# The Languages of Strings and Arrows

# LANGUAGES OF STRINGS & ARROWS TABLE OF CONTENTS

# THE LANGUAGES OF STRINGS & ARROWS  INTRODUCTION

Two fundamental modes of thought for understanding the world around us are the classification of objects into sets and the study of relationships among objects. In everyday life, we classify cars by brand (Ford, Chevrolet, Toyota, and so on) and we study relationships among people (Sally is Mark's sister, Nancy is Mark's cousin). Chemists classify elements by properties, and zoologists study predator-prey relationships. Similarly, mathematicians sort numbers by primeness, and they employ functions to model predicted relationships, for example, between inflation and unemployment.

Many of a child's earliest learning experiences involve attempts to classify and to discern relationships. A child classifies people by roles (the teacher, the doctor), and creates relationships between the smell and taste of foods. Part of language development depends on a child's repeated attempts to sort objects by function, and to relate words with things or events.

The role of sets and relations is so pervasive in mathematics, that perhaps the simplest definition of mathematics is "the study of sets and relations principally involving numbers and geometrical objects." Given the equally pervasive presence of these two notions in everyday life and in a child's experiences, it is natural that they should play a key role in an elementary mathematics curriculum. Yet the inclusion of classification and the study of relations require an appropriate language for representing and studying them. For that reason, *CSMP* develops the nonverbal languages of strings and of arrows.

The pictorial language of strings represents the grouping together of objects into sets. The pictorial language of arrows represents relations among objects of the same or different sets. Each of these languages permeates the different content strands of the *CSMP* curriculum, providing unity both pedagogically and mathematically. With continual use, the languages become versatile student tools for modeling situations, for posing and solving problems, and for investigating mathematical concepts.

The general aim of this strand is to present situations that are inherently interesting and thought provoking, and that involve classification or the analysis of relations. The activities emphasize the role of logical thinking in problem solving rather than the development of specific problem-solving techniques. In addition to a varied assortment of lessons concerning sets and relations, this strand includes lessons involving systematic methods for solving combinatorial (counting) problems.

## Classification: The Language of Strings

As the word implies, classifying means putting things into classes, or as the mathematician says, sets. The mathematics of sets can help students to understand and use the ideas of classification. The basic idea is simple: Given a set $S$ and any object $x$, either $x$ belongs to $S$ ($x$ is in $S$) or $x$ does not belong to $S$ ($x$ is not in $S$). We represent this simple act of sorting—in or out—by using pictures to illustrate in and out in a dramatic way. Objects to be sorted are represented by dots, and the sets into which they are sorted are represented by drawing strings around dots. A dot inside the region delineated by a set's string is for an object in the set, and a dot outside a set's string is for an object not in the set.

This language of strings and dots provides a precise (and nonverbal) way of recording and communicating thoughts about classification. The ability to classify, to reason about classification, and to extract information from a classification are important skills for everyday life, for intellectual activity in general, and for the pursuit and understanding of mathematics in particular. The unique quality of the language of strings is that it provides a nonverbal language that is particularly suited to the mode of thinking involved in classification. It frees young minds to think logically and creatively about classes, and to report their thinking long before they have extensive verbal skills.

## Relations: The Language of Arrows

Relations are interesting and important to us in our everyday lives, in our careers, in school, and in scientific pursuits. We are always trying to establish, explore, and understand relations. In mathematics it is the same; to study mathematics means to study the relations among mathematical objects like numbers or geometric shapes. The tools we use to understand the everyday world are useful to understand the world of mathematics. Conversely, the tools we develop to help us think about mathematical things often serve us in non-mathematical situations.

A serious study of anything requires a language for representing the things under investigation. The language of arrows provides an apt language for studying and talking about relations. Arrow diagrams are a handy graphic representation of a relation, somewhat the same way that a blueprint is a handy graphic representation of a house. By means of arrow diagrams, we can represent important facts about a given relation in a simple, suggestive, pictorial way—usually more conveniently than the same information could be presented in words.

The convenience of arrow diagrams has important pedagogical consequences for introducing children to the study of relations in the early grades. A child can read—and also draw—an arrow diagram of a relation long before he or she can read or present the same information in words. The difficulty of presenting certain ideas to children lies not in their intellectual inability to grasp the ideas; rather, the limitations are often mechanical. Arrow diagrams have all the virtues of a good notation: they present information in a clear, natural way; they are attractive, colorful things to look at; they are easy and fun for children to draw. Students may use arrow pictures to study, test, and explain their thinking about concepts or situations under consideration. Discussion about an arrow picture often aids the teacher in clarifying a student's solution or misunderstanding of a problem.

Another educational bonus occurs when an arrow diagram spurs students' curiosity to investigate variations or extensions of the original problem. A minor change in an arrow picture sometimes reinforces a pattern already discussed and at other times suggests new problems to explore.

One of the purposes of this strand is to use the power of the language of arrows to help children think logically about relations. Again it must be remembered that our goals concern the thinking process and not the mechanism. The ability to draw prescribed arrows is not the objective in itself, just another format for drill problems in arithmetic.

## Content Overview

# THE LANGUAGES OF STRINGS & ARROWS INTRODUCTION

## Games with Strings

*IG-V* continues the experiences with *The String Gam*e, thoroughly familiar to students from preceding parts of *CSMP Mathematics for the Intermediate Grades*, and introduces *The Divisors Game*, a relative of the former. As in *The String Game*, the object in this new game is to identify the strings, but this time knowing that each string is for the positive divisors of a nonzero whole number less than or equal to 50. *The String Game* situations involve students in thinking critically about the relations "is a divisor of," "is a multiple of," "is greater than," and so on, as well as challenge them to think logically about the relationships between various sets and to give persuasive arguments. In *The Divisors Game*, students specifically investigate the relation "is a divisor of." Strategies involve thinking about which numbers are most beneficial to locate in the string picture, about where multiples of a number already located could or do belong in the picture, and about the implications of knowing that the secret number must be 50 or less.

The games with strings continue to provide rich situations that encourage logical reasoning, and further develop familiarity and ease with numerical relations.

Lessons: L1, 4, 7, 8, and 13.

## The Table Game

Another game is introduced in *IG-V*, *The Table Game*. The emphasis in *The Table Game* is on operations—on finding the images of ordered pairs of numbers under an operation, whereas in *The String Game* the emphasis is on classification—on building sets of numbers. Both games serve a common purpose, to develop strategic thinking. In *The Table Game* students jump to the analysis level quickly because of the extensive analysis already done in *The String Game*. The correct placement of a number immediately limits the possibilities for the string labels in *The String Game* or for the secret operation in *The Table Game*. However, in *The Table Game* the incorrect placement of a number also immediately limits the possibilities for the operation, whereas in *The String Game* this is generally not the case.

*The Table Game*, as its name suggests, involves the table for an operation. The secret operation is one of 12 possibilities, and entries made during the game gradually reveal its identity. The rules of the game encourage careful analysis of entries made early in the game, and reward knowledge of the operation and correct reading of an operation table later in the game.

Lessons: L10, 11, 14, and 15.

## Multiples and Divisors

The relations "is a multiple of" and "is a divisor of" play a major role in the games with strings and in *The Table Game*. Another lesson this semester reviews the greatest common divisor and least common multiple operations, and then uses these operations to present clues in a detective story. In still another lesson, students use prime divisors to explore the prime factor relation and the distance between numbers using the relation.

Lesson: L1, 2, 4, 7, 8, 9, 10, 11, 13, 14, and 15.

## Codes and Counting

# THE LANGUAGES OF STRINGS & ARROWS  INTRODUCTION

Often in mathematics we find a model or framework in which to better examine a problem. In the Necklaces lessons interesting combinatorics questions are posed and, by using simple arithmetic models, one not only suspects the solutions but is totally convinced of them.

The first of the two Necklaces lessons asks how many distinct necklaces can be made with exactly three red beads and seven white beads. Experimentation leads to a simple coding technique, and through arithmetic observations the solution becomes evident. The problem is extended to find how many ways the beads could be put on a pole with one end affixed to the ground. Students discover that the number of possibilities increases.

The second lesson poses a variation of the original question: How many distinct necklaces can be made with exactly four red beads and six white beads? The experimentation is similar, the coding is related, but the arithmetic considerations are far more complex.

Two other lessons in this strand involve students in some counting problems with relations. In one, a binary code helps to count the number of different relations on a three-element set. In the other, a grid representation helps to find the number of permutations of six objects, and then to count the permutations with connected arrow pictures.

Lessons: L3, 5, 6, and 12.

## Capsule Lesson Summary

Put numbers named in a variety of ways (for example, 75, 3 x 17, $\widehat{5}$, 16 + 88) into a two-string picture where each string is for the multiples of some known number. Play *The String Game* with two strings.

### Materials

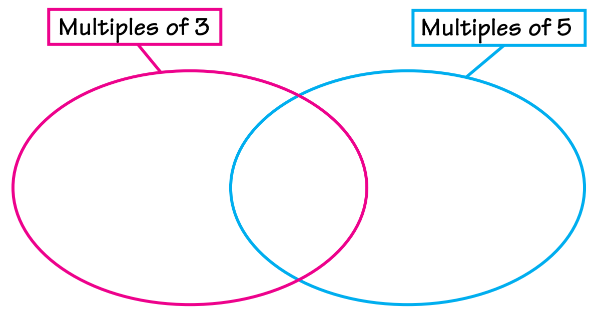
**Teacher**
- Colored chalk
- Numerical String Game kit

**Student**
- Paper
- Colored pencils, pens, or crayons

## Description of Lesson

### Exercise 1

Put a string picture on the board, as shown here. Suggest that students copy and label a similar string picture on their papers. You may use a generic blackline to make copies of a string picture for students to color and label. To one side of the string picture, draw a team board with the indicated numbers on the respective sides of the board. You may prefer to use three or four teams and divide the numbers accordingly.



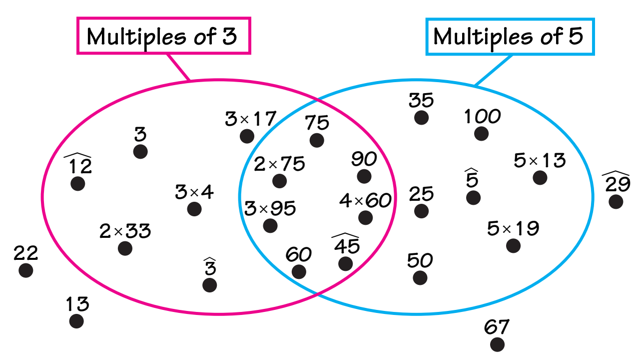

Prepare the class to play a game with teams.

| TEAM A | | TEAM B | |
|---|---|---|---|
| 3×17 | $\widehat{29}$ | 3×4 | $\widehat{3}$ |
| 75 | 2×33 | 60 | 4×60 |
| 3×95 | 90 | 2×75 | 13 |
| 100 | 5×13 | 50 | 5×19 |
| 25 | 3 | 35 | $\widehat{5}$ |
| $\widehat{12}$ | 67 | $\widehat{45}$ | 22 |

**T:** ***We are going to play a string game with these numbers. During your turn, you will have only one chance to put a number into the string picture. The first team to place all the numbers from its side of the board correctly in the string picture wins.***

Alternating teams, call on students to locate numbers in the picture and to write them in chalk. If a number is correctly placed in the picture, erase it from the team board; if a number is incorrectly placed, erase it from the string picture. The game should proceed quickly. Individual students can follow the play of the game by placing numbers in their string pictures.

This illustration shows the correct placement of all the numbers from both sides of the team board.

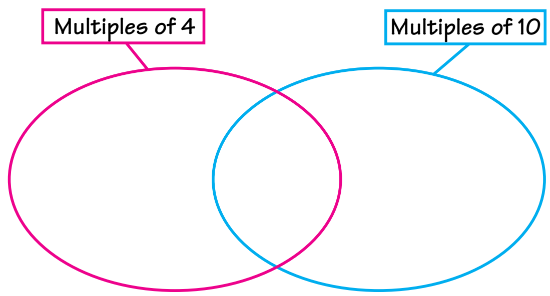


When one team has won the game, consider any numbers not yet placed in the picture from the other team's side of the board. Let students discuss how they know where various numbers belong in the picture. The following questions will focus on various ways to locate numbers in the picture.

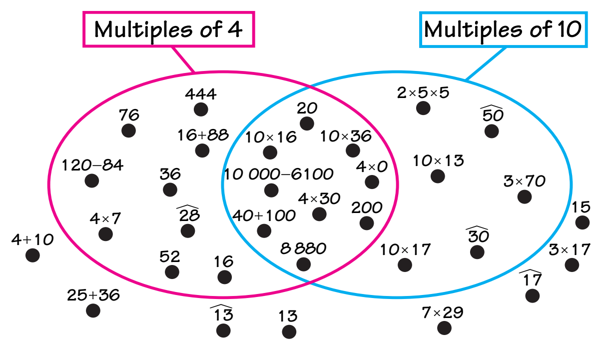
- Where does three times any integer go in the picture? (In the red string)
- Which multiples of 3 are in the intersection? (Three times any multiple of 5)
- Which multiples of 3 are in the red string but outside the blue string? (Three times any integer which is not a multiple of 5)
- Where does five times any integer go in the picture? (In the blue string)
- Which multiples of 5 are in the intersection? (Five times a multiple of 3)
- Which multiples of 5 are in the blue string but outside the red string? (Five times a number that is not a multiple of 3)
- Which numbers are outside both strings? (Numbers which are neither multiples of 3 nor multiples of 5)

Repeat this activity for the situation pictured below, or invite students to locate the numbers individually or with a partner.

| TEAM A | | TEAM B | |
|---|---|---|---|
| 20 | 15 | 16 | $\widehat{28}$ |
| 4+10 | 4×30 | 200 | 3×70 |
| $\widehat{13}$ | 36 | 10×36 | 52 |
| 10 × 16 | 7×29 | 13 | 3×17 |
| 4 × 7 | $\widehat{50}$ | 25 +36 | $\widehat{30}$ |
| 76 | 10×17 | 4×0 | 40+100 |
| 2×5×5 | 16+88 | $\widehat{17}$ | 10×13 |
| 10 000−6100 | 44 | 8 880 | 120−84 |

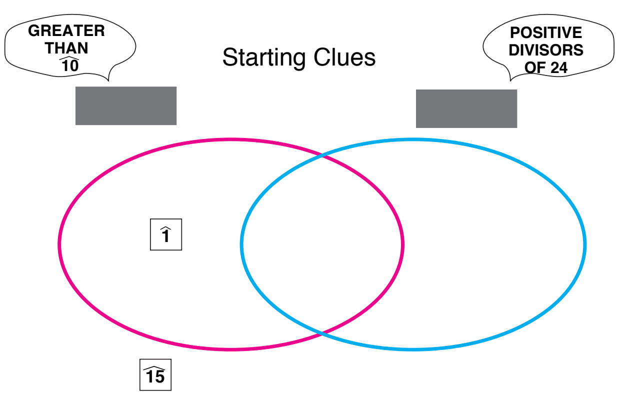


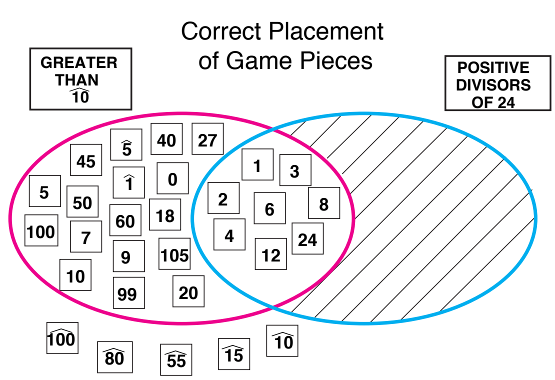

This illustration shows the correct placement of all the numbers from both sides of the team board.




## Exercise 2

For the remainder of the lesson, play *The String Game*. (Appendix D: The String Game, Version C.) A possible game is suggested below.



# L2 THE RELATIONS $\sqcap$ AND $\sqcup$

## Capsule Lesson Summary

Review the greatest common divisor operation $\sqcap$ and the least common multiple operation $\sqcup$ using string pictures. Present a detective story with clues involving these two relations and a string picture telling how many positive divisors the secret number has in common with another specified number.

## Materials

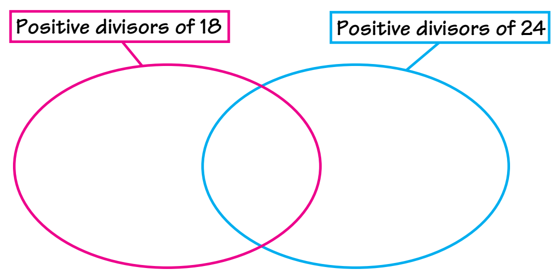
**Teacher**
- Colored chalk

**Student**
- Paper
- Colored pencils, pens, or crayons
- Worksheets L2*, **, and ***

## Description of Lesson

### Exercise 1

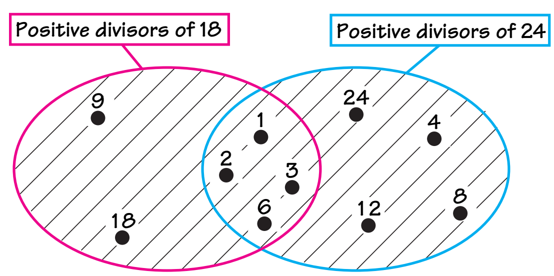
Draw this string picture on the board, and instruct students to copy it on their papers.




**T:** ***Put all of the positive divisors of 18 and the positive divisors of 24 in the picture.***

Allow a few minutes for individual work. When many students have finished, check the work collectively by inviting students to complete the picture at the board.

When the class agrees that all of the divisors of 18 and of 24 are in the picture, hatch the strings.




**T:** ***What are the common divisors of 18 and 24?***

**S:** ***1, 2, 3, and 6.***

**T:** ***What is the greatest common divisor of 18 and 24?***

**S:** ***6.***

Write this number sentence on the board.

$$18 \sqcap 24 = 6$$

Point out that the symbol $\sqcap$ is for the greatest common divisor operation.

**T:** ***The greatest common divisor of 18 and 24 is 6. Look at the common divisors of 18 and 24 again. What do you notice about the common divisors?***

**S:** ***They are the positive divisors of 6 (the greatest common divisor).***

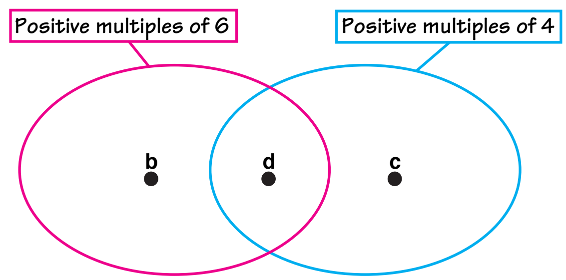
# L2

Write the following problems on the board and ask students to solve them. (Answers are in boxes.) Feel free to adjust the problems to the numerical abilities of your students. Encourage students to use string pictures if they are having difficulty. After a few minutes, check the work collectively.

$$12 \sqcap 15 = \boxed{3} \qquad 28 \sqcap 42 = \boxed{14}$$
$$24 \sqcap 40 = \boxed{8} \qquad 75 \sqcap 135 = \boxed{15}$$
$$4 \sqcap 9 = \boxed{1} \qquad 16 \sqcap 32 = \boxed{16}$$

## Exercise 2

Draw this string picture on the board.




Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

**T** (pointing to **b**): ***Which numbers could be here?***

**S:** ***6, 18, 30, 42, 54, and so on. Any positive multiple of 6 that is not a multiple of 4.***

**T** (pointing to **c**): ***Which numbers could be here?***

**S:** ***4, 8, 16, 20, 28, 32, 40, and so on. Any positive multiple of 4 that is not a multiple of 6.***

**T** (pointing to **d**): ***Which numbers could be here?***

**S:** ***12, 24, 36, 48, and so on.***

List the numbers on the board as they are suggested by the students.

12, 24, 36, 48, 60, 72, ...

**T:** ***What do you notice about these numbers?***

**S:** ***They are multiples of 6 and of 4.***

**S:** ***They are multiples of 12.***

**T:** ***Yes, these numbers are common multiples of 6 and 4. Which positive number is the least common multiple of 6 and 4?***

**S:** ***12.***

Write this number sentence on the board.

$$6 \sqcup 4 = 12$$

Point out that the symbol $\sqcup$ is for the least common (positive) multiple operation.

**T:** ***The least common multiple of 6 and 4 is 12. Look at the common multiples of 6 and 4 again. What do you notice about them?***

**S:** ***Each common multiple is a multiple of 12 (the least common multiple).***

Write these problems on the board and ask students to solve them. (Answers are in boxes.) Feel free to adjust the problems to the numerical abilities of your students. Encourage students to use string pictures if they are having difficulty. Check the work collectively.

$$10 \;\square\; 15 = \boxed{30} \qquad 7 \;\square\; 7 = \boxed{7}$$
$$4 \;\square\; 7 = \boxed{28} \qquad 3 \;\square\; 6 = \boxed{6}$$
$$12 \;\square\; 10 = \boxed{60} \qquad 14 \;\square\; 21 = \boxed{42}$$

## Exercise 3

Present this detective story about a secret number Fa.

Clue 1

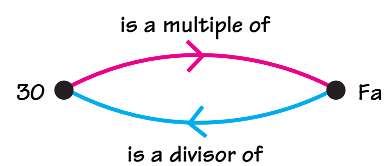
Write this number sentence on the board. $10 \;\square\; \text{Fa} = 30$

**T:** ***Fa is a secret whole number. What information does this clue give us about Fa?***

**S:** ***30 is the least common multiple of 10 and Fa.***

**S:** ***Fa is a divisor of 30.***

If no one makes this last observation, prompt it by using a return arrow.




**T:** ***Write the positive divisors of 30 on your paper.***

List the positive divisors of 30 on the board as students name them.

1, 2, 3, 5, 6, 10, 15, 30

**T:** ***Which of these numbers cannot be Fa?***

**S:** ***Fa cannot be 1, 2, 5, or 10. The least common multiple of 10 and each of these numbers is 10, not 30.***

Cross off the numbers that Fa cannot be.

**T:** ***Do you notice anything interesting about the numbers that we crossed off?***

**S:** ***Each number that we crossed off is a positive divisor of 10.***

The students should conclude that Fa could be 3, 6, 15, or 30.

Clue 2

Write this information on the board. $24 \;\square\; \text{Fa} = 3$

**T:** ***What information does this clue give us about Fa?***

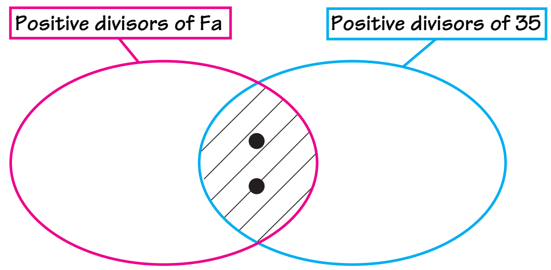
**S:** ***3 is the greatest common divisor of 24 and Fa.***

**S:** ***Fa is a multiple of 3.***

Let students work independently, testing the possibilities for Fa in this clue. The class should conclude that Fa could be 3 or 15.

Clue 3

Draw this string picture on the board.




**T:** ***What new information does this string picture give us?***

**S:** ***Fa and 35 have exactly two common divisors.***

**T:** ***Who is Fa?***

Allow a minute or two for students to test the possibilities for Fa in this clue.

**S:** ***Fa is 15, because 15 and 35 have exactly two positive divisors in common: 1 and 5.***

**S:** ***Fa cannot be 3, because 3 and 35 have only one common positive divisor: 1.***

The class should conclude that Fa is 15.

Worksheets L2*, **, and *** are available for individual work.

## Writing Activity

Suggest that students try to write a detective story with clues that involve the greatest common divisor relation or the least common multiple relation.

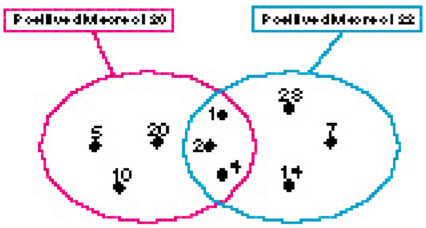
You may like students to take lesson notes on some, most, or even all their math lessons. The “lesson Notes” section in the Notes to the Teacher gives suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students can describe the greatest common divisor ($\sqcap$) and least common multiple ($\sqcup$) relations.

## Home Activity

This is a good time to send a letter to parents/guardians about the use of strings. Blackline L2 has a sample letter.

Name________________ L2 ★

Put all of the positive divisors of 20 and of 28 in this string picture.




Complete these number sentences.

20 ⊓ 28 = 4

10 ⊓ 28 = 2

35 ⊓ 28 = 7

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Name________________ L2 ★★

Zim is a secret whole number.

Clue 1

Zim ⊔ 24 = 72

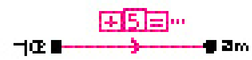
Zim could be 9, 18, 36, or 72.

Clue 2

Zim ⊓ 30 = 6

Zim could be 18, 36, or 72.

Clue 3




Who is Zim? 18

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Name________________ L2 ★★★

Pom is a secret whole number.

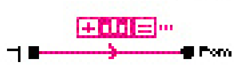
Clue 1

Pom ⊓ 28 = 7

Find a pattern for the numbers that Pom could be.

Pom could be 7, 21, 35, 49, 63, 77, 91, 105, 119, 133, 147, and so on. (Pom is an odd multiple of 7.)

Clue 2




Find a pattern for the numbers that Pom could be.

Pom could be 49, 119, 189, 259, 329, 399, and so on. (Pom is a multiple of 7 with 9 in the ones place, or Pom could be 49 or any whole number gotten by adding 70s to 49.)

Clue 3

In this list, Pom is the greatest number less than 1000.

Who is Pom? 959

## Capsule Lesson Summary

Count the number of different relations on a set of three objects by using a binary code to assign a number to each relation.

### Materials

**Teacher**
- Colored chalk
- Magnetic checkers

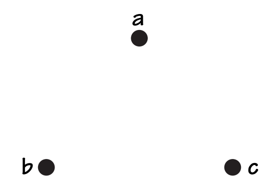
**Student**
- Paper
- Colored pencils, pens, or crayons
- Worksheets L3* and **

## Description of Lesson

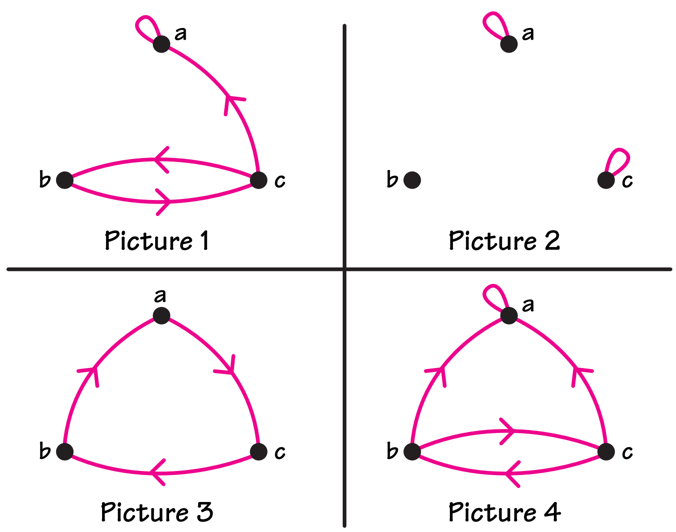
### Exercise 1

Draw this picture on the board, and instruct students to do the same on their papers.

**T:** ***Draw three dots on your paper and label them*** **a**, **b**, ***and*** **c**. ***Then draw any arrow picture you wish with these three dots. When you have completed one picture, draw several more. Try to make each arrow picture different from the others.***



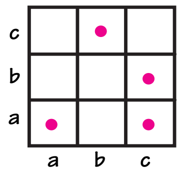

Allow several minutes for individual work. If necessary, indicate that it would be redundant to have two arrows start at the same dot and end at the same dot. You could also suggest to students that they may consider arrows that start and end at the same dot (loops). After a while, invite several students to draw one of their pictures on the board. The sample pictures below will be used in the following discussion. You will need to adapt the discussion to fit the arrow pictures you have on the board.

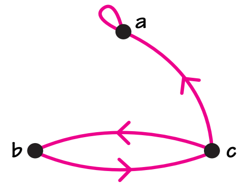



**T:** ***Right now we have pictures for four different relations on a set of three objects. How many different relations like this do you think there are?***

Write some students' guesses on the board. At this point accept any guess; the rest of the lesson will be directed at finding the exact number of relations on a set of three objects.

Draw a 3-by-3 grid on the board near one of the pictures. Represent each arrow or loop in the picture with a checker (magnetic or drawn dot) on the grid. For example:

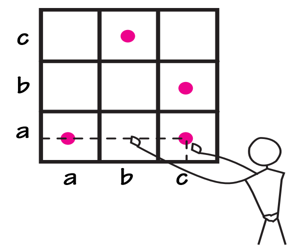



Picture 1

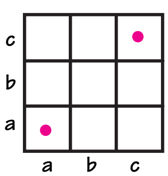
**T:** ***To help find the number of relations, we can use codes. The first code involves putting checkers on a 3-by-3 grid. Do you understand this grid for Picture 1? Can you explain this code?***

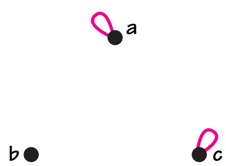
Let the class study the picture for a few minutes.



