Geometry & Measurement

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GEOMETRY & MEASUREMENT INTRODUCTION

Geometry is useful and interesting, an integral part of mathematics. Geometric thinking permeates mathematics and relates to most of the applications of mathematics. Geometric concepts provide a vehicle for exploring the rich connections between arithmetic concepts (such as numbers and calculations) and physical concepts (such as length, area, symmetry, and shape). Even as children experience arithmetic notions in their daily lives, they encounter geometric notions. Common opinion recognizes that each child should begin developing a number sense early. There is no sound argument to postpone the development of geometric sense. These are compelling reasons why the systematic development of geometric thinking should be a substantial component of an elementary school curriculum.

A benefit of early development of geometric thinking is that it provides opportunities for all learners. Children develop concepts at individual rates. We are careful to allow students a great deal of time to explore quantitative relationships among physical objects before studying arithmetic. In the same spirit, students should be allowed to explore geometric relationships through free play and guided experiences. Before studying geometric properties of lines and shapes, they should have free play with models and with drawing instruments, using both to create new figures and patterns. Before considering parallelism and perpendicularity as relations on geometric lines, they should have free play in creating patterns with parallel and perpendicular lines. Before learning formulas for finding measurements, they should have informal experiences to establish and sharpen their intuitive notions. These early experiences are aesthetically rewarding, and just as stimulating and suggestive of underlying concepts as early experiences leading to arithmetic. Such experiences are a crucial first step in the development of a geometric sense.

A variety of constructions forms the basis for the geometry of *CSMP Mathematics for the Intermediate Grades*. Students use tools to explore geometric concepts, directly discovering their properties and interrelationships. The tools include a straightedge, a compass, mirrors, angle templates, and a translator for drawing parallel lines. The constructions foster insights into the properties of shapes, independent of the measurement of those properties. Only after students are familiar with the shapes do they begin to use rulers and protractors to measure lengths and angles. In this sequential development of geometric ideas, measurement is viewed as the intersection of geometric concepts and arithmetic concepts.

The focus of this strand is experience. The measurement activities guide students to refine their ability to measure accurately lengths of line segments and areas of polygons, as well as to begin a study of the area of a circle. Another sequence of lessons allows students to explore geometric constructions through use of a common tool, a compass. As a natural consequence of their involvement in these activities, students develop their knowledge and skills in geometry.
The effects of this informal approach should be judged by the long-term effects on students' knowledge, confidence, intuition, and interest in the world of geometry and measurement.

Content Overview

Geometric Constructions

Most of our geometry so far has been concerned with points and lines. Parallelism was the central concept of our study of points and lines. Through a variety of informal experiences, students developed their geometric intuition of points and lines and their relations in constructions based on parallelograms. The circle is another fundamental geometric shape. The lessons on geometric constructions this semester provide experience with circles and their relations with points and lines. In earlier lessons, students learned to perform accurate constructions, using a translator. In *IG-V* students learns care and precision in drawing circles using a compass. The compass is an ancient geometric tool with a tradition predating the earliest works on geometry. Its power in geometric study increases with the sophistication of the student.

The first activities reinforce the concepts of *center* and *radius* of a circle. A series of constructions present an opportunity to practice using a compass and to develop a sense of the dependence among the radii of various circles.

While earlier lessons on parallelism provided several applications related to other strands of the curriculum—especially in uses of the number line construction—no direct comparison of lengths of line segments was possible, unless the lines were parallel. In *IG-V* the compass provides a tool for such comparisons. The compass activities proceed with the classical construction of the midpoint of a line segment and of perpendicular line segments, first introduced for constructions by use of a folded paper square corner. As in their work with the translator, students' dynamic interaction with the compass provides a basis for the growth of spatial intuition and an understanding of informal geometric relationships.

Two lessons are based on constructions of polygons under restrictive conditions. The ultimate objective is to bring students to some realization of a relationship between the lengths of given line segments and the number and type of shapes that can be constructed from them. The simple construction of duplicating a given line segment leads to the problem of constructing polygons with a specified number of sides, all the same length. Next, two line segments are given with which to construct quadrilaterals. Discussion of the contrast between the small number of essentially different solution shapes and the infinite variety of areas and orientations that can occur supports students' developing sense of space.

Lessons: G1, 2, 3, 4, 11, and 12.

Measurement

Consensus calls for measurement activities in the elementary curriculum, but with no agreement on the form or scope of these activities. Rather than stress mastery of formulas for perimeter, area, and volume, or for comparison of standard units, the lessons of this strand provide open-ended experiences within rich problem-solving situations. Measurement becomes a means for investigating problems and developing concepts, rather than an end in itself. Direct experiences with the concepts and tools is central. The carefully designed problems do lead to insights concerning accurate measurement and involving area of various shapes, especially circles. The emphasis is on the development of ideas and understanding rather than on the memorization of rules. The measurement activities become the means for investigating problems and developing concepts rather than an end in themselves.

Two lessons this semester involve estimating the areas of various circles and, in the process, developing a formula for the area of a circle which introduces a new (irrational) number π. These lessons, together with a third lesson on area, investigate area patterns for circles related to similar area patterns for polygons. Another lesson this semester looks for a relationship between the diameter and the circumference of a circle, thereby finding another way to estimate π.

Two lessons this semester review calculating volumes of boxes. The first involves finding the volume of an open box constructed from a cardboard square by cutting smaller squares from each corner, and then examining the effect on the box's volume by changing the size of the cut-out squares. The second explores volume patterns for boxes, similar to area patterns for polygons.

The lessons in this strand are supplemented regularly with pages in the *Arcade of Problems* workbooks. In particular, the workbooks contain practice with linear measure and with finding area and perimeter of polygons.

Lessons: G5, 6, 7, 8, 9, and 10.

Note on Grids

Several lesson call for demonstration on a grid board. A portable, commercially made, permanent grid chalkboard is ideal. However, if such a grid board is not available, try one of these alternatives.

- Make a grid transparency for use on an overhead projector and use erasable overhead markers. The Blackline section includes several size grids to use in making a grid transparency.
- Printed grids on large sheets of paper are provided in classroom sets of materials for *CSMP*. Tape one of these sheets to the board, use thick felt-tipped pens for drawings, and discard the sheet after use. You may prefer to laminate or mount one of the sheets on cardboard and cover it with clear contact paper to provide a protective surface. Then use erasable markers or wax crayons for drawings. These grid sheets also are useful as graphing mats.
- For each lesson, carefully draw a grid on the chalkboard using an easy-to-make tool. Either (1) use a music staff drawer with chalk in the first, third, and fifth positions; or (2) clip five pen caps, each having the diameter to hold a piece of chalk tightly, to a ruler at intervals of 4–8 centimeters, and secure them with tape.

G1 COMPASS CONSTRUCTIONS #1

Capsule Lesson Summary

Review some of the language associated with circles and the use of a compass for constructing circles. Practice using the compass by drawing many circles to some specification. Make some observations about the radii of circles in the resulting drawings.

Materials

Teacher
- Chalkboard compass
- Colored chalk
- Straightedge

Student
- Unlined paper
- Colored pencils, pens, or crayons
- Compass

Description of Lesson

In this lesson students will again meet concepts related to circles. The focus is on practice in using a compass to draw circles carefully and accurately, a skill which will be important in the later lessons of this strand. Students' compasses should be of good quality. The joint of the legs should be tight enough to prevent slippage while drawing a circle, but not so tight as to make setting radii difficult. A sharp pencil in the compass will aid accuracy. To help prevent the compass point from slipping on the desk, place a piece of cardboard or a section of newspaper beneath the paper. A safety or pointless compass may be a good alternative, but they have some limitations with respect to creating a circle of a given radius.

Exercise 1

Draw a red dot on the board and display a straightedge.

T: ***Let's suppose we are looking at a treasure map and this dot represents a large rock on an island. The length of this stick (straightedge) represents 20 paces on the treasure map. If a pirate buried a treasure chest 20 paces from this rock, who can show us a place on the map where the treasure could be?***

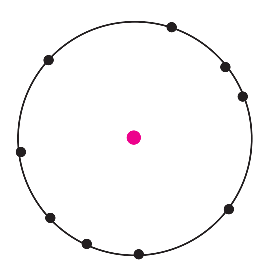
Allow several students to demonstrate where the treasure could be by drawing dots the length of the straightedge away from the red dot.

T: ***How could we describe all of the points where the treasure could be?***

S: ***They are on a circle around the red dot.***

T: ***A compass will help us draw a circle showing all of the places where the treasure might be.***

Demonstrate how to use the compass to draw the circle.



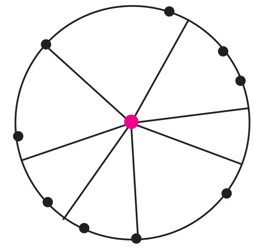
T: ***What do we call the red dot where the rock is in this picture?***

S: ***The center of the circle.***

Draw a radius of the circle.

T: ***A line segment from the center to a point on the circle is a* radius *of the circle.***

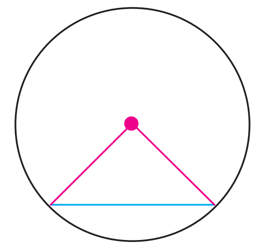
G1



Write *radius* on the board and invite several students to draw radii in the picture.

T: ***Do all of these line segments have the same length?*** (Yes)

Note: You may want to comment that the term *radius* may refer either to a specific line segment or to the length of the line segment. A circle has many radii when we refer to particular line segments, but only one radius when we mean the common length of those line segments. Context usually distinguishes these two uses of the word.



Draw a red dot on the board and invite a student to construct a circle whose center is at the dot. Then draw this triangle in the circle.

T: ***What is special about these two red line segments?***

S: ***They are radii of the circle.***

T: ***Which is greater, the length of the blue line segment or the sum of the lengths of the two red line segments?***

S: ***The sum of the lengths of the two red line segments.***

Erase the red line segments and draw another blue line segment with endpoints on the circle.

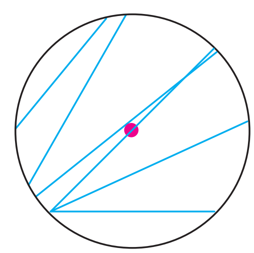
T: ***Which is greater, the length of this blue line segment or twice the length of a radius?***

S: ***Twice the radius.***

If necessary, draw the two radii from the endpoints of the blue line segment to the center of the circle. Continue this activity by asking students to draw a few more blue line segments.

T: ***What is the longest line segment that we can draw with endpoints on this circle? Why?***

S: ***A line segment that goes through the center of the circle will be twice as long as a radius. Any line segment that does not go through the center of the circle will be shorter than twice the length of a radius.***



Invite a student to draw a blue line segment through the center of the circle. Write *diameter* on the board.

T: ***A line segment with endpoints on the circle and passing through the center of the circle is a* diameter.**

Draw another circle on the board, indicating its center with a red dot. Ask several students to draw diameters of this circle. Emphasize that the length of a diameter is twice the length of a radius by asking each student to indicate the two line segments that are part of the diameter and are radii.

Note: As with *radius*, the term *diameter* may refer either to a specific line segment or to the length of the line segment. When we are talking about the lengths of particular line segments, we say that the diameter of a circle is twice the radius.

Exercise 2

For this exercise each student will need a compass and unlined paper. Erase the board and draw a red circle, marking its center with a red dot. After drawing the circle, display your compass with the radius unchanged.

T: ***Look at the compass with which I drew this circle. What is the distance between the point of the compass and the chalk?***

S: ***It is the same as the radius of the circle.***

T: ***Yes. We call that distance the radius of the compass. Of course, we can easily change the radius of the compass by moving the legs closer together or further apart.***

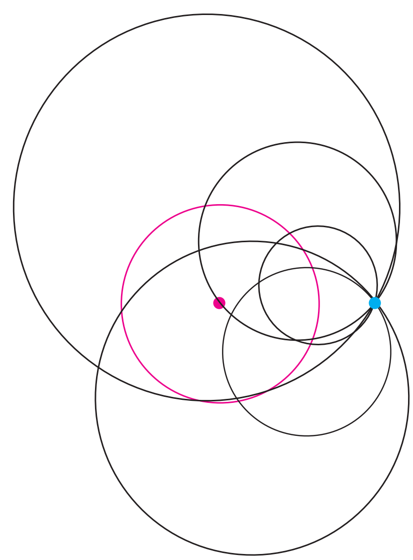
Add a blue dot to your picture about 10 cm outside the red circle and to one side. Change the color of chalk in your board compass.

T: ***Let's draw some more circles on the board. The rule is that the center of each new circle must be on the original red circle and the blue dot must be on each new circle.***

Invite several students to draw circles (new color or white) in the picture, and encourage discussion of the emerging picture.

T: ***Can any point on the red circle be the center of a new circle?*** (Yes)

Let any skeptical student choose a point on the red circle. Then invite another student to construct a new circle according to the rule. When five or six circles have been drawn, the class will be aware of the technique of locating a center and then setting the radius of the compass by placing its point on the prospective center and its chalk on the blue dot.

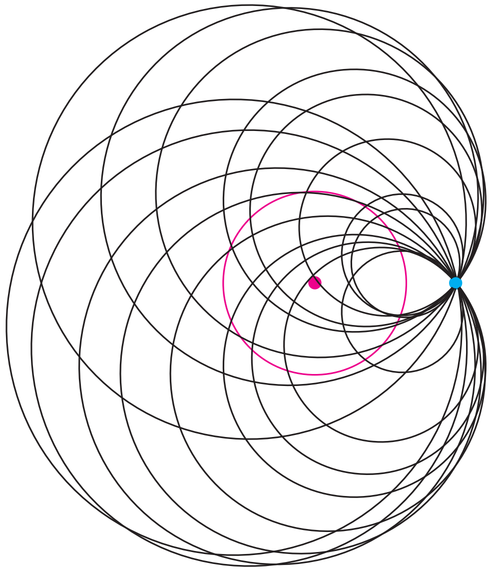


T: ***Now each of you draw a similar picture. Mark a red dot in the center of your paper. Draw a medium-sized circle with the red dot as center, and mark a blue dot outside the circle. Then draw more circles according to this rule: the center of each new circle is on the original circle and the blue dot is on each new circle. Make the red and blue dots small and draw your circles carefully and accurately.***

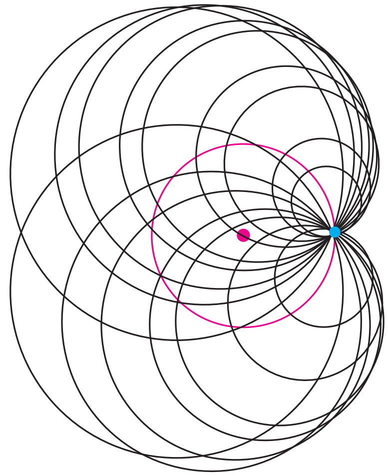
Observe student use of the compass and encourage accuracy of construction. Suggest that students draw many circles, and solicit comments on the emerging picture. After a while, describe a second construction similar to the first, but with the blue dot on the original circle. Begin a similar drawing on the board while the class works, and occasionally ask students to add circles to your picture.

The illustrations on the next page show possible resulting pictures for both constructions. It is not necessary that your students include so many circles in their drawings.

First Construction



Second Construction



During the last five to ten minutes of the class period, lead a discussion of the collection of circles in each picture. Raise questions about the radii of the new circles in each picture. For example:

T: ***In the first picture, where is the center of the circle with the shortest radius?***

S: ***On the line segment connecting the red and blue dots.***

T: ***Where is the center of the circle with the longest radius?***

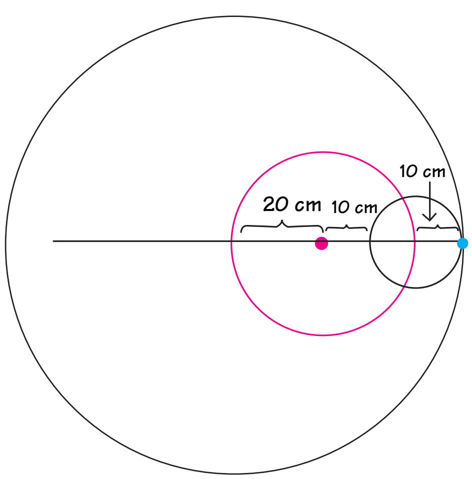
S: ***On the line connecting the red and blue dots but not between the dots.***

T: ***How many circles can be drawn in the picture with each possible radius?***

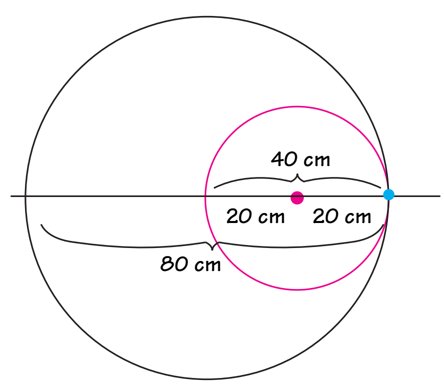
S: ***One each for the longest and the shortest possible radii, and two each for all other possible radii.***

To demonstrate this fact, ask students to find the two circles for several possible radii.

T: ***If the radius of the red circle on the left is 20 cm and the radius of the smallest new circle is 10 cm, what is the radius of the largest new circle?*** (50 cm)



T: ***If the radius of the red circle on the right is 20 cm, what is the radius of the largest new circle?*** (40 cm) ***The diameter of the red circle?*** (40 cm) ***The diameter of the largest new circle?*** (80 cm)



Capsule Lesson Summary

Review the circle constructions of Lesson G1 and present two more similar constructions. Again make observations about the radii in the resulting drawings. Construct a lattice by connecting the centers of a family of circles drawn to a certain specification.

