The World of Numbers

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WORLD OF NUMBERS INTRODUCTION

By now, veteran *CSMP* students have had a rich variety of experiences in the World of Numbers. They have met and become familiar with various kinds of numbers, and with operations and relations on them. They have encountered positive and negative integers, decimal numbers, fractions, numerical functions (such as 5x, +3, ÷10, –5, ⅔x), order relations (such as < and >), and the notions of multiples and divisors of a given number. They have been introduced to paper-and-pencil algorithms for addition, subtraction, multiplication, and division with whole numbers; for addition, subtraction, and multiplication with decimal numbers; and for multiplication of fractions. They have had extensive experiences involving division in preparation for a more general algorithm for division with decimal numbers. They have used several models for addition and subtraction of fractions in preparation for algorithms. Topics from combinatorics and number theory have provided many interesting problems.

In *CSMP Mathematics for the Intermediate Grades*, *Part V*, these earlier numerical experiences will be revisited, extended, and deepened through familiar games and activities, as well as in fascinating new situations. As always, *CSMP* stresses the unity and continuity of growth of mathematical ideas and concepts. The program's spiral approach does not require mastery of each lesson, but rather allows students to encounter the elements of each content strand in different situations throughout the year. It is important to recognize this approach consciously. If you strive for mastery of each single lesson, you will find yourself involved in a great deal of redundancy as the year progresses.

Further, *CSMP* presents the content in a situational framework. That is, a "pedagogy of situations" engages students in rich problem-solving activities as they construct mathematical ideas. These situations offer opportunities both to develop necessary numerical skills and to gain deeper understanding of mathematical concepts in the world of numbers. At the same time, the situations presented encourage students to develop patterns of logical thinking and strategies for approaching problems.

Perhaps the most important embodiments of the *CSMP* approach are the nonverbal languages and tools used throughout the program. These are vehicles that allow students to investigate the contexts in which the content is presented and to explore new mathematical ideas. It is hard to overstate the value of developing languages and tools that are not confined to one area of mathematical content or to one level of the development of content; that aid in attacking problems as well as in representing situations. Equipped with the universally applicable languages of the *CSMP* curriculum, students grow more and more familiar with the syntax of these languages and are free to explore new content as extensions rather than think of each new mathematical idea as tied to a certain new language. This is not to say that *CSMP* students do not learn the usual descriptive language of mathematics; naturally, they do. However, in the *CSMP* approach the usual descriptive language is not a requisite for learning new concepts, but only a means for succinctly describing those ideas as they are being explored.

The Minicomputer, calculator, strings, and arrows embody three fundamental concepts of mathematics: binary and decimal number systems, sets, and functions. Using these tools and pictorial languages to highlight unifying themes counteracts the tendency to fragment mathematics instruction into a large set of independent topics. For specific examples of the situations and the ways *CSMP* uses instructional tools and nonverbal languages in this strand, we refer you to the brief topic summaries later in this introduction and, in particular, to the lessons themselves.

Standard Algorithms

CSMP seeks to develop basic numerical skills as well as an understanding of the underlying mathematical ideas. We are fully in agreement with the thesis that, along with the growth of understanding of the world of numbers, there must be a concomitant growth of familiarity and facility with numbers and operations on them. But facility should not be confused with understanding; they are partners in the growth of mathematical maturity. A balanced growth of each must be maintained, neither being sacrificed for the other.

Students must eventually learn mechanical algorithms for the basic operations (addition, subtraction, multiplication, division). However, the development of these methods should occur only after students have had many experiences with prerequisite concepts. Premature presentation of these algorithms may actually inhibit a student's desire and ability to develop alternative algorithms, to do mental arithmetic, or to estimate.

CSMP believes that students should be able to solve a problem such as 672 ÷ 32 using models, pictures, or mental arithmetic before being introduced to a division algorithm. Even after students have mastered an algorithm, they should be aware that alternative methods are often more appropriate. For example, consider the problem of calculating 698 x 9. Rather than using a standard multiplication algorithm, it may be easier and more efficient to note that 700 x 9 = 6 300, so that 698 x 9 = 6 300 – 18 = 6 282. Indeed, built into this way of approaching the problem is an excellent estimate (6 300) of the product. To insist on a mechanistic response to such a problem would be to encourage inefficiency and might also inhibit the development of the flexibility necessary for problem solving. On the other hand, a rich array of situations in which students interact with numbers provides them with opportunities to gain the necessary facility with standard algorithmic procedures while retaining the openness required to respond creatively to new situations in the world

Content Overview

Multiplication

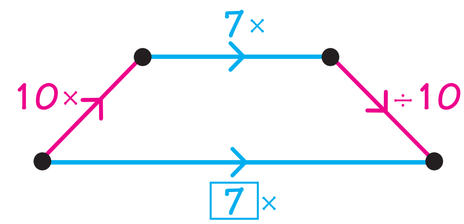
By this time your *CSMP* students are quite familiar with the concept of multiplication and with paper-and-pencil algorithms for multiplying whole numbers, decimals, and fractions. Here, in *IG-V*, both familiar and new situations present many opportunities to review and apply multiplication. Arrow pictures provide an ideal vehicle for developing methods of multiplying both decimal numbers and fractions. Estimation and pattern work strengthen multiplication concepts.

As review, students encounter multiplication in activities such as *Minicomputer Golf*, *Guess My Rule*, detective stories, and calculator puzzles. Multiplication becomes a tool for investigating new topics, for example, counting relations and permutations, Cartesian graphs of various linear relations, or the least common multiple operation. It is also used to decode numbers on a new abacus, and to find ways to display numbers on a calculator with key restrictions. The extent and range of these activities reflect the students' growing confidence with multiplication.

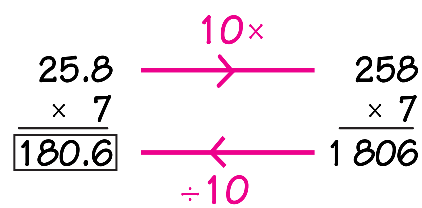
Earlier work with the Minicomputer examined sequences of calculations such as this, introducing multiplication with decimal numbers:

$7 \times 60 = 420$ $\quad$ $7 \times 6 = 42$ $\quad$ $7 \times 0.6 = 4.2$ $\quad$ $7 \times 0.06 = 0.42$

An arrow picture serves to generalize this process. Students first determine that a 7x arrow can be drawn from the beginning dot to the ending dot in this picture. So the result is unchanged by the combined action of the 10x and ÷10 arrows.



This suggests a technique for multiplying with decimal numbers; for example, 7 x 25.8. In other words, the problem 7 x 25.8 can be solved by performing the more familiar calculation 7 x 258, and then dividing the result by 10. While slightly less efficient than the rule "count the decimal places" (which *CSMP* students discover), this technique works well as it enhances understanding of multiplication with decimal numbers.



Lessons on multiplication of rational numbers review the development of a standard algorithm for the multiplication of two fractions, namely,

$$\frac{3}{4} \times \frac{2}{5} = \frac{3 \times 2}{4 \times 5} = \frac{6}{20} = \frac{3}{10}$$

This algorithm is then used to confirm results in multiplication calculations involving decimals. For example,

$$0.3 \times 1.4 = \frac{3}{10} \times \frac{14}{10} = \frac{42}{100} = 0.42$$

Lessons: N1, 2, 3, 5, 6, 8, 9, 11, 13, 17, 18, 20, 21, 25, 31, 34, and 35.

Division

CSMP students already have experience with division as a sharing process (sharing 108 books equally among three classes), as repeated subtraction (finding how many 12s are in 200), and as a multiplication inverse. They have been introduced to an efficient paper-and-pencil algorithm for division with whole numbers. The lessons in *IG-V* extend students' experiences with division to new patterns and applications, and extend the algorithm for division to include decimals.

Division by a fraction may again be viewed as a sharing process or repeated subtraction. For example, students might interpret $6 \div \frac{3}{4}$ as asking how many $\frac{3}{4}$s there are in 6. Through the use of arrow pictures students examine several methods of dividing $6 \div \frac{3}{4}$, such as counting by $\frac{3}{4}$s (with $+\frac{3}{4}$ arrows) from 0 to 6 on a number line or rewriting the problem to multiply $6 \times \frac{4}{3}$. Such experiences lead to the familiar division process for fractions, namely, rewriting a division calculation in an equivalent form with multiplication. For example:

$$\frac{4}{5} \div \frac{5}{6} = \frac{4}{5} \times \frac{6}{5} = \frac{24}{25}$$

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Learning division algorithms is only one part of developing an understanding of the concept of division. Therefore, your students' exposure to division in many contexts continues with activities involving calculators, arrow roads, number lines, fractions, and numerical patterns.

Lessons: N2, 5, 7, 8, 9, 11, 12, 18, 26, 31, 33, 34, and 35.

Negative Integers

CSMP introduces negative integers in first grade through a story about Eli the Elephant and magic peanuts. The story leads to a model for adding integers, first by pictures, then also on the Minicomputer. By the end of fourth grade, *CSMP* students have encountered negative numbers in games, in reading outdoor temperatures, in arrow roads, and in calculator activities. These experiences extend the concept of order from whole numbers to negative numbers, and provide models for the addition of negative numbers.

The activities in this strand increase students' familiarity with negative numbers in many contexts. The goal is to portray negative integers not as a strange new set of numbers, but as a natural and necessary extension of counting numbers. Therefore, few lessons focus on negative numbers but many lessons include them. Negative numbers appear regularly on the Minicomputer in *Minicomputer Golf,* in *The String Game* with numbers, in detective stories, in Cartesian graphs, in *Guess My Rule* activities, in arrow roads, and in calculator activities. Of particular note this semester is the Cartesian graph lesson (N19) that extends the line graphs to include, for example, $\widehat{3}\mathbf{x}$. This model reinforces the fact that a negative number times a negative number is a positive number.

CSMP employs a special notation for representing negative numbers. Traditional approaches to arithmetic often make no distinction on the printed page between the function "subtract 3" and the number "negative 3"; both are denoted by "–3." Only by context can a person discern the intended meaning of "–3." In *CSMP*, negative numbers are distinguished from subtraction in the following ways:

- The minus sign "–" is reserved for subtraction. Thus, for example, "–14" denotes the function "subtract 14."
- The $\frown$ symbol denotes a negative number. Thus, "$\widehat{14}$" denotes the number "negative 14." This symbol was introduced first in the story about Eli the Elephant.
- A raised minus sign may be used when recording a negative number, especially for results obtained from using a calculator. For example, $^{-}14$.

We recommend that you continue to use both the $\frown$ and raised minus notations for negative numbers and recognize alternative notations as students encounter them in other contexts (calculators, temperature, tests, and so on).

Lessons: N1, 3, 12, 13, 17, 19, 21, 27, 28, and 29.

Decimal Numbers

Just as students' confidence with whole numbers requires several years of growth, so must the development of decimal number concepts proceed gradually. The introductory activities in second, third, fourth, and fifth grades rely on money, on the Minicomputer, and on the number line as models for decimal numbers. These three models complement each other. Whereas all facilitate computation, the Minicomputer highlights patterns while the number line and money focus on order and relative magnitude of decimal numbers.

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Reflecting and furthering the students' growing confidence, decimal numbers appear in this semester in activities involving calculators, Cartesian coordinates, *Guess My Rule* situations, arrow diagrams, string pictures, *Minicomputer Golf*, and money problems. These activities require students to perform many computations involving decimal numbers, relying on the various models to confirm their results. For example:

$3 \times (7 \div 2) = 10.5$ $\quad$ $((5 + 6) \div 5) - 2 = 0.2$

$(1.5 \times 0.20) + (2.5 \times 0.40) = 1.30$ $\quad$ $\frac{4}{5} \times 2 = 1.6$

The ability to perform such calculations as well as to order decimal numbers indicate that students can discover and become familiar with the subtleties of decimal numbers without a too early reliance on rules and mechanical manipulation of numbers.

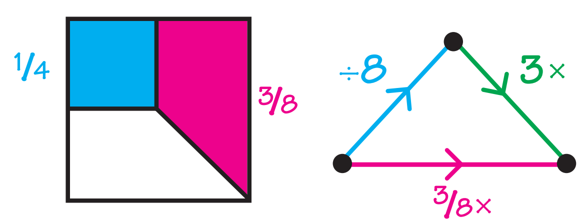
Building on the students' knowledge of various models for fractions, decimal numbers, and division, a goal of this strand is to identify the relationships among these concepts by observing equalities such as $7 \div 5 = \frac{7}{5} = 1.4$.

Students continue to build their understanding of decimal numbers by encountering them in a variety of situations involving estimation and patterns. Moreover, the lessons emphasize relationships among decimal numbers, fractions, and division.

Lessons: N1, 2, 5, 6, 7, 8, 11, 12, 13, 18, 19, 20, 21, 24, 26, 27, 28, 29, 30, and 34.

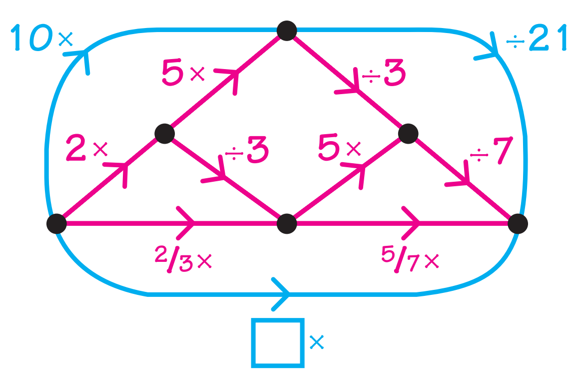
Fractions

The activities involving fractions in *IG-V* reflect *CSMP*'s belief in the spiral approach. Early exposure to fractions began in first grade. From second through fifth grade students have gradually become familiar with two concepts involving fractions: fractional parts of a whole and certain composite functions (for example, $\frac{3}{8}\times$ is the same as $3\times$ followed by $\div 8$).



A variety of activities involving these models has increased student understanding of fractions. This background has prepared students to compute (add, subtract, multiply, and divide) with fractions in *IG-V*. One technique for multiplying fractions relies on the composition of multiplication functions._

Labeling the blue arrow in the diagram below requires the drawing of several detour arrows and the appropriate compositions.

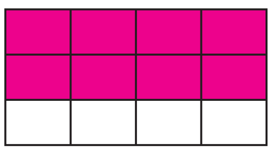


The unlabeled blue arrow is the composition of 10x followed by ÷21 or $\frac{10}{21}$x. Therefore $\frac{2}{3}$x followed by $\frac{5}{7}$x is $\frac{10}{21}$x or, analogously, $\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$. The arrow picture suggests a generalization to $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$, the standard algorithm for multiplying fractions, for example, $\frac{5}{6} \times \frac{11}{20} = \frac{55}{120} = \frac{11}{24}$. Furthermore, students learn to calculate problems such as $5\frac{3}{4} \times 8$ by considering $5 \times 8 = 40$ and $\frac{3}{4} \times 8 = 6$.

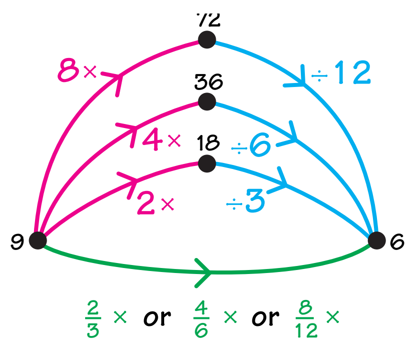
The section on Division in this introduction describes the development of methods for dividing by a fraction. This development follows, of course, the established algorithm for multiplying fractions. With arrow pictures, the students observe the equivalence, for example, of $\frac{2}{3}$x and ÷$\frac{3}{2}$. Thus, they begin to use the familiar algorithm for dividing by fractions, for example,

$$\frac{3}{4} \div \frac{3}{2} = \frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{1}{2}$$

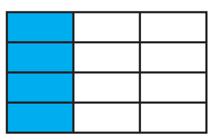
A prerequisite for adding fractions is an understanding of equivalent fractions. Both the area model and arrow pictures suggest that numbers can have different fractional names.



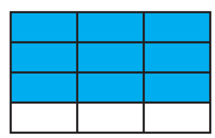
$\frac{2}{3} = \frac{8}{12}$



With the concept of equivalent fractions in hand, students recall a cutting "cakes" method for adding fractions introduced in *IG-IV*. The area model emphasizes the need for equal-sized regions, i.e., for a common denominator.



$\frac{1}{3} = \frac{4}{12}$



$\frac{3}{4} = \frac{9}{12}$

$$\frac{1}{3} + \frac{3}{4} = \frac{4}{12} + \frac{9}{12} = \frac{13}{12} = 1\frac{1}{12}$$

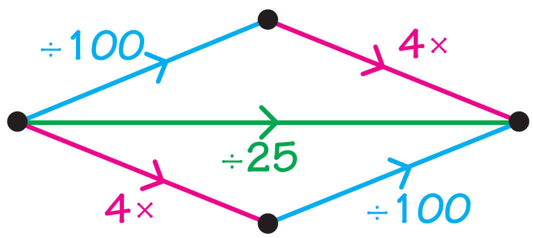
In addition, in order to focus on a common error in adding fractions, students provide arguments based on area or estimation to prove that, for example, $\frac{2}{5} + \frac{3}{4} \neq \frac{2+3}{5+4}$ or $\frac{5}{9}$. In *IG-V*, students also review calculations such as $1\frac{4}{5} + \frac{3}{5} = 2\frac{2}{5}$ and $4\frac{1}{5} - 1\frac{3}{5} = 2\frac{3}{5}$.

The section on Decimals in this introduction mentions methods for changing fractions to decimal numbers and vice-versa; for example, $1.4 = 1\frac{2}{5} = \frac{7}{5} = 7 \div 5$. These equivalences are used continuously to reinforce results of calculations with fractions and with decimals.

Lessons: N4, 6, 8, 9, 10, 12, 15, 20, 22, 24, 28, 31, 32, 33, and 35.

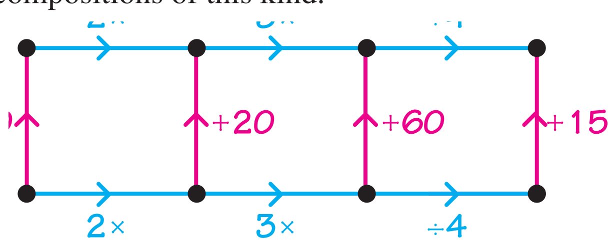
Composition of Functions and Percent

Several lessons in this strand deal with what happens when you compose a sequence of functions, that is, when you apply the functions in order, one at a time. These compositions lead to many powerful insights into the properties of numbers and operations. Arrow diagrams provide a concrete means to study this abstract but practical concept. For example, if you divide any number by 100 and then multiply the result by 4, the net effect is the same as dividing the original number by 25.



Besides depicting the composition, the arrow picture also suggests that an easy way to divide any number by 25 is to divide by 100 and then multiply by 4 or vice-versa; the two operations can often be performed mentally. For example, 4 x 63 = 252 and 252 ÷ 100 = 2.52, so 63 ÷ 25 = 2.52.

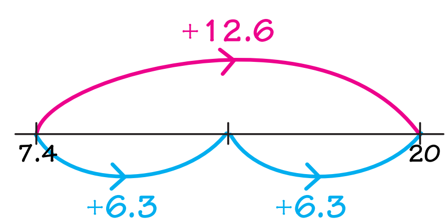
Many pairs of functions *commute*. That is, they produce the same effect regardless of the order in which they are applied. $\frac{2}{3}$x can be interpreted as 2x followed by ÷3 or as ÷3 followed by 2x; +98 can be interpreted as +100 followed by –2 or as –2 followed by +100. However, other pairs of functions, for example, +10 and 2x, do not commute. Yet patterns do exist—students find, for example, that +10 followed by 2x has the same effect as 2x followed by +20. The following arrow picture depicts several compositions of this kind.



From the students' perspective, they are solving challenging problems and discovering new numerical patterns. From the mathematical viewpoint, they are investigating the commutative and distributive properties.

The composition of functions can lead to insights in many problem-solving situations. In *Minicomputer Golf*, the use of composition aids in finding solutions to problems requiring that students move two checkers to produce a specified change. In the section on Fractions in this introduction, an arrow picture involving compositions supports an algorithm for multiplying fractions.

The idea of composition also facilitates finding the midpoint of two numbers on a number line.



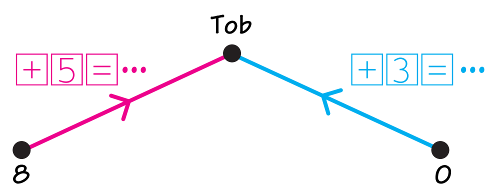
The composition of functions shows how the language of arrows is able to visually highlight rich and practical mathematical concepts and techniques. For example, three lessons this semester introduce the concept of percent as a composite function; that is, "n% of" is nx followed by ÷100, or ÷100 followed by nx. In this context, the exercises investigate many useful names for certain percents (for example, $20\% = \frac{2}{10} = \frac{1}{5} = 0.2$), several interesting patterns, and some helpful properties. The lessons all include some applications of percent in the solution of real life problems.

Lessons: N1, 5, 6, 9, 11, 15, 21, 23, 30, 31, 34, and 35.

Multiples and Divisors

The study of multiples and divisors leads to practical applications such as the addition of fractions, as well as to investigations of many fascinating properties of numbers. Your students have had many earlier experiences finding multiples and divisors of whole numbers, and through string pictures they have encountered the notions of common multiples and common divisors. In *IG-V*, calculator activities, arrow diagrams, and string pictures provide further opportunities to explore common multiples and common divisors. Calculator relations, for example $\boxed{+}\ \boxed{7}\ \boxed{=}$ … and $\boxed{-}\ \boxed{4}\ \boxed{=}$ . . ., provide a means for posing and studying problems suggested in arrow pictures.

Students use calculators to generate a list of possibilities for Tob, namely, 18, 33, 48, 63, 78, 93, …, and they note that consecutive numbers in the list differ by 15, a common multiple of 5 and 3. Further investigation reveals that this is not a coincidence.



To gain insight into the least common multiple and greatest common divisor operations, students study problems such as $10 \sqcup \square = 30$ and $24 \sqcap \square = 3$ where $\sqcup$ and $\sqcap$ are the symbols for "least common multiple" and "greatest common divisor," respectively.

Common multiples also appear in the study of fractions and in the introduction to modular arithmetic. As preparation for the addition of fractions, arrow pictures and activities involving the fair division of rectangular cakes both lead students to generate lists of equivalent fractions. Students recognize the role of common multiples in determining the numerators and denominators of equivalent fractions.

Problems involving adding two multiples of 3 or two non-multiples of 3, for example, and a new relation involving the sum of a number's digits lead to common tests for divisibility by 3 and by 9. Properties of multiples and divisors contribute to an examination of several other divisibility tests. A probability problem suggests that certain divisibility tests be used in its solution.

A *Guess My Rule* game introduces exponential notation and leads to a beginning study of prime factorization. In another *Guess My Rule* context, students receive their first introduction to rounding as finding the closest multiple of some number. Later this experience with rounding is extended to include rounding to the nearest n, where n can be any rational number.

Lessons: N1, 3, 14, 16, 21, 25, 27, 29, and 32.

Cartesian Graphs

Several lessons this semester review the idea of graphing relations in a two dimensional coordinate system. The resulting graphs are called *Cartesian graphs* because one commonly refers to the coordinate system as the *Cartesian plane*†. Graphs for several kinds of relations, for example, +n, nx, "distance from 0," and "is less than," provide opportunities to make observations which distinguish the Cartesian graphs of functions from those of non-functions, and which describe the effect of modifying an algebraic expression. In connection with studying the rounding operation, students graph ®4 and, thereby, discover a step function.

The lesson on the linear relations (nx) describes a model for studying these multiplication functions and, in particular, reinforces the fact that a negative number times a negative number is a positive number.

†*Cartesian* is a name honoring the French mathematician René Descartes (1596-1650) for his unification of algebra and geometry in the creation of analytic geometry.

Capsule Lesson Summary

Decide which numbers in a given list can be put on the Minicomputer using exactly one of the weighted checkers: ②, ③, ④, ⑤, ⑥, ⑦, ⑧, and ⑨. Put the other numbers in the list on the Minicomputer using exactly two specified checkers. Following an arrow road, make changes to a number on the Minicomputer by moving, for each arrow, exactly one checker from one square to another. Play *Minicomputer Golf*.

Materials

Teacher
- Minicomputer set
- Weighted Checker set
- Colored chalk

Student
- Paper

Description of Lesson

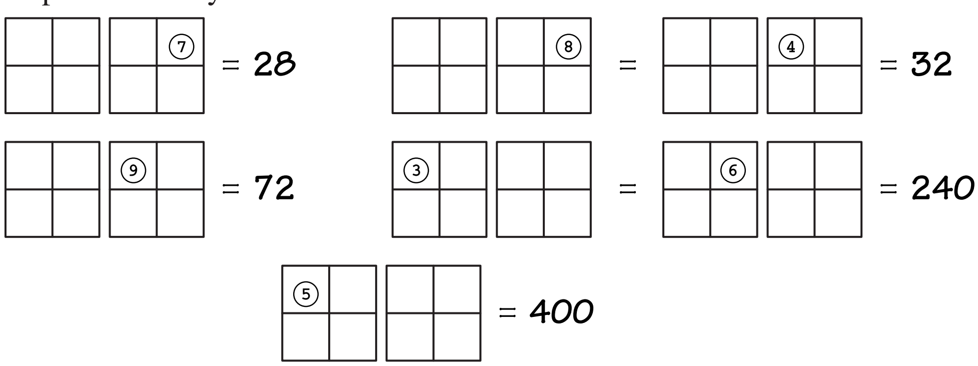
Exercise 1

Display two Minicomputer boards and the weighted checkers: ②, ③, ④, ⑤, ⑥, ⑦, ⑧, and ⑨. Write this list of numbers on the chalkboard.

28 32 54 72 240 400 420

T: ***Find all of the numbers in this list that can be put on the Minicomputer using exactly one of these weighted checkers.***

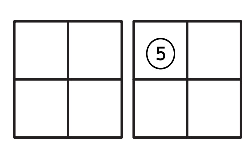
Invite students to put numbers on the Minicomputer using exactly one of the displayed checkers. Each time, ask the rest of the class to check that the number displayed is indeed one of the numbers in the list. Continue until the class finds that five numbers (28, 32, 72, 240, and 400) can be put on the Minicomputer this way.



The class should conclude that 54 and 420 cannot be put on the Minicomputer with exactly one of the displayed checkers.

T: ***On your paper show how 54 can be put on the Minicomputer using a ⑤-checker and a ⑦-checker.***

Let students work independently for a couple minutes; then ask a student to place one checker on the Minicomputer. Suppose it is a ⑤-checker.



T: ***Which number is on the Minicomputer?***

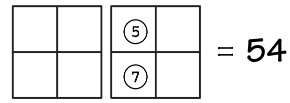
S: ***40, because 5 x 8 = 40.***

T: ***How much greater is 54 than 40?***

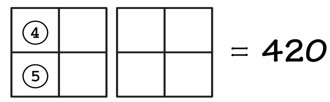
S: ***14 more.***

T: ***Can someone put 14 on the Minicomputer using a ⑦-checker?***

Invite a student to place the ⑦-checker.



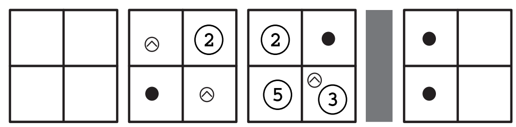
Continue by instructing students to show 420 on the Minicomputer using a ④-checker and a ⑤-checker.



Exercise 2

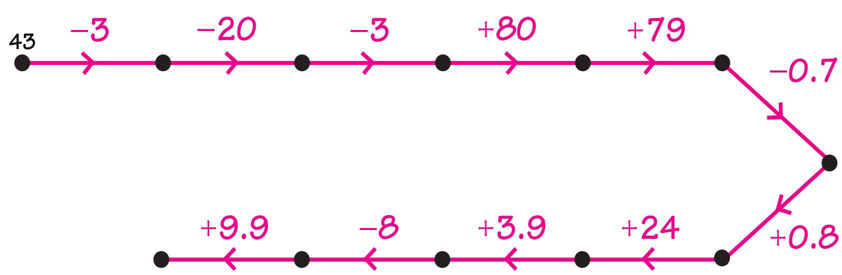
Put this configuration on the Minicomputer.

T: ***Write this number on your paper.***



Allow a few minutes for students to decode the number on the Minicomputer while you draw the following arrow road on the board.

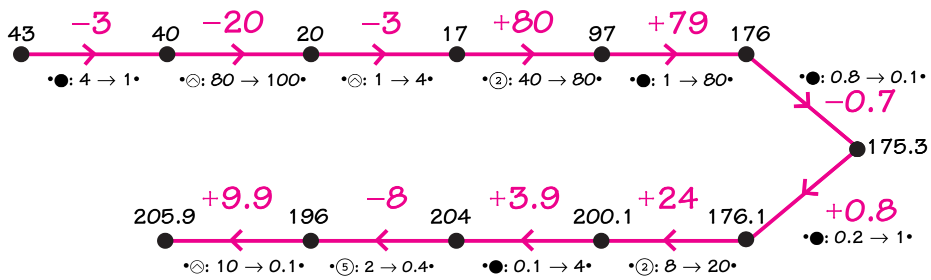
Check several papers before asking a student to announce the number. (43) You may ask the class to collectively decode the number on the Minicomputer. Label the first dot of the arrow road with this number.



T: ***This arrow road starts at the number on the Minicomputer. Each arrow tells us how to change the number, but we will only move one checker to make a change.*** (Point to the first arrow.) ***What move will decrease the number on the Minicomputer by 3, that is, make it 3 less?***

Invite a student to make a move on the Minicomputer. Then ask the class for verification and for the new number on the Minicomputer. Label the next dot, and continue in the same manner until you reach the end of the arrow road.

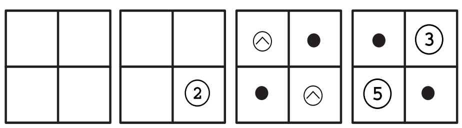
The following illustration has all the dots labeled and indicates possible moves for each arrow. For example, (⊗: 1→4) indicates moving a negative checker from the 1-square to the 4-square.



Suggest that students check the configuration on the Minicomputer to verify that the ending number is 205.9.

Exercise 3

Play *Minicomputer Golf* with this starting configuration and goal.



= 201 Goal: 1000

Writing Activity

You may like students to take lesson notes on some, most, or even all their math lessons. The "Lesson Notes" section in Notes to the Teacher gives some suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students may explain how to decide which numbers can be put on the Minicomputer using exactly one weighted

Practice/Assessment Activity

You may like to give students additional fact practice with weighted checkers on the Minicomputer. Blank Minicomputer pages are available in the Blackline section and may be used to prepare additional problems.

Home Activity

If you believe parents/guardians would benefit from a reminder about the Minicomputer, you may like to send home an introductory letter. Blacklines N1(a), (b), and (c) have a sample letter together with a home Minicomputer. Send home a description of *Minicomputer Golf*, and suggest that students play the game with family members.

N2 DECIMALS #1

Capsule Lesson Summary

Play a secret number game on the number line with the secret number having a non-zero hundredths place. Label parts of a number line with graduations in hundredths and thousandths. Observe patterns involving division by 10 to support calculations with decimals. Use estimation procedures and a calculator to solve missing addend problems.

Materials

Teacher
- Colored chalk
- Straightedge

Student
- Paper
- Colored pencils, pens, or crayons
- Calculator
- Worksheets N2(no star), N2*, **, and ***

Description of Lesson

Exercise 1

Draw this part of a number line on the board. Place a dot approximately between 14 and 15.

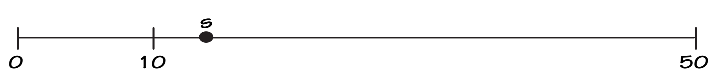
T: ***This dot is for a secret number. What number do you think is the secret number?***

Note: The secret number is 14.62.

For each incorrect guess, announce whether the guess is more or less than the secret number, and locate approximately the guess on the number line. For example:

S: ***10.***

T: ***10 is less than my secret number.***



Continue taking guesses until the class finds that the secret number is between 14 and 15.

T: ***What are some numbers between 14 and 15?***

S: ***14.1, 14.5, and 14.8.***

S: ***$14\frac{1}{3}$ and $14\frac{1}{4}$.***

Draw an enlargement of the part of the number line between 14 and 15.

T: ***I have enlarged the part of the number line between 14 and 15 and have indicated the location of the secret number. Try again to guess the secret number. Please give your guesses as decimal numbers.***

As before, announce how incorrect guesses compare (more or less) to the secret number, and locate them on the number line. Continue until the class finds that the secret number is between 14.6 and 14.7. Then enlarge that part of the number line on the board.

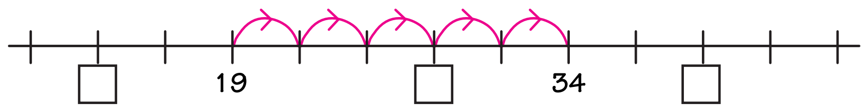
Continue in a similar manner until a student guesses the secret number, 14.62.

Exercise 2

Distribute Worksheet N2 (no star). Draw the first number line picture from the worksheet on the board. Point to the box between 19 and 34.

T: ***What number is this?***

Let students discuss the problem. Once they realize that they must find five equal increments between 19 and 34, draw these arrows.



S: ***28 is in that box (between 19 and 34).***

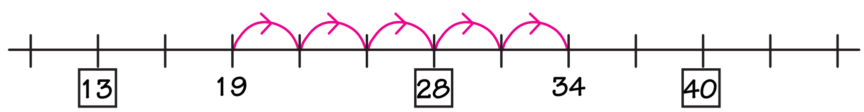
T: ***Why?***

S: ***Red arrows are for +3, so the marks between 19 and 34 are for 22, 25, 28, and 31.***

T: ***How did you decide that the red arrows are for +3?***

S: ***19 + 15 = 34. Therefore, five red arrows must add 15. Since 5 x 3 = 15, the red arrows are for +3.***

Invite students to fill in the other boxes.



Instruct students to complete the worksheet individually or with a partner. As they work, you may wish to offer the following hints:

- **Problem 2** (given $\overset{\frown}{7}$ and 13): Suggest drawing a blue arrow from $\overset{\frown}{7}$ to 13. Label the blue arrow (+20), and then find that red arrows between successive marks would be for +4.

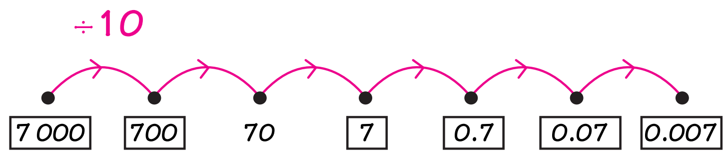
- **Problem 3** (given 5.4 and 5.6): Interpret 5.4 as $5.40 and 5.6 as $5.60.

- **Problem 4** (given 4.33 and 4.34): Rename 4.33 and 4.34 as 4.330 and 4.340, respectively.

After a while, solve the problems collectively at the board. Ask various students to explain their methods of solution. An answer key for this worksheet is at the end of the lesson description.

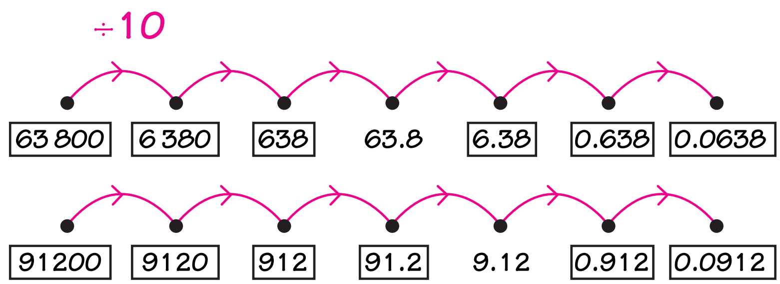
Exercise 3

Draw the following arrow road on the board and call on students to label the dots. (Answers are in boxes.)

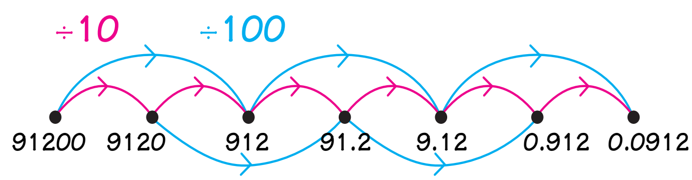


If appropriate, refer to money to support calculations such as 7 ÷ 10 = 0.7 and 0.7 ÷ 10 = 0.07; that is \$7.00 ÷ 10 = \$0.70 and \$0.70 ÷ 10 = \$0.07. Note that the return arrow for ÷10 is 10x. Encourage students to describe their methods for dividing by 10 and for multiplying by 10.

Present the following arrow roads in a similar manner. (Answers are in boxes.)



After labeling the dots on the arrow road that passes through 9.12, invite students to draw blue arrows for ÷100.



Point out that ÷100 is the same as ÷10 followed by ÷10. Review this idea with two more problems.

T: ***What number is 46 000 ÷ 100?***

S: ***460, since 46 000 ÷ 10 = 4 600 and 4 600 ÷ 10 = 460.***

T: ***What number is 591 ÷ 100?***

S: ***5.91, since 591 ÷ 10 = 59.1 and 59.1 ÷ 10 = 5.91.***

$$46\,000 \div 100 = 460$$

$$591 \div 100 = 5.91$$

Exercise 4

Write this set of problems on the board and ask students to copy and complete them on a piece of paper. (Answers are in boxes.)

$$5 \times 6 = \boxed{30} \qquad 5 \times 0.006 = \boxed{0.03}$$
$$5 \times 0.6 = \boxed{3} \qquad 50 \times 0.6 = \boxed{30}$$
$$5 \times 0.06 = \boxed{0.3} \qquad 50 \times 0.06 = \boxed{3}$$

If appropriate, encourage some students to think of money. For example, a student could interpret 5 x 0.6 as "five times six dimes" or "five times sixty cents." Also suggest that students look for patterns among the problems and the solutions.

Check the problems collectively, inviting students to explain their solutions. Encourage a variety of explanations; for example:

S: ***5* x *0.06 = 0.30 or 0.3 since five times six cents equals thirty cents.***

S: ***5* x *0.006 = 0.03. I noticed a ÷10 pattern in the problems 6 → 0.6 → 0.06 → 0.006 and in the solutions 30 → 3 → 0.3 → 0.03.***

S: ***50* x *0.6 = 30. 50* x *6 = 300, so 50* x *0.6 = 300 ÷ 10 = 30.***

S: ***50* x *0.06 = 3 since 5* x *0.06 = 0.3 and 10* x *0.3 = 3. 50*x *is the same as 5*x *followed by 10*x.**

Exercise 5

Write these problems on the board.

$4.3 + \square = 10$
$4.36 + \square = 10$
$4.362 + \square = 10$

T: ***Use your calculator to help fill in the boxes.***

Let students work independently or with partners. Encourage estimation and trial-and-error methods. For example, in the first problem, students might note that since 4.3 + 5 = 9.3 and 4.3 + 6 = 10.3, then the answer must be between 5 and 6. You may suggest that students think about the calculations as money problems.

After a while, check the problems collectively (solutions are 5.7, 5.64, and 5.638, respectively), and invite a variety of explanations. For example,

S: ***4.3 + 5.7 = 10. Think of 4.3 as $4.30 and 5.7 as $5.70; then $4.30 + $5.70 = $10.00.***

S: ***4.3 + 5.7 = 10 since 4 + 5 = 9, 0.3 + 0.7 = 1, and 9 + 1 = 10.***

Give students an opportunity to discuss other methods for solving these problems. For example:

- A student might suggest writing a problem in a vertical format.

$$\begin{array}{r} 4.36 \\ +\boxed{5.64} \\ \hline 10.00 \end{array}$$

S: ***Start from the right. The right-most digit must be 4 since 6 + 4 = 10. The next two column sums must be 9, so 6 (3 + 6 = 9) and 5 (4 + 5 = 9) are needed.***

- A student might suggest using subtraction to solve the problems.

$10 - 4.3 = 5.7$
$10 - 4.36 = 5.64$
$10 - 4.362 = 5.638$

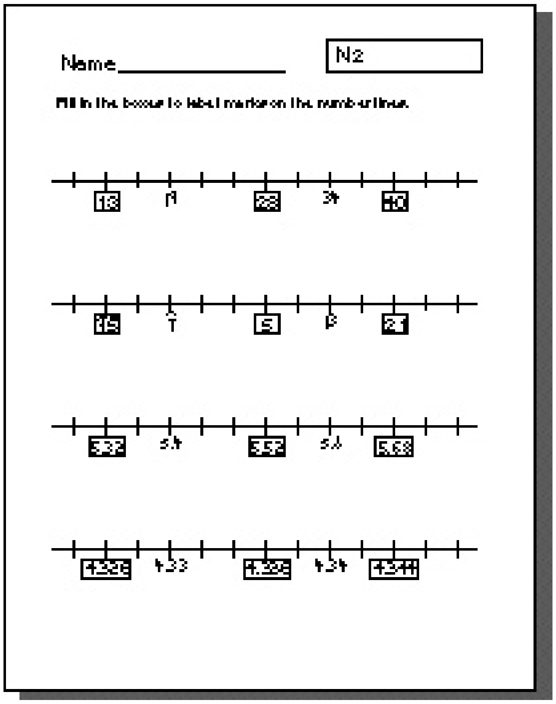
Worksheets N2*, **, and *** are available for individual work.

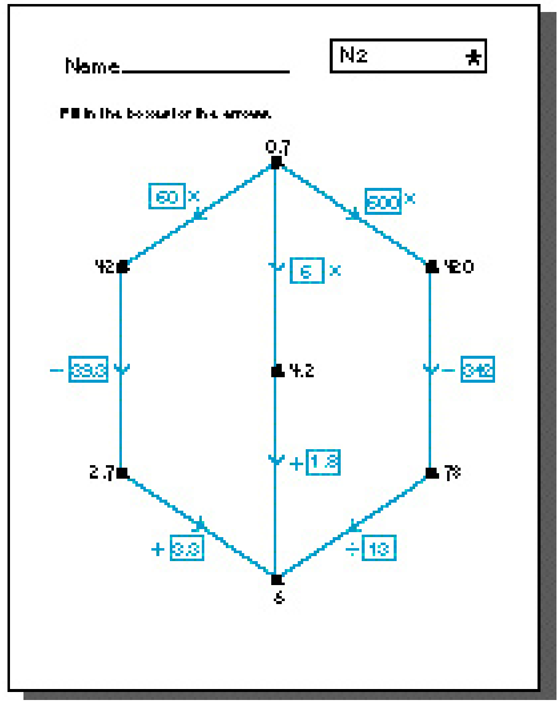
Additional Practice

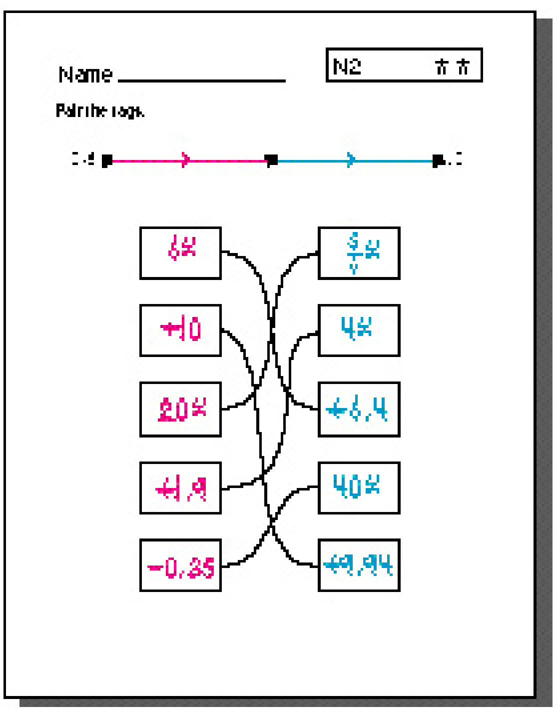
Variations of Exercise 1, 3, or 4 may be good practice at times when the class has a few minutes to fill.

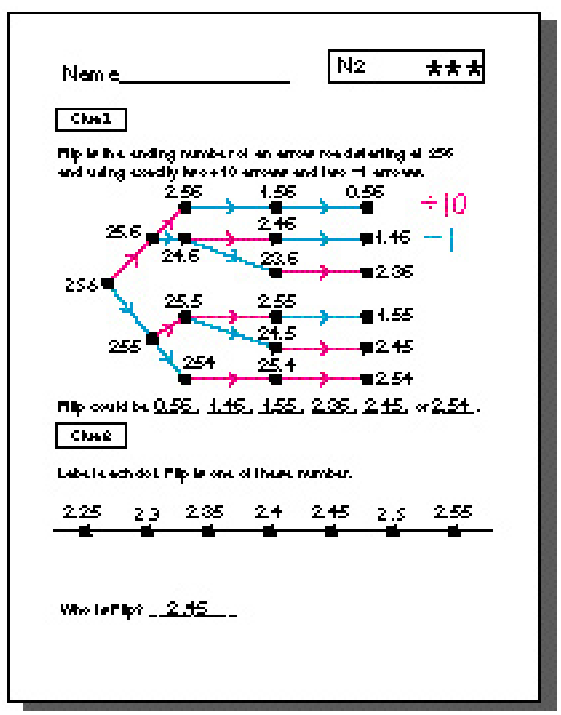
Home Activity

This would be a good time to send a letter to parents/guardians about mental arithmetic. Blackline N2 has a sample letter.


Name
N2
Fill in the boxes to label marks on the number lines.


Name
N2
Fill in the boxes for the arrows.


Name
N2


Name
N2
Clue 1
Clue 2
Who is Flip? 2.45

N3 WHO IS TOB?

Capsule Lesson Summary

Find the greatest number and the least number that can result from combining three numbers and two operations. Present a detective story with clues involving calculator relations, parentheses, and weighted checkers on the Minicomputer.

Materials

Teacher
- Colored chalk
- Minicomputer set
- Weighted checker set
- Calculator

Student
- Paper
- Calculator
- Worksheets N3*, **, and ***

Description of Lesson

Exercise 1

Write these numerals and symbols on the board.

25 8 6 × – ( )

T: ***What is a number we can name using all of these symbols and using each exactly once?***

S: ***50; (8 – 6) x 25 = 50.***

Record a student's suggestion on the board.

$$(8 - 6) \times 25 = 50$$

T: ***What is the greatest number we can name with these symbols? … the least?***

Allow a few minutes for students to work on these problems. If many students stop looking for a number less than 23, announce that 23 is not the least possible number.

For your information, there are twelve numbers that can be named with the given symbols, each used exactly once. One way to name each number is given below.

$194 = (8 \times 25) - 6$	$\widehat{194} = 6 - (25 \times 8)$	$102 = 6 \times (25 - 8)$	$\widehat{102} = (8 - 25) \times 6$
$152 = (25 - 6) \times 8$	$\widehat{152} = 8 \times (6 - 25)$	$50 = 25 \times (8 - 6)$	$\widehat{50} = 25 \times (6 - 8)$
$142 = (25 \times 6) - 8$	$\widehat{142} = 8 - (25 \times 6)$	$23 = (8 \times 6) - 25$	$\widehat{23} = 25 - (8 \times 6)$

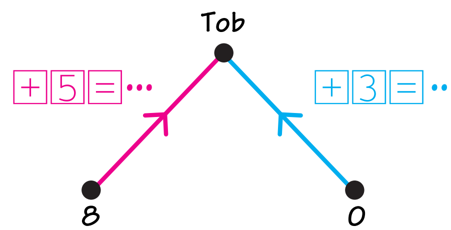
Once the class has discovered the greatest such number (194) and the least ($\widehat{194}$), ask for another negative number that can be named with the symbols and for the least positive number (23).

Exercise 2

Present the following detective story about a secret number named Tob. You may need to begin with a brief review of calculator relations. Do this by asking students to explain the red and blue arrows in the first clue.

Clue 1

Draw this arrow picture on the board.



T: ***Tob is a secret number. Our first clue is this arrow picture. Who could Tob be?***

S: ***33.***

T: ***Let's check. If we put 8 on the display of the calculator and press*** $\boxed{+}\ \boxed{5}\ \boxed{=}$ ***…, how many times do we need to press*** $\boxed{=}$ ***to get 33?***

S: ***Five times.***

Put 33 next to Tob in the picture.

T: ***If we put 0 on the display of the calculator and press*** $\boxed{+}\ \boxed{3}\ \boxed{=}$ ***…, how many times do we need to press*** $\boxed{=}$ ***to get 33?***

S: ***Eleven times.***

Continue taking suggestions for numbers Tob could be. List the suggested numbers in increasing order. After several have been recorded, interrupt the activity to ask the class what they know about Tob. From the red arrow we know Tob's ones digit is 3 or 8, and from the blue arrow we know that Tob must be a multiple of 3. If students notice a +15 pattern in the list, ask them to use it to predict the next possibility for Tob; then check the number using a calculator.

Continue until the list extends to at least 138.

Tob: 18, 33, 48, 63, 78, 93, 108, 123, 138, …

Clue 2

$$9 \times 4 \times 8 - 2$$

Write this expression on the board.

T: ***By adding parentheses to this expression, we can get a name for Tob. Let's find some numbers Tob could be. There are three operations in the expression. Which do you want to do first?***

A sample dialogue follows.

S: ***Multiplication.***

$$9 + (4 \times 8) - 2$$

T: ***So let's put red parentheses around 4 x 8.***

	Now what?
S:	*Addition.*
T:	*Let's put blue parentheses to indicate we want to do addition next.*

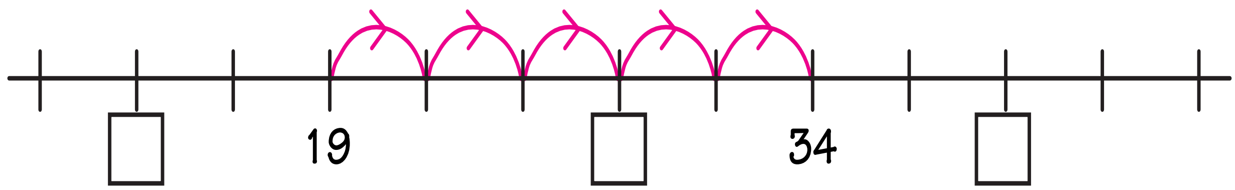
$$(9 + (4 \times 8)) - 2$$

Subtraction is the only choice left. It is not necessary to put in more parentheses. What number does this expression name?

$$(9 + (4 \times 8)) - 2 = 39$$

S: *39.*

Continue finding all of the possible names for Tob that could result from this clue. Always ask which operation a student wishes to do first, to do second, and to do third. A complete list is shown below.



T: *Using what we know about Tob from the first clue, which of these numbers could be Tob?*

S: *33 and 78.*

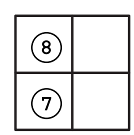
Clue 3

Display one Minicomputer board and these three checkers: ⑦ ⑧ ⑨

T: *Tob can be put on the ones board of the Minicomputer using exactly two of these checkers.*

S: *Tob is 78.*

Invite a student to put Tob on the Minicomputer.



= 78

T: *Can Tob be 33? Why or why not?*

S: *No, we would have to put ⑦ or ⑨ on the 1-square to get an odd number. 33 – 7 = 26 and 33 – 9 = 24. There is no way to put either 26 or 24 on the Minicomputer with any one of those three checkers. 26 is not a multiple of 8 or of 9; 24 is a multiple of 8, but there is no 3-square on the Minicomputer.*

You may need to help the class see why Tob cannot be 33.

Worksheets N3*, **, and *** are available for individual work.

Writing Activity

Students who like to write detective stories may like to write a **** worksheet with a detective story involving weighted checkers on the Minicomputer, expressions with parentheses, or calculator relations.

Name________________ N3 ★

Complete these number sentences.

$(8 \times 6) + (4 \div 2) = \underline{50}$

$(8 \times 6) + (4 \div 2) = \underline{50}$

$(8 \times (6 + 4)) \div 2 = \underline{40}$

$((8 \times 6) + 4) \div 2 = \underline{26}$

$8 \times ((6 + 4) \div 2) = \underline{40}$

$8 \times (6 + (4 \div 2)) = \underline{64}$

Name________________ N3 ★★

Fig is a secret number.

Clue 1

Parentheses are missing from this name for Fig.

$2 \times 3 + 4 \times 5$

Show all of the possible ways to put parentheses in this expression and find the numbers that Fig could be.

$2 \times (3 + (4 \times 5)) = 46$

$2 \times ((3 + 4) \times 5) = 70$

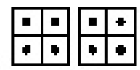
$((2 \times 3) + 4) \times 5 = 50$

$(2 \times (3 + 4)) \times 5 = 70$

$(2 \times 3) + (4 \times 5) = 26$

Fig could be __26__, __46__, __50__, or __70__.

Clue 2



By moving exactly two of these checkers to other squares, you will find Fig.

Fig could be __70__ or __50__.

Clue 3

●: 80 ÷ 1 ●: 80 ÷ 1
●: 20 ÷ 4 ●: 40 ÷ 4

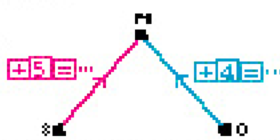
Fig is not a positive divisor of 200.

Who is Fig? __70__

Name________________ N3 ★★★

Fil is a secret number.

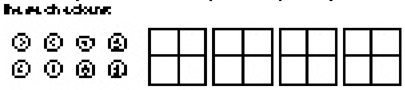
Clue 1



Fil could be 28, 48, 68, 88, 108, 128, 148, 168, and so on.

Clue 2

Fil can be put on the Minicomputer using exactly one of these checkers:



Fil could be __28__ or __48__.

Clue 3

A name for Fil can be written using four 7s and these symbols:

$\times \quad - \quad (\quad)$

Who is Fil? __28__

Write a name for Fil using four 7s and these symbols. $\underline{77 - (7 \times 7)}$

Capsule Lesson Summary

Locate fractions on a number line using two different names, for example, $\frac{23}{5} = 4\frac{3}{5}$. Order a set of numbers given as fractions with different denominators. Use a circle model to support adding and subtracting fractions.

Materials

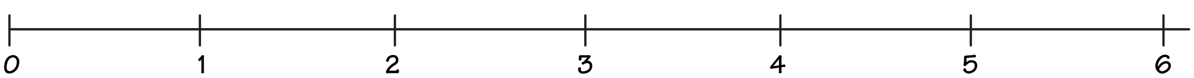
Teacher
- Colored chalk
- Metric ruler

Student
- Colored pencils, pens, or crayons
- Worksheets N4*, **, and ***

Description of Lesson

Exercise 1

Draw this part of a number line on the board, making the distance between marks a multiple of 5 cm. For example, 30 cm between marks is a convenient length.



T: ***Who can locate $\frac{3}{5}$ on this number line?***

If someone suggests that $\frac{3}{5}$ is between 3 and 4, point out that $\frac{3}{5}$ is not 3.5.

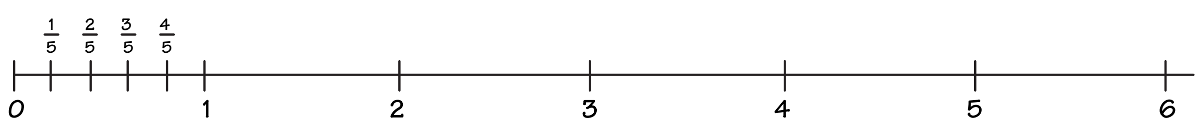
S: ***$\frac{3}{5}$ is between 0 and 1.***

S: ***Divide the segment from 0 to 1 into five equal parts. The third mark to the right of 0 is for $\frac{3}{5}$.***

T: ***The distance between 0 and 1 is 30 cm. Where is $\frac{3}{5}$?***

S: ***18 cm from 0. Since $30 \div 5 = 6$, make marks at 6 cm, 12 cm, 18 cm, and 24 cm from 0. Since $3 \times 6 = 18$, $\frac{3}{5}$ is at 18 cm.***

Label the mark for $\frac{3}{5}$, and then invite students to label the other marks between 0 and 1.



T: ***Where is $\frac{5}{5}$?***

S: ***$\frac{5}{5} = 1$.***

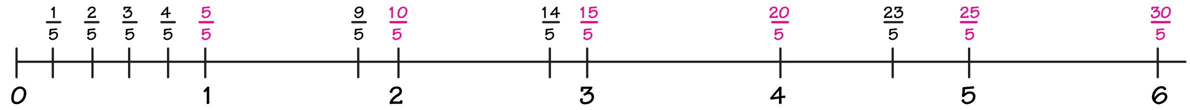
Label the 1 mark for $\frac{5}{5}$ above the line.

Write these fractions on the board.

$$\frac{15}{5} \qquad \frac{23}{5} \qquad \frac{14}{5}$$

N4

Call on students to select a fraction, accurately locate it on the number line, and explain how they found the location. Encourage class discussion and, at an appropriate time, lead to the idea of labeling (above the line) the 2 mark for $\frac{10}{5}$, the 3 mark for $\frac{15}{5}$, the 4 mark for $\frac{20}{5}$, and so on.



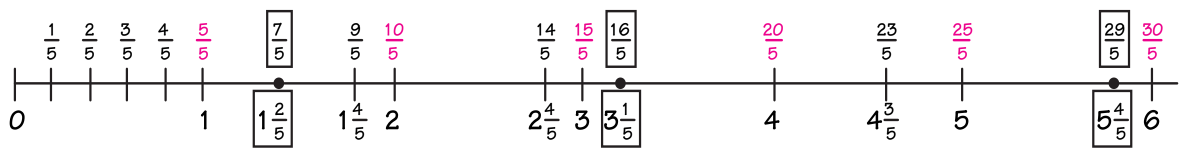
T: ***On the number line, we have two labels for some marks: 1 = $\frac{5}{5}$, 2 = $\frac{10}{5}$, 3 = $\frac{15}{5}$, and so on. Similarly, fractions like $\frac{9}{5}$, $\frac{14}{5}$, and $\frac{23}{5}$ have other names. What is another name for $\frac{23}{5}$***

S: ***$4\frac{3}{5}$, since $\frac{23}{5}$ is $\frac{3}{5}$ more than 4.***

Of course there are many names for $\frac{23}{5}$, but use the number line to suggest this "mixed" name with as great a whole number part as possible.

Similarly, let students determine that $\frac{9}{5} = 1\frac{4}{5}$, and $\frac{14}{5} = 2\frac{4}{5}$.

Use a ruler to accurately place dots for $1\frac{2}{5}$, $3\frac{1}{5}$, and $5\frac{4}{5}$ on the number line. Call on students to give at least two labels for each dot. (Answers are in boxes.)

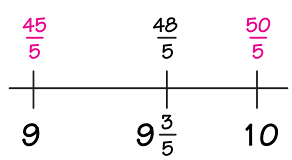


T: ***Between what two whole numbers is $\frac{48}{5}$ located?***

S: ***Between 9 and 10 since 9 = $\frac{45}{5}$ and 10 = $\frac{50}{5}$.***

S: ***Between 9 and 10 since 48 ÷ 5 is 9 with a remainder of 3.***

Draw part of a number line between 9 and 10 on the board. Invite students to locate $\frac{48}{5}$ and to suggest another label ($9\frac{3}{5}$) for the mark.



T: ***In locating $\frac{48}{5}$ on the number line, what does the denominator† 5 suggest to us?***

S: ***Consider five equal length "parts" or "steps" between each two consecutive whole numbers.***

S: ***Mark the number line for counting by one-fifths.***

T: ***What does the numerator 48 suggest to us?***

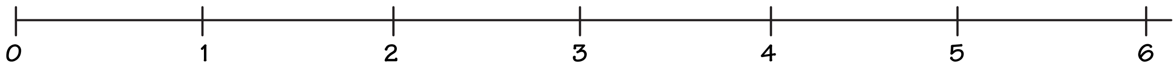
S: ***Count 48 steps from 0.***

---

†Define *numerator* and *denominator* if your students are unfamiliar with the terminology. Use these terms throughout the lessons on fractions.

Exercise 2

Erase all but the whole number marks on the number line on the board. List five fractions near the number line but not on the number line.



$$\frac{7}{8} \qquad \frac{23}{10} \qquad \frac{3}{2} \qquad \frac{13}{4} \qquad \frac{13}{3}$$

Ask which of these five fractions is the least and which is the greatest. Let students discuss the questions but do not reveal correct answers yet. After a while, write this expression under the list of fractions.

$$0 < \boxed{} < 1 < \boxed{} < 2 < \boxed{} < 3 < \boxed{} < 4 < \boxed{} < 5$$

T: ***Each of these five fractions belongs in one of the boxes. For example, one of these fractions is between 0 and 1, another is between 1 and 2, and so on.***

Invite students to select a fraction, put it in one of the boxes, and explain why it is correct. Encourage using the number line for an explanation; for example:

S: ***For $\frac{13}{4}$ think of marking the number line for counting by one-fourths.***

S: ***$\frac{13}{4}$ is between 3 and 4, since $\frac{12}{4} = 3$ and $\frac{16}{4} = 4$. In fact, $\frac{13}{4} = 3\frac{1}{4}$.***

Continue until all five boxes are correctly filled.

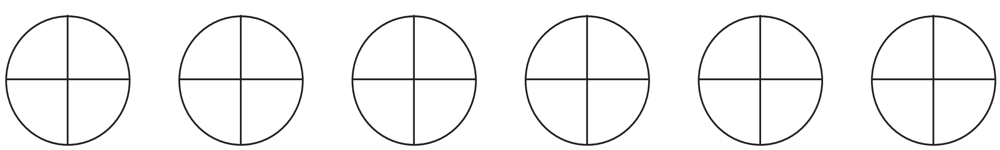
$$0 < \boxed{\frac{7}{8}} < 1 < \boxed{\frac{3}{2}} < 2 < \boxed{\frac{23}{10}} < 3 < \boxed{\frac{13}{4}} < 4 < \boxed{\frac{13}{3}} < 5$$

Confirm that $\frac{7}{8}$ is the least and $\frac{13}{3}$ is the greatest of the five fractions. You may wish to discuss students' methods for comparing fractions.

Exercise 3

Note: You may prefer to use a fraction manipulative with circles divided into quarters rather than to draw them on the board.

Draw these divided circles on the board.

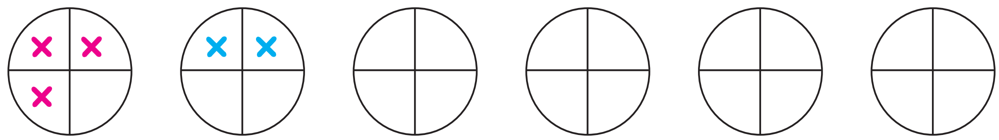


T: ***Let's pretend these are six pies each cut fairly into four pieces. What fractional part of a whole pie is each piece?***

S: ***$\frac{1}{4}$.***

T: ***We say that the pies are divided into fourths. A simple picture like this can be very helpful in adding and subtracting fractions. How could we use the pieces to help us add $\frac{3}{4} + \frac{1}{2}$?***

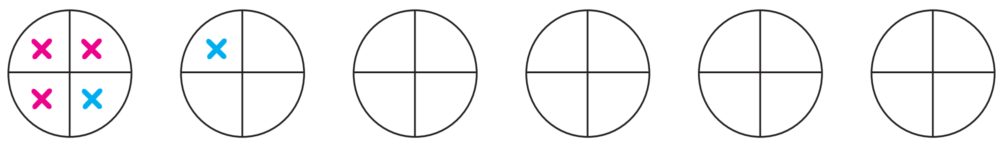
Let students express their opinions. Invite one student to draw red marks in three-fourths of a pie and another student to draw blue marks in one-half or two-fourths of a pie. For example:



T: ***How much pie is marked?***

S: ***$\frac{5}{4}$ or $1\frac{1}{4}$.***

S: ***To see $1\frac{1}{4}$ more clearly, we could move one blue mark from the second pie to the first pie.***



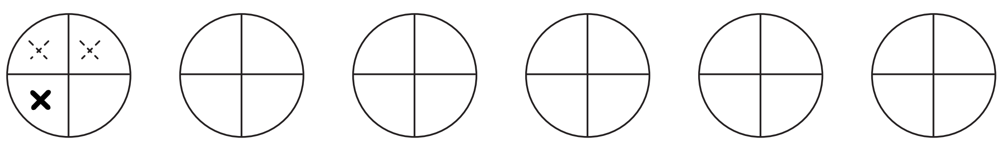
Complete the equation: $\frac{3}{4} + \frac{1}{2} = \frac{3}{4} + \frac{2}{4} = \frac{5}{4} = 1\frac{1}{4}$.