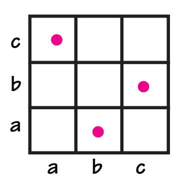

**S:** ***There is an arrow starting at* c *and ending at* a *in Picture 1. On the grid there is a checker in the square above* c *and to the right of* a *(across to* c *then up to* a*). The same is true for each of the other arrows.***

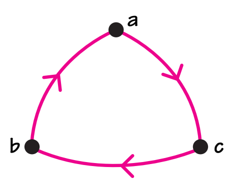
Ask students to code each of the other arrow pictures on a grid. Then invite students to draw the grid pictures on the board near the corresponding arrow pictures. For example:



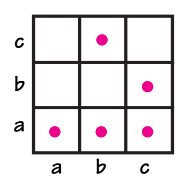



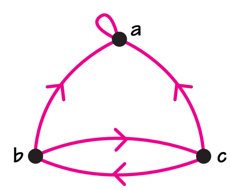

Picture 2




Picture 3



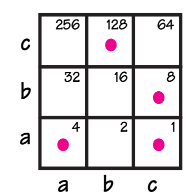




Picture 4

**T:** ***Next, we can use a binary code. Let's label the squares of a 3-by-3 grid as binary places: 1, 2, 4, 8, 16, and so on.***

Label the squares of the grid picture for Picture 1.

Code Number 141

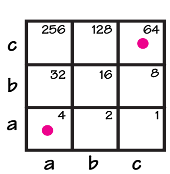


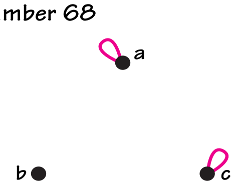

**T:** ***With a binary code, we can assign exactly one number to each grid picture (or arrow picture). Which number do we assign to Picture 1?***

**S:** ***141, because 1 + 4 + 8 + 128 = 141.***

Ask the class to find the code numbers for other pictures you have on the board. Allow a few minutes for individual or partner work while you observe and help students having difficulty.
Call on a student who has a correct code number to explain the coding at the board.

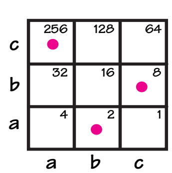
Code Number 68

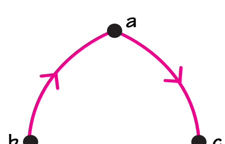



Picture 2

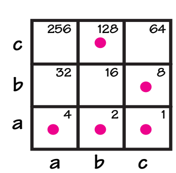
Code Number 266

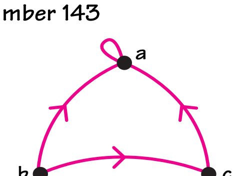



Picture 3

Code Number 143



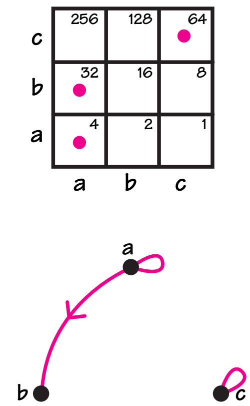

Picture 4

## Exercise 2

**T:** ***100 is the code number of a relation on the set {a, b, c}. Can you draw its arrow picture?***

**S:** ***First draw a grid and label the squares like a binary abacus. Then put checkers on the appropriate squares.***

Code Number 100




**T:** ***Where do we put the checkers for 100?***

**S:** ***Put one checker on the 64-square, one checker on the 32-square, and one checker on the 4-square.***

**T:** ***How do we draw the arrow picture?***

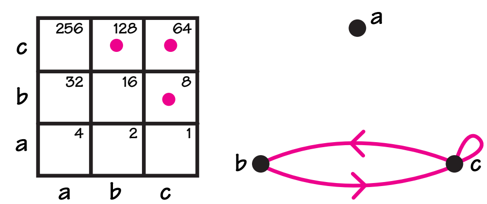
**S:** ***Draw three dots labeled* a*,* b*, and* c*. Look at the grid. There are two checkers in the first column. That means there are two arrows starting at* a*—one arrow ends at* a *(it's a loop) and one arrow ends at* b*. Also, there is one checker in the third column, so there is an arrow starting at* c *and ending at* c *(a loop).***

**T:** ***Very good. Now let's find the arrow picture with code number 200.***

**S:** ***200 is the double of 100, so move the checker on the 4-square to the 8-square, move the checker on the 32-square to the 64-square, and move the checker on the 64-square to the 128-square.***

Invite a student to draw the corresponding arrow picture on the board.

Code Number 200

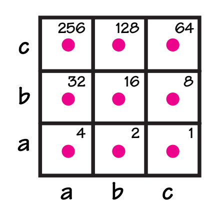



**T:** ***How many different relations do you think there are on a set of three objects?***

Compare students' guesses now with those made at the beginning of the lesson. Erase any guess the class believes is unreasonable (such as three or four) and write the new guesses on the board.

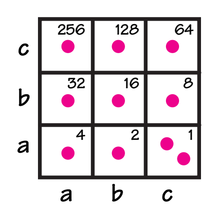
**T:** ***We know each relation has one code number and each code number is for exactly one relation. What is the greatest possible code number for a relation on a set of three objects?***

**S:** ***The greatest possible code number is the one we get by placing a checker on each square of the grid.***

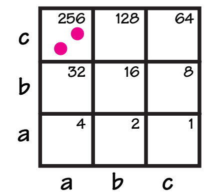



**T:** ***Can you find this number without doing much addition?***

If no student suggests putting an extra checker on the 1-square, do so yourself.




**T:** ***Which number is this?***




**S:** ***512. Trade the two checkers on the 1-square for one checker on the 2-square. Then we have two checkers on the 2-square, which we can trade for one checker on the 4-square. Keep doing this until there are two checkers on the 256-square, which (on the binary abacus) is the same as one checker on a 512-square.***

If you wish, invite a student to make the trades.

**S:** ***The greatest code number is 511; 512 – 1 = 511.***

If necessary, remind the class that an extra checker was put on the grid to get 512, so that a checker on every square is for 511.

**T:** ***What is the least code number?***

**S:** ***0, the code number for the relation with no arrows.***

**T:** ***Do you think there is a relation for each code number between 0 and 511?***

**S:** ***Yes. We can always use the grid picture to draw the arrow picture.***

**T:** ***For each relation there is exactly one code number, and for each code number there is exactly one relation.***

***How many relations are there on a set of three objects?***

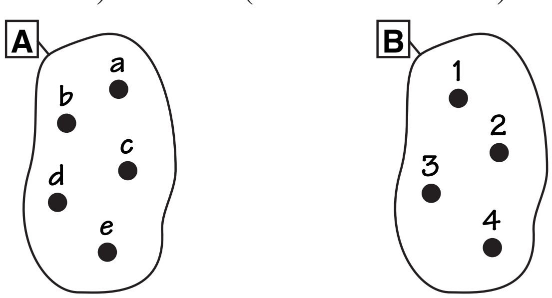
**S:** ***There is exactly one relation for each code number, 0 to 511, so there are 512 relations.***

Compare students' estimates to 512.

Worksheets L3* and ** are available for individual work.

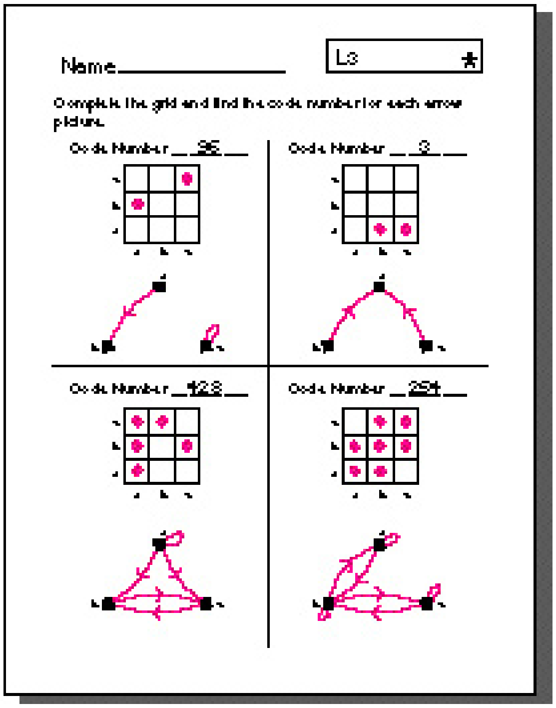
## Extension Activity

Invite students to solve a similar counting problem to find the number of different relations from a set **A** (with five members) to a set **B** (with four members).

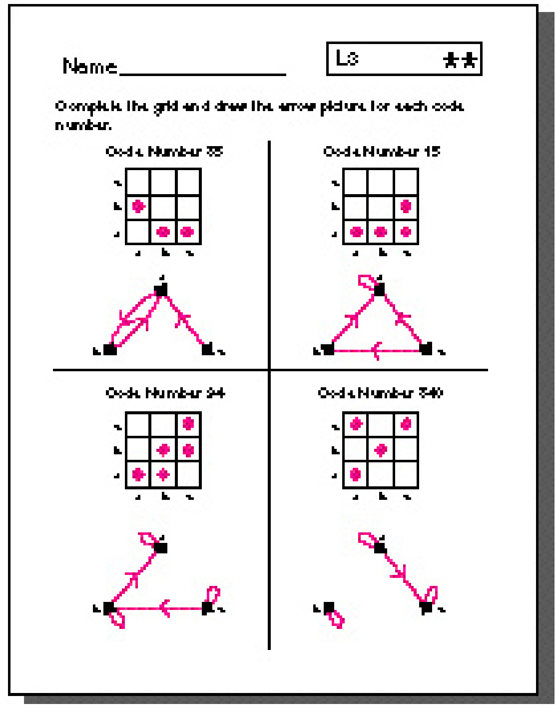



**Answer: 1,048,576**

Blackline L3 has a description of this problem for students to use. The solution technique suggested is similar to the one used in this lesson. Some students may have or want to use different methods.



Name
L3 ★
Complete the grid and find the code number for each arrow picture.
Code Number 35
Code Number 3
Code Number 428
Code Number 254

Name
L3 ★★
Complete the grid and draw the arrow picture for each code number.
Code Number 35
Code Number 15
Code Number 24
Code Number 340

## Capsule Lesson Summary

Introduce *The Divisors Game*, a type of string game in which each of two strings is for the positive divisors of an unknown number. Through selection and placement of numbers in the string picture, discover the unknown numbers. Play a variation of the game involving a three-string picture, or put numbers into a two-string picture with given string labels: **Positive divisors of 36** and **Positive divisors of 42**.

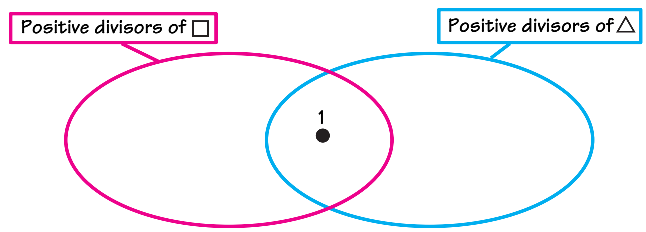
### Materials

| Teacher | • Colored chalk | Student | • None |
|---|---|---|---|

## Description of Lesson

### Exercise 1

Draw this string picture on the board.




Introduce a type of string game in which students eventually determine the string labels. Announce that the numbers in the square and in the triangle may be the same or different but are restricted to whole numbers from 1 to 50. Place 1 in the middle region because 1 is a positive divisor of any whole number.

Divide the class into teams. The teams alternate, and the members take turns within each team. During a turn, a student either selects a whole number from 2 to 50 for you to locate in the picture or tries to identify one of the strings. If a student correctly identifies a string label, you record the appropriate number in the square or in the triangle. Otherwise, simple say that the identification is incorrect.

Points are gained or lost during the game in the following ways.

- A team gains one point for each string inside which the selected number belongs when a member selects a number for you to locate. For example, if a member selects a number belonging in both the red string and the blue string, the team gains two points. If a member selects a number belonging in only one of the strings, the team gains one point. If a member selects a number belonging outside both strings, the team gains no point. **Note:** The game starts with 1 placed in the intersection so that the first player does not have the advantage of gaining two points automatically.
- A team gains four points if a member correctly identifies one of the strings.
- A team loses one point if a member incorrectly identifies one of the strings.

The game is over when both strings have been determined. Tally the scores for the teams; the team with the most points wins the game.

# L4

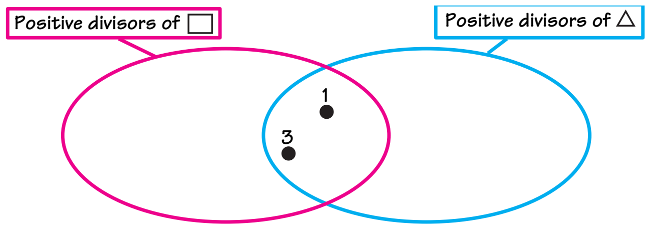
The following dialogue is a condensed description of a game played by two teams. In this game the red string is for **Positive divisors of 30** and the blue string is for **Positive divisors of 42**.

**S** (Team A)**:** *3.*

**T:** *3 is in the intersection. Two points.*

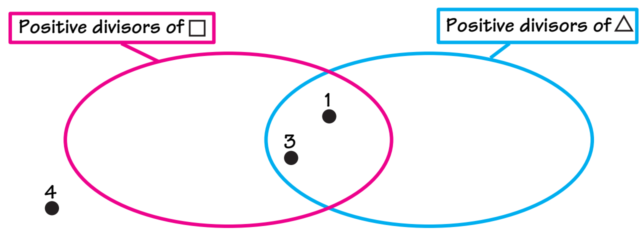
Score

| Team A | Team B |
|---|---|
| 2 | |




**S** (Team B)**:** *4.*

**T:** *4 is outside both strings.*
*No points.*

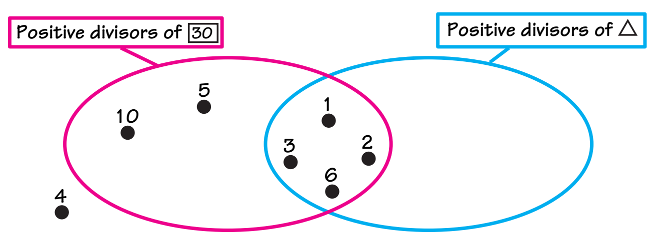



Students next select 5, 2, 10, and 6.

**S** (Team A)**:** *The red string is for* **Positive divisors of 30.**

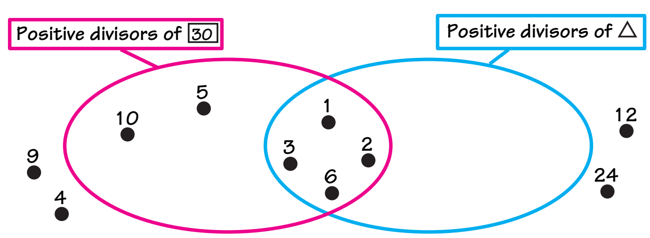
Score

| Team A | Team B |
|---|---|
| 2 | 2 |
| 1 | 2 |
| 1 | |
| 4 | |




Students next select 24, 9, and 12.

**S** (Team A)**:** *The blue string is for* **Positive divisors of 6.**




**T:** *No. Your team loses a point.*

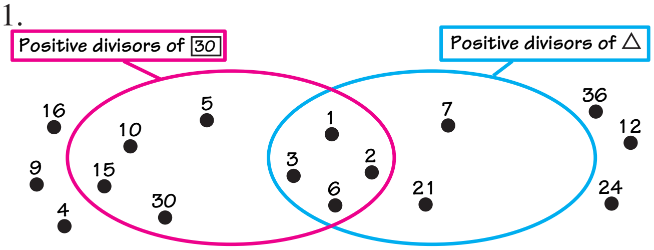
Score

| Team A | Team B |
|---|---|
| 2 | 2 |
| ~~1~~ | 2 |
| 1 | |
| 4 | |

Students next select 16, 15, 30, 36, 7, and 21.

Score

| Team A | Team B |
|---|---|
| 2 | 2 |
| ~~1~~ | 2 |
| 1 | 1 |
| 4 | 1 |
| 1 | |
| 1 | |

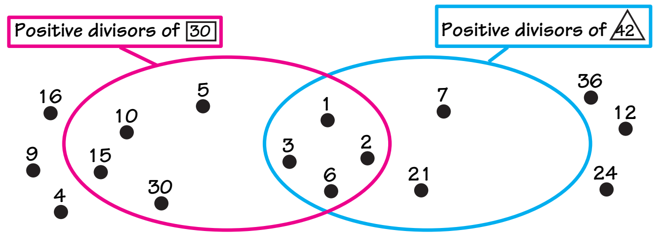



**S** (Team B)**:** ***The blue string is for* Positive divisors of 42.**

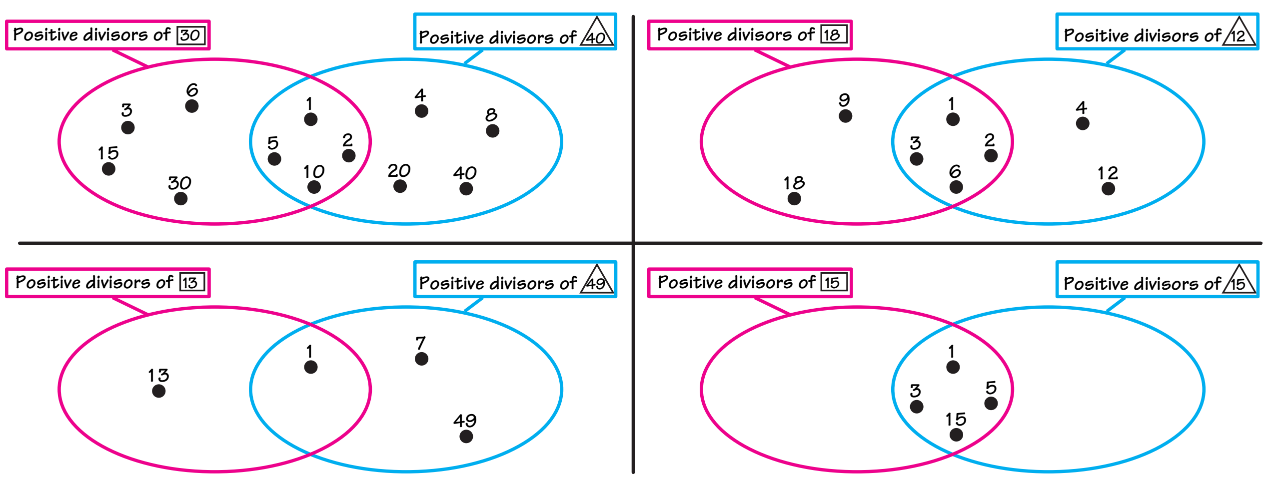
**T:** *That's correct!*

Score

| Team A | Team B |
|---|---|
| 2 | 2 |
| ~~1~~ | 2 |
| 1 | 1 |
| 4 | 1 |
| 1 | 4 |
| 1 | 10 |
| 9 | |

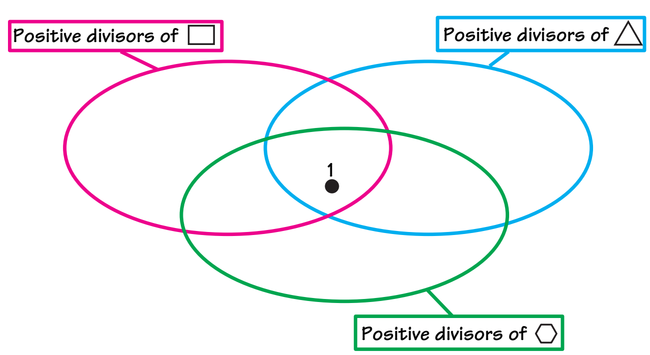



Play this game a couple more times with your class. Partial crib sheets for four possible games are given below. Note that only the numbers inside the strings are shown; assume that any whole number not in the pictures is outside both strings. Remember to place 1 in the intersection before starting a game.




Choose Exercise 2a or 2b according to the ability of your class. Exercise 2 is more challenging than Exercise 2b.

## Exercise 2a




Play *The Divisors Game* with three strings. Locate 1 in the intersection of the three strings before starting the game.

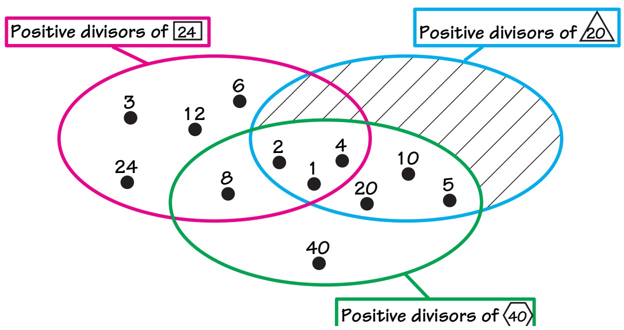
Points are gained or lost during the game in the following ways.

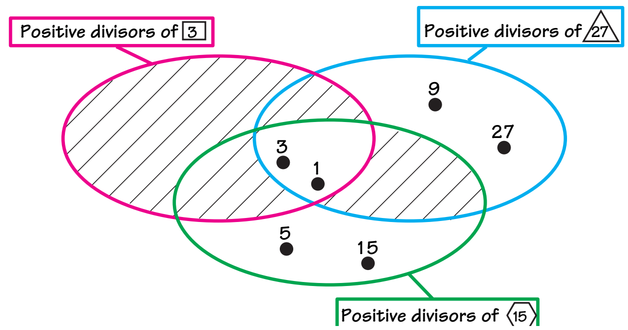
- A team gains one point for each string inside which a selected number belongs (for example, three points if a number belongs inside all three strings; two points if a number belongs inside only two strings, and so on).

- A team gains five points if a member correctly identifies one of the string labels.
- A team loses one point if a member incorrectly identifies one of the string labels.

The game is over when all three strings have been determined. The team with the most points wins the game.

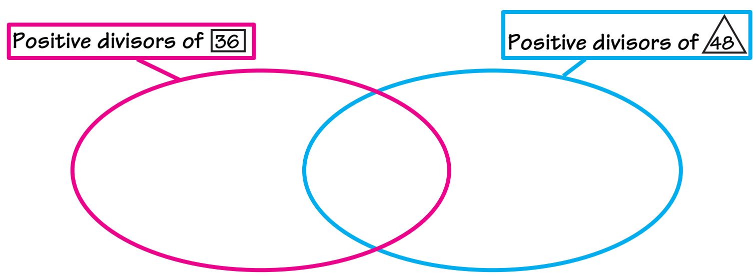
Play this game once or twice with your class. Partial crib sheets for two possible games are given below. Note that only the numbers inside the strings are shown; assume that any whole number not in the pictures is outside the strings. The hatching is included for your reference and should not be given as a clue.




## Exercise 2b

Label the strings in the picture from Exercise 1 in this way.



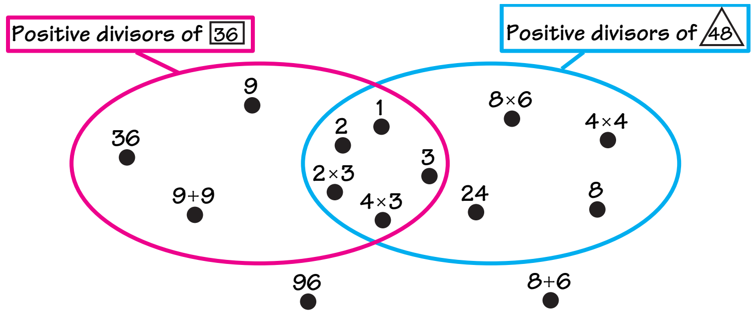

Ask students to explain where each of the following numbers belongs in the picture.

1 2 3 9 24

2 × 3 96 4 × 4 8 × 6 9 × 9

36 4 × 3 8 8 × 6

The following illustration shows the correct placement of these numbers.



## Capsule Lesson Summary

Determine how many ways there are to arrange seven white beads and three beads in a necklace. Then determine how many ways there are to arrange them on a pole.

### Materials

**Teacher**
- Blackline L5
- Colored chalk
- Tape
- Red markers

**Student**
- Colored pencils, pens, or crayons
- Worksheet L5

**Advance Preparation:** Use Blackline L5 to make 12–15 copies of the necklace picture for use in Exercise 1.

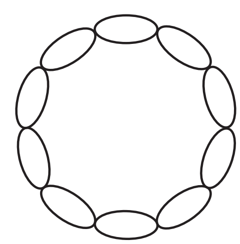
## Description of Lesson

### Exercise 1

Introduce the counting problem in a story context.

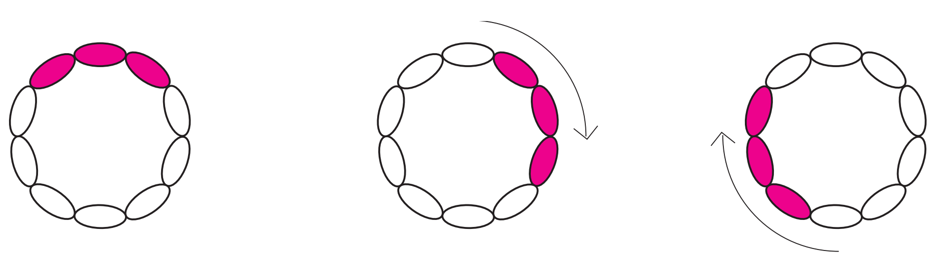
**T:** ***Theophilus is a boy who lives in a village where each young person wears a necklace of ten beads around the neck. The necklace has all white beads when the person receives it.***

Tape one picture of a necklace to the board.



**T:** ***When a young person has performed a special task or has demonstrated a certain skill, he or she can trade a white bead for a red bead. Theophilus has done well and has earned three red beads for his necklace. He wonders how to arrange the beads; he could put the three red beads together or space then apart.***

Color three beads red in the picture so that they are together. Take the picture off the board and rotate it several times as you say,



**T:** ***This describes the same necklace as this one … and this one.***

Tape the picture back to the board.

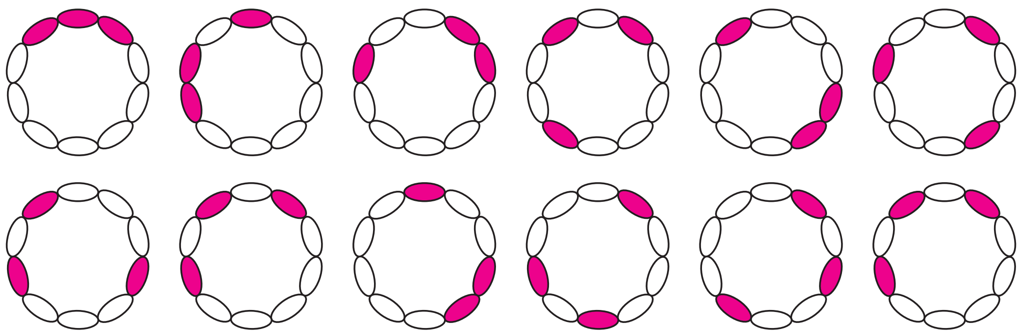
**T:** ***How many different arrangements of the beads do you think there are?***

Record estimates from several students. Distribute copies of Worksheet L5 and red pencils.

**T:** ***By coloring three beads red in these pictures, show the different ways Theophilus could arrange the beads in his necklace.***

**Note:** There are more than enough pictures on the worksheet to show the complete set of possibilities, but do not tell this to students.

While students are working independently, tape 10 or 11 more pictures of the necklace to the board. After a short while, invite students to color beads in the pictures on the board. Students should use a red marker to color the beads so the color will show through to the other side of the paper. Your board might look like this:



**T:** ***Are any two of these colorings the same necklace?***

If beads are colored in more than eight pictures, there will be some duplication. When someone claims that a pair of colorings are duplicates, take one of them off the board. Holding it in the same position as it was on the board, ask,

**T:** ***Is this the same necklace as this one*** (point to the suggested duplicate)***?***

**S:** ***Yes, just turn it.***

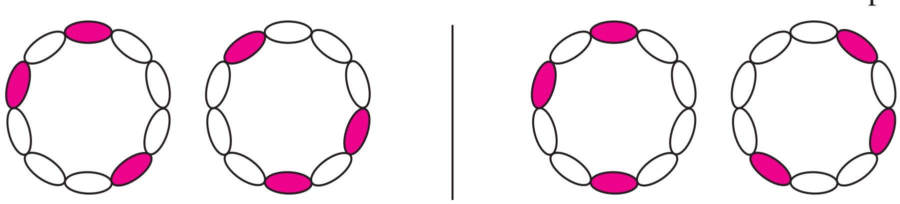
Rotate the picture and let the class see if the red beads can be in the same position. If one picture is not simply a rotation of the other, ask,

**T:** ***Some of you think these are the same necklace, but just turning one of them does not show that. Is there something else we could do?***

**S:** ***Turn the picture over.***

If no one suggests flipping the picture, suggest it yourself. In the preceding illustration …,

…these necklaces are rotations of one another … …and these necklaces are flips on one another.



