L Strand

The Languages of Strings and Arrows

LANGUAGES OF STRINGS & ARROWS TABLE OF CONTENTS

THE LANGUAGES OF STRINGS & ARROWS INTRODUCTION

Two fundamental modes of thought for understanding the world around us are the classification of objects into sets and the study of relationships among objects. In everyday life, we classify cars by brand (Ford, Chevrolet, Toyota, and so on) and we study relationships among people (Sally is Mark's sister, Nancy is Mark's cousin). Chemists classify elements by properties, and zoologists study predator-prey relationships. Similarly, mathematicians sort numbers by primeness, and they employ functions to model predicted relationships, for example, between inflation and unemployment.

Many of a child's earliest learning experiences involve attempts to classify and to discern relationships. A child classifies people by roles (the teacher, the doctor), and creates relationships between the smell and taste of foods. Part of language development depends on a child's repeated attempts to sort objects by function, and to relate words with things or events.

The role of sets and relations is so pervasive in mathematics, that perhaps the simplest definition of mathematics is "the study of sets and relations principally involving numbers and geometrical objects." Given the equally pervasive presence of these two notions in everyday life and in a child's experiences, it is natural that they should play a key role in an elementary mathematics curriculum. Yet the inclusion of classification and the study of relations require an appropriate language for representing and studying them. For that reason, *CSMP* develops the nonverbal languages of strings and of arrows.

The pictorial language of strings represents the grouping together of objects into sets. The pictorial language of arrows represents relations among objects of the same or different sets. Each of these languages permeates the different content strands of the *CSMP* curriculum, providing unity both pedagogically and mathematically. With continual use, the languages become versatile student tools for modeling situations, for posing and solving problems, and for investigating mathematical concepts.

The general aim of this strand is to present situations that are inherently interesting and thought provoking, and that involve classification or the analysis of relations. The activities emphasize the role of logical thinking in problem solving rather than the development of specific problem-solving techniques. In addition to a varied assortment of lessons concerning sets and relations, this strand includes lessons involving systematic methods for solving combinatorial (counting) problems.

Classification: The Language of Strings

As the word implies, classifying means putting things into classes, or as the mathematician says, sets. The mathematics of sets can help students to understand and use the ideas of classification. The basic idea is simple: Given a set S and any object x, either x belongs to S (x is in S) or x does not belong to S (x is not in S). We represent this simple act of sorting—in or out—by using pictures to illustrate in and out in a dramatic way. Objects to be sorted are represented by dots, and the sets into which they are sorted are represented by drawing strings around dots. A dot inside the region delineated by a set's string is for an object in the set, and a dot outside a set's string is for an object not in the set.

This language of strings and dots provides a precise (and nonverbal) way of recording and communicating thoughts about classification. The ability to classify, to reason about classification, and to extract information from a classification are important skills for everyday life, for intellectual activity in general, and for the pursuit and understanding of mathematics in particular. The unique quality of the language of strings is that it provides a nonverbal language that is particularly suited to the mode of thinking involved in classification. It frees young minds to think logically and creatively about classes, and to report their thinking long before they have extensive verbal skills.

Relations: The Language of Arrows

Relations are interesting and important to us in our everyday lives, in our careers, in school, and in scientific pursuits. We are always trying to establish, explore, and understand relations. In mathematics it is the same; to study mathematics means to study the relations among mathematical objects like numbers or geometric shapes. The tools we use to understand the everyday world are useful to understand the world of mathematics. Conversely, the tools we develop to help us think about mathematical things often serve us in non-mathematical situations.

A serious study of anything requires a language for representing the things under investigation. The language of arrows provides an apt language for studying and talking about relations. Arrow diagrams are a handy graphic representation of a relation, somewhat the same way that a blueprint is a handy graphic representation of a house. By means of arrow diagrams, we can represent important facts about a given relation in a simple, suggestive, pictorial way—usually more conveniently than the same information could be presented in words.

The convenience of arrow diagrams has important pedagogical consequences for introducing children to the study of relations in the early grades. A child can read—and also draw—an arrow diagram of a relation long before he or she can read or present the same information in words. The difficulty of presenting certain ideas to children lies not in their intellectual inability to grasp the ideas; rather, the limitations are often mechanical. Arrow diagrams have all the virtues of a good notation: they present information in a clear, natural way; they are attractive, colorful things to look at; they are easy and fun for children to draw. Students may use arrow pictures to study, test, and explain their thinking about concepts or situations under consideration. Discussion about an arrow picture often aids the teacher in clarifying a student’s solution or misunderstanding of a problem.

Another educational bonus occurs when an arrow diagram spurs students’ curiosity to investigate variations or extensions of the original problem. A minor change in an arrow picture sometimes reinforces a pattern already discussed and at other times suggests new problems to explore.

One of the purposes of this strand is to use the power of the language of arrows to help children think logically about relations. Again it must be remembered that our goals concern the thinking process and not the mechanism. The ability to draw prescribed arrows is not the objective in itself, nor is viewing an arrow diagram just another format for drill problems in arithmetic.

THE LANGUAGES OF STRINGS & ARROWS INTRODUCTION

Content Overview

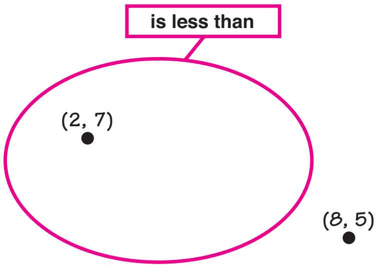
The String Game

The String Game with numbers was introduced in *IG-I* and has been used throughout *CSMP Mathematics for the Intermediate Grades*. Intuitive playing has developed into strategic thinking through exercises in which students carefully analyze the kinds of information obtained from plays and starting clues. In *IG-VI* the focus is on the analysis involved in using a clue that tells how many numbers belong in one region of a string picture. To use this type of clue, all of the strings must be considered simultaneously; every combination of possible labels must be checked. The burden of this task is lightened by prior use of other clues that limit the number of possibilities for each string's label and thus limit the number of combinations.

Lessons: L1, 5, and 9

The Relations Game

A relation is a set of ordered pairs. For example, "is less than" is the set of ordered pairs of numbers with the first component less than the second component. (2, 7) belongs to that relation because 2 is less than 7; (8, 5) does not belong to it because 8 is not less than 5.



Five lessons this semester introduce *The Relations Game*, a game resembling *The String Game* but involving ordered pairs of numbers. In *The Relations Game*, players determine which of 21 relations a string is for by observing the placement of ordered pairs in the string picture. Unlike *The String Game*, *The Relations Game* has no specified set of playing pieces (ordered pairs to be placed in the picture). Because it is not so restricted, the game provides an opportunity for arithmetic practice with a variety of numbers, including rational numbers.

During a game, students very quickly learn to distinguish ordered pairs with reverse order, for example, (2, 7) and (7, 2), since their placements in the string picture convey different information about the secret relation(s). To play *The Relations Game* well, students must exercise their knowledge and intuition about relations, which they have been studying throughout the *CSMP* curriculum with the aid of arrow representations.

Lessons: L6, 7, 8, 10, and 11

Codes and Counting

Often in mathematics we find a model or framework in which to better examine a problem. Sometimes we even find a model can be used to relate seemingly quite different problems, and thereby solve multiple problems. The Codes lessons present several very different looking combinatorics situations in which the binary number system provides a useful scheme for counting. The situations all can be seen to relate to one another with an appropriate "translation" of the symbols (**0** and **1**) of binary code words.

The Codes lessons make use of one special positional number system, and several lessons in the World of Numbers strand will review and extend these concepts.

L1 THE STRING GAME #1

Capsule Lesson Summary

Analyze some starting clues and the clues given by making plays in a string game. Select plays that give as much new information as possible. Introduce *The String Game* with special scoring rules.

Materials

Teacher
- Colored chalk
- Numerical String Game kit
- Numerical 3-String Game posters
- Colored markers or crayons

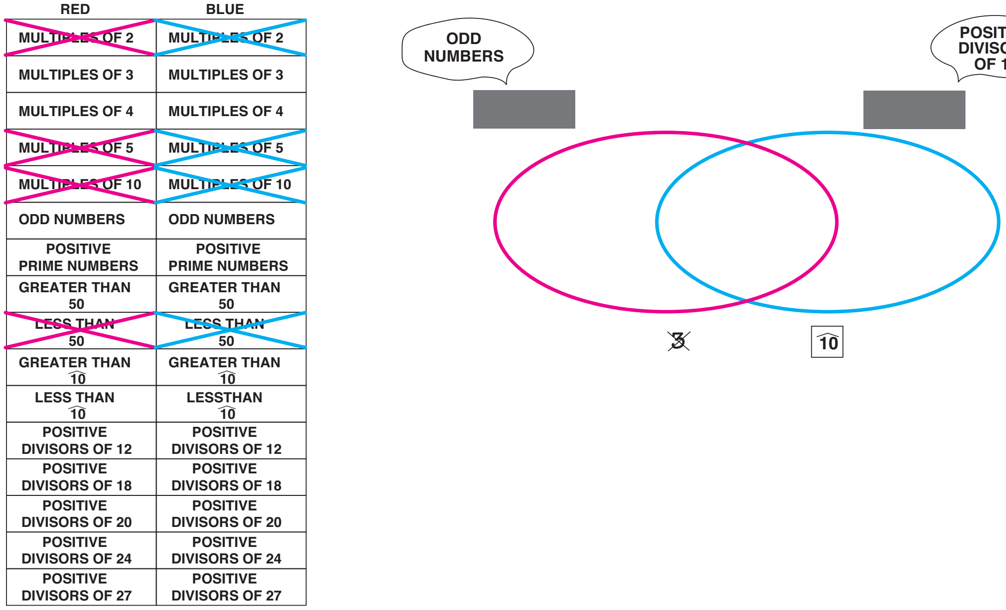
Student
- 3-String Game analysis sheet

Note: A 3-String Game poster and a 3-String Game analysis sheet can be used in a 2-string situation by crossing out or folding under the third column.

Description of Lesson

Exercise 1

Display a clean String Game poster with the string game situation shown below. Ask students to explain the crossed-out number clue (3 must be inside one or both of the strings). Collectively analyze the clues and find that they eliminate the same four possibilities for both strings.



RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
MULTIPLES OF 3	MULTIPLES OF 3
MULTIPLES OF 4	MULTIPLES OF 4
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	ODD NUMBERS
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS
GREATER THAN 50	GREATER THAN 50
~~LESS THAN 50~~	~~LESS THAN 50~~
GREATER THAN $\widehat{10}$	GREATER THAN $\widehat{10}$
LESS THAN $\widehat{10}$	LESSTHAN $\widehat{10}$
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27

The object of this exercise is for students to determine the string labels with as few additional clues as possible.

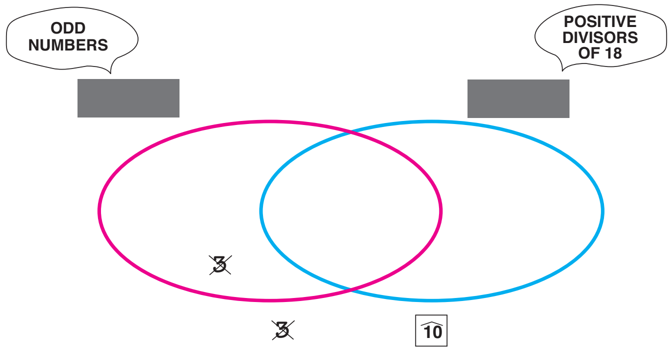
T: ***Let's make some plays as if we were playing The String Game. We would like to determine the strings as quickly as possible, so try to make a play that will give us a lot of new information.***

Invite a student to make a play. If the placement is correct, let the number remain in the picture. If the placement is incorrect, record the play with a crossed-out number in the picture. In either case, collectively analyze the situation after each play.

We cannot predict the plays your students will choose. Therefore, the following description is just an example of how the activity may proceed. In the illustrations, previously eliminated labels are indicated in black, and the labels that are eliminated as a result of the current play are indicated in color.

- A student puts 3 inside the red string but outside the blue string. (Incorrect)

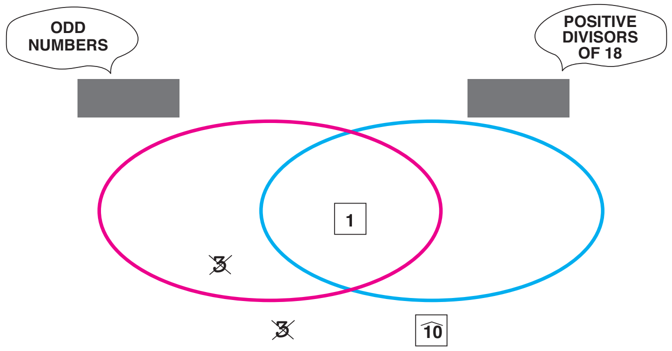
RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
MULTIPLES OF 3	MULTIPLES OF 3
MULTIPLES OF 4	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	ODD NUMBERS
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS
GREATER THAN 50	~~GREATER THAN 50~~
~~LESS THAN 50~~	~~LESS THAN 50~~
GREATER THAN 10	GREATER THAN 10
LESS THAN 10	~~LESSTHAN 10~~
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27



This clue tells us that 3 belongs inside the blue string, so it eliminates four more labels from the Blue list and no more from the Red list.

- A student puts 1 inside both strings. (Correct)

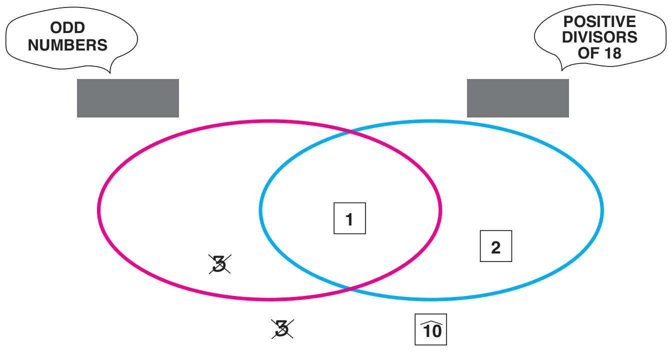
RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	ODD NUMBERS
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
~~GREATER THAN 50~~	~~GREATER THAN 50~~
~~LESS THAN 50~~	~~LESS THAN 50~~
GREATER THAN 10	GREATER THAN 10
~~LESS THAN 10~~	~~LESSTHAN 10~~
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	~~POSITIVE DIVISORS OF 20~~
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27



This clue eliminates five more labels from the Red list and two more from the Blue list.

- A student puts 2 inside the blue string but outside the red string. (Correct)

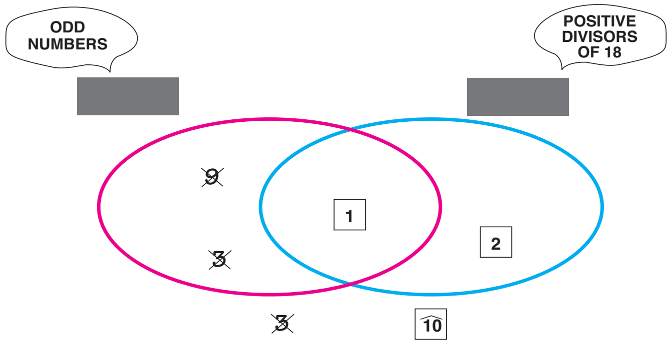
RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
~~GREATER THAN 50~~	~~GREATER THAN 50~~
~~LESS THAN 50~~	~~LESS THAN 50~~
~~GREATER THAN 10~~	GREATER THAN 10
~~LESS THAN 10~~	~~LESS THAN 10~~
~~POSITIVE DIVISORS OF 12~~	POSITIVE DIVISORS OF 12
~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 18
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~



This clue eliminates five more labels from the Red list and two more from the Blue list.

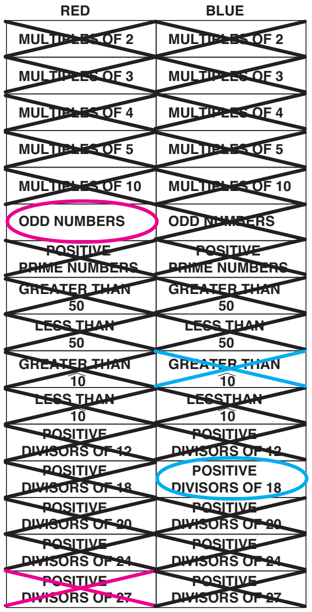
- A student puts 9 inside the red string but outside the blue string. (Incorrect)

RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
~~GREATER THAN 50~~	~~GREATER THAN 50~~
~~LESS THAN 50~~	~~LESS THAN 50~~
~~GREATER THAN 10~~	GREATER THAN 10
~~LESS THAN 10~~	~~LESS THAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 18
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~

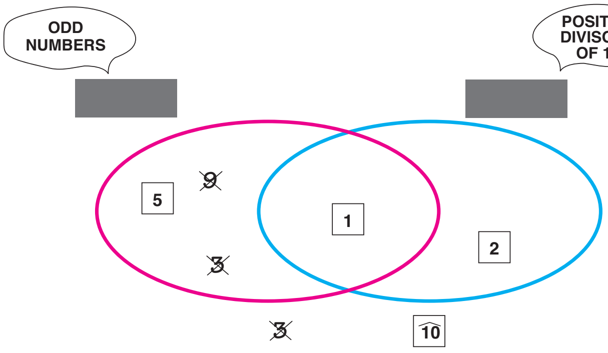


This clue tells us that 9 belongs inside both strings because, with both of the two remaining possibilities for the red string, 9 must be inside the red string. This clue eliminates two more labels from the Blue list.