Erase the marks from the previous problem. Using students to mark the pies, present the following problems in a similar manner. (Answers are in boxes.)

$$2\frac{3}{4} + \frac{3}{4} = \boxed{3\frac{2}{4} = 3\frac{1}{2}} \qquad 1\frac{1}{4} + 3\frac{3}{4} = \boxed{4\frac{4}{4} = 5}$$

T: ***How could we use the pies to help us subtract $\frac{3}{4} - \frac{1}{2}$?***

Let students express opinions. Invite one student to draw marks in three-fourths of a pie and let a second student erase marks from one-half or two-fourths of a pie. For example:



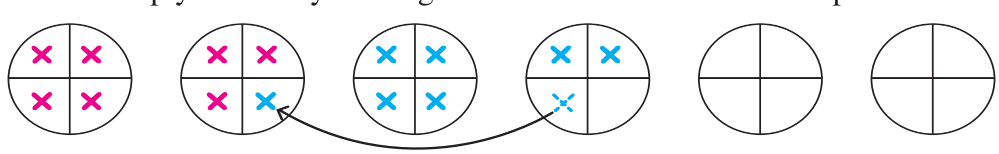
T: ***What number is $\frac{3}{4} - \frac{1}{2}$?*** ($\frac{1}{4}$)

Complete the equation: $\frac{3}{4} - \frac{1}{2} = \frac{3}{4} - \frac{2}{4} = \frac{1}{4}$.

Present the following problems in a similar manner. (Answers are in boxes.)

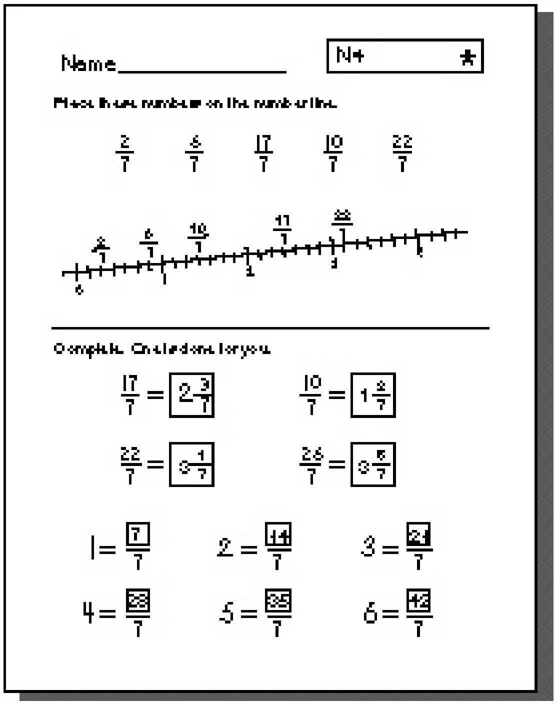
$$2 - \frac{3}{4} = \boxed{1\frac{1}{4}} \qquad 3\frac{1}{4} - 1\frac{3}{4} = \boxed{1\frac{2}{4} = 1\frac{1}{2}} \qquad 4\frac{1}{2} - 2\frac{3}{4} = \boxed{1\frac{3}{4}}$$

Use the circles to multiply 2 x $1\frac{3}{4}$ by inviting two students to each mark $1\frac{3}{4}$ pies.



Observe that 2 x $1\frac{3}{4} = 2\frac{6}{4} = 3\frac{2}{4} = 3\frac{1}{2}$.

Worksheets N4*, **, and *** are available for individual work.



Name________________ N4 ★

Place these numbers on the number line.

$\frac{2}{7}$ $\frac{6}{7}$ $\frac{17}{7}$ $\frac{10}{7}$ $\frac{22}{7}$

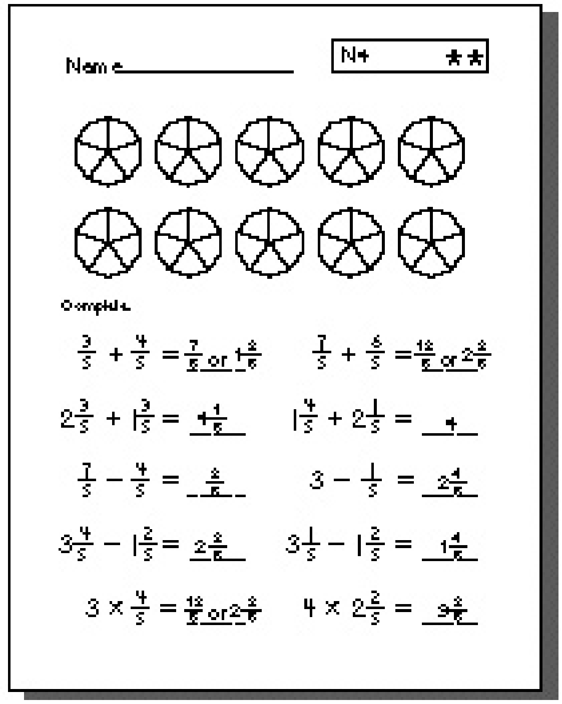
Complete. One is done for you.

$\frac{17}{7} = \boxed{2\frac{3}{7}}$ $\frac{10}{7} = \boxed{1\frac{3}{7}}$

$\frac{22}{7} = \boxed{3\frac{1}{7}}$ $\frac{26}{7} = \boxed{3\frac{5}{7}}$

$1 = \frac{\boxed{7}}{7}$ $2 = \frac{\boxed{14}}{7}$ $3 = \frac{\boxed{21}}{7}$

$4 = \frac{\boxed{28}}{7}$ $5 = \frac{\boxed{35}}{7}$ $6 = \frac{\boxed{42}}{7}$



Name________________ N4 ★★

Complete.

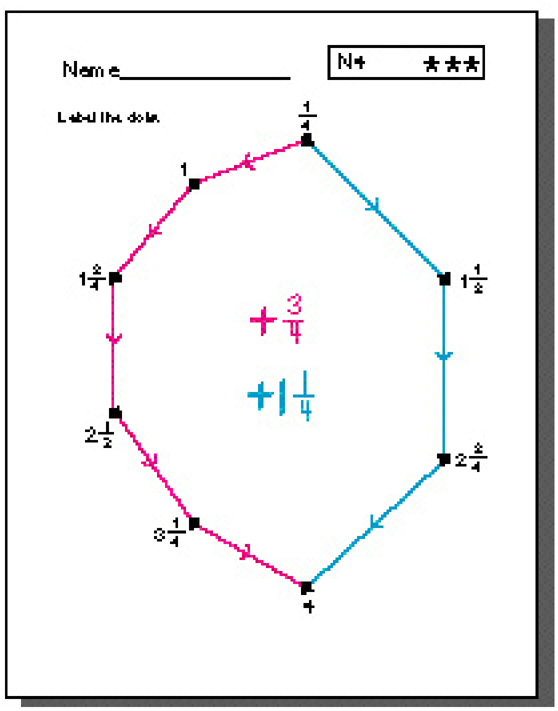
$\frac{3}{5} + \frac{4}{5} = \frac{7}{5}$ or $1\frac{2}{5}$ $\frac{7}{5} + \frac{6}{5} = \frac{13}{5}$ or $2\frac{3}{5}$

$2\frac{3}{5} + 1\frac{3}{5} =$ $4\frac{1}{5}$ $1\frac{4}{5} + 2\frac{1}{5} =$ 4

$\frac{7}{5} - \frac{4}{5} =$ $\frac{3}{5}$ $3 - \frac{1}{5} =$ $2\frac{4}{5}$

$3\frac{4}{5} - 1\frac{2}{5} =$ $2\frac{2}{5}$ $3\frac{1}{5} - 1\frac{2}{5} =$ $1\frac{4}{5}$

$3 \times \frac{4}{5} = \frac{12}{5}$ or $2\frac{2}{5}$ $4 \times 2\frac{2}{5} =$ $9\frac{3}{5}$



N5 CALCULATOR PUZZLES #1

Capsule Lesson Summary

Put selected numbers on the display of a calculator using a restricted set of keys. Find solutions that require pressing keys fewer than ten times. Between two given numbers, build arrow roads in which each arrow is for +, –, x, or ÷ some number with digits limited to 2 and 8. Try to find shortest such arrow roads.

Materials

Teacher	• Calculator • Colored chalk	Student	• Calculator • Paper • Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

List these calculator keys on the board, and refer to them as you give the following directions.

[2] [8] [=]
[+] [–] [×] [÷]

T: ***Today we are going to solve some calculator puzzles. The puzzles require that you use only certain keys on the calculator. You may use these keys in any way you like, but these are the only keys you may use. Try to put 3 on the display of your calculator.***

Allow several minutes for students to experiment. You may like to talk to students individually about their attempts, and encourage some students to find several solutions to the puzzle. After a few minutes, call for the class's attention and ask for solutions. When a student offers a solution, invite another student to check it. Make a list of solutions on the board. There are many solutions; here is a sampling:

3: [8] [–] [2] [÷] [2] [=]
[2] [÷] [2] [+] [2] [=]
[2] [2] [+] [2] [÷] [8] [=]
[8] [+] [2] [=] [=] [÷] [2] [=] [=]
[2] [×] [8] [+] [8] [÷] [8] [=]

This activity offers an excellent opportunity for students to use calculators in a very open and creative way. Some students will find solutions that make use of interesting features of the calculator. In such cases, discuss how the calculator operates to get the solution. Try to get a variety of solutions. Sometimes one student's solution will result in several similar solutions from other students.

Note: The list of solutions here assumes the calculator does chain operations and has an automatic constant feature (see "Role and Use of Calculators" in Section One: Notes to the Teacher). As necessary, make adjustments for the calculators in use by your students. For example, in several solutions above you may need to insert another [=] if your calculators do operations in a priority order (x, ÷ +, –) rather than a chain order of entry.

Continue this activity putting on other numbers such as 5, 7, 9, 25, 0.5, and 5.5. Challenge the class to think about shorter solutions by introducing the condition that it costs 1¢ (or $1) to press a key and asking for solutions that cost no more than 10¢ ($10).

You may want to encourage some students to find solutions using as many different operations as possible (even if they cost more than 10¢). Allow a few minutes for independent work, then collectively record solutions on the board. Below are several possible solutions for each suggested number.

5:	$\boxed{8}\boxed{+}\boxed{2}\boxed{\div}\boxed{2}\boxed{=}$	6¢
	$\boxed{2}\boxed{\div}\boxed{2}\boxed{+}\boxed{2}\boxed{=}\boxed{=}$	7¢
	$\boxed{2}\boxed{2}\boxed{\div}\boxed{2}\boxed{-}\boxed{2}\boxed{=}\boxed{=}\boxed{=}$	9¢
7:	$\boxed{2}\boxed{8}\boxed{\div}\boxed{2}\boxed{=}\boxed{=}$	6¢
	$\boxed{2}\boxed{8}\boxed{\times}\boxed{2}\boxed{\div}\boxed{8}\boxed{=}$	7¢
	$\boxed{2}\boxed{\times}\boxed{8}\boxed{-}\boxed{2}\boxed{\div}\boxed{2}\boxed{=}$	8¢
9:	$\boxed{2}\boxed{\div}\boxed{2}\boxed{+}\boxed{8}\boxed{=}$	6¢
	$\boxed{8}\boxed{+}\boxed{2}\boxed{\div}\boxed{2}\boxed{+}\boxed{2}\boxed{=}\boxed{=}$	9¢
	$\boxed{8}\boxed{\times}\boxed{2}\boxed{+}\boxed{2}\boxed{\div}\boxed{2}\boxed{=}$	8¢

25:	$\boxed{8}\boxed{\div}\boxed{8}\boxed{+}\boxed{8}\boxed{=}\boxed{=}\boxed{=}$	8¢
	$\boxed{2}\boxed{2}\boxed{+}\boxed{2}\boxed{8}\boxed{\div}\boxed{2}\boxed{=}$	8¢
	$\boxed{8}\boxed{-}\boxed{2}\boxed{\div}\boxed{2}\boxed{+}\boxed{2}\boxed{2}\boxed{=}$	9¢
0.5:	$\boxed{2}\boxed{\div}\boxed{2}\boxed{=}\boxed{=}$	5¢
	$\boxed{8}\boxed{\div}\boxed{8}\boxed{\div}\boxed{2}\boxed{=}$	6¢
	$\boxed{2}\boxed{+}\boxed{2}\boxed{\div}\boxed{8}\boxed{=}$	6¢
5.5:	$\boxed{2}\boxed{\div}\boxed{8}\boxed{\times}\boxed{2}\boxed{2}\boxed{=}$	7¢
	$\boxed{2}\boxed{2}\boxed{\div}\boxed{2}\boxed{=}\boxed{=}$	6¢
	$\boxed{8}\boxed{8}\boxed{\div}\boxed{8}\boxed{\div}\boxed{2}\boxed{=}$	7¢

Exercise 2

Pose a different type of calculator puzzle, still using just the keys $\boxed{2}$, $\boxed{8}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$ and $\boxed{=}$.

T: ***In our next problem, an arrow can be for add, subtract, multiply, or divide by some number with digits limited to 2 and 8. Give us an example of such an arrow.***

S: *÷8.*

S: *+28.*

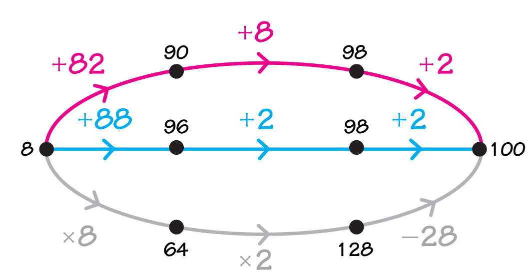
S: *–888.*

If students do not suggest immediately examples such as +28 or –888, do so yourself.

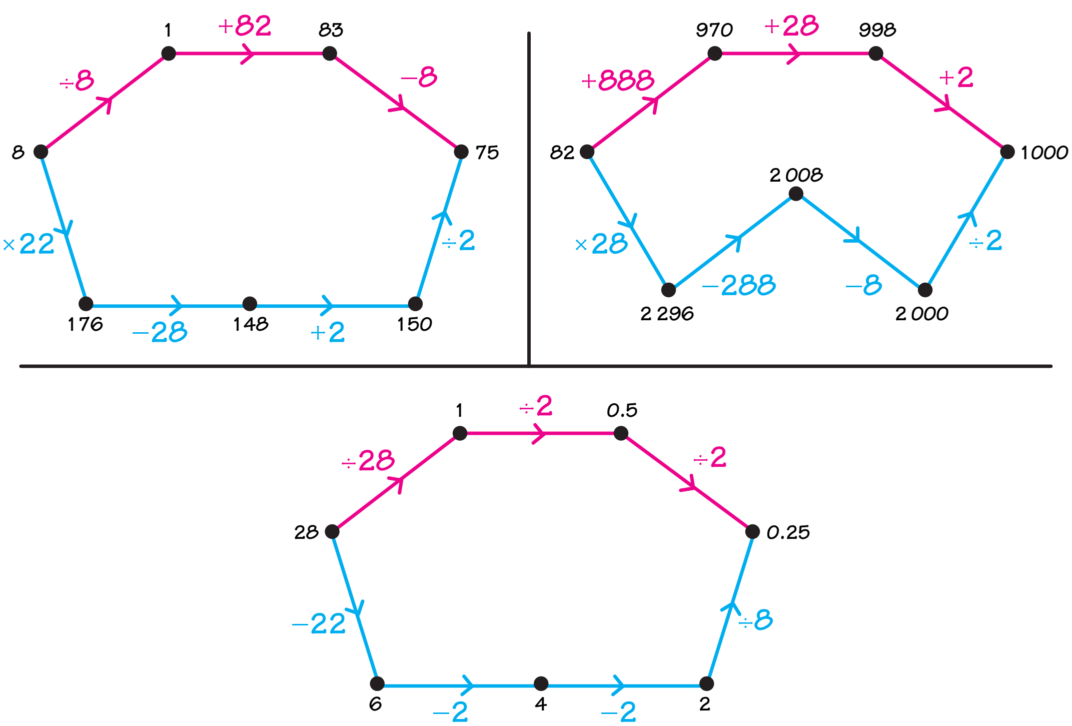
T: ***Now the problem is to build an arrow road from 8 to 100 using only these kinds of arrows.***

Allow several minutes for students to work independently on this problem. Encourage each student to find at least one solution. Challenge some students to find more than one solution or solutions with fewer arrows.

When most students have found at least one solution, invite some students to draw their roads on the board. There are many possible solutions, three of which are given here. With the class, observe which solution has the fewest arrows, and challenge the class to find a shorter (fewer arrows) solution, if appropriate.



Continue this activity by asking students to draw other roads, for example, from 8 to 75, from 82 to 1000, or from 28 to 0.25. You may suggest they try to find solutions with as few arrows as possible.



Home Activity

This would be a good time to send a letter to parents/guardians about the use of calculators. Blackline N5 has a sample letter.

Create calculator puzzles for students to work on at home with family members. For example:

> The only keys you may use are $\boxed{4}$, $\boxed{7}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, and $\boxed{=}$. You may use the keys in any way you like. Start at 0 and try to put 100 (or 150 or 0.5) on the display.

Capsule Lesson Summary

Review multiplication by a fraction as a composition of relations. Observe that the return arrow for a $\frac{3}{4}$x arrow is a $\frac{4}{3}$x arrow. Use composition of multiplication arrows to help calculate $\frac{2}{3} \times \frac{4}{5} \times \frac{3}{4} = \frac{24}{60} = \frac{2}{5}$. Relate multiplication of decimal numbers to multiplication of fractions.

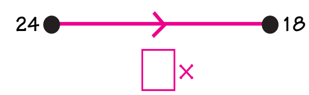
Materials

Teacher		Student	
	• Colored chalk		• Paper • Colored pencils, pens, or crayons • Worksheets N6 (no star), N6*, **, and ***

Description of Lesson

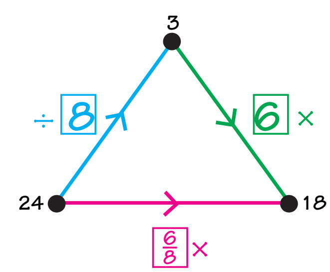
Exercise 1

Draw this arrow picture on the board.



T: ***What could the red arrow be for?***

After receiving several suggestions, draw composite arrows in the picture to serve as a hint. Then label the dots and arrows as suggested by a student.



S: ***The red arrow could be for $\frac{6}{8}$x, because $\frac{6}{8}$ is the same as ÷8 followed by 6x. 24 ÷ 8 = 3 and 3 x 6 = 18.***

T: ***How do we know that the red arrow is for $\frac{6}{8}$x, not $\frac{8}{6}$x?***

S: ***You always divide by the denominator (bottom number) and multiply by the numerator (top number) of a fraction, so $\frac{6}{8}$x is the same as ÷8 followed 6x (or 6x followed by ÷8).***

T: ***Are there other names for the red arrow?***

S: ***$\frac{3}{4}$x. 24 ÷ 4 = 6 and 3 x 6 = 18.***

S: ***$\frac{18}{24}$x. 24 ÷ 24 = 1 and 18 x 1 = 18.***

With each suggestion, relabel the arrow picture as a check. Make a list of possible labels for the red arrow on the board, such as given here.

$\frac{3}{4}\times \quad \frac{6}{8}\times \quad \frac{9}{12}\times \quad \frac{18}{24}\times$

$$\frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{18}{24}$$

T: ***These are different names for the same relation. They suggest names for equivalent fractions.***

Do you see any patterns? Can you use a pattern to find more fractions equivalent to these?

S: ***If you double both the numerator and the denominator you get equivalent fractions; for example, $\frac{3}{4} = \frac{6}{8}$ and $\frac{9}{12} = \frac{18}{24}$. Therefore,***

$$\frac{3}{4} = \frac{6}{8} = \frac{12}{16} = \frac{24}{32} = \frac{48}{64} = \ldots = \frac{9}{12} = \frac{18}{24} = \frac{36}{48} = \frac{72}{96} = \ldots .$$

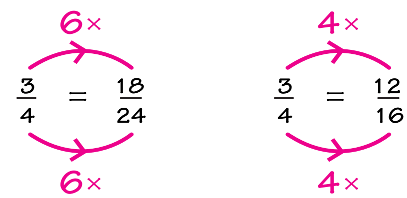
S: ***With $3/4$, $6/8$, $9/12$, the numerators increase by 3 while the denominators increase by 4. So $3/4 = 6/8 = 9/12 = 12/16 = 15/20 = 18/24 = 21/28 = \ldots$.***

Accept other workable patterns students observe. List the equivalent fractions they generate.

If students have not suggested the following method, do so yourself.

T: ***Notice that $3/4 = 18/24 = \frac{4 \times 3}{4 \times 4}$ and $3/4 = 12/16 = \frac{4 \times 3}{4 \times 4}$.***

Draw pictures on the board to illustrate this pattern.

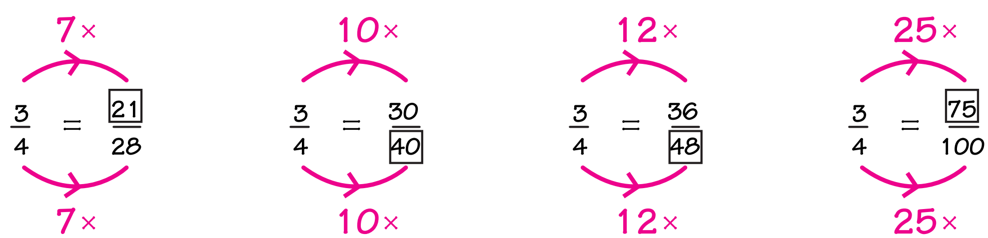


T: ***If you multiply (or divide) both the numerator and the denominator of a fraction by the same number, you get an equivalent fraction.***

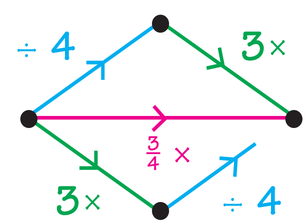
Pose some problems similar to the following. Attempt to generate equivalent fractions your students have not yet found.

$$\frac{3}{4} = \frac{\square}{28} \qquad \frac{3}{4} = \frac{30}{\square} \qquad \frac{3}{4} = \frac{36}{\square} \qquad \frac{3}{4} = \frac{\square}{100}$$

Let students use any pattern they wish to solve the problems, but point out the method of multiplying both the numerator and the denominator by the same number.



Draw this arrow picture on the board.



T: ***How should we complete the lower blue arrow for ÷4?***

S: ***It should end here (at the end of the red arrow), because ÷4 followed by 3x is the same as 3x followed by ÷4.***

S: ***If we start at 8 here (start of the red arrow), $8 \div 4 = 2$ and $3 \times 2 = 6$. Also, $3 \times 8 = 24$ and $24 \div 4 = 6$.***

Let students check a few more possibilities for the beginning dot. Conclude that the lower blue arrow should indeed end at the same place as upper green arrow.

Write the following problems on the board and ask students to copy them on their papers. Allow a few minutes for individual work; then solve the problems collectively. (Answers are in boxes.) Note that $4/3$x is the opposite of $3/4$x because $4/3$x is ÷3 followed by 4x, the opposites of 3x and ÷4.

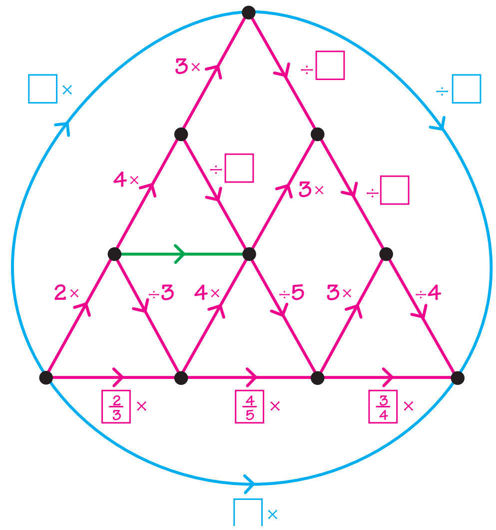
$$\frac{3}{4} \times 12 = \boxed{9} \qquad \frac{4}{3} \times 12 = \boxed{16}$$

$$\frac{3}{4} \times 36 = \boxed{27} \qquad \frac{4}{3} \times 36 = \boxed{48}$$

Exercise 2

Distribute Worksheet N6 (no star). While students work independently, draw the arrow picture from the worksheet on the board.

After a while, invite students to label the red arrows along the bottom of the arrow picture. Then refer to another unlabeled red arrow and draw a "helping" green arrow, as illustrated here.



T: ***What could this green arrow be for?***

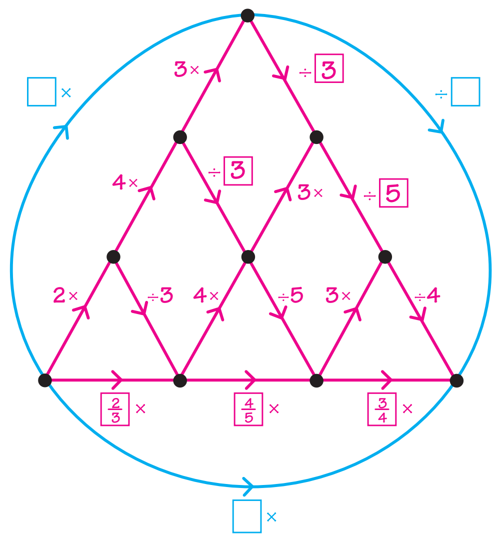
S: ***$\frac{4}{3}$x, because it is the same as ÷3 followed by 4x.***

Label the green arrow, and point to the unlabeled red arrow ending where the green arrow ends.

T: ***How should this red arrow be labeled?***

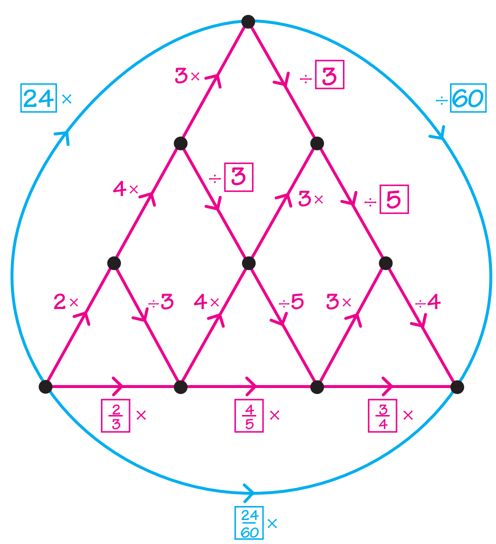
S: ***÷3, because 4x followed by ÷3 is also the same as $\frac{4}{3}$x.***

As necessary use other "helping" green arrows to label the remaining red arrows. Then erase the green arrows.



T: ***Now let's label the blue arrows. We are especially interested in the blue arrow at the bottom which is for $\frac{2}{3}$x followed by $\frac{4}{5}$x followed by $\frac{3}{4}$x. The blue arrow at the bottom is also the same as the blue arrow on the left followed by the blue arrow on the right.***

Accept student explanations of how to label blue arrows, and fill in the boxes as correct labels are suggested.



S: ***The blue arrow on the left is for 2x followed by 4x followed by 3x. It is for 24x because 2 x 4 x 3 = 24.***

S: ***The blue arrow on the right is for ÷3 followed by ÷5 followed by ÷4. It is for ÷60 because 3 x 5 x 4 = 60.***

S: ***On the right, return arrows for the three red arrows are 4x, 5x, and 3x. 4x followed by 5x followed by 3x is 60x since 4 x 5 x 3 = 60. The return arrow for 60x is ÷ 60, so the blue arrow on the right is for ÷60.***

S: ***The blue arrow at the bottom is for 24x followed by ÷60. It must be for $\frac{24}{60}$x.***

T: ***Therefore, $\frac{2}{3}$x followed by $\frac{4}{5}$x followed by $\frac{3}{4}$x is the same as $\frac{24}{60}$x. This suggests that $\frac{2}{3} \times \frac{4}{5} \times \frac{3}{4} = {}^{24}/$***

T: ***Where does the numerator 24 come from?***

S: ***2** x **4** x **3 = 24.***

T: ***Where does the denominator 60 come from?***

S: ***3** x **5** x **4 = 60.***

Write this information on the board.

$$\frac{2}{3} \times \frac{4}{5} \times \frac{3}{4} = \frac{2 \times 4 \times 3}{3 \times 5 \times 4} = \frac{24}{60}$$

T: ***Can you suggest some other names for $^{24}/_{60}$?***

Accept several equivalent fractions, for example, $^{24}/_{60} = {}^{12}/_{30} = {}^{4}/_{10} = {}^{2}/_{5} = {}^{1}/_{2.5} = {}^{0.5}/_{1.25}$

T: ***All of these fractions are equivalent to $^{24}/_{60}$. The one with smallest whole number numerator and denominator is often preferred, here it is $^{2}/_{5}$. We could say $^{2}/_{5}$ is in*** **lowest terms.**

Write the following problems on the board. Instruct students to copy the problems and to do the calculations on a piece of paper. Encourage finding "preferred" names for the products, that is, fractions in lowest terms. (Answers are in boxes.)

$$\frac{3}{4} \times \frac{10}{5} = \boxed{\frac{30}{20} = \frac{3}{2}}$$

$$\frac{2}{3} \times \frac{2}{6} \times \frac{5}{2} = \boxed{\frac{20}{36} = \frac{5}{9}}$$

$$\frac{7}{10} \times \frac{13}{10} = \boxed{\frac{91}{100}}$$

$$\frac{3}{5} \times \frac{15}{4} \times \frac{2}{3} = \boxed{\frac{90}{60} = \frac{3}{2}}$$

Refer to $^{7}/_{10}$ x $^{13}/_{10}$ as you ask,

T: ***What is a decimal name for $^{7}/_{10}$? For $^{13}/_{10}$?***

S: ***$^{7}/_{10}$ = 7 ÷ 10 = 0.7. $^{13}/_{10}$ = 13 ÷ 10 = 1.3.***

Write this information under the problem on the board.

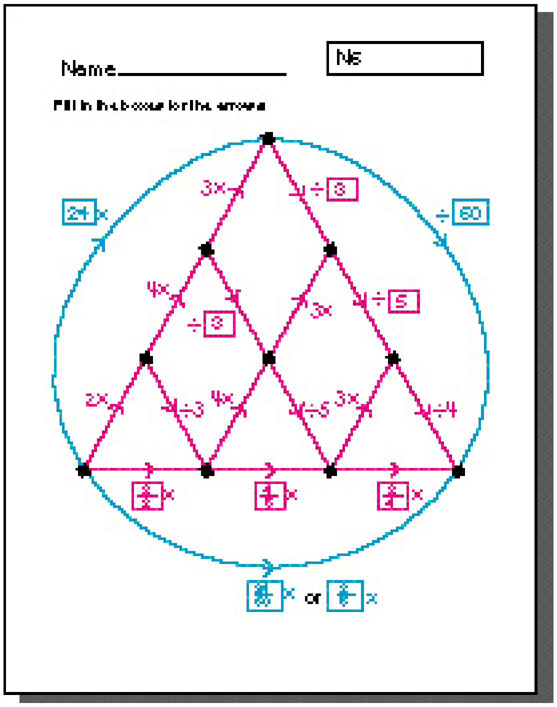
$$\frac{7}{10} \times \frac{13}{10} = \frac{91}{100}$$

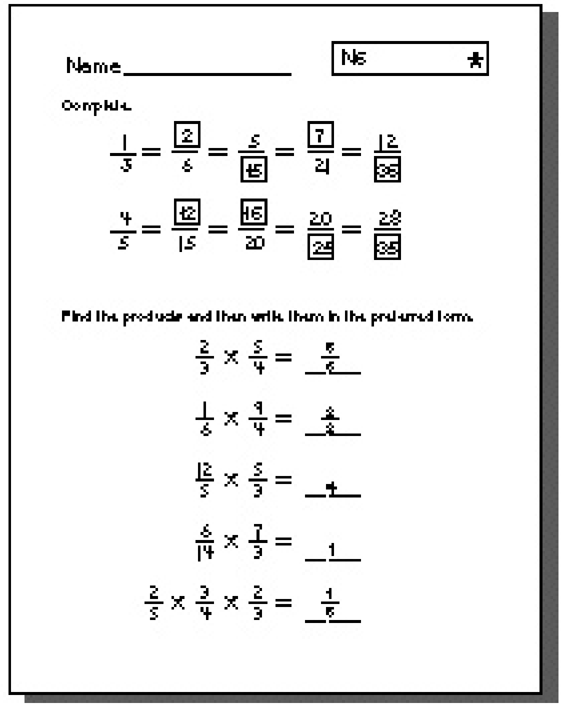
$$0.7 \times 1.3 = 0.91$$

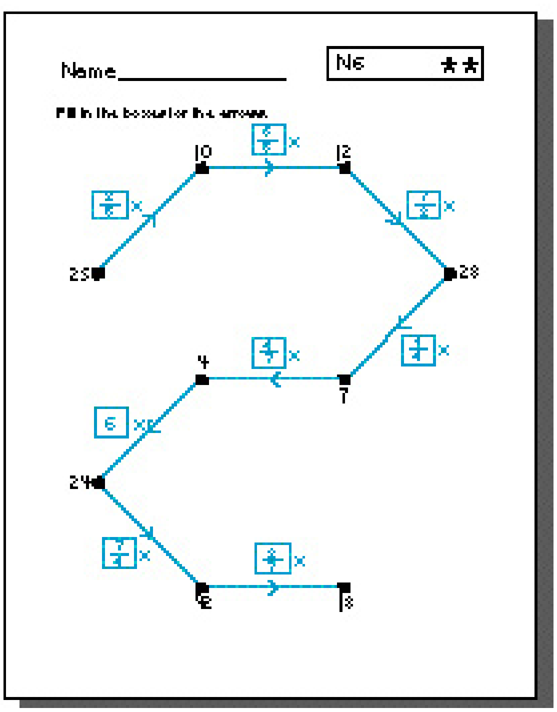
T: ***What number is 0.7** x **1.3?***

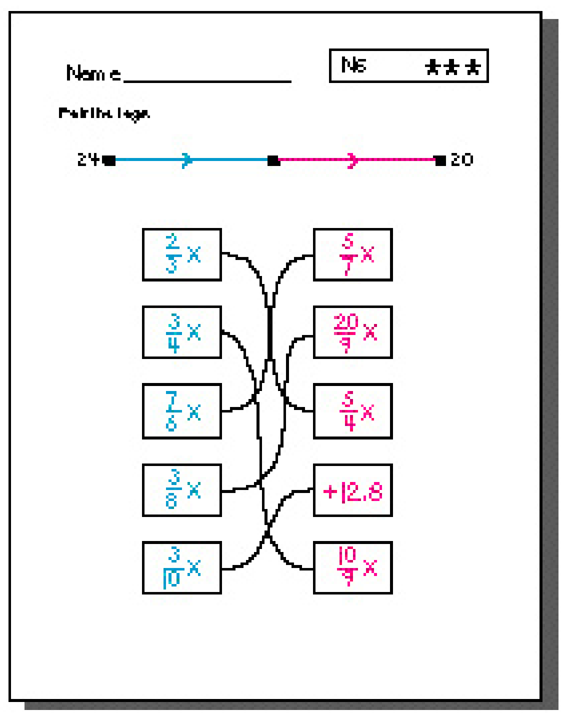
S: ***0.91, since $^{91}/_{100}$ = 91 ÷ 100 = 0.91.***

Worksheets N6*, **, and *** are available for individual work.


Name
N6
Fill in the boxes for the arrows.
24×
÷60
3×
÷3
4×
÷3
3×
÷5
2×
÷3
4×
÷5
3×
÷4
or


Name
N6 ★
Complete.
1/3 = 2/6 = 5/15 = 7/21 = 12/36
4/5 = 12/15 = 16/20 = 20/25 = 28/35
Find the products and then write them in the preferred form.
2/3 × 5/4 = 5/6
6/14 × 7/3 = 1


Name
N6 ★★
Fill in the boxes for the arrows.
10
12
25
28
4
7
6×
24


Name
N6 ★★★
Pair the legs.
24
20
2/3×
5/7×
3/4×
20/9×
7/6×
5/4×
3/8×
+12.8
3/10×
10/9×

Capsule Lesson Summary

In cooperative groups, solve several comparison problems, and present or write explanations for methods of solution.

Materials

Teacher

- Colored chalk
- Cubes, counters, or other props

Student

- Paper
- Colored pencils, pens, or crayons
- Calculator
- Cubes, counters, or other props
- Worksheets N7(a), (b), and (c)

Description of Lesson

Organize the class into small cooperative groups for problem solving. Each group should have supplies such as paper, colored pencils, a calculator, and props or manipulatives. Distribute copies of Worksheets N7(a), (b), and (c).

Direct the groups to work on the three problems on the worksheets. Encourage students to find methods of solving the problems that everyone in their group can explain. You may want students to answer questions and write explanations on their worksheets individually, but suggest that each group prepare to present their solutions to the class.

As you observe group work, look for different solution techniques so that you can arrange that these different approaches are presented to the whole class.

For your information, some different ways to solve each problem are shown on the following pages (solutions are shaded in gray). Your students are likely to find their own methods and explanations.

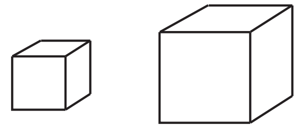
Writing Activity

Suggest that students or groups write problems similar to the ones in this lesson. You may like to let groups or individual students exchange problems, or you may suggest students take home one of their problems to do with a family member.

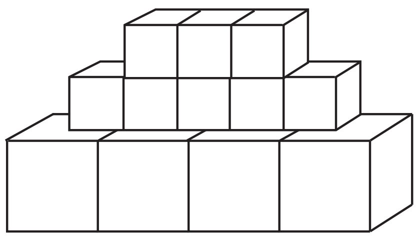
Name ____________________

N7(a)

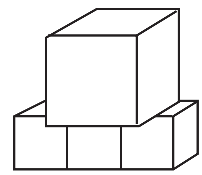
Small cubes are all of equal weight.
Large cubes are all of equal weight.



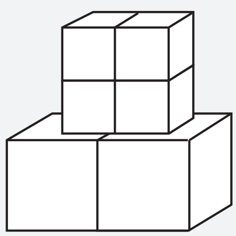
A structure with 4 large and 8 small cubes weighs 10 pounds.



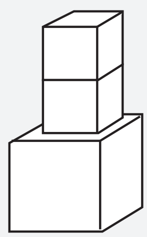
A structure with 3 small and 1 large cubes weighs 3 pounds.



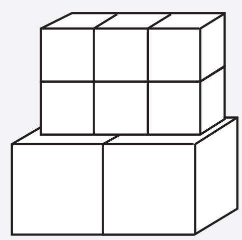
Find the weights of some different combinations-structures-with small and large cubes.



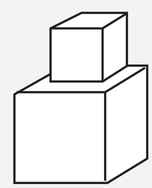
2 large and 4 small cubes weigh 5 pounds



1 large and 2 small cubes weigh 2.5 pounds



2 large and 6 small cubes weigh 6 pounds



1 large and 1 small cube weigh 2 pounds

Many solutions are possible.

How many small cubes would balance a large cube? Three

How much does each kind of cube weight? small 0.5 pounds

large 1.5 pounds

- Trial and error methods may be used, especially along with different combinations above.
- 1 large and 2 small cubes weigh 2.5 pounds.
  1 large and 1 small cube weigh 2 pounds.
  So the small cube weighs 0.5 pounds, and the large cube weighs 1.5 pounds.

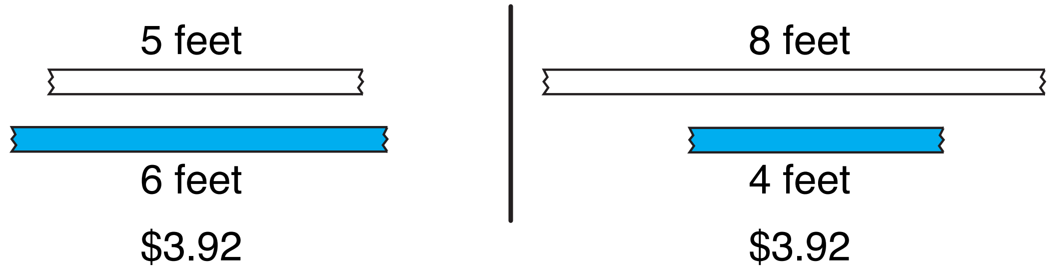
Name ____________________

N7(b)

The fabric store sells white and blue ribbon by the foot.

Dora spent $3.92 on ribbon. She got 5 feet of white and 6 feet of blue ribbon.

Ted also spent $3.92 on ribbon. He got 8 feet of white and 4 feet of blue ribbon.



Which color ribbon costs more per foot? Blue

- On the left 11 feet of ribbon cost $3.92 and on the right 12 feet of ribbon cost $3.92. Since there is more blue ribbon on the left, blue must be more expensive
- Consider: 5 ft. white, 6 ft. blue $\overset{\text{cost same}}{=}$ 8 ft. white, 4 ft. blue

  If you take 5 ft. white away… 6 ft. blue = 3 ft. white, 4 ft. blue

  and 4 ft. blue away … 2 ft. blue = 3 ft. white …so blue costs more.

Find the cost of some other quantities of white and blue ribbon.

13 feet white 10 feet white cost $7.84	4 feet white 2 feet blue cost $1.96	2 feet white 1 foot blue cost $0.98	1 foot white 4 feet blue cost $1.96	6 feet white 3 feet blue cost $2.94

- Use the second quantity above and the fact that 2 feet blue costs the same as 3 feet white to find that 7 feet white costs $1.96. $1.96 ÷ 7 = $0.28. So 1 foot white ribbon costs 28¢.
- Use the fourth quantity above and the fact that 1 foot white costs $0.28 to find that 4 feet blue costs $1.68. $1.68 ÷ 4 = $0.42. So 1 foot blue ribbon costs 42¢.

Find the cost of one foot of white ribbon. 28¢

Find the cost of one foot of blue ribbon. 42¢

Students may use different quantities and methods to find the costs of white and blue ribbon.

Name ___________________

N7(c)

Two hamburgers and two colas cost $4.20.

Three orders of french fries and two colas cost $3.19.

One hamburger, one order of french fries, and one cola cost $2.65.

What is the individual cost of each item?

Hamburger $1.33

Cola $0.77

French Fries $0.55

Several different methods may lead to the individual cost of each item. One possible sequence of steps is given here.

Since two hamburgers, two colas cost $4.20
then one hamburger, one cola cost $2.10

Since one hamburger, one order french fries, one cola cost $2.65
and one hamburger, one cola cost $2.10
then one order french fries costs $2.65 – $2.10 = $0.55

Since three orders french fries and two colas cost $3.19
and three orders french fries cost 3 x $0.55 = $1.65
then two colas cost $3.19 – $1.65 = $1.54
and one cola costs $1.54 ÷ 2 = $0.77

Since two hamburgers, two colas cost $4.20
and two colas cost $1.54
then two hamburgers cost $4.20 – $1.54 = $2.66
and one hamburger costs $1.33

Capsule Lesson Summary

Put decimal points in given numbers to make their locations in a string picture correct. Use patterns to complete division calculations with decimal numbers. Review equivalences such as $\frac{3}{10} = 3 \div 10 = 0.3$. Play a calculator game involving multiplication of decimal numbers.

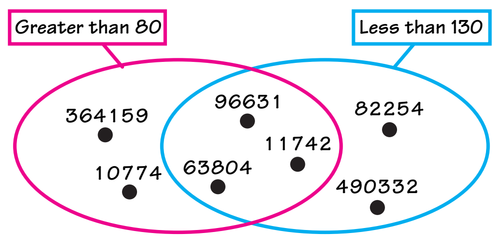
Materials

Teacher		Student	
	• Colored chalk		• Paper • Calculator • Worksheets N8*, **, ***, and ****

Description of Lesson

Exercise 1

Draw this string picture on the board.



T: ***What is wrong in this string picture?***

S: ***The numbers in the blue string are not less than 130.***

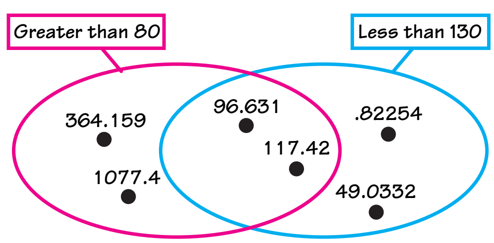
T: ***Yes; that is because all of these numbers are missing decimal points. Who can put a decimal point in a number so that the number is correctly placed in the string picture?***

Suppose a student puts a decimal point in 490332 between 9 and 0.

T: ***Why is 49.0332 correctly placed?***

S: ***49.0332 is less than 130 but is not greater than 80.***

S: ***4.90332 or .490332 would also be correct.***



Let students put decimal points in other numbers to make them correctly placed in the picture. There are multiple solutions for numbers not in the middle region; 364159 and 10774 are correctly placed as they are, but they could have decimal points too. It is not possible to correct the placement of 63804 by putting in a decimal point. 6380.4 and 638.04 are both more than 130 while 63.804, 6.3804, and .63804 are all less than 80. Erase 63804 from the picture.

Draw a dot outside of both strings.

T: ***What number could be here?***

Let students suggest possibilities until they conclude that there is no number outside both strings.

T: ***Why is there no number outside both strings?***

S: ***A number outside both strings must be less than 80 and more than 130 which is impossible.***

S: ***All numbers belong inside at least one of the strings since all numbers are greater than 80 or less than 130 or both.***

Exercise 2

Write this list of problems on the board.

$4\,200 \div 7 = \boxed{}$

$420 \div 7 = \boxed{}$

$42 \div 7 = \boxed{}$

T: ***What number is 4 200 ÷ 7?***

S: ***600, 42 ÷ 7 = 6, so 4 200 ÷ 7 = 600.***

Invite students to complete the other calculations.
Add information to the list as shown here.

$4\,200 \div 7 = \boxed{600}$

$420 \div 7 = \boxed{60}$

$42 \div 7 = \boxed{6}$

$\boxed{} \div 7 = \boxed{}$

$\boxed{} \div 7 = \boxed{}$

$\boxed{} \div 7 = \boxed{}$

T: ***What could the red arrows be for?***

S: ***÷10. 4 200 ÷ 10 = 420 and 420 ÷ 10 = 42.
Also, 600 ÷ 10 = 60 and 60 ÷ 10 = 6.***

T: ***Can we continue the pattern?***

S: ***4.2 ÷ 7 = 0.6 since 42 ÷ 10 = 4.2 and 6 ÷ 10 = 0.6.***

Let students fill in the boxes to complete the list of problems.

÷10

$4\,200 \div 7 = \boxed{600}$

$420 \div 7 = \boxed{60}$

$42 \div 7 = \boxed{6}$

$\boxed{4.2} \div 7 = \boxed{0.6}$

$\boxed{0.42} \div 7 = \boxed{0.06}$

$\boxed{0.042} \div 7 = \boxed{0.006}$

Point out that you have done many calculations based on a single fact, 42 ÷ 7 = 6.

Write these problems on the board and invite students to solve them. (Answers are in boxes.)

$270 \div 3 = \boxed{90}$

$2.7 \div 3 = \boxed{0.9}$

$27\,000 \div 3 = \boxed{9\,000}$

$0.027 \div 3 = \boxed{0.009}$

Encourage students to explain their solution techniques. Some students might use patterns as in the previous problems, for example, 27 ÷ 3 = 9 so 2.7 ÷ 3 = 0.9. Other students might extend a pattern; for example, 2.7 ÷ 100 = 0.027 and 0.9 ÷ 100 = 0.009 so 0.027 ÷ 3 = 0.009.

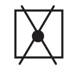
Write the following problems on the board and instruct the students to solve them. Encourage the use of patterns. (Answers are in boxes.)

$200 \div 5 = \boxed{40}$ $\qquad 2\,400 \div 6 = \boxed{400}$

$0.2 \div 5 = \boxed{0.04}$ $\qquad 0.024 \div 6 = \boxed{0.004}$

Exercise 3

Write this information on the board.



$\frac{3}{10}$

T: ***How could you put $3/10$ on the display of a calculator without using the decimal point key?***

S: ***Since $3/10 = 3 \div 10$, press [3] [÷] [1][0] [=].***

T: ***What decimal number is $3 \div 10$?***

S: ***0.3.***

$$\frac{3}{10} = 3 \div 10 = 0.3$$

Ask students to use their calculators to confirm that $3 \div 10 = 0.3$, and write the equivalences on the board.

T: ***We usually read 0.3 as "zero point three." Since $0.3 = 3/10$, we also say "three tenths."***

Present these problems in a similar manner. (Answers are in boxes.)

$$\frac{12}{10} = \boxed{12 \div 10 = 1.2} \qquad \frac{463}{10} = \boxed{463 \div 10 = 46.3}$$

$$\frac{36}{10} = \boxed{36 \div 10 = 3.6} \qquad \frac{36}{100} = \boxed{36 \div 100 = 0.36}$$

Explain how to read the decimal numbers; for example, read 0.36 as "zero point three six" or as "thirty-six hundredths."

$$\frac{4}{5} = \boxed{0.8}$$

T: ***What is a decimal name for $4/5$?***

S: ***0.8, since $4/5 = 8/10 = 8 \div 10 = 0.8$.***

S: ***0.8, since $4/5 = 4 \div 5$ and $40 \div 5 = 8$ so $4 \div 5 = 0.8$.***

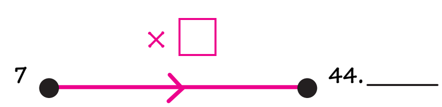
Present the following problems in a similar manner. (Answers are in boxes.)

$$\frac{7}{5} = \boxed{1.4} \qquad \frac{28}{50} = \boxed{0.56}$$

$$\frac{7}{2} = \boxed{3.5} \qquad \frac{13}{20} = \boxed{0.65}$$

Exercise 4

Draw this arrow picture on the board.



T: ***We are looking for a label for the red arrow so that the ending dot is for forty-four point something. Seven times what number is near 44?***

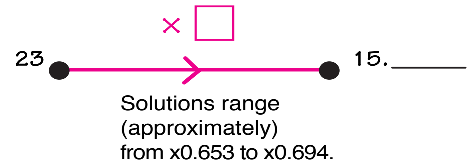
S: ***$7 \times 6 = 42$ and $7 \times 7 = 49$.***

S: ***The red arrow must be for times some number between 6 and 7; that is, six point something.***

T: ***Use your calculator to help find a solution.***

Let students work independently. After a while, ask for a possible label for the red arrow. Students may find that 7 x 6.3 = 44.1 and 7 x 6.4 = 44.8. Of course many solutions are possible, ranging (approximately) from x6.29 to x6.428.

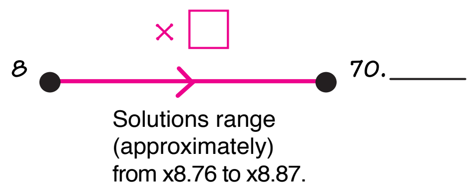
Present this problem in a similar manner. First lead the class to observe that the red arrow must be for times some number between 0 and 1.

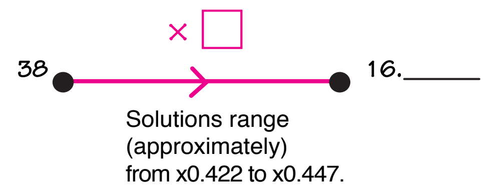


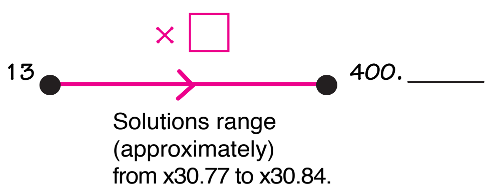
T: ***Let's use this type of problem to play a game.***

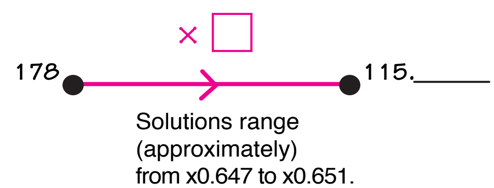
Pair students and write the problems shown below on the board. Give the following instructions:

- Students in a pair take turns to try to solve a problem.
- Record each attempt on a piece of paper for both to see.
- The first student to find a solution is the winner.







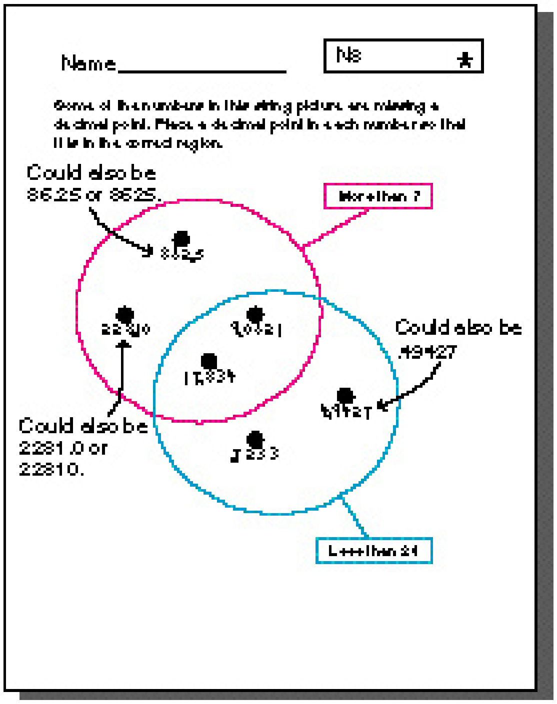


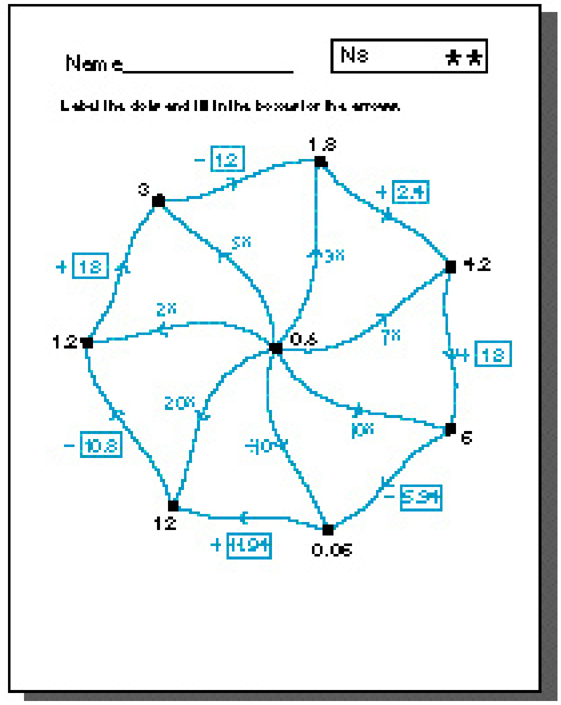
Start with the first problem and proceed to other problems as time allows.

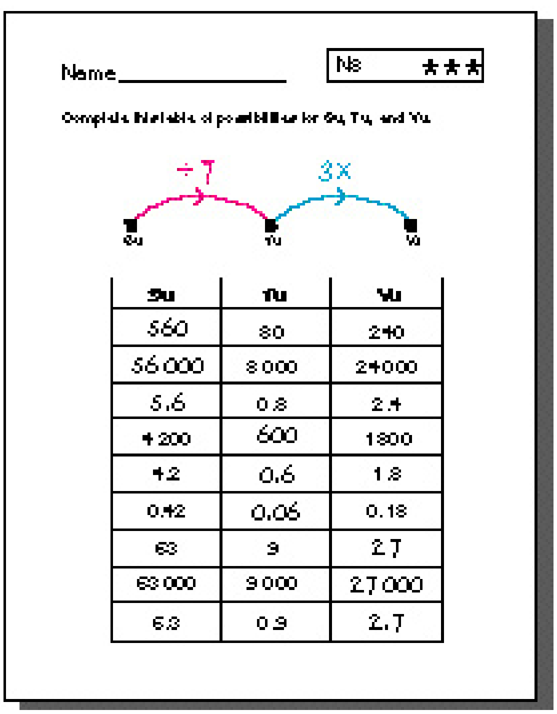
Worksheets N8*, **, ***, and **** are available for individual work.

Home Activity

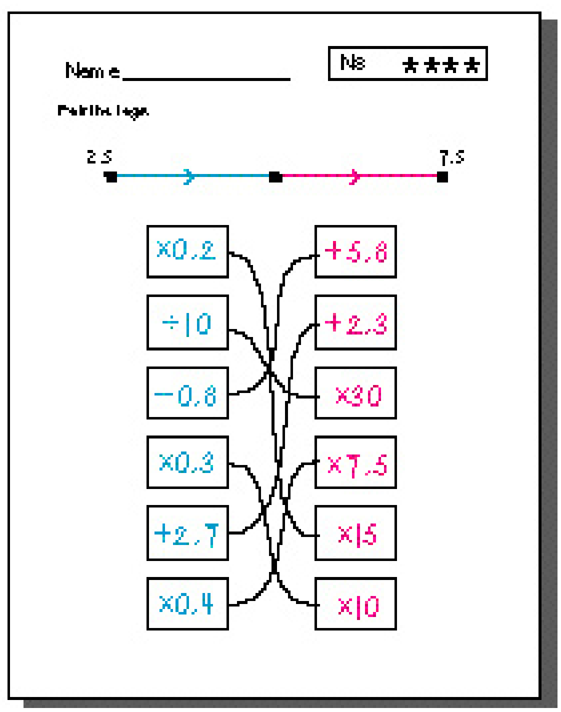
Send home a description of the game in Exercise 4, and suggest that students play the game with family members.







Su	Tu	Vu
560	80	240
56 000	8000	24000
5.6	0.8	2.4
4200	600	1800
4.2	0.6	1.8
0.42	0.06	0.18
63	9	27
63 000	9000	27 000
6.3	0.9	2.7



Capsule Lesson Summary

Label arrows for multiplication and for division in an arrow picture. Compare the composition 2x followed by +20 with the composition +20 followed by 2x. Use composition to explain a numerical magic trick.

Materials

Teacher	• Colored chalk	Student	• Paper • Colored pencils, pens, or crayons • Worksheets N9*, **, and ***

Description of Lesson

Exercise 1

Tell the class that you have a magic trick with numbers. Ask each student to select a secret number between 10 and 100.[†] State the following instructions, one at a time, allowing time for students to do the calculations. You may want to write the sequence of calculations on the board.

T: ***Double your secret number.***
Add 14 to that answer.
Divide by 2.
Subtract 5.

2×
×14
÷2
–5

Tell me your ending number and I will announce instantaneously your starting number.

To calculate a student's starting number, subtract 2 from the ending number. Do not reveal your method. For example:

S: ***My ending number is 34.***

T (thinking 34 – 2)**:** ***Your starting number is 32.***

Similarly, ask about five other students for their ending numbers and tell them their starting numbers. Ask students who disagree to check their calculations.

T: ***I'll explain the trick later.***

You may ask students to write the sequence of calculations in a corner of their papers for reference in Exercise 4.

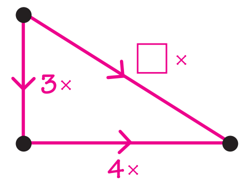
Exercise 2

In this exercise, emphasize the importance of arrow direction and the idea of using a "detour" in order to decide how to label an arrow. If you label dots to check a disputed answer, check it with at least two sets of numbers.

[†]This restriction is not necessary. The method works with any starting number.

N9

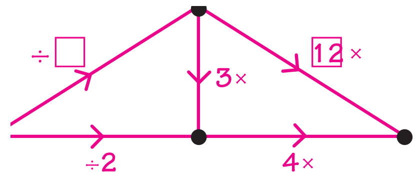
Draw this arrow picture on the board, and invite students to fill in the box for the unlabeled arrow.



S: ***The arrow is for 12x. 3x followed by 4x is 12x.***

If necessary, check the answer by labeling the dots. Fill in the box, and point out that 3x followed by 4x is a kind of detour for 12x.

Add two more arrows to the picture.

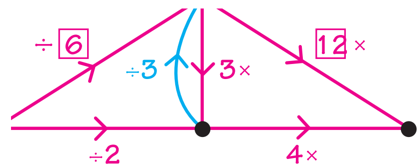


T (tracing the ÷□ arrow)**:** ***What is a detour for this arrow?***

S: ***÷2 followed by against 3x.***

S: ***Against 3x is the same as ÷3 since the return arrow for 3x is ÷3.***

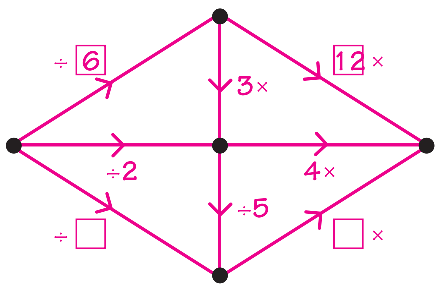
Draw an appropriate return arrow in blue.



S: ***÷2 followed by ÷3 is ÷6. The unlabeled arrow is for ÷6.***

If necessary, label dots to check this result. Then fill in the box.

Erase the blue arrow, and add three more arrows to the picture.

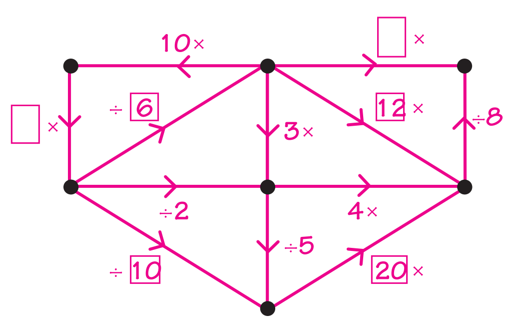


S: ***This arrow (lower left) is for ÷10. ÷2 followed by ÷5 is the same as ÷10.***

S: ***The other arrow (lower right) is for 20x. A detour is against ÷5 followed by 4x which is the same as 5x followed by 4x or 20x.***

Draw the 5x arrow as a return arrow for the ÷5 arrow, and illustrate that the composite 5x followed by 4x is 20x. Fill in the boxes and then erase the 5x arrow.

Add four more arrows to the picture. Trace the upper right arrow as you ask,



T: ***Do you see a detour in the picture for this arrow?***

S: ***12x followed by ÷8 is $\frac{12}{8}$x.***

T: ***Are there other names for $\frac{12}{8}$x?***

S: ***$\frac{6}{4}$x, $\frac{24}{16}$x $\frac{3}{2}$x, and so on.***

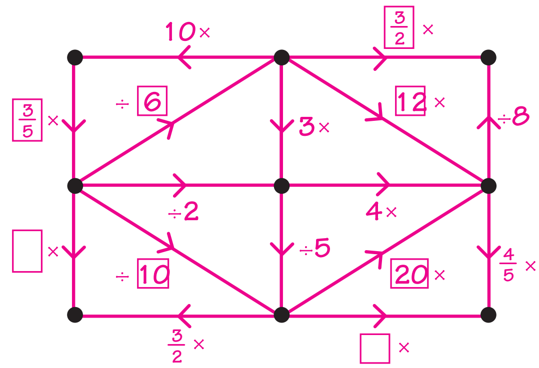
Use $\frac{3}{2}$x to fill in the box for the arrow.

S (tracing the other unlabeled arrow)*:* ***A detour for this arrow is against 10x followed by against ÷6.***

Draw appropriate return arrows.