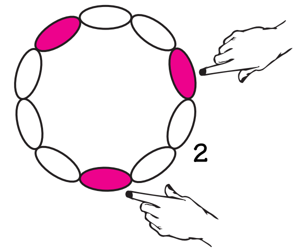
**Note:** In some cases, you may need to both flip and rotate a picture to show that it is a duplicate of another picture.

Continue in this manner until all duplicates are found. Leave only one from a set of duplicates on the board.

**T:** ***How do we find duplicates? What do we look at?***

**S:** ***I check the number of white beads between pairs of red beads.***

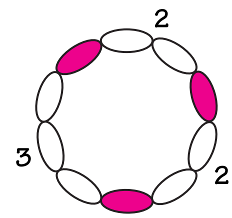
Consider any one of the necklaces pictured on the board.



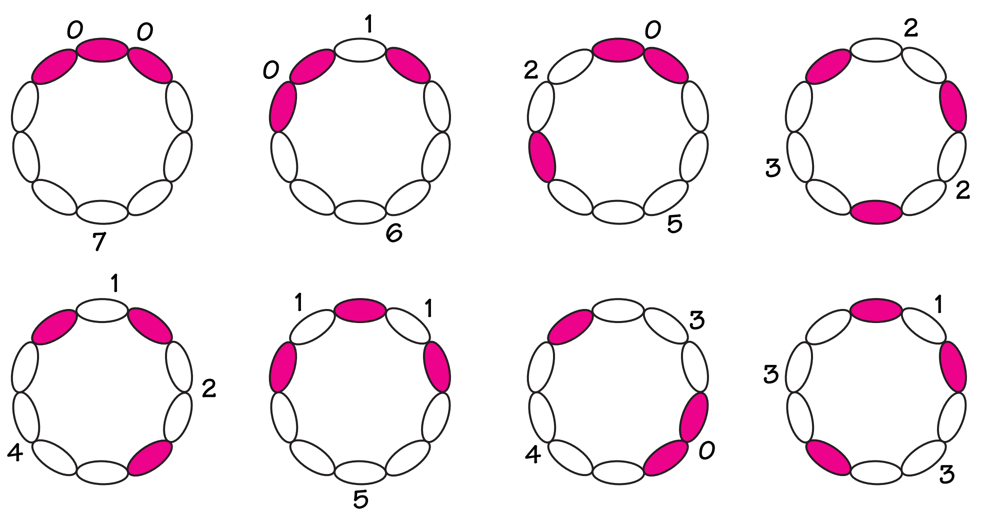

**T:** ***How many white beads are between these two red beads?***

**S:** ***Two.***

Similarly consider the other groups of white beads between two red beads in the necklace.




Invite students to record the number of white beads in each group for the necklaces pictured on the board. Remove any additional duplicates found.




**T:** ***What do you notice about these numbers?***

**S:** ***There are three numbers for each necklace.***

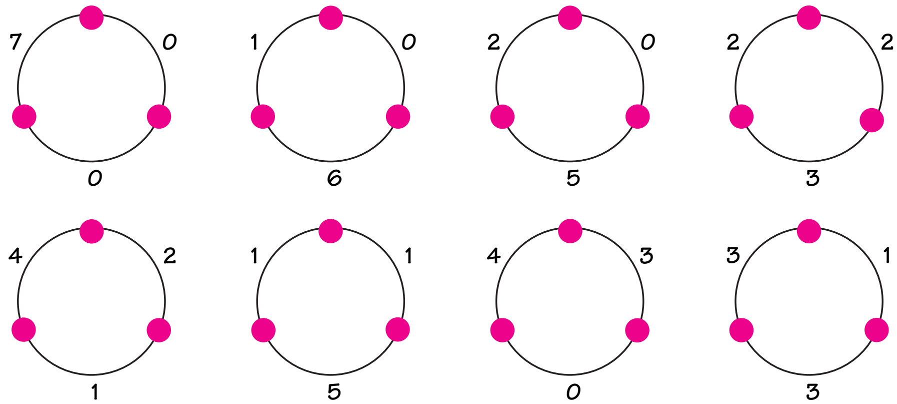
**S:** ***The sum of the three numbers for any necklace is 7.***

**T:** ***How do we know whether or not we have all of the possibilities? What could we do to check?***

**S:** ***Look at the numbers.***

**T:** ***We know there are three numbers and their sum is 7. Turn your worksheet over and write down all of the possible combinations of three (whole) numbers whose sum is 7.***

After a couple minutes, ask students to announce the combinations, and record each one as shown below. Make sure your students understand that each dot is for a red bead and the number between two dots is the number of white beads between them.



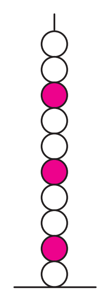

Compare the possible number combinations to those recorded by the necklace pictures. Conclude that eight distinct necklaces are possible, or use the number combinations to find necklaces that are missing. Take the pictures off the board and record the number of solutions on the side of the board. Compare the actual number of ways (8) to students' estimates.

**necklace : 8 ways**

## Exercise 2

**T:** ***Theophilus has a friend in another village. A similar thing is done for the young people in that village. Ten beads are put on a pole, and the pole is stuck in the ground in front of the young person's house. Just as Theophilus, his friend has earned three red beads.***

On the board, draw a picture of a pole with ten beads.



**T:** ***Who can suggest one way of arranging the beads?***

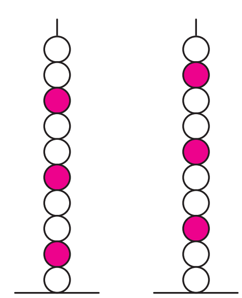
As a student gives the color of the beads, color the picture on the board accordingly. For example:

**S:** ***White, red, white, white, red, white, white, red, white, white.***

**T:** ***Let's find how many ways we can arrange the three red and seven white beads on a pole. Do you think there are more or less than eight? Why?***

**S:** ***I think there are more than eight, because if you put the beads on one way and turn it upside down, you have a different arrangement. With an arrangement on the necklace, no matter how you turn it, it is basically the same necklace.***

Draw another ten-bead pole, and color it to be the 180° rotation of the one already on the board.



**T:** ***If the pole of beads is turned upside down or the beads are put on in the reverse order, we get another arrangement.***

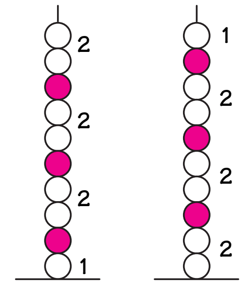
Accept other comments on the number of possible arrangements of the beads on the pole, as compared to eight for the necklace.

**T:** ***Can you suggest a method we could use to count the various arrangements?***

**S:** ***We could count the white beads in between red beads like we did with the necklace.***

**S:** ***We should also count the white beads at the bottom and at the top of the pole.***

In each of the two poles pictured on the board, count the white beads at the bottom of the pole before the first red bead, the white beads between the red beads (two sets), and the white beads at the top of the pole after the last red bead. Record the four numbers, some of which might be zeroes, near the pictures.

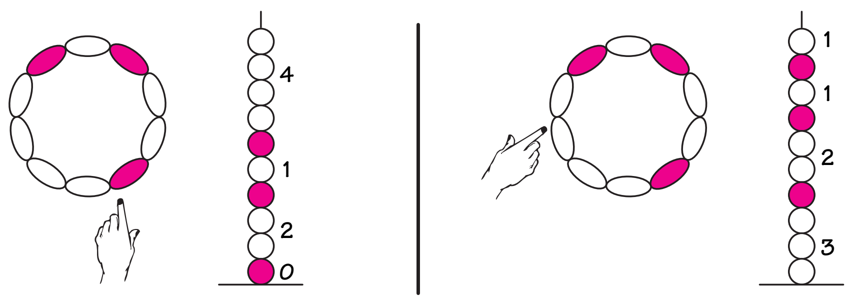



**T:** ***Looking at the number of white beads between red beads and the number at the bottom and at the top of the pole certainly helps distinguish arrangements on a pole. Do the necklace arrangements suggest pole arrangements?***

**S:** ***Yes, if you break a necklace at any of ten places and hold the string of beads straight up and down, it resembles a pole.***

Take one of the necklace pictures and ask a student to indicate a breaking place.

Hold your finger in that place and ask the student to draw a corresponding pole arrangement. The student might go clockwise or counterclockwise on the necklace. Code the picture. Ask another student to split it in another way and to draw a corresponding pole arrangement. Code the picture. For example:




**T:** ***One of these pictures is coded 0-2-1-4. Do you think we can use the same four numbers to get a different arrangement other than this one turned top to bottom?***

**S:** ***Let's try 0-1-2-4.***

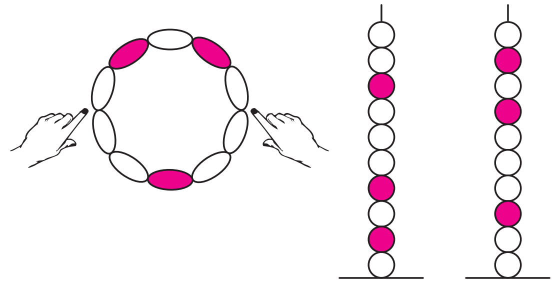
**T:** ***How many possible pole arrangements do you think there are?***

**S:** ***There are eight different arrangements for the necklace. Since there are ten different places to break a necklace arrangement to make a pole arrangement, I think there are 80 different pole arrangements.***

**S:** ***But when you break a necklace, you can hold the beads one way or turn them top to bottom. So I think there are 160 different pole arrangements.***

**T:** ***But there might be some duplicates. Maybe we can say that there are at most 160 different pole arrangements.***

Use the following demonstration to show there will be some duplicates among 160 arrangements. Hold up the picture of this necklace.



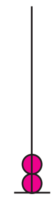
**T:** ***Suppose we break it in either of these two places. We would get the same two pole arrangements for each.***

Clear a space on the board to build a large picture.

**T:** ***Let's try to count systematically all of the possibilities, being careful to avoid duplicates.***

***Suppose there is no white bead at the bottom of the pole. We could start with two red beads at the bottom.***

***Where can we put the third red bead? In how many different places? Remember that there are ten beads altogether.***

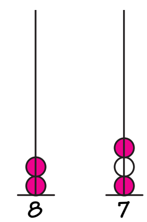


Eight more beads must be put on the pole. Any one of them could be the third red bead.

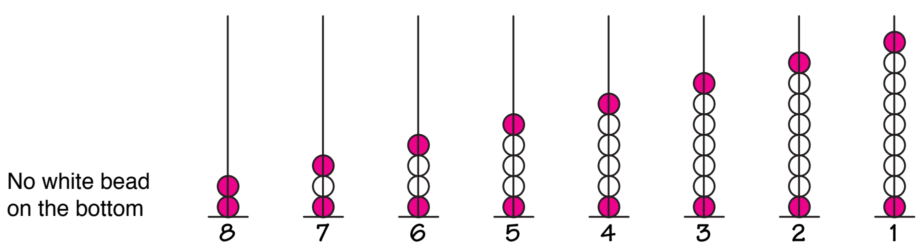
Record 8 under the pole.

**T:** ***Now suppose there is no white bead at the bottom of the pole, and we start with one red bead, one white bead, and then a second red bead. Where can we put the third red bead? In how many different places?***

**S:** ***Seven more beads must be put on the pole. Any one of them could be the third red bead.***




Record 7 under the pole. Next consider a red bead followed by two white beads followed by a red bead. Continue by increasing the number of white beads between the first two beads to seven. Label the row "No white bead on the bottom."




**T:** ***Now suppose there is one white bead at the bottom of the pole. Then we could put on two red beads.***

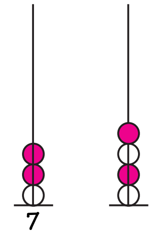
***Where can we put the third red bead? In how many different places? Remember that there are ten beads altogether.***

**S:** ***Seven more beads have to be put on the pole. Any one of them could be the third red bead.***

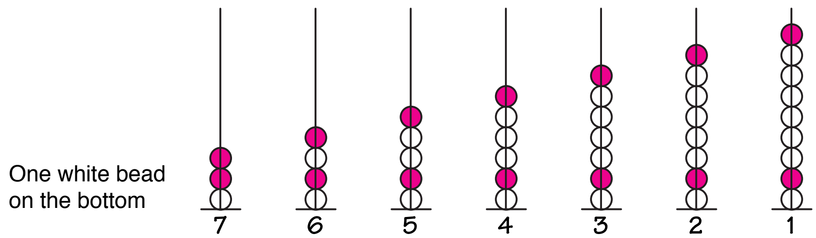
Record 7 under the pole.

**T:** ***Suppose there is one white bead at the bottom of the pole, and then a red bead, one white bead, and another red bead. In how many different places could we put the third red bead?***

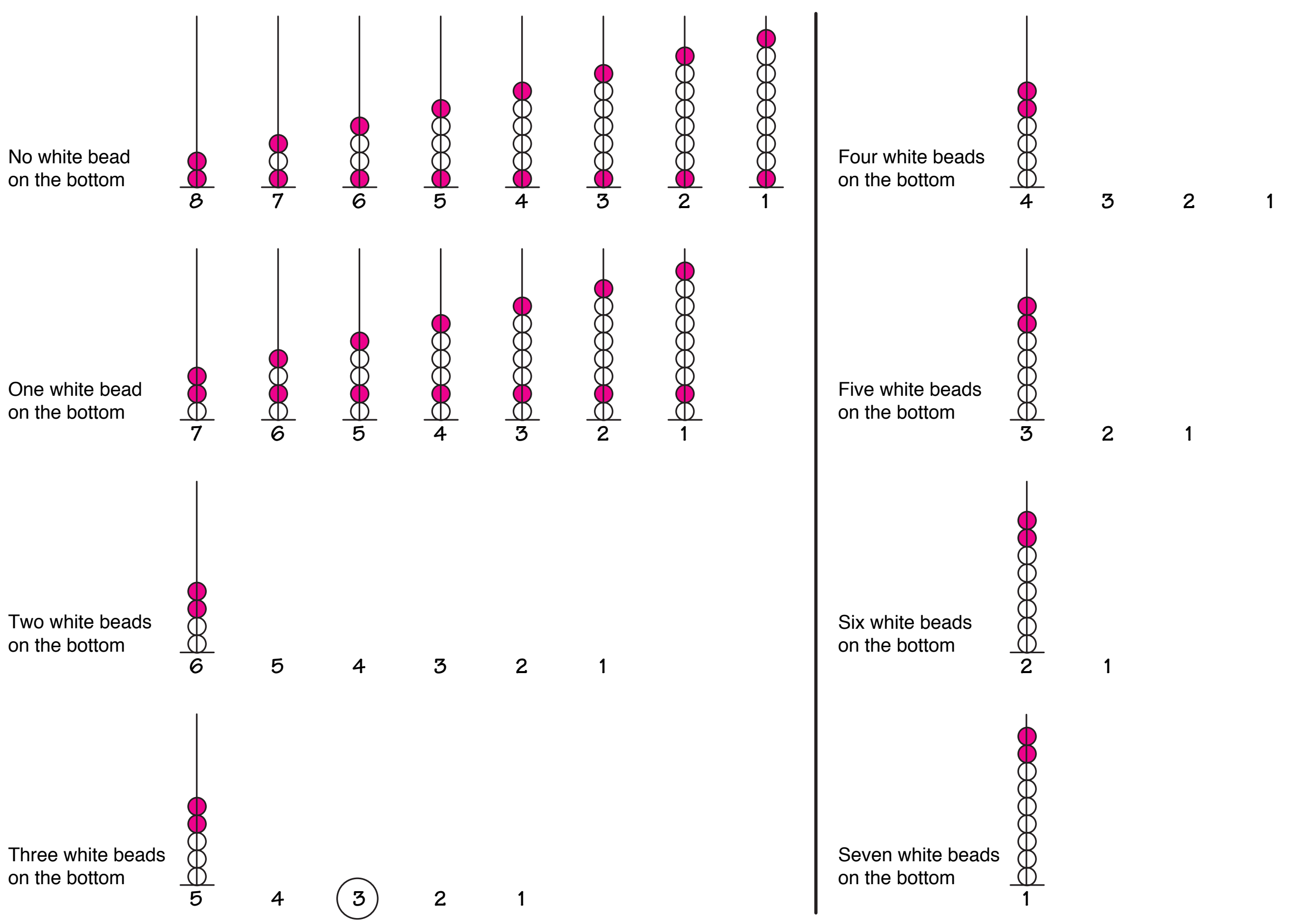
**S:** ***Six more beads have to be put on the pole. Any one of them could be the third red bead.***




Record 6 under the pole. Continue until all seven cases in which there is exactly one white bead at the bottom of the pole have been considered. Notice that since there are only seven white beads and one is on the bottom of the pole, the greatest number of white beads that can be between the first two red beads is six.




Instruct students to work with partners to consider the six cases in which there are exactly two white beads at the bottom of the pole before the first red bead. Students are likely getting accustomed to the analysis and the structure the chart is taking on. Drawing all of the poles at this time may be unnecessary. You may find it sufficient to draw only the first pole in the third row and each row thereafter. In all other cases, enter only the number of ways to complete a particular pole in the chart. Continue adding rows to the chart, each time increasing the number of white beads below the first red bead by one. The chart is shown below.



You may check that students understand the structure of the chart by asking the class which case a particular entry pertains to, as in the following dialogue (see the circled entry in the preceding illustration).

**T:** ***What does 3 mean here?***

**S:** ***There are three different places to put the third red bead on a pole that already has three white beads followed by one red bead followed by two white beads followed by a red bead.***

**T:** ***How do we find the total number of possible pole arrangements?***

**S:** ***Add all those numbers.***

**T:** ***Do we need to be concerned about duplicates?***

**S:** ***No, in any row the number of white beads between two of the red beads is different, and in different rows the numbers of white beads on the bottom are different.***

**T:** ***Does anyone see an easy way to add all of these numbers?***

**S:** ***There are eight 1s and one 8, seven 2s and two 7s, six 3s and three 6s, five 4s and four 5s.***

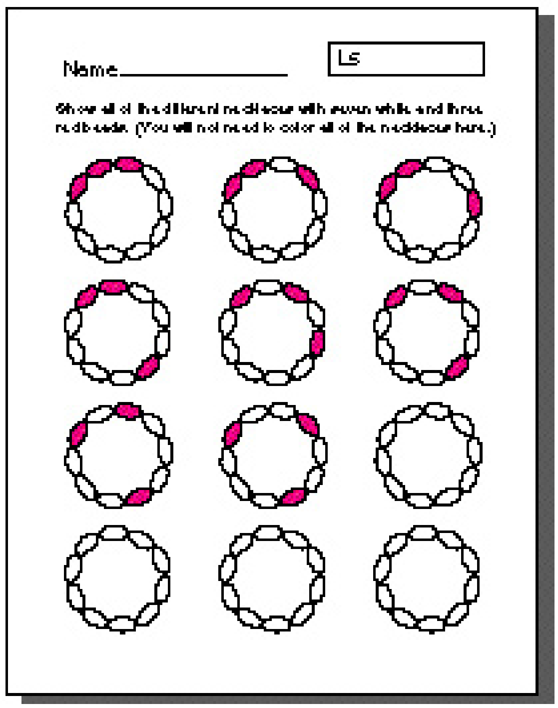
If this method is suggested, do the calculation with the class.

$$(8\times 1) + (1\times 8) + (7\times 2) + (2\times 7) + (6\times 3) + (3\times 6) + (5\times 4) + (4\times 5)$$

$$\underbrace{\underbrace{8 + 8}_{16} + \underbrace{14 + 14}_{28} + \underbrace{18 + 18}_{36} + \underbrace{20 + 20}_{40}}_{120}$$

Otherwise, let students do the calculation in any way they like. Conclude that while there are only eight different ways to arrange seven white beads and three red beads on a necklace, there are 120 ways to arrange them on a pole.

$$\frac{\text{necklace : 8 ways}}{\text{pole : 120 ways}}$$


Name
L5
Show all of the different necklaces with seven white and three red beads. (You will not need to color all of the necklaces here.)

# L6 NECKLACES #2

## Capsule Lesson Summary

Determine how many ways there are to arrange six white beads and four red beads in a necklace. Then determine how many ways there are to arrange them on a pole.

### Materials

**Teacher**
- Blackline L5
- Colored chalk
- Tape
- Red markers

**Student**
- Colored pencils, pens, or crayons
- Paper
- Worksheet L6(a) and (b)

**Advance Preparation:** Use Blackline L5 to make several copies of the necklace picture for use in Exercise 1.

## Description of Lesson

### Exercise 1

Ask the class to recall the story of Theophilus from Lesson L5 *Necklaces #1*.

**T:** ***Do you remember how many distinct ways there are to arrange three red and seven white beads in a necklace?***

**S:** ***Eight.***

Record this information on the board.

**Necklace**

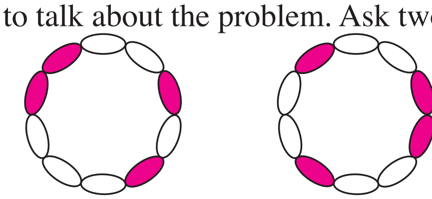
| | |
|---|---|
| 3 red – 7 white | : 8 ways |
| 4 red – 6 white | : ? |

**T:** ***Theophilus has earned a fourth red bead for his necklace and now must decide how to arrange four red beads and six white beads. How many arrangements do you think there are? More or less than eight?***

Students may argue either way. Perhaps someone will notice that there can be at most 56 arrangements; in any of the eight three-red-bead arrangements, seven changes could be made. Let students give estimates and record them on the board. These estimates will be referred to later in the lesson.

**T:** ***Try to solve the problem on your own (or with your partner). Use any method you like to solve it.***

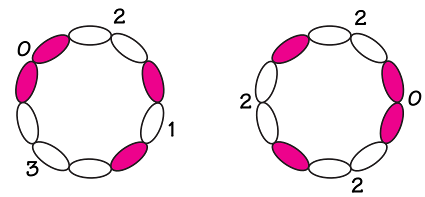
Distribute copies of Worksheet L6(a) and red pencils. Students may want to have paper as well. Allow about ten minutes and then call for the class's attention to talk about the problem. Ask two students each to color one picture of a possible necklace at the board. Suppose these colorings are given:



**T:** ***Are these pictures of the same necklace or not? How can you tell?***

**S:** ***They are different; the one on the left has a group of two white beads, a single white bead, and a group of three white beads. The one on the right has three groups of two white beads.***

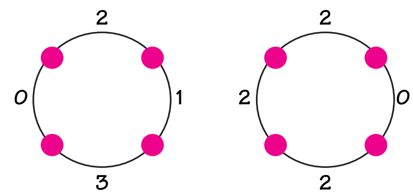
Record the number of white beads in each group, as in this illustration. Emphasize that 0 is recorded when two red beads are next to one another.




**T:** ***What is the sum of the numbers in each case?***

**S:** ***6, because there are six white beads.***

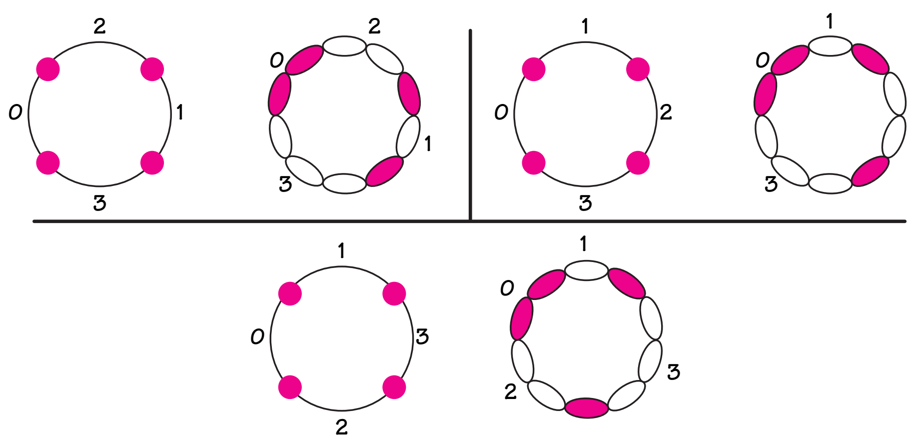
Represent each combination of numbers in a circle code as shown here.



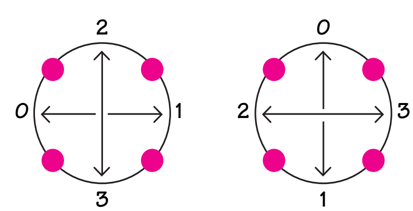

**T** (holding a copy of Worksheet L6(b)**:** ***On this worksheet, list in coded form any necklace arrangement that you found, and then try to use the code to find other arrangements.***

Let students work individually or with a partner. You may like to comment occasionally on the number of arrangements you have seen on different students' papers. After about ten minutes, discuss the various arrangements collectively.

Point to each of the two circle codes recorded earlier in the lesson and ask the class if there could be a different arrangement of the beads with the same numbers in its circle code. For example, three 2s and a 0 can only be used to describe one necklace. Whereas the numbers 0, 1, 2, and 3 can be used to describe three different necklaces, as shown below.

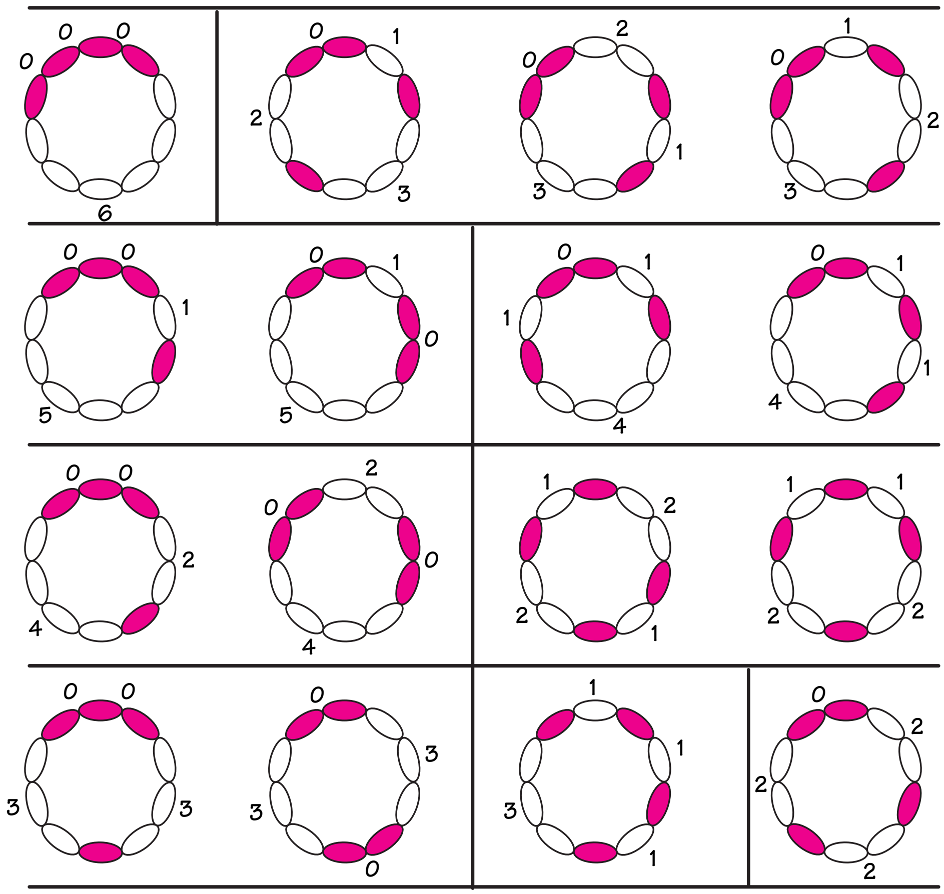


**Note:** One way to check to see if two circle codes describe the same necklace is to see if the same numbers are across from one another in both.




Continue writing circle codes for various arrangements. After several colorings have been made, most students will begin to recognize flips and rotations by simply looking at two circle codes and they no longer will need the colorings. Your students should find 16 arrangements of the beads.

The next chart is for your information; it shows colorings for the 16 arrangements, and it groups together those having the same four numbers in their circle codes. You should have the circle codes recorded on the board, but it is unlikely all the colorings will have been made.



Record the number of arrangements of four red and six white beads, and compare this number (16) with the estimates recorded earlier in the lesson.

**Necklace**

| | |
|---|---|
| 3 red – 7 white | 8 ways |
| 4 red – 6 white | 16 ways |

## Exercise 2

**T:** ***Do you remember what they do with their beads in the village where Theophilus's friend lives?***

**S:** ***They put the beads on poles in front of young people's houses.***

**T:** ***Do you remember how many ways there are to arrange three red and seven white beads on a pole?***

**S:** ***120.***

Make a chart similar to the one in Exercise 1.