- A student puts 5 inside the red string but outside the blue string.



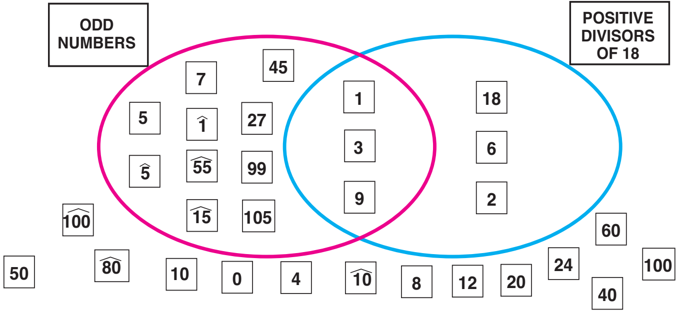
RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~
~~GREATER THAN 50~~	~~GREATER THAN 50~~
~~LESS THAN 50~~	~~LESS THAN 50~~
~~GREATER THAN 10~~	~~GREATER THAN 10~~
~~LESS THAN 10~~	~~LESSTHAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~
~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 18
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~



This clue eliminates one more label from each of the two lists and, hence, the strings are determined.

For your information, the correct placement for all of the game pieces is shown here.

Correct Placement of Game Pieces

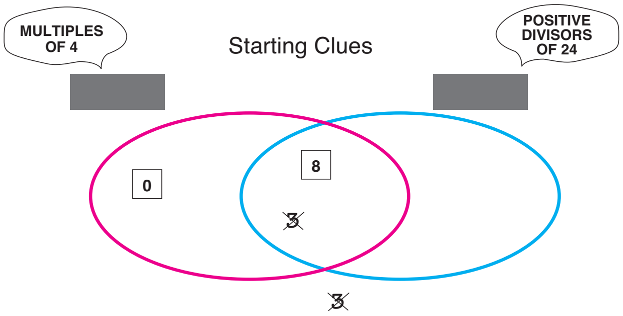


Exercise 2

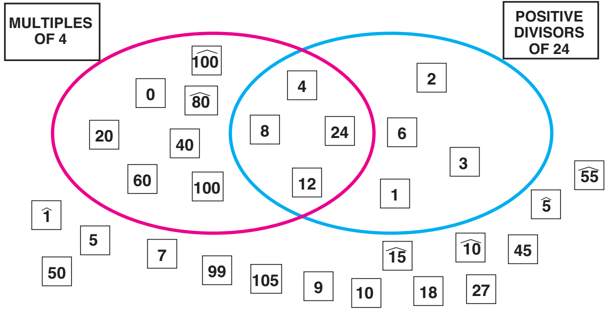
Play *The String Game* with special scoring as described in Appendix D: *The String Game*, Version D. Explain the special scoring rules to the class.

A possible game is suggested below. One of the starting clues is that 3 does not go in the intersection or in the region outside both strings.

Note: The starting clues for this game determine the blue string and eliminate all but two possibilities (multiples of 2 and multiples of 4) for the red string. This is for your information only; do not tell the class.



Correct Placement of Game Pieces



L2 CODES #1

Capsule Lesson Summary

Use a binary code to describe a given path between two points in a grid and to interpret a message given by a string picture. Find a new way to send a secret message and interpret it in binary code.

Materials

Teacher
- Neighborhood map transparency
- Colored chalk
- Eight small boxes and eight pennies (optional)
- Blacklines L2(a) and (b)

- Colored pencils, pens, or crayons
- Neighborhood map
- String picture

Student
- Paper

Advance Preparation: Use Blackline L2(a) to make a transparency of the neighborhood map (grid) for projection and to make copies for students. You may prefer to put the directions on a grid board for display. Use Blackline L2(b) to make copies of a string picture for students.

Exercise 1

Display the neighborhood map (grid) from Blackline L2(a).

T: ***Spike and Spane are detectives. Their office is here*** (point to **S**)***. The streets in their neighborhood are all like this, north-south or east-west. Each time Spike goes to a house in the neighborhood, he takes a shortest route and leaves a coded message for Spane. The message describes the route Spike plans to follow.***

Students may recall Spike's adventures in the storybook *The Hidden Treasure*, used in *CSMP Mathematics for the Intermediate Grades, Part IV*.

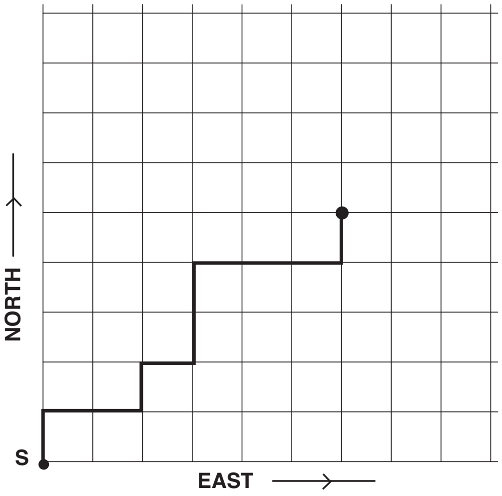
Write this binary code word on the board.

1 0 0 1 0 1 1 0 0 0 1

T: ***One day Spike leaves this message for Spane. Can you trace the route that Spike plans to take on this map?***

Let students respond to the question; do not comment yourself. Some students will observe that this question cannot be answered until some agreement concerning **1** and **0** is reached.

T: ***Let's agree that* 1 *indicates that Spike travels one block north and that* 0 *indicates that he travels one block east.***



Once this agreement is made, ask students to draw the path on their copy of the neighborhood grid; call on one student to do the same at the board.

Write these binary code words on the board.

11100001001   101010010110

T: ***One of these messages is for a route that leads to the same house as before. How can we tell without actually tracing the routes?***

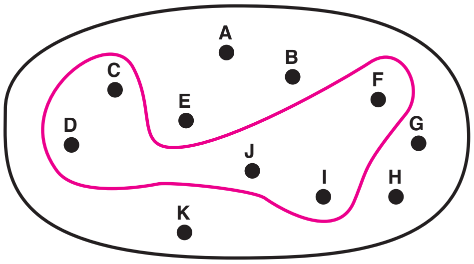
Allow a minute or two for students to study the messages.

S: *For the first message (***10010110001***), there are five* **1***s and six* **0***s. Any shortest path from* **S** *to the house at the end of this path has five* **1***s and six* **0***s in some order. So* **11100001001** *describes a path from* **S** *to the same house;* **101010010110** *does not.*

Ask students to trace the two routes in the neighborhood to verify the result.

Draw this string picture on the board as you continue the story.

T: ***On another day, Spike leaves a coded message in a string picture. It also describes the route he plans to take to a house. On your paper, try to decode the message by writing a binary code word for this picture.***

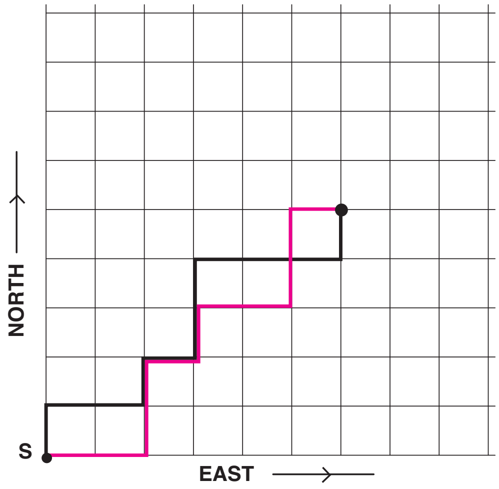


There are two usual responses. **11001011001** and **00110100110**. These binary code words arise naturally if the dots inside the red strings are assigned **0** (the former case) or **1** (the latter case). After some discussion, tell the class that Spike assigns **1** to each letter inside the red string so this message is **00110100110**.

Write the letters **A** through **K** in alphabetical order on the board. Ask a student to write the binary code word **00110100110** appropriately below the letters and to trace the path (shown in red).

The class should find that the route leads to the same house as before.

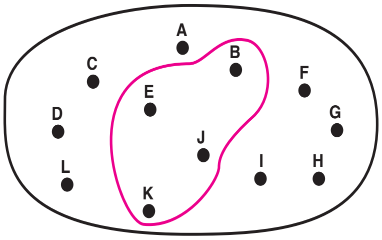
A	B	C	D	E	F	G	H	I	J	K
0	0	1	1	0	1	0	0	1	1	0



Reverse the problem by writing a binary code word on the board and asking for its string picture message. For example, use the code word given here and direct students to draw a red string in the picture on Blackline L2(b).

Invite one student to draw the string picture message on the board.

0 1 0 0 1 0 0 0 0 1 1 0

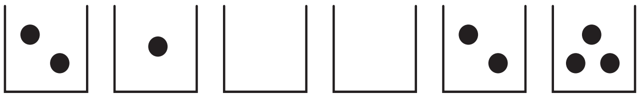


Exercise 2

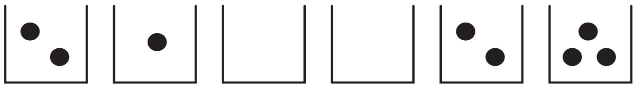
If you like, display a row of six small boxes and lay out eight pennies.

T: ***Spike is worried that the secret messages he has been leaving may be too easy to decode. So Spike tries to disguise them even more. He changes his message from a binary code word or a string picture to a row of boxes with some pennies in them. Here is a secret message; try to find a binary code word for the message.***

Draw this picture of six boxes and eight pennies on the board.



Let students comment about this situation. Perhaps no one will find a good way to decode this first message, in which case write the binary code word beneath the picture yourself and ask for observations.



0 0 1 0 1 1 1 0 0 1 0 0 0

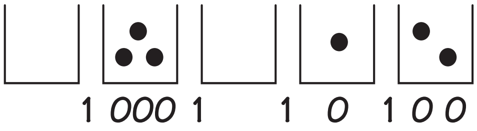
Note: For your information, pennies represent **0**s and spaces between boxes represent **1**s. Let students make this discovery.

S: ***There are eight pennies, and there are eight 0s.***

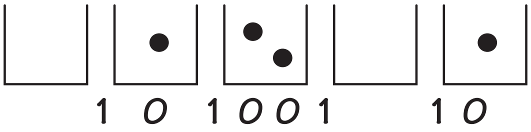
S: ***There are six boxes, but there are only five 1s.***

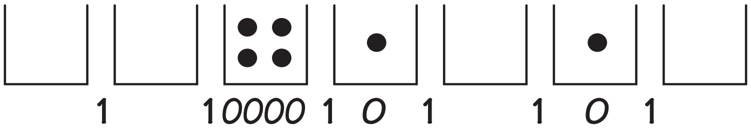
Continue by asking the class to try to decode another message.

If they need a hint, write the binary code word below the boxes, spaced as shown below.



Continue the decoding, using the "boxes and pennies" messages below. Allow students to announce the binary code words as they discover the secret. When a correct binary code word is given, write it below the boxes, spaced as shown below.

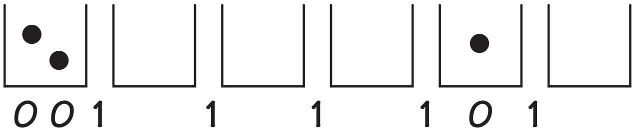


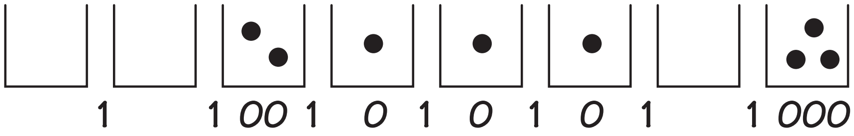


When many students understand the code, ask for an explanation.

S: ***The pennies are for 0s and the spaces between boxes are for 1s.***

To check that everyone understands the code, ask students to decode the "boxes and pennies" messages below. Invite students to write the binary code words below the boxes.





T: ***Now I'll write a binary code word, and you draw its picture of boxes and pennies.***

Write this binary code word on the board, and ask students to draw the "boxes and pennies" message on their papers.

110 010 0 0111 0 01

If some students need help getting started, ask the following questions.

T: ***How many pennies will we need?***

S: ***Seven pennies, because there are seven 0s.***

T: ***How many boxes?***

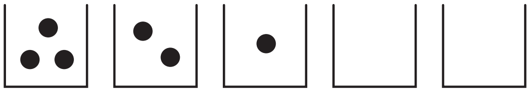
S: ***There are seven 1s, so there must be seven spaces between boxes. That means we must have eight boxes.***

Invite a student to draw the picture on the board.

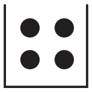
1 1 0 0 1 0 0 0 1 1 1 0 0 1

Continue with individual work, asking students to draw the "boxes and pennies" pictures for the following binary code words. (Answers are on the right.)

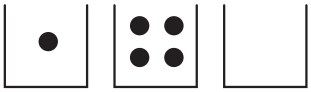
0 0 0 1 0 0 1 0 1 1



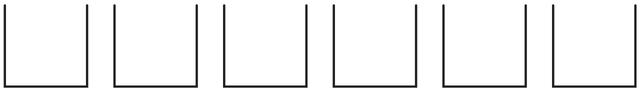
0 0 0 0



0 1 0 0 0 0 1



1 1 1 1 1



Writing Activity

You may like students to take lesson notes on some, most, or even all their math lessons. The "Lesson Notes" section in Notes to the Teacher gives suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students may note how the binary code, the string picture, and the "boxes and pennies" messages all describe routes in Spike's neighborhood.

L3 CODES #2

Capsule Lesson Summary

Do some mental arithmetic relating kilometers and meters. Review the messages from Lesson L2. Count the number of routes from **S** to various intersections of the grid. Locate grid points at a certain taxi-distance from **S**, and count the collective routes from **S** to these grid points. Observe a relationship between taxi-distance and the number of routes. Relate the counting scheme in this lesson to counting the ways to put a certain number of pennies into a given number of boxes.

Materials

Teacher

- Neighborhood map transparency
- Labeled neighborhood map transparency
- Colored chalk and markers
- Blackline L3

Student

- Paper
- Colored pencils, pens, or crayons
- Labeled neighborhood map

Advance Preparation: Use the neighborhood map transparency from Lesson L2 or prepare it on a grid board for Exercise 2. Use Blackline L3 to make a transparency of the labeled neighborhood map (grid) for projection and to make copies for students. You may prefer to write the number labels on your grid board just before starting Exercise 3.

Description of Lesson

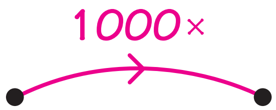
Exercise 1

Begin the lesson by briefly reviewing that one kilometer is 1000 meters, and that the prefix *kilo* in the metric system means *thousand.*

Draw a 1000**x** arrow on the board

T: ***2 kilometers is how many meters?*** (2000 meters)

Continue with other problems, giving the label for one of the dots and asking for the label of the other dot. Suggested problems with answers in boxes are shown below. Continue to phrase the questions in terms of meters and kilometers.



2	$\boxed{2\,000}$
60	$\boxed{60\,000}$
0.4	$\boxed{400}$
$\boxed{12}$	12 000
$\boxed{9.5}$	9 500
$\boxed{1.62}$	1620
6.08	$\boxed{6\,080}$
2.4	$\boxed{2\,400}$

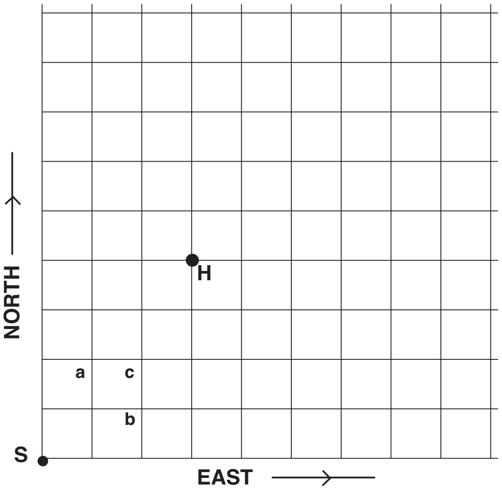
Exercise 2

Display the neighborhood grid with dots labeled **S** and **H**. Briefly review the coded messages from the story in Lesson L2.

T: ***In addition to writing the secret codes for routes, Spike is also concerned about the number of different routes from his office (*****S*****) to a house, for example, here*** (point to **H**). ***How many shortest paths are there that Spike could take from*** **S** ***to*** **H*****?***

Write some estimates on the board.

T: ***How could we find this number?***



Do not write the letters **a**, **b**, and **c** on the board. They are here just to make the description of the lesson easier to follow

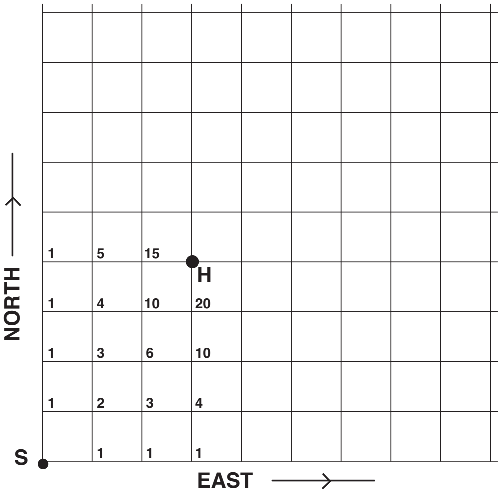
Perhaps some students will remember this kind of problem in the storybook *The Hidden Treasure* and will suggest a systematic procedure for counting paths. You may need to reintroduce students to such a procedure, as in the following dialogue.