Materials

Teacher
- Chalkboard compass
- Colored chalk
- Straightedge

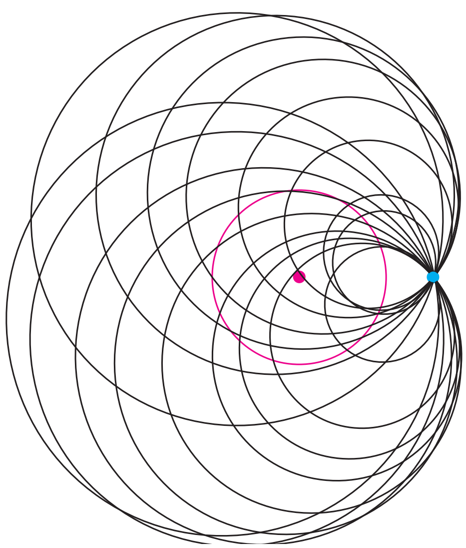
Student
- Unlined paper
- Colored pencils, pens, or crayons
- Compass
- Straightedge

Description of Lesson

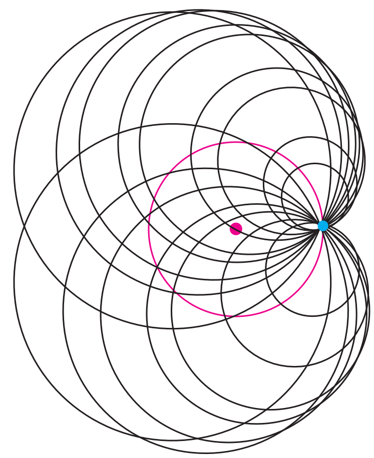
Exercise 1

Lead the class in a collective reconstruction of the drawings of Lesson G1 as shown below.

First Construction

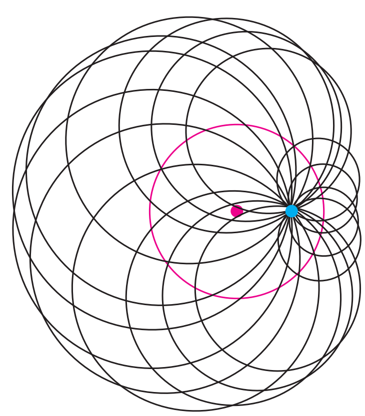


Second Construction



T: ***Now let's make a third drawing. We will start the same way with a red circle and a blue dot, but this time the blue dot will be inside the red circle. Again, we will draw circles with the rule that the center of each new circle must be on the original red circle and the blue dot must be on each new circle.***

With student assistance, add several circles to the third drawing. You need not include as many circles as shown here, but continue letting students draw circles until the class has a good feeling for the emerging picture. You may want to suggest that students make a similar construction on their papers while others are adding to the picture on the board.



G2

As at the end of Lesson G1, lead a discussion of the collection of circles in this third picture. Raise similar questions about the radii of the new circles.

T: ***Where is the center of the circle with the shortest radius?***

S: ***On the line through the red and blue dots but closer to the blue dot.***

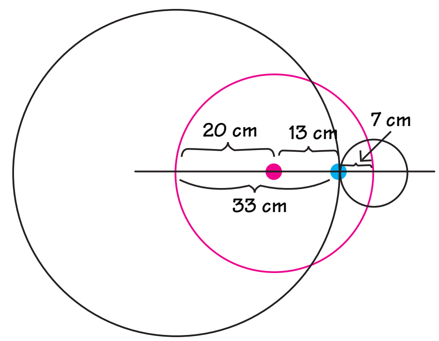
T: ***Where is the center of the circle with the longest radius?***

S: ***On the line through the red and blue dots but closer to the red dot.***

T: ***How many circles can be drawn with each possible radius?***

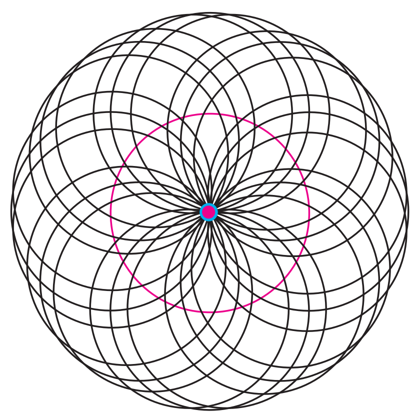
S: ***One each for the shortest and the longest radii and two each for all other possible radii.***

T: ***If the radius of the red circle is 20 cm and the radius of the smallest new circle is 7 cm, what is the radius of the largest new circle?*** (33 cm)

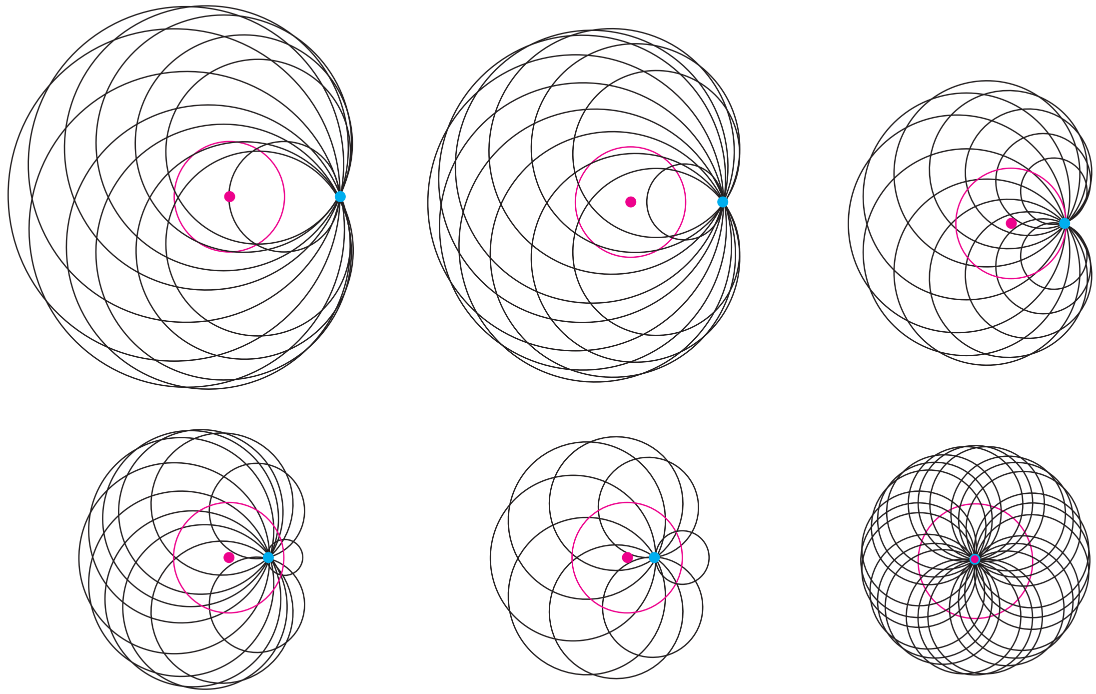


T: ***Suppose we start again, but this time put the blue dot at the center of the red circle; that is, the red and blue dots are at the same point.***

Begin the drawing at the board and invite students to add circles. You may suggest that students also do the construction on their papers. The class should quickly recognize that all of the new circles have the same radius and in fact have the same radius as the red circle.



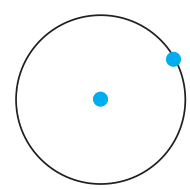
A group project can be made of producing a series of circle constructions similar to those drawn on the board. All original circles should be the same size and located in the same position on each page. The blue dot should be shifted slightly from picture to picture. The resulting sequence of pictures will emphasize the effect of the distance between the red and blue dots on the configuration and sizes of the added circles. The illustration on the next page shows such a sequence of pictures.



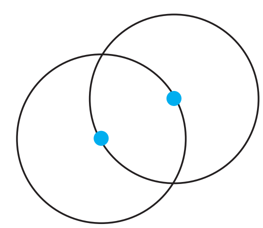
Exercise 2

Each student will need a compass, unlined paper, colored pencils, and a straightedge for this exercise. Begin the following construction on the board as you describe it.

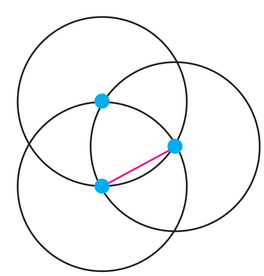
T: ***Here is another construction. First make a dot in blue. Then draw a circle with the blue dot for the center. Next mark another blue dot anywhere on the circle.***



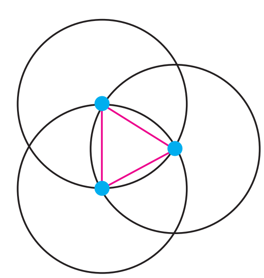
Draw another circle with the second blue dot for center, but do not change the radius of the compass.



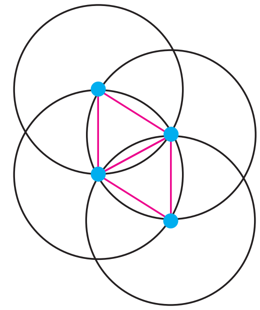
Use a straightedge to connect the blue dots with a red line segment. Now pick a point where the two circles meet, and draw a blue dot. Then draw another circle using the new blue dot for center and the same radius as the first two circles.



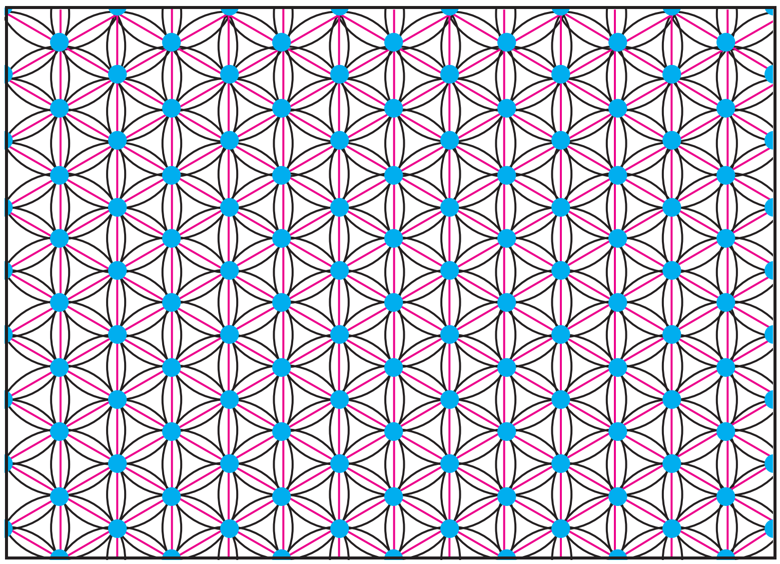
Draw red line segments from the new center to the other blue dots on its circle.



T: ***Continue by picking a point where any two circles meet, drawing a blue dot, and drawing a new circle with that blue dot as center and with the same radius as the other circles. Then draw red line segments from the center to the blue dots on its circle.***



Instruct students to make this drawing on their papers and to try to fill the whole page with circles. Encourage accurate and careful constructions by students to create a design as pictured below.



At the end of the lesson, allow a few minutes for discussing the drawings. Include these observations:

- All of the red segments have the same length and that length is the radius of all the circles.
- There are three families of parallel lines.
- Each circle has at most six blue dots on it.
- Two adjacent segments that lie on a straight line form a diameter of a circle.

Extension Activity

Students may enjoy creating other designs with compass constructions. The ones in these lessons or other creations could be used as cover designs for booklets, or put in a class or school newspaper.

Writing Activity

You may like students to take lesson notes on some, most, or even all their math lessons. The "Lesson Notes" section in Notes to the Teacher gives some suggestions and refers to forms in the Blacklines you may provide for this purpose. With lessons G1 and G2, for example, students may note some designs they made from compass constructions and describe the conditions/properties of the constructions.

G3 PERPENDICULARITY #1

Capsule Lesson Summary

Review the notion of a square corner and produce one by folding paper. Use a square corner to construct line segments perpendicular to given line segments and containing given points. Construct shapes having all square corners.

Materials

Teacher
- Construction paper
- Translator
- Straightedge

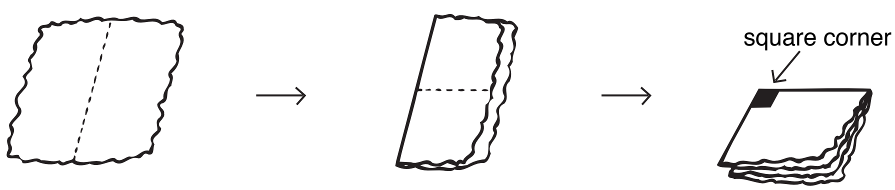
Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets G3(a) and (b)

Note: If you do not have a translator, borrow one from a *CSMP* fourth grade class. A translator is a construction tool for drawing parallel line segments or for checking parallelism. See the lessons on parallelism in the Geometry strand of the *CSMP Mathematics for the Intermediate Grades, Part I.*

Description of Lesson

Exercise 1

Begin by discussing square corners. Illustrate by providing one example, such as the corner of a piece of paper, and call on students to provide other examples in the classroom. Raise the question of how to check suggested examples. Demonstrate folding a piece of construction paper to produce a square corner, as illustrated here. You may wish to start with a ragged piece of paper to emphasize how and where you get a square corner.

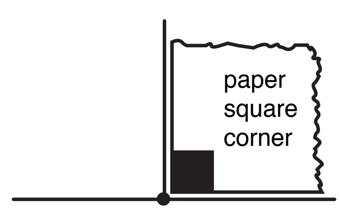


Instruct each student to produce a square corner using a piece of paper. Ask students to mark the square corner, as shown above.

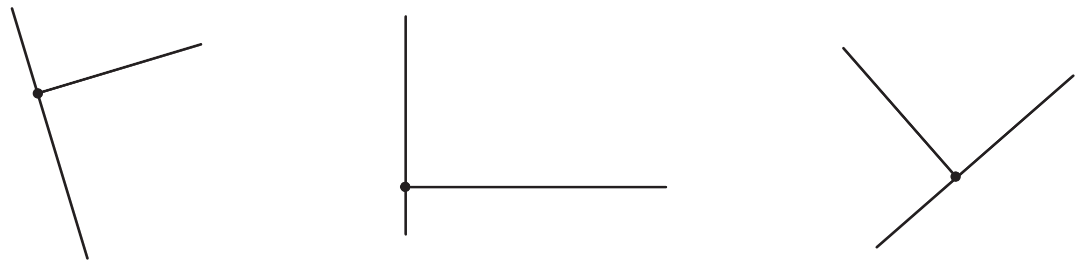
On the board, draw a line segment with a dot on it.

T: ***Let's make a square corner at this point with this line segment for one side of the corner. We can use a paper square corner as a tool.***

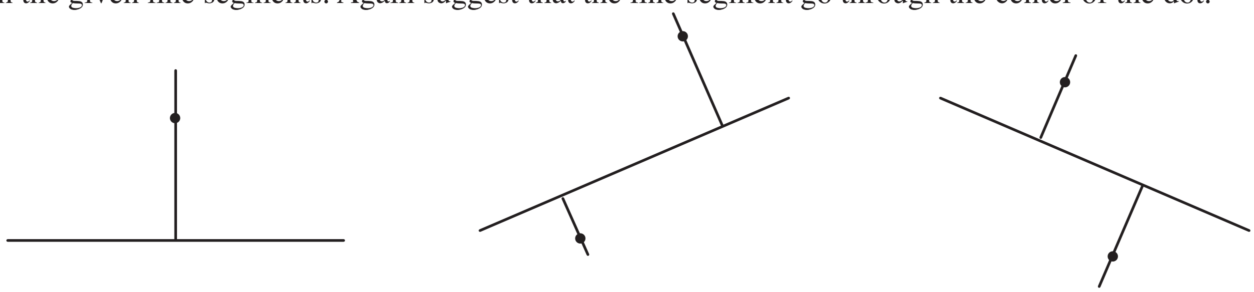
Ask a student to perform the construction using your paper square corner which, being made of construction paper, should survive repeated use. Several other students might check the construction with their paper square corners.



Similarly, ask students to construct square corners in the following situations created by differing orientations of line segments and placements of dots. In each case, suggest that the line segments forming a square corner meet at the center of the indicated dot.



Next, ask students to construct square corners by drawing line segments through dots which are not on the given line segments. Again suggest that the line segment go through the center of the dot.



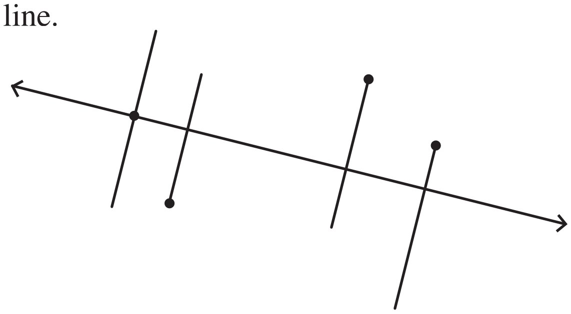
T: ***Two lines or line segments that meet in a square corner are called* perpendicular.**

Write *perpendicular* on the board and call on students to identify several pairs of perpendicular line segments on the board or in the classroom in order to reinforce the word as well as the concept.

Distribute copies of Worksheets G3(a) and (b). Instruct students to use their paper square corners to draw line segments through the given points and perpendicular to the specified line segments. When most students finish at least Worksheet G3(a), continue with Exercise 2.

Exercise 2

Draw a long line segment and several dots on the board, as shown below. Make the segment very long to suggest a line rather than a line segment. For each dot, call on a student to construct a line segment through that point and perpendicular to the given line. Be sure students extend their line segments to cross the given line.

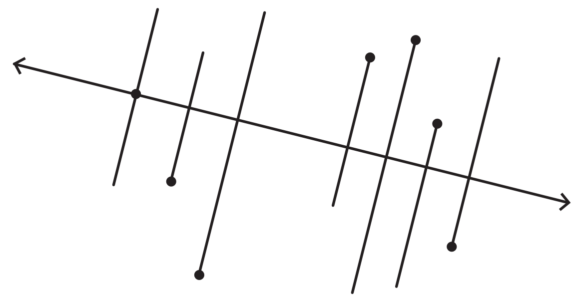


T: ***How many square corners do we have in the drawing so far?***

S: ***16.***

Lead the class to observe that four square corners result from each perpendicular intersection. Moreover, all of the new line segments are parallel. Use a translator to check parallelism.