**Pole**

| | |
|---|---|
| 3 red – 7 white | 120 ways |
| 4 red – 6 white | ? |

**T:** ***Now we want to find the number of ways there are to arrange four red and six white beads on a pole. How can we find that number?***

**S:** ***We could count them like we did with three red and seven white beads.***

It is true that the ways could be counted systematically similar to that which was done in Lesson L5, although the extra red bead would make the analysis more complex. Let students discuss ways, but solve the problem by making use of the "three-red-beads/seven-white-beads" solution as described here.

Draw ten beads on a pole and invite a student to color any three of them red.

**T:** ***How many ways are there to arrange three red beads and seven white beads on a pole?***

**S:** ***120.***

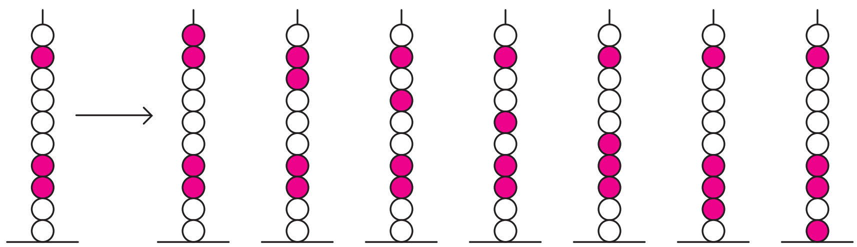
**T:** ***One of those ways is shown on the board. How can we change this picture to have four red beads and six white beads?***

**S:** ***Trade one of the white beads for a red bead. Color one of the white beads red.***

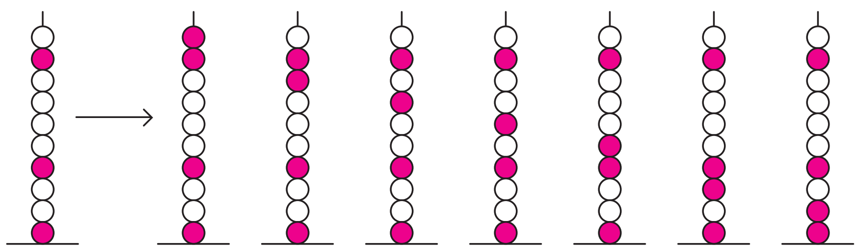
**T:** ***How many different choices do we have?***

**S:** ***Seven; we can trade any of the seven white beads for a red bead.***

With students' help draw the seven possibilities. For example, with the arrangement shown below on the left, the seven pictures that can result if one white bead is traded for a red bead are shown on the right.



Change the position of only one of the three red beads in the students' three-red-beads arrangement and repeat the activity.



**T:** ***Can we generate seven four-red-beads arrangements with any of the other three-red-beads arrangements?*** (Yes) ***If we do so with each of the 120 possible arrangements, how many four-red-beads arrangements will we get?***

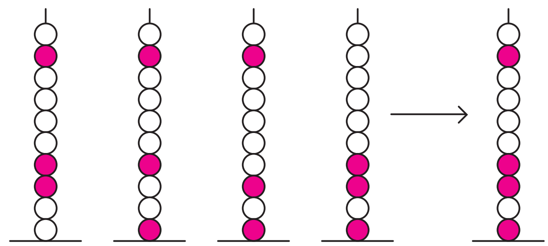
**S:** ***840; 7*** x ***120 = 840.***

**T:** ***Will they all be different?*** (No)

By changing the position of only one red bead in the student's three-red-beads arrangement for the second example, there must be duplicates on the board. If the students do not notice the duplicates, point them out yourself.

**T** (pointing to the duplicate four-red-beads arrangements)**:** ***We can get this arrangement from either of these three-red-beads arrangements. Could we get it by trading a white bead for a red bead in any other three-red-beads arrangement?***

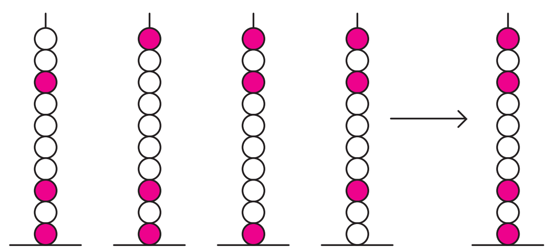
Allow a few minutes for students to consider the situation. Any four-red-beads arrangement can result from four different three-red-beads arrangements. When your students find the other two three-red-beads arrangements, display them on the board. For the example used in this description, the picture would look like the one below.



Invite a student to draw a four-red-beads arrangement on the board, different than the one just considered.

**T:** ***What three-red-beads arrangements could this have come from?***

Let students draw their suggestions on the board and indicate which white bead would be traded for a red bead to make the given four-red-beads arrangement. For example, in the picture below, the four-red-beads arrangement on the right could have come from any of the three-red-beads arrangements on the left.



**T:** ***Do you think any four-red-beads arrangement could have come from four different three-red-beads arrangements?***

**S:** ***Yes, to find them just trade one of the four red beads for a white bead.***

**T:** ***So we start with 120 different three-red-beads arrangements and generate 840 four-red-beads arrangements by adding a red bead to each in seven ways. But in doing so, we not only get every possible four-red-beads arrangement, we get four copies of each of them. If we keep only one out of each set of four, how many different four-red-beads arrangements will there be?***

**S:** ***210;*** $\frac{1}{4} \times 840 = 210$.

**S:** $840 \div 4 = 210$.

Record the number of four-red-beads arrangements on the board.

**T:** ***If we increase the number of red beads, do you think there would continue to be more and more different arrangements?***

Pole

| |
|---|
| 3 red – 7 white : 120 ways |
| 4 red – 6 white : 210 ways |

Let the students comment freely. If they think the number would continue to increase, ask them about the case of ten red beads and no white beads (one possible arrangement). You might wish to give students this chart to fill in the number of ways as a class project. The chart is symmetrical. For example, if you take all of the three-red-beads/seven-white-beads arrangements and reverse the colors, you get all of the seven-red-beads/three-white-beads arrangements.

**Pole**

| | |
|---|---|
| 0 red – 10 white : 1 way | 10 red – 0 white : 1 way |
| 1 red – 9 white : 10 ways | 9 red – 1 white : 10 ways |
| 2 red – 8 white : 45 ways | 8 red – 2 white : 45 ways |
| 3 red – 7 white : 120 ways | 7 red – 3 white : 120 ways |
| 4 red – 6 white : 210 ways | 6 red – 4 white : 210 ways |
| 5 red – 5 white : 252 ways | |

## Extension Activity

Give students other combinations of ten beads (red and white) to find the number of ways to make necklaces. If students look at all the combinations, they should notice symmetry. The chart below is a complete listing.

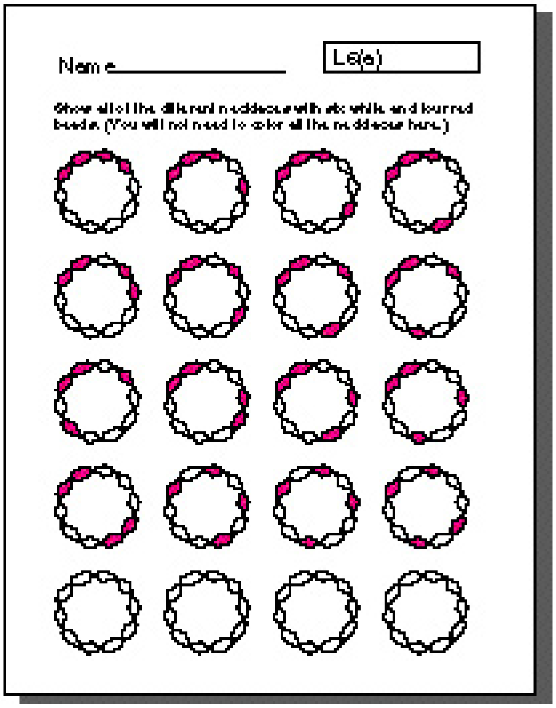
**Necklace**

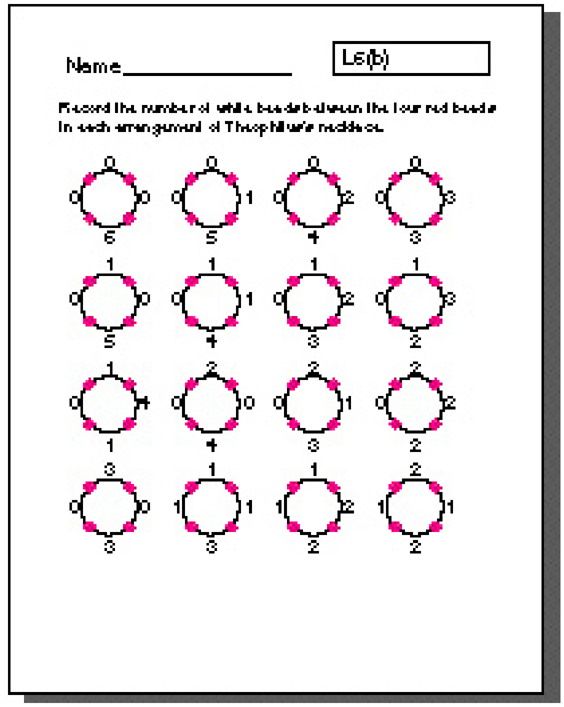
| | |
|---|---|
| 0 red – 10 white : 1 way | 10 red – 0 white : 1 way |
| 1 red – 9 white : 1 way | 9 red – 1 white : 1 way |
| 2 red – 8 white : 5 ways | 8 red – 2 white : 5 ways |
| 3 red – 7 white : 8 ways | 7 red – 3 white : 8 ways |
| 4 red – 6 white : 16 ways | 6 red – 4 white : 16 ways |
| 5 red – 5 white : 7 ways | |

## Home Activity

Students may like to explain the Necklace and the Pole problems to family members, asking which they predict would have more possible arrangements. A simpler case for students to tell family members about might be with five beads—two red and three white.

| **Necklace** | **Pole** |
|---|---|
| 2 red-3 white : 2 ways | 2 red-3 white : 10 ways |


Name
L6(a)
Show all of the different necklaces with six white and four red beads. (You will not need to color all the necklaces here.)


Name
L6(b)
Record the number of white beads between the four red beads in each arrangement of Theophilus's necklace.

## Capsule Lesson Summary

Play *The Divisors Game* with either two or three strings. Play a round of *The Divisors Game* in which the class analyzes the placement of each number in the picture. In a game situation, use clues involving the placement of a certain numbers to determine the labels of two strings.

### Materials

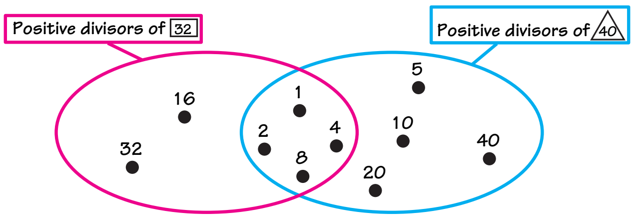
**Teacher**
- Colored chalk

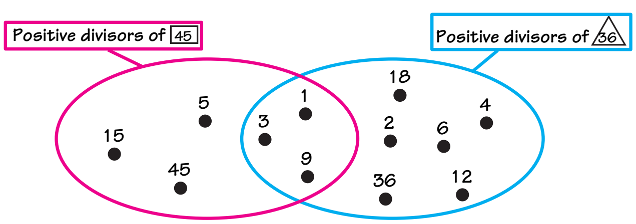
**Student**
- Paper
- Colored pencils, pens, or crayons

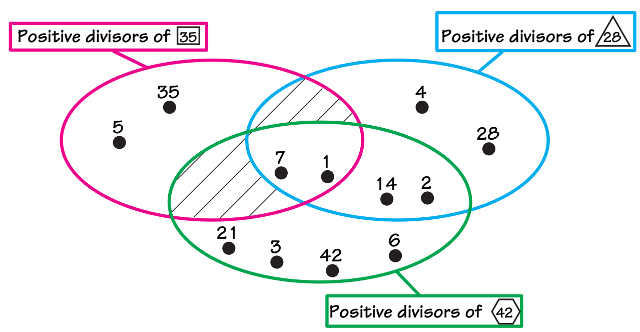
## Description of Lesson

### Exercise 1: The Divisors Game

Divide the class into teams and play *The Divisors Game* (described in Lesson L4) once or twice. The next illustration provides partial crib sheets for three possible games; choose from these or make up similar ones. Note that only the numbers inside the strings are shown in the crib sheets; assume that any whole number not shown in the pictures is outside the strings. The hatching is included for your reference and should not be given as a clue.

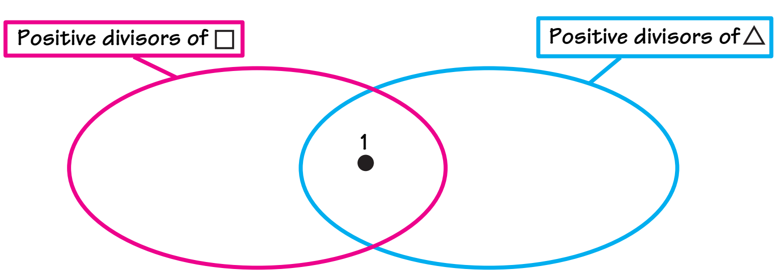



### Exercise 2

Prepare the board for another round of *The Divisors Game*, and list the whole numbers from 1 to 50 next to the string picture.




| | | | | |
|---|---|---|---|---|
| 1 | 11 | 21 | 31 | 41 |
| 2 | 12 | 22 | 32 | 42 |
| 3 | 13 | 23 | 33 | 43 |
| 4 | 14 | 24 | 34 | 44 |
| 5 | 15 | 25 | 35 | 45 |
| 6 | 16 | 26 | 36 | 46 |
| 7 | 17 | 27 | 37 | 47 |
| 8 | 18 | 28 | 38 | 48 |
| 9 | 19 | 29 | 39 | 49 |
| 10 | 20 | 30 | 40 | 50 |

# L7

Play a cooperative game where the class tries to identify the strings as quickly as possible. When a number is located in the picture, collectively discuss which numbers cannot be in □, and which numbers cannot be in △. If a number cannot be in □, cross the numeral in the list with a red slash in one direction; if a number cannot be in △, cross the numeral in the opposite direction in blue. Three types of analysis involved are explained in the examples illustrated below.

**Note:** Here the unknown numbers are 48 for □ and 39 for △.

If 2 is located inside the red string but outside the blue string, then the number in □ must be a multiple of 2. Odd numbers are not multiples of 2 and so can be slashed in red. The number in △ cannot be a multiple of 2, so the even numbers can be slashed in blue.

| | | | | |
|---|---|---|---|---|
| ~~1~~ | ~~11~~ | ~~21~~ | ~~31~~ | ~~41~~ |
| ~~2~~ | ~~12~~ | ~~22~~ | ~~32~~ | ~~42~~ |
| ~~3~~ | ~~13~~ | ~~23~~ | ~~33~~ | ~~43~~ |
| ~~4~~ | ~~14~~ | ~~24~~ | ~~34~~ | ~~44~~ |
| ~~5~~ | ~~15~~ | ~~25~~ | ~~35~~ | ~~45~~ |
| ~~6~~ | ~~16~~ | ~~26~~ | ~~36~~ | ~~46~~ |
| ~~7~~ | ~~17~~ | ~~27~~ | ~~37~~ | ~~47~~ |
| ~~8~~ | ~~18~~ | ~~28~~ | ~~38~~ | ~~48~~ |
| ~~9~~ | ~~19~~ | ~~29~~ | ~~39~~ | ~~49~~ |
| ~~10~~ | ~~20~~ | ~~30~~ | ~~40~~ | ~~50~~ |

If 3 is located inside both the red and blue strings, then the number in □ and the number in △ must both be multiples of 3. Any number which is not a multiple of 3 can be slashed in red and blue.

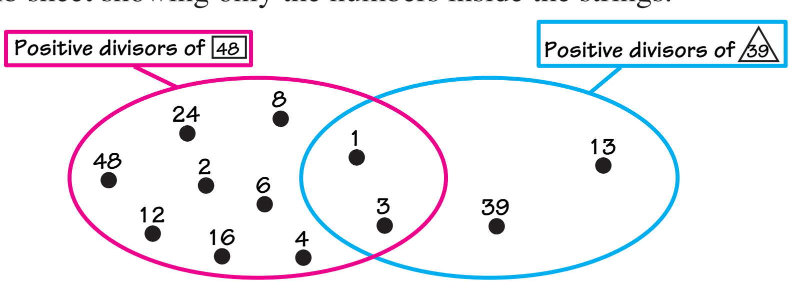
| | | | | |
|---|---|---|---|---|
| ~~1~~ | ~~11~~ | ~~21~~ | ~~31~~ | ~~41~~ |
| ~~2~~ | 12 | ~~22~~ | ~~32~~ | 42 |
| 3 | ~~13~~ | ~~23~~ | 33 | ~~43~~ |
| ~~4~~ | ~~14~~ | 24 | ~~34~~ | ~~44~~ |
| ~~5~~ | 15 | ~~25~~ | ~~35~~ | 45 |
| 6 | ~~16~~ | ~~26~~ | 36 | ~~46~~ |
| ~~7~~ | ~~17~~ | 27 | ~~37~~ | ~~47~~ |
| ~~8~~ | 18 | ~~28~~ | ~~38~~ | 48 |
| 9 | ~~19~~ | ~~29~~ | 39 | ~~49~~ |
| ~~10~~ | ~~20~~ | 30 | ~~40~~ | ~~50~~ |

If 5 is located outside both strings, then the number in □ and the number in △ cannot be multiples of 5. Any number which is a multiple of 5 can be slashed in red and in blue.

| | | | | |
|---|---|---|---|---|
| 1 | 11 | 21 | 31 | 41 |
| 2 | 12 | 22 | 32 | 42 |
| 3 | 13 | 23 | 33 | 43 |
| 4 | 14 | 24 | 34 | 44 |
| ~~5~~ | ~~15~~ | ~~25~~ | ~~35~~ | ~~45~~ |
| 6 | 16 | 26 | 36 | 46 |
| 7 | 17 | 27 | 37 | 47 |
| 8 | 18 | 28 | 38 | 48 |
| 9 | 19 | 29 | 39 | 49 |
| ~~10~~ | ~~20~~ | ~~30~~ | ~~40~~ | ~~50~~ |

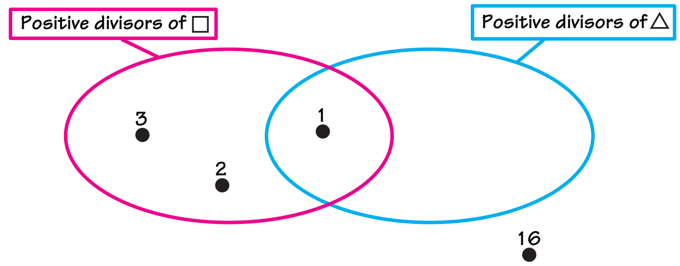
When a student uses a turn to identify the label for one of the strings, ask the student to convince the class, using the numbers in the picture. Continue until both strings are determined and correctly identified.

Here is a partial crib sheet showing only the numbers inside the strings.



## Exercise 3

Erase the board except leave the strings for another round of *The Divisors Game*. Put the numbers 1, 2, 3, and 16 in the picture, as shown here.

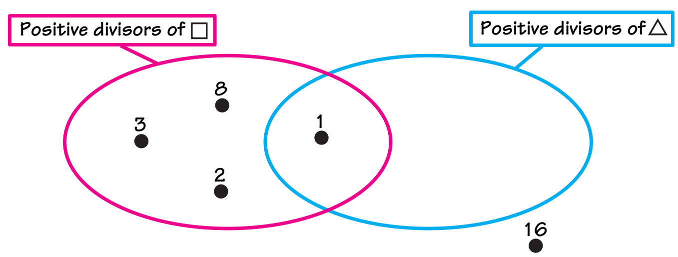



**T:** ***Suppose we are playing The Divisors Game and so far these numbers have been located in the picture. Also, suppose I stop the game and tell you that the numbers 8 and 11 do not belong outside both strings. Where do they belong? Can you determine what the strings are for? Remember that in The Divisors Game the numbers that can be in the square and in the triangle are restricted to whole numbers from 1 to 50.***

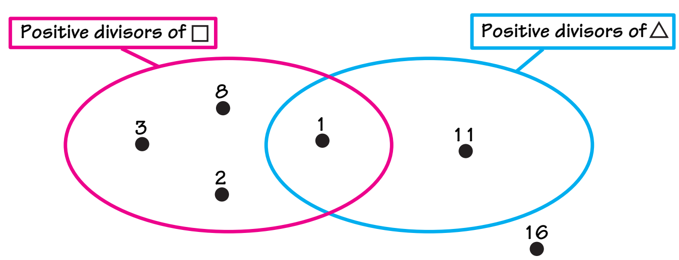
Instruct students to copy the picture on their papers, to locate 8 and 11, and to determine the string labels (there is sufficient information to do this). You may like to organize the class into cooperative groups for this exercise.

After a few minutes, invite students to draw and label dots for 8 and 11 in the picture on the board.

**S:** ***2 must be in the same string(s) that 8 is in. Since we know where 2 belongs, we also know where 8 belongs.***




**S:** ***11 cannot be in the red string; otherwise that string would be for the positive divisors of a number greater than 50. If 11, 2, and 3 were in the same string, that string would be for the positive divisors of a multiple of 66 because 66 is the least common multiple of 11, 2, and 3.***




Collectively discuss how to determine the string labels. A sample analysis is given below.

- 8 is in the red string so the number in the square must be a multiple of 8 between 1 and 50, namely:

  8 16 24 32 40 48

- 16 is outside the red string so the number in the square cannot be a multiple of 16. Cross out 16, 32. and 48.

  ~~16~~ 24 ~~32~~ 40 ~~48~~

- 3 is in the red string so the number in the square must be a multiple of 3 between 1 and 50. Cross out 8 and 40.

~~16~~ 24 ~~32~~ ~~40~~ ~~48~~

- 11 is in the blue string so the number in the triangle must be a multiple of 11 between 1 and 50, namely:

11 22 33 44

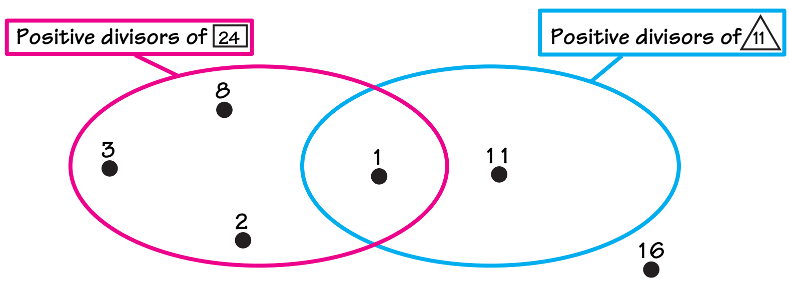
- 2 is not in the blue string so the number in the triangle cannot be a multiple of 2. Cross out 22 and 44.

11 ~~22~~ 33 ~~44~~

- 3 is not in the blue string so the number in the triangle cannot be a multiple of 3. Cross out 33.

11 ~~22~~ ~~33~~ ~~44~~

This picture shows the strings correctly labeled.




Ask students if there are other numbers which belong inside either string. (24, 12, 4, and 6 belong in the red string but outside the blue string.)

## Capsule Lesson Summary

In the context of *The String Game* with numbers, use clues involving the placement of certain numbers (where each does or does not belong) to determine the labels of two strings. Play *The String Game* with numbers.

### Materials

**Teacher**
- Colored chalk
- Numerical String Game kit
- Numerical 3-String Game Poster
- Colored markers

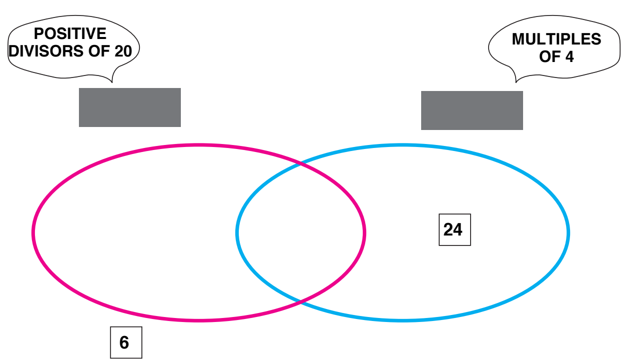
**Student**
- 3-String Game analysis sheets
- Colored pencils, pens, or crayons

**Note:** The Numerical 3-String Game poster and 3-String Game analysis sheet can be used for either a 2-string or a 3-string situation. With only 2-strings, simply cross out (or fold under) the third column.

## Description of Lesson

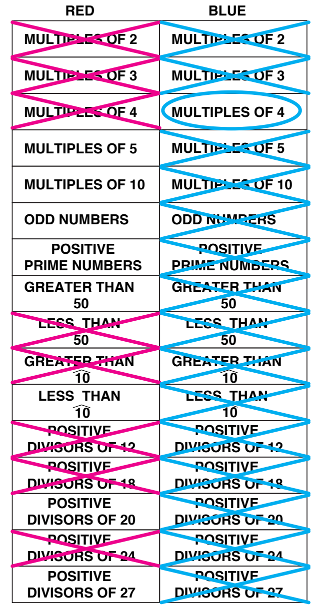
### Exercise 1

Set up your board for *The String Game* with numbers, as shown below. (Bubbles indicate what is on the hidden labels.)

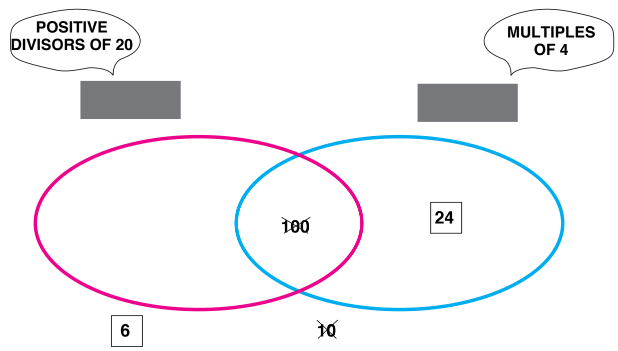



| RED | BLUE |
|---|---|
| MULTIPLES OF 2 | MULTIPLES OF 2 |
| MULTIPLES OF 3 | MULTIPLES OF 3 |
| MULTIPLES OF 4 | MULTIPLES OF 4 |
| MULTIPLES OF 5 | MULTIPLES OF 5 |
| MULTIPLES OF 10 | MULTIPLES OF 10 |
| ODD NUMBERS | ODD NUMBERS |
| POSITIVE PRIME NUMBERS | POSITIVE PRIME NUMBERS |
| GREATER THAN 50 | GREATER THAN 50 |
| LESS THAN 50 | LESS THAN 50 |
| GREATER THAN 10 | GREATER THAN 10 |
| LESS THAN 10 | LESS THAN 10 |
| POSITIVE DIVISORS OF 12 | POSITIVE DIVISORS OF 12 |
| POSITIVE DIVISORS OF 18 | POSITIVE DIVISORS OF 18 |
| POSITIVE DIVISORS OF 20 | POSITIVE DIVISORS OF 20 |
| POSITIVE DIVISORS OF 24 | POSITIVE DIVISORS OF 24 |
| POSITIVE DIVISORS OF 27 | POSITIVE DIVISORS OF 27 |

Let students suggest labels to cross out on the lists. Before asking for a class discussion, you may want to allow a couple minutes for students to do this on their analysis sheets. Continue until students find that the placement of these two pieces leaves eight possibilities for the red string but determines the blue string.




Add these clues in the picture. Point to each new clue in turn as you explain,




**T:** ***100 is not in the middle region and 10 is not outside both strings. Where could 100 be? Where could 10 be?***

**S:** ***100 is a multiple of 4, so 100 is inside the blue string. 100 must be inside the blue string but outside the red string.***

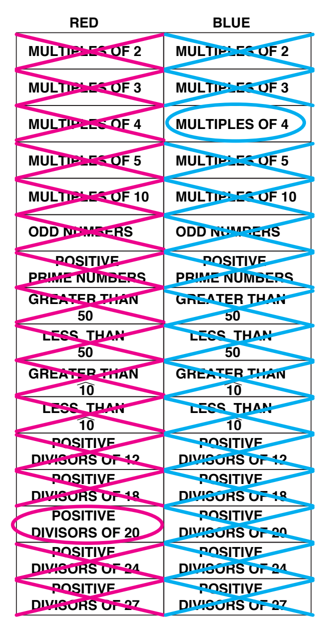
**S:** ***10 is not a multiple of 4, so 10 is not in the blue string. 10 must be inside the red string but outside the blue string.***

Invite students to use these new clues to cross out additional labels on the red list. For example:

**S:** ***Since 10 is inside the red string, the red string cannot be for*** **ODD NUMBERS** ***or*** **POSITIVE PRIME NUMBERS.**

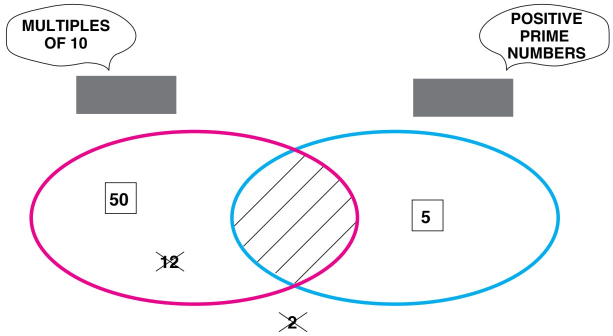
**S:** ***Since 100 is outside the red string, the red string cannot be for*** **MULTIPLES OF 5** ***or*** **MULTIPLES OF 10.**

The class should find that with these two new clues, both strings are determined.