T (pointing to **c**)**:** ***How many paths are there from*** **S** ***to this intersection?***

S (pointing to the appropriate intersections)**:** ***The only way to get here (*****c*****) is to travel through this corner (*****a*****) or this one (*****b*****). Since there are three ways to get to this corner (*****a*****) and three ways to get to this corner (*****b*****), there must be six ways (3 + 3 = 6) to get here (*****c*****).***

Put numbers next to intersections to indicate the number of shortest routes. Continue until **H** is labeled 35.

Call on students to label several other corners of the grid, letting each student have a chance to label at least one corner. Then display and distribute copies of a completely labeled neighborhood grid (Blackline L3). Let students check that the corner numbers they found are the same as on the labeled grid.



Exercise 3

T: ***Suppose that in Spike's neighborhood each block is 200 m long. One day Spike drives to a secret house 1.6 km away. How many blocks does Spike drive?***

S: ***Eight blocks, because 1 600 ÷ 200 = 8.***

T: ***So we say that the taxi-distance from Spike's office to the secret house is 8.***

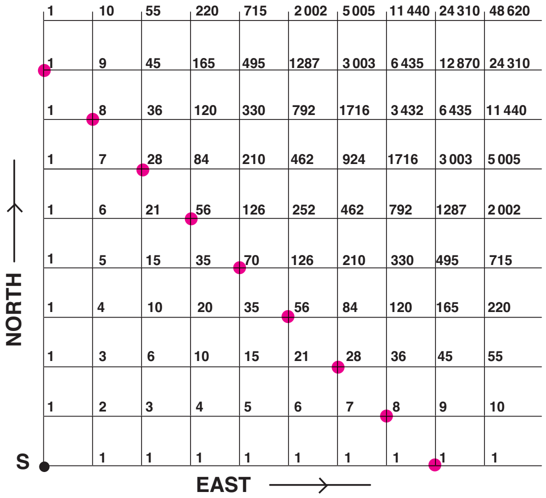
Record the taxi-distance on the board.

T(S, Secret House) = 8

T: ***On your maps, find some places where the secret house could be and draw red dots at those corners.***

Allow several minutes for individual work before asking volunteers to locate the possibilities for the secret house on the displayed copy of the map.

T: ***There are nine places where the secret house could be. How many different routes are there altogether from* S *to a place where the secret house could be, one of these nine dots?***



1	10	55	220	715	2 002	5 005	11 440	24 310	48 620
1	9	45	165	495	1287	3 003	6 435	12 870	24 310
1	8	36	120	330	792	1716	3 432	6 435	11 440
1	7	28	84	210	462	924	1716	3 003	5 005
1	6	21	56	126	252	462	792	1287	2 002
1	5	15	35	70	126	210	330	495	715
1	4	10	20	35	56	84	120	165	220
1	3	6	10	15	21	28	36	45	55
1	2	3	4	5	6	7	8	9	10
	1	1	1	1	1	1	1	1	1

Allow a few minutes for students to do the calculation.

S: *1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 = 256.*

T: ***Good. There are 256 paths altogether from* S *to these nine houses.***

On the board, begin a table in which to summarize this information.

Repeat this activity, replacing 1.6 km with 1 km, with 1.2 km, and with 1.4 km.

Distance (km)	Taxi- istance (blocks)	Number of routes from S
1.6	8	256

Encourage students to observe number patterns in the table.

T: ***Look at the numbers in the third column. Do you see a way to name these numbers using exponents?***

S: *32 = 2 x 2 x 2 x 2 x 2 =* 2^5.

If this information is not forthcoming, suggest it yourself. Record the numbers of routes from **S** as powers of 2.

Distance (km)	Taxi- istance (blocks)	Number of routes from S
1.6	8	256
1	5	32
1.2	6	64
1.4	7	128

T: ***What do you notice about the exponents in the numbers of routes?***

S: ***The exponents in the numbers of routes are exactly the same as the corresponding taxi-distances.***

T: *Yes, let's see if we can explain this fact. Suppose we look again at the houses a taxi-distance of 8 away from* S*. Choose one of the red dots and write the binary code word for a route from* S *to that house.*

After a few minutes, ask several students to write their binary code words on the board. For example:

11100010 11111111
10101010 10001100

T: *What do you notice about these numbers in the binary code?*

S: *Each number has eight digits.*

T: *What is the least number using eight digits in binary code?*

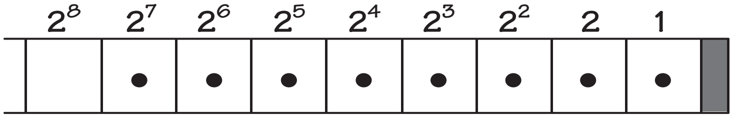
S: **00000000***, or just* **0***.*

T: *What is the greatest number using eight digits in binary code?*

S: **11111111.**

T: *Which number is this in decimal writing?* (255)

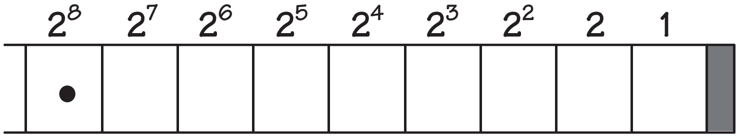
Draw a binary abacus on the board and put eight checkers on it for **11111111**.



T (pointing to the abacus)**:** *Here is the greatest eight-digit binary number on the binary abacus. Do you know an easy way to calculate this number?*

S: *Put another checker on the ones board and then make trades.*

Call on a student to make trades, or make them yourself.



S: *So the number is 255; $2^8 - 1 = 255$.*

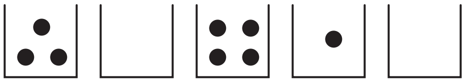
T: *How many binary numbers with eight digits are there?*

S: *There are 256 numbers from 0 to 255 (inclusive); $256 = 2^8$.*

T: *So there are 2^8 (or 256) binary numbers with eight digits, and there are 2^8 (or 256) routes going from* S *to one of the houses at a taxi-distance of 8 from* S*. We could check this for each taxi-distance.*

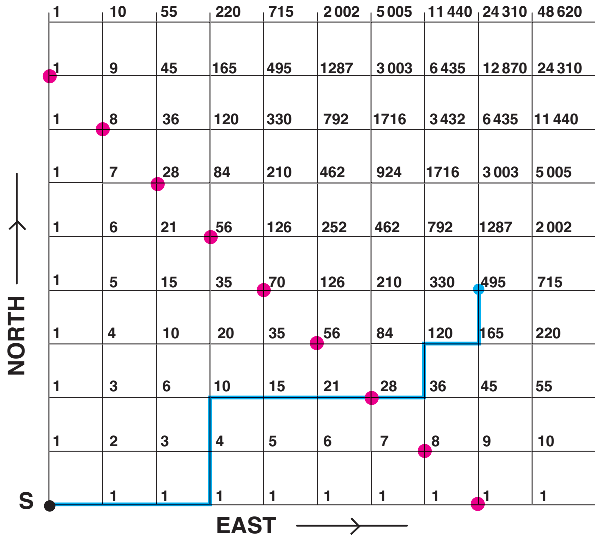
Exercise 4

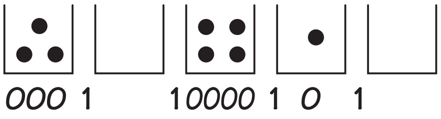
Draw this "boxes and pennies" picture on the board near the map.



T: *One day Spike left this message for his partner, Spane. Can you decode it and locate the corresponding house on your map?*

Invite a student to write the binary code word corresponding to the message, to trace the indicated route on the map, and to draw a blue dot at the location of the house.





T: *On your paper, make up another message using exactly five boxes and eight pennies. Then write the corresponding binary code word and find the house on your map.*

Invite several students to write their binary code words on the board and to locate the corresponding houses. For example:

1111 0 0 0 0 0 0 0 0   0 0 0 11 0 1 0 0 1 0 0

0 0 1 0 0 1 0 0 1 0 0 1   1 0 0 0 1 0 0 1 0 0 0 1

T: *What do you notice?*

Students may be surprised to discover that each message involving five boxes and eight pennies names a path to the same house.

S: *Each binary code word takes us to the house four blocks north and eight blocks east of* **S**.

T: *How can we explain this?*

S: *Each binary code word has four* **1***s and eight* **0***s; each path must go four blocks north and eight blocks east.*

T: *How many ways can we put eight pennies into five boxes?*

S: *495, the number of paths from* **S** *to this house.*

Draw a picture of six boxes on the board.

T: ***How many ways can we put seven pennies into six boxes?***

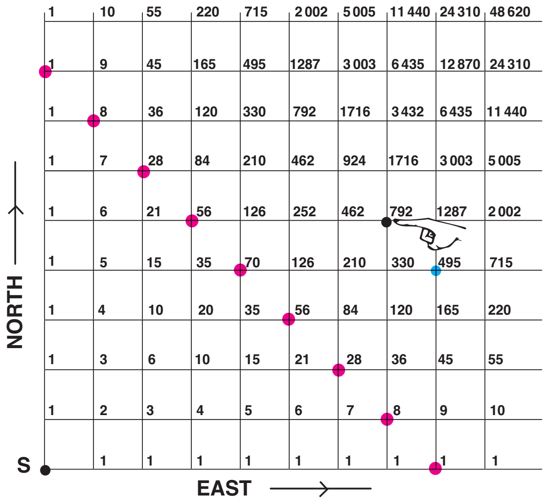
S: ***There are seven pennies, so there will be seven*** **0*****s in each binary code word.***

S: ***There are six boxes, so there will be five*** **1*****s in each binary code word. The*** **1*****s are the spaces between the boxes, and with six boxes there are five spaces.***

S: ***Each binary code word with seven*** **0*****s and five*** **1*****s represents a path from*** **S** ***to the house that is five blocks north and seven blocks east.***

Ask a student to find that house on the map of Spike's neighborhood.

T: ***How many paths are there from*** **S** ***to this house?*** (792)
How many ways can we put seven pennies into six boxes? (792)



1	10	55	220	715	2 002	5 005	11 440	24 310	48 620
1	9	45	165	495	1287	3 003	6 435	12 870	24 310
1	8	36	120	330	792	1716	3 432	6 435	11 440
1	7	28	84	210	462	924	1716	3 003	5 005
1	6	21	56	126	252	462	792	1287	2 002
1	5	15	35	70	126	210	330	495	715
1	4	10	20	35	56	84	120	165	220
1	3	6	10	15	21	28	36	45	55
1	2	3	4	5	6	7	8	9	10
	1	1	1	1	1	1	1	1	1

Home Activity

Some students may enjoy showing family members or friends how to use codes in counting problems like the ones in this lesson. Suggest they start with an easier problem such as the number of ways to put five pennies into three boxes in a row. To save time, they can take home a copy of the labeled grid.

L4 CODES #3

Capsule Lesson Summary

Relate problems of finding the number of ways of putting checkers on the Minicomputer to those of putting pennies into boxes (see Lessons L2 and L3). Answer a probability question in this context.

Materials

Teacher
- Minicomputer set
- Labeled neighborhood map transparency
- Blackline L4

Student
- Minicomputer worksheet

Advance Preparation: Use the labeled neighborhood map transparency from Lesson L3, or prepare it on a grid board. Use Blackline L4 to make copies of the Minicomputer worksheet for students.

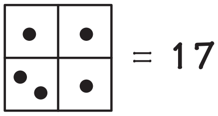
Description of Lesson

Exercise 1

Display one Minicomputer board and four regular checkers.

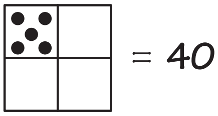
T: ***Let's put a number on the ones board of the Minicomputer using exactly five regular (positive) checkers.***

Invite a student to put such a number on the Minicomputer. For example:



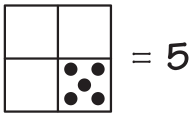
T: ***What is the greatest number we can put on the Minicomputer with exactly five regular checkers?***

S: ***40.***



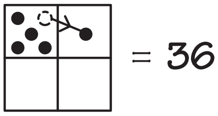
T: ***And the least number?***

S: ***5.***



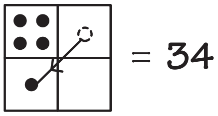
T: ***Can we put every number between 5 and 40 on the Minicomputer with exactly five regular checkers?***

Allow a couple minutes for students to think about the question.



S: ***We cannot put on 37, 38, or 39, because below 40 the next greatest number we can put on is 36.***

S: ***Also, we cannot put on 35. Below 36, the next greatest number is 34.***



There are five numbers between 5 and 40 that cannot be put on the Minicomputer with exactly five regular checkers: 31, 35, 37, 38, and 39. Accept any of these numbers that students suggest. It is not necessary for the class to find them all.

T: *How many different ways can these five checkers be put on the ones board of the Minicomputer?*

Again, allow time for students to contemplate the question.

S: *Putting these five checkers on the Minicomputer is like putting five pennies into four boxes.*

If no one suggests this connection between the two problems, present it yourself.

S: *The five checkers are like five pennies. That means there should be five* 0*s in a binary code word.*

S: *Four boxes produce three spaces between boxes, so there should be three* 1*s in a binary code word.*

S: *Each binary code word for the "five pennies into four boxes" problem has five* 0*s and three* 1*s.*

T: *How many different ways can five pennies be put into four boxes?*

S: *Finding the number of ways to put five pennies into four boxes is like finding the number of paths to a house in Spike's neighborhood.*

T: *Yes, it is the same problem stated another way. How can we determine which corner (house) to look at for this problem?*

S: *We should look at the corner five blocks east and three blocks north of* S*.*

Display the labeled neighborhood map (grid) and let students find the corner point.

S: *There are 56 routes from* S *to the house at this corner, so there are 56 ways to put five pennies into four boxes. There must also be 56 ways to put these five checkers on the ones board of the Minicomputer.*

Encourage students to contribute as much as possible to the class's collective solution to the problem, but be prepared to guide the discussion when necessary.

T: *If the five checkers are placed randomly on the Minicomputer, what is the probability that the number is between 10 and 20?*

Write students' estimates on the board.

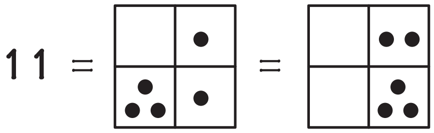
T: *How might we solve this problem?*

S: *We must find out how many ways the numbers between 10 and 20 can be put on the Minicomputer (with exactly five regular checkers).*

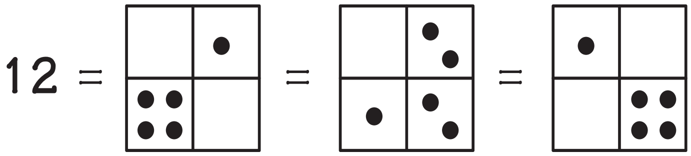
Distribute copies of the Minicomputer worksheet (Blackline L4) for students' use.

T: *Let's start with 11. How can we put 11 on the Minicomputer with exactly five regular checkers?*

Invite a student to put 11 on the Minicomputer. The students should conclude that there are two ways to do this.



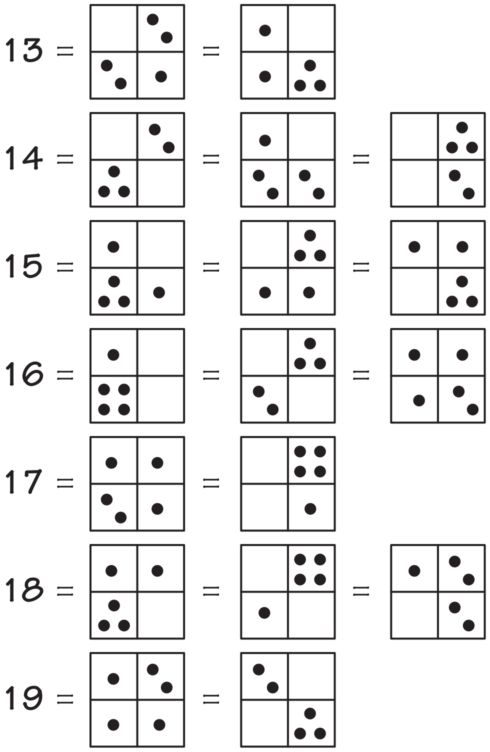
Continue by asking how to put 12 on the Minicomputer with exactly five regular checkers.



Begin a table on the board in which to record this information.

Number	Number of ways to be put on the Minicomputer
11	2
12	3

Direct students to continue independently or with a partner to find the number of ways that each of the integers 13 through 19 can be put on the Minicomputer. After a while, complete the table with the class, and invite a student to sum the numbers in the right column.



Number	Number of ways to be put on the Minicomputer
11	2
12	3
13	2
14	3
15	3
16	3
17	2
18	3
19	2
	23

T: ***There are 23 ways to put the numbers between 10 and 20 on the Minicomputer with exactly five regular checkers. What is the probability of getting one of these numbers when we randomly place the five checkers on the Minicomputer?***

S: ***23⁄56, because there are 56 ways to put the four checkers on the Minicomputer, and 23 of the ways give a number between 10 and 20.***

Exercise 2

Display two Minicomputer boards and six regular checkers. Invite several students to put numbers of their choice on their Minicomputer, using exactly six regular checkers. After seeing three or four examples, ask for the greatest such number (480) and the least such number (6).

T: ***Are there some numbers between 6 and 480 that we cannot put on this Minicomputer with exactly six regular checkers?***

It is quite easy to argue that the numbers between 440 and 480 and those between 420 and 440 cannot be put on in this way. It is not necessary (nor very easy) to find all such numbers.