S: ***That is the same as ÷10 followed by 6x or $\frac{6}{10}$x. $\frac{6}{10}$x is the same as $\frac{3}{5}$x.***

Fill in the boxes for the unlabeled arrows, and add four more arrows to the picture. Trace the lower right arrow as you ask,

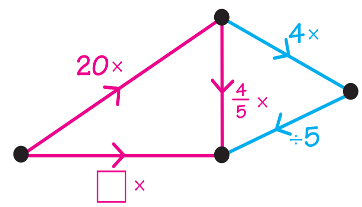


T: ***Do you see a detour for this arrow?***

S: ***20*x *followed by* $\frac{4}{5}$x.**

S: ***20*x *followed by* $\frac{4}{5}$x *is (20* x *$\frac{4}{5}$)*x *which is* $\frac{80}{5}$ *or 16*x.**

Another method for considering the above composite is to add a detour for the $\frac{4}{5}$x arrow.

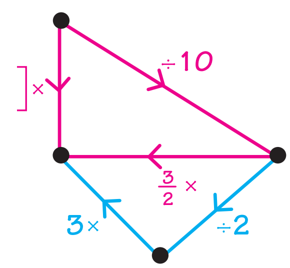


Then, a longer detour for the unlabeled arrow is 20x followed by 4x followed by ÷5. This detour is the same as 80x followed by ÷5 which is $\frac{80}{5}$x or 16x.

Fill in the box for the lower right arrow.
Refer to the remaining unlabeled arrow.

T: ***Do you see a detour for this arrow?***

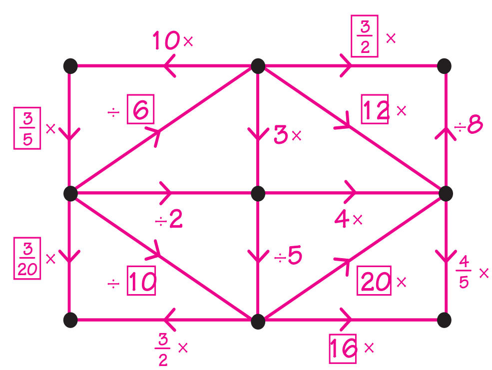
S: ***÷10 followed by* $\frac{3}{2}$x.**



There are two ways to consider this composite.

- ÷10 followed by $\frac{3}{2}$x is the same as $\frac{1}{10}$x followed by $\frac{3}{2}$x which is ($\frac{1}{10}$ x $\frac{3}{2}$)x or $\frac{3}{20}$x.
- Use a detour for $\frac{3}{2}$x. Then, ÷10 followed by ÷2 followed by 3x is the same as ÷20 followed by 3x which is $\frac{3}{20}$x.

Choose whichever method you prefer to fill in the box for this arrow.



Exercise 3

You may wish to select two students to act out the following story. If so, tell the students that they will play two genies, Ali Baba and Jeanne. If necessary, explain that a genie is a spirit in Arab folklore that lives in a magical lamp and grants its owner's wishes.

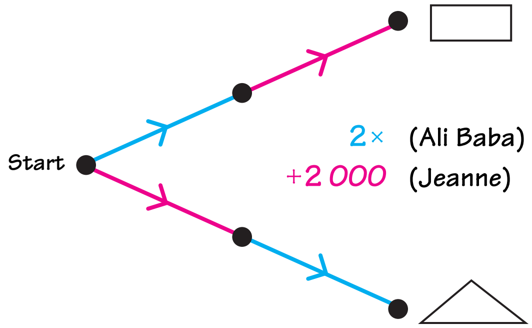
T: ***Pretend that we have found the magic lamps of two genies, Ali Baba and Jeanne. These two genies are very specialized. Ali Baba always offers to double your money. Jeanne always offers to give you $2 000.***

Write this information on the board.

2× (Ali Baba)
+2 000 (Jeanne)

T: ***You may go to each genie only once. Who would you go to first? Or does it matter?***

Let students discuss the choice and explain their answers. Some may think it depends on how much money they start with. After a while, draw this arrow picture and table on the board.



Start	□	△

Trace the blue arrow followed by the red arrow as you ask,

T: ***Suppose you start with \$50. How much money would you end with if you go first to Ali Baba and then to Jeanne?***

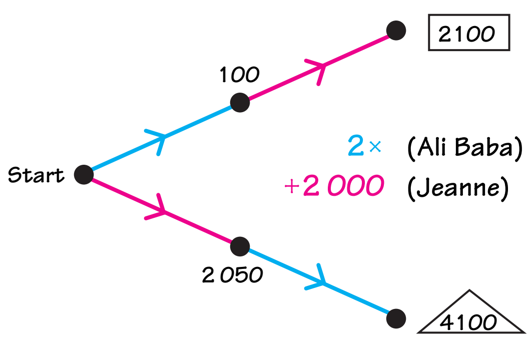
S: ***\$2 100. 2 x 50 = 100 and 100 + 2 000 = 2100.***

Trace the red arrow followed by the blue arrow as you ask,

T: ***What if you go first to Jeanne and then to Ali Baba?***

S: ***You end with \$4 100. 50 + 2 000 = 2 050 and 2 x 2 050 = 4 100.***

Label the dots and record the results in the table.



Start	□	△
50	2100	4100

Let students suggest other possible starting amounts and record the results in the table, for example:

Start	□	△
50	2100	4100
2	2 004	4 004
1000	4 000	6 000
10 000	22 000	24 000

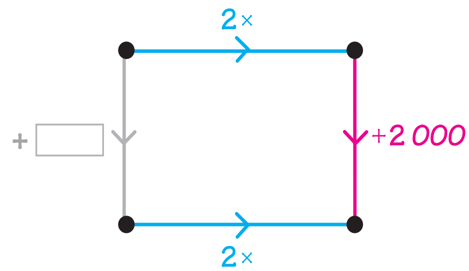
T: ***What do you notice?***

S: ***You always end with \$2 000 more if you go first to Jeanne.***

S: ***That must be because Ali Baba doubles the \$2 000 you receive from Jeanne as well as doubling your starting money.***

Observe that it does matter which genie you go to first because 2x followed by +2 000 is not the same at +2 000 followed by 2x.

Draw this arrow picture on the board.



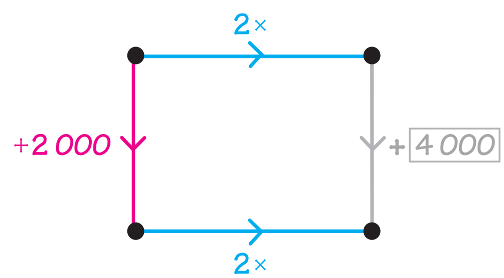
T: ***2x followed by +2 000 is the same as + what number followed by 2x?***

S: ***+1 000. +2 000 followed by 2x always gave us 2 000 more than 2x followed by +2 000, so use +1000 (2 x 1000 = 2 000).***

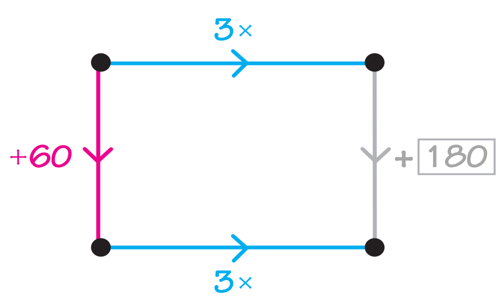
S: ***+1 000. Try different starting numbers and label the dots.***

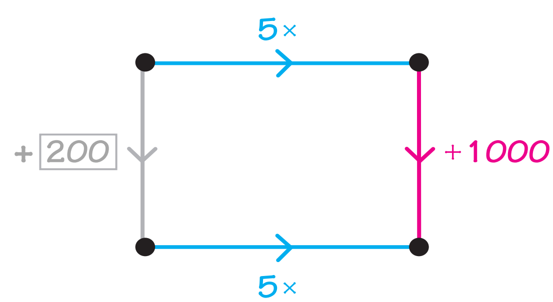
Check that the gray arrow could be for +1 000 by starting with several different numbers at the upper left dot.

Reverse the red and gray arrows in the picture above. In a similar manner, find that +2 000 followed by 2x is the same as 2x followed by +4 000 and label the gray arrow.



Present a couple more similar problems as shown below, and fill in the boxes for the gray arrows.

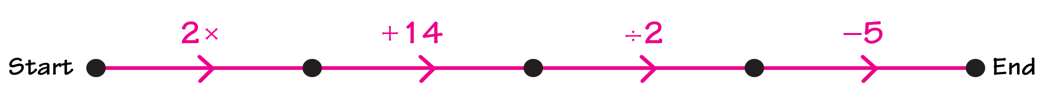




Exercise 4

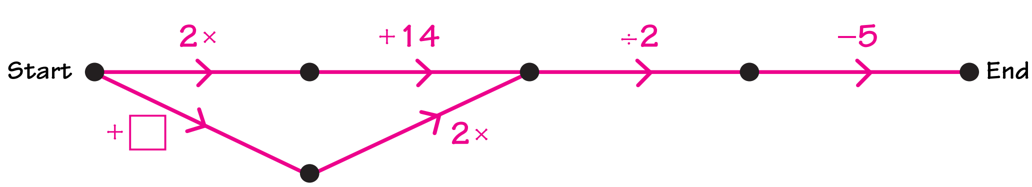
T: ***Now I'm ready to explain my magic trick. You each chose a starting number. Then what calculations did you do to get an ending number?***

Draw the following arrow road on the board as students recall the sequence of calculations.

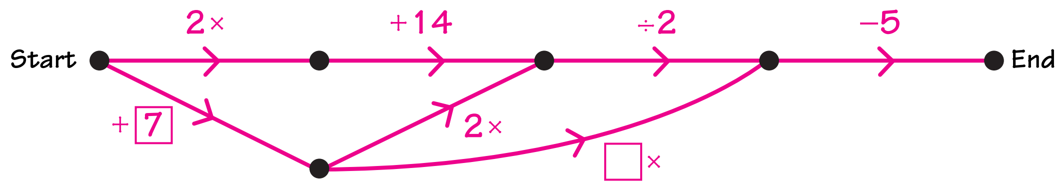


T: ***To explain my trick, let's try to build a shorter, but equivalent, arrow road between the starting number and ending number.***

Add these arrows to the picture.



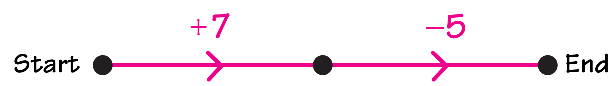
As in Exercise 3, direct students to find that the unlabeled arrow is for +7. Fill in the box and add an arrow, as shown below.



T: ***What could this arrow be for?***

S: *$\frac{2}{2}$x or 1x, since $\frac{2}{2} = 1$.*

Fill in the box and then observe that 1x is really a loop. Decide that a shorter arrow road from start to end could be drawn, as shown here. Some students may have suggested a similar picture earlier, having noted that a 2x arrow and a ÷2 arrow are returns of each other.

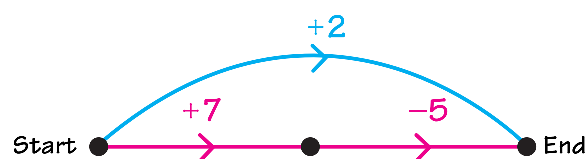


Draw an arrow from the starting number to the ending number.

T: ***What could this arrow be for?***

S: ***+2. +7 followed by –5 is +2.***

Label the arrow.



T: ***We have that the original arrow road, 2x followed by +14 followed by ÷2 followed by –5, is the same as +2. Can you explain my trick?***

S: ***To find a secret starting number, you simply subtract 2 from the ending number.***

Confirm that this method works by either asking students to recall their starting and ending numbers or by repeating the trick. Let one student state an ending number and ask another student to announce the starting number.

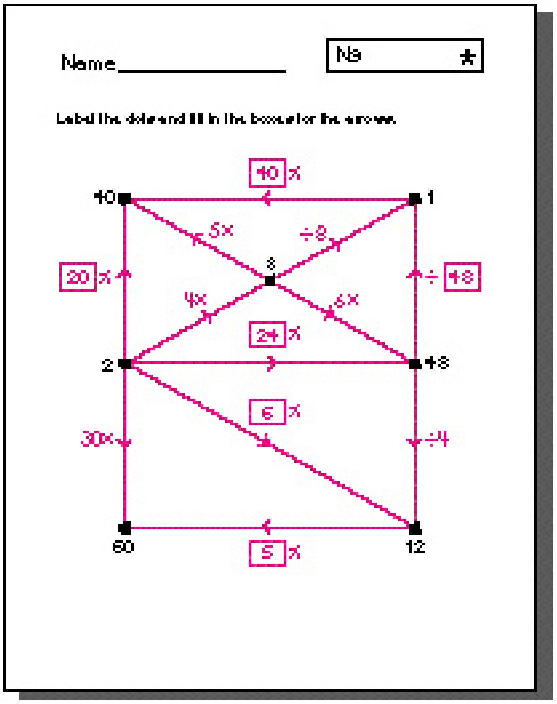
Suggest to students that they try this trick on their friends or that they create another magic number trick.

Worksheets N9*, **, and *** are available for individual work.

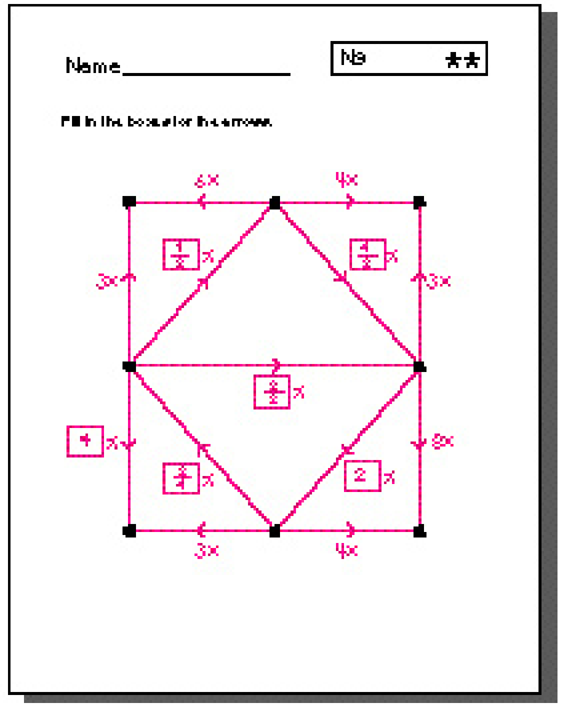
Writing Activity

Suggest that students create a magic number trick like the one in this lesson and then write a description of how to use the trick. The description should include an explanation of why it works.

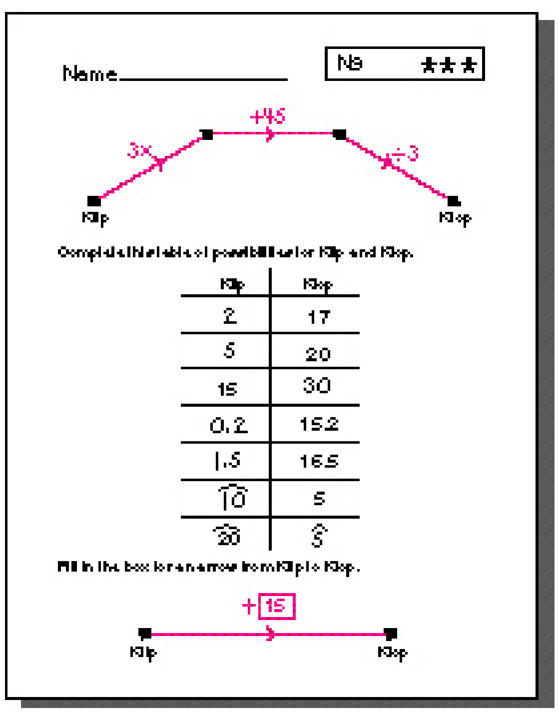
Home Activity


Name__________ N9 ★

Label the dots and fill in the boxes for the arrows.


Name__________ N9 ★★

Fill in the boxes for the arrows.


Name__________ N9 ★★★

Complete this table of possibilities for Klip and Klop.

Klip	Klop
2	17
5	20
15	30
0.2	15.2
1.5	16.5
-10	5
-20	-5

Fill in the box for an arrow from Klip to Klop.

N10 ADDITION OF RATIONAL NUMBERS

Capsule Lesson Summary

Use a circle model to find mixed names for fractions (for example, $\frac{9}{5} = 1\frac{4}{5}$, and to add and subtract fractions with like denominators. Use an area model to add fractions with unlike denominators.

Materials

Teacher
- Meter stick
- Colored chalk

Student
- Metric ruler
- Colored pencils, pens, or crayons
- Worksheets N10*, **, ***, and ****

Description of Lesson

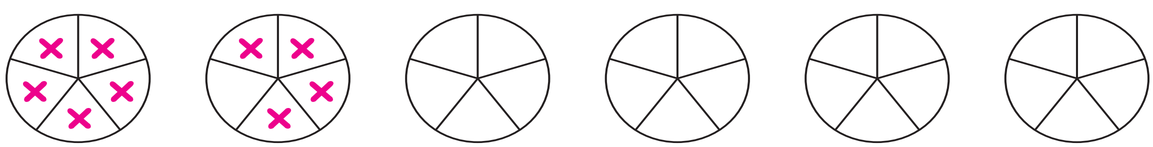
Exercise 1

Note: You may prefer to use a fraction manipulative with circles divided into fifths rather than to draw circles on the board.

Draw six circles divided into fifths on the board.

T: ***Let's pretend that these are six pies each cut fairly into five pieces. What fractional part of a whole pie is each piece?*** ($\frac{1}{5}$) ***We say that the pies are divided into fifths. Who can mark*** $\frac{9}{5}$ ***pies?***

Invite a student to mark pieces for $\frac{9}{5}$; for example:



T: ***What other name for*** $\frac{9}{5}$ ***does this picture suggest?***

S: $1\frac{4}{5}$.

Write the equation on the board.

$$\frac{9}{5} = 1\frac{4}{5}$$

Erase only the marks, and continue the activity of finding mixed† names for the following fractions. Student may, but need not, mark the circles for each problem. (Answers are in boxes.)

$$\frac{13}{5} = \boxed{2\frac{3}{5}} \qquad \frac{21}{5} = \boxed{4\frac{1}{5}} \qquad \frac{30}{5} = \boxed{6} \qquad \frac{39}{5} = \boxed{7\frac{4}{5}}$$

T: ***How did you find that*** $\frac{39}{5} = 7\frac{4}{5}$***?***

S: ***I counted by fives to determine the number of whole pies and the number of extra pieces (fifths).***

S: ***I used division; 39 ÷ 5 is 7 with a remainder of 4.***

---

†*Mixed* here refers to having a whole and a fractional part. Casually use the term *mixed name* throughout the lesson.

N10

Write these fractions on the board, and invite students to circle those that are greater than 1. Student responses are indicated. Encourage students to discuss how one knows when a fraction is greater than 1.

$$\frac{5}{2} \qquad \frac{2}{5} \qquad \frac{7}{5} \qquad \frac{7}{8} \qquad \frac{13}{6} \qquad \frac{19}{3} \qquad \frac{21}{25}$$

S: ***For $^{13}/_{6}$ consider marking pies each cut into six equal pieces. Thirteen pieces would be more than two pies.***

S: ***If the numerator of a fraction is greater than its denominator, the fraction is greater than 1.***

Emphasize this last observation. Then for each of the fractions greater than 1, ask students to give mixed names. (Answers are in boxes.)

$$\frac{5}{2} = \boxed{2\tfrac{1}{2}} \qquad \frac{7}{5} = \boxed{1\tfrac{2}{5}} \qquad \frac{13}{6} = \boxed{2\tfrac{1}{6}} \qquad \frac{19}{3} = \boxed{6\tfrac{1}{3}}$$

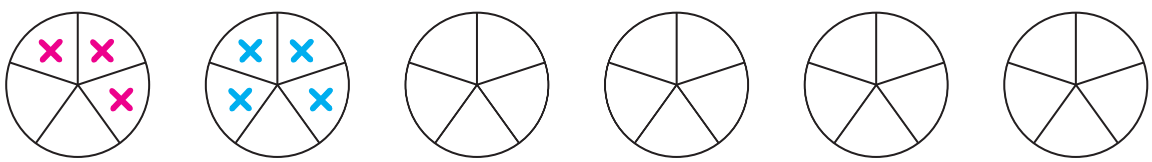
Exercise 2

Erase the board except for the divided circles. Write an addition problem under the circles.

T: ***What number is $^{3}/_{5} + ^{4}/_{5}$?***

$$\frac{3}{5} + \frac{4}{5} =$$

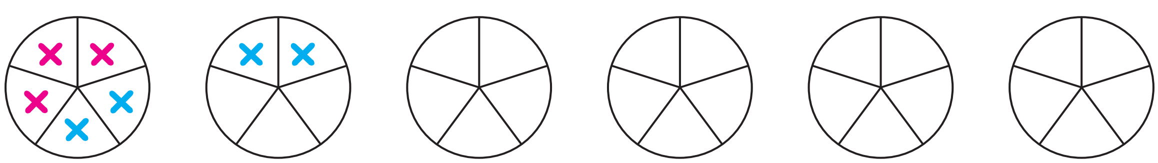
Let students answer and then check with the circles. Invite one student to mark in red $^{3}/_{5}$ pie and a second student to mark in blue $^{4}/_{5}$ pie; for example:



T: ***How much pie is marked?***

S: ***$^{7}/_{5}$ or $1^{2}/_{5}$.***

If necessary, suggest moving two blue marks to the first pie to see $1^{2}/_{5}$ more clearly.



$$\frac{3}{5} + \frac{4}{5} = \frac{7}{5} = 1\frac{2}{5}$$

If some students thought that $^{3}/_{5} + ^{4}/_{5} = ^{7}/_{10}$, you may wish to discuss why adding numerators and denominators in not a good method. Observe that $^{7}/_{10}$ is less than 1, but clearly $^{3}/_{5} + ^{4}/_{5}$ is more than 1. (Use the circles to illustrate this.) Also, point out that in using the circles we add three pieces and four pieces to get seven pieces, but that each piece is a fifth, not a tenth.

Solve the following problems in a similar manner using the circles model. Let students calculate and discuss the results before marking circles to find the sums. (Answers are in boxes.)

$$2\tfrac{1}{5} + 1\tfrac{1}{5} = \boxed{3\tfrac{2}{5}} \qquad 1\tfrac{2}{5} + \tfrac{4}{5} = \boxed{1\tfrac{6}{5} = 2\tfrac{1}{5}}$$

$$3\tfrac{3}{5} + 2\tfrac{2}{5} = \boxed{5\tfrac{5}{5} = 6} \qquad 2\tfrac{4}{5} + 2\tfrac{4}{5} = \boxed{4\tfrac{8}{5} = 5\tfrac{3}{5}}$$

Present the following subtraction problems in a similar manner.

$$3 - \tfrac{1}{5} = \boxed{2\tfrac{4}{5}} \qquad 4\tfrac{1}{5} - 1\tfrac{4}{5} = \boxed{2\tfrac{2}{5}}$$

$$4\tfrac{3}{5} - 3\tfrac{1}{5} = \boxed{1\tfrac{2}{5}} \qquad 3\tfrac{2}{5} - 1\tfrac{3}{5} = \boxed{1\tfrac{4}{5}}$$

Observe that addition and subtraction of fractions with equal denominators are fairly easy, especially when you use the circles. There is nothing special about fifths, the same method works with other fractions. For example:

T: ***What number is $\frac{7}{8} + \frac{5}{8}$?***

S: ***$\frac{12}{8}$ is $1\frac{4}{8}$ or $1\frac{1}{2}$.***

$$\frac{7}{8} + \frac{5}{8} = \frac{12}{8} = 1\frac{4}{8} = 1\frac{1}{2}$$

Mention how you could use circles cut into eighths to help with the addition.

T: ***What number is $2\frac{7}{10} - \frac{9}{10}$.***

Without drawing circles, accept some students' answers. For example:

S: ***$1\frac{8}{10}$. First take $\frac{7}{10}$ from each number; then $2\frac{7}{10} - \frac{9}{10} = 2 - \frac{2}{10} = 1\frac{8}{10}$.***

S: ***$1\ \frac{8}{10}$. $\frac{9}{10} + \frac{1}{10} = 1$ and $2\frac{7}{10} - 1 = 1\frac{7}{10}$, so $2\frac{7}{10} - \frac{9}{10} = 1\frac{7}{10} + \frac{1}{10} = 1\frac{8}{10}$.***

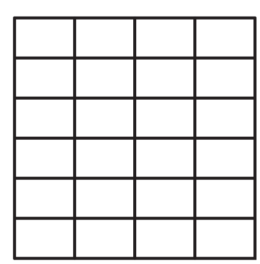
S: ***$1\frac{4}{5}$. $2\frac{7}{10} = \frac{27}{10}$, $\frac{27}{10} - \frac{9}{10} = \frac{18}{10} = \frac{9}{5} = 1\frac{4}{5}$.***

Complete the equation on the board. You may wish to use circles to check the result or to explain $\frac{8}{10} = \frac{4}{5}$.

$$2\frac{7}{10} - \frac{9}{10} = 1\frac{8}{10} = 1\frac{4}{5}$$

Exercise 3

On the board, draw a large square divided equally into 24 pieces.



Trace three columns of the square.

T: ***If these three columns were shaded, how much of the square would be shaded?***

S: ***$\frac{3}{4}$. Three of our four columns would be shaded.***

S: ***$\frac{18}{24}$. 18 out of 24 small pieces would be shaded.***

Write the equivalence on the board.

$$\frac{3}{4} = \frac{18}{24}$$

Trace three rows of the square.

T: ***If these three rows were shaded, how much of the square would be shaded?***

S: ***$\frac{3}{6}$. Three out of six rows.***

S: ***$\frac{12}{24}$. 12 out of 24 small pieces.***

S: ***$\frac{1}{2}$.***

Write the equivalences on the board.

$$\frac{3}{6} = \frac{12}{24} = \frac{1}{2}$$

In a similar manner, referring to appropriate parts of the large square, generate the following equivalent fractions.

$$\frac{1}{4} = \frac{6}{24} \qquad \frac{2}{6} = \frac{8}{24} = \frac{1}{3} \qquad \frac{2}{4} = \frac{1}{2} = \frac{12}{24}$$

T: ***How could we shade four-sixths of the square?***

S: ***Shade four of the six rows.***

T: ***What other name for $\frac{4}{6}$ would this suggest?***

S: ***$\frac{16}{24}$.***

Note the equivalences $\frac{4}{6} = \frac{16}{24} = \frac{2}{3}$. Similarly, find that $\frac{3}{4} = \frac{18}{24}$ and that $\frac{5}{6} = \frac{20}{24}$.

Write this addition problem on the board and let students suggest answers. Record their suggestions on the board, but do not be concerned about a correct answer yet. If some students suggest that $\frac{2}{3} + \frac{4}{5} = \frac{6}{8}$ (or any number less than 1), make the following observations.

$$\frac{2}{3} + \frac{4}{5} =$$

T: ***Is $\frac{2}{3}$ more or less than $\frac{1}{2}$?*** (More)
Is $\frac{4}{5}$ more or less than $\frac{1}{2}$? (More)
If $\frac{2}{3}$ and $\frac{4}{5}$ are each more than $\frac{1}{2}$, what do we know about $\frac{2}{3} + \frac{4}{5}$?

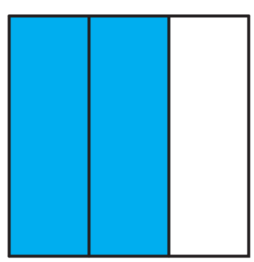
S: ***It must be more than 1.***

Erase any answers less than 1. Draw two squares on the board, side by side. 60 cm by 60 cm is a convenient size.

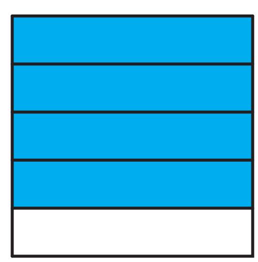
T: ***Since the denominators are not the same, it is not easy to use pies to add $\frac{2}{3} + \frac{4}{5}$. Let's solve this addition problem using the method of dividing these squares into rows and columns.***

Veteran *CSMP* students were introduced to this method in fifth grade. As much as possible, let them lead you through the following procedure.

Ask a student to use a meter stick to divide the left square into three columns in order to shade $\frac{2}{3}$. Ask another student to divide the right square into five rows in order to shade $\frac{4}{5}$.



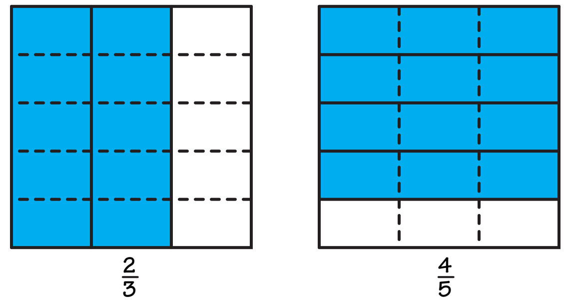
$\frac{2}{3}$



$\frac{4}{5}$

T: ***It is still hard to add $\frac{2}{3} + \frac{4}{5}$ because the pieces of the two squares are not the same size. One way to make smaller pieces all the same size is to draw the same rows and columns in both squares.***

Use a meter stick to accurately divide the squares.



T: ***Do these pictures suggest other names for $\frac{2}{3}$ and for $\frac{4}{5}$?***

S: ***$\frac{2}{3} = \frac{10}{15}$.***

S: ***$\frac{4}{5} = \frac{12}{15}$.***

Include this information in the problem on the board.

T: ***We've changed a difficult problem into an easier problem. What number is $\frac{10}{15} + \frac{12}{15}$?***

S: ***$\frac{22}{15}$ or $1\frac{7}{15}$.***

Complete the calculation on the board.

$$\frac{2}{3} + \frac{4}{5} = \frac{10}{15} + \frac{12}{15} = \frac{22}{15} = 1\frac{7}{15}$$

Recall whether any students correctly predicted that $\frac{2}{3} + \frac{4}{5} = 1\frac{7}{15}$.

Worksheets N10*, **, ***, and **** are available for individual work.

If several students have trouble getting started on N10*, collectively review this area method for adding fractions.

Home Activity

You may like to send home worksheets, one like either N10* or ** and one like either N10*** or ****, to illustrate for parents these methods of adding fractions with unlike denominators.

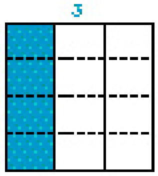
N10

Name______________ N10 ★

$\frac{1}{3} + \frac{3}{4}$

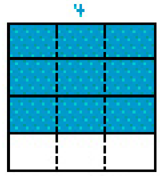
Use the pictures and a ruler to help with the addition problem.

$\frac{1}{3}$



$\frac{1}{3} = \frac{4}{12}$

$\frac{3}{4}$



$\frac{3}{4} = \frac{9}{12}$

Complete the calculation.

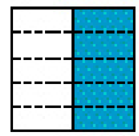
$\frac{1}{3} + \frac{3}{4} = \frac{\boxed{4}}{12} + \frac{\boxed{9}}{12} = \frac{13}{12} = 1\frac{1}{12}$

Name______________ N10 ★★

$\frac{1}{2} + \frac{2}{5}$

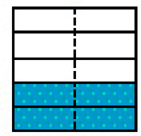
Use the pictures and a ruler to help with the addition problem.

$\frac{1}{2}$



$\frac{1}{2} = \frac{5}{10}$

$\frac{2}{5}$



$\frac{2}{5} = \frac{4}{10}$

Complete the calculation.

$\frac{1}{2} + \frac{2}{5} = \frac{\boxed{5}}{10} + \frac{\boxed{4}}{10} = \frac{9}{10}$

Name______________ N10 ★★★

$\frac{2}{3} + \frac{3}{5}$

Write at least three other fractional names for each fraction. Make sure you include names for $\frac{2}{3}$ and $\frac{3}{5}$ with the same denominator.

$\frac{2}{3}$	$\frac{3}{5}$
$\frac{4}{6}$	$\frac{6}{10}$
$\frac{6}{9}$	$\frac{9}{15}$
$\frac{8}{12}$	$\frac{12}{20}$
$\frac{10}{15}$	$\frac{15}{25}$
$\frac{12}{18}$	$\frac{18}{30}$

Many fractional names are possible.

Complete the calculation.

$\frac{2}{3} + \frac{3}{5} = \frac{10}{15} + \frac{9}{15} = \frac{19}{15} = 1\frac{4}{15}$

Name______________ N10 ★★★★

$\frac{1}{4} + \frac{3}{10}$

Write at least three other fractional names for each fraction. Make sure you include names for $\frac{1}{4}$ and $\frac{3}{10}$ with the same denominator.

$\frac{1}{4}$	$\frac{3}{10}$
$\frac{2}{8}$	$\frac{6}{20}$
$\frac{3}{12}$	$\frac{9}{30}$
$\frac{5}{20}$	$\frac{12}{40}$
$\frac{6}{24}$	$\frac{15}{50}$
$\frac{10}{40}$	$\frac{30}{100}$

Many fractional names are possible.

Complete the calculation.

$\frac{1}{4} + \frac{3}{10} = \frac{5}{20} + \frac{6}{20} = \frac{11}{20}$

Capsule Lesson Summary

Put selected numbers on the display of a calculator using a restricted set of keys. Find solutions which require pressing keys at most ten times. Build arrow roads between two given numbers with arrows for +, –, x, or ÷ numbers whose digits are 4 or 6. Try to find shortest such arrow roads.

Materials

Teacher		Student	
	• Calculator • Colored chalk		• Calculator • Colored pencils, pens, or crayons • Paper

Description of Lesson

Exercise 1

List these calculator keys on the board, and refer to them as you give the following directions.

$\boxed{4}$ $\boxed{6}$ $\boxed{=}$
$\boxed{+}$ $\boxed{-}$ $\boxed{\times}$ $\boxed{\div}$
$\boxed{M+}$ $\boxed{M-}$ $\boxed{MRC}$

T: ***Today we are going to solve some calculator puzzles using only certain keys on the calculator. The only number keys you may press are $\boxed{4}$ and $\boxed{6}$. Also, you can add, subtract, multiply,***
or divide. You can, of course, press $\boxed{=}$ at any time, and you can use the memory keys $\boxed{M+}$, ȣ , and $\boxed{MRC}$. Try to put 9 on the display of your calculator.

9:
6 ÷ 6 + 4 = =
6 ÷ 6 M+ 6 + 4 – MRC =
+ 6 = = = = = = = ÷ 4 =
4 4 ÷ 4 – 6 + 4 =
6 6 – 6 ÷ 4 – 6 =
6 + 4 M+ 4 ÷ 4 M- MRC

Allow several minutes for students to explore this situation. There are many solutions; encourage students to find several solutions. When a student offers a solution invite another student to check it. On the board, make a list of solutions your students offer, as illustrated here.

Continue this activity, putting on other numbers such as 23, 99, 0.5, and 0.1. Feel free to adjust your choice of numbers to the abilities of your students. Write all the numbers on the board with sufficient space between them to record several solutions for each number. Provide time for individual work, allowing students to choose the order in which they work on these numbers. You may challenge the class to think about shorter solutions by introducing the condition that it costs a penny (or a dollar) to press a key. Ask for solutions that cost 10¢ or less.

23:	6 6 ÷ 6 + 6 = =	8¢
	6 – 4 M+ 4 6 ÷ MRC =	9¢
	+ 4 6 = = ÷ 4 =	8¢
	6 ÷ 6 M+ 4 × 6 – MRC =	10¢
99:	4 4 ÷ 4 + 4 4 = =	9¢
	6 6 ÷ 4 × 6 =	7¢
	6 ÷ 4 × 6 6 =	7¢
	6 6 ÷ 6 M+ 6 6 + MRC = = =	12¢

0.5:	6 – 4 ÷ 4 =	6¢
	+ 4 = = ÷ 4 = =	8¢
	4 ÷ 4 M+ 6 ÷ 4 – MRC =	10¢
	6 ÷ 4 M+ + MRC ÷ 6 =	9¢
0.1:	6 + 4 ÷ = =	5¢
	6 6 – 6 M+ 6 ÷ MRC =	9¢
	6 6 M+ 6 M- ÷ MRC =	8¢
	6 + 4 M+ 4 ÷ 4 ÷ MRC =	10¢

Exercise 2

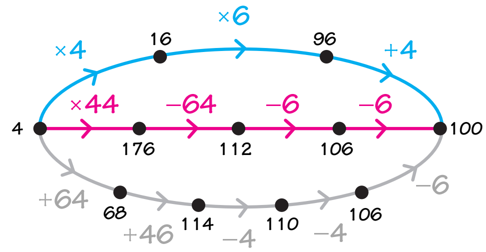
Pose a different type of calculator puzzle using just the keys 4, 6, +, −, ×, ÷, and =.

T: ***For our next problem, an arrow can be for add, subtract, multiply, or divide by some number whose digits are 4 or 6.***

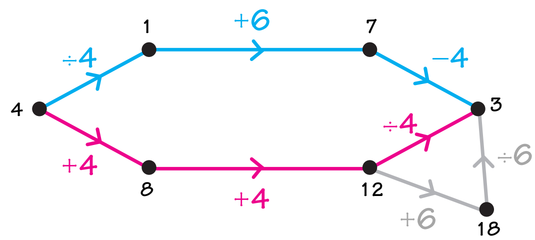
Ask for examples of such arrows. Include possibilities such as ÷6, +46, –666, and x464.

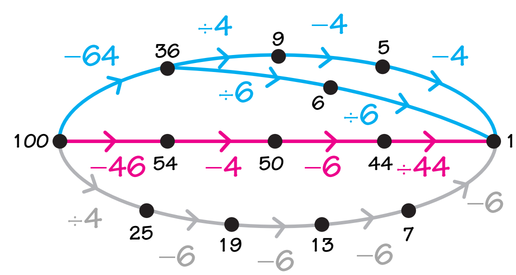
T: ***Now, the problem is to build an arrow road from 4 to 100 using only these kinds of arrows. Try to find a shortest road; that is, a road with the fewest number of arrows.***

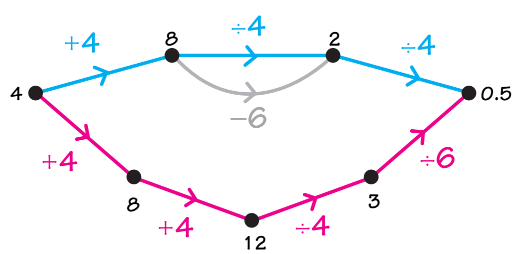
Allow several minutes for students to work independently on this problem. Encourage each student to find at least one solution even if it is not a shortest road; challenge some students to find more than one solution. When most students have found at least one solution, send volunteers to draw their roads on the board. There are many possible solutions, three of which are given here. Notice that a shortest solution has three arrows.



Continue this activity by inviting students to draw other roads, for example, from 4 to 3, from 100 to 1, or from 4 to 0.5. Some solutions are given below.







Home Activity

This would be a good time to send a letter to parents/guardians about basic facts. Blackline N11 has a sample letter.

N12 DECIMALS #3

Capsule Lesson Summary

Play a *Guess My Rule* game with a secret operation ∗ where a ∗ b = (a x b) + 1. Determine the conditions on n for n x 12 to be more (less) than 12. Estimate the product of two decimal numbers. Multiply decimal numbers using equivalences such as 0.3 = $\frac{3}{10}$ and 0.8 = $\frac{8}{10}$ to get 0.3 x 0.8 = 0.24.

Materials

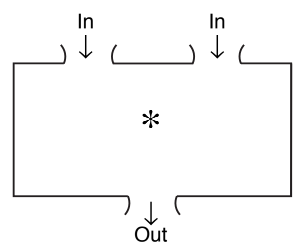
Teacher	• Colored chalk	**Student**	• Paper • Worksheets N12*, **, ***, and ****

Description of Lesson

Exercise 1: Guess My Rule

When playing *Guess My Rule*, encourage students to think about what the rule could be without saying it out loud. This will allow other students a chance to discover the rule on their own. You may let students who think that they know the rule test it on numbers suggested by you or by other students. Each time, confirm or deny the result.

For this exercise the rule is for an operation. You may like to use a machine picture to explain how an operation works.



T: ***I have a secret rule for *. This rule is like a machine. If I put two (a pair of) numbers into the machine, it sends one number out.***

I'll give you some clues using a star (∗) for the operation. Try to figure out the secret rule for ∗.

Write several number sentences on the board as clues. Then write an open sentence and see if anyone can predict which number goes in the box.

$3 * 4 = 13$
$4 * 3 = 13$
$5 * 5 = 26$
$1 * 9 = 10$
$7 * 8 = \boxed{}$

Note: The rule is a ∗ b = (a x b) + 1. In this case, the number in the box is 57 because 7 ∗ 8 = (7 x 8) + 1 = 57.

You might suggest that students write their guesses on paper for you to check. After a moment, let a student announce that 57 is in the box. Otherwise, put 57 in the box yourself and continue with other open sentences. For example:

$8 * 7 = \boxed{57}$
$4 * 0 = \boxed{1}$
$9 * 2 = \boxed{19}$

Continue the activity with the following problems. (Answers are in boxes.)

$4 * \boxed{6} = 25$ $\quad$ $\boxed{6} * 7 = 43$ $\quad$ $\boxed{8} * \boxed{8} = 65$

(The same number must go in each box)

When many students know the rule, let one explain it to the class. Insist on a clear explanation, and write the rule on the board as described.

S: ***Multiply the two numbers and add 1 to the product. For example, 4 * 6 = (4 x 6) + 1 = 24 + 1 = 25.***

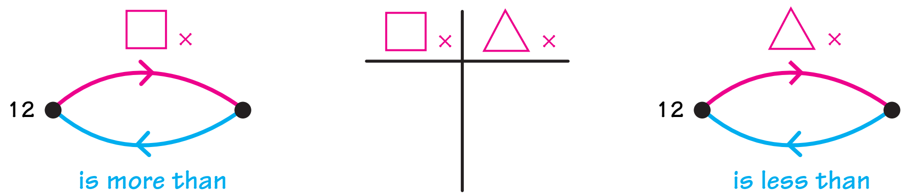
$$\boxed{a * b = (a \times b) + 1}$$

Let students practice with the rule by solving some more problems. Include problems involving decimal numbers. (Answers are in boxes.)

$2 * 7 = \boxed{15}$	$2 * 0.4 = \boxed{1.8}$	$6 * 9 = \boxed{55}$
$9 * 8 = \boxed{73}$	$\boxed{6.5} * 2 = 14$	$6 * 0.9 = \boxed{6.4}$
$\boxed{9} * 5 = 46$	$7 * \boxed{\widehat{4}} = \widehat{27}$	$6 * 0.09 = \boxed{1.54}$

Exercise 2

At the board, draw these arrow pictures and the table.



Let students suggest numbers to go in the box and in the triangle, for example,

S: ***3 could be in the box, since 3 x 12 = 36 and 36 is more than 12.***

S: ***¾ could be in the triangle, since ¾ x 12 = 9 and 9 is less than 12.***

Record suggestions in the table, and encourage students to consider various types of numbers: whole numbers, negative numbers, fractions, and decimal numbers. Lead the class to generalize the results. You may use false starts to gradually find complete solutions; for example:

S: ***Any whole number can go in the box.***

T: ***Can any other numbers go in the box?***

S: ***Yes, 2.5 x 12 = 30 and 30 is more than 12. I think that any number 2 or more can go in the box.***

S: ***Also, any number between 1 and 2 can go in the box; for example, 1.5 x 12 = 18 and 18 is more than 12.***

Continue until students conclude that any number greater than 1 can go in the box and any number less than 1, including all negative numbers, can go in the triangle.

T: ***What about 1?***

S: ***1 cannot be in the box nor in the triangle since 1 x 12 = 12 and 12 is not more nor less than 12. You could draw a loop at 12 for 1x.***

Exercise 3

Write this expression on the board. $3.7 \times 42.28 = 1\,5\,6\,4\,3\,6$

T: ***This number sentence is correct except that a decimal point is missing in the product.***

S: ***Make it 156.436, since 3.7 x 42.28 is approximately 4 x 40 or 160 which is close to 156.436.***

Other estimates, for example, between 3 x 40 and 4 x 50, also lead to the solution, 156.436.

Present these problems in a similar manner, encouraging estimation. The arrows indicate correct placement of a decimal point.

$$0.84 \times 46.19 = 38\;7996$$
$$16.95 \times 3.8 = 64\;41$$
$$8.7 \times 19.5 = 169\;65$$

Pose another similar problem with this expression. $421 \times 617 = 25.9757$

T: ***Put decimal points in the two numbers on the left to make this number sentence true.***

Using estimation, lead students to find three possible solutions. Students may comment on patterns they notice or on methods other than estimation they use to find solutions.

$$4.21 \times 6.17 = 25.9757$$
$$.421 \times 61.7 = 25.9757$$
$$42.1 \times .617 = 25.9757$$

Exercise 4

Write equivalences on the board as you ask,

T: ***What division calculation does 9/10 suggest?***

S: ***9 ÷ 10.***

T: ***What decimal number is 9 ÷ 10?***

S: ***0.9.***

$$\frac{9}{10} = 9 \div 10 = 0.9$$

Present the following equivalences in a similar manner. (Answers are in boxes.)

$$\frac{56}{10} = \boxed{56 \div 10 = 5.6}$$
$$\frac{56}{100} = \boxed{56 \div 100 = 0.56}$$
$$\frac{56}{1000} = \boxed{56 \div 1000 = 0.056}$$
$$\frac{9}{100} = \boxed{9 \div 100 = 0.09}$$
$$\frac{8}{5} = \boxed{\frac{16}{10} = 16 \div 10 = 1.6}$$
or
$$\frac{8}{5} = \boxed{8 \div 5 = 1.6}$$

Let students give a decimal name for each of the following fractions without necessarily stating a suggested division calculation. (Answers are in boxes.)

$$\frac{82}{100} = \boxed{0.82} \qquad \frac{8}{100} = \boxed{0.08}$$
$$\frac{256}{100} = \boxed{2.56} \qquad \frac{361}{10} = \boxed{36.1}$$

N12

Write this problem on the board.

T: ***What number is 0.3* x *0.8?*** (0.24)

$$0.3 \times 0.8 =$$

Ask for an explanation of how to do this calculation. Try to include a discussion of fractional equivalents in the explanation.

T: ***What are fractions for 0.3 and 0.8?***

S: ***$\frac{3}{10}$ and $\frac{8}{10}$.***

T: ***What number is $\frac{3}{10}$* x *$\frac{8}{10}$?***

S: ***$\frac{24}{100}$.***

T: ***What is the decimal name for $\frac{24}{100}$?***

S: ***0.24.***

$$.3 \times 0.8 = \frac{3}{10} \times \frac{8}{10} = \frac{24}{100} = 0.24$$

Worksheets N12*, **, ***, and **** are available for individual work.

Writing/Home Activity

Ask students to write their own missing decimal points problems, similar to those in Exercise 3 or on the * worksheet, to challenge family members or classmates.

Name________________ N12 ★

One number in each number sentence is missing a decimal point. Put a decimal point in the number to make the equation true.

6.32 + 23.9 = 30.22

71.4 − 32.615 = 38.785

209.68 × 3.38 = 708.7184

5.85 + .674 = 6.524

386.17 − 381.77 = 4.4

7.5 × 5.84 = 43.8

Put decimal points in the numbers that are missing them to make the equations true.

21.6 × 2.84 = 61.344

9.2 × .85 = 7.82

Other solutions are possible.

Name________________ N12 ★★

$$a \# b = (a \times b) + 1$$

Complete.

$5 \# 8 = \boxed{41}$ $\quad 70 \# 3 = \boxed{211}$

$7 \# 70 = \boxed{491}$ $\quad 9 \# 7 = \boxed{64}$

$\boxed{4} \# 8 = 33$ $\quad 3 \# \boxed{5} = 16$

$6 \# \boxed{7} = 43$ $\quad 70 \# \boxed{40} = 2801$

$\boxed{6} \# \triangle{4} = 25$ $\quad \boxed{2.4} \# \triangle{10} = 25$

Many solutions are possible for □ # △ = 25.

Name________________ N12 ★★★

Complete.

Fractional Name		Decimal Name	Fractional Name		Decimal Name
$\frac{9}{10}$	=	0.9	$\frac{7}{10}$	=	0.7
$\frac{24}{10}$	=	2.4	$\frac{62}{10}$	=	6.2
$\frac{27}{100}$	=	0.27	$\frac{87}{100}$	=	0.87
$\frac{9}{100}$	=	0.09	$\frac{431}{100}$	=	4.31
$\frac{3}{5}$	=	0.6	$\frac{13}{20}$	=	0.65

Other names are possible.

Name________________ N12 ★★★★

$$a \# b = (a \times b) + 1$$

Complete.

$2 \# 0.3 = \boxed{1.6}$ $\quad 5 \# 0.9 = \boxed{5.5}$

$0.9 \# 8 = \boxed{8.2}$ $\quad 0.9 \# 0.8 = \boxed{1.72}$

$8 \# \boxed{0.3} = 3.4$ $\quad 0.3 \# \boxed{0.6} = 1.18$

Put the same number in each box.

$\boxed{11} \# \boxed{11} = 122$ $\quad \boxed{0.2} \# \boxed{0.2} = 1.04$

Label the dots.

#2

1.125 3.25 7.5 16 33

Capsule Lesson Summary

Introduce the notion of a Cartesian graph of a relation and graph several relations including the +2 function, the distance-from-0 function, and the 2x function.

Materials

Teacher
- Colored chalk
- Coordinate grid
- Blackline N13

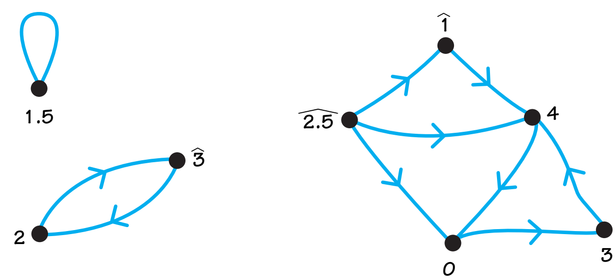
Student
- Colored pencils, pens, or crayons
- Coordinate grids

Advance Preparation: Use Blackline N13 to prepare a coordinate grid for display or prepare a grid board with this coordinate grid. Make copies of coordinate grid sheets for student use in Exercise 2 and 3.

Description of Lesson

Exercise 1

Display a coordinate grid and draw this arrow picture on the board.

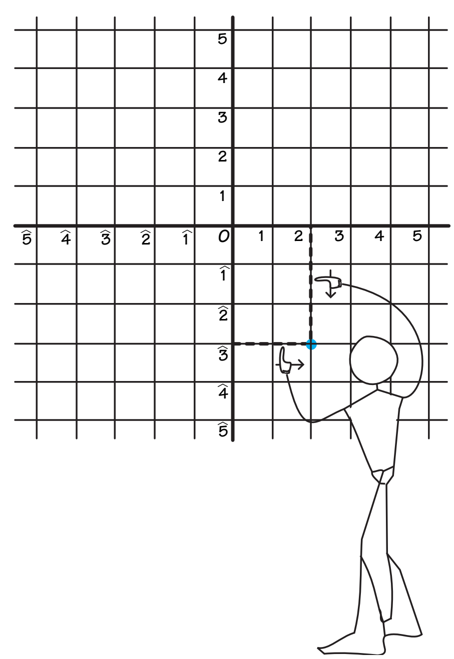


T (tracing the arrow from 2 to $\widehat{3}$)**:** ***This arrow is for which ordered pair of numbers?***

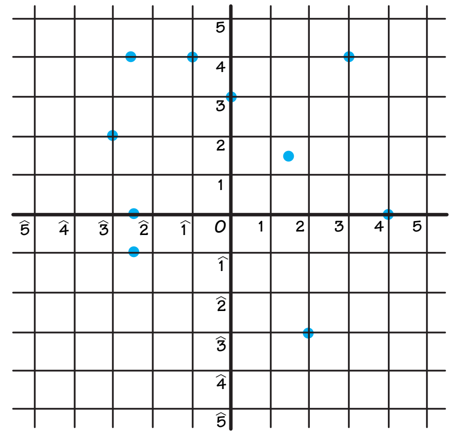
S: ***(2, $\widehat{3}$), because 2 is the starting number and $\widehat{3}$ is the ending number.***

T: ***Where is the point on the grid for (2, $\widehat{3}$)?***

Invite a student to explain how to locate the point.

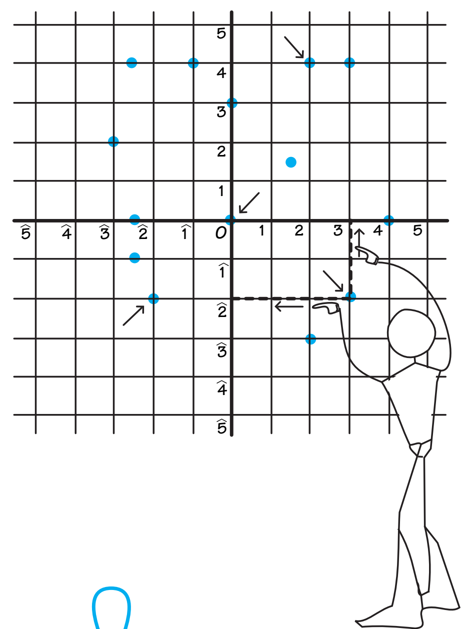


Continue this activity by asking students to locate points for all of the arrows and the loop. Emphasize that the starting number is located along the horizontal axis and the ending number is located along the vertical axis. Upon completion of this activity, there should be ten dots on your grid, as shown here.



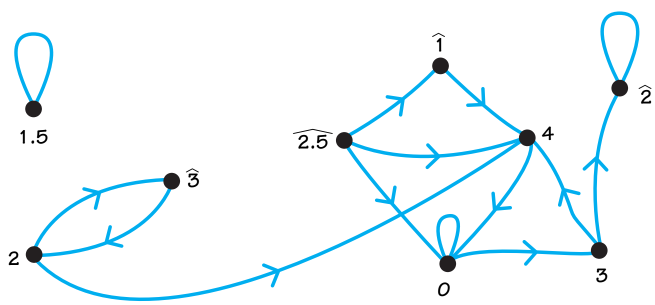
You may refer to the dots on the grid as a *Cartesian graph* for the arrow picture.

Reverse the situation by drawing, one at a time, four new blue dots (as indicated here) on your coordinate grid. Ask students to draw the corresponding arrows and to explain how they know which is the appropriate arrow.



S: ***This point is for the pair (3, $\hat{2}$), so I draw an arrow starting at 3 and ending at $\hat{2}$. We need a dot for $\hat{2}$ in our arrow picture.***

The completed arrow picture is shown below.



Point out that there are several arrows in the picture ending at 4.

T: ***What do you notice about the dots for these arrows on our grid (in the Cartesian graph)?***

S: ***They all are on the horizontal line through 4.***

Check that another dot on this horizontal line in the Cartesian graph would be for an arrow ending at 4.

Point out that there are several arrows in the picture starting at $\widehat{2.5}$.

T: ***What do you notice about the dots for these arrows on our grid (in the Cartesian graph)?***

S: ***They all are on a vertical line through $\widehat{2.5}$.***

Again check that another dot on this vertical line in the Cartesian graph would be for an arrow starting at $\widehat{2.5}$.

Indicate the three dots in the Cartesian graph on a diagonal line through the origin (0, 0).

T: ***How are these dots represented in the arrow picture?***

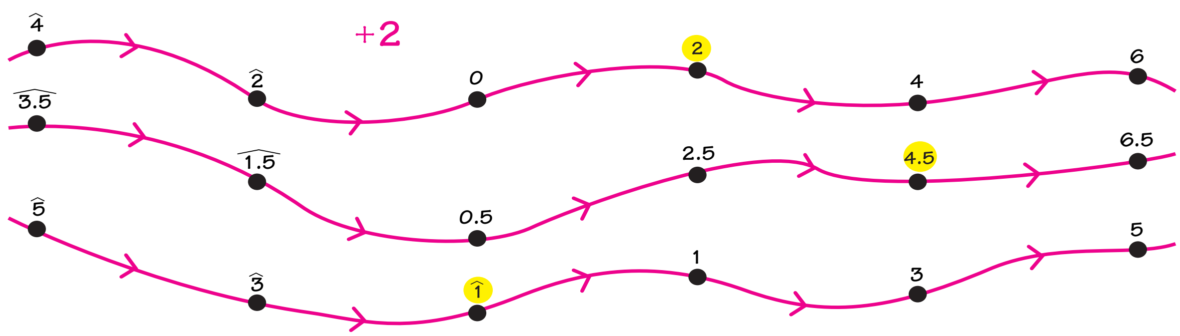
S: ***Each is for a loop; a loop at 1.5, a loop at 0, and a loop at $\hat{2}$.***

Exercise 2

Erase the board and the grid board. Then draw several pieces of a +2 arrow picture, labeling just one dot in each piece (see the highlighted numbers in the picture below).

T: ***This is part of an arrow picture for the +2 relation.***

Ask students to label other dots and to explain how the pieces (roads) could go on and on.



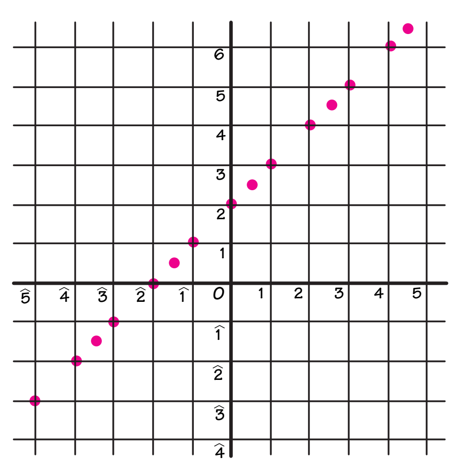
T: ***Do you notice anything interesting about this picture?***

S: ***Even numbers are all on one road and odd numbers are on another road.***

Let the class comment freely about the picture. You may ask students to describe, without drawing, some other pieces of the +2 arrow picture.

T: ***Let's make a Cartesian graph for the +2 relation. We can locate some points on our grid corresponding to the arrows of the +2 relation. Then we will see if there are any interesting patterns in the grid picture (Cartesian graph). This arrow*** (point to the arrow from 2 to 4) ***is for the ordered pair (2, 4). Where is the point on the grid for this arrow?***

Call on a student to draw a red dot at the point for (2, 4). Invite students to locate points for several more arrows. Then direct everyone to draw red dots for the arrows of the +2 relation on their copy of the coordinate grid. When most students have about ten dots on their grid, complete the picture at the board.



T: ***Did we draw a dot for every arrow of the +2 relation?***

S: ***No, there are many other arrows that could be drawn; for example, an arrow starting at 1.5 and ending at 3.5, or an arrow starting at 100 and ending at 102.***

T: ***Could we draw dots at all possible points for arrows?***

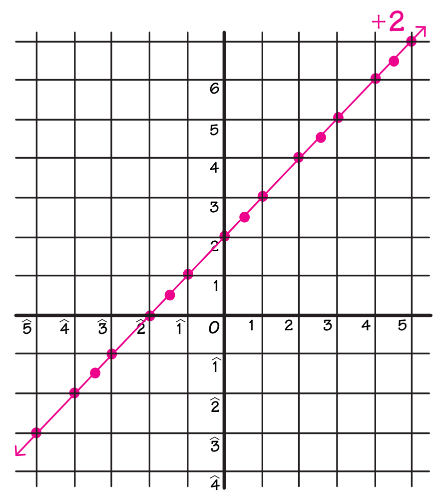
S: ***No, there is an endless number of points.***

T: ***Where are the points for arrows that are missing?***

S: ***Along this line; they are the points on a line connecting the ones we have marked already.***

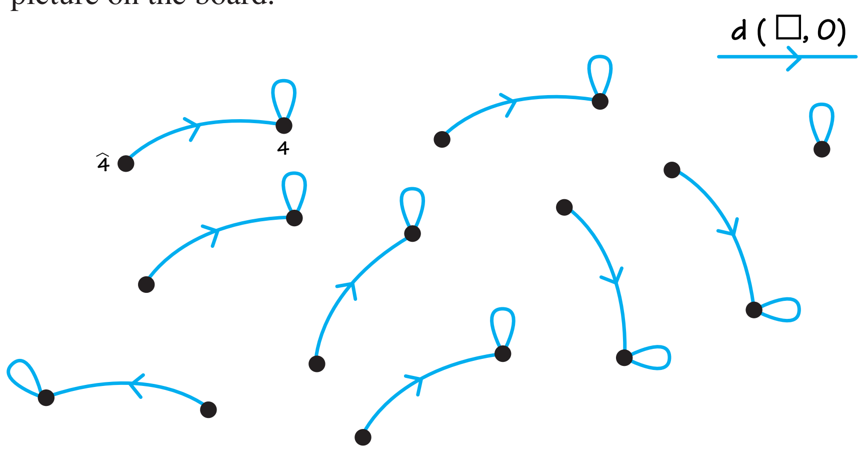
Draw the line in red for the +2 relation.

T: ***This line is the* Cartesian graph *of the +2 relation.***



Exercise 3

Draw this arrow picture on the board.



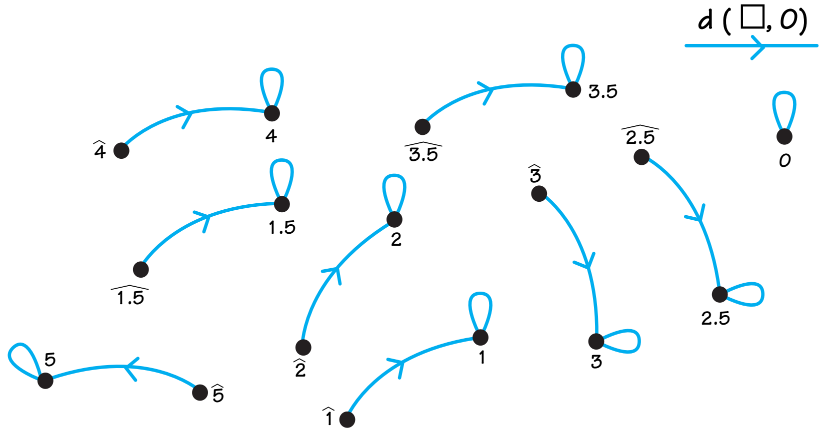
T: ***This is part of an arrow picture for the distance-from-0 relation. Can you explain the distance-from-0 relation?***

Let students give their own interpretation of the relation. Then illustrate the relation with d ($\widehat{4}$, 0). Use the number line to demonstrate.

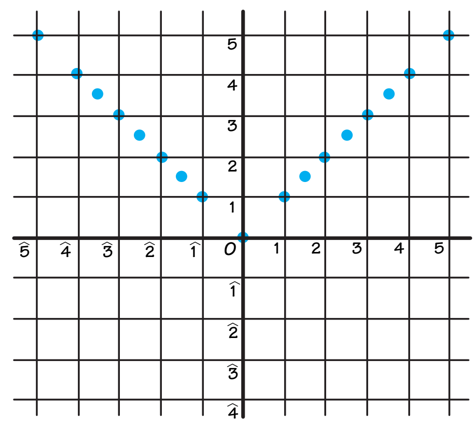
T: ***There is an arrow from $\widehat{4}$ to 4*** (trace the arrow) ***because $\widehat{4}$ is a distance of 4 away from 0 on the number line.***

Also, 4 is at a distance of 4 from 0 on the number line, so there is an arrow from 4 to 4 (trace the loop at 4)**.**

Invite students to provide labels for the remaining dots. Occasionally, ask for an explanation of an arrow or a loop. Encourage students to use numbers between $\hat{8}$ and 8 or numbers that can be located on your coordinate grid. Your arrow picture might look like this upon completion.



Instruct students to locate many points for the distance-from-0 relation on another of their copies of the coordinate grid.



After a short period of individual work, collectively locate points on your coordinate grid.

T: ***Is this a complete picture of the distance-from-0 relation?***

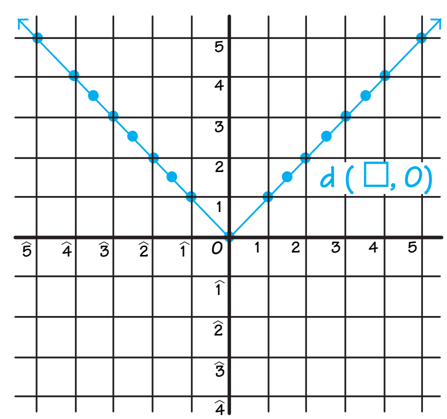
S: ***No, there are infinitely many arrows that could be drawn and so infinitely many points could be found.***

T: ***What do you think the picture would look like if we could draw dots at all possible points?***

S: ***There would be "half" of two slanting lines starting at (0, 0).***

S: ***There would be a square corner at (0, 0).***

Draw the graph in blue and label it.



T: ***Do you notice anything interesting about the distance-from-0 relation?***

Students may make observations about symmetry, or about the fact that distance is never negative.

Capsule Lesson Summary

Explore a new rule which assigns to a number the sum of its digits. Use the arrow picture for this relation to discover a test for divisibility by 9.

Materials

Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N14(a) and (b)

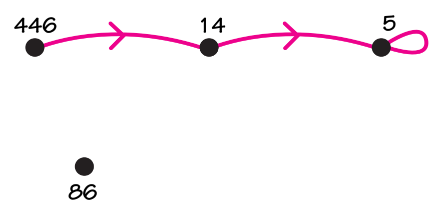
Advance Preparation: There are several related arrow pictures in the two exercises of this lesson. Ideally, you should not have to erase one picture to fit another arrow picture on the board. Plan the spacing so that all of the arrow pictures of both Exercise 1 and Exercise 2 fit on the board at the same time.

Description of Lesson

In this lesson use only whole numbers. You may want to announce to the class that today you will work only with whole numbers.

Exercise 1

Draw this part of an arrow picture on the board.



T: ***I have a secret rule for red arrows. These red arrows and the red loop are clues. Try to figure out the rule for red arrows, but do not tell us the rule yet. Instead, tell us where a red arrow starting at 86 would end.***

Note: The rule assigns to a number the sum of its digits. For example, in the picture, there is an arrow from 446 to 14 since 4 + 4 + 6 = 14. Do not announce the rule at this time.

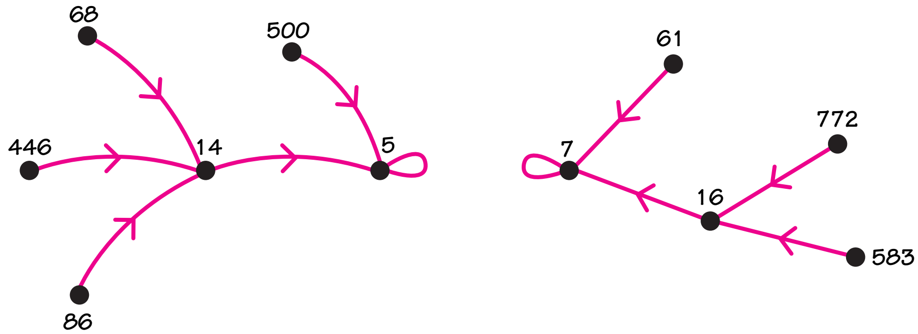
Allow a few minutes for students to study the arrow picture. Then ask students to write their guesses on a piece of paper. If a guess is correct, simply say, "Yes, that is correct," without announcing the number. If incorrect, tell the class, for example, "No, an arrow does not go from 86 to 5." After a while, let a student draw a red arrow from 86 to 14.

Draw a dot for 61 and ask,

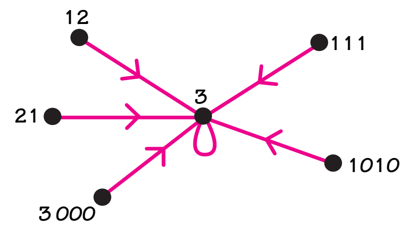
T: ***Where does a red arrow starting at 61 end?***

Check several students' answers before letting a student announce the correct answer (7). If necessary, give the answer yourself and draw an arrow from 61 to 7.

In a similar manner ask, one at a time, about arrows starting at 16, 500, 772, 68, 583, and 7. The following arrow picture should result.



Draw several arrows ending at a dot for 3 as well as a loop at that dot. Do not erase the preceding arrow picture. Without revealing the rule yet, invite students to label the dots, such as illustrated here.

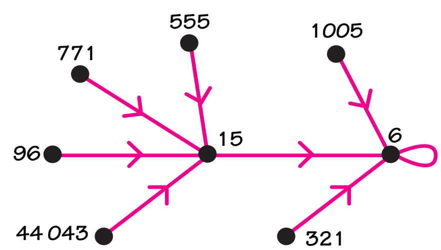


T: ***Can you explain the secret rule for red arrows?***

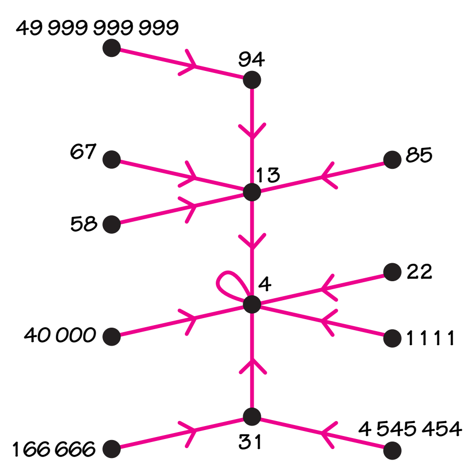
S: ***A red arrow goes from a number to the sum of that number's digits. For example, there is an arrow from 772 to 16 because 7 + 7 + 2 = 16.***

Check that the rule works for other numbers in the arrow pictures on the board. If no one guesses the rule at this point, give further clues or explain it yourself.

Draw this arrow picture on the board with only the dot for 15 labeled. Invite students to label the other dots. Many solutions are possible, such as illustrated here.



Distribute copies of Worksheet N14(a). Instruct students to work independently or with partners on N14(a) while you draw the arrow picture from the worksheet on the board. As they are working, invite students to the board to label the dots. Many solutions are possible, such as illustrated here.



With one possible solution (labeling for the dots) for the worksheet on the board, invite students to suggest alternate solutions, especially for the dots that are starting numbers for arrows ending at 94 and 31.

S: ***9,191,919,191,919,191,914 could also be the starting number for the arrow ending at 94.***

S: ***1,555,555 could also be the starting number for an arrow ending at 31.***

S: ***There are an infinite number of possibilities. For example, arrows from 40; 400; 4 000; 40 000; and so on all end at 4.***

S: ***Also, 31; 301; 3 001; 30 001; and so on all end at 4.***

T: ***Does every whole number have an infinite number of arrows ending at it?***

S: ***All whole numbers except 0. 0 has a loop, but no other arrow ends at 0.***

Draw a dot for 0 and a loop on the board.

T: ***The arrow picture for this rule seems to be in several pieces. On the board, for example, there is a piece in which all numbers are part of arrow roads that lead to 5. In another piece, all the numbers are part of arrow roads that lead to 6. On the worksheet all the numbers are on arrow roads that lead to 4. How many pieces do you think there would be in a full arrow picture of this rule?***

S: ***Ten pieces. A piece with just 0 in it; a piece with 1 in it, with 2, with 3, and so on to 9.***

Let students discuss this situation until the class concludes that there would be exactly ten pieces.

Exercise 2

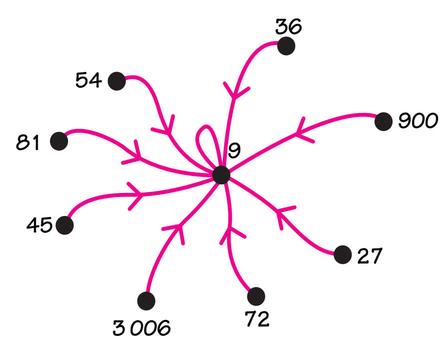
Direct students to complete Worksheet N14(b). Suggest they look for patterns or unusual results.

On the board, draw a dot for 9, a loop at that dot, and many arrows ending at that dot.

When many students have completed the worksheet, hold a class discussion. Let students comment on what numbers they put in their pictures. Perhaps some will notice that the numbers are all multiples of 9.

T: ***Let's label the dots in the picture on the board.***

Invite students, one at a time, to label the dots in the picture. For example:



T: ***What do you notice?***

S: ***All of those numbers are multiples of 9.***

Check any number for divisibility by 9 if there is a question. For example, 3 006 = 999 + 999 + 999 + 9, and the sum of multiples of 9 is a multiple of 9.

T: ***Could all of the multiples of 9 be in this picture?***

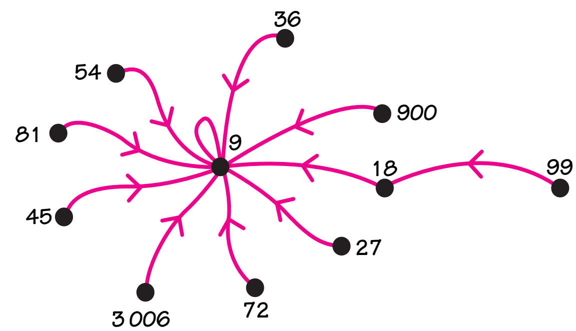
S: ***99 is a multiple of 9, but 9 + 9 = 18.***

T: ***How could we put 99 in this picture?***

S: ***Draw an arrow from 99 to 18 and an arrow from 18 to 9.***

N14

Include the appropriate arrows in the picture.



Let students suggest several other multiples of 9 and show how they connect into the picture. Note that 0 cannot be in the picture although 0 is a multiple of 9.

T: ***So the positive multiples of 9 are all in this piece of the arrow picture with 9 in it. Look at numbers in the piece with 4 in it on Worksheet N14(a). What do you notice about the numbers in this piece?***

S: ***Each number is 4 more than a multiple of 9. For example, 94 is 4 more than 90, and 22 is 4 more than 18.***

T: ***Now look at numbers in the piece with 5 in it (on the board). What do you notice about the numbers in this piece?***

S: ***Each number is 5 more than a multiple of 9. For example, 14 is 5 more than 9, and 86 is 5 more than 81.***

T: ***We have looked at three pieces of the picture. What do you think is true of numbers belonging to the other pieces?***

Allow a few minutes for students to consider the question.

S: ***Numbers that belong to the same piece are all multiples of 9 plus the same number. For example, the numbers belonging to the piece with 7 in it are each 7 more than a multiple of 9.***

You may wish to check a few examples with the class.

List these numbers on the board.

9181
45 009
638
745 457

T: ***In what piece of the picture are each of these numbers?***

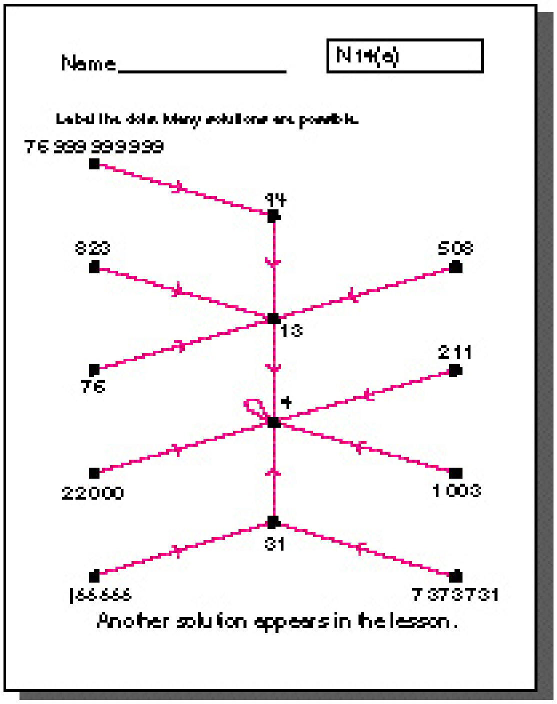
S: ***9 + 1 + 8 + 1 = 19 and 1 + 9 = 10 and 1 + 0 = 1. So 9 181 is in the piece with 1.***

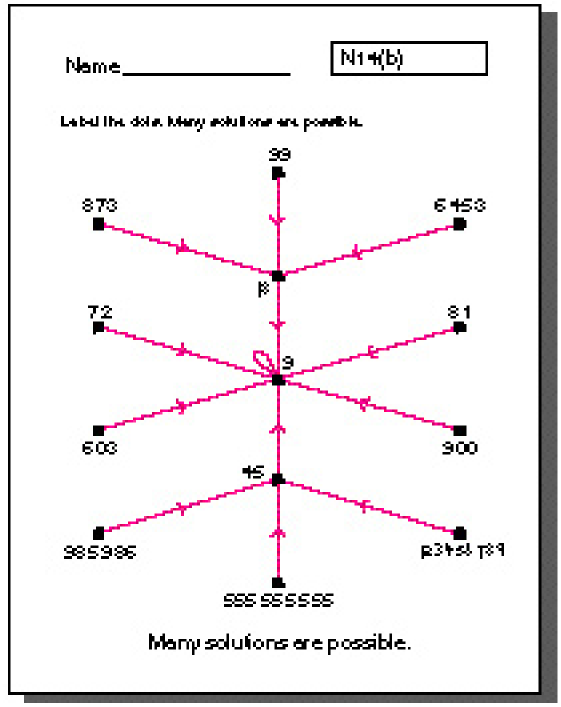
Likewise, let students explain (by adding digits) in which piece each of the other three numbers belongs.

45 009: 4 + 5 + 9 = 18 and 1 + 8 = 9: piece with 9
638: 6 + 3 + 8 = 17 and 1 + 7 = 8: piece with 8
745 457: 7 + 4 + 5 + 4 + 5 + 7 = 32 and 3 + 2 = 5: piece with 5

T: ***If a positive number is a multiple of 9, in what piece of the picture does it belong?***

S: ***In the piece with 9.***


Name
N14(a)
Label the dots. Many solutions are possible.
76 999 999999
14
823
508
13
76
211
4
22000
1003
31
166666
7 373731
Another solution appears in the lesson.


Name
N14(b)
Label the dots. Many solutions are possible.
99
873
6453
18
72
81
9
603
900
45
985986
123456789
555 555555
Many solutions are possible.

Capsule Lesson Summary

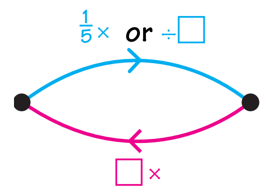
Using an arrow picture, solve some problems involving $\frac{1}{5}$x and $\frac{1}{3}$x. Find labels for several pairs of arrows, one ÷☐ and one ☐x, from 45 to 27. Label the composite arrow in different ways, and introduce *percent* as a new name. Use patterns to do percent calculations.

Materials

Teacher		Student	
	• Colored chalk		• Paper • Worksheets N15* and **

Description of Lesson

Exercise 1



Draw this arrow picture on the board.

T: ***What division name does*** $\frac{1}{5}$x ***have?*** (÷5)
What is the opposite of $\frac{1}{5}$x ***or ÷5?*** (5x)

Fill in the boxes for the arrows. Refer to the arrow picture when completing the following problems. (Answers are in boxes.)

$$\frac{1}{5} \times 20 = \boxed{4} \qquad \frac{1}{5} \times \boxed{60} = 12 \qquad \frac{1}{5} \times 2\,500 = \boxed{500}$$

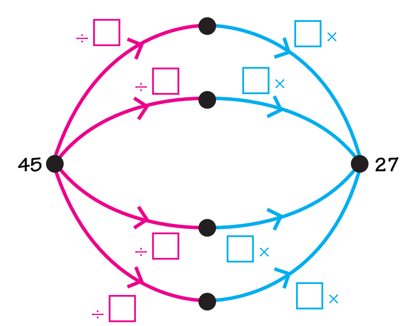
$$\frac{1}{5} \times \boxed{35} = 7 \qquad \frac{1}{5} \times 360 = \boxed{72} \qquad \frac{1}{5} \times \boxed{125} = 25$$

Change the arrows in the picture to $\frac{1}{3}$x or ÷3 and 3x. Then solve these problems. (Answers are in boxes.)

$$\frac{1}{3} \times 27 = \boxed{9} \qquad \frac{1}{3} \times \boxed{36} = 12 \qquad \frac{1}{3} \times 2\,400 = \boxed{800}$$

$$\frac{1}{3} \times 42 = \boxed{14} \qquad \frac{1}{3} \times \boxed{180} = 60 \qquad \frac{1}{3} \times \boxed{1\,200} = 400$$

Exercise 2

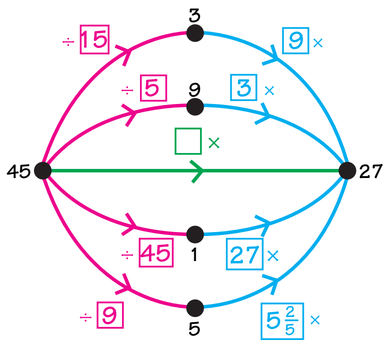


Draw this arrow picture on the board.

Invite students to find labels for pairs of arrows, red followed by blue. Record correct solutions in a table, such as the one shown on the next page.

S: ***If a red arrow is for ÷15, then the following blue arrow is for 9x; 45 ÷ 15 = 3, and 9 x 3 = 27.***

After several solutions have been found, draw a green arrow from 45 to 27.



÷15	9×
÷5	3×
÷45	27×
÷10	6×
÷20	12×
÷9	$5\frac{2}{5}$×

T: ***What could this green arrow be for?***

S: ***$\frac{9}{15}$x, because it is ÷15 followed by 9x.***

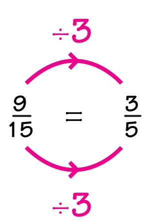
Make a list of possible names for the green arrow. Most will be suggested by the picture and the table, but students may give others. The table can be extended at the same time with more entries for the red and blue arrows.

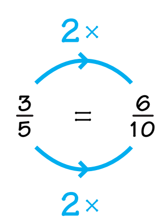
$$\frac{9}{15}\times \quad \frac{3}{5}\times \quad \frac{27}{45}\times \quad \frac{6}{10}\times \quad \frac{12}{20}\times \quad \frac{18}{30}\times \quad \frac{5\frac{2}{5}}{9}\times$$

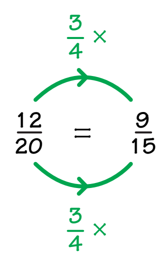
T: ***All of the fractions in these names for the green arrow are equivalent.***

$$\frac{9}{15} = \frac{3}{5} = \frac{27}{45} = \frac{6}{10} = \frac{12}{20} = \frac{18}{30} = \frac{5\frac{2}{5}}{9}$$

With the class, discuss how to check that fractions are equivalent. You may draw arrows between some pairs of equivalent fractions to illustrate a technique for checking equality. For example:





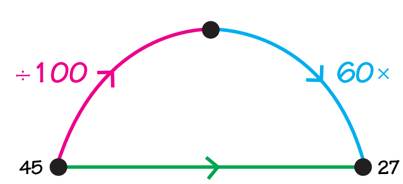


Present each of the following problems, asking students to provide the missing numerator or denominator. (Answers are in boxes.)

$$\frac{3}{5} = \frac{\boxed{30}}{50} \qquad \frac{3}{5} = \frac{\boxed{60}}{100} \qquad \frac{3}{5} = \frac{300}{\boxed{500}} \qquad \frac{3}{5} = \frac{1}{\boxed{\frac{5}{3}}}$$

Return to the table and the arrow picture to include ÷100 followed by 60x, if it is not already listed. Erase all of the other arrows leaving this picture.

T: ***We already know many names for the green arrow. What fraction times do these arrows suggest for the green arrow?*** ($\frac{60}{100}$x) ***Another name for the green arrow is "60% of"*** (read as "sixty percent of")***. "60% of" is a percent name for ÷100 followed by 60x, or for 60x followed by ÷100.***



Include $\frac{60}{100}$x and "60% of" in the list of equivalent names for the green arrows.

Exercise 3

Begin this exercise by asking the class where they sometimes hear percents being used. The discussion might include the following:

- weather (percent chance of rain)
- sports (percent wins or success)
- sales (percent discounts or percent off original price)
- test results (percent correct)

Use these examples to give context to some of the percents mentioned in this exercise.

Draw this table on the board and invite students to complete it. Some students might envision arrow pictures similar to the one at the end of Exercise 2. For example:

- 50% of a number is the same as ½x the number, because 50x followed by ÷100 is the same as ÷2 or ½x.
- 20% = $\frac{20}{100}$ = $\frac{2}{10}$ = $\frac{1}{5}$, because 20% of is ÷100 followed by 20x.

Fraction	Percent
	50%
	25%
$\frac{3}{4}$	
	20%
$\frac{4}{5}$	

Other students may rely on their intuitive knowledge of percent and on the use of patterns; for example:

- 50% attendance means that one-half on the people are present.
- ¾ = 75%, since ¼ = 25% and 3 x 25 = 75.

Highlight this information from the table.

$$\frac{1}{2} = 50\% \quad \frac{1}{4} = 25\% \quad \frac{1}{5} = 20\%$$

T: ***What do you notice?***

S: ***The numerator in each fraction is 1.***

S: ***2* x *50, 4* x *25, and 5* x *20 all equal 100.***

Write the following problems on the board, and ask students to do the calculations using fraction equivalences for the percents or using patterns. (Answers are in boxes.)

$$50\% \text{ of } 60 = \boxed{30} \qquad 100\% \text{ of } 60 = \boxed{60}$$
$$25\% \text{ of } 60 = \boxed{15} \qquad 10\% \text{ of } 60 = \boxed{6}$$
$$75\% \text{ of } 60 = \boxed{45} \qquad 5\% \text{ of } 60 = \boxed{3}$$

Encourage students to explain each calculation in several ways. For example:

S: ***25% of 60 = 15, because 50% of 60 = 30 and ½* x *30 = 15.***

S: ***25% = ¼, so 25% of 60 = ¼* x *60 = 15.***

S: ***75% of 60 = 45, because 25% of 60 = 15 and 3* x *15 = 45.***

S: ***50% of 60 = 30 and 25% of 60 = 15, so 75% of 60 = 30 + 15 = 45.***

S: ***5% of 60 = 3, because 10% of 60 = 6 and ½* x *6 = 3.***

S: ***50% of 60 = 30 and 30 ÷ 10 = 3, so 5% of 60 = 3.***

S: ***"5% of" is ÷100 followed by 5x or $^5/_{100}$x.***
$^5/_{100}$x = $^1/_{20}$x = ÷20, so 5% of 60 = 60 ÷ 20 = 3.

Suggest that students use results already on the board to do these percent calculations. (Answers are in boxes.)

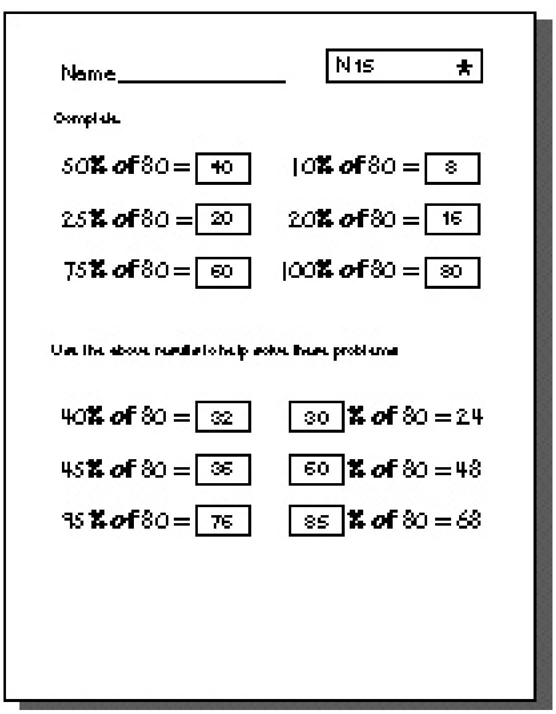
55% of 60 = $\boxed{33}$ | 95% of 60 = $\boxed{57}$
80% of 60 = $\boxed{48}$ | 1% of 60 = $\boxed{0.6}$
40 % of 60 = $\boxed{24}$ | 125 % of 60 = $\boxed{75}$
35% of 60 = $\boxed{21}$ | 8 % of 60 = $\boxed{4.8}$

Accept several explanations for each calculation; for example:

S: ***95% of 60 = 57 because 100% of 60 = 60 and 5% of 60 = 3, so 95% of 60 = 60 – 3 = 57.***

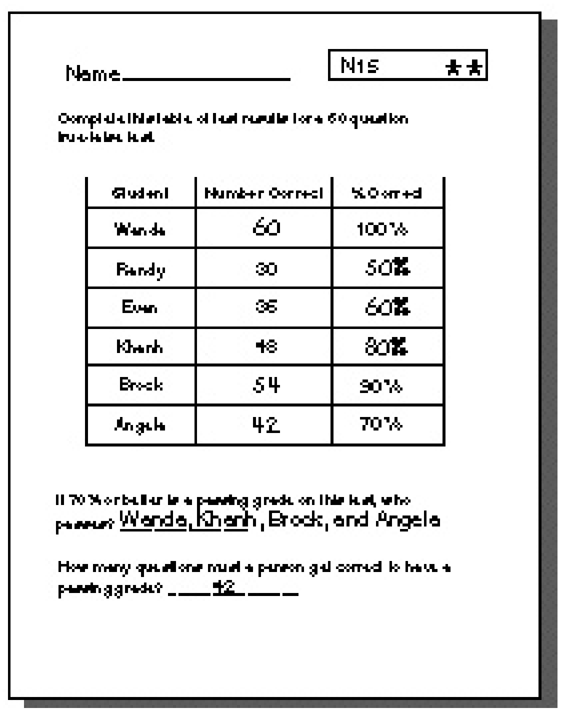
S: ***75% of 60 = 45 and 10% of 60 = 6, so 95% of 60 = 45 + (2 x 6) = 57.***

Worksheets N15* and ** are available for individual work.


Name____________ N15 *

50% of 80 = 40 10% of 80 = 8
25% of 80 = 20 20% of 80 = 16
75% of 80 = 60 100% of 80 = 80

40% of 80 = 32 30 % of 80 = 24
45% of 80 = 36 60 % of 80 = 48
95% of 80 = 76 85 % of 80 = 68


Name____________ N15 **

Student	Number Correct	% Correct
Wanda	60	100%
Randy	30	50%
Evan	36	60%
Khanh	48	80%
Brock	54	90%
Angela	42	70%

Wanda, Khanh, Brock, and Angela

42

Capsule Lesson Summary

Observe that multiples of 9 are also multiples of 3. For 356 = □ + △, find the least possible whole number to put in the triangle when the number in the box must be some multiple of 9. Relate this activity to the arrow pictures in Lesson N14, and find a test for deciding whether or not a number is a multiple of 9 or a multiple of 3. A number is a multiple of 9 (3) if the sum of its digits is a multiple of 9 (3). Provide missing digits in numbers to insure their divisibility by 9.

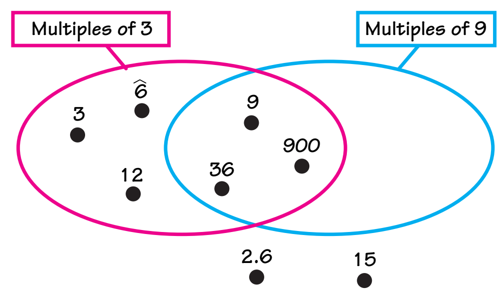
Materials

Teacher		Student	
	• Colored chalk		• Paper • Colored pencils, pens, or crayons • Worksheets N16* and **

Description of Lesson

Exercise 1

Draw two overlapping strings on the board, one for multiples of 3 and one for multiples of 9. Ask students for numbers belonging in each region of the picture. Your picture might look similar to this one.



T (pointing to the empty region)**:** ***Are there numbers that belong in this region?***

S: ***No, numbers in the blue string must also be in the red string since all multiples of 9 are also multiples of 3.***

T: ***Why?***

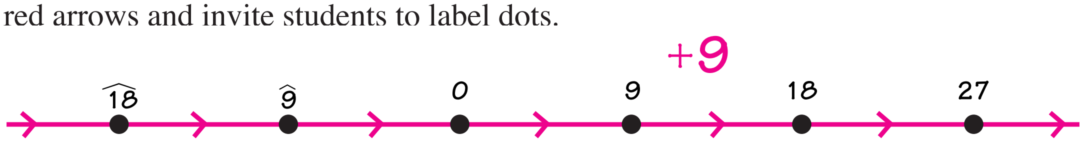
Let students explain. You may wish to illustrate the situation as follows.

Draw an arrow road on the board.

T: ***If we want this arrow road to include all multiples of 9 and no other numbers, what could the red arrows be for?***

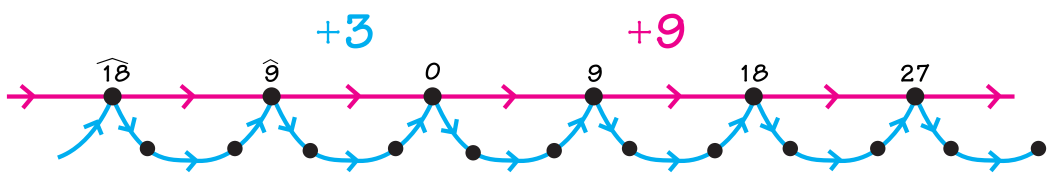
S: ***+9.***

Key the red arrows and invite students to label dots.



T: ***How can we use this arrow road to see that all multiples of 9 are also multiples of 3?***

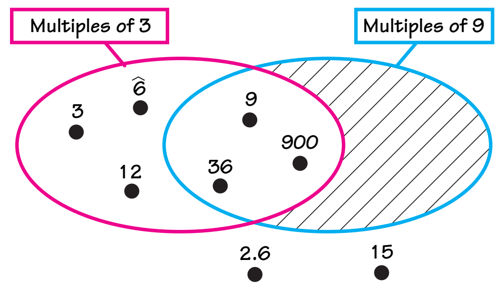
S: ***Draw a +3 arrow road going through 0 to include all the multiples of 3. Since 3 x 3 = 9, every third +3 arrow meets a multiple of 9. Therefore, every multiple of 9 is also a multiple of 3.***



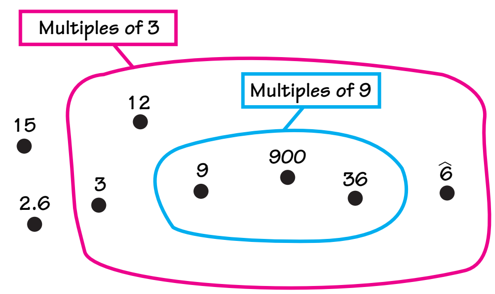
T: ***How can we show an empty region in the string picture?***

S: ***Hatch that region.***

T: ***There is another way of drawing a string picture to show that all multiples of 9 are also multiples of 3. Instead of hatching a region, let's try moving the strings.***



Lead to the idea of drawing the string for multiples of 9 completely inside the string for multiples of 3. Invite students to put the numbers from the first string picture into this one. For example:



Exercise 2

Note: This exercise will examine number sentences such as 10 = □ + △ in which some multiple of 9 goes in the box and a whole number goes in the triangle. It may be necessary for you to remind students frequently that the number in the box is always some multiple of 9.

Write this open sentence on the board.

10 = □ + △

T: ***Some multiple of 9 goes in the box and some other whole number goes in the triangle; 10 is equal to some multiple of 9 plus some whole number. What is the least possible whole number that could be in the triangle?***

S: ***1, because 10 = 9 + 1.***

T: ***Yes, 10 is 1 more than a multiple of 9.***

Consider another open sentence,

10 = $\boxed{9}$ + △1

40 = □ + △

T: ***40 is equal to some multiple of 9 plus some whole number. What is the least whole number that could be in the triangle?***

S: ***4, because 40 = 36 + 4.***

T: ***Yes. Could you use the first number sentence to solve this problem?***

S: ***Multiply all of the numbers in the first equation by 4. Four times a multiple of 9 plus 1 is some multiple of 9 plus 4.***

Include this information on the board.

$$\left. \begin{aligned} 10 &= \boxed{9} + 1 \\ 40 &= \boxed{36} + 4 \end{aligned} \right\} 4\times$$

Continue this activity by asking for the least whole number that could be in the triangle for 100; 1 000; 70; 700; 300; 50; 6; and 356. You may let students first work individually on the problems; then after a few minutes, check the solutions collectively.

$$\begin{aligned} 100 &= \boxed{99} + 1 & \qquad 300 &= \boxed{297} + 3 \\ 1\,000 &= \boxed{999} + 1 & 50 &= \boxed{45} + 5 \\ 70 &= \boxed{63} + 7 & 6 &= \boxed{0} + 6 \\ 700 &= \boxed{693} + 7 & 356 &= \boxed{351} + 5 \end{aligned}$$

Some students might answer that 356 is 14 more than a multiple of 9. For example:

S: ***356 = 300 + 50 + 6, so 356 is a multiple of 9 plus 3 + 5 + 6 or a multiple of 9 plus 14.***

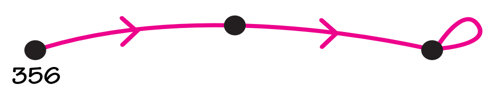
T: ***It is good that you used the previous work to help you, but are you sure that 14 is the least possible number that could be in the triangle?***

S: ***14 = 9 + 5, so 356 is a multiple of 9 plus 5.***

T: ***Yes, 5 is the least whole number that could be in the triangle.***

Draw this arrow picture on the board near the last number sentence.

$$356 = \boxed{351} + 5$$



T: ***Do you remember the rule where we drew an arrow from a whole number to the sum of its digits? We can draw an arrow from 356 to what number?***

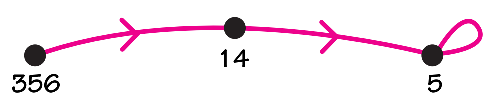
S: ***14, because 3 + 5 + 6 = 14.***

Label the dot for 14.

T: ***And we can draw an arrow from 14 to what number?***

S: ***5; 1 + 4 = 5.***

$$356 = \boxed{351} + 5$$



T: ***Do you see any connection between the arrow picture and the number sentence about 356?***

S: ***356 is 5 more than a multiple of 9 just like every number that would be in this piece of the arrow picture.***

T: ***We found there were ten pieces of the arrow picture for this rule: one piece with 0, one piece with 1, and so on to one piece with 9. What kind of numbers were in the piece with 9?***

S: ***Multiples of 9.***

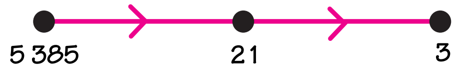
N16

T: ***Is 5 385 a multiple of 9?***

S: ***No; 5 + 3 + 8 + 5 = 21, which is not a multiple of 9.***

S: ***No; 5 + 3 + 8 + 5 = 21 and 2 + 1 = 3, which is not a multiple of 9.***

Draw an arrow road starting at 5 385 as the students explain how to check that it is not a multiple of 9. Write a box and triangle expression near the arrow road.

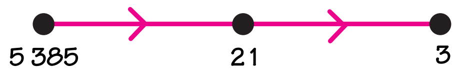


$$5\,385 = \square + \triangle$$

T: ***I want to find two numbers whose sum is 5 385. The number in the box must be a multiple of 9. What is the least whole number that could go in the triangle?***

S: ***3. 5 382 + 3 = 5 385; 5 382 is a multiple of 9 since 5 + 3 + 8 + 2 = 18 and 18 is a multiple of 9.***

Record the solution.

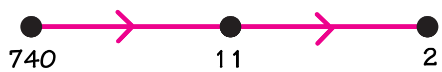


$$5\,385 = \boxed{5\,382} + 3$$

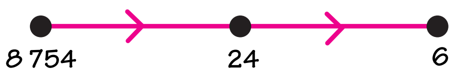
T: ***Is 5 385 a multiple of 3?***

S: ***Yes; each multiple of 9 is a multiple of 3, and adding 3 to a multiple of 3 gives another multiple of 3.***

Present these problems in a similar manner. 740 is not a multiple of 3, and 8 754 is a multiple of 3. Neither are multiples of 9.



$$740 = \boxed{738} + 2$$



$$8\,754 = \boxed{8\,748} + 6$$

T: ***If a number is a multiple of 3, what kind of number do we get when we add its digits?***

S: ***A multiple of 3.***

Note: Any number 1, 2, 4, 5, 7, or 8 more than a multiple of 9 is that much more than a multiple of 3; therefore, they are not multiples of 3. Any number 3, 6, or 9 more than a multiple of 9 is that much more than a multiple of 3; a multiple of 3 plus a multiple of 3 is also a multiple of 3.

Exercise 3

On the board, write the following numbers, each with a missing digit. Ask students to copy them on a piece of paper and to insert the missing digits to produce multiples of 9. (Answers are in boxes.) After a short while, invite students to present and explain their solutions.

$4\,\boxed{2}\,12$ $\qquad$ $7\,\boxed{0}\,92$ or $7\,\boxed{9}\,92$

$7\,777\,3\boxed{4}1$ $\qquad$ $87\boxed{8}\,769$

Erase the digits in the boxes and restate the problem to produce multiples of 3 rather than multiples of 9.

T: ***Now fill in the boxes so that each number is a multiple of 3.***

Allow a few minutes for individual work before checking the solutions collectively. Students should recognize that the solutions for multiples of 9 are also multiples of 3, but encourage them to find other solutions as well.

$4\boxed{2}12$ or $4\boxed{5}12$ or $4\boxed{8}12$ $\quad$ $7\boxed{0}92$ or $7\boxed{3}92$ or $7\boxed{6}92$ or $7\boxed{9}92$

$7\,777\,3\boxed{4}1$ or $7\,777\,3\boxed{1}1$ or $7\,777\,3\boxed{7}1$ $\quad$ $87\boxed{8}769$ or $87\boxed{5}769$ or $87\boxed{2}769$

$165\boxed{}7\triangle7$

Repeat the exercise with a number having two missing digits.

Invite students to list all of the possible solutions on the board. Since 1 + 6 + 5 + 7 + 7 = 26, the two inserted digits must have a sum of 1 (26 + 1 = 27) or 10 (26 + 10 = 36).

1650 717
1651 707
1655 757

1651 797
1659 717
1652 787
1658 727

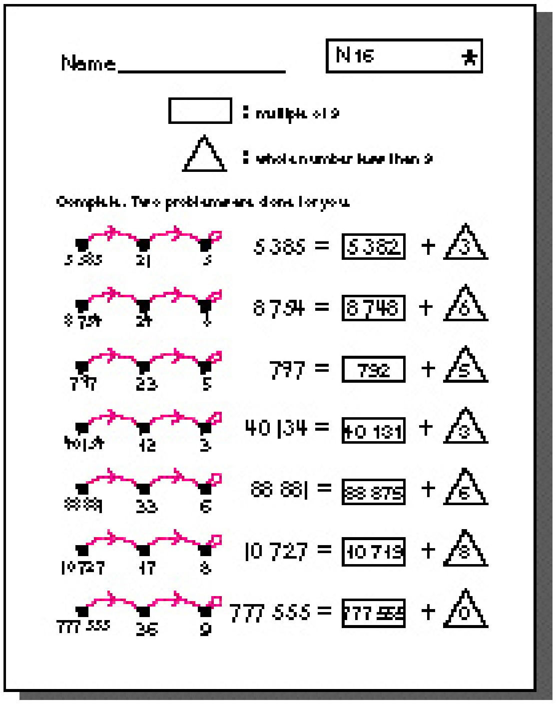
1653 777
1657 737
1654 767
1656 747

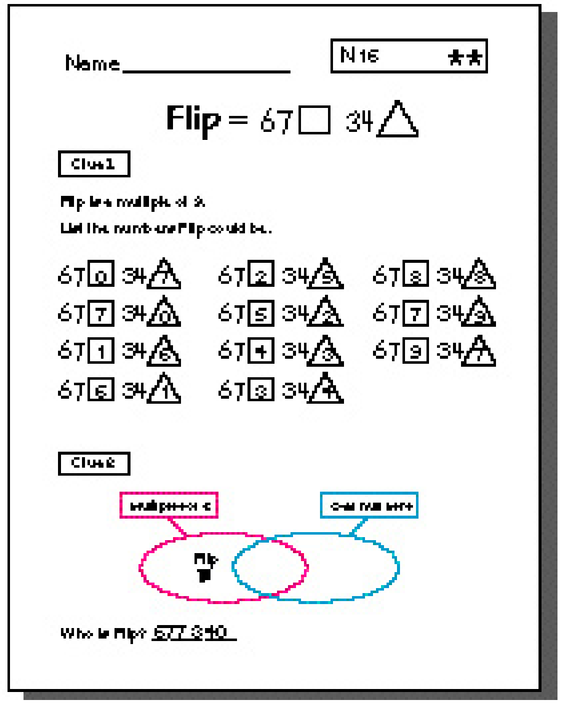
T: ***In this lesson we only considered tests (add digits) for multiples of 9 and 3. Does a similar test work for multiples of 8?***

S: ***No; 48 is a multiple of 8, but 4 + 8 is not a multiple of 8.***

S: ***No, 2 + 6 = 8, but 26 is not a multiple of 8.***

Worksheets N16* and ** are available for individual work.


Name
N16
5 385 = 5 382 + 3
8 754 = 8 748 + 6
797 = 792 + 5
40 134 = 40 131 + 3
88 881 = 88 875 + 6
10 727 = 10 719 + 8
777 555 = 777 555 + 0


Name
N16
Flip = 67□ 34△
Clue 1
Clue 2
Who is Flip? 677 340

Capsule Lesson Summary

Construct a Cartesian graph model for whole number multiplication. Extend the model to include multiplication by integers and rational numbers.

Materials

Teacher
- Colored chalk
- Coordinate grid
- Blackline N17

Student
- Coordinate grid sheets
- Colored pencils, pens, or crayons

Advance Preparation: Use Blackline N17 to prepare a coordinate grid for display, or prepare a grid board with this coordinate grid. Make copies of coordinate grid sheets for students.

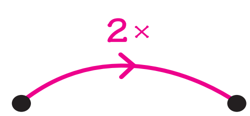
Description of Lesson

Exercise 1

Display a coordinate grid with horizontal and vertical axes extending to about 10 and $\widehat{10}$.

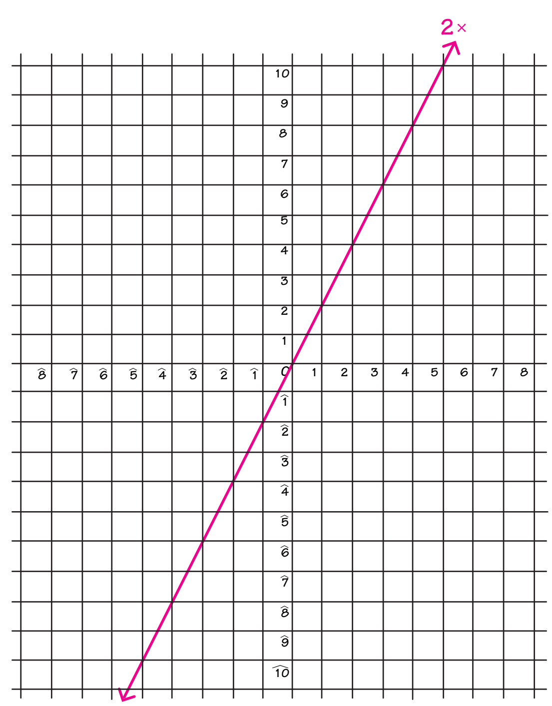
T: ***In an earlier lesson we drew Cartesian graphs for different relations. Let's make a Cartesian graph for the 2x relation.***

Draw a 2x arrow on the board, and ask students to suggest some starting and ending numbers. Record these numbers in ordered pairs. For example:



(2, 4) (0.5, 1) (1, 2)
(3, 6) (5, 10) (10, 20)
$(\hat{1}, \hat{2})$ $(\widehat{2.5}, \hat{5})$ $(\hat{5}, \widehat{10})$

Call on students to locate points for several ordered pairs. Then instruct students to complete the graph on their coordinate grid paper.



After several minutes of independent work, complete the picture at the board.

T: ***What do you notice about the 2x relation?***

S: ***Its graph is a straight line.***

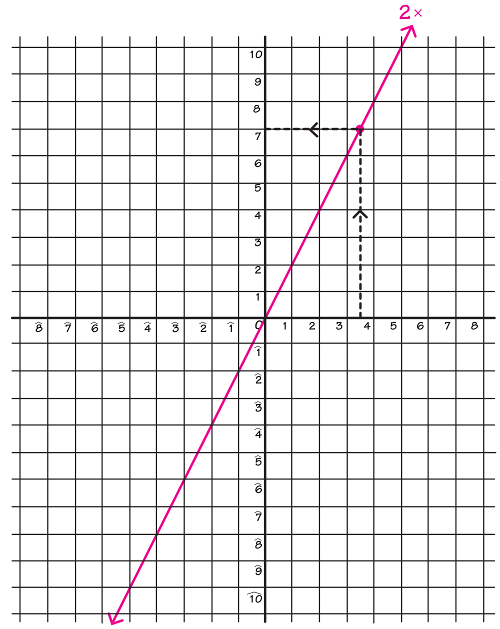
S: ***The line goes through (0, 0).***

T: ***Why is (0, 0) on the 2x line?***

S: ***2 x 0 = 0.***

T: ***How could we use the 2x line to find 2 x 3.5?***

S (tracing the horizontal axis)**:** ***Find 3.5 on this line; then go straight up until you hit the 2*x *line; then go straight across to see what is the second number in the ordered pair for that point on the 2*x *line.***

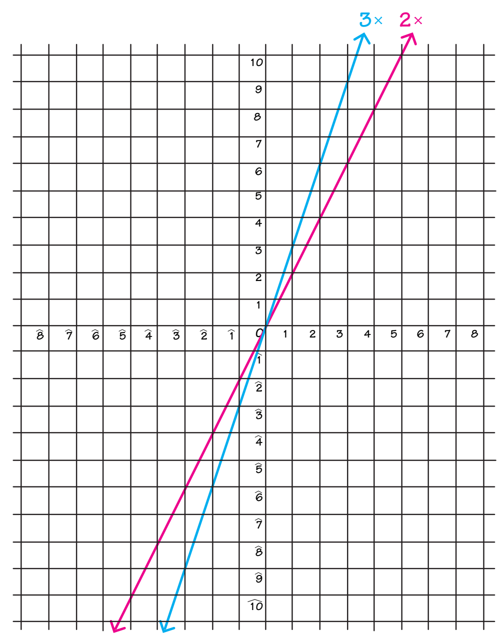


Ask other students to show how to use the graph to find 2 x 2.5, 2 x $\widehat{3}$, 2 x 0.5, and 2 x $\widehat{4.5}$.

T: ***Next, let's graph the 3*x *relation on this same copy of the grid.***

Let students work independently or with partners. You may recommend that they use a second color for this graph.

After a while collectively complete the picture at the board.



T: ***What do you notice about the 3*x *relation?***

S: ***Its graph is a line going through (0, 0). The 3*x *line is steeper than the 2*x *line.***

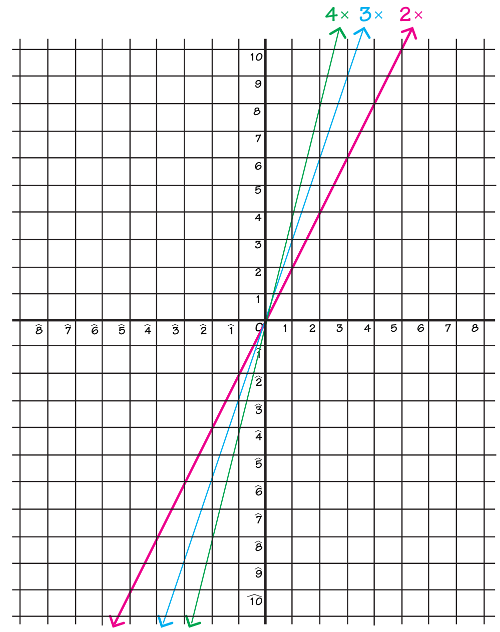
T: ***Let's graph the 4*x *relation. How do you think its graph will look?***

S: ***It will be a straight line a little steeper than the 3*x *line.***

Invite several students to trace where they think a 4x line would be.

T: ***Do we need to draw dots for many ordered pairs from the 4*x *relation, or can you think of a quick way to draw the graph?***

S: ***The 4*x *line will go through (0, 0) and through (2, 8). One point other than (0, 0) is enough to determine how to draw the 4*x *line.***



Let students draw the 4x line on their papers while you do so on the board.

T: ***What do you notice about all of these lines?***

S: ***They all go through (0, 0).***

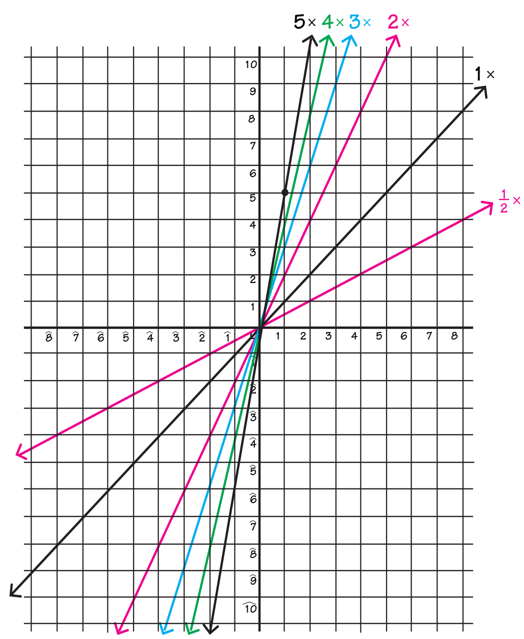
S: ***They get steeper as we multiply by greater numbers.***

S: ***They slant up from left to right.***

T: ***How could we tell these lines apart if they were not labeled?***

S: ***Look at the (vertical) line through 1. It hits the 2x line at (1, 2), the 3x line at (1, 3), and the 4x line at (1, 4).***

Verify this observation. You may need to make it yourself.



Check students' understanding of this idea by drawing a 1x line, a 5x line, and a ½x line and asking students to identify them.

T: ***Where is the 0x line?***

A student should trace the horizontal axis at the board.

S: ***It is this horizontal line.***

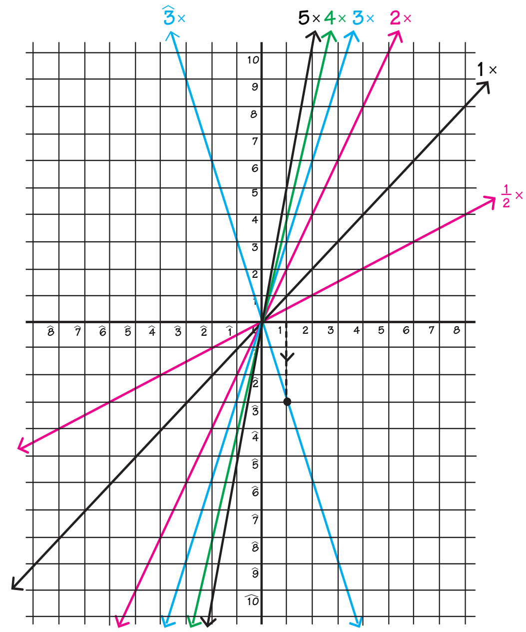
T: ***Let's graph the $\widehat{3}$x relation. How could we do this without first plotting points for a lot of ordered pairs in the $\widehat{3}$x relation?***

Allow a few minutes for students to consider the $\widehat{3}$x relation and how its graph might look.

S: ***It's graph will be a straight line through (0, 0) like the other lines. $\widehat{3} \times 0 = 0$.***

S: ***We can find another point on the $\widehat{3}$x line by looking at the vertical line through 1. This line should hit the $\widehat{3}$x line at (1, $\widehat{3}$); $\widehat{3} \times 1 = \widehat{3}$.***

Invite students to first trace the $\hat{3}$x line; then draw it in the picture at the board. Instruct students to add the $\hat{3}$x line to their graphs.

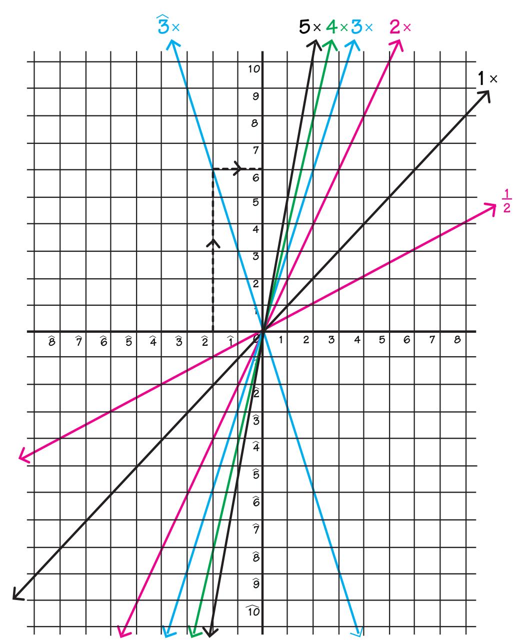


Write these problems on the board. (Answers are in boxes.)

T: ***You can use the $\hat{3}$x line to help with all these $\hat{3}$x calculations.***

$\hat{3} \times 2 = \boxed{\hat{6}}$	$\hat{3} \times 0 = \boxed{0}$	$\hat{3} \times \hat{2} = \boxed{6}$
$\hat{3} \times 1 = \boxed{\hat{3}}$	$\hat{3} \times \hat{1} = \boxed{3}$	$\hat{3} \times \hat{3} = \boxed{9}$

Let students work independently for a few minutes. Ask students to demonstrate at the board how the $\hat{3}$x line can be used for each calculation. This illustration shows using the $\hat{3}$x line for $\hat{3}$ x $\hat{2}$ = 6.



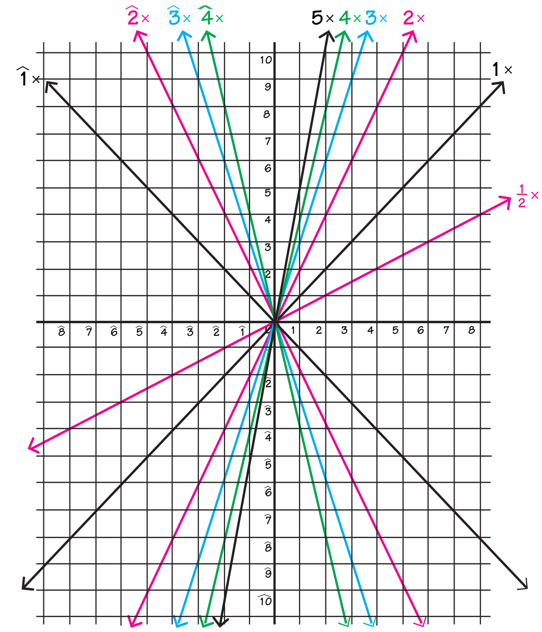
T: ***What do you notice about the graph of $\hat{3}$x?***

S: ***It is a straight line through (0, 0).***

S: ***It's a line with the same steepness as the 3x line but slanted in the opposite direction.***

Continue this activity by asking students to draw the $\hat{2}$x line, the $\hat{4}$x line, and the $\hat{1}$x line.

While students are completing this work, occasionally ask them to use their graphs to find various products of integers; for example, $\widehat{4}$ x 2, $\widehat{4}$ x $\widehat{1}$, $\widehat{2}$ x $\widehat{5}$, 3 x $\widehat{3}$, and so on.



N18 DIVISION ALGORITHM #1

Capsule Lesson Summary

Use patterns to do mental arithmetic problems involving multiplication of decimal numbers. Solve division problems in which a given amount of money is shared equally among some specified number of people. Present a standard algorithm for dividing a decimal by a whole number.

Materials

Teacher
- Colored chalk

Student
- Paper
- Worksheets N18* and **

Description of Lesson

Exercise 1

Write the following problems on the board, and ask students to copy them on their papers. Let students explain how to find the products, emphasizing the technique of starting with a known fact (for example, 9 x 3 = 27) and using patterns. To involve all students, you may ask them to write solutions on their papers and check several papers before inviting students to announce results. (Answers are in boxes.)

$$9 \times 0.03 = \boxed{0.27} \qquad 6 \times 0.008 = \boxed{0.048}$$
$$15 \times 400 = \boxed{6\,000} \qquad 12 \times 6\,000 = \boxed{72\,000}$$
$$8 \times 0.7 = \boxed{5.6} \qquad 35 \times 0.02 = \boxed{0.7}$$

S: ***9 x 3 = 27, so 9 x 0.3 = 2.7 and 9 x 0.03 = 0.27.***

S: ***15 x 4 = 60, so 15 x 40 = 600 and 15 x 400 = 6 000.***

Exercise 2

Begin this table on the board.

Amount of Money	Number of People	Amount of Money for Each Person	Amount of Money Left Over
$20.80	4		

T: ***Four people are going to share $20.80 equally. How much money does each person receive and how much money is left over, if any?***

S: ***Each person receives $5.00, since 4 x $5.00 is $20.00. $0.80 is left over.***

T: ***Can the four people share $0.80 equally?***

S: ***Yes, each person receives an additional $0.20.***

S: ***So altogether each person receives $5.20.***

Add this information to your table. Continue the activity by giving the information in the first two columns of the table below and asking students to provide the numbers in the boxes. Feel free to adjust the level of difficulty to the numerical abilities of your students. You may prefer to ask students to copy and fill in the table on their papers. Then you can check several responses before inviting students to fill in the table on the board.

Amount of Money	Number of People	Amount of Money for Each Person	Amount of Money Left Over
$20.80	4	$5.20	0
$110.75	10	$11.07	$0.05
$36.80	6	$6.13	$0.02
$67.26	8	$8.40	$0.06

Put $4 809.68 and 15 in the first two columns of the table, respectively.

T: ***15 people are sharing equally $4 809.68. To determine how much money each person receives, what calculation could we do?***

S: ***4 809.68 ÷ 15.***

Write the calculation on the board.

$$15\overline{)4\,809.68}$$

T: ***Does each person receive at least $10? At least $100? At least $1 000?***

S: ***Between $100 and $1 000. 15 x 100 = 1 500 and 15 x 1000 = 15 000.***

T: ***How many hundreds of dollars could we give to each person?***

Do not insist on the optimal response, $300. If a student suggests $200, for example, record a student's answer in the division problem on the board as you work through the calculation.

T: ***If we give $200 to each person, how much money is that?***

S: ***$3 000 (15 x $200).***

T: ***How much money remains to share?***

S: ***$1 809.68 (4 809.68 – 3 000.00).***

S: ***Give $100 more to each person since 15 x 100 = 1 500.***

$$\begin{array}{r|l} 15\overline{)4\,809.68} & \\ -3\,000.00 & 200 \\ \hline 1809.68 & \\ -1500.00 & 100 \\ \hline 309.68 & \end{array}$$

If someone suggests an amount such as $110 or $150 for each person, agree but point out that multiplying by 100 is much easier than multiplying by 110 or 150. Recommend considering multiples of 10 after giving $300 to each person.

Proceed in a similar manner to complete the calculation. Stop occasionally during the process to ask how much money each person receives so far. Perhaps your class will solve the problem as shown here.

T: ***How much money does each person receive?***

S: ***$320.64 (200 + 100 + 20 + 0.50 + 0.10 + 0.04) and there is $0.08 left over.***

Record the quotient and the remainder in the division problem and in the third and fourth columns, respectively, of the table.

$$
\begin{array}{r|r}
\multicolumn{2}{r}{320.64 \;\; R = 0.08} \\
15\overline{)4809.68} & \\
-3000.00 & 200 \\ \hline
1809.68 & \\
-1500.00 & 100 \\ \hline
309.68 & \\
-300.00 & 20 \\ \hline
9.68 & \\
-7.50 & 0.50 \\ \hline
2.18 & \\
-1.50 & 0.10 \\ \hline
0.68 & \\
-0.60 & 0.04 \\ \hline
0.08 &
\end{array}
$$

Review and discuss this division method. Then write the following problems on the board, and ask students to do the calculations on their papers. (Answers are in boxes.)

$$\begin{array}{r} \boxed{26.48 \quad R = 0.02} \\ 6\overline{)158.90} \end{array} \qquad \begin{array}{r} \boxed{22.10 \quad R = 0} \\ 5\overline{)110.50} \end{array}$$

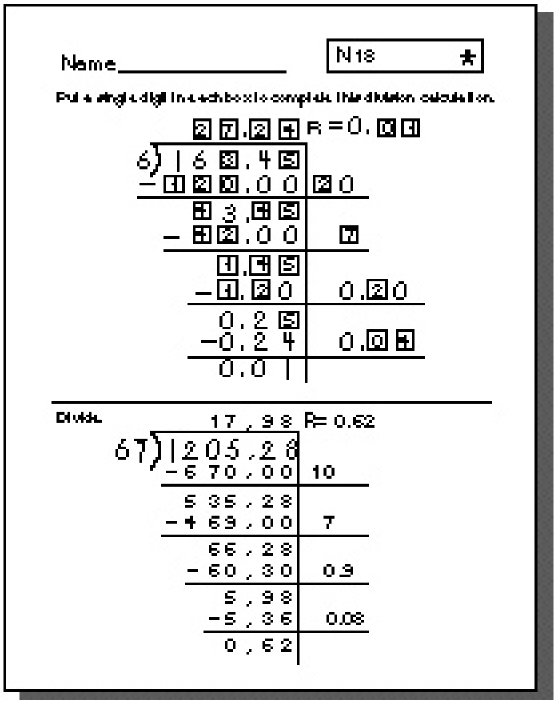
$$\begin{array}{r} \boxed{120.04 \quad R = 0.04} \\ 8\overline{)960.36} \end{array} \qquad \begin{array}{r} \boxed{71.26 \quad R = 0.02} \\ 23\overline{)1639.00} \end{array}$$

As you observe students' work, especially watch for students who either spend too much time finding "best" estimates or, at the other extreme, always use only 100, 10, 1, 0.1, or 0.01. When appropriate, suggest alternatives to these students. You may wish to solve one of the problems collectively at the board.

Worksheets N18* and ** are available for additional individual work.

Home Activity

This is a good time to send a letter to parents/guardians about a division algorithm. Blackline N18 has a sample letter.



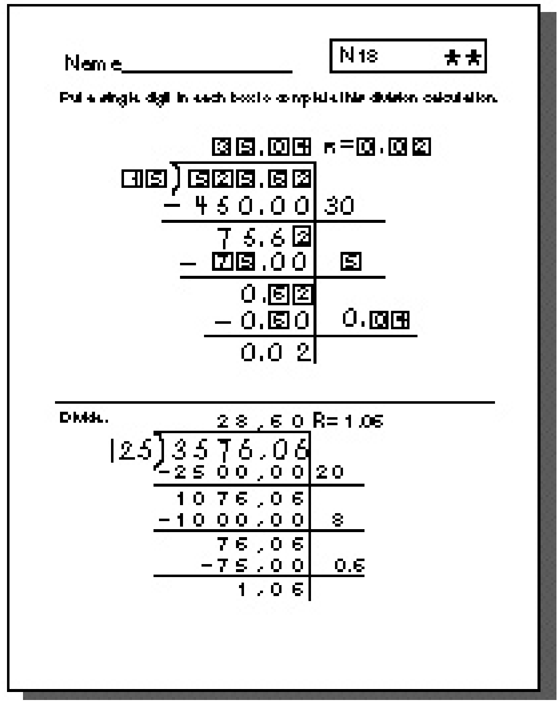
Name__________ N18 ★

Put a single digit in each box to complete this division calculation.

```
      27.24 R=0.01
 6)  163.45
   - 120.00    20
      43.45
   -  42.00     7
       1.45
   -   1.20  0.20
       0.25
   -   0.24  0.04
       0.01
```

Divide.

```
          17,98 R= 0.62
 67)   1205,28
     -  670,00   10
        535,28
     -  469,00    7
         66,28
     -   60,30  0.9
          5,98
     -    5,36  0.08
          0,62
```



Name__________ N18 ★★

Put a single digit in each box to complete this division calculation.

```
      35.04 R=0.02
 15) 525.62
   - 450.00    30
      75.62
   -  75.00     5
       0.62
   -   0.60  0.04
       0.02
```

Divide.

```
           28,60 R= 1.06
 125)   3576,06
      - 2500,00   20
        1076,06
      - 1000,00    8
          76,06
      -   75,00  0.6
           1,06
```

N19 CARTESIAN GRAPHS #3

Capsule Lesson Summary

Review the notion of a Cartesian graph, and graph several relations including the "is less than" relation. Introduce an in-out convention to indicate points which do or do not belong to a graph.

Materials

Teacher
- Colored chalk
- Coordinate grid
- Blackline N19

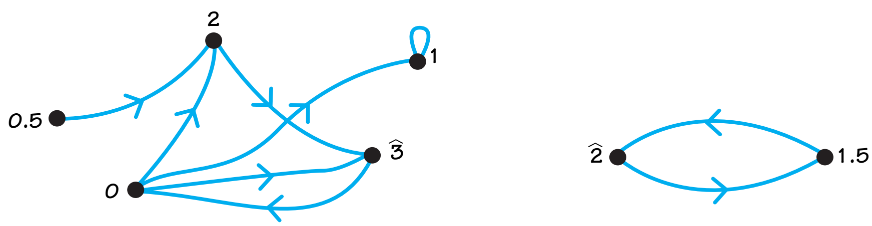
Student
- Paper
- Colored pencils, pens, or crayons
- Coordinate gridsheets

Advance Preparation: Use Blackline N19 to prepare a coordinate grid for display, or prepare a grid board with this coordinate grid. Make copies of coordinate grid sheets for students' use in Exercise 3.

Description of Lesson

Exercise 1

Display a coordinate grid and draw this arrow picture on the board.



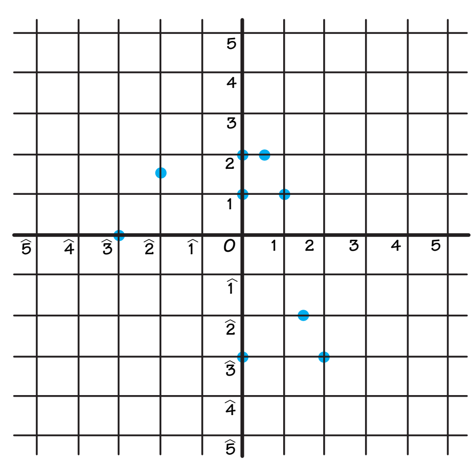
Trace an arrow, for example from 0.5 to 2, as you ask,

T: ***This arrow is for which ordered pair of numbers?***

S: ***(0.5, 2), because 0.5 is the starting number and 2 is the ending number.***

T: ***Where is the point on the grid for (0.5, 2)?***

Invite a student to explain how to locate the point.



Continue this activity, asking students to locate points for the eight arrows and the loop. Emphasize that the starting number is located along the horizontal axis and the ending number is located along the vertical axis. Upon completion of this activity, there should be nine dots on your coordinate grid. Mention that points on the coordinate grid make up the *Cartesian graph* of the relation in the arrow picture.

T: ***How can we tell by looking at its Cartesian graph that the relation has three arrows starting at 0?***

S: ***Look at the vertical line through 0; there are three points on it corresponding to the three arrows that start at 0.***

T: ***How can we tell by looking at its Cartesian graph that the relation has two arrows ending at $\hat{3}$?***

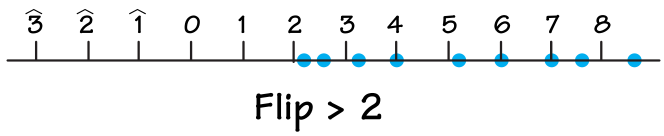
S: ***Look at the horizontal line through $\hat{3}$; there are three points on it corresponding to the two arrows ending at $\hat{3}$.***

Exercise 2

On the board, draw part of a number line, as shown below. Near the number line, indicate that Flip is a number greater than 2.

T: ***Let's draw blue dots on the number line for numbers that Flip could be.***

Invite several students to draw blue dots. After several dots are drawn, students should notice that all of the dots are to the right of 2. Your number line might look like this.



T: ***What is the least number Flip could be? What is the greatest?***

Of course, there is no least nor greatest number that Flip could be but it may be the first time some students have given much thought to these questions. Lead a class discussion to observe that there is no least number nor greatest number. For example:

S: ***Flip could be 2.1.***

T: ***Can you think of a number less than 2.1 that Flip could be?***

S: ***2.01.***

T: ***Is this the least number Flip could be?***

S: ***No.***

T: ***Can you find a number less than 2.01?***

S: ***2.0000 … 1—with 125 zeroes!***

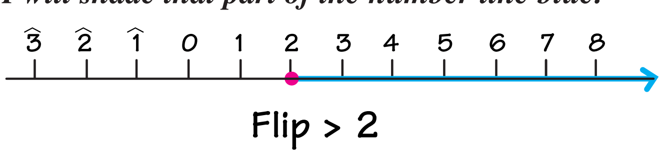
T: ***Is this the least?***

S: ***No, not even if there were 1 000 zeroes before 1.***

S: ***There is no least number that Flip could be.***

S: ***Flip could be numbers closer and closer to 2, but Flip cannot be 2.***

T: ***To show that Flip cannot be 2, I will draw a red dot at 2. Since Flip could be any number to the right of 2, I will shade that part of the number line blue.***



T: ***Let's agree that blue shows numbers that Flip could be and, when we need to make the picture clear, red shows numbers that Flip could not be.***

Erase the red dot at 2 on the number line.

T: ***Look again at the number line. Suppose I draw a blue dot at 2. How would this change the information (number sentence) about Flip?***

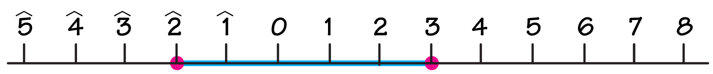
Allow a minute for students to consider the question.

S: ***Flip could be any number greater than or equal to 2. (Flip ≥ 2.)***

Erase and redraw the number line. Write a new number sentence about Flip, $\hat{2}$ < Flip < 3.

T: ***Using our red-blue convention, let's indicate which numbers Flip could be.***

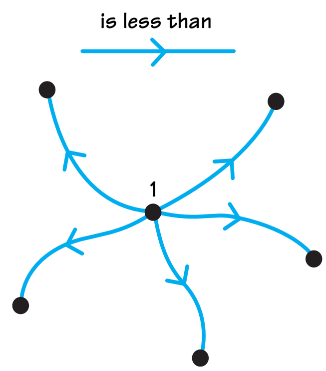
Suggest that students do this on their papers first, and then invite a student to draw the picture on the board.



$\hat{2}$ < Flip < 3

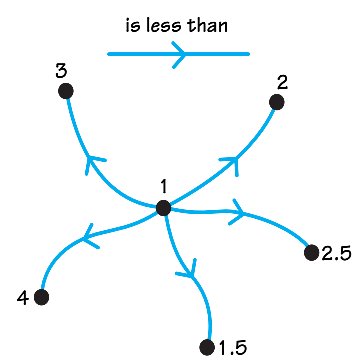
Exercise 3

Draw this arrow picture on the board.



T: ***Now blue arrows are for the relation "is less than." How could we label the dots?***

Invite some students to label the dots. Of course, there are many possible labels. Encourage choice of labels so that the arrows are for ordered pairs that can be plotted on your coordinate grid.

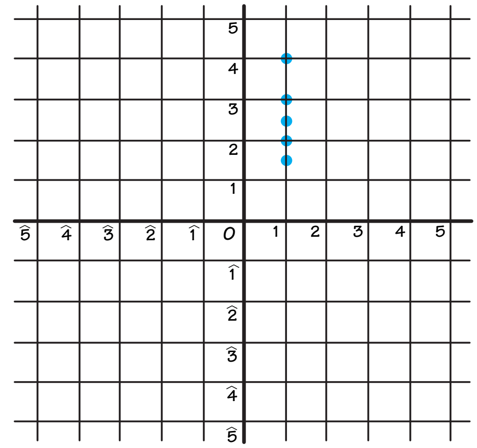


Refer students to their coordinate grid sheets.

T: ***On your grid, draw dots at points for these arrows.***

N19

Allow a few minutes for students to work independently. Then, ask some students to draw the dots on the demonstration grid.



T: ***Could we draw more arrows starting at 1?***

S: ***Yes, infinitely many of them.***

T: ***What can we say about the ending numbers of these arrows?***

S: ***Each ending number will be greater than 1.***

T: ***Is there a least possible number?***

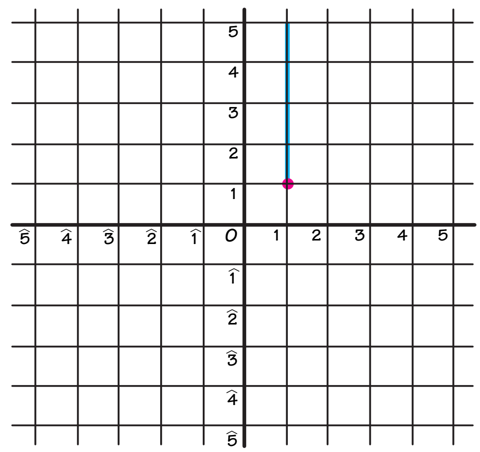
S: ***No, it's like the problem about Flip. The numbers are greater than 1 but can be very close to 1.***

T: ***How can we show this situation on the grid?***

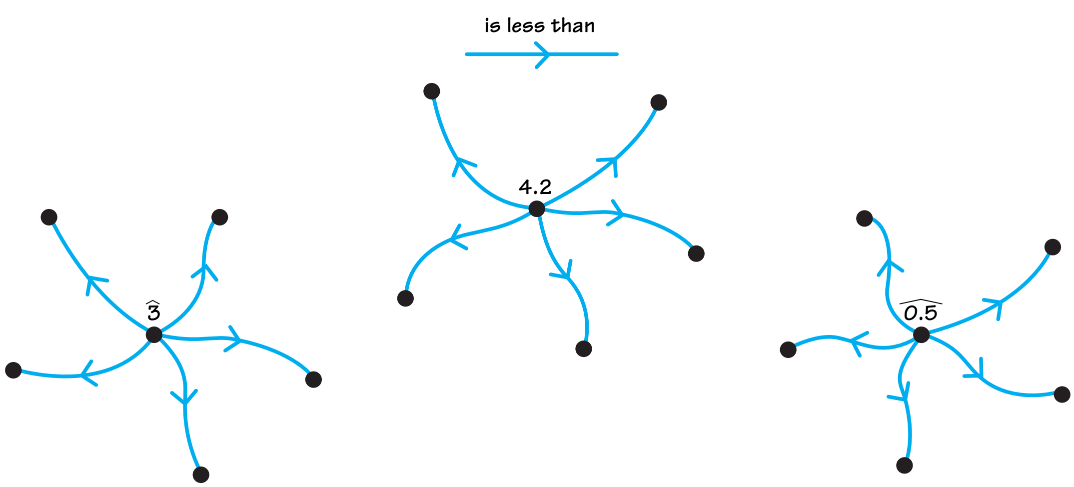
S: ***It's like the problem of showing the numbers Flip could be on the number line. Imagine that the vertical line at 1 is a number line. Color in blue to show numbers greater than 1.***

S: ***Draw a red dot at (1, 1) to show that 1 is not less than 1.***

Invite students to draw the picture.

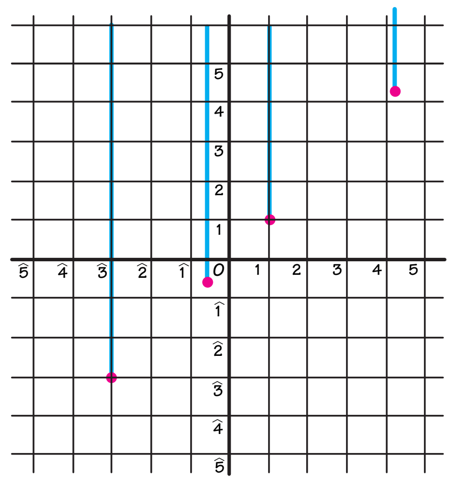


Add more pieces to your arrow picture, such as shown here.



In a similar manner, let the students analyze the situation for all of the arrows starting at $\hat{3}$, at 4.2, and at $\widehat{0.5}$. When most students have finished, ask some students to put their drawings on the board.

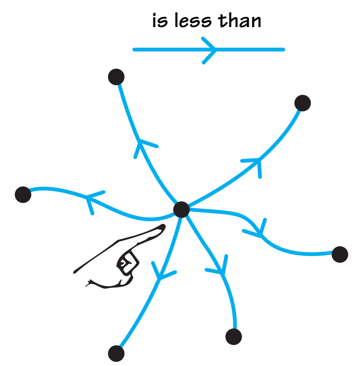
T: ***Look at the grid picture. Do you notice anything interesting?***



Perhaps students will observe that the red dots lie on a line through (0, 0) and this line includes all ordered pairs of the form (n, n); that is, the 1**x** line. They may also comment that the entire part of a line (open half-line) above each red dot is colored blue. Accept any reasonable comments.

Draw a new piece in the arrow picture and point to the center dot.

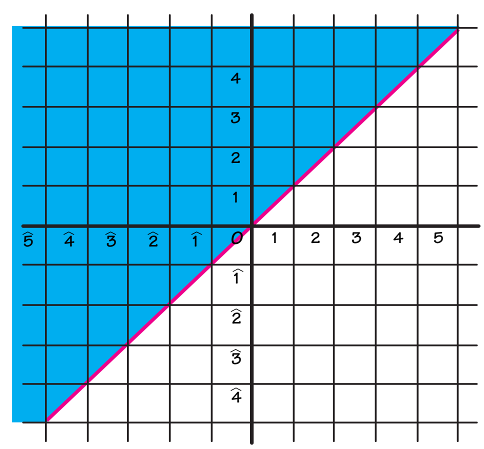
T: ***Any number could be here. On your grid, find other points that belong to the graph of the "is less than" relation by thinking about lots of possibilities for numbers at this dot. Try to find all the points that belong to the Cartesian graph of the "is less than" relation.***



Provide time for students to work independently or with partners on this problem. When several students have the full solution, invite one student to draw the picture on the board.

S: ***The blue lines run together like this.***

T (tracing the blue region)**:** ***This region is the Cartesian graph of the "is less than" relation. Where is a point that does not belong to the "is less than" relation?***



Ask a student to indicate such a point, for example, $(1, \widehat{2})$.

T: ***What is the ordered pair of numbers for this point?***

S: ***$(1, \widehat{2})$.***

T: ***What do you notice about these numbers?***

S: ***1 is greater than $\widehat{2}$.***

Repeat this activity several more times. The class should conclude that for ordered pairs of numbers which do not belong to the "is less than" relation, the first number is equal to or greater than the second number.

Capsule Lesson Summary

Estimate and do addition and subtraction calculations with decimal numbers. Use fractions to confirm results such as 0.3 x 1.4 = 0.42. Play a calculator game emphasizing estimation of products of whole numbers and decimal numbers.

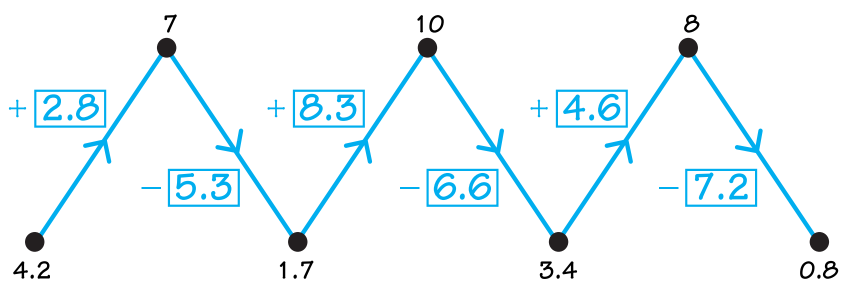
Materials

Teacher		Student	
	• Colored chalk		• Calculator • Paper • Worksheets N20*, **, ***, and ****

Description of Lesson

Exercise 1

On the board, draw this road one arrow at a time, and invite students to label the arrows as you go. (Answers are in boxes.)



Encourage discussion of a variety of methods for doing the calculations, for example,

S: ***4.2 + 3 = 7.2 and 3 – 0.2 = 2.8, so 4.2 + 2.8 = 7.***

S: ***7 – 5 = 2 and 2 – 0.3 = 1.7, so 7 – 5.3 = 1.7.***

S: ***1.7 + 8 = 9.7 and 9.7 + 0.3 = 10, so 1.7 + 8.3 = 10.***

S: ***To solve 10 – □ = 3.4, think 3.4 + □ = 10.***

S: ***Use the addition algorithm:***

$$\begin{array}{r} 34 \\ +\boxed{46} \\ \hline 80 \end{array}$$

Exercise 2

Write these problems on the board.

$$14.6 \times 8.437 \times 9.65 =$$
$$7.29 \times 46 \times 0.847 =$$
$$23.46 - 8.6 =$$
$$16.3 - 7.59 =$$

Ask students to copy and complete the calculations on their papers. After several minutes, begin a collective discussion of the first problem.

T: ***If we consider only the whole number parts of these numbers, what would the sum be?***

S: ***31; (14 + 8 + 9) = 31.***

T: ***So if your result is not near but more than 31, do the calculation again.***

Allow students to continue working individually. Encourage using estimation to check results. When many students are finished with the problems, continue the collective discussion.

T: ***What is the sum in the first problem?*** (32.687) ***How did you do the calculation?***

S: ***I added the whole number parts (14 + 8 + 9 = 31) and the decimal parts (0.6 + 0.437 + 0.65 = 1.687). Then the sum is 31 + 1.687 or 32.687.***

S: ***I wrote the numbers in a column and used the addition algorithm.***

T: ***Why is it important when adding decimal numbers to line up the decimal points?***

S: ***Then the whole number parts and the decimal parts line up properly for addition.***

$$\begin{array}{r} 14.6 \\ 8.437 \\ +\ 9.65 \\ \hline 32.687 \end{array}$$

Emphasize the importance of lining up decimal points when adding or subtracting decimal numbers. You may wish to allow time for students to check and correct their other calculations. Then ask for the remaining results, and let students explain or show how they did the calculations. For example:

$$\begin{array}{r} 7.290 \\ 46.000 \\ +\ 0.847 \\ \hline 54.137 \end{array} \qquad \begin{array}{r} 23.46 \\ -\ 8.60 \\ \hline 14.86 \end{array} \qquad \begin{array}{r} 16.30 \\ -\ 7.59 \\ \hline 8.71 \end{array}$$

Some students may find it helpful to put in 0s as place holders as shown above.

Exercise 3

Write the equivalence on the board as you ask,

T: ***What is a decimal name for $\frac{7}{10}$?***

S: ***$\frac{7}{10} = 7 \div 10 = 0.7$.***

$$\frac{7}{10} = 0.7$$

Find the following equivalences in a similar manner. (Answers are in boxes.)

$$\frac{27}{10} = \boxed{2.7} \qquad \frac{18}{100} = \boxed{0.18} \qquad 0.4 = \boxed{\frac{4}{10} = \frac{2}{5}}$$

$$\frac{456}{10} = \boxed{45.6} \qquad \frac{7}{100} = \boxed{0.07} \qquad 7.13 = \boxed{\frac{713}{100}}$$

Point out that when a fraction's denominator is 10 or 100, it is easy to determine its decimal name.

T: ***What is a decimal name for $\frac{3}{5}$?***

S: ***$\frac{3}{5} = \frac{6}{10} = 6 \div 10 = 0.6$.***

S: ***$\frac{3}{5} = 3 \div 5$ and $3 \div 5 = 0.6$, since $30 \div 5 = 6$.***

$$\frac{3}{5} = \frac{6}{10} = 0.6$$

Encourage students to use equivalent fractions or division patterns to find decimal names in the following problems. (Answers are in boxes.)

$$\frac{11}{5} = \boxed{\frac{22}{10} = 2.2} \qquad \frac{7}{25} = \boxed{\frac{28}{100} = 0.28}$$

$$\frac{9}{20} = \boxed{\frac{45}{100} = 0.45} \qquad \frac{3}{4} = \boxed{\frac{75}{100} = 0.75}$$

Write this multiplication problem on the board.

T: ***What number is 0.3* x *1.4?*** (0.42)

$$0.3 \times 1.4 =$$

Allow students to explain how to do the calculation, leading to the following explanation.

T: ***What are fractions for 0.3 and 1.4?***

S: ***0.3 = 3⁄10 and 1.4 = 14⁄10.***

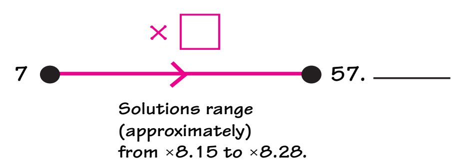
S: $\frac{3}{10} \times \frac{14}{10} = \frac{3 \times 14}{10 \times 10} = \frac{42}{100}$.

S: ***42⁄100 = 42 ÷ 100 = 0.42.***

$$\times \frac{14}{10} = \frac{42}{100} = 0.42$$

Exercise 4

Draw this arrow picture on the board.

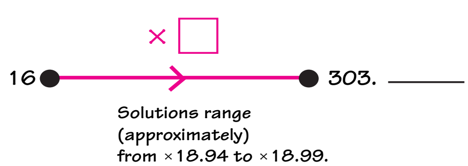


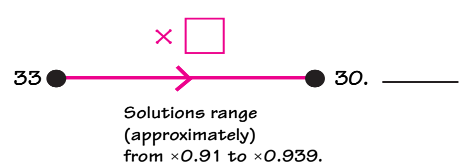
Pair students, and direct each pair to use one calculator. Review the rules of the game introduced in Exercise 4 of Lesson N8 *Decimals #2*.

- Students in a pair take turns trying to solve the problem of finding a label for the arrow so that the ending dot is for fifty-seven point something.
- Students record each attempt on a piece of paper so that both players see all of the previous attempts. The first student to find a solution is the winner.

Direct the partners to first play the game with this problem. When they solve it, instruct them to proceed to one of the following problems or make up their own problems.

Put two additional problems on the board for partners to choose from.

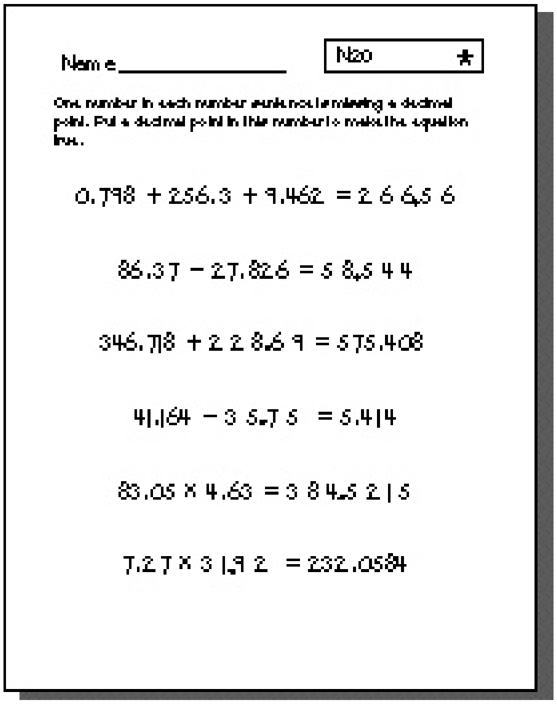


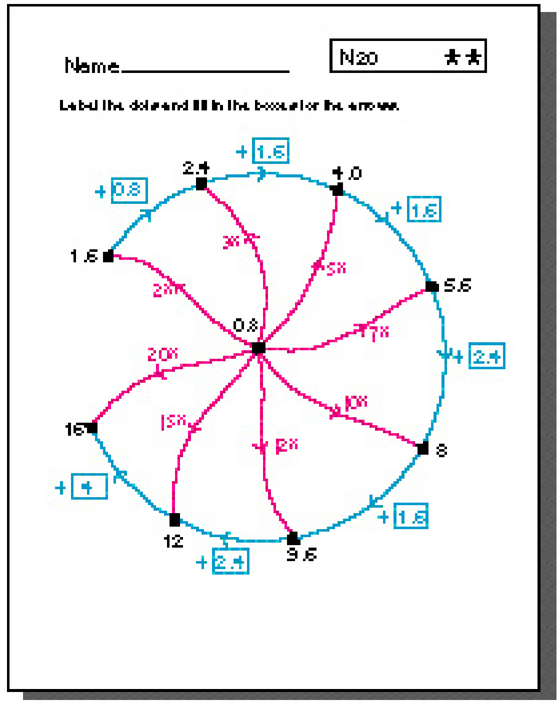


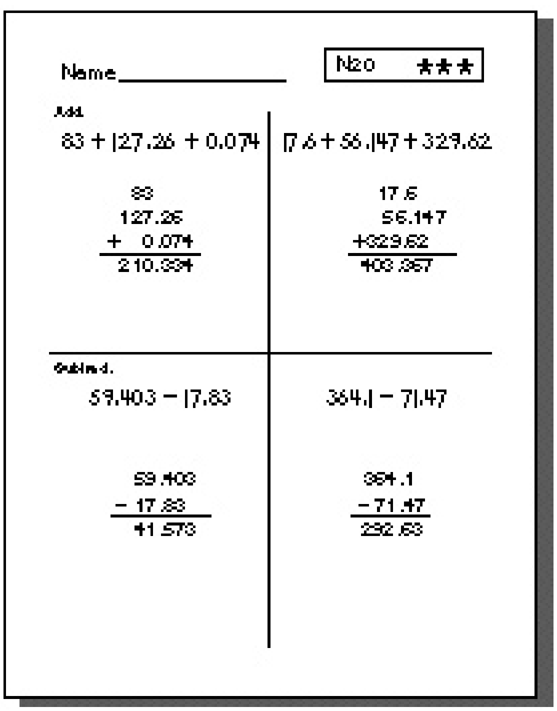
Worksheets N20*, **, ***, and **** are available for individual work.

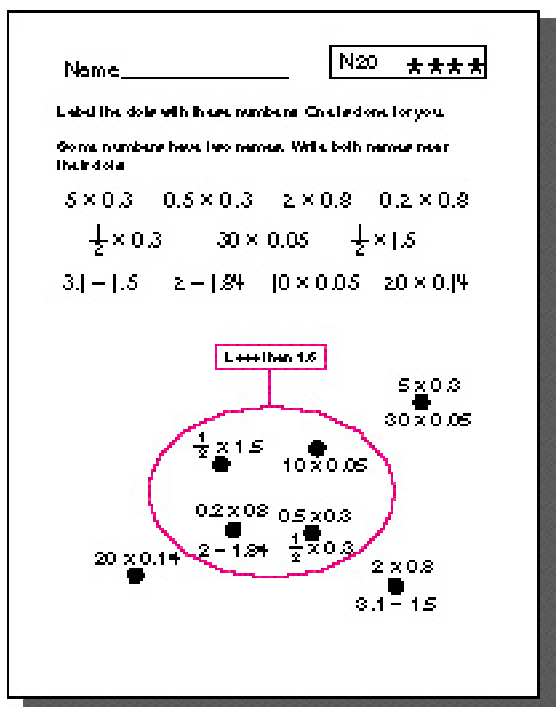
Home Activity

Send home a description of the calculator game in Exercise 4 and suggest that students play the game with family members.


Name
N20 ★
One number in each number sentence is missing a decimal point. Put a decimal point in this number to make the equation true.
0.798 + 256.3 + 9.462 = 2 6 6,5 6
86.37 − 27.826 = 5 8,5 4 4
346.718 + 2 2 8,6 9 = 575.408
41.164 − 3 5,7 5 = 5.414
83.05 × 4.63 = 3 8 4,5 2 1 5
7,2 7 × 3 1,9 2 = 232.0584


Name
N20 ★★
Label the dots and fill in the boxes for the arrows.
+ 0.8
2.4
+ 1.6
4.0
+ 1.6
1.6
3×
5×
5.6
2×
0.8
7×
+ 2.4
20×
10×
16
15×
8
12×
+ 4
+ 1.6
12
9.6
+ 2.4


Name
N20 ★★★
Add.
83 + 127.26 + 0.074
17.6 + 56.147 + 329.62
83
127.26
+ 0.074
210.334
17.6
56.147
+329.62
403.367
Subtract.
59.403 − 17.83
364.1 − 71.47
59.403
− 17.83
41.573
364.1
− 71.47
292.63


Name
N20 ★★★★
Label the dots with these numbers. One is done for you.
Some numbers have two names. Write both names near the dots.
5 × 0.3 0.5 × 0.3 2 × 0.8 0.2 × 0.8
½ × 0.3 30 × 0.05 ½ × 1.5
3.1 − 1.5 2 − 1.84 10 × 0.05 20 × 0.14
Less than 1.5
5 × 0.3
30 × 0.05
½ × 1.5
10 × 0.05
0.2 × 0.8
0.5 × 0.3
2 − 1.84
½ × 0.3
20 × 0.14
2 × 0.8
3.1 − 1.5

Capsule Lesson Summary

Remove various checkers, one at a time, from a configuration on the Minicomputer. Each time, record what happens to the number on the Minicomputer. Following an arrow road, make changes to a number on the Minicomputer by moving, for each arrow, exactly one checker from one square to another. Play *Minicomputer Golf* with a decimal number as the goal.

Materials

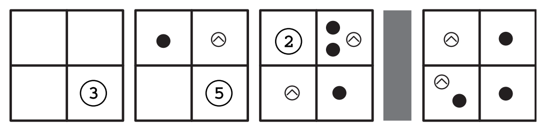
Teacher
- Minicomputer set
- Weighted checker set
- Colored chalk

Student
- Paper

Description of Lesson

Exercise 1

Put this configuration on the Minicomputer.



T: ***What number is on the Minicomputer? Write it on your paper.***

Allow a few minutes for students to decode the number. Check several papers before asking a student to announce the number. (408.7) If necessary, decode the number collectively.

Remove the ③-checker from the 100-square.

T: ***What number is on the Minicomputer now?*** (108.7) ***Describe what happens to the number on the Minicomputer when I remove the ③-checker.***

S: ***Since the ③-checker was on the 100-square, removing it takes away 300. 408.7 – 300 = 108.7.***

Record this in a number sentence on the board. Then return the ③-checker to the 100-square. Continue in a similar manner with the removal of several more checkers. Each time, replace the checker before removing another. This is a possible sequence with corresponding number sentences.

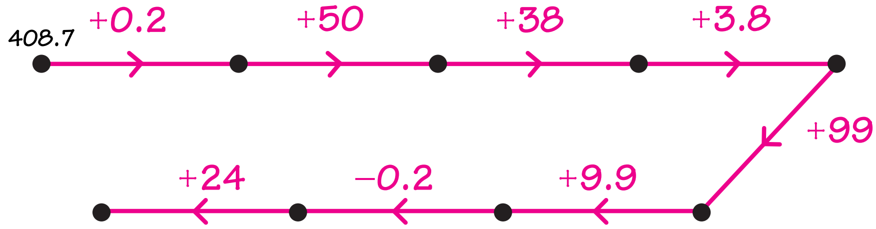
Remove	Number Sentence
② from the 8-square	$408.7 - 16 = 392.7$
⊙ from the 40-square	$408.7 - \widehat{40} = 448.7$ or $408.7 + 40 = 448.7$
● from the 0.4-square	$408.7 - 0.4 = 408.3$
⑤ from the 10-square	$408.7 - 50 = 358.7$
⊙ from the 0.2-square	$408.7 - \widehat{0.2} = 408.9$ or $408.7 + 0.2 = 408.9$

Encourage students to give both an addition and a subtraction sentence for the removal of a negative checker.

Exercise 2

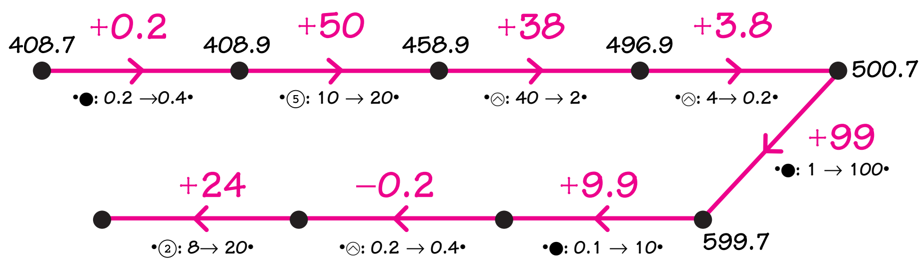
Check that the configuration on the Minicomputer is the same as at the start of Exercise 1.

Draw the following arrow road on the board.



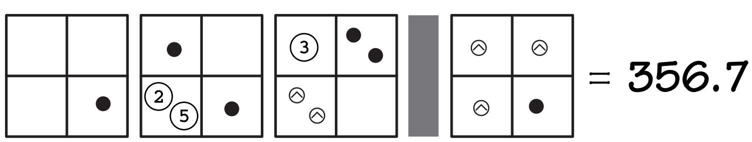
T: ***This arrow road starts at the number on the Minicomputer. Let's make changes to the number on the Minicomputer following this arrow road. For each arrow we will move exactly one checker from the square it's on to another square.*** (Point to the +0.2 arrow.) ***What move will increase the number on the Minicomputer by 0.2?***

Invite a student to make a move on the Minicomputer. Then ask the class for verification and for the new number on the Minicomputer. Label the next dot, and continue in the same manner until you reach the end of the arrow road. The following illustration has all the dots labeled and indicates a possible move for each arrow.



Exercise 3

Play *Minicomputer Golf*. The following is a possible game using a starting configuration for 356.7 and with a goal of 999.9.



Goal: 999.9

N22 FRACTIONS #2

Capsule Lesson Summary

Find other fractional names for $\frac{12}{18}$. Add $\frac{7}{6} + \frac{3}{8}$ by using equivalent fractions with like denominators. Play a *Guess My Rule* game involving a square configuration of four whole numbers. The rule can be stated in several ways in terms of products or equivalent fractions.

Materials

Teacher
- Colored chalk

Student
- Paper
- Worksheets N22*, **, and ***

Description of Lesson

Exercise 1

Ask the class for fractions equivalent to $\frac{12}{18}$, but do not ask for explanations at this moment. For example:

$$\frac{12}{18} = \frac{6}{9} \qquad \frac{12}{18} = \frac{24}{36} \qquad \frac{12}{18} = \frac{2}{3}$$

T: ***How did you find these equivalent fractions?***

S: ***I took one half of both 12 and 18 to get $\frac{6}{9}$.***

S: ***I divided both 12 and 18 by 6 to get $\frac{2}{3}$.***

S: ***I multiplied both 12 and 18 by 2 to get $\frac{24}{36}$.***

Draw arrows to describe students' methods.

$$\frac{12}{18} \underset{\frac{1}{2}\times}{\overset{\frac{1}{2}\times}{=}} \frac{6}{9} \qquad \frac{12}{18} \underset{2\times}{\overset{2\times}{=}} \frac{24}{36} \qquad \frac{12}{18} \underset{\div 6}{\overset{\div 6}{=}} \frac{2}{3}$$

Then ask for several more fractions equivalent to $\frac{12}{18}$. Try to get another fraction with a denominator greater than 18 and another with a denominator less than 18. For example:

$$\frac{12}{18} \underset{\div 3}{\overset{\div 3}{=}} \frac{4}{6} \qquad \frac{12}{18} \underset{10\times}{\overset{10\times}{=}} \frac{120}{180} \qquad \frac{12}{18} \underset{4\times}{\overset{4\times}{=}} \frac{48}{72}$$

Write the addition problem on the board.

T: ***How can we add $\frac{7}{6} + \frac{3}{8}$?***

$$\frac{7}{6} + \frac{3}{8} =$$

If some students suggest that $\frac{7}{6} + \frac{3}{8} = \frac{7+3}{6+8} = \frac{10}{14}$, observe that $\frac{7}{6}$ is more than 1, and therefore, $\frac{7}{6} + \frac{3}{8}$ must be more than 1. But $\frac{10}{14}$ is less than 1. Point out that adding numerators and denominators
is not a good method for adding fractions.

S: ***Draw rectangles with six rows and eight columns.***

S: ***Find fractions equivalent to $\frac{7}{6}$ and $\frac{3}{8}$ with the same denominator.***

T: ***Yes, it is usually easy to add fractions with the same denominator. For example, what number is $\frac{7}{13} + \frac{2}{13}$?*** ($\frac{9}{13}$) ***$\frac{9}{8} + \frac{3}{8}$?*** ($\frac{12}{8}$ or $1\frac{4}{8}$ or $1\frac{1}{2}$)

Let's try to rewrite our addition problem using fractions with the same denominator. What are some fractions equivalent to $\frac{7}{6}$ and to $\frac{3}{8}$?

Continue listing equivalent fractions suggested by students until you have several fractions equivalent to each of $\frac{7}{6}$ and $\frac{3}{8}$ and you have at least one pair of fractions from the two lists with the same denominator. Circle a pair of fractions with the same denominator. For example:

$$\frac{7}{6} = \frac{14}{12} = \frac{28}{24} = \frac{70}{60} = \frac{56}{48} \qquad \frac{3}{8} = \frac{6}{16} = \frac{18}{48} = \frac{30}{80} = \frac{9}{24}$$

T: ***What number is $\frac{7}{6} + \frac{3}{8}$?***

S: ***$1\frac{13}{24}$, since $\frac{7}{6} + \frac{3}{8} = \frac{28}{24} + \frac{9}{24} = \frac{37}{24} = 1\frac{13}{24}$.***

$$\frac{7}{6} + \frac{3}{8} = \frac{28}{24} + \frac{9}{24} = \frac{37}{24} = 1\frac{13}{24}$$

Record this information on the board.

Exercise 2

When playing *Guess My Rule*, encourage students to think about what the rule could be without saying it out loud. In this exercise, the rule is a relationship among four numbers arranged in a square.

Put these squares on the board.

5	10
15	30

4	6
6	9

T: ***I have a secret rule for putting numbers in four parts of a square. These two completed squares are clues. Try to figure out my rule.***

Draw another 2-by-2 square on the board and leave one part empty. Ask students to try to predict which number goes in the empty box.

3	5
12	

Note: There are several ways to formulate the rule. The four numbers in a completed square must satisfy these equivalent conditions:

a	b
c	d

$$a \times d = b \times c$$

$$\frac{a}{b} = \frac{c}{d} \qquad \frac{a}{c} = \frac{b}{d}$$

In this case, the number in the empty box is 20 because 3 x 20 = 5 x 12, or $\frac{3}{12} = \frac{5}{20}$, or $\frac{3}{5} = \frac{12}{20}$.

You might suggest that students write their guesses on paper for you to check. Acknowledge any correct guess and deny any incorrect guess. After a while, invite a student to complete the square, but do not give away the rule. If necessary, announce that 20 goes in the empty part of the square.

Continue the activity, using configurations with a different part of the square empty. In each case, ask students to fill in the number in red.

6	9
8	12

5	15
7	21

4	20
8	40

3	4
6	8

6	24
7	28

T: ***Who can explain the rule?***

Accept various explanations, each time checking the method on several squares.

S: ***When you multiply diagonal numbers, you get the same result. For example, in the square*** $\begin{array}{|c|c|}\hline 3&5\\\hline 12&20\\\hline\end{array}$***, 3 x 20 = 60 and 5 x 12 = 60.***

S: ***The columns of numbers form equivalent fractions. For example, in the square*** $\begin{array}{|c|c|}\hline 5&10\\\hline 15&30\\\hline\end{array}$***,*** $\frac{5}{15} = \frac{10}{30} = \frac{1}{3}$.

S: ***You form equivalent fractions using the rows of numbers. For example, in the square*** $\begin{array}{|c|c|}\hline 3&5\\\hline 12&20\\\hline\end{array}$***,*** $\frac{3}{5} = \frac{12}{20}$.

S: ***You can multiply the numbers in the top row by the same number to get the numbers in the bottom row. For example, in the square*** $\begin{array}{|c|c|}\hline 3&5\\\hline 12&20\\\hline\end{array}$***, 4 x 3 = 12 and 4 x 5 = 20.***

S: ***You can multiply the numbers in the first column by the same number to get the numbers in the second column. For example, in the square*** $\begin{array}{|c|c|}\hline 5&10\\\hline 15&30\\\hline\end{array}$***, 2 x 5 = 10 and 2 x 15 = 30.***

The last two observations may not be mentioned because it is harder to find a number in a case like $\begin{array}{|c|c|}\hline 6&9\\\hline 8&12\\\hline\end{array}$.

Here, for example, going from the top row to the bottom row you can multiply by $\frac{4}{3}$, and going from the first column to the second column you can multiply by $\frac{3}{2}$. If students do not mention the first two explanations above, suggest them yourself. Then, invite students to complete the following squares. Use several methods to confirm each result. Solutions are in red.

3	6
9	18

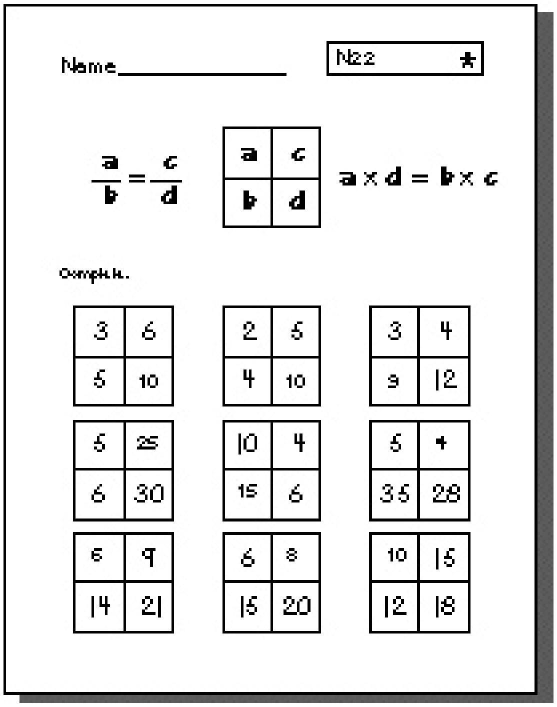
8	10
12	15

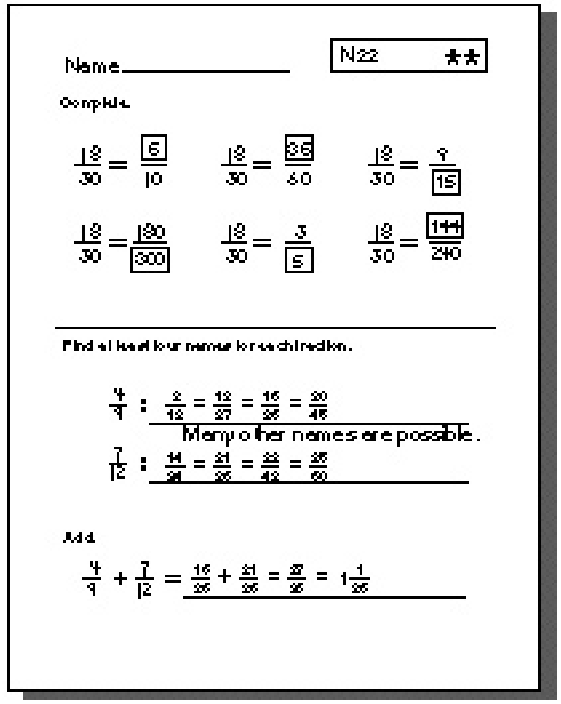
30	20
12	8

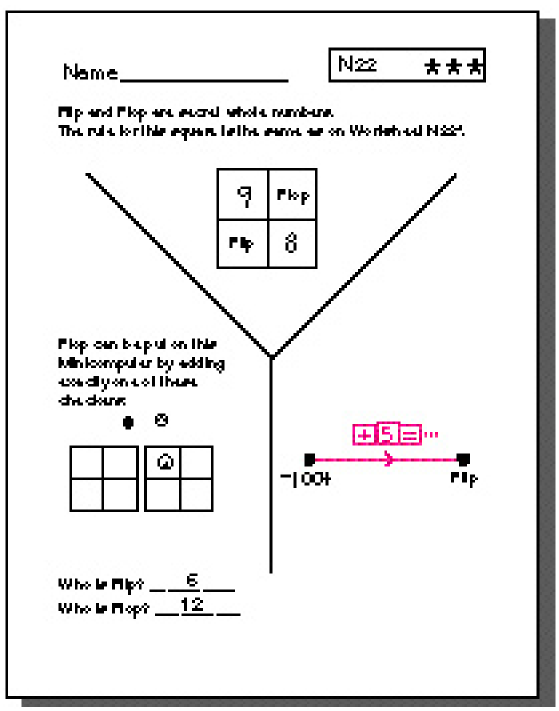
6	10
30	50

10	15
6	9

Worksheets N22*, **, *** are available for individual work.


Name
N22 ★
$\frac{a}{b} = \frac{c}{d}$
a c
b d
a × d = b × c
Complete.
3 6
5 10
2 5
4 10
3 4
9 12
5 25
6 30
10 4
15 6
5 4
35 28
6 9
14 21
6 8
15 20
10 15
12 18


Name
N22 ★★
Complete.
$\frac{18}{30} = \frac{6}{10}$
$\frac{18}{30} = \frac{36}{60}$
$\frac{18}{30} = \frac{9}{15}$
$\frac{18}{30} = \frac{180}{300}$
$\frac{18}{30} = \frac{3}{5}$
$\frac{18}{30} = \frac{144}{240}$
Find at least four names for each fraction.
$\frac{4}{9}$: $\frac{8}{18} = \frac{12}{27} = \frac{16}{36} = \frac{20}{45}$
Many other names are possible.
$\frac{7}{12}$: $\frac{14}{24} = \frac{21}{36} = \frac{28}{48} = \frac{35}{60}$
Add.
$\frac{4}{9} + \frac{7}{12} = \frac{16}{36} + \frac{21}{36} = \frac{37}{36} = 1\frac{1}{36}$


Name
N22 ★★★
Flip and Flop are secret whole numbers.
The rule for this square is the same as on Worksheet N22*.
9 Flop
Flip 8
Flop can be put on this Minicomputer by adding exactly one of these checkers.
+5=
−100
Flip
Who is Flip? 6
Who is Flop? 12

Capsule Lesson Summary

Estimate the percent of a square region covered by a hand print. Use a 10 x 10 square grid the same size as the square region (the square divided into 100 smaller squares) to find a better estimate. Use similar materials to find the percent of given squares that are shaded, or to shade a given percent of a square region.

Materials

Teacher
- Blacklines N23(a)–(c)
- Square transparency
- 10 x 10 square grid transparency
- Scissors
- Overhead projector

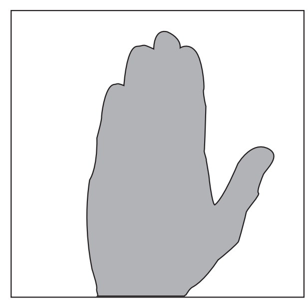
Student
- Transparent 10 x 10 square grid
- Worksheets N23(a) and (b)

Advance Preparation: Use Blacklines N23(a) and (b) to make transparencies of a square and a 10 x 10 square grid the same size. Make copies of Blackline N23(c) on transparency film or tracing paper, and cut out the 10 x 10 square grids.

Description of Lesson

Exercise 1

Display the 18 cm square on Blackline N23(a). Use your hand or call on a student to use a hand to cover part of the square.



T: ***About what percent of this square did I (or the student) cover with my (his or her) hand?***

Let students estimate the percent covered, and encourage explanations of their estimates. You are likely to get a wide range of estimates. Trace around your (the student's) hand, and shade in the region covered.

T: ***How could we get a better estimate or a way to know if an estimate is close to the percent of the square covered?***

Encourage a variety of suggestions. Then display the transparent 10 x 10 square grid the same size as the 18 cm square. Demonstrate for the class that the squares are the same size.

T: ***This square is divided into 100 smaller squares. How could we use it to get a better estimate, or to check our estimates of percent covered?***

You may want to suggest students first talk with a partner or in small groups about how to use the 10 x 10 square grid. Then lead a collective discussion that observes the following:

- *Percent* means "per hundred" or "out of a hundred."
- The bigger square is divided into 100 smaller squares by the grid.
- Counting how many smaller squares are covered, or making a good estimate, will lead to a good estimate of the percent covered—that is, how many small squares out of 100 are covered.

N23

Let the class find an estimate of how many grid squares are covered by the shaded hand print, and then note the percent of the square covered by the hand. Compare this result to some of the earlier estimates.

Exercise 2

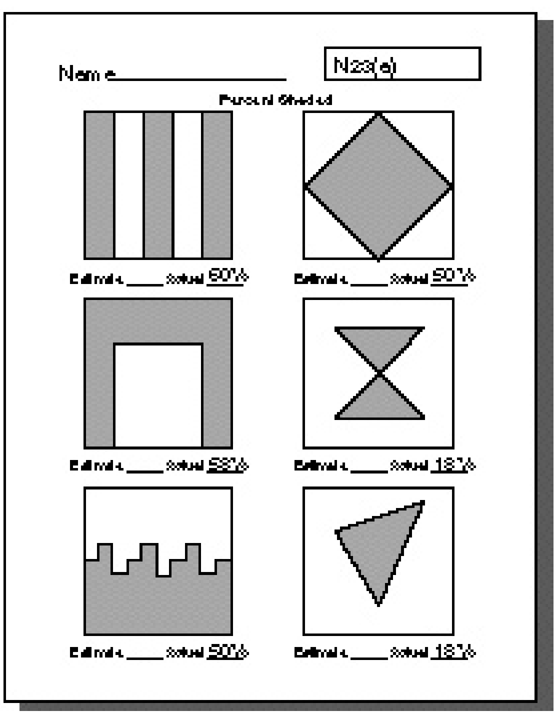
Distribute copies of Worksheets N23(a) and (b) and a transparent 10 x 10 square grid (which is the same size square as those on the worksheets). Instruct students first to estimate what percent of each square is shaded and then to find the actual percent shaded using the grid to measure.

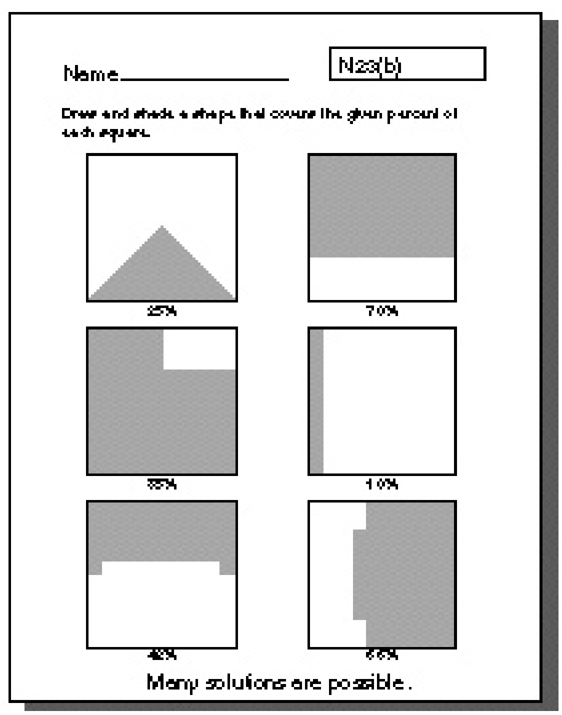
When several students have completed Worksheet N23(a), you may like to organize groups where students can compare their results. A group should agree on an actual percent shaded for each square and be ready to give explanations. You may like to have a collective discussion of a few of the squares, especially to share different methods for finding the actual percent shaded.

Direct students to complete Worksheet N23(b). Again in groups, students could compare and check each others' work. They should observe the variety of solutions and begin to get a visual of 25% shaded; of 70% shaded, and so on.

Home Activity

This is a good time to send a letter to parents/guardians about percent. Blackline N23(d) has a sample letter.





Capsule Lesson Summary

Given a rectangular cake cut into six pieces, determine the fraction of the cake each piece represents. Give a cost for the whole cake, and decide what the corresponding cost would be for each piece. Cut a rectangle into six pieces, all different sizes.

Materials

Teacher
- Rectangle transparency
- Colored pencils, pens, or crayons
- Overhead projector (optional)
- Blackline N24

Student
- Metric ruler
- Unlined paper
- Worksheets N24(no star), *, and **

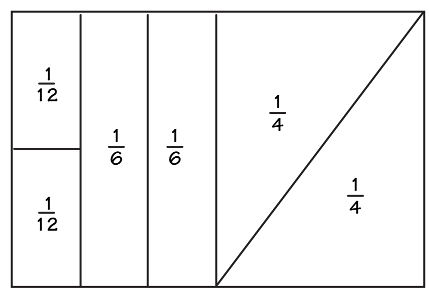
Advance Preparation: Use Blackline N24 to make a display copy (transparency) of a rectangle cut into six pieces for Exercise 1. You may prefer to carefully draw this rectangle on the board.

Description of Lesson

If your students recall the story of Sara and Amelia (buying parts of cakes) used to develop a model for adding fractions, you may use that situation to present the problems in this lesson. The story was used in several lessons in *CSMP Mathematics for the Intermediate Grades, Part IV*.

Exercise 1

Display a rectangular shape cut into six pieces, as shown here. The fractions will be added by the class.

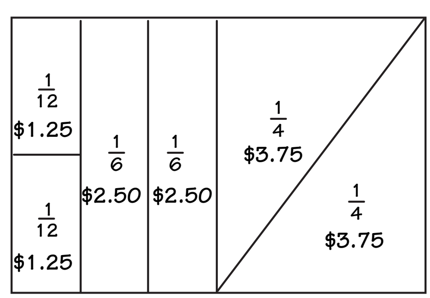


T: ***One day when Sara and Amelia arrive at the bakery, Mr. Booker (the owner) announces that he can only sell them pieces of a cake he has already cut. This is the cake cut into six pieces. What fractional part of the cake is each piece?***

Invite students to describe each piece as a fractional part of the whole cake. Ask for explanations and class agreement for each piece.

T: ***Suppose Mr. Booker decides he wants to collect a total of $15 for the whole cake. How much should he charge for each piece?***

Let students explain how Mr. Booker can determine how much to charge for each piece. For example:



S: ***If he wants to get $15 for the whole cake, then half of the cake would be $7.50. Each triangle piece is half of a half, and ½ x $7.50 = $3.75.***

S: ***The other half of the cake must total $7.50 too. So each long rectangular piece (⅙) should be $2.50; 3 x $2.50 = $7.50.***

S: ***The smaller rectangular pieces are each half of a $2.50 piece, so they should each cost $1.25.***

As appropriate, write some number sentences on the board to highlight the explanations.
For example:

$\frac{1}{2} \times \$15.00 = \$7.50$ $\frac{1}{2} \times \$7.50 = \$3.75$	$\frac{1}{3} \times \$7.50 = \$2.50$ $\frac{1}{2} \times \$2.50 = \$1.25$
$\frac{1}{4} \times \$15.00 = \$3.75$	$\frac{1}{6} \times \$15.00 = \$2.50$ $\frac{1}{12} \times \$15.00 = \$1.25$

Highlight the pieces of cake in your picture as you explain,

T: ***Sara decides to get one triangular piece and one long rectangular piece. What fractional part of the whole cake does Sara get? How much will Sara pay for this much cake?***

As students respond, write fraction facts on the board to describe the situation.

S: ***Sara gets ¼ + ⅙ of the cake.***
¼ + ⅙ = 3⁄12 + 2⁄12 = 5⁄12.

Sara: $\frac{1}{4} + \frac{1}{6} = \frac{3}{12} + \frac{2}{12} = \frac{5}{12}$

S: ***Sara will pay $3.75 + $2.50 = $6.25.***

$\frac{5}{12} \times \$15.00 = \$6.25$

T: ***Does Sara get more or less than one-half of the cake?*** (Less)

Note that 5⁄12 is 1⁄12 less than one-half of the cake.

T: ***Amelia announces that she has just $5.00 to spend. What is the largest fractional part of the cake she can get? Which pieces could she get?***

S: ***Amelia can get 2⁄6 or ⅓ of the cake. Each ⅙ piece costs $2.50, so 2⁄6 costs $5.00.***

S: ***Amelia can get ⅓ of the cake because***
⅓ x $15.00 = $5.00.

$\frac{1}{3} \times \$15.00 = \$5.00$

S: ***Amelia could get the one long and the two smaller rectangular pieces.***

Distribute copies of Worksheet N24(no star). Organize students in small groups or partners, and instruct the groups to repeat this exercise with a square cake cut into six pieces, as on the worksheet. Note that this time the cake is to sell for a total of $20. After a short while, you may call on groups to present their findings or to compare results with other groups.

Exercise 2

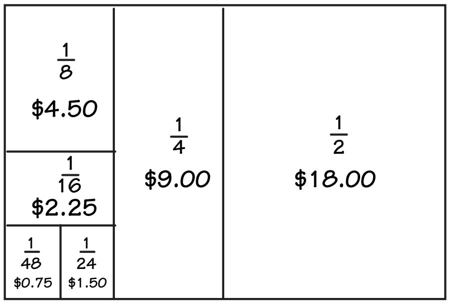
Draw a rectangle on the board, and present a new problem to the groups.

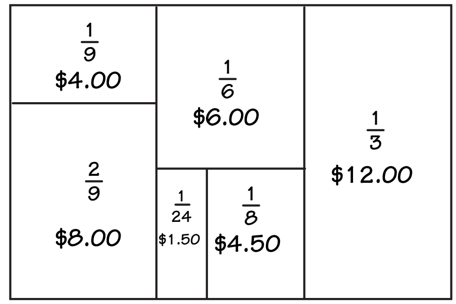
T: ***Both of these cakes were cut into six pieces, and in each case there were sometimes two pieces of the same size. Mr. Booker has another rectangular cake, but this time he wants to cut it into six pieces that are all different sizes. Mr. Booker also must know what fraction of the cake each piece is so he can determine what to charge for it.***

Provide groups with unlined paper, and suggest they draw a rectangle 16 cm by 24 cm. Direct the groups to cut the rectangle into six pieces, all different sizes, and then to label each piece to show what fraction of the cake it is. Warn the groups that when they cut the cake they must be able to determine the fraction of the cake for each piece.

There are, of course, many solutions to this problem. Some groups may end up with pieces for which it is difficult to determine the fraction. You may suggest they begin again with a new copy of the 16 cm by 24 cm rectangle.

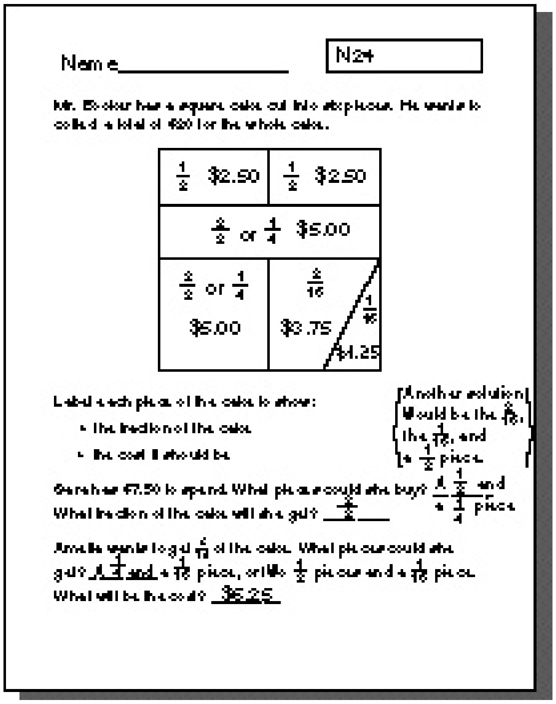
Choose solutions from one or two of the groups to display on the board. Then assign a total cost for the whole cake, and determine the cost of each piece. For example:





Worksheets N24* and ** are available for individual work.

N24



Name________________ N24

Mr. Booker has a square cake cut into six pieces. He wants to collect a total of $20 for the whole cake.

Label each piece of the cake to show:

- the fraction of the cake
- the cost it should be

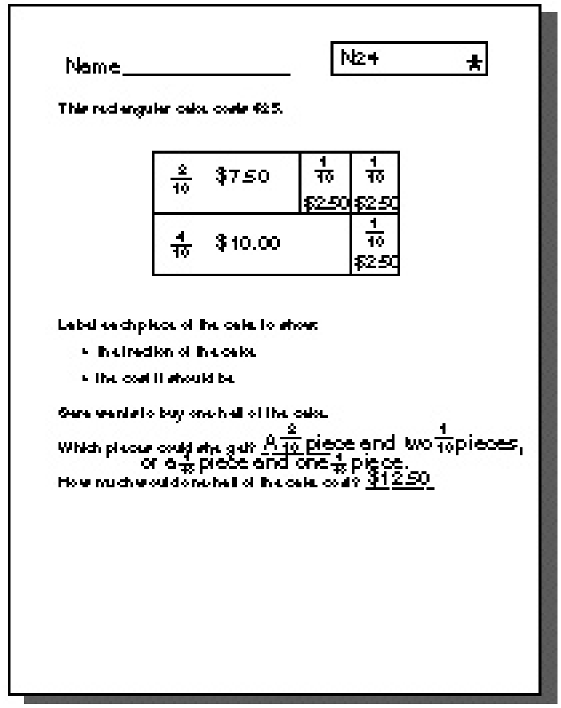
Another solution would be the $\frac{3}{16}$, the $\frac{1}{16}$, and a $\frac{1}{8}$ piece.

Sara has $7.50 to spend. What pieces could she buy? A $\frac{1}{8}$ and a $\frac{1}{4}$ piece

What fraction of the cake will she get? $\frac{3}{8}$

Amelia wants to get $\frac{5}{16}$ of the cake. What pieces could she get? A $\frac{1}{4}$ and a $\frac{1}{16}$ piece, or the $\frac{1}{8}$ pieces and a $\frac{1}{16}$ piece.

What will be the cost? $6.25



Name________________ N24 ★

This rectangular cake costs $25.

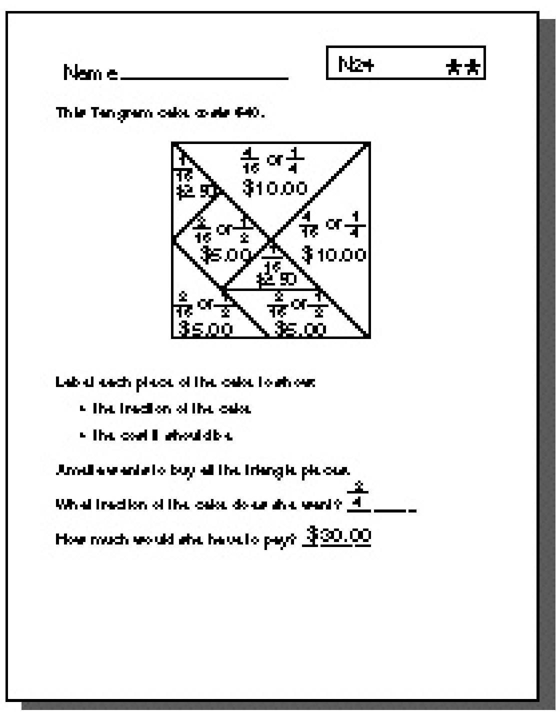
Label each piece of the cake to show:

- the fraction of the cake
- the cost it should be

Sara wants to buy one-half of the cake.

Which pieces could she get? A $\frac{3}{10}$ piece and two $\frac{1}{10}$ pieces, or a $\frac{4}{10}$ piece and one $\frac{1}{10}$ piece.

How much would one-half of the cake cost? $12.50



Name________________ N24 ★★

This Tangram cake costs $40.

Label each piece of the cake to show:

- the fraction of the cake
- the cost it should be

Amelia wants to buy all the triangle pieces.

What fraction of the cake does she want? $\frac{3}{4}$

How much would she have to pay? $30.00

Capsule Lesson Summary

Introduce exponential notation using a *Guess My Rule* game for an operation $*$ where $a * b = a^b$. Write various numbers as products of prime numbers, making use of the exponential notation. Solve a detective story with clues involving prime divisors, a string picture, and the calculator relation $\boxed{-}$ $\boxed{6}$ $\boxed{=}$ … .

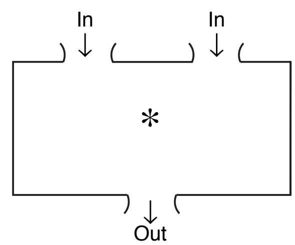
Materials

Teacher		Student	
	• Colored chalk		• Paper • Colored pencils, pens, or crayons

Description of Lesson

Exercise 1: Guess My Rule

This exercise uses an operation rule to introduce exponential notation. You may like to use a "machine" picture to review how an operation rule works.



T: ***I have a secret for rule for $*$. I'll give you some clues about my rule. Try to figure out the rule for $*$.***

Write several number sentences on the board as clues. Then write an open sentence, and see if anyone can predict which number goes in the box.

$2 * 3 = 8$
$3 * 2 = 9$
$3 * 1 = 3$
$1 * 3 = 1$
$5 * 2 = \boxed{}$

Note: The rule is $a * b = a^b$. The number in the box is 25 because $5 * 2 = 5^2 = 5 \times 5 = 25$.

You might suggest that students write their guesses on paper for you to check. Acknowledge aloud correct guesses and reject incorrect ones, for example, "No, $5 * 2$ is not 10." After a moment, let a student announce to the class that 25 is in the box, without giving away the rule. If no one guesses correctly, put 25 in the box yourself.

Continue the activity with these problems. (Answers are in boxes.)

$10 * 2 = \boxed{100}$
$4 * 2 = \boxed{16}$
$6 * 2 = \boxed{36}$
$8 * 2 = \boxed{64}$
$9 * 2 = \boxed{81}$

T: ***Who can explain $*2$?***

S: ***$10 * 2 = 100$ because $10 \times 10 = 100$; $4 * 2 = 16$ because $4 \times 4 = 16$; and so on.***

S: ***For $*2$, you multiply the first number by itself (you square the number).***

T: ***Let's look at $*3$.***

Write these problems on the board, and instruct students to complete them on their papers. (Answers are in boxes.)

$5 * 3 = \boxed{125}$
$3 * 3 = \boxed{27}$
$4 * 3 = \boxed{64}$

T: ***Who can explain $*3$?***

S: ***For $*3$, you multiply the first number by itself three times; $5 * 3 = 5 \times 5 \times 5 = 125$.***

Accept any reasonable explanation.

T: ***Now you know part of my rule.***

Write these problems on the board and instruct students to complete them. (Answers are in boxes.)

$2 * 4 = \boxed{16}$ $\quad 1 * 10 = \boxed{1}$ $\quad 10 * 3 = \boxed{1000}$

$3 * 4 = \boxed{81}$ $\quad 10 * 1 = \boxed{10}$ $\quad 2 * 6 = \boxed{64}$

T: ***Who can explain my rule?***

S: ***$2 * 4$ means multiply 2 by itself four times; $2 \times 2 \times 2 \times 2 = 16$.***

Accept any reasonable explanations that students offer. As this may be a difficult idea to verbalize, students might prefer to explain the rule by writing the equivalent multiplication problems.

$$2 * 4 = 2 \times 2 \times 2 \times 2 = \boxed{16}$$
$$3 * 4 = 3 \times 3 \times 3 \times 3 = \boxed{81}$$
$$1 * 10 = 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 = \boxed{1}$$
$$10 * 1 = 10 = \boxed{10}$$
$$10 * 3 = 10 \times 10 \times 10 = \boxed{1000}$$
$$2 * 6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = \boxed{64}$$

T: ***As you can see, it is sometimes quite cumbersome to indicate many multiplications, so there is a shorthand notation we will use.***

Extend one of the number sentences on the board.

$$2 * 4 = 2 \times 2 \times 2 \times 2 = 2^4 = \boxed{16}$$

T: ***For $2 \times 2 \times 2 \times 2$ write 2^4*** (read as "two to the fourth power").

Invite students to write the other expressions with exponential notation. Then write these problems on the board. Direct students to copy them on their papers. Encourage students who have difficulty with this notation to write the associated multiplication calculations. Check the problems collectively. (Answers are in boxes.)

$2^6 = \boxed{64}$ $\quad 7^2 = \boxed{49}$

$3^5 = \boxed{243}$ $\quad 4^3 = \boxed{64}$

T: ***What number is $2^5 \times 2^3$? Why?***

$$2^5 \times 2^3 =$$

Allow a few minutes for students to consider the question.

S: ***256. $2^5 = 32$, and $2^3 = 8$. $32 \times 8 = 256$.***

S: ***$2^5 \times 2^3 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8$, and $2^8 = 256$.***

Record the equivalent problem and the result.

$$2^5 \times 2^3 = 2^8 = 256$$

Pose a new problem.

T: ***What number is $3^2 \times 3^3$?***

$$3^2 \times 3^3 =$$

S: ***$3^2 \times 3^3 = 3^5 = 243$.***

T: ***How did you conclude that $3^2 \times 3^3 = 3^5$?***

S: ***$3^2 = 3 \times 3$ and $3^3 = 3 \times 3 \times 3$, so $3^2 \times 3^3 = 3 \times 3 \times 3 \times 3 \times 3 = 3^5$.***

T: ***Do you notice anything interesting about these problems?***

S: ***$2^5 \times 2^3 = 2^{5+3} = 2^8$. $3^2 \times 3^3 = 3^{2+3} = 3^5$.***

Exercise 2

Pose the following problem for students to do individually or with a partner.

T: ***Write 36 as a product of prime numbers. You may use a prime number more than once. Try to use our shorthand notation when a prime number occurs more than once in the product.***

Allow a couple minutes for independent work. Then call on a volunteer to write a product on the board.

$$36 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$$

Write these numbers on the board, and instruct students to write each number as a product of prime numbers. Feel free to adjust the problems to the numerical abilities of your students.

24 42 70 84 100 124 200

Check students' work collectively. Solutions are given below.

$$42 = 2 \times 3 \times 7$$
$$24 = 2^3 \times 3$$
$$70 = 2 \times 5 \times 7$$
$$84 = 2^2 \times 3 \times 7$$

$$100 = 2^2 \times 5^2$$
$$120 = 2^3 \times 3 \times 5$$
$$200 = 2^3 \times 5^2$$

T: ***Many of you have devised your own way to solve these problems. Let me show you one way I like to find the prime numbers.***

T: ***First give me one positive prime divisor of 360.***

S: ***2; $2 \times 180 = 360$, and 2 is prime.***

$$\begin{array}{r|l} 360 & \\ 180 & 2 \end{array}$$

As students name prime divisors, organize this information on the board, as shown here.

T: ***Now give me on me one positive prime divisor of 180.***

S: ***2; $2 \times 90 = 180$, and 2 is prime.***

Continue this process until the last number in the left column is 1.

$$\begin{array}{r|l} 360 & \\ 180 & 2 \\ 90 & 2 \\ 45 & 2 \\ 15 & 3 \\ 5 & 3 \\ 1 & 5 \end{array}$$

Invite a student to write 360 as a product of prime numbers on the board.

$$360 = 2^3 \times 3^2 \times 5$$

Exercise 3: Detective Story

Announce that Zog is a secret number.

Clue 1

You may like to abbreviate this information about Zog on the board as you announce,

T: ***Zog is a whole number less than 100. The only possible prime divisors of Zog are 2 and 3. That is, 2 or 3 might be a divisor of Zog, but no other positive prime number is a divisor of Zog. Which numbers could Zog be?***

Zog < 100

Prime divisors of Zog: 2 or 3

S: ***Zog could be 4 because 2 x 2 = 4.***

S: ***Zog could be 27 because 3 x 3 x 3 = 27.***

S: ***Zog could be 6 because 2 x 3 = 6.***

On the board, list numbers that Zog could be as students suggest them. It is not necessary to get a complete list at this point. For example:

4, 27, 6, 12, 9, 3, 8, 2, …

T: ***We have found some of the numbers that Zog could be, but we may not have found all of the possibilities. Let's try to be more systematic in our search for these numbers.***

Draw this table on the board.

X	1	3	3^2	3^3	3^4
1					
2					
2^2					
2^3					
2^4					
2^5					
2^6					

T: ***Why do you think I stopped at 3^4 and 2^6?***

S: ***3^5 and 2^7 are both greater than 100.***

T: ***How can this table help us in our search?***

Allow time for students to consider the question.

S: ***In the first row, there will be the numbers that have only 3 as a prime divisor. In the first column, there will be the numbers that have only 2 as a prime divisor. In the rest of the table, there will be numbers that have both 2 and 3 as prime divisors.***

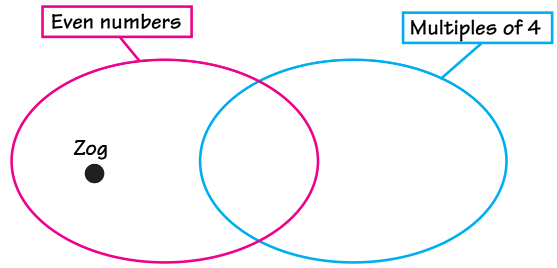
Invite students to complete the relevant portion of the table. Since Zog is less than 100, it is not necessary to fill in numbers greater than 100.

The class should conclude that Zog could be any number (less than 100) that appears in the table.

12)4 444	
−3 600	300
844	
−480	40
364	
−360	30
4	

Clue 2

Draw this string picture on the board and ask,



T: ***What new information about Zog does this clue give us?***

S: ***Zog is an even number, but Zog is not a multiple of 4.***

S: ***Zog is an even number between two multiples of 4; that is, 2 more than a multiple of 4.***

T: ***Zog is one of the numbers in the table. Who is Zog?***

Allow a couple minutes for students to check the numbers in the table as possibilities for Zog. Encourage various approaches.

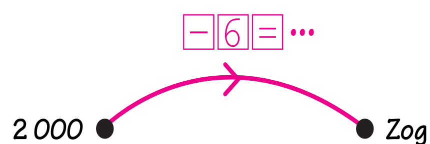
S: ***Since Zog is even, we can eliminate the numbers in the first row (1, 3, 9, 27, 81).***

S: ***4, 12, 36, and all of the numbers that appear below those numbers in the table can be eliminated because they are all multiples of 4.***

S: ***2, 6, 18, and 54 are all possibilities for Zog.***

Clue 3

Draw this arrow picture on the board.



T: ***The third clue is given by this arrow picture. Who is Zog?***

Let students work independently or with partners to find Zog. Ask a volunteer to explain the solution.

S: ***If we press $\boxed{=}$ 300 times, we subtract 1 800; 2 000 – 1800 = 200. If we press $\boxed{=}$ 30 more times, we subtract 180; 200 – 180 = 20. Then press $\boxed{=}$ three more times, subtracting 18, and we get 2. So Zog is 2.***

The class should check that Zog cannot be 6, 18, or 54, and conclude that Zog is 2.

N26 DIVISION ALGORITHM #2

Capsule Lesson Summary

Observe patterns in a sequence of division problems. Determine many division problems with solutions that can be written as 2 R = 3. Practice a standard algorithm for dividing a decimal by a whole number.

Materials

Teacher	• Colored chalk	Student	• Paper

Description of Lesson

Exercise 1

Write this division calculation on the board.

$$43.38 \div 6 =$$

T: ***A six-player volleyball team wins first prize of $43.38. The players share the prize equally. How much money does each player receive?***

Encourage students to use mental arithmetic and estimation techniques. For example:

S: ***Each player receives at least $7 since 6 x 7 = 42.***

S: ***Then $1.38 ($43.38 – $42) remains to share.***

S: ***Give each player $0.20 more since 6 x 0.20 = 1.20.***

S: ***$0.18 ($1.38 – $1.20) remains, so give each player $0.03 more.***

S: ***Therefore, each player receives $7.23 ($7.00 + $0.20 + $0.03).***

Record the solution, and then pose a new related problem.

$$\begin{array}{rl} +\square & 43.38 \div 6 = 7.23 \\ & 49.38 \div 6 = \boxed{} \quad +\square \end{array}$$

T: ***What decimal number is 49.38 ÷ 6? Explain your answer.***

S: ***8.23. If we think of sharing money, there is $6 more to share since 43.38 + 6.00 = 49.38. So each player receives an additional $1.00.***

Record this information and extend the sequence of problems.

$$\begin{array}{rl} +\boxed{6} & 43.38 \div 6 = 7.23 \quad +\boxed{1} \\ +\square & 49.38 \div 6 = \boxed{8.23} \quad +\square \\ & 79.38 \div 6 = \boxed{} \end{array}$$

T: ***How much more money is there to share?***

S: ***$30 ($79.38 – $49.38).***

S: ***Each person receives $5 more because 6 x 5 = 30.***

S: ***Now each player receives $13.23 ($8.23 + $5.00).***

N26

Record the responses. Continue in a similar manner with the following sequence of problems. (Answers are in boxes.)

$$\begin{array}{rlr}
 & 43.38 \div 6 = 7.23 & \\
+\boxed{6} & 49.38 \div 6 = \boxed{8.23} & +\boxed{1} \\
+\boxed{30} & 79.38 \div 6 = \boxed{13.23} & +\boxed{5} \\
+\boxed{12} & 91.38 \div 6 = \boxed{15.23} & +\boxed{2} \\
+\boxed{0.60} & 91.98 \div 6 = \boxed{15.33} & +\boxed{0.10} \\
+\boxed{2.40} & 94.38 \div 6 = \boxed{15.73} & +\boxed{0.40} \\
+\boxed{0.54} & 94.92 \div 6 = \boxed{15.82} & +\boxed{0.09} \\
+\boxed{4.20} & 99.12 \div 6 = \boxed{16.52} & +\boxed{0.70}
\end{array}$$

Exercise 2

Write a division problem on the board in two ways.

$$11 \div 4 = \qquad 4\overline{)11}$$

Mention that these are different ways to write the same division calculation. Ask for the result, attempting to get three responses:

$$2\tfrac{3}{4} \qquad\qquad 2.75 \qquad\qquad 2 \text{ R} = 3$$

T: ***$2\frac{3}{4} = 2.75$. Both are standard ways of writing the result of the calculation $11 \div 4$. But when we write $4\overline{)11}$, we often write the result as 2 R = 3.***

Begin this table on the board.

2 R = 3
$11 \div 4 = 2\frac{3}{4}$

T: ***Can you think of other division calculations with results that we could write as 2 R = 3?***

S: ***$2\frac{3}{\square}$ or $17 \div 7$.***

T: ***What is a fractional name for $17 \div 7$?***

S: ***$2\frac{3}{7}$.***

Record this information in the table on the board and solicit several more answers. For example:

2 R = 3
$11 \div 4 = 2\frac{3}{4}$
$17 \div 7 = 2\frac{3}{7}$
$15 \div 6 = 2\frac{3}{6} = 2\frac{1}{2}$
$43 \div 20 = 2\frac{3}{20}$
$63 \div 30 = 2\frac{3}{30} = 2\frac{1}{10}$

T: ***Do you notice any patterns?***

S: ***Each fraction has the form $2\frac{3}{\square}$ where $\square$ is for a whole number greater than 3.***

Let students describe other patterns they observe.

T: ***Are all these numbers—$2\frac{3}{4}$, $2\frac{3}{7}$, $2\frac{3}{6}$, …—equal?*** (No)
In our list which number is the greatest? ($2\frac{3}{4}$) ***The least?*** ($2\frac{1}{10}$)

Could we find division calculations with results more or less than these that fit this pattern?

S: ***$103 \div 50 = 2\frac{3}{50}$, and $2\frac{3}{50}$ is less than $2\frac{1}{10}$.***

S: ***We can find a result less than any other by using a greater denominator. For example, $203 \div 100 = 2\frac{3}{100}$ and $2\frac{3}{100}$ is still less than $2\frac{3}{50}$.***

S: ***We cannot find a greater result.***

Consider the list of suggested results in order, for example:

$2\frac{3}{4}$ $\quad 2\frac{3}{6}$ $\quad 2\frac{3}{7}$ $\quad 2\frac{3}{20}$ $\quad 2\frac{3}{30}$ $\quad 2\frac{3}{50}$ $\quad 2\frac{3}{100}$

T: ***We do not have $2\frac{3}{8}$ in our list. Can we find a division calculation with the result $2\frac{3}{8}$?***

S: ***$19 \div 8$. Double 8 and add 3 to get 19.***

Ask for division calculations for a few other fractions of the form $2\frac{3}{\square}$ that are not in your list.

T: ***We have found that 2 R = 3 can come from many division calculations. 2 R = 3 is only understood when the problem is there; it does not name a number. Often results like $2\frac{3}{4}$ or 2.75 are preferred because they are numbers and are understood as such.***

Exercise 3

Write this division calculation on the board.

$$12\overline{)4\,444}$$

T: ***Twelve people are sharing \$4 444. Does each person receive at least \$10? \$100? \$1 000?***

Direct solving this problem in the manner suggested in Exercise 2 of Lesson N18. You need not insist on best estimates at each stage. Proceed until only \$4 remains to share.

12)4 444	
−3 600	300
844	
−480	40
364	
−360	30
4	

T: ***So far, how much does each person receive?***

S: ***\$370. There is \$4 left over.***

T: ***How could we write the result of this calculation?***

S: ***370 R = 4.***

S: ***$370\frac{4}{12}$ or $370\frac{1}{3}$.***

T: ***We could leave the result as 370 R = 4 or we could write $4\,444 \div 12 = 370\frac{1}{3}$, but also we can share \$4 further. With money we usually write results as decimals rather than fractions like $\frac{1}{3}$. Let's continue the division process by using decimal numbers.***

Add zeros to the calculation as shown below.

T: ***Suppose we share \$4.00 among the 12 people. How much could each person receive?***

S: ***\$0.30, since $12 \times 0.30 = 3.60$.***

Continue until only \$0.04 remains to share.

$$\begin{array}{r|l} 12\overline{)4\,444.00} & \\ -3\,600 & 300 \\ \hline 844 & \\ -480 & 40 \\ \hline 364 & \\ -360 & 30 \\ \hline 4.00 & \\ -3.60 & 0.30 \\ \hline 0.40 & \\ -0.36 & 0.03 \\ \hline .04 & \end{array}$$

T: ***Could we continue this process?***

S: ***Yes, if we don't just think of money. Write 0.04 as 0.040. Then 12 x 0.003 = 0.036.***

S: ***The process would never stop since there is always a remainder like 0.04 or 0.004.***

T: ***If we think of money, how might we write the result?***

S: ***370.33 R = 0.04.***

T: ***Think of the process never stopping. What number is 4 444 ÷ 12?***

S: ***370.33 … or*** $370.\overset{*}{3}$ ***or 370⅓.***

Mention that all three forms of the result are acceptable, and record them on the board.

$$\begin{array}{r} 370.33 \text{ R} = 0.04 \\ 12\overline{)4\,444.00} \end{array} \qquad 4\,444 \div 12 = 370.\overset{*}{3} = 370\tfrac{1}{3}$$

As you write the following problems on the board, instruct students to copy them. Request that the calculations be carried out to at least two decimal places. (Answers are in boxes; other forms are possible.)

$$\begin{array}{r} \boxed{207.36 \quad \text{R} = 0.04} \\ 8\overline{)1658.92} \end{array} \qquad \begin{array}{r} \boxed{46.17} \\ 3\overline{)138.51} \end{array}$$

$$\begin{array}{r} \boxed{204.22 \quad \text{R} = 0.02} \\ 9\overline{)1838.00} \end{array} \qquad \begin{array}{r} \boxed{150.43 \quad \text{R} = 0.14} \\ 22\overline{)3\,309.60} \end{array}$$

$$\begin{array}{r} \boxed{3\,072.36 \quad \text{R} = 0.24} \\ 41\overline{)125\,967.00} \end{array}$$

If you wish, solve one or two of the above problems collectively. You may want to demonstrate carrying the division process beyond two decimal places.

Writing Activity

Suggest that students write "word problems" similar to the one in Exercise 1 that involve division with non-whole number decimals.

N27 ROUNDING OPERATION #1

Capsule Lesson Summary

Play *Guess My Rule* using a rounding operation as the secret operation. That is, the rule is “a ® b is the multiple of b closest to a.” Solve some problems involving ®. Extend the rounding operation to make possible rounding when b is not an integer.

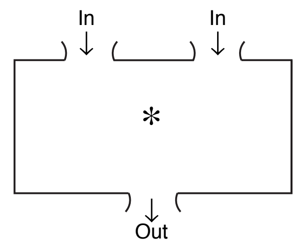
Materials

Teacher		Student	
	• None		• Paper • Calculator

Description of Lesson

Exercise 1: Guess My Rule

This exercise uses an operation rule to introduce a rounding operation. You may like to use a “machine” picture to review how an operation rule works.



T: ***I have a secret rule for ∗. I’ll give you some clues about my rule. Try to figure out the rule for ∗.***

Write several number sentences on the board as clues. Then write an open sentence and see if anyone can predict which number goes in the box.

$26 * 10 = 30$
$12 * 5 = 10$
$5 * 12 = 0$
$3 * 5 = 5$
$5 * 3 = 6$
$18 * 6 = 18$
$33 * 7 = \boxed{}$

Note: The rule is “ a ∗ b is the multiple of b closest to a.” The number in the box is 35 because 35 is the multiple of 7 closest to 33.

Suggest that students write their guesses on paper for you to check. Acknowledge aloud correct guesses and reject incorrect ones. For example, “No, I do not get 30 using my rule on 33 ∗ 7.” Let a student who guesses correctly tell the class that 35 goes in the box, but do not give away the rule. If no one guesses correctly, announce that 35 goes in the box yourself. This rule may be more difficult than usual for students to discover. Using several sentences with the same second number may focus on the idea that the result is a multiple of the second number. Then the question is, which multiple?

Continue the activity with several more examples. (Answers are in boxes.)

$16 * 7 = \boxed{14}$	$16 * 3 = \boxed{15}$
$17 * 7 = \boxed{14}$	$24 * 8 = \boxed{24}$
$18 * 7 = \boxed{21}$	$21 * 1 = \boxed{21}$
$20 * 7 = \boxed{21}$	$24 * 10 = \boxed{20}$

If necessary, give the class a hint about the rule by asking,

T: ***What do you notice about the result compared to second number each time?***

S: ***The result is always a multiple of the second number.***

N27

T: ***Yes, and how does the rule decide which multiple of the second number to take as the result? Who can explain my rule?***

S: ***You always take the multiple of the second number closest to the first number.***

S: ***26 ∗ 10 = 30, because 30 is a multiple of 10 as close as possible to 26.***

Discuss one or two other examples and be sure the class understands the rule for ∗.

T: ***This operation is called a* rounding operation*. Rather than ∗, Let's use ® (read as, "circle R") for the operation to suggest the idea of rounding.***

Write several rounding problems on the board, and instruct the students to solve them on their papers. (Answers are in boxes.)

$$17 \circledR 5 = \boxed{15} \qquad 125 \circledR 100 = \boxed{100}$$
$$24 \circledR 20 = \boxed{20} \qquad 695 \circledR 100 = \boxed{700}$$
$$4 \circledR 12 = \boxed{0} \qquad 1568 \circledR 1000 = \boxed{2\,000}$$

After a few minutes, check the work collectively. Read several of the problems so that students will begin to become familiar with such expressions as "… rounded to the nearest 10 is …" and "…rounded to the nearest 100 is …." For example, 1 568 rounded to the nearest 1000 is 2000.

Write this problem on the board, and let the class discuss it.

$$15 \circledR 6 = \boxed{}$$

S: ***The problem is to round 15 to the nearest 6.***

S: ***Multiples of 6 are 0, 6, 12, 18, and so on; 15 is in the middle between 12 and 18. Both 12 and 18 are nearest multiples of 6 in this case.***

S: ***15 is 3 away from 12 and 3 away from 18.***

T: ***In order to have just one solution to problems such as these, we must agree to always take just one of the two nearest multiples. What do you suggest?***

Let students make suggestions. There are good reasons to make either choice. The usual agreement, and the one you should eventually accept, is to take the greater of the two nearest multiples.

T: ***We say that we round up in cases like this. So 15 ® 6 = 18.***

Exercise 2

Write several rounding problems on the board and instruct students to find solutions whenever possible. (Answers are in boxes.) Discuss the last three problems collectively.

$$8.6 \circledR 4 = \boxed{8} \qquad 6\,213 \circledR 100 = \boxed{6\,200}$$
$$\widehat{3} \circledR 5 = \boxed{\widehat{5}} \qquad 7.5 \circledR 4 = \boxed{8}$$
$$73 \circledR 7 = \boxed{70} \qquad \boxed{} \circledR 6 = 36$$
$$25 \circledR 10 = \boxed{30} \qquad 16 \circledR \boxed{} = 15$$
$$\widehat{10} \circledR 6 = \boxed{\widehat{12}} \qquad 14 \circledR \boxed{} = 9$$

T: ***Which number rounded to the nearest 6 is 36?***

S: ***34, because 34 is closer to 36 than to 30.***

S: ***Also 36, because the nearest multiple of 6 to 36 is 36.***

List solutions on the board as they are suggested. If 33 is not mentioned, ask,

T: ***What is the least number that could go in the box?***

S: ***33; 32 ® 6 = 30 and 33 ® 6 = 36; we agreed to take the greater multiple when the first number is exactly in the middle between two multiples.***

$\square$ ® 6 = 36

33, 34, 35, 36, 37, 38

T: ***Are there other numbers that could go in the box?***

Allow a few minutes for students to think about your question.

S: ***Those are all of the whole numbers that could go in the box.***

S: ***34.5, because 34.5 is closer to 36 than to 30.***

S: ***37.26.***

S: ***33.12486.***

T: ***Yes, there are many other numbers that could go in the box. Is there a greatest number that could go in the box?***

S: ***38.9.***

S: ***38.9999.***

S: ***There is no greatest number.***

If students conjecture that 39 is the greatest number that could go in the box, remind them that 39 ® 6 = 42.

T: ***What about 16 ® □ = 15; which numbers could go in the box?***

S: ***15 must be a multiple of the number in the box.***

S: ***3 or 5 or 15.***

If students suggest that a number like 2.5 could go in the box, discuss why this is reasonable. Counting by 2.5 you get … 10, 12.5, 15, 17.5, 20, … , so 15 is the nearest multiple† of 2.5 to 16. However, do not force such a solution at this time. In Exercise 3 you will discuss such situations.

T: ***Can you find a solution for this last problem, 14 ® □ = 9?***

S: ***I tried 3 and 9, but they don't work.***

Let the class discuss this problem. They should conclude that there is no whole number solution.

†In the past, we have insisted that the "is a multiple of " relation only involves integers. At this point we really have only integer solutions to 16 ® □ = 15.

Exercise 3

Erase the board and write this problem.

$2.6 ® \frac{1}{2} = \boxed{}$

T: ***Think of a calculator and imagine we start at 0 and count by ½s (0.5s). How could we do this?***

S: ***Start with 0 on the display and press [+] [0] [.] [5] [=] [=] [=] and so on. The display would show 0.5, 1, 1.5, 2, 2.5, and so on.†***

T: ***Which of the numbers that would appear on the display is closest to 2.6?***

S: ***2.5, because 2.6 – 2.5 = 0.1 and 3 – 2.6 = 0.4.***

T: ***2.6 rounded to the nearest ½ is 2.5.***

$2.6 ® \frac{1}{2} = \boxed{2.5}$

Write these problems on the board, and ask students to solve them. (Answers are in boxes.)

$4.7 ® \frac{1}{2} = \boxed{4.5}$

$8.67 ® 1 = 9$

$10 ® 1\frac{1}{2} = \boxed{10.5}$

$7.342 ® 0.01 = \boxed{7.34}$

$3.68 ® 0.1 = \boxed{3.7}$

$\frac{1}{3} ® \frac{1}{10} = \boxed{\frac{3}{10} = 0.3}$

Check the problems collectively, using a calculator discussion as necessary.

†For some calculators, you can put 0.5 on the display and press [+] [=] [=] [=] ….

Capsule Lesson Summary

Practice adding, subtracting, and multiplying fractions. Locate rational numbers given with non-standard names (such as $\frac{2}{3} \times \frac{3}{2}$, $\frac{3}{8} + \frac{1}{4}$, and $2 - \frac{5}{4}$) in or out of a string for **More than ½**. Play a game where the object is to label the dots in a two-string picture using a set of non-standard names for the numbers.

Materials

Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N28*, **, and ***

Description of Lesson

Exercise 1

Review multiplication and addition of fractions by doing a couple problems collectively. Results for two suggested problems are given below.

$$\frac{4}{5} \times \frac{2}{3} = \boxed{\frac{8}{15}} \qquad \frac{4}{5} + \frac{2}{3} = \boxed{\frac{12}{15} + \frac{10}{15} = \frac{22}{15} = 1\frac{7}{15}}$$

Continue with this set of subtraction problems. (Answers are in boxes.)

$$\frac{4}{5} - \frac{1}{5} = \boxed{\frac{3}{5}} \qquad 1\frac{4}{5} - \frac{1}{5} = \boxed{1\frac{3}{5}}$$

$$1\frac{4}{5} - 1\frac{2}{5} = \boxed{\frac{2}{5}} \qquad 1 - \frac{2}{5} = \boxed{\frac{3}{5}}$$

$$1\frac{3}{5} - 1 = \boxed{\frac{3}{5}} \qquad 1\frac{1}{5} - \frac{2}{5} = \boxed{\frac{4}{5}}$$

$$1\frac{1}{5} - \frac{1}{5} = \boxed{1} \qquad 1\frac{1}{5} - \frac{4}{5} = \boxed{\frac{2}{5}}$$

T: ***What number is $\frac{4}{5} - \frac{2}{3}$? Why is this a harder problem?***

$$\frac{4}{5} - \frac{2}{3} =$$

If students suggest that $\frac{4}{5} - \frac{2}{3} = \frac{2}{2} = 1$, observe that $\frac{4}{5}$ is less than 1 so certainly $\frac{4}{5} - \frac{2}{3}$ is less than 1. Point out that subtracting numerators and denominators is not a good method for subtracting fractions.

S: ***The fractions in this subtraction problem have different denominators.***

T: ***Addition problems involving fractions with different denominators are also harder. What might we do then?***

S: ***Find equivalent fractions with the same denominator and rewrite the problem.***

T: ***There are many pairs of equivalent fractions for $\frac{4}{5}$ and $\frac{2}{3}$ with the same denominator. Which pair did we use to add $\frac{4}{5} + \frac{2}{3}$?***

S: ***$\frac{12}{15}$ and $\frac{10}{15}$.***

Restate the problem in terms of these equivalent fractions.

$$\frac{4}{5} - \frac{2}{3} = \frac{12}{15} - \frac{10}{15}$$

T: ***What number is $^{12}/_{15} - {}^{10}/_{15}$?***

S: ***$^{2}/_{15}$.***

Record the result and continue with some related problems. (Answers are in boxes.)

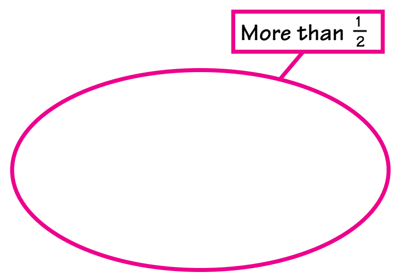
$$\frac{4}{5} - \frac{2}{3} = \boxed{\frac{12}{15} - \frac{10}{15} = \frac{2}{15}} \qquad \frac{2}{3} - \frac{4}{5} = \boxed{\frac{10}{15} - \frac{12}{15} = \frac{\hat{2}}{15}}$$

$$1\frac{4}{5} - \frac{2}{3} = \boxed{1\frac{2}{15}} \qquad 1\frac{2}{3} - \frac{4}{5} = \boxed{1 + \frac{\hat{2}}{15} = \frac{15}{15} + \frac{\hat{2}}{15} = \frac{13}{15}}$$

$$5\frac{4}{5} - 2\frac{2}{3} = \boxed{3\frac{2}{15}} \qquad 4\frac{2}{3} - 1\frac{4}{5} = \boxed{3 + \frac{\hat{2}}{15} = 2\frac{13}{15}}$$

Exercise 2

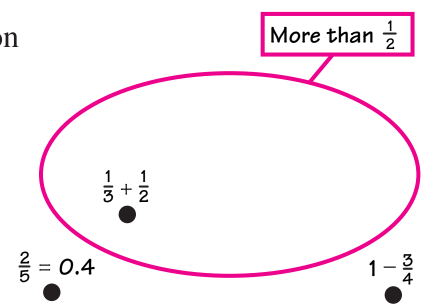
Draw this string picture on the board, and instruct students to copy it on their papers.



T: ***I'll write some numbers on the board. Place these numbers correctly in your picture.***

$$\frac{2}{5} \qquad \frac{1}{3} + \frac{1}{2} \qquad 0.4 \qquad 1 - \frac{3}{4}$$

After a few minutes of individual work, lead a collective discussion to locate the numbers in the picture.



S: ***$^{2}/_{5} = {}^{4}/_{10}$, and $^{1}/_{2} = {}^{5}/_{10}$; so $^{2}/_{5}$ is less than $^{1}/_{2}$.***

S: ***$^{1}/_{3} + {}^{1}/_{2}$ is more than $^{1}/_{2}$, because $^{1}/_{3}$ is added to $^{1}/_{2}$.***

S: ***$^{1}/_{3} + {}^{1}/_{2} = {}^{5}/_{6}$, and $^{5}/_{6}$ is more than $^{1}/_{2}$.***

S: ***$0.4 = {}^{4}/_{10} = {}^{2}/_{5}$, and $^{2}/_{5}$ is already in the picture.***

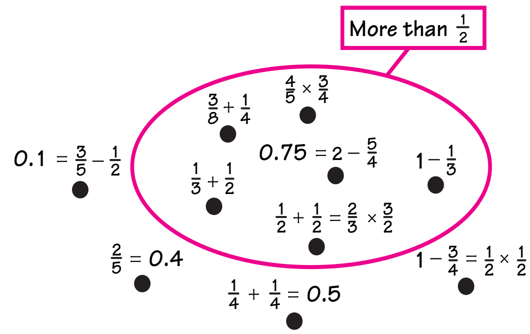
S: ***$1 - {}^{3}/_{4} = {}^{1}/_{4}$, and $^{1}/_{4}$ is less than $^{1}/_{2}$.***

Continue this activity by asking students to place these numbers, or others more appropriate to the numerical abilities of your class, in the string picture.

0.75	$\frac{1}{2} + \frac{1}{2}$	$1 - \frac{1}{3}$	$\frac{1}{2} \times \frac{1}{2}$
0.1	$\frac{1}{4} + \frac{1}{4}$	$2 - \frac{5}{4}$	$\frac{2}{3} \times \frac{3}{2}$
0.5	$\frac{3}{8} + \frac{1}{4}$	$\frac{3}{5} - \frac{1}{2}$	$\frac{4}{5} \times \frac{3}{4}$

It may be necessary to remind students that a dot may be labeled in more than one way.

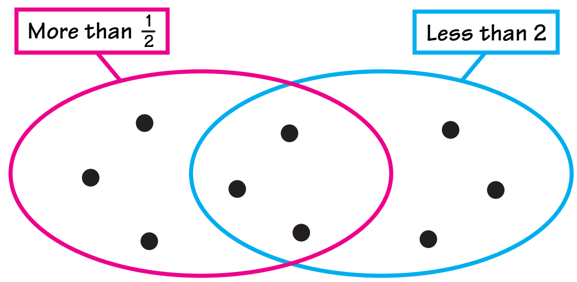
Check the work collectively by calling on volunteers to locate the numbers in the picture at the board. Be prepared to discuss the location of a number with the class. This activity offers an opportunity for students to become better acquainted with rational numbers. A completed string picture is shown here.



Exercise 3

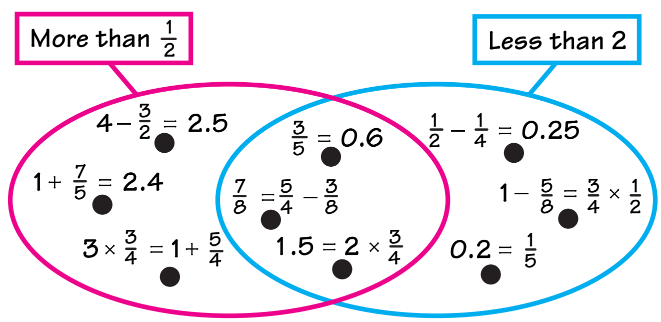
Draw this picture on the board, and divide the class into teams.

Team A	Team B	Team C
$\frac{1}{2}-\frac{1}{4}$	$1+\frac{5}{4}$	0.25
0.6	$\frac{3}{5}$	$3\times\frac{3}{4}$
$\frac{7}{8}$	1.5	2.5
$1-\frac{5}{8}$	$\frac{3}{4}\times\frac{1}{2}$	$\frac{1}{5}$
2.4	0.2	$\frac{5}{4}-\frac{3}{8}$
$2\times\frac{3}{4}$	$4-\frac{3}{2}$	$1+\frac{7}{5}$

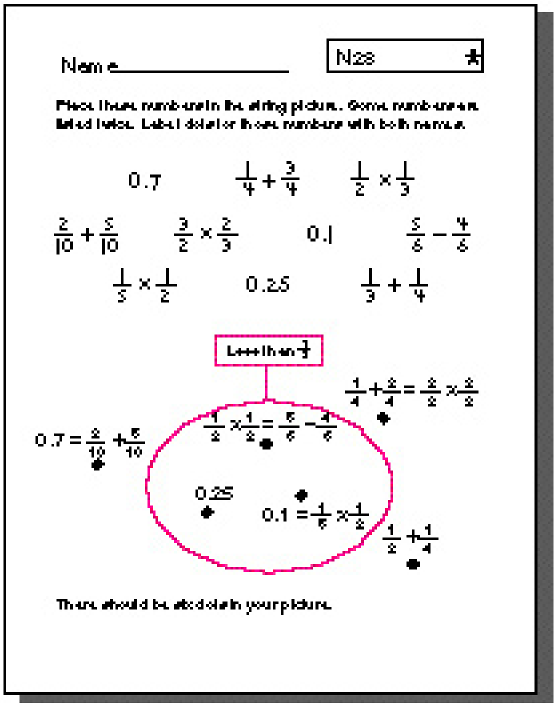


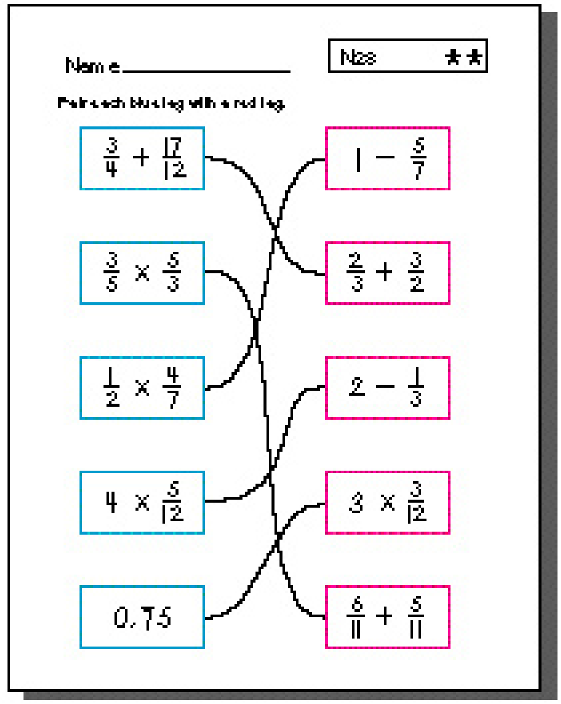
T: ***We are going to play a game. The teams will take turns trying to label correctly the dots in the string picture. You may only use the number labels in your section of the Team Board. The team that correctly locates all of its numbers first is the winner.***

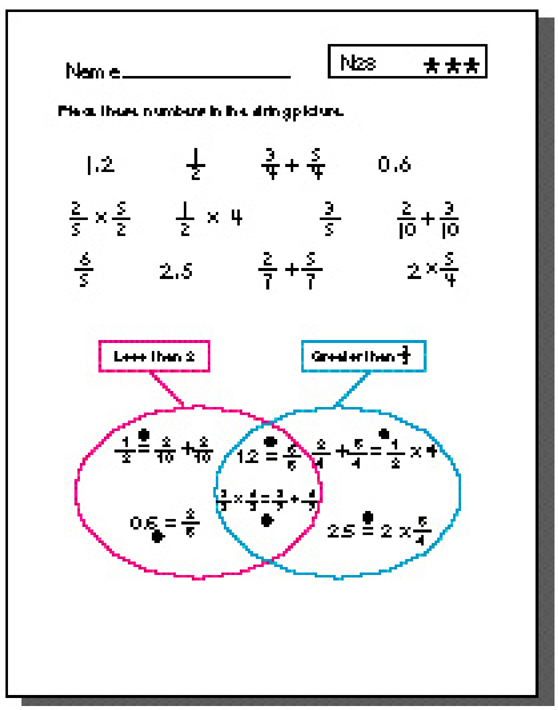
Teams alternate and members of a team take turns playing. When a number is correctly placed in the string picture, erase it from the Team Board. An incorrectly placed number remains on the Team Board and is erased from the string picture. When the game is over, you may wish to invite students to place the remaining numbers in the picture. A completed picture is shown below.



Worksheets N28*, **, and *** are available for individual work.


Name
N28
0.7
$\frac{1}{4}+\frac{3}{4}$
$\frac{1}{2}\times\frac{1}{3}$
$\frac{2}{10}+\frac{5}{10}$
$\frac{3}{2}\times\frac{2}{3}$
0.1
$\frac{5}{6}-\frac{4}{6}$
$\frac{1}{5}\times\frac{1}{2}$
0.25
$\frac{1}{3}+\frac{1}{4}$


Name
N28
Pair each blue leg with a red leg.
$\frac{3}{4}+\frac{17}{12}$
$\frac{3}{5}\times\frac{5}{3}$
$\frac{1}{2}\times\frac{4}{7}$
$4\times\frac{5}{12}$
0.75
$1-\frac{5}{7}$
$\frac{2}{3}+\frac{3}{2}$
$2-\frac{1}{3}$
$3\times\frac{3}{12}$
$\frac{6}{11}+\frac{5}{11}$


Name
N28
1.2
$\frac{1}{2}$
$\frac{3}{4}+\frac{5}{4}$
0.6
$\frac{2}{5}\times\frac{5}{2}$
$\frac{1}{2}\times 4$
$\frac{3}{5}$
$\frac{2}{10}+\frac{3}{10}$
$\frac{6}{5}$
2.5
$\frac{2}{7}+\frac{5}{7}$
$2\times\frac{5}{4}$

N29 ROUNDING OPERATION #2

Capsule Lesson Summary

Review the rounding operation. Label dots in an arrow picture for the ®4 function . Graph the ®4 function.

Materials

Teacher
- Colored chalk
- Coordinate grid
- Blackline N29

Student
- Paper
- Coordinate grid sheets
- Colored pencils, pens, or crayons
- Worksheets N29(a) and (b)

Advance Preparation: Use Blackline N29 to prepare a coordinate grid for display, or prepare a grid board with this coordinate grid. Make copies of coordinate grid sheets for students' use in Exercise 3.

Description of Lesson

Exercise 1

Write this open sentence on the board.

17 ® 4 = $\boxed{}$

T: ***Do you remember the rounding operation?***

S: ***For the problem on the board, we must find the multiple of 4 closest to 17.***

S: ***17 ® 4 = 16.***

Continue with a second problem.

17 ® 4 = $\boxed{16}$
27 ® 6 = $\boxed{}$

S: ***27 is exactly in the middle between 24 and 30; it is 3 away from 24 and 3 away from 30.***

S: ***We agreed to take the greater multiple in such a case, so 27 ® 6 = 30.***

Instruct students to copy and solve these problems. After a few minutes, check solutions collectively. (Answers are in boxes.)

36 ® 10 = $\boxed{40}$
3 ® 10 = $\boxed{0}$
129 ® 100 = $\boxed{100}$
350 ® 100 = $\boxed{400}$
1186 ® 100 = $\boxed{1200}$

2 657 ® 1000 = $\boxed{3\,000}$
10 986 ® 1000 = $\boxed{11\,000}$
8.36 ® 0.1 = $\boxed{8.4}$
14.52 ® 0.1 = $\boxed{14.5}$
3.093 ® 0.01 = $\boxed{3.09}$

Note: You may wish to go directly to Exercise 2 at this time in order to leave plenty of time for the graphing activity in Exercise 3.

T: ***Suppose we have a rounding problem with a sum of two numbers. Do you think it will matter whether we first add the numbers and then do the rounding, or first round each number and then add the results? Let me give you an example of what I mean.***

N29

Write these expressions on the board.

(5 + 3) ® 4 (5 ® 4) + (3 ® 4)

Refer to the expression on the left first as you explain,

T: ***The parentheses in this problem tell us to first add 5 and 3, and then to round the sum to the nearest multiple of 4. In the other problem, the parentheses tell us first to round each number, 5 and 3, to the nearest multiple of 4, and then to add the results. Do both problems and see what happens.***

Allow a minute or two for independent work. Then call on volunteers to solve the problems at the board.

(5 + 3) ® 4 = 8 (5 ® 4) + (3 ® 4) = 8

4 + 4

S: ***They are the same.***

T: ***Do you think it always happens this way?***

Encourage students to comment about this situation. Then ask the class to carry this experiment further by working a few more pairs of problems. (Answers are in boxes.)

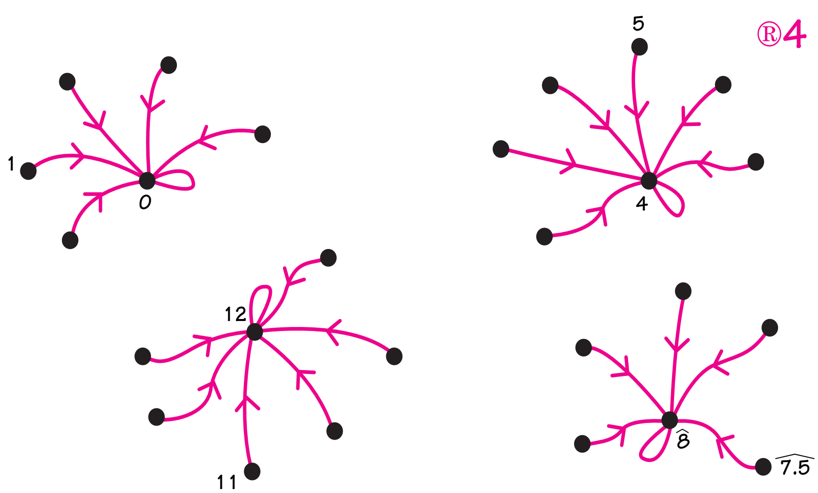
(15 + 3) ® 7 = $\boxed{21}$ (15 ® 7) + (3 ® 7) = $\boxed{14}$
(16 + 9) ® 5 = $\boxed{25}$ (16 ® 5) + (9 ® 5) = $\boxed{25}$
(16 + 9) ® 6 = $\boxed{24}$ (16 ® 6) + (9 ® 6) = $\boxed{30}$

S: ***Sometimes it does make a difference.***

Encourage students to make observations about when there is a difference.

Exercise 2

Draw the arrow picture from Worksheet N29(a) on the board.

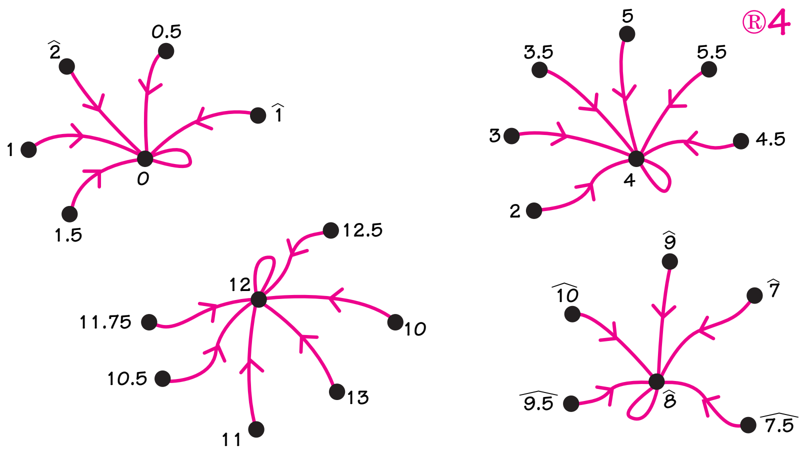


T: ***This is part of an arrow picture for the ®4 relation. It is like the one on your worksheet. Can you label some other dots in this picture?***

Invite students to label dots of their choice at the board. When several dots have been labeled, trace an arrow for which both the starting and ending dots are labeled. For example:

T (tracing the arrow from 3 to 4)**:** ***This dot could be for 3 because 3 ® 4 = 4. On your worksheet label all of the dots.***

Periodically ask various students to label dots at the board. Upon completion, the arrow picture might look similar to the one below.



T (pointing to the upper left piece of the arrow picture)**:** ***Could I draw more red arrows in this piece of the picture?***

S: ***Yes, there could be many more arrows in each piece.***

T: ***Think about the ending number for all of the arrows in one piece. What can we say about the center number of each piece?***

S: ***There will be a separate piece for each multiple of 4, and the ending numbers for all of the arrows will be multiples of 4.***

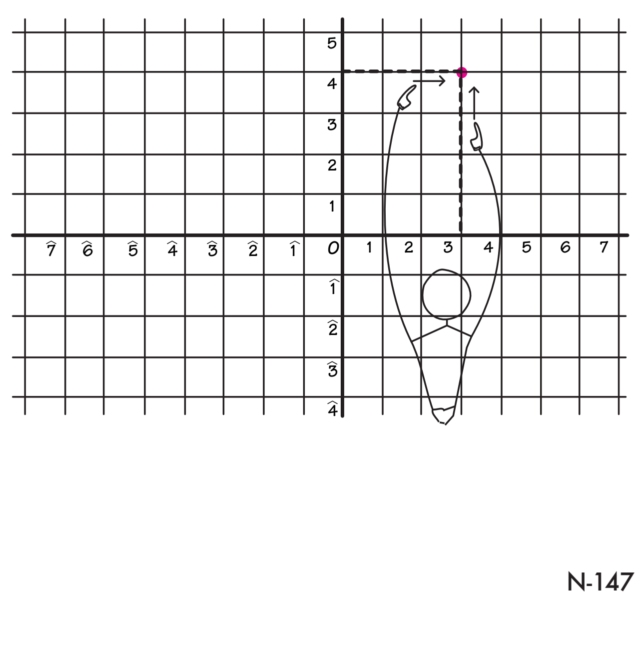
Instruct students to label dots on Worksheet N29(b), observing that this worksheet has more pieces of the ®4 arrow picture.

Exercise 3

Display a coordinate grid and distribute coordinate grid sheets to students.

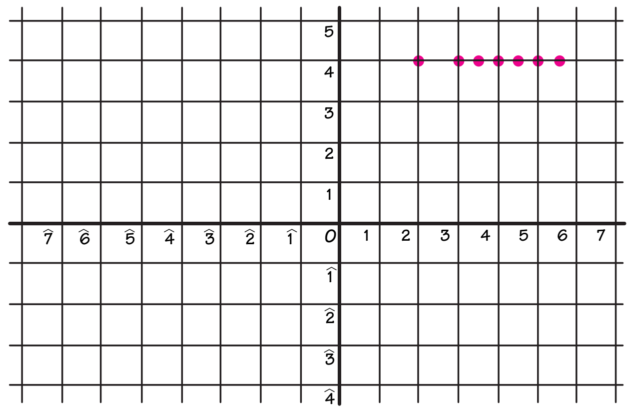
T: ***Let's draw the Cartesian graph for the ®4 relation. How do we locate the point for the arrow that starts at 3 and ends at 4*** (trace this arrow)***?***

Invite a student to locate the point.



N29

Call on students to locate the other points for arrows in the piece of the picture centered at 4; all arrows ending at 4.



T: ***We said before that there are many arrows, in fact infinitely many, that could be drawn in each piece. How can we reflect this fact in our graph of ®4?***

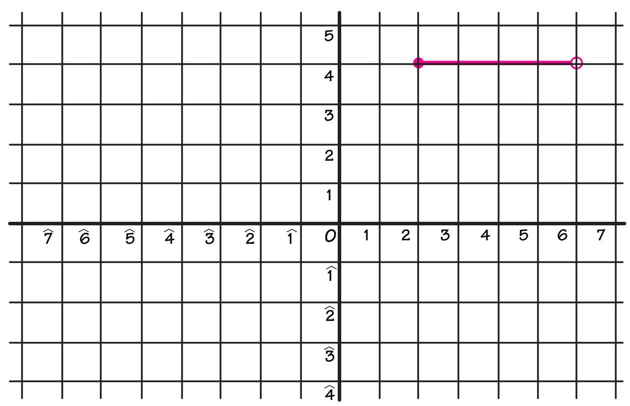
S: ***We can draw a line segment*** (fill in the gaps) ***from 2 to 6.***

T: ***What about at 6?***

S: ***We can put a dot at 5.9, 5.99, 5.999, and so on, but not at 6 because 6 ® 4 = 8.***

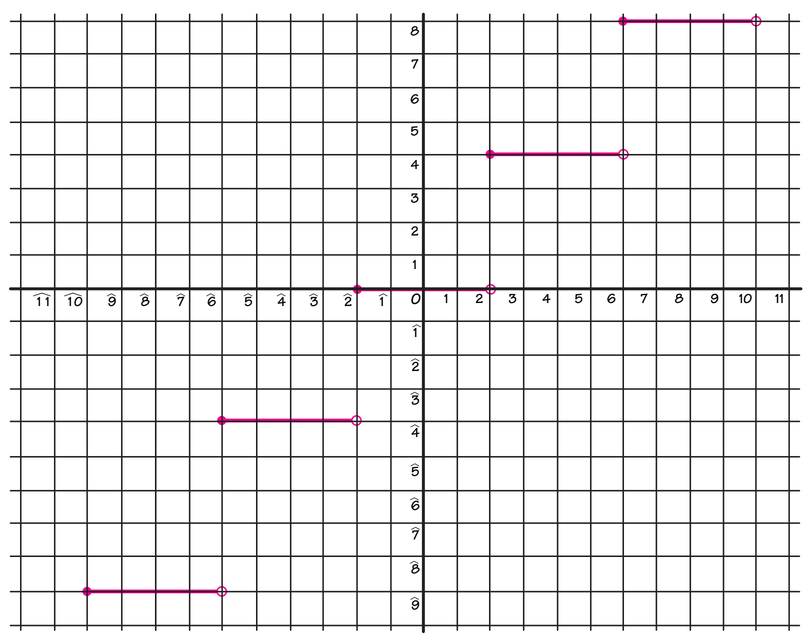
T: ***Let's put an open circle at 6 to show that it is left out.***

Complete this part of the graph.



Instruct students to locate points for the remaining arrows (as many as possible) on the grid on the board or on their worksheets. Encourage students to fill in the graph for the pieces of the picture on the board and to think about how the picture would look if the grid were extended.

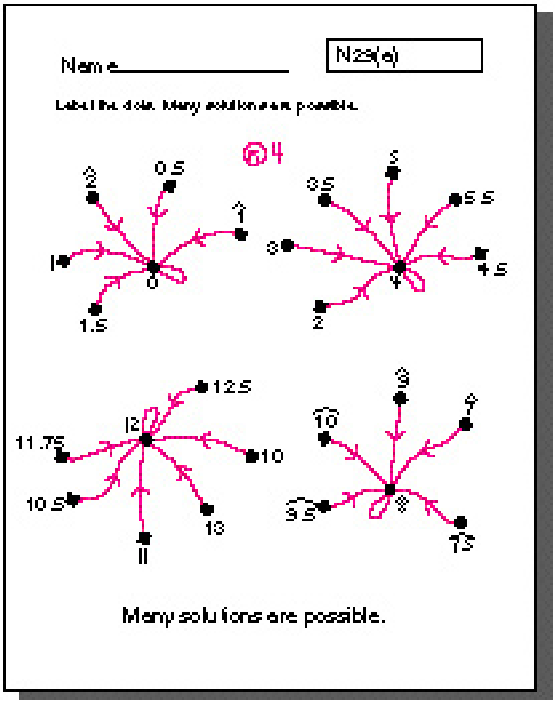
When many students are finished, complete the graph at the board.

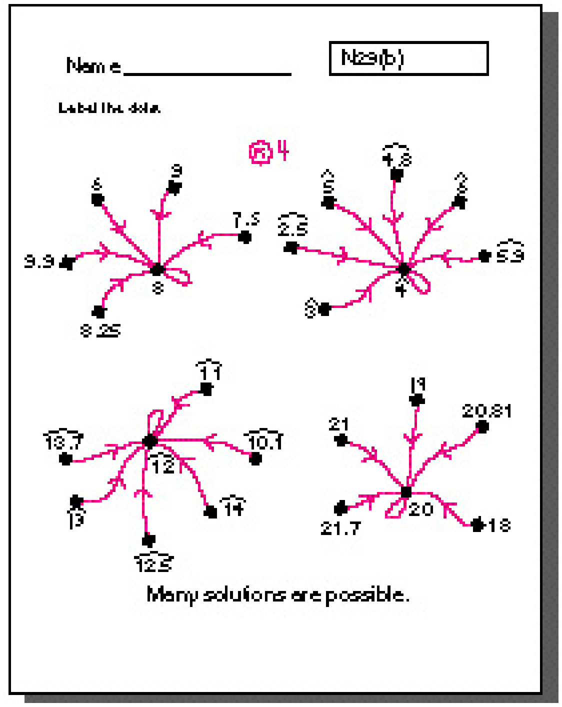


T: ***What do you notice about our graph?***

S: ***It looks like stair steps.***

Let students comment on the graph and discuss how it would go on if the grid were bigger. You may also like to discuss what a Cartesian graph for ®5 or ®10 or ®½ would look like.


Name
N29(a)
Label the dots. Many solutions are possible.
Many solutions are possible.


Name
N29(b)
Label the dots.
Many solutions are possible.

Capsule Lesson Summary

Allowing use of only the number keys [5], [6], [8], and [9], the operation keys, the memory keys, and [=], put numbers on the display of a calculator. Then, using keys other than [5], [6], [8], and [9], find keystrokes that will multiply a number on the display by 9, by 8, by 6, by 5, by 60, by 36, by 15, and by 45. Play *Calculator Golf*.

Materials

Teacher
- Calculator
- Colored chalk

Student
- Calculator
- Paper
- Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

List these calculator keys on the board, and refer to them as you give the following directions.

[5] [6] [8] [9]
[+] [–] [×] [÷]
[M+] [M–] [MRC] [=]

T: ***Today we are going to solve some calculator puzzles using only certain keys on the calculator. The only number keys you may press are [5], [6], [8], and [9]. You may use any of the four operation keys and any of the memory keys, and of course [=]. Suppose it costs a penny (or a dollar) each time you press a key. Try to put 61 on your display for less than 10¢.***

Several solutions and their prices are given here.

61:	[6] [5] [9] [–] [5] [9] [8] [=]	8¢
	[6] [9] [–] [8] [=]	5¢
	[8] [×] [8] [+] [6] [–] [9] [=]	8¢
	[5] [6] [+] [5] [=]	5¢
	[9] [×] [9] [–] [5] [=] [=] [=] [=]	9¢

Continue this activity, putting on other numbers such as 42, 343, 0.5, 0.25, and 7.5. Write all the numbers on the board with sufficient space between them to record several solutions for each. Provide time for individual work, allowing students to work on the numbers in any order they choose. After a while, record several solutions for each number. The following lists give some possible solutions for each suggested number.

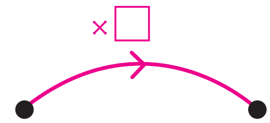
42:	[6] [×] [8] [–] [6] [=]	6¢
	[5] [6] [–] [9] [–] [5] [=]	7¢
	[9] [8] [–] [5] [6] [=]	6¢
	[5] [×] [8] [+] [8] [–] [6] [=]	8¢
343:	[6] [6] [×] [5] [+] [8] [+] [5] [=]	9¢
	[9] [9] [9] [–] [6] [5] [6] [=]	8¢
	[+] [9] [6] [=] [=] [=] [+] [5] [5] [=]	10¢
	[5] [×] [8] [×] [9] [–] [9] [–] [8] [=]	10¢
	[8] [–] [6] [+] [5] [×] [=] [=]	8¢

0.5:	[9] [–] [5] [÷] [8] [=]	6¢
	[6] [+] [6] [M+] [6] [÷] [MR] [=]	8¢
	[6] [×] [8] [÷] [9] [6] [=]	7¢
0.25:	[8] [–] [6] [÷] [8] [=]	6¢
	[9] [+] [9] [÷] [9] [÷] [8] [=]	8¢
	[9] [÷] [6] [=] [=]	5¢
7.5:	[9] [×] [5] [÷] [6] [=]	6¢
	[5] [5] [+] [5] [÷] [8] [=]	7¢
	[9] [÷] [6] [+] [6] [=]	6¢

Note: The lists of solutions here assume the calculator does chain operations and has an automatic constant feature (see “Role and Use of Calculators” in Section One: Notes to the Teacher). As necessary, make adjustments for the calculators in use by your students. For example, in several solutions above you may need to insert another $\boxed{=}$ if your calculator does operations in a priority order (x, ÷. +, –) rather than a chain order of entry.

Exercise 2

Draw an unlabeled arrow on the board.



T: ***This arrow is for times some number and we can use our calculators to multiply by this number. If*** $\boxed{5}$, $\boxed{6}$, $\boxed{8}$, ***and*** $\boxed{9}$ ***are the only number keys we can press, what could this arrow be for?***

S: x*5.*

T: ***What keys would we press to multiply by 5?***

S: ***Press*** $\boxed{\times}\boxed{5}\boxed{=}$.

Record this response on the board and continue until many possibilities are listed. There are infinitely many correct answers, for example:

×5: $\boxed{\times}\ \boxed{5}\ \boxed{=}$

×6: $\boxed{\times}\ \boxed{6}\ \boxed{=}$

×8: $\boxed{\times}\ \boxed{8}\ \boxed{=}$

×9: $\boxed{\times}\ \boxed{9}\ \boxed{=}$

×56: $\boxed{\times}\ \boxed{5}\ \boxed{6}\ \boxed{=}$

×30: $\boxed{\times}\ \boxed{6}\ \boxed{\times}\ \boxed{5}\ \boxed{=}$

×275: $\boxed{\times}\ \boxed{5}\ \boxed{5}\ \boxed{\times}\ \boxed{5}\ \boxed{=}$

×7: $\boxed{\times}\ \boxed{5}\ \boxed{6}\ \boxed{\div}\ \boxed{8}\ \boxed{=}$

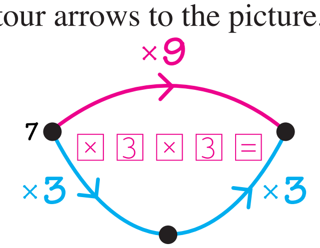
Note: Students may suggest that you can multiply by any whole number by pressing $\boxed{+}\ \boxed{=}\ \boxed{=}$ … $\boxed{=}$ where $\boxed{=}$ is pressed the desired whole number of times. Accept this response, but suggest it would be rather inefficient. Also, be careful of suggestions such as x10: $\boxed{\times}\ \boxed{5}\ \boxed{+}\ \boxed{5}$.

In this case, the calculator might not multiply the number on the display by 10 because it would first multiply by 5 and then add 5; for example, starting with 8 on the display, the result would be 45 not 80.

Label the arrow for x9.

T: ***Now, instead of using the number keys*** $\boxed{5}$, $\boxed{6}$, $\boxed{8}$, ***and*** $\boxed{9}$, ***suppose we can use any number key except these. Can we still find a way to multiply by 9?***

If your students find this question difficult, give a hint by adding some detour arrows to the picture.



T: ***What could these detour arrows be for?***

S: x*3 followed by* x*3.*

T: ***What keys would we press?***

S: ***Press*** $\boxed{\times}\ \boxed{3}\ \boxed{\times}\ \boxed{3}\ \boxed{=}$.

T: ***Let’s check if that works. Put 7 on your display. Press*** $\boxed{\times}\ \boxed{3}\ \boxed{\times}\ \boxed{3}\ \boxed{=}$. (Point to the ending dot of the x9 arrow.) ***Do we get 7*** x ***9?***

S: ***Yes, 63.***

Perform a couple more checks by starting with different numbers on the display (starting dot of the x9 arrow), pressing [×][3][×][3][=], and observing that the result is what should be at the ending dot.

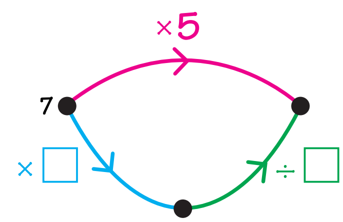
Repeat this activity by asking students to find keystrokes (without using [5], [6], [8], and [9]) for x8 and x6. Record solutions on the board and check them using the arrow picture. Some solutions are listed below; many other solutions are possible.

×9: [×] [3] [×] [3] [=]

×8: [×] [4] [×] [2] [=]
[×] [2] [×] [2] [×] [2] [=]

×6: [×] [3] [×] [2] [=]

T: ***Can we multiply by 5 without using [5], [6], [8], and [9]?***



If necessary, give a composition hint.

S: ***Press*** [×] [1][0] [÷] [2] [=].

S: ***Press*** [×] [2][0] [÷] [4] [=].

S: ***Press*** [×] [3][0] [÷] [2] [÷] [3] [=].

Check these solutions using the arrow picture. Then invite the class to find solutions for x60, x36, x15, and x45. Allow several minutes for individual or partner work. Then collectively record solutions at the board. Encourage students to check their solutions, starting with many different numbers on their displays.

The following lists give several solutions for each kind of arrow.

×60:
[×] [3] [0] [×] [2] [=]
[×] [2] [0] [×] [3] [=]
[×] [1] [2] [0] [÷] [2] [=]

×36:
[×] [3] [×] [3] [×] [4] [=]
[×] [7] [2] [÷] [2] [=]
[×] [1] [2] [×] [3] [=]

×15:
[×] [3] [0] [÷] [2] [=]
[×] [1] [0] [÷] [2] [×] [3] [=]
[×] [2] [0] [÷] [4] [×] [3] [=]

×45:
[×] [3] [0] [÷] [2] [×] [3] [=]
[×] [3] [×] [3] [×] [1] [0] [÷] [2] [=]
[×] [2] [0] [÷] [4] [×] [3] [×] [3] [=]

Exercise 3: Calculator Golf

Ask the class to recall some of what they remember about golf and the *Minicomputer Golf* game.

T: ***Today we are going to play* Calculator Golf. *We start with a number on the calculator display and set a goal.***

Put this information on the board.

Starting Number: 83
Goal: 500
Keys: [+] [−] [×] [÷]
[5] [6] [=]

T: ***Suppose we start with 83*** (put 83 on the calculator display) ***and make 500 the goal. When you play this golf game, you can press an operation key ([+], [−], [×], or [÷]) followed by number keys (only [5] or [6]) and then [=]. Play continues until 500 is on the display.***

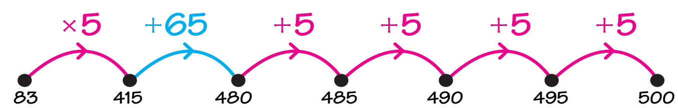
Put 83 on your display. Try to get 500. You can add, subtract, multiply, or divide by any number using only the digits 5 and 6.

Allow a few minutes for students to work independently or with partners on this problem. Then ask students to share their solutions.

Note: You may ask students to keep track of their steps, using an arrow picture where each step has an operation, a number, and then $\boxed{=}$. Then, recall that in golf one tries to reach the goal with as few strokes as possible. Students should try to reach the goal with few steps.

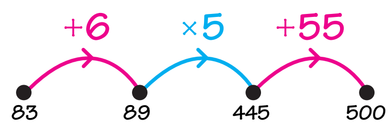
As a student describes a solution, draw a corresponding arrow road on the board.

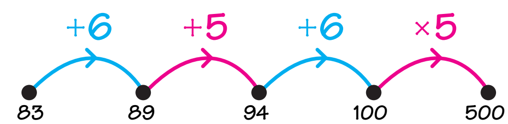
S: ***I pressed*** $\boxed{\times}$ $\boxed{5}$ $\boxed{=}$***,***
then $\boxed{+}$ $\boxed{6}\boxed{5}$ $\boxed{=}$***,***
then $\boxed{+}$ $\boxed{5}$ $\boxed{=}$ ***four times.***

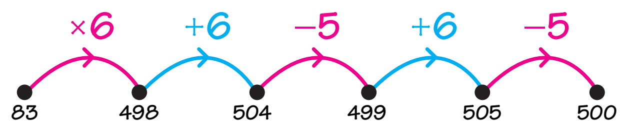


T: ***This solution has six steps.***
Did anyone find a solution with fewer steps? Is it possible?

Continue this activity until several solutions are on the board. For your information, several shorter solutions are shown here.







Variations: Play the game with other number keys allowed during play, with different starting numbers and different goals, or in teams. In a team game, the object is not necessarily to use few steps. Rather, alternating teams, the winner is the first team to put the goal on the display under the same play conditions. You or your students may find other creative ways to vary the game.

Home Activity

Send home a description of *Calculator Golf* and suggest that students play the game with family members.

Capsule Lesson Summary

Review multiplication of rational numbers, a fraction times a whole number and a fraction times a fraction. Illustrate the equivalence of $\frac{2}{3}$x and ÷$\frac{3}{2}$ with an arrow picture. Calculate
30 ÷ $\frac{3}{2}$ and $\frac{4}{5}$ ÷ $\frac{5}{6}$ by rewriting them in an equivalent form 30 x $\frac{2}{3}$ and $\frac{4}{5}$ x $\frac{6}{5}$, respectively. Perform calculations involving addition, multiplication, and division of rational numbers in order to solve a probability problem concerning the random formation of numerical expressions from given numbers and operations.

Materials

Teacher
- Colored chalk
- 3 boxes (optional)
- 7 papers (optional)

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheet N35

Advance Preparation: For Exercise 3 you may like to prepare three boxes in the following way. On each piece of paper write one of these symbols: $\frac{2}{3}$, $\frac{3}{2}$, x, ÷, +, $\frac{1}{2}$, $\frac{3}{4}$. Then fold the papers and put the first two numbers in Box I, the three operations in Box II, and the last two numbers in Box III.

Description of Lesson

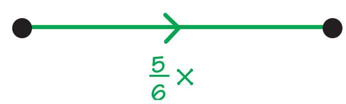
Exercise 1

Put these problems on the board, and instruct students to do the calculations on a piece of paper. (Answers are in boxes.) Collectively check and discuss the solutions.

$$\frac{1}{7} \times 28 = \boxed{4} \qquad \frac{3}{4} \times 24 = \boxed{18}$$

$$\frac{5}{7} \times 28 = \boxed{20} \qquad \frac{4}{9} \times 36 = \boxed{16}$$

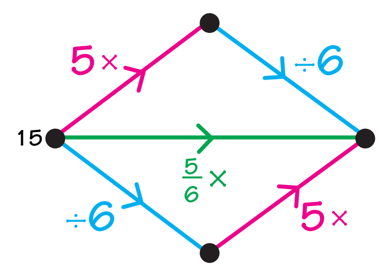
Draw this arrow on the board.



T: ***How can we calculate $\frac{5}{6}$x a number?***

S: ***Use a composition (detour): 5x followed by ÷6, or ÷6 followed by 5x.***

Draw composite arrows in the picture and label the dot at the left.



T: ***Let's use this picture to calculate $\frac{5}{6}$ x 15. Can you label any of the other dots?***

S: ***The top dot is for 75 (5 x 15 = 75).***

S: ***75 ÷ 6 is 12 with a remainder of 3.***

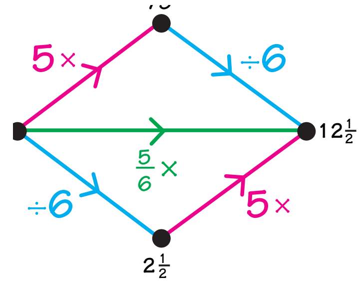
S: ***75 ÷ 6 = $\frac{75}{6}$ = 12$\frac{3}{6}$ = 12$\frac{1}{2}$.***

If necessary, point out that 12 with a remainder of 3 can be interpreted in a division situation but it does not name a number. The label for a dot should name a number.

S: ***The bottom dot is 15 ÷ 6 = $^{15}\!/_6$ = $2\frac{1}{2}$.***

S: ***Then 5 x $^{15}\!/_6$ = $^{75}\!/_6$ = $12\frac{3}{6}$ = $12\frac{1}{2}$.***

S: ***5 x $2\frac{1}{2}$ = $12\frac{1}{2}$.***



Label the arrow picture appropriately, and conclude that $^5\!/_6$ x 15 = $12\frac{1}{2}$.

Put these problems on the board, and instruct students to do the calculations on a piece of paper. (Answers are in boxes.) Collectively check students' work, and discuss various forms their solutions might have.

$$\frac{3}{2} \times \frac{7}{5} = \boxed{\frac{21}{10} = 2\frac{1}{10}} \qquad \frac{5}{6} \times \frac{11}{4} = \boxed{\frac{55}{24} = 2\frac{7}{24}}$$

$$\frac{4}{5} \times \frac{3}{4} = \boxed{\frac{12}{20} = \frac{3}{5}} \qquad \frac{3}{5} \times \frac{10}{3} = \boxed{\frac{30}{15} = 2}$$

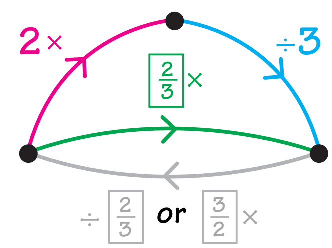
Exercise 2

Draw this arrow picture on the board. Let students explain how to fill in the boxes for the green and the gray arrows.

S: ***The green arrow is for $^2\!/_3$x; 2x followed by ÷3 is the same as $^2\!/_3$x.***

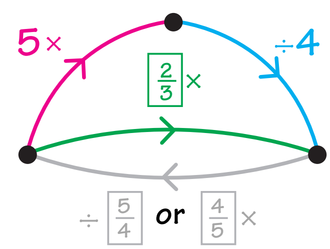
S: ***The return arrow for $^2\!/_3$x is ÷$^2\!/_3$.***

S: ***The return arrow for ÷3 is 3x and the return arrow for 2x is ÷2. 3x followed by ÷2 is the same as $^3\!/_2$x.***



Point out the equivalence of ÷$^2\!/_3$ and $^3\!/_2$x for the gray arrow.

Relabel the arrow picture, as in this illustration, and repeat the above activity. As you do this, record the results in a table near the arrow picture.



Erase the labels from the arrow picture and enter $^2\!/_7$x in the column for the green arrow.

T: ***If the green arrow were for $^2\!/_7$x, what would the return arrow be for?***

S: ***÷$^2\!/_7$ or $^7\!/_2$x.***

Record these answers in the table. Encourage students to comment on patterns they observe. Similarly, fill in two or three more lines of the table.

Erase the first column for the green arrow.

$\square\times$	$\div\square$	$\square\times$
$\frac{2}{3}\times$	$\div\frac{2}{3}$	$\frac{3}{2}\times$
$\frac{5}{4}\times$	$\div\frac{5}{4}$	$\frac{4}{5}\times$
$\frac{2}{7}\times$	$\div\frac{2}{7}$	$\frac{7}{2}\times$
$\frac{3}{2}\times$	$\div\frac{3}{2}$	$\frac{2}{3}\times$
$\frac{11}{5}\times$	$\div\frac{11}{5}$	$\frac{5}{11}\times$

T: *We have found many pairs of equivalent names for the gray arrow. What do you notice about these pairs?*

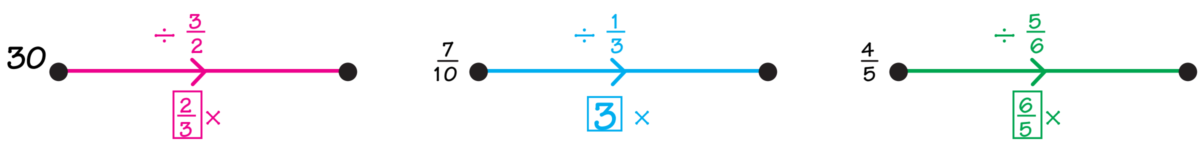
S: *In each pair, the numerators and the denominators are switched, for example, ÷⁵⁄₄ and ⁴⁄₅x.*

S: *In each pair, one is division and the other is multiplication.*

T: *What is another name for ÷²⁄₉?* (⁹⁄₂x) *For ³⁄₇x?* (÷⁷⁄₃) *We already know how to multiply fractions. Now we can change a division problem involving fractions into a multiplication problem.*

Put the following problems and arrow pictures on the board. Ask students to explain how to do the calculations and to fill in the boxes for the arrows. (Answers are in boxes.)

$30 \div \frac{3}{2} = \boxed{20}$ $\qquad \frac{7}{10} \div \frac{1}{3} = \boxed{\frac{21}{10} = 2\frac{1}{10}}$ $\qquad \frac{4}{5} \div \frac{5}{6} = \boxed{\frac{24}{25}}$

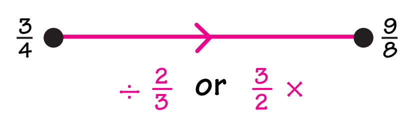


Ask students to copy and complete the following calculations. As necessary, remind students to change a division problem to a multiplication problem. After a while check the problems collectively. (Answers are in boxes.)

$12 \div \frac{2}{3} = \boxed{12 \times \frac{3}{2} = 18}$ $\qquad \frac{3}{4} \div \frac{2}{3} = \boxed{\frac{3}{4} \times \frac{3}{2} = \frac{9}{8} = 1\frac{1}{8}}$

$6 \div \frac{5}{8} = \boxed{6 \times \frac{8}{5} = \frac{48}{5} = 9\frac{3}{5}}$ $\qquad \frac{2}{3} \div \frac{4}{9} = \boxed{\frac{2}{3} \times \frac{9}{4} = \frac{18}{12} = \frac{3}{2} = 1\frac{1}{2}}$

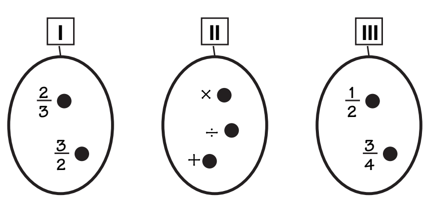
A common student error is to invert both fractions, for example, changing ¾ ÷ ⅔ into ⁴⁄₃ x ³⁄₂. If this occurs, draw an appropriate arrow picture and point out that the first number, ¾, labels the dot and does not change, as illustrated here.

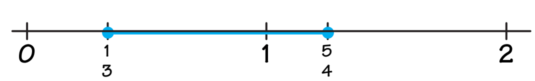


Exercise 3

Note: For this exercise you may like to use boxes prepared with pieces of paper, as described below. Then the class can actually do the experiment, rather than just imagine it.

On the board, draw this string picture and this part of a number line nearby. Ask students to locate ⅓ and ⁵⁄₄. Color the segment from ⅓ to ⁵⁄₄ blue.





T: *Imagine three boxes with some pieces of paper in them. Each piece of paper has a symbol written on it and is folded so that the symbol is hidden. The string picture shows what is in the boxes. Suppose we randomly choose one piece of paper from each box. What number could we choose from the first box?*

S: *⅔ (or ³⁄₂).*

N35

Continue in a similar manner with Box II and Box III. Write the symbols in order on the board as the students announce what you could choose. Suppose the choices are $\frac{2}{3}$, x, and $\frac{3}{4}$. (Adapt the following discussion to your class's choices.)

$$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} = \frac{1}{2}$$

T: ***Complete the calculation. What number is $\frac{2}{3} \times \frac{3}{4}$?***

S: ***$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} = \frac{1}{2}$.***

T (pointing to the number line)**:** ***Does $\frac{1}{2}$ belong to the blue segment?***

S: ***Yes, $\frac{1}{2}$ is more than $\frac{1}{3}$ and less than $\frac{5}{4}$.***

Ask a student to point to where $\frac{1}{2}$ is located on the number line.

Repeat this activity once more. Perhaps the choices will be $\frac{3}{2}$, +, and $\frac{1}{2}$ this time, in which case you get $\frac{3}{2} + \frac{1}{2} = 2$, and 2 does not belong to the blue segment.

T: ***We got one number that is on the blue segment and one that is not on the blue segment. If we randomly choose a paper from each box, what is the probability of getting a number that is on the blue segment?***

Let the class discuss how to calculate this probability. Conclude that you need to find all of the possible numbers that could be formed and then determine how many of those numbers are on the blue segment.

Distribute Worksheet N35. Explain that completing the tables is a way to find all of the possible numbers that could be formed. While students are working, draw the three tables from the worksheet on the board.

Invite students to complete the tables on the board (see the next illustration). Discuss the results, especially those that are disputed.

T: ***Now what should we do?***

S: ***Decide which numbers in the tables are on the blue segment.***

T: ***On your worksheet, circle in blue all the numbers that are on the blue segment.***

After a few minutes, invite students to circle the numbers in the tables on the board. Be prepared to discuss numbers the students question.

x	$\frac{1}{2}$	$\frac{3}{4}$
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$
$\frac{3}{2}$	$\frac{3}{4}$	$1\frac{1}{8}$

÷	$\frac{1}{2}$	$\frac{3}{4}$
$\frac{2}{3}$	$1\frac{1}{3}$	$\frac{8}{9}$
$\frac{3}{2}$	3	2

+	$\frac{1}{2}$	$\frac{3}{4}$
$\frac{2}{3}$	$1\frac{1}{6}$	$1\frac{5}{12}$
$\frac{3}{2}$	2	$2\frac{1}{4}$

T: ***What is the probability of getting a number that is on the blue segment?***

S: ***$\frac{6}{12}$ or $\frac{1}{2}$. According to the tables, 6 out of 12 possible results are on the blue segment.***

Capsule Lesson Summary

Discuss everyday uses of percent and percents that are easy to use in calculations. Combine easy percents to do calculations such as 35% of 80. Use composition of ÷100 and □x to define □% of, and find equivalent names for □% using equivalent compositions. Observe patterns for doing percent calculations, and apply this to calculating a 15% tip on a meal in a restaurant.

Materials

Teacher
- Colored chalk

Student
- Paper
- Worksheets 30*, **, and ***

Description of Lesson

Exercise 1

Begin the lesson by asking students about some of the ways they hear percents used in everyday situations. Elicit a variety of responses and discuss these uses of percent. For example:

S: ***The weather report might say there is a 20% chance of rain.***

T: ***What does that mean to you? Would you plan to have an umbrella or raincoat with you? Do you think the weather is likely to force a sporting event to be cancelled? Is it likely to rain?***

S: ***A test score was 75%.***

T: ***What does 75% on a test indicate to you?***
If the test had 20 items, how many were correct? (75% of 20 = 15)

S: ***A store might advertise a sale with 40% off.***

T: ***Is that a good sale? Is it better than a half-price sale? Approximately what would you expect to pay for an item that originally sold for $50?*** ($30)

You may like to make some statements using percents and ask whether or not they are reasonable. For example,

T: ***Susan spends 90% of her allowance on ice cream. Do you think this is smart?***

When a family eats out they usually spend about $35. Would a 50% tip be too much to leave?

The Central basketball team makes about 10% of the baskets they attempt. What do you think about the shooting skills on the team?

After discussing a few examples of the use of percents in everyday contexts, ask,

T: ***What percents are quite easy to use in calculations?***

Accept suggestions such as 50%, 25%, 100%, 10%, and 1%. For example:

S: ***50%, because it is the same as ½ or ÷2.***

T: ***What is 50% of 14?*** (7) ***of 40?*** (20) ***of 3,000?*** (1 500)

Continue until it is clear that most students can readily find 50% of a number. In a similar manner, ask students to do calculations involving 25%, 10%, and 1%.

Suggest students solve the following problems by combining easy percent calculations. (Answers are in boxes.)

75% of $48 = \boxed{36}$ $\qquad$ 35% of $80 = \boxed{28}$

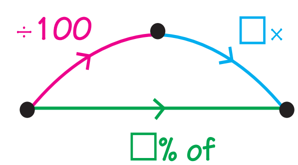
60% of $20 = \boxed{12}$ $\qquad$ 11% of $200 = \boxed{22}$

S: ***50% of 48 = 24 and 25% of 48 = 12, so 75% of 48 = 24 + 12 = 36.***

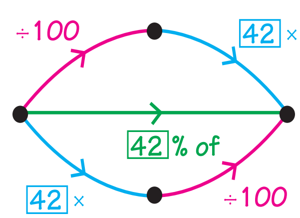
S: ***50% of 20 = 10 and 10% of 20 = 20 ÷ 10 = 2, so 60% of 20 = 10 + 2 = 12.***

Exercise 2

Draw this arrow picture on the board.



T: ***Percent can be defined as a composition of two arrows, as in the picture. With the red arrow for ÷100, the same number goes in each box. For example, if the green arrow is for 42%, then the blue arrow is for 42x.***

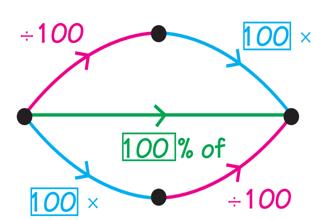


Label the arrows, and point out that the red and blue arrows could be reversed. So, 42% of is ÷100 followed by 42x as well as 42x followed by ÷100.

Erase the numbers in the boxes.

T: ***If the blue arrow is for 17x, then what is the green arrow for?*** (17% of)
If the green arrow is for 53% of, then what is the blue arrow for? (53x)
If the green arrow is for 100% of, then what is the blue arrow for? (100x)

Label the arrows.



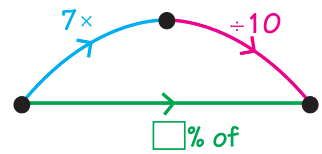
T: ***What is 100% of a number?***

S: ***The same number.***

T: ***How does the arrow picture show that 100% of a number is the same number?***

S: ***100x followed by ÷100 is the same as ¹⁰⁰⁄₁₀₀x or 1x. An arrow for 100% of should be a loop like 1x.***

Repeat this line of questioning to observe that 200% of a number is 2x the number, and 300% of a number is 3x the number.

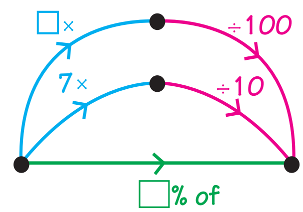


Draw this arrow picture on the board.

T: ***How should we fill in the box for the green arrow?***

If some students respond, "7%," observe,

T: ***If the red arrow were for ÷100, then the green arrow would be for 7% of. But the red arrow is for ÷10.***



Add these arrows to the picture.

T: ***Let's consider another composition for the green arrow where the red arrow is for ÷100. How should we fill in the box for this blue arrow?***

S: ***70x. 70x followed by ÷100 is the same as 7x followed by ÷10 since $^{70}/_{100}$x = $^{7}/_{10}$x.***

T: ***What is the green arrow for?***

S: ***70% of. 70x followed by ÷100 is 70% of.***

Label the green arrow.

Exercise 3

Write the following information on the board.

□% of 38 < 38 □% of 38 = 38 □% of 38 > 38

T: ***How could we fill in the boxes to make these statements true?***

S: ***□% of 38 = 38 has only one solution, 100, since 100% of 38 = 38.***

S: ***Any number less than 100 makes □% of 38 < 38 true.***

S: ***Any number more than 100 makes □% of 38 > 38 true.***

T: ***What number is 150% of 38?***

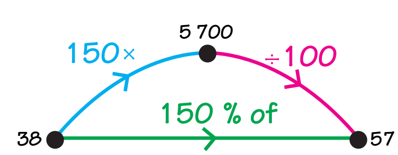
S: ***57. 100% of 38 = 38 and 50% of 38 = 19, so 150% of 38 = 38 + 19 = 57.***

100% of 38 = 38
50% of 38 = 19
150% of 38 = 57

Indicate this method of calculating 150% of 38 on the board.

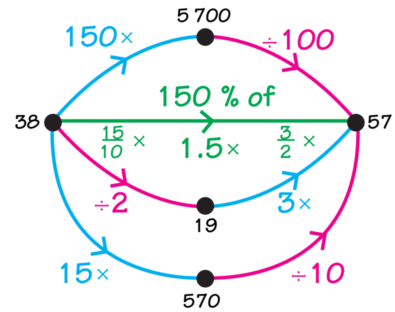
S: ***We could use arrows. 150% of is the same as 150x followed by ÷100.***

Draw this arrow picture on the board, and label the dots with student direction.



N30

You may also find several equivalent compositions and equivalent names for 150%, as illustrated here.



Leave the calculation for 150% of 38 on the board when you proceed to Exercise 4.

Exercise 4

Lead a brief discussion of the practice of tipping waitresses, waiters, and other service workers. Explain that tipping is a major part of the income of many such people. Ask whether any student knows how much is an appropriate tip for a waiter or a waitress.

T: ***For good service, one might tip 15% of the total cost of the meal. If a family's meal costs $45, how much would a 15% tip be?***

Lead the class to use the following method of calculating 15% of $45.

10% of \$45 = \$4.50	…since 10% = $\frac{1}{10}$, $\frac{1}{10}$x = ÷10, and 45 ÷ 10 = 4.5
5% of \$45 = \$2.25	…since 5% is half of 10% and ½ x 4.5 = 2.25
15% of \$45 = \$6.75	

Ask students to calculate 15% tips for meals costing $38 and $26. Encourage similar methods of solution.

15% of \$38 = \$3.80 × \$1.90 = \$5.70

15% of \$26 = \$2.60 × \$1.30 = \$3.90

Note that 150% of 38 = 57 and 15% of 38 = 5.7.

Worksheets N30*, **, and *** are available for individual work.

Home Activity

Send home percent problems like those on Worksheets N30* and ** for students to do with family help. Especially note for parents/guardians the use of patterns in solving many percent calculations.

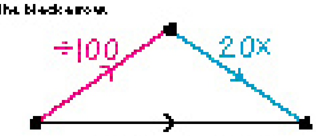
Name ________ N30 ★

Complete.

50% of 120 = 60	100% of 32 = 32
25% of 120 = 30	50% of 32 = 16
10% of 120 = 12	150% of 32 = 48
5% of 120 = 6	75% of 32 = 24
15% of 120 = 18	25% of 32 = 8
35% of 120 = 42	125% of 32 = 40
10% of 40 = 4	50% of 68 = 34
5% of 40 = 2	25% of 68 = 17
15% of 40 = 6	75% of 68 = 51
20% of 40 = 8	10% of 68 = 6.8
40% of 40 = 16	35% of 68 = 23.8
45% of 40 = 18	85% of 68 = 57.8

Name ________ N30 ★★

Label the black arrow.



$\boxed{\frac{20}{100}}$ x or $\boxed{20}$ % of

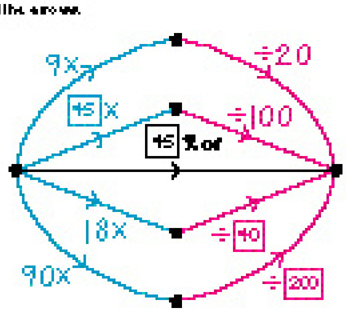
Give two other names for the black arrow: $\boxed{\frac{2}{10}}$ x or $\boxed{\frac{1}{5}}$ x

Complete.

20% of 15 = $\boxed{3}$	20% of 35 = $\boxed{7}$
20% of 60 = $\boxed{12}$	20% of 42 = $\boxed{8.4}$
20% of $\boxed{300}$ = 60	20% of $\boxed{210}$ = 42

Name ________ N30 ★★★

Label the arrows.



Complete.

$\frac{9}{20}$ x 200 = $\boxed{90}$	45% of 80 = $\boxed{36}$
45% of 200 = $\boxed{90}$	45% of 18 = $\boxed{8.1}$
45% of $\boxed{600}$ = 270	45% of $\boxed{40}$ = 18

Capsule Lesson Summary

Use patterns to solve a sequence of division problems. Interpret division of a whole number by a fraction as a measurement process. Examine several methods to divide $36 \div \frac{3}{4}$, such as rewriting the problem to multiply $36 \times \frac{4}{3}$.

Materials

Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheet 31

Description of Lesson

Exercise 1

Begin this table on the board.

Number of Cans	Number of Full Baskets (15 cans per basket)	Number of Extra Cans
600		

Center the following discussion on any school or local organization that distributes food baskets to needy families for Thanksgiving or Christmas.

T: ***Our school collects cans of food to give to needy families for Thanksgiving (or Christmas). Suppose we decide to put 15 cans of food into each food basket. If we collect 600 cans, how many baskets can we fill?***

S: ***40. 60 ÷ 15 = 4, so 600 ÷ 15 = 40.***

S: ***40. 40 x 15 = 600.***

Add this information to the table.

Number of Cans	Number of Full Baskets (15 cans per basket)	Number of Extra Cans
+ 30 600	40	0

T: ***Suppose we collect 30 more cans of food. How many cans do we have now?*** (630)

T: ***How many baskets could we fill with 630 cans?***

S: ***42. The additional 30 cans will fill two more baskets (30 ÷ 2 = 15). 40 + 2 = 42.***

Record this information, and continue to complete this table in a similar manner.

Number of Cans	Number of Full Baskets (15 cans per basket)	Number of Extra Cans
600 + 30	40 +2	0
630 +15	42 +1	0
$\boxed{645}$ +150	$\boxed{43}$ +10	$\boxed{0}$
$\boxed{795}$ + 30	$\boxed{53}$ +2	$\boxed{0}$
$\boxed{825}$ +4	$\boxed{55}$ +0	$\boxed{0}$
$\boxed{829}$ +6	$\boxed{55}$ +0	$\boxed{4}$
$\boxed{835}$ +10	$\boxed{55}$ +1	$\boxed{10}$
$\boxed{845}$ +12	$\boxed{56}$ +1	$\boxed{5}$
$\boxed{857}$	$\boxed{57}$	$\boxed{1}$

Exercise 2

T: ***A local bakery donates pies for the food baskets. If the bakery gives us 36 large pies and we make about 100 baskets, about how much pie could we put in each basket?***

Let students discuss the problem. Accept estimates and then proceed to consider more specific cases.

T: ***If the bakery gives us 36 pies and we put ½ pie in each basket, how many baskets will get some pie?***

S: ***72. Each pie is for two baskets and 36 x 2 = 72.***

T: ***What number facts can we write about this situation?***

Accept several possibilities, and write these two sentences (a multiplication and a division) on the board. If necessary, suggest the second one yourself.

$$36 \times 2 = 72$$
$$36 \div \frac{1}{2} = 72$$

T: ***What does 36 ÷ ½ mean?***

S: ***Just like 36 ÷ 2 asks how many twos are in 36, 36 ÷ ½ asks how many one halves are in 36. In our situation, 36 ÷ ½ asks how many ½ pies are in 36 pies.***

T: ***If the bakery gives us 36 large pies and we put ⅓ pie in each basket, how many baskets will get some pie?***

S: ***108, because 36 x 3 = 108.***

T: ***What number facts can we write about this situation?***

Accept several possibilities, and write these two number sentences on the board.

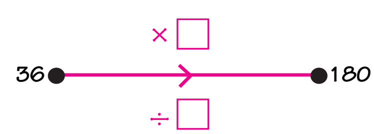
$$36 \times 3 = 108$$
$$36 \div \frac{1}{3} = 108$$

Note: At this point, students may observe that between ½ and ⅓ pie (closer to ⅓ pie) could be used to put into 100 baskets.

In a similar manner, generate the following pairs of number sentences.

$36 \times 4 = 144$	$36 \times 8 = 288$
$36 \div \frac{1}{4} = 144$	$36 \div \frac{1}{8} = 288$

Draw this arrow picture on the board.



T: ***How should we label the red arrow?***

S: ***x5, since 36 x 5 = 180.***

S: ***÷⅕, since 36 x 5 = 36 ÷ ⅕ = 180.***

Ask students to complete the following calculations. (Answers are in boxes.) As necessary, interpret a problem in the context of putting parts of pies into food baskets.

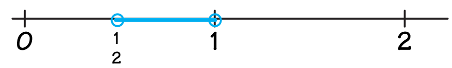
$17 \div \frac{1}{2} = \boxed{34}$	$28 \div \frac{1}{2} = \boxed{56}$
$17 \div \frac{1}{4} = \boxed{68}$	$28 \div \frac{1}{3} = \boxed{84}$
$17 \div \frac{1}{6} = \boxed{102}$	$28 \div \frac{1}{7} = \boxed{196}$

Exercise 3

T: ***Suppose the bakery gives us 36 pies for the food baskets. If we put 1 pie in each basket, 36 baskets will receive some pie. If we put ½ pie in each basket, 72 baskets will receive some pie. But if we want to fill just 50 baskets, then what part of a pie can go in each basket?***

Let students discuss this question. They should conclude that between ½ pie and 1 whole pie can go in each basket.

On the board, draw part of a number line and highlight the segment between ½ and 1.



Ask students to suggest fractions between ½ and 1, for example, ⅔, ¾, and ⅞.

Distribute Worksheet N31.

T: ***We are assuming that the bakery gives us 36 pies. Suppose we put ¾ pie in each food basket. How many baskets would receive some pie? You may use the pies on the worksheet to solve this problem, or solve it another way if you wish.***

Allow about five minutes for students to work individually.

T: ***How many baskets will get some pie?***

S: ***48. Every three pies go in four baskets. 36 ÷ 3 = 12 and 4 x 12 = 48.***

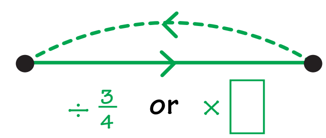
S: ***48. I divided each circle into fourths and counted how many ¾ pie shares there are.***

S: ***48. If each basket got ¼ pie, we could fill 4 x 36 or 144 baskets. But since each basket gets ¾ pie, 144 ÷ 3 or 48 baskets will get some pie.***

Note: At this point students may observe that if 48 (close to 50) baskets will each get ¾ pie, then 96 (close to 100) baskets will each get ⅜ pie.

Exercise 4

Draw this arrow picture on the board.



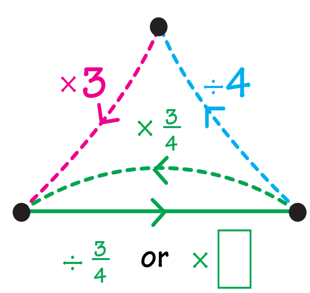
T: ***If the green arrow is for ÷¾, what would the opposite (return) arrow be for?***

S: ***x¾.***

T: ***What composition (detour) could be used for x¾?***

S: ***÷4 followed by x3.***

Draw this composition, and then observe that ÷¾ is the same as an opposite or return composition.

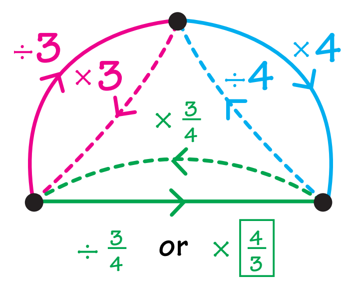


N31

S: *So ÷¾ is ÷3 followed by* x*4.*

T: *What is another name for ÷¾?*

S: x*⁴⁄₃, since ÷3 followed by* x*4 is also* x*⁴⁄₃.*



Point out the equivalence between ÷¾ and x⁴⁄₃.

Write this information on the board.

$36 \div \frac{3}{4} = 48$

$39 \div \frac{3}{4} =$

T: *What number is 39 ÷ ¾?*

S: *52. 39 ÷ 3 = 13 and 4* x *13 = 52.*

S: *52. 39 ÷ ¾ = 39* x *⁴⁄₃ = 52.*

S: *52. 39 = 36 + 3. With 3 more pies, we can put ¾ pie into 4 more baskets. 48 + 4 = 52.*

Continue with these problems. (Answers are in boxes.)

$36 \div \frac{3}{4} = 48$

$39 \div \frac{3}{4} = 52$

$42 \div \frac{3}{4} = \boxed{56}$

$45 \div \frac{3}{4} = \boxed{60}$

$51 \div \frac{3}{4} = \boxed{68}$

T: *Do you notice any pattern?*

S: *As the numbers on the left increase by 3, the solutions on the right increase by 4.*

Use arrows to illustrate this pattern.

$$\begin{array}{rcl} & 36 \div \frac{3}{4} = 48 & \\ +3 & 39 \div \frac{3}{4} = 52 & +4 \\ +3 & 42 \div \frac{3}{4} = 56 & +4 \\ +3 & 45 \div \frac{3}{4} = 60 & +4 \\ +3 & 51 \div \frac{3}{4} = 68 & +4 \end{array}$$

T: *Let's calculate 43 ÷ ¾.*

S: *57 with a remainder of ¼. 42 ÷ ¾ = 56. With 1 more pie, we can put ¾ pie in 1 more basket and ¼ pie will be left over.*

S: *43 ÷ 3 = 14⅓ and 14 ⅓* x *4 = 57⅓.*

S: *43 ÷ ¾ = 43* x *⁴⁄₃ = ¹⁷²⁄₃ = 57⅓.*

Note: We can interpret 43 ÷ ¾ as 57 with a remainder of ¼ but 43 ÷ ¾ ≠ 57¼. A student might observe that ¼ pie would make a basket ⅓ full; therefore 43 ÷ ¾ = 57⅓.

T: *Let's calculate 60 ÷ ⅔.*

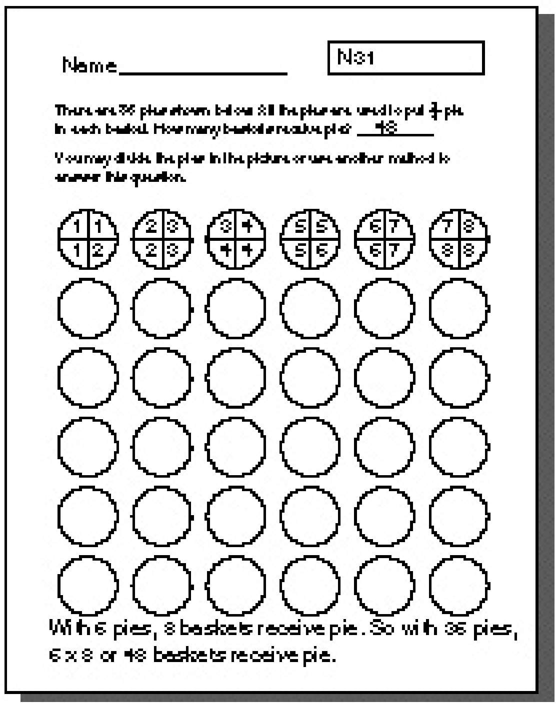
S: *90. 60 ÷ 2 = 30 and 30* x *3 = 90.*

S: *90. 60 ÷ ⅔ = 60* x *³⁄₂ = 90.*

T: *Those methods are correct. Could you also convince me using the idea of distributing pies in food baskets?*

S: *If we put ⅔ pie in each basket, then 2 pies are enough for 3 baskets. Since 60 ÷ 2 = 30 and 30* x *3 = 90, 60 ÷ ⅔ = 90.*

Accept other explanations, and confirm that 60 ÷ ⅔ = 90.



Name________________ N31

There are 36 pies shown below. All the pies are used to put $\frac{3}{4}$ pie in each basket. How many baskets receive pie? 48

You may divide the pies in the picture or use another method to answer this question.

With 6 pies, 8 baskets receive pie. So with 36 pies, 6 x 8 or 48 baskets receive pie.

Capsule Lesson Summary

Find all of the positive divisors from 1 to 10 of some selected numbers. Review tests for deciding whether or not a given number is divisible by 2, by 3, by 4, by 5, by 6, by 8, by 9, or by 10. Consider the numbers that can be formed by randomly arranging four given digits, and find the probabilities of getting numbers divisible by 2, by 3, by 4, by 8, by 9, by 5, and by 6.

Materials		
	Teacher	• 4 index cards
	Student	• Paper • Worksheets 32*, **, and ***

Advance Preparation: Write each of the four digits 0, 3, 4, and 5 on a separate index card.

Description of Lesson

Exercise 1

Write the corresponding number sentences on the board as you ask these questions.

T: ***Is 300 a multiple of 3?*** (Yes)
What number times 3 equals 300? (100) — $100 \times 3 = 300$
Is 24 a multiple of 3? (Yes)
What number times 3 equals 24? (8) — $8 \times 3 = 24$
Is 324 a multiple of 3? (Yes)
What number times 3 equals 324? (108) — $108 \times 3 = 324$

Is 55 a multiple of 5? (Yes)
What number times 5 equals 55? (11) — $11 \times 5 = 55$
Is 550 a multiple of 5? (Yes)
What number times 5 equals 550? (110) — $110 \times 5 = 550$
Is 5 500 a multiple of 5? (Yes)
What number times 5 equals 5 500? (1 100) — $1100 \times 5 = 5500$

Is 100 a multiple of 4? (Yes)
What number times 4 equals 100? (25) — $25 \times 4 = 100$
Is 200 a multiple of 4? (Yes)
What number times 4 equals 200? (50) — $50 \times 4 = 200$
Is 800 a multiple of 4? (Yes)
What number times 4 equals 800? (200) — $200 \times 4 = 800$
Is 900 a multiple of 4? (Yes)
What number times 4 equals 900? (225) — $225 \times 4 = 900$
Name a multiple of 4 (36)
Is 936 a multiple of 4? (Yes) — $9 \times 4 = 36$
What number times 4 equals 936? (234)
Is 915 a multiple of 4? (No, 15 is not a multiple of 4 so 15 more than a multiple of 4 is not a multiple of 4) — $234 \times 4 = 936$
How can we easily check to see whether a number is a multiple of 4?

Lead the class to see that if the last two digits of a number are divisible by 4, then the number is divisible by 4. Adding hundreds to a one- or two-digit multiple of 4 yields a multiple of 4, no matter how many hundreds are added. Adding hundreds to a one- or two-digit number that is not a multiple of 4 never yields a number that is a multiple of 4.

T: ***Is 10 a multiple of 8?*** (No)
Is 100 a multiple of 8? (No, 96 and 104 are multiples of 8)
Is 1000 a multiple of 8? (Yes, 800 is a multiple of 8 and 200 is a multiple of 8, so 800 + 200 is a multiple of 8)
What number times 8 equals 1 000? (125) — $125 \times 8 = 1000$
Is 2 000 a multiple of 8? (Yes)
What number times 8 equals 2 000? (250) — $250 \times 8 = 2\,000$
Is 3 000 a multiple of 8? (Yes)
What number times 8 equals 3 000? (375) — $375 \times 8 = 3\,000$
Name a multiple of 8. (72)
Is 3 072 a multiple of 8? (Yes)
What number times 8 equals 3 072? (384) — $384 \times 8 = 3\,072$
Is 3 020 a multiple of 8? (No, 20 is not a multiple of 8 so 20 more than a multiple of 8 is not a multiple of 8)
How can we easily check to see whether a number is a multiple of 8?

Lead the class to see that if the last three digits of a number are divisible by 8, then the number is divisible by 8. Adding thousands to a one-, two-, or three-digit multiple of 8 yields a multiple of 8, no matter how many thousands are added. Adding thousands to a one-, two-, or three-digit number that is not a multiple of 8 never yields a number that is a multiple of 8.

Erase the board before going on to Exercise 2.

Exercise 2

Put this information on the board.

24 630 is divisible by:

1 2 3 4 5 6 7 8 9 10

T: ***Let's circle the numbers in this list that are divisors of 24 630 and cross off those that are not.***

Encourage students to explain why a number gets circled or why it gets crossed off.

S: ***10 is a divisor of 24 630 because 24 630 ends in 0.***

S: ***1 is a divisor of any integer.***

S: ***24 630 is even, so 2 is a divisor.***

S: ***Any number that ends in 5 or 0 is a multiple of 5.***

S: ***2 + 4 + 6 + 3 + 0 = 15; 15 is a multiple of 3 but not a multiple of 9. Therefore, 24 630 is divisible by 3 but is not divisible by 9.***

S: ***30 is not a multiple of 4; so 24 630 is not a multiple of 4.***

S: ***630 is not a multiple of 8. 640 is a multiple of 8 and 632 (640 – 8) is a multiple of 8, but 630 is not. So , 24 630 is not divisible by 8.***

S: ***24 630 is a multiple of 6. 24 000 is a multiple of 6; 600 is a multiple of 6; and 30 is a multiple of 6.***

S: ***If 2 is a divisor of a number and 3 is also a divisor, then 6 must be a divisor. 24 630 is divisible by both 2 and 3, and thus by 6.***

T: ***7 is the only number left to consider. Is 7 a divisor of 24 630?***

Lead the class to see that 24 630 is not divisible by 7. For example:

$$\begin{array}{r} 24\,630 \\ -21\,000 \\ \hline 3\,630 \\ -3\,500 \\ \hline 130 \end{array}$$

T: ***What is a number close to 24 630 that is divisible by 7?*** (21 000)
What number is 24 630 – 21 000? (3 630)
What is a number close to 3 630 that is a multiple of 7? (3 500)
What number is 3 630 – 3 500? (130)
Is 130 a multiple of 7? (No. 140 is a multiple of 7, 140 – 7 = 133, and 133 – 7 = 126; therefore, 130 is not a multiple of 7.)
So is 24 630 a multiple of 7? (No)

Conclude that 24 630 is divisible by 1, 2, 3, 5, 6, and 10.

24 630 is divisibile by:

1 2 3 ~~4~~ 5 6 ~~7~~ ~~8~~ ~~9~~ 10

Repeat the activity with 33 376.

33 376 is divisibile by:

1 2 ~~3~~ 4 ~~5~~ ~~6~~ 7 8 ~~9~~ ~~10~~

Exercise 3

Display the four index cards with these digits.

4	3	0	5

T: ***On your paper, write any number you wish using each of these digits exactly once. Decide if your number is divisible by 2, 3, 4, 5, 6, 7, 8, 9, and/or 10.***

Allow several minutes for individual or small group work.

Ask for a number, and let the class find which numbers from 2 to 10 are positive divisors of the suggested number. For example:

5 304 is divisible by 2, 3, 4, and 6
3 045 is divisible by 3, 5, and 7

Repeat this activity with one or two more suggested numbers.

T: ***Let's mix up these four cards and then randomly arrange them to get a four-digit number. What is the probability that we get a number divisible by 2?***

Allow some time for students to consider the question. You may like to act out the mixing and random arrangement of the four cards.

S: ***Divisibility by 2 is determined by the ones digit of a number. In this case, the ones digit may be 0 or 3 or 4 or 5. If the ones digit is 0 or 4, the number is even and divisible by 2. If the ones digit is 3 or 5, the number is odd and not divisible by 2. So, there are two chances out of four that the number is divisible by 2; the probability is $\frac{1}{2}$.***

Do not expect a student to give as detailed an explanation as the above. Instead, direct the discussion along the lines outlined in the above response. Try to involve as many students as possible in such a discussion. Conclude that the probability of the number being divisible by 2 is $\frac{1}{2}$.

T: ***What is the probability that we get a number divisible by 3?***

S: ***$4 + 3 + 0 + 5 = 12$ and 12 is a multiple of 3, so no matter how the digits are arranged, the sum of the digits is always 12. Such a number is always divisible by 3. The probability is 1.***

T: ***What is the probability that we get a number divisible by 4?***

Begin the discussion by recalling that since multiples of 100 are also multiples of 4, only the possibilities for the last two digits need be examined.

T: ***How can we tell if a number is divisible by 4? Remember, multiples of 100 are divisible by 4.***

S: ***Look only at the last two digits. If that number is divisible by 4, then the number itself is divisible by 4.***

T: ***How does this help us to find the probability that we get a number divisible by 4?***

S: ***We can look at all possibilities for the last two digits.***

The class should find 12 possible two-digit ending numbers. List these numbers on the board as they are suggested.

50	53	54	40	43	30
05	35	45	04	34	03

S: ***In this list, only 40 and 4 are divisible by 4. So if the number ends in 40 or 04, then it is divisible by 4. The probability is $\frac{2}{12}$ or $\frac{1}{6}$.***

T: ***What is the probability that the number is divisible by 8?***

After the discussion for divisibility by 4, recall that since 1 000 is a multiple of 8, only the possibilities for the last three digits need be examined. There are 24 possibilities; however, the work can be shortened if students recall that a number divisible by 8 must also be divisible by 4. Keeping this in mind, it is only necessary to look at these four of the 24 possibilities for the last three digits.

540 504 340 304

Two of these numbers, 504 and 304, are divisible by 8; so the class should conclude that the probability of getting a number that is divisible by 8 is $\frac{2}{24}$, or $\frac{1}{12}$.

Continue this exercise by asking students, perhaps in small groups, to calculate the probabilities that the number is divisible by 9, by 5, and by 6. The following paragraphs give a brief explanation of the analysis for each of these situations.

Divisibility by 9: Since 4 + 3 + 0 + 5 = 12 and 12 is not a multiple of 9, any number with these four digits is not divisible by 9. The probability is 0.

Divisibility by 5: A number is divisible by 5 if and only if its ones digit is 0 or 5. There are two chances out of four that the ones digit of the randomly arranged number will be 0 or 5. The probability is $\frac{2}{4}$, or $\frac{1}{2}$.

Divisibility by 6: A number is divisible by 6 if and only if it is divisible by both 2 and 3. Previously, we found that any four-digit number with digits 4, 3, 0, and 5 is divisible by 3. If the digits are randomly arranged, there are two chances out of four of getting a number that is divisible by 2. So, the probability of getting a number that is divisible by 6 is also $\frac{2}{4}$, or $\frac{1}{2}$.

T: ***Could a number with these four digits be prime?***

S: ***No; such a number is divisible by 3.***

You may challenge students further by asking if the digits 0 through 9 could be arranged to get a prime number. The answer is no. The sum of the digits would be 45, and so such number would be divisible by 3 and by 9.

Worksheets N32*, **, and *** are available for individual work.

Extension Activity

Challenge students to find a four-digit number with digits 0, 3, 4, and 5 that is divisible by as many numbers from 1 to 10 as possible.

N32

Name_______________ N32 ★

Zot is a secret number.

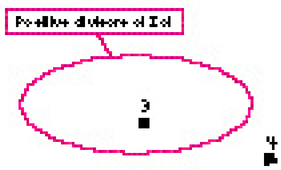
Clue 1

Zot is an even number and Zot's name can be completed by putting a single digit in the box.

Zot = 96 73☐

Zot could be 96730, 96732, 96734, 96736, or 96738.

Clue 2



Who is Zot? 96738

Name_______________ N32 ★★

For each number on the left check whether it is divisible by 2, 3, 4, 5, 6, 8, 9, or 10. For example, the X's in the table show 5560 is divisible by 2, 4, 5, 8, and 10.

Is divisible by	2	3	4	5	6	8	9	10
5560	X		X	X		X		X
320 787		X					X	
82 386	X	X			X		X	
11 881								
1080	X	X	X	X	X	X	X	X
118 652	X		X					
12 115				X				
21 060	X	X	X	X	X		X	X
102 246	X	X			X			
10072	X		X			X		
394 581		X						

Name_______________ N32 ★★★

Flop = 5 6 0 8☐

Flop's name can be completed by choosing at random exactly one of the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 to put in the box.

Find the probability that

Flop is divisible by 2 $\frac{1}{2}$

Flop is divisible by 3 $\frac{2}{10}$

Flop is divisible by 4 $\frac{2}{10}$

Flop is divisible by 5 $\frac{1}{5}$

Flop is divisible by 6 $\frac{1}{5}$

Flop is divisible by 7 $\frac{1}{10}$

Flop is divisible by 8 $\frac{1}{5}$

Flop is divisible by 9 $\frac{1}{10}$

Flop is divisible by 10 $\frac{1}{10}$

N33 DIVISION OF RATIONAL NUMBERS

Capsule Lesson Summary

Count by thirds and then by two-thirds. Use a number line model to solve division problems where the divisor is a fraction. Illustrate the equivalence of $\div\frac{3}{5}$ and $\frac{5}{3}$x with an arrow picture. Solve division problems involving fractions by rewriting them as multiplication problems.

Materials

Teacher
- Meter stick
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N33*, **, ***, and ****

Description of Lesson

Exercise 1

Use an order natural to the seating arrangement in your classroom for this exercise. This dialogue assumes there are 28 students.

T: ***Let's count by thirds. If we simply say $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$, $\frac{4}{3}$, and so on, with what fraction will we end?***

S: ***$\frac{28}{3}$, because there are 28 of us.***

T: ***Okay, but let's say the numbers a little differently. What is another name for $\frac{3}{3}$?*** (1) ***Instead of $\frac{3}{3}$ you will say 1. What would come next after 1?***

S: ***$1\frac{1}{3}$, which is the same as $\frac{4}{3}$.***

T: ***If we say the numbers in this way, what name for $\frac{28}{3}$ will the last person say?***

S: ***$9\frac{1}{3}$.***

Conduct the counting by thirds activity, and check that the last person does indeed say $9\frac{1}{3}$.

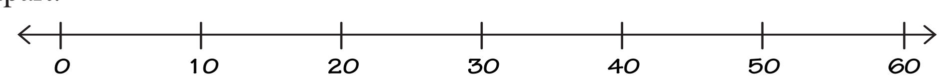
T: ***Suppose instead we count by two-thirds. That is we say $\frac{2}{3}$, $1\frac{1}{3}$, 2, $2\frac{2}{3}$, and so on. At what number will we end?***

S: ***$18\frac{2}{3}$. $\frac{2}{3}$ is the double of $\frac{1}{3}$ so we will end at the double of $9\frac{1}{3}$.***

Conduct the counting by two-thirds, and check that the last person does indeed say $18\frac{2}{3}$.

Exercise 2

Use a meter stick to draw this part of a number line on the board. Accurately space the marks, say 20 cm apart.

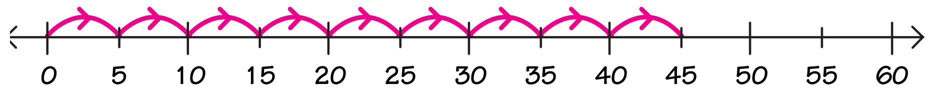


T: ***What number is 45 ÷ 5?*** (9) ***Can you use this number line to convince us that 45 ÷ 5 = 9?***

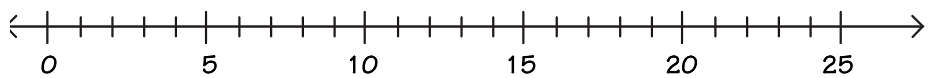
N33

S: ***Start at 0 and count by fives. Make marks for 5, 15, 25, 35, and 45 halfway between the other marks. There are nine fives to get to 45.***

Invite a student to make marks for counting by fives, or for the ending point of successive +5 arrows starting at 0. Confirm that there are nine fives, or nine +5 arrows, from 0 to 45.



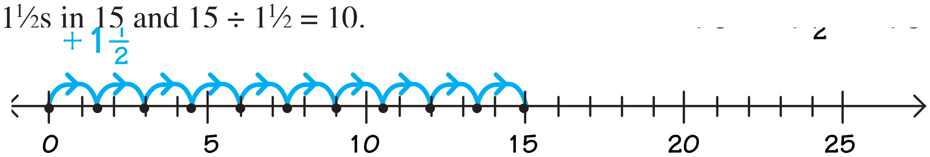
Draw this part of a number line, and write a new division problem on the board.



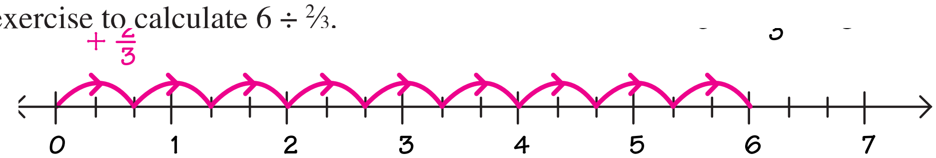
T: ***What number is $15 \div 1\frac{1}{2}$? How can we use the number line for this calculation?***

S: ***Start at 0 and count by $1\frac{1}{2}$s. Find how many $+1\frac{1}{2}$ arrows there are from 0 to 15.***

Invite students to draw the $+1\frac{1}{2}$ arrows or mark the points on the number line while counting by $1\frac{1}{2}$s. The class should find ten $+1\frac{1}{2}$ arrows from 0 to 15, or count ten $1\frac{1}{2}$s to reach 15. That is, there are ten $1\frac{1}{2}$s in 15 and $15 \div 1\frac{1}{2} = 10$.



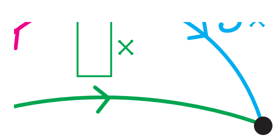
Repeat this exercise to calculate $6 \div \frac{2}{3}$.



Write the following problems on the board, asking students to copy and solve them. (Answers are in boxes.) You may want to encourage students to use the number line model, but also allow other methods, such as changing a division problem to a multiplication problem.

Exercise 3

Draw this ~~arrow picture on the~~ board.



T: ***What could the green arrow be for?***

S: $\frac{3}{5}$x. ***÷5 followed by 3x is the same as*** $\frac{3}{5}$x.

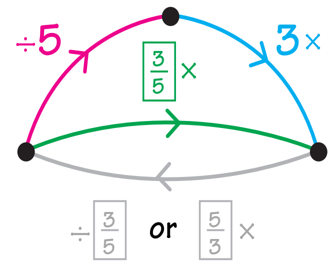
Draw a return or opposite arrow for the green arrow

T: *What is the return (or opposite) of* $\frac{3}{5}$x.

S: $\div\frac{3}{5}$.

T: *Could we give this arrow a* $\square$x *label?*

S: *If we use the opposites of 3*x *and ÷5, it is ÷3 followed by 5*x *which is the same as* $\frac{5}{3}$x.



You may want to draw the opposite arrows to better view this composition.

Use the arrow picture to solve the following problems. (Answers are in boxes.)

$\frac{3}{5} \times 25 = \boxed{15}$ $\qquad$ $30 \div \frac{3}{5} = \frac{5}{3} \times 30 = \boxed{50}$

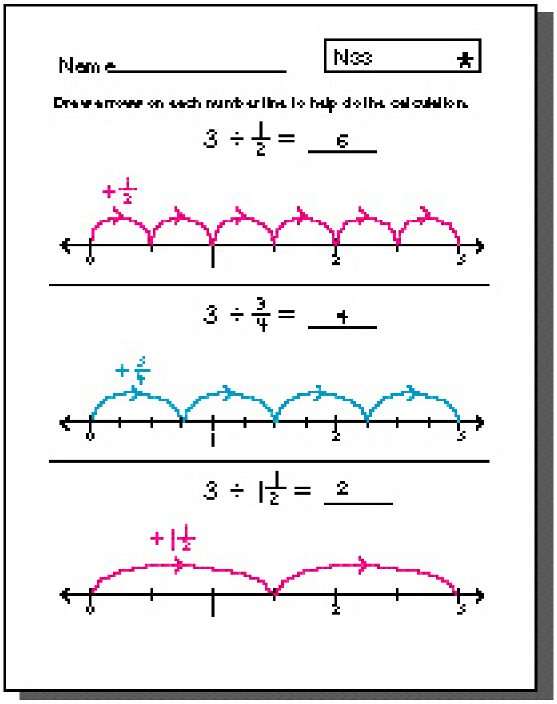
$\frac{3}{5} \times 3\frac{1}{3} = \boxed{2}$ $\qquad$ $1\frac{1}{2} \div \frac{3}{5} = \frac{5}{3} \times 1\frac{1}{2} = \boxed{2\frac{1}{2}}$

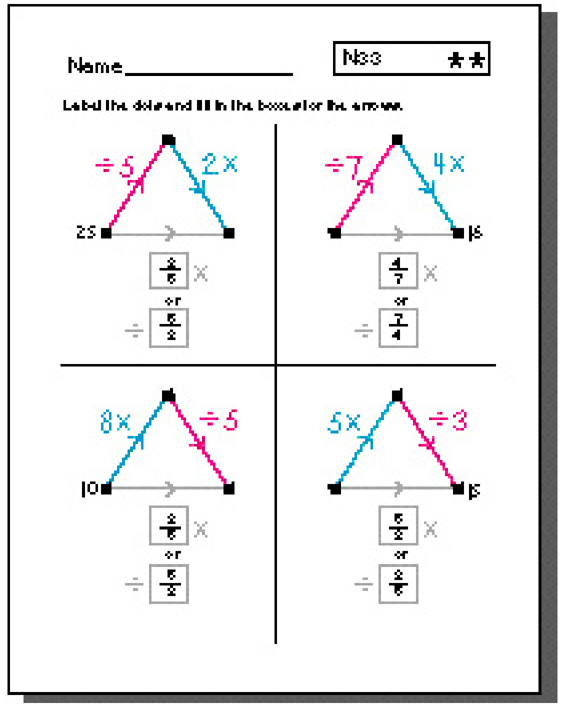
Write the following problems on the board, asking students to copy and solve them.

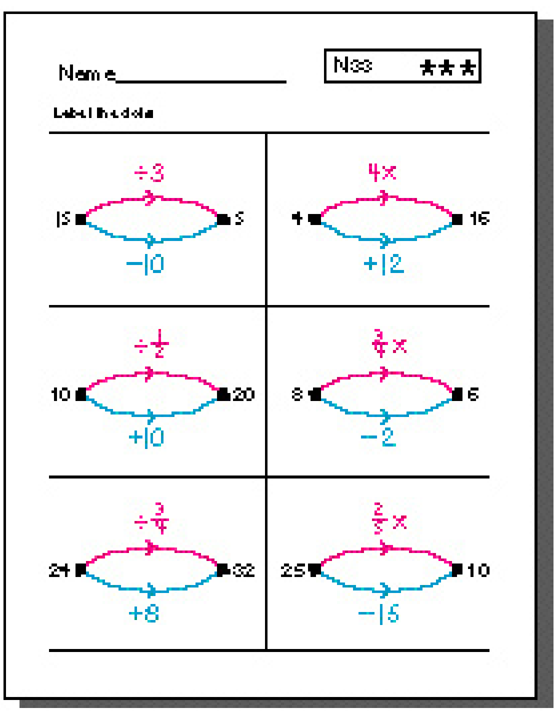
$12 \div \frac{3}{5} = \boxed{20}$ $\qquad$ $5 \div \frac{3}{5} = \boxed{\frac{5}{3} \times 5 = \frac{25}{3} = 8\frac{1}{3}}$ $\qquad$ $\frac{3}{4} \div \frac{3}{5} = \boxed{\frac{5}{3} \times \frac{3}{4} = \frac{5}{4} = 1\frac{1}{4}}$

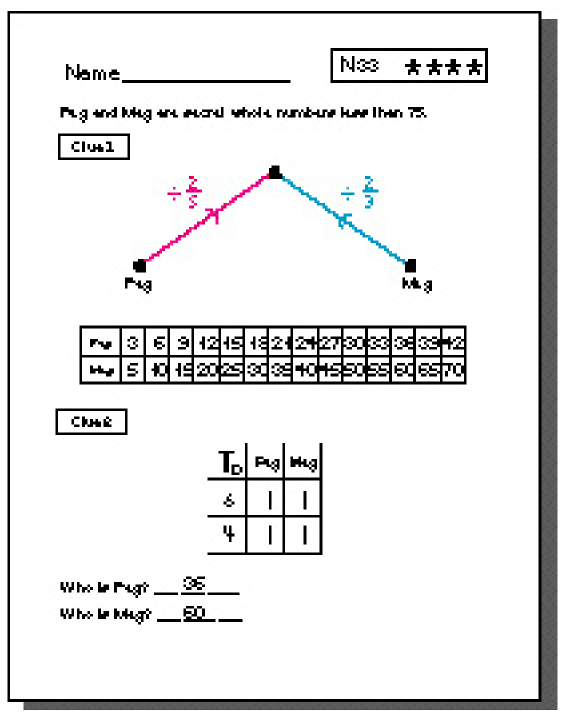
Worksheets N33*, **, ***, and **** are available for individual work.

N33









Peg	3	6	9	12	15	18	21	24	27	30	33	36	39	42
Meg	5	10	15	20	25	30	35	40	45	50	55	60	65	70

T_D	Peg	Meg
6	I	I
4	I	I

Capsule Lesson Summary

Using an arrow picture, solve some problems involving $\frac{1}{5}$x and $\frac{1}{3}$x. Find labels for several pairs of arrows, one ÷☐ and one ☐x, from 45 to 27. Label the composite arrow in different ways, and introduce *percent* as a new name. Use patterns to do percent calculations.

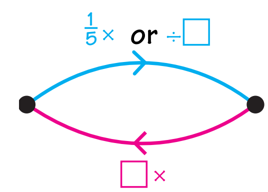
Materials

Teacher	• Colored chalk	Student	• Paper • Worksheets N15* and **

Description of Lesson

Exercise 1

Draw this arrow picture on the board.



T: ***What division name does*** $\frac{1}{5}$x ***have?*** (÷5)
What is the opposite of $\frac{1}{5}$x ***or ÷5?*** (5x)

Fill in the boxes for the arrows. Refer to the arrow picture when completing the following problems. (Answers are in boxes.)

$\frac{1}{5} \times 20 = \boxed{4}$ $\quad \frac{1}{5} \times \boxed{60} = 12$ $\quad \frac{1}{5} \times 2\,500 = \boxed{500}$

$\frac{1}{5} \times \boxed{35} = 7$ $\quad \frac{1}{5} \times 360 = \boxed{72}$ $\quad \frac{1}{5} \times \boxed{125} = 25$

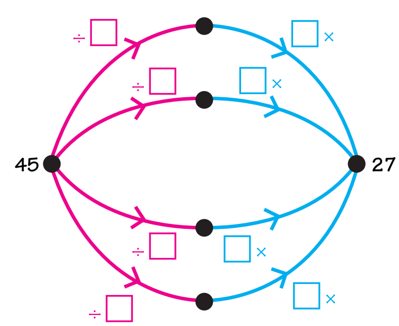
Change the arrows in the picture to $\frac{1}{3}$x or ÷3 and 3x. Then solve these problems. (Answers are in boxes.)

$\frac{1}{3} \times 27 = \boxed{9}$ $\quad \frac{1}{3} \times \boxed{36} = 12$ $\quad \frac{1}{3} \times 2\,400 = \boxed{800}$

$\frac{1}{3} \times 42 = \boxed{14}$ $\quad \frac{1}{3} \times \boxed{180} = 60$ $\quad \frac{1}{3} \times \boxed{1200} = 400$

Exercise 2

Draw this arrow picture on the board.

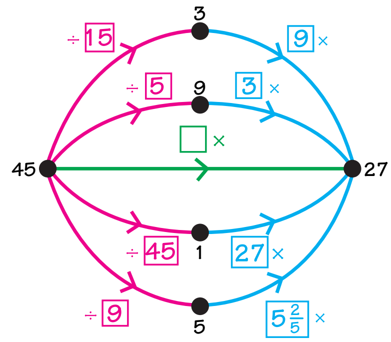


Invite students to find labels for pairs of arrows, red followed by blue. Record correct solutions in a table, such as the one shown on the next page.

S: ***If a red arrow is for ÷15, then the following blue arrow is for 9x; 45 ÷ 15 = 3, and 9 x 3 = 27.***

N15

After several solutions have been found, draw a green arrow from 45 to 27.



→	→
÷15	9×
÷5	3×
÷45	27×
÷10	6×
÷20	12×
÷9	$5\frac{2}{5}$×

T: ***What could this green arrow be for?***

S: ***⁹⁄₁₅x, because it is ÷15 followed by 9x.***

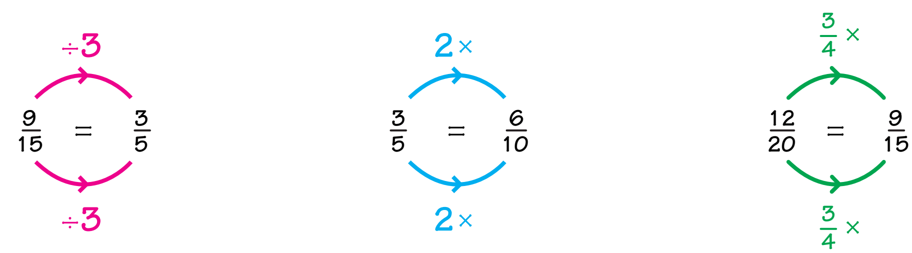
Make a list of possible names for the green arrow. Most will be suggested by the picture and the table, but students may give others. The table can be extended at the same time with more entries for the red and blue arrows.

$$\frac{9}{15}\times \quad \frac{3}{5}\times \quad \frac{27}{45}\times \quad \frac{6}{10}\times \quad \frac{12}{20}\times \quad \frac{18}{30}\times \quad \frac{5\frac{2}{5}}{9}\times$$

T: ***All of the fractions in these names for the green arrow are equivalent.***

$$\frac{9}{15} = \frac{3}{5} = \frac{27}{45} = \frac{6}{10} = \frac{12}{20} = \frac{18}{30} = \frac{5\frac{2}{5}}{9}$$

With the class, discuss how to check that fractions are equivalent. You may draw arrows between some pairs of equivalent fractions to illustrate a technique for checking equality. For example:

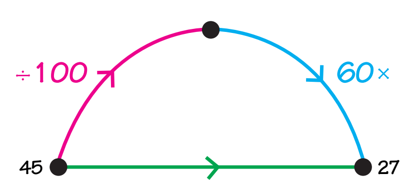


Present each of the following problems, asking students to provide the missing numerator or denominator. (Answers are in boxes.)

$$\frac{3}{5} = \frac{\boxed{30}}{50} \qquad \frac{3}{5} = \frac{\boxed{60}}{100} \qquad \frac{3}{5} = \frac{300}{\boxed{500}} \qquad \frac{3}{5} = \frac{1}{\boxed{\frac{5}{3}}}$$

Return to the table and the arrow picture to include ÷100 followed by 60x, if it is not already listed. Erase all of the other arrows leaving this picture.

T: ***We already know many names for the green arrow. What fraction times do these arrows suggest for the green arrow?*** (⁶⁰⁄₁₀₀x) ***Another name for the green arrow is "60% of"*** (read as "sixty percent of")***. "60% of" is a percent name for ÷100 followed by 60x, or for 60x followed by ÷100.***



Include ⁶⁰⁄₁₀₀x and "60% of" in the list of equivalent names for the green arrows.

Exercise 3

Begin this exercise by asking the class where they sometimes hear percents being used. The discussion might include the following:

- weather (percent chance of rain)
- sports (percent wins or success)
- sales (percent discounts or percent off original price)
- test results (percent correct)

Use these examples to give context to some of the percents mentioned in this exercise.

Draw this table on the board and invite students to complete it. Some students might envision arrow pictures similar to the one at the end of Exercise 2. For example:

- 50% of a number is the same as $\frac{1}{2}$x the number, because 50x followed by ÷100 is the same as ÷2 or $\frac{1}{2}$x.
- 20% = $\frac{20}{100}$ = $\frac{2}{10}$ = $\frac{1}{5}$, because 20% of is ÷100 followed by 20x.

Fraction	Percent
	50%
	25%
$\frac{3}{4}$	
	20%
$\frac{4}{5}$	

Other students may rely on their intuitive knowledge of percent and on the use of patterns; for example:

- 50% attendance means that one-half on the people are present.
- $\frac{3}{4}$ = 75%, since $\frac{1}{4}$ = 25% and 3 x 25 = 75.

Highlight this information from the table.

$$\frac{1}{2} = 50\% \quad \frac{1}{4} = 25\% \quad \frac{1}{5} = 20\%$$

T: ***What do you notice?***

S: ***The numerator in each fraction is 1.***

S: ***2* x *50, 4* x *25, and 5* x *20 all equal 100.***

Write the following problems on the board, and ask students to do the calculations using fraction equivalences for the percents or using patterns. (Answers are in boxes.)

$$50\% \text{ of } 60 = \boxed{30} \qquad 100\% \text{ of } 60 = \boxed{60}$$
$$25\% \text{ of } 60 = \boxed{15} \qquad 10\% \text{ of } 60 = \boxed{6}$$
$$75\% \text{ of } 60 = \boxed{45} \qquad 5\% \text{ of } 60 = \boxed{3}$$

Encourage students to explain each calculation in several ways. For example:

S: ***25% of 60 = 15, because 50% of 60 = 30 and ½* x *30 = 15.***

S: ***25% = ¼, so 25% of 60 = ¼* x *60 = 15.***

S: ***75% of 60 = 45, because 25% of 60 = 15 and 3* x *15 = 45.***

S: ***50% of 60 = 30 and 25% of 60 = 15, so 75% of 60 = 30 + 15 = 45.***

S: ***5% of 60 = 3, because 10% of 60 = 6 and ½* x *6 = 3.***

S: ***50% of 60 = 30 and 30 ÷ 10 = 3, so 5% of 60 = 3.***

S: ***"5% of" is ÷100 followed by 5x or $^{5}/_{100}$x.***
$^{5}/_{100}$x = $^{1}/_{20}$x = ÷20, so 5% of 60 = 60 ÷ 20 = 3.

Suggest that students use results already on the board to do these percent calculations. (Answers are in boxes.)

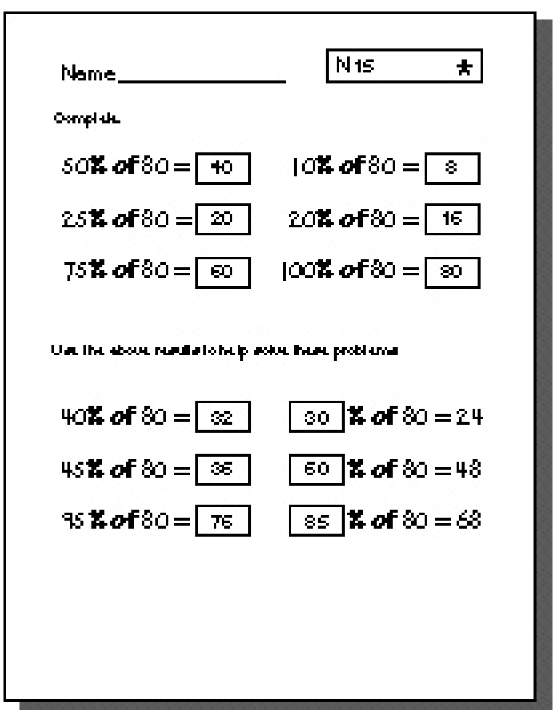
55% of 60 = $\boxed{33}$	95% of 60 = $\boxed{57}$
80% of 60 = $\boxed{48}$	1% of 60 = $\boxed{0.6}$
40 % of 60 = $\boxed{24}$	125 % of 60 = $\boxed{75}$
35% of 60 = $\boxed{21}$	8 % of 60 = $\boxed{4.8}$

Accept several explanations for each calculation; for example:

S: ***95% of 60 = 57 because 100% of 60 = 60 and 5% of 60 = 3, so 95% of 60 = 60 – 3 = 57.***

S: ***75% of 60 = 45 and 10% of 60 = 6, so 95% of 60 = 45 + (2 x 6) = 57.***

Worksheets N15* and ** are available for individual work.



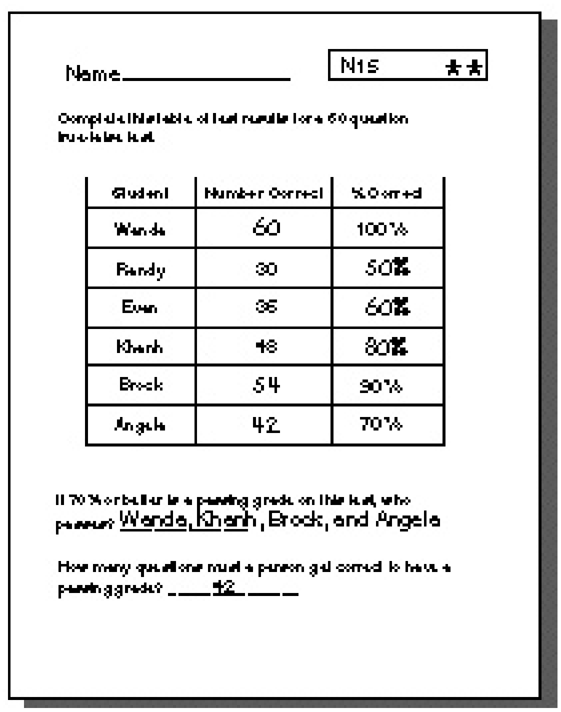
Name______________ N15 *

Complete.

50% of 80 = $\boxed{40}$	10% of 80 = $\boxed{8}$
25% of 80 = $\boxed{20}$	20% of 80 = $\boxed{16}$
75% of 80 = $\boxed{60}$	100% of 80 = $\boxed{80}$

Use the above results to help solve these problems.

40% of 80 = $\boxed{32}$	$\boxed{30}$ % of 80 = 24
45% of 80 = $\boxed{36}$	$\boxed{60}$ % of 80 = 48
95% of 80 = $\boxed{76}$	$\boxed{85}$ % of 80 = 68



Name______________ N15 **

Complete this table of test results for a 60 question true-false test.

Student	Number Correct	% Correct
Wanda	60	100%
Randy	30	50%
Evan	36	60%
Khanh	48	80%
Brock	54	90%
Angela	42	70%

If 70% or better is a passing grade on this test, who passed? Wanda, Khanh, Brock, and Angela

How many questions must a person get correct to have a passing grade? 42

N16 DIVISIBILITY BY 9 #2

Capsule Lesson Summary

Observe that multiples of 9 are also multiples of 3. For 356 = □ + △, find the least possible whole number to put in the triangle when the number in the box must be some multiple of 9. Relate this activity to the arrow pictures in Lesson N14, and find a test for deciding whether or not a number is a multiple of 9 or a multiple of 3. A number is a multiple of 9 (3) if the sum of its digits is a multiple of 9 (3). Provide missing digits in numbers to insure their divisibility by 9.

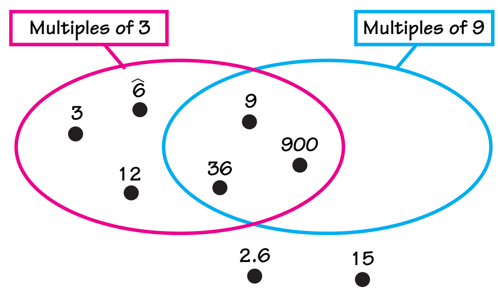
Materials

Teacher		Student	
	• Colored chalk		• Paper • Colored pencils, pens, or crayons • Worksheets N16* and **

Description of Lesson

Exercise 1

Draw two overlapping strings on the board, one for multiples of 3 and one for multiples of 9. Ask students for numbers belonging in each region of the picture. Your picture might look similar to this one.



T (pointing to the empty region)**:** ***Are there numbers that belong in this region?***

S: ***No, numbers in the blue string must also be in the red string since all multiples of 9 are also multiples of 3.***

T: ***Why?***

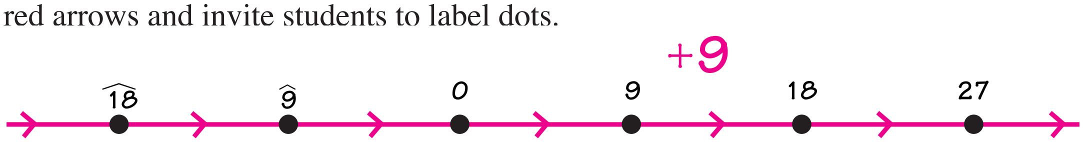
Let students explain. You may wish to illustrate the situation as follows.

Draw an arrow road on the board.

T: ***If we want this arrow road to include all multiples of 9 and no other numbers, what could the red arrows be for?***

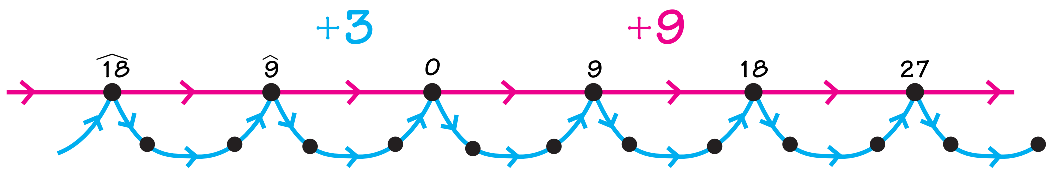
S: ***+9.***

Key the red arrows and invite students to label dots.



T: ***How can we use this arrow road to see that all multiples of 9 are also multiples of 3?***

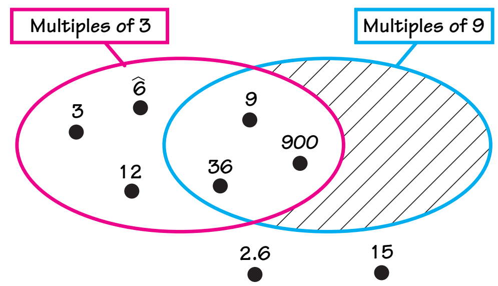
S: ***Draw a +3 arrow road going through 0 to include all the multiples of 3. Since 3 x 3 = 9, every third +3 arrow meets a multiple of 9. Therefore, every multiple of 9 is also a multiple of 3.***



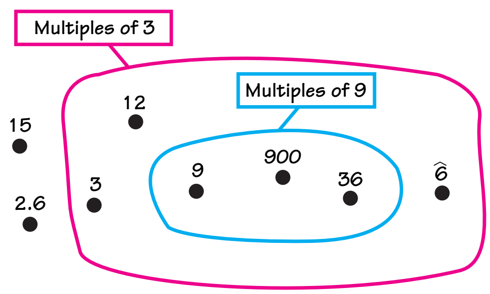
T: ***How can we show an empty region in the string picture?***

S: ***Hatch that region.***

T: ***There is another way of drawing a string picture to show that all multiples of 9 are also multiples of 3. Instead of hatching a region, let's try moving the strings.***



Lead to the idea of drawing the string for multiples of 9 completely inside the string for multiples of 3. Invite students to put the numbers from the first string picture into this one. For example:



Exercise 2

Note: This exercise will examine number sentences such as 10 = □ + △ in which some multiple of 9 goes in the box and a whole number goes in the triangle. It may be necessary for you to remind students frequently that the number in the box is always some multiple of 9.

Write this open sentence on the board.

10 = □ + △

T: ***Some multiple of 9 goes in the box and some other whole number goes in the triangle; 10 is equal to some multiple of 9 plus some whole number. What is the least possible whole number that could be in the triangle?***

S: ***1, because 10 = 9 + 1.***

T: ***Yes, 10 is 1 more than a multiple of 9.***

Consider another open sentence,

10 = $\boxed{9}$ + △1

40 = □ + △

T: ***40 is equal to some multiple of 9 plus some whole number. What is the least whole number that could be in the triangle?***

S: ***4, because 40 = 36 + 4.***

T: ***Yes. Could you use the first number sentence to solve this problem?***

S: ***Multiply all of the numbers in the first equation by 4. Four times a multiple of 9 plus 1 is some multiple of 9 plus 4.***

Include this information on the board.

$$10 = \boxed{9} + \triangle 1$$
$$40 = \boxed{36} + \triangle 4$$

4×

Continue this activity by asking for the least whole number that could be in the triangle for 100; 1 000; 70; 700; 300; 50; 6; and 356. You may let students first work individually on the problems; then after a few minutes, check the solutions collectively.

$$100 = \boxed{99} + \triangle 1$$
$$1\,000 = \boxed{999} + \triangle 1$$
$$70 = \boxed{63} + \triangle 7$$
$$700 = \boxed{693} + \triangle 7$$
$$300 = \boxed{297} + \triangle 3$$
$$50 = \boxed{45} + \triangle 5$$
$$6 = \boxed{0} + \triangle 6$$
$$356 = \boxed{351} + \triangle 5$$

Some students might answer that 356 is 14 more than a multiple of 9. For example:

S: ***356 = 300 + 50 + 6, so 356 is a multiple of 9 plus 3 + 5 + 6 or a multiple of 9 plus 14.***

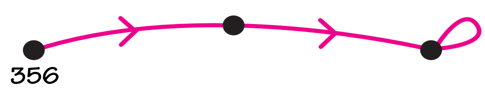
T: ***It is good that you used the previous work to help you, but are you sure that 14 is the least possible number that could be in the triangle?***

S: ***14 = 9 + 5, so 356 is a multiple of 9 plus 5.***

T: ***Yes, 5 is the least whole number that could be in the triangle.***

Draw this arrow picture on the board near the last number sentence.

$$356 = \boxed{351} + \triangle 5$$



T: ***Do you remember the rule where we drew an arrow from a whole number to the sum of its digits? We can draw an arrow from 356 to what number?***

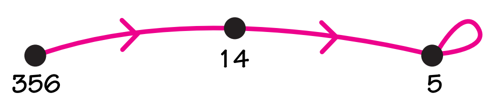
S: ***14, because 3 + 5 + 6 = 14.***

Label the dot for 14.

T: ***And we can draw an arrow from 14 to what number?***

S: ***5; 1 + 4 = 5.***

$$356 = \boxed{351} + \triangle 5$$



T: ***Do you see any connection between the arrow picture and the number sentence about 356?***

S: ***356 is 5 more than a multiple of 9 just like every number that would be in this piece of the arrow picture.***

T: ***We found there were ten pieces of the arrow picture for this rule: one piece with 0, one piece with 1, and so on to one piece with 9. What kind of numbers were in the piece with 9?***

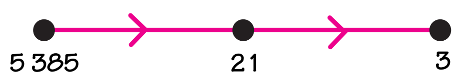
S: ***Multiples of 9.***

T: ***Is 5 385 a multiple of 9?***

S: ***No; 5 + 3 + 8 + 5 = 21, which is not a multiple of 9.***

S: ***No; 5 + 3 + 8 + 5 = 21 and 2 + 1 = 3, which is not a multiple of 9.***

Draw an arrow road starting at 5 385 as the students explain how to check that it is not a multiple of 9. Write a box and triangle expression near the arrow road.

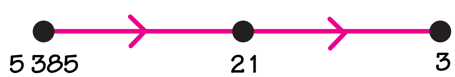


$$5\,385 = \square + \triangle$$

T: ***I want to find two numbers whose sum is 5 385. The number in the box must be a multiple of 9. What is the least whole number that could go in the triangle?***

S: ***3. 5 382 + 3 = 5 385; 5 382 is a multiple of 9 since 5 + 3 + 8 + 2 = 18 and 18 is a multiple of 9.***

Record the solution.

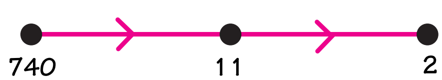


$$5\,385 = \boxed{5\,382} + 3$$

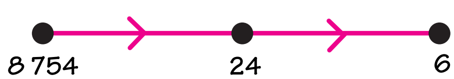
T: ***Is 5 385 a multiple of 3?***

S: ***Yes; each multiple of 9 is a multiple of 3, and adding 3 to a multiple of 3 gives another multiple of 3.***

Present these problems in a similar manner. 740 is not a multiple of 3, and 8 754 is a multiple of 3. Neither are multiples of 9.



$$740 = \boxed{738} + 2$$



$$8\,754 = \boxed{8\,748} + 6$$

T: ***If a number is a multiple of 3, what kind of number do we get when we add its digits?***

S: ***A multiple of 3.***

Note: Any number 1, 2, 4, 5, 7, or 8 more than a multiple of 9 is that much more than a multiple of 3; therefore, they are not multiples of 3. Any number 3, 6, or 9 more than a multiple of 9 is that much more than a multiple of 3; a multiple of 3 plus a multiple of 3 is also a multiple of 3.

Exercise 3

On the board, write the following numbers, each with a missing digit. Ask students to copy them on a piece of paper and to insert the missing digits to produce multiples of 9. (Answers are in boxes.) After a short while, invite students to present and explain their solutions.

$4\boxed{2}12$ $\qquad$ $7\boxed{0}92$ or $7\boxed{9}92$

$7\,777\,3\boxed{4}1$ $\qquad$ $87\boxed{8}\,769$

Erase the digits in the boxes and restate the problem to produce multiples of 3 rather than multiples of 9.

T: ***Now fill in the boxes so that each number is a multiple of 3.***

Allow a few minutes for individual work before checking the solutions collectively. Students should recognize that the solutions for multiples of 9 are also multiples of 3, but encourage them to find other solutions as well.

4 $\boxed{2}$ 12 or 4 $\boxed{5}$ 12 or 4 $\boxed{8}$ 12

7 $\boxed{0}$ 92 or 7 $\boxed{3}$ 92 or 7 $\boxed{6}$ 92 or 7 $\boxed{9}$ 92

7 777 3 $\boxed{4}$ 1 or 7 777 3 $\boxed{1}$ 1 or 7 777 3 $\boxed{7}$ 1

87 $\boxed{8}$ 769 or 87 $\boxed{5}$ 769 or 87 $\boxed{2}$ 769

165 $\square$ 7 $\triangle$ 7

Repeat the exercise with a number having two missing digits.

Invite students to list all of the possible solutions on the board. Since 1 + 6 + 5 + 7 + 7 = 26, the two inserted digits must have a sum of 1 (26 + 1 = 27) or 10 (26 + 10 = 36).

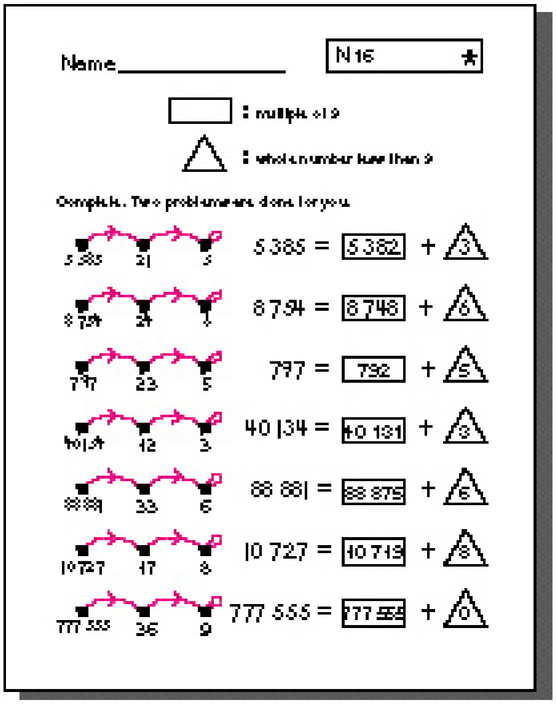
1650 717	1651 797	1653 777
1651 707	1659 717	1657 737
1655 757	1652 787	1654 767
	1658 727	1656 747

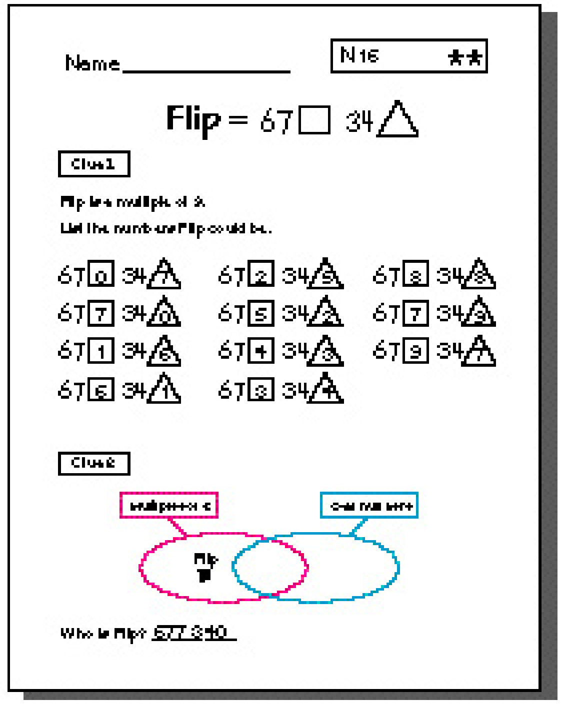
T: ***In this lesson we only considered tests (add digits) for multiples of 9 and 3. Does a similar test work for multiples of 8?***

S: ***No; 48 is a multiple of 8, but 4 + 8 is not a multiple of 8.***

S: ***No, 2 + 6 = 8, but 26 is not a multiple of 8.***

Worksheets N16* and ** are available for individual work.


Name
N16 ★
5 385 = 5 382 + 3
8 754 = 8 748 + 6
797 = 792 + 5
40 134 = 40 131 + 3
88 881 = 88 875 + 6
10 727 = 10 719 + 8
777 555 = 777 555 + 0


Name
N16 ★★
Flip = 67□ 34△
Clue 1
Clue 2
Flip

Capsule Lesson Summary

Construct a Cartesian graph model for whole number multiplication. Extend the model to include multiplication by integers and rational numbers.

Materials

Teacher
- Colored chalk
- Coordinate grid
- Blackline N17

Student
- Coordinate grid sheets
- Colored pencils, pens, or crayons

Advance Preparation: Use Blackline N17 to prepare a coordinate grid for display, or prepare a grid board with this coordinate grid. Make copies of coordinate grid sheets for students.

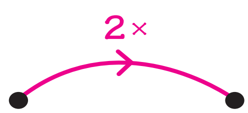
Description of Lesson

Exercise 1

Display a coordinate grid with horizontal and vertical axes extending to about 10 and $\widehat{10}$.

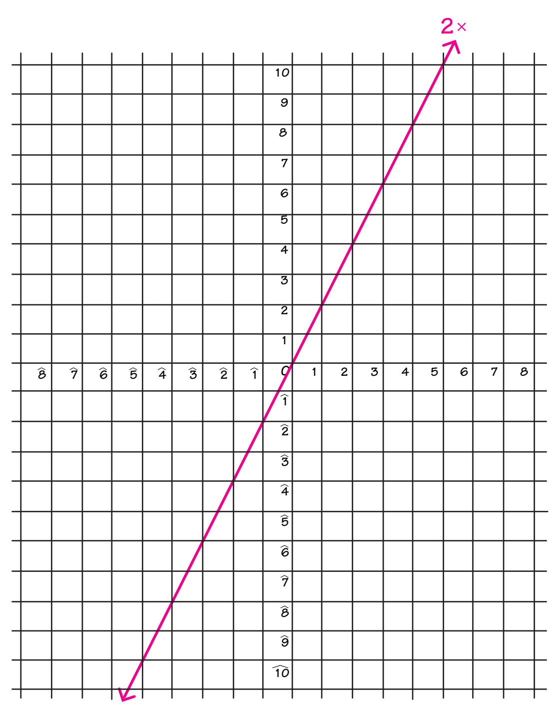
T: ***In an earlier lesson we drew Cartesian graphs for different relations. Let's make a Cartesian graph for the 2x relation.***

Draw a 2x arrow on the board, and ask students to suggest some starting and ending numbers. Record these numbers in ordered pairs. For example:



(2, 4) (0.5, 1) (1, 2)
(3, 6) (5, 10) (10, 20)
($\widehat{1}$, $\widehat{2}$) ($\widehat{2.5}$, $\widehat{5}$) ($\widehat{5}$, $\widehat{10}$)

Call on students to locate points for several ordered pairs. Then instruct students to complete the graph on their coordinate grid paper.



After several minutes of independent work, complete the picture at the board.

T: ***What do you notice about the 2x relation?***

S: ***Its graph is a straight line.***

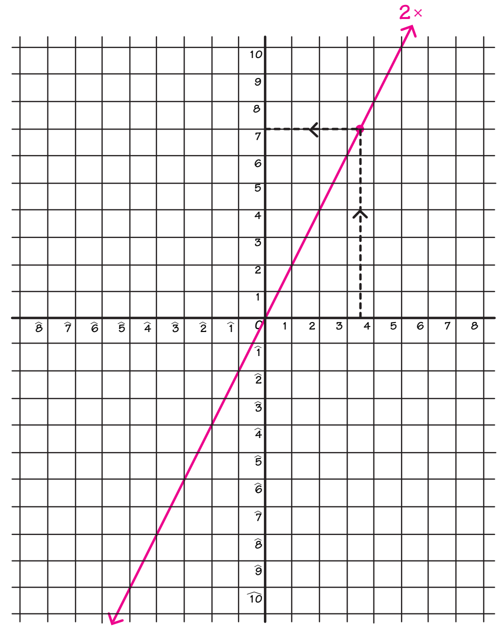
S: ***The line goes through (0, 0).***

T: ***Why is (0, 0) on the 2x line?***

S: ***2 x 0 = 0.***

T: ***How could we use the 2x line to find 2 x 3.5?***

S (tracing the horizontal axis)**:** ***Find 3.5 on this line; then go straight up until you hit the 2x line; then go straight across to see what is the second number in the ordered pair for that point on the 2x line.***

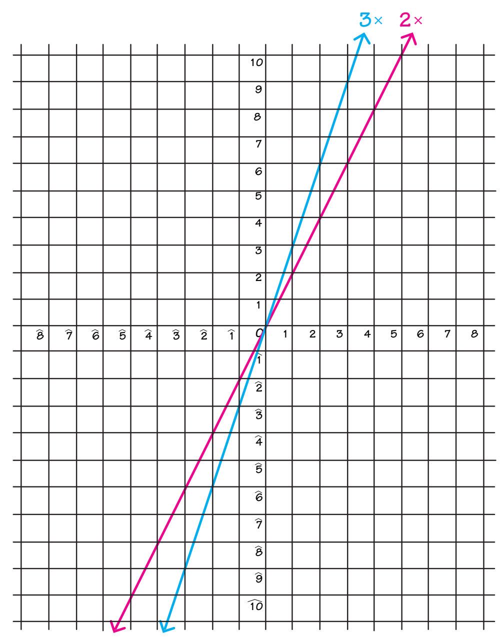


Ask other students to show how to use the graph to find 2 x 2.5, 2 x $\widehat{3}$, 2 x 0.5, and 2 x $\widehat{4.5}$.

T: ***Next, let's graph the 3x relation on this same copy of the grid.***

Let students work independently or with partners. You may recommend that they use a second color for this graph.

After a while collectively complete the picture at the board.



T: ***What do you notice about the 3x relation?***

S: ***Its graph is a line going through (0, 0). The 3x line is steeper than the 2x line.***

T: ***Let's graph the 4x relation. How do you think its graph will look?***

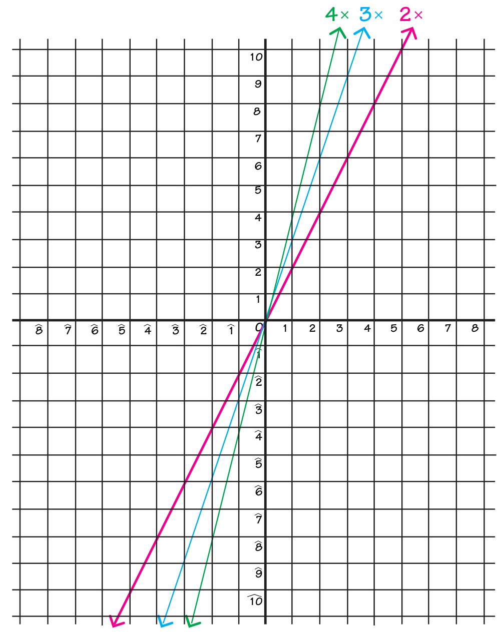
S: ***It will be a straight line a little steeper than the 3x line.***

Invite several students to trace where they think a 4x line would be.

T: ***Do we need to draw dots for many ordered pairs from the 4x relation, or can you think of a quick way to draw the graph?***

S: ***The 4x line will go through (0, 0) and through (2, 8). One point other than (0, 0) is enough to determine how to draw the 4x line.***

Let students draw the 4x line on their papers while you do so on the board.



T: ***What do you notice about all of these lines?***

S: ***They all go through (0, 0).***

S: ***They get steeper as we multiply by greater numbers.***

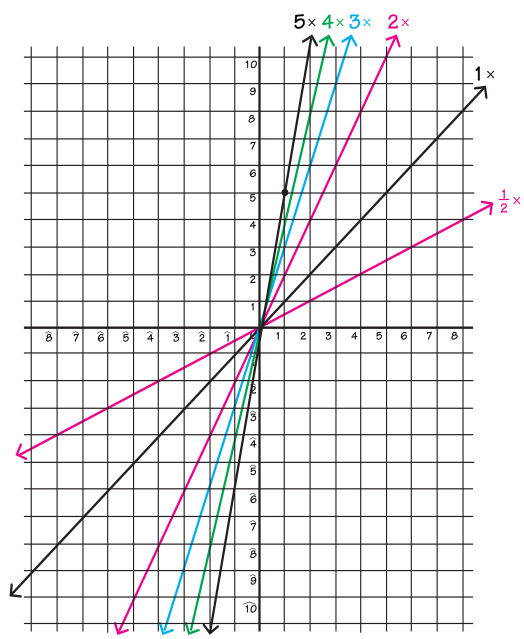
S: ***They slant up from left to right.***

T: ***How could we tell these lines apart if they were not labeled?***

S: ***Look at the (vertical) line through 1. It hits the 2x line at (1, 2), the 3x line at (1, 3), and the 4x line at (1, 4).***

Verify this observation. You may need to make it yourself.

Check students' understanding of this idea by drawing a 1x line, a 5x line, and a ½x line and asking students to identify them.



T: ***Where is the 0x line?***

A student should trace the horizontal axis at the board.

S: ***It is this horizontal line.***

T: ***Let's graph the $\widehat{3}$x relation. How could we do this without first plotting points for a lot of ordered pairs in the $\widehat{3}$x relation?***

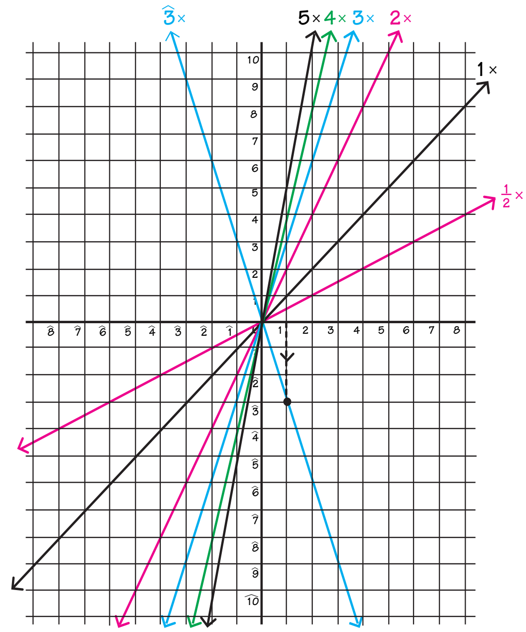
Allow a few minutes for students to consider the $\widehat{3}$x relation and how its graph might look.

S: ***It's graph will be a straight line through (0, 0) like the other lines. $\widehat{3} \times 0 = 0$.***

S: ***We can find another point on the $\widehat{3}$x line by looking at the vertical line through 1. This line should hit the $\widehat{3}$x line at (1, $\widehat{3}$); $\widehat{3} \times 1 = \widehat{3}$.***

N17

Invite students to first trace the $\widehat{3}x$ line; then draw it in the picture at the board. Instruct students to add the $\widehat{3}x$ line to their graphs.

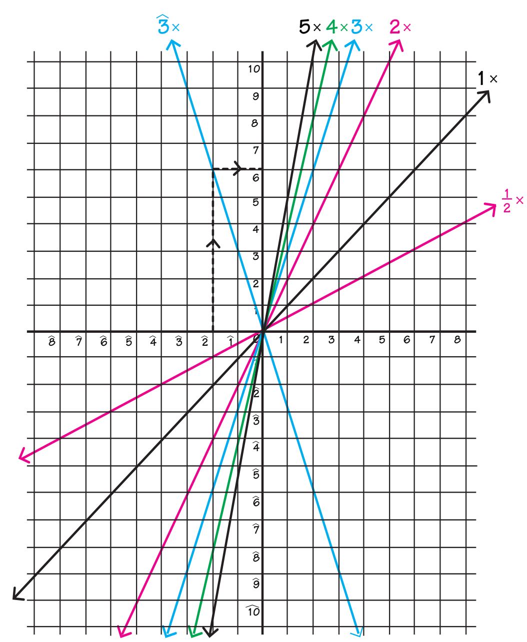


Write these problems on the board. (Answers are in boxes.)

T: ***You can use the $\widehat{3}x$ line to help with all these $\widehat{3}x$ calculations.***

$\widehat{3} \times 2 = \boxed{\widehat{6}}$	$\widehat{3} \times 0 = \boxed{0}$	$\widehat{3} \times \widehat{2} = \boxed{6}$
$\widehat{3} \times 1 = \boxed{\widehat{3}}$	$\widehat{3} \times \widehat{1} = \boxed{3}$	$\widehat{3} \times \widehat{3} = \boxed{9}$

Let students work independently for a few minutes. Ask students to demonstrate at the board how the $\widehat{3}x$ line can be used for each calculation. This illustration shows using the $\widehat{3}x$ line for $\widehat{3} \times \widehat{2} = 6$.



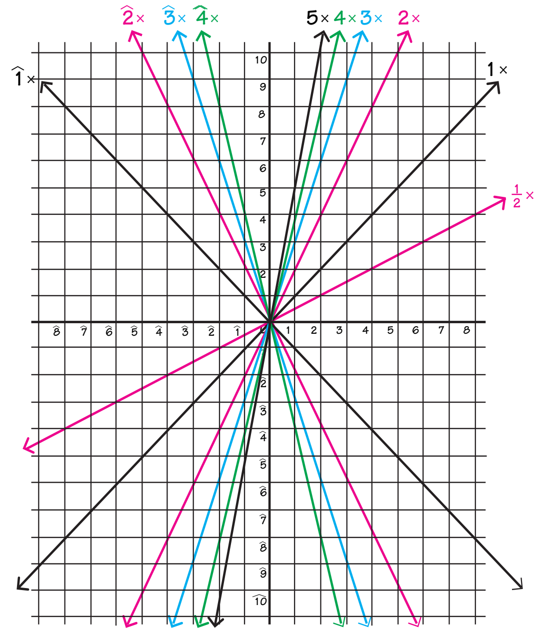
T: ***What do you notice about the graph of $\widehat{3}x$?***

S: ***It is a straight line through (0, 0).***

S: ***It's a line with the same steepness as the 3x line but slanted in the opposite direction.***

Continue this activity by asking students to draw the $\widehat{2}x$ line, the $\widehat{4}x$ line, and the $\widehat{1}x$ line.

While students are completing this work, occasionally ask them to use their graphs to find various products of integers; for example, $\widehat{4} \times 2$, $\widehat{4} \times \widehat{1}$, $\widehat{2} \times \widehat{5}$, $3 \times \widehat{3}$, and so on.



N18 DIVISION ALGORITHM #1

Capsule Lesson Summary

Use patterns to do mental arithmetic problems involving multiplication of decimal numbers. Solve division problems in which a given amount of money is shared equally among some specified number of people. Present a standard algorithm for dividing a decimal by a whole number.

Materials

Teacher
- Colored chalk

Student
- Paper
- Worksheets N18* and **

Description of Lesson

Exercise 1

Write the following problems on the board, and ask students to copy them on their papers. Let students explain how to find the products, emphasizing the technique of starting with a known fact (for example, 9 x 3 = 27) and using patterns. To involve all students, you may ask them to write solutions on their papers and check several papers before inviting students to announce results. (Answers are in boxes.)

$9 \times 0.03 = \boxed{0.27}$ $\quad$ $6 \times 0.008 = \boxed{0.048}$

$15 \times 400 = \boxed{6\,000}$ $\quad$ $12 \times 6\,000 = \boxed{72\,000}$

$8 \times 0.7 = \boxed{5.6}$ $\quad$ $35 \times 0.02 = \boxed{0.7}$

S: *9 x 3 = 27, so 9 x 0.3 = 2.7 and 9 x 0.03 = 0.27.*

S: *15 x 4 = 60, so 15 x 40 = 600 and 15 x 400 = 6 000.*

Exercise 2

Begin this table on the board.

Amount of Money	Number of People	Amount of Money for Each Person	Amount of Money Left Over
$20.80	4		

T: *Four people are going to share $20.80 equally. How much money does each person receive and how much money is left over, if any?*

S: *Each person receives $5.00, since 4 x $5.00 is $20.00. $0.80 is left over.*

T: *Can the four people share $0.80 equally?*

S: *Yes, each person receives an additional $0.20.*

S: *So altogether each person receives $5.20.*

Add this information to your table. Continue the activity by giving the information in the first two columns of the table below and asking students to provide the numbers in the boxes. Feel free to adjust the level of difficulty to the numerical abilities of your students. You may prefer to ask students to copy and fill in the table on their papers. Then you can check several responses before inviting students to fill in the table on the board.

Amount of Money	Number of People	Amount of Money for Each Person	Amount of Money Left Over
\$20.80	4	\$5.20	0
\$110.75	10	$\boxed{\$11.07}$	$\boxed{\$0.05}$
\$36.80	6	$\boxed{\$6.13}$	$\boxed{\$0.02}$
\$67.26	8	$\boxed{\$8.40}$	$\boxed{\$0.06}$

Put \$4 809.68 and 15 in the first two columns of the table, respectively.

T: ***15 people are sharing equally \$4 809.68. To determine how much money each person receives, what calculation could we do?***

S: ***4 809.68 ÷ 15.***

Write the calculation on the board.

$$15\overline{)\,4\,809.68}$$

T: ***Does each person receive at least \$10? At least \$100? At least \$1 000?***

S: ***Between \$100 and \$1 000. 15 x 100 = 1 500 and 15 x 1000 = 15 000.***

T: ***How many hundreds of dollars could we give to each person?***

Do not insist on the optimal response, \$300. If a student suggests \$200, for example, record a student's answer in the division problem on the board as you work through the calculation.

T: ***If we give \$200 to each person, how much money is that?***

S: ***\$3 000 (15 x \$200).***

T: ***How much money remains to share?***

S: ***\$1 809.68 (4 809.68 – 3 000.00).***

S: ***Give \$100 more to each person since 15 x 100 = 1 500.***

$$\begin{array}{r|l} 15\overline{)\,4\,809.68} & \\ -3\,000.00 & 200 \\ \hline 1809.68 & \\ -1500.00 & 100 \\ \hline 309.68 & \end{array}$$

If someone suggests an amount such as \$110 or \$150 for each person, agree but point out that multiplying by 100 is much easier than multiplying by 110 or 150. Recommend considering multiples of 10 after giving \$300 to each person.

Proceed in a similar manner to complete the calculation. Stop occasionally during the process to ask how much money each person receives so far. Perhaps your class will solve the problem as shown here.

320.64 R = 0.08	
15) 4 809.68	
−3 000.00	200
1809.68	
−1500.00	100
309.68	
−300.00	20
9.68	
−7.50	0.50
2.18	
−1.50	0.10
0.68	
− 0.60	0.04
0.08	

T: ***How much money does each person receive?***

S: ***$320.64 (200 + 100 + 20 + 0.50 + 0.10 + 0.04) and there is $0.08 left over.***

Record the quotient and the remainder in the division problem and in the third and fourth columns, respectively, of the table.

Review and discuss this division method. Then write the following problems on the board, and ask students to do the calculations on their papers. (Answers are in boxes.)

$\boxed{26.48 \quad R = 0.02}$
6) 158.90

$\boxed{22.10 \quad R = 0}$
5) 110.50

$\boxed{120.04 \quad R = 0.04}$
8) 960.36

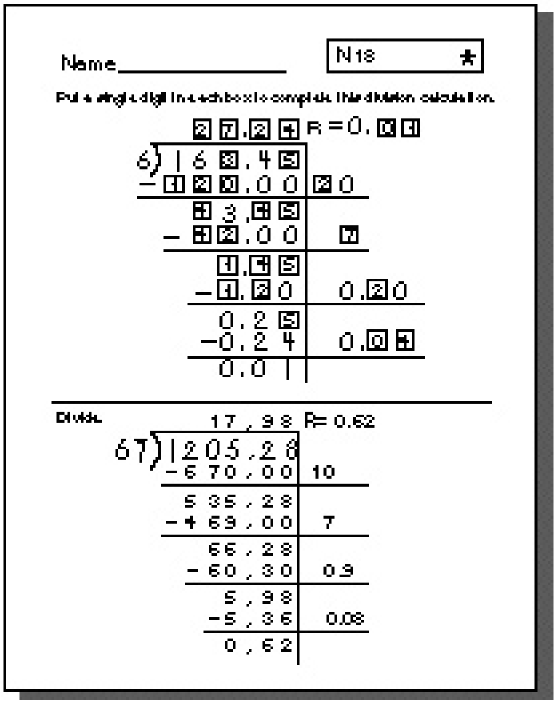
$\boxed{71.26 \quad R = 0.02}$
23) 1639.00

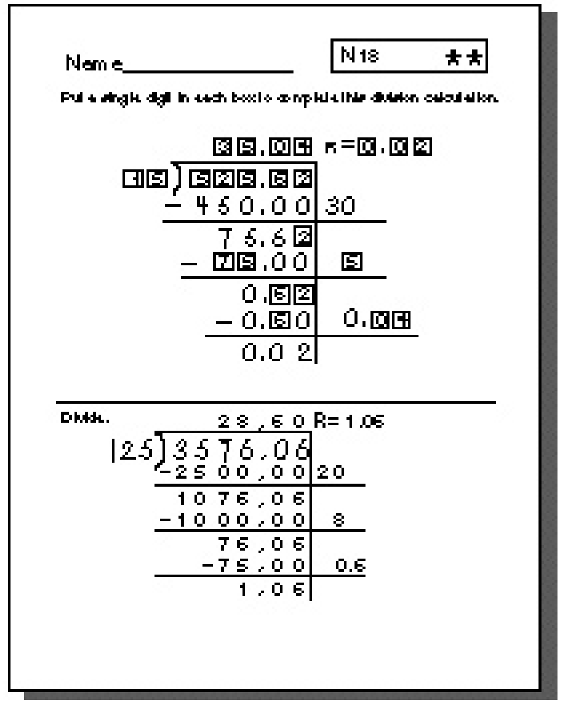
As you observe students' work, especially watch for students who either spend too much time finding "best" estimates or, at the other extreme, always use only 100, 10, 1, 0.1, or 0.01. When appropriate, suggest alternatives to these students. You may wish to solve one of the problems collectively at the board.

Worksheets N18* and ** are available for additional individual work.

Home Activity

This is a good time to send a letter to parents/guardians about a division algorithm. Blackline N18 has a sample letter.


Name
N18 ★
Put a single digit in each box to complete this division calculation.
Divide.
17 , 98 R= 0.62
67)1205.28
- 6 70 , 00 10
5 35 , 28
- 4 69 , 00 7
66 , 28
- 60 , 30 0.9
5 , 98
- 5 , 36 0.08
0 , 62


Name
N18 ★★
Put a single digit in each box to complete this division calculation.
Divide.
28 , 60 R= 1.06
125)3576.06
- 25 00 , 00 20
10 76 , 06
- 10 00 , 00 8
76 , 06
- 75 , 00 0.6
1 , 06

Capsule Lesson Summary

Review the notion of a Cartesian graph, and graph several relations including the "is less than" relation. Introduce an in-out convention to indicate points which do or do not belong to a graph.

Materials

Teacher
- Colored chalk
- Coordinate grid
- Blackline N19

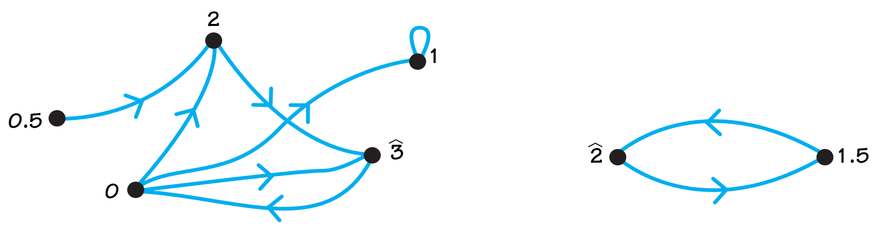
Student
- Paper
- Colored pencils, pens, or crayons
- Coordinate gridsheets

Advance Preparation: Use Blackline N19 to prepare a coordinate grid for display, or prepare a grid board with this coordinate grid. Make copies of coordinate grid sheets for students' use in Exercise 3.

Description of Lesson

Exercise 1

Display a coordinate grid and draw this arrow picture on the board.



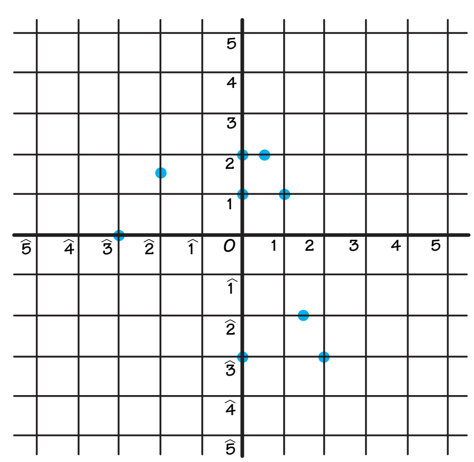
Trace an arrow, for example from 0.5 to 2, as you ask,

T: ***This arrow is for which ordered pair of numbers?***

S: ***(0.5, 2), because 0.5 is the starting number and 2 is the ending number.***

T: ***Where is the point on the grid for (0.5, 2)?***

Invite a student to explain how to locate the point.



Continue this activity, asking students to locate points for the eight arrows and the loop. Emphasize that the starting number is located along the horizontal axis and the ending number is located along the vertical axis. Upon completion of this activity, there should be nine dots on your coordinate grid. Mention that points on the coordinate grid make up the *Cartesian graph* of the relation in the arrow picture.

T: ***How can we tell by looking at its Cartesian graph that the relation has three arrows starting at 0?***

S: ***Look at the vertical line through 0; there are three points on it corresponding to the three arrows that start at 0.***

T: ***How can we tell by looking at its Cartesian graph that the relation has two arrows ending at $\widehat{3}$?***

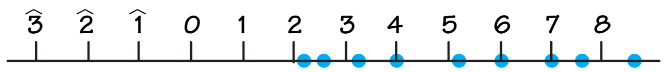
S: ***Look at the horizontal line through $\widehat{3}$; there are three points on it corresponding to the two arrows ending at $\widehat{3}$.***

Exercise 2

On the board, draw part of a number line, as shown below. Near the number line, indicate that Flip is a number greater than 2.

T: ***Let's draw blue dots on the number line for numbers that Flip could be.***

Invite several students to draw blue dots. After several dots are drawn, students should notice that all of the dots are to the right of 2. Your number line might look like this.



Flip > 2

T: ***What is the least number Flip could be? What is the greatest?***

Of course, there is no least nor greatest number that Flip could be but it may be the first time some students have given much thought to these questions. Lead a class discussion to observe that there is no least number nor greatest number. For example:

S: ***Flip could be 2.1.***

T: ***Can you think of a number less than 2.1 that Flip could be?***

S: ***2.01.***

T: ***Is this the least number Flip could be?***

S: ***No.***

T: ***Can you find a number less than 2.01?***

S: ***2.0000 … 1—with 125 zeroes!***

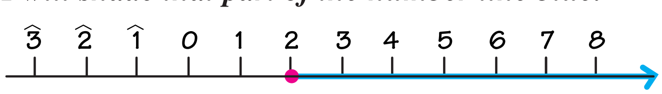
T: ***Is this the least?***

S: ***No, not even if there were 1 000 zeroes before 1.***

S: ***There is no least number that Flip could be.***

S: ***Flip could be numbers closer and closer to 2, but Flip cannot be 2.***

T: ***To show that Flip cannot be 2, I will draw a red dot at 2. Since Flip could be any number to the right of 2, I will shade that part of the number line blue.***



Flip > 2

T: ***Let's agree that blue shows numbers that Flip could be and, when we need to make the picture clear, red shows numbers that Flip could not be.***

Erase the red dot at 2 on the number line.

T: ***Look again at the number line. Suppose I draw a blue dot at 2. How would this change the information (number sentence) about Flip?***

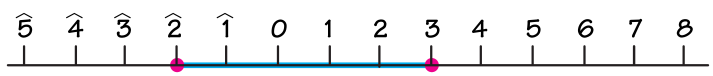
Allow a minute for students to consider the question.

S: ***Flip could be any number greater than or equal to 2. (Flip ≥ 2.)***

Erase and redraw the number line. Write a new number sentence about Flip, $\widehat{2}$ < Flip < 3.

T: ***Using our red-blue convention, let's indicate which numbers Flip could be.***

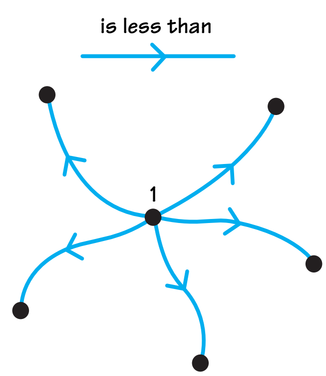
Suggest that students do this on their papers first, and then invite a student to draw the picture on the board.



$\widehat{2}$ < Flip < 3

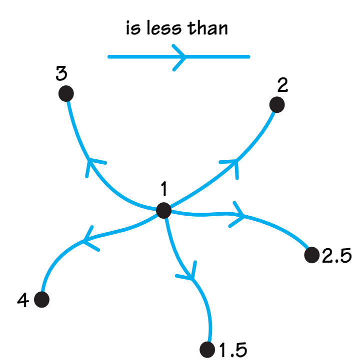
Exercise 3

Draw this arrow picture on the board.



T: ***Now blue arrows are for the relation "is less than." How could we label the dots?***

Invite some students to label the dots. Of course, there are many possible labels. Encourage choice of labels so that the arrows are for ordered pairs that can be plotted on your coordinate grid.

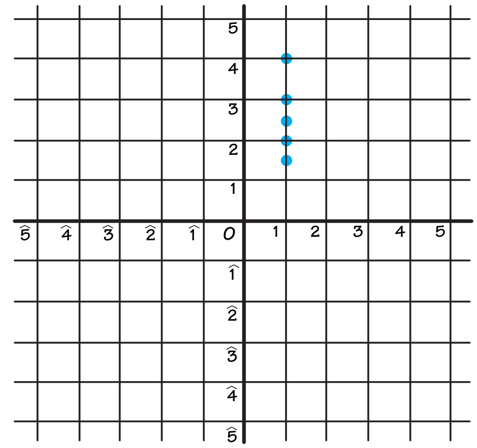


Refer students to their coordinate grid sheets.

T: ***On your grid, draw dots at points for these arrows.***

N19

Allow a few minutes for students to work independently. Then, ask some students to draw the dots on the demonstration grid.



T: ***Could we draw more arrows starting at 1?***

S: ***Yes, infinitely many of them.***

T: ***What can we say about the ending numbers of these arrows?***

S: ***Each ending number will be greater than 1.***

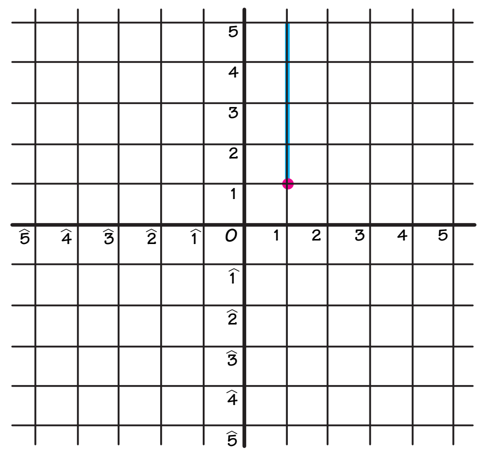
T: ***Is there a least possible number?***

S: ***No, it's like the problem about Flip. The numbers are greater than 1 but can be very close to 1.***

T: ***How can we show this situation on the grid?***

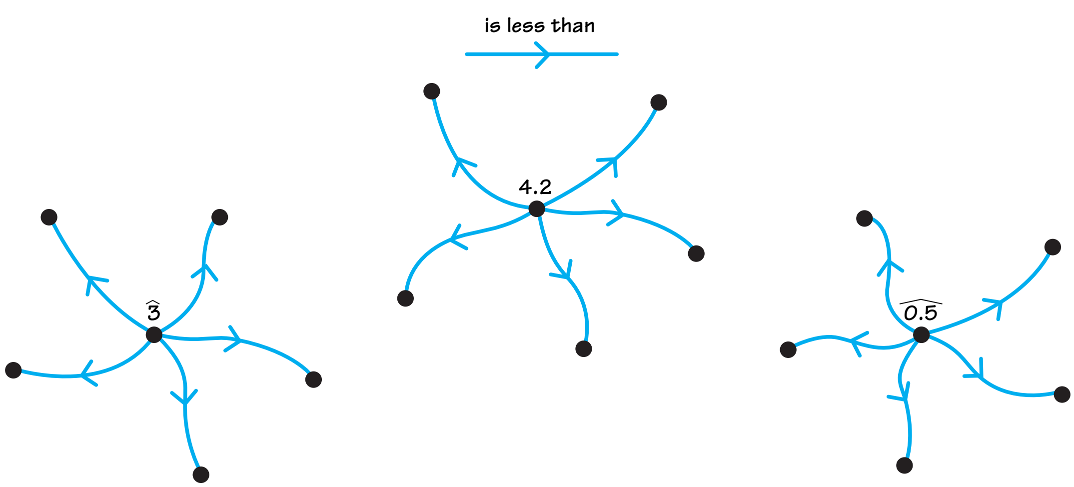
S: ***It's like the problem of showing the numbers Flip could be on the number line. Imagine that the vertical line at 1 is a number line. Color in blue to show numbers greater than 1.***

S: ***Draw a red dot at (1, 1) to show that 1 is not less than 1.***

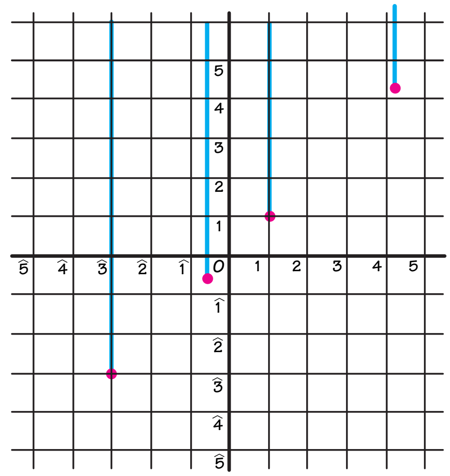


Invite students to draw the picture.

Add more pieces to your arrow picture, such as shown here.



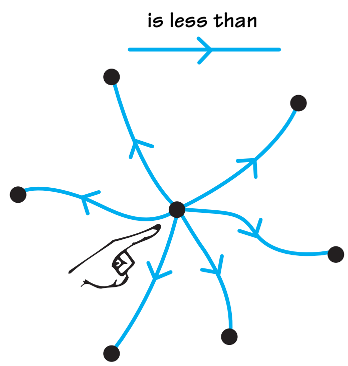
In a similar manner, let the students analyze the situation for all of the arrows starting at $\widehat{3}$, at 4.2, and at $\widehat{0.5}$. When most students have finished, ask some students to put their drawings on the board.



T: ***Look at the grid picture. Do you notice anything interesting?***

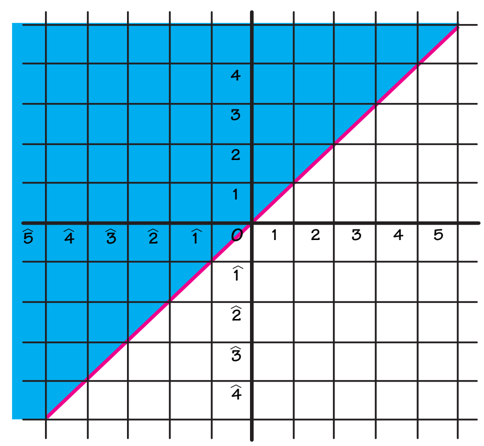
Perhaps students will observe that the red dots lie on a line through (0, 0) and this line includes all ordered pairs of the form (n, n); that is, the 1x line. They may also comment that the entire part of a line (open half-line) above each red dot is colored blue. Accept any reasonable comments.

Draw a new piece in the arrow picture and point to the center dot.



T: ***Any number could be here. On your grid, find other points that belong to the graph of the "is less than" relation by thinking about lots of possibilities for numbers at this dot. Try to find all the points that belong to the Cartesian graph of the "is less than" relation.***

Provide time for students to work independently or with partners on this problem. When several students have the full solution, invite one student to draw the picture on the board.



S: ***The blue lines run together like this.***

T (tracing the blue region)**:** ***This region is the Cartesian graph of the "is less than" relation. Where is a point that does not belong to the "is less than" relation?***

Ask a student to indicate such a point, for example, (1, $\widehat{2}$).

T: ***What is the ordered pair of numbers for this point?***

S: ***(1, $\widehat{2}$).***

T: ***What do you notice about these numbers?***

S: ***1 is greater than $\widehat{2}$.***

Repeat this activity several more times. The class should conclude that for ordered pairs of numbers which do not belong to the "is less than" relation, the first number is equal to or greater than the second number.