## Exercise 2

Reset the board with the following analysis problem.




| RED | BLUE |
|---|---|
| MULTIPLES OF 2 | MULTIPLES OF 2 |
| MULTIPLES OF 3 | MULTIPLES OF 3 |
| MULTIPLES OF 4 | MULTIPLES OF 4 |
| MULTIPLES OF 5 | MULTIPLES OF 5 |
| MULTIPLES OF 10 | MULTIPLES OF 10 |
| ODD NUMBERS | ODD NUMBERS |
| POSITIVE PRIME NUMBERS | POSITIVE PRIME NUMBERS |
| GREATER THAN 50 | GREATER THAN 50 |
| LESS THAN 50 | LESS THAN 50 |
| GREATER THAN 10 | GREATER THAN 10 |
| LESS THAN 10 | LESS THAN 10 |
| POSITIVE DIVISORS OF 12 | POSITIVE DIVISORS OF 12 |
| POSITIVE DIVISORS OF 18 | POSITIVE DIVISORS OF 18 |
| POSITIVE DIVISORS OF 20 | POSITIVE DIVISORS OF 20 |
| POSITIVE DIVISORS OF 24 | POSITIVE DIVISORS OF 24 |
| POSITIVE DIVISORS OF 27 | POSITIVE DIVISORS OF 27 |

Instruct students to work individually or with partners to cross out labels the strings cannot have and to circle the correct labels for the strings. After about ten minutes, you may want to collectively analyze the situation.

Using the information of where 5 and 50 are in the picture, cross off the labels.

| RED | BLUE |
|---|---|
| MULTIPLES OF 2 | ~~MULTIPLES OF 2~~ |
| ~~MULTIPLES OF 3~~ | ~~MULTIPLES OF 3~~ |
| ~~MULTIPLES OF 4~~ | ~~MULTIPLES OF 4~~ |
| ~~MULTIPLES OF 5~~ | ~~MULTIPLES OF 5~~ |
| MULTIPLES OF 10 | ~~MULTIPLES OF 10~~ |
| ~~ODD NUMBERS~~ | ODD NUMBERS |
| ~~POSITIVE PRIME NUMBERS~~ | POSITIVE PRIME NUMBERS |
| ~~GREATER THAN 50~~ | ~~GREATER THAN 50~~ |
| ~~LESS THAN 50~~ | LESS THAN 50 |
| ~~GREATER THAN 10~~ | ~~GREATER THAN 10~~ |
| ~~LESS THAN 10~~ | ~~LESS THAN 10~~ |
| ~~POSITIVE DIVISORS OF 12~~ | ~~POSITIVE DIVISORS OF 12~~ |
| ~~POSITIVE DIVISORS OF 18~~ | ~~POSITIVE DIVISORS OF 18~~ |
| ~~POSITIVE DIVISORS OF 20~~ | POSITIVE DIVISORS OF 20 |
| ~~POSITIVE DIVISORS OF 24~~ | ~~POSITIVE DIVISORS OF 24~~ |
| ~~POSITIVE DIVISORS OF 27~~ | ~~POSITIVE DIVISORS OF 27~~ |

Using the information that 12 does not belong in the red string, decide that the red string is for **MULTIPLES OF 10.**

| RED | BLUE |
|---|---|
| ~~MULTIPLES OF 2~~ | ~~MULTIPLES OF 2~~ |
| ~~MULTIPLES OF 3~~ | ~~MULTIPLES OF 3~~ |
| ~~MULTIPLES OF 4~~ | ~~MULTIPLES OF 4~~ |
| ~~MULTIPLES OF 5~~ | ~~MULTIPLES OF 5~~ |
| MULTIPLES OF 10 | ~~MULTIPLES OF 10~~ |
| ~~ODD NUMBERS~~ | ODD NUMBERS |
| ~~POSITIVE PRIME NUMBERS~~ | POSITIVE PRIME NUMBERS |
| ~~GREATER THAN 50~~ | ~~GREATER THAN 50~~ |
| ~~LESS THAN 50~~ | LESS THAN 50 |
| ~~GREATER THAN 10~~ | ~~GREATER THAN 10~~ |
| ~~LESS THAN 10~~ | ~~LESS THAN 10~~ |
| ~~POSITIVE DIVISORS OF 12~~ | ~~POSITIVE DIVISORS OF 12~~ |
| ~~POSITIVE DIVISORS OF 18~~ | ~~POSITIVE DIVISORS OF 18~~ |
| ~~POSITIVE DIVISORS OF 20~~ | ~~POSITIVE DIVISORS OF 20~~ |
| ~~POSITIVE DIVISORS OF 24~~ | ~~POSITIVE DIVISORS OF 24~~ |
| ~~POSITIVE DIVISORS OF 27~~ | ~~POSITIVE DIVISORS OF 27~~ |

Using the information that there is no number in the middle region, cross off two more possibilities for the blue string.

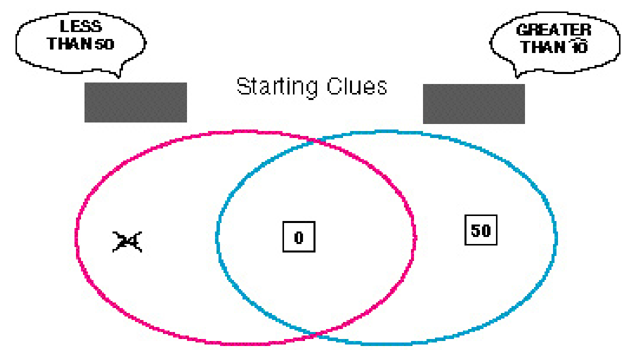
| RED | BLUE |
|---|---|
| ~~MULTIPLES OF 2~~ | ~~MULTIPLES OF 2~~ |
| ~~MULTIPLES OF 3~~ | ~~MULTIPLES OF 3~~ |
| ~~MULTIPLES OF 4~~ | ~~MULTIPLES OF 4~~ |
| ~~MULTIPLES OF 5~~ | ~~MULTIPLES OF 5~~ |
| MULTIPLES OF 10 | ~~MULTIPLES OF 10~~ |
| ~~ODD NUMBERS~~ | ODD NUMBERS |
| ~~POSITIVE PRIME NUMBERS~~ | POSITIVE PRIME NUMBERS |
| ~~GREATER THAN 50~~ | ~~GREATER THAN 50~~ |
| ~~LESS THAN 50~~ | ~~LESS THAN 50~~ |
| ~~GREATER THAN 10~~ | ~~GREATER THAN 10~~ |
| ~~LESS THAN 10~~ | ~~LESS THAN 10~~ |
| ~~POSITIVE DIVISORS OF 12~~ | ~~POSITIVE DIVISORS OF 12~~ |
| ~~POSITIVE DIVISORS OF 18~~ | ~~POSITIVE DIVISORS OF 18~~ |
| ~~POSITIVE DIVISORS OF 20~~ | ~~POSITIVE DIVISORS OF 20~~ |
| ~~POSITIVE DIVISORS OF 24~~ | ~~POSITIVE DIVISORS OF 24~~ |
| ~~POSITIVE DIVISORS OF 27~~ | ~~POSITIVE DIVISORS OF 27~~ |

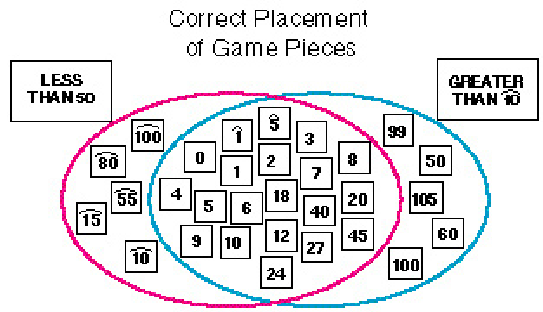
Using the information that 2 does not belong outside the strings, find that 2 must be in the blue string and, therefore, the blue string is for **POSITIVE PRIME NUMBERS**.

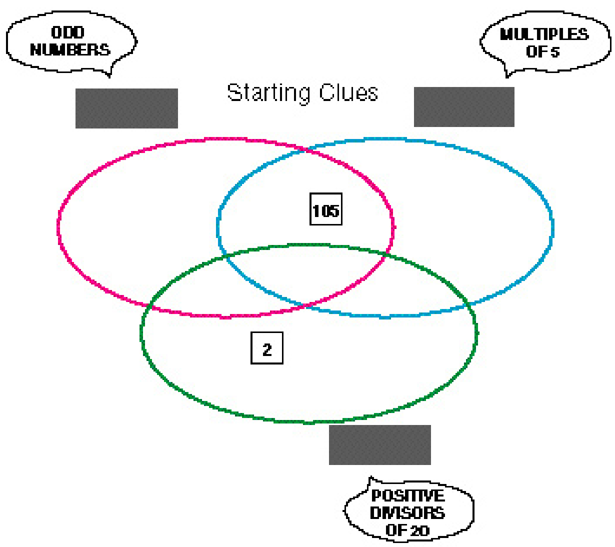
| RED | BLUE |
|---|---|
| ~~MULTIPLES OF 2~~ | ~~MULTIPLES OF 2~~ |
| ~~MULTIPLES OF 3~~ | ~~MULTIPLES OF 3~~ |
| ~~MULTIPLES OF 4~~ | ~~MULTIPLES OF 4~~ |
| ~~MULTIPLES OF 5~~ | ~~MULTIPLES OF 5~~ |
| MULTIPLES OF 10 | ~~MULTIPLES OF 10~~ |
| ~~ODD NUMBERS~~ | ~~ODD NUMBERS~~ |
| ~~POSITIVE PRIME NUMBERS~~ | POSITIVE PRIME NUMBERS |
| ~~GREATER THAN 50~~ | ~~GREATER THAN 50~~ |
| ~~LESS THAN 50~~ | ~~LESS THAN 50~~ |
| ~~GREATER THAN 10~~ | ~~GREATER THAN 10~~ |
| ~~LESS THAN 10~~ | ~~LESS THAN 10~~ |
| ~~POSITIVE DIVISORS OF 12~~ | ~~POSITIVE DIVISORS OF 12~~ |
| ~~POSITIVE DIVISORS OF 18~~ | ~~POSITIVE DIVISORS OF 18~~ |
| ~~POSITIVE DIVISORS OF 20~~ | ~~POSITIVE DIVISORS OF 20~~ |
| ~~POSITIVE DIVISORS OF 24~~ | ~~POSITIVE DIVISORS OF 24~~ |
| ~~POSITIVE DIVISORS OF 27~~ | ~~POSITIVE DIVISORS OF 27~~ |

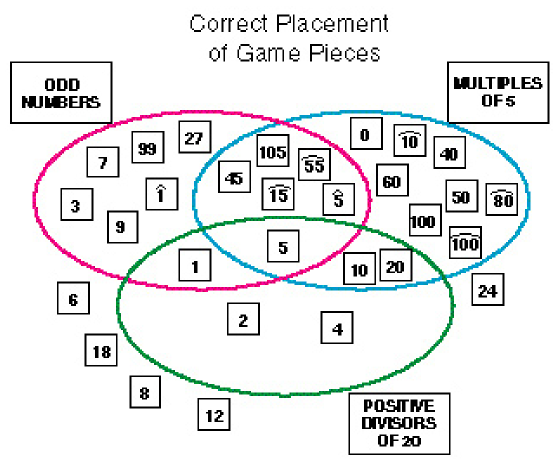
## Exercise 3

Play *The String Game* with numbers in the usual way. If you believe your class needs the challenge, choose a 3-string game. Two possible games, one with 2-strings and the other with 3-strings, are suggested here.



## Capsule Lesson Summary

Review the prime factor relation, and find the prime factor distance between several pairs of numbers. Solve a detective story with clues involving prime factor distance and the calculator relations $\boxed{+}\ \boxed{5}\ \boxed{=}\ \cdots$ and $\boxed{+}\ \boxed{3}\ \boxed{=}\ \cdots$ .

### Materials

**Teacher**
- Colored chalk

**Student**
- Unlined paper
- Colored pencils, pens, or crayons
- Calculator

## Description of Lesson

### Exercise 1

Ask the class to name all of the positive prime numbers less than 50. List them on the board.

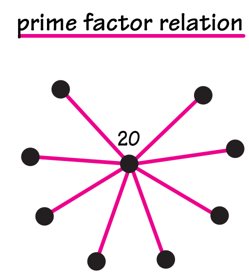
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

Put a key on the board indicating that a red cord is for the prime factor relation.

**T:** ***Red cords are for the prime factor relation. This relation works with whole numbers only. Do you remember the rule for this relation?***

**S:** ***Two whole numbers are joined by a red cord if and only if one number equals a prime number times the other number.***

Draw this cord picture on the board as you ask,




**T:** ***Which numbers can be joined to 20 with a red cord?***

**S:** ***4, because 5 x 4 = 20 and 5 is a prime number.***

**S:** ***220, because 11 x 20 = 220 and 11 is a prime number.***

Label the dots in the picture with numbers suggested by students.
For three of the dots make more specific requests.

**T** (pointing to a dot)**:** ***Is there a number between 100 and 200 that could be here?***

**S:** ***140; 7 x 20 = 140.***

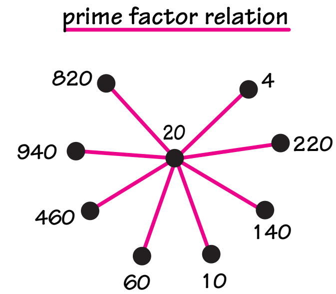
**T:** ***What is the greatest number less than 500 that could be here?***

**S:** ***460; 23 x 20 = 460. 29 is the next prime number after 23, but 29 x 20 = 580 (which is greater than 500).***

**T:** ***What is the least number more than 750 that could be in our picture?***

**S:** ***820; 41 x 20 = 820. 37 is the next smaller prime, but 37 x 20 = 740 (which is less than 750).***

Continue until all of the dots in the picture are labeled, as illustrated here.



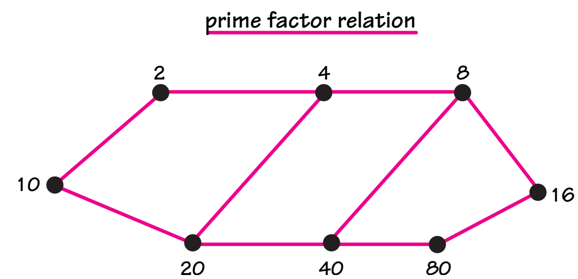

This exercise can be repeated by replacing 20 with another number, such as 15 or 36.

## Exercise 2

Draw dots for 10 and 16 on the board as you pose this problem.

**T:** ***On your paper, use red cords to build a road between 10 and 16. Try to build a shortest road. Build more than one road if you have time.***

Let students work independently or with partners for a few minutes. While others are still working, invite some students to draw their solutions on the board. Attempt to find students who offer differing solutions, not only the shortest roads. For example, your picture might look similar to this one.




**T:** ***What is the shortest length of any of the roads on the board?***

**S:** ***4.***

**T:** ***Did anyone find a shorter road between 10 and 16?***

**S:** ***No.***

**T:** ***We say that the prime factor distance between 10 and 16 is 4. The least number of cords needed to build a road between 10 and 16 is 4.***

Write pfd (10, 16) = 4, and then write several more problems on the board.

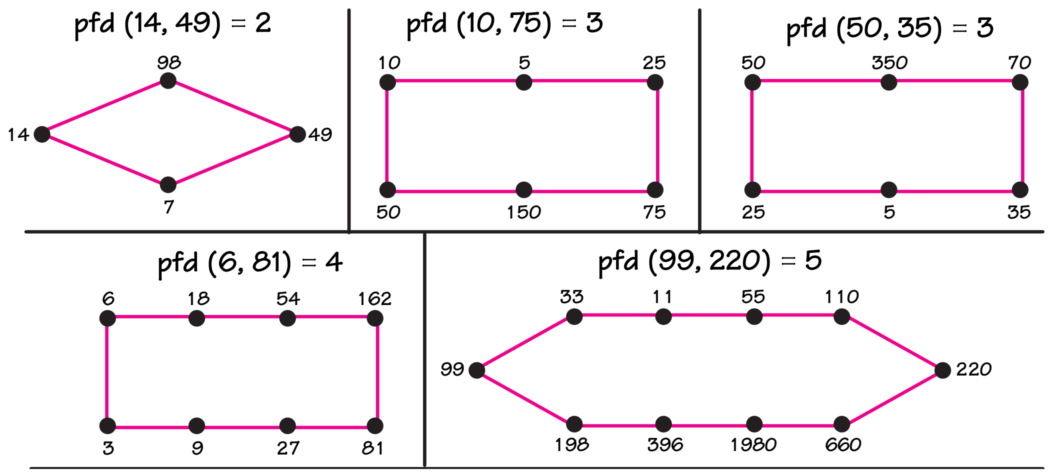
| | |
|---|---|
| pfd (10, 16) = 4 | pfd (14, 49) = ____ |
| pfd (10, 75) = ____ | pfd (50, 35) = ____ |
| pfd (6, 81) = ____ | pfd (99, 220) = ____ |

**T:** ***For each of these problems, find the prime factor distance by drawing a shortest road between the two numbers.***

Let students work independently or in pairs on these problems.

As students find roads, invite them to draw their roads on the board. Encourage finding more than one road for a problem. Do not insist that only shortest roads be drawn, but let students who find shorter roads add them to the picture. After a while, ask for the prime factor distance between each pair of numbers.

Two possible shortest roads for each problem are presented below. Many other roads are possible.




## Exercise 3 ______________

Present this detective story about a secret whole number.

Clue 1

Write this information on the board.

Blick < 100
pfd (12, Blick) = 2

**T:** ***Blick is a whole number less than 100. The prime factor distance between 12 and Blick is 2. Which numbers could Blick be?***

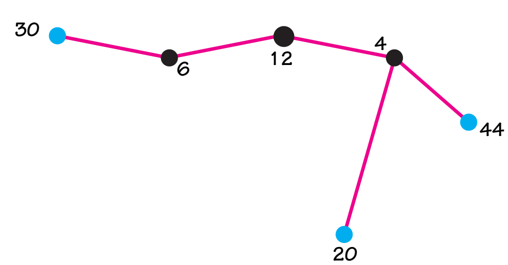
Adapt the following discussion to your students' suggestions.

**S:** ***Blick could be 20.***

**T:** ***Why?***

**S:** ***Draw a red cord from 12 to 4, because 3 x 4 = 12 and 3 is a prime. Then draw a red cord from 4 to 20, because 5 x 4 = 20 and 5 is prime.***

Draw a picture on the board.




**Note:** Blue dots are used to highlight the numbers that Blick could be.

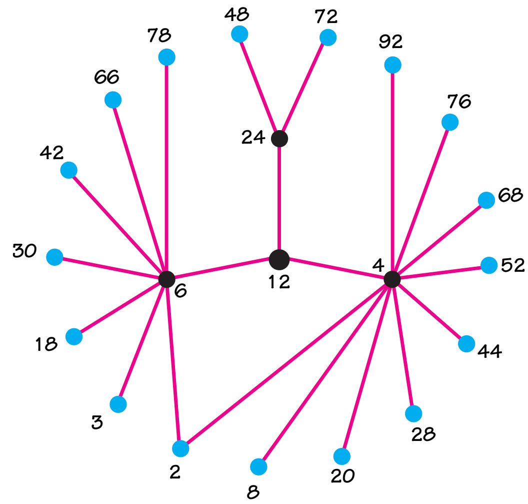
**T:** ***Are there other numbers that Blick could be?***

**S:** ***Blick could be 44, because 11 x 4 = 44 and 11 is prime.***

**S:** ***Blick could be 30, because you can draw a red cord from 12 to 6 (2 x 6 = 12), and then from 6 to 30 (5 x 6 = 30).***

After the class finds three or four numbers that Blick could be, instruct students to copy the picture from the board and to find all of the whole numbers less than 100 at a prime factor distance of 2 from 12. Let students work independently or with partners for about five to ten minutes.

Invite students to complete the picture on the board. There are many ways to draw the paths to the 17 numbers that Blick could be. A completed picture is shown here.

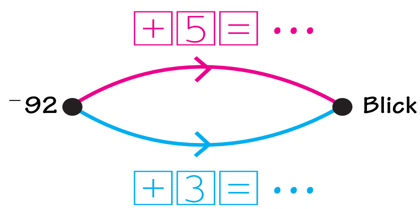



**T:** ***Which numbers could Blick be?***

**S:** ***2, 3, 18, 30, 42, 66, 78, 48, 72, 92, 76, 68, 52, 44, 28, 20, or 8.***

Write the second clue on the board, and let the students solve the detective story independently or with partners.

$\boxed{\text{Clue 2}}$




When many students have solved the detective story, lead a brief discussion of the second clue.

**T:** ***What information does this clue give us about Blick?***

**S:** ***The difference between Blick and ⁻92 is a multiple of 5 and a multiple of 3.***

**S:** ***If you start at ⁻92 and press*** $\boxed{+}\ \boxed{5}\ \boxed{=}$ ***… you can add 95 and get 3. Blick could be 3 or 8 or a whole number that has 3 or 8 in the ones place.***

**S:** ***If you start at ⁻92 and press*** $\boxed{+}\ \boxed{3}\ \boxed{=}$ ***…, you can get ⁻2 and then 1 and then 4 and so on. Blick must be a multiple of 3 plus 1.***

**S:** ***The only number in the picture (first clue) that Blick could be is 28.***

The class should conclude that Blick is 28.

# L10 THE TABLE GAME #1

## Capsule Lesson Summary

Introduce a truth table for the relation "is a divisor of." Review the greatest common divisor operation $\sqcap$. Introduce the minimum operation $\downarrow$ and the maximum operation $\uparrow$. Given a list of seven operations, determine which is the operation for a 9-by-9 table.

### Materials

**Teacher**
- 9-by-9 operation table transparency
- Blackline L10

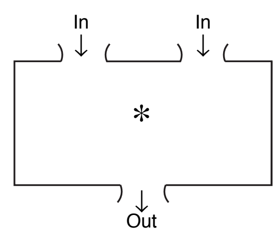
**Student**
- Paper
- Two 9-by-9 operation tables
- Worksheets L10* and **

**Advance Preparation:** Use Blackline L10 to make both a transparency of a 9-by-9 operation table for display and copies for students. You may prefer to prepare the table on the chalkboard or a grid board.

## Description of Lesson

### Exercise 1

This exercise uses an operation rule to introduce a truth table for the relation "is a divisor of." As necessary, review how an operation rule works.




**T:** ***I have a secret rule for *. I'll give you some clues about my rule. Try to guess my rule.***

Write several number sentences on the board as clues.

$2 * 4 = 1$
$4 * 2 = 0$
$3 * 9 = 1$
$3 * 8 = 0$
$6 * 2 = 0$
$2 * 6 = \square$

Then write an open sentence and see if anyone can predict what number goes in the box.

**T:** ***If you think you know my rule, what number is 2 * 6 (read as "two star six")? Write it on a piece of paper.***

Check many answers before letting someone answer aloud, but do not announce the rule yet.

**S:** ***2 * 6 = 1.***

Continue with a few more open sentences before asking for a description of the rule.

$2 * 6 = \boxed{1}$
$5 * 5 = \boxed{1}$
$4 * 7 = \boxed{0}$
$4 * 8 = \boxed{1}$

**T:** ***What is my rule?***

**S:** ***If the first number is a divisor of the second, the answer is 1. If it is not, the answer is 0.***

Rephrase students' explanations of the rule to emphasize true and false statements, as in the following dialogue. Point to the appropriate number sentences as you explain,

**T:** ***"2 is a divisor of 4" is true, so 2 * 4 = 1.***
***"4 is a divisor of 2" is false, so 4 * 2 = 0.***
***"3 is a divisor of 9" is true, so 3 * 9 = 1.***
⋮ ⋮
***"4 is a divisor of 7" is false, so 4 * 7 = 0.***
***"4 is a divisor of 8" is true, so 4 * 8 = 1.***

Display a 9-by-9 operation table, and distribute copies of the table for students.

**T:** ***We are going to make a table for this operation. Since the operation gives a truth value (a statement is true or false), let's call the operation $T_D$ meaning truth table for "is a divisor of."***

Instruct students to put $T_D$ in the upper left corner of their tables.

Make several entries in the table with the class. For example, here entries are made for the results on the board.

| $T_D$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | 1 | | | | | |
| 2 | | | | 1 | | 1 | | | |
| 3 | | | | | | | | 0 | 1 |
| 4 | | 0 | | | | | 0 | 1 | |
| 5 | | | | | 1 | | | | |
| 6 | | 0 | | | | | | | |
| 7 | | | | | | | | | |
| 8 | | | | | | | | | |
| 9 | | | | | | | | | |

Read another entry, such as "1 is a divisor of 4" is true.

Then direct students to work individually or with partners to complete the table.

After many students have completed the table, discuss patterns in the table.

**T:** ***Do you notice any patterns in the table?***

**S:** ***There are 1s along the diagonal (from upper left to lower right).***

Record 1s along that diagonal in the table on display.

**S:** ***1 is a divisor of every number, so there are all 1s in the first row.***

Record 1s in the first row of the table.

**T:** ***Are there any 1s in the table below the diagonal of 1s?***

**S:** ***No, because the first number is greater than the second number in those cases. A number cannot be a divisor of a number less than itself.***

Record 0s in the table below the diagonal of 1s.

**T:** ***Are there any more 1s in the table?***

**S:** ***Yes, a few more.***

Quickly complete the table with the class.

| $T_D$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **2** | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| **3** | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| **4** | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| **5** | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| **6** | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| **7** | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| **8** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| **9** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

**T:** ***In this table we put 1 for true (the first number is a divisor of the second), and we put 0 for false (the first number is not a divisor of the second). We call this a truth table for the relation "is a divisor of."***

## Exercise 2

Erase the board and then write this expression.

$8 \sqcap 12$

**T:** ***Do you remember what this symbol ($\sqcap$) means?***

**S:** ***Greatest common divisor.***

**T:** ***Let's find the greatest common divisor of 8 and 12.***
***What are the positive divisors of 8?*** (1, 2, 4, and 8)
***What are the positive divisors of 12?*** (1, 2, 3, 4, 6, and 12)

List the divisors of 8 and 12 on the board.

| 8 $\sqcap$ | 12 |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 4 | 3 |
| 8 | 4 |
| | 6 |
| | 12 |

**T:** ***What positive divisors do 8 and 12 have in common?*** (1, 2, and 4)
***Which of these common divisors is the greatest?*** (4)
***So the greatest common divisor of 8 and 12 is 4.***

Record the result and pose several more problems. (Answers are in boxes.)

$8 \sqcap 12 = 4$ $\qquad$ $15 \sqcap 45 = \boxed{15}$

$14 \sqcap 49 = \boxed{7}$ $\qquad$ $56 \sqcap 64 = \boxed{8}$

Provide students with another copy of a 9-by-9 operation table and display a cleared table. Instruct students to put $\sqcap$ in the upper left corner of this table, and then begin collectively completing this operation table for $\sqcap$. You may want to stop and let students complete the table individually. As before, look for patterns in the table.

| $\sqcap$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **2** | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 |
| **3** | 1 | 1 | 3 | 1 | 1 | 3 | 1 | 1 | 3 |
| **4** | 1 | 2 | 1 | 4 | 1 | 2 | 1 | 4 | 1 |
| **5** | 1 | 1 | 1 | 1 | 5 | 1 | 1 | 1 | 1 |
| **6** | 1 | 2 | 3 | 2 | 1 | 6 | 1 | 2 | 3 |
| **7** | 1 | 1 | 1 | 1 | 1 | 1 | 7 | 1 | 1 |
| **8** | 1 | 2 | 1 | 4 | 1 | 2 | 1 | 8 | 1 |
| **9** | 1 | 1 | 3 | 1 | 1 | 3 | 1 | 1 | 9 |

## Exercise 3

Erase the board and clear the table.