T: ***How many ways are there to put the six checkers on this Minicomputer?***

Write students' estimates on the board for future reference. The class should observe the equivalence of this problem to the problem of putting six pennies into eight boxes. Let students discuss this situation.

S: ***There are six* 0*s and seven* 1*s in a binary code word for a solution to the six pennies and eight boxes problem.***

S: ***All binary code words with six* 0*s and seven* 1*s give ways to put six pennies into eight boxes.***

S: ***We should look at the corner that is six blocks east and seven blocks north of* S.**

Let students consult the labeled map of Spike's neighborhood and conclude that there are 1 716 routes from **S** to the house at the appropriate corner and, therefore, 1 716 ways to put six checkers on two Minicomputer boards.

Extension Activity

A challenging probability question can extend the counting problem in Exercise 2. For example:

> If the six checkers are placed randomly on the Minicomputer (two boards), what is the probability that the number will be an odd number?

Answer: For an odd number, there must be one, three, or five checkers in the 1-square.

- With one checker in the 1-square, the other five checkers can be placed on the other seven squares in 462 ways.
- With three checkers in the 1-square, the other three checkers can be placed on the other seven squares in 84 ways.
- With five checkers in the 1-square, the other checker can be placed on the other seven squares in seven ways.

Therefore, the probability of an odd number is $\frac{462 + 84 + 7}{1\,716} = \frac{553}{1\,716}$, close to ⅓.

L5 THE STRING GAME #2

Capsule Lesson Summary

Collectively analyze some starting clues and the clues given by making plays in a string game. Select plays that give as much new information as possible. Play *The String Game* with special scoring rules.

Materials

Teacher

- Colored chalk
- Numerical String Game kit
- Numerical 3-String Game posters
- Colored markers or crayons

Student

- 3-String Game analysis sheet
- Worksheets L5* and **

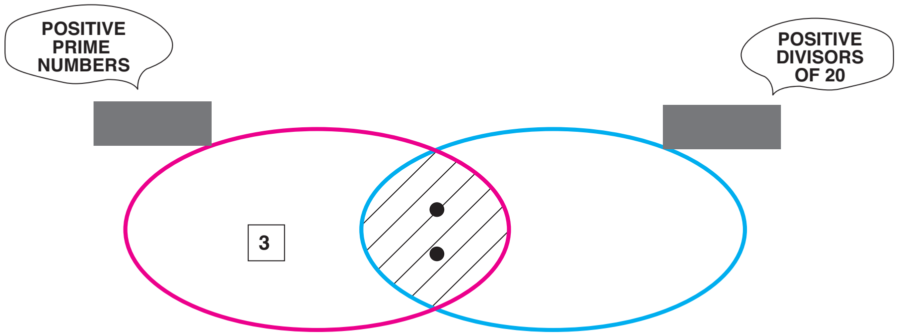
Note: A 3-String Game poster and a 3-String Game analysis sheet can be used in a 2-string situation by crossing out or folding under the third column.

Description of Lesson

Exercise 1

Prepare for a 2-string game analysis activity, as shown below. Ask a student to explain the hatching and the two dots in the middle region. This is a clue indicating that the strings have exactly two numbers in common.

RED	BLUE
MULTIPLES OF 2	MULTIPLES OF 2
MULTIPLES OF 3	MULTIPLES OF 3
MULTIPLES OF 4	MULTIPLES OF 4
MULTIPLES OF 5	MULTIPLES OF 5
MULTIPLES OF 10	MULTIPLES OF 10
ODD NUMBERS	ODD NUMBERS
POSITIVE PRIME NUMBERS	POSITIVE PRIME NUMBERS
GREATER THAN 50	GREATER THAN 50
LESS THAN 50	LESS THAN 50
GREATER THAN 10	GREATER THAN 10
LESS THAN 10	LESS THAN 10
POSITIVE DIVISORS OF 12	POSITIVE DIVISORS OF 12
POSITIVE DIVISORS OF 18	POSITIVE DIVISORS OF 18
POSITIVE DIVISORS OF 20	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	POSITIVE DIVISORS OF 24
POSITIVE DIVISORS OF 27	POSITIVE DIVISORS OF 27



Collectively analyze this starting situation. first, consider the location of 3 in the picture; then, use the information that there are only two numbers in the intersection of the strings. Three types of argument contribute to the analysis. Eliminations made using the respective type of argument are shown in color below each; previous eliminations are shown in black.

Note: The type 2 and type 3 arguments can be reversed. In this case, successive eliminations would be different, but the final result would be the same.

1

Eliminate possible labels for the strings using the information that 3 is inside the red string but outside the blue string.

RED	BLUE
~~MULTIPLES OF 2~~	MULTIPLES OF 2
MULTIPLES OF 3	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	MULTIPLES OF 4
~~MULTIPLES OF 5~~	MULTIPLES OF 5
~~MULTIPLES OF 10~~	MULTIPLES OF 10
ODD NUMBERS	~~ODD NUMBERS~~
POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
~~GREATER THAN 50~~	GREATER THAN 50
LESS THAN 50	~~LESS THAN 50~~
GREATER THAN $\widehat{10}$	~~GREATER THAN $\widehat{10}$~~
~~LESS THAN $\widehat{10}$~~	LESS THAN $\widehat{10}$
POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~

2

Check that each label remaining on the Blue list matches with at least one label remaining on the Red list to give exactly two numbers in the intersection. If not, cross out that label on the Blue list.

RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
MULTIPLES OF 3	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	MULTIPLES OF 4
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~
POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
~~GREATER THAN 50~~	~~GREATER THAN 50~~
LESS THAN 50	~~LESS THAN 50~~
GREATER THAN $\widehat{10}$	~~GREATER THAN $\widehat{10}$~~
~~LESS THAN $\widehat{10}$~~	~~LESS THAN $\widehat{10}$~~
POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20
POSITIVE DIVISORS OF 24	~~POSITIVE DIVISORS OF 24~~
POSITIVE DIVISORS OF 27	~~POSITIVE DIVISORS OF 27~~

3

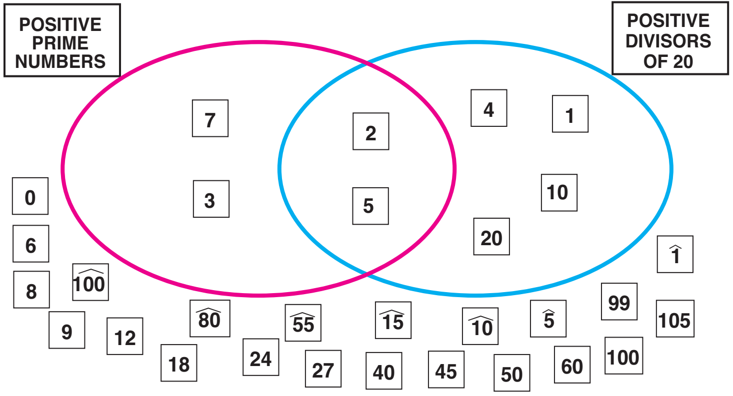
Check that each label remaining on the Red list matches with at least one label remaining on the Blue list to give exactly two numbers in the intersection. If not, cross out that label on the Red list.

RED	BLUE
~~MULTIPLES OF 2~~	~~MULTIPLES OF 2~~
~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	MULTIPLES OF 4
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~
POSITIVE PRIME NUMBERS	~~POSITIVE PRIME NUMBERS~~
~~GREATER THAN 50~~	~~GREATER THAN 50~~
~~LESS THAN 50~~	~~LESS THAN 50~~
~~GREATER THAN $\widehat{10}$~~	~~GREATER THAN $\widehat{10}$~~
~~LESS THAN $\widehat{10}$~~	~~LESS THAN $\widehat{10}$~~
POSITIVE DIVISORS OF 12	~~POSITIVE DIVISORS OF 12~~
POSITIVE DIVISORS OF 18	~~POSITIVE DIVISORS OF 18~~
~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

Do not be concerned if your class does not reduce the lists to six possibilities: four for the red and two for the blue. Simply proceed as in Exercise 1 of Lesson L1, asking students to suggest plays that give new information about the strings.

Collectively analyze the clues given by plays until the strings are determined. The following picture shows the correct placement for all of the game pieces.

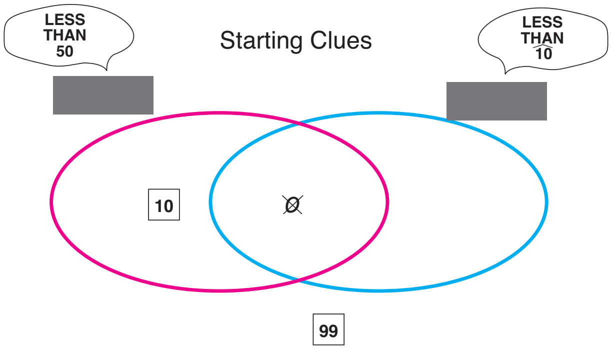
Correct Placement
of Game Pieces

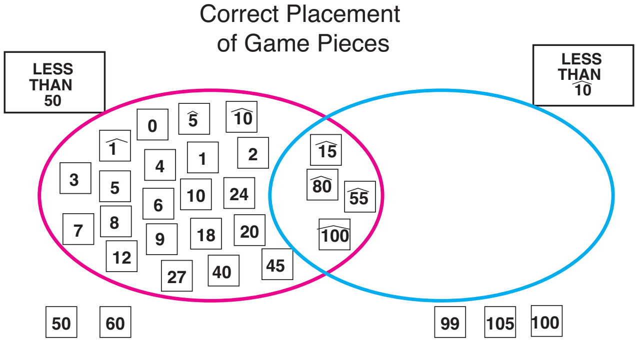


Exercise 2

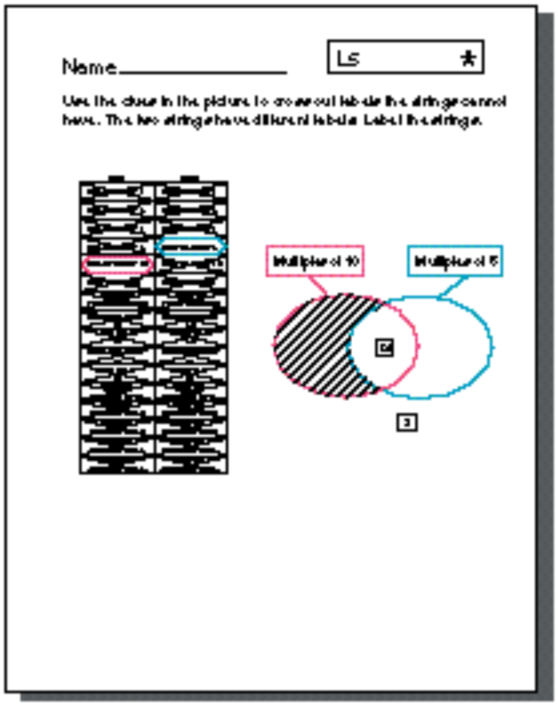
Play *The String Game* with special scoring, as described in Appendix D: *The String Game*, Version D. Review the special scoring rules for the class. A possible game is suggested below.

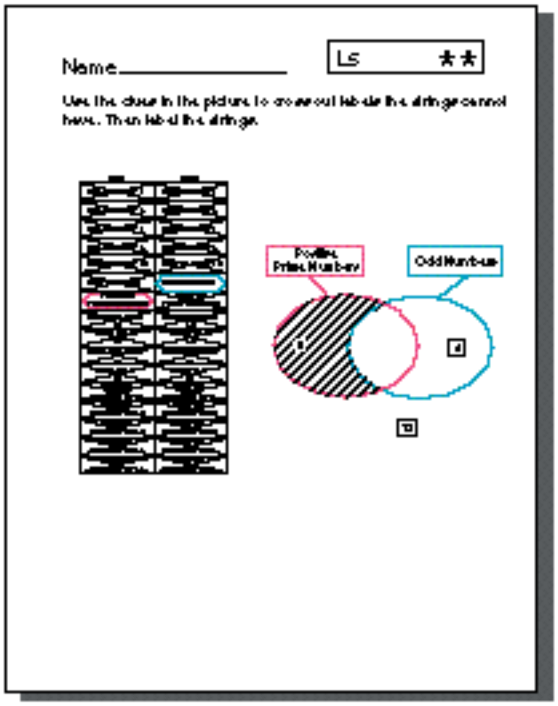
Note: The starting clues for this game determine the red string and eliminate all but five possibilities
for the blue string (**LESS THAN $\widehat{10}$**, **POSITIVE DIVISORS OF 12**, **POSITIVE DIVISORS OF 18**, **POSITIVE DIVISORS OF 24**, and **POSITIVE DIVISORS OF 27**). This is for your information only; do not tell the class.





Worksheets L5* and ** are available for individual work.





L6 THE RELATIONS GAME #1

Capsule Lesson Summary

Using an arrow picture for support, locate ordered pairs of numbers in a string picture involving the relation **is a multiple of**. Do a similar activity with the relation x10. Let students ask for the location of ordered pairs in a one-string picture to help them decide which of ten relations the string is for.

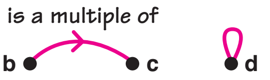
Materials

Teacher	• Colored chalk	**Student**	• Paper • Colored pencils, pens, or crayons

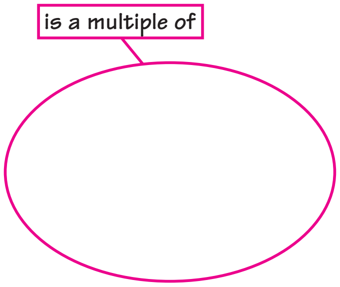
Description of Lesson

Exercise 1

Draw these pictures on the board.

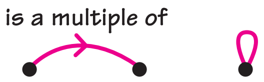


Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.



T: ***What numbers could these dots*** (point to **b** and **c**) ***be for?***

S: ***4 and 2. 4 is a multiple of 2.***



4	2
100	5
30	3
18	6
36	12

Record correct responses in lists below the dots. Continue the activity until four or five possibilities for **b** and **c** have been suggested, as shown here.

Ask for possibilities satisfying certain conditions. For example:

- **b** is more than 30. (32, 8)
- **b** is between 40 and 48. (45, 5)
- **b** is negative. ($^{-}9$, 3)
- **b** is 0. (If **b** is 0, **c** can be any integer)
- **c** is more than 50. (204, 102)
- **c** is 17. (51, 17)
- **c** is negative. ($^{-}63$, $^{-}7$)
- **c** is 1. (If **c** is 1, **b** can be any integer)

Note: In the series of lessons on *The Relations Game*, we will use raised minus signs for negative numbers since the game involves many calculator relations.

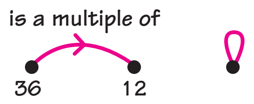
T: ***This string is related to this arrow.***

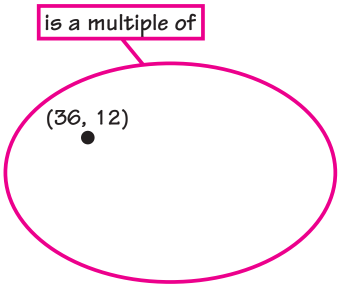
Point to one possibility for **b** and **c**, for example (36, 12).

T: ***36 is a multiple of 12. We can show this in the string picture.***

Draw a dot for (36, 12) inside the string.

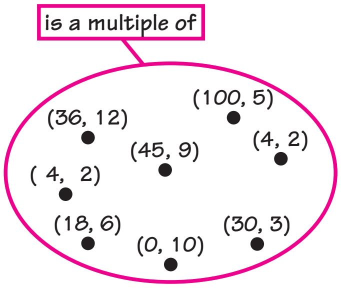
L6





T: ***We have a first number*** (indicate 36 in each picture) ***and a second number*** (indicate 12 in each picture)***. (36, 12) is called an* ordered pair *or a* couple. *What are some other ordered pairs that belong inside the string?***

Let students offer several ordered pairs belonging to the relation **is a multiple of**, as illustrated here.



T (tracing the loop)**: *What number could this dot (*****d*****) be for?***

Any integer is a multiple of itself, so the dot could be for any integer. A sample dialogue follows. Put the appropriate ordered pairs inside the string as they are mentioned.

S: ***0.***

T (putting (0,0) inside the string)**: *Yes, 0 is a multiple of 0.***

S: ***1.***

T: ***That's also right; 1 is a multiple of 1. So what ordered pair belongs inside the string?***

S: ***(1, 1).***

S: ***The dot with a loop could be for 9 because 9 is a multiple of 9, so (9, 9) belongs inside the string.***

T: ***What about ($^{-}7$. $^{-}7$)?***

S: ***It belongs inside the string because $^{-}7$ is a multiple of $^{-}7$.***

T: ***How do you know?***

S: ***1 x $^{-}7$ = $^{-}7$, and 1 is an integer.***

T (pointing to **d**)**: *Could any integer be here?*** (Yes)

Review what numbers are integers by checking numbers such as these.

2.0	(Yes)	$^{-}3.0$	(Yes)	0.4	(No)
$\frac{1}{2}$	(No)	$\frac{2}{3}$	(No)	$\frac{4}{2}$	(Yes)
100	(Yes)	$^{-}3143$	(Yes)	$\frac{^{-}5}{3}$	(No)

T: ***Both numbers in an ordered pair for the relation* is a multiple of *are integers.***

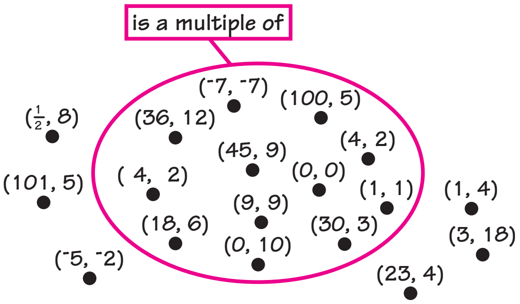
In *The Relations Game*, it will be important for students to remember this restriction.

Note: Although 4 x $\frac{1}{3} = \frac{4}{3}$ and 4 is an integer, the ordered pair ($\frac{4}{3}$, $\frac{1}{3}$) does not belong to this relation because the numbers are not integers. Likewise, (2, $\frac{1}{2}$) does not belong to this relation.

T: ***Where does (1, 4) belong in this string picture?***

S: ***Outside the string.***

Let students suggest other ordered pairs that belong outsi the string and put them in the picture, as illustrated here.



T: ***Let's find some ordered pairs in which neither number is an integer.***

S: ***(0.4, 96.6).***

S: ***($\frac{1}{2}$, $\frac{1}{3}$).***

T: ***Where do such ordered pairs belong in the string picture?***

S: ***Outside the string.***

T: ***What about ($\frac{1}{4}$, $\frac{1}{8}$)?*** (Outside) ***(1, $\frac{1}{2}$)?*** (Outside)
What about ($\frac{18}{3}$, $\frac{6}{3}$)?

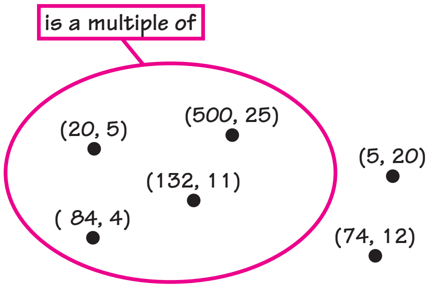
S: ***Inside the string, because $\frac{18}{3}$ = 6, $\frac{6}{3}$ = 2, and 6 is a multiple of 2.***

Erase all of the ordered pairs from the string picture; then write a list of ordered pairs nearby. For example:

(5, 20)	(500, 25)	(74, 12)
(20, 5)	(132, 11)	(84, 4)

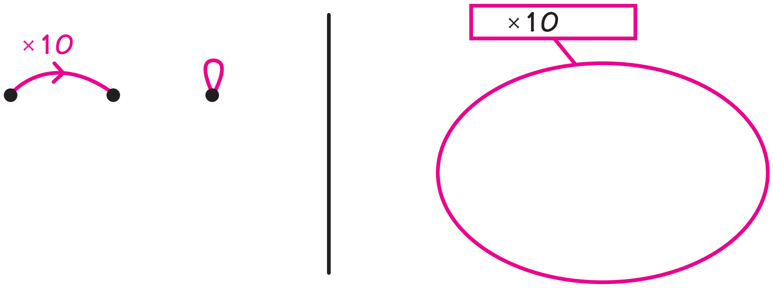
T: ***On a piece of paper, draw a string for the relation* is a multiple of *and locate these ordered pairs in your string picture.***

After a few minutes of individual work, discuss the correct location of each ordered pair.



Exercise 2

Change the pictures in Exercise 1 to consider the relation x10.



T: ***Let's consider the relation* x10*. This relation is not restricted to integers.***

Proceed in the following manner.

- Ask for ordered pairs that belong inside the string. Use the arrow picture for support. Be sure that some ordered pairs with negative components are included as well as some with positive components.
- Give one component of an ordered pair belonging to the relation x10 and ask for the other. Some examples are given below with answers in boxes. Take advantage of this opportunity to review multiplication of fractions and to find equivalent names for fractions.

$(\frac{1}{3}, \boxed{3\frac{1}{3}})$ $(\frac{3}{2}, \boxed{15})$ $(\boxed{\frac{4}{30}}, \frac{4}{3})$

$(\frac{2}{5}, \boxed{4})$ $(\boxed{\frac{2}{3}}, \frac{20}{3})$ $(\boxed{^{-}9}, ^{-}90)$

Refer to the opposite relation (return arrow) ÷10, or x$^1/_{10}$, whenever it would be helpful.

- Ask what number is at the dot with the loop. (0)
- Ask for ordered pairs that belong outside the string.
- List ordered pairs such as these on the board, and ask students to locate them in a string picture for x10 on their papers.

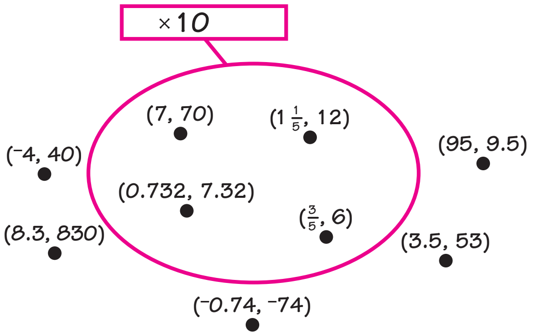
(7, 70)
(95, 9.5)
(8.3, 830)

$(^{-}0.74, ^{-}74)$
(0.732, 7.32)
$(^{-}4, 40)$

$(\frac{3}{5}, 6)$
$(1\frac{1}{5}, 12)$
(3.5, 53)

Allow a few minutes for individual work before discussing the correct location of each ordered pair.

Erase everything on the board except the string before going on to Exercise 3.



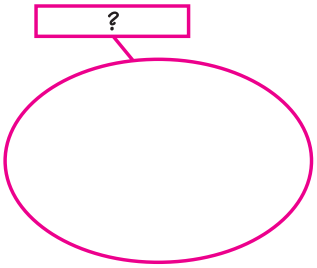
Exercise 3

List these relations on the board. Review them with the class. Mention that the relation **is a positive divisor of** is restricted to integers.

Note: The string is for + 5 = ···, but do not reveal this fact to students yet.

T: ***This string is for one of these ten relations. I'll give you a clue as to which one.***

is less than
is at least 20 less than
is a multiple of
is a positive divisor of
×10
×2
2 = ···
+ 3 = ···
+ 4 = ···
+ 5 = ···



Draw a dot for (8, 88) inside the string.

T: ***(8, 88) belongs to the relation. Which of these relations*** (point to the list) ***can we cross off?***

S: **Is a multiple of** *because 8 is not a multiple of 88.*

S: **x10***, because 8* x *10 = 80, not 88.*

S: **x2***, because 8* x *2 = 16, not 88.*

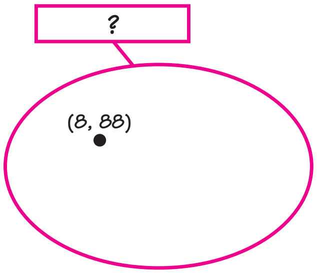
S: *Cross off* $\boxed{+}\boxed{3}\boxed{=}\cdots$*, because the difference between 8 and 88 is 80, and 80 is not a multiple of 3.*

S: *When you add 3s starting at 8, you can get 89 but not 88.*

S: *8 is two more than a multiple of 3, but 88 is one more than a multiple of 3.*

With the class, check that (8, 88) belongs to the remaining six relations.

is less than
is at least 20 less than
~~is a multiple of~~
is a positive divisor of
~~×10~~
~~×2~~
$\boxed{+}\boxed{2}\boxed{=}\cdots$
~~$\boxed{+}\boxed{3}\boxed{=}\cdots$~~
$\boxed{+}\boxed{4}\boxed{=}\cdots$
$\boxed{+}\boxed{5}\boxed{=}\cdots$



T: *Suggest an ordered pair whose location, whether inside or outside the string, would eliminate some of these relations as possibilities.*

When a student suggests an ordered pair, place it correctly in the picture. For example:

S: *(10, 20).*

T: *(10, 20) goes inside the string. Does this eliminate some relations as possibilities for the string? What can we cross off the list?*

S: **Is at least 20 less than***; 10 is only 10 less than 20, not at least 20 less than 20.*

S: $\boxed{+}\boxed{4}\boxed{=}\cdots$ *because 10 + 4 = 14, 14 + 4 = 18, and 18 + 4 = 22. 20 is skipped.*

T: *We need some more information. What is another ordered pair whose location would help us eliminate some other of these relations?*

S: *(1, 2).*

T: *(1, 2) goes outside the string. What can we cross off the list?*

S: **Is less than** *because 1 is less than 2.*

S: **Is a positive divisor of** *because 1 is a positive divisor of 2.*

T: *The relation is either* $\boxed{+}\boxed{2}\boxed{=}\cdots$ *or* $\boxed{+}\boxed{5}\boxed{=}\cdots$ *.*

S: *(7, 9).*

T: *(7, 9) goes outside the string.*

S: *The string is for* $\boxed{+}\boxed{5}\boxed{=}\cdots$*.*

~~is less than~~

~~is at least 20 less than~~

~~is a multiple of~~

~~is a positive ivisor of~~

~~×10~~

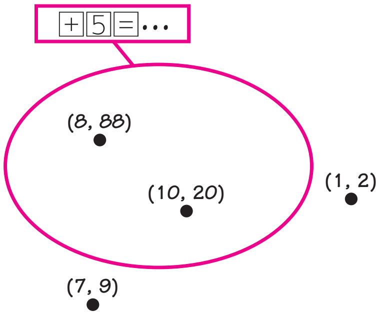
~~×2~~

~~$\boxed{+}\boxed{2}\boxed{=}\ldots$~~

~~$\boxed{+}\boxed{3}\boxed{=}\ldots$~~

~~$\boxed{+}\boxed{4}\boxed{=}\ldots$~~

$\boxed{+}\boxed{5}\boxed{=}\ldots$



L7 THE RELATIONS GAME #2

Capsule Lesson Summary

Review the idea of locating ordered pairs of numbers in a string for a relation. Let students ask for the location of ordered pairs in a one-string picture to help them decide which of ten relations the string is for. Introduce *The Relations Game*.

Materials

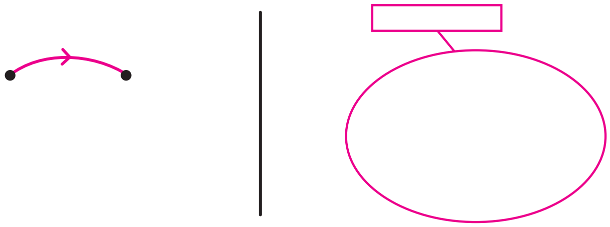
Teacher	- Colored chalk - Relations Game poster - Relations Game scoring sheet - Tape - Blacklines 7(a) and (b)	**Student**	- Relations Game analysis sheet

Advance Preparation: Use Blackline L7(a) to make copies of *The Relations Game* analysis sheet for students. Use Blackline L7(b) to make copies of a scoring sheet for use in play of the game (Exercise 3).

Description of Lesson

Exercise 1

Draw these pictures on the board.



T: *Last week we used both arrows and strings for relations. Can someone explain what the string has to do with the arrow?*

S: *Suppose the red arrow is for* **x10**. *If the beginning dot is for 2, then the ending dot is for 20. The ordered pair (2, 20) goes inside the string.*

T: *(2, 20) goes inside the string for* **x10**. *Would (20, 2) also go inside the string?*

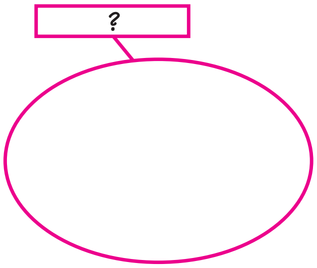
S: *No, because 20* x *10 = 200, not 2.*

Review this idea with a couple other examples of relations; for example, $\boxed{+}\boxed{3}\boxed{=}\cdots$ … and $\times\frac{2}{3}$.

Exercise 2

List these relations on the board near the string picture.

is less than
is at least 20 less than
is a positive divisor of
$\times 10$
$\times 2$
$\times \frac{2}{3}$
$\boxed{+}\boxed{2}\boxed{=}\cdots$
$\boxed{+}\boxed{3}\boxed{=}\cdots$
$\boxed{+}\boxed{4}\boxed{=}\cdots$
$\boxed{+}\boxed{5}\boxed{=}\cdots$



T: *This string is for one of these ten relations. I'll give you a clue as to which one.*

Note: The string is for the relation **is at least 20 less than**. Do not reveal the relation to the students at this time. The students will discover it during the activity.

Draw a dot inside the string for (3.5, 63½).

T: ***(3.5, 63½) belongs to the relation. Are there any of these relations*** (point to the list) ***that the string cannot be for?***

S: **x10** ***and*** **x2*****. 3.5*** x ***10 = 35 and 3.5*** x ***2 = 7, not 63½.***

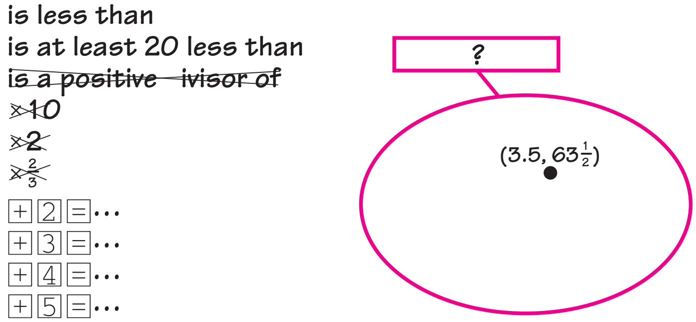
Cross a relation off the list when a student explains why the string cannot be for that relation.

S: $\times\frac{2}{3}$. ***3.5*** x ***⅔ = ⁷⁄₃.***

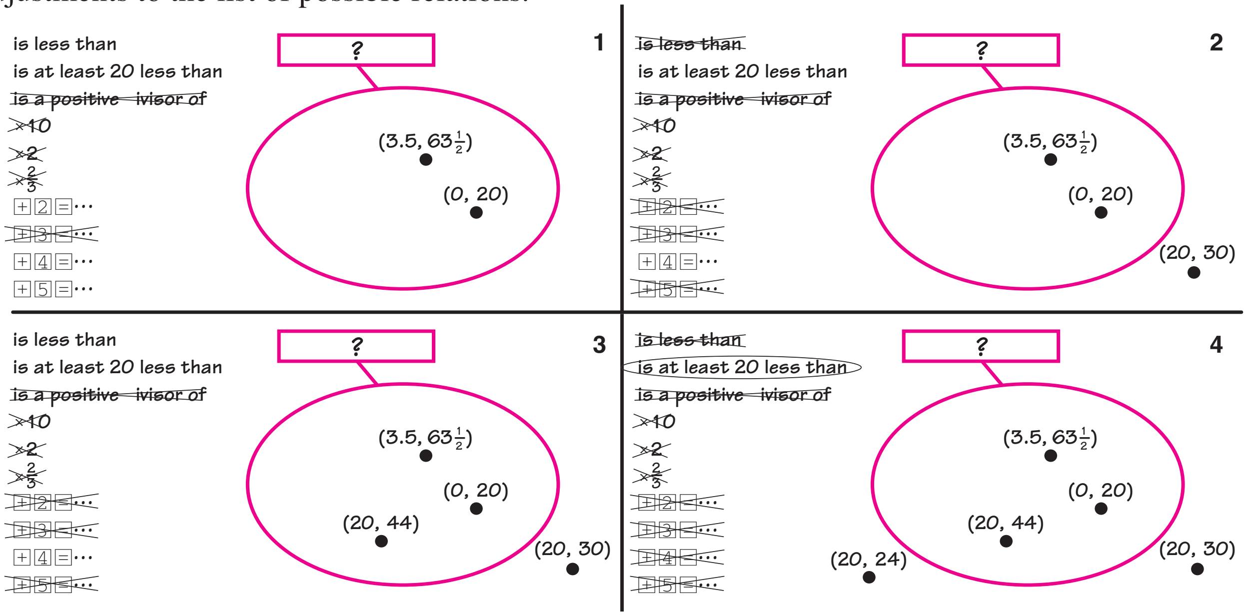
S: ***The relation*** **is a positive divisor of** ***has only to do with integers, so we can cross it off.***

With the class, check that (3.5, 63½) belongs to the remaining relations.

T: ***Can you suggest an ordered pair whose location, whether inside or outside the string, would eliminate some of these other possibilities?***



In the string picture, locate whichever ordered pair is suggested. The class should use the information of where the ordered pair belongs to eliminate, if possible, some of the remaining relations. Continue asking for ordered pairs and eliminating possibilities until the relation is determined. The illustrations that follow record a sequence of suggested ordered pairs and the corresponding adjustments to the list of possible relations.



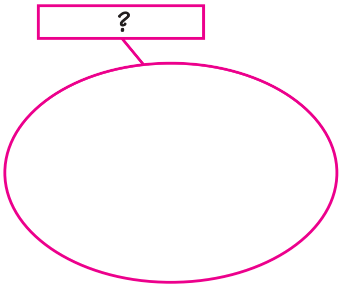
The location of (20, 44) gives no new information.

Exercise 3: The Relations Game

The rules of *The Relations Game* are similar to the rules of *The Table Game*. In *The Relations Game*, players attempt to determine which of 21 given relations (see the poster) a string[†] is for. The game is divided into two parts. In both parts players select ordered pairs, but only in the second part do they predict the location of ordered pairs before they are put correctly in the picture. Scoring rules change from one part to the next. To play the usual version of *The Relations Game*, proceed as follows.

On the board draw a large string and display a copy of *The Relations Game* poster near the string.

is the square of	The Relations Game		
is a multiple of	is a positive divisor of	is less than	is greater than
x2	x10	x$\frac{2}{3}$	is at least 20 greater than
÷2	÷10	÷$\frac{2}{3}$	is at least 20 less than
$\boxed{+}\boxed{2}\boxed{=}\cdots$	$\boxed{+}\boxed{3}\boxed{=}\cdots$	$\boxed{+}\boxed{4}\boxed{=}\cdots$	$\boxed{+}\boxed{5}\boxed{=}\cdots$
$\boxed{-}\boxed{2}\boxed{=}\cdots$	$\boxed{-}\boxed{3}\boxed{=}\cdots$	$\boxed{-}\boxed{4}\boxed{=}\cdots$	$\boxed{-}\boxed{5}\boxed{=}\cdots$



Select a student who will be a reliable scorekeeper[††]. Divide the rest of the class into four (or two) teams. In the following example, the four teams are referred to as Team A, Team B, Team C, and Team D. Distribute Relations Game analysis sheets to students.

The game begins with one to three ordered pairs in the string picture as starting clues. Such ordered pairs can belong either inside or outside the string. Players, either independently or with their team members, use the clues to try to eliminate some of the 21 possibilities for the relation. Before further play, give students the option to turn in their analysis sheets with one of the relations circled. A team gains 15 points each time a member correctly identifies the relation before additional plays are made. If team members are consulting among themselves, you may prefer to ask the team to turn in just one sheet. The scorekeeper tallies the points for each team but does not reveal the score at this time.

After a few minutes, proceed by letting each team make a play. A play consists of selecting an ordered pair to put in the picture as an additional clue. After each play, students or teams may turn in their analysis sheets with a relation circled. Correctly identifying the relation gains a certain number of points (see the scoring sheet) depending upon how many plays (additional clues) have been given. With two teams, each team makes two plays. The last time students (or teams) may turn in their analysis sheets is after the fourth play is made.

Note: You may find reason to vary the number of additional clues—for example, if time is running short, if most players have turned in their analysis sheets after two additional clues, or if very few players have turned in their analysis sheets after four additional clues, and so on.

[†] See Lesson L11 for a description of how to play this game with two strings.

[††] The scorekeeper can be an adult who is available during the class time, such as a teacher's aide or student teacher.

L7

Now the scoring changes. During the second part of the game, each player (alternating teams) selects an ordered pair and says whether it belongs inside or outside the string. Announce whether or not the play is correct, and put the suggested ordered pair correctly in the picture. The scoring for this part of the game is as follows:

- A team gains two points each time a member correctly locates an ordered pair inside the string.
- A team gains one point each time a member correctly locates an ordered pair outside the string.
- A team neither gains nor loses points when a member incorrectly locates an ordered pair.

The second part of the game ends after each player has had a turn. Call on a student to identify the secret relation while the scorekeeper computes the scores. It is a good idea to provide a scoring sheet for the scorekeeper to use. Blackline L7 has a sample.

Secret Relation: ________________

Points		**TEAM A**	**TEAM B**	**TEAM C**	**TEAM D**
First part of game					
For **correctly** identifying the relation					
• before additional plays	15 pts.				
• after first play	10 pts.				
• after second play	9 pts.				
• after third play	8 pts.				
• after fourth play	7 pts.				
Second part of game					
For **correctly** locating an ordered pair					
• inside the string	2 pts.				
• outside the string	1 pt.				

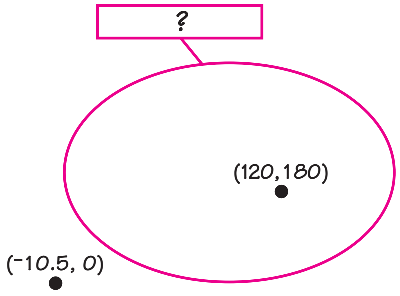
Optional Rule: In the second part of the game, you can encourage students to suggest many kinds of numbers by awarding an extra point for each non-whole number in an ordered pair correctly located inside the string. For a game played with the optional rule, modify the scoring sheet as shown below.

Secret Relation: ________________

Points		**TEAM A**	**TEAM B**	**TEAM C**	**TEAM D**
First part of game					
For **correctly** identifying the relation					
• before additional plays	15 pts.				
• after first play	10 pts.				
• after second play	9 pts.				
• after third play	8 pts.				
• after fourth play	7 pts.				
Second part of game					
For **correctly** locating an ordered pair					
• inside the string	2 pts.				
(bonus for non-whole numbers	1 pt. each)				
• outside the string	1 pt.				

Play the game with your class. The rest of this lesson description presents a possible game. The secret relation is $\div\frac{2}{3}$; there are two ordered pairs—(120, 180) and (⁻10.5, 0)—located as starting clues.

is the square of	The Relations Game		
is a multiple of	is a positive divisor of	is less than	is greater than
x2	x10	$\times\frac{2}{3}$	is at least 20 greater than
÷2	÷10	$\div\frac{2}{3}$	is at least 20 less than
$\boxed{+}\boxed{2}\boxed{=}\cdots$	$\boxed{+}\boxed{3}\boxed{=}\cdots$	$\boxed{+}\boxed{4}\boxed{=}\cdots$	$\boxed{+}\boxed{5}\boxed{=}\cdots$
$\boxed{-}\boxed{2}\boxed{=}\cdots$	$\boxed{-}\boxed{3}\boxed{=}\cdots$	$\boxed{-}\boxed{4}\boxed{=}\cdots$	$\boxed{-}\boxed{5}\boxed{=}\cdots$



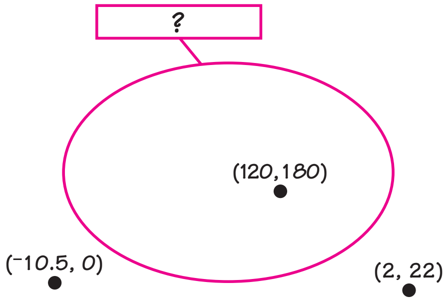
T: ***Knowing that (120, 180) belongs inside the string and that (⁻10.5, 0) belongs outside the string should help you to eliminate some of the possibilities for the relation. On your analysis sheet cross out relations the string cannot be for.***

Allow a few minutes for analysis. Then ask if anyone wants to turn in an analysis sheet for a possible 15 points.

Note: For your information only, there are six remaining possibilities for the secret relation: **is at least 20 less than**, $\div\frac{2}{3}$, $\boxed{+}\boxed{2}\boxed{=}\cdots$, $\boxed{+}\boxed{3}\boxed{=}\cdots$, $\boxed{+}\boxed{4}\boxed{=}\cdots$, and $\boxed{+}\boxed{5}\boxed{=}\cdots$.

No one turns in an analysis sheet.

Call on a player from Team A to suggest an ordered pair for you to locate.



S: *(2, 22).*

T: ***(2, 22) belongs outside the string.***

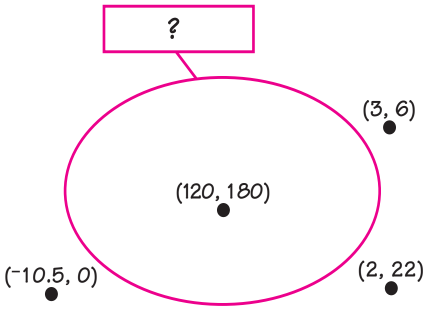
Allow a few minutes for players to analyze the situation.

Note: This clue eliminates four more possibilities and there are two remaining possibilities: $\div\frac{2}{3}$ and $\boxed{+}\boxed{3}\boxed{=}\cdots$.

T: ***Does anyone want to turn in their sheets for a possible ten points?***

No one turns in an analysis sheet.

Call on a player from Team B to suggest another ordered pair.



S: *(3, 6).*

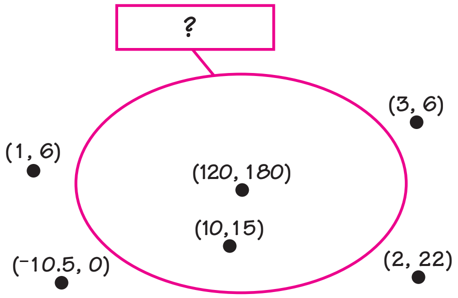
T: ***(3, 6) belongs outside the string.***

Allow a few minutes for the analysis.

T: ***Does anyone want to turn in their sheets for a possible nine points?***

Several students turn in their analysis sheets. (The relation is actually determined at this point.)

Next, players from Team C and Team D suggest (10, 15) and (1, 6) respectively. More students turn in their analysis sheets for eight points or for seven points.



The scorekeeper tallies the points in the game so far but does not reveal the totals to the class.

Team A	Team B	Team C	Team D
34	33	30	33

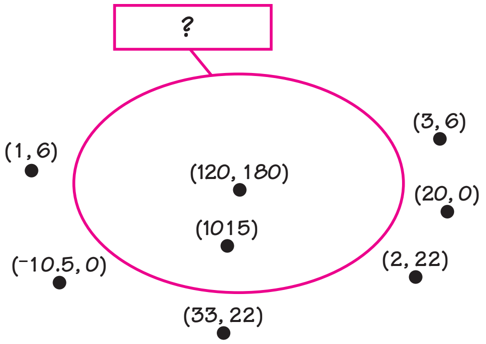
Now the scoring changes. Players select ordered pairs and state whether each belongs inside or outside the string.

S (Team A)**:** ***(20, 0) belongs outside the string.***

T: ***Yes. Team A gains one point.***

S (Team B)**:** ***(33, 22) belongs inside the string.***

T: ***No, (33, 22) belongs outside. Team B neither gains nor loses points.***



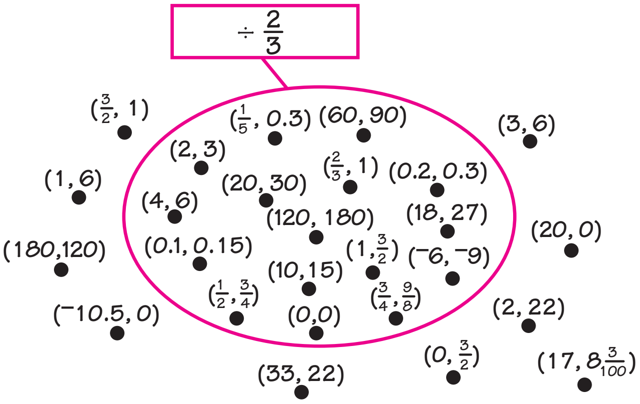
The game continues until every player has had a turn.

T: ***While the scorekeeper is computing the scores, who can tell us which relation the string is for?***

S: $\div\frac{2}{3}$.

The scorekeeper reveals the scores. Team B wins.

The string picture and the scorekeeper's account at the end of the game are shown below. (In this example, the optional rule was in effect.)



Team A	Team B	Team C	Team D
34	33	30	33
1	0	4	1
0	2	2	0
4	4	0	4
2	2	2	3
1	4	4	3
42	45	42	44

L8 THE RELATIONS GAME #3

Capsule Lesson Summary

Complete ordered pairs of numbers, some which belong and some which do not belong to the relation **is the square of**. In a relations game context, use three clues to eliminate possibilities for a string's label. Then investigate how the location of various ordered pairs would distinguish the four remaining possibilities. Play *The Relations Game*.

Materials

Teacher
- Colored chalk
- Relations Game poster
- Relations Game scoring sheet
- Markers
- Blacklines L7(a) and (b)

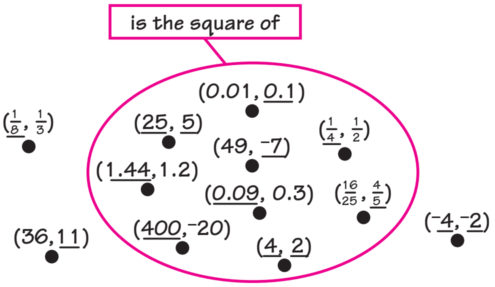
Student
- Relations Game analysis sheet
- Paper
- Colored pencils, pens, or crayons

Advance Preparation: Use Blackline L7(a) to make copies of *The Relations Game* analysis sheet for students. Use Blackline L7(b) to make copies of a scoring sheet for use in play of the game (Exercise 3).

Description of Lesson

Exercise 1

Draw this string picture on the board, but leave blanks where there are underlined numerals. Ask students to copy the picture and to fill in the blanks. Allow about five minutes for students to work individually or with partners; then review possible answers collectively. The illustration here shows one of the many possible ways to complete the picture.

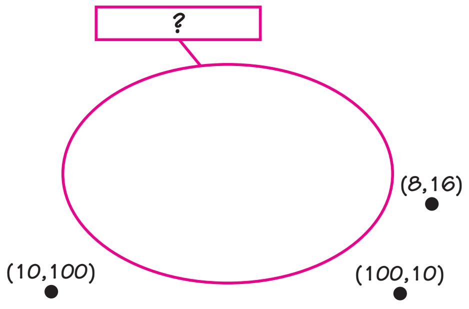


Comments

- Be aware that there are many correct ways to fill in each blank since every number has many names. For example, the square of $\frac{1}{2}$ may be written as $\frac{1}{4}$, $\frac{2}{8}$, 0.25, and so on.
- When multiplying decimals, students may prefer to do the calculation with fractions. For example, $0.3 \text{ x } 0.3 = \frac{3}{10} \text{ x } \frac{3}{10} = \frac{9}{100} = 0.09$.
- Observe that there is only one number that is the square of another number, but that a positive number is the square of two different numbers. For example, in this exercise there is only one possibility for the first number of the ordered pair (_____, 1.2), namely 1.44. But there are two possibilities for the second number of the ordered pair (49, _____), namely 7 and $^-7$.
- Notice that the square of a number is not always greater than the number; for example, $(\frac{1}{2})^2 = \frac{1}{4}$ and $\frac{1}{4} < \frac{1}{2}$. For your information, the square of a number between 0 and 1 is always less than the number.

Exercise 2

Draw this string picture on the board and display a Relations Game poster nearby.



is the square of	The Relations Game		
is a multiple of	is a positive divisor of	is less than	is greater than
x2	x10	$\times\frac{2}{3}$	is at least 20 greater than
÷2	÷10	$\div\frac{2}{3}$	is at least 20 less than
$\boxed{+}\boxed{2}\boxed{=}\cdots$	$\boxed{+}\boxed{3}\boxed{=}\cdots$	$\boxed{+}\boxed{4}\boxed{=}\cdots$	$\boxed{+}\boxed{5}\boxed{=}\cdots$
$\boxed{-}\boxed{2}\boxed{=}\cdots$	$\boxed{-}\boxed{3}\boxed{=}\cdots$	$\boxed{-}\boxed{4}\boxed{=}\cdots$	$\boxed{-}\boxed{5}\boxed{=}\cdots$

T: *The string is for one of these relations* (point to the poster)*. There are three clues in the picture. Which relations can we cross off, and why?*

S: *Cross off* x10 *and* ÷10*. 10* x *10 = 100 and (10, 100) is outside the string; 100 ÷ 10 = 10 and (100, 10) is outside the string.*

S: *100 is 90 greater than 10 so cross off* **is at least 20 greater than** *and* **is greater than.**

S: *Since 90 (the difference between 100 and 10) is a multiple of 2, 3, and 5, we can cross off* $\boxed{-}\boxed{2}\boxed{=}\cdots$*,* $\boxed{-}\boxed{3}\boxed{=}\cdots$*, and* $\boxed{-}\boxed{5}\boxed{=}\cdots$*. But we cannot cross off* $\boxed{-}\boxed{4}\boxed{=}\cdots$ *because 90 is not a multiple of 4.*

S: *Likewise 10 is 90 less than 100. So we can cross off …***is at least 20 less than***,* **is less than***,* $\boxed{+}\boxed{2}\boxed{=}\cdots$*,* $\boxed{+}\boxed{3}\boxed{=}\cdots$*, and* $\boxed{+}\boxed{5}\boxed{=}\cdots$*.*

Continue the collective analysis until only four labels remain as possibilities.

T: *There are still four possibilities for the string's label. If we knew an ordered pair that belonged inside the string, would we know what the string is for?*

~~is the square of~~	The Relations Game		
~~is a multiple of~~	~~is a positive divisor of~~	~~is less than~~	~~is greater than~~
~~x2~~	~~x10~~	$\times\frac{2}{3}$	~~is at least 20 greater than~~
÷2	~~÷10~~	$\div\frac{2}{3}$	~~is at least 20 less than~~
~~$\boxed{+}\boxed{2}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{3}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{4}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{5}\boxed{=}\cdots$~~
~~$\boxed{-}\boxed{2}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{3}\boxed{=}\cdots$~~	$\boxed{-}\boxed{4}\boxed{=}\cdots$	~~$\boxed{-}\boxed{5}\boxed{=}\cdots$~~

Allow a few minutes for students to consider the situation before discussing it. Perhaps their initial reaction will be that the location of any ordered pair inside the string would determine the string label. To investigate, ask for ordered pairs that could belong inside the string, and ask students what their location tells about the string label. For example:

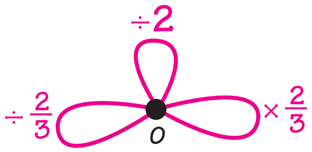
S: *If (6, 4) belongs inside the string, then the string is for the relation* $\times\frac{2}{3}$*; 6* x *⅔ = 4. But 6 ÷ 2 = 3, not 4. 6 ÷ ⅔ = 9, not 4. And* $\boxed{6}\ \boxed{-}\ \boxed{4}\ \boxed{=}$ *… never equals 4.*

S: *If (30, 10) belongs inside the string, then the string is for the relation* $\boxed{-}\boxed{4}\boxed{=}\cdots$ *because the difference between 30 and 10 is 20, a multiple of 4. But 30 ÷ 2 = 15, not 10. Also, 30* x *⅔ = 20, not 10, and 30 ÷ ⅔ = 45, not 10.*

T: *Can an ordered pair ever belong to more than one of these four relations?*

If no one suggests (0, 0), do so yourself.

The class should find that (0, 0) belongs to three of the four relations.



T: ***Can an ordered pair ever belong to both ÷2 and $\boxed{-}\boxed{4}\boxed{=}\cdots$? To both ×$\frac{2}{3}$ and $\boxed{-}\boxed{4}\boxed{=}\cdots$? To both ÷$\frac{2}{3}$ and $\boxed{-}\boxed{4}\boxed{=}\cdots$?***

While students consider the situation, draw this table on the board.

÷2	×$\frac{2}{3}$	÷$\frac{2}{3}$

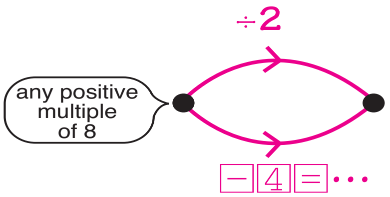
T: ***Let's find some ordered pairs that belong to each of these relations, trying to find some that also belong to the relation $\boxed{-}\boxed{4}\boxed{=}\cdots$.***

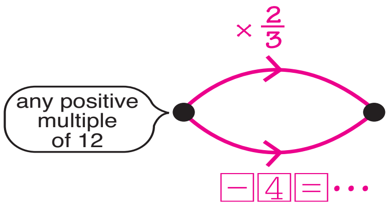
Accept and record ordered pairs for each relation. Each time an ordered pair is suggested, ask the class to check if it also belongs to the relation $\boxed{-}\boxed{4}\boxed{=}\cdots$; circle an ordered pair that does.

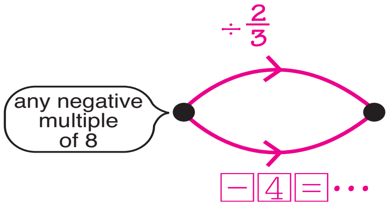
Continue until the class finds at least two ordered pairs for each of the three relations that also belong to the relation $\boxed{-}\boxed{4}\boxed{=}\cdots$. Your table might look similar to this one.

÷2	×$\frac{2}{3}$	÷$\frac{2}{3}$
(10, 5)	$(1, \frac{2}{3})$	(10, 15)
$\enclose{circle}{(8, 4)}$	(6, 4)	(2, 3)
(12, 6)	(9, 6)	(4, 6)
$(7, 3\frac{1}{2})$	$\enclose{circle}{(12, 8)}$	(⁻4, ⁻6)
$\enclose{circle}{(16, 8)}$	(15, 10)	$\enclose{circle}{(^-8, ^-12)}$
$\enclose{circle}{(24, 12)}$	$\enclose{circle}{(24, 16)}$	$\enclose{circle}{(^-16, ^-24)}$
(⁻8, ⁻4)	(⁻12, ⁻8)	$\enclose{circle}{(^-24, ^-36)}$

Note: During the activity, the class may begin to see that there are infinitely many possibilities in each case, as summarized in the pictures below.







T: ***Getting back to our original question, would we necessarily know which of the four relations the string is for if we knew an ordered pair that belonged inside the string?***

S: ***No; knowing that (8, 4) belongs inside the string, for example, would not tell us whether the string is for ÷2 or for $\boxed{-}\boxed{4}\boxed{=}\cdots$.***

Any of the circled ordered pairs in the table could be used in an explanation.

Exercise 3

Play *The Relations Game* if there is sufficient time remaining in the class period. (See Lesson L7, Exercise 3 for a description of the game.) For a possible game, let the string be for the relation **is a positive divisor of**, and locate (1.5, 21.5) outside the string as a starting clue.

L9 THE STRING GAME #3

Capsule Lesson Summary

Analyze some starting clues and the clues given by making plays in a 3-string game. Select plays that give as much new information as possible. Play *The String Game* with special scoring rules and three strings.

Materials

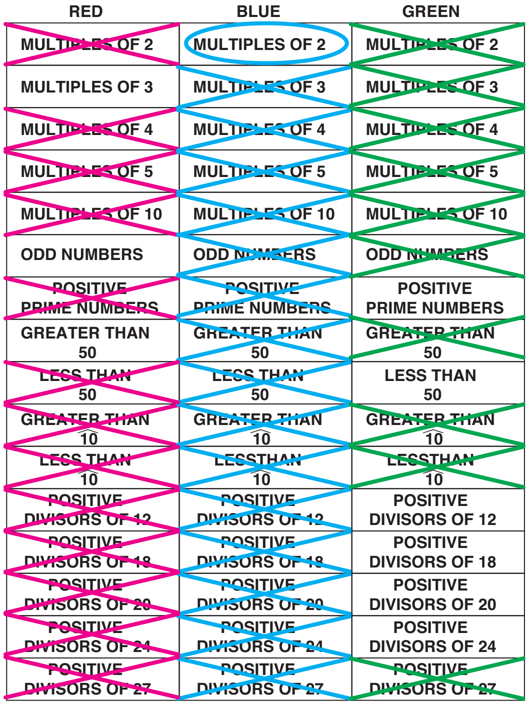
Teacher
- Colored chalk
- Numerical String Game kit
- Numerical 3-String Game posters
- Colored markers or crayons

Student
- 3-String Game analysis sheet
- Worksheets L9* and **

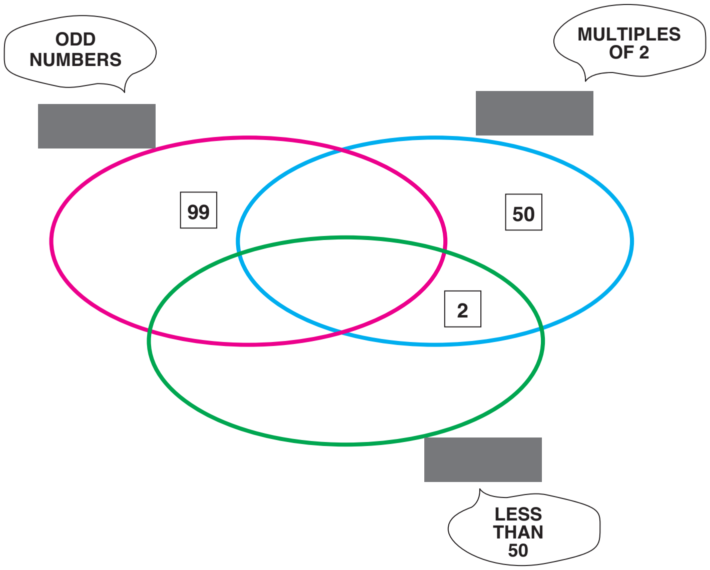
Description of Lesson

Exercise 1

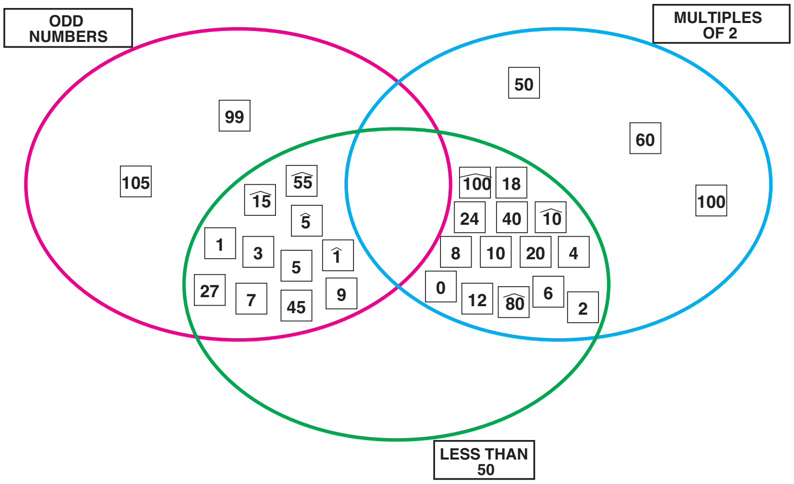
Prepare for a 3-string game analysis activity as shown below. Distribute 3-String Game analysis sheets, and ask students to cross out labels the strings cannot have. After several minutes of individual or small group work, collectively analyze the starting clues. Encourage students to eliminate several labels at a time. The result of the analysis is shown below.



RED	BLUE	GREEN
~~MULTIPLES OF 2~~	MULTIPLES OF 2	~~MULTIPLES OF 2~~
MULTIPLES OF 3	~~MULTIPLES OF 3~~	~~MULTIPLES OF 3~~
~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~	~~MULTIPLES OF 4~~
~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~	~~MULTIPLES OF 5~~
~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~	~~MULTIPLES OF 10~~
ODD NUMBERS	~~ODD NUMBERS~~	~~ODD NUMBERS~~
~~POSITIVE PRIME NUMBERS~~	~~POSITIVE PRIME NUMBERS~~	POSITIVE PRIME NUMBERS
GREATER THAN 50	~~GREATER THAN 50~~	~~GREATER THAN 50~~
~~LESS THAN 50~~	~~LESS THAN 50~~	LESS THAN 50
~~GREATER THAN 10~~	~~GREATER THAN 10~~	~~GREATER THAN 10~~
~~LESS THAN 10~~	~~LESSTHAN 10~~	~~LESSTHAN 10~~
~~POSITIVE DIVISORS OF 12~~	~~POSITIVE DIVISORS OF 12~~	POSITIVE DIVISORS OF 12
~~POSITIVE DIVISORS OF 18~~	~~POSITIVE DIVISORS OF 18~~	POSITIVE DIVISORS OF 18
~~POSITIVE DIVISORS OF 20~~	~~POSITIVE DIVISORS OF 20~~	POSITIVE DIVISORS OF 20
~~POSITIVE DIVISORS OF 24~~	~~POSITIVE DIVISORS OF 24~~	POSITIVE DIVISORS OF 24
~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~	~~POSITIVE DIVISORS OF 27~~

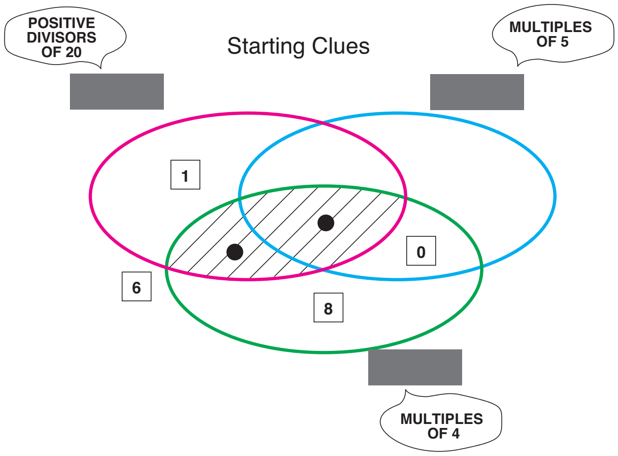


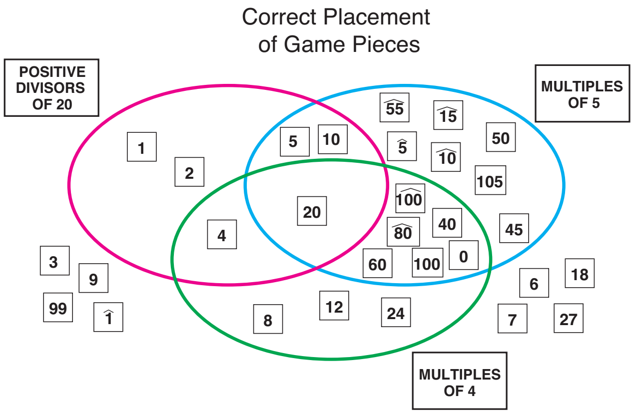
Invite students to make some plays and to analyze the resulting clues, as in Lesson L1 Exercise 1. Challenge the class to determine the strings with as few plays as possible. For your information, the correct placement for all of the game pieces is shown below.



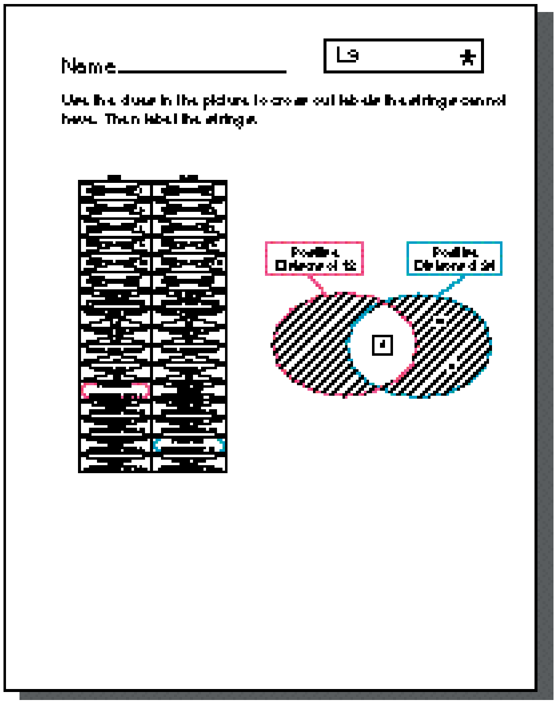
Exercise 2

Play *The String Game* with special scoring as described in Appendix D: *The String Game*, Version D. Use a game with three strings, such as the one suggested below. Here the starting clues determine the red and green strings, and leave only two possibilities for the blue string.



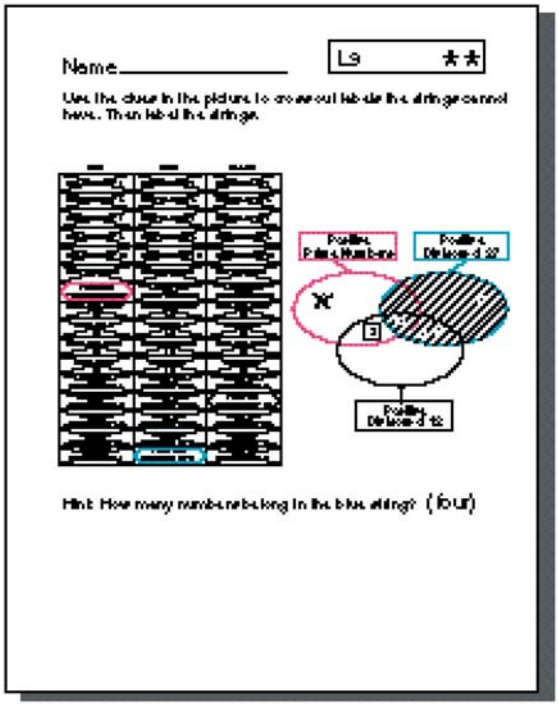


Worksheets L9* and ** are available for individual work or for group analysis. You may wish to have a class discussion about the hatching on Worksheet L9*, namely that the two sets are identical except for two elements. Both worksheets are difficult.



Name________________ L9 ★

Use the clues in the picture to cross out labels the strings cannot have. Then label the strings.



Name________________ L9 ★★

Use the clues in the picture to cross out labels the strings cannot have. Then label the strings.

Hint: How many numbers belong in the blue string? (four)

L10 THE RELATIONS GAME #4

Capsule Lesson Summary

Play *The Relations Game*. Locate some ordered pairs in each of the four regions of a two-string picture in which one string is for the relation $\boxed{+}\boxed{5}\boxed{=}\cdots$ and the other is for the relation **is at least 20 less than**.

Materials

Teacher

- Colored chalk
- Relations Game poster
- Relations Game scoring sheet
- Colored markers or crayons
- Blacklines L7(a) and (b)

Student

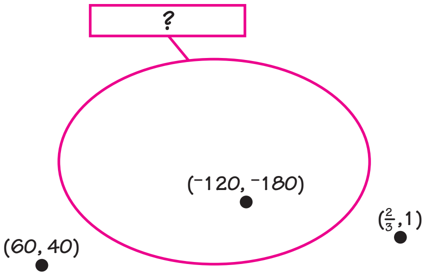
- Relations Game analysis sheet
- Paper
- Colored pencils, pens, or crayons

Advance Preparation: Use Blackline L7(a) to make copies of *The Relations Game* analysis sheet for students. Use Blackline L7(b) to make a scoring sheet for use in play of the game (Exercise 1).

Description of Lesson

Exercise 1

Play *The Relations Game*. For a possible game, let the string be for the relation $\boxed{-}\boxed{3}\boxed{=}\cdots$, and locate three ordered pairs in the string picture as starting clues. You may want to tell the class that the relation is determined by these clues.



is the square of	The Relations Game		
is a multiple of	is a positive divisor of	is less than	is greater than
x2	x10	x $\frac{2}{3}$	is at least 20 greater than
÷2	÷10	÷ $\frac{2}{3}$	is at least 20 less than
$\boxed{+}\boxed{2}\boxed{=}\cdots$	$\boxed{+}\boxed{3}\boxed{=}\cdots$	$\boxed{+}\boxed{4}\boxed{=}\cdots$	$\boxed{+}\boxed{5}\boxed{=}\cdots$
$\boxed{-}\boxed{2}\boxed{=}\cdots$	$\boxed{-}\boxed{3}\boxed{=}\cdots$	$\boxed{-}\boxed{4}\boxed{=}\cdots$	$\boxed{-}\boxed{5}\boxed{=}\cdots$

After the game, collectively analyze the starting situation to find that the relation indeed is determined by the three clues.

Using the information that ($^{-}$120, $^{-}$180) is inside the string, cross off these relations.

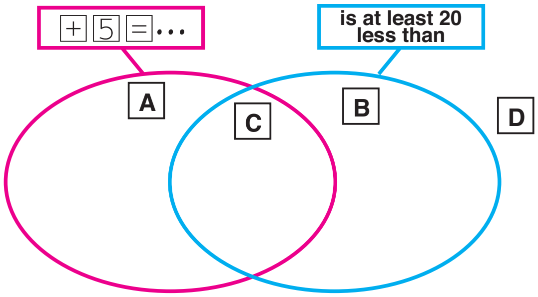
~~is the square of~~	The Relations Game		
~~is a multiple of~~	~~is a positive divisor of~~	~~is less than~~	is greater than
~~x2~~	~~x10~~	~~x $\frac{2}{3}$~~	is at least 20 greater than
~~÷2~~	~~÷10~~	÷ $\frac{2}{3}$	~~is at least 20 less than~~
~~$\boxed{+}\boxed{2}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{3}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{4}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{5}\boxed{=}\cdots$~~
$\boxed{-}\boxed{2}\boxed{=}\cdots$	$\boxed{-}\boxed{3}\boxed{=}\cdots$	$\boxed{-}\boxed{4}\boxed{=}\cdots$	$\boxed{-}\boxed{5}\boxed{=}\cdots$

Using the information that (60, 40) and (⅔, 1) are outside the string, cross off additional relations.

~~is the square of~~	The Relations Game		
~~is a multiple of~~	~~is a positive divisor of~~	~~is less than~~	~~is greater than~~
~~x2~~	~~x10~~	~~x $\frac{2}{3}$~~	~~is at least 20 greater than~~
~~÷2~~	~~÷10~~	~~÷ $\frac{2}{3}$~~	~~is at least 20 less than~~
~~$\boxed{+}\boxed{2}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{3}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{4}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{5}\boxed{=}\cdots$~~
~~$\boxed{-}\boxed{2}\boxed{=}\cdots$~~	$\boxed{-}\boxed{3}\boxed{=}\cdots$	~~$\boxed{-}\boxed{4}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{5}\boxed{=}\cdots$~~

Your students might not use the clues in any particular order nor exhaust each clue completely. Allow the analysis to flow in a style that your students initiate.

Exercise 2



Draw this string picture on the board.

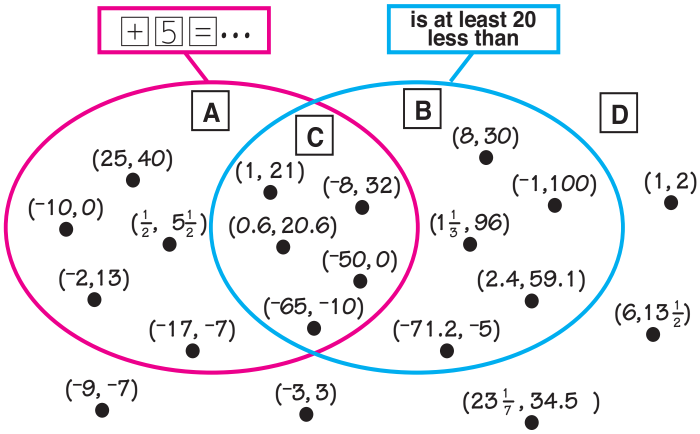
Ask for ordered pairs that belong to each of the four regions. Sometimes place conditions on the numbers, for example:

- Both numbers in the ordered pair are whole numbers.
- Only one number in the ordered pair is a whole number.
- Neither number in the ordered pair is a whole number.
- One number in the ordered pair is negative.
- Both numbers in the ordered pair are negatives.

When a student suggests an ordered pair, ask why it belongs in the region being discussed at that time. A good response might be the following.

S: ***(10, 15) belongs in region* A. *10 + 5 = 15, so (10, 15) belongs in the red string. 10 is not at least 20 less than 15, so (10, 15) does not belong in the blue string.***

After a while, your picture might look similar to this one.

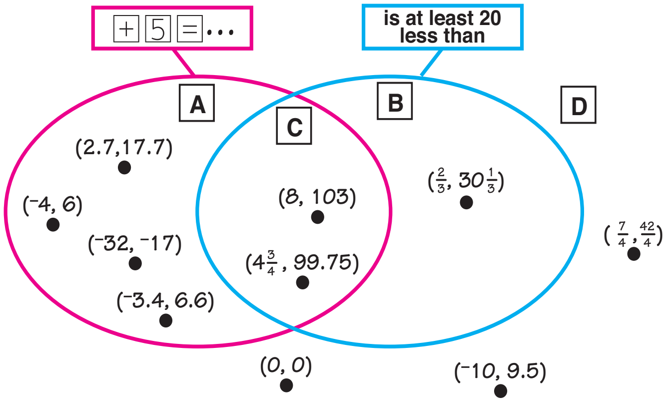


Erase all of the ordered pairs in the string picture; then write this list of ordered pairs near the picture.

(8, 103)	(⁻4, 6)	(⁻32, ⁻17)	$(\frac{2}{3}, 30\frac{1}{3})$	(⁻10, 9.5)
(2.7, 17.7)	(0, 0)	(⁻3.4, 6.6)	$(\frac{7}{2}, \frac{42}{4})$	$(4\frac{3}{4}, 99.75)$

T: ***Copy the string picture on a piece of paper and locate these ordered pairs*** (point to the list).

After about five minutes of individual or partner work, discuss the location of the pairs.



Accept any correct method, but encourage looking at the difference between the numbers of an ordered pair to decide whether or not the ordered pair belongs inside the red string. For example, the difference between ⁻4 and 6 is 10, a multiple of 5, so (⁻4, 6) belongs inside the red string. Also, since ⁻4 is not at least 20 less than 6, (⁻4, 6) belongs outside the blue string.

L11 THE RELATIONS GAME #5

Capsule Lesson Summary

Determine relations for two strings when given three clues and placement of other ordered pairs suggested by the class. Play *The Relations Game* with two-strings, explaining minor changes in the rules when extending to a two-string game.

Materials

Teacher
- Colored chalk
- Relations Game posters
- Relations Game scoring sheet
- Colored markers
- Blacklines L11(a) and (b)

Student
- Relations Game analysis sheet
- Worksheets L11* and **

Advance Preparation: Use Blackline L11(a) to make copies of *The Relations Game* analysis sheet for students. Use Blackline L11(b) to make a scoring sheet for use in play of a two-string game (Exercise 2).

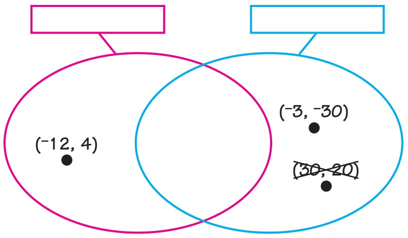
Description of Lesson

Exercise 1

Draw this string picture on the board and display two Relations Game posters, one on each side of the string picture.

Red

is the square of	The Relations Game		
is a multiple of	is a positive divisor of	is less than	is greater than
x2	x10	x$\frac{2}{3}$	is at least 20 greater than
÷2	÷10	÷$\frac{2}{3}$	is at least 20 less than
+2=···	+3=···	+4=···	+5=···
−2=···	−3=···	−4=···	−5=···



Blue

is the square of	The Relations Game		
is a multiple of	is a positive divisor of	is less than	is greater than
x2	x10	x$\frac{2}{3}$	is at least 20 greater than
÷2	÷10	÷$\frac{2}{3}$	is at least 20 less than
+2=···	+3=···	+4=···	+5=···
−2=···	−3=···	−4=···	−5=···

Note: The red string is for **is a multiple of** and the blue string is for **is at least 20 greater than.** Do not reveal the string labels at this time. Students will discover them during the activity.

T: ***There are three clues as to which of these relations*** (point to a poster) ***the strings are for. What does it mean to have (30, 20) crossed out?***

S: ***It means that (30, 20) does not belong in that region of the picture.***

T: ***Could (30, 20) belong in the blue string?***

S: ***Yes, but then it would be in the middle region belonging also in the red string.***

T: ***What can we cross off the Red list?***

S: ***All of the "minus" relations.***

T: ***Why?***

S: ***Because 4 is greater than ⁻12. If we subtract a positive number from ⁻12, we get a negative number less than ⁻12.***

Cross −2=···, −3=···, −4=···, and −5=··· off the Red list.

S: *Cross off* $\times\frac{2}{3}$*, because* $^-12 \times \frac{2}{3} = {}^-8$*.*

Continue the analysis until the class finds that only four possibilities remain for the red string. The location of ($^-$12, 4) is sufficient to eliminate 18 of the possibilities for the red string.

Red

~~is the square of~~	The Relations Game		
is a multiple of	~~is a positive divisor of~~	is less than	~~is greater than~~
~~x2~~	~~x10~~	~~x$\frac{2}{3}$~~	~~is at least 20 greater than~~
~~÷2~~	~~÷10~~	~~÷$\frac{2}{3}$~~	~~is at least 20 less than~~
$\boxed{+}\boxed{2}\boxed{=}\cdots$	~~$\boxed{+}\boxed{3}\boxed{=}\cdots$~~	$\boxed{+}\boxed{4}\boxed{=}\cdots$	~~$\boxed{+}\boxed{5}\boxed{=}\cdots$~~
~~$\boxed{-}\boxed{2}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{3}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{4}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{5}\boxed{=}\cdots$~~

T: *What can we cross off the Blue list?*

Let students suggest relations that can be crossed off. Using the location of ($^-$3, $^-$30), students will be able to eliminate all but four of the possibilities for the blue string.

Blue

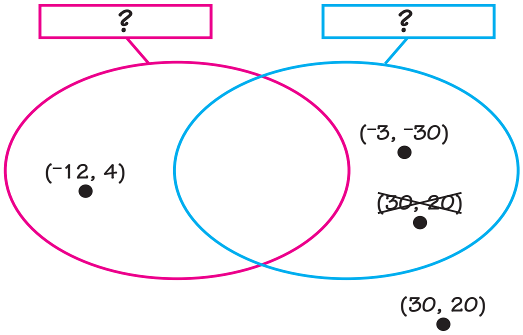
~~is the square of~~	The Relations Game		
~~is a multiple of~~	~~is a positive divisor of~~	~~is less than~~	is greater than
~~x2~~	x10	~~x$\frac{2}{3}$~~	is at least 20 greater than
~~÷2~~	~~÷10~~	~~÷$\frac{2}{3}$~~	~~is at least 20 less than~~
~~$\boxed{+}\boxed{2}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{3}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{4}\boxed{=}\cdots$~~	~~$\boxed{+}\boxed{5}\boxed{=}\cdots$~~
~~$\boxed{-}\boxed{2}\boxed{=}\cdots$~~	$\boxed{-}\boxed{3}\boxed{=}\cdots$	~~$\boxed{-}\boxed{4}\boxed{=}\cdots$~~	~~$\boxed{-}\boxed{5}\boxed{=}\cdots$~~

Determining the location of (30, 20) will allow students to eliminate one more possibility. You may need to prompt the use of this clue, as in the following dialogue.

T: *Where does (30, 20) belong? Could it be inside the red string?*

S: *(30, 20) cannot be inside the red string because there are only four possibilities left for the red string and (30, 20) does not belong to any of them: 30 is not a multiple of 20; 30 is not less than 20, and using* $\boxed{+}\boxed{2}\boxed{=}\cdots$ *or* $\boxed{+}\boxed{4}\boxed{=}\cdots$ *from 30 we get greater numbers, never 20.*

S: *(30, 20) belongs outside both strings.*



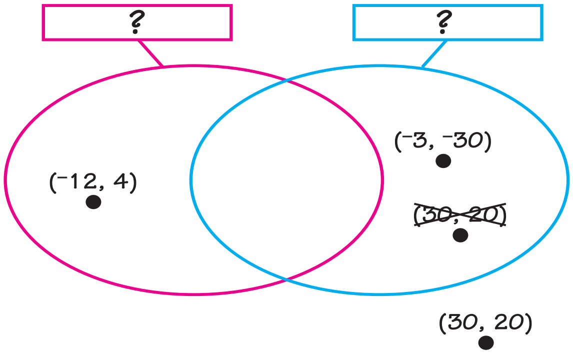
T: *Knowing that (30, 20) belongs outside both strings, what else can we cross off the Blue list?*

S: **Is greater than** *because 30 is greater than 20.*

T: *There are four possibilities left for the red string and three possibilities left for the blue string.*

Suggest some ordered pairs whose locations would help us determine what the string labels are.

A sample dialogue follows.



S: *(15, 25).*

T: *(15, 25) belongs outside both strings.*

S: *The red string cannot be for* $\boxed{+}\boxed{2}\boxed{=}\cdots$*, because we can add 2s to 15 and get 25.*

S: *The red string cannot be for* **is less than** *because 15 is less than 25.*

S: *We cannot cross any more relations off the Blue list.*

T: *So now we have two possibilities left for the red string and still three possibilities left for the blue string.*

Red

~~is the square of~~	The Relations Game		
is a multiple of	~~is a positive divisor of~~	~~is less than~~	~~is greater than~~
~~x2~~	~~x10~~	~~x $\frac{2}{3}$~~	~~is at least 20 greater than~~
~~÷2~~	~~÷10~~	~~÷ $\frac{2}{3}$~~	~~is at least 20 less than~~
~~+2=···~~	~~+3=···~~	+4=···	~~+5=···~~
~~−2=···~~	~~−3=···~~	~~−4=···~~	~~−5=···~~

Blue

~~is the square of~~	The Relations Game		
~~is a multiple of~~	~~is a positive divisor of~~	~~is less than~~	~~is greater than~~
~~x2~~	x10	~~x $\frac{2}{3}$~~	is at least 20 greater than
~~÷2~~	~~÷10~~	~~÷ $\frac{2}{3}$~~	~~is at least 20 less than~~
~~+2=···~~	~~+3=···~~	~~+4=···~~	~~+5=···~~
~~−2=···~~	−3=···	~~−4=···~~	~~−5=···~~

S: *(1, 10).*

T: *(1, 10) belongs outside both strings.*

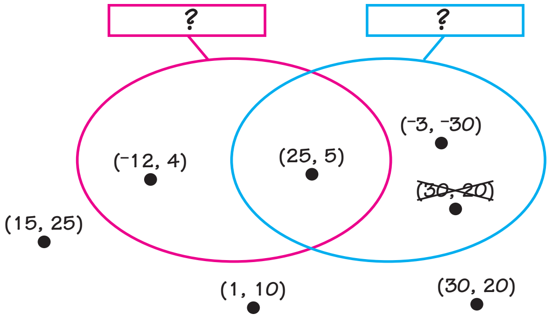
S: *The blue string cannot be for* **x10** *because 10* x *1 = 10.*

S: *The two possibilities for the red string are still okay.*

S: *(25, 5).*

T: *(25, 5) belongs inside both strings.*

S: *The red string cannot be for* +4=···, *and the blue string cannot be for* −3=···.



Red

~~is the square of~~	The Relations Game		
is a multiple of	~~is a positive divisor of~~	~~is less than~~	~~is greater than~~
~~x2~~	~~x10~~	~~x $\frac{2}{3}$~~	~~is at least 20 greater than~~
~~÷2~~	~~÷10~~	~~÷ $\frac{2}{3}$~~	~~is at least 20 less than~~
~~+2=···~~	~~+3=···~~	~~+4=···~~	~~+5=···~~
~~−2=···~~	~~−3=···~~	~~−4=···~~	~~−5=···~~

Blue

~~is the square of~~	The Relations Game		
~~is a multiple of~~	~~is a positive divisor of~~	~~is less than~~	~~is greater than~~
~~x2~~	~~x10~~	~~x $\frac{2}{3}$~~	is at least 20 greater than
~~÷2~~	~~÷10~~	~~÷ $\frac{2}{3}$~~	~~is at least 20 less than~~
~~+2=···~~	~~+3=···~~	~~+4=···~~	~~+5=···~~
~~−2=···~~	~~−3=···~~	~~−4=···~~	~~−5=···~~

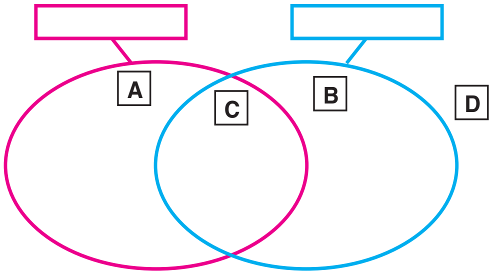
Exercise 2: The Relations Game with Two Strings

Minor modifications need to be made in the rules for *The Relations Game* (see Exercise 3 of Lesson L7) in order to extend it to a two-string game.

First Part of the Game:

In the first part of the game, the point system stays the same except points are awarded only when both relations are correctly identified.

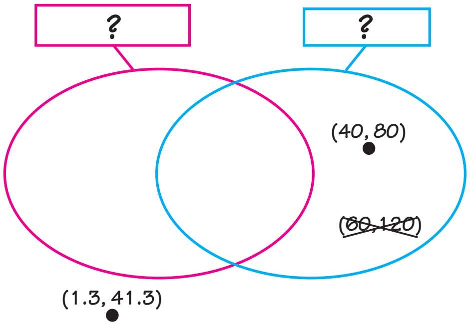
Second Part of the Game:



In the second part of the game, points are awarded as follows:

- A team gains four points each time a member correctly locates an ordered pair in region **C**.
- A team gains two points each time a member correctly locates an ordered pair in regions **A** or **B**.
- A team gains one point each time a member correctly locates an ordered pair in region **D**.
- A team neither gains nor loses points when a member incorrectly locates an ordered pair.

Use a scoring sheet that incorporates these changes (see Blackline L11(b)).



Play a two-string game with your class. As a possible game, let the red string be for $\boxed{+}\boxed{3}\boxed{=}\cdots$ and the blue string be for **x2**. Give three starting clues as shown here. You may want to tell the class that the red string (relation) is determined by these clues.

Worksheets L11* and ** are available for individual work.