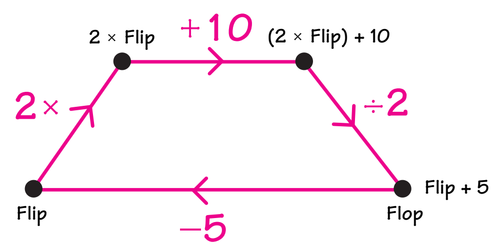
T (tracing the ÷2 arrow)**:** ***What can you tell me about Flop? Be careful!***

S: ***Flop is Flip + 5.***

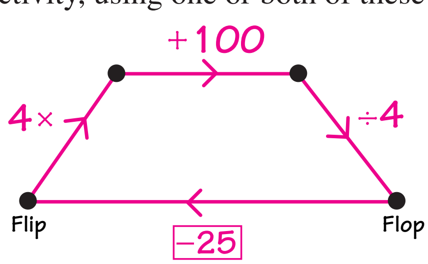
S: ***Divide 2 x Flip by 2, and divide 10 by 2.***

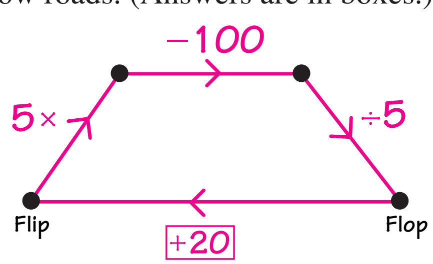
T (tracing the unlabeled arrow)**:** ***What could this arrow be for?***

S: ***–5.***

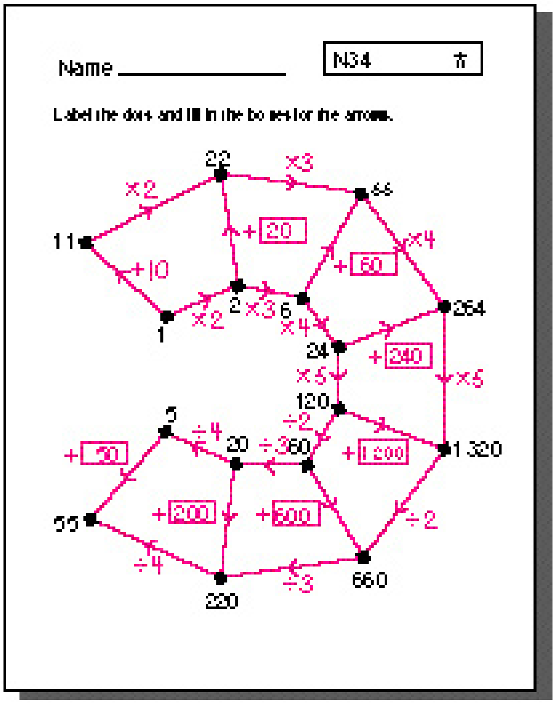


Repeat this activity, using one or both of these two arrow roads. (Answers are in boxes.)

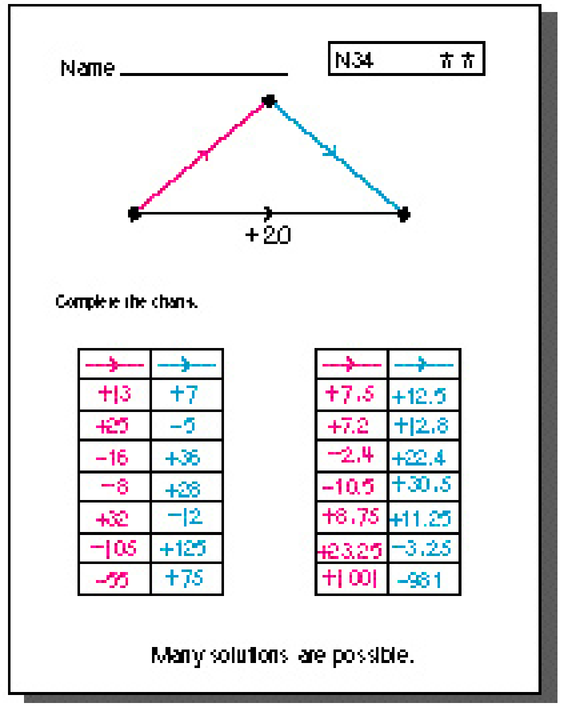




Worksheets N34*, **, and *** are available for individual work.


Name ____________ N34 ☆

Label the dots and fill in the boxes for the arrows.

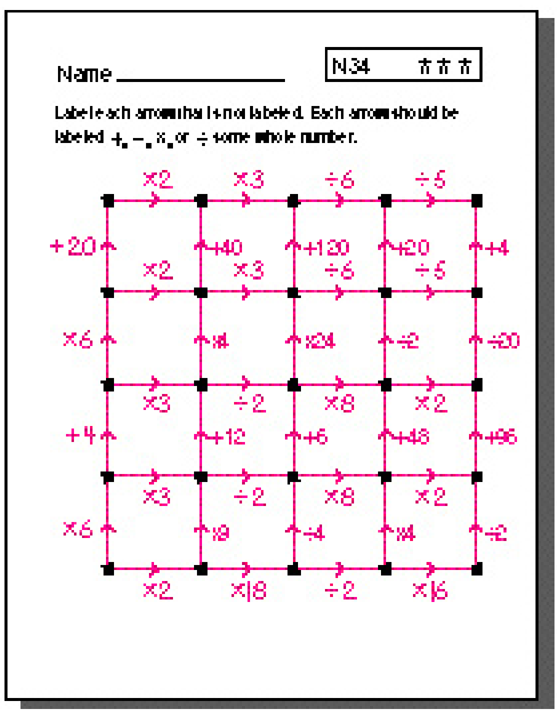

Name ____________ N34 ☆☆

Complete the charts.

→	→
+13	+7
+25	−5
−16	+36
−8	+28
+32	−12
−105	+125
−55	+75

→	→
+7.5	+12.5
+7.2	+12.8
−2.4	+22.4
−10.5	+30.5
+8.75	+11.25
+23.25	−3.25
+1001	−981

Many solutions are possible.


Name ____________ N34 ☆☆☆

Label each arrow that is not labeled. Each arrow should be labeled +, −, ×, or ÷ some whole number.

Capsule Lesson Summary

Examine characteristics of the sum of two multiples of 3, the sum of a multiple of 3 and a non-multiple of 3, and the sum of two non-multiples of 3. Solve a detective story in which the clues involve multiples of 3, patterns generated by moving one checker in a configuration on the Minicomputer, a string picture, and an arrow picture.

Materials

Teacher		Student	
	• Colored chalk		• Colored pencils, pens, or crayons • Paper • Worksheets N35*, **, ***, and ****

Description of Lesson

Exercise 1

Write this expression on the board.

□ + △

T: ***Suppose we put a multiple of 3 in the box and a multiple of 3 in the triangle. When we add the two numbers, what can we say about the sum?***

Allow a few minutes for students to consider the situation.

S: ***The sum is a multiple of 3. A multiple of 3 plus a multiple of 3 is a multiple of 3.***

Invite students to check this by choosing multiples of 3 to put in the box and in the triangle. Then have them add the numbers and note that indeed the sum is a multiple of 3. For example, 9 + 36 = 45;
12 + 15 = 27; and 27 + 45 = 72.

Repeat this activity with other characteristics for the numbers in the box and the triangle. Two situations are outlined below. Be sure to illustrate these facts with several examples.

□ **: Multiple of 3** △ **: Non-multiple of 3** **Sum:** Non-multiple of 3

Examples

6 + 2 = 8
12 + 7 = 19
21 + 11 = 32
24 + 13 = 37

□ **: Non-multiple of 3** △ **: Non-multiple of 3** **Sum:** The sum can be either a multiple of 3 or a non-multiple of 3 depending on the choice of non-multiples of 3.

Examples

5 + 1 = 6 (Multiple of 3)
5 + 2 = 7 (Non-multiple of 3)
16 + 22 = 38 (Non-multiple of 3)
25 + 38 = 63 (Multiple of 3)
17 + 26 = 43 (Non-multiple of 3)

Exercise 2

Present this detective story about a secret number, Clip.

Clue 1

T: ***Clip is a multiple of 3. Name some multiples of 3.***

Invite students to name several multiples of 3. You may request some specific examples such as the following:

T: ***What is the least number greater than 1 000 that Clip could be?***

S: ***1 002. 999 is a multiple of 3, and 999 + 3 is the next greater multiple of 3. 999 + 3 = 1002.***

T: ***What is the greatest number less than 2 000 that Clip could be?***

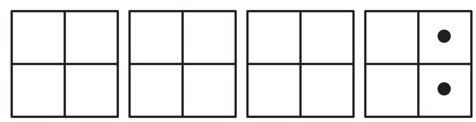
S: ***1 998. 999 + 999 = 1998.***

T: ***What is a number between 230 and 235 that Clip could be?***

S: ***231 or 234.***

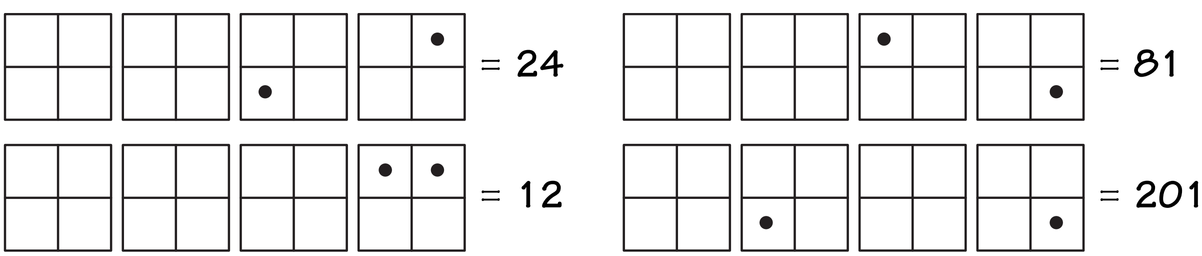
Clue 2

Display four Minicomputer boards with the standard configuration for 5.



T: ***If we move one of these checkers to another square, we can get Clip. Which numbers could Clip be?***

Invite students to move one checker on the Minicomputer and find possibilities for Clip. As necessary, remind students that Clip is a multiple of 3. Some possibilities for Clip are shown below.



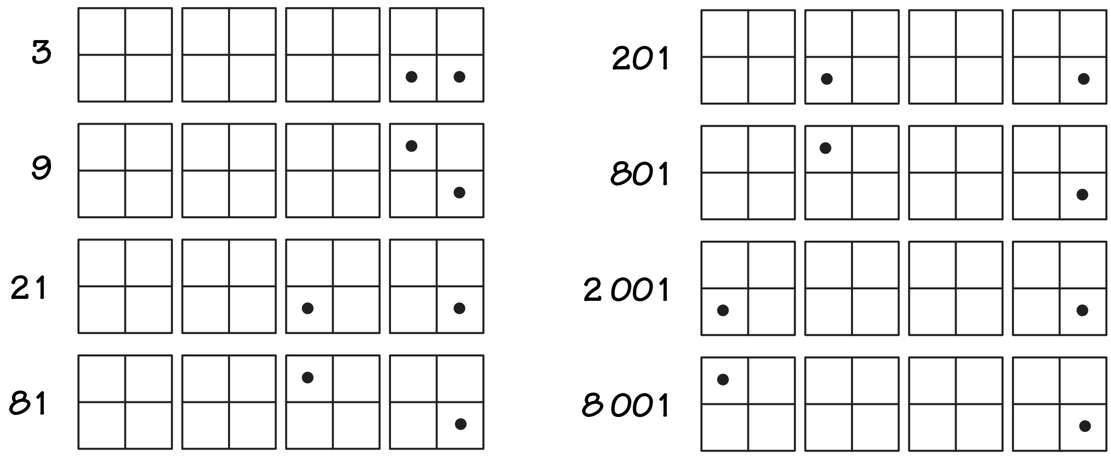
When it is obvious that most students understand the clue, continue as follows.

T: ***Let's try to be more systematic about finding numbers that Clip could be. Perhaps we will observe a pattern.***

Let's move the checker on the 4-square to other squares. Each time the number is a multiple of 3, raise your hand.

Start by moving the checker on the 4-square to the 1-square, then to the 2-square, the 8-square, the 10-square, the 20-square, and so on.

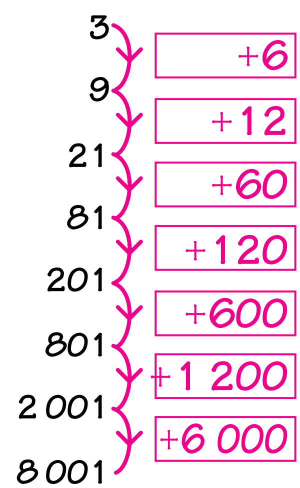
Each time, check the number and see whether or not students raise their hands. On the board, make a list of the multiples of 3 you get this way. Continue until the checker that was originally on the 4-square has been moved to all of the other squares on the four boards. Your list of numbers for Clip should have these eight entries.



T: ***What patterns do you notice?***

S: ***The number is a multiple of 3 when the checker is moved to a red square or a brown square.***

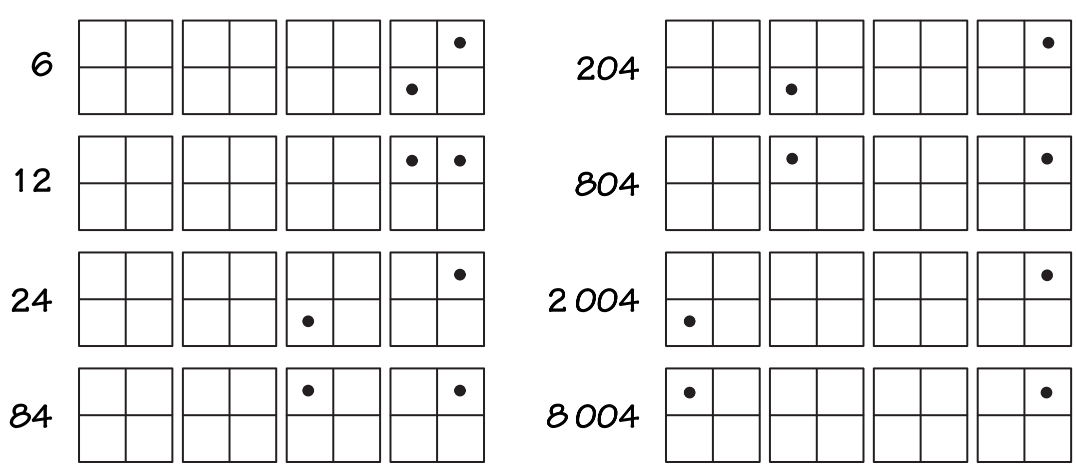
Add these arrows to your list and ask students to label them. (Answers are in boxes.)



T: ***Now, I will return the checker to the 4-square and move the checker on the 1-square. Again, raise your hand when the number is a multiple of 3.***

Start by moving the checker on the 1-square to the 2-square, then to the 4-square, the 8-square, the 10-square, the 20-square, and so on.

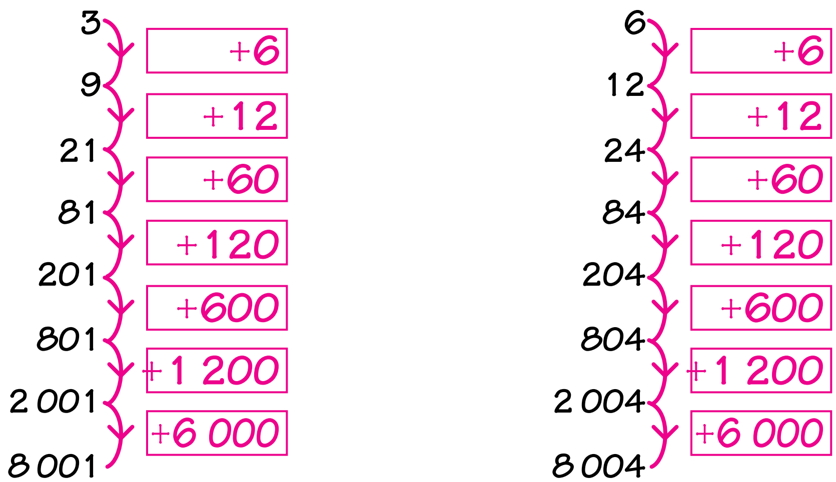
As before, make a list of the numbers for Clip (multiples of 3) you obtain. Students should observe a similar pattern.



N35

S: ***The number is a multiple of 3 when the checker is moved to a red square or a brown square.***

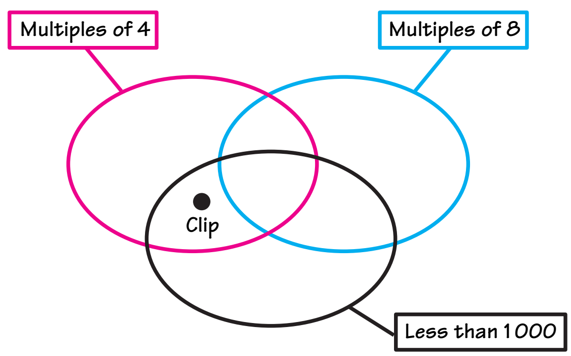
Draw red arrows in the same manner as for the first list and ask students to label them. (Answers are in boxes.)



Clue 3

Draw this string picture on the board.

T: ***Clip is a multiple of 4 and less than 1 000, but Clip is not a multiple of 8. Who could Clip be?***



Of the numbers in the lists from the second clue, students should conclude that Clip is one of these numbers.

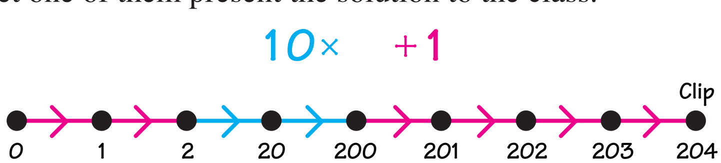
Clue 4

Clip: 12, 84, 204, 804

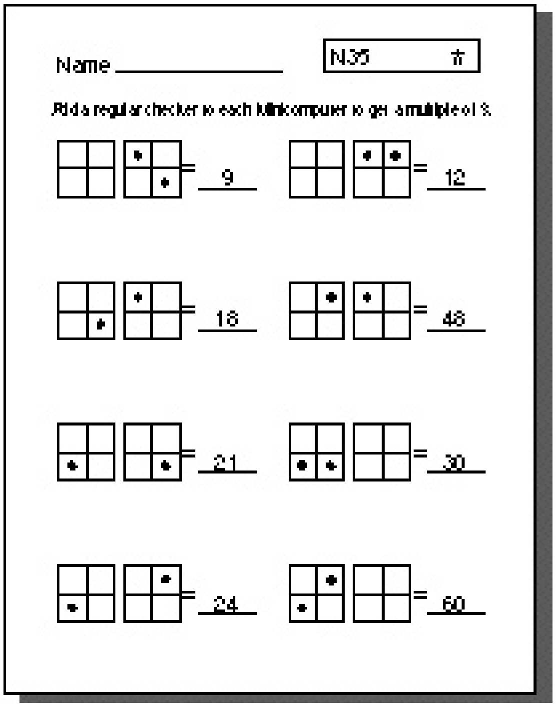
On the board, draw a dot for 0 and arrow keys for 10x and +1.

T: ***Clip is the ending number of an arrow road that has exactly eight arrows; the arrows can be either 10*x *or +1.***

Instruct students to work on this clue with a partner. When several students have discovered the secret number (204), let one of them present the solution to the class.



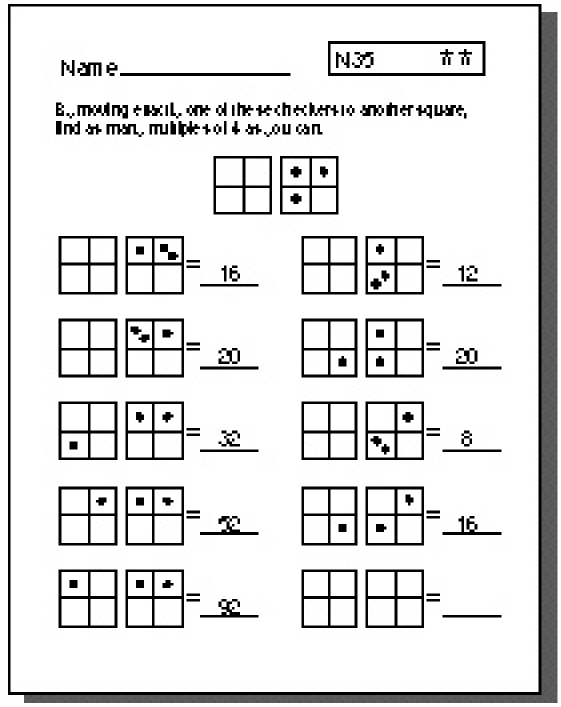
Worksheets N35*, **, ***, and **** are available for individual work.



Name ____________ N35 ☆

Put a regular checker on each Minicomputer to get a multiple of 3.

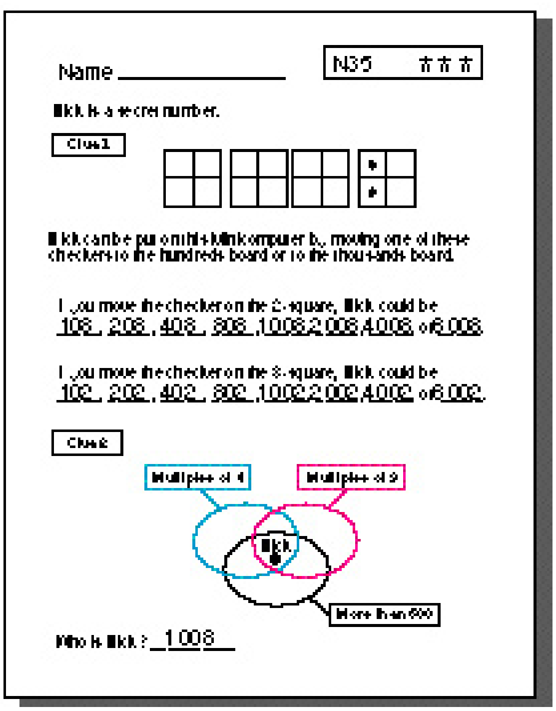
= 9
= 12
= 18
= 48
= 21
= 30
= 24
= 60



Name ____________ N35 ☆☆

By moving exactly one of these checkers to another square, find as many multiples of 4 as you can.

= 16
= 12
= 20
= 20
= 32
= 8
= 52
= 16
= 92
= ____



Name ____________ N35 ☆☆☆

Mick is a secret number.

Clue 1

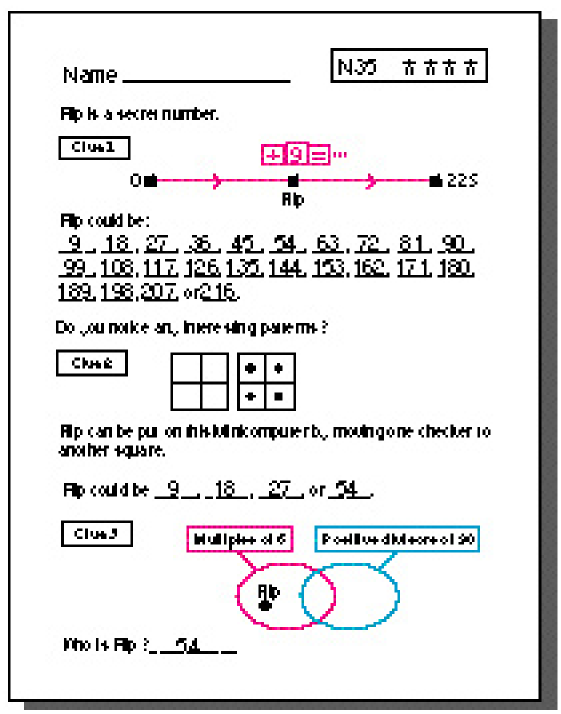
Mick can be put on this Minicomputer by moving one of these checkers to the hundreds board or to the thousands board.

If you move the checker on the 2-square, Mick could be 108, 208, 408, 808, 1008, 2008, 4008, or 8008.

If you move the checker on the 8-square, Mick could be 102, 202, 402, 802, 1002, 2002, 4002, or 8002.

Clue 2

Who is Mick? 1008



Name ____________ N35 ☆☆☆☆

Flip is a secret number.

Clue 1

Flip could be: 9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, 108, 117, 126, 135, 144, 153, 162, 171, 180, 189, 198, 207, or 216.

Do you notice any interesting patterns?

Clue 2

Flip can be put on this Minicomputer by moving one checker to another square.

Flip could be 9, 18, 27, or 54.

Clue 3

Who is Flip? 54

N36 CALCULATOR PUZZLES #3

Capsule Lesson Summary

Put selected numbers on the display of a calculator using a restricted set of keys. Explore ways to use this restricted calculator to multiply by numbers less than 100. Play *Calculator Golf*.

Materials

Teacher
- Calculator
- Colored chalk

Student
- Calculator
- Paper
- Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

List these calculator keys on the board, and refer to them as you explain the following:

T: ***Today the puzzles require that you use only these keys on the calculator. Try to put these numbers on the display of your calculator, and try to do so as cheaply as possible. It costs 1¢ each time you press a key.***

2 3 8 9

+ − × ÷ =

Put several numbers on the board; for example, 41, 171, 0.5, and 22.5. Let students work independently or with partners for several minutes. Encourage students to find several solutions. Some possible solutions are presented below.

Number	Keys	Cost
41:	2 2 + 2 2 − 3 =	8¢
	8 2 ÷ 2 =	5¢
	2 × 8 = = + 9 =	8¢
	3 9 + 2 =	5¢
171:	8 8 × 2 − 3 − 2 =	9¢
	3 3 3 + 9 ÷ 2 =	8¢
	3 2 8 ÷ 2 + 9 − 2 =	10¢
	3 9 3 − 2 2 2 =	8¢
	9 × 9 × 2 + 9 =	8¢

Number	Keys	Cost
0.5:	8 ÷ 2 ÷ 8 =	6¢
	2 ÷ 2 = =	5¢
	2 ÷ 8 × 2 =	6¢
	9 ÷ 2 − 2 = =	7¢
22.5:	2 + 3 × 9 ÷ 2 =	8¢
	2 ÷ 2 = = + 2 2 =	9¢
	8 9 ÷ 2 − 2 2 =	8¢
	3 3 ÷ 2 + 3 = =	8¢
	2 3 + 2 2 ÷ 2 =	8¢

Exercise 2

Pose a new puzzle with only these four keys available.

T: ***Suppose we have only these keys to use. What are some numbers we can multiply by?***

2 3 × ÷ =

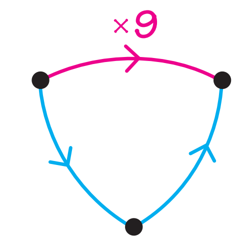
Let students respond freely. Most likely the students will suggest numbers such as 2, 23, 322, and so on. Record some of their responses on the board.

T: ***Can we multiply by 9?***

A likely response might be this:

S: ***No,*** $\boxed{9}$ ***is not one of the keys we can use.***

Draw this arrow picture on the board.



T: ***It is true that we are not using the*** $\boxed{9}$ ***key, but let's see if we can find a way to do x9. What could these blue arrows be for?***

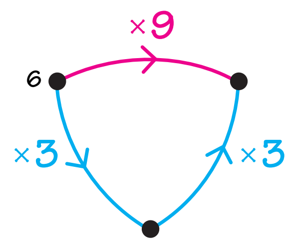
S: x*3 followed by* x*3.*

T: ***How could we do that on a calculator?***

S: ***Press*** $\boxed{\times}\ \boxed{3}\ \boxed{\times}\ \boxed{3}\ \boxed{=}$.

T: ***Let's see if that works.***

Label the starting dot of the x9 arrow with any number, perhaps one suggested by the class; for example:



T: ***Put 6 on your calculator.*** (Point to the ending dot of the x9 arrow.) ***What is this dot for?***

S: ***54.***

T: ***Press*** $\boxed{\times}\ \boxed{3}\ \boxed{\times}\ \boxed{3}\ \boxed{=}$. ***Does it work?***

S: ***Yes.***

Verify that these keystrokes work by trying other numbers. Repeat this activity, asking for ways to multiply by 6, by 11, and by 8.

×6	$\boxed{\times}\ \boxed{3}\ \boxed{\times}\ \boxed{2}\ \boxed{=}$ or $\boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{3}\ \boxed{=}$
×11	$\boxed{\times}\ \boxed{2}\ \boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$ or $\boxed{\times}\ \boxed{3}\ \boxed{3}\ \boxed{\div}\ \boxed{3}\ \boxed{=}$
×8	$\boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{2}\ \boxed{=}$

Exercise 3

Ask the class to recall some of what they remember about golf and the *Minicomputer Golf* game.

T: ***Today we are going to play*** **Calculator Golf*****, a game similar to*** **Minicomputer Golf*****. We start with a number on the display of the calculator and set a goal.***

Put this information on the board.

Starting Number: 17
Goal: 500
Keys: $\boxed{2}$ $\boxed{3}$
$\boxed{+}$ $\boxed{-}$ $\boxed{\times}$ $\boxed{\div}$ $\boxed{=}$

T: ***Suppose we start with 17*** (put 17 on the calculator display) ***and make 500 the goal. When you play this golf game, you can press an operation key ($\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, or $\boxed{\div}$) followed by number keys (only $\boxed{2}$ or $\boxed{3}$) and then $\boxed{=}$. Play continues until 500 is on the display.***

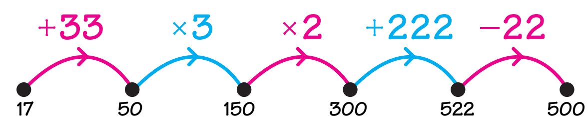
Put 17 on your display. Try to get 500. You can add, subtract, multiply, or divide by any number using only the digits 2 and 3.

Allow a few minutes for students to work independently or with partners on this problem. Then ask students to share their solutions.

Note: You may ask students to keep track of their steps using an arrow picture where each step has an operation, a number, and then $\boxed{=}$. Then, recall that in golf one tries to reach the goal with as few strokes as possible. Here some students may try to reach the goal with few steps.

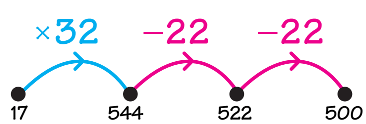
As a student describes a solution, draw a corresponding arrow road on the board.

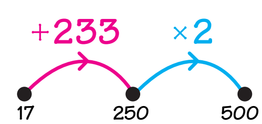
S: ***I pressed $\boxed{+}$ $\boxed{3}\boxed{3}$ $\boxed{=}$,***
then $\boxed{\times}$ $\boxed{3}$ $\boxed{=}$,
then $\boxed{\times}$ $\boxed{2}$ $\boxed{=}$,
then $\boxed{+}$ $\boxed{2}\boxed{2}\boxed{2}$ $\boxed{=}$,
and finally $\boxed{-}$ $\boxed{2}\boxed{2}$ $\boxed{=}$.

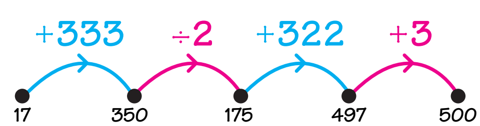


T: ***This solution has five steps. Did anyone find a solution with fewer steps? Is it possible?***

Continue this activity until several solutions are on the board. Several shorter solutions are shown here for your information.







Variations: Play the game with other number keys allowed during play, with different starting numbers and different goals, or in teams. In a team game, the object is not necessarily to use few steps. Rather, alternating teams, the winner is the first team to put the goal on the display under the same play conditions. You or your students may find other creative ways to vary the game.

Home Activity

Send home a description of *Calculator Golf* and suggest that students play the game with family members.