**T:** ***I have another secret rule for *. Here are some hints.***

$$3 * 5 = 5$$
$$9 * 4 = 9$$
$$4 * 9 = 9$$
$$6 * 6 = 6$$
$$1 * 8 = \boxed{\phantom{0}}$$

**Note:** The rule assigns the greater of the two numbers in a pair.

**T:** ***If you think you know my rule, what is 1 * 8?***

**S:** ***8.***

**T:** ***What is my rule?***

**S:** ***You always take the greater of the two numbers. If the two numbers are the same, you give that number.***

Draw an arrow pointing upward on the board.

$$\uparrow$$

**T:** ***Let's use an arrow pointing upward to indicate this operation, and call the operation* maximum.**

To reinforce the notation, call on students to complete these number sentences. (Answers are in boxes.)

$$2 \uparrow 5 = \boxed{5}$$
$$3 \uparrow 3 = \boxed{3}$$
$$0 \uparrow 1 = \boxed{1}$$

**T:** ***What do you think an arrow pointing down means?***

Draw an arrow pointing down on the board.

$$\downarrow$$

**S:** ***Minimum.***

**S:** ***Take the smaller of the two numbers.***

**T:** ***And if the numbers are the same?***

**S:** ***The result is that number.***

Call on students to complete these number sentences. (Answers are in boxes.)

$$7 \downarrow 3 = \boxed{3}$$
$$6 \downarrow 9 = \boxed{6}$$
$$8 \downarrow 8 = \boxed{8}$$

With class input, observe how to complete operation tables for these two operations.

| ↑ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **2** | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **3** | 3 | 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **4** | 4 | 4 | 4 | 4 | 5 | 6 | 7 | 8 | 9 |
| **5** | 5 | 5 | 5 | 5 | 5 | 6 | 7 | 8 | 9 |
| **6** | 6 | 6 | 6 | 6 | 6 | 6 | 7 | 8 | 9 |
| **7** | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 8 | 9 |
| **8** | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 9 |
| **9** | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

| ↓ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **2** | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| **3** | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| **4** | 1 | 2 | 3 | 4 | 4 | 4 | 4 | 4 | 4 |
| **5** | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 5 | 5 |
| **6** | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 6 | 6 |
| **7** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 7 |
| **8** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 8 |
| **9** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

## Exercise 4

List these operations on the board and display a cleared 9-by-9 table.

$T_D$
$\uparrow$
$\downarrow$
$\sqcap$
+
–
x

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 2 | | | | | | | | | |
| 3 | | | | | | | | | |
| 4 | | | | | | | | | |
| 5 | | | | | | | | | |
| 6 | | | | | | | | | |
| 7 | | | | | | | | | |
| 8 | | | | | | | | | |
| 9 | | | | | | | | | |

**T:** ***We're going to play a kind of Guess My Rule game with this table. I'll choose one of these seven operations*** (point to the list) ***to be *; you will try to figure out which one.***

**Note:** Choose $\downarrow$ as the secret operation, but do not tell the class. Record 1 in the table for 1 * 1.

**T:** ***Your first clue is that 1 * 1 = 1. Are there any operations that we know * cannot be?***

**S:** **** cannot be + or –. 1 + 1 = 2 and 1 – $\widehat{1}$ = 0***

Eliminate + and – from the list of possible operations.

As a second clue, record 9 in the table for 9 * 9.

**S:** **** cannot be*** x ***because 9*** x ***9 = 81.***

**S:** ***The table cannot be a truth table because it has a number other than 0 and 1 in it.***

Eliminate $T_D$, and x from the list of operations. At this point, observe that there are still three possibilities for the table: $\uparrow$, $\downarrow$, or $\sqcap$.

**T:** ***With these three possibilities for *, are there any entries you know for sure we could make in the table?***

**S:** ***A diagonal entry, because for all three possibilities, a number starred with itself is still that number.***

Record the entries along the diagonal from the upper left square to the lower right square.

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | |
| 2 | | 2 | | | | | | | |
| 3 | | | 3 | | | | | | |
| 4 | | | | 4 | | | | | |
| 5 | | | | | 5 | | | | |
| 6 | | | | | | 6 | | | |
| 7 | | | | | | | 7 | | |
| 8 | | | | | | | | 8 | |
| 9 | | | | | | | | | 9 |

**T:** ***Suppose you could ask for any entry in the table to help you discover *. What would you ask for?***

Consider students' comments concerning their choices.

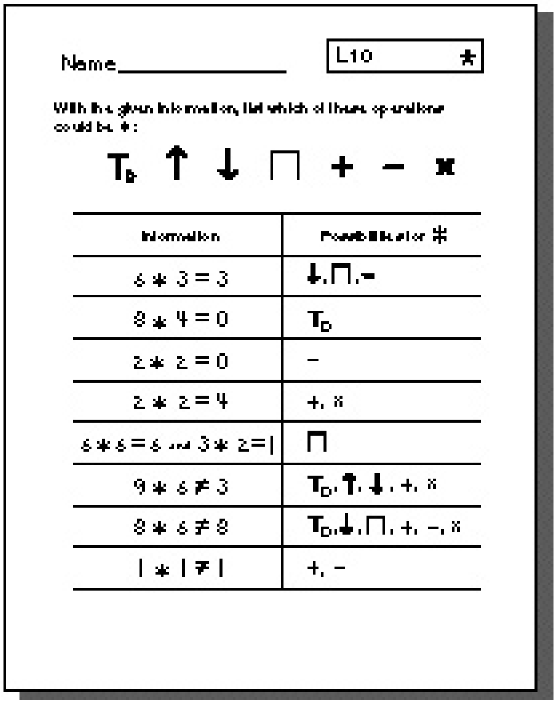
**S:** ***1 * 5. If 1 * 5 = 5, we know * is $\uparrow$. If 1 * 5 = 1, we could eliminate $\uparrow$.***

**S:** ***4 * 6, because 4 $\uparrow$ 6 = 6 and 4 $\downarrow$ 6 = 4, and 4 $\sqcap$ 6 = 2, three different answers.***

Any entry that differentiates the three possibilities would be a good choice. Make an entry as requested by a student, for example, 4 * 6 = 4. The students should identify the operation as $\downarrow$.

Worksheets L10* and ** are available for individual work.

# L10

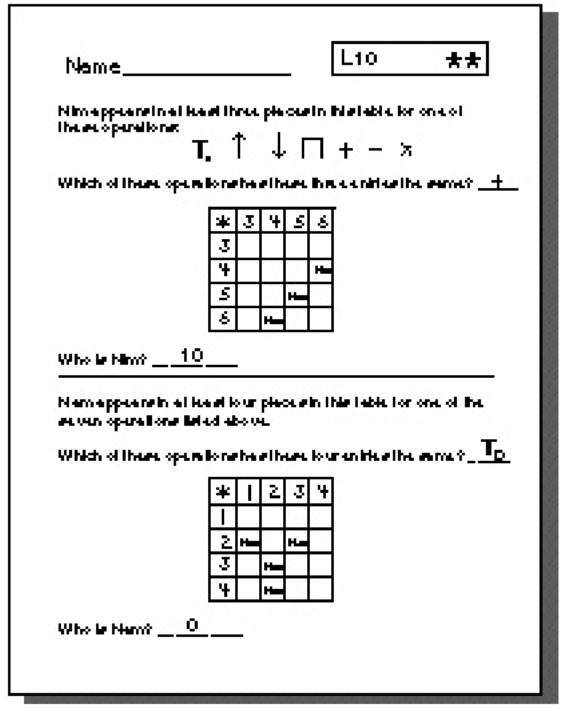


Name______________ L10 ★

With the given information, list which of these operations could be $*$:

$T_D$ ↑ ↓ ⊓ + − ×

| Information | Possibilities for $*$ |
|---|---|
| $6 * 3 = 3$ | ↓, ⊓, − |
| $8 * 4 = 0$ | $T_D$ |
| $2 * 2 = 0$ | − |
| $2 * 2 = 4$ | +, × |
| $6 * 6 = 6$ and $3 * 2 = 1$ | ⊓ |
| $9 * 6 \neq 3$ | $T_D$, ↑, ↓, +, × |
| $8 * 6 \neq 8$ | $T_D$, ↓, ⊓, +, −, × |
| $1 * 1 \neq 1$ | +, − |



Name______________ L10 ★★

Nim appears in at least three places in this table for one of these operations:

$T_D$ ↑ ↓ ⊓ + − ×

Which of these operations has these three entries the same? __+__

| $*$ | 3 | 4 | 5 | 6 |
|---|---|---|---|---|
| 3 | | | | |
| 4 | | | | Nim |
| 5 | | | Nim | |
| 6 | | Nim | | |

Who is Nim? __10__

Nim appears in at least four places in this table for one of the seven operations listed above.

Which of these operations has these four entries the same? __$T_D$__

| $*$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | | | | |
| 2 | Nim | | Nim | |
| 3 | | Nim | | |
| 4 | | Nim | | |

Who is Nim? __0__

# L11 THE TABLE GAME #2

## Capsule Lesson Summary

Introduce truth tables for the relations “is a multiple of,” <, and >. Review the least common multiple operation $\sqcup$. Review addition, subtraction, and multiplication modulo 10 using games with ten number friends. Provided with a list of 14 possible operations, determine the operation for:

- a 2-by-2 table by analyzing its given entries;
- a 9-by-9 table by making trial entries until the operation is evident.

## Materials

**Teacher**
- 9-by-9 operation table transparency
- Blackline L10

**Student**
- Paper
- 9-by-9 operation tables
- Worksheets L11* and **

**Advance Preparation:** Use Blackline L10 to make both a transparency of a 9-by-9 operation table for display and copies for students. You may prefer to prepare the table on the chalkboard or a grid board.

## Description of Lesson

### Exercise 1

Display a 9-by-9 operation table. Remind the class about how to construct a truth table for the relation “is a divisor of” (see Exercise 1 of Lesson L10).

| $T_D$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **2** | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| **3** | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| **4** | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| **5** | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| **6** | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| **7** | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| **8** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| **9** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

Write the symbol $T_M$ on the board.

**T:** ***We can also make a truth table for the relation “is a multiple of.” In the table 1 means true, so when the first number is a multiple of the second, assign 1. 0 means false, so assign 0 when the first number is not a multiple of the second.***

| $T_M$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **2** | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **3** | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| **4** | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| **5** | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| **6** | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| **7** | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| **8** | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| **9** | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |

Invite students to help complete this truth table, look for patterns, and compare it to the truth table for $T_D$.

# L11

Write the symbols $T_<$ and $T_>$ on the board.

**T:** ***We can also make truth tables for the relations < and >. In the $T_<$ table, assign 1 if it is true that the first number is less than the second and assign 0 if it is not. For example, 2 * 8 would be 1 and 8 * 2 would be 0.***

Invite students to help complete these truth tables. Look for patterns and compare the tables.

| $T_<$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **2** | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **3** | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| **4** | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| **5** | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| **6** | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| **7** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| **8** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| **9** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| $T_>$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **2** | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **3** | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **4** | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| **5** | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| **6** | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| **7** | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| **8** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| **9** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |

Erase the board and then write this expression.

$$9 \sqcup 6$$

**T:** ***What does this symbol*** (point to $\sqcup$) ***mean?***

**S:** ***Least common multiple.***

**T:** ***When we ask for the least common multiple of two numbers, it is understood that we want the least common positive multiple of the numbers. What are some positive multiples of 9?***

**S:** ***9, 18, 27, 36, and so on.***

**T:** ***What are some positive multiples of 6?***

**S:** ***6, 12, 18, 24, 30, and so on.***

List some of the multiples of 9 and of 6 on the board.

$9 \sqcup 6$

| 9 | 6 |
|---|---|
| 18 | 12 |
| 27 | 18 |
| 36 | 24 |
| 45 | 30 |
| ⋮ | 36 |
| | ⋮ |

**T:** ***What is the least common multiple of 9 and 6?***

**S:** ***18.***

Record the answer and pose several more problems. (Answers are in boxes.)

$\sqcup 6 = 18$ $\qquad$ $16 \sqcup 8 = \boxed{16}$

$\sqcup 4 = \boxed{20}$ $\qquad$ $12 \sqcup 15 = \boxed{60}$

Direct students to complete a 9-by-9 operation table for $\sqcup$.

| ⊔ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **2** | 2 | 2 | 6 | 4 | 10 | 6 | 14 | 8 | 18 |
| **3** | 3 | 6 | 3 | 12 | 15 | 6 | 21 | 24 | 9 |
| **4** | 4 | 4 | 12 | 4 | 20 | 12 | 28 | 8 | 36 |
| **5** | 5 | 10 | 15 | 20 | 5 | 30 | 35 | 40 | 45 |
| **6** | 6 | 6 | 6 | 12 | 30 | 6 | 42 | 24 | 18 |
| **7** | 7 | 14 | 21 | 28 | 35 | 42 | 7 | 56 | 63 |
| **8** | 8 | 8 | 24 | 8 | 40 | 24 | 56 | 8 | 72 |
| **9** | 9 | 18 | 9 | 36 | 45 | 18 | 63 | 72 | 9 |

## Exercise 2

If appropriate, recall with the class the storybook *Dancing Friends*.† In the story, the boy invites the ten whole numbers 0 through 9 over to his house to play games.

Write these numerals on the board.

0 1 2 3 4 5 6 7 8 9

**T:** ***The ten number friends are upset at first because they don't know any games that just the ten of them can play. They can't do multiplication. Do you know why?***

**S:** ***8 x 0 = 72, and 72 is not one of the ten friends.***

A student may resist and say that 7 and 2 are present and therefore that 72 is present. If so, point out that 7, 2, and 72 are three different numbers.

**T:** ***Are there other operations they cannot do?***

The class should rule out addition, subtraction, and division. Consider any other operations or relations that your students suggest, and decide with the class whether or not the ten numbers could play them. For example, a student could suggest that the numbers play ⊓ (greatest common divisor) in which the greatest common divisor of any two of these numbers is again one of the ten numbers. Note that any whole number 1 to 9 is a divisor of 0.

**T:** ***Well, 0 invents several games that the ten friends can play. First 0 shows the ten friends how to do addition with ten friends, or addition modulo 10 ($+_{10}$).***

Write these number sentences on the board.

$2 \times_{10} 4 = 6$
$4 \times_{10} 7 = 1$
$6 \times_{10} 6 = 2$
$5 \times_{10} 3 = 8$
$7 \times_{10} 6 =$

**T:** ***Do you understand 0's game? What number is 7 $+_{10}$ 6? Who can explain the game?***

**S:** ***7 $+_{10}$ 6 = 3.***

**S:** ***Add in the usual way and keep only the ones digit.***

**S:** ***Add the two numbers. If the sum is greater than 10, subtract 10.***

Pose several problems for students to practice addition modulo 10. (Answers are in boxes.)

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†It is not necessary that students have read this storybook prior to this lesson.

$8 +_{10} 4 = \boxed{2}$ $\quad$ $\boxed{7} +_{10} 8 = 5$ $\quad$ $\boxed{4} +_{10} \boxed{4} = 8$

$5 +_{10} 9 = \boxed{4}$ $\quad$ $9 +_{10} \boxed{6} = 5$ $\quad$ or

$\boxed{9} +_{10} \boxed{9} = 8$

Write this problem on the board.

$3 \times_{10} 4 =$

**T:** ***The ten numbers also play a multiplication game. What do you think $3 \times_{10} 4$ (read as "three times four modulo 10") is?***

**S:** ***2, because $3 \times 4 = 12$ and we get 2 if we keep only the ones digit in 12.***

Pose several more problems for students to practice multiplication modulo 10. (Answers are in boxes.)

$4 \times_{10} 5 = \boxed{0}$ $\quad$ $\boxed{6} \times_{10} 9 = 4$ $\quad$ $\boxed{3} \times_{10} \boxed{3} = 9$

$7 \times_{10} 3 = \boxed{1}$ $\quad$ $8 \times_{10} \boxed{4 \text{ or } 9} = 2$ $\quad$ $\boxed{7} \times_{10} \boxed{7} = 9$

$\boxed{5} \times_{10} \boxed{5} = 5$

**Note:** Explanations most likely will be in terms of keeping the ones digit. If a student suggests subtracting 10 from the result of the usual multiplication, observe that they may need to subtract 10 more than once to get a number from 0 to 9.

Write this problem on the board.

$7 -_{10} 3 =$

**T:** ***The ten numbers might also play a subtraction game. What is $7 -_{10} 3$?***

**S:** ***4, because $7 - 3 = 4$ and 4 is one of the ten numbers.***

**T:** ***When would there be a problem playing the subtraction game?***

**S:** ***When the first number is less than the second number, such as $3 -_{10} 7$.***

$3 -_{10} 7 =$

**T:** ***How would the ten friends find $3 -_{10} 7$ ?***

**S:** ***$3 - 7 = \hat{4}$, but $\hat{4}$ is not one of the ten friends so maybe they would use 4.***

**S:** ***$3 - 7 = \hat{4}$. Since $\hat{4}$ is not one of the ten friends maybe they add 10 (rather than subtract 10) to get 6.***

Let students express their ideas on how to define subtraction modulo 10. Eventually agree that they should add 10 when the result of usual subtraction is negative. The reason for this is to make addition and subtraction modulo 10 opposites. That is,

$$\text{if } 3 -_{10} 7 = \boxed{6} \text{ then } \boxed{6} +_{10} 7 = 3$$

$$\text{just as } 3 - 7 = \boxed{\hat{4}} \text{ and } \boxed{\hat{4}} + 7 = 3$$

Distribute copies of 9-by-9 operation tables, and direct students to complete tables for $+_{10}$, $-_{10}$, and $\times_{10}$. You may like to organize this work in small groups.

| $+_{10}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 |
| **2** | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 |
| **3** | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 |
| **4** | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 |
| **5** | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 |
| **6** | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 |
| **7** | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| **8** | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| **9** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

| $-_{10}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 0 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
| **2** | 1 | 0 | 9 | 8 | 7 | 6 | 5 | 4 | 3 |
| **3** | 2 | 1 | 0 | 9 | 8 | 7 | 6 | 5 | 4 |
| **4** | 3 | 2 | 1 | 0 | 9 | 8 | 7 | 6 | 5 |
| **5** | 4 | 3 | 2 | 1 | 0 | 9 | 8 | 7 | 6 |
| **6** | 5 | 4 | 3 | 2 | 1 | 0 | 9 | 8 | 7 |
| **7** | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 9 | 8 |
| **8** | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 9 |
| **9** | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

| $\times_{10}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **2** | 2 | 4 | 6 | 8 | 0 | 2 | 4 | 6 | 8 |
| **3** | 3 | 6 | 9 | 2 | 5 | 8 | 1 | 4 | 7 |
| **4** | 4 | 8 | 2 | 6 | 0 | 4 | 8 | 2 | 6 |
| **5** | 5 | 0 | 5 | 0 | 5 | 0 | 5 | 0 | 5 |
| **6** | 6 | 2 | 8 | 4 | 0 | 6 | 2 | 8 | 4 |
| **7** | 7 | 4 | 1 | 8 | 5 | 2 | 9 | 6 | 3 |
| **8** | 8 | 6 | 4 | 2 | 0 | 8 | 6 | 4 | 2 |
| **9** | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

## Exercise 3

Draw this 2-by-2 table on the board and list these operations.

| * | 3 | 4 |
|---|---|---|
| **6** | 1 | 0 |
| **7** | 0 | 0 |

$T_D$ $\quad T_M$ $\quad T_<$ $\quad T_>$

$\downarrow$ $\quad\uparrow$ $\quad\sqcap$ $\quad\sqcup$

$+_{10}$ $\quad -_{10}$ $\quad \times_{10}$

**T:** ***This table is for one of these operations. Which of these operations cannot be *? Why?***

**S:** **** cannot be $\sqcap$ because 0 is not a divisor of any number.***

**S:** ***It might be a truth table because it has only 0s and 1s in it.***

**S:** **** cannot be $T_<$ or $T_>$. $6 > 3$, $6 > 4$, $7 > 3$, $7 > 4$, so all of the entries in the table would be 1.***

**S:** **** cannot be $\uparrow$ or $\downarrow$ because if it were there would be no 0 entry.***

**S:** **** cannot be $\times_{10}$ because $6 \times 3 = 18$, so $6 \times_{10} 3 = 8$.***

Cross out operations that * cannot be. Continue until the class finds that * is $T_M$.

## Exercise 4

Draw a 9-by-9 table on the board, and next to it list the same operations as in Exercise 3.

**T:** ***I am thinking of one of these operations*** (point to the list)***. Try to figure out which one. You choose numbers and tell me where to put them in the table. I'll tell you whether or not the entries are correct.***

Choose a secret operation, for example $-_{10}$, but do not tell the class.

Invite students to suggest numbers and tell you where to put them in the table. Record an entry in the table and announce when it is correct; put X through an incorrect entry. After each entry discuss which operations can be crossed off the list. Continue until the operation is determined.

The following sequence shows how the operation could be determined after three trial entries.

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | $\cancel{1}$ | | | | | |
| 2 | | | | | | | | | |
| 3 | | | | | | | | | |
| 4 | | | | | | | | | |
| 5 | | | | | | | | | |
| 6 | | | | | | | | | |
| 7 | | | | | | | | | |
| 8 | | | | | | | | | |
| 9 | | | | | | | | | |

$\cancel{T_D}$ $T_M$ $\cancel{T_<}$ $T_>$

$\cancel{\downarrow}$ $\uparrow$ $\cancel{\sqcap}$ $\sqcup$

$+_{10}$ $-_{10}$ $x_{10}$

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | $\cancel{1}$ | | | | | |
| 2 | | | | | | | | | |
| 3 | | | 0 | | | | | | |
| 4 | | | | | | | | | |
| 5 | | | | | | | | | |
| 6 | | | | | | | | | |
| 7 | | | | | | | | | |
| 8 | | | | | | | | | |
| 9 | | | | | | | | | |

$\cancel{T_D}$ $\cancel{T_M}$ $\cancel{T_<}$ $T_>$

$\cancel{\downarrow}$ $\cancel{\uparrow}$ $\cancel{\sqcap}$ $\cancel{\sqcup}$

$\cancel{+_{10}}$ $-_{10}$ $\cancel{x_{10}}$

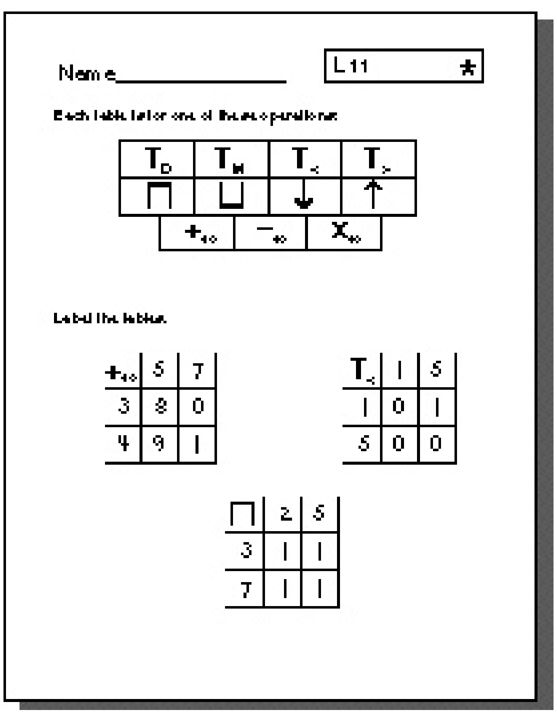
| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | $\cancel{1}$ | | | | | |
| 2 | | | | | | | | | |
| 3 | | | 0 | | | | | | |
| 4 | | | | | | | | | |
| 5 | | | | | | | | | |
| 6 | | | | | $\cancel{0}$ | | | | |
| 7 | | | | | | | | | |
| 8 | | | | | | | | | |
| 9 | | | | | | | | | |

$\cancel{T_D}$ $\cancel{T_M}$ $\cancel{T_<}$ $\cancel{T_>}$

$\cancel{\downarrow}$ $\cancel{\uparrow}$ $\cancel{\sqcap}$ $\cancel{\sqcup}$

$\cancel{+_{10}}$ $-_{10}$ $\cancel{x_{10}}$

Worksheets L11* and ** are available for individual work.



Name__________ L 11 *

Each table is for one of these operations:

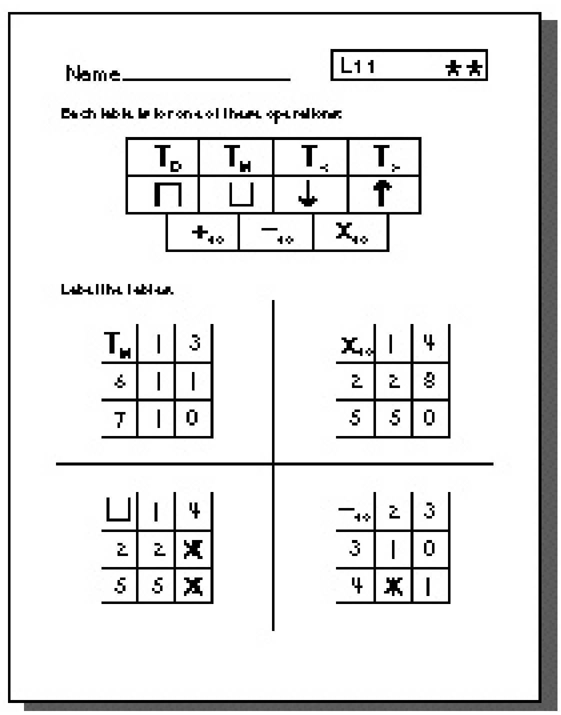
| $T_D$ | $T_M$ | $T_<$ | $T_>$ |
|---|---|---|---|
| $\sqcap$ | $\sqcup$ | $\downarrow$ | $\uparrow$ |
| $+_{10}$ | $-_{10}$ | $x_{10}$ | |

Label the tables.

| $+_{10}$ | 5 | 7 |
|---|---|---|
| 3 | 8 | 0 |
| 4 | 9 | 1 |

| $T_<$ | 1 | 5 |
|---|---|---|
| 1 | 0 | 1 |
| 5 | 0 | 0 |

| $\sqcap$ | 2 | 5 |
|---|---|---|
| 3 | 1 | 1 |
| 7 | 1 | 1 |



Name__________ L 11 **

Each table is for one of these operations:

| $T_D$ | $T_M$ | $T_<$ | $T_>$ |
|---|---|---|---|
| $\sqcap$ | $\sqcup$ | $\downarrow$ | $\uparrow$ |
| $+_{10}$ | $-_{10}$ | $x_{10}$ | |

Label the tables.

| $T_M$ | 1 | 3 |
|---|---|---|
| 6 | 1 | 1 |
| 7 | 1 | 0 |

| $x_{10}$ | 1 | 4 |
|---|---|---|
| 2 | 2 | 8 |
| 5 | 5 | 0 |

| $\sqcup$ | 1 | 4 |
|---|---|---|
| 2 | 2 | X |
| 5 | 5 | X |

| $-_{10}$ | 2 | 3 |
|---|---|---|
| 3 | 1 | 0 |
| 4 | X | 1 |

## Capsule Lesson Summary

Find the number of arrow pictures that can be drawn with six dots for people and using arrows for a name exchange (a permutation). Observe that the arrow picture for such a permutation may have from one to six pieces. Count the permutations with connected (one-piece) arrow pictures, and determine the probability of getting a connected arrow picture.

### Materials

**Teacher**
- Six slips of paper with student names
- Colored chalk
- 6-by-6 grid transparency (optional)
- Blackline L12

**Student**
- Colored pencils, pens, or crayons
- Unlined paper
- 6-by-6 grid (optional)
- Worksheets L12(a) and (b)

**Advance Preparation:** Choose six students in your class whose first names start with different letters. Write each name on a separate piece of paper and fold it so the name is not visible. You may want to use Blackline L12 to make both a 6-by-6 grid transparency and copies for students.

## Description of Lesson

To make the description of the lesson easier to follow, these six names will be used: Alice, Brad, Charlene, Darrel, Erin, and Felix.

### Exercise 1

Ask six students (the ones whose names are on your papers) to stand at the front of the classroom. Draw a dot for each of them on the board. Label the dots with the first letters of the students' names. Tell the class that they are to imagine that these six students are in a family (or club) that is exchanging names for a gift exchange.

**T:** ***I have six pieces of paper with one of your names written on each piece. I will mix up the pieces of paper and let each of you take one piece. Do not open it yet. When I say "go," unfold the paper and read the name. Then, with your right hand, point to the person whose name is on your piece of paper.***

Distribute the papers and say "go." Watch the students to make sure they follow instructions.

**T:** ***Let's draw red arrows to show the relation "I got your name" or "I point to you."***

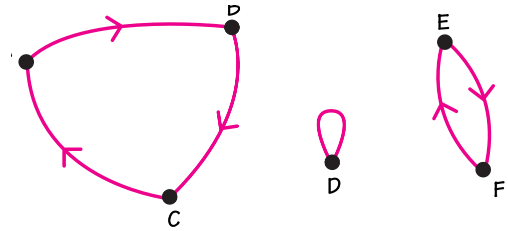
Invite students not participating in the game to show where to draw arrows on the board.

**S:** ***Charlene has Alice's name, so we can draw an arrow from* C *to* A.**

**S:** ***Darrel got his own name—there should be a loop at* D.**

# L12

Complete the arrow picture, perhaps as illustrated here.