Continue by adding several more points to the picture and drawing lines perpendicular to the original line. This will emphasize that all of the constructed line segments are parallel to each other as well as perpendicular to the original line. Use a translator to check parallelism.

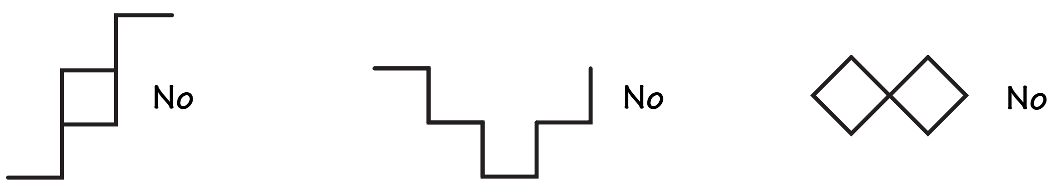


Exercise 3

Organize the class into groups of two to four students. As the exercise progresses, ask each group to write observations about what they find, to justify their conclusions, and to generalize if possible.

T: ***Rectangles are four-sided shapes with only square corners. Can you draw an eight-sided shape that has only square corners? Try it.***

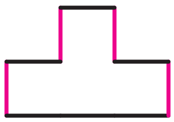
As groups work, elaborate on the rules for the construction: The shape must be closed with no dangling pieces, and each side can meet only two other sides.

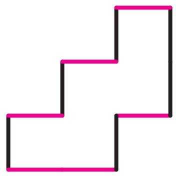


Introduce these rules only when necessary, thereby allowing students greater freedom to explore this problem. As groups discover solutions, invite them to draw their solutions on the board using the square corner made from construction paper. Continue in the same manner by asking for shapes with other even numbers of sides: 6, 10, 12, 14, 16, 18, or 20. To introduce greater variety for the collective discussion, you may ask different groups to use different numbers. Groups that produce several shapes with even numbers of sides may be asked to draw a shape with 7, 9, 11, or 13 sides. There is no solution to the problem for these numbers, but the attempt will be instructive and may lead to some generalizations.

G3

Finish the lesson with a collective discussion of the results. Ask students to draw their solutions on the board and to make a list of numbers with which they were successful (all even numbers more than 2) and unsuccessful (all odd numbers). A good observation to elicit is that in any solution, regardless of the number of sides, there are two families of parallel sides with half of the sides in each family. In the following examples, red sides show one family of parallel sides and black sides show another.


8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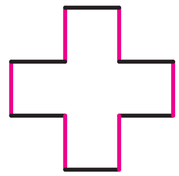
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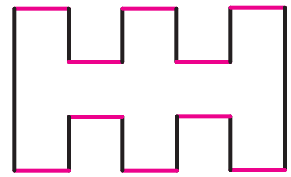
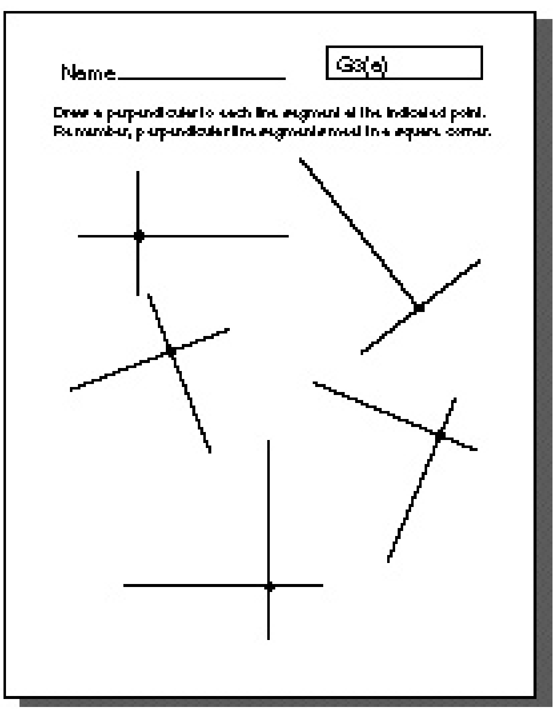
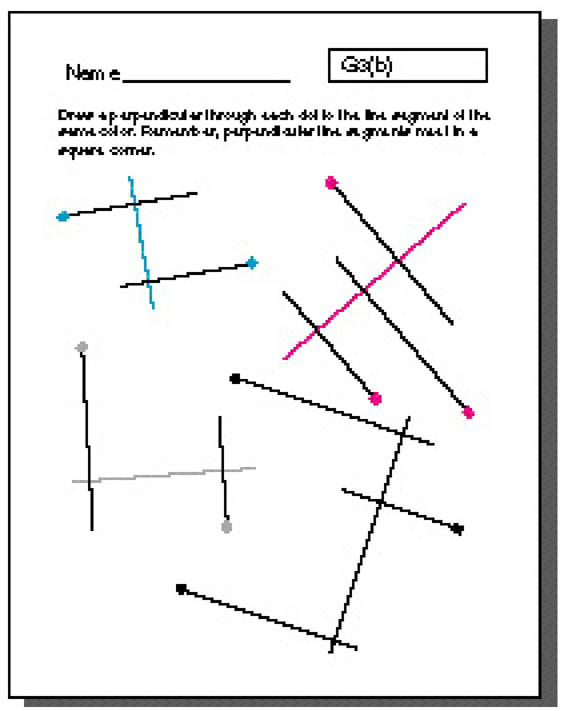

20

Writing Activity

Ask students, perhaps with their groups, to write about some observations they made while attempting to construct shapes with only square corners. Perhaps some students can include an explanation of why it is impossible to have an odd number of sides in a shape with only square corners.





Capsule Lesson Summary

Construct pairs of intersecting circles of the same radius whose centers are two given points on a line segment, and find that the intersection points for pairs of circles are all on a line. Use this construction to locate the midpoint of a line segment and to construct a perpendicular to a line segment through a given point.

Materials

Teacher
- Chalkboard compass
- Colored chalk
- Straightedge
- Paper square corner

Student
- Paper
- Compass
- Straightedge
- Paper square corner
- Colored pencils, pens, or crayons
- Worksheets G4(a), (b), and (c)

Description of Lesson

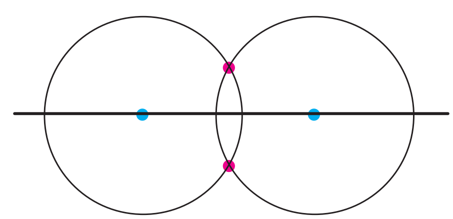
Exercise 1

For this exercise, you may suggest that students follow the construction on the board with a similar construction on their papers.

On the board, draw a line segment about 60 cm long. Draw two dots in blue on the segment; then open the chalkboard compass to more than half the distance between the two blue dots.

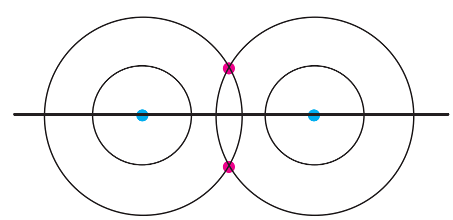
T: ***Can we construct two circles whose centers are these blue dots and each with a radius equal to the radius of this compass?***

Invite a student to perform the construction at the board. Ask another student to indicate where the circles intersect with red dots.



T: ***If we change the radius of this compass, could we construct two circles with the new radius and with centers at the blue dots but that do not intersect?***

Invite a student to perform such a construction.



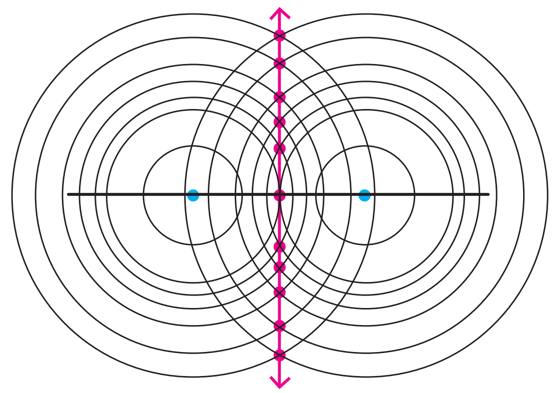
T: ***How long must the radius be to have the two circles intersect?***

S: ***Longer than half the distance between the two blue dots.***

T: ***What will happen if the radius is equal to half the distance between the blue dots?***

S: ***The circles will intersect at one point, a point on the line segment between the blue dots.***

Invite a student to construct this pair of circles intersecting at one point. Then continue to let students construct other pairs of intersecting circles with the same radius and centers at the blue dots. Mark the intersection points with red dots. Your construction should be similar to the picture below.



T: ***What do you notice about the red dots?***

S: ***They are on a straight line.***

Use a straightedge to draw a red line through the red dots.

T: ***What do you notice about the red line and the original line segment?***

S: ***They form square corners where they intersect.***

Verify this observation using a paper square corner.

S: ***They are perpendicular.***

S: ***The red line crosses the original line segment exactly halfway between the two blue dots.***

T: ***The blue dots are the endpoints of a line segment. What do we call the point exactly in the middle of a line segment?***

S: ***The midpoint.***

T: ***Point to the midpoint of the line segment between the blue dots.***

After a student indicates the midpoint, use the compass to verify that it is the midpoint of the line segment between the blue dots. Set the radius of the compass to the distance from one blue dot to the midpoint and compare this radius to the distance from the midpoint to the other blue dot. If necessary, suggest these relationships yourself.

Distribute copies of Worksheet G4(a) and instruct students to construct pictures like the one on the board, using the line segments and the blue dots on the worksheet. After a few minutes, ask students to use a paper square corner to check whether, in each case, the red line is perpendicular to the black line segment. Then ask them to use their compasses to verify that the red line intersects the line segment between the blue dots at its midpoint.

T (pointing to any red dot on the board)**:** ***Is this point closer to the blue dot on the left or on the right?***

S: ***The distance is the same, because the red dot is on two circles with the same radius whose centers are the blue dots.***

Emphasize this fact by checking the distance of several of the red dots to each of the blue dots with the demonstration compass.

Exercise 2

Leave the picture constructed in Exercise 1 on the board. The students will use it later to discover constructions for locating the midpoint of a line segment and drawing a perpendicular to a line segment.

Draw a line segment about 40 cm long on the board.

T: ***In the construction on the board, we found a midpoint of a line segment. The red line crosses at the midpoint of the segment between the two blue dots.***

Suppose we want to find the midpoint of this new line segment without measuring. Where should we draw blue dots?

S: ***At the endpoints of the line segment.***

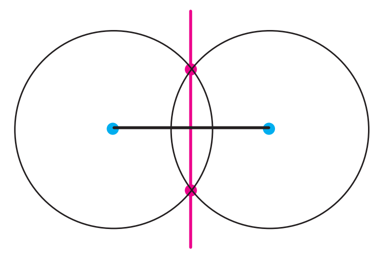
T: ***How should we use these blue dots to locate the midpoint of this line segment?***

Allow students to tell you how to do the construction.

S: ***Draw two circles with centers at the blue dots. They should have the same radius and they should intersect.***

S: ***Draw red dots where the circles intersect, and then draw a red line through the red dots.***

S: ***The red line will cross the original line segment at its midpoint.***



Repeat this activity by drawing another line segment on the board and inviting students to demonstrate the construction for locating the midpoint.

Instruct students to turn to Worksheet G4(b) and to find the midpoint of each line segment on the worksheet. When every student has completed at least one construction, continue the lesson by referring back to the construction of Exercise 1.

T: ***What else did we notice about the red line?***

S: ***It is perpendicular to the original line segment.***

On the board, draw a line segment with a dot several centimeters above it.

T: ***This is a different problem. Let's use a compass to construct a line that is perpendicular to this line segment and that passes through this dot. Have we drawn lines perpendicular to line segments in any of our pictures?***

S: ***In the picture on the board and in each picture on Worksheet G4(a), the red line is perpendicular to the line segment with blue dots on it.***

Color the dot above the line segment red.

T (pointing to the red dot)**:** ***Why do you suppose I made this dot red?***

S: ***Because we are going to draw a perpendicular (red line) through it.***

T: ***What did we notice about the distance from a red dot to each of the two blue dots in our other constructions?***

S: ***A red dot is the same distance from each of the blue dots.***

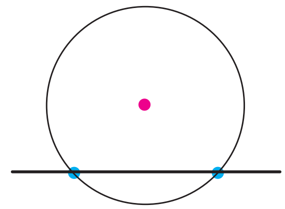
T: ***Let's locate two blue dots on this line segment. How can we do this?***

G4

There are several ways to locate points on the line segment that are equidistant from the red dot. Use any correct method that your class suggests. For example:

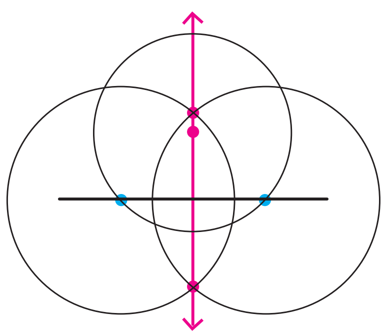
S: ***Draw a circle with its center at the red dot and intersecting the line segment at two points.***

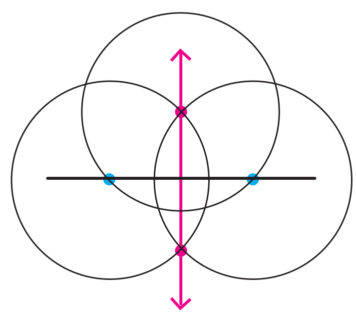
Ask a student to do the construction at the board and to locate two blue dots on the line segment.



T: ***Now that we have located two blue dots on the line segment, how do we draw a line passing through the red dot and perpendicular to this line segment?***

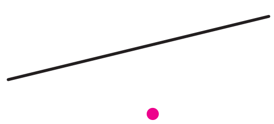
Invite a student to finish the construction by drawing intersecting circles with the same radius and with centers at the blue dots. Then draw a line through their intersection points to obtain the requested perpendicular line that passes through the original red dot. **Note:** Students may suggest that the radius of the intersecting circles with centers at the blue dots be the same as the radius of the circle with center at the red dot. In this case, your construction would look like the one on the right.



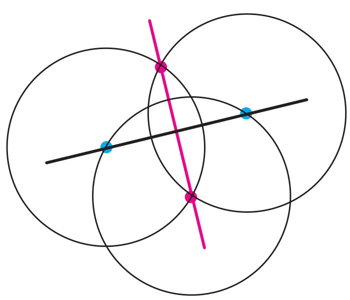


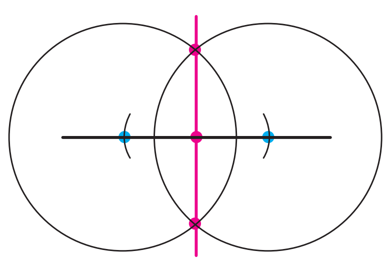
Use a paper square corner to demonstrate that the red line is perpendicular to the original line segment.

Repeat this activity with the following line segment and dot configurations.

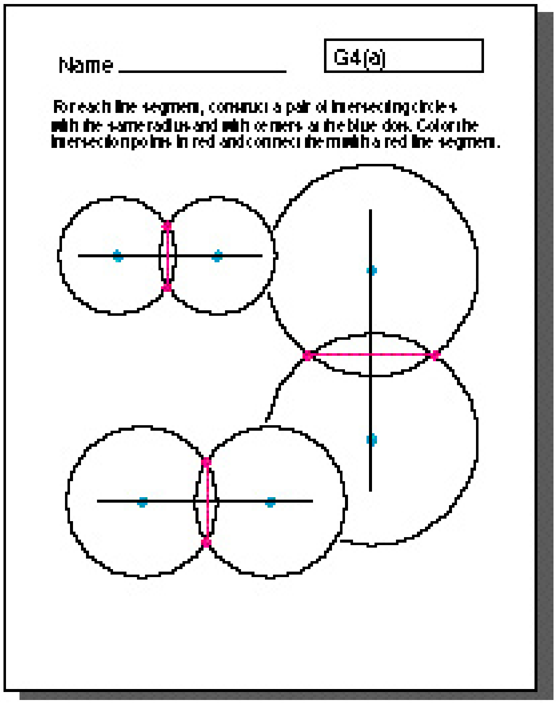


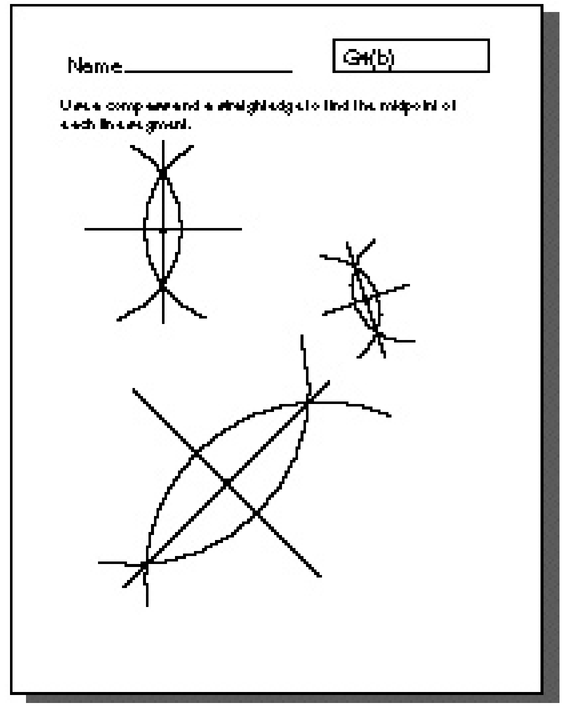
Completed constructions may resemble those below.

