Geometry and Measurement

GEOMETRY AND MEASUREMENT TABLE OF CONTENTS

GEOMETRY AND MEASUREMENT INTRODUCTION

Geometry is useful and interesting, an integral part of mathematics. Geometric thinking permeates mathematics and relates to most of the applications of mathematics. Geometric concepts provide a vehicle for exploring the rich connections between arithmetic concepts (such as numbers and calculations) and physical concepts (such as length, area, symmetry, and shape). Even as children experience arithmetic notions in their daily lives, they encounter geometric notions. Common opinion recognizes that each child should begin developing a number sense early. There is no sound argument to postpone the development of geometric sense. These are compelling reasons why the systematic development of geometric thinking should be a substantial component of an elementary school curriculum.

A benefit of early development of geometric thinking is that it provides opportunities for all learners. Children develop concepts at individual rates. We are careful to allow students a great deal of time to explore quantitative relationships among physical objects before studying arithmetic. In the same spirit, students should be allowed to explore geometric relationships through free play and guided experiences. Before studying geometric properties of lines and shapes, they should have free play with models and with drawing instruments, using both to create new figures and patterns. Before considering parallelism and perpendicularity as relations on geometric lines, they should have free play in creating patterns with parallel and perpendicular lines. Before learning formulas for finding measurements, they should have informal experiences to establish and sharpen their intuitive notions. These early experiences are aesthetically rewarding, and just as stimulating and suggestive of underlying concepts as early experiences leading to arithmetic. Such experiences are a crucial first step in the development of a geometric sense.

A variety of constructions forms the basis for the geometry of *CSMP Mathematics for the Intermediate Grades*. Students use tools to explore geometric concepts, directly discovering their properties and interrelationships. The tools include a straightedge, a compass, mirrors, angle templates, and a translator for drawing parallel lines. The constructions foster insights into the properties of shapes, independent of the measurement of those properties. Only after students are familiar with the shapes do they begin to use rulers and protractors to measure lengths and angles. In this sequential development of geometric ideas, measurement is viewed as the intersection of geometric concepts and arithmetic concepts.

The focus of this strand is experience. The measurement activities guide students to refine their ability to measure accurately lengths of line segments and areas of polygons, as well as to begin a study of angle measurement. Prior to measuring angles, students explore angle concepts in geometric constructions making use of angle templates. As a natural consequence of their involvement in these activities, students develop their knowledge and skills in geometry. The effects of this informal approach should be judged by the long-term effects on students' knowledge, confidence, intuition, and interest in the world of geometry and measurement.

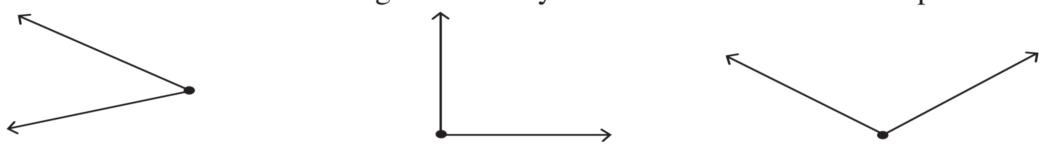
Geometry of Angles

The geometry of *CSMP Mathematics for the Intermediate Grades* is based on a variety of constructions. Students use tools to explore geometric concepts, experiencing directly their properties and interrelationships. This semester angles and their measurement are the central ideas of a series of geometry lessons. In keeping with an established pattern, our approach is to provide tools with which students study geometric constructions and concepts.

GEOMETRY AND MEASUREMENT INTRODUCTION

Many treatments of angles simultaneously introduce the idea of angle and the measurement of an angle with a protractor. With such an approach, students often identify angle with its measure. We take the position that the two should be realized separately, just as we learn about line segments before measuring them with a ruler. In *IG-VI* nine lessons are devoted to providing students an opportunity to explore angles and some of their properties through constructions before a protractor and angle measurement are introduced.

The lessons introduce the idea of an angle as two rays which have a common endpoint.



Students observe angles implicit in physical objects and notice the prevalence of right angles, defined simply through a paper-folding construction. In the process, an informal comparison of angles with right angles leads naturally to the idea of acute and obtuse angles.

Using only a right angle template to determine acute and obtuse angles, students attempt to construct polygons with various numbers of sides and acute angles. Through this exercise, they focus on angles as controlling elements in the structure of a polygon, whereas their previous experience is likely to have emphasized the line segments which appear as sides.

Given two supplementary pairs of angles, students attempt to construct quadrilaterals under the condition that each angle of the quadrilateral is congruent to one of the four given angles, and vice versa. They discover several solutions and note that two pairs (the supplementary pairs) together can be used to construct a quadrilateral, but no other pairing will work. This informal introduction to supplementary angles is carried through three more lessons. In the lessons students use eight angle templates, four supplementary pairs, to perform a series of constructions designed to emphasize supplementary pairs. Through the constructions, students develop an appreciation of some basic facts of Euclidean geometry. For example, three angles of a quadrilateral determine the fourth.

Triangle constructions are more tightly controlled since they admit fewer choices. Working with seven angle templates, students discover all triangles with each angle congruent to one of the seven. While performing the constructions, students are alert for patterns among the successes and the failures and discover a relation similar to the idea of supplementary pairs of angles. Thus, they discover the classical result that, in terms of measurement, the sum of the measures of the angles of a triangle is 180°.

The use of templates in this series of constructions provides an opportunity to develop an understanding of angles without the necessity to develop skill with a protractor. Experience in constructing polygons and a growing sense of angle relations, which are independent of angle measures, are preliminary to learning how to use a protractor. Its use is foreshadowed by the use of the angle templates, and the idea of angle measurement is a natural extension of the informal comparison of angles in the earlier lessons. From experience with the templates comes an appreciation of the ease, efficiency, and versatility of the protractor as a tool for constructing angles. The lessons on angles conclude with a reprise of some the earlier problems recouched in terms of measurement.

Lessons: G1, 2, 3, 4, 5, 6, 7, 8, 9, 11, and 12

Measurement

GEOMETRY AND MEASUREMENT INTRODUCTION

Consensus calls for measurement activities in the elementary curriculum, but with no agreement on the form or scope of these activities. Rather than stress mastery of formulas for perimeter, area, and volume, or for comparison of standard units, the lessons of this strand provide open-ended experiences within rich problem-solving situations. Measurement becomes a means for investigating problems and developing concepts, rather than an end in itself. Direct experiences with the concepts and tools is central. The carefully designed problems do lead to insights concerning accurate measurement and involving area of various shapes, especially circles. The emphasis is on the development of ideas and understanding rather than on the memorization of rules. The measurement activities become the means for investigating problems and developing concepts rather than an end in themselves.

Two lessons this semester introduce measure for angles using a protractor. These lessons are included in the above discussion on Geometry of Angles. A third lesson uses the fact that area is additive (i.e., the sum of the areas of the parts in a decomposition of a shape is the area of the shape) to informally investigate the Pythagorean Theorem for right triangles.

The lessons in this strand are supplemented regularly with pages in the *Collage of Problems* workbooks. In particular, the workbooks contain practice with linear measure and with finding area and perimeter of polygons.

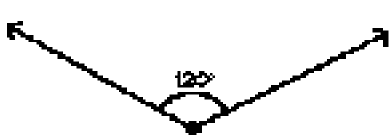
Lessons: G10, 11, and 12

Breaking a Stick

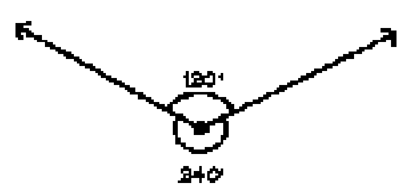
Four lessons this semester study a problem in geometry and probability: If a stick is randomly broken at two points, what is the probability that the three pieces will form a triangle? In preparation for this problem, students examine some triangle constructions. Constructing triangles with two given line segments leads to a recognition that the relative lengths of the line segments control the result of one or two essentially different solutions. In this way students discover a special case of the Triangle Inequality in a context in which the alternatives are both productive, rather than in the general case in which the alternative solutions are one or none. This realization is followed by the problem of constructing a triangle using three given segments, which leads to a formulation of the Triangle Inequality itself.

In studying the breaking-a-stick problem, students have an opportunity to relate their work in several strands of the curriculum, numerical and probabilistic as well as geometrical.

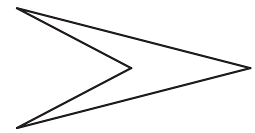
It is important to understand that only one measure between 0˚ and 180˚ is associated with an angle in the lessons on angles. This may be a difficult viewpoint for those who are accustomed to always associating two measures with an angle. For example, given this angle we assign one measure, namely 120°.



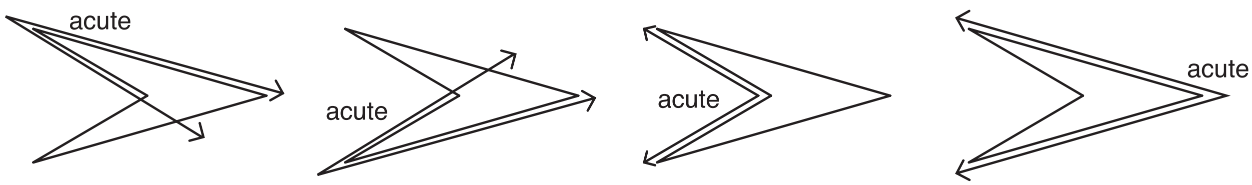
rather than



Now suppose we ask, What are the angles determined by this polygon?



To answer, we recall that an angle consists of two rays with a common point and are emphatic in the identification which results from taking the four corners in turn as vertices (see the illustration).

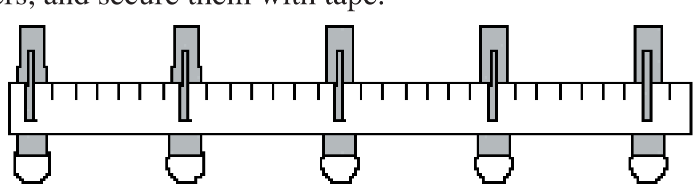


However, some might say that one angle is obtuse. This apprehension results from the dual measurement system. To assist in setting aside this perception, cover the extraneous sides of the polygon momentarily and trace the pertinent sides. Then examine the angle in isolation. The angles determined by a polygon can be viewed independently of the polygon. If necessary, extract them from the polygon for inspection.

Note on Grids

Several lessons call for demonstration on a grid board. A portable, commercially made, permanent grid chalkboard is ideal. However, if such a grid board is not available, try one of these alternatives.

- Make a grid transparency for use on an overhead projector and use erasable overhead markers. The Blackline section includes several size grids to use in making a grid transparency.
- Printed grids on large sheets of paper are provided in classroom sets of materials for *CSMP*. Tape one of these sheets to the board, use thick felt-tipped pens for drawings, and discard the sheet after use. You may prefer to laminate or mount one of the sheets on cardboard and cover it with clear contact paper to provide a protective surface. Then use erasable markers or wax crayons for drawings. These grid sheets also are useful as graphing mats.
- For each lesson, carefully draw a grid on the chalkboard using an easy-to-make tool. Either (1) use a music staff drawer with chalk in the first, third, and fifth positions; or (2) clip five pen caps, each having the diameter to hold a piece of chalk tightly, to a ruler at intervals of 4–8 centimeters, and secure them with tape.



G1 INTRODUCTION TO ANGLES

Capsule Lesson Summary

Introduce angles and practice recognizing the angles implicit in geometric objects. Define a right angle as a square corner, and compare a square corner to other angles, classifying them as *acute* (smaller), *obtuse* (larger), or *right*.

Materials

Teacher
- Colored chalk
- Straightedge
- Construction paper

Student
- Construction paper
- Worksheets G1(a) and (b)

Description of Lesson

Begin with a discussion to review the ideas of line and line segment, and to introduce angles and related concepts. Distinguishing between line and line segment is worthwhile as it helps students to organize their understanding of the geometry. The following dialogue suggests how the discussion might proceed.

Draw a line segment on the board, emphasizing the endpoints with small dots.

T: ***What do we call this?***

S: ***A line segment.***

T: ***Tell us about a line.***

S: ***It is straight.***

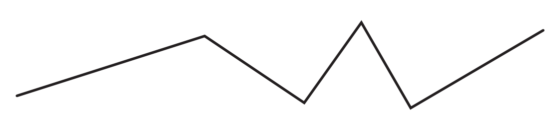
S: ***It goes on and on, but it does not curve.***

S: ***It can be vertical or horizontal or diagonal.***

T: ***What is a zigzag?***

S: ***A zigzag is a bunch of line segments connected to each other at endpoints.***

Draw a zigzag on the board.



S: ***A zigzag is not a line nor a single line segment.***

S: ***A curve is not a line either.***

S: ***A line is a straight segment that can be as long as you like.***

T: ***Like a kilometer long?***

S: ***No, a line goes on and on.***

S: ***A line segment has a beginning and an ending, but a line has no beginning and no ending.***

The class should observe that a line is straight and has no endpoints, whereas a line segment is part of a line and has endpoints.

T (pointing to the line segment on the board)**:** ***Can you indicate a line that is closely related to this line segment?***

A volunteer should indicate with gestures the extension of the line segment in both directions without end.

T: ***Since we cannot draw all of a line on the board, we draw part of it and then use arrowheads to indicate it goes on in both directions.***

How many endpoints does a line segment have? (Two) ***And how many endpoints does a line have?*** (None)

How about something that has only one endpoint, like this? Is it a line segment?

S: ***No it has only one endpoint.***

T: ***Is it a line?***

S: ***No, it keeps going but in only one direction.***

S: ***It has one endpoint and a line has none.***

T: ***Think about a flashlight. When you turn it on, the light beams out in one direction. We call this a*** **ray*****.***

Now we have three concepts: **line segment*****,*** **line*****, and*** **ray*****. And we have a way to show each in a drawing.***

Refer to the line on the board.

T: ***I am thinking of a ray that is part of this line. What do you need to know so that you can find it?***

S: ***Where is its endpoint?***

Mark a point on the line.

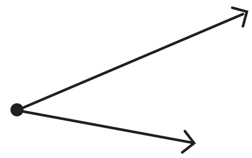
S: ***You could be thinking of one of two different rays.***

Invite a student to indicate the two rays determined by the chosen point. Emphasize that the two rays are different even though they have the same endpoint and are part of the same line. Repeat the activity with one or two other points until the class understands that a line can be broken at any point into two different rays.

Refer to the line segment on the board.

T: ***Can we draw a ray that includes this line segment and has one of the endpoints of the line segment as its endpoint?***

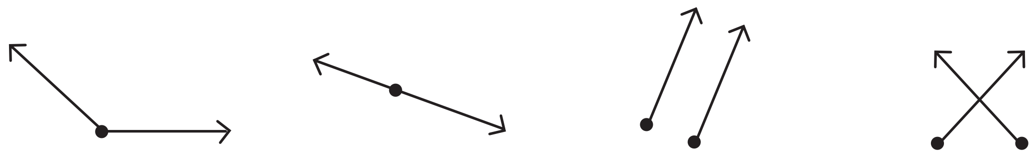
Invite students to extend the segment to produce the two rays that it determines. They are indicated here in red and in blue.



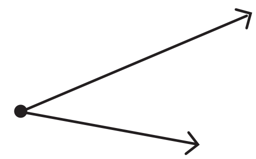
Draw another pair of rays, this time with the same endpoint but not on one line.

Call on students to indicate the rays by tracing them. Emphasize, if necessary, that there are two rays explicit in the drawing.

Repeat the activity, looking at several other pairs of rays.

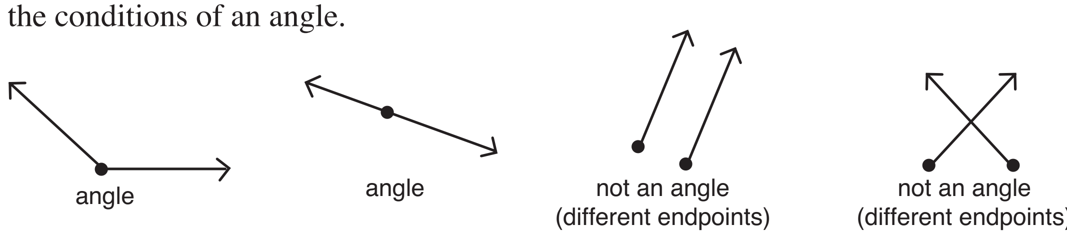


T: ***Let's look at these two rays again.***

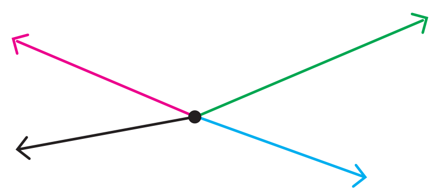


We call this an *angle*. An angle is two different rays with the same endpoint. The common endpoint is called the *vertex* of the angle. Which of the other pairs of rays that we looked at form angles?

Invite students to explain which pairs are angles and to indicate how the other pairs of rays fail to meet the conditions of an angle.



Draw this picture of several rays on the board.



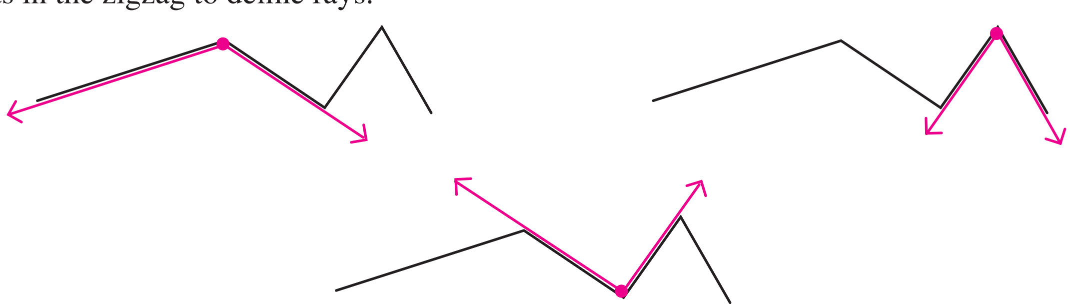
Invite students to indicate angles and their common vertex in the picture. For example:

S: ***The red ray with the green ray form an angle.***

Draw a zigzag with several line segments on the board. (See the next illustration.)

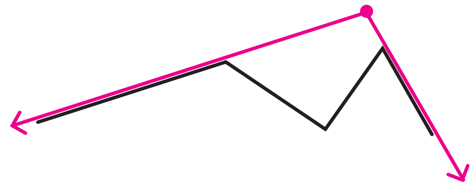
T: ***Show us an angle that includes line segments in this zigzag.***

Ask students to indicate angles by putting dots at the corners of the zigzag and by extending line segments in the zigzag to define rays.

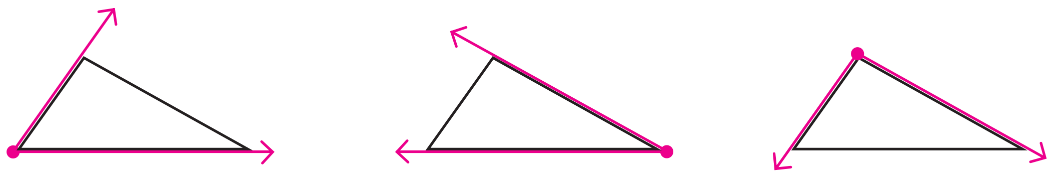


G1

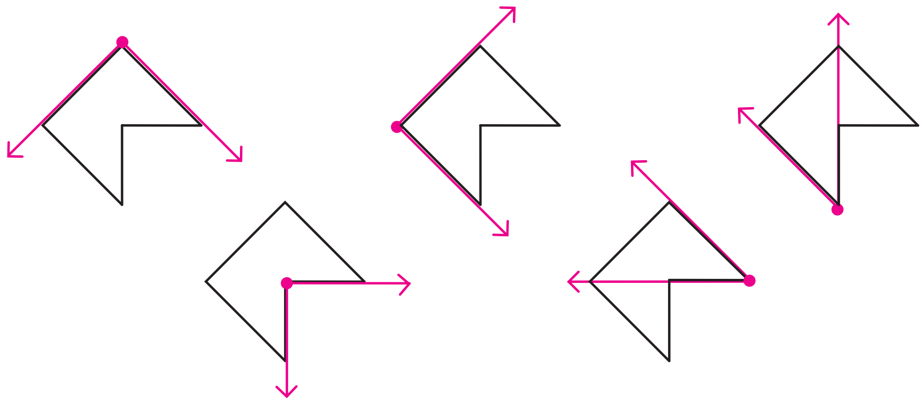
Note: The angles shown in the previous illustration are the ones that arise most naturally. However, other angles are possible but their vertices are not corners of the zigzag, as illustrated in this example.



Repeat this activity with a triangle.

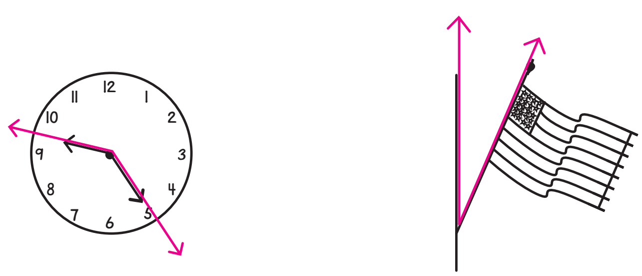


Repeat the activity again with a shape similar to the following.