**Note:** Each dot in the picture must have exactly one arrow starting at it and exactly one arrow ending at it.

**T:** ***Each dot has exactly one arrow starting at it or has a loop. Why did that happen?***

**S:** ***Each person in the game got one other person's name.***

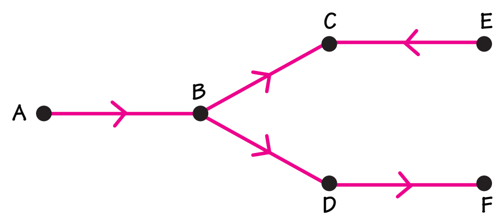
**T:** ***Each dot has exactly one arrow ending at it or has a loop. Why?***

**S:** ***The name of each person in the game is on exactly one piece of paper.***

Instruct students to draw six dots on a clean piece of unlined paper. Then tell them to label the dots with the same letters that were used on the board.

**T:** ***Imagine that the students have to exchange names a second time. Draw an arrow picture for this imaginary exchange. Make your picture different than the one on the board. If we did another exchange, the result would most likely be different.***

Let students work independently on their arrow pictures for a few minutes. As you observe students' work, give help to those having difficulty getting started, and correct obvious errors. For example, you may need to discuss why this is not a good picture.




There are several things wrong with the picture.

- Two arrows end at **C**, which would mean that two people got Charlene's name.
- Two arrows start at **B**, which would mean that Brad received two names.
- No arrow ends at **E**, which would mean that no one got Erin's name.
- No arrow starts at **C**, which would mean that Charlene received no name.

Suggest that students study their arrow pictures to see if they made any similar errors.

**T:** ***Look at our first arrow picture on the board. Into how many groups have the six people been divided?***

**S:** ***Three.***

**T:** ***On your papers, into how many groups have you divided the six people? Does anyone have an arrow picture with just one group?***

**S:** ***Yes; they are all in one large circle.***

**T:** ***What is the greatest number of groups that you could have?***

**S:** ***Six. There could be six loops.***

**T:** ***Do you think that any two of you drew exactly the same arrow picture?***

Let students briefly discuss the possibility.

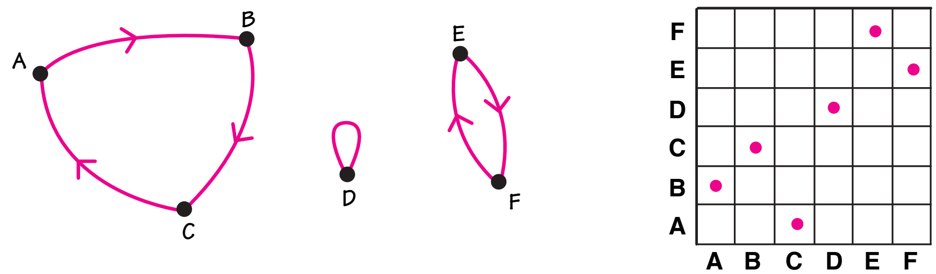
**T:** ***Can you estimate the number of different arrow pictures we could draw for six people exchanging names?***

Record students' estimates and save them for future reference.

**T:** ***We could try to draw all of the pictures, but that might take a long time and it would be difficult to check for duplicates. Instead, let's see if we can use a grid to make the problem easier.***

Draw a 6-by-6 grid on the board, and instruct students to draw the grid on their papers. You may prefer to display a prepared grid and distribute copies to students.

Refer back to the first arrow picture on the board, and use the letters (dot labels) in the grid. Call on students to represent each arrow or loop in the arrow picture with a checker (magnetic or drawn dot) on the grid. As they do this, ask for explanations of where to place checkers (dots).




Instruct students to draw checkers in their grid for the arrow picture on their papers.

**T:** ***Do you notice anything interesting about the placement of checkers in the grid?***

**S:** ***There is exactly one checker in each row and in each column.***

**T:** ***One arrow starts at each dot so there is one checker in each column. One arrow ends at each dot so there is one checker in each row.***

Distribute copies of Worksheets L12(a) and (b). With the class, check that the arrow picture on L12(a) is a good one for the exchanging names game. Then instruct students to complete these worksheets. You may like to let students work with partners.

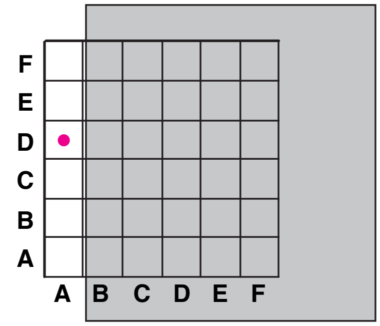
After a few minutes, check the grid on L12(a) and observe that it has exactly one checker in each row and in each column. Invite a student to draw the arrow picture for the grid on L12(b) on the board. Check that it is an exchanging names picture.

**T:** ***How do you suppose the grid can help us count the number of different arrow pictures?***

**S:** ***We can count the ways to put checkers in the grid so that there is exactly one in each row and in each column.***

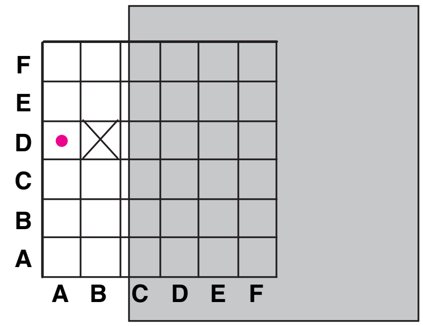
Lead the class to view this situation as a multiplication problem. On a clear copy of a 6-by-6 grid, cover all but column **A** with a paper.

**T:** ***First, let's put a checker in column* A. *How many choices do we have?*** (Six)




Invite a student to place a checker in column **A**.

Uncover column **B**.




**T:** ***Now, let's put a checker in column* B. *How many choices do we have?***

**S:** ***Five. We cannot put it in row* D*, because then there would be two dots in that row.***

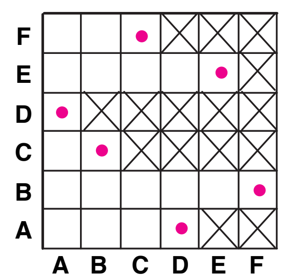
**T:** ***Yes, each column and each row must have exactly one checker in it.***

You may indicate the forbidden space with an X, and then invite a student to place a checker in one of the other five squares of column **B**.

**T:** ***Now, altogether how many ways are there to place checkers in the first two columns.***

**S:** ***6 x 5 = 30. There are 30 ways; six ways to put one checker in column* A*, and for each of those ways, five ways to put one checker in column* B.**

Continue in this manner, analyzing each column. Your grid might look like this. The class should conclude that there are 720 (6 x 5 x 4 x 3 x 2 x 1 = 720) different ways to place checkers in the grid, so also 720 different arrow pictures. Compare students' estimates to 720.




## Exercise 2

**T:** ***Earlier we noticed that sometimes the arrow picture is in several pieces. In fact, we said that there could be as few as one piece or as many as six pieces. On your paper, draw a one-piece arrow picture we could get when exchanging names with six people. Make your picture different from the one-piece picture on Worksheet L12(b).***

You may like to let students work with partners to draw two different one-piece arrow pictures.

**T:** ***When we exchange names, randomly giving out the pieces of paper, what is the probability of getting a one-piece arrow picture?***

Record students' estimates on the board, saving them for later comparison.

**T:** ***How can we calculate this probability?***

Allow students to express their opinions.

**S:** ***We need to know how many different one-piece arrow pictures there are among the 720 arrow pictures.***

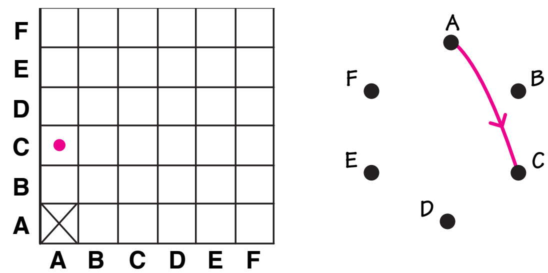
**T:** ***Very good. Let's look at the grid and try to put six checkers in the grid in such a way that the corresponding arrow picture is connected (has one piece).***

Clear the grid and draw six labeled dots on the board.

**T:** ***Where could we place a dot in column* A*?***

**S:** ***Anywhere except in row* A*. We cannot put the dot in row* A *because we would have a loop and eventually more than one piece. There are only five possibilities.***

Cross out the forbidden square, and invite a student to put a checker in any one of the five remaining squares. Also draw the corresponding arrow, as shown here.

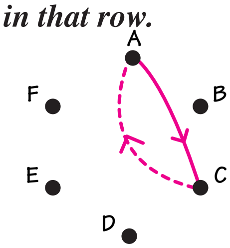



**T:** ***Now, where could we put a checker in column* C*? Why do I ask about column* C *next?***

**S:** ***The first arrow ends at* C *so, to keep the connection, we look next for the arrow starting at* C.

**T:** ***Good. Can we put this checker anywhere in column* C*?***

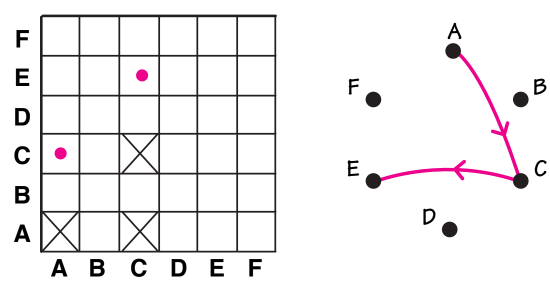
**S:** ***No. We cannot put it in row* C *because then there would be two checkers in that row.***




**S:** ***Also, we cannot put it in row* A*, because then there would be an arrow starting at* C *and ending at* A*. That would make one piece and the other dots would be in another piece.***

**S:** ***There are only four ways to place the checker.***

Cross out the two forbidden squares, and place a checker in one of the four remaining squares. Draw the corresponding arrow, as shown here.




**T:** ***How many ways are there to place the first two checkers in this grid?***

**S:** ***20 ways; 5 x 4 = 20.***

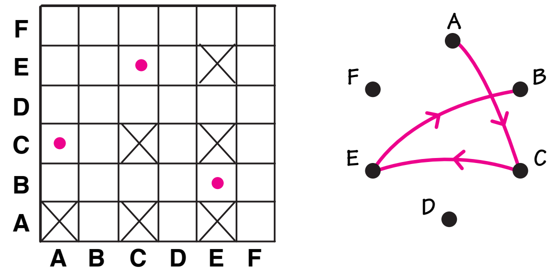
**T:** ***In which column should we put the next checker?*** (**E**) ***Can we put the checker anywhere in column* E*?***

**S:** ***Not in row* A*—that would make a cycle with only three dots. The other three dots would then be in another piece.***

**S:** ***Not in row* C *or* E*, because then there would be two checkers in a row. There are just three ways to place a checker in column* E.

# L12

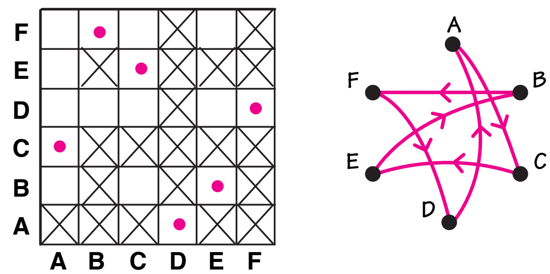
Cross out the three forbidden squares, place a checker in one of the three remaining squares, and draw the corresponding arrow, as shown here.




**T:** ***How many ways are there to place the first three checkers in this grid?***

**S:** ***60 ways; 5 x 4 x 3 = 60.***

Continue the analysis in this manner for the three remaining columns. The class should conclude that there are two possibilities for column **B** and one possibility each for columns **D** and **F**. So there are 120 (5 x 4 x 3 x 2 x 1 = 120) different ways to have a one-piece arrow picture. Your grid might look like this one.



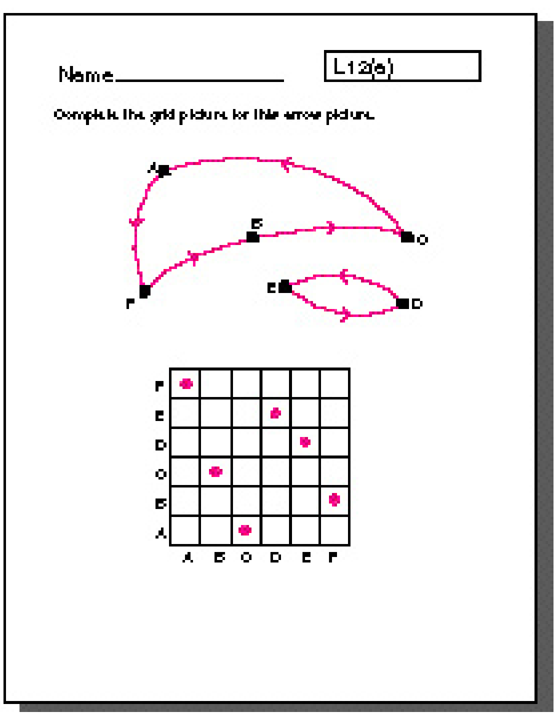

**T:** ***What is the probability of getting a connected (one-piece) arrow picture?***

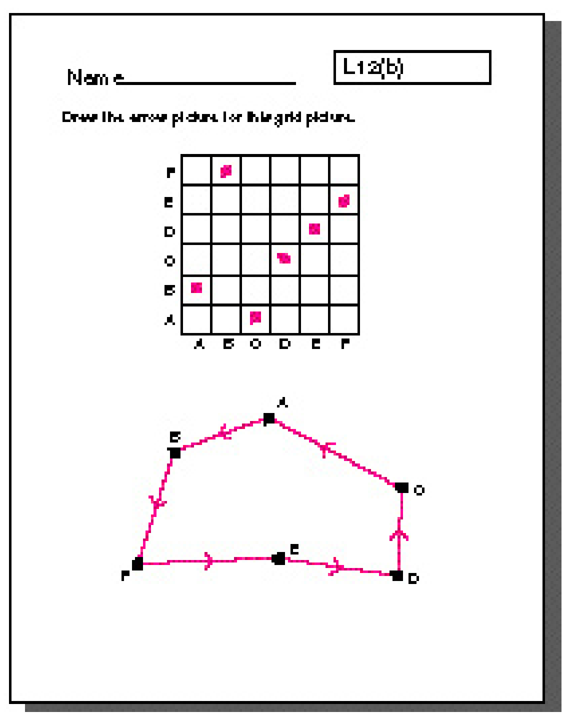
**S:** ***120 chances out of 720.***

Write this information on the board, and ask for a simpler fraction for $^{120}/_{720}$.

$$\frac{120}{720} = \frac{1}{6}$$

**T:** ***So there is about one chance out of six of getting a connected arrow picture.***



## Capsule Lesson Summary

In a string game situation, use clues involving the placement of numbers to determine the labels of three strings. Play a 3-string version of *The String Game* with numbers.

## Materials

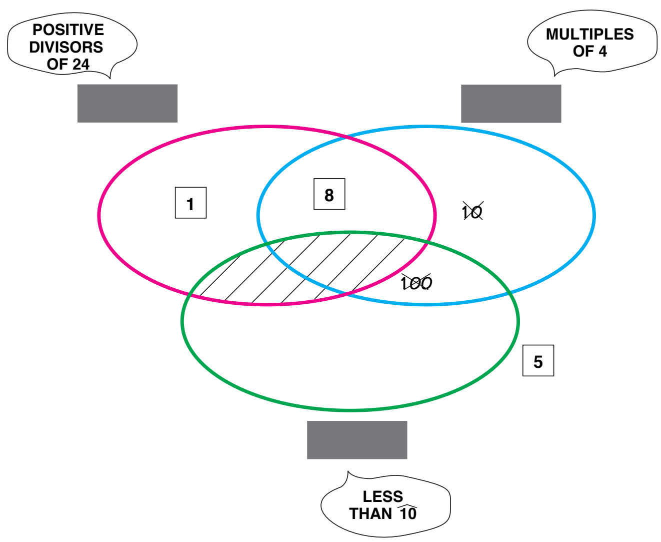
**Teacher**
- Colored chalk
- Numerical String Game kit
- Numerical 3-String Game poster
- Colored markers

**Student**
- Paper
- Colored pencils, pens, or crayons
- 3-String Game analysis sheet
- Worksheets L13* and **

# Description of Lesson

### Exercise 1

Prepare your board for a 3-string analysis activity, as shown below. Bubbles indicate what is on the hidden labels.




| RED | BLUE | GREEN |
|---|---|---|
| MULTIPLES OF 2 | MULTIPLES OF 2 | MULTIPLES OF 2 |
| MULTIPLES OF 3 | MULTIPLES OF 3 | MULTIPLES OF 3 |
| MULTIPLES OF 4 | MULTIPLES OF 4 | MULTIPLES OF 4 |
| MULTIPLES OF 5 | MULTIPLES OF 5 | MULTIPLES OF 5 |
| MULTIPLES OF 10 | MULTIPLES OF 10 | MULTIPLES OF 10 |
| ODD NUMBERS | ODD NUMBERS | ODD NUMBERS |
| POSITIVE PRIME NUMBERS | POSITIVE PRIME NUMBERS | POSITIVE PRIME NUMBERS |
| GREATER THAN 50 | GREATER THAN 50 | GREATER THAN 50 |
| LESS THAN 50 | LESS THAN 50 | LESS THAN 50 |
| GREATER THAN 10 | GREATER THAN 10 | GREATER THAN 10 |
| LESS THAN 10 | LESSTHAN 10 | LESS THAN 10 |
| POSITIVE DIVISORS OF 12 | POSITIVE DIVISORS OF 12 | POSITIVE DIVISORS OF 12 |
| POSITIVE DIVISORS OF 18 | POSITIVE DIVISORS OF 18 | POSITIVE DIVISORS OF 18 |
| POSITIVE DIVISORS OF 20 | POSITIVE DIVISORS OF 20 | POSITIVE DIVISORS OF 20 |
| POSITIVE DIVISORS OF 24 | POSITIVE DIVISORS OF 24 | POSITIVE DIVISORS OF 24 |
| POSITIVE DIVISORS OF 27 | POSITIVE DIVISORS OF 27 | POSITIVE DIVISORS OF 27 |

Instruct students to work individually or with partners to cross out labels the strings cannot have. They should find that the clues determine the string so they can circle what the strings are for on their lists.

After about ten minutes, you may want to collectively analyze the situation.

Using the information of where 1, 5, and 8 are in the picture, cross off these labels. Decide that the red string is for **POSITIVE DIVISORS OF 24**.

| RED | BLUE | GREEN |
|---|---|---|
| ~~MULTIPLES OF 2~~ | MULTIPLES OF 2 | ~~MULTIPLES OF 2~~ |
| ~~MULTIPLES OF 3~~ | ~~MULTIPLES OF 3~~ | MULTIPLES OF 3 |
| ~~MULTIPLES OF 4~~ | MULTIPLES OF 4 | ~~MULTIPLES OF 4~~ |
| ~~MULTIPLES OF 5~~ | ~~MULTIPLES OF 5~~ | ~~MULTIPLES OF 5~~ |
| ~~MULTIPLES OF 10~~ | ~~MULTIPLES OF 10~~ | MULTIPLES OF 10 |
| ~~ODD NUMBERS~~ | ~~ODD NUMBERS~~ | ~~ODD NUMBERS~~ |
| ~~POSITIVE PRIME NUMBERS~~ | ~~POSITIVE PRIME NUMBERS~~ | ~~POSITIVE PRIME NUMBERS~~ |
| ~~GREATER THAN 50~~ | ~~GREATER THAN 50~~ | GREATER THAN 50 |
| ~~LESS THAN 50~~ | ~~LESS THAN 50~~ | ~~LESS THAN 50~~ |
| ~~GREATER THAN $\widehat{10}$~~ | ~~GREATER THAN $\widehat{10}$~~ | ~~GREATER THAN $\widehat{10}$~~ |
| ~~LESS THAN $\widehat{10}$~~ | ~~LESS THAN $\widehat{10}$~~ | LESS THAN $\widehat{10}$ |
| ~~POSITIVE DIVISORS OF 12~~ | POSITIVE DIVISORS OF 12 | ~~POSITIVE DIVISORS OF 12~~ |
| ~~POSITIVE DIVISORS OF 18~~ | ~~POSITIVE DIVISORS OF 18~~ | ~~POSITIVE DIVISORS OF 18~~ |
| ~~POSITIVE DIVISORS OF 20~~ | ~~POSITIVE DIVISORS OF 20~~ | ~~POSITIVE DIVISORS OF 20~~ |
| POSITIVE DIVISORS OF 24 | ~~POSITIVE DIVISORS OF 24~~ | ~~POSITIVE DIVISORS OF 24~~ |
| ~~POSITIVE DIVISORS OF 27~~ | ~~POSITIVE DIVISORS OF 27~~ | ~~POSITIVE DIVISORS OF 27~~ |

100 is a multiple of 2 and 4, so it must be in the blue string. 100 is not a divisor of 24, so it is not in the red string. Using the information that 100 is not in the indicated region, find that 100 must be in the blue string only. This elminates two more possibilities for the green string.

| RED | BLUE | GREEN |
|---|---|---|
| ~~MULTIPLES OF 2~~ | MULTIPLES OF 2 | ~~MULTIPLES OF 2~~ |
| ~~MULTIPLES OF 3~~ | ~~MULTIPLES OF 3~~ | MULTIPLES OF 3 |
| ~~MULTIPLES OF 4~~ | MULTIPLES OF 4 | ~~MULTIPLES OF 4~~ |
| ~~MULTIPLES OF 5~~ | ~~MULTIPLES OF 5~~ | ~~MULTIPLES OF 5~~ |
| ~~MULTIPLES OF 10~~ | ~~MULTIPLES OF 10~~ | ~~MULTIPLES OF 10~~ |
| ~~ODD NUMBERS~~ | ~~ODD NUMBERS~~ | ~~ODD NUMBERS~~ |
| ~~POSITIVE PRIME NUMBERS~~ | ~~POSITIVE PRIME NUMBERS~~ | ~~POSITIVE PRIME NUMBERS~~ |
| ~~GREATER THAN 50~~ | ~~GREATER THAN 50~~ | ~~GREATER THAN 50~~ |
| ~~LESS THAN 50~~ | ~~LESS THAN 50~~ | ~~LESS THAN 50~~ |
| ~~GREATER THAN $\widehat{10}$~~ | ~~GREATER THAN $\widehat{10}$~~ | ~~GREATER THAN $\widehat{10}$~~ |
| ~~LESS THAN $\widehat{10}$~~ | ~~LESS THAN $\widehat{10}$~~ | LESS THAN $\widehat{10}$ |
| ~~POSITIVE DIVISORS OF 12~~ | POSITIVE DIVISORS OF 12 | ~~POSITIVE DIVISORS OF 12~~ |
| ~~POSITIVE DIVISORS OF 18~~ | ~~POSITIVE DIVISORS OF 18~~ | ~~POSITIVE DIVISORS OF 18~~ |
| ~~POSITIVE DIVISORS OF 20~~ | ~~POSITIVE DIVISORS OF 20~~ | ~~POSITIVE DIVISORS OF 20~~ |
| POSITIVE DIVISORS OF 24 | ~~POSITIVE DIVISORS OF 24~~ | ~~POSITIVE DIVISORS OF 24~~ |
| ~~POSITIVE DIVISORS OF 27~~ | ~~POSITIVE DIVISORS OF 27~~ | ~~POSITIVE DIVISORS OF 27~~ |

Since the intersection of the red and green strings is empty, the green string must be for **LESS THAN $\widehat{10}$**.

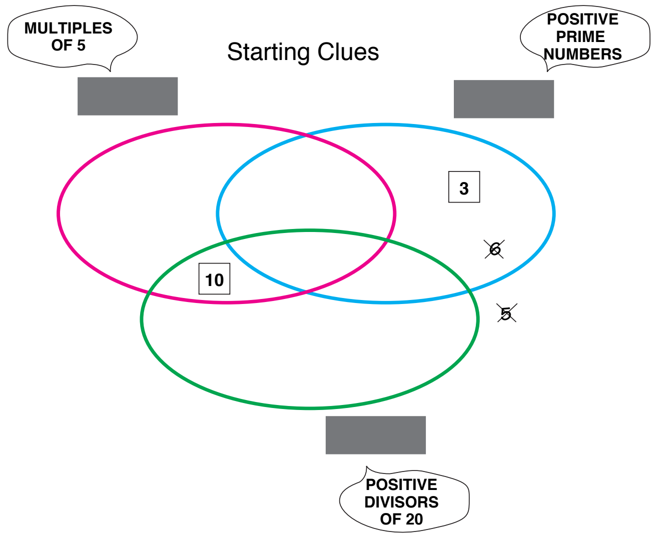
| RED | BLUE | GREEN |
|---|---|---|
| ~~MULTIPLES OF 2~~ | MULTIPLES OF 2 | ~~MULTIPLES OF 2~~ |
| ~~MULTIPLES OF 3~~ | ~~MULTIPLES OF 3~~ | ~~MULTIPLES OF 3~~ |
| ~~MULTIPLES OF 4~~ | MULTIPLES OF 4 | ~~MULTIPLES OF 4~~ |
| ~~MULTIPLES OF 5~~ | ~~MULTIPLES OF 5~~ | ~~MULTIPLES OF 5~~ |
| ~~MULTIPLES OF 10~~ | ~~MULTIPLES OF 10~~ | ~~MULTIPLES OF 10~~ |
| ~~ODD NUMBERS~~ | ~~ODD NUMBERS~~ | ~~ODD NUMBERS~~ |
| ~~POSITIVE PRIME NUMBERS~~ | ~~POSITIVE PRIME NUMBERS~~ | ~~POSITIVE PRIME NUMBERS~~ |
| ~~GREATER THAN 50~~ | ~~GREATER THAN 50~~ | ~~GREATER THAN 50~~ |
| ~~LESS THAN 50~~ | ~~LESS THAN 50~~ | ~~LESS THAN 50~~ |
| ~~GREATER THAN $\widehat{10}$~~ | ~~GREATER THAN $\widehat{10}$~~ | ~~GREATER THAN $\widehat{10}$~~ |
| ~~LESS THAN $\widehat{10}$~~ | ~~LESS THAN $\widehat{10}$~~ | LESS THAN $\widehat{10}$ |
| ~~POSITIVE DIVISORS OF 12~~ | POSITIVE DIVISORS OF 12 | ~~POSITIVE DIVISORS OF 12~~ |
| ~~POSITIVE DIVISORS OF 18~~ | ~~POSITIVE DIVISORS OF 18~~ | ~~POSITIVE DIVISORS OF 18~~ |
| ~~POSITIVE DIVISORS OF 20~~ | ~~POSITIVE DIVISORS OF 20~~ | ~~POSITIVE DIVISORS OF 20~~ |
| POSITIVE DIVISORS OF 24 | ~~POSITIVE DIVISORS OF 24~~ | ~~POSITIVE DIVISORS OF 24~~ |
| ~~POSITIVE DIVISORS OF 27~~ | ~~POSITIVE DIVISORS OF 27~~ | ~~POSITIVE DIVISORS OF 27~~ |

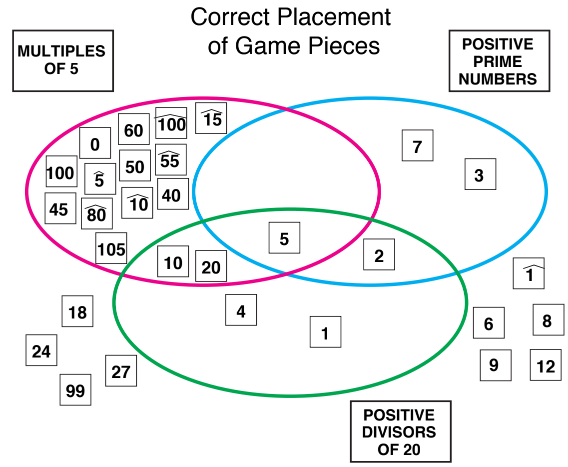
Using the information that 10 is not in the indicated region, 10 must not be in the blue string, and the blue string must be for **MULTIPLES OF 4**.

| RED | BLUE | GREEN |
|---|---|---|
| ~~MULTIPLES OF 2~~ | ~~MULTIPLES OF 2~~ | ~~MULTIPLES OF 2~~ |
| ~~MULTIPLES OF 3~~ | ~~MULTIPLES OF 3~~ | ~~MULTIPLES OF 3~~ |
| ~~MULTIPLES OF 4~~ | MULTIPLES OF 4 | ~~MULTIPLES OF 4~~ |
| ~~MULTIPLES OF 5~~ | ~~MULTIPLES OF 5~~ | ~~MULTIPLES OF 5~~ |
| ~~MULTIPLES OF 10~~ | ~~MULTIPLES OF 10~~ | ~~MULTIPLES OF 10~~ |
| ~~ODD NUMBERS~~ | ~~ODD NUMBERS~~ | ~~ODD NUMBERS~~ |
| ~~POSITIVE PRIME NUMBERS~~ | ~~POSITIVE PRIME NUMBERS~~ | ~~POSITIVE PRIME NUMBERS~~ |
| ~~GREATER THAN 50~~ | ~~GREATER THAN 50~~ | ~~GREATER THAN 50~~ |
| ~~LESS THAN 50~~ | ~~LESS THAN 50~~ | ~~LESS THAN 50~~ |
| ~~GREATER THAN $\widehat{10}$~~ | ~~GREATER THAN $\widehat{10}$~~ | ~~GREATER THAN $\widehat{10}$~~ |
| ~~LESS THAN $\widehat{10}$~~ | ~~LESS THAN $\widehat{10}$~~ | LESS THAN $\widehat{10}$ |
| ~~POSITIVE DIVISORS OF 12~~ | POSITIVE DIVISORS OF 12 | ~~POSITIVE DIVISORS OF 12~~ |
| ~~POSITIVE DIVISORS OF 18~~ | ~~POSITIVE DIVISORS OF 18~~ | ~~POSITIVE DIVISORS OF 18~~ |
| ~~POSITIVE DIVISORS OF 20~~ | ~~POSITIVE DIVISORS OF 20~~ | ~~POSITIVE DIVISORS OF 20~~ |
| POSITIVE DIVISORS OF 24 | ~~POSITIVE DIVISORS OF 24~~ | ~~POSITIVE DIVISORS OF 24~~ |
| ~~POSITIVE DIVISORS OF 27~~ | ~~POSITIVE DIVISORS OF 27~~ | ~~POSITIVE DIVISORS OF 27~~ |

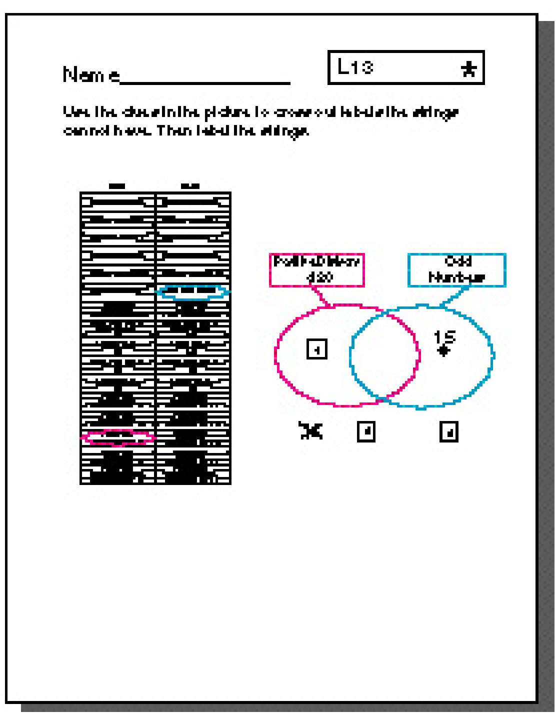
## Exercise 2

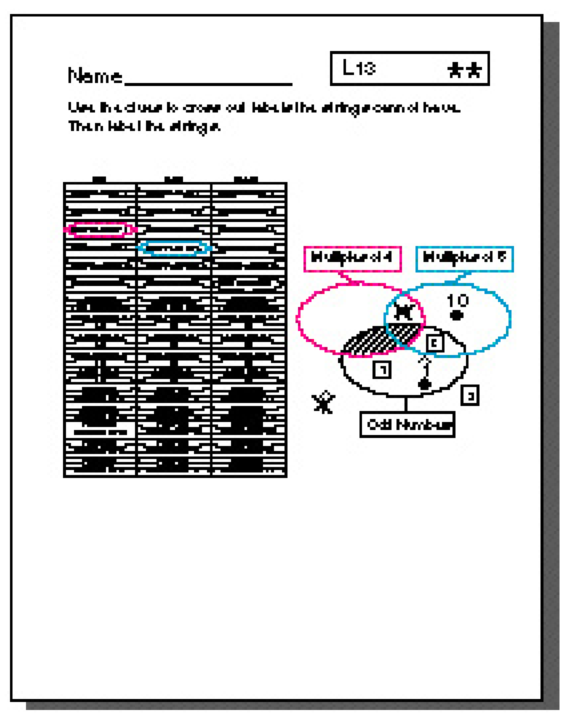
Play *The String Game* with numbers in the usual way. A possible 3-string game is suggested below.