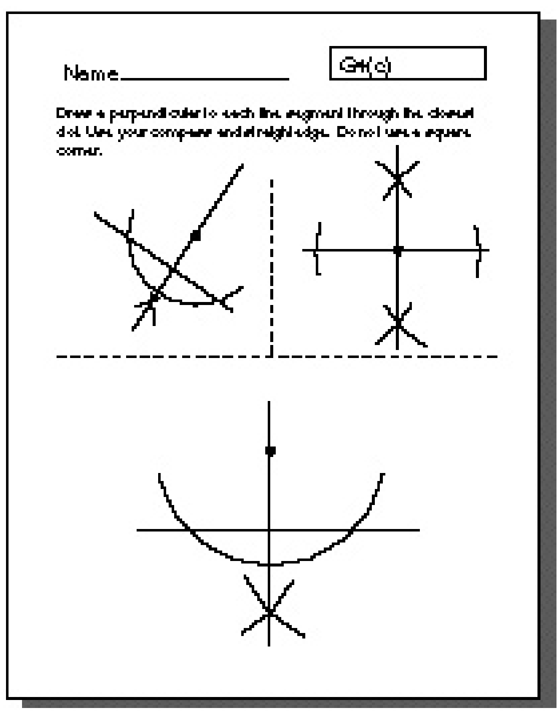




Worksheet G4(c) is available for students to practice this construction.


Name
G4(a)
For each line segment, construct a pair of intersecting circles with the same radius and with centers at the blue dots. Color the intersection points in red and connect them with a red line segment.


Name
G4(b)
Use a compass and a straightedge to find the midpoint of each line segment.


Name
G4(c)
Draw a perpendicular to each line segment through the closest dot. Use your compass and straightedge. Do not use a square corner.

G5 AREA OF A CIRCLE #1

Capsule Lesson Summary

Through experimentation, find that when the lengths of all of the sides of certain polygons are doubled, the areas of those polygons are quadrupled. Estimate the areas of circles drawn on a grid by counting squares inside the circles.

Materials

Teacher
- Translator (optional)
- Meter stick
- Grid board
- Chalkboard compass (optional)
- Colored chalk
- Blackline G5

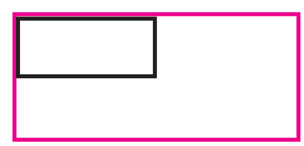
Student
- Translator (optional)
- 1 cm grid paper
- Compass
- Metric ruler
- Calculator
- Paper
- Colored pencils, pens, or crayons
- Worksheets G5(a) and (b)

Advance Preparation: Use Blackline G5 to make copies of 1 cm grid paper for students' use in Exercise 2.
If possible, you may want to borrow a classroom set of translators from a *CSMP* fourth grade class

Description of Lesson

Exercise 1

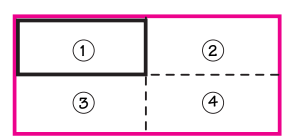
Draw a rectangle on the board. Use a meter stick to draw a red rectangle with sides twice as long as the sides of the first rectangle, as shown here.



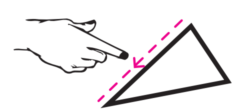
T: ***The red rectangle has sides twice as long as and parallel to the sides of the smaller rectangle. How many times greater in area is the red rectangle than the smaller rectangle?***

S: ***Four times. Four of the smaller rectangles could fit in the red rectangle.***

Let a student illustrate this at the board.



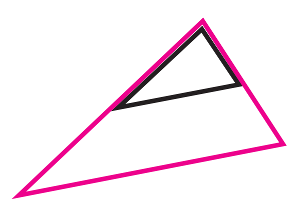
Invite a student to use a straightedge to draw a triangle on the board. Select and trace one side of the triangle.



T: ***Let's draw a red segment twice as long along this side. Start the red segment at one end.***

Let a student draw an appropriate red segment.

Similarly, let a student double the other side of the triangle starting at one end of the red segment. Then complete a red triangle.

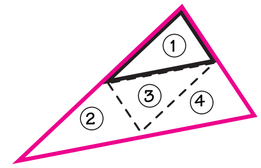


Invite students to use a meter stick to check that the third side of the red triangle is twice as long as the third side of the original triangle. Also, ask students to check that those two sides are parallel (using a translator).

Instruct students to make triangle drawings on their papers similar to the one done on the board.

T: ***Draw any triangle on your paper. Then draw a red triangle with sides twice as long as and parallel to the sides of the first triangle. When you finish, find how many smaller triangles fit into the larger red triangle.***

Allow several minutes for individual work. Then invite a student to illustrate at the board how many smaller triangles fit into the red triangle. All of the students should check that four of their smaller triangles fit into their larger red triangle.

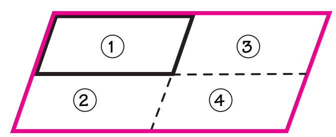


T: ***What happens to the area of a triangle when we double the length of each of its sides?***

S: ***The area is multiplied by four.***

Invite a student to use a translator (or straightedge) to draw a parallelogram on the board. With the class, recall that a parallelogram has opposite sides parallel and of equal length.

Tell each student to use a translator (or straightedge) to draw a parallelogram on his or her paper. Then instruct each student to draw the corresponding red parallelogram with sides twice as long as and parallel to the sides of the smaller parallelogram. After several minutes of individual work, invite a student to do the construction at the board. Note that the area of the red parallelogram is four times the area of the smaller parallelogram.



T: ***When we double the lengths of the sides of a rectangle, a triangle, or a parallelogram, what happens to the area of each of these shapes?***

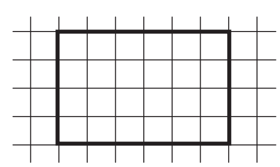
S: ***When we double the lengths of the sides of a shape, we multiply its area by four.***

T: ***What do you think happens to the area of a shape if we triple the lengths of all its sides?***

Let students express their opinions. Then distribute copies of worksheets G5(a) and (b). You may like to let students work with partners to find that when you triple the lengths of the sides of a parallelogram or a triangle, the resulting parallelogram or triangle has area nine times the original.

Exercise 2

Draw a four-by-six rectangle on a grid board.



Area: 24 cm^2

T: ***Suppose the squares of this grid are each one centimeter on a side. What would be the area of this rectangle?***

S: ***24 square centimeters. There are four rows of six squares each.***

S: ***4 x 6 or 24 square centimeters since the rectangle is 4 cm by 6 cm.***

Write the area near the rectangle on the grid board. Remind students of the exponential notation for square centimeters.

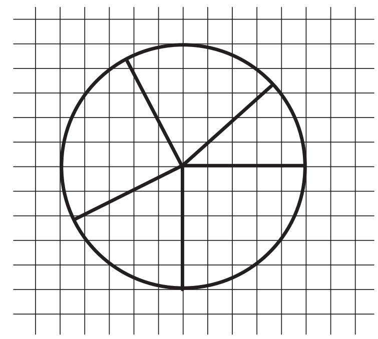
Draw a few more squares and rectangles on the grid, and call on students to calculate their areas.

Use a demonstration compass or a string with a piece of chalk tied to one end to draw a circle with a radius of length 5 on the grid. (See the illustration below.)

T: ***What is a radius of a circle?***

S: ***A line segment between the center of the circle and a point on the circle.***

Invite several students to draw radii of the circle.



T: ***Let's continue to assume that the squares of this grid are 1 cm on a side. What lengths are these radii?***

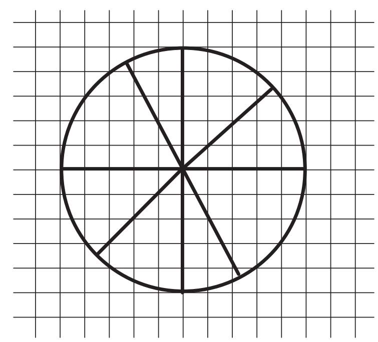
S: ***They are all the same, 5 cm.***

Use a piece of string to confirm that the radii all have the same length.

T: ***What is a diameter of a circle?***

S: ***A line segment across the circle (between two points on the circle) that goes through the center of the circle.***

Invite several students to draw diameters of the circle.



T: ***What lengths are these diameters?***

S: ***They are all the same, 10 cm.***

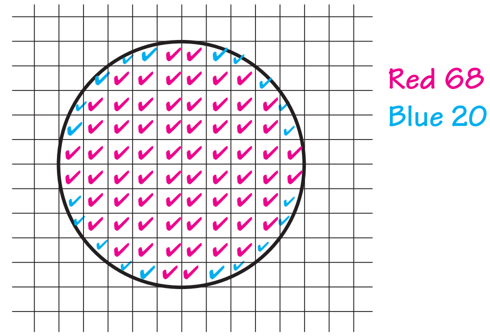
T: ***Our next problem is to find the area of this circle. Finding the area of a square or a rectangle is quite easy; we simply multiply the lengths of adjacent sides (length times width). It is harder to find the area of a circle. How might we find the area of this circle?***

S: ***Count the number of grid squares inside the circle.***

Discuss the difficulty of counting those squares not entirely within the circle. Lead to the following method of estimating the area of a circle.

1) Mark red all squares that are totally inside (or very close to totally inside) the circle and count them.
2) Mark blue all squares that are split by the circle and count them.
3) To estimate the area, add the number of red squares and one-half of the number of blue squares.

Invite several students to help use this procedure with the circle on the grid board.



Note: Here there are eight red squares that are so close to being totally inside the circle that we included them in the red square count. If the numbers of red squares and blue squares in your drawing differ a little from 68 and 20 because of the accuracy of the drawing or because you decide not to include as red the eight squares mentioned above, do not be concerned.

T: ***What is the approximate area of this circle?***

S: ***78 cm^2; count all of the red squares and one-half of the blue squares to get 68 + ($\frac{1}{2}$ x 20) = 78.***

Begin a table on the board.

Radius	Approximate Area of Circle
3 cm	
4 cm	
5 cm	78 cm^2
6 cm	
7 cm	
8 cm	
9 cm	

Equip students with 1 cm grid paper and a compass. Organize the class into five or six groups, and assign each group a different length radius from 3 cm to 9 cm. Instruct students in each group to draw a circle with the specified length radius on the grid paper and then to use the above method to estimate its area. Ask groups to get an approximate area for their circles by agreement or perhaps by averaging individual results. As you observe group work you may suggest that, for larger circles, an efficient way to count red squares and blue squares is to count just the squares in one half (or one-fourth) on the circle and then multiply by two (or four).

After a while, collect results from the groups in the table For example:

Radius	Approximate Area of Circle
3 cm	28 cm^2
4 cm	50 cm^2
5 cm	78 cm^2
6 cm	112 cm^2
7 cm	152 cm^2
8 cm	198 cm^2
9 cm	252 cm^2

T: ***Do you see any patterns in the table? Given the length of a radius of a circle, can you predict its approximate area? What do you think is the approximate area of a circle with a 10 cm radius? 15 cm radius?***

Let students express their ideas. They probably will not reach definite conclusions about the area of a circle. Do not force any relationship between the length of a radius of a circle and its area.

T: ***At the beginning of this lesson, what did we observe happens to the area of a shape when we double the lengths of its sides?***

S: ***The area becomes four times as large.***

T: ***Does it appear that the same thing happens to the area of a circle when we double the length of a radius?***

S: ***In our table, there are examples of doubling the length of a radius: 3 cm to 6 cm or 4 cm to 8 cm. The area of the circle with a 6 cm radius is four times larger than that of the circle with a 3 cm radius; 4 x 28 = 112. The area of the circle with an 8 cm radius is about four times larger than that of the circle with a 4 cm radius; 4 x 50 = 200 and 200 is close to 198.***

T: ***If this pattern is true, what might we expect to be the approximate area of a circle with a 10 cm radius?***

S: ***About 312 cm^2 since the approximate area of a circle with a 5 cm radius is about 78 cm^2 and 4 x 78 = 312.***

Note that the predicted area, 312 cm^2, appears plausible as it is greater than 252 cm^2, the approximate area of a circle with a 9 cm radius.

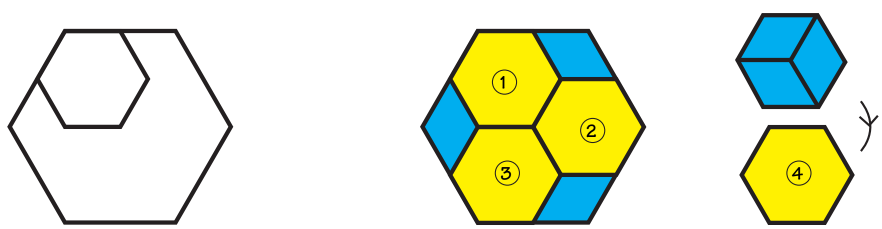
If you have an example in the table, ask also about what happens to the area when you triple the length of the radius.

T: ***In a later lesson, we will learn how to calculate the area of a circle when we know the length of a radius.***

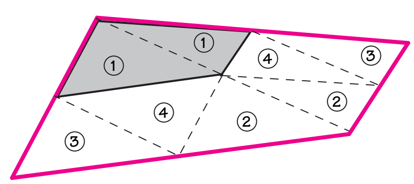
Note: Record the data from the table on a piece of paper and save it for use in Lesson G6 *Area of a Circle #2*.

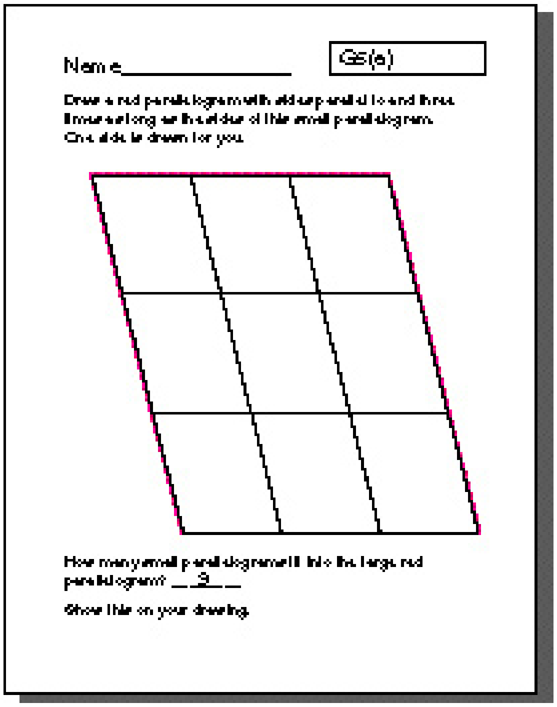
Extension Activity

Suggest that students use pattern blocks to investigate further the effect on area of doubling or tripling the side lengths of a shape, keeping corresponding sides parallel. For example, the following illustration shows a hexagon.



Also, students might experiment making drawings with irregular shapes. For example:



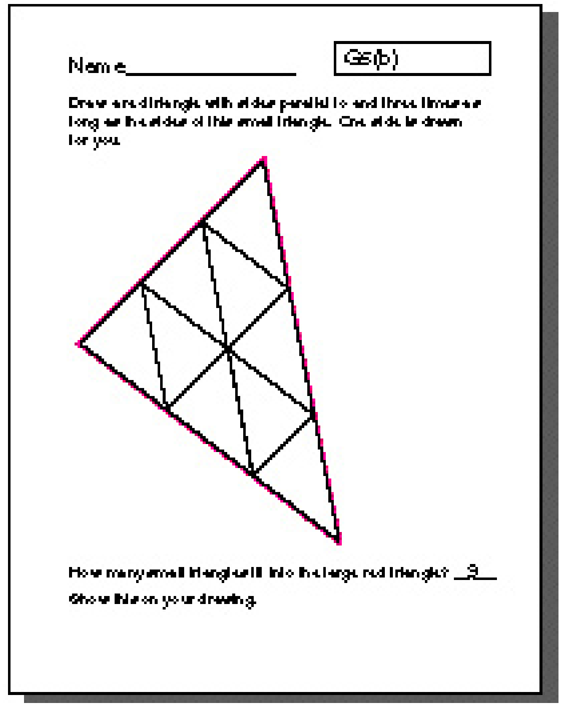


Name________________ G5(a)

Draw a red parallelogram with sides parallel to and three times as long as the sides of this small parallelogram. One side is drawn for you.

How many small parallelograms fit into the large red parallelogram? __9__

Show this on your drawing.



Name________________ G5(b)

Draw a red triangle with sides parallel to and three times as long as the sides of this small triangle. One side is drawn for you.

How many small triangles fit into the large red triangle? __9__

Show this on your drawing.

Capsule Lesson Summary

Given a circle with a radius of length r, determine how many squares with sides of length r will fit in the circle. Use this investigation to develop a formula for the area of a circle: area = $\pi \times r^2$. Determine the best buy among three sizes of pizza.

Materials

Teacher
- Data collected in Lesson G5 *Area of a Circle #1*
- Colored chalk
- Chalkboard compass (optional)

Student
- Paper
- Calculator

Description of Lesson

Exercise 1

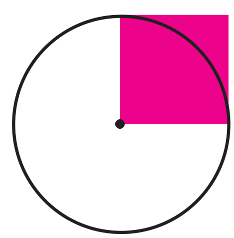
Put the table of data found by your class during Lesson G5, *Area of a Circle #1* on the board. If the data from your class is not available, use this data:

Radius	Approximate Area of Circle
3 cm	28 cm^2
4 cm	50 cm^2
5 cm	78 cm^2
6 cm	112 cm^2
7 cm	152 cm^2
8 cm	198 cm^2
9 cm	252 cm^2

Draw a circle on the board, marking its center with a small dot. Ask students to summarize Lesson G5 by describing how they counted squares on grid paper to estimate the areas of circles of various length radii.

T: ***One way to estimate the area of a circle is to draw the circle on grid paper and to count the number of squares within the circle. We found that this method is quite slow for large circles. Also, it may not be very accurate because of the many partial squares involved. Today let's look for a more efficient way to approximate the area of a circle.***

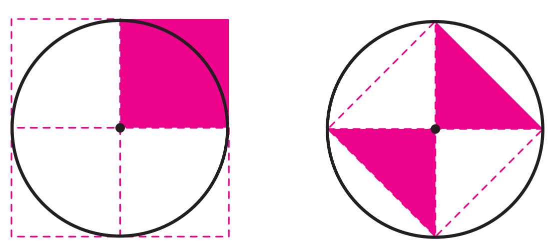
Draw a red square overlapping the circle, as shown here.



T: ***Instead of counting small centimeter squares, consider this large red square. In terms of area, how many red squares would be the same as the circle?***

S: ***Less than four because four red squares would more than cover the circle.***

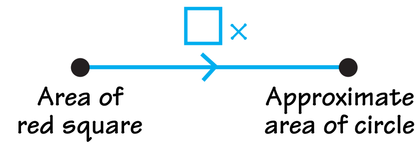
S: ***More than two. Here's a way to fit two red squares inside the circle. The four red triangles have the same area as two red squares.***



Students might conclude that approximately three red squares would have the same area as the circle.

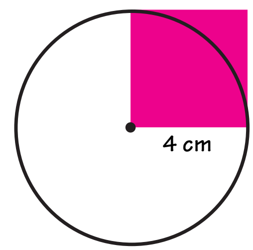
G6

Draw this arrow picture on the board.



T: ***If we know exactly the number of red squares that have the same area as the circle, then finding the area of the circle is quite easy. We only have to find the area of the red square and then multiply that area by our special number. Let's try to find that special number today.***

Add a third column to the table and assign a length of 4 cm to a radius of the circle drawn on the board.



T: ***If a circle has a 4 cm radius, we found that its area is approximately 50 cm^2. What is the area of that circle's red square?***

S: ***16 cm^2. Each side of the red square is 4 cm and 4 x 4 = 16.***

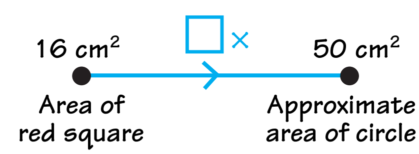
Similarly, find the area of the red square for the circles with other length radii and enter this information in the table.

Radius	Approximate Area of Circle	Area of Red Square
3 cm	28 cm^2	9 cm^2
4 cm	50 cm^2	16 cm^2
5 cm	78 cm^2	25 cm^2
6 cm	112 cm^2	36cm^2
7 cm	152 cm^2	49 cm^2
8 cm	198 cm^2	64 cm^2
9 cm	252 cm^2	81 cm^2

T: ***We think that the area of each circle is about three times the area of its red square. Is that true?***

Let students confirm that 3x is a good approximation for each of the circles.

Label the dots in the arrow picture with your data for the circle with a 4 cm radius. For example:



T: ***Let's try to find a better estimate of □x for each circle. 3 x 16 = 48. What number might we multiply times 16 to get even closer to 50?***

For each response, suggest students do the calculation on their calculators and record the result on the board. For example:

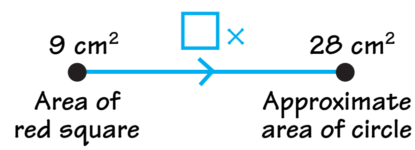
$3 \times 16 = 48$	$3.2 \times 16 = 51.2$	$3.15 \times 16 = 50.4$
$4 \times 16 = 64$	$3.1 \times 16 = 49.6$	$3.14 \times 16 = 50.24$
$3.5 \times 16 = 56$		$3.13 \times 16 = 50.08$

Stop the process when the information on the board allows estimating to the nearest hundredth (for example, 3.13x). You might get an exact response (for example, 3.125x) in which case use it.

Add a new column in the table for this information.

Radius	Approximate Area of Circle	Area of Red Square	Estimate of $\square\times$
3 cm	28 cm^2	9 cm^2	
4 cm	50 cm^2	16 cm^2	3.13×
5 cm	78 cm^2	25 cm^2	
6 cm	112 cm^2	36 cm^2	
7 cm	152 cm^2	49 cm^2	
8 cm	198 cm^2	64 cm^2	
9 cm	252 cm^2	81 cm^2	

Relabel the dots in the arrow picture with your data for the circle with a 3 cm radius. For example:

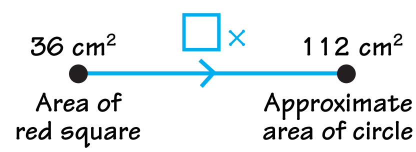


Instruct students to work with a partner to find □x so as to get as close to 28 as possible. After a few minutes, record some of the closest results on the board. For example:

$$3 \times 9 = 27 \qquad 3.2 \times 9 = 28.8 \qquad 3.13 \times 9 = 28.17$$
$$3.5 \times 9 = 31.5 \qquad 3.1 \times 9 = 27.9 \qquad 3.11 \times 9 = 27.99$$

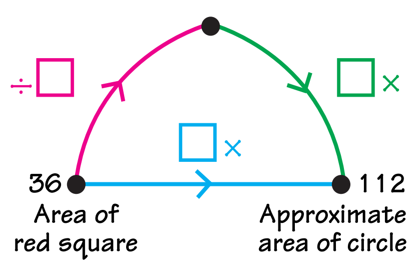
As before, select the result estimated to the nearest hundredth and record it in the table. Similarly, guide the class to find a good multiplier for your data for a circle with a 5 cm radius.

Relabel the dots in the arrow picture according to your data for a circle with a 6 cm radius. For example:



T: ***We could continue using trial-and-error methods to estimate □x. Has anyone thought of a more efficient method?***

Students might suggest using division (112 ÷ 36 or a fraction $^{112}/_{36}$). You may need to prompt this observation with a detour in the arrow picture.



T: ***Let's find a detour that will suggest a fractional name for the blue arrow.***

S: ***÷ 36 followed by 112x since 36 ÷ 36 = 1 and 1 x 112 = 112. Then the blue arrow is for $^{112}/_{36}$x.***

Equivalent fractions may be suggested, but emphasize the simplicity of finding this name for the blue arrow.

T: ***How can we find a decimal name for $^{112}/_{36}$?***

S: ***$^{112}/_{36}$ = 112 ÷ 36.***

Let students use their calculators to do the division calculation.

S: ***When I calculate 112 ÷ 36 on my calculator, I get 3.1111111.***

Note that the result is between 3.11 and 3.12 but closer to 3.11.

In a similar manner use fractions and/or division to complete the table.

Radius	Approximate Area of Circle	Area of Red Square	Estimate of □x
3 cm	28 cm^2	9 cm^2	3.11x
4 cm	50 cm^2	16 cm^2	3.13x
5 cm	78 cm^2	25 cm^2	3.12x
6 cm	112 cm^2	36 cm^2	3.11x
7 cm	152 cm^2	49 cm^2	3.10x
8 cm	198 cm^2	64 cm^2	3.09x
9 cm	252 cm^2	81 cm^2	3.11x

T: ***We have six estimates for the multiplier we need. What should we do with these estimates?***

Let students discuss the question. Several answers are acceptable. If necessary, suggest finding the mean and using it to estimate □x for all cases. Let students use calculators to find the mean. For example, (3.11 + 3.13 + 3.12 + 3.11 + 3.10 + 3.09 + 3.11) ÷ 7 = 3.11

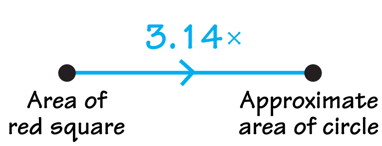
Write this expression on the board.

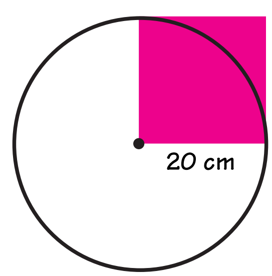
$$\pi = 3.14159 \cdots$$

T: ***The number we are looking for is a very special number in mathematics called "pi" (read as "pie"). The symbol used for pi is π, a letter in the Greek alphabet. The value of π to five decimal places is 3.14159, but a decimal name for π would never end and never be repeating. With the help of computers, people have calculated over a million digits of π, but nobody can write a complete decimal name. Let's use 3.14 as an estimate of π.***

Compare the class's estimate for □ to 3.14. Let students discuss possible sources of error in their data and estimates for □x. The most significant error arises from the inherent inaccuracy of approximating the area of a circle by counting grid squares inside the circle.

Label the blue arrow, and assign a new length radius to the circle.





T: ***Now we know that approximately 3.14 red squares fill the circle. How can we use that information to find the area of a circle with a 20 cm radius?***

S: ***The area of the red square is 400 cm^2 (20 x 20 = 400). So the area of the circle is approximately 1256 cm^2 (3.14 x 400 = 1256).***

Ask students to find the area of several other circles with radii of the lengths given in this table. (Answers are in boxes.)

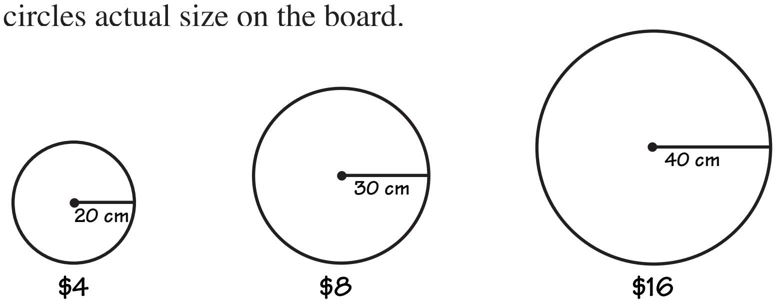
Radius	Area of Red Square	Approximate Area of Circle
25 cm	$625\ cm^2$	$1962.5\ cm^2$
15 cm	$225\ cm^2$	$706.5\ cm^2$
4 cm	$16\ cm^2$	$50.24\ cm^2$
5 cm	$25\ cm^2$	$78.5\ cm^2$
8 cm	$64\ cm^2$	$200.96\ cm^2$

After completing the above calculations, make several observations:

- For circles with 4 cm, 5 cm, and 8 cm radii, compare these calculated areas with your experimental estimates of the areas.
- This method for finding the area of a circle can be expressed in a formula: area = $\pi \times r^2$, where $\pi \approx 3.14$ and r is the length of a radius.
- Doubling the length of a radius does multiply the area of a circle by four.
- Tripling the length of a radius does multiply the area of a circle by nine.

Exercise 2

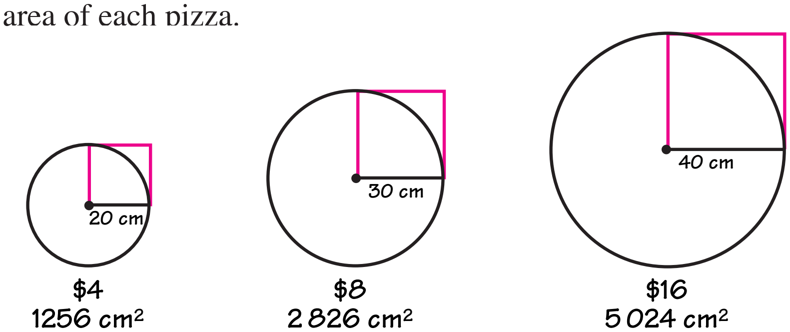
Draw the following circles actual size on the board.



T: ***These circles represent three pizzas with the same selection of ingredients and costing the indicated amounts. Which pizza is the best buy?***

Let students express their opinions. Lead the class to suggest comparing the areas of the circles and the respective pizza costs.

Following directions from students, draw the appropriate red squares and use 3.14 as an estimate of π to determine the area of each pizza.



T: ***Which pizza is the best buy?***

S: ***The medium pizza is a better buy than the small pizza because the medium pizza costs twice as much (2 x 4 = 8), but it is more than twice as big (2 826 > 2 x 1 256).***

S: ***The medium pizza is also a better buy than the large pizza. The medium pizza costs one half as much ($\frac{1}{2}$ x 16 = 8), but it is more than one half as big (2 826 > $\frac{1}{2}$ x 5 024).***

S: ***So the medium pizza is the best buy.***

Extension Activity

Suggest that students get size and cost information from local pizza shops to check for best buys

Writing/Home Activity

Direct students to write a description of how the class found an estimate for the number π and how this led to a method for calculating the area of a circle. This description could be written to parents, and could accompany a letter to parents about the development of a formula for the area of a circle.

G7 ESTIMATION OF π

Capsule Lesson Summary

Review the relationship between multiplication functions and lines on a Cartesian graph. Measure a diameter and the circumference of a variety of round objects, and plot the values on a graph. Use the graph to obtain an estimate of π.

Materials

Teacher
- Cartesian graph transparency
- Overhead projector
- *IG-V Geometry Poster* #1 (optional)
- Piece of string longer than one meter
- Several circular objects
- Can of three tennis balls (optional)
- Blackline G7

Student
- Cartesian graph
- One circular object
- Metric measuring tape
- Paper

Advance Preparation: Ask each student to bring one circular object to class (for example, a can, a dish, a flower pot, a frisbee, a toy wheel, etc.). Encourage a variety of sizes with diameters varying in length between 3 cm and 30 cm. Bring a few extra circular objects yourself.

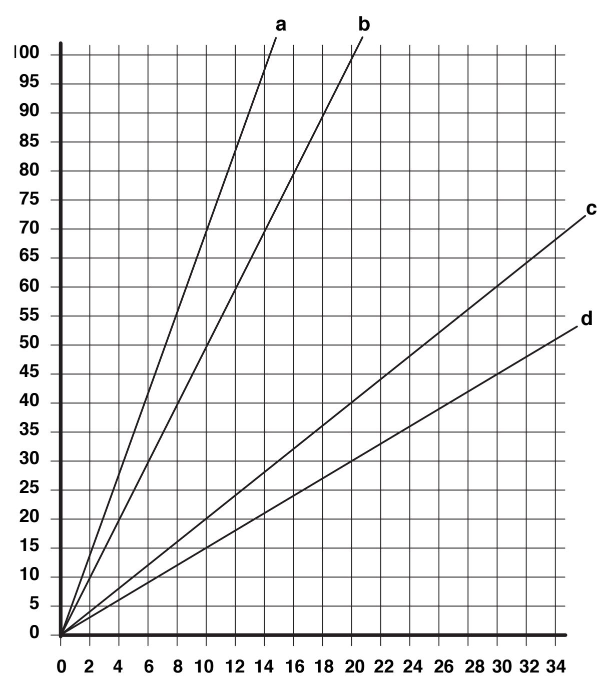
If possible, borrow a set of metric measuring tapes from a second or third grade *CSMP* teacher. If metric measuring tapes are not available, students can use string and metric rulers.

Use Blackline G7 to make an overhead transparency of the Cartesian graph for Exercise 1. Also make copies for students. As an alternative to making an overhead transparency of this Cartesian graph, carefully draw the lines on a grid board or use *IG-V Geometry Poster* #1.

Description of Lesson

Exercise 1

Display *IG-V Geometry Poster* #1 or the Cartesian graph, and refer students to their copies.



Do not write the letters on the graph. They are here just to make the description of the lesson easier to follow.

T (pointing to line **c**)**:** ***Name some points on this line.***

List points as students name them; for example:

(20, 40)	(10, 20)
(8,16)	(16, 32)
(24, 48)	(30, 60)

T: ***What do you notice about these pairs of numbers?***

S: ***The second number is always double the first .***

Check that all of the points students name meet this criteria. If a student names a point close to being on the line, admit that it is difficult to be sure by looking at the graph but that this point is not on the line. Label line **c** 2x.

In a similar manner, determine that **b** is the 5x line, **a** is the 7x line, and **d** is the 1.5x or $\frac{3}{2}$x line. Observe that only one or two points on a line need to be checked to determine the line.

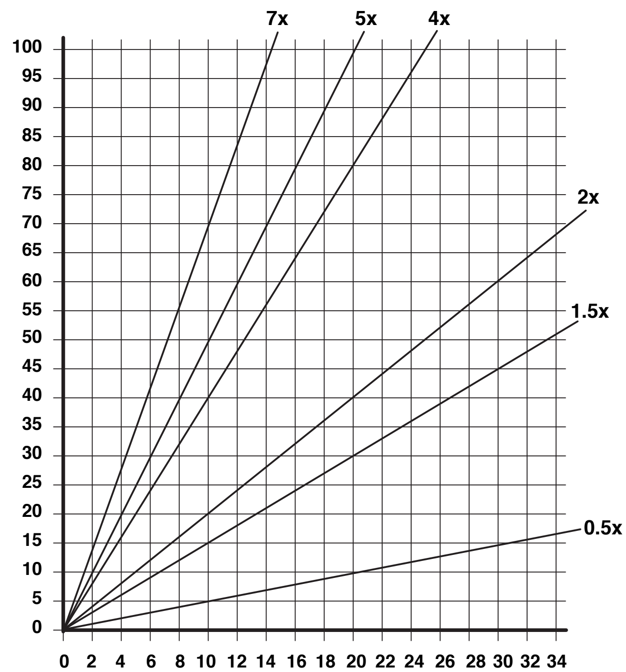
T: ***Let's draw a 4x line. Where do you think the 4x line will be?***

S: ***Below the 5x line.***

S: ***Between the 2x and 5x line.***

S: ***The line from (0, 0) to (10, 40).***

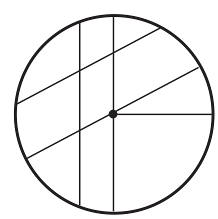
Invite students to locate several points on the 4x line and then draw the line. In a similar manner, let students direct you in drawing a 0.5x or $\frac{1}{2}$x line.



Exercise 2

Draw a circle on the board, marking its center. Draw several radii, diameters, and other cords.

Let students identify radii and diameters. Observe that some of the segments are neither radii nor diameters.



T: ***Does anyone know what the circumference of a circle is?***

S: ***The distance around the circle.***

If necessary, define *circumference* yourself.

Draw this table on the board.

Object	Diameter	Circumference
can		

T: ***Let's measure a diameter and the circumference of the circular objects we brought to class.***

Display a measuring tape and a can or other circular object. Indicate its circular bottom (or top).

T: ***How can we measure a diameter of the bottom (top) of this can?***

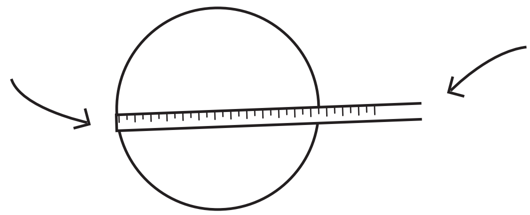
S: ***Measure the distance across the bottom (top).***

T: ***But I cannot tell exactly where there is a diameter since the center of the circle is not marked. If I move my tape measure a little the measurement changes. How do I know which measurement to use?***

S: ***Use the longest measurement. A diameter is a longest line segment across a circle.***

Demonstrate the following method of finding the length of a diameter of a circle.

Hold the tape with 0 at the edge of the circle.



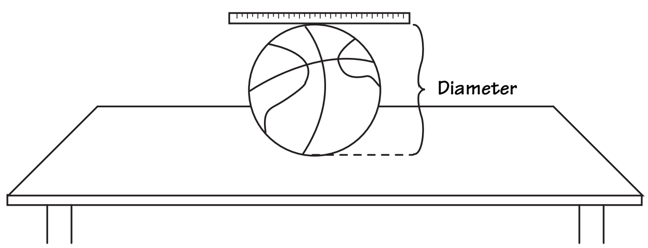
Move the tape measure to obtain maximum length which is the length of a diameter.

T: ***How can we measure the circumference of this circle?***

S: ***Wrap the tape measure around the can and read the measurement where the tape overlaps, meeting 0 again.***

Demonstrate the technique, and record the measurements in the table on the board.

Instruct students to measure the length of a diameter and the circumference of the objects they brought to class. You may wish to pair students or organize them in small groups so that they can check each other's measurements. As you observe, emphasize the importance of making accurate measurements. Some objects may require special measurement techniques. For example, you can measure a diameter of a ball by measuring its height above the table (see the next illustration). The circumference will require looking for the greatest distance around the ball.



For some very small objects, it may help to trace the circle and then measure a diameter and the circumference of the circle drawn on paper.

Record several results in the table on the board, as illustrated here.

Object	Diameter	Circumference
can	7.4 cm	22.6 cm
dish	14.2 cm	46.8 cm
clock	5.8 cm	16.1 cm
ball	24.3 cm	79.8 cm
film cannister	3 cm	9.4 cm

T: ***Do you notice any relationship between the length of a diameter and the circumference of a circle?***

Lead to the observation that the circumference is about three times the length of a diameter.

T: ***Let's graph our data and use the graph to determine more accurately how many times greater the circumference is than the length of a diameter.***

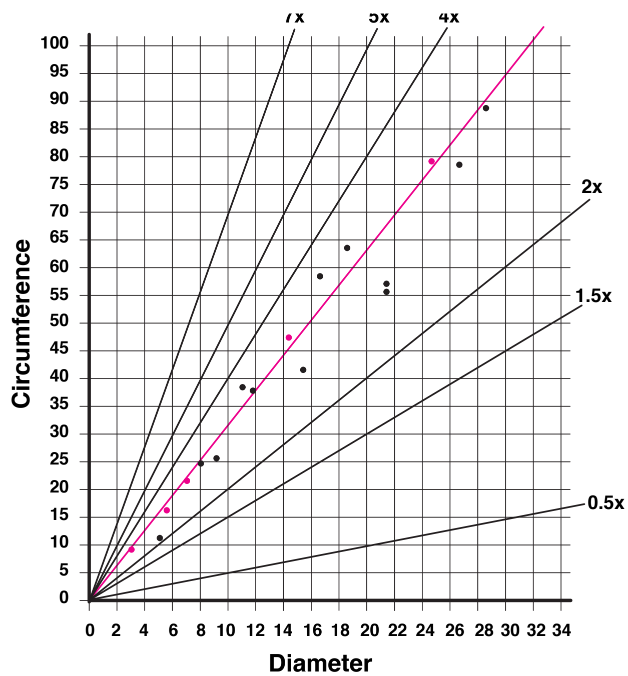
Using the graph from Exercise 1, label the axes **Diameter** and **Circumference**. Invite students to draw a dot for each object already listed in the table. Then ask for the data from other students' measurements and draw a dot for each object. For example, your graph might look similar to the next illustration.

T: ***What do you notice about these dots?***

Accept several observations but suggest that although the dots do not lie on a straight line, they are close to a line. If a few dots are far from a line, cross them out and ask those students to recheck their measurements.

T: ***Let's draw a line through (0, 0) and as close to as many of these dots as possible.***

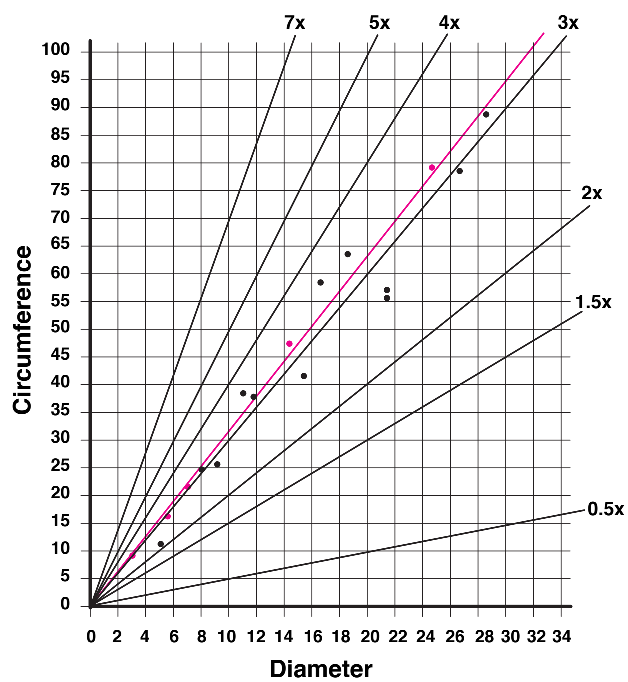
Take a piece of string and ask a student to hold one end at (0, 0). Hold the other end near the upper right part of the graph. Move that end of the string according to students' directives until they feel it is as close to as many dots as possible. Draw a line in red on the graph. For example:



T: ***This straight line through (0, 0) should be for some number times. In fact it will tell us about how many times greater the circumference is than the length of a diameter of a circle. Which other line for some number times is our line near?***

S: ***3*x*. (10, 30) is on the 3*x* line and (10, 30) is near our line.***

S: ***3*x*. Our data suggests that the circumference is about three times the length of a diameter.***



Call on students to name a few points on the 3x line, and use those points to draw the 3x line.

T: ***Our line is between the 3*x* line and the 4*x* line. What line could it be?***

S: ***3.5*x*.***

Ask students to name a few points on the 3.5x line and then draw the 3.5x line. If necessary, suggest calculations like 3.5 x 2, 3.5 x 10, and 3.5 x 20 that are fairly easy to do. In a similar manner, draw one or two more lines between the 3x line and the 3.5x line.

T: ***Which of these lines is closest to the red line?*** (This dialogue assumes that the 3.2x line is closest.)

S: ***The 3.2x line.***

T: ***So the circumference of a circle is about 3.2 times the length of a diameter. This gives us an easy way to estimate the circumference once we know the length of a diameter. For example, if the length of a diameter of a circle is 20 cm, then what is its approximate circumference?***

S: ***64 cm (3.2 x 20 = 64).***

Write this expression for π on the board.

$$\pi = 3.14159265 \cdots$$

T: ***In fact, the exact number we are looking for is a very special number in mathematics called "pi" (read as "pie"). The symbol used for pi is π, a letter in the Greek alphabet. The value of π to eight decimal places is 3.14159265. We cannot write a complete decimal name for π because it would never end and never be repeating. With the help of computers, scientists have calculated π to over a million decimal places.***

Briefly review the relationship among the diameter of a circle, the circumference, and π. Also review the relationship among the radius, the area, and π found in Lesson G6.

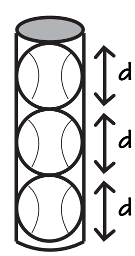
Exercise 3 (optional)

Empty a tennis ball can in front of the class and leave the three tennis balls on the desk. Indicate the height and the circumference of the can as you ask,

T: ***Which do you think is greater, the height of this can or the circumference of the can? Try to answer the question without measuring anything.***

Let students express their opinions. After most students have made a prediction, lead a discussion bringing out the following information.

- The height of the can is almost equal to three times the length of a diameter of a tennis ball. This is because three tennis balls fit snugly into the can.
- The circumference of the can is almost equal to the circumference of a tennis ball. Therefore, the circumference of the can is about 3.14 times the length of a diameter of a tennis ball.
- The circumference of the can is greater than the height of the can, although the difference is not much.



Writing Activity

Ask students to write a short article about the number π explaining its use in circle measurements. Some students may like to do a little more research on π.

Home Activity

Suggest that students pose the tennis ball can problem to someone at home.

Capsule Lesson Summary

Construct an open-topped box from a paper square by cutting out smaller squares from each corner of the paper squares and folding. Find the volume of the resulting box. Change the size of the cut-out squares and examine the effect on the box's volume. Graph the results, and determine the box of largest possible volume.

Materials

Teacher
- Paper square
- Tape
- Scissors
- Volume graph transparency
- Blacklines G8(a) and (b)

Student
- 18 cm by 18 cm square
- Tape
- Scissors
- Volume graph
- Calculator
- Centimeter cubes (optional)

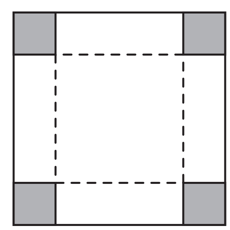
Advance Preparation: Use Blacklines G8(a) and (b) to make copies of an 18 cm by 18 cm square and a volume graph for students, and a volume graph transparency.

Description of Lesson

Hold up your paper square as you ask the class,

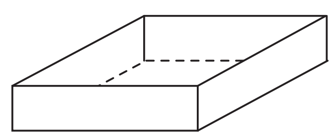
T: ***How could I make a box (open on the top) out of this paper square.***

Let students make suggestions and eventually describe the following steps:



1. Cut a square from each corner making all four square cut-outs the same size.
2. Fold the edges (on the dotted lines shown here) to form a box; then tape.

Do this construction with your paper square to make an open box.



T: ***How could we find the volume of this box?***

S: ***See how many cubes fit in it.***

S: ***Measure the length and width of its bottom, and its height. Then multiply these three numbers.***

Let students make the appropriate measurements. They should notice that the bottom is a square, so the length and width are the same. The height is the side length of the square cut-outs.

T: ***Suppose we wanted to make the volume of this box as large as possible. What size square should we cut out from each corner?***

Let students guess at the size of the square cut-out that will produce a box of greatest volume. Then organize the class into groups of three to five students each to work on this problem. Provide each group with multiple copies of an 18-by-18 centimeter square, scissors, tape, and paper.

Instruct the groups to experiment constructing different boxes from the 18-by-18 centimeter squares. The group should record the dimensions for each box they construct and calculate its volume in cubic centimeters. You may like to provide centimeter cubes for students to use to fill their boxes and calculate volume.

As you observe group work, suggest that groups organize their results in a convenient way such as in a table. For example:

Side length of cut-out square	Dimensions of open box	Volume
1 cm	1 cm × 16 cm × 16 cm	256 cm^3
2 cm	2 cm × 14 cm × 14 cm	392 cm^3
3 cm	3 cm × 12 cm × 12 cm	432 cm^3
⋮	⋮	⋮

You may want to ask some groups to try cutting out squares with side lengths of non-whole number rational numbers, such as 2½ cm.

When many groups have found the volume for quite a few different boxes (cutting out various-sized squares) and believe they have found the greatest possible volume, hold a collective discussion of the results. You may do this by collecting the data from all the groups in one table on the board.

For example:

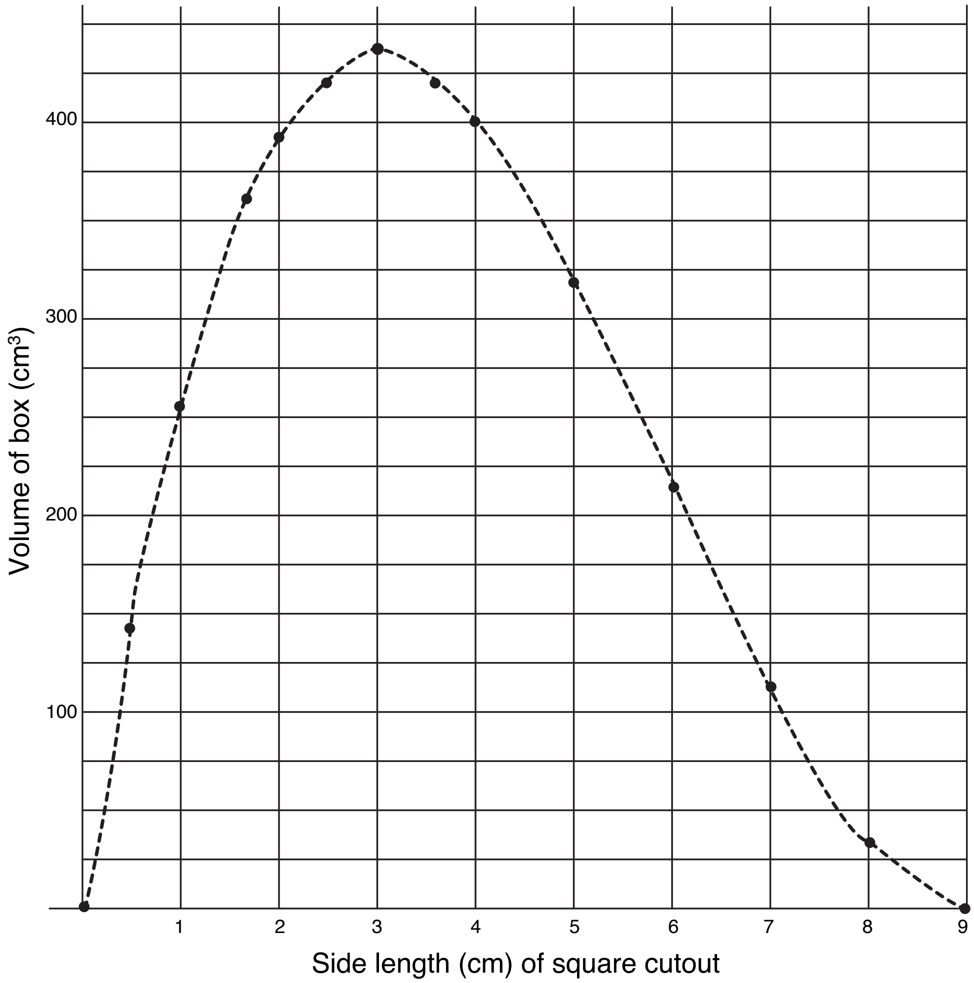
Side length of cut-out square	Dimensions of open box	Volume
$\frac{1}{2}$ cm	$\frac{1}{2}$ cm × 17 cm × 17 cm	144.5 cm^3
1 cm	1 cm × 16 cm × 16 cm	256 cm^3
2 cm	2 cm × 14 cm × 14 cm	392 cm^3
$2\frac{1}{2}$ cm	$2\frac{1}{2}$ cm × 13 cm × 13 cm	422.5 cm^3
3 cm	3 cm × 12 cm × 12 cm	432 cm^3
$3\frac{1}{2}$ cm	$3\frac{1}{2}$ cm × 11 cm × 11 cm	423.5 cm^3
4 cm	4 cm × 10 cm × 10 cm	400 cm^3
5 cm	5 cm × 8 cm × 8 cm	320 cm^3
6 cm	6 cm × 6 cm × 6 cm	216 cm^3
7 cm	7 cm × 4 cm × 4 cm	112 cm^3
8 cm	8 cm × 2 cm × 2 cm	32 cm^3
9 cm	9 cm × 0 cm × 0 cm	0 cm^3

If no group found the volume for boxes when the cut-out square has side length greater than 4 cm or 5 cm, add these to the table as a class. This data will be used in a graph.

Display and distribute copies of the graph from Blackline G8(b).

T: ***Let's make a Cartesian graph of our results.***

Observe that the graph on Blackline G8(b) has side length along the horizontal axis and volume of the box along the vertical axis. Choose two or three lines from the table of data to graph collectively. Then suggest students continue individually or in their groups.



Students may want to test other side lengths close to 3 cm for the square cut-out to be sure that the greatest volume results with the 3 cm square cut-out.

Extension Activity

Suggest that students try this same problem starting with a different sized paper square, such as a 24-by-24 centimeter square or a 30-by-30 centimeter square.

As a further extension, use a paper rectangle rather than a square.

Capsule Lesson Summary

Study the effect on the area of a square when the length of the sides is increased. Approximate the length of the side of a square with area 2 cm^2. Notice similarities when the activity is repeated with a triangle and with a parallelogram.

Materials

Teacher

- Colored chalk
- Grid board
- Tape
- Colored markers or crayons
- Blacklines G9(a) and (b)
- Triangle grid transparency
- *IG-V Geometry Poster #2* (optional)

Student

- Colored pencils, pens, or crayons
- Straightedge
- Centimeter square grid paper
- Calculator
- Triangle grid paper
- Worksheet G9

Advance Preparation: Use Blackline G9(a) to make centimeter square grid paper for students. Use Blackline G9(b) to prepare a triangle grid transparency and triangle grid paper for students.

Description of Lesson

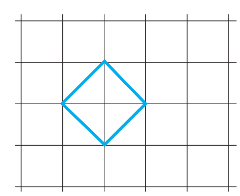
Exercise 1

Draw several squares on the grid board. Ask the class for the length of the sides of each square and for its area. Do as many examples as you feel necessary for students to recall that the area of a square is equal to the length of a side times itself, as illustrated in this chart.

Length of a side	Area
2 grid lengths	4 grid squares
4 grid lengths	16 grid squares
$4\frac{1}{2}$ grid lengths	$20\frac{1}{4}$ grid squares
$5\frac{1}{2}$ grid lengths	$30\frac{1}{4}$ grid squares
9 grid lengths	81 grid squares

Exercise 2

Provide centimeter square grid paper and colored pencils for this exercise. Ask students to build the same picture on their grids as you are building at the board. Clear the grid board and draw this square.



T: ***Your grid paper has 1 cm squares, so let's pretend that the squares on the grid board are that size also. What is the area of this blue square? How do you find its area?***

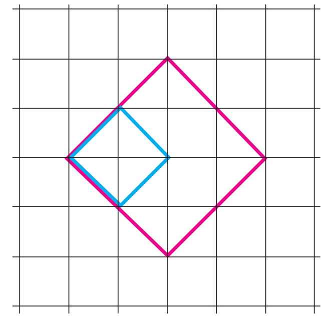
S: ***2 cm^2; each grid square is 1 cm^2 and the blue square has four half grid squares or two whole squares.***

Record the area of the blue square on the board.

T: ***Let's build a square with sides twice as long as the sides of the blue square. How can the grid help us to draw a line segment twice as long as a side of the blue square?***

S: ***A side of the blue square cuts diagonally across a grid square. We need to cut diagonally across each of two grid squares.***

Perhaps this is easier traced than described verbally. When suggested, draw such a line segment starting at one corner of the blue square. Ask how to complete the square, letting a student trace it and then drawing it yourself.



2 cm²

T: *What is the area of this red square? How do you find its area?*

S: *8 cm²; there are four whole grid squares and eight half grid squares inside the red square.*

S: *8 cm²; four of the blue squares would fit inside it. 4 x 2 = 8.*

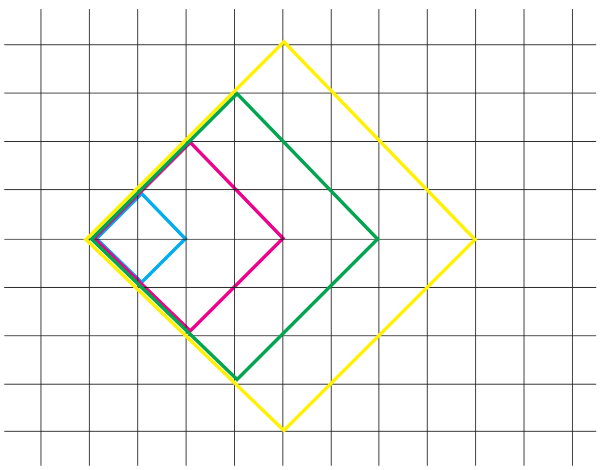
S: *8 cm²; I counted the grid squares and half grid squares that get added on to what's in the blue square. The red square is 6 cm² larger in area than the blue square.*

T: *So when we double the length of the sides of the blue square, we get a square with four times the area.*

+6 2 cm²
8 cm²

Change in the length of a side	Change in area
2×	4×

Continue the activity by asking for the area of a square with sides three times as long as the blue square's sides, and the area of a square with sides four times as long as the blue square's sides.



+6 2 cm²
+10 8 cm²
+14 18 cm²
32 cm²

Change in the length of a side	Change in area
2×	4×
3×	9×
4×	16×

T: *What do you think the area of a square with sides five times as long as those of the blue square would be?*

S: *50 cm²; the area should be 25 times larger.*

S: *I think it will be 18 cm² larger than the area of the yellow square.*

+6 2 cm²
+10 8 cm²
+14 18 cm²
+18 32 cm²
50 cm²

Change in the length of a side	Change in area
2×	4×
3×	9×
4×	16×
5×	25×

Draw a square with sides five times as long as the sides of the blue square while students do the same on their grids. They should confirm that the area is 50 cm².

Ask students about patterns they notice, and then use those patterns to predict what would be the area of a square with sides seven times as long as the blue square's sides. (98 cm²)

T: ***How can we find the area of a square if we know the length of the sides?***

S: ***Multiply the length of a side times itself.***

T: ***The blue square has area 2 cm². How long is one of its sides*** (trace one of them)***?***

S: ***Between 1 cm and 2 cm.***

Take a piece of string and hold a grid length of string between your fingers. Compare it to a side of the blue square. Do the same holding the string two grid lengths between your fingers. The class should conclude that the blue square has sides longer than 1 cm but shorter than 2 cm.

T: ***How long do you think the sides of the blue square are? How can we check?***

Take predictions. A student could argue that it is about 1.4 cm, because when you multiply the side length by 5 you get a square with area 50 cm² which is close to 49 cm². When the area is 49 cm², the side length is 7 cm and 7 ÷ 5 = 1.4.

The class can find successive estimations as follows. When someone suggests looking for a number that when multiplied by itself yields 2, write $\square \times \square = 2$ on the board. Then direct students to experiment with calculators to see how close they can get to 2 by multiplying a number times itself.

S: ***1.4*** x ***1.4 = 1.96.***

S: ***1.45*** x ***1.45 = 2.1025.***

Record students' suggestions on the board. Briefly review the use of exponents. Continue the activity until the class gets very close to 1.4142. Remind students that the calculator's display is limited to eight digits. Perhaps your list will look similar to the one below.

$(1.4)^2 = 1.96 < 2$

$(1.45)^2 = 2.1025 > 2$

$(1.41)^2 = 1.9881 < 2$

$(1.425)^2 = 2.030625 > 2$

$(1.424)^2 = 2.027776 > 2$

$(1.414)^2 = 1.999396 < 2$

T: ***So the length of each side of the blue square is very close to 1.414. Approximately how long is each side of the red square?***

S: ***2.828; each side of the red square is twice as long as each side of the blue square, so I multiplied 1.414 by 2.***

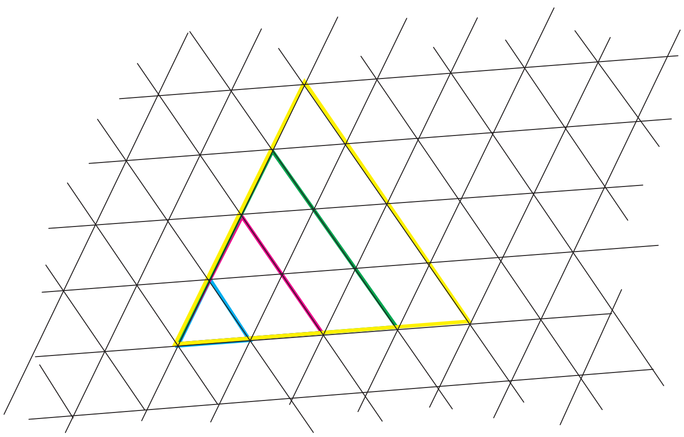
T: ***Let's check how close 2.828 is. We know the area of the red square is 8 cm². Multiply 2.828 times itself and see how close the result is to 8.***

S: ***The calculator displays 7.997584.***

You may wish to check that $(4.242)^2$ is close to 18 for the green square also. Save the data on the different colored squares and the picture on the grid board so that you can refer to it in Exercise 3.

Exercise 3

Display a triangle grid and provide copies of a similar one for students to use. Color one grid triangle blue and tell students to suppose its area is 2 cm^2. Double, triple, quadruple the length of a side of the blue triangle to build larger and larger triangles. Before you build each triangle, ask the class to predict its area; afterward, confirm or deny any prediction. Students should be building the same picture on their triangle grids.



2 cm^2
+6
8 cm^2
+10
18 cm^2
+14
32 cm^2

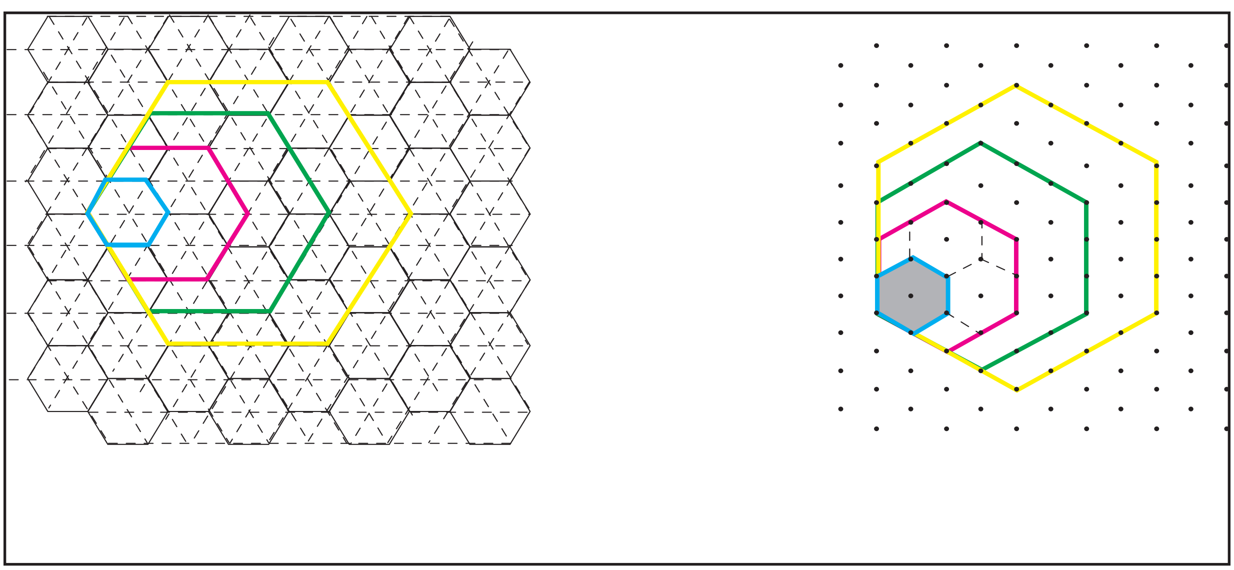
Change in the length of a side	Change in area
2×	4×
3×	9×
4×	16×

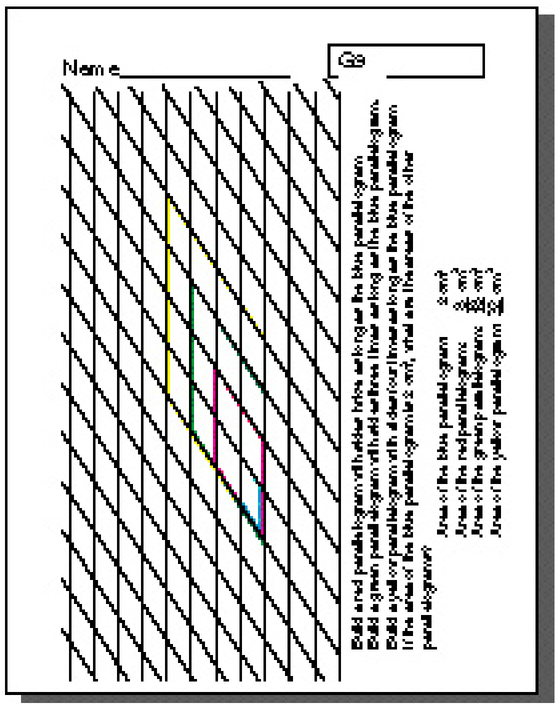
The class should notice similarities between this situation and that of Exercise 2.

Ask students to repeat this kind of activity individually with the parallelogram grid on Worksheet G9.

Extension Activity

Do a similar activity with hexagons. Blackline G9(c) has a hexagon grid and Blackline G9(d) has isometric dot paper to guide drawing hexagons. It is more complicated to draw hexagons with double, triple, or quadruple side length, as illustrated below. Students should find similar effects on the area of hexagons when the length of the sides is increased.

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G9

G10 VOLUME PATTERNS

Capsule Lesson Summary

Study the effect on the volume of a cube when the length of the edges is increased. Observe similar results when a non-cubical box is used in the activity. Design a cereal box with constraints about its volume and surface area.

Materials

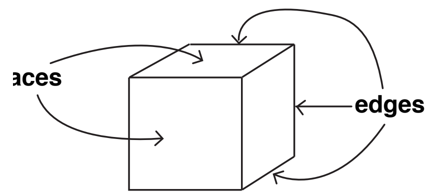
Teacher
- Approx. 200 centimeter cubes
- 30 cubes all the same size (½ inch, 2 cm, or 1 inch)
- Large box
- Cereal box
- Metric ruler

Student
- Paper
- Construction paper
- Scissors
- Tape
- Metric ruler

Description of Lesson

Exercise 1

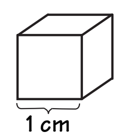
Display a cube. With the class discuss, the special features of a cube—its six square faces and its 12 edges of equal length. Display a centimeter cube.



T: ***The edge length of this cube is 1 cm.***
What is the area of one of its faces? (1 cm^2)
What is its volume (the space it fills)?

S: ***One cubic centimeter.***

T: ***The cubic centimeter is a unit we can use to measure volume.***

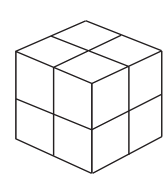


volume: 1 cm^3

Pass your centimeter cube around the room so that students can gain a sense of its size.

Using a large box as a pedestal, display approximately 30 cubes on top. Hold up one of the cubes and announce its volume. Use a number that closely approximates the volume of one of your cubes. For example, if your cubes are 2-cm cubes, say the volume is 8 cm^3. If your cubes are 1-inch cubes, say the volume is about 16 cm^3. ½-inch cubes have volume approximately 2 cm^3. This lesson description is based on using 1-inch cubes.

T: ***The volume of this cube is about 16 cm^3. Let's build a cube with edges twice as long as the edges of this cube.***



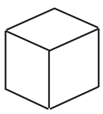
Invite students to make the cube with edge length twice as long on your pedestal.

T: ***What is the volume of this cube with edge length twice as long?***

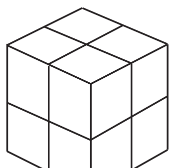
S: ***128 cm^3. We needed eight cubes to build it, and each cube has volume 16 cm^3. 8 x 16 = 128.***

Continue the activity by asking students to build a cube with edge length three times as long as the individual cube, and then imagine the cube with edge length four times as long. Each time, note the volume and how much change.

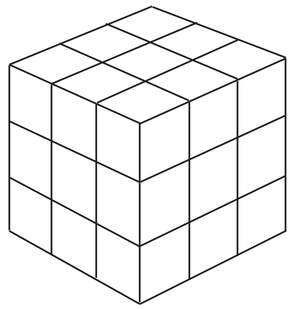
Change in the length of an edge	Change in volume
2×	8×
3×	27×
4×	64×



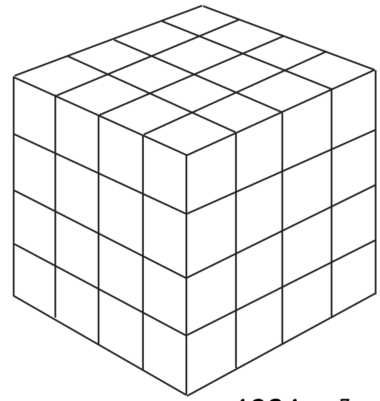
16 cm³



128 cm³



432 cm³



1024cm³

Begin to build a cube with edge length five times as long, and ask students to predict what its volume will be. (2 000 cm^3; 125 x 16 = 2 000)

T: ***What patterns do you notice in the change in volume when we change the length of an edge of the cube?***

S: ***If we change the edge length a number times, then the volume changes by the cube of that number times. For a change in edge length of 5x, the volume change is 5 x 5 x 5 = 125 times.***

Use this pattern to predict the change in volume for a cube with edges seven times or ten times as long as the original cube.

Exercise 2

Use centimeter cubes for this exercise.

Call on a student to build a box shape on your pedestal with dimensions 1 cm by 2 cm by 3 cm.

T: ***What is the volume of this box?***

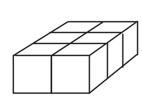
S: ***6 cm^3.***

T: ***Let's build a box with all its edges (dimensions) twice as long as this box.***

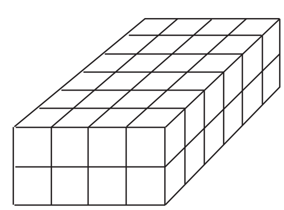
Let a student build a 2 cm by 4 cm by 6 cm box next to the 1 cm by 2 cm by 3 cm box. Call on students to check that all the edges are twice as long.

T: ***What is the volume of this new box?***

S: ***48 cm^3.***



6 cm³



48 cm³

Repeat this activity, asking for a box with all its edges (dimensions) three times as long as the 1 cm by 2 cm by 3 cm box; that is, a 3 cm by 6 cm by 9 cm box. Students should find that the volume of such a box is 162 cm^3.

Begin a table similar to that in Exercise 1 in which to record the change in volume with a uniform change in edge lengths. Students should notice quickly that the change pattern is the same as for the cube in Exercise 1.

Change in the length of an edge	Change in volume
2×	8×
3×	27×
4×	64×
5×	125×

Exercise 3

Display your cereal box. Ask students to measure the edges of the box to the nearest centimeter. Then record its dimensions on the board. For example, 16 cm by 25 cm by 6 cm.

T: ***What is the volume of this box?***

S: ***2 400 cm^3. 16 x 25 x 6 = 2 400.***

T: ***Suppose the cereal company wants to make a box that holds twice as much cereal. What could they do?***

Be prepared for someone to suggest doubling all the edges. Note that in this case the volume would be 8 x 2 400, so that the box would hold eight times as much cereal.

S: ***They could double just one dimension on the box. For example, a box 32 cm by 25 cm by 6 cm has volume 4 800 cm^3. It would hold twice as much cereal as this box with volume 2 400 cm^3.***

Let students make several suggestions for boxes with twice the volume of the original cereal box.

Indicate the faces of the cereal box as you explain,

T: ***The company has another concern. They don't want to have to use too much more cardboard to make the box. What do we call the measurement of the amount of cardboard used to make the box?***

S: ***The surface area of the box.***

With the class, review how to find the surface area of a box. Note that the box has six faces, and two opposite faces have the same area. In this case, the surface area can be found as follows:

$$2 \times [(16\text{ cm} \times 25\text{ cm}) + (16\text{ cm} \times 6\text{ cm}) + (25\text{ cm} \times 6\text{ cm})]$$
$$2 \times [400\text{ cm}^2 + 96\text{ cm}^2 + 150\text{ cm}^2]$$
$$2 \times 646\text{ cm}^2 = 1292\text{ cm}^2$$

Organize the class in groups to work on this problem:

> Design a cereal box for the company that has twice the volume of the original box but can be made using as little cardboard as possible.

Once a group believes they have a good design, ask them to make the box with construction paper for comparison to the original box.

Note: Do not expect that all groups will find a best design. For the cereal box described in the lesson, the following are possible designs. The best design with whole number centimeter dimensions is 16 cm by 20 cm by 15 cm, requiring 1 720 cm^2 of cardboard.

Cereal Box: 16 cm × 25 cm × 6 cm   Volume: 2 400 cm^3 Surface Area: 1292 cm^2

Dimensions	Volume	Surface Area
32 cm by 25 cm by 6 cm	4 800 cm^3	2 284 cm^2
16 cm by 50 cm by 6 cm	4 800 cm^3	2 392 cm^2
16 cm by 25 cm by 12 cm	4 800 cm^3	1784 cm^2
20 cm by 20 cm by 12 cm	4 800 cm^3	1760 cm^2
16 cm by 20 cm by 15 cm	4 800 cm^3	1720 cm^2
16 cm by 16 cm by $18\frac{3}{4}$ cm	4 800 cm^3	1712 cm^2
16 cm by 18 cm by $16\frac{2}{3}$ cm	4 800 cm^3	$1709\frac{1}{3}$ cm^2

Writing/Home Activity

Suggest that students write a report to the cereal company on their suggestions for solving the company's problem. This report can be taken home and the problem explained to a family member.

G11 POLYGONS #1

Capsule Lesson Summary

Use a compass and a straightedge to duplicate a line segment. Discuss a variety of polygons, and then use a compass and a straightedge to construct equilateral, eight-sided polygons.

Materials

Teacher
- Chalkboard compass
- Straightedge

Student
- Unlined paper
- Compass
- Straightedge

Description of Lesson

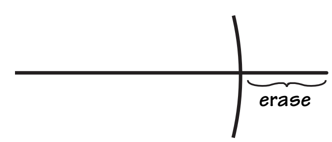
Exercise 1

Draw a line segment about 40 cm long on the board.

T: ***I have here a compass and a straightedge. The straightedge will only be used to draw line segments. It will not be used to measure. The problem is to construct another line segment the same length as this one.***

Invite students to experiment with constructions. Someone may suggest making a mark on the straightedge. Respond with a prohibition and a remark against making the straightedge serve as a ruler, that is, a measuring tool. Guide the discussions to one of two constructions.

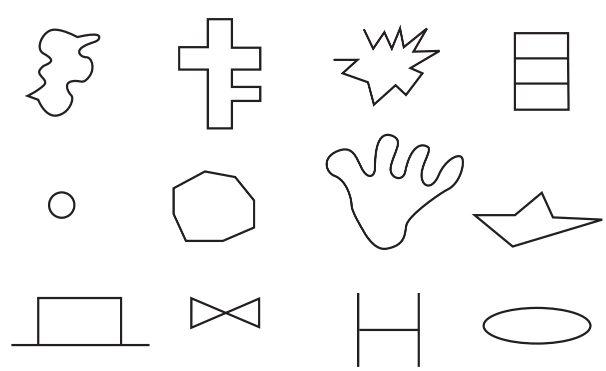
- Draw a second long line segment. Set the compass radius to the length of the original segment, and then place the compass point at one endpoint of the second segment. Draw a portion of a circle intersecting the segment. Erase the excess.



- Set the compass to the length of the given segment. Draw a dot, set the compass point at the dot, and draw a portion of a circle. Use the straightedge to draw a segment from the dot to any point on the circle.

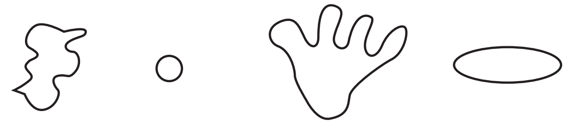
Exercise 2

Invite students to draw shapes of their own choosing on the board. Continue until there are at least ten different shapes, such as those illustrated here.



T: ***Today we are going to construct shapes with special characteristics. One rule is that a shape must have only line segments for sides. Which of these shapes do not follow this rule?***

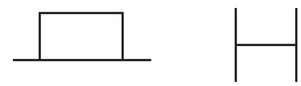
Let students tell you to erase all of the shapes on the board that have curved edges (boundaries). These shapes in the previous illustration would be excluded by this rule.



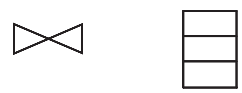
Continue in the same manner with three more rules. The shapes in the illustration that would be excluded by each rule are shown on the right.

T: ***A second rule is that a shape must be closed.***

A third rule is that a shape can have no dangling pieces.



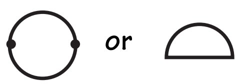
A fourth rule is that each side of a shape can meet exactly two other sides.



Repeat the four rules for special shapes, and check again that all of the shapes remaining on the board follow these rules.

T: ***Suppose we were to draw some more shapes following these rules. How many sides could one of these special shapes have?***

Lead students to realize that any number of sides is possible, in principle, except two or one. Students may offer several solutions for two-sided shapes, but these should be rejected by the class.



S: ***No; our special shapes may only have line segments for sides.***

S: ***No; it is not closed.***

S: ***No; this shape is not closed either.***

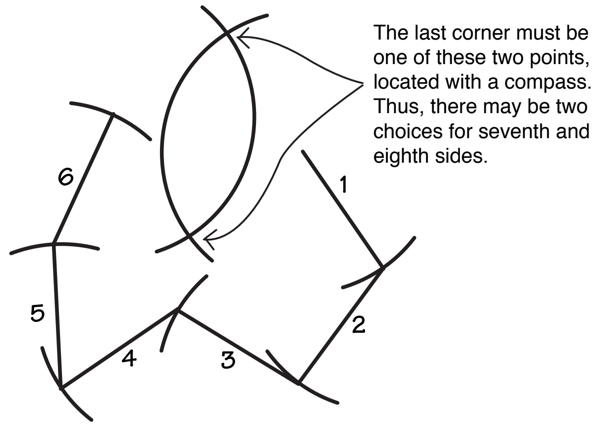
S: ***No; dangling pieces are not allowed.***

Draw a line segment about 30 cm long on the board.

T: ***Let's use a compass and straightedge to draw a shape with eight sides, all the same length, that satisfies all of our rules. Suppose each side of the shape has the same length as this segment. Who would like to construct a first side?***

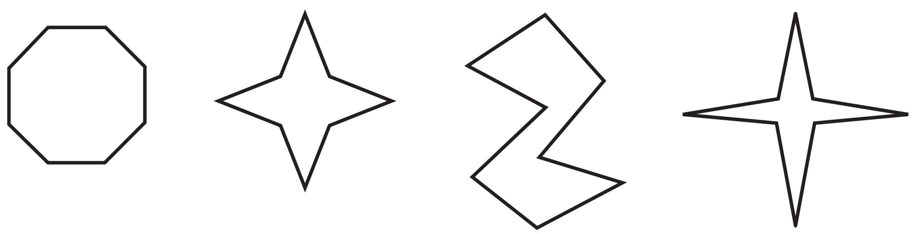
Invite students, in turn, to add a side to the drawing, and encourage class discussion of the process. Each time a side is drawn, discuss where the next side could be drawn before it is constructed. Some adjustment of the sides already drawn may be necessary to allow the shape to close.

Note that the key to the construction lies in bringing the ends of the chain of sides close together when the sixth side is added. Then the drawing of the seventh and eighth sides is forced to close the shape. There are usually two choices for the location of the last corner, and these are found at the intersections of the circles with the given radius centered on the free endpoints, as shown here.



Discuss this technique of closing the shape after the sixth side is constructed.

Instruct students to use a compass and a straightedge to construct their own eight-sided shapes where all of the sides are the same length. Monitor individual work. As students produce solutions, ask for another with a smaller or larger area, but using the same length side. Some solutions are shown here.



Students who construct two or three eight-sided shapes where all the sides are the same length may be asked to do a similar construction for six-sided or nine-sided shapes. In fact, you may suggest constructing shapes with from three to ten sides.

Extension Activity

Invite students to produce a poster or other display of “different” equilateral shapes with from three to ten sides.

G12 POLYGONS #2

Capsule Lesson Summary

Use a compass and a straightedge to construct an equilateral, six-sided polygon; an equilateral, four-sided polygon; and an equilateral triangle. Given two line segments, use a compass and a straightedge to construct as many essentially different four-sided shapes as possible with each side the same length as one of the given line segments. Make some observations concerning the variety of shapes that can be constructed and the possible diagonal lengths of certain shapes.

Materials

Teacher
- Chalkboard compass
- Straightedge
- Colored chalk

Student
- Unlined paper
- Compass
- Straightedge
- Worksheet G12

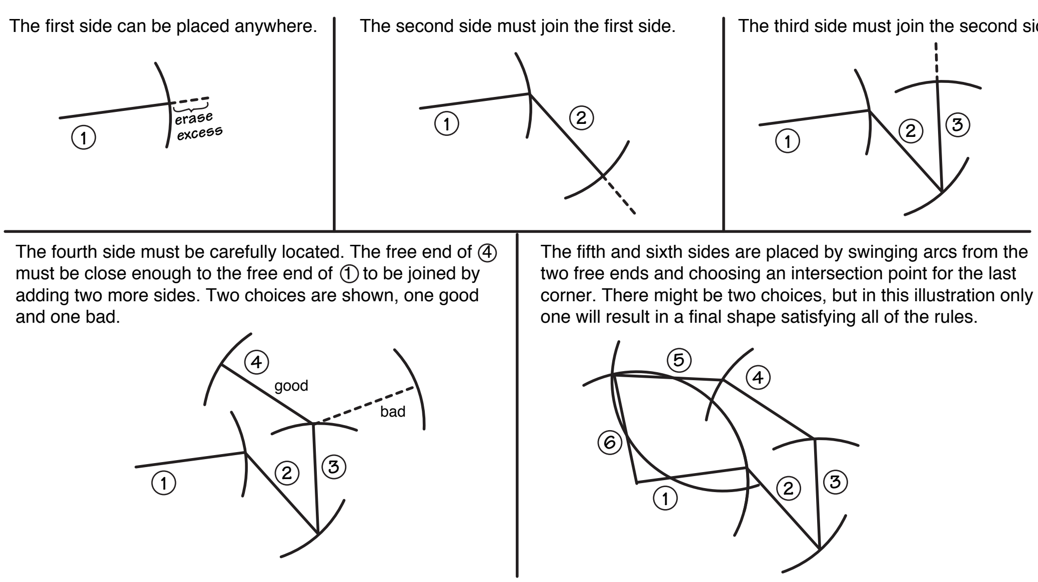
Description of Lesson

Exercise 1

Draw a line segment about 40 cm long on the board.

T: ***Let's draw a six-sided shape whose sides are each the length of this line segment. We will use only a compass and a straightedge.***

Invite students, in turn, to add a side to the drawing, and encourage class discussion of the process. Review special shapes that follow the rules presented in Lesson G11. Enforce these rules as the drawing progresses. Some students may just want to guess at locating corners of the shape being drawn and may need prompting from their peers on the use of the compass to assist in the construction. At each step ask for some indication of any restriction on the availability of choices. For example, the first side can be drawn anywhere. The second side need only be connected to the first, and the third side to the second. There are infinitely many choices in each case, but the fourth side cannot be placed arbitrarily if the shape is to be closed. See the sequence of illustrations below.



The first side can be placed anywhere.

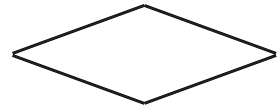
The second side must join the first side.

The third side must join the second side.

The fourth side must be carefully located. The free end of ④ must be close enough to the free end of ① to be joined by adding two more sides. Two choices are shown, one good and one bad.

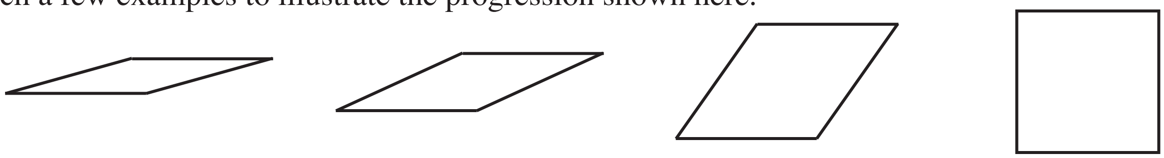
The fifth and sixth sides are placed by swinging arcs from the two free ends and choosing an intersection point for the last corner. There might be two choices, but in this illustration only one will result in a final shape satisfying all of the rules.

Repeat the activity by constructing a four-sided shape, using the same length for each side. For example:



T: ***This shape is called a* rhombus. *We could draw many different rhombi with the same length side.***

Sketch a few examples to illustrate the progression shown here.



T: ***One of these rhombi has a special name.***

S: ***The square.***

T: ***Right; a square is a rhombus with square corners. Now let's draw a three-sided shape with each side this length.***

S: ***You mean a triangle.***

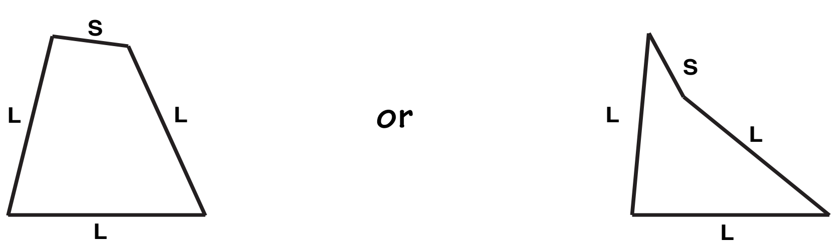
In the process of constructing the triangle, the class should realize that while many rhombi (all sides the same length) can be drawn with a given length side, only one equilateral triangle can be drawn with a given length side. Emphasize this fact.

Exercise 2

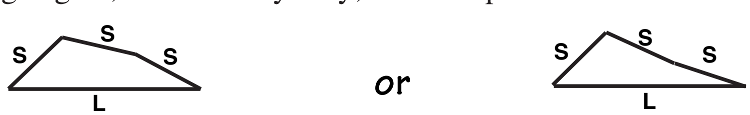
Distribute Worksheet G12.

T: ***On this worksheet there are two line segments. Use your compass and straightedge to draw as many different four-sided shapes as you can. Each side of a shape must be the same length as one or the other of these two segments.***

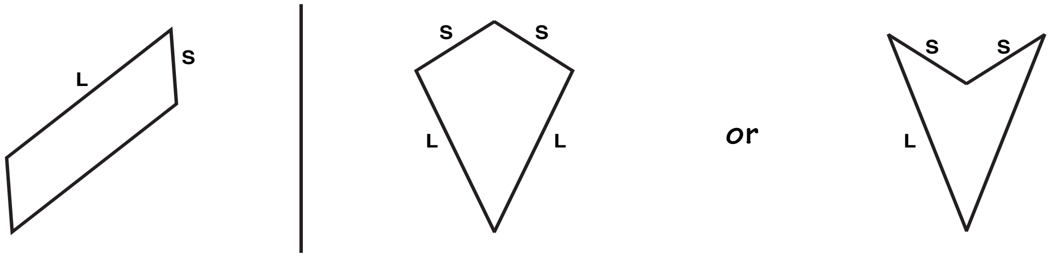
There are five possible combinations of sides: all short; all long; one short and three long; three short and one long; and two short and two long. In the first two cases, the resulting shape is a rhombus, as already discussed. There are many different-sized rhombi in each case. In the case of one short side and three long sides, there is, again, essentially only one solution (with respect to the arrangement of sides); that is, short-long-long-long. However, the area may vary, for example:



The case of three short sides and one long side is similar; there is essentially one solution, short-short-short-long. Again, the area may vary; for example:



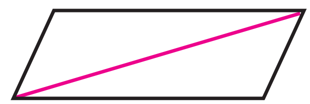
The case of two short sides and two long sides is more interesting since there are essentially two solutions, depending on the order of the sides in rotation: short-long-short-long or short-short-long-long. The first sequence results in a parallelogram, and the second results in a kite or wedge.



Encourage students to find examples of each type rather than to repeat examples from the same case. The answer key following the lesson shows a full set of possibilities.

Near the end of the lesson, draw examples of the various possibilities on the board so that students can compare their drawings and decide which shapes are of the same type. Also, let students spend some time comparing drawings among themselves to note the different sizes possible within each type.

For the final discussion, focus on a parallelogram. Draw the longer diagonal in red.



T: ***What happens to this red line segment as you change the size of the parallelogram?***

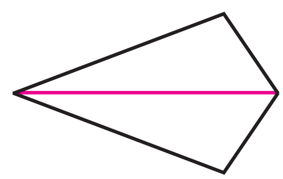
S: ***It gets longer if you squeeze the parallelogram together.***

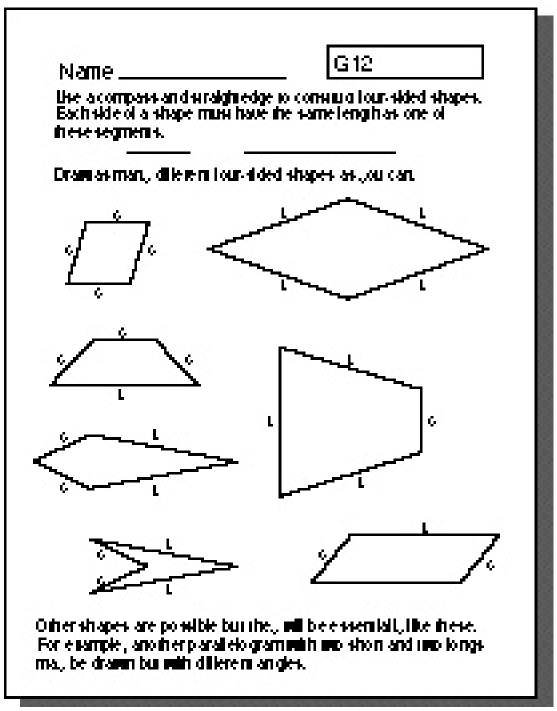
T: ***How long can it be?***

S: ***No longer than the sum of the lengths of the short and the long sides.***

Refer to a kite, and draw the longer diagonal in red.

T: ***Now look at this kite.***





Name ____________ G 12

Use a compass and straightedge to construct four-sided shapes. Each side of a shape must have the same length as one of these segments.

Draw as many different four-sided shapes as you can.

Other shapes are possible but they will be essentially like these. For example, another parallelogram with two short and two longs may be drawn but with different angles.