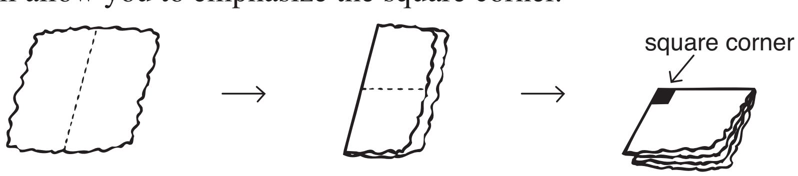
Distribute copies of Worksheet G1(a). Allow only a few minutes for students to complete the worksheet, checking their understanding of the various terms and geometric elements of this lesson. As you observe individual work, correct any evident misconceptions.

Initiate a discussion of the angles that are implicit in objects of the classroom (there are many, of course). Let students indicate the rays and the vertex of each angle under consideration. For example:



T: ***Many of the angles found in the classroom are special. Watch as I fold this piece of paper.***

Fold a piece of construction paper as shown below. It would be best to use a scrap (irregular) piece of paper as this will allow you to emphasize the square corner.



T: ***The corner of the paper can be used to draw a special kind of angle.***

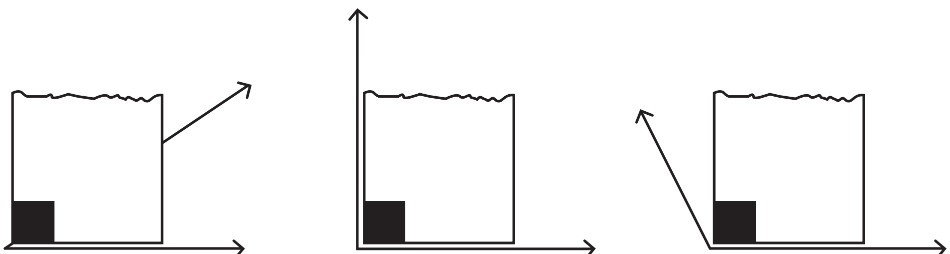
Draw some angles on the board, using your paper square corner as a template.

T: ***Each of these angles is called a* square corner *or* right angle*. Why are they called square corners?***

S: ***Each is like the corner of a square. The angle at each corner of a square is a right angle.***

T: ***We can also compare this paper square corner to other angles, checking if an angle is a right angle or not.***

Draw some angles on the board and decide whether or not they are right angles. Proper technique consists of matching the vertex of the paper square corner with the vertex of the angle being compared and then aligning one edge of the square corner to one ray of the angle. If the second ray of the angle also matches an edge of the square corner, then the angle is a right angle.

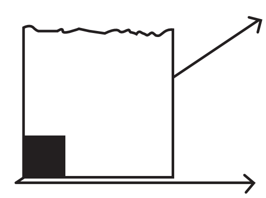


Instruct students to make their own paper square corners. When all are equipped, instruct them to check the angles on Worksheet G1(b). While the class is working, check student use of the paper square corner to compare angles. After a while, review the answers collectively, asking students to recheck their work, if necessary.

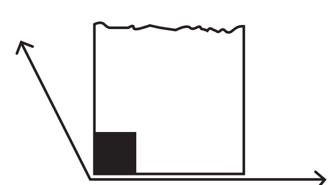
T: ***The paper square corner may also be used to decide if an angle is smaller or larger than a right angle.***

As you describe the technique of checking an angle with the paper square corner, demonstrate it at the board.

T: ***Match an edge and the vertex of the paper square corner with a ray and the vertex of an angle. If the other ray of the angle is partially covered by the paper, the angle is smaller than a right angle. An angle which is smaller than a right angle is called* acute*.***



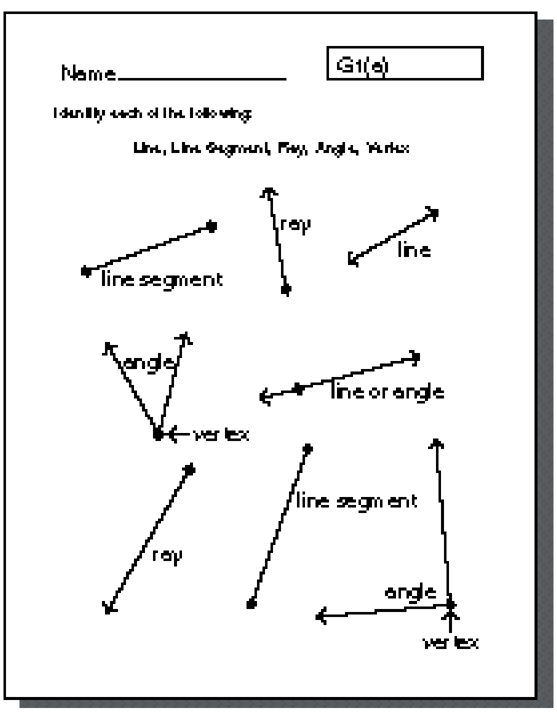
If the other ray is not partially covered and does not line up with the edge of the paper square corner, then the angle is larger than a right angle. An angle which is larger than a right angle is called* obtuse*.

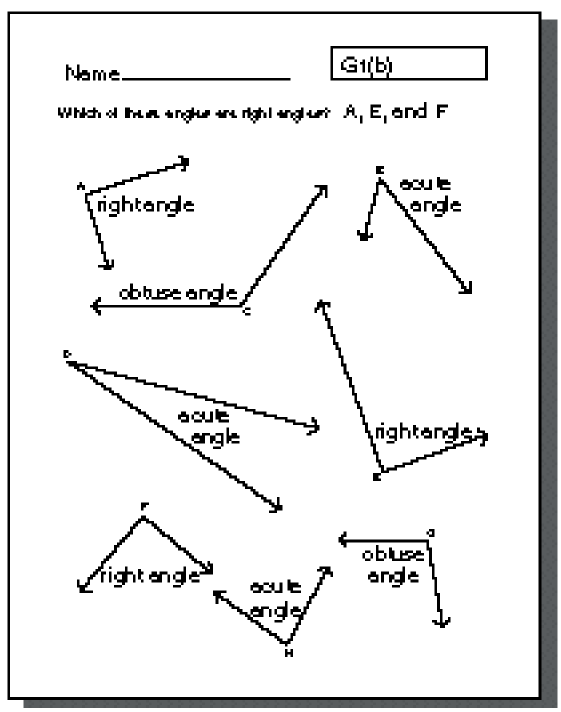


Draw several other angles on the board, and collectively decide what kind (right, acute, or obtuse) each angle is. Then instruct students to do the same with the angles on Worksheet G1(b).

Writing Activity

You may like students to take lesson notes on some, most, or even all their math lessons. The "Lesson Notes" section in Notes to the Teacher gives some suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students may write descriptions of the various geometry terms: line, line segment, ray, angle, right angle, acute angle, and obtuse angle.





G2 CONSTRUCTION WITH ACUTE

Capsule Lesson Summary

Review the concepts introduced in Lesson G1. Construct polygons with a prescribed number of sides and acute angles.

Materials

Teacher
- Colored chalk
- Straightedge
- Paper square corner

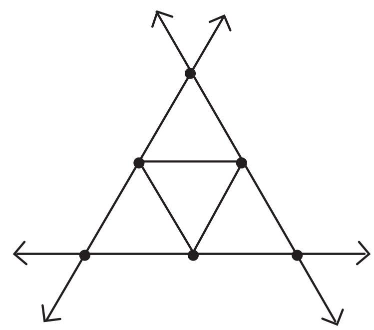
Student
- Unlined paper
- Straightedge
- Paper square corner

Description of Lesson

Note: You may wish to begin the lesson by leading the students in folding paper for a square corner as described in Lesson G1.

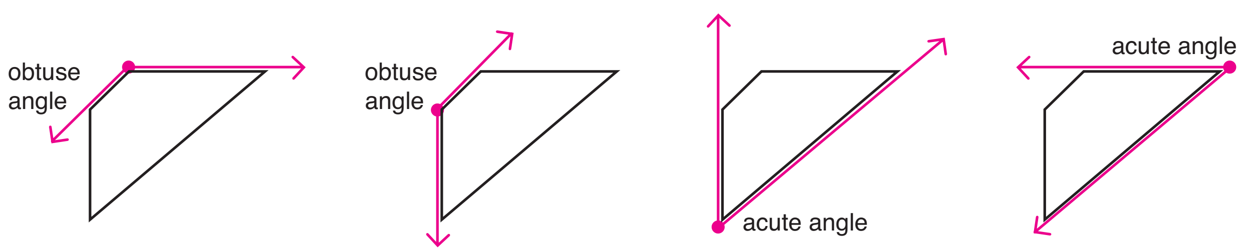
Exercise 1

Draw this picture on the board.

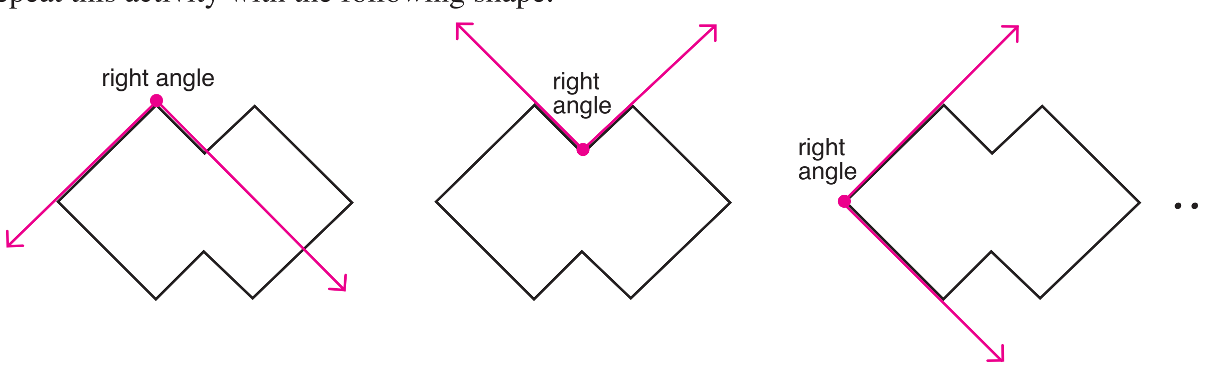


Review the concepts from Lesson G1 by inviting students to point out some of the line segments, lines, rays, and angles in the picture. Then ask students to identify several angles that are smaller than a right angle (acute) and several angles that are larger than a right angle (obtuse). Check the responses with a paper square corner.

Draw a quadrilateral on the board and invite students to indicate its corner angles. Call on students to use a paper square corner to identify each angle as acute, obtuse, or right. For example:

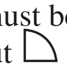
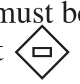


Repeat this activity with the following shape.



Exercise 2

Instruct students to sketch shapes with specified numbers of angles (for example, four, seven, ten) on a piece of paper. Students should use a straightedge to draw straight line segments. Invite several students to copy their drawings on the board. Keep as many of the drawings on the board as possible. The following restrictions need only be introduced if they are violated in students' drawings.

- Sides must be line segments
  rules out 
- No dangling edges
  rules out 
- Shapes must be closed
  rules out 
- Sides do not cross
  rules out 
- Three different sides do not meet
  rules out 
- Shapes must be connected
  rules out 

Following these rules, students will consistently construct simple (closed) polygons. Shapes of this type will be the central focus of the geometry lessons and exercises to follow. In the discussion of each polygon, make a point to observe that the number of sides equals the number of angles.

Note: Students should begin to learn the standard names for polygons with specific numbers of sides from your regular usage of the terms.

triangle: 3 sides
quadrilateral: 4 sides
pentagon: 5 sides
hexagon: 6 sides
septagon: 7 sides
octagon: 8 sides
nonagon: 9 sides
decagon: 10 sides

After students have drawn several polygons on their papers, ask them to use paper square corners to designate the angles of their shapes appropriately as acute, obtuse, or right.

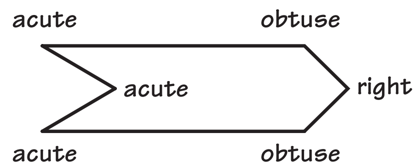
While students are working, draw this table on the board. Make the cells large enough to draw polygons within them.

		Number of acute angles					
		0	1	2	3	4	5
Number of sides	3						
	4						
	5						
	6						

Invite students to label the angles of the polygons on the board. Encourage class discussion and several checks of any disputed labeling. This exercise should provide an opportunity to emphasize the importance of careful, neat drawings, especially of straight line segments and precisely defined vertices.

Note: When appropriate, mention that *vertices* is the plural of vertex.

Choose one of the polygons on the board with six or less sides. For example:



T: ***How many sides does this shape have?*** (Six)
How many acute angles? (Three)
Where would this shape belong in the table?

Invite a student to indicate the appropriate cell for the shape. Then draw a sketch of the shape in that cell of the table and label the acute angles **a**.

T: ***Let's see if we can find an example of each type of shape to fill in the cells of this table.***

Allow students to work on this problem in small groups for most of the remaining class time. As groups find appropriate examples, invite them to sketch their shapes in the table on the board.

Of course, the number of sides of a polygon equals the number of angles, so some of the constructions are automatically impossible. For example, a polygon with three sides cannot have four or more acute angles. In addition, a triangle must have two or three acute angles. The experience of attempting these constructions will be valuable.

For each possible construction, there are many solutions. One set of solutions is shown below.

Number of acute angles

Number of sides	0	1	2	3	4	5
3	impossible	impossible	a a	a a a	impossible	impossible
4		a	a a	a a a	a a a a	impossible
5		a	a a	a a a	a a a a	a a a a a
6		a	a a	a a a	a a a a	a a a a a

Close the lesson by discussing why certain shapes are impossible to construct.

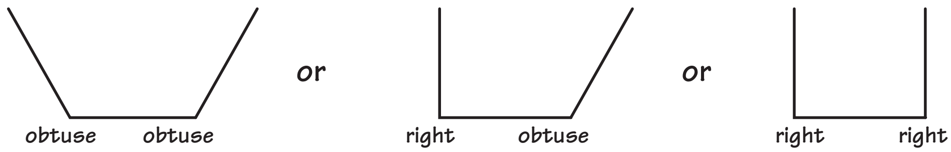
T: ***Which of the constructions were you not able to do? Do you think they are impossible?***

S: ***A triangle cannot have more than three angles nor can a quadrilateral have more than four angles; a polygon has the same number of angles as it has sides.***

S: ***It is impossible to have a triangle with no acute angles or with just one acute angle.***

T: ***Why?***

S: ***If I draw a side of a triangle that is not part of an acute angle, then the other sides can never meet to finish the triangle.***



T: ***What can we say about these two other line segments when we draw a side with right angles at both of its endpoints?***

S: ***They are parallel.***

Capsule Lesson Summary

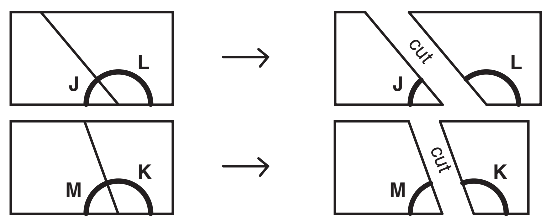
Construct quadrilaterals using four given angles. Determine the number of different quadrilaterals with the four given angles.

Materials

Teacher
- Straightedge
- Four cardboard angle templates

Student
- Straightedge
- Tracing paper
- Worksheet G3

Advance Preparation: Cut four templates out of cardboard, one to match each of the angles on Worksheet G3. The angle measures are: **J** 50°, **K** 110°, **L** 130°, and **M** 70°. This is for your information only; do not tell the class. Use a protractor and a straightedge to lay off each angle as shown here.



Description of Lesson

Each student should have a copy of Worksheet G3 and two sheets of tracing paper. First, ask students to make several copies of the angles on the worksheet, using a straightedge and a piece of tracing paper. Instruct them not to label the angles with letters as on the worksheet. Then, direct students to trade papers and to label the angles on the tracing paper they receive with the corresponding letters from the worksheet. When everyone finishes, let students trade back their papers and check that the angles are labeled correctly. Continue with the following discussion.

T: ***There are four angles on this worksheet. What kind of polygon has four angles?***

S: ***Four angles, four sides—that is a quadrilateral.***

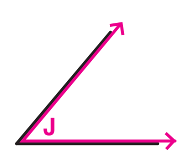
T: ***I have four templates, one to match each of the angles on the worksheet. Which angles are acute? Which are obtuse? Which are right angles?***

S: ***J and M are acute; K and L are obtuse; there is no right angle on the worksheet.***

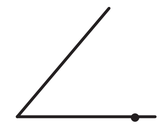
T: ***Can you construct a quadrilateral on a piece of tracing paper using all four of the angles shown on the worksheet? Use the tracing paper to trace the angles as you build the quadrilateral.***

As you observe individual work, check that students use a straightedge to make accurate, neat drawings. Some students may need special attention in developing a technique for building quadrilaterals on tracing paper. One procedure is as follows:

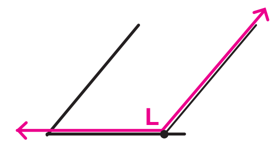
- Trace one of the angles using a straightedge, and make the line segments as long as desired.



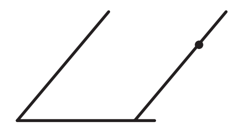
- Pick a point on one of the line segments to be a second corner of the quadrilateral and, hence, the vertex of the second angle to be traced.



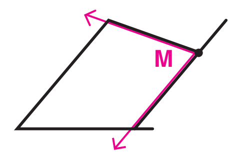
- Lay the tracing paper on a second angle so that the vertex is under the chosen point and so that one ray points toward the first corner.



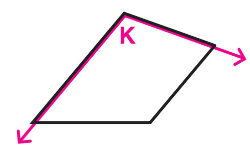
- Trace along the other ray as long as desired. Pick a point for the third corner.



- Lay the tracing paper on a third angle so that the vertex is under the chosen point and so that one ray points toward the second corner. Trace along the other ray, completing the third and fourth corner.



- Check that the fourth corner (angle) matches the fourth angle on the worksheet, and erase extensions of line segments beyond the corners.



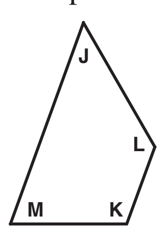
If many students have difficulty getting started, demonstrate this procedure at the board. First use the angle templates to draw a set of angles on the board as on the worksheet.

As students succeed in constructing quadrilaterals, ask them to build others which are “different.” Let students decide for themselves what might be considered different among quadrilaterals. Invite some students to draw their solutions on the board using the templates. When most students have several solutions, initiate a class discussion of solutions to develop criteria for deciding when two solutions differ.

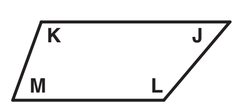
There are several ways in which solutions may differ:

- orientation on the page
- size
- relative position of the angles

Orientation on the page is not important and should be readily discarded by the class. Size is more significant but is still not very interesting. Lead the discussion to consider the relative position of the angles in each solution. Beneath each quadrilateral write the sequence of angles beginning at **J** and reading clockwise. For example:



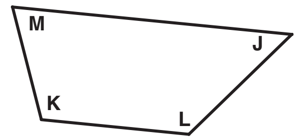
J - L - K - M



J - L - M - K

If two quadrilaterals have the same sequence, conclude that they must be the same except for size or orientation on the board. For example:

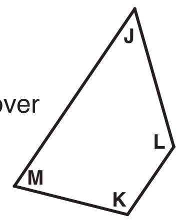
J - L - K - M



J - L - K - M

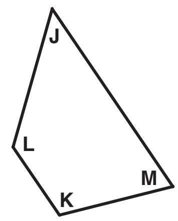
If two quadrilaterals have reverse sequences (that is, the clockwise sequence of one is the same as the counterclockwise sequence of the other), they too are the same except for size and orientation. Notice that by turning over the tracing paper, you can reverse the sequence of angles.

turning over



J - L - K - M

yields



J - M - K - L

Students may comment that shapes of the same size with reverse sequences of angles are reflections of each other. Observe that any of the quadrilaterals has a reflective twin which may at first glance look different.

Conclude that two quadrilaterals are essentially different if their sequences of angles are different and are not reverse sequences.

Ask students to label their individual quadrilaterals in this way and to decide how many different solutions they have found. Then invite students to draw solutions on the board which are not already there.

Note: A complete set can be found as follows. In any solution, either angles **J** and **K** will be opposite each other or they will be next to each other. The possibilities are: **J-?-K-?** and **J-K-?-?**.
Now L and M can be inserted in the remaining positions to yield only four sequences of angles:
J-L-K-M; **J-M-K-L**; **J-K-L-M**; and **J-K-M-L**. Notice that the first two are reverse sequences and, hence, for similar solutions. The class should find that there are only three essentially different solutions.

Capsule Lesson Summary

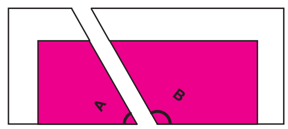
Use two angle templates to construct polygons with three, four, five, and six sides.

Materials

Teacher
- Demonstration angle templates corresponding to **A** and **B** in the student set
- Straightedge
- Poster board

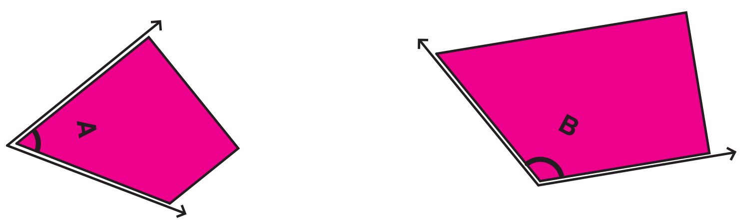
Student
- Large unlined paper
- Straightedge
- Angle template set

Advance Preparation: Use a large, sturdy piece of cardboard to cut two demonstration angle templates corresponding to **A** and **B** of the student set. Angle **A** measures 60° and angle **B** measures 120°. This is for your information only; do not tell the class. You can use templates **A** and **B** from a student set as a guide for making your own.



Description of Lesson

Begin the lesson by introducing the class to the angle templates **A** and **B**. Make clear to students that the templates are for the angles marked with heavy black arcs. They will use the templates to make copies of those angles in some construction problems. At the board, demonstrate how to use the templates to make copies of angle **A** and of angle **B**.



Note: In the lesson descriptions, the same name is used for both a template and the angle indicated on that template.

You may like to pair students to work on the following construction problem:

Construct as many different quadrilaterals as possible, subject only to the condition that each angle correspond to, or be *congruent* to **A** or **B**. An angle traced using template **A** is congruent to angle **A**, and an angle traced using template **B** is congruent to angle **B**.

Remind the class of their experience in Lesson G3 in which quadrilaterals were distinguished by the relative position of their angles, rather than by size or by orientation on the page.

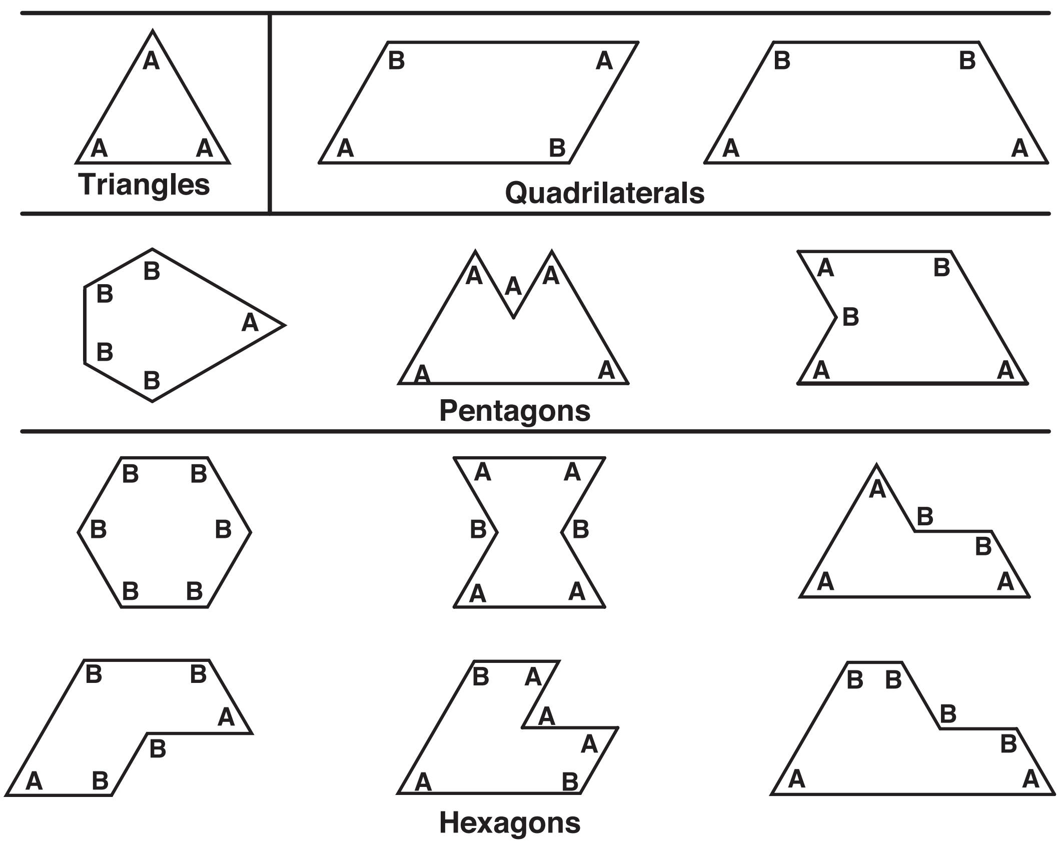
Note: There are essentially two quadrilaterals to be found: a parallelogram and a trapezoid. Each has two angles congruent to **A** and two angles congruent to **B**. In the parallelogram, the angles congruent to **A** are diagonally opposite; in the trapezoid, they are adjacent. We will use this important characterization of the solutions in later lessons.

Invite students to put the solutions on the board using your templates. Under each quadrilateral, write its sequence of angles.

As students discover the quadrilateral solutions, ask them to use only templates **A** and **B** to construct polygons with three, five, and six sides. Invite students to copy solutions on the board using your templates. Label each angle appropriately as **A** or **B**. A collection of solutions is shown below.

With many solutions displayed on the board, lead a collective discussion of them. Your students should notice that there are only two quadrilateral solutions, one of which is a parallelogram. Introduce the other quadrilateral as a *trapezoid*. A trapezoid is a quadrilateral with just one pair of parallel sides. In the discussion, call attention to whether or not solutions have indentations. In the following illustration, all of the possible solutions without indentations are shown, along with some of the many solutions with indentations. There are no other solutions with three or four sides, with or without indentations.

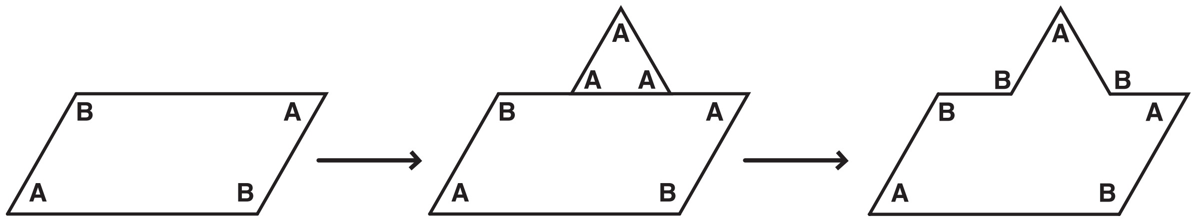
Polygons with each angle congruent either to A or to B



Preserve students' drawings for Lesson G5. The class may enjoy making a large poster of solutions and the poster can serve well for reference in later lessons.

Optional Activity

For an advanced problem, ask a student to find a way of building solutions with any given number of sides. Having constructed several solutions with three, four, five, and six sides, the student can look for solutions with seven, eight, nine, ten sides, and so on. The idea is to find a general process for building solutions with more and more sides from solutions already found. One approach is to build a triangle with all angles congruent to **A** along one side of a solution. This procedure is successful since **A** and **B** are closely related; by constructing an angle congruent to **A** along a side, we automatically get a neighboring angle congruent to **B**. For example:



Capsule Lesson Summary

Use angle templates to construct quadrilaterals. Recognize the special pairs of angles which give parallelograms and trapezoids.

Materials

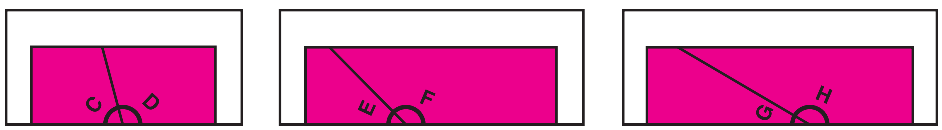
Teacher

- Poster of solutions from Lesson G4 (if available)
- Demonstration angle templates corresponding to **A**, **B**, **C**, **D**, **E**, **F**, **G**, and **H** in the student set
- Straightedge
- Poster paper

Student

- Drawings from Lesson G4
- Large unlined paper
- Straightedge
- Angle template set

Advance Preparation: As you did with templates **A** and **B** in preparation for Lesson G4, cut out large sturdy cardboard angle templates corresponding to **C**, **D**, **E**, **F**, **G**, and **H** of the student set. The angles measure: **C** 75°, **D** 105°, E 45°, **F** 135°, **G** 30°, and **H** 150°. This is for your information only; do not tell the class.



Description of Lesson

In this lesson, students use four angle templates (**C**, **D**, **E**, and **F**) to construct quadrilaterals. The objective is their guided discovery that these four angles fall naturally into two pairs: **C** with **D**, and **E** with **F**. The discovery of this pairing comes through recognizing that these pairs each yield two different quadrilaterals, and further, that they are related as mutually supplementary angles. The aim is for students to discover the idea of supplementary angles without its being defined in terms of angle measure (two angles are mutual supplements if the sum of their measures is 180°). In this way, students can develop a feeling for angles and their fundamental relationships without the potential distraction of measurement.

Display the poster of solutions from Lesson G4, if you have one. Otherwise, ask students to consult their drawings from that lesson.

T: ***Last week we used templates* A *and* B *to construct polygons with three, four, five, and six sides. You found many solutions but only two solutions with four sides, two different quadrilaterals. Do you remember what we called the two different quadrilaterals?***

S: ***One is a* parallelogram*, and the other is a* trapezoid.**

Hold up angle templates **C**, **D**, **E**, and **F**. Ask students to locate these four among their templates.

T: ***Today we will use four other templates. Construct as many quadrilaterals as you can such that each angle is congruent to one of these four angles:* C*,* D*,* E*, and* F*. Label the angles in your drawing.***

You may like to let students work in pairs on this task with the four templates. For the purpose of class discussion, make a record of solutions by inviting students to draw their solutions on the board. Make sure each angle of the quadrilaterals is labeled.

There are nine different solutions in all, each fitting into one of three categories.

- Type 1: Four angles, none of which is congruent to any other
- Type 2: Exactly three of the four angles are congruent to any other
- Type 3: Two pairs of congruent angles

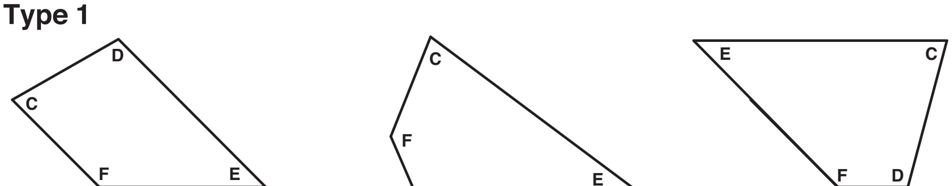
The solutions of Type 1 each have a reflective twin which may at first appear to be different as did the solutions to the problem of Lesson G3. The following illustrations show a complete set of solutions arranged by type.

Recall the system of describing a quadrilateral by its sequence of angles and write this sequence below each quadrilateral solution on the board. For example:

C - D - E - F   **C - D - C - D**

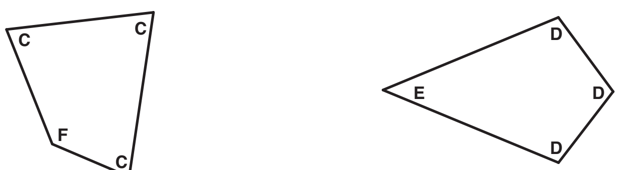
Direct the discussion toward recognizing and eliminating duplications in the solutions on the board. If necessary, recall that size and orientation on the page are not to be considered in deciding whether solutions differ. The corner from which the angle sequence starts is not important; thus **C-D-E-F** is the same as **E-F-C-D**. Also reflective pairs have sequences in opposite order; thus **C-D-E-F** and **F-E-D-C** are for essentially the same quadrilateral.

Type 1



C - D - E - F   **C - E - D - F**   **C - D - F - E**

Type 2



C - C - C - F   **D - D - D - E**

Type 3

E - F - E - F   **E - E - F - F**

C - D - C - D   **C - C - D - D**

Refer students to their drawings from Lesson G4 for comparison with the solutions on the board.

T: ***Do any of these quadrilaterals resemble the quadrilaterals that we constructed using templates* A *and* B*?***

Students should observe that the four Type 3 quadrilaterals on the previous page resemble the two solutions they found using templates **A** and **B**.

T: ***What do you notice about these four Type 3 quadrilaterals?***

S: ***Two of them use* C *and* D *and two of them use* E *and* F.**

T: ***What kinds of quadrilaterals are the Type 3 solutions?***

S: ***Two are parallelograms and two are trapezoids.***

S: ***One of each for each pair of angles.***

T: ***Look at your papers showing the polygons using templates* A *and* B. *What kinds of quadrilaterals did we construct with these angles?***

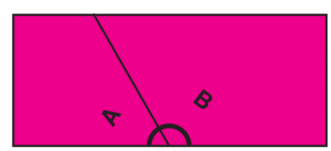
S: ***We found a parallelogram and a trapezoid with* A *and* B*, too.***

Refer to the six templates **A**, **B**, **C**, **D**, **E**, and **F**.

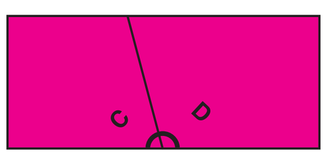
T: ***So we have three special pairs:* A *with* B*,* C *with* D*, and* E *with* F. *Each pair gives us a parallelogram and a trapezoid. Is there anything else that you notice about these special pairs of angles?***

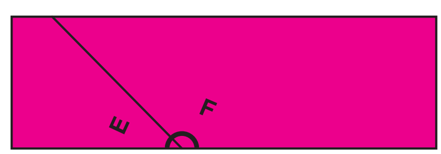
S: ***We can fit templates* A *and* B *together so that two sides are on a line.***

This is the important observation. Call on students to demonstrate with the templates.



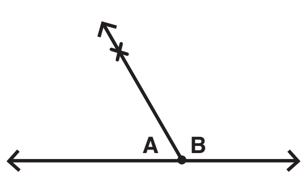
T: ***What about the other pairs? Do they fit together so that two sides are on a line? Arrange them on your tables if you can.***

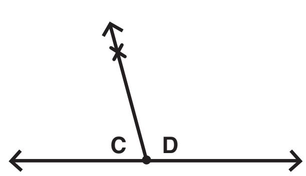


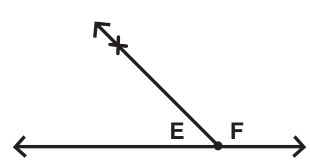


T: ***Let's draw these three pairs of angles on the board.***

Use templates to make drawings showing the pairs of angles juxtaposed. In each, mark the common ray with an × .



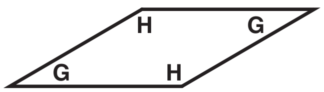


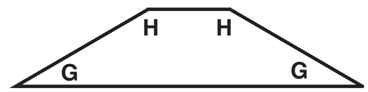


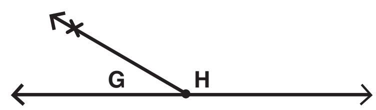
Instruct students to look for other pairs of templates that behave like these pairs. They should find that **G** and **H** also fit together so that two sides are on a line.

T: ***Try to build a parallelogram and a trapezoid with* G *and H.***

While students are doing their constructions with **G** and **H**, draw the two quadrilaterals on the board. Also draw the juxtaposed angle pair (**G** and **H**) on the board next to the three already there.







When most students have completed their constructions with **G** and **H**, continue the discussion.

T: ***A pair of angles which fit together[†] so that two rays are in line this way is called a* supplementary pair*.* A *and* B *are supplementary;* C *and* D *are supplementary;* E *and* F *are supplementary; and* G *and* H *are supplementary.***

On a piece of poster paper, make a record of the different quadrilaterals constructed in this lesson and preserve students’ work for reference in Lesson G6 and again in Lesson G9.

[†]We rely on how templates fit together to convey what we mean by angles “fitting together.” This is not precise technology, but it is clear and natural.

Capsule Lesson Summary

Explore supplementary pairs of angles. Compare the template angles according to an intuitive idea of size.

Materials

Teacher
- Poster of solutions from Lesson G5
- Straightedge
- Demonstration angle templates corresponding to **A**, **B**, **C**, **D**, **E**, **F**, **G**, and **H** in the student set
- Index card
- Scissors
- Tracing paper
- Black marker

Student
- Drawings from Lesson G5
- Tracing paper
- Large unlined paper
- Index card
- Scissors
- Straightedge
- Angle template set
- Worksheet G6

Description of Lesson

Review the constructions of Lesson G5, referring to your poster of the solutions and the students' drawings from that lesson.

T: ***We found that among the eight angles* A, B, C, D, E, F, G, *and* H *there are four special pairs. What makes them special?***

S: ***We can construct quadrilaterals with these pairs.***

S: ***We made a parallelogram and a trapezoid with each pair.***

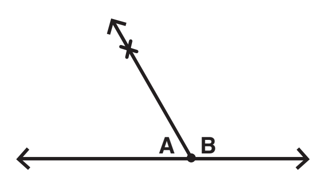
S: ***The special pairs of angles fit together so that two of their rays are on a line.***

T: ***Which are the pairs?***

S: **A *with* B, C *with* D, E *with* F, *and* G *with* H.**

T: ***What do we call these pairs?***

S: ***Supplementary pairs of angles.***



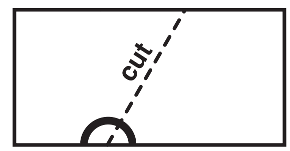
Invite students to demonstrate that **A** and **B** form a supplementary pair. They should match the vertex of **A** with the vertex of **B** and match one ray of each. Mark the common ray with ×. The other two rays form a line.

T: ***Test other pairs of the angles to see if they fit together with two sides on a line.***

Allow some time for students to test other pairs and to build confidence that there are only four supplementary pairs.

Exercise 1

T: ***Do you think any pair of supplementary angles can be used to construct a parallelogram and a trapezoid?***



Provide each student with an index card and a pair of scissors. Instruct students to cut the index card into two parts to create two new templates for angles that are supplementary. Illustrate how to mark the supplementary angles with an arc. Encourage a variety of angle sizes.

Direct students to try to construct a parallelogram and a trapezoid using their new angle templates. You may like to let students work with a partner and to check that everyone is successful.

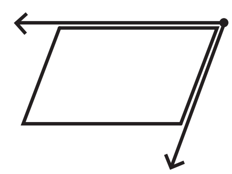
T: ***Each of you used a pair of supplementary angles, and each of you constructed a parallelogram and a trapezoid. It appears that any pair of supplementary angles can be used to construct a parallelogram and a trapezoid.***

Exercise 2

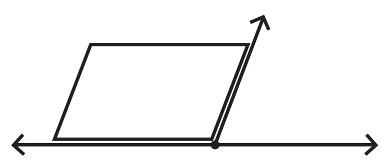
Draw a parallelogram on the board.

T: ***Can we find a pair of supplementary angles in this parallelogram?***

Invite a student to indicate two angles. Then illustrate how to use tracing paper to check that the pair of angles is indeed supplementary.



- Place a large piece of tracing paper over one of the angles, and trace the angle using a straightedge and a black marker.



- Move the tracing paper to shift the vertex of the angle on the tracing paper to another corner (the vertex of an adjacent angle) of the parallelogram. Then trace the angle at the second corner.

T: ***Are these two angles supplementary?***

S: ***Yes; when we fit them together two of their rays are on a line.***

Distribute Worksheet G6. Instruct students to use tracing paper and a straightedge to check for supplementary pairs of angles, just as you have illustrated. When most students have finished, discuss the results.

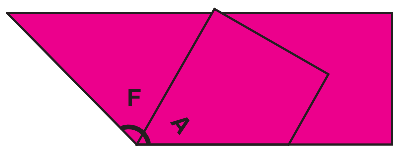
An answer key for this worksheet appears at the end of the lesson. Notice the special case of the rectangle. All of its angles are right angles and hence congruent. Any right angle is supplementary to another right angle. Perhaps someone will notice that there are four different angles forming two pairs of supplementary angles in one of the shapes and that it is a trapezoid.

Exercise 3

Conclude the lesson with a brief discussion of the relative sizes of angles. Refer to angle templates **A**, **B**, **C**, **D**, **E**, **F**, **G**, and **H** in the student set.

T: ***Look at the eight templates. Which angles are acute?*** (**A**, **C**, **E**, and **G**)
Which are obtuse? (**B**, **D**, **F**, and **H**)
Which angle is the smallest? (**G**)
Which angle is smaller,* A *or* C*? (**A**) **D** ***or* F*?*** (**D**) **B** ***or* D*?*** (**D**)
Which angle is largest? (**H**)
Which angle is larger,* C *or* E*? (**C**) ***E or G?*** (**E**) **G** ***or* B*?*** (**B**)

Note: The idea of comparing angles was introduced in Lesson G1 where angles were compared with a paper square corner. We can extend that informal idea to a comparison of any two angles. With the templates in hand, it is easy to compare the angles directly by placing one on top of the other, matching the vertices, and aligning one edge of each template. The following illustration shows **A** on top of **F**; angle **A** is smaller than angle **F**.



T: ***Line up the templates on your desk in order of size.***

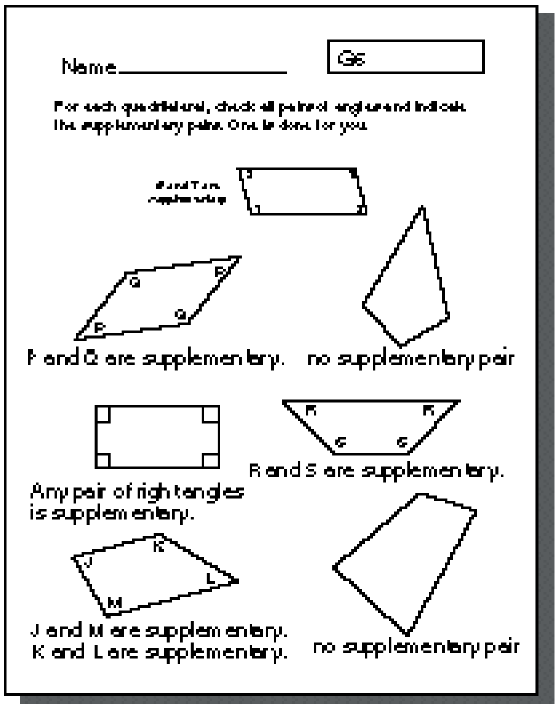
Note: The order sequence is **G**, **E**, **A**, **C**, **D**, **B**, **F**, **H**.

T: ***Which angle is the smallest?*** (**G**)
And which angle is the largest? (**H**)
What do you notice about the supplementary pairs?

S: ***The largest and the smallest are supplementary; the second smallest and the second largest are also supplementary; and so on.***

T: ***One angle in each pair is acute and the other is obtuse.***

S: ***That would be true for any supplementary pair, unless they were both right angles.***



Name________________ G6

For each quadrilateral, check all pairs of angles and indicate the supplementary pairs. One is done for you.

P and Q are supplementary.

no supplementary pair

Any pair of right angles is supplementary.

R and S are supplementary.

J and M are supplementary.
K and L are supplementary.

no supplementary pair

G7 TRIANGLE CONSTRUCTIONS #1

Capsule Lesson Summary

Use seven angle templates to construct triangles.

Materials

Teacher
- Straightedge
- Demonstration angle templates corresponding to **A**, **B**, **G**, **W**, **X**, **Y**, and **Z** in the student set

Student
- Straightedge
- Large unlined paper
- Angle template set

Advance Preparation: For earlier lessons you made templates corresponding to **A**, **B** and **G**. Prepare similar demonstration angle templates (see Lessons G4 and G5) corresponding to **W**, **X**, **Y**, and **Z** in the student's set. The angle measures are: **W** 40°, **X** 50°, **Y** 70°, **Z** 90°. This is for your information only; do not tell the class.

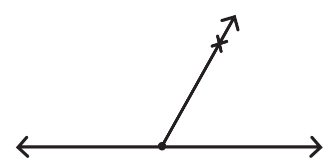
Description of Lesson

Begin the lesson with a brief review of the work of Lesson G6.

T: ***Last week we used eight different angle templates in constructing quadrilaterals. How many supplementary pairs did we have among the eight angles?***

S: ***There were four supplementary pairs.***

Ask a student to draw a supplementary pair of angles on the board, marking the common ray with ×.



T: ***Are there other supplementary pairs of angles? How many are there with different angle sizes?***

S: ***Infinitely many; we could point the common ray in any direction.***

T: ***Today we will use angle templates again, but some of them will be ones we have not used before.***

Direct students to take out templates **A**, **B**, **G**, **W**, **X**, **Y**, and **Z**.

T: ***Which of these angles are acute?*** (**A**, **G**, **W**, **X**, and **Y**)
And which are obtuse? (Only **B**)
What about Z? (**Z** is a right angle)
Which is the smallest of the seven angles? (**G**)
And which is the largest? (**B**)
Arrange them on your desk in order of size from smallest to largest.

The order sequence is: **G**, **W**, **X**, **A**, **Y**, **Z**, **B**.

T: ***The problem today is to use these angles to construct triangles. There are many possibilities. How many?***

Let students discuss this question but do not expect a definitive answer at this time.

S: ***We can start with any of the seven angles.***

S: ***May we use an angle twice?***

T: ***In constructing quadrilaterals, we often used an angle more than once; we may be able to do the same with triangles.***

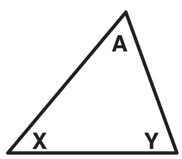
T: ***Then we have seven choices for the second angle and seven choices for the third angle; that is 7 x 7 x 7 or 343 choices in all.***

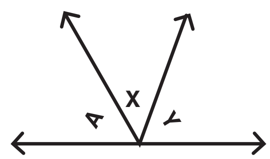
T: ***Do you think that we can construct 343 different triangles?***

S: ***Probably some of the combinations will not work. Not all of the combinations worked when we were constructing quadrilaterals.***

T: ***As you try different combinations, label the angles on your papers. Keep a record of the combinations that work and those that do not work. Also be looking for something special about the triples of angles that do work.***

Note: The special relationship of successful triples is similar to that of supplementary pairs; namely, they can be juxtaposed to form a straight line. Leave this important fact for students to discover.

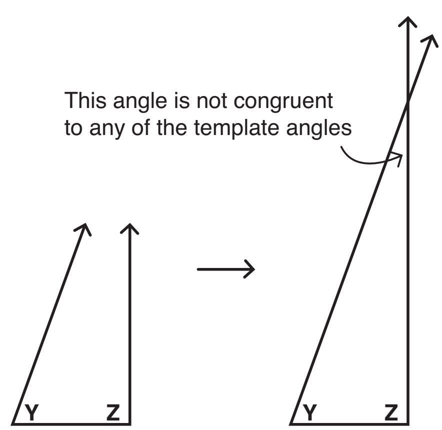




In pairs or small groups, allow students to work on the problem of constructing triangles for most of the remainder of the lesson. Many students will not immediately form a well-organized, systematic program for testing triples of angles. Instead they are more likely to choose triples almost at random. Encourage groups to keep a record of successes and failures, as illustrated here.

B - G - G	yes
A - B - G	no
Z - Y - G	no

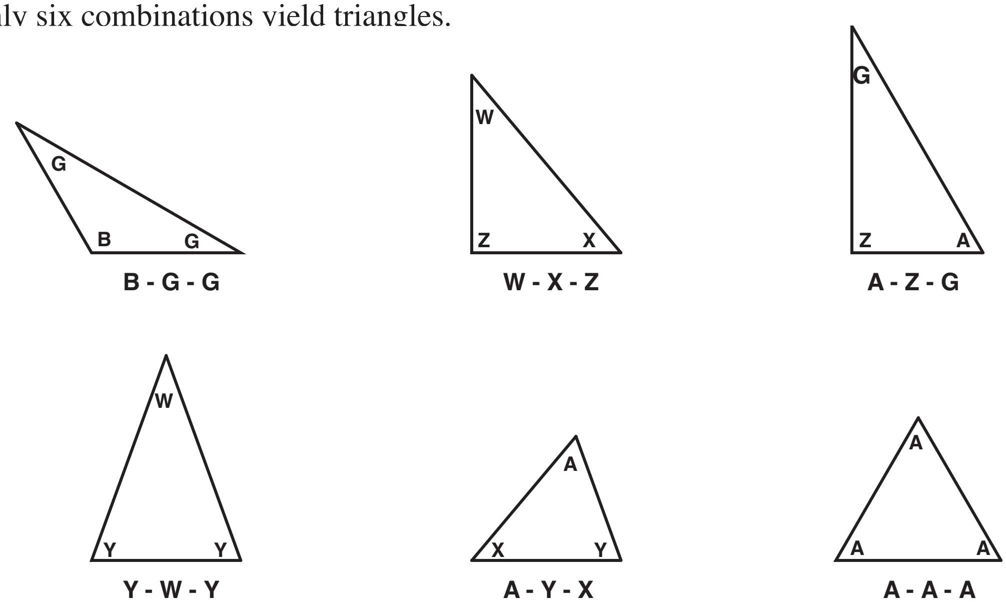
Students should discover through experimentation that once two angles are chosen and drawn in the construction of a triangle, then the third angle is already determined. For example, if **Y** and **Z** are chosen first and drawn, extending their rays completes a triangle. Then the question is whether the third angle is congruent to any of the seven among the templates. In this case, the answer is "no."



When students in a group make this observation, either independently or with prompting, pursue it. In particular, ask how this idea can be used to improve the class's estimates of the number of triangles. For example, with this information, there are seven choices for the first and seven choices for the second angle, but only one choice for the third angle. So the total number of combinations is at most 49 (7 x 7 x 1), which is much less than 343. A better estimate appears naturally in the next lesson on this topic, Lesson G8.

Reserve some time at the end of the lesson to begin separate lists of successes and failures to use as references in Lesson G8. Some suggestions for the list may not be correct and there may be some disagreement. However, list all suggestions for further discussion and checking, which will continue in Lesson G8. Also, save students' drawings for that lesson so students can continue to experiment with triples of angles.

Note: Only six combinations yield triangles.



G8 TRIANGLE CONSTRUCTIONS #2

Capsule Lesson Summary

Complete a collection of all of the possible triangles that can be constructed with seven angle templates. Explore a relationship among the triples of angles which yields triangles.

Materials

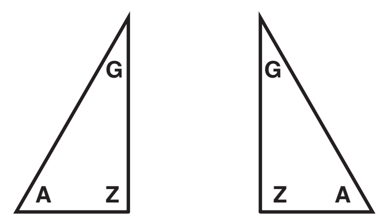
Teacher
- Record of work from G7
- Straightedge
- Demonstration angle templates corresponding to **A**, **B**, **G**, **W**, **X**, **Y**, and **Z** in the student set

Student
- Drawings from Lesson G7
- Straightedge
- Angle template set

Description of Lesson

Direct students to take out angle templates **A**, **B**, **G**, **W**, **X**, **Y**, and **Z**. Review the problem of constructing as many triangles as possible using these seven angles, and return students' drawings from Lesson G7. Then copy on the board the record of successes and failures already found in Lesson G7. Instruct students to check the lists for accuracy and for other triples not listed. Remind the class that they also were looking for some special property of the successful triples. Initiate discussion of the constructions and of the lists to lead the class toward completing the problem. For example:

- When there is disagreement about a proposed successful triple, collectively check its accuracy.
- Observe that when two angles of a triangle are drawn, there is no choice for the third. (See the end of Lesson G7 for a discussion of this fact.) Therefore, a pair of angles can appear on the success list at most once.
- Note that order in a triple is insignificant. For example, **A**-**G**-**Z** and **A**-**Z**-**G** give essentially the same triangle. One triangle is similar to a reflection of the other.

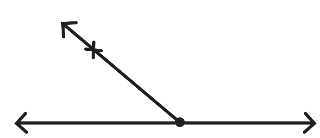


Note: A complete solution to the problem appears on the last page of Lesson G7.

When the class is confident that all of the possible triangles have been found, return to the question of whether there is some special property of successful triples. If no one makes the observation that a successful triple of angles can be juxtaposed to form a straight line, initiate a discussion as follows.

T: ***When we constructed quadrilaterals, we discovered the idea of supplementary pairs of angles.***

Draw the defining picture for a supplementary pair of angles on the board.



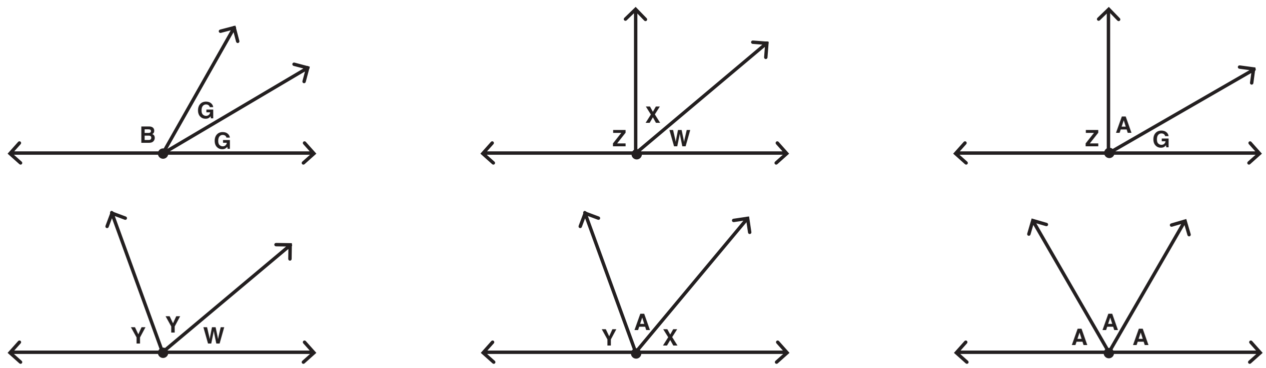
T: ***Choose one of your triangles and put its three angle templates together in a similar way. What do you find?***

S: ***We get a line with them.***

Demonstrate with a drawing on the board. The six combinations are shown below. Note that the order of the angles in each triple is not important.

T: ***Check the other successful triples with your templates.***

Allow several minutes for individual or group work. Conclude that all the successful triples form a line.



If any incorrect triple remains in the success list, students should find that its angles do not have this property. Use this opportunity to collectively check that the triple cannot be used to construct a triangle.

Record all of the successful triples (supplementary triples) as shown below for reference in Lesson G9.

B - G - G	**Z - X - W**	**Z - A - G**
Y - Y - W	**Y - A - X**	**A - A - A**

T: ***Now let's use the templates to check a triple that does not yield a triangle.***

S: ***The angles do not fit together to make a straight line either.***

G9 TRIANGLE CONSTRUCTIONS #3

Capsule Lesson Summary

Explore a relationship among the three angles of a triangle. Extend the experimentation to the four angles of a quadrilateral.

Materials

Teacher
- Straightedge
- Demonstration angle templates corresponding to **A**, **B**, **G**, **W**, **X**, **Y**, and **Z** in the student set
- Results from Lessons G5 and G8

Student
- Straightedge
- Several index cards
- Large unlined paper
- Angle template set
- Scissors

Description of Lesson

Begin with a review of the triangle construction problem from Lesson G7 and G8.

T: ***We have been using these templates*** (refer to angle templates **A**, **B**, **G**, **W**, **X**, **Y**, and **Z**) ***to construct triangles. What did we discover about the angles?***

S: ***We had seven angle templates to use, but not all combinations of the angles gave us a triangle.***

S: ***Some triples of the angles could successfully be used in a triangle, but most could not.***

S: ***We could construct only six different triangles with the seven templates.***

Refer to the results of Lesson G8, and copy the successful triples on the board.

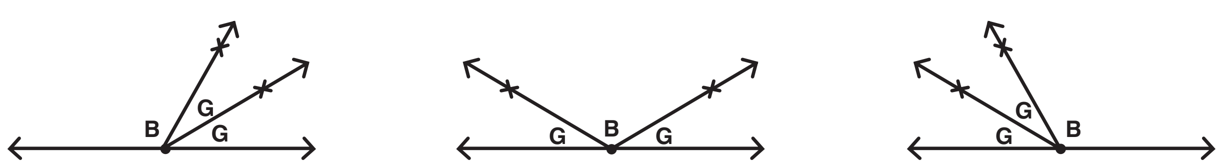
B - G - G	**Z - X - W**	**Z - A - G**
Y - Y - W	**Y - A - X**	**A - A - A**

T: ***These are the triples with which we were successful in constructing triangles. What special property did we find for these triples of angles?***

S: ***If we fit the three angles together, they formed a line.***

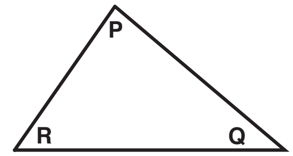
Invite students to choose one of the triples of angles for a triangle, and then fit them together.

Let students use the demonstration templates to produce an appropriate drawing on the board. For example, choosing **B**-**G**-**G** would produce one of these three drawings. Regardless of the order of the angles, a straight line is determined by juxtaposition.



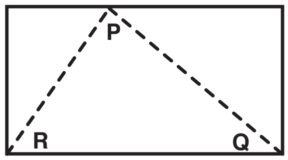
Exercise 1

Draw a triangle on the board and label the angles.

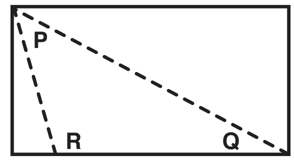


T: ***Cut out a triangle from an index card. Label the angles of your triangle* P*,* Q*, and* R *as I have done on the board.***

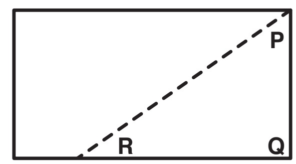
Suggest that students do this with two straight line cuts. For example:



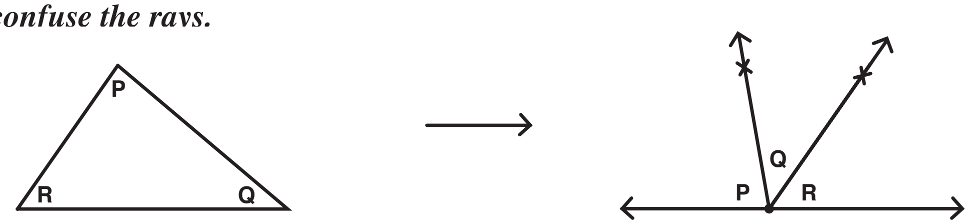
or



Of course, one straight line cut is sufficient but then the triangle necessarily has a right angle.



T: ***Trace angles* P*,* Q*, and* R *with a common vertex just as we did with the templates. Each time you add another angle to the drawing, mark the common ray with a small so you do not confuse the rays.***



Allow a few minutes for individual or partner work. Then discuss the results collectively.

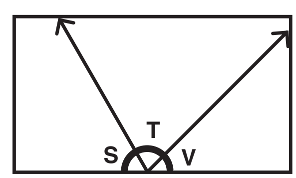
T: ***Do the angles of your triangles have the same property?***

S: ***Yes, we always get a straight line.***

Exercise 2

Illustrate the following task as you describe it.

T: ***Use another index card. Pick a point near the middle of a long edge, and use your straightedge to draw two rays from that point. Label the three angles as I am doing on the board.***

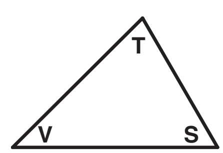


Cut along the rays to get three angle templates. Can you construct a triangle with angles* S*,* T*, and* V*?

Allow several minutes for individual or partner work. Again discuss the results collectively.

S: ***Mine works!***

S: ***We can always construct a triangle.***



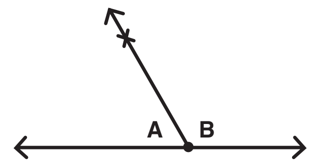
Exercise 3

T: ***Now let's look at a parallelogram.***

Draw a parallelogram on the board using angle templates **A** and **B**.

T: ***What do we know about angles A and B?***

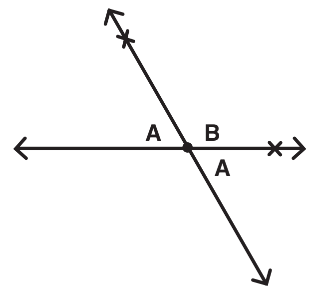
S: ***They are supplementary.***



Using **A** and **B**, draw the defining picture for a supplementary pair.

T: ***Suppose we add another angle congruent to A to this drawing. Match the vertex and one ray of template A to the second ray of B, and mark the common ray.***

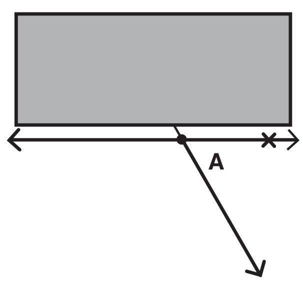
What can we say about the fourth angle, the angle formed by the unmarked rays?



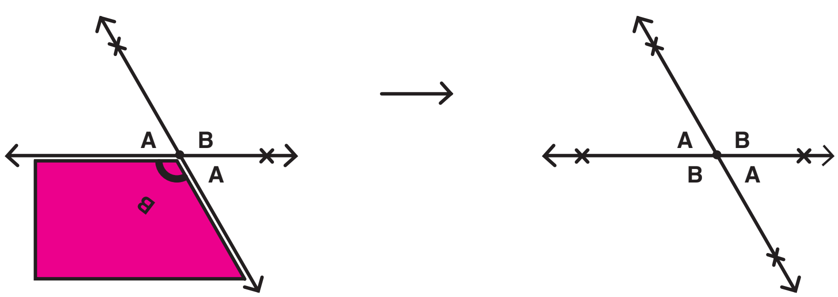
S: ***It is congruent to B.***

S: ***If we cover the top part of the drawing, then we see a supplementary pair. One angle in this pair is A so the other must by B.***

S: ***We could test it with template B.***



Illustrate that template **B** fits this fourth angle and label it.



S: ***The four angles of the parallelogram bring us around in a circle.***

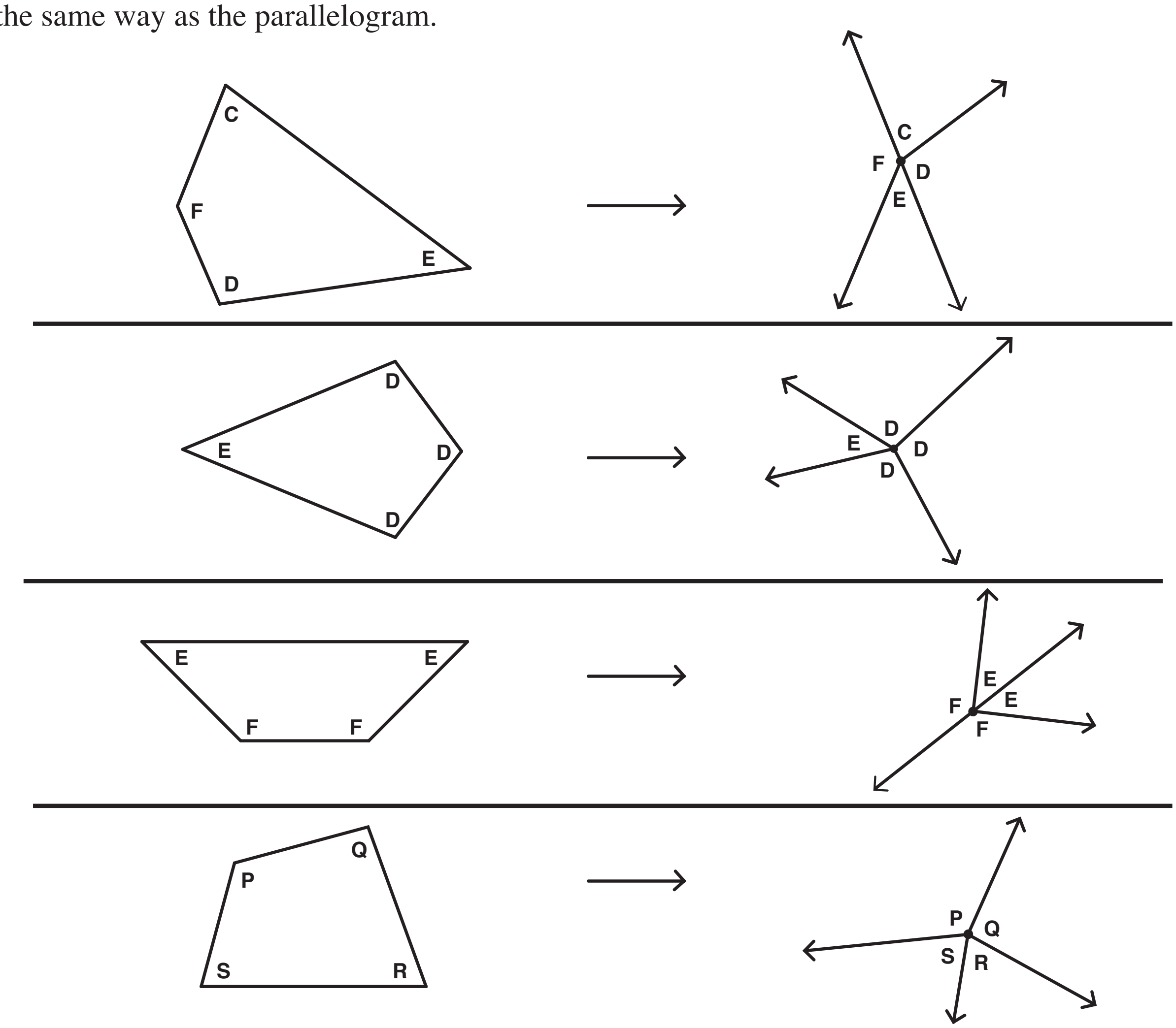
Display the list of quadrilaterals constructed in Lesson G5.

T: ***These are some quadrilaterals we were able to construct using certain of the angle templates. Choose one and check whether its four angles bring you around in a circle.***

Assist students in selecting a quadrilateral and the appropriate templates for checking its angles. Suggest that adding angles consistently in a clockwise direction around the vertex and marking the common rays after each addition will help to organize the work. Encourage careful, accurate drawings, since even a slight error may cause the experiment to fail. Some examples of completed work are shown on the next page.

Ask students who finish quickly to test arbitrary quadrilaterals. They can do this by cutting quadrilaterals any way they like out of index cards, and then proceeding to check the angles. The last example below is of this kind.

The collective experience will support the inference that all quadrilaterals behave in this construction in the same way as the parallelogram.



Home Activity

Suggest that students work with a family member to check that any four angles that fit together around a point (bring one around in a circle) can be used to construct a quadrilateral.

Capsule Lesson Summary

Observe that area is preserved when a shape is cut into pieces and all of the pieces are used to form a new shape. Solve some tangram-type puzzles where two squares are cut into five pieces and the five pieces can be put together to form one square. Find that the sides of the three squares can be used to form a right triangle. Discover that when a right triangle is formed by the sides of three squares, the sum of the areas of the two smaller squares equals the area of the largest square.

Materials

Teacher
- Demonstration 2-by-4 rectangle
- Scissors
- Tape
- Overhead projector
- Worksheet G10(a)

Student
- Scissors
- Tape
- Worksheets G10(a) and (b)

Advance Preparation: Use Blackline G10(s) to prepare (cut out) a 2-by-4 rectangle for use in Exercise 1. This rectangle has 6 cm grid squares. If you prefer a larger demonstration rectangle, use ledger size paper to make a 2-by-4 rectangle with 10 cm grid squares or cut out two copies of Blackline G10(l) and tape the squares together. Note in Exercise 1 that with the larger 20 cm by 40 cm rectangle, the area is 800 cm^2.

Cut out demonstration copies of the squares on Worksheet G10(a) for use in Exercise 2.

Description of Lesson

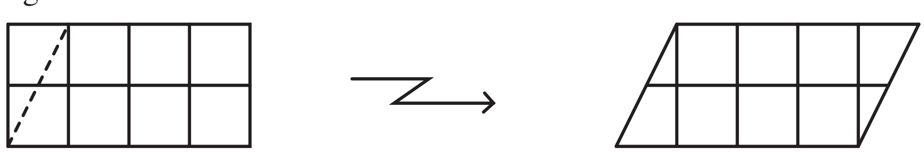
Exercise 1

Display the 2-by-4 rectangle.

T: ***Each of these grid squares is 6 cm on a side. What is the area in square centimeters of this rectangle?***

S: ***288 cm^2; each grid square is 36 cm^2 and there are eight grid squares in the rectangle. 8 x 36 = 288.***

Cut off a corner of the rectangle, as shown below with a dotted line. Then reposition the corner to form a parallelogram.



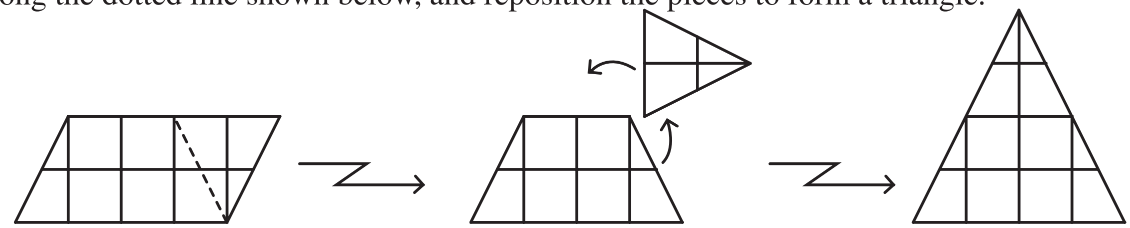
T: ***What is the area of this parallelogram? Why?***

S: ***288 cm^2. The parallelogram has the same area as the rectangle.***

Move the cut-off corner back and forth, forming the rectangle and the parallelogram, to emphasize that the area is unchanged. Then tape the cut-off corner so that you now have a parallelogram.

G10

Cut along the dotted line shown below, and reposition the pieces to form a triangle.



T: ***What is the area of this triangle? Why?***

S: ***288 cm². The triangle has the same area as the rectangle and the parallelogram.***

T: ***When a shape is cut into pieces and the pieces are repositioned to form a new shape, what can we say about the area of the new shape?***

S: ***The new shape has the same area as the original shape.***

T: ***What could we do if we wanted to get a shape with a different area?***

S: ***We could add some other pieces or we could take some pieces away (not use some pieces in forming the new shape).***

Exercise 2

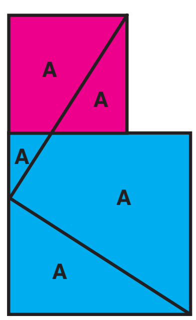
Direct students to cut out the squares on Worksheet G10(a).

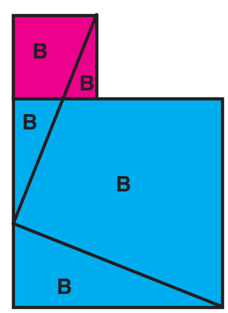
T: ***What do you notice about the squares?***

S: ***There are four squares marked with the letter* A *and four with the letter* B*.***

S: ***There are two red squares with the same area for each letter, and there are two blue squares with the same area for each letter.***

S: ***There are black line segments on some of the squares. If you put the red* A *square with lines on it on top of the blue* A *square with lines on it and line up their left sides, then the black line segments also line up. The same is true for the* B *squares.***

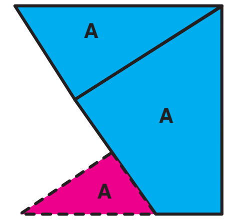




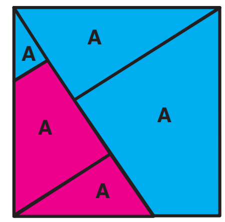
T: ***Put the squares without lines on them to one side. Carefully cut along the black lines of the remaining squares so that you have five* A *pieces and five* B *pieces.***

While students are working, cut your squares along the black line segments. Lay the five non-square **A** pieces on the overhead, and project them on the board. When everyone is finished, ask students to put aside the **B** pieces and refer to just the five non-square **A** pieces.

T: ***These five pieces all have* A *on them. See if you can find a way to put these five pieces together to form one big square.***



Allow a few minutes for individual or partner work. Students who finish quickly can be asked to do the same with five non-square **B** pieces. After a while, if students need some help, begin putting the pieces together on the overhead. Two or three pieces of the final square configuration, as illustrated here, should be enough of a hint.

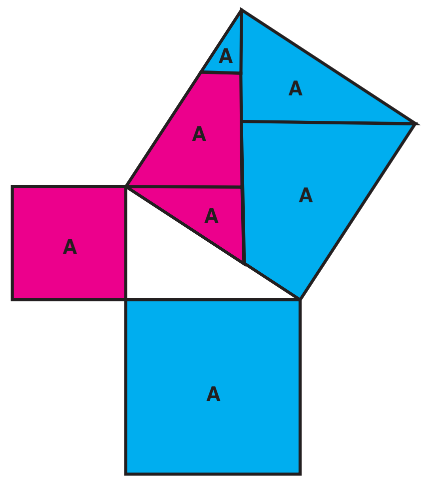


As students complete the square, ask them to tape the pieces together as a square. Invite a student to complete the square configuration on the overhead. Then tape the pieces together.

Refer students to their uncut red and blue **A** squares. Put these squares on the overhead as well.

T: ***What can we say about the area of this new big square?***

S: ***Its area is equal to the area of the red* A *square plus the area of the blue* A *square.***



T: ***Watch as I put certain corners of these three* A *squares together. Try it with your squares too.***

T: ***What do you notice?***

S: ***There is a triangle in the middle.***

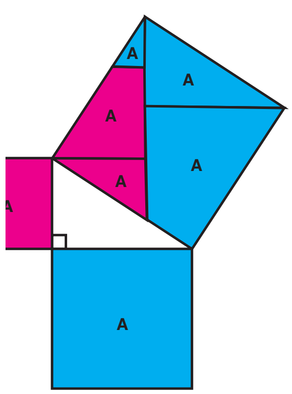
S: ***The sides of the three squares, one from each, form a triangle.***

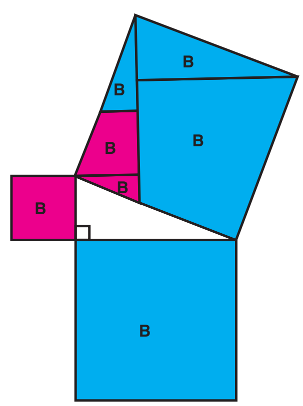
S: ***The triangle has a square corner (right angle).***

Invite a student to trace the triangle formed by sides of the respective squares and to point out the right angle. Mark the right angle in the projection on the board. Comment that triangles with a right angle are called *right triangles*.

Instruct students to push this final **A** square picture to one side on their desks or tables, keeping it together.

Repeat this activity using the **B** squares. Forming a square with the five non-square **B** pieces should be easier since it is very similar to the configuration with **A** pieces. The final picture using the **B** squares is shown below.





G10

Refer to the final pictures with the **A** squares and with the **B** squares.

T: ***Where is the right angle in these pictures?***

S: ***Between the all red square and the all blue square.***

T: ***What is special about the areas of the squares?***

S: ***In each case (A or B), the area of the red square plus the area of the blue square equals the area of the new square.***

Exercise 3

Distribute copies of Worksheet G10(b); instruct students to carefully cut out the squares.

T: ***Using these squares, try to form the right triangles like we did with the three A squares and the three B squares. Some combinations of three squares will work; some will not. Make a list of the triples of squares that can be used to form right triangles. The numbers on the squares are their areas in square centimeters. Use these numbers to identify the squares.***

Allow several minutes for individual or partner work. As students discover combinations of three squares that can be used to form right triangles, record the areas in triples on the board.
A complete list is given here.

(4, 9, 13)	(9, 16, 25)	(13, 36, 39)
(4, 45, 49)	(9, 36, 45)	

T: ***What do you notice about any triple of squares that can be used to form a right triangle?***

S: ***The sum of the areas of the two smaller squares equals the area of the largest square in the triple.***

G11 INTRODUCTION TO THE PROTRACTOR #1

Capsule Lesson Summary

Introduce the protractor and its use in measuring angles and in constructing angles of given measures. Discover that the sum of the measures of the angles of a triangle is 180°.

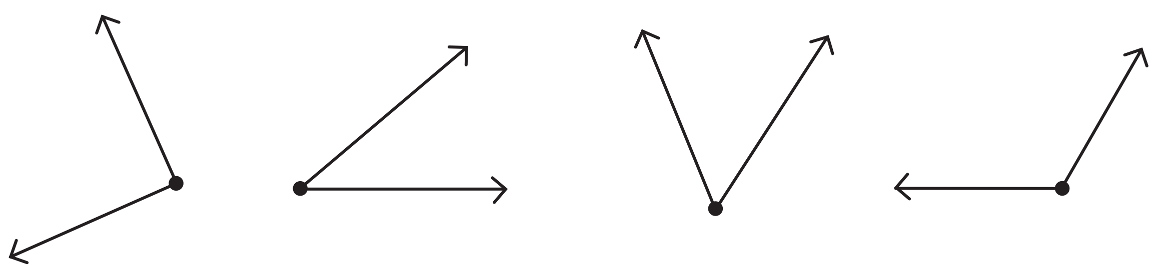
Materials

Teacher
- Demonstration protractor
- Straightedge
- Paper square corner

Student
- Paper
- Protractor
- Paper square corner
- Worksheets G11(a) and (b)

Description of Lesson

Before the lesson begins, use a protractor to draw four angles on the board. They should measure 90°, 40°, 55°, and 120°.



T: ***Here are some angles. How can we decide which is the largest and which is the smallest?***

S: ***Trace them on tracing paper and then compare them.***

S: ***Make templates for them so that we can compare two of them by putting one template over another.***

T: ***Those are good ideas, but they require some construction work either with tracing paper or with cardboard to make templates. If we draw two line segments on the board and want to know which of these line segments is longer, what could we do?***

S: ***Use a ruler to measure their lengths.***

T: ***Of course. A ruler is marked with a scale so that we can measure the length of one line segment, say in centimeters, and then measure the length of the second line segment, also in centimeters. Then we only need to compare the measures.***

Here is a tool for measuring angles. It is called a **protractor.**

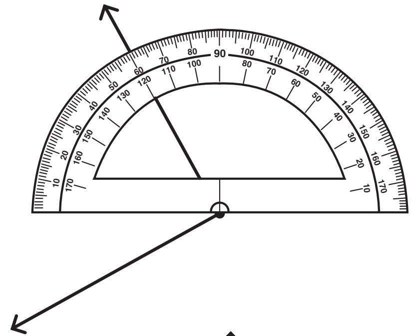
Display your demonstration protractor.

T: ***The protractor has a scale from 0 to 180. Each division is one degree. The degree is the unit for measuring angles. Do you recognize any of these angles on the board?***

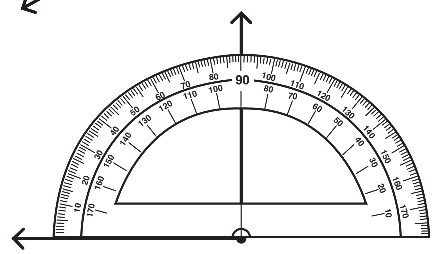
S: ***One of them is a right angle, two are acute angles, and one is an obtuse angle.***

G 11

Verify this observation using a paper square corner. Ask students to use a paper square corner to draw a right angle on their papers. Then, as you demonstrate how to measure an angle using the protractor, ask students to follow your actions to measure the right angle on their papers.



T: ***Let's use the protractor to measure this right angle. Look at your protractor. There is a small circle (or other mark) at the center of the straight edge of the protractor. Place the circle (mark) over the vertex of the angle.***



Then align the straight edge or line that passes through the circle (mark) on your protractor with one of the rays of the angle. Now look where the other ray cuts across the scale of the protractor.

Some students may need to extend the rays in their drawings to cut across the scale of the protractor.

T: ***What is the measure of a right angle?***

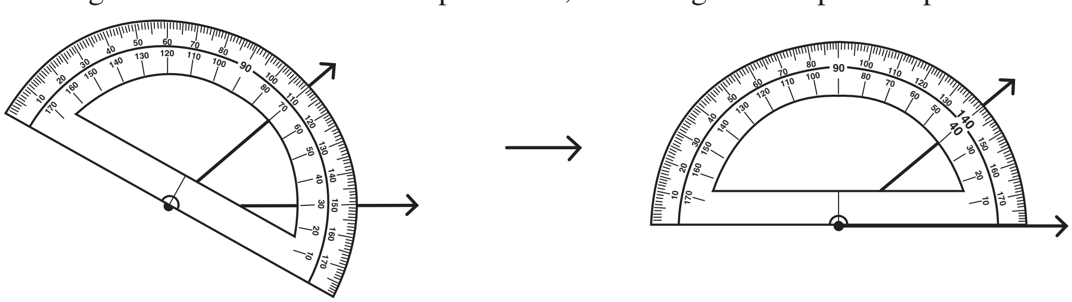
S: ***90°.***

Point to the angle whose measure is 40°.

T: ***Is this angle larger or smaller than a right angle?*** (Smaller)
What kind of angle is this? (Acute)
Do you think its measure will be more or less than 90°?

S: ***Less, because it is smaller than a right angle.***

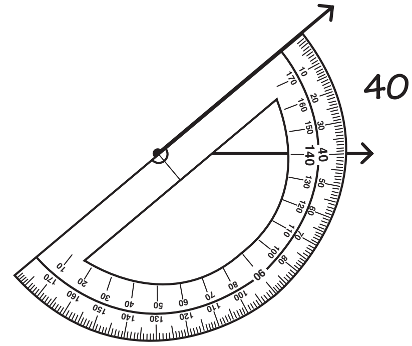
Measure the angle with the demonstration protractor, reviewing each step of the procedure.



T: ***There are numbers on the scale of a protractor to help count the degrees. In fact, there are two rows of numbers and we must be careful to use the correct row. In this case the second ray crosses the scale at a mark for both 40 and 140. Which number do you think we should use as the measure of this angle?***

S: ***40; because 40 is less than 90.***

Demonstrate the alternative alignment and measurement.



Repeat this activity, measuring the remaining two angles on the board. Before measuring an angle with the protractor, decide whether it is acute or obtuse. Then use this information to choose the proper scale when measuring.

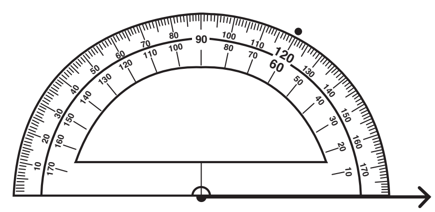
You may wish to point out that an alternative to first observing whether an angle is acute or obtuse is to first observe which row of numbers has the first ray (aligned along the edge) at 0. Use that row of numbers in measuring the angle. There are two rows of numbers because we can align the protractor with either of the two rays of an angle.

Instruct students to measure each of the angles on Worksheet G11(a). Monitor use of the protractor and correct faulty technique. When most students finish this task, check the angle measurements collectively. An answer key for this worksheet is at the end of the lesson. Then continue a discussion of the protractor.

T: ***In addition to measuring angles, we can use the protractor to construct angles. Suppose we want to construct an angle measuring 60°. How might we start?***

S: ***Pick a point for the vertex, and draw a ray with that point as its endpoint.***

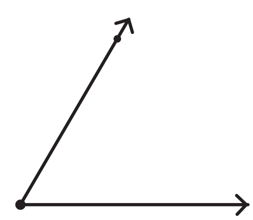
Draw a ray on the board.



S: ***Now put the center mark of the protractor on the endpoint of the ray and align the protractor with the ray.***

S: ***Next, count from 0 to 60 on the scale and make a dot on the chalkboard at 60.***

S: ***Draw a ray from the vertex through the dot.***



Refer to Worksheet G11(b).

T: ***On this worksheet you are to construct several angles with your protractor.***

Again monitor angle construction techniques. A common mistake is to use the wrong scale. Simply remind students, as necessary, to first decide whether an angle will be acute or obtuse. Alternatively, remind students to begin counting degrees at 0. When most students finish the constructions, resume the discussion.

T: ***Now draw a triangle. Measure all three angles of the triangles and add the measures.***

G11

Record each student's total on the board. A typical set of results is shown here.

175°	181°	183°	180°	183°	186°	182°
180°	181°	185°	181°	180°	182°	180°
182°	178°	187°	187°	180°	180°	179°
176°	181°	180°	184°	178°		

T: ***What do you think of these totals?***

S: ***They are all 180° or close to 180°.***

T: ***Can you give any reason for the differences?***

S: ***We all drew different triangles.***

S: ***I think that it is hard to measure accurately. I had trouble deciding whether to read 42° or 43° for one of my angles.***

T: ***Different readings in measuring the angles would account for differences in the totals.***

S: ***All of the totals are 180° or close to 180°, and the protractor scale is divided into 180 degrees.***

S: ***Following the scale from one side to the other on the protractor is like going halfway around a circle.***

T: ***When we were using templates, what did we discover when we fit the three angles of a triangle together?***

S: ***They made a straight line, just like the straight edge of the protractor.***

S: ***So the three angles of a triangle together measure 180°.***

T: ***Can you construct a triangle with angles measuring 40°, 60°, and 110°?***

S: ***No. The total is 210° and that is more than 180°.***

T: ***If two angles of a triangle measure 40° and 60°, what does the third angle measure?*** (80°)

A while back we used seven templates to construct triangles. How could we use a protractor to find all of the possible triangles?

S: ***First we could measure all seven angles. Then we could find which triples of angles have a total measure of 180°.***

T: ***These are the measures of the seven angles from the templates.***

Write them on the board.

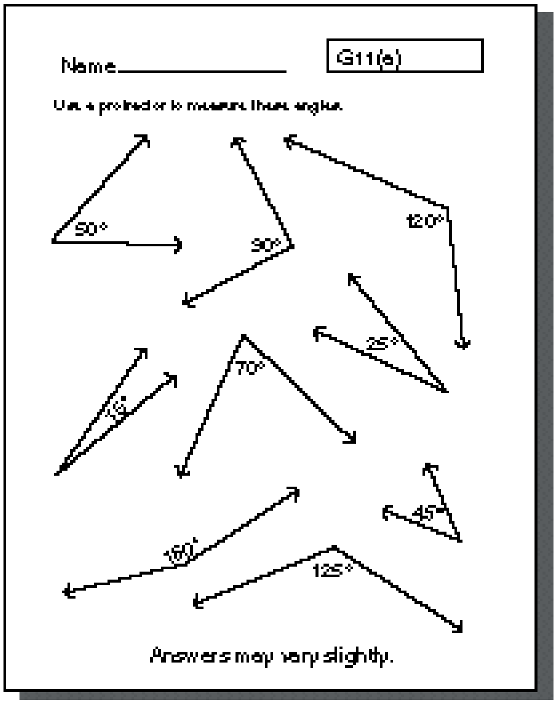
G	W	X	A	Y	Z	B
30°	40°	50°	60°	70°	90°	120°

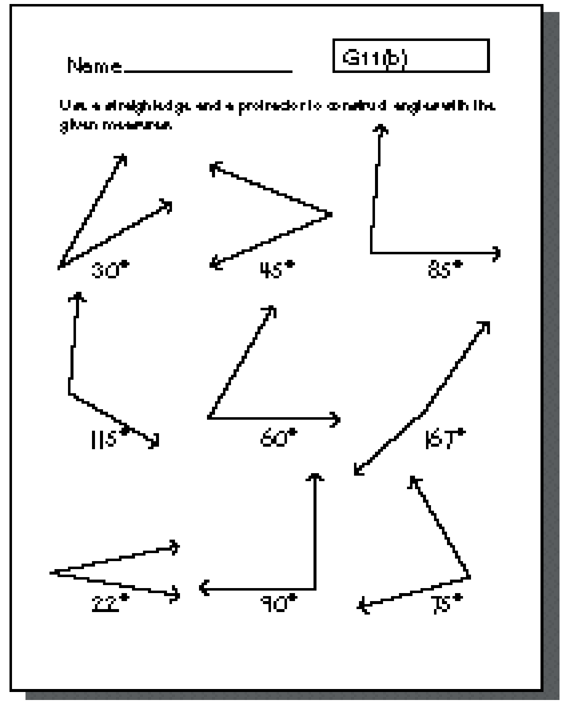
T: ***Which triples give triangles?***

End the discussion by listing the six triples which add to 180°.

B - G - G   **Z - X - W**   **Z - A - G**

Y - Y - W   **Y - A - X**   **A - A - A**


Name
G11(a)
Use a protractor to measure these angles.
50°
90°
120°
25°
15°
70°
160°
45°
125°
Answers may vary slightly.


Name
G11(b)
Use a straightedge and a protractor to construct angles with the given measures.
30°
45°
85°
115°
60°
167°
22°
90°
75°

G12 INTRODUCTION TO THE PROTRACTOR #2

Capsule Lesson Summary

Discover relations among the sums of the measures of angles of polygons without indentations.

Materials

Teacher
- Demonstration protractor
- Straightedge
- Colored chalk

Student
- Paper
- Protractor

Description of Lesson

Begin the lesson with a brief review of the results of Lesson G11.

T: ***Last week we used a protractor to measure angles. What did we learn about the three angles of a triangle?***

S: ***If we measure the three angles and add the measures, the sum is 180°.***

T: ***Did you all get a sum of 180° when you measured the angles of the triangle that you drew?***

S: ***No, but that is because measurements are not always exact. We were all close to 180°.***

T: ***Do the same exercise again, but this time, instead of a triangle, draw a quadrilateral with no indentation.***

Allow time for individual work. Then record each student's total on the board. A typical set of results is shown here.

360°	358°	357°	356°	359°	362°
355°	363°	360°	358°	361°	360°
365°	362°	360°	364°	360°	360°
358°	360°	350°	360°	362°	359°

S: ***Most of the results are 360° or close to 360°.***

T: ***Why do you think that 360° appears to be important for quadrilaterals?***

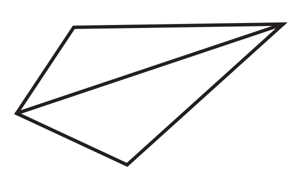
S: ***We get 180° with a triangle and 360° with a quadrilateral.***

S: ***The four angles of a quadrilateral bring us completely around in a circle.***

S: ***We can always divide a quadrilateral into two triangles by drawing a line segment between opposite corners. So a quadrilateral has the same angle sum as two triangles. 2 x 180 = 360.***

Sketch a drawing for this explanation on the board.

T: ***Each of you draw a diagonal in your quadrilateral to make two triangles. Measure all of the angles of the triangles and add the measures together.***



Allow time for individual work.

T: ***Do you get the same total as before?*** (Yes) ***What angle sum do you think we get if we add the measures of the angles of a pentagon with no indentations?***

S: ***540.***

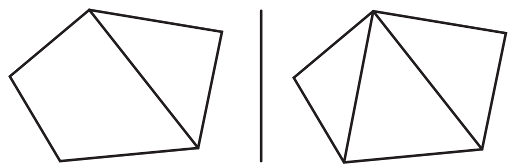
T: ***Each of you draw a pentagon without indentations and find the total of the measures of its angles.***

Allow time for individual work. Then let students share their results with the class.

T: ***Why do you think we are getting sums of 540° or close to 540°?***

S: ***If you connect two (non-adjacent) corners of a pentagon you make a triangle and a quadrilateral. The angles of the triangle contribute 180° and the angles of the quadrilateral contribute 360°. So the sum of the measures of the angles of a pentagon should be 540°.***

Sketch a drawing for this explanation.



Note: Another explanation might involve dividing the pentagon into three triangles. Then multiply 3 x 180 to get the total of 540°, as illustrated here.

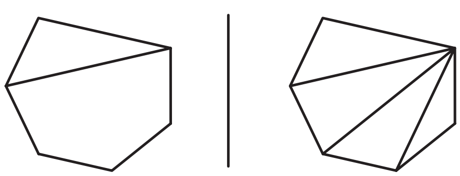
T: ***Let's make a table of the information we have found.***

Number of sides	Sum of angle measures
3	180
4	360
5	540
6	

What sum do you expect for a hexagon with no indentations?

S: ***720°; add 180° to the sum for a pentagon because we can connect two corners of a hexagon to make a pentagon and a triangle.***

S: ***4*** x ***180° = 720°. A hexagon can be divided into four triangles.***



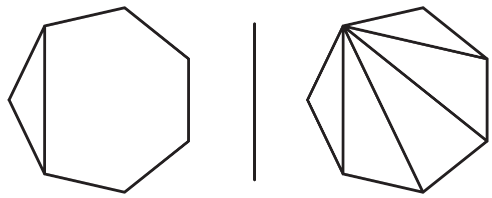
Number of sides	Sum of angle measures
3	180
4	360
5	540
6	720

S: ***4*** x ***180° = 720°. A hexagon can be divided into four triangles.***

T: ***And what about the sum for a septagon with no indentations?***

S: ***900°; add another 180°.***

S: *5* x *180° = 900°.*



Number of sides	Sum of angle measures
3	180
4	360
5	540
6	720
7	900

T: ***Do you see a pattern?***

S: ***For the sum of the measures, subtract 2 from the number of sides and then multiply times 180.***

S: ***Take the number of sides times 180 and then subtract 360°.***

T: ***What about the sum for a kilogon (1000 sides)?***

S: ***For a kilogon, 998*** x ***180 = 179 640.***

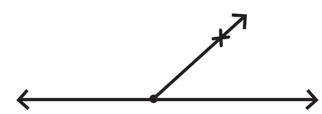
S: ***1000*** x ***180 = 180 000, and 180 000 – 360 = 179 640.***

Number of sides	Sum of angle measures
3	180
4	360
5	540
6	720
7	900
1000	179 640

Add this information to the table, and save the table for later reference.

T: ***Now let's construct a supplementary pair of angles.***

Invite a student to draw a supplementary pair of angles on the board.

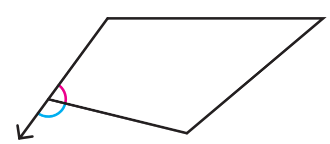


T: ***What does this picture suggest about the measures of a supplementary pair?***

S: ***The sum of their measures is 180°.***

T: ***What is the measure of a supplement to a 50° angle?*** (130°)
Of a supplement to a 135° angle? (45°)

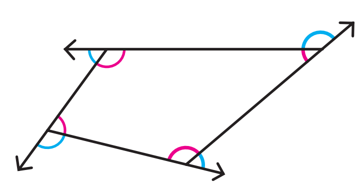
On the board, draw a quadrilateral with no indentation. Then alter the drawing as in the following pictures. Describe the alterations as you make them.



T: ***Extend one side of this quadrilateral to indicate a ray.***

Now, mark the angle of the quadrilateral in red and its supplement in blue.

T: ***Do the same again at the other three corners.***



S: ***The rays make the drawing look like a pinwheel.***

T: ***How many supplementary pairs did we indicate here?***

S: ***Four.***

T: ***And what is the sum of their measures?***

S: ***720°; 4 x 180 = 720.***

Add another column to the table.

Number of sides	Sum of angle measures	
3	180	
4	360	720
5	540	
6	720	
7	900	
1000	179 640	

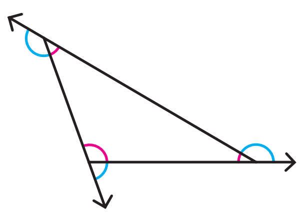
T: ***Let's do the same thing with a triangle.***

Draw a triangle and repeat the process, as pictured below.

T: ***How many supplementary pairs are there?***

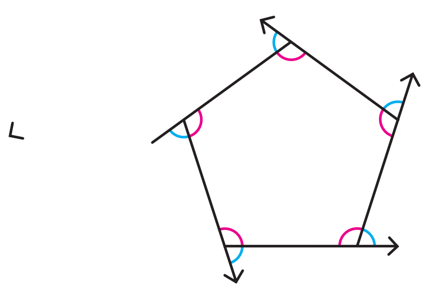
S: ***Three, and the sum of their measures is 540°; 3 x 180 = 540.***

Add the information to the table.



Number of sides	Sum of angle measures	
3	180	540
4	360	720
5	540	
6	720	
7	900	
1000	179 640	

T: ***Let's do the same thing with a pentagon having no indentations.***



Number of sides	Sum of angle measures	
3	180	540
4	360	720
5	540	900
6	720	
7	900	
1000	179 640	

T: ***What do you predict for a hexagon (without indentations)?***

S: ***The sum is equal to the number of sides times 180, because there are as many corners as sides and we build a supplementary pair at each corner.***

Complete the third column in the table.

T: ***Now let's go back to the quadrilateral. What is the sum of the measures of the blue angles? Use the table to help answer this question.***

S: ***The red angles are the angles of the quadrilateral and their measures add to 360°. The measures of the red and blue angles together add to 720°. So the measures of the blue angles must add to 360°.***

Add another column to the table, as in the next illustration. Then, fill in this column for a quadrilateral.

T: *What is the sum of the measures of the triangle's blue angles?*

S: *Red and blue together measure 540°. Red angles alone measure 180°. So the blue angles measure 360°.*

S: *The sum of the measures of the blue angles is 360° for the pentagon and for the hexagon.*

Number of sides	Sum of angle measures		Sum of blue angle measures
3	180	540	360
4	360	720	360
5	540	900	360
6	720	1080	360
7	900	1260	
1000	179 640	180 000	

S: *The same is always true. The number in the second column gives the total number of degrees for the red angles, and the number in the third column gives the total number of degrees for the red and blue angles together. We only have to subtract. So we always get 360° as the total number of degrees for the blue angles.*

S: *Even for the kilogon!*

Complete the table.

Capsule Lesson Summary

Given two line segments, construct as many essentially different triangles as possible with each side the same length as one of the line segments. Discover a rule relating the length of the segments to the number of different triangles. Repeat this activity with three given line segments and discover the Triangle Inequality.

Materials

Teacher
- Chalkboard compass
- Straightedge

Student
- Unlined paper
- Straightedge
- Compass
- Metric ruler
- Worksheets G13(a) and (b)

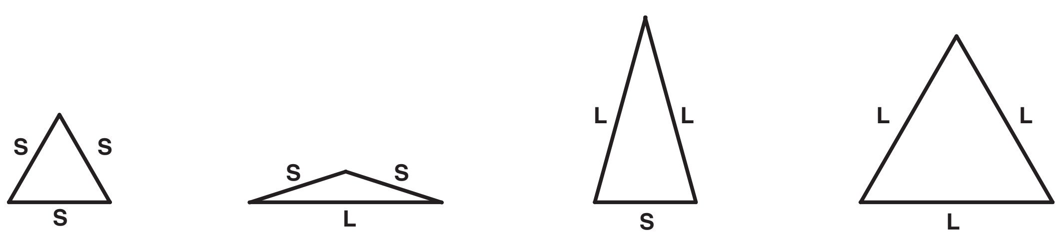
Description of Lesson

Exercise 1

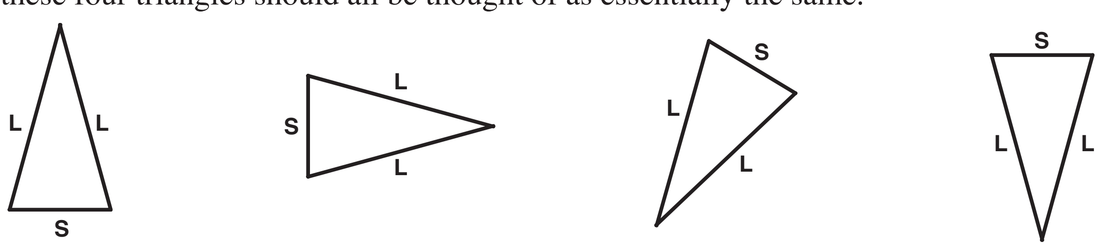
Draw two line segments on the board, one about 25 cm long and the other about 40 cm long.

T: ***These two line segments have different lengths. Let's draw a triangle using a compass and a straightedge. Each side of the triangle must be the same length as one of these line segments.***

Invite students to perform the construction at the board while the class discusses the technique. (See *IG-V* Lesson G12 Polygons #2 for a description of compass and straightedge constructions.) Be sure students realize that although a meter stick or metric ruler may be used for drawing straight line segments in these constructions, it should not be used for measuring. There are four essentially different triangles that can be constructed depending on the number of each length side chosen: three short; two short and one long; one short and two long; and three long.



If an equilateral triangle is constructed either with three short or with three long sides, comment on its correctness, and then introduce the restriction that each of the two lengths must be used at least once in each triangle. During the discussion, introduce the idea that the orientation of the triangle on the board does not alter it in any way important to its identity with respect to the choice of sides. Thus, these four triangles should all be thought of as essentially the same.



Once the class is secure in the triangle construction technique and there are two triangles (short-short-long and long-long-short) on the board, distribute copies of Worksheet G13(a) which shows five pairs of line segments. Direct students to construct as many triangles as possible with each pair of segments. Each side of a triangle must be the same length as one member of the pair, and each line segment of the related pair must be used at least once. The pairs are denoted **A**, **B**, **C**, **D**, and **E**. Instruct students to mark each triangle with the same letter as the pair of segments used to construct it. As students fill up the space on the worksheet, provide them with unlined paper.

Encourage accurate and careful constructions. Allow time for experimentation and for the conviction to grow that in no case are more than two triangles possible. Encourage students to try to formulate a rule to predict the possibility of one or two triangles, and then to draw pairs of segments to test the rule. Now students may use the metric rulers for measuring in order to further test a rule based on relative lengths of the line segments.

Ask several students to measure, to the nearest centimeter, the lengths of the line segments on the worksheet. Collect the results in a table on the board.

	Short segment	Long segment	Number of triangles
A	3 cm	5 cm	2
B	3 cm	7 cm	1
C	5 cm	7 cm	2
D	2 cm	4 cm	1
E	4 cm	9 cm	1

Lead the discussion to elicit a rule for deciding when two triangles are possible. There are at least two good ways to state this rule:

- The short segment must be more than half as long as the long segment.

OR

- Twice the length of the short segment must be more than the length of the long segment.

Check students' understanding of the rule by listing several pairs of lengths and asking for the number of possible triangles.

Short	Long	Number of triangles
4 cm	5 cm	2
15 cm	25 cm	2
15 cm	30 cm	1
15 cm	40 cm	1
1007 cm	2 003 cm	2
4.5 cm	8 cm	2

Exercise 2

Pose a triangle construction problem where three different length line segments are given.

T: ***Now let's use three segments of different lengths to draw triangles. Each segment must be used once in a triangle.***

Distribute copies of Worksheet G13(b). Proceed as with Worksheet G13(a). After the individual work, collect the results on the board.

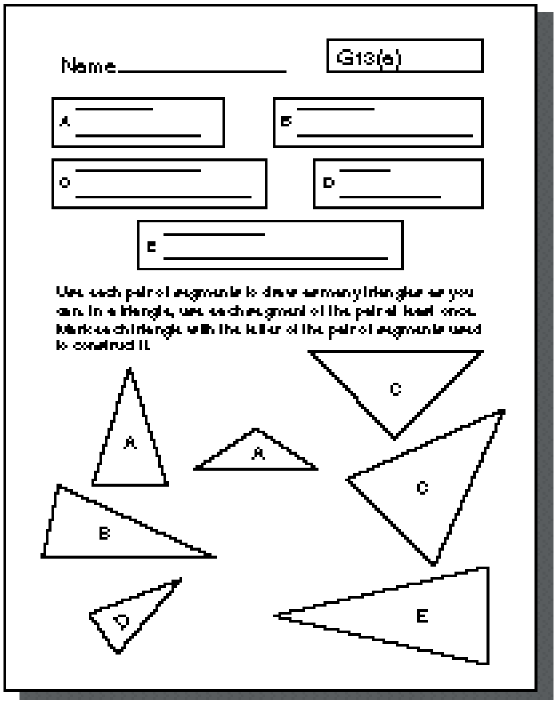
	Lengths			Number of triangles
A	3 cm	4 cm	5 cm	1
B	2 cm	4 cm	7 cm	0
C	1.5 cm	3.5 cm	4 cm	1
D	2.5 cm	3.5 cm	6 cm	0

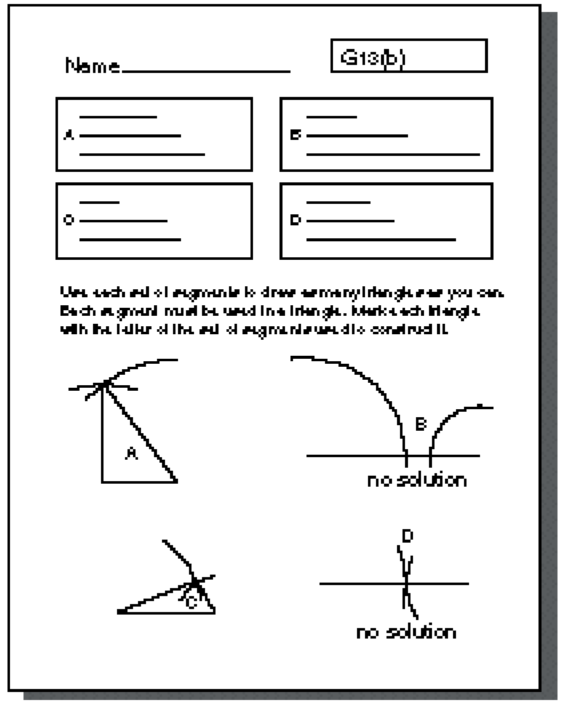
Discuss the results with the purpose of formulating a rule for the possibility of constructing a triangle. This is a good formulation:

- The sum of the lengths of the two shorter segments must be more than the length of the longest segment. (Triangle Inequality)

Ask students to apply the rule to several sets of lengths, deciding whether or not a triangle can be formed.

Lengths			Triangle?
1 cm	6 cm	8 cm	No
2 cm	6 cm	8 cm	No
3 cm	6 cm	8 cm	Yes
107 cm	412 cm	500 cm	Yes
4.3 cm	9.8 cm	15 cm	No


Name
G13(a)
A
B
C
D
E
Use each pair of segments to draw as many triangles as you can. In a triangle, use each segment of the pair at least once. Mark each triangle with the letter of the pair of segments used to construct it.
A
A
C
C
B
D
E


Name
G13(b)
A
B
C
D
Use each set of segments to draw as many triangles as you can. Each segment must be used in a triangle. Mark each triangle with the letter of the set of segments used to construct it.
A
B
no solution
C
D
no solution

Capsule Lesson Summary

Pose the problem of finding the probability that if we break a stick in two places, choosing those places randomly, the three resulting pieces will form a triangle. Examine many cases to decide conditions for success and for failure. Rephrase the Triangle Inequality.

Materials

Teacher
- 20 cm stick or straw
- Straightedge
- Blacklines G14(a) and (b)

Student
- Colored pencils
- Compass
- Metric ruler
- Unlined paper
- Sheet with 20-cm line segments
- Breaking points table
- Worksheets G14* and **

Advance Preparation: Locate a stick that breaks easily (e.g., balsa wood) or a straw that you can cut to 20 cm length. For students, use Blackline G14(a) to make copies of a sheet with 20 cm line segments; use Blackline G14(b) to makes copies of the table for recording broken points.

Description of Lesson

Display a stick (or straw) 20 cm long.

T: ***Suppose we have a stick 20 cm long and we break it in two places. What is the probability of being able to make a triangle with the three pieces if the breaking points are chosen at random?***

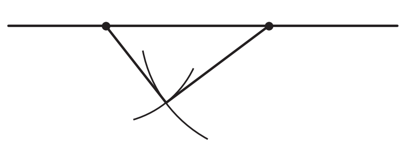
Let students freely discuss the problem. Some students may wish to estimate the probability; others to suggest ways of finding the probability; and others to discuss how to break the stick at two randomly chosen breaking points. At some time during the discussion, let the class select two breaking points on the stick. Make the breaks and try to make a triangle with the three pieces.

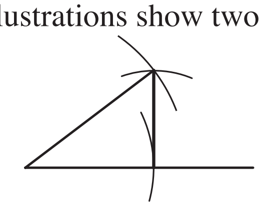
Distribute copies of Blackline G14(a) to pairs of students.

T: ***There are five 20-centimeter line segments drawn on this sheet. Each has two breaking points indicated by dots. In each case, decide whether a triangle can be formed with the three pieces. Use a compass and a ruler for a straightedge.***

Note: If you prefer, give student pairs five 20-cm pieces of string to cut at the breaking points indicated on the sheet. Then they can attempt to make triangles with the pieces of string straightened as line segments.

Help students who have difficulty getting started. When most student pairs have completed four or five of the problems, discuss them collectively. The following illustrations show two similar methods of construction.





T: ***You were able to make triangles with the segments in A and in C but not with those in B, in D, and in E. Why?***

S: ***The two short pieces together must be longer than the longest piece.***

T: ***Let's measure to check what you are saying.***

S: ***The three lengths in A are 5 cm, 8 cm, and 7 cm. 5 + 7 is more than 8, so we can make a triangle.***

S: ***The three lengths in B are 3 cm, 5 cm, and 12 cm. 3 + 5 is less than 12, so we cannot make a triangle.***

T: ***In C, what are the three lengths?***

S: ***9 cm, 2 cm, and 9 cm.***

T: ***So we have to compare 9 + 2 to 9 since we must use a 9 cm segment as a longest piece.***

S: ***9 + 2 is more than 9, so we can make a triangle. It will be a narrow triangle.***

In a similar manner, discuss **D** and **E**.

Begin a table on the board similar to that on Blackline G14(b). Provide students with a copy. Draw a 20-cm line segment on the board, marking one end 0 and the other 20. Then place breaking points at 5 cm and 14 cm.

T: ***If we break the 20-cm stick at 5 cm and at 14 cm, what would the lengths of the three pieces be?***

S: ***5 cm, 9 cm, and 6 cm.***

T: ***Could we make a triangle?***

S: ***Yes, 5 + 6 > 9.***

Brea ing points		Lengths	Triangle?
5 cm	14 cm	5 cm 9 cm 6 cm	yes

Record the information in the table. Instruct students to use the five 20-cm line segments on Blackline G14(a) to enter data in the table. Continue the table with other choices for the breaking points, occasionally letting students choose them. Be sure to include an example in which one of the three lengths is 10 cm and, hence, the sum of the lengths of the other two segments is exactly 10 cm, giving a failure.

Your table should resemble this one.

Brea ing points		Lengths			Triangle?
5 cm	14 cm	5 cm	9 cm	6 cm	yes
5 cm	13 cm	5 cm	8 cm	7 cm	yes
3 cm	8 cm	3 cm	5 cm	12 cm	no
9 cm	11 cm	9 cm	2 cm	9 cm	yes
11 cm	15 cm	11 cm	4 cm	5 cm	no
2 cm	$12\frac{1}{2}$ cm	2 cm	$10\frac{1}{2}$ cm	$7\frac{1}{2}$ cm	no
6 cm	12 cm	6 cm	6 cm	8 cm	yes
3 cm	13 cm	3 cm	10 cm	7 cm	no
1 cm	4 cm	1 cm	3 cm	16 cm	no
$6\frac{2}{3}$ cm	$13\frac{1}{3}$ cm	$6\frac{2}{3}$ cm	$6\frac{2}{3}$ cm	$6\frac{2}{3}$ cm	yes
1.4 cm	14.2 cm	1.4 cm	12.8 cm	5.8 cm	no

T: ***If one of the breaks is at 4 cm, where could the other break be so that we could form a triangle? Why?***

S: ***At 11 cm. The lengths would be 4 cm, 7 cm, and 9 cm. 4 + 7 > 9.***

S: ***At 12 cm. The lengths would be 4 cm, 8 cm, and 8 cm. 4 + 8 > 8.***

Any breaking point between 10 cm and 14 cm would be a correct answer. Record this information in the table.

Consider other possibilities such as having one breaking point at 13 cm, one at 10.5 cm, one at 10.25 cm, and one at 10 cm. Only in the last case is it impossible to construct a triangle. Record the information in separate lines in the table, as illustrated here.

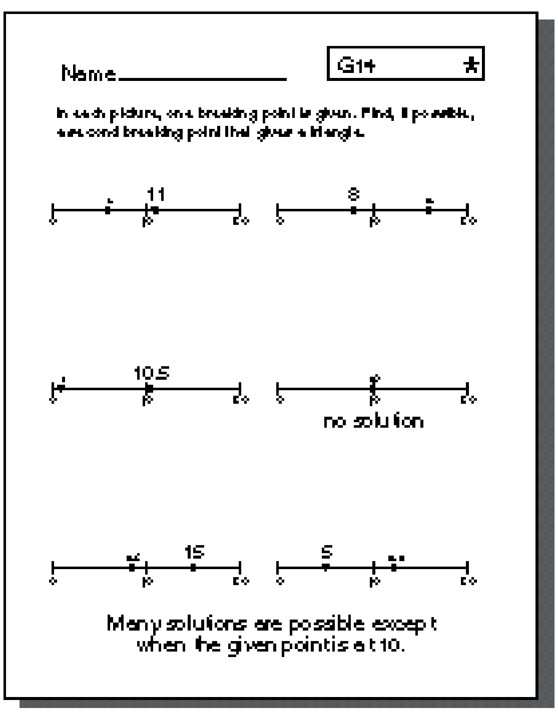
Brea ing points		Lengths			Triangle?
4 cm	11 cm	4 cm	7 cm	9 cm	yes
4 cm	12 cm	4 cm	8 cm	8 cm	yes
13 cm	7.5 cm	7.5 cm	5.5 cm	7 cm	yes
10.5 cm	5.5 cm	5.5 cm	5 cm	9.5 cm	yes
10.25 cm	2 cm	2 cm	8.25 cm	9.75 cm	yes
10 cm					no

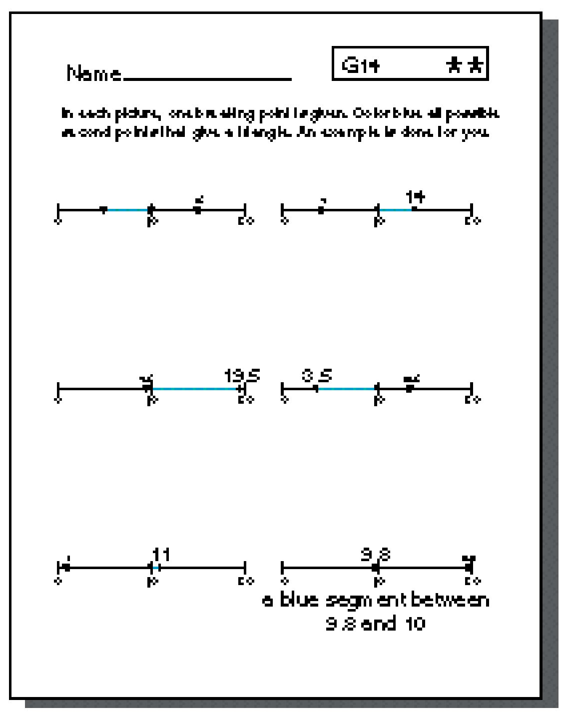
T: ***When can we make a triangle? How long can the pieces be for success?***

Students may state the Triangle Inequality (as given in Lesson G13), but lead them to also notice the equivalent rule that each piece must be shorter than half the length of the stick (in this case, 10 cm). To elicit this rule, direct attention to the list of lengths in the table. Successes occur when exactly all lengths are less than 10 cm. In the discussion, someone might comment that if the longest length is more than 10 cm, then the other two pieces together must be shorter than 10 cm since the stick is 20 cm long. So if one piece is longer than 10 cm, there is a failure.

G14

Worksheets G14* and ** are available for individual work to provide practice determining successful choices of points. On the ** worksheet, explain that the open dots in the example are used only to indicate that certain points *cannot* be breaking points, whereas every point *between* the open dots could be a second breaking point.





Capsule Lesson Summary

Establish the one-to-one correspondence between a point in a triangle and two breaking points on a stick. Use this correspondence to provide a means for randomly choosing two breaking points.

Materials

Teacher
- Colored chalk
- Meter stick
- Grid board

Student
- Worksheets G15* and**

Advance Preparation: Use Blackline G15 to make a grid, or prepare your grid board as indicated on this Blackline.

Description of Lesson

On the board, draw a line segment about 1 m long, and refer to this as a "stick" in the following discussion.

T: ***What is the "breaking a stick" problem?***

S: ***If you break a stick in two places to make three pieces, sometimes the three pieces can be used to form a triangle, and sometimes they cannot. We want to know the probability of forming a triangle if the breaking points are chosen at random.***

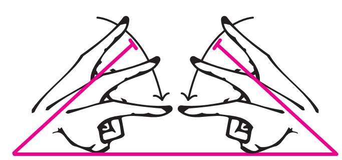
T: ***When will the three pieces form a triangle?***

S: ***When the shorter two pieces together are longer than the longest piece.***

T: ***With our 20-cm stick, could one of the pieces be 10 cm long?***

S: ***No, because together the other two pieces would be 10 cm long. When we try to make a triangle, the two shorter pieces collapse to a line segment 10 cm long.***

Students often use their hands in describing this situation.



T: ***Could one of the pieces be longer than 10 cm?***

S: ***No, there would be even less of the stick left for the other two pieces.***

T: ***So what can we say about the length of each of three pieces that will form a triangle?***

S: ***Each is shorter than 10 cm.***

Write this requirement on the board for emphasis. Mark the midpoint of the line segment on the board and label it 10 cm.

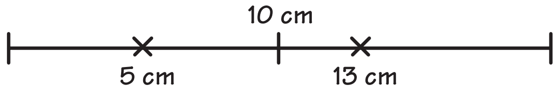
T: ***The problem involves a question about probability. If we choose the breaking points at random, what is the probability that the three pieces will make a triangle? How can we choose two breaking points at random?***

Let students suggest and discuss devices (spinners, darts, and so on) that might be used for choosing the breaking points.

T: ***Name some possible breaking points and let's see if they are successes or failures.***

S: ***5 cm and 13 cm.***

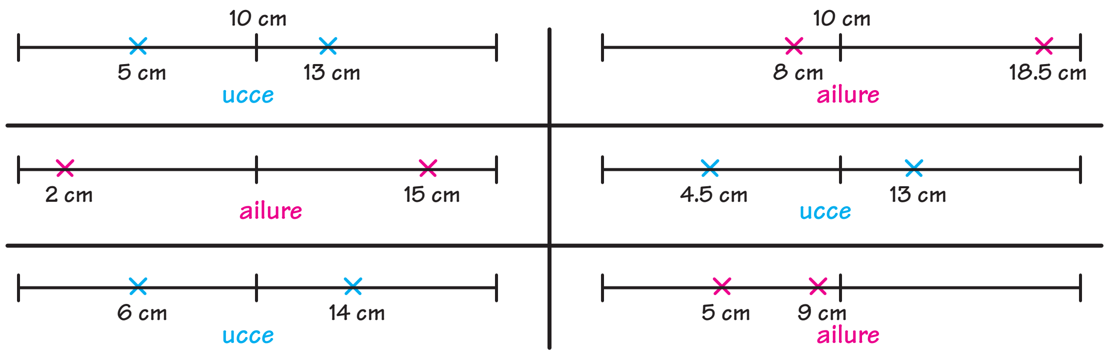
Locate the breaking points. Mark them with ×s and label them.



T: ***If we break the stick at 5 cm and at 13 cm, would the resulting pieces form a triangle?***

S: ***Yes; the three pieces would be 5 cm, 8 cm, and 7 cm long—each is less than 10 cm long.***

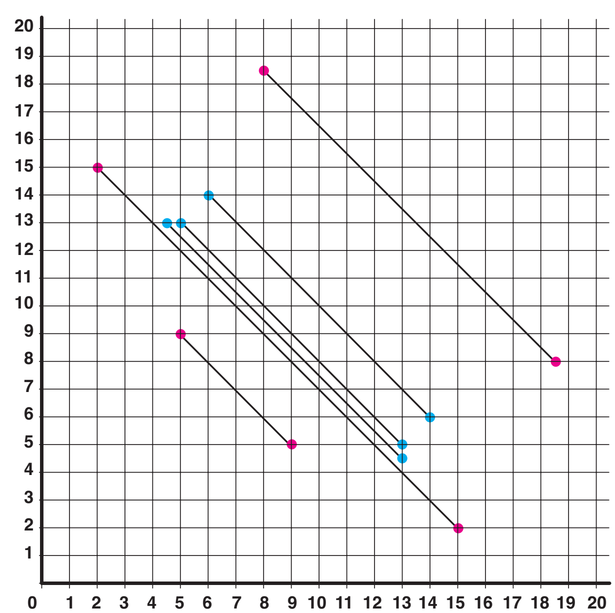
Continue the activity until five or six pairs of breaking points have been suggested and discussed. In the next illustration, success (you *can* form a triangle) is shown in blue and failure (you *cannot* form a triangle) is shown in red.



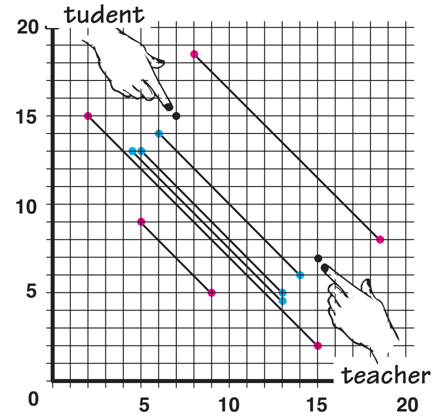
Refer to the first pair of breaking points (5 cm and 13 cm) listed on the board.

T: ***Let's record this pair of breaking points with a blue dot at a point on the grid. Where should we put the dot?***

There are two points that would be natural to use: (5, 13) and (13, 5). Mark each with a blue dot and connect the two dots with a segment. Continue by graphing the other examples of breaking points listed on the board. Use blue or red dots according to whether the points give a success or a failure.



Choose a student to come to the board. Every time you touch a point in the picture, ask the student to touch the other point that could represent the same breaking points. Repeat the activity several times.

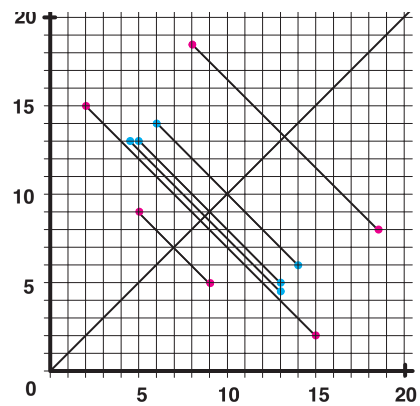


T: ***What do you notice about these pairs of points?***

S: ***The picture is symmetrical.***

T: ***Where would we place a mirror to see the symmetry?***

Invite a student to show where one would place a mirror. The student should trace the diagonal line segment that passes through the points (0, 0) and (20, 20).



T: ***So that we have only one point for each pair of breaking points, let the first coordinate be for the rightmost breaking point*** (label the horizontal axis) ***and let the second coordinate be for the leftmost breaking point*** (label the vertical axis)***. Where then would dots for pairs of breaking points for the stick be?***

S: ***Below the diagonal line.***

T: ***But what about points on the diagonal line?***

S: ***They would be for the two breaking points being the same.***

T: ***That could happen if the breaking points were chosen at random. When the breaking points are the same, can we form a triangle with the pieces?***

S: ***No, there are only two pieces.***

Redraw the diagonal line in red. Erase the connecting cords and the dots above the diagonal. Trace the line segment from (20, 0) to (20, 20).

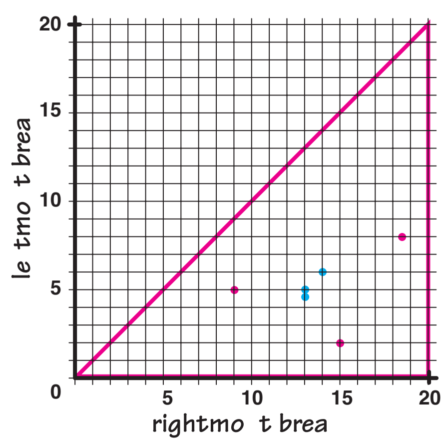
T: ***What can we say about these points?***

S: ***They are for the rightmost breaking point being 20 cm. There would really be no break, because 20 cm is at the end of the stick. Also, there would not be three pieces to form a triangle.***

Draw the line segment from (20, 0) to (20, 20) in red. Likewise, conclude that all points along the line segment from (0, 0) to (20, 0) should be red.

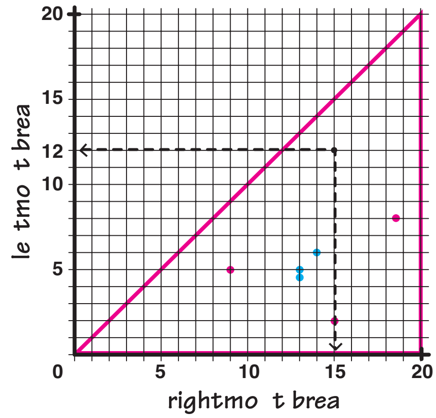
Point to a grid point below the diagonal.

T: ***How can we find which breaking points this point on the graph represents?***



G 15

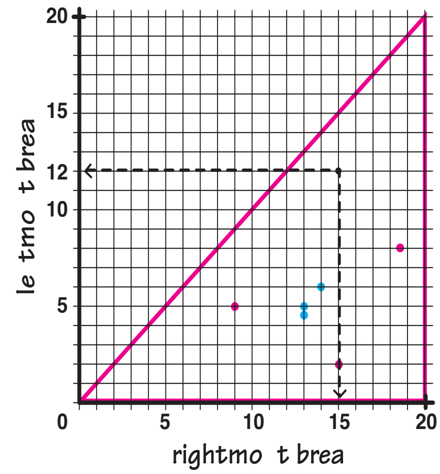
Invite a student to demonstrate the technique at the board. The student should project the point onto each axis. For example:



T: ***The stick is broken at 12 cm and at 15 cm. Let me show you a way in which we can represent the stick in the picture and show the breaking points on the stick.*** (Erase the labels of the axes.)

Let this be the stick.

Project down as we did before to get one breaking point. Project to the left, but stop at the diagonal line and project straight down to get the second breaking point.



Be sure the students observe that this method of projection gives the same set of breaking points.

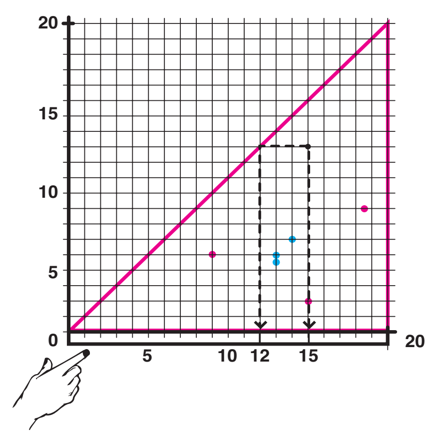
T: ***Does this dot represent a success or a failure for making a triangle?***

S: ***Failure*** (the answer depends on the example)***.***

Color the dot red for failure or blue for success.

Demonstrate the technique with several other points, tracing without drawing.

T: ***Now let's take two breaking points on the stick and find the point in the triangle that represents them.***



Mark two breaking points on the stick, and ask for a volunteer to show how to find the corresponding point in the red triangle, or do so yourself.

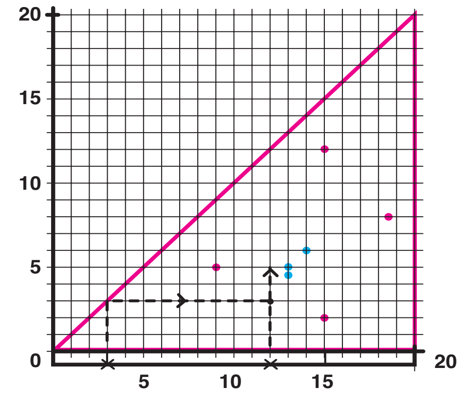
Repeat the activity with another pair of breaking points.

T: ***For every pair of breaking points we can find a point in the triangle, and for every point in the triangle we can find a pair of breaking points. This gives us a way to choose two breaking points randomly—we just need to choose one point in the triangle randomly. How could we choose a point in the triangle randomly?***

Let students make suggestions.

Shade the interior of the (red) triangle as you say,

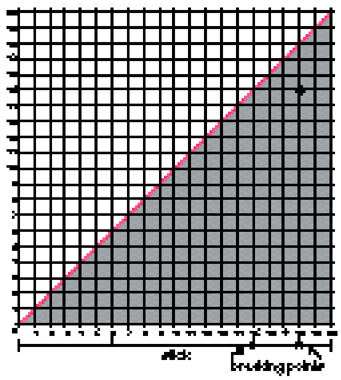
T: ***We could smear honey everywhere inside the triangle and set a fly loose in the room. The fly will choose a point at random on which to land.***



Worksheets G15* and ** are available for individual practice making projections.

Name________________ G15 ★

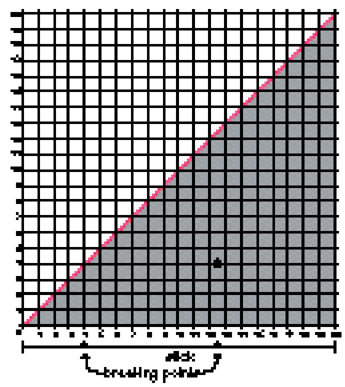
Find the breaking points on the stick that correspond to the point in the honey triangle that is marked with the dot.



Will the three resulting pieces form a triangle? No

(over)

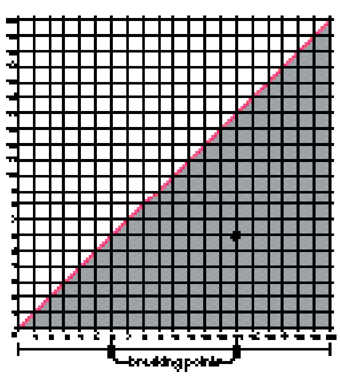
Find the breaking points on the stick that correspond to the point in the honey triangle marked with a dot.



Will the three resulting pieces form a triangle? Yes

Name________________ G15 ★★

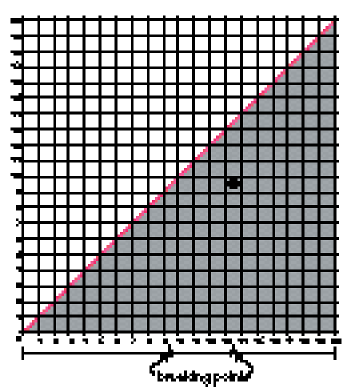
Find the point in the honey triangle that corresponds to the breaking points on the stick that are indicated.



Will the three resulting pieces form a triangle? Yes

(over)

Find the point in the honey triangle that corresponds to the breaking points on the stick that are indicated.



Will the three resulting pieces form a triangle? Yes

G16 BREAKING A STICK #4

Capsule Lesson Summary

Recall the problem of breaking a stick. Using the fact that no piece can be as long or longer than half of the stick and the idea of the "honey triangle" from Lesson G15, find the probability of getting three pieces that form a triangle when two breaking points are selected at random.

Materials

Teacher		Student	
	• Colored chalk • Meter stick • Grid board		• Colored pencils • Worksheet G16

Advance Preparation: Use Blackline G15 to make a grid, or prepare your grid board as indicated on this Blackline.

Description of Lesson

Draw the "honey triangle" from the end of Lesson G15 on the grid.

T: ***Who can recall the "breaking the stick" problem?***

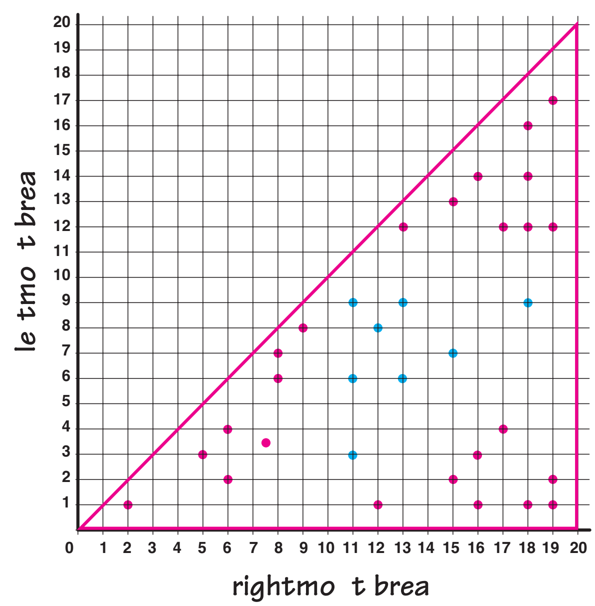
S: ***If we break a stick at two points chosen at random, what is the probability that we can make a triangle with the resulting three pieces?***

T: ***How can we choose the two points at random?***

S: ***By letting a fly land on the honey triangle.***

Refer students to their copies of Worksheet G16 and as you give these directions.

T: ***Last week we found a one-to-one correspondence between pairs of breaking points on a stick and points in the honey triangle. Let's mark some points in the triangle with blue or red dots. If a point corresponds to breaks resulting in three pieces that will form a triangle, draw a blue dot. Otherwise, draw a red dot.***

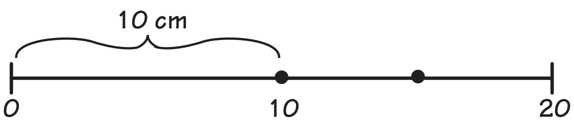


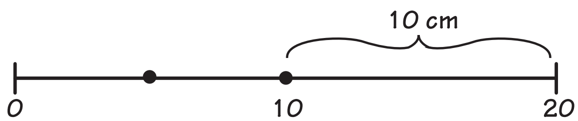
Allow about ten minutes for the students to mark red and blue points in the triangle on their worksheets. Invite students to mark red and blue points in the graph on the board. General areas of red dots and blue dots will become obvious. Encourage students to comment. If a point looks to be incorrectly colored, question the student who drew it and make necessary changes in color. After a while your picture will look similar to this one.

T: ***Let's look closer at the situation. If one of the breaking points is at 10 cm, can we ever make a triangle?***

S: ***No, because one of the pieces would be 10 cm long.***

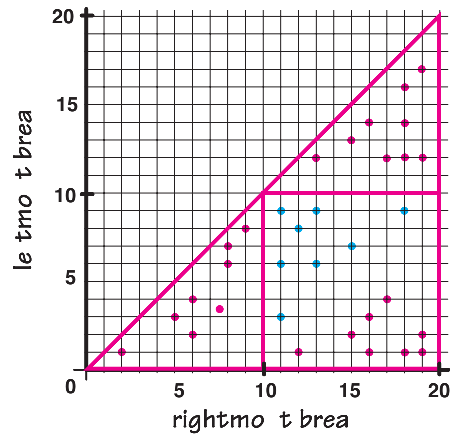
Illustrate the two possible situations on the board.





T: ***Which points in the triangle correspond to having one breaking point at 10 cm?***

Ask a student to show them in the picture on the board. The points lie on two line segments. Draw them in red.



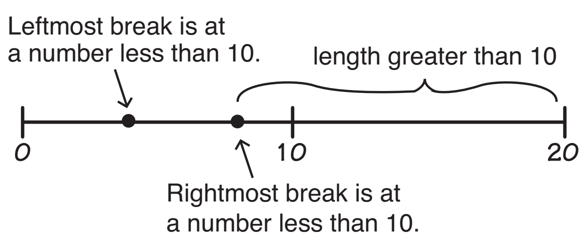
Point to the small triangle at the lower left.

T: ***There are red dots in this region. Could there be a blue dot in this region?***

S: ***No; the rightmost break would be at a number less than 10, making the piece on the right longer than 10 cm.***

S: ***If we are to get a triangle, both breaks cannot be on the same half of the stick.***

Illustrate the situation.



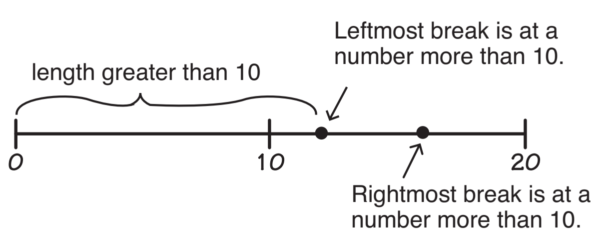
If some students are having difficulty, choose several points in the lower left triangular region and illustrate where the breaks would be in each case. When the class is convinced that no blue dot belongs in that small triangle, color its interior red.

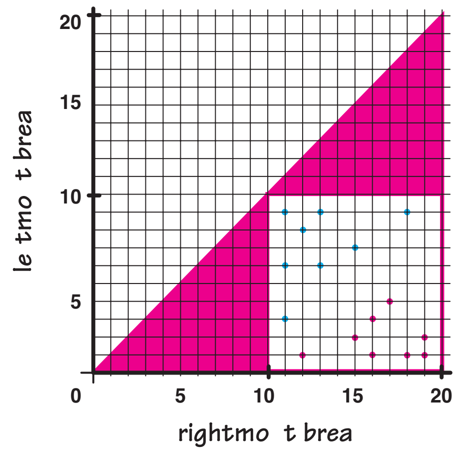
Refer to the small triangle at the upper right.

T: ***There are red dots in this region. Could there be a blue dot in this region?***

S: ***No; the leftmost break would be at a number more than 10, making the piece on the left longer than 10 cm.***

S: ***Again, if we are to get a triangle, both breaks cannot be on the same half of the stick.***





When the class is convinced that no blue dot belongs in the small upper-right triangle, color its interior red.

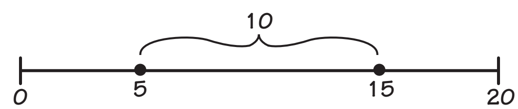
Point to the square inside the large triangle.

T: ***In this region, we have some blue dots and some red dots. Is there any pattern?***

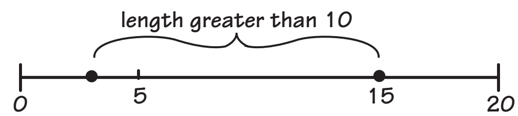
Perhaps a student will indicate that blue dots seem to fall above the diagonal from (10, 0) to (20, 0) and red dots fall below it.

T: ***Let's look at the situation more carefully. If the rightmost break is at 15, where could the leftmost break be to yield a success?***

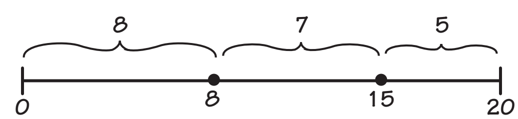
Illustrate the cases as students discuss them.



S: ***The leftmost breaking point cannot be at 5 because then the middle segment would be 10 cm long.***

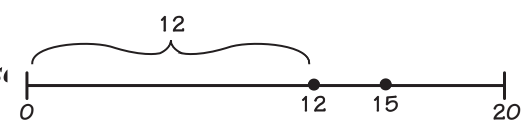


S: ***The leftmost breaking point cannot be at a number less than 5 because then the middle segment would be longer than 10 cm.***



S: ***The leftmost breaking point can be at any number more than 5 but less than 10. For example, if the leftmost break were at 8, then the three pieces would be 8 cm, 7 cm, and 5 cm long.***

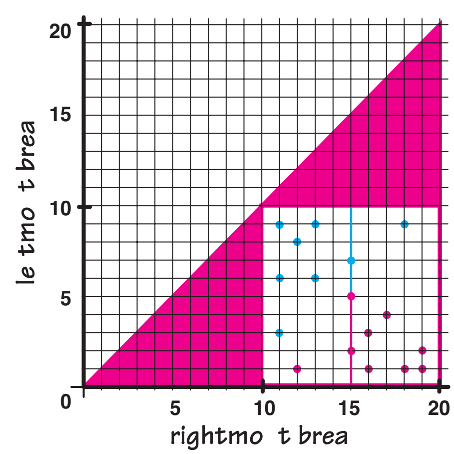
T: ***Why does the leftmost breaking point have to be at a number less than 10? Why not at 10? Why not at 12?***



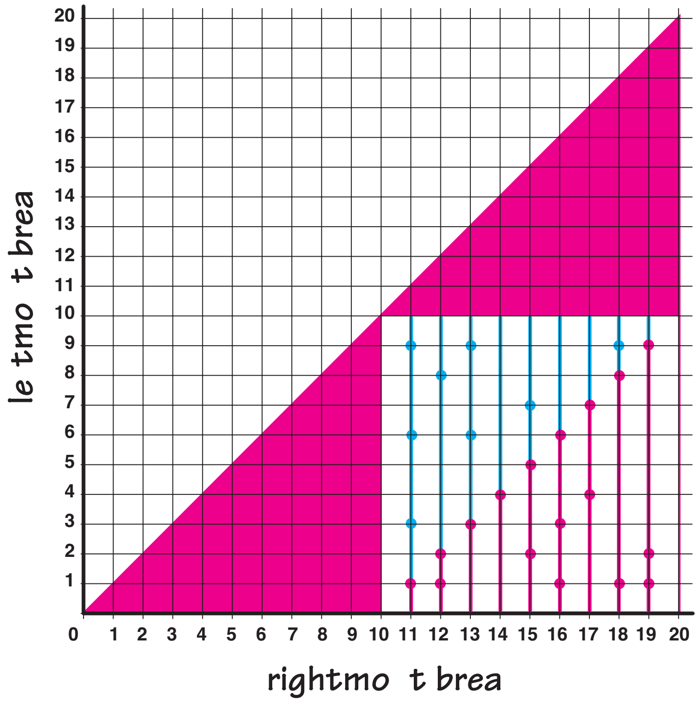
S: ***The leftmost breaking point cannot be at 10 becaus long.***

S: ***The leftmost break cannot be at 12 because then the left piece would be longer than 10 cm.***

For a situation in which the rightmost break is at 15, indicate success and failure points with a line segment partially red and partially blue. Mark the point (15, 5) with a red dot.



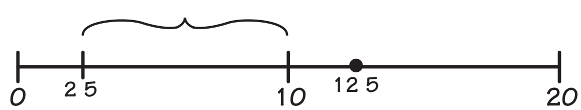
Continue the activity, considering each whole number between 10 and 20 as a possible rightmost break. Your picture should look like the one below.



Try other choices for the rightmost breaking point, such as 12.5 and 16.25.

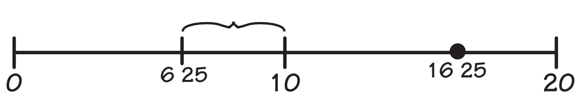
A success when the rightmost break is at 12.5 results if the choice for the leftmost break is a number more than 2.5 but less than 10.

The leftmost break can be anywhere between 2.5 and 10.



A success when the righmost break is at 16.25 results if the choice for the leftmost break is a number more than 6.25 but less than 10.

The leftmost break can be anywhere between 6.25 and 10.



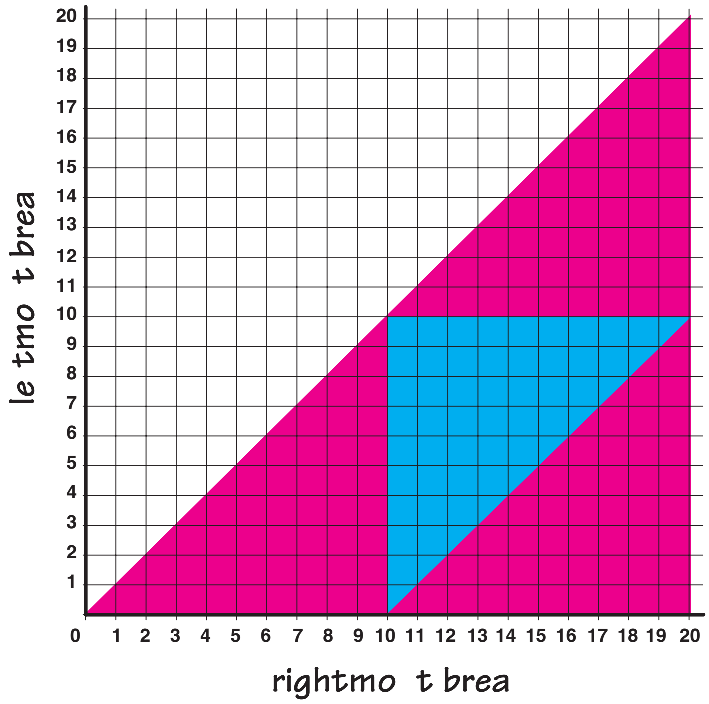
Add the information to the graph.

Point to the red dots at (11, 1), (12, 2), (12.5, 2.5), (13, 3), (14, 4), …, and (20, 10).

T: ***Why are the dots along this diagonal red?***

S: ***They are for the cases where the middle piece of the stick would be exactly 10 cm long.***

By now the class should suspect that solid areas of red and blue, as suggested by the red-blue segments, can be colored in. Shade the appropriate regions.



T: ***What is the probability that a fly landing in the honey triangle will land in the blue region? What is the probability that if two breaking points are chosen randomly, the pieces will form a triangle?***

S: ***¼; the area of the blue region is one fourth of the area of the honey triangle.***