Worksheets L13* and ** are available for individual work.



# L14 THE TABLE GAME #3



## Capsule Lesson Summary

Introduce a truth table for the relation "has the same parity." Provide a list of 12 operations, and ask which of those operations have tables with various characteristics. Introduce *The Table Game*.

### Materials

**Teacher**
- 9-by-9 operation table transparency
- Table Game posters
- Table Game scoring sheet
- Tape
- Markers
- Blacklines L14(a) and (b)

**Student**
- Table Game analysis sheet

**Advance Preparation:** Use Blackline L10 to make a transparency of a 9-by-9 operation table. You may prefer to prepare the table on the chalkboard or a grid board. Use Blackline L14(a) to make copies of the Table Game analysis sheet for students. Use Blackline L14(b) to make copies of a scoring sheet for use in play of the game (Exercise 2).

## Description of Lesson

### Exercise 1

Tape a copy of *The Table Game* poster to the board, and display a 9-by-9 operation table nearby.

**The Table Game**

| | | |
|---|---|---|
| $+_{10}$ | $-_{10}$ | $x_{10}$ |
| $\sqcap$ | $\sqcup$ | $\uparrow$ |
| $T_D$ | $T_M$ | $\downarrow$ |
| $T_<$ | $T_>$ | $T_P$ |

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 2 | | | | | | | | | |
| 3 | | | | | | | | | |
| 4 | | | | | | | | | |
| 5 | | | | | | | | | |
| 6 | | | | | | | | | |
| 7 | | | | | | | | | |
| 8 | | | | | | | | | |
| 9 | | | | | | | | | |

Remind the class that in previous lessons they have constructed tables for all the operations on the Table Game poster except $T_P$.

**T:** ***This last symbol ($T_P$) on the poster is for another truth table. The P means* parity*. Do you know how we use this word with numbers?***

Some students may recognize the word as having to do with even and odd. Let students express their ideas first, and then explain that two numbers have the same parity if they are both even or both odd.

**T:** ***Let's examine the truth table for the relation "has the same parity." Remember, in a truth table 1 means true and 0 means false. Where will there be 1s in the table? Where will there be 0s?***

Label the table for $T_P$, and invite students to make some entries. As entries are made reiterate that 1 is for true and 0 is for false.

| $T_P$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| **2** | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| **3** | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| **4** | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| **5** | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| **6** | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| **7** | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| **8** | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| **9** | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

**T:** ***"2 has the same parity as 6" is true because 2 and 6 are both even.***
***"2 has the same parity as 5" is false because 2 is even and 5 is odd.***
***"5 has the same parity as 3" is true because 5 and 3 are both odd.***

The class should quickly find patterns that make the table easy to complete.

Clear the display table, and ask the following questions to stimulate students to think about other operations and their tables.† Refer to the table on the board. Each time a question is asked, list the possibilities on the board as students suggest them and the class agrees. The last three questions will require the most analysis. Encourage students to rule out several possibilities at a time or to consider related operations as well.

- Which of these operations have no zero in their tables? ($\downarrow$, $\uparrow$, $\sqcap$, $\sqcup$)
- Do any of these operations have negative numbers in their tables? (No)
- Do any of these operations have numbers greater than 9 in their tables? ($\sqcup$)
- Which of these operations have zeros everywhere along this diagonal (trace the diagonal from the upper left corner to the lower right corner) in their tables? ($-_{10}$, $T_<$, $T_>$)
- Which of these operations have ones everywhere along this diagonal (trace the diagonal from the upper left corner to the lower right corner) in their tables? ($T_D$, $T_M$, $T_P$)
- Which of these operations have all ones in the first row (the 1-row)? ($\sqcap$, $\downarrow$, $T_D$)
- Which of these operations have all zeros in the first row (the 1-row)? ($T_>$)

## Exercise 2: The Table Game

Select one student who will be a reliable scorekeeper††. Divide the rest of the class into four (or two) teams. In the following example, the four teams are referred to as Team A, Team B, Team C, and Team D. Distribute Table Game analysis sheets to students.

Play a game involving a table for an unknown operation *. The object is to determine which of 12 operations listed on the poster is *. Although all of the plays in the game are made in the same way, the scoring rules divide the game into two parts with different but related goals.

The game begins with one or two entries already in the table as starting clues. Such an entry can be correct or incorrect (indicated by an X through it). Players, either independently or with their team members, then use the clues to try to eliminate some of the 12 possibilities for * listed on their analysis sheets. After a few minutes, a player from Team A suggests an entry to the table; for example,

†"Their tables" refers to their completed 9-by-9 tables. Completed tables for all 12 operations in *The Table Game* can be found at the end of Lesson L15.

††The scorekeeper could be an adult who is available during the class time, such as a teacher's aide or a student teacher.

the player might suggest that 2 * 3 = 6. Make the entry in the table. If correct, tell the class; if incorrect, say "no" and put an X through the entry. Allow a few minutes for players to use the information gained from the play to help identify the operation *. Then give players the option to turn in their analysis sheets with one of the operations circled. A team gains ten points each time a member correctly identifies the operation. If team members are consulting among themselves, you may ask the team to turn in just one sheet. The scorekeeper tallies the points for each team but does not reveal the score at this time.

Proceed in the same manner for plays from the other three teams, but each time decrease the number of points that can be gained (9-8-7). With two teams, each team makes two plays. The last time students (or teams) may turn in their analysis sheets is after the fourth play is made.

Now the scoring changes. The remainder of the game allows each player (alternating teams) to suggest an entry in the table in the same manner as before. Only now players earn points for their team based on whether or not their entries are correct. The scoring is as follows.

- A team gains one point if a member suggests a correct entry in a square on the main diagonal (shaded squares).
- A team gains two points if a member suggests a correct entry in a square other than on the main diagonal.

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 2 | | | | | | | | | |
| 3 | | | | | | | | | |
| 4 | | | | | | | | | |
| 5 | | | | | | | | | |
| 6 | | | | | | | | | |
| 7 | | | | | | | | | |
| 8 | | | | | | | | | |
| 9 | | | | | | | | | |

A team is not penalized for an incorrect suggestion; the entry is simply made and crossed out.

The game ends after each player has had a turn. Call on a player to identify * while the scorekeeper computes the scores. It is a good idea to give the scorekeeper a tally sheet similar to the one on Blackline L14(b).

The secret operation is _____.

| **Points** | TEAM A | TEAM B | TEAM C | TEAM D |
|---|---|---|---|---|
| First part of game<br>For **correctly** identifying *<br>• after first play 10 pts.<br>• after second play 9 pts.<br>• after third play 8 pts.<br>• after fourth play 7 pts.<br>First part of game<br>For **correctly** making an entry<br>• on the diagonal 1 pt.<br>• other than on the diagonal 2 pts. | | | | |

Play the game a couple of times. These are some possible choices for the unknown operation * with appropriate starting clues.

| | **Operation** * | **Starting Clue** |
|---|---|---|
| **First game:** | $x_{10}$ | $3 * 7 = 1$ |
| **Second game:** | $T_M$ | $8 * 5 = 0$ |
| **Third game:** | $\uparrow$ | $5 * 5 \neq 0$ |

The rest of this lesson description describes a possible game. The operation * is $x_{10}$ and $3 * 7 = 1$ is a starting clue.

**T:** ***Knowing that 3 * 7 = 1 should help you to eliminate some of the possibilities for the operation. Cross out as many operation symbols as you can on your analysis sheet.***

Allow a few minutes for analysis.

Note: For your information only, four possibilities for * remain: $x_{10}$, $\sqcap$, $T_<$, and $T_P$.

The first player from Team A suggests an entry.

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | |
| 2 | | | | | | | | | |
| 3 | | | | | | | 1 | | |
| 4 | | | | | | | | | |
| 5 | | | | | | | | | |
| 6 | | | | | | | | | |
| 7 | | | | | | | | | |
| 8 | | | | | | | | | |
| 9 | | | | | | | | | |

**S:** ***1 * 1 = 1.***

**T:** ***That's correct.***

Allow a few minutes for analysis.

**T:** ***Does anyone want to turn in their sheet for a possible ten points?***

Nobody turns in an analysis sheet.

The first player from Team B suggests an entry.

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | |
| 2 | | | | | | | | | |
| 3 | | | | | | | 1 | | |
| 4 | | | | | | | | | |
| 5 | | | | | | ~~1~~ | | | |
| 6 | | | | | | | | | |
| 7 | | | | | | | | | |
| 8 | | | | | | | | | |
| 9 | | | | | | | | | |

**S:** ***5 * 6 = 1.***

**T:** ***No; that's not correct.***

Allow a few minutes for analysis.

**T:** ***Does anyone want to turn in their sheet for a possible nine points?***

A couple players (teams) turn in their sheets. Now there are two possibilities for *: $x_{10}$, or $T_P$. Those who turned in their sheets may have been taking a chance. They get the nine points only if they circled $x_{10}$.

The next two players make suggestions, and some more players turn in their sheets for eight points and for seven points.

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | | | | | | | | |
| **2** | | | | | | | | | |
| **3** | | | | | | | 1 | | |
| **4** | | | | ~~1~~ | | | | | |
| **5** | | | | | | ~~1~~ | | | |
| **6** | | | | | 0 | | | | |
| **7** | | | | | | | | | |
| **8** | | | | | | | | | |
| **9** | | | | | | | | | |

**S** (Team C)**:** ***4 * 4 = 1.***

**T:** ***No, that's not correct.***

**S** (Team D)**:** ***6 * 5 = 0.***

**T:** ***Yes, that's correct.***

**Note:** The operation is determined when Team C's entry is made. Because 1 is crossed out, meaning 4 * 4 ≠ 1, * cannot be $T_P$.

The scorekeeper tallies the points so far but does not reveal the results to the class.

Now the scoring changes. Students suggest entries and earn points for their team based on whether or not the entries are correct.

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | ~~0~~ | | | | | | | |
| **2** | | | | | 0 | | | | |
| **3** | | | | | | | 1 | | |
| **4** | | | | ~~1~~ | | | | | |
| **5** | | 0 | | | 5 | ~~1~~ | | | |
| **6** | | | | | 0 | | | | |
| **7** | | | | | | | | | |
| **8** | | | | | | | | | |
| **9** | | | | | | | | | |

**S** (Team A)**:** ***2 * 5 = 0.***

**T:** ***Correct—two points for Team A.***

**S** (Team B)**:** ***1 * 2 = 0.***

**T:** ***Not correct—no point for Team B.***

**S** (Team C)**:** ***5 * 5 = 5.***

**T:** ***Correct—one point for Team C.***

**S** (Team D)**:** ***5 * 2 = 0.***

**T:** ***Correct—two points for Team D.***

The game continues until every player has had a turn.

**T:** ***While the scorekeeper is computing the scores, who can tell us which operation * is?***

**S:** $\times_{10}$.

The scores are revealed and the winning team is announced.

**Note:** You may like to vary the game by letting students turn in their analysis sheets for a possible 15 points after the starting clues alone. The operation in the game sometimes can be determined by only one or two entries. If you choose to play in this way, adjust the scorekeeper's sheet accordingly.

# L15 THE TABLE GAME #4

## Capsule Lesson Summary

Given a 2-by-2 table with all four of its entries, identify which of 12 operations the table is for. Given three entries in a 9-by-9 table, determine which of 12 operations the table is for. Play *The Table Game*.

### Materials

**Teacher**
- 9-by-9 operation table transparency
- Table Game posters
- Tape
- Markers

**Student**
- Table Game analysis sheet
- Worksheets L15* and **

**Advance Preparation:** Use Blackline L10 to make a transparency of a 9-by-9 operation table. You may prefer to prepare the table on the chalkboard or on a grid board.

## Description of Lesson

### Exercise 1

Display a copy of *The Table Game* poster and draw this 2-by-2 table nearby.

**The Table Game**

| | | |
|---|---|---|
| $+_{10}$ | $-_{10}$ | $\times_{10}$ |
| $\sqcap$ | $\sqcup$ | $\uparrow$ |
| $T_D$ | $T_M$ | $\downarrow$ |
| $T_<$ | $T_>$ | $T_P$ |

| * | 1 | 4 |
|---|---|---|
| 2 | 0 | 1 |
| 5 | 1 | 0 |

**T:** ***This table is for one of these operations. Looking at the entries in this table, you probably have some idea about which operation the table is for. Take a few minutes to test some operations that you think could be *.***

The emphasis in this exercise is on using intuition and not systematic consideration of each possibility for *. After two or three minutes, let students suggest operations that might be *. Most likely, students will suggest looking at truth tables first since the table has only 0s and 1s. Each time someone suggests an operation, ask the class if any of the four entries in the table prohibits * from being that operation. Continue until the class identifies the operation as $T_P$.

Repeat this exercise with a couple more 2-by-2 tables. Use the tables suggested below or others of your choice.

| $\uparrow$ | 7 | 8 |
|---|---|---|
| 5 | 7 | 8 |
| 6 | 7 | 8 |

| $-_{10}$ | 3 | 1 |
|---|---|---|
| 7 | 4 | 6 |
| 2 | ~~11~~ | 1 |

| $T_D$ | 1 | 6 |
|---|---|---|
| 1 | 1 | 1 |
| 3 | 0 | 1 |

# L15

## Exercise 2

Erase the 2-by-2 table and display a 9-by-9 table. Make the entries shown below.

**The Table Game**

| | | |
|---|---|---|
| $+_{10}$ | $-_{10}$ | $x_{10}$ |
| $\sqcap$ | $\sqcup$ | $\uparrow$ |
| $T_D$ | $T_M$ | $\downarrow$ |
| $T_<$ | $T_>$ | $T_P$ |

| * | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | $\cancel{1}$ |
| 2 | | | | | | | | | |
| 3 | | | | | | | | | |
| 4 | | | | | | | | | |
| 5 | | | | | | | | | |
| 6 | | | | | | | | | |
| 7 | | | | | | | | | |
| 8 | | | | | | | | | |
| 9 | 0 | | | | | | | | |

**T:** ***This table is for one of these operations. There are two clues. What can we eliminate from the list of possible operations.***

**S:** **** cannot be $\uparrow$, $\downarrow$, $\sqcap$, or $\sqcup$ because those operations do not have 0s in the table.***

**S:** **** cannot be $T_P$ because 1 and 9 are both odd; they are the same parity so 1 * 9 would be 1.***

**S:** **** cannot be $T_M$ because 9 is a multiple of 1. 9 * 1 would be 1.***
**** cannot be $T_D$ because 1 is a divisor of 9. 1 * 9 would be 1.***

**S:** **** cannot be $T_<$ because 1 < 9 is true. 1 * 9 would be 1.***
**** cannot be $T_>$ because 9 > 1 is true. 9 * 1 would be 1.***

The class should determine that the table is for $+_{10}$.

## Exercise 3

Play *The Table Game* in the usual way. (See Lesson L14 for a description of the game.) A possible game is described below. Indicate only the starting clues in the 9-by-9 table on the board at the beginning of the game.

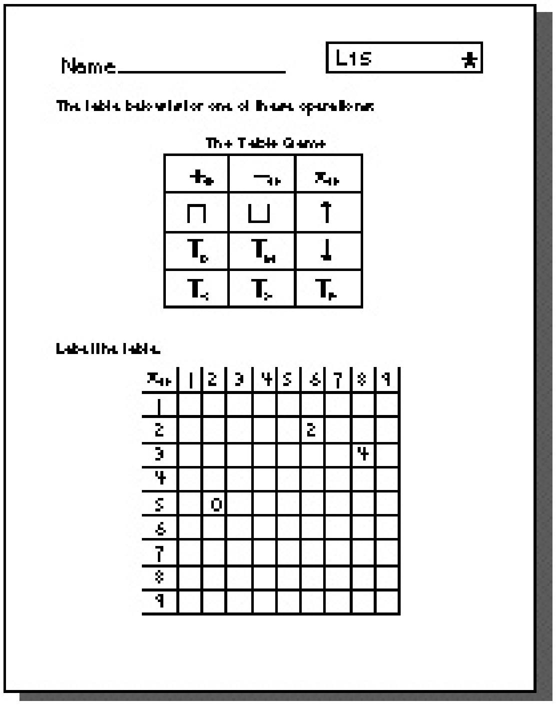
| Operation * | Starting Clues |
|---|---|
| $T_M$ | $1 * 2 \neq 1$ |
| | $3 * 3 \neq 0$ |

Worksheets L15*, **, and *** are available for individual work.

**Note:** With these starting clues 6 of the 12 possible operations can be eliminated; namely, $-_{10}$, $\sqcap$, $T_D$, $\downarrow$, $T_<$, and $T_>$.

Worksheets L15*, **, and *** are available for individual work.

For your convenience, the page following the worksheet keys has completed tables for all the operations in *The Table Game*.



Name________________ L15 ★

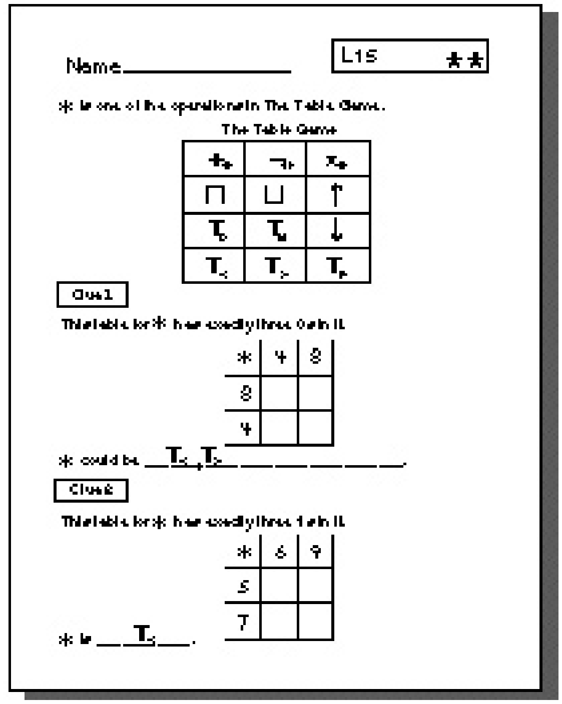
The table below is for one of these operations.

The Table Game

| | | |
|---|---|---|
| $+_8$ | $\neg_{11}$ | $x_{11}$ |
| ⊓ | ⊔ | ↑ |
| $T_b$ | $T_M$ | ↓ |
| $T_<$ | $T_>$ | $T_F$ |

Label the table.

| $x_{11}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 2 | | | | | | 2 | | | |
| 3 | | | | | | | | 4 | |
| 4 | | | | | | | | | |
| 5 | 0 | | | | | | | | |
| 6 | | | | | | | | | |
| 7 | | | | | | | | | |
| 8 | | | | | | | | | |
| 9 | | | | | | | | | |



Name________________ L15 ★★

$*$ is one of the operations in The Table Game.

The Table Game

| | | |
|---|---|---|
| $+_8$ | $\neg_{11}$ | $x_8$ |
| ⊓ | ⊔ | ↑ |
| $T_b$ | $T_M$ | ↓ |
| $T_<$ | $T_>$ | $T_F$ |

Clue 1

This table for $*$ has exactly three 0s in it.

| $*$ | 4 | 8 |
|---|---|---|
| 8 | | |
| 4 | | |

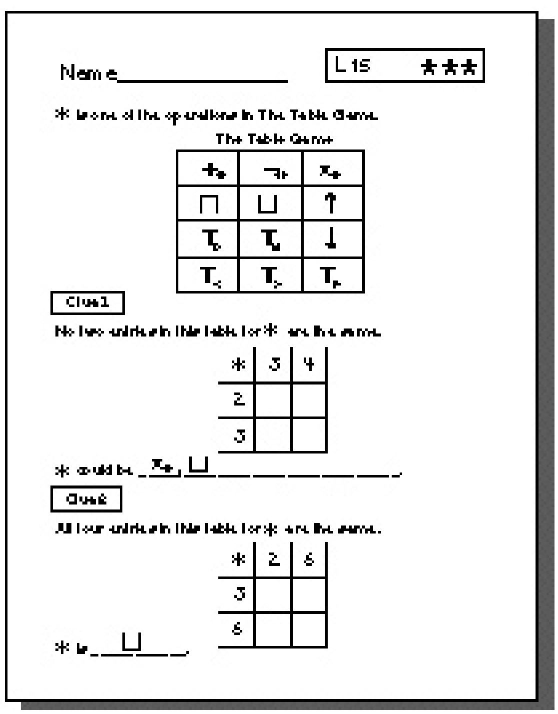
$*$ could be __$T_<$, $T_>$__________________.

Clue 2

This table for $*$ has exactly three 1s in it.

| $*$ | 6 | 9 |
|---|---|---|
| 5 | | |
| 7 | | |

$*$ is ___$T_<$___.



Name________________ L15 ★★★

$*$ is one of the operations in The Table Game.

The Table Game

| | | |
|---|---|---|
| $+_8$ | $\neg_{11}$ | $x_8$ |
| ⊓ | ⊔ | ↑ |
| $T_b$ | $T_M$ | ↓ |
| $T_<$ | $T_>$ | $T_F$ |

Clue 1

No two entries in this table for $*$ are the same.

| $*$ | 3 | 4 |
|---|---|---|
| 2 | | |
| 3 | | |

$*$ could be _$x_8$, ⊔__________________.

Clue 2

All four entries in this table for $*$ are the same.

| $*$ | 2 | 6 |
|---|---|---|
| 3 | | |
| 6 | | |

$*$ is ___⊔___.

# L15

| $+_{10}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 |
| 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 |
| 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 |
| 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 |
| 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

| $-_{10}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
| 2 | 1 | 0 | 9 | 8 | 7 | 6 | 5 | 4 | 3 |
| 3 | 2 | 1 | 0 | 9 | 8 | 7 | 6 | 5 | 4 |
| 4 | 3 | 2 | 1 | 0 | 9 | 8 | 7 | 6 | 5 |
| 5 | 4 | 3 | 2 | 1 | 0 | 9 | 8 | 7 | 6 |
| 6 | 5 | 4 | 3 | 2 | 1 | 0 | 9 | 8 | 7 |
| 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 9 | 8 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 9 |
| 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

| $x_{10}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 2 | 4 | 6 | 8 | 0 | 2 | 4 | 6 | 8 |
| 3 | 3 | 6 | 9 | 2 | 5 | 8 | 1 | 4 | 7 |
| 4 | 4 | 8 | 2 | 6 | 0 | 4 | 8 | 2 | 6 |
| 5 | 5 | 0 | 5 | 0 | 5 | 0 | 5 | 0 | 5 |
| 6 | 6 | 2 | 8 | 4 | 0 | 6 | 2 | 8 | 4 |
| 7 | 7 | 4 | 1 | 8 | 5 | 2 | 9 | 6 | 3 |
| 8 | 8 | 6 | 4 | 2 | 0 | 8 | 6 | 4 | 2 |
| 9 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

| $\sqcap$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 |
| 3 | 1 | 1 | 3 | 1 | 1 | 3 | 1 | 1 | 3 |
| 4 | 1 | 2 | 1 | 4 | 1 | 2 | 1 | 4 | 1 |
| 5 | 1 | 1 | 1 | 1 | 5 | 1 | 1 | 1 | 1 |
| 6 | 1 | 2 | 3 | 2 | 1 | 6 | 1 | 2 | 3 |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 7 | 1 | 1 |
| 8 | 1 | 2 | 1 | 4 | 1 | 2 | 1 | 8 | 1 |
| 9 | 1 | 1 | 3 | 1 | 1 | 3 | 1 | 1 | 9 |

| $\sqcup$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 2 | 2 | 6 | 4 | 10 | 6 | 14 | 8 | 18 |
| 3 | 3 | 6 | 3 | 12 | 15 | 6 | 21 | 24 | 9 |
| 4 | 4 | 4 | 12 | 4 | 20 | 12 | 28 | 8 | 36 |
| 5 | 5 | 10 | 15 | 20 | 5 | 30 | 35 | 40 | 45 |
| 6 | 6 | 6 | 6 | 12 | 30 | 6 | 42 | 24 | 18 |
| 7 | 7 | 14 | 21 | 28 | 35 | 42 | 7 | 56 | 63 |
| 8 | 8 | 8 | 24 | 8 | 40 | 24 | 56 | 8 | 72 |
| 9 | 9 | 18 | 9 | 36 | 45 | 18 | 63 | 72 | 9 |

| $\uparrow$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 3 | 3 | 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 4 | 4 | 4 | 4 | 5 | 6 | 7 | 8 | 9 |
| 5 | 5 | 5 | 5 | 5 | 5 | 6 | 7 | 8 | 9 |
| 6 | 6 | 6 | 6 | 6 | 6 | 6 | 7 | 8 | 9 |
| 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 8 | 9 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

| $T_D$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 4 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 5 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

| $T_M$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 5 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 6 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| 7 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 8 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 9 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |

| $\downarrow$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 1 | 2 | 3 | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 5 | 5 |
| 6 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 6 | 6 |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 7 |
| 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 8 |
| 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

| $T_<$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 5 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| $T_>$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 6 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| 9 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |

| $T_P$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 7 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 8 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 9 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |