N Strand

The World of Numbers

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WORLD OF NUMBERS INTRODUCTION

By now, veteran *CSMP* students have had a rich variety of experiences in the World of Numbers. They have met and become familiar with various kinds of numbers, and with operations and relations on them. They have encountered positive and negative integers, decimal numbers, fractions, numerical functions (such as 5x, +3, ÷10, –5, $\frac{2}{3}$x), order relations (such as < and >), and the notions of multiples and divisors of a given number. They have been introduced to paper-and-pencil algorithms for addition, subtraction, multiplication, and division with whole numbers; for addition, subtraction, and multiplication with decimal numbers; and for multiplication and division of fractions. They have had extensive experiences involving division in preparation for a more general algorithm for division with decimal numbers. They have used several models for addition and subtraction of fractions in preparation for algorithms. Topics from combinatorics and number theory have provided many interesting problems.

In *CSMP Mathematics for the Intermediate Grades, Part VI*, these earlier numerical experiences will be revisited, extended, and deepened through familiar games and activities, as well as in fascinating new situations. As always, *CSMP* stresses the unity and continuity of growth of mathematical ideas and concepts. The program's spiral approach does not require mastery of each lesson, but rather allows students to encounter the elements of each content strand in different situations throughout the year. It is important to recognize this approach consciously. If you strive for mastery of each single lesson, you will find yourself involved in a great deal of redundancy as the year progresses.

Further, *CSMP* presents the content in a situational framework. That is, a "pedagogy of situations" engages students in rich problem-solving activities as they construct mathematical ideas. These situations offer opportunities both to develop necessary numerical skills and to gain deeper understanding of mathematical concepts in the world of numbers. At the same time, the situations presented encourage students to develop patterns of logical thinking and strategies for approaching problems.

Perhaps the most important embodiments of the *CSMP* approach are the nonverbal languages and tools used throughout the program. These are vehicles that allow students to investigate the contexts in which the content is presented and to explore new mathematical ideas. It is hard to overstate the value of developing languages and tools that are not confined to one area of mathematical content or to one level of the development of content; that aid in attacking problems as well as in representing situations. Equipped with the universally applicable languages of the *CSMP* curriculum, students grow more and more familiar with the syntax of these languages and are free to explore new content as extensions rather than think of each new mathematical idea as tied to a certain new language. This is not to say that *CSMP* students do not learn the usual descriptive language of mathematics; naturally, they do. However, in the *CSMP* approach the usual descriptive language is not a requisite for learning new concepts, but only a means for succinctly describing those ideas as they are being explored.

The Minicomputer, calculator, strings, and arrows embody three fundamental concepts of mathematics: binary and decimal number systems, sets, and functions. Using these tools and pictorial languages to highlight unifying themes counteracts the tendency to fragment mathematics instruction into a large set of independent topics. For specific examples of the situations and the ways *CSMP* uses instructional tools and nonverbal languages in this strand, we refer you to the brief topic summaries later in this introduction and, in particular, to the lessons themselves.

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Standard Algorithms

CSMP seeks to develop basic numerical skills as well as an understanding of the underlying mathematical ideas. We are fully in agreement with the thesis that, along with the growth of understanding of the world of numbers, there must be a concomitant growth of familiarity and facility with numbers and operations on them. But facility should not be confused with understanding; they are partners in the growth of mathematical maturity. A balanced growth of each must be maintained, neither being sacrificed for the other.

Students must eventually learn mechanical algorithms for the basic operations (addition, subtraction, multiplication, division). However, the development of these methods should occur only after students have had many experiences with prerequisite concepts. Premature presentation of these algorithms may actually inhibit a student's desire and ability to develop alternative algorithms, to do mental arithmetic, or to estimate.

CSMP believes that students should be able to solve a problem such as 672 ÷ 32 using models, pictures, or mental arithmetic before being introduced to a division algorithm. Even after students have mastered an algorithm, they should be aware that alternative methods are often more appropriate. For example, consider the problem of calculating 698 x 9. Rather than using a standard multiplication algorithm, it may be easier and more efficient to note that 700 x 9 = 6 300, so that 698 x 9 = 6 300 – 18 = 6 282. Indeed, built into this way of approaching the problem is an excellent estimate (6 300) of the product. To insist on a mechanistic response to such a problem would be to encourage inefficiency and might also inhibit the development of the flexibility necessary for problem solving. On the other hand, a rich array of situations in which students interact with numbers provides them with opportunities to gain the necessary facility with standard algorithmic procedures while retaining the openness required to respond creatively to new situations in the world of numbers.

Content Overview

Multiplication

By this time your *CSMP* students are quite familiar with the concept of multiplication and with paper-and-pencil algorithms for multiplying whole numbers, decimals, and fractions. Here, in *IG-VI*, both familiar and new situations present many opportunities to review and apply multiplication. Arrow pictures provide an ideal vehicle for developing methods of multiplying both decimal numbers and fractions. Estimation and pattern work strengthen multiplication concepts. Of special note this semester is a lesson using a multiplication square with patterns that reinforce the rule that a negative number times a negative number is a positive number.

As review, students encounter multiplication in activities such as *Minicomputer Golf*, *Guess My Rule*, detective stories, and calculator puzzles. Multiplication becomes a tool for investigating new topics, for example, percent, Cartesian graphs of quadratic relations, finding ways to operate with a restricted calculator, and prime factorization. The extent and range of these activities reflect students' growing confidence with multiplication.

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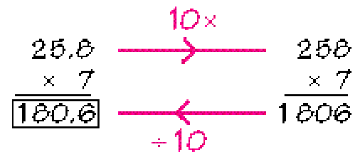
Earlier work with the Minicomputer and arrow pictures examined multiplication patterns with decimal numbers. The fact that the product is unchanged by the combined action of 10x and ÷10 suggests a technique for multiplying with decimal numbers.

For example, the problem 7 x 25.8 can be solved by performing the more familiar calculation 7 x 258 and then dividing the result by 10. Techniques such as this and estimation work well to enhance understanding of multiplication with decimal numbers. This semester students examine and use the more efficient rule of "counting decimal places."



The lessons on fractions review the development of a standard algorithm for the multiplication of two fractions, namely,

$$\frac{3}{4} \times \frac{2}{5} = \frac{3 \times 2}{4 \times 5} = \frac{6}{20} = \frac{3}{10}$$

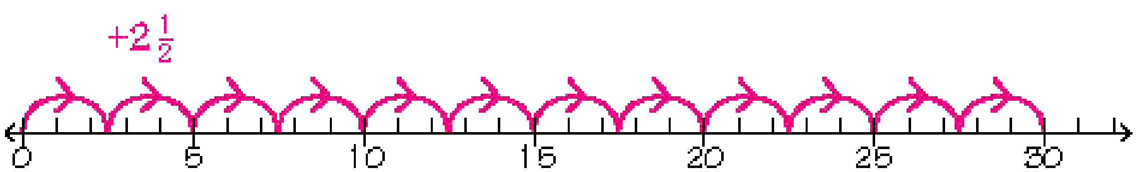
This algorithm is then used to confirm results in multiplication calculations involving decimals. For example, $0.3 \times 1.4 = \frac{3}{10} \times \frac{14}{10} = \frac{42}{100} = 0.42$.

Lessons: N1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 15, 16, 17, 18, 19, 21, 22, 26, 27, 29, 31, 33, 34, 35, and 36

Division

CSMP students already have experience with division as a sharing process (sharing 108 books equally among three classes), as repeated subtraction (finding how many 12s are in 200), and as a multiplication inverse. They have been introduced to an efficient paper-and-pencil algorithm for division with whole numbers. This algorithm has been extended to include division of a decimal by a whole number. The lessons this semester continue students' experiences with division in patterns and applications, and further extend the algorithm for division to include decimal divisors.

Division by a fraction may again be viewed as a sharing process or repeated subtraction. For example, students might interpret 30 ÷ 2½ as asking how many 2½s there are in 30. Using +2½ arrows on a number line can display 30 ÷ 2½ = 12.



Through the use of composition arrow pictures students examine other methods of dividing 30 ÷ 2½, such as rewriting the problem to multiply 30 x ⅖. Such experiences lead to a familiar division process for fractions, namely, rewriting a division calculation in an equivalent form with multiplication. For example:

$$\frac{4}{5} \div \frac{5}{6} = \frac{4}{5} \times \frac{6}{5} = \frac{24}{25}$$

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Learning division algorithms is only one part of developing an understanding of the concept of division. Therefore, your students' exposure to division in many contexts continues with activities involving calculators, arrow roads, number lines, numerical patterns, and estimation.

Lessons: N1, 2, 5, 6, 8, 9, 10, 15, 16, 17, 19, 21, 25, 26, 30, and 32

Negative Numbers

CSMP introduces negative integers in first grade through a story about Eli the Elephant and magic peanuts. The story leads to a model for adding integers, first in pictures, then also on the Minicomputer. By the end of fourth grade, *CSMP* students have encountered negative numbers in games, in reading outdoor temperatures, in arrow roads, and in calculator activities. These experiences extend the concept of order from whole numbers to negative numbers, and provide models for the addition of negative numbers.

The activities in this strand increase students' familiarity with negative numbers in many contexts. The goal is to portray negative integers not as a strange new set of numbers, but as a natural and necessary extension of counting numbers. Therefore, few lessons focus on negative numbers but many lessons include them. Negative numbers appear regularly on the Minicomputer in *Minicomputer Golf,* in detective stories, in Cartesian graphs, and in calculator activities. Of particular note this semester are two lessons on positional systems that investigate the possibility of having a negative base, namely Base $\widehat{2}$.

CSMP employs a special notation for representing negative numbers. Traditional approaches to arithmetic often make no distinction on the printed page between the function "subtract 3" and the number "negative 3"; both are denoted by "–3." Only by context can a person discern the intended meaning of "–3." In *CSMP*, negative numbers are distinguished from subtraction in the following ways:

- The minus sign "–" is reserved for subtraction. Thus, for example, "–14" denotes the function "subtract 14."
- The $\frown$ symbol denotes a negative number. Thus, "$\widehat{14}$" denotes the number "negative 14." This symbol was introduced first in the story about Eli the Elephant.
- A raised minus sign may be used when recording a negative number, especially for results obtained from using a calculator. For example, $^{-}14$.

We recommend that you continue to use both the $\frown$ and raised minus notations for negative numbers and recognize alternative notations as students encounter them in other contexts (calculators, temperature, tests, and so on).

Lessons: N1, 2, 4, 7, 8, 12, 13, 14, 16, 20, 28, 30, and 31

Decimal Numbers

Just as students' confidence with whole numbers requires several years of growth, so must the development of decimal number concepts proceed gradually. The introductory activities in second, third, fourth, and fifth grades rely on money, on the Minicomputer, and on the number line as models for decimal numbers. These three models complement each other. Whereas all facilitate computation, the Minicomputer highlights patterns while the number line and money focus on order and relative magnitude of decimal numbers.

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Reflecting and furthering the students' growing confidence, decimal numbers appear in this semester in activities involving calculators, Cartesian coordinates, arrow diagrams, string pictures, *Minicomputer Golf*, and positional systems. These activities require students to perform many computations involving decimal numbers, relying on the various models to confirm their results. For example:

$3 \times (7 \div 2) = 10.5$	$((5 + 6) \div 5) - 2 = 0.2$
$(1.5 \times 0.20) + (2.5 \times 0.40) = 1.30$	$\frac{4}{5} \times 2 = 1.6$

The ability to perform such calculations as well as to order decimal numbers indicate that students can discover and become familiar with the subtleties of decimal numbers without a too early reliance on rules and mechanical manipulation of numbers.

Building on students' knowledge of various models for fractions, decimal numbers, and division, a goal of this strand is to identify the relationships among these concepts by observing equalities such as $7 \div 5 = \frac{7}{5} = 1.4$.

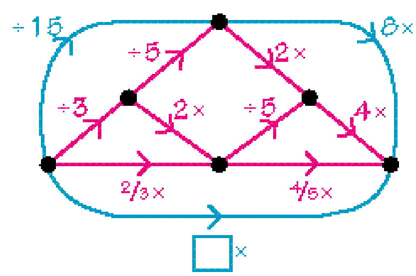
Students continue to build their understanding of decimal numbers by encountering them in a variety of situations involving estimation and patterns. Moreover, the lessons emphasize relationships among decimal numbers, fractions, and division.

Lessons: N1, 4, 6, 7, 9, 10, 11, 12, 13, 14, 17, 19, 21, 25, 26, 30, 32, 33, 34, and 36

Fractions

The activities involving fractions in *IG-VI* reflect *CSMP*'s belief in the spiral approach. Early exposure to fractions began in first grade. From second through fifth grades students have gradually become familiar with two concepts involving fractions: fractional parts of a whole and certain composite functions (for example, $\frac{3}{8}$x is the same as 3x followed by ÷8). This background prepares students to compute (add, subtract, multiply, and divide) with fractions.

One technique for multiplying fractions relies on the composition of multiplication functions. The unlabeled blue arrow is the composition of ÷15 followed by 8x or $\frac{8}{15}$x. Therefore, $\frac{2}{3}$x followed by $\frac{4}{5}$x is $\frac{8}{15}$x or, analogously $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$. The arrow picture suggests a generalization to the standard algorithm for multiplying fractions: $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.



Furthermore, students learn to calculate problems such as $5\frac{3}{4} \times 8$ by considering $5 \times 8 = 40$ and $\frac{3}{4} \times 8 = 6$.

The section on Division in this introduction describes the development of methods for dividing by a fraction. This development follows, of course, the established algorithm for multiplying fractions. With arrow pictures, the students observe the equivalence, for example, of $\frac{2}{3}$x and $\div\frac{3}{2}$. Thus, they begin to use the familiar algorithm for dividing by fractions, for example,

$$\frac{3}{4} \div \frac{3}{2} = \frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{1}{2}$$

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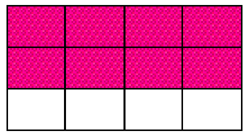
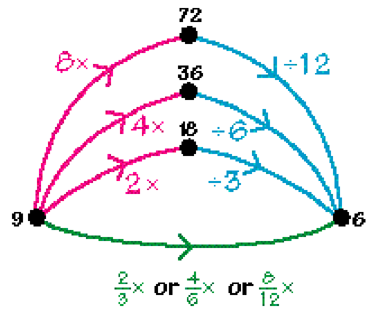
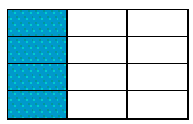
A prerequisite for adding fractions is an understanding of equivalent fractions. Both the area model and arrow pictures suggest that numbers can have different fractional names.



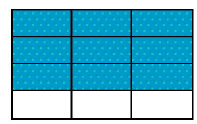
$\frac{2}{3} = \frac{8}{12}$



With the concept of equivalent fractions in hand, students recall a cutting "cakes" method for adding fractions introduced in *IG-IV*. The area model emphasizes the need for equal-sized regions, i.e., for a common denominator.



$\frac{1}{3} = \frac{4}{12}$



$\frac{3}{4} = \frac{9}{12}$

$$\frac{1}{3} + \frac{3}{4} = \frac{4}{12} + \frac{9}{12} = \frac{13}{12} = 1\frac{1}{12}$$

There are many opportunities this semester to use properties and patterns in adding, for example, $2\frac{4}{5} + \frac{3}{5} = 3\frac{2}{5}$, so also $1\frac{4}{5} + 1\frac{3}{5} = 3\frac{2}{5}$.

In a *Guess My Rule* context students discover a special case of the general addition algorithm for fractions, namely that for unit fractions: $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{a \times b}$. The lesson continues with an exploration of an early Egyptian problem of decomposing fractions as sums of distinct unit fractions.

The section on Decimals in this introduction mentions methods for changing fractions to decimal numbers and vice-versa; for example, $1.4 = 1\frac{2}{5} = \frac{7}{5} = 7 \div 5$. These equivalences are used continuously to reinforce results of calculations with fractions and with decimals.

Lessons: N5, 6, 7, 10, 12, 14, 15, 16, 21, 23, 24, 26, 28, 30, 31, and 32

Positional Systems

Positional systems were introduced in parts *II*, *III*, *IV*, and *V* of *CSMP for the Intermediate Grades*. This semester we review and extend this prior work with various positional systems giving special attention to the binary (Base Two) system. Problems involve binary calculations and the binary number line. Three lessons in the L strand present several very different looking combinatorics situations in which the binary code provides a useful scheme for counting. The situations can all be seen to relate to one another with an appropriate "translation" of the symbols (**0** and **1**) of binary code "words." Of special note are two lessons in this strand that introduce a new positional system, Base $\hat{2}$. Here comparison is made to the binary system and some unique characteristics of this system are investigated.

Lessons: N4, 12, 14, 24, 28, 30, and 35

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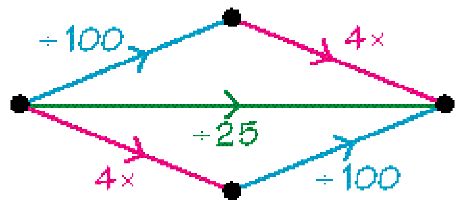
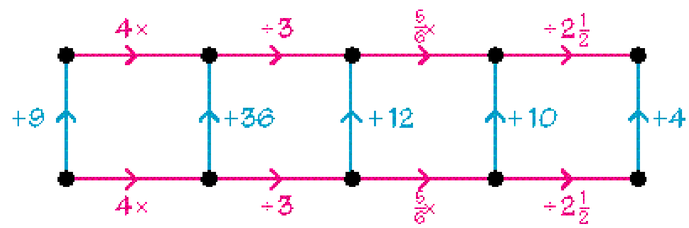
Composition of Functions and Percent

Several lessons in this strand deal with what happens when you compose a sequence of functions, that is, when you apply the functions in order, one at a time. These compositions lead to many powerful insights into the properties of numbers and operations. Arrow diagrams provide a concrete means to study this abstract but practical concept. For example, if you divide any number by 100 and then multiply the result by 4, the net effect is the same as dividing the original number by 25.



Besides depicting the composition, the arrow picture also suggests that an easy way to divide any number by 25 is to divide by 100 and then multiply by 4 or vice-versa; the two operations can often be performed mentally. For example, 4 x 63 = 252 and 252 ÷ 100 = 2.52, so 63 ÷ 25 = 2.52.

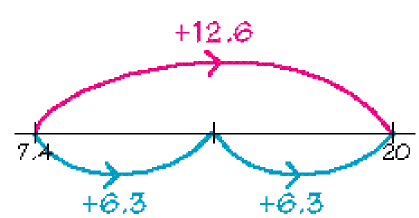
Many pairs of functions *commute*. That is, they produce the same effect regardless of the order in which they are applied. $\frac{2}{3}$x can be interpreted as 2x followed by ÷3 or as ÷3 followed by 2x; +98 can be interpreted as +100 followed by –2 or as –2 followed by +100. However, other pairs of functions, for example, +9 and 4x, do not commute. Yet patterns do exist—students find, for example, that +9 followed by 4x has the same effect as 4x followed by +36. The following arrow picture depicts several compositions of this kind.



From the students' perspective, they are solving challenging problems and discovering new numerical patterns. From a mathematical viewpoint, they are investigating the commutative and distributive properties.

The composition of functions can lead to insights in many problem-solving situations. In *Minicomputer Golf*, the use of composition aids in finding solutions to problems requiring that students move two checkers to produce a specified change. In the section on Fractions in this introduction, an arrow picture involving compositions supports an algorithm for multiplying fractions.

The idea of composition also facilitates finding the midpoint of two numbers on a number line.



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The composition of functions shows how the language of arrows is able to visually highlight rich and practical mathematical concepts and techniques. For example, three lessons this semester introduce the concept of percent as a composite function; that is, "n% of" is nx followed by ÷100, or ÷100 followed by nx. In this context, the exercises investigate many useful names for certain percents (for example, 20% = $\frac{2}{10}$ = $\frac{1}{5}$ = 0.2), several interesting patterns, and some helpful properties. The lessons all include some applications of percent in the solution of real life problems.

Lessons: N1, 2, 6, 8, 9, 10, 16, 21, 26, and 31

Multiples and Divisors

The study of multiples and divisors leads to practical applications such as the addition of fractions, as well as to investigations of many fascinating properties of numbers. Your students have had many earlier experiences finding multiples and divisors of whole numbers, and through string pictures they have encountered the notions of common multiples and common divisors. In *IG-VI*, calculator activities, arrow diagrams, and string pictures provide further opportunities to explore common multiples and common divisors.

Common multiples also appear in the study of fractions and in the introduction to modular arithmetic. As preparation for the addition of fractions, arrow pictures and activities involving the fair division of rectangular cakes both lead students to generate lists of equivalent fractions. Students recognize the role of common multiples in determining the numerators and denominators of equivalent fractions.

Several lessons this semester review exponential notation and continue a study of prime factorization. The goal of these lessons is to make use of prime factorization to introduce a technique for counting the positive divisors of a whole number. A probability problem employs the prime factorization of a number in its solution.

Lessons: N3, 11, 15, 16, 18, 19, 20, 22, 27, 29, 32, 33, and 35

Cartesian Graphs and Linear Programming

Several lessons this semester review the idea of graphing relations in a two dimensional coordinate system. The resulting graphs are called *Cartesian graphs* because one commonly refers to the coordinate system as the *Cartesian plane*†. Graphs for several quadratic relations provide opportunities to make observations which describe the effect of modifying an algebraic expression. In connection with studying the notion of relatively prime, students graph both "is relatively prime" and "is not relatively prime" relations to answer questions about which is more common.

Two lessons this semester investigate a linear programming problem in which cost must be minimized on a commodity subject to certain restrictions and requirements. The solution makes use of the Cartesian graphs for several linear functions.

Lessons: N13, 20, 34, and 36

†*Cartesian* is a name honoring the French mathematician René Descartes (1596-1650) for his unification of algebra and geometry in the creation of analytic geometry.

Capsule Lesson Summary

Put selected numbers on the display of a calculator using a restricted set of keys. With the same limited use of keys, find keystrokes that will multiply any number on the display by a specified number. Use the calculator to assist in finding a number to multiply by 19 so that the product is between 500 and 501.

Materials

Teacher	• Calculator • Colored chalk	**Student**	• Calculator • Paper • Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

List these calculator keys on the board, and refer to them as you give directions.

$\boxed{2}\ \boxed{3}\ \boxed{5}\ \boxed{6}$
$\boxed{+}\ \boxed{-}\ \boxed{\times}\ \boxed{\div}$
$\boxed{=}$

T: ***Today you are going to do some calculator puzzles using only certain keys on the calculator. The only number keys you may press are*** $\boxed{2}$, $\boxed{3}$, $\boxed{5}$, ***and*** $\boxed{6}$. ***You may use any of the four operation keys and you may press*** $\boxed{=}$ ***at any time. Try to put 45 on your calculator display.***

Allow several minutes for students to explore this situation. There are many solutions; encourage students to find several solutions. When a student offers a solution, invite another student to check it. On the board, make a list of solutions your students offer, as illustrated here.

45: $\boxed{2}\ \boxed{2}\ \boxed{+}\ \boxed{2}\ \boxed{3}\ \boxed{=}$
$\boxed{3}\ \boxed{\times}\ \boxed{3}\ \boxed{\times}\ \boxed{5}\ \boxed{=}$
$\boxed{6}\ \boxed{5}\ \boxed{+}\ \boxed{2}\ \boxed{5}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$
$\boxed{2}\ \boxed{5}\ \boxed{\times}\ \boxed{2}\ \boxed{-}\ \boxed{5}\ \boxed{=}$
$\boxed{2}\ \boxed{2}\ \boxed{5}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$

Continue this activity, putting on other numbers such as 202, 4.5, 8.4, and $^{-}17$. Feel free to adjust your choice of numbers to the abilities of your students. Write all the numbers on the board with sufficient space between them to record several solutions for each number. Provide time for individual work, allowing students to choose the order in which they work on these numbers. You may challenge the class to think about shorter solutions by introducing the condition that it costs a penny (or a dollar) to press a key. Ask for solutions that cost 10¢ or less.

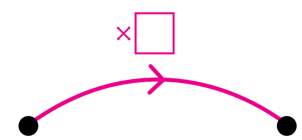
N 1

Note: The list of solutions here assumes the calculator does chain operations and has an automatic constant feature (see “Role and Use of Calculators” in Section One: Notes to the Teacher). As necessary, make adjustments for the calculators in use by your students. For example, in several solutions you may need to insert another $\boxed{=}$ if your calculator does operations in a priority order (x, ÷, +, –) rather than a chain order of entry.

202:	$\boxed{2}\ \boxed{3}\ \boxed{5}\ \boxed{-}\ \boxed{3}\ \boxed{3}\ \boxed{=}$	7¢
	$\boxed{6}\ \boxed{3}\ \boxed{6}\ \boxed{\div}\ \boxed{3}\ \boxed{-}\ \boxed{5}\ \boxed{=}\ \boxed{=}$	9¢
	$\boxed{6}\ \boxed{5}\ \boxed{\times}\ \boxed{3}\ \boxed{+}\ \boxed{5}\ \boxed{+}\ \boxed{2}\ \boxed{=}$	9¢
4.5:	$\boxed{6}\ \boxed{+}\ \boxed{3}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$	6¢
	$\boxed{3}\ \boxed{\div}\ \boxed{6}\ \boxed{\times}\ \boxed{5}\ \boxed{+}\ \boxed{2}\ \boxed{=}$	8¢
	$\boxed{3}\ \boxed{\div}\ \boxed{2}\ \boxed{+}\ \boxed{3}\ \boxed{=}$	6¢
8.4:	$\boxed{3}\ \boxed{2}\ \boxed{\div}\ \boxed{5}\ \boxed{+}\ \boxed{2}\ \boxed{=}$	7¢
	$\boxed{6}\ \boxed{+}\ \boxed{6}\ \boxed{\div}\ \boxed{5}\ \boxed{+}\ \boxed{6}\ \boxed{=}$	8¢
	$\boxed{6}\ \boxed{\div}\ \boxed{5}\ \boxed{+}\ \boxed{3}\ \boxed{\times}\ \boxed{2}\ \boxed{=}$	8¢
⁻17:	$\boxed{-}\ \boxed{5}\ \boxed{-}\ \boxed{6}\ \boxed{=}\ \boxed{=}$	6¢
	$\boxed{5}\ \boxed{-}\ \boxed{2}\ \boxed{2}\ \boxed{=}$	5¢
	$\boxed{2}\ \boxed{-}\ \boxed{3}\ \boxed{6}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$	7¢

Exercise 2

Draw an unlabeled arrow on the board.



T: ***This arrow is for times some number. We can use our calculators to multiply by this number. If $\boxed{2}$, $\boxed{3}$, $\boxed{5}$, and $\boxed{6}$ are the only number keys we can press, what could this arrow be for?***

S: x***5.***

S: x***23.***

S: x***10.***

T: ***What keys would we press to multiply by 10?***

S: ***Press $\boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{5}\ \boxed{=}$.***

Check that the suggested keystrokes multiply any starting number by, in this case, 10. Ask students each to start with different numbers on the display of their calculators, then to press the suggested sequence of keys, and finally to verify that the result is the starting number times 10.

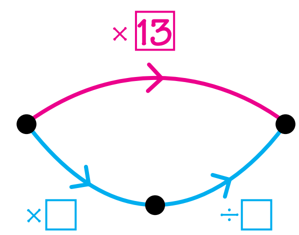
Continue asking for other possibilities for the arrow. Verify the correctness of solutions and record them on the board. For example:

×4:	$\boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{2}\ \boxed{=}$
×20:	$\boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{5}\ \boxed{=}$
×30:	$\boxed{\times}\ \boxed{5}\ \boxed{\times}\ \boxed{6}\ \boxed{=}$
	$\boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{5}\ \boxed{\times}\ \boxed{3}\ \boxed{=}$
×100:	$\boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{5}\ \boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{5}\ \boxed{=}$
×18:	$\boxed{\times}\ \boxed{3}\ \boxed{\times}\ \boxed{3}\ \boxed{\times}\ \boxed{2}\ \boxed{=}$
	$\boxed{\times}\ \boxed{6}\ \boxed{\times}\ \boxed{3}\ \boxed{=}$
	$\boxed{\times}\ \boxed{3}\ \boxed{6}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$
×75:	$\boxed{\times}\ \boxed{2}\ \boxed{5}\ \boxed{\times}\ \boxed{3}\ \boxed{=}$
	$\boxed{\times}\ \boxed{5}\ \boxed{\times}\ \boxed{5}\ \boxed{\times}\ \boxed{3}\ \boxed{=}$

Note: Students may suggest that you can multiply by any whole number by pressing $\boxed{+}\ \boxed{=}\ \boxed{=}$ … $\boxed{=}$ where $\boxed{=}$ is pressed the desired whole number of times. Accept this response, but suggest it would be rather inefficient. Also, be careful of suggestions such as x12: $\boxed{\times}\ \boxed{6}\ \boxed{+}\ \boxed{6}$. In this case, the calculator might not multiply the number on the display by 12 because it would first multiply by 6 and then add 6; for example, starting with 5 on the display, the result would be 36 not 60.

If students limit themselves to numbers whose digits are 2, 3, 5, and 6 or to numbers whose only factors are numbers with these digits, ask for a specific number as the multiplier. For example:

T: ***Could this arrow be for* x*13?***



You may need to give the class a hint by drawing a detour for the red arrow.

S: ***Press*** $\boxed{\times}\ \boxed{2}\ \boxed{6}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$.

S: ***Press*** $\boxed{\times}\ \boxed{6}\ \boxed{5}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$.

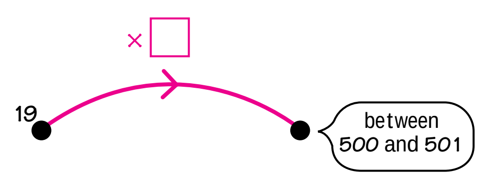
After you have a good variety of solutions on the board, you may wish to challenge the class to find solutions for all the multipliers from 2 to 25. One possible sequence of keystrokes for each such multiplier is given below.

×2: $\boxed{\times}\ \boxed{2}\ \boxed{=}$
×3: $\boxed{\times}\ \boxed{3}\ \boxed{=}$
×4: $\boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{2}\ \boxed{=}$
×5: $\boxed{\times}\ \boxed{5}\ \boxed{=}$
×6: $\boxed{\times}\ \boxed{6}\ \boxed{=}$
×7: $\boxed{\times}\ \boxed{3}\ \boxed{5}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$
×8: $\boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{2}\ \boxed{=}$
×9: $\boxed{\times}\ \boxed{3}\ \boxed{\times}\ \boxed{3}\ \boxed{=}$
×10: $\boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{5}\ \boxed{=}$
×11: $\boxed{\times}\ \boxed{2}\ \boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$
×12: $\boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{6}\ \boxed{=}$
×13: $\boxed{\times}\ \boxed{2}\ \boxed{6}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$
×14: $\boxed{\times}\ \boxed{5}\ \boxed{6}\ \boxed{\div}\ \boxed{2}\ \boxed{=}\ \boxed{=}$
×15: $\boxed{\times}\ \boxed{5}\ \boxed{\times}\ \boxed{3}\ \boxed{=}$
×16: $\boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{2}\ \boxed{=}$
×17: $\boxed{\times}\ \boxed{2}\ \boxed{5}\ \boxed{5}\ \boxed{\div}\ \boxed{3}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$
×18: $\boxed{\times}\ \boxed{3}\ \boxed{\times}\ \boxed{6}\ \boxed{=}$
×19: $\boxed{\times}\ \boxed{6}\ \boxed{6}\ \boxed{5}\ \boxed{\div}\ \boxed{3}\ \boxed{5}\ \boxed{=}$
×20: $\boxed{\times}\ \boxed{5}\ \boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{2}\ \boxed{=}$
×21: $\boxed{\times}\ \boxed{6}\ \boxed{3}\ \boxed{\div}\ \boxed{3}\ \boxed{=}$
×22: $\boxed{\times}\ \boxed{2}\ \boxed{2}\ \boxed{=}$
×23: $\boxed{\times}\ \boxed{2}\ \boxed{3}\ \boxed{=}$
×24: $\boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{2}\ \boxed{\times}\ \boxed{6}\ \boxed{=}$
×25: $\boxed{\times}\ \boxed{2}\ \boxed{5}\ \boxed{=}$

Exercise 3

Inform students that in this exercise they are free to use any of the number keys on the calculator.

Draw this arrow picture on the board.



T: ***Put 19 on the display of your calculator. Try to multiply 19 by some number to get a number between 500 and 501.***

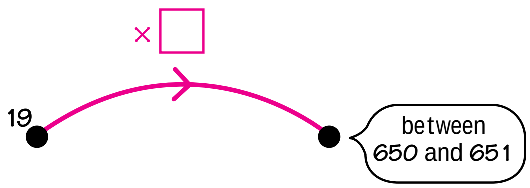
N 1

Allow several minutes for students to work on the problem individually or with partners. Then accept some solutions, recording them on the board. For example:

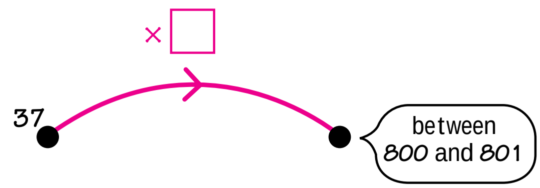
$$19 \times 26.33 = 500.27 \qquad 19 \times 26.35 = 500.65$$
$$19 \times 26.316 = 500.004 \qquad 19 \times 26.368 = 500.992$$

Note: To four decimal places, 26.3158 ≤ the multiplier ≤ 26.3684.

Repeat this activity, starting at 19 and trying to get a number between 650 and 651, or starting at 37 and trying to get a number between 800 and 801. For example:



Solutions range(approximately)
from ×34.211 to ×34.263



Solutions range(approximately)
from ×21.622 to ×21.648

Writing Activity

You may like students to take lesson notes on some, most, or even all their math lessons. The “Lesson Notes” section in Notes to the Teacher gives some suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students may explain how to use a restricted set of keys on the calculator and still multiply by almost any whole number.

Capsule Lesson Summary

Build arrow roads with each arrow for +, –, x, or ÷ some whole number from 1 to 9. Define the one-digit distance between two integers as the number of arrows in a shortest such road between them. Determine some of the whole numbers whose one-digit distance from 100 is 2.

Materials

Teacher	• Colored chalk	**Student**	• Unlined paper • Colored pencils, pens, or crayons

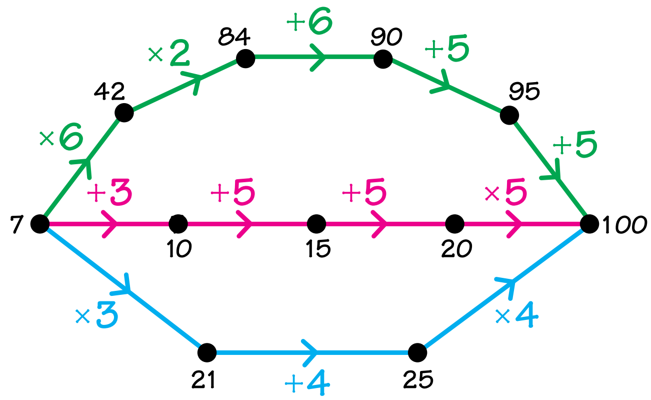
Description of Lesson

Exercise 1

Draw a dot for 7 and a dot for 100 on the board. Separate the dots to allow several arrows to be drawn between them. (See the next illustration.)

T: ***On your paper, build an arrow road from 7 to 100 using only arrows for +, –, x, or ÷ some whole number from 1 to 9. You may use more than one type of arrow, but you must get only integers at the dots. Only integers are allowed in this situation.***

Be sure students understand the directions and restrictions. If necessary, ask for examples of what the arrows could be for: +5, –3, x9, ÷2, and so on. Note that the first arrow in the road could not be ÷2 because 7 ÷ 2 = 3.5 and 3.5 is not an integer. Let students work individually or with partners for a few minutes. Encourage every student to find a solution; some students can be asked to find more than one solution or a shortest road. Invite several students to put their solutions on the board. Include a road with three arrows, such as the blue road below.



T: ***We have a five-arrow road, a four-arrow road, and a three-arrow road from 7 to 100. Do you think it's possible to build a road with fewer than three arrows from 7 to 100?***

After several attempts to build a road with only one or two arrows, the class should suspect that a three-arrow road is the shortest.

N 2

T: ***With the information we have on the board, is it easy to find an arrow road from 100 to 7? What is the length of a shortest such road?***

S: ***Just use return arrows.***

S: ***A shortest arrow road from 100 to 7 has length 3 (three arrows).***

T: ***A shortest road from 7 to 100 has length 3 and, using return arrows, we see that a shortest road from 100 to 7 also has length 3. Therefore, the one-digit distance between 7 and 100 is 3. We write it this way.***

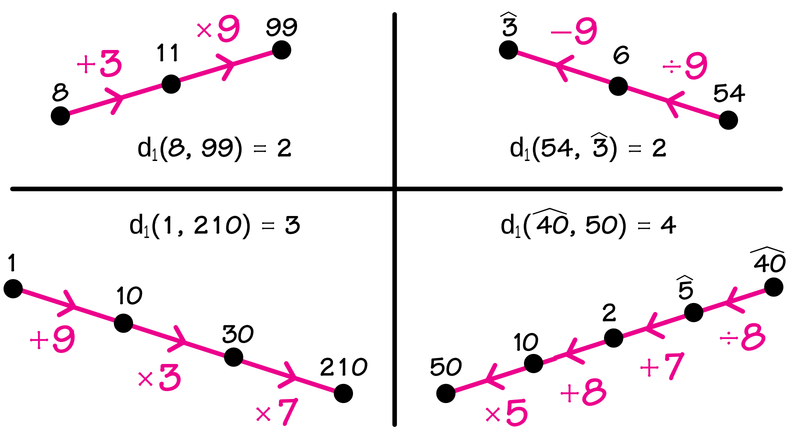
$$d_1(7, 100) = 3$$

Write these problems on the board and assign them for students to do individually or with partners.

$$d_1(8, 99) \qquad d_1(54, \hat{3})$$

$$d_1(1, 210) \qquad d_1(\widehat{40}, 50)$$

When most students have drawn at least two roads, invite students with shortest roads to put their solutions on the board. The one-digit distance and a shortest road for each pair of numbers are given here. Other shortest roads are possible.



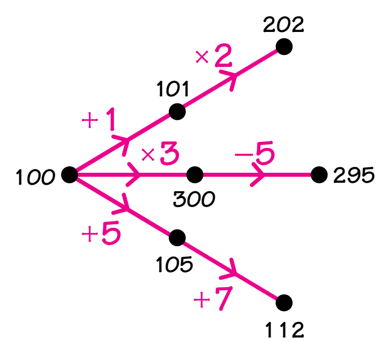
Exercise 2

Write this number sentence on the board.

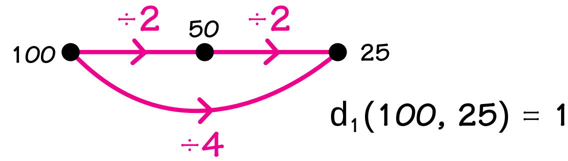
$$d_1(100, \square) = 2$$

T: ***Find some whole numbers whose one-digit distance from 100 is 2.***

Allow several minutes for independent or partner work. Put several examples that students suggest on the board.

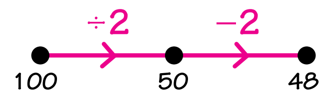


Be alert for common mistakes such as roads that are not shortest. For example:



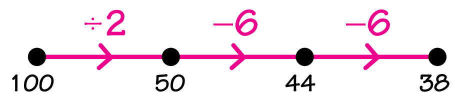
T: ***Is 48 at a one-digit distance of 2 from 100?***

Allow a few minutes for students to draw a two-arrow road from 100 to 48.



S: ***Yes, the one-digit distance from 100 to 48 is 2.***

T: ***Is 38 at a distance of 2 from 100?***

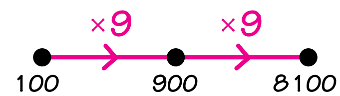


S: ***No, the one-digit distance from 100 to 38 is 3.***

T: ***What about 25?***

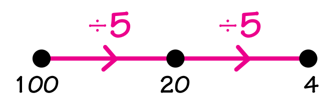
S: ***25 is at a one-digit distance of 1 from 100, because 100 ÷ 4 = 25.***

T: ***What is the greatest whole number at a distance of 2 from 100?*** (8100)



Invite a student to draw the appropriate road on the board.

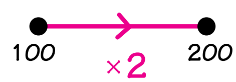
Ask students to find the least whole number (4) at a one-digit distance of 2 from 100.



T: ***Let's try to find all the whole numbers between 200 and 300 that are at a one-digit distance of 2 from 100. first, we choose a number at a distance of 1 from 100. How could we start?***

S: ***100*** x ***2 = 200, so 200 is at a distance of 1 from 100.***

Draw a x2 arrow starting at 100.

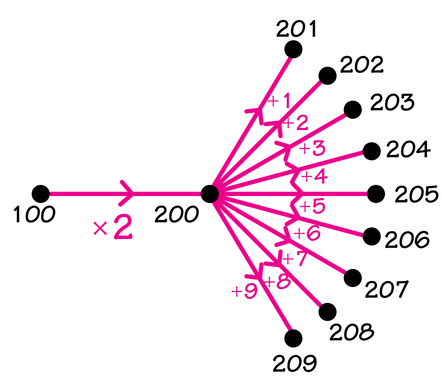


T: ***Now can we find some numbers that are at a distance of 2 from 100 and are between 200 and 300?***

S: ***200 + 1 = 201, so 201 is at a distance of 2 from 100.***

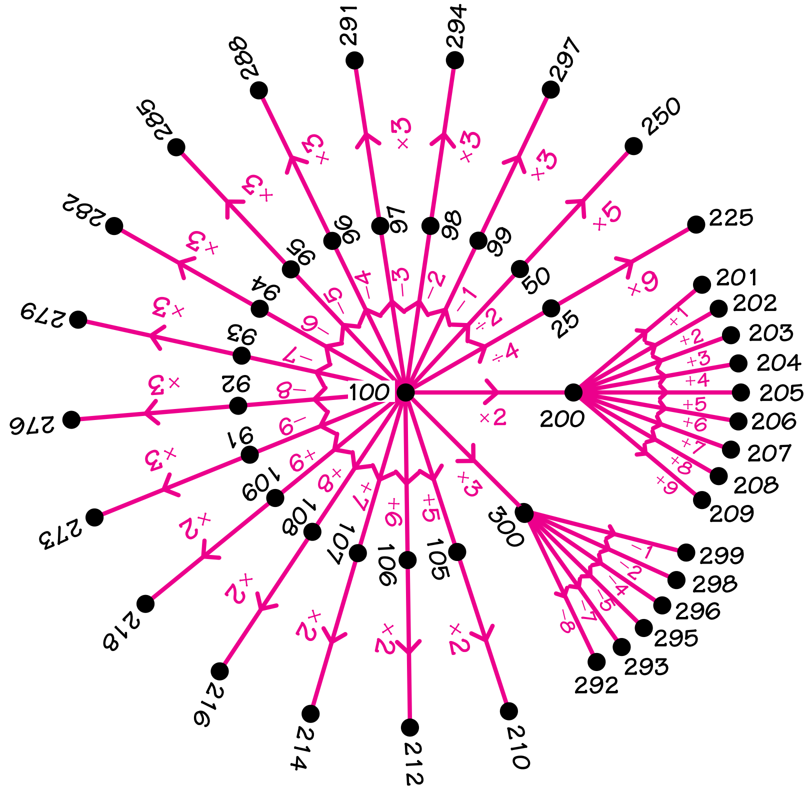
S: ***You could add 1 or 2 or 3 or 4 or 5 or 6 or 7 or 8 or 9 and get a number between 200 and 300 that is a distance of 2 from 100.***

Add this information to the picture.



N 2

When it is obvious that most students understand the problem, let them work individually or with partners for a while. List the numbers as they are found on the board. A complete solution is shown below; finding a complete solution might be assigned as a class project.



Although the preceding illustration shows all the integers between 200 and 300 that are at a one-digit distance of 2 from 100, it does not show all the possible ways to arrive at those numbers. For example, to build a road from 100 to 206, you can use a x2 arrow followed by a +6 arrow or you can use a +3 arrow followed by a x2 arrow.

Capsule Lesson Summary

Decide which numbers in a given list can be put on the Minicomputer using exactly one of the weighted checkers ②, ③, …, ⑨. Put the other numbers in the list on the Minicomputer using two specified weighted checkers. Present a detective story with clues involving the prime numbers between 50 and 100, moving one checker on the Minicomputer, and the one-digit distance relation.

Materials

Teacher
- Minicomputer set
- Weighted checker set
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N3*, **, and ***

Description of Lesson

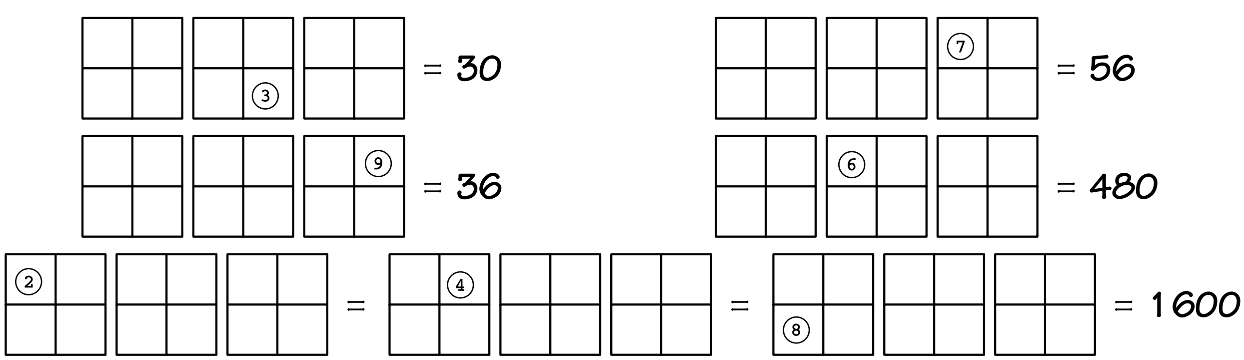
Exercise 1

Display three Minicomputer boards and the weighted checkers ②, ③, ④, ⑤, ⑥, ⑦, ⑧, and ⑨. Write this list of numbers on the chalkboard.

30 36 46 56 350 480 1 600

T: ***Find all of the numbers in this list that can be put on the Minicomputer using exactly one of these weighted checkers.***

Invite students to put numbers on the Minicomputer using exactly one of the displayed checkers. Each time, ask the rest of the class to check that the number displayed is indeed one of the numbers in the list. Continue until the class finds that five numbers (30, 36, 56, 480, and 1 600) can be put on the Minicomputer this way.

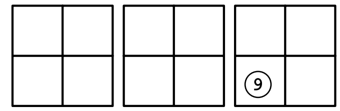


The class should conclude that 46 and 350 cannot be put on the Minicomputer with exactly one of the displayed checkers.

T: ***On your paper show how 46 can be put on the Minicomputer using a ⑦-checker and a ⑨-checker.***

N 3

Let students work independently for a couple minutes; then ask a student to place one checker on the Minicomputer. Suppose it is a ⑨-checker.



T: ***Which number is on the Minicomputer?***

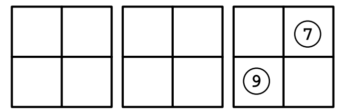
S: ***18, because 9*** x ***2 = 18.***

T: ***How much more than 18 do we need to get 46?***

S: ***28 more.***

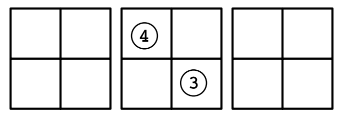
T: ***Can someone put 28 on the Minicomputer using a ⑦-checker?***

Invite a student to place the ⑦-checker.



= 46

Continue by asking students to show 350 on the Minicomputer using a ③-checker and a ④-checker.



= 350

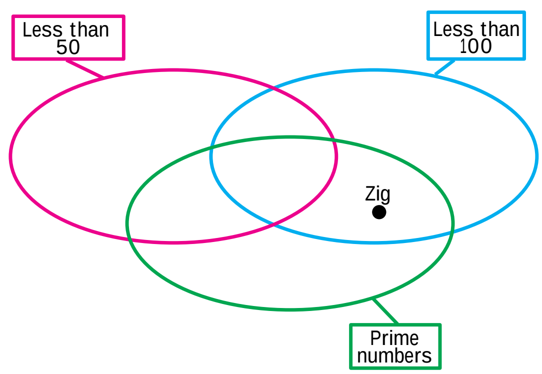
Note: Students may insist that you use a ⑤-checker instead of a ④-checker because they think about 350 = 300 + 50. Here they must use 350 = 320 + 30.

Exercise 2

Present the following detective story about a secret whole number named Zig.

Clue 1

Draw this string picture on the board and ask,



T: ***What do we learn about Zig from this picture?***

S: ***Zig is a prime number between 50 and 100.***

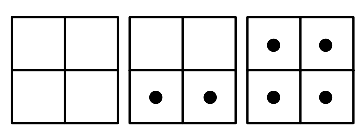
T: ***Which numbers could Zig be?***

On the board, list the numbers that Zig could be as students announce them. Organize the list of numbers in order to help students be more systematic in their search for primes.

Zig: 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

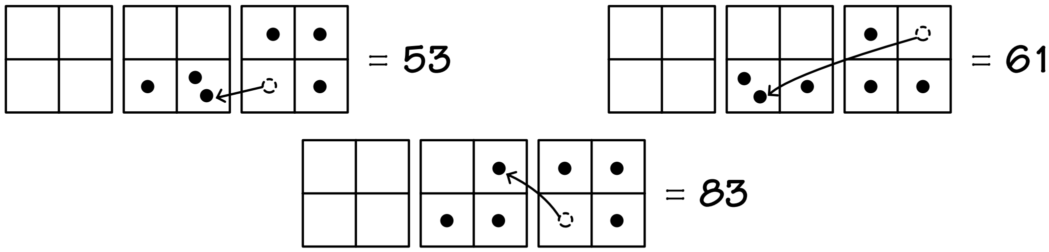
Clue 2

Display three Minicomputer boards with this configuration of checkers.



T: ***What number is on the Minicomputer?*** (45) ***If you move exactly one of these checkers to another square, you can get Zig.***

Invite students to show possibilities for Zig by moving one checker on the Minicomputer. The numbers that Zig could be are shown below.



Clue 3

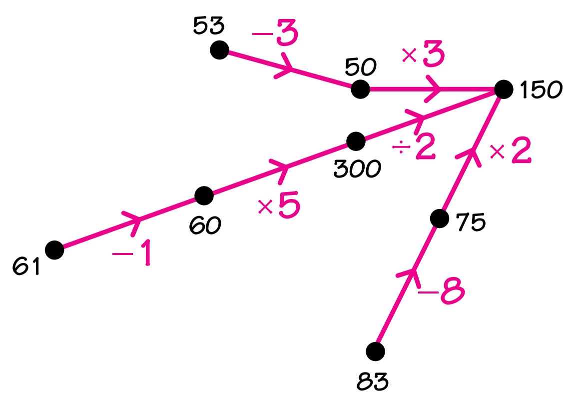
Write this information on the board.

$$d_1(Zig, 150) = 3$$

T: ***What new information about Zig does this clue give us?***

S: ***The one-digit distance from Zig to 150 is 3.***

Instruct students to draw some appropriate roads from possibilities for Zig to 150 on their papers using only +, –, x, or ÷ a one-digit positive number. Remind them that they are looking for shortest such roads to find the one-digit distance. You may need to help some students get started as the one-digit distance relation is a relatively new idea. After several minutes of individual or partner work, ask several students to draw their roads on the board. A shortest possible road is shown below from each of three possibilities (from Clue 2) for Zig to 150.



S: ***The one-digit distance from 61 to 150 is 3, and the other two numbers are each a one-digit distance of 2 from 150. Therefore, Zig is 61.***

Worksheets N3*, **, and *** are available for individual work.

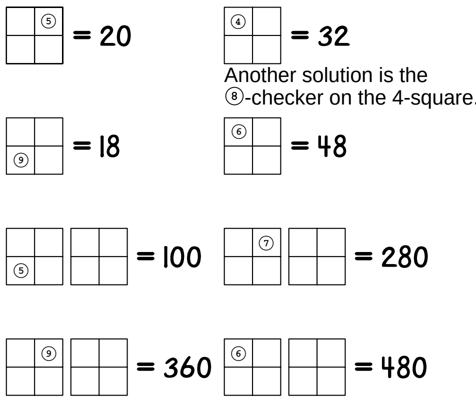
Writing Activity

Students who like to write detective stories may like to write a **** worksheet with a detective story involving weighted checkers on the Minicomputer, prime numbers, or the one-digit distance relation.

Name ________________ N3 *

Put each number on the Minicomputer using exactly one of these weighted checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



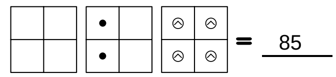
Name ________________ N3 **

Clue 1

Paf is a prime number between 60 and 80.

Paf could be 61, 67, 71, 73, or 79.

Clue 2

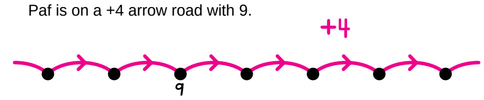


Paf can be shown here on the Minicomputer by moving exactly one checker to another square.

Paf could be 67, 73, or 79.

Clue 3

Paf is on a +4 arrow road with 9.



Who is Paf? 73

Name ________________ N3 ***

Rif and Raf are secret whole numbers.

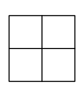
Clue 1

Rif is a prime number between 100 and 120.

Rif could be 101, 103, 107, 109, or 113.

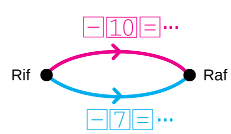
Clue 2

Raf can be put on the ones board of the Minicomputer using just a ③-checker and a ⑦-checker.



Raf could be 10, 13, 19, 31, 17, 20, 26, 38, 34, 40, 52, 59, 62, 68, 80

Clue 3



Who is Rif? 101 Who is Raf? 31

Capsule Lesson Summary

Review the binary abacus and binary writing. Do some addition calculations in the binary system. Label a binary number line.

Materials

Teacher
- Colored chalk
- Blackline N4
- Minicomputer checkers (optional)

Student
- Binary abacus
- Checkers
- Paper

Advance Preparation: Use Blackline N4 to make copies of a binary abacus for students.

Description of Lesson

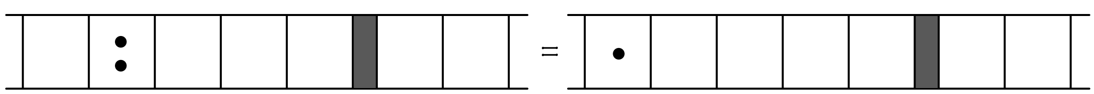
Exercise 1

On the chalkboard, draw part of a binary abacus with eight to ten places, including two or three to the right of the bar.

Note: If your chalkboard is magnetic, use Minicomputer checkers on the abacus. Otherwise, draw and erase checkers as needed.

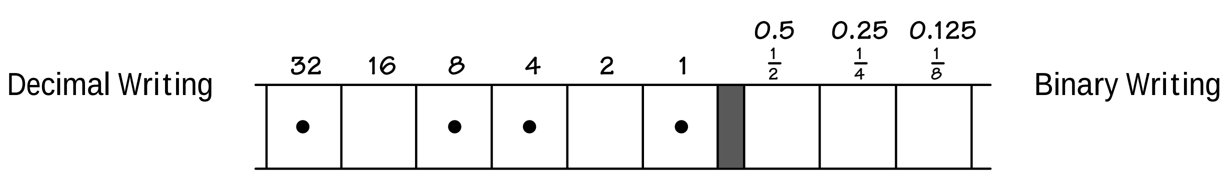
T: ***Do you remember the binary abacus and its rule?***

Invite a student to explain the binary abacus to the class.



Two checkers on a board
trade for
one checker on the next board to the left.

Ask students to label the boards of the binary abacus. Put this configuration of checkers on the abacus and write the two headings close to either side, as illustrated below.



T: ***What number is this?***

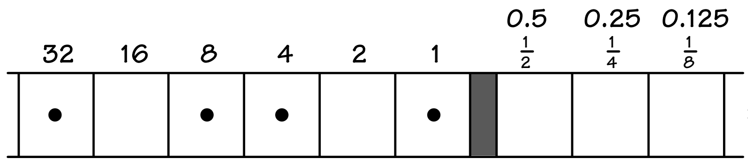
S: ***45.***

T: ***How do we represent this number in binary writing?***

N 4

Invite a student to write the binary name for the number on the board.

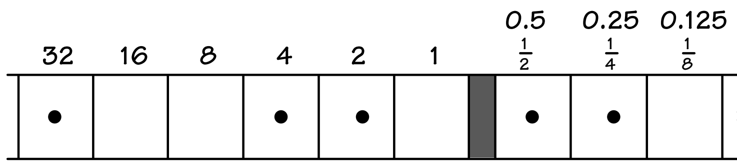
Decimal Writing 45 =



= Binary Writing 101101

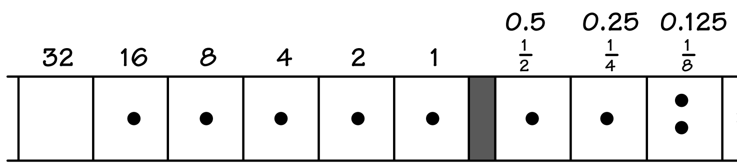
Continue this activity with a few more examples, such as the ones below.

Decimal Writing 38.75 =



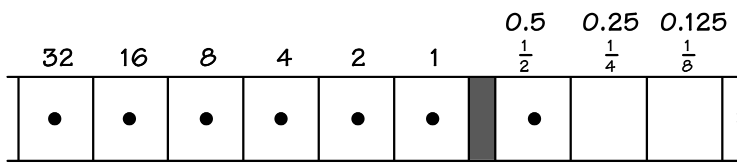
= Binary Writing 100110.11

Decimal Writing 32 =



= Binary Writing 100000

Decimal Writing 63.5 =

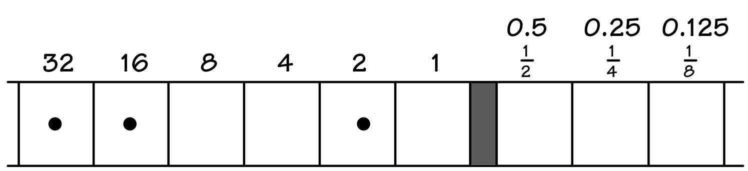


= Binary Writing 111111.1

Observe that when putting a number on the binary abacus one can always do it with at most one checker on any board. Discuss why this is so and how the binary writing of numbers uses only two digits, 0 and 1.

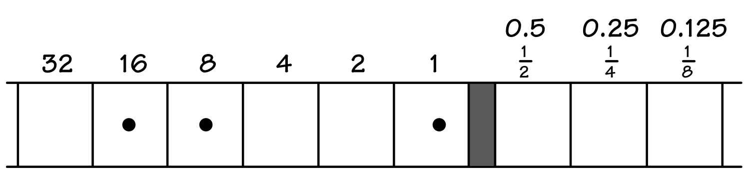
Ask students to put 50 on their binary abaci using at most one checker on any board, and then to write both the decimal and the binary names for 50.

Decimal Writing 50 =



= Binary Writing 110010

T: ***I'll move each checker on the abacus one board to the right. Now what number is on the abacus?***

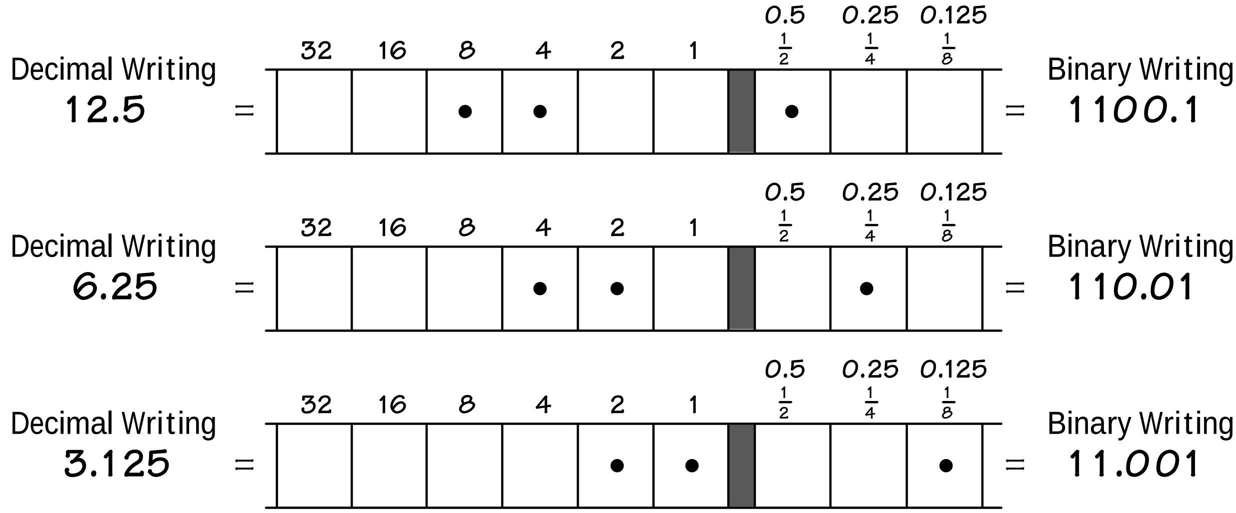


S: ***25; moving the checkers one board to the right halves the number.***

T: ***Good. Now represent 25 in binary writing.***

Decimal Writing		Binary Writing
50	=	110010
25	=	11001

Repeat this activity several times, each time moving checkers one board to the right, and asking for decimal and binary names for the number.



T: ***Do you see an easy way to calculate one half of a number in the binary system?***

S: ***If you are writing in binary, just move the digits one place to the right.***

Let students try this rule by starting with **110010** and halving repeatedly.

$$110010 \xrightarrow[\frac{1}{2}\times]{} 11001 \xrightarrow[\frac{1}{2}\times]{} 1100.1 \xrightarrow[\frac{1}{2}\times]{} 110.01$$

Exercise 2

Write this addition problem on the board.

$$\begin{array}{r} 11001 \\ +\ \ 1101.1 \\ \hline \end{array}$$

T: ***I have written an addition problem in the binary system. How do you suggest we solve this problem?***

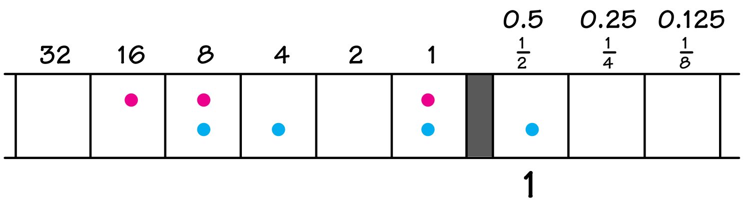
Some students might suggest rewriting the problem in the decimal system, adding in the decimal system, and then converting back to binary. Indicate that such a method is overly cumbersome, however, that rewriting into the decimal system might be a good way to check your work. Others may be able to solve the problem in the binary system directly (without the support of the binary abacus). For the benefit of those who need the visual support of the binary abacus, recommend that the class solve the problem in the binary system and make use of the abacus as follows.

Put the addends on the abacus, using checkers of a different color for each. In the next illustration, 11001 is in red and 1101.1 is in blue.

T: ***Look carefully at each board of the abacus. There is only one checker on the halves board, so no trade needs to be made with that checker.***

N 4

As you make this observation, write the corresponding information in the appropriate places under the abacus and in the addition calculation.

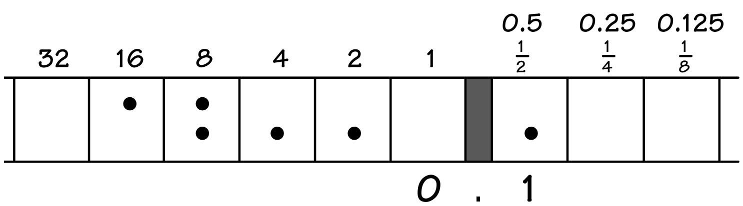


$$\begin{array}{r} 11001 \\ +\ 1101.1 \\ \hline 1 \end{array}$$

T: ***Now look at the ones board. There are two checkers on this board so we can make a trade; this is the same as adding 1 and 1.***

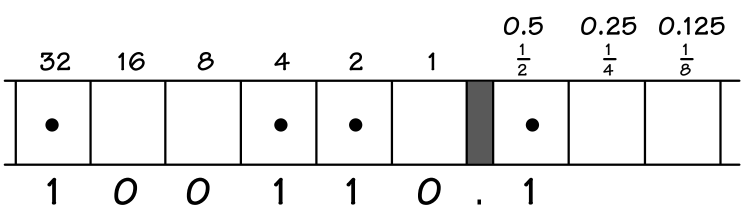
Make the trade and write the corresponding information in the appropriate places under the abacus and in the addition calculation.

Note: After the initial placement of checkers on the abacus, the colors are no longer relevant. Therefore, the illustrations now will have black checkers.



$$\begin{array}{r} \overset{1}{} \\[-4pt] 1100\overset{}{1} \\ +\ 1101.1 \\ \hline 0.1 \end{array}$$

Continue working from the abacus to the calculation until you reach this configuration and complete the calculation.



$$\begin{array}{r} {}^{1\,1}\ \ {}^{1} \\[-4pt] 11001 \\ +\ 1101.1 \\ \hline 100110.1 \end{array}$$

If you or the class would like, check the calculation in the decimal system.

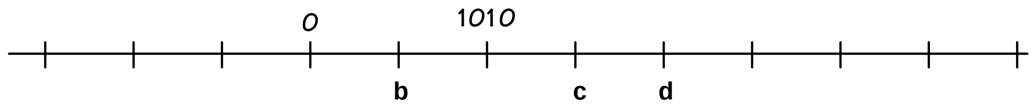
Binary Writing		Decimal Writing
11001	=	25
+ 1101.1	=	+13.5
100110.1		38.5

Continue the activity with these problems. Allow time for students to do the calculations on their papers; then check the solutions collectively. (Answers are in boxes.)

$$\begin{array}{r} 101010 \\ +\ 11101 \\ \hline \boxed{1000111} \end{array} \qquad \begin{array}{r} 110011.001 \\ +\ 11001.101 \\ \hline \boxed{1001100.110} \end{array} \qquad \begin{array}{r} 111.011 \\ +\ 10.101 \\ \hline \boxed{1010.000} \end{array}$$

Exercise 3

Draw this part of a binary number line on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: ***This is part of a number line with the marks labeled in the binary system. How should we label this mark (*****point to b*****)? Remember, we want to use binary writing.***

Allow a few minutes for students to study the given information. As with calculations in the binary system, some students might suggest converting to the decimal system, determining the appropriate decimal label, and then converting back. Indicate that this method is okay but unnecessarily involved. Encourage students to notice that the mark (b) is halfway between **0** and **1010**. Then ask about halving in the binary system.

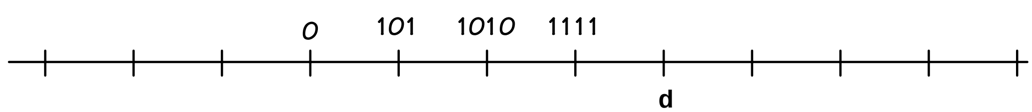
S (pointing to b)**:** ***This number is* 101 *because it is one half of* 1010.**

The class can do the calculation (½ x 1010) on the binary abacus to verify the result.

T (pointing to c)**:** ***How should we label this mark? Again, we want to use binary writing.***

Allow a few minutes for students to study the situation. They should observe that the first mark to the right of 0 is for 101 and the second mark to the right of 0 is for 1010 (101 + 101). So to find labels for the other marks in consecutive order, they should continue adding 101.

S: ***Just add*** 101 ***to*** 1010***;*** 1010 + 101 = 1111**.**

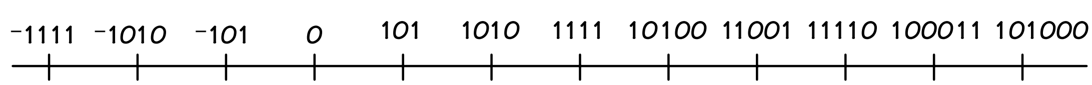


Explain that, on this number line, we label consecutive marks counting by **101** (in the binary system). So each time you want to label a mark, add **101** to the number at the mark directly to its left.

T (pointing to d)**:** ***How should we label this mark?***

S: ***Since* 1111 + 101 = 10100, *label it* 10100.**

Invite students to finish labeling the marks on the number line. A complete labeling is shown below.



Capsule Lesson Summary

Count backward by $\frac{1}{4}$s and then forward by $1\frac{1}{2}$s. Use a number line model to solve division problems where the divisor is a fraction. Illustrate the equivalence of $\div\frac{3}{4}$ and $\frac{4}{3}$x with an arrow picture. Solve division problems involving fractions by rewriting them as multiplication problems.

Materials

Teacher
- Meter stick
- Colored chalk
- Fraction pieces (optional)

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets N5*, **, and ***

Description of Lesson

Exercise 1

Use an order natural to the seating arrangement in your classroom for this exercise.

T: ***Let's start at 10 and count backward by fourths. I'll start. 10. What number is 10 – $\frac{1}{4}$?***

S: ***$9\frac{3}{4}$.***

Call on students one, at a time, to continue the count backward by fourths. As necessary, give hints referring to a number line or a fraction manipulative. Encourage responses in mixed form with simplest fractions; for example, $9\frac{3}{4}$, $9\frac{2}{4}$ or $9\frac{1}{2}$, $9\frac{1}{4}$, 9, $8\frac{3}{4}$, $8\frac{1}{2}$, and so on. Continue until every student contributes to the count.

T: ***Suppose instead we start at 0 and count by $1\frac{1}{2}$s. I'll start. 0.***

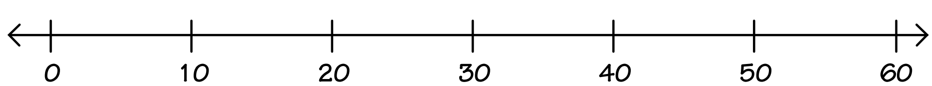
S: ***$1\frac{1}{2}$.***

S: ***3.***

Conduct the counting by $1\frac{1}{2}$s until every student contributes to the count. You may like to compare the last number said to the number of students in class.

Exercise 2

Use a meter stick to draw this part of a number line on the board. Accurately space the marks, for example, at 20 cm apart.



T: ***What number is 54 ÷ 6?*** (9) ***Can you use this number line to convince us that 54 ÷ 6 = 9?***

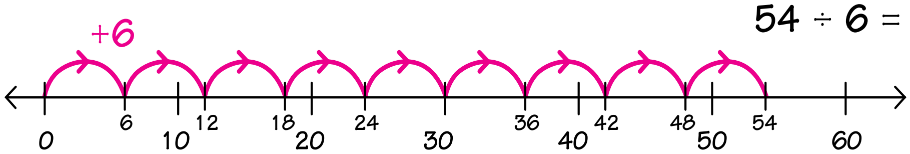
S: ***If you divide the segment from 0 to 54 into six equal parts, each part will have nine units.***

S: ***Start at 0 and count by sixes. Make marks for 6, 12, 18, 24, and so on until we get to 54.***

N 5

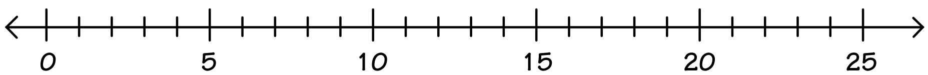
Invite a student to make marks for counting by sixes, or for the ending points of successive +6 arrows starting at 0. Confirm that there are nine 6s, or nine +6 arrows, from 0 to 54.

$54 \div 6 = 9$



Draw this part of a number line, and write a new division problem on the board.

$20 \div 2\frac{1}{2} =$

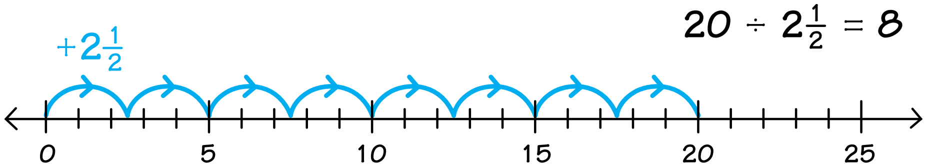


T: ***What number is 20 ÷ 2½? How can we use the number line for this calculation?***

S: ***Start at 0 and count by 2½s. Find how many +2½ arrows there are from 0 to 20.***

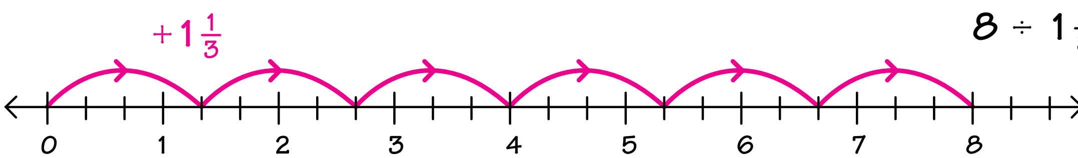
Invite students to draw the +2½ arrows or to mark the points on the number line while counting by 2½s. The class should find eight +2½ arrows from 0 to 20, or count eight 2½s to reach 20. That is, there are eight 2½s in 20 and 20 ÷ 2½ = 8.

$20 \div 2\frac{1}{2} = 8$

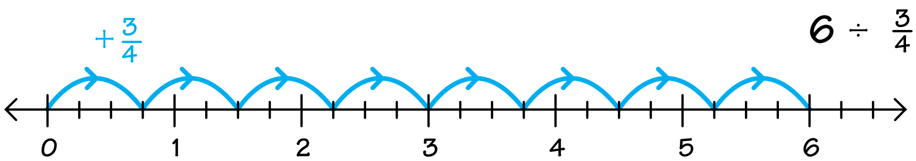


Repeat this exercise to calculate 8 ÷ 1⅓ (6) and 6 ÷ ¾ (8).

$8 \div 1\frac{1}{3} = 6$



$6 \div \frac{3}{4} = 8$



Write the following problems on the board, asking students to copy and solve them. (Answers are in boxes.) You may want to encourage students to use the number line model, but also allow other methods, such as changing a division problem to a multiplication problem.

$4 \div \frac{1}{2} = \boxed{4 \times 2 = 8}$ $\qquad 9 \div \frac{3}{5} = \boxed{15}$ $\qquad 12 \div 1\frac{1}{2} = \boxed{8}$

S: ***There are two halves in 1, so there are 4*** x ***2 or 8 halves in 4.***

T: ***When the divisor is a unit fraction (a fraction with 1 as numerator), it is easy to solve a division problem by changing it into a multiplication problem with the same result. Use this technique on these problems.***

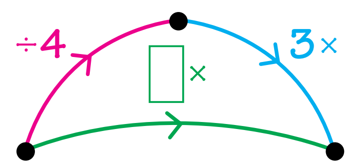
Write the following problems on the board, asking students to copy and solve them. (Answers are in the boxes.)

$6 \div \frac{1}{3} = \boxed{6 \times 3 = 18}$ $\qquad$ $4\frac{2}{7} \div \frac{1}{2} = \boxed{4\frac{2}{7} \times 2 = 8\frac{4}{7}}$

$9 \div \frac{1}{6} = \boxed{9 \times 6 = 54}$ $\qquad$ $2\frac{1}{5} \div \frac{1}{3} = \boxed{2\frac{1}{5} \times 3 = 6\frac{3}{5}}$

Exercise 3 _______________

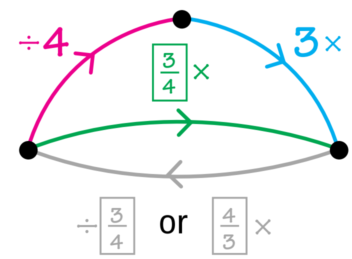
Draw this arrow picture on the board.



T: ***What could the green arrow be for?***

S: ***$\frac{3}{4}$x. ÷ 4 followed by 3x is the same as $\frac{3}{4}$x.***

Draw a return or opposite arrow for the green arrow.



T: ***What is the return (or opposite) of $\frac{3}{4}$x.***

S: ***÷$\frac{3}{4}$.***

T: ***Could we give this arrow a □ x label?***

S: ***If we use the opposites of 3x and ÷4, it is ÷3 followed by 4x which is the same as $\frac{4}{3}$x.***

You may want to draw the opposite arrows to better view this composition.

Use the arrow picture to solve the following problems. (Answers are in boxes.)

$\frac{3}{4} \times 20 = \boxed{15}$ $\qquad$ $30 \div \frac{3}{4} = \frac{4}{3} \times 30 = \boxed{40}$

$\frac{3}{4} \times 2\frac{2}{3} = \boxed{2}$ $\qquad$ $1\frac{1}{2} \div \frac{3}{4} = \frac{4}{3} \times 1\frac{1}{2} = \boxed{2}$

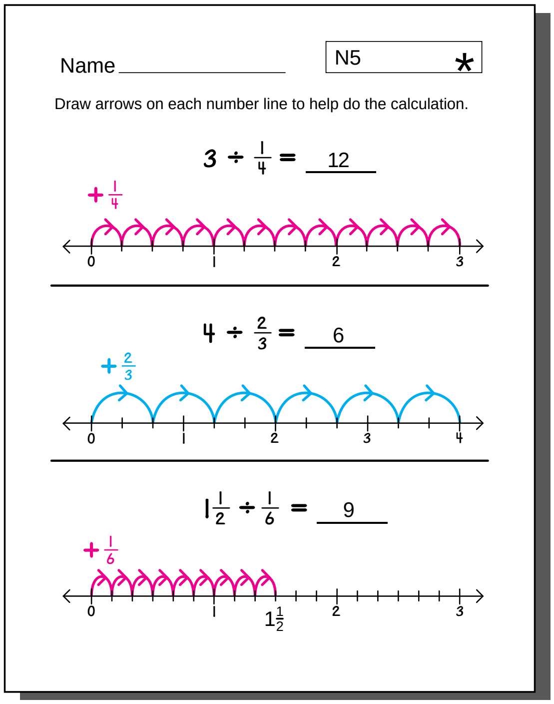
Write the following problems on the board, asking students to copy and solve them.

$12 \div \frac{3}{4} = \boxed{16}$ $\qquad$ $5 \div \frac{3}{4} = \boxed{\frac{4}{3} \times 5 = \frac{20}{3} = 6\frac{2}{3}}$ $\qquad$ $\frac{3}{5} \div \frac{3}{4} = \boxed{\frac{4}{3} \times \frac{3}{5} = \frac{4}{5}}$

Worksheets N5*, **, and *** are available for individual work.

Home Activity

This is a good time to send a letter to parents/ guardians about division with fractions. Blackline N5 has a sample letter.



Name ________ N5 *

Draw arrows on each number line to help do the calculation.

$3 \div \frac{1}{4} = \underline{12}$

$+\frac{1}{4}$

0 1 2 3

$4 \div \frac{2}{3} = \underline{6}$

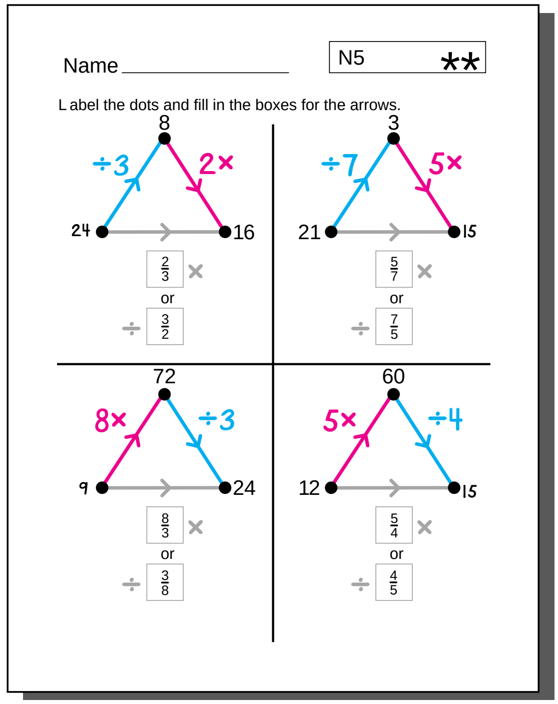
$+\frac{2}{3}$

0 1 2 3 4

$1\frac{1}{2} \div \frac{1}{6} = \underline{9}$

$+\frac{1}{6}$

0 1 $1\frac{1}{2}$ 2 3



Name ________ N5 **

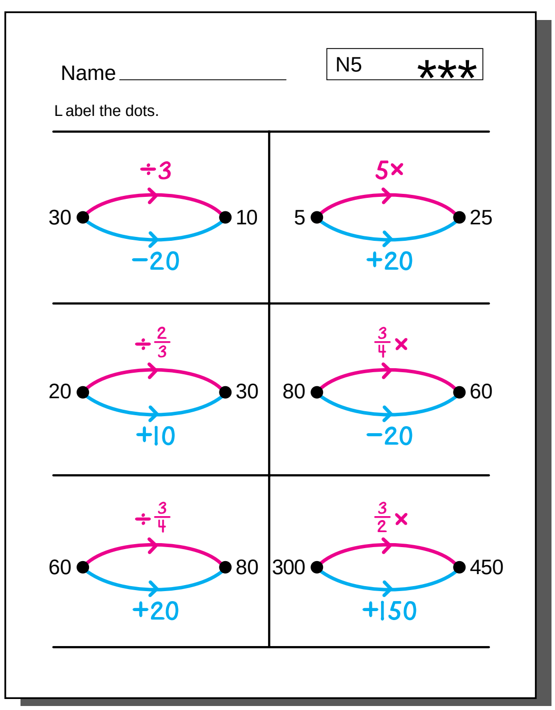
Label the dots and fill in the boxes for the arrows.

8; ÷3; 2×; 24; 16; $\boxed{\frac{2}{3}}$ × or ÷ $\boxed{\frac{3}{2}}$

3; ÷7; 5×; 21; 15; $\boxed{\frac{5}{7}}$ × or ÷ $\boxed{\frac{7}{5}}$

72; 8×; ÷3; 9; 24; $\boxed{\frac{8}{3}}$ × or ÷ $\boxed{\frac{3}{8}}$

60; 5×; ÷4; 12; 15; $\boxed{\frac{5}{4}}$ × or ÷ $\boxed{\frac{4}{5}}$



Name ________ N5 ***

Label the dots.

÷3; 30; 10; −20

5×; 5; 25; +20

$\div\frac{2}{3}$; 20; 30; +10

$\frac{3}{4}$×; 80; 60; −20

$\div\frac{3}{4}$; 60; 80; +20

$\frac{3}{2}$×; 300; 450; +150

N6 PERCENT #1 N 6

Capsule Lesson Summary

Label marks on a number line that has been scaled in two different ways. Use a double scaled number line with one scale a percent scale to solve various types of percent problems.

Materials

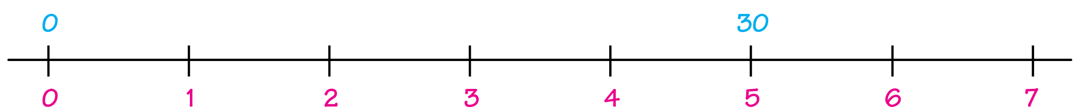
Teacher	• Colored chalk • Blackline N6	**Student**	• Paper • Colored pencils, pens, or crayons • Percent number lines

Advance Preparation: Use Blackline N6 to make several copies of the percent number lines for students' use in Exercises 2 and 3.

Description of Lesson

Exercise 1

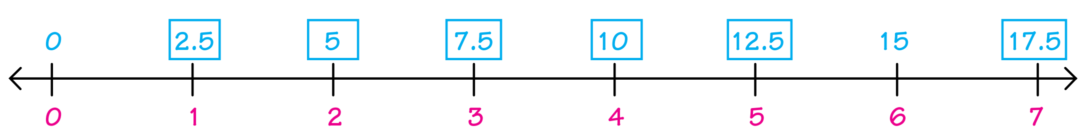
Draw a section of a double scaled number line on the board.



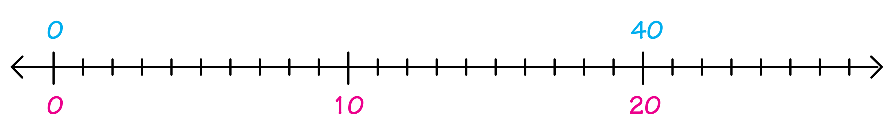
T: ***This number line has two scales. You have seen this kind of situation before with centimeters and inches on a ruler, or with degrees centigrade and degrees Fahrenheit on a thermometer. How should we label the marks using the blue scale?***

Invite students to label the marks at the board.

Repeat this activity using another blue scale. (Answers are in boxes.)



Erase the scales on the number line and remark it with approximately 20 divisions. Then label marks as shown below.

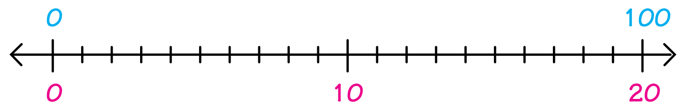


T: ***How would the marks be labeled in the red scale?*** (1, 2, 3, 4, …)
In the blue scale? (2, 4, 6, 8, …)

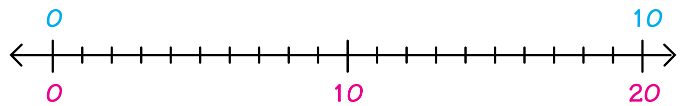
N 6

Keeping the red scale with marks for 0, 1, 2, 3, 4, …, change the blue scale several times by putting a different label above 20 in the red scale. Each time, ask what the marks would be labeled in the new blue scale. For example:

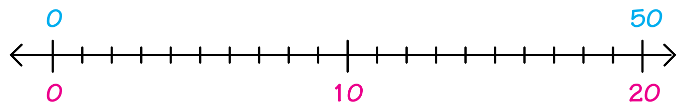
Blue scale: 0, 5, 10, 15, 20, ⋯



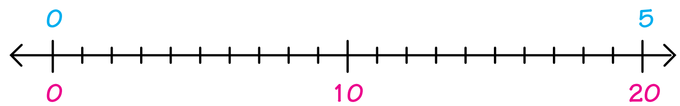
Blue scale: 0, $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, ⋯



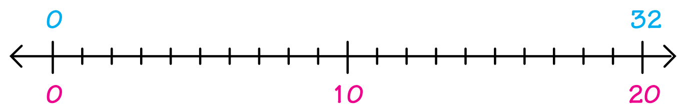
Blue scale: 0, 2.5, 5, 7.5, 10, 12.5, ⋯



Blue scale: 0, 0.25, 0.5, 0.75, 1, 1.25, ⋯



Blue scale: 0, 1.6, 3.2, 4.8, 6.4, 8, ⋯



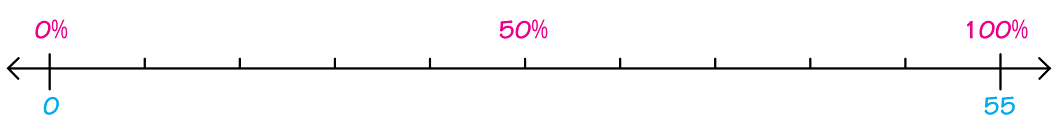
Exercise 2

Announce to the class that you would like to make a percent number line and then use it to calculate percents of numbers. Draw a section of a number line with a percent scale in red.

Pose a percent problem such as 70% of 55.

T: ***Let's start another scale on this number line from 0 to 55. Where should we put 0? Where should we put 55?***

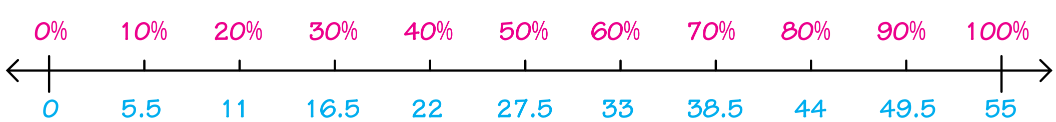
S: ***Put 0 at the mark for 0%. 0% is always 0. Put 55 at the mark for 100% because 100% of 55 = 55.***



Invite students to determine and then label the other marks in the blue scale.

T: ***Now, how does the number line help us calculate 70% of 55?***

S: ***Find the mark for 70% in the red scale. Look beneath at the label for this mark in the blue scale; that is, 70% of 55, or 38.5.***



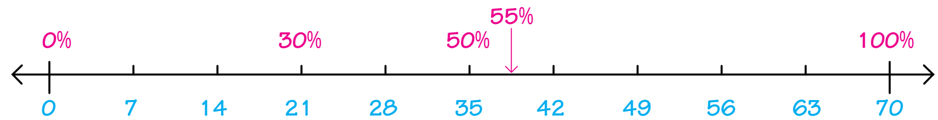
T: ***Can you use this number line to find 45% of 55?***

S: ***45% is halfway between 40% and 50%. So 45% of 55 is halfway between 22 and 27.5. 45% of 55 = 24.75.***

Pose several percent problems and instruct students to draw their own number lines to solve the problems. Of course, students may want to do the calculations in other ways, but ask if they can show how to use the number line model. The following are sample problems.

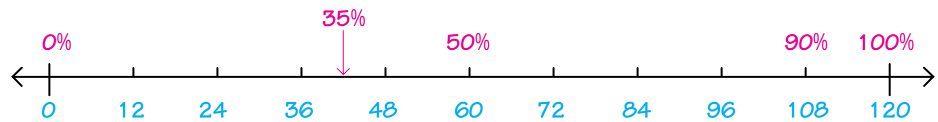
30% of 70 = $\boxed{21}$

55% of 70 = $\boxed{38.5}$



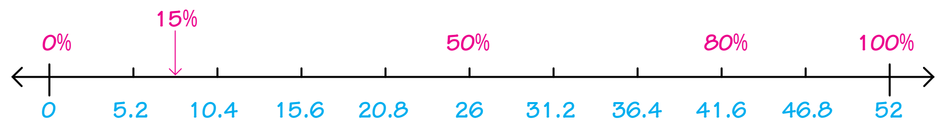
90% of 120 = $\boxed{108}$

35% of 120 = $\boxed{42}$



80% of 52 = $\boxed{41.6}$

15% of 52 = $\boxed{7.8}$



At this point, you may like to extend the number line with a percent scale and the blue scale to find, for example, 110% of 52 = 57.2.

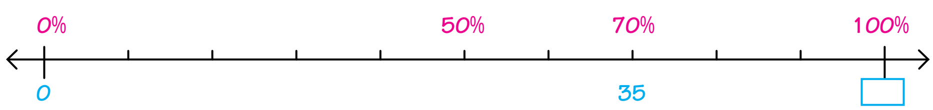
Exercise 3 ______________

Write this percent problem on the board as you ask, 70% of $\boxed{}$ = 35

T: ***Do you think we could use our percent number line to solve this problem?***

Let students describe how to label marks on the percent number line.

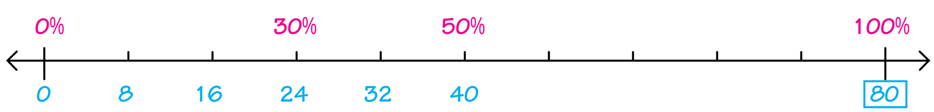
S: ***On the blue scale, put 0 at 0% and 35 at 70%. The number in the box will then be at 100%.***



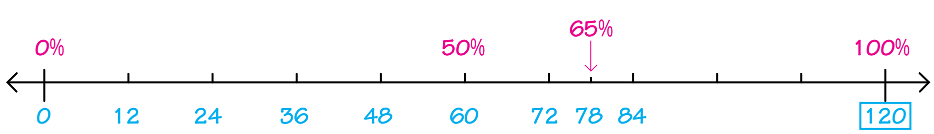
S: ***The marks on the blue scale are 0, 5, 10, 15, and so on. So the number in the box is 50. 70% of 50 = 35.***

Present one or two similar percent problems for students to solve with number lines they draw. For example:

30% of $\boxed{80}$ = 24



65% of $\boxed{120}$ = 78



N 6

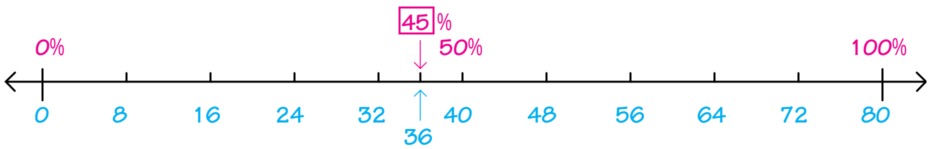
Write this percent problem on the board as you ask,

$\square$% of 80 = 36

T: ***Could we use our percent number line to solve this problem?***

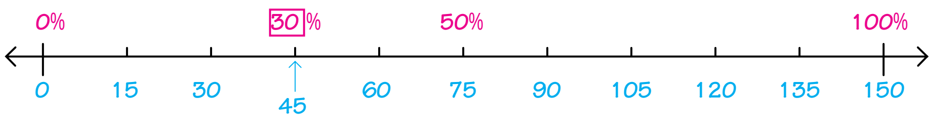
Let students describe how to label marks on the percent number line.

S: ***On the blue scale, put 0 at 0% and 80 at 100%. Then locate 36 on the blue scale. $\square$% will be above 36 on the red scale. 45% of 80 = 36.***

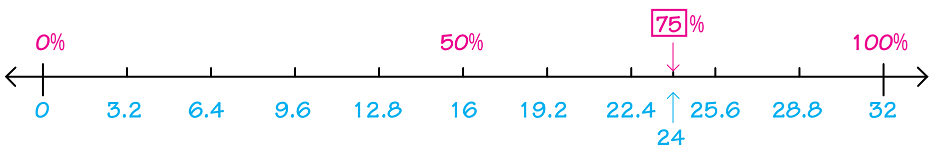


Pose one or two similar percent problems for students to solve with number lines they draw. For example:

$\boxed{30}$% of 150 = 45



$\boxed{75}$% of 32 = 24



Capsule Lesson Summary

Investigate patterns in a multiplication square, and generate a rule that relates to area calculations. By extending the rule to include negative numbers, give credence to the rule that a negative number times a negative number is a positive number.

Materials

Teacher	• Calculator	**Student**	• Paper • Calculator

Description of Lesson

Begin the lesson with mental arithmetic involving negative numbers. The following are suggested sequences of problems. (Answers are in boxes).

$8 + \widehat{2} = \boxed{6}$	$70 + \widehat{12} = \boxed{58}$	$312 + \widehat{412} = \boxed{\widehat{100}}$
$80 + \widehat{20} = \boxed{60}$	$700 + \widehat{120} = \boxed{580}$	$312 + \widehat{411} = \boxed{\widehat{99}}$
$\widehat{80} + 20 = \boxed{\widehat{60}}$	$7\,000 + \widehat{1200} = \boxed{5\,800}$	$312 + \widehat{410} = \boxed{\widehat{98}}$
$800 + \widehat{200} = \boxed{600}$	$\widehat{7\,000} + 1200 = \boxed{\widehat{5\,800}}$	$312 + \widehat{409} = \boxed{\widehat{97}}$

Exercise 1

Draw this multiplication square on the board and let students give the four entries. Then invite a student to the board to find the sum of the four entries.

×	3	12
3	9	36
12	36	144

$$\begin{array}{r} 9 \\ 36 \\ 36 \\ +\ 144 \\ \hline 225 \end{array}$$

Erase all but the grid.

T: ***Let's make another multiplication square. Give us two other whole numbers.***

Label the multiplication square according to student response. Then request the four entries, and invite a student to find the sum of the four entries. For example:

×	10	7
10	100	70
7	70	49

$$\begin{array}{r} 100 \\ 70 \\ 70 \\ +\ 49 \\ \hline 289 \end{array}$$

T: ***Now each of you choose two whole numbers and make a multiplication square for them on your paper.***

×	a	b
a		
b		

Give individual assistance as needed. Then collect some of the results in a table on the board. For example:

a	b	Sum of four entries
3	12	225
10	7	289
2	8	100
6	7	169
9	4	169
5	5	100
11	14	625
20	25	2 025
3	4	49
2	3	25

Ask if other students found the same sum as one of those recorded in the table. Ask if their choice of numbers was the same. Notice duplicate sums already in the table. Then pick a four-entry sum from the table, for example, 100, and ask students to find as many possibilities as they can that would yield the same four-entry sum.

Extend the table and record some of the student' s suggestions. For example:

a	b	Sum of four entries
⋮	⋮	
1	9	100
3	7	100
4	6	100
0	10	100

The class should soon discover that pairs of numbers with the same sum have multiplication squares with the same four-entry sum. For example, any two numbers whose sum is 10 have a multiplication square with 100 as the four-entry sum. Refer to the table on the board and ask about patterns.

S: ***The sum of the four entries is always a square number.***

S: ***Add the two numbers* a *and* b *for the multiplication square. Square that number and you get the sum of the four entries. For example, 1 + 9 = 10 and*** $10^2 = 100$***. 9 + 4 = 13 and*** $13^2 = 169$***.***

Add a column to the table and label it **a** + **b**. Complete this column with the class and see that the sum and square rule works in each case.

a	b	Sum of four entries	a + b
3	12	225	15
10	7	289	17
2	8	100	10
6	7	169	13
9	4	169	13
5	5	100	10
11	14	625	25
20	25	2 025	45
3	4	49	7
2	3	25	5
1	9	100	10
3	7	100	10
4	6	100	10
0	10	100	10

T: ***What do you predict the sum of the four entries is in a multiplication square for 17 and 13? Why?***

S: ***900, because 17 + 13 = 30 and*** $30^2 = 900$***.***

×	17	13
17	289	221
13	221	169

$$\begin{array}{r} 289 \\ 221 \\ 221 \\ +\ 169 \\ \hline 900 \end{array}$$

Check with the class that 900 is correct.

T: ***Name two numbers that you know for sure have a multiplication square with a four-entry sum greater than 900.***

S: ***18 and 13. 18 + 13 is greater than 17 + 13, so*** $(18 + 13)^2$ ***is greater than*** $(17 + 13)^2$***.***

Repeat the activity with 9 and 16, asking students to predict the sum of the four entries and then to do the calculations to check their predictions.

T: ***Now let's try two positive numbers that are not whole numbers, for example, $1\frac{1}{4}$ and $2\frac{1}{2}$. What do you predict the sum of the four entries is in a multiplication square for $1\frac{1}{4}$ and $2\frac{1}{2}$?***

S: ***$1\frac{1}{4} + 2\frac{1}{2} = 3\frac{3}{4}$, and $3\frac{3}{4} \times 3\frac{3}{4} = \frac{15}{4} \times \frac{15}{4} = \frac{225}{16} = 14\frac{1}{16}$. So the four-entry sum should be $14\frac{1}{16}$.***

Accept students' predictions without comment except to collectively check the suggested calculations. Complete the multiplication square with the class.

×	$1\frac{1}{4}$	$2\frac{1}{2}$
$1\frac{1}{4}$	$1\frac{9}{16}$	$3\frac{1}{8}$
$2\frac{1}{2}$	$3\frac{1}{8}$	$6\frac{1}{4}$

T: ***We need to add the four entries. Let's first add $3\frac{1}{8} + 3\frac{1}{8}$.***

S: ***$3\frac{1}{8} + 3\frac{1}{8} = 6\frac{2}{8} = 6\frac{1}{4}$.***

T: ***Now let's add $6\frac{1}{4} + 6\frac{1}{4}$.***

S: ***$6\frac{1}{4} + 6\frac{1}{4} = 12\frac{2}{4} = 12\frac{1}{2}$.***

T: ***Now let's add $12\frac{1}{2} + 1\frac{9}{16}$.***

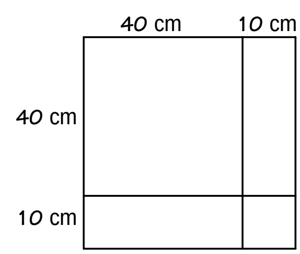
$$\begin{aligned} 12\tfrac{1}{2} + 1\tfrac{9}{16} &= (12 + 1) + \left(\tfrac{1}{2} + \tfrac{9}{16}\right) \\ &= 13 + \left(\tfrac{8}{16} + \tfrac{9}{16}\right) \\ &= 13 + \tfrac{17}{16} = 14\tfrac{1}{16} \end{aligned}$$

Write the problem on the board and ask students to tell you how to complete the calculations.

T: ***Again the rule works. We add the two numbers: $1\frac{1}{4} + 2\frac{1}{2} = 3\frac{3}{4}$. Then we square that result to find the sum of the four entries: $3\frac{3}{4} \times 3\frac{3}{4} = 14\frac{1}{16}$.***

Exercise 2

Draw this square picture accurately on the board.

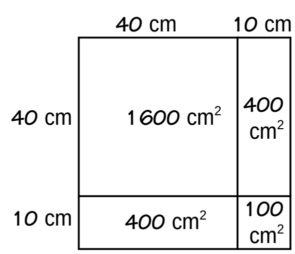


T: ***What is the area of the whole large square? How can you find the answer?***

S: ***2 500cm^2; I multiplied 50 times itself since each side of the large square is 50 cm long.***

S: ***2 500 cm^2; I added the areas of the four regions inside the large square.***

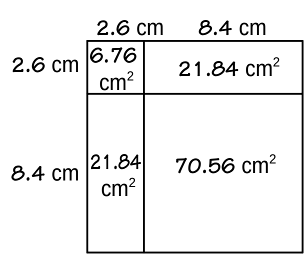
Invite students to record the areas of the four regions in the picture.



$$50 \times 50 = 2\,500$$

$$\begin{array}{r} 1\,600 \\ 400 \\ 400 \\ +\ 100 \\ \hline 2\,500 \end{array}$$

Repeat the activity to describe two ways of finding the area of another large square. Let students use calculators for the multiplication calculations.



$$2.6 + 8.4 = 11$$
$$11 \times 11 = 121$$

$$\begin{array}{r} 6.76 \\ 21.84 \\ 21.84 \\ +\ 70.56 \\ \hline 121.00 \end{array}$$

T: ***Do you notice similarities between these two problems and the multiplication squares?***

The connection should be obvious, since the area problems look like the multiplication squares.

S: ***So the rule we found with multiplication squares is not coincidental. Why it works is clear when you look at area pictures.***

Exercise 3

Draw a new multiplication square on the board as you explain,

×	8	$\widehat{5}$
8		
$\widehat{5}$		

T: ***Let's return to multiplication squares, but this time let's choose a positive number and a negative number. Because of the negative number, it will not help us to think of area.***

What number is 8 x ***8?*** (64)
What number is 8 x ***$\widehat{5}$?*** ($\widehat{40}$; the sum of eight $\widehat{5}$s is $\widehat{40}$)
What number is $\widehat{5}$ x ***8?*** ($\widehat{40}$; $\widehat{5}$ x 8 = 8 x $\widehat{5}$ = $\widehat{40}$)

Make these three entries.

×	8	$\widehat{5}$
8	64	$\widehat{40}$
$\widehat{5}$	$\widehat{40}$	

T: ***We still have one entry to make for $\widehat{5}$*** x ***$\widehat{5}$. What number do you think $\widehat{5}$*** x ***$\widehat{5}$ is? Why?***

There might be some disagreement as to whether the answer is 25 or $\widehat{25}$.

T: ***Most of you think $\widehat{5}$*** x ***$\widehat{5}$ is either $\widehat{25}$ or 25. What would you expect the sum of the four entries in this multiplication square to be?***

S: ***9; 8 + $\widehat{5}$ = 3 and 3^2 = 9.***

T: ***Let's see what sum we have so far in the square. What number is $\widehat{40}$ + $\widehat{40}$?*** ($\widehat{80}$)
And $\widehat{80}$ + 64? ($\widehat{16}$)

Record additions on the board.

$$\widehat{40} + \widehat{40} + 64 = \widehat{16}$$
$$\widehat{16} + \boxed{} = 9$$

T: ***Now what would the fourth entry have to be to get a four-entry sum of 9?***

S: ***25; $\widehat{16}$ + 25 = 9.***

Enter 25 in the number sentence and in the multiplication square. Repeat the activity with 17 and $\widehat{12}$.

×	17	$\widehat{12}$
17	289	$\widehat{204}$
$\widehat{12}$	$\widehat{204}$	144

$$17 + \widehat{12} = 5$$
$$5 \times 5 = 25$$
$$289 + \widehat{204} + \widehat{204} = \widehat{119}$$
$$\widehat{119} + \boxed{144} = 25$$

Invite students to pose other numbers for a multiplication square, and then let the class complete the calculations.

Note: This activity does not prove that a negative number times a negative number is positive, but it does provide another opportunity to let the extension of patterns suggest this rule.

Capsule Lesson Summary

Build arrow roads with each arrow for +, –, x, or ÷ some whole number from 1 to 9. Revisit the idea of one-digit distance between integers. Solve a detective story in which clues involve prime numbers and the one-digit distance between numbers.

Materials

Teacher		Student	
	• Colored chalk		• Paper • Colored pencils, pens, or crayons

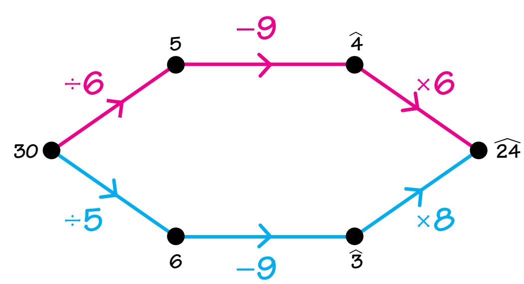
Description of Lesson

Exercise 1

Draw a dot for 30 and a dot for $\widehat{24}$ on the board. Separate the dots to allow several arrows to be drawn between them. (See the next illustration.)

T: ***On your paper, build an arrow road from 30 to $\widehat{24}$ using only arrows for +, –, x, or ÷ some whole number from 1 to 9. You may use more than one type of arrow, but in this situation only integers are allowed at the dots.***

Let students work individually or with partners for a few minutes. Students who find at least one appropriate road can be asked to find a shortest road. Invite several students to put their shortest roads on the board.



T: ***What is the one-digit distance between 30 and 24?***

S: ***3, because a shortest road from 30 to $\widehat{24}$ has three arrows and a shortest road from $\widehat{24}$ to 30 has three arrows also.***

Record the distance on the board.

$$d_1(30, \widehat{24}) = 3$$

Pose several one-digit distance problems, such as the ones listed here, for students to solve individually or with partners.

T: ***For each problem, find the one-digit distance by drawing a shortest road between the two numbers.***

Note: The last two problems here are more difficult.

$$d_1(\widehat{10}, 63) =$$

$$d_1(17, 175) =$$

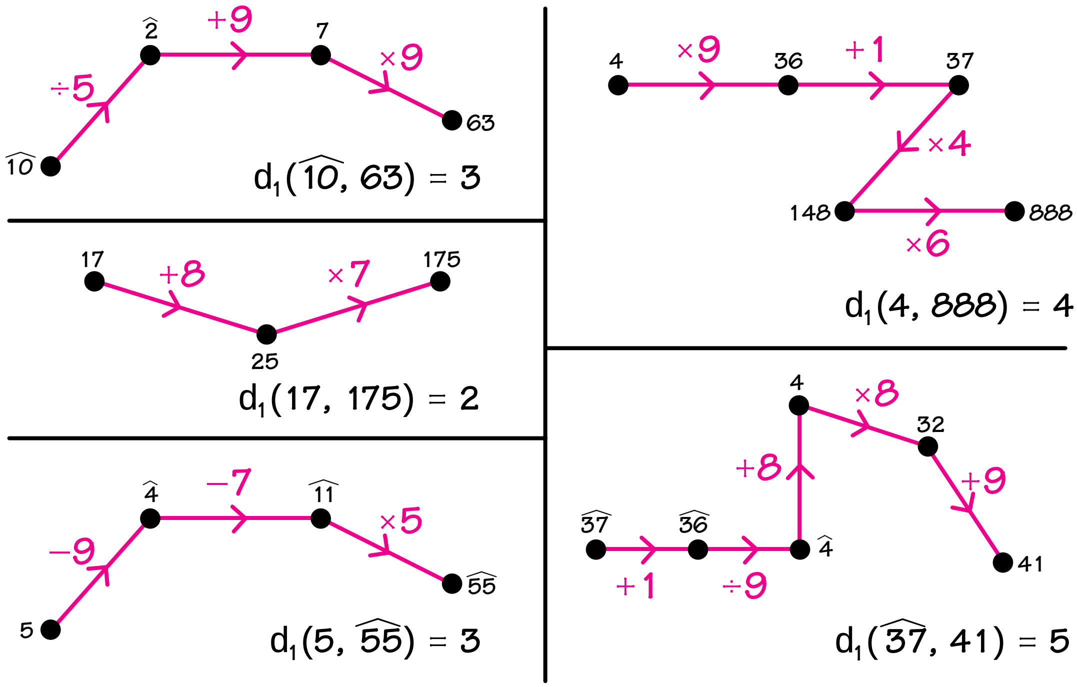
$$d_1(5, \widehat{55}) =$$

$$d_1(4, 888) =$$

$$d_1(\widehat{37}, 41) =$$

N 8

When most students have solved at least two problems, invite students with shortest roads to draw them on the board. The one-digit distance and a shortest road for each problem are given below. Many other shortest roads are possible.



Exercise 2

Present the following detective story about a secret whole number named Kip.

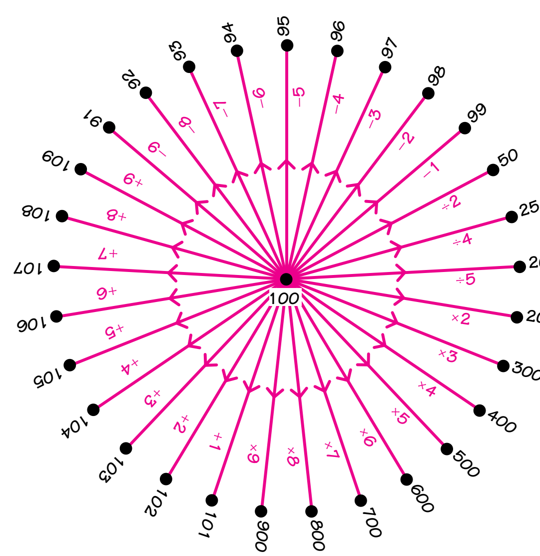
Clue 1

Write this information on the board.

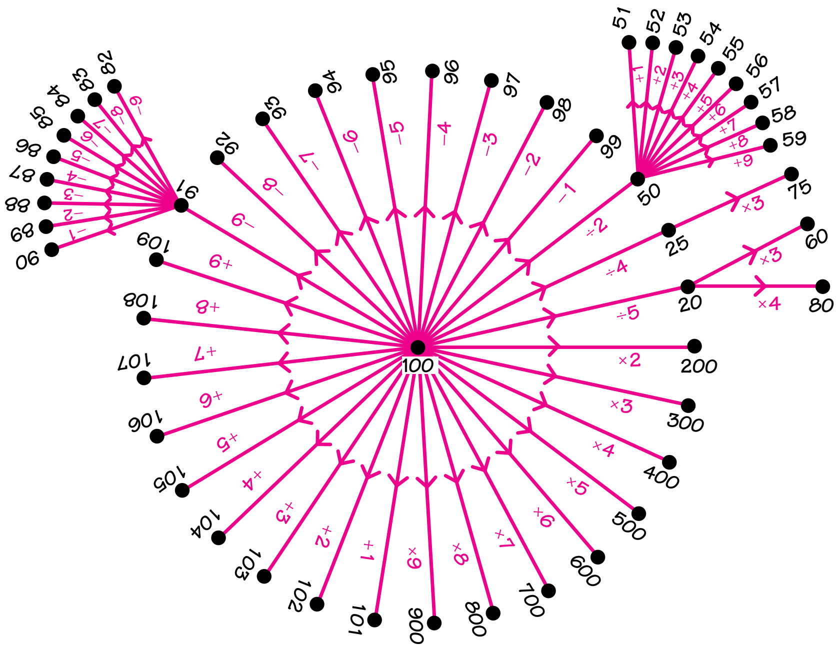
$$50 < \textbf{Kip} < 100$$
$$d_1(100, \textbf{Kip}) = 2$$

T: ***Kip is a whole number between 50 and 100. The one-digit distance between 100 and Kip is 2. Which numbers could Kip be?***

Let students work on the problem individually or with partners for a few minutes; then discuss the clue collectively. Help students develop a systematic method of finding numbers for Kip by first listing all of the numbers at a distance of 1 from 100.



Once the numbers at a one-digit distance of 1 from 100 have been found, then whole numbers at a distance of 2 from 100 (between 50 and 100) can be determined more readily. One possible complete solution is shown below.



The students should conclude that Kip could be one of the numbers in the arrow picture at the end of a two-arrow road starting at 100.

Note: In some cases there are other ways to find a two-arrow road from 100 to one of the numbers shown above. For example, $100 \xrightarrow{+4} 104 \xrightarrow{\div 2} 52$ is another road to get 52.

Clue 2

You may like to list the 21 possibilities for Kip in order on the board.

T: ***The second clue is that Kip is a prime number. Which numbers could Kip be?***

Let students eliminate from the list any number that is not prime. This activity should go rather quickly. Discuss any number that is disputed. The class should conclude that Kip could be 53, 59, 83, or 89.

Clue 3

Write this number sentence on the board.

$$d_1(\text{Kip}, 30) \neq d_1(\text{Kip}, 46)$$

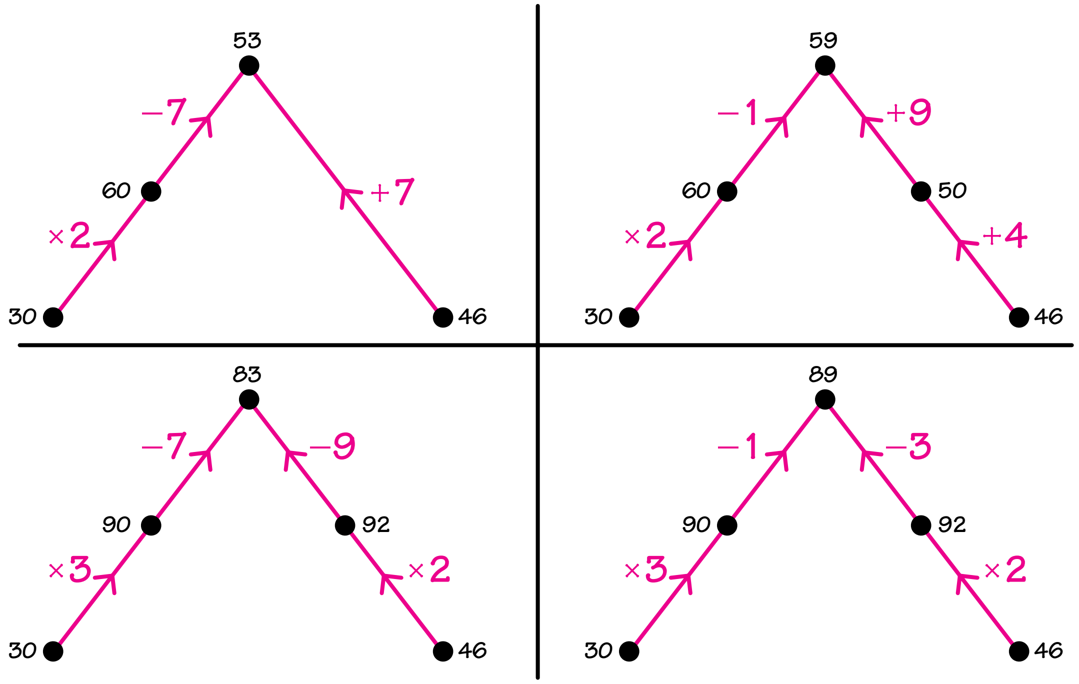
T: ***What new information does this clue give us about Kip?***

S: ***The one-digit distance between Kip and 30 is different that the one-digit distance between Kip and 46.***

T: ***Can you find which number is Kip?***

N 8

Let students work on this problem for several minutes. When most students have finished, invite several of them to draw their roads on the board.



The students should conclude that Kip is 53.

N9 DECIMALS #1

Capsule Lesson Summary

Write decimal names for fractions, and vice versa. Use estimation to place the decimal point in the product of two decimal numbers. Practice multiplying decimal numbers and look for a rule for placing the decimal point in the product of two decimal numbers.

Materials

Teacher	• None	**Student**	• Paper • Worksheets N9* and **

Description of Lesson

Exercise 1

Write these fractions on the board and ask students to find the equivalent decimal names. (Answers are in boxes.)

$$\frac{12}{3} = \boxed{4} \qquad \frac{24}{4} = \boxed{6} \qquad \frac{13}{2} = \boxed{6.5}$$

T: ***What calculations do these equations suggest?***

S: ***Division calculations: 12 ÷ 3 = 4; 24 ÷ 4 = 6; and 13 ÷ 2 = 6.5.***

S: ***Multiplication calculations: 4 x 3 = 12; 6 x 4 = 24; and 2 x 6.5 = 13.***

T: ***What is a decimal name for $^{6}/_{10}$?***

S: ***0.6, since 6 ÷ 10 = 0.6.***

$$\frac{6}{10} = 6 \div 10 = 0.6$$

Remind students that one way to determine the decimal name for a fraction is to divide the numerator by the denominator. Observe also that $^{13}/_{2}$ = 13 ÷ 2 = 6.5.

Ask students to first express each of the following fractions as a division calculation and then to find a decimal name. (Answers are in boxes.)

$$\frac{316}{10} = \boxed{316 \div 10 = 31.6} \qquad \frac{48}{10} = \boxed{48 \div 10 = 4.8}$$

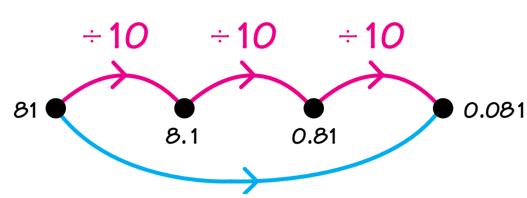
$$\frac{6}{100} = \boxed{6 \div 100 = 0.06} \qquad \frac{48}{100} = \boxed{48 \div 100 = 0.48}$$

$$\frac{6}{1\,000} = \boxed{6 \div 1\,000 = 0.006} \qquad \frac{81}{1\,000} = \boxed{81 \div 1\,000 = 0.081}$$

You may wish to suggest that students use patterns to do calculations like 81 ÷ 1 000; for example:

$81 \div 10 = 8.1$
$81 \div 100 = 0.81$
$81 \div 1000 = 0.081$

or



N 9

T: ***What is a fraction for 1.7?***

S: ***$^{17}/_{10}$, because $^{17}/_{10}$ = 17 ÷ 10 = 1.7.***

S: ***1.7 = $1^{7}/_{10}$.***

For students who have difficulty, write 1.7 = 17 ÷ □ = $^{17}/_{\square}$ on the board and ask them to fill in the boxes with the same number.

Similarly, ask students to find fractions equivalent to the following decimals. Emphasize the corresponding division calculations. (Answers are in boxes.)

$$0.8 = \boxed{\frac{8}{10}} \qquad 0.08 = \boxed{\frac{8}{100}} \qquad 4.3 = \boxed{\frac{43}{10}}$$

$$28.39 = \boxed{\frac{2\,839}{100}} \qquad 0.987 = \boxed{\frac{987}{1000}} \qquad 0.043 = \boxed{\frac{43}{1000}}$$

T: ***How do you know whether the denominator could be 10 or 100 or 1 000 or some other number?***

S: ***It depends on the division calculation you use.***

S: ***I see a pattern. You can always find a fraction whose denominator is 10 or 100 or 1 000 and so on. The number of digits to the right of the decimal point is the same as the number of zeroes in that denominator.***

Check this pattern by asking students to give the decimal name for fractions with such a denominator, or vice versa. (Answers are in boxes.)

$$\frac{73}{10} = \boxed{7.3} \qquad 0.42 = \boxed{\frac{42}{100}}$$

$$\frac{73}{100} = \boxed{0.73} \qquad 17.1 = \boxed{\frac{171}{10}}$$

$$\frac{73}{1000} = \boxed{0.073} \qquad 4.016 = \boxed{\frac{4\,016}{1000}}$$

Exercise 2

Write this information on the board.

$$4.26 \times 7.48 = 3\ 1\ 8\ 6\ 4\ 8$$
$$106.3 \times 3.7 = 3\ 9\ 3\ 3\ 1$$
$$468.306 \times 0.95 = 4\ 4\ 4\ 8\ 9\ 0\ 7$$
$$72.8 \times 0.46 = 3\ 3\ 4\ 8\ 8$$

T: ***These calculations were correct, but now the decimal point in each number on the right has been erased. Where should the missing decimal points be?***

Encourage students to use estimation in placing decimal points, but do not insist on a particular method of estimation. For example:

S: ***4.26* x *7.48 = 31.8648. 4* x *7 = 28 and 5* x *8 = 40 so the product of 4.26* x *7.48 is between 28 and 40.***

S: ***106.3* x *3.7 = 393.31. 106.3 is close to 100 and 3.7 is close to 4. 100* x *4 = 400 and 393.31 is close to 400.***

S: ***468.306* x *0.95 = 444.8907. 0.95 is close to 1 so this product should be close to 468.***

S: ***72.8* x *0.46 = 33.488. 0.46 is close to ½ so the product should be close to ½* x *72.8, or 36.4.***

Write this information on the board. $156 \times 329 = 51.324$

T: ***This time decimal points were erased from both or one of the numbers on the left. Where could we place decimal points to make a correct calculation?***

S: ***15.6* x *3.29 = 51.324, because 16* x *3 = 48 and 48 is close to 51.324.***

S: ***1.56* x *32.9 = 51.324, because 1.5* x *30 = 45 and 45 is close to 51.324.***

S: ***0.156* x *329 = 51.324, because 0.2* x *300 = 60 and 60 is close to 51.324.***

S: ***156* x *0.329 = 51.324, because 150* x *0.3 = 45 and 45 is close to 51.324.***

Write this problem on the board. $1.4 \times 0.3 =$

T: ***One way to do this calculation is to first ignore the decimal points and simply calculate 14* x *3. What number is 14* x *3?*** (42)

Add this information to the calculation. $1.4 \times 0.3 = 42$

T: ***Now we know that 42 has the correct digits, but we must place the decimal point. Where does the decimal point go?***

S: ***0.42. 1.4* x *0.3 must be less than 1.4 since 0.3 is less than 1.***

S: ***0.42. 0.3 is about ⅓ and 1.4 is close to 1.5. If we think about money, one third of $1.50 is $0.50. So ⅓* x *1.5 = 0.5 and that is close to 0.42.***

S: ***3* x *1.4 = 4.2, so 0.3* x *1.4 = 0.42.***

Write the following problems on the board. Ask students to copy and do the calculations. (Answers are in boxes.) Encourage the technique of first multiplying whole numbers and then placing the decimal point with estimation or patterns.

$$6 \times 0.8 = \boxed{4.8}$$
$$42 \times 0.2 = \boxed{8.4}$$
$$3.2 \times 0.3 = \boxed{0.96}$$

$$\begin{array}{r} 2.06 \\ \times\ 6.9 \\ \hline \boxed{14.214} \end{array} \qquad \begin{array}{r} 7.8 \\ \times\ 3.6 \\ \hline \boxed{28.08} \end{array}$$

After a while, invite students to present and explain their solutions. Accept a variety of explanations for determining where to place the decimal point in each product, for example:

S: ***6*** x ***8 = 48, so 6*** x ***0.8 = 4.8.***

S: ***42*** x ***1 = 42 and 42*** x ***0.1 = 4.2, so 42*** x ***0.2 = 8.4.***

S: ***3.2*** x ***3 = 9.6, so 3.2*** x ***0.3 = 0.96.***

S: ***206*** x ***69 = 14 214. 2.06 is near 2; 6.9 is near 7; and 2*** x ***7 = 14. So 2.06*** x ***6.9 = 14.214.***

Suggest that students study the five problems and look for a rule for placing the decimal point in the product of decimal numbers.

T: ***Look at the number of digits to the right of the decimal point in each product. Sometimes it would be nice if we could predict the number of decimal places in the product without using estimation or patterns. By looking at the numbers being multiplied in each problem, can you predict the number of decimal places in the product?***

S: ***The number of digits to the right of the decimal point in the product equals the sum of the number of decimal places in the two numbers being multiplied. For example, 2.06 has two decimal places, 6.9 has one decimal place, and their product, 14.214, has three decimal places.***

Let students confirm that this rule works for each of the other four problems.

Write these problems on the board. Point out that 372 x 16 = 5 952 is correct. Then ask students to find the other products. Remind them to check the rule. (Answers are in boxes.)

$$
\begin{aligned}
372 \times 16 &= 5\,952 \\
3.72 \times 16 &= \boxed{59.52} \\
372 \times 1.6 &= \boxed{595.2} \\
0.372 \times 1.6 &= \boxed{0.5952} \\
0.372 \times 0.16 &= \boxed{0.05952} \\
0.0372 \times 0.0016 &= \boxed{0.00005952}
\end{aligned}
$$

Many students may find the last two problems difficult.

S: ***0.372 has three decimal places and 0.16 has two decimal places. So the product needs five decimal places, but 5 952 has only four digits.***

T: ***We know from the previous problem that 0.372*** x ***1.6 = 0.5952. Should 0.372*** x ***0.16 be more or less than 0.5952?***

S: ***Less.***

S: ***In fact, since 1.6 ÷ 10 = 0.16, the result is 0.5952 ÷ 10 or 0.05952.***

S: ***And 0.05952 does have five decimal places.***

Worksheets N9* and ** are available for individual work.

Home Activity

Make a sheet of problems similar to those on Worksheet N9* for students to take home. Or, suggest that students make their own page of such problems to challenge a family member.

Name ________________ N9 *

One number in each number sentence is missing a decimal point.
Put a decimal point in this number to make the calculation correct.

1.26 × 26.98 = 3 3.9 9 4 8

17.35 × 0.5 = 8.6 7 5

9.84 × 6.95 = 6 8.3 8 8

4 2.7 4 × 2.29 = 97.8746

19.638 × 7 2.0 7 = 1415.31066

9.1 7 5 × 7.6 = 69.73

Name ________________ N9 **

Write a decimal name for each fraction.

Example: $\frac{23}{10} = 23 \div 10 = 2.3$

$\frac{9}{10} =$ 0.9 $\frac{9}{100} =$ 0.09

$\frac{85}{10} =$ 8.5 $\frac{85}{100} =$ 0.85

$\frac{436}{10} =$ 43.6 $\frac{436}{100} =$ 4.36

Multiply. Show your work in the space provided.

7.19	0.074
×8.6	×0.48
4.314	0.00592
57.520	0.02960
61.834	0.03552

There are other ways to show work. Students may multiply with whole numbers and then use estimation (or counting decimal places) to place the decimal point in the result.

Capsule Lesson Summary

Estimate and then calculate the percent of whole numbers less than 100 with certain characteristics. Review the definition of □% of as a composition of ÷100 and □x. Use arrow pictures to find different names for percents such as 12%, and to solve problems such as 12% of 75 = $\boxed{9}$ and 12% of $\boxed{175}$ = 21 . Write story problems using percents.

Materials

Teacher	• Colored chalk • Blacklines N10(a) and (b)	**Student**	• Paper • Colored pencils, pens, or crayons • Percent estimation problems • 0–99 numeral chart

Advance Preparation: Use Blackline N10(a) to make copies of the percent estimation problems; use Blackline N10(b) to make copies of a 0–99 numeral chart for students.

Description of Lesson

Exercise 1

Refer to a number line or other number display as you ask,

T: ***How many whole numbers less than 100 are there?*** (100)
What percent of the whole numbers less than 100 are multiples of 10?

S: ***10%.***

T: ***How did you determine that 10% of the whole numbers less than 100 are multiples of 10?***

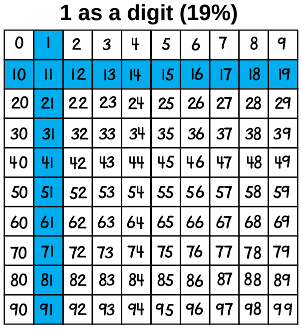
S: ***There are 10 multiples of 10 (0, 10, 20, … 90) less than 100, and 10 out of 100 is 10%.***

Distribute copies of Blackline N10(a). Direct students to work with a partner to estimate percents as indicated. Do not give students a 0–99 numeral chart yet, though some may request it. Suggest that at the moment they should be thinking about an estimate.

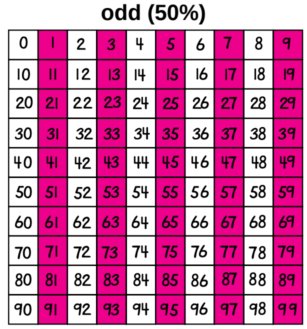
N 10

When many students have completed the page of estimates, distribute copies of the 0–99 numeral chart. Instruct partners to use the chart to help calculate the actual percents and to compare these to their estimates. You may like to discuss some of the problems collectively.

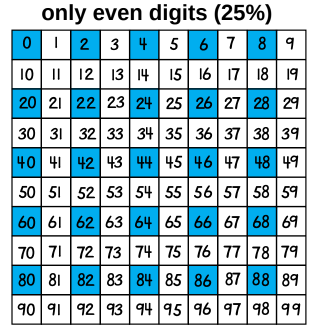
1 as a digit (19%)



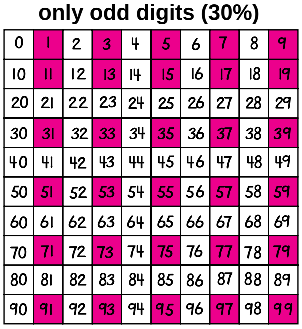
odd (50%)



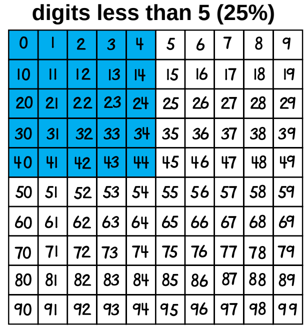
only even digits (25%)



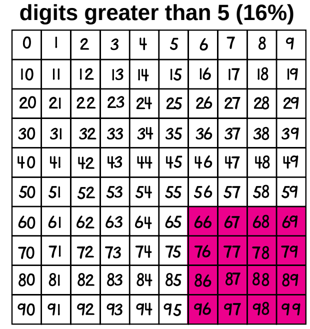
only odd digits (30%)



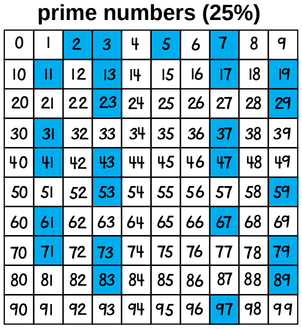
digits less than 5 (25%)



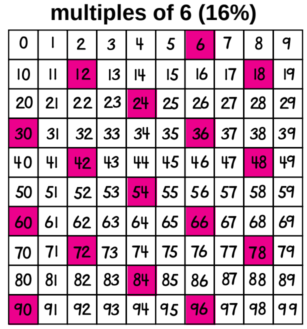
digits greater than 5 (16%)



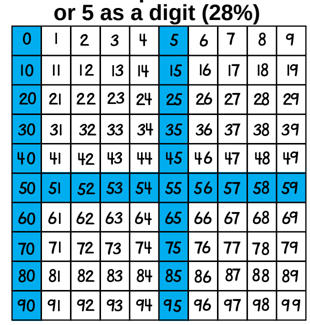
prime numbers (25%)



multiples of 6 (16%)

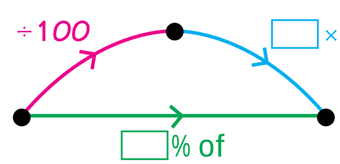


multiples of 5 or 5 as a digit (28%)



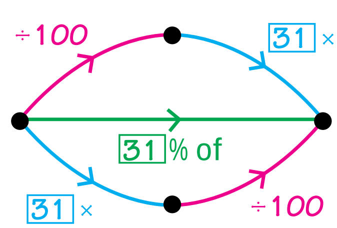
Exercise 2

Draw this arrow picture on the board.



T: ***Remember how we used a composition of two arrows to define percent? With the red arrow for ÷100, the same number goes in each box. For example, if the green arrow is for "31% of," then the blue arrow is 31*x*.***

Label the arrows, and point out that the red and blue arrows could be reversed. So, 31% of is 31 x followed by ÷100, as well as ÷100 followed by 31x.

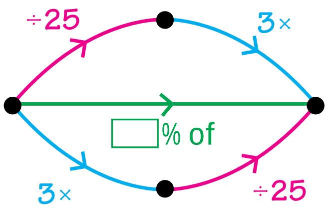


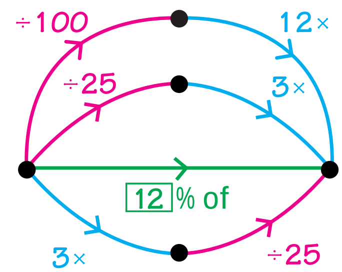
Erase the numbers in the boxes and ask,

T: ***If the blue arrows are for 73x, then what is the green arrow?*** (73% of)
If the green arrows is for 19% of, then what is the blue arrow? (19x)
If the green arrow is for 115% of, then what is the blue arrow? (115x)

Erase the arrow labels in the picture and relabel as shown here.

T: ***How should we fill in the box for the green arrow?***

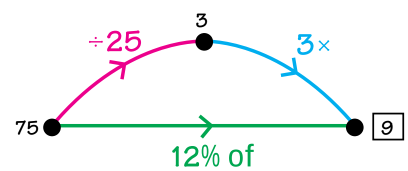




Students should observe that another composition equivalent to ÷25 followed by 3x is ÷100 followed by 12x. They may consider that the green arrow is $\frac{3}{25}$x and $\frac{3}{25} = \frac{12}{100}$. From this composition one finds that the green arrow is for “12% of.”

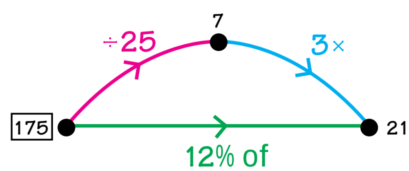
Put a number at the dot on the left, and use the arrow picture to calculate 12% of the number. For example:

12% of 75 = $\boxed{9}$



Put a number at the dot on the right, and use the arrow picture to calculate 12% of what number equals the given number. For example:

12% of $\boxed{175}$ = 21



Repeat this activity with other compositions and percents. For example:

- $\frac{7}{20}$x $= \frac{35}{100}$x = 35%

  35% of 140 = $\boxed{49}$
  35% of $\boxed{160}$ = 56

- $\frac{5}{4}$x $= \frac{125}{100}$x = 125%

  125% of 92 = $\boxed{115}$
  125% of $\boxed{60}$ = 75

Exercise 3

Begin this exercise by asking the class for situations when they might use percents. Then direct students to choose a situation of interest to them and to write their own percent problems for that situation.

You may like to impose some criteria on the story problems students write. For example:

- The problem must have a question or an implied question, and the answer must require percents.
- Appropriate information must be given in the statement of the problem.
- The student must be able to solve his or her own problem.

After a while, let students exchange story problems and solve each other’s problems. Students can then compare their solutions and, if necessary, explain their methods to each other.

You may want to collect the story problems written by students as they are likely to give you information about what your students understand about percent.

N11 MINICOMPUTER GOLF WITH DECIMALS #1

Capsule Lesson Summary

Put some decimal numbers on the Minicomputer using a ③-checker, a ④-checker, and a ⑤-checker. Put numbers on the Minicomputer using exactly two of the weighted checkers ②, ③, ④, …, ⑨. Moving exactly one checker, change a number on the Minicomputer by a specified amount. Play Minicomputer Golf with decimals.

Materials

Teacher
- Minicomputer set
- Weighted checkers
- Colored chalk

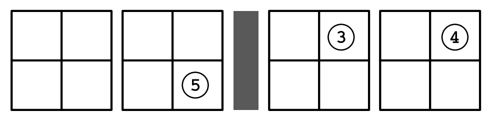
Student
- Paper
- Minicomputer sheet (optional)
- Worksheets N11*, and **

Advance Preparation: You may like to make copies of Blackline N11 for students' use in Exercise 1.

Description of Lesson

Exercise 1

Display four Minicomputer boards and put on this configuration of checkers.



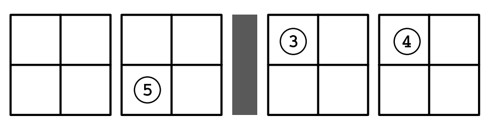
T: ***What number is on the Minicomputer?***

S: ***6.36.***

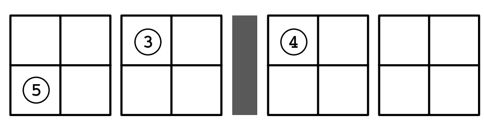
T: ***Who can put 12.72 on the Minicomputer using the same three checkers?***

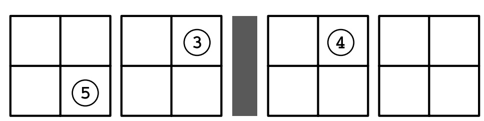
S: ***12.72 = 2 x 6.36, so move each checker to the square with twice the value; that doubles the number.***

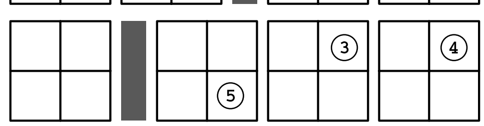
Invite a student to move the checkers to this configuration.


= 12.72

Continue this activity by asking different students to put the numbers suggested here on the Minicomputer using only a ③-checker, a ④-checker, and a ⑤-checker. Emphasize the 10x, ½x, and ÷100 functions.


= 127.2
(10 × 12.72)

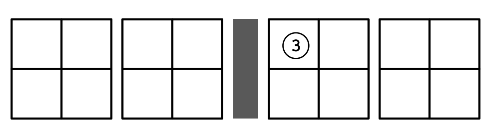

= 63.6
($\frac{1}{2}$ × 127.2)


= 0.636
(63.6 ÷ 100)

Remove all of the checkers from the Minicomputer and display the weighted checkers: ②, ③, ④, ⑤, ⑥, ⑦, ⑧, and ⑨.

T: ***Can you put 10.4 on the Minicomputer using exactly two of these checkers?***

Invite a student with a solution to place one of the checkers. Suppose a ③-checker is placed first.



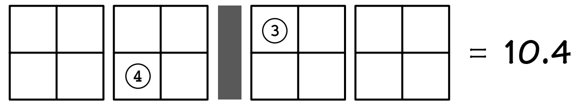
T: ***What number is on the Minicomputer?***

S: ***2.4, because 3* x *0.8 = 2.4.***

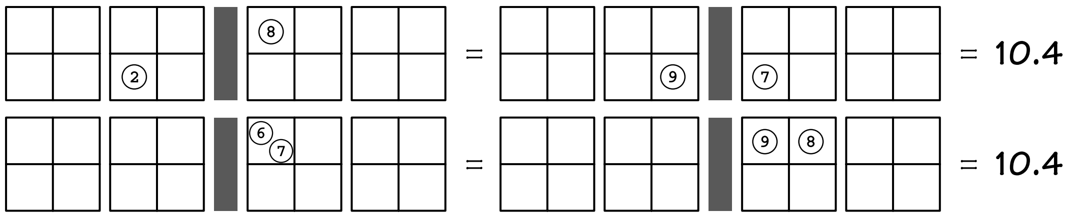
T: ***10.4 is how much more than 2.4?***

S: ***8.***

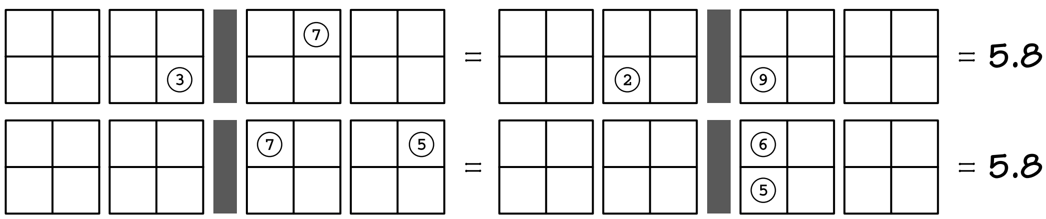
T: ***Who can place the second checker?***



Let students find other solutions. There are many possibilities, including the following:

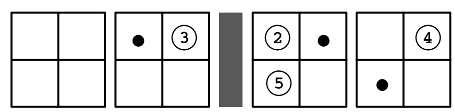


Repeat this activity with 5.8. Several solutions are possible, including the following:



Exercise 2

Display four Minicomputer boards with this configuration of checkers.



T: ***Watch as I move some checkers. Each time I move a checker, tell me if I increase or decrease the number. Also, tell how much more or how much less is the new number.***

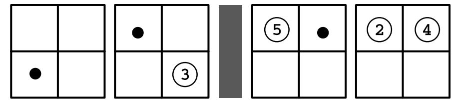
Move the ③-checker from the 4-square to the 1-square.

S: ***A decrease, the new number is 9 less because 12 – 9 = 3 (or 9 less because moving a regular checker would make the number 3 less and 3* x *3 = 9).***

Continue in this manner, making the following moves:

- Move the ⑤-checker from the 0.2-square to the 0.8-square. (An increase of 3; 1 + 3 = 4 or 5 x 0.6 = 3)
- Move the regular checker from the 0.02-square to the 20-square. (An increase of 19.98; 0.02 + 19.98 = 20)
- Move the ②-checker from the 0.8 square to the 0.08-square. (A decrease of 1.44; 1.6 – 1.44 = 0.16, or 2 x 0.72 = 1.44)

After making the above moves, the configuration on the Minicomputer will be as shown here.



T: ***Can you increase the number by 21 by moving a checker?***

S: ***Move the ③-checker from the 1-square to the 8-square.***

Make the move as a student suggests and check with the class that the move is correct. Continue the activity by asking similar questions. Feel free to adjust the level of difficulty of the questions to the numerical abilities of your students.

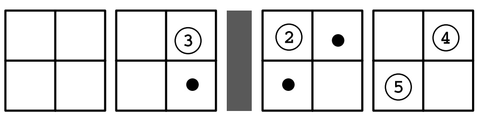
T: ***Can you decrease the number by 0.3?***
(Move the regular checker from the 0.4-square to the 0.1-square; 0.4 – 0.3 = 0.1)

Can you increase the number by 1?
(Move the ⑤-checker from the 0.8 square to the 1-square; 4 + 1 = 5 or 5 x 0.2 = 1.)

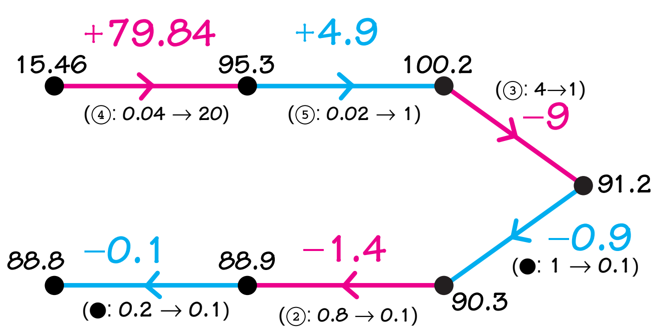
Can you decrease the number by 0.12?
(Move the ②-checker from the 0.08-square to the 0.02-square, or move the ④-checker from the 0.04-square to the 0.01-square; 0.16 – 0.12 = 0.04.)

Exercise 3

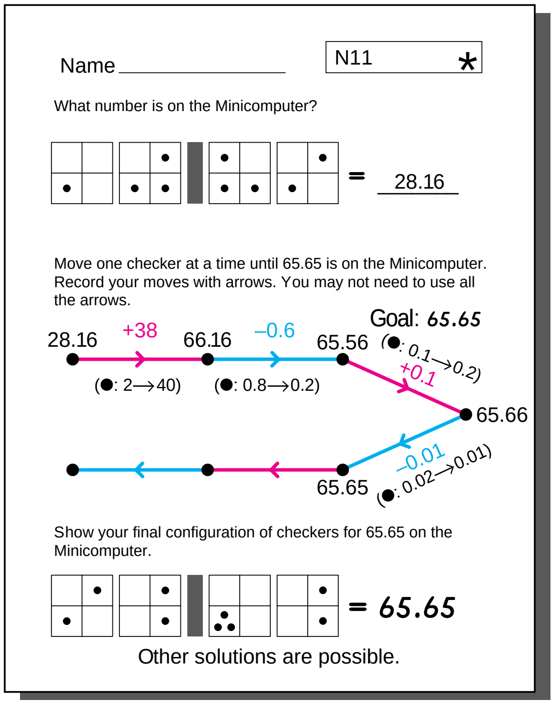
Play Minicomputer Golf. The following is a possible game using a starting configuration for 15.46 and with a goal of 88.8.


= 15.46 Goal: 88.8

You may choose to play the game with two teams; the first team to reach the goal wins the game. Or you may like to play a cooperative game with the class trying to reach the goal with as few moves as possible. The following illustration could be from a cooperative game that was completed in six moves. See the answer key for Worksheet N11** for a shorter solution.



Worksheets N11* and ** are available for individual or small group work. Notice that the situation on N11** is identical to the situation in the game just played in class. Challenge students to find a shorter solution than that found in class.



Name ____________ N11 *

What number is on the Minicomputer?

= 28.16

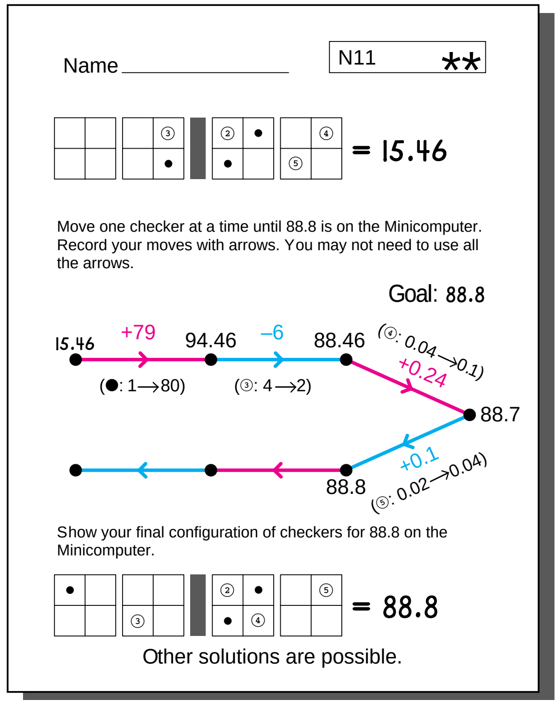
Move one checker at a time until 65.65 is on the Minicomputer. Record your moves with arrows. You may not need to use all the arrows.

Goal: 65.65

Show your final configuration of checkers for 65.65 on the Minicomputer.

= 65.65

Other solutions are possible.



Name ____________ N11 **

= 15.46

Move one checker at a time until 88.8 is on the Minicomputer. Record your moves with arrows. You may not need to use all the arrows.

Goal: 88.8

Show your final configuration of checkers for 88.8 on the Minicomputer.

= 88.8

Other solutions are possible.

Capsule Lesson Summary

Review the binary abacus and binary writing. Label a binary number line. Use binary notation to list some numbers that belong to a given interval of the number line.

Materials

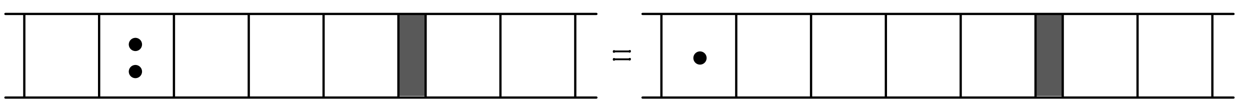
Teacher	• Colored chalk • Minicomputer checkers (optional) • Blackline N4	**Student**	• Paper • Binary abacus • Checkers

Advance Preparation: Use Blackline N4 to make copies of a binary abacus for students.

Description of Lesson

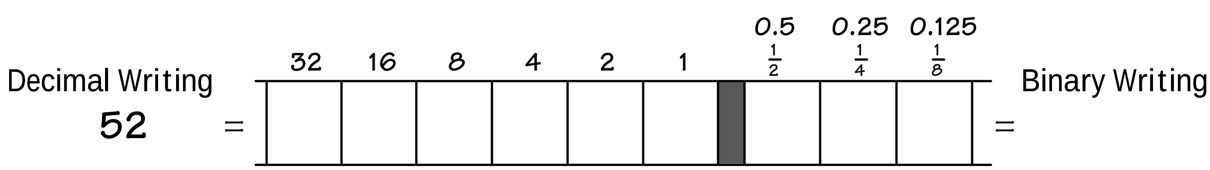
Exercise 1

Draw a binary abacus on the chalkboard and briefly review the rule for the binary abacus.



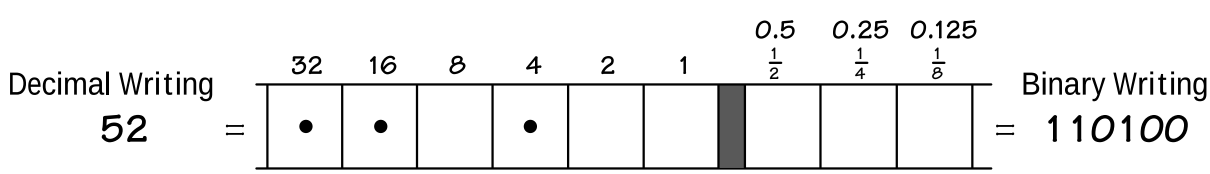
Two checkers on a board
trade for
one checker on the next board to the left.

Label the boards of the abacus and put headings on either side of the it, as shown below.



T: ***On your paper, express this number (52) in binary writing. Use the abacus as an aid.***

If necessary, point out that using the binary abacus means putting on the number with at most one checker on a board. After a couple minutes, ask a student to put 52 on the binary abacus and to write its binary name.



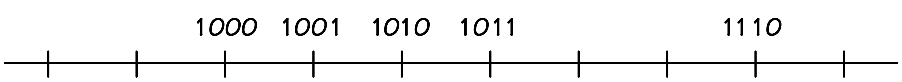
N 12

Continue this activity using some of the problems below. Keep the pace of this activity brisk so that you will have time for the remaining exercises.

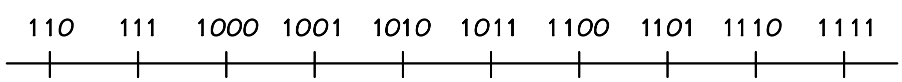
Decimal Writing		32	16	8	4	2	1	0.5 ($\frac{1}{2}$)	0.25 ($\frac{1}{4}$)	0.125 ($\frac{1}{8}$)		Binary Writing
$\boxed{26}$	=		•	•		•					=	11010
13	=			•	•		•				=	$\boxed{1101}$
8.25	=			•					•		=	$\boxed{1000.01}$
$\boxed{5}$	=				•		•				=	101
$\boxed{0.625}$	=							•		•	=	0.101

Exercise 2

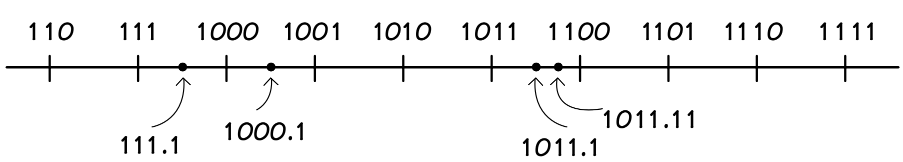
Draw this part of a number line with labels in binary writing on the board.



Invite students to label any mark they wish until all the marks are labeled. The important observation to make is that consecutive marks have labels which differ by 1.

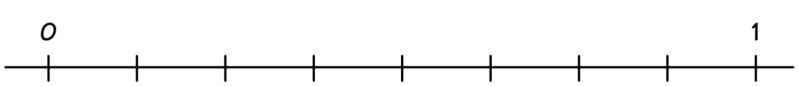


Add the points shown below to your drawing and invite students to label them.



Exercise 3

Draw this part of a binary number line on the board.



T: ***Copy this part of a number line on your paper and label the marks in binary writing.***

Allow several minutes for independent work before collectively labeling the number line. Students may label the marks in any order they wish. Perhaps some will observe that one method involves repeated halving; thus, label the mark halfway between 0 and 1 first, the mark halfway between 0 and 0.1(½) second , the mark halfway between 0 and 0.01 (¼) third, and then label the other marks by adding.

S: ***The mark halfway between* 0 *and* 1 *is for ½, and its binary writing is* 0.1*.***

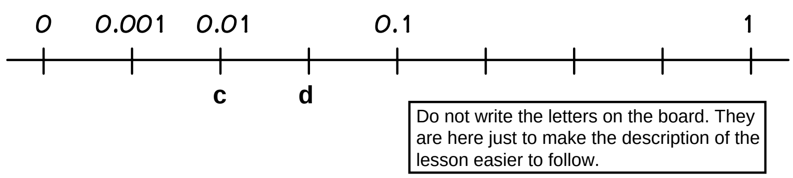
S: ***Halfway between* 0 *and* 0.1 *the mark is for* 0.01*; you just move the digit* 1 *over one place to the right, because that is halving in the binary system.***

S: **0.01 *is the binary writing for ¼ and that mark is one-fourth of the way from* 0 *to* 1*.***

S: ***Halfway between* 0 *and* 0.01 *the mark is for* 0.001*.* 0.001 *is the binary writing for ⅛ and that mark is one-eighth of the way from* 0 *to*1*.***

T: ***The marks on this number line show counting by how much?***

S: ***⅛ or* 0.001*.***

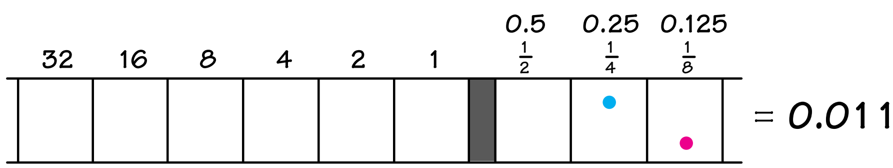


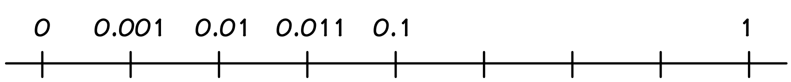
T (pointing to **d**)**:** ***How much more is this number than*** (point to **c**) **0.01*?***

S: **0.001 *more, or ⅛ more.***

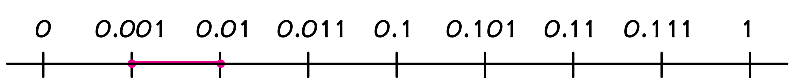
T: ***Calculate* 0.01 + 0.001.**

Encourage students to do this calculation mentally or using the binary abacus.





Continue in this manner until the number line is completely labeled; then indicate a red segment, as shown below.



T: ***Suggest a number in binary writing that belongs to the red segment but is not one of the endpoints.***

N 12

Allow a few minutes for students to find such numbers; then make a list of the numbers suggested. For example:

0.0011
0.0011101
0.00101010101 ...
0.001111

T: ***Do you notice anything interesting about these numbers?***

S: ***Each number starts like* 0.001*.***

S: ***Any number starting with* 0.001 *and followed by as many* 0*s and* 1*s as you wish will belong to the red segment. Also* 0.01 *belongs to the red segment, but it is an endpoint.***

Choose another segment, for example, from **0.101** to **0.11**, and repeat this activity. The class should conclude that any number starting with **0.101** and also the endpoint **0.11** belongs to this segment.

T: ***Where does* 0.1111 *belong on this number line?***

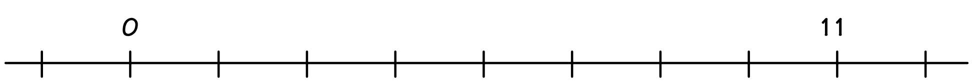
S: **0.1111 *is between* 0.111 *and* 1*.***

Ask students to locate a few more numbers between two given marks on the number line. For example:

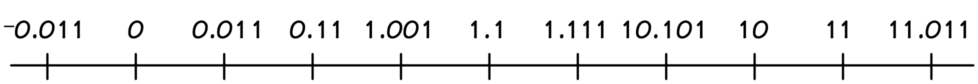
0.0110101	between 0.011 and 0.1
0.11011	between 0.11 and 0.111
0.000101	between 0 and 0.001

If you wish, assign the following problem to be done in class or as homework.

Label this part of a number line in binary writing.



Solution:



Capsule Lesson Summary

Use a Guess My Rule activity to explore a squaring relation: $a \longrightarrow a^2$. Graph this relation and other similar relations, such as $a \longrightarrow (a^2 + 3)$, on the same set of axes.

Materials

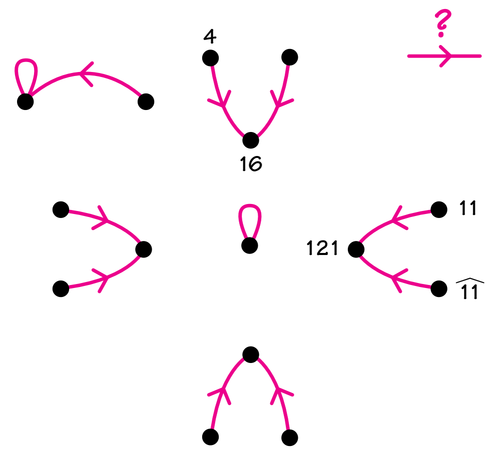
Teacher	• Colored chalk • Coordinate grid • Blackline N13	**Student**	• Paper • Colored pencils, pens, or crayons • Coordinate grid sheets

Advance Preparation: Use Blackline N13 to prepare a coordinate grid for display, or prepare a grid board with this coordinate grid. Make copies of coordinate grid sheets for students.

Description of Lesson

Exercise 1

Draw this arrow picture on the board.



T: ***Can you guess what the secret rule is for the red arrows? There are some hints in this arrow picture. Try to find the rule for red arrows.***

Note: The rule for red arrows is $a \longrightarrow a^2$. That is, the ending number of a red arrow is the square of the starting number.

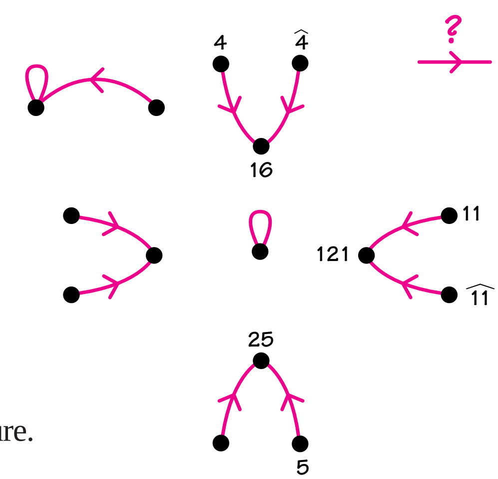
Allow a couple minutes for the class to study the picture, and then invite students to label dots of their choice without yet announcing the rule. When a student labels a dot according to the rule, announce that it is correct. Erase labels that do not follow the rule with the explanation that the label is not consistent with the rule. For example:

S: ***The starting number for this other arrow ending at 16 is $\widehat{4}$.***

T: ***Yes, that fits the rule for the red arrows.***

S (pointing to a red arrow at the bottom of the picture)**:** ***Here the starting number could be 5 when the ending number is 25.***

T: ***Yes, that also fits the rule.***



Note: If a student suggests that a starting number could be 2 and the ending number 4, point out that 4 is already in the picture and, therefore, an arrow starting at 2 would have to end there. You may want to add such an arrow to the picture.

Continue until it is clear that many students know the rule, and ask students to explain the rule to the class.

S: ***The ending number is the square of the starting number.***

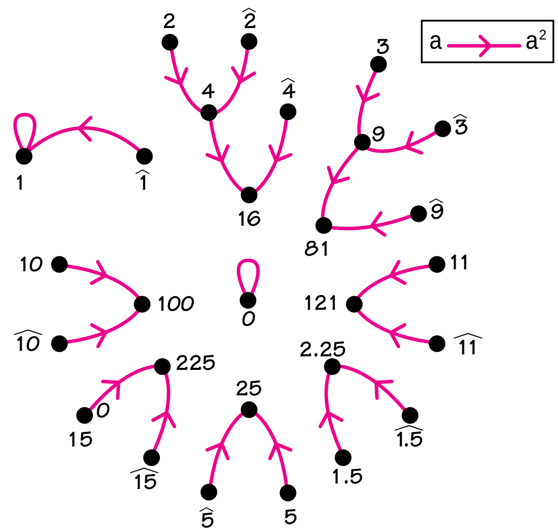
S: ***Multiply the starting number by itself to get the ending number.***

S: ***The red arrows are for "You are my square."***

Note: Avoid saying that the relation is "is the square of." The problem comes in trying to read as you trace an arrow; for example, you don' t want to say, "5 is the square of 25."

You may need to remind students that a negative number times a negative number is a positive number. They should notice that for any whole number *n*, the squares of *n* and $\hat{n}$ are the same.

Once the rule is established, record it on the board. Continue to label dots until all the dots have labels. You may like to invite students to add more arrows (or pieces) to the picture.

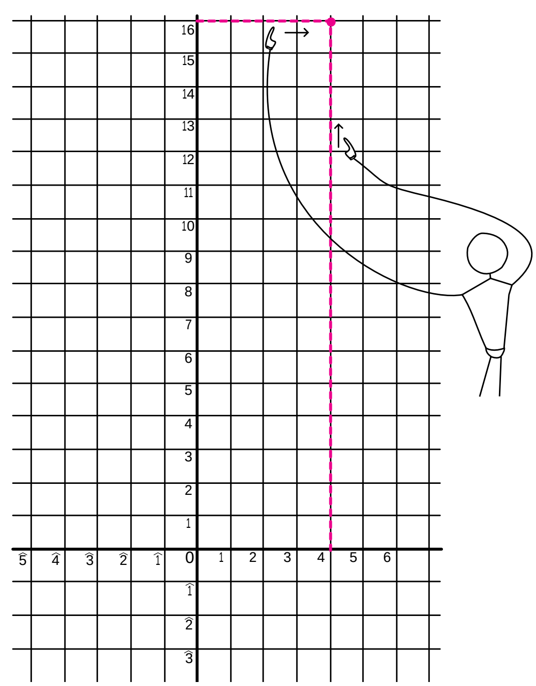


Exercise 2

Display a coordinate grid, as shown here.

T: ***Let's draw the Cartesian graph for the squaring relation. How do we locate the point for this arrow starting at 4 and ending at 16*** (trace the appropriate arrow)***?***

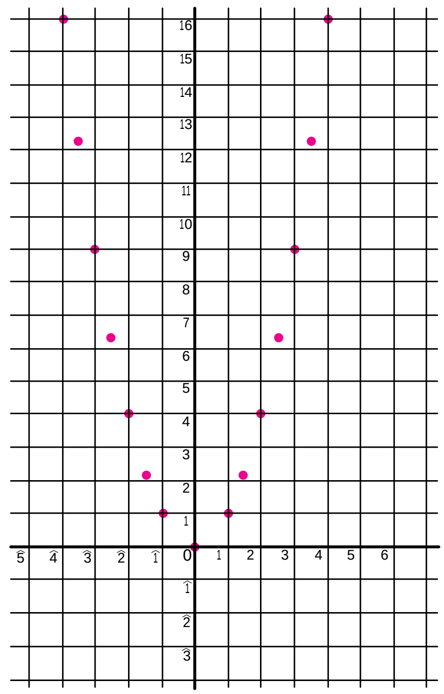
S: ***Find the starting number going across from 0 (on the horizontal axis) and the ending number going up from 0 (on the vertical axis). Then see where those grid lines meet.***



Invite a student to locate the point.

Ask students to locate points for two or three more arrows collectively; for example, (0,0) and ($\hat{3}$, 9). Then instruct students to draw on their grids a red dot for each arrow on the board, if possible within the limits of the axes. Encourage students to find other points on their grids for arrows in the squaring relation that are not in the arrow picture.

After several minutes of independent work, solicit student help to complete the picture at the board.



T: ***Did we draw a dot for every arrow of the squaring relation?***

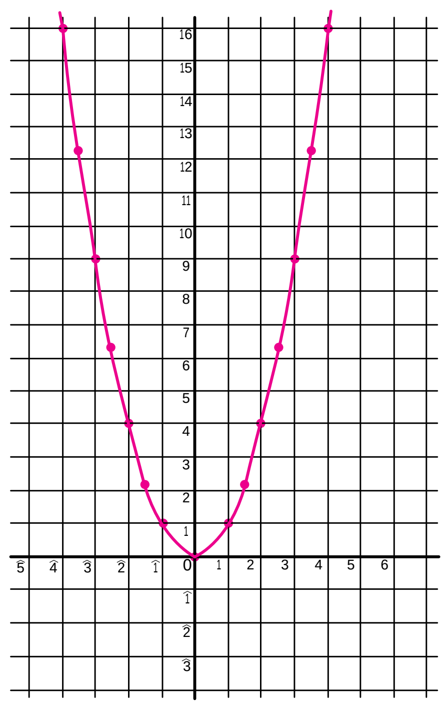
S: ***No, there are many other arrows that could be drawn; for example, an arrow starting at ½ and ending at ¼.***

S: ***There are infinitely many arrows in the squaring relation.***

T: ***Where are there points for other arrows in the squaring relation?***

S: ***Along this curve.***

Draw the curve (parabola) as shown here.



Exercise 3 ______________

Announce that you have a new secret rule, and write these ordered pairs on the board.

$(2, 7)$ $(7, 52)$

$(\hat{4}, 19)$ $(5, 28)$

$(6, \square)$

T: ***Can you guess the rule for these ordered pairs? Which number could go in the box?***

Note: The rule is that the second component is 3 more than the square of the first component. The number in the box is 39 because $6^2 + 3 = 39$.

N 13

Suggest that students write their guesses on paper for you to check. After a while, let a student who guesses correctly tell the class that 39 is in the box, but do not announce the rule yet.

Continue this activity with several more ordered pairs. (Answers are in boxes.)

$(3, \boxed{12})$ $(\hat{3}, \boxed{12})$ $(10, \boxed{103})$

$(\hat{2}, \boxed{7})$ $(12, \boxed{147})$ $(1.2, \boxed{4.44})$

T: ***Who can explain the rule?***

S: ***It is similar to the squaring relation except 3 is added to the square of the first number to get the second number.***

S: ***Start with a number, square it, then add 3 to the result.***

Summarize the rule on the board.

$$a \longrightarrow a^2 + 3$$

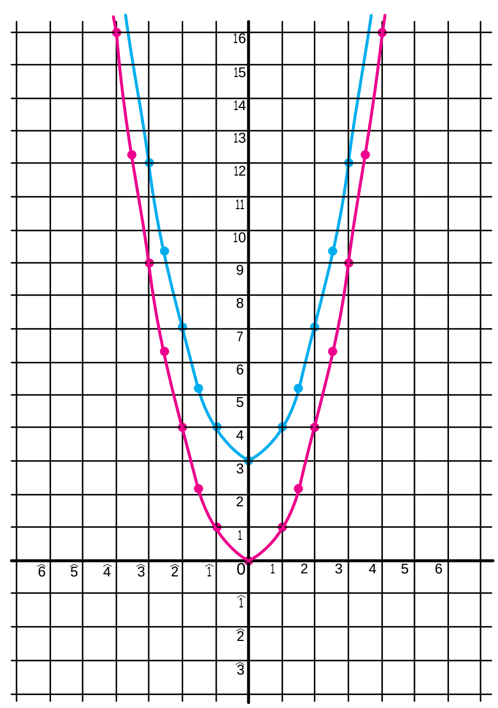
Ask students to complete several more ordered pairs using this rule. Here boxes indicate what is to be filled in. The class should notice that there are two solutions to some problems because the square of a number is the same as the square of its opposite.

$(\boxed{8}, 67)$ or $(\boxed{\hat{8}}, 67)$ $(4, \boxed{19})$ $(\boxed{1}, 4)$ or $(\boxed{\hat{1}}, 4)$

$(1.5, \boxed{5.25})$ $(\boxed{0}, 3)$ $(\boxed{2.5}, 9.25)$ or $(\boxed{\widehat{2.5}}, 9.25)$

T: ***Let's draw the Cartesian graph for this (blue) relation. We can use the same grid as we did for the squaring relation to locate its points.***

Call on students to go to the board to locate a couple ordered pairs in the blue relation; then let students continue locating points on their grids. Suggest that students first locate the ordered pairs listed on the board; then encourage some students to find points for ordered pairs in the blue relation that have not been discussed. When most students have several points in their graphs, complete the picture at the board.



T: ***Do you notice anything interesting about these graphs?***

S: ***They look alike.***

S: ***Each point on the blue curve is three units higher than a corresponding point on the red curve.***

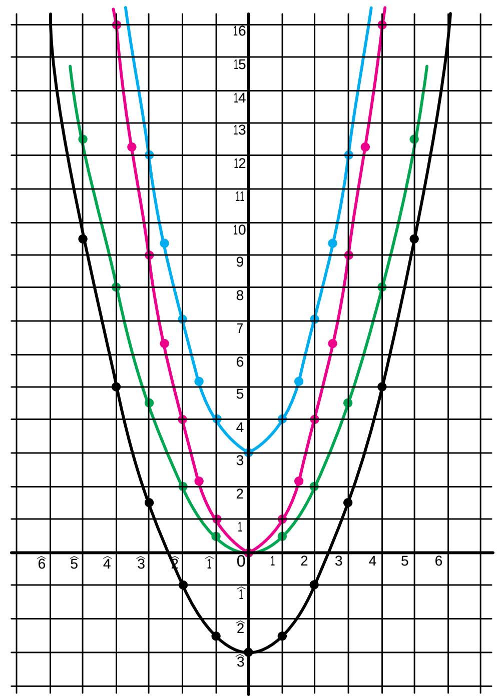
Exercise 4

$a \rightarrow \frac{1}{2} \times a^2$

$a \rightarrow (\frac{1}{2} \times a^2) - 3$

Write these rules for relations on the board. Ask students to draw the Cartesian graphs of these relations on their grids.

After a while, invite students to help draw the Cartesian graphs of the green and the black relations on the board. When many students have finished, encourage them to comment on the graphs.



Extension Activity

Suggest that students draw Cartesian graphs for the following relations on a single grid.

$a \rightarrow (a+1)^2 \qquad a \rightarrow (a-1)^2 \qquad a \rightarrow (a+1)^2 + 1 \qquad a \rightarrow (a+1)^2 - 1$

Capsule Lesson Summary

Review the binary abacus and binary writing. Label a binary number line. Discover a secret number given a sequence of clues involving the location of the number within intervals on a binary number line.

Materials

Teacher
- Meter stick
- Colored chalk
- Minicomputer checkers (optional)
- Blackline N4

Student
- Paper
- Binary abacus
- Checkers

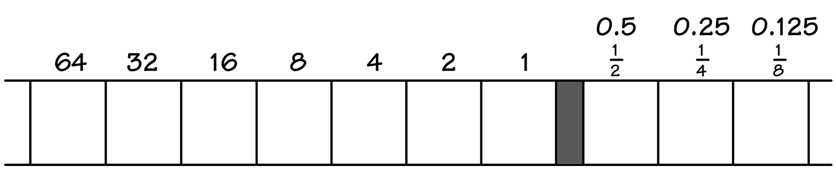
Advance Preparation: Use Blackline N4 to make copies of a binary abacus for students.

Description of Lesson

Exercise 1

Draw this part of an abacus on the chalkboard, review the rule for the binary abacus, and use the rule to label the boards.

Two checkers on a board
trade for
one checker on the next board to the left.

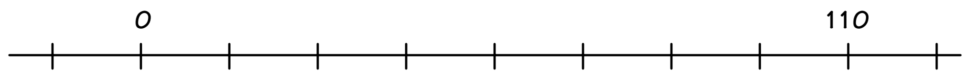


Begin a list of numbers written both in binary and decimal. Give one form and ask the class to supply the other. (Answers are in boxes.)

Binary Writing	Decimal Writing
1000100	$\boxed{68}$
$\boxed{100010}$	34
10001	$\boxed{17}$
$\boxed{1000.1}$	8.5
100.01	$\boxed{4.25}$
$\boxed{101}$	5
0.101	$\boxed{\frac{5}{8} = 0.625}$
0.0101	$\boxed{\frac{5}{16} = 0.3125}$

Exercise 2

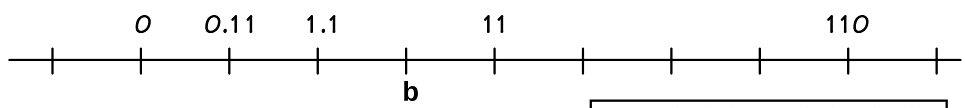
Draw this part of a binary number line on the board.



T: ***Let's label the other marks on this number line in binary writing.***

Invite students to label the marks in any order they choose; however, a likely first choice will be to label the midpoint of 0 and 110.

If your students are hesitant, encourage them to recall the ease of halving in the binary system and to label the midpoints of segments from 0 to some other labeled point. As necessary, use the abacus to verify calculations. Continue until the mark to the right of 0 is labeled.



Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

T (tracing the segment from **0** to **0.11**)**:** ***The marks on this number line show counting by how much? Remember, we are thinking of the binary writing of numbers here.***

S: **0.11.**

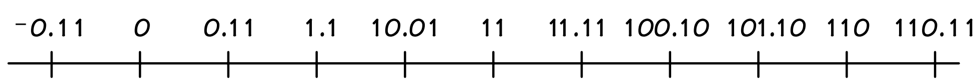
T (pointing to **b**)**:** ***How much more is this number than* 1.1*?***

S: **0.11** ***more.***

T: ***Calculate* 1.1 + 0.11.**

$$\begin{array}{r} \scriptstyle 1 \\ 1.1 \\ +\ 0.11 \\ \hline 10.01 \end{array}$$

Use the result of the calculation to label **b**. Continue until all the marks on the number line are labeled.

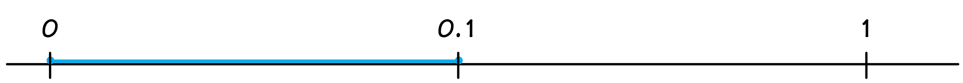


Exercise 3

Draw a line segment about one meter long on the board, and put marks for **0** and **1** near the ends.

T: ***I am thinking of a secret number. My number is located somewhere in this part of the binary number line.***

Indicate this blue segment on your number line.



T: ***My secret number is in this blue segment. What can we say about the blue segment?***

S: ***The blue segment starts at* 0 *and ends at the midpoint of* 0 *and* 1, 0.1.**

T: ***So what does the clue tell you about my secret number?***

S: ***Your secret number is between* 0 *and* 0.1*, or it could be one of the endpoints* 0 *or* 0.1*.***

T: ***Yes. If we were to express my secret number in binary writing, how would we start?***

S: 0.0**.**

S: ***Or just* 0.1*.***

Record the information on the board in this manner.

0.0 … **or** 0.1

T: ***The three dots indicate that there could be other* 0*s and* 1*s that we have not yet determined.***

Indicate this red segment on your number line.

T: ***My secret number is also in this red segment. What do you notice?***

S: ***The red segment starts at* 0.01 *and ends at* 0.1*. It is the right half of the blue segment.***

S: ***The secret number is between* 0.01 *and* 0.1*, or it could be one of the endpoints* 0.01 *or* 0.1*.***

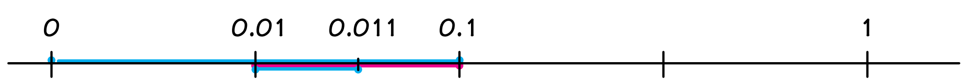
T: ***What more does this clue tell us about the binary writing of my secret number?***

S: ***It still could be just* 0.1*, or it starts like* 0.01*.***

Record the new information on the board.

0.0 … **or** 0.1
0.01 … **or** 0.1

Indicate another blue segment on your number line, as shown below.



T: ***My secret number is in this smaller blue segment. What do you notice?***

S: ***This time you took the left half of the red segment.***

S: ***Your secret number is between* 0.01 *and* 0.011*, or it could be one of the endpoints,* 0.01 *or* 0.011*.***

S: ***The secret number cannot be* 0.1*.***

T: ***Good. What more does this clue tell us about the binary writing of my secret number?***

S: ***It starts like* 0.010 *or it is just* 0.011*.***

0.0 … **or** 0.1
0.01 … **or** 0.1
0.010 … **or** 0.011

T: ***My secret number is also in this smaller red segment. What do you think?***

S: ***You took the right half of the blue segment—left, right, left, right.***

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S: ***The secret number is between*** 0.0101 ***and*** 0.011***, or it could be one of the endpoints,*** 0.0101 ***or*** 0.011.

T: ***What about the binary writing of this number?***

S: ***It starts like*** 0.0101 ***or it is just*** 0.011.

0.0 …	or	0.1
0.01 …	or	0.1
0.010 …	or	0.011
0.0101 …	or	0.011

If the class has not yet caught on to your method, continue to give clues, each time taking either the left or right half of the previous segment.

T: ***Can you predict what my next clue will be?***

S: ***A blue segment which is the left half of the last red segment.***

S: ***You always choose the left half with blue, then the right half with red.***

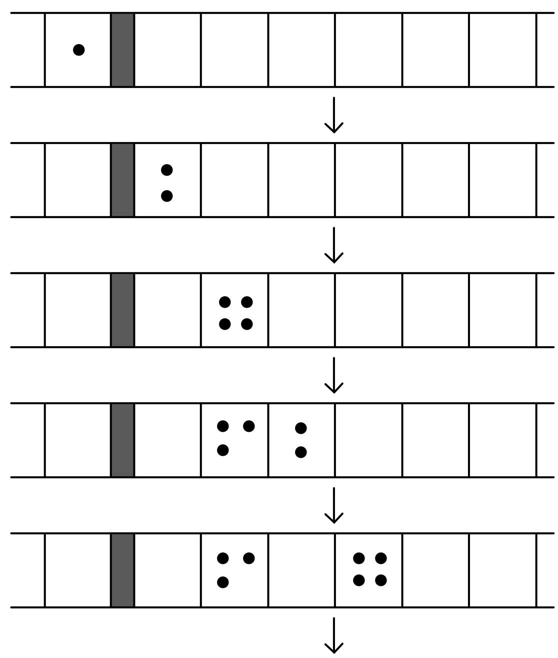
S: ***Your clues go on and on. The segments get shorter and shorter.***

T: ***Yes. If we could go on forever, we would find the secret number. Do you know what my secret number is?***

Allow time for students to consider the situation.

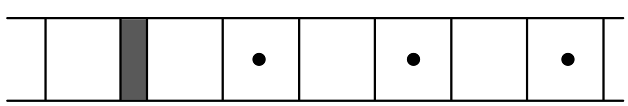
S: ***It is*** 0.01010101 … .

Your students may have seen ⅓ in binary writing, though it was some time ago. If time and interest permit, your class might enjoy checking this by calculating ⅓ x 1 on the binary abacus, as demonstrated here. The backward trades being made are trying to get three or zero checkers on a board.



and so on

Remove two checkers wherever there is a group of three, leaving one. Then ⅓ is on the binary abacus.

 $= \frac{1}{3}$

Write the binary name for ⅓.

Binary Writing

$$\frac{1}{3} = 0.010101\ \ldots$$

Capsule Lesson Summary

Use an area method to add and subtract fractions in mixed form (for example, $1\frac{4}{5} + 1\frac{3}{5} = 3\frac{2}{5}$). Discuss methods of deciding which is the greater of two fractions. Add, subtract, multiply, and divide a pair of fractions, and compare the four numbers.

Materials

Teacher	• Meter stick • Colored chalk • Fraction pieces (optional)	**Student**	• Paper • Colored pencils, pens, or crayons • Worksheets N15*, **, ***, and ****

Description of Lesson

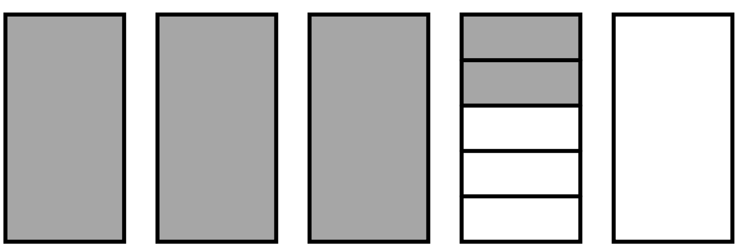
Exercise 1

Note: You may prefer to use a fraction manipulative rather than rectangles drawn on the board for this exercise.

Draw five rectangles on the board. 30 cm by 50 cm is a convenient size. Invite students to shade $3\frac{2}{5}$ rectangles, using a meter stick to divide accurately the fourth rectangle. Then write the equation $\square + \triangle = 3\frac{2}{5}$ nearby.

T: ***Let's find some pairs of numbers whose sum is $3\frac{2}{5}$.***

Record suggestions in a table, and use the rectangles to confirm calculations.



$\square + \triangle = 3\frac{2}{5}$

$\square$	$\triangle$
3	$\frac{2}{5}$
$1\frac{1}{5}$	$2\frac{1}{5}$

Extend the table with the following or similar problems. (Answers are in boxes.)

Encourage students to use the rectangles or patterns to support their answers; for example:

S: ***$2\frac{4}{5} + \frac{3}{5} = 3\frac{2}{5}$, so also $1\frac{4}{5} + 1\frac{3}{5} = 3\frac{2}{5}$.***

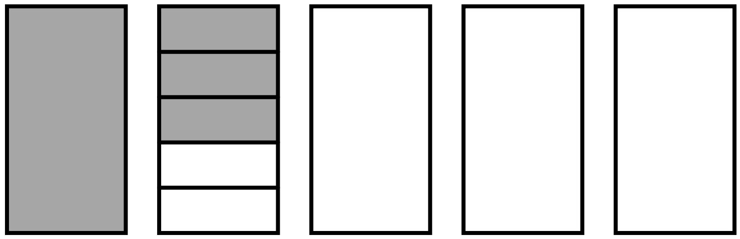
S: ***$2\frac{4}{5} + \frac{3}{5} = 3\frac{2}{5}$, so also $\frac{4}{5} + 2\frac{3}{5} = 3\frac{2}{5}$.***

$\square + \triangle = 3\frac{2}{5}$

$\square$	$\triangle$
$2\frac{4}{5}$	$\boxed{\frac{3}{5}}$
$\boxed{1\frac{4}{5}}$	$1\frac{3}{5}$
$1\frac{2}{5}$	$\boxed{2}$
$\boxed{2\frac{3}{5}}$	$\frac{4}{5}$

N 15

Erase the board and redraw the five rectangles. Invite students to shade 1⅗ rectangles. Write the following information near the shaded rectangles.



$\square - \triangle = 1\frac{3}{5}$

As in the previous activity, invite students to find pairs of numbers whose difference is 1⅗. Encourage students to explain their suggestions using the rectangles or patterns.

After a while, present some problems by extending the table. (Answers are in boxes.) Feel free to change the problems here if the indicated pairs of numbers are already in your table.

$\square - \triangle = 1\frac{3}{5}$

2	$\boxed{\frac{2}{5}}$
$3\frac{1}{5}$	$\boxed{1\frac{3}{5}}$
$\boxed{4\frac{4}{5}}$	$3\frac{1}{5}$
$\boxed{4\frac{1}{5}}$	$2\frac{3}{5}$
$\boxed{3}$	$1\frac{2}{5}$

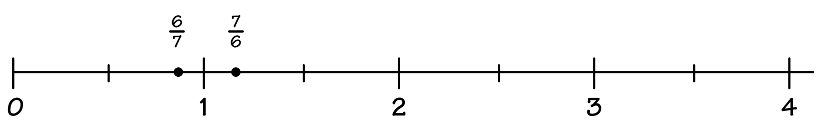
Exercise 2

Draw a number line from 0 to 4 on the board. Write the fractions 7⁄6 and 6⁄7 nearby.

T: ***Which is greater, 7⁄6 or 6⁄7?***

S: ***7⁄6. 7⁄6 is more than 1, while 6⁄7 is less than 1.***

Invite students to locate (approximately) 7⁄6 and 6⁄7 on the number line, and complete the inequality.



$$\frac{7}{6} > \frac{6}{7}$$

Present the following inequalities in a similar manner.

$$\frac{2}{7} < \frac{6}{7} \qquad \frac{11}{3} > \frac{3}{11}$$

T: ***Sometimes it is not so easy to decide which of two fractions is greater. Which is greater, 2⁄3 or 5⁄8? Why?***

Encourage students to suggest a variety of methods to compare 2⁄3 and 5⁄8. Students who attempt to draw circles or rectangles and then to shade the appropriate regions may find it difficult to be sufficiently accurate.

S: ***Multiply the same number by 2⁄3 and by 5⁄8; then compare the results. For example, 2⁄3 x 24 = 16 and 5⁄8 x 24 = 15. Since 16 > 15, we know 2⁄3 > 5⁄8.***

S: ***Find equivalent fractions with the same denominator and compare them. 2⁄3 = 16⁄24 and 5⁄8 = 15⁄24. Since 16⁄24 > 15⁄24, we know 2⁄3 > 5⁄8.***

S: ***Find equivalent fractions with the same numerator and compare them. 2⁄3 = 10⁄15 and 5⁄8 = 10⁄16. Since 10⁄15 > 10⁄16, we know 2⁄3 > 5⁄8.***

If students do not suggest methods involving equivalent fractions which are easier to compare, ask for and list fractions equivalent to ⅔ and to ⅝. For example:

$$\frac{2}{3} = \frac{4}{6} = \frac{8}{12} = \frac{16}{24} = \frac{20}{30} = \frac{10}{15} = \cdots$$

$$\frac{5}{8} = \frac{10}{16} = \frac{20}{32} = \frac{15}{24} = \frac{25}{40} = \frac{50}{80} = \cdots$$

T: ***Can you find two fractions, one equivalent to ⅔ and one equivalent to ⅝, with the same denominator? Which is greater?***

S: ***$\frac{2}{3} = \frac{16}{24}$ and $\frac{5}{8} = \frac{15}{24}$. $\frac{16}{24} > \frac{15}{24}$, so $\frac{2}{3} > \frac{5}{8}$.***

T: ***Can you find two fractions, one equivalent to ⅔ and one equivalent to ⅝, with the same numerator? Which is greater?***

S: ***$\frac{2}{3} = \frac{20}{30}$ and $\frac{5}{8} = \frac{20}{32}$. $\frac{20}{30} > \frac{20}{32}$, so $\frac{2}{3} > \frac{5}{8}$.***

Leave the lists of equivalent fractions on the board for the next exercise.

Exercise 3

Write these four calculations on the board.

T: ***Before doing the calculations, try to predict the order of these four numbers from least to greatest.***

$$\frac{2}{3} + \frac{5}{8} \qquad \frac{2}{3} \times \frac{5}{8}$$

$$\frac{2}{3} - \frac{5}{8} \qquad \frac{2}{3} \div \frac{5}{8}$$

Allow a few minutes for students to write their lists. Then, invite students to do the calculations and to explain their methods. For example:

S: ***$\frac{2}{3} = \frac{16}{24}$ and $\frac{5}{8} = \frac{15}{24}$, so $\frac{2}{3} + \frac{5}{8} = \frac{16}{24} + \frac{15}{24} = \frac{31}{24} = 1\frac{7}{24}$.***

S: ***$\frac{2}{3} = \frac{16}{24}$ and $\frac{5}{8} = \frac{15}{24}$, so $\frac{2}{3} - \frac{5}{8} = \frac{16}{24} - \frac{15}{24} = \frac{1}{24}$.***

S: ***$\frac{2}{3} \times \frac{5}{8} = \frac{2 \times 5}{3 \times 8} = \frac{10}{24} = \frac{5}{12}$.***

S: ***$\frac{2}{3} \div \frac{5}{8} = \frac{8}{5} \times \frac{2}{3} = \frac{16}{15} = 1\frac{1}{15}$.***

Write the results on the board, and invite students to order them from least to greatest. Write the appropriate calculation beneath each number.

$$\frac{1}{24} < \frac{10}{24} < 1\frac{1}{15} < 1\frac{7}{24}$$

$$\left(\frac{2}{3} - \frac{5}{8}\right) < \left(\frac{2}{3} \times \frac{5}{8}\right) < \left(\frac{2}{3} \div \frac{5}{8}\right) < \left(\frac{2}{3} + \frac{5}{8}\right)$$

Write four more calculations on the board, and again ask students to predict the order of the four numbers before doing the calculations. Then ask students to complete the calculations independently. After a few minutes, invite students to explain and record their solutions on the board.

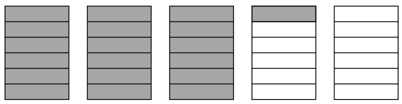
$$\frac{1}{4} < \frac{3}{8} < 1\frac{1}{4} < 1\frac{1}{2}$$

$$\left(\frac{3}{4} - \frac{1}{2}\right) < \left(\frac{3}{4} \times \frac{1}{2}\right) < \left(\frac{3}{4} + \frac{1}{2}\right) < \left(\frac{3}{4} \div \frac{1}{2}\right)$$

Worksheets N15*, **, ***, and **** are available for individual work.

Name ____________ N15 *

Shade $3\frac{1}{6}$ rectangles.



Find pairs of numbers whose sum is $3\frac{1}{6}$.

$\square + \triangle = 3\frac{1}{6}$

1	$2\frac{1}{6}$
$\frac{1}{6}$	3
$1\frac{5}{6}$	$1\frac{2}{6}$ or $1\frac{1}{3}$
2	$1\frac{1}{6}$
$2\frac{5}{6}$	$\frac{2}{6}$ or $\frac{1}{3}$
$1\frac{1}{2}$	$1\frac{4}{6}$ or $1\frac{2}{3}$
$2\frac{1}{2}$	$\frac{2}{3}$

Many solutions are possible.

Name ____________ N15 **

Write at least five equivalent fractions for $\frac{4}{5}$.

$\frac{8}{10}, \frac{16}{20}, \frac{32}{40}, \frac{40}{50}, \frac{12}{15}, \frac{24}{30}$

Many other fractions are possible.

Write at least five equivalent fractions for $\frac{3}{4}$.

$\frac{6}{8}, \frac{12}{16}, \frac{24}{32}, \frac{30}{40}, \frac{15}{20}, \frac{9}{12}$

Many other fractions are possible.

Complete the calculations.

$\frac{4}{5} + \frac{3}{4} = \frac{16}{20} + \frac{15}{20} = \frac{31}{20} = 1\frac{11}{20}$

$\frac{4}{5} \times \frac{3}{4} = \frac{12}{20} = \frac{3}{5}$

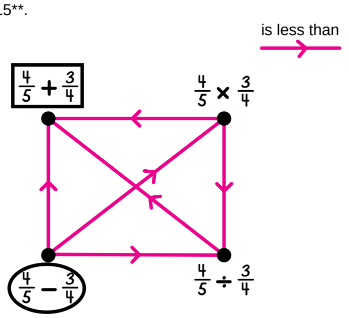
$\frac{4}{5} - \frac{3}{4} = \frac{32}{40} - \frac{30}{40} = \frac{2}{40} = \frac{1}{20}$

$\frac{4}{5} \div \frac{3}{4} = \frac{4}{5} \times \frac{4}{3} = \frac{16}{15} = 1\frac{1}{15}$

Other uses of equivalent fractions are possible.

Name ____________ N15 ***

Draw as many red arrows as possible. Use your calculations from N15**.



Circle the least number and draw a box around the greatest number.

Name ____________ N15 ****

Complete the calculations. Show your work.

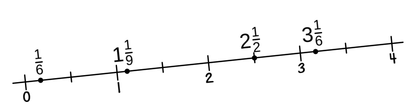
$\frac{5}{3} + \frac{3}{2} = \frac{10}{6} + \frac{9}{6} = \frac{19}{6} = 3\frac{1}{6}$

$\frac{5}{3} - \frac{3}{2} = \frac{10}{6} - \frac{9}{6} = \frac{1}{6}$

$\frac{5}{3} \times \frac{3}{2} = \frac{15}{6} = 2\frac{3}{6} = 2\frac{1}{2}$

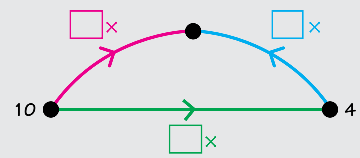
$\frac{5}{3} \div \frac{3}{2} = \frac{5}{3} \times \frac{2}{3} = \frac{10}{9} = 1\frac{1}{9}$

For each number above, draw and label a dot to show its approximate location on this number line.



N16 COMPOSITION #1 — N 16

Capsule Lesson Summary



Do some mental arithmetic involving multiplication of a whole number times a fraction. Find many different pairs of red and blue arrows for this picture and, thereby, label the green arrow in a variety of ways.

Use compositions of multiplication, division, and addition to label arrows in arrow pictures. (Optional: Use an arrow picture to explain a trick for guessing a secret number.)

Materials

Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Calculator
- Worksheets N16*, **, and ***

Description of Lesson

Begin the lesson with mental arithmetic such as in the following sequences of multiplication problems. You may wish to write the sequences on the board to emphasize the patterns. If students respond with a fraction greater than 1, ask for a simple mixed name for the number.

7×4	(28)	6×6	(36)	5×7	(35)	9×8	(72)
$7 \times \frac{1}{2}$	$(3\frac{1}{2})$	$6 \times \frac{1}{3}$	(2)	$5 \times \frac{1}{4}$	$(1\frac{1}{4})$	$9 \times \frac{2}{5}$	$(3\frac{3}{5})$
$7 \times 4\frac{1}{2}$	$(31\frac{1}{2})$	$6 \times 6\frac{1}{3}$	(38)	$5 \times 7\frac{1}{4}$	$(36\frac{1}{4})$	$9 \times 8\frac{2}{5}$	$(75\frac{3}{5})$

Write the following problems on the board, and ask students to do the calculations mentally. (Answers are in boxes.)

$4 \times 3\frac{1}{2} = \boxed{14}$

$7 \times 8\frac{1}{4} = \boxed{57\frac{3}{4}}$

$8 \times 6\frac{1}{3} = \boxed{50\frac{2}{3}}$

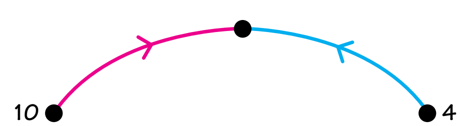
$8 \times 8\frac{3}{4} = \boxed{70}$

$6 \times 7\frac{2}{3} = \boxed{46}$

$20 \times 3\frac{3}{5} = \boxed{72}$

Exercise 1

Draw this arrow picture on the board.

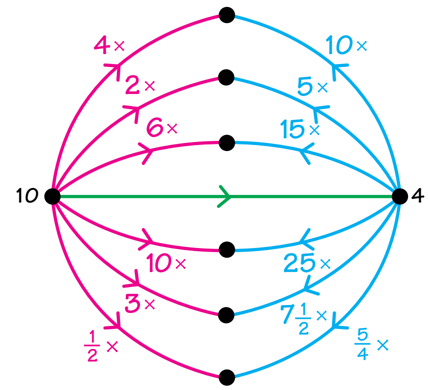


T: *Each arrow is for some number times. What could the red and blue arrows be for?*

S: *The red arrow could be for 6*x *and then the blue arrow would be for 15*x*, since 6* x *10 = 60 and 15* x *4 = 60.*

N 16

As students suggest different pairs of red and blue arrows, draw and label arrows in the picture on the board. Some of the many possibilities are shown here. Draw a green arrow from 10 to 4.



T: ***This green arrow is also for some number times. What could it be for?***

S: ***$\frac{6}{15}$x. 6x followed by "against" 15x is the same as 6x followed by ÷15, and that is $\frac{6}{15}$x.***

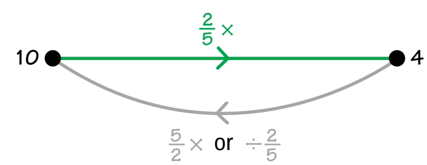
S: ***$\frac{2}{5}$x. A return arrow for 5x is ÷5. 2x followed by ÷5 is the same as $\frac{2}{5}$x.***

Make a list of possible names for the green arrow. Most will be suggested by the arrow picture, but students may find others. For example:

$\frac{6}{15}\times \quad \frac{2}{5}\times \quad \frac{4}{10}\times \quad \frac{10}{25}\times \quad \frac{3}{7\frac{1}{2}}\times \quad \frac{\frac{1}{2}}{\frac{5}{4}}\times \quad \frac{40}{100}\times$

T: ***All of these are correct names for the green arrow, but often people prefer the simplest name, $\frac{2}{5}$x.***

Label the green arrow, and draw a return arrow from 4 to 10.

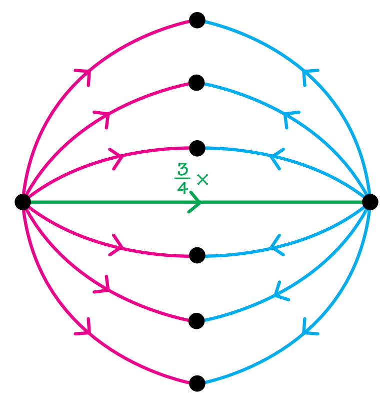


T: ***What could this return arrow be for?***

S: ***$\frac{5}{2}$x or ÷$\frac{2}{5}$.***

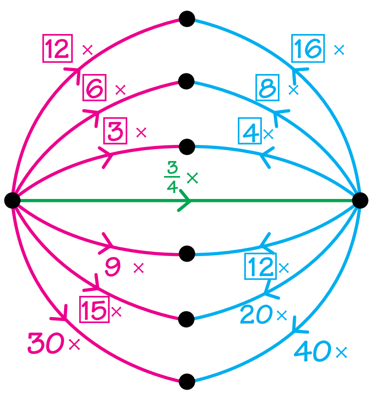
Exercise 2

Erase all the labels in the arrow picture generated in Exercise 1. Label the green arrow $\frac{3}{4}$x, and solicit possible labels for the red and blue arrows (in pairs).



Occasionally label a red arrow and ask for its paired blue arrow. Or, label a blue arrow and ask for its paired red arrow.

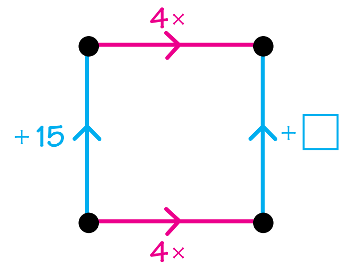
Some of the many possibilities are shown here. (Boxes indicate labels given by students.) From the picture, generate a list of other possible names for the green arrow.



$\frac{3}{4}\times \quad \frac{6}{8}\times \quad \frac{12}{16}\times \quad \frac{9}{12}\times \quad \frac{15}{20}\times \quad \frac{30}{40}\times$

Exercise 3

Draw this arrow picture on the board.

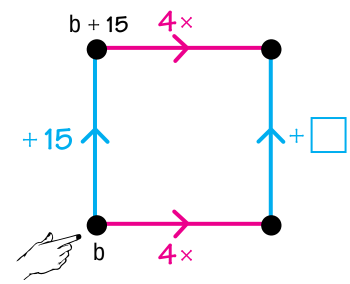


T (tracing the appropriate arrows)**:** ***+15 followed by 4*x**
is the same as 4*x *followed by +☐.
What number goes in the box?

Let the class discuss the question, and check some answers by labeling the dots. Lead the discussion to this analysis, starting at the lower left dot.

T: ***Suppose we put some number here, call it* b.** (Trace the +15 arrow and point to its ending dot.) ***What do we know about the number here?***

S: ***It is* b *+ 15.***

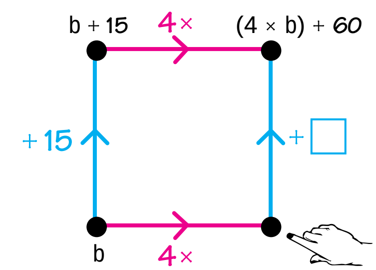


Trace the 4x arrow starting at **b + 15** and point to its ending dot.

T: ***What do we know about this number?***

S: ***It is 4*** x **(b** ***+ 15).***

S: ***You multiply both* b *and 15 by 4, so it is (4*** x **b*) + 60 because 4*** x ***15 = 60.***



Trace the 4x arrow starting at **b** and point to its ending dot.

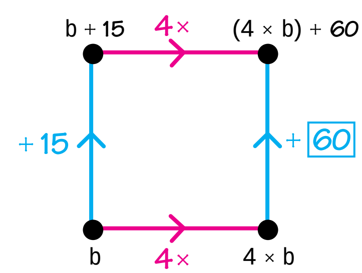
T: ***What do we know about this number?***

S: ***It is 4*** x **b.**

Trace the arrow from **4** x **b** to **(4** x **b) + 60**.

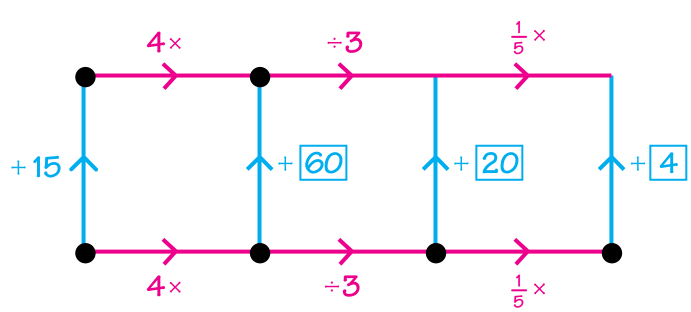
T: ***What is this arrow for?***

S: ***+60, since the ending number, (4*** x **b*) + 60, is 60 more than the starting number, 4*** x **b.**



Erase the labels for the dots. If students wish, let them label the dots with numbers to check that +60 is correct.

Extend the arrow picture three arrows at a time (forming another square), and each time ask students to label the new arrow on the right. (Answers are in boxes.)



N 16

Exercise 4

Instruct students to select any whole number they like, to write the number on their papers, but not to let you see the number.

T: ***Put your number on the display of your calculator. Using the calculator, add 60, divide by 4, subtract 15, and then multiply by 2. Press = to see the result on your display.***

Repeat the sequence of four operations: +60, ÷4, –15, and x2. Allow time for students to complete the calculations on their calculators.

T: ***If you tell me the number now on the display of your calculator, I can tell you your secret starting number.***

S: ***47.***

T: ***Your starting number was 94.***

Note: To calculate a student's starting number, simply double the student's ending number. Do not tell the class this rule.

Let five or more students tell you their ending numbers, and you tell them their starting numbers. Students may or may not notice the 2x pattern. In any case, record several students' starting and ending numbers in a table on the board, as illustrated here.

Starting Number	Ending Number
94	47
33	16.5
7180	3590
2	1
175	87.5
16	8

T: ***Do you notice any pattern? Can you guess my trick for finding the secret starting number?***

S: ***An ending number is always one-half of the starting number.***

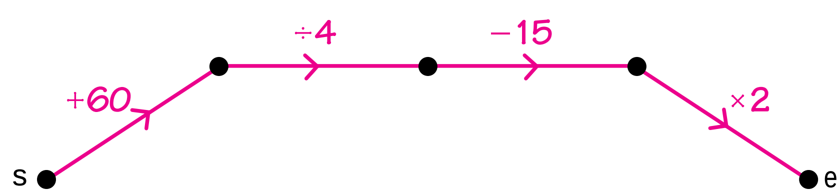
S: ***You double an ending number to find the starting number.***

Announce to the class that you will draw an arrow picture to explain the trick. Start with a dot.

T: ***This dot (s) is for your starting number. What operations did you do on the calculator?***

S: ***+60 followed by ÷4 followed by –15 followed by x2.***

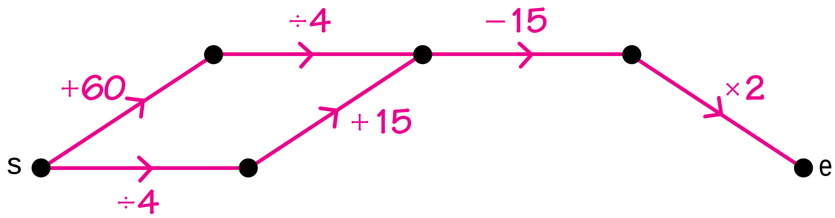
Draw an arrow road to indicate the sequence of steps.



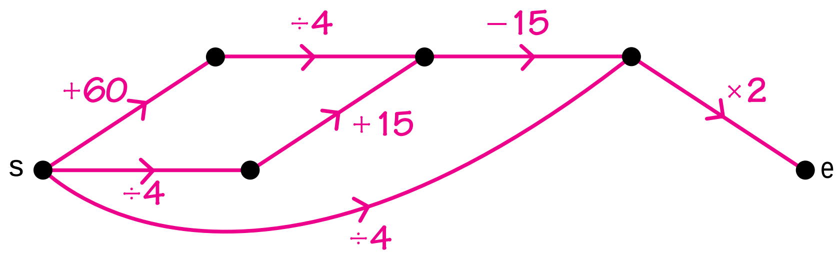
T: ***This last dot (e) is for your ending number. Let's try to shorten the arrow road from s to e.***

The following sequence of pictures shows how to shorten the arrow road to an equivalent single arrow from **s** to **e**.

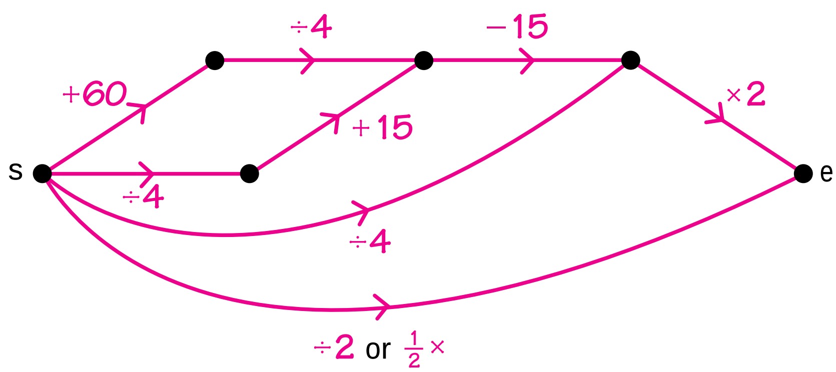
- +60 followed by ÷4 is the same as ÷4 followed by +15.



- +15 and –15 are opposites, so the starting number for the +15 arrow is the same as the ending number of the –15 arrow.



- ÷4 followed by x2 is the same as ÷2 or ½x.



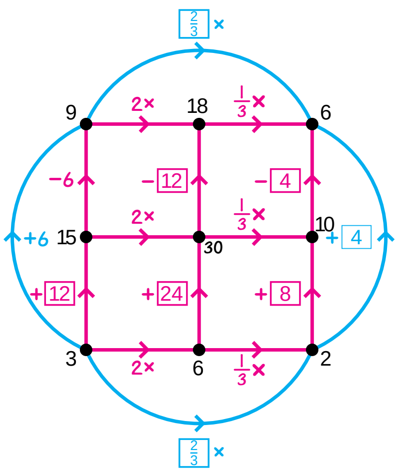
T: ***Do you see how this picture explains my method of calculating your secret starting number when you tell me your ending number?***

S: ***An ending number is always one-half of the starting number, so you simply multiply the ending number by 2.***

Worksheets N16*, **, and *** are available for individual work.

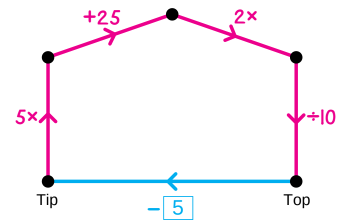
Name ________________ N16 *

Label the dots and fill in the boxes for the arrows.



Name ________________ N16 **

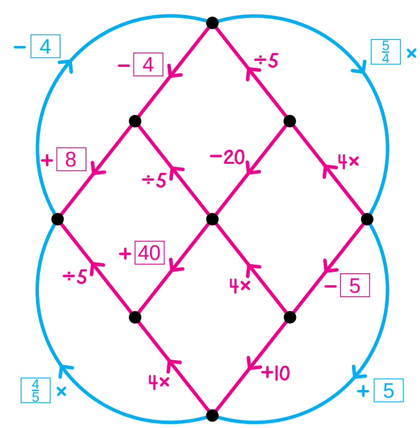
Fill in the box for the arrow and complete the table.



Tip	Top
3	8
10	15
3	2
0.4	5.4
0.2	5.2
5	10

Name ________________ N16 ***

Fill in the boxes for the arrows.



Capsule Lesson Summary

Given the result of a division calculation, use multiplication to produce other division problems with the same result. Use a similar technique to transform a division problem such as $3.5\overline{)463.75}$ into an easier problem such as $35\overline{)4\,637.5}$ or $7\overline{)927.5}$.

Materials

Teacher	• None	**Student**	• Paper

Description of Lesson

Begin the lesson with mental arithmetic involving division patterns. Feel free to adjust the problems here to the numerical abilities of your students.

15 ÷ 5 (3)	60 ÷ 4 (15)	0.27 ÷ 3 (0.09)
100 ÷ 5 (20)	600 ÷ 4 (150)	48 ÷ 3 (16)
115 ÷ 5 (23)	660 ÷ 4 (165)	48.27 ÷ 3 (16.09)
1150 ÷ 5 (230)	6 600 ÷ 4 (1 650)	4.827 ÷ 3 (1.609)

Exercise 1

T: ***What number is 256 ÷ 4*** (64)

Write this division fact on the board, and begin a list of related problems below it. Each time, ask students to use the previous information to help solve the new problem. (Answers are in boxes.)

$$256 \div 4 = 64$$
$$512 \div 8 = \boxed{64}$$
$$128 \div 2 = \boxed{64}$$
$$384 \div 6 = \boxed{64}$$

The class should notice that each calculation has the same result: 64.

T: ***Why are these all division facts for 64? Do you see a pattern?***

S: ***For 512 ÷ 8, 2 x 256 = 512 and 2 x 4 = 8.***

S: ***For 128 ÷ 2, ½ x 256 = 128 and ½ x 4 = 2.***

S: ***For 384 ÷ 6, 3 x 128 = 384 and 3 x 2 = 6.***

S: ***It appears that if you multiply (or divide) both numbers in a division calculation by the same number, the result does not change.***

S: ***For 384 ÷ 6, I noticed that 256 + 128 = 384 and 4 + 2 = 6. It appears that if we add the corresponding numbers in two division calculations with the same result, then the new calculation also has that same result.***

If students do not notice the addition pattern, continue without observing it. Add two more problems to the list on the board.

$$1024 \div 16 = \boxed{}$$
$$1280 \div 20 = \boxed{}$$

T: ***Should the result for each of these calculations also be 64?*** (Yes) ***Why?***

S: ***For 1 024 ÷ 16, look at 512 ÷ 8 = 64. 2 x 512 = 1 024 and 2 x 8 = 16.***

S: ***For 1280 ÷ 20, look at 128 ÷ 2 = 64. 10* x *128 = 1280 and 10* x *2 = 20.***

Complete the calculations on the board and add these problems to the list.

$768 \div \boxed{} = 64$

$25.6 \div \boxed{} = 64$

T: ***Can you use patterns to complete these division facts for 64?***

S: ***768 ÷ 2 = 64. Use 384 ÷ 6 = 64 and multiply by 2: 2* x *384 = 768 and 2* x *6 = 12.***

S: ***768 ÷ 12 = 64. Use 512 ÷ 8 = 64 and 256 ÷ 4 = 64. Notice that 512 + 256 = 768 and 8 + 4 = 12.***

S: ***25.6 ÷ 0.4 = 64. Use 256 ÷ 4 = 64 and divide by 10: 256 ÷ 10 = 25.6 and 4 ÷ 10 = 0.4.***

T: ***Can you use patterns to create more division facts for 64?***

Let several students explain how they find division facts for 64. For example:

S: ***3 840 ÷ 60 = 64. Use 384 ÷ 6 = 64 and multiply both 384 and 6 by 10; 10* x *384 = 3 840 and 10* x *6 = 60.***

S: ***896 ÷ 14 = 64. Use 512 ÷ 8 = 64 and 384 ÷ 6 = 64, and add corresponding numbers. 512 + 384 = 896 and 8 + 6 = 14.***

S: ***1.28 ÷ 0.02 = 64. Use 128 ÷ 2 = 64 and divide both 128 and 2 by 100; 128 ÷ 100 = 1.28 and 2 ÷ 100 = 0.02.***

Erase the board except for the original fact. Write this problem under it.

$256 \div 4 = 64$

$256 \div 8 = \boxed{}$

T: ***Could 256 ÷ 8 = 64?***

S: ***No; if you divide 256 by a number other than 4, then you do not get 64.***

T: ***Is 256 ÷ 8 more or less than 64?***

S: ***Less, because you are dividing 256 by a greater number.***

S: ***Less. Think about sharing $256 among more (eight rather than four) people. Each person would receive less money.***

T: ***What number is 256 ÷ 8?***

S: ***½* x *64, or 32, since 2* x *4 = 8.***

Complete the calculation, and continue the activity by asking students to complete these calculations. (Answers are in boxes.)

$256 \div 16 = \boxed{16}$

$25.6 \div 8 = \boxed{3.2}$

$25.6 \div 4 = \boxed{6.4}$

Exercise 2

Write this problem on the board.

$3.5\overline{)463.75}$

T: ***We used patterns to find many division facts for the same number. Let's try to use a pattern to change this division problem into another with the same result. If we are thoughtful, we may find an easier problem. In particular, it would be nice if the divisor*** (point to 3.5) ***were a whole number.***

T: ***Can you think of an easier division problem with the same result?***

S: ***Multiply each number by 10.*** $35\overline{)4\,637.5}$

S: ***Multiply each number by 2.*** $7\overline{)927.5}$

S: ***Multiply each number by 20.*** $70\overline{)9\,275}$

There are other possibilities. Try to get at least two with whole number divisors. If necessary, suggest them yourself. Let the class decide which new problem they would prefer doing. For example, suppose the class chooses $35\overline{)4\,637.5}$.

T: ***Let's do this division problem together. How should we start?***

S: ***100* x *35 = 3 500 and 200* x *35 = 7 000, so start with 100.***

Let students direct the solution until the remainder is 0. If necessary, suggest the use of decimal numbers less than 1 as multipliers. Students may suggest some different steps than those shown here, but the result should be the same.

$$\begin{array}{r|r}
35\overline{)4\,637.5} & \\
-3\,500.0 & 100 \\ \hline
1\,137.5 & \\
-1\,050.0 & 30 \\ \hline
87.5 & \\
-70.0 & 2 \\ \hline
17.5 & \\
-17.5 & 0.5 \\ \hline
0 &
\end{array}$$

T: ***What number is 4 637.5 ÷ 35?***

S: ***132.5 (100 + 30 + 2 + 0.5).***

T: ***Therefore, what number is 463.75 ÷ 3.5?***

S: ***Also 132.5.***

Record these two results.

$$\begin{array}{r} 132.5 \\ 3.5\overline{)463.75} \end{array} \qquad \begin{array}{r} 132.5 \\ 35\overline{)4\,637.5} \end{array}$$

Assign students to check that another problem on the board has the same result by doing the long division. For example:

$$\begin{array}{r|r}
7\overline{)927.5} & \\
-700.0 & 100 \\ \hline
227.5 & \\
-140.0 & 20 \\ \hline
87.5 & \\
-70.0 & 10 \\ \hline
17.5 & \\
-14.0 & 2 \\ \hline
3.5 & \\
-3.5 & 0.5 \\ \hline
0 & 132.5
\end{array}$$

N 17

Write the following division problems on the board, and use them as you would worksheets.

*	$2.5\overline{)161.25}$	$0.6\overline{)8.22}$
**	$1.4\overline{)40.25}$	$3.7\overline{)77.33}$
***	$7.06\overline{)1634.39}$	$4.15\overline{)51.46}$

T: ***Copy these problems. Before starting a calculation, you may want to change the problem into an easier division problem. Usually it is easier to divide by a whole number. Continue each calculation until the remainder is 0. Your results will include decimal parts.***

The illustration below has results in boxes and suggested multiplication functions in parentheses. Of course, any multiplication function except 0x can be used, but the suggested ones are convenient because they yield whole number divisors.

*	$\begin{array}{r}\boxed{64.5}\\ 2.5\overline{)161.25}\end{array}$	$\begin{pmatrix}10\times\\ 4\times\\ 2\times\end{pmatrix}$	$\begin{array}{r}\boxed{13.7}\\ 0.6\overline{)8.22}\end{array}$	$\begin{pmatrix}10\times\\ 5\times\end{pmatrix}$
**	$\begin{array}{r}\boxed{28.75}\\ 1.4\overline{)40.25}\end{array}$	$\begin{pmatrix}10\times\\ 5\times\end{pmatrix}$	$\begin{array}{r}\boxed{20.9}\\ 3.7\overline{)77.33}\end{array}$	$(10\times)$
***	$\begin{array}{r}\boxed{231.5}\\ 7.06\overline{)1634.39}\end{array}$	$(100\times)$	$\begin{array}{r}\boxed{12.4}\\ 4.15\overline{)51.46}\end{array}$	$(100\times)$

Capsule Lesson Summary

Review exponential notation. Write positive divisors and multiples of 60 as products of prime numbers. Write various numbers as products of prime numbers, making use of exponential notation.

Materials

Teacher	• None	**Student**	• Paper

Description of Lesson

Exercise 1

Write 2^3 on the board.

T: ***What number is this? What calculation does 2^3*** (read as "two to the third power") ***indicate?***

S: ***$2^3 = 2 \times 2 \times 2 = 8$.***

T: ***Right; the raised 3, the*** **exponent**, ***indicates how many times to multiply 2 by itself.***

Write this information on the board.

$$2^3 = 2 \times 2 \times 2 = 8$$

T: ***What number is 3^2?***

S: ***9, because $3^2 = 3 \times 3 = 9$.***

$$3^2 = 3 \times 3 = 9$$

Write these problems on the board.

$$3^{\square} = 81$$
$$2^{\square} = 211$$

T: ***Copy these problems on your paper and, if possible, fill in each box with a whole number.***

Give students a few minutes to consider the problems.

S: ***$3^2 = 9$ and $9 \times 9 = 81$, so $3 \times 3 \times 3 \times 3 = 81$; $3^4 = 81$.***

S: ***$2^7 = 128$ and $2^8 = 256$, so there is no (whole number) solution to $2^{\square} = 211$.***

S: ***2 to any (whole number) power is an even number and 211 is odd, so there is no (whole number) solution to the second problem.***

Write these problems on the board, and instruct students to complete them whenever possible.

$6^3 = \square$	$2^{\square} = 512$	$\square^3 = 125$
$7^3 = \square$	$5^{\square} = 610$	$\square^{\square} = 27$
		$\square^{\triangle} = 256$

N 18

Remind students that for a problem like $\square^{\square} = 27$, the same shapes indicate that the same number must go in both boxes. Whereas, for a problem like $\square^{\triangle} = 256$, the different shapes indicate that different numbers may go in the box and in the triangle; however, one could put the same number in the box and in the triangle if there is one that works.

When many students have finished the problems, check the work collectively. (Answers are in boxes.)

$6^3 = \boxed{216}$ $\quad$ $2^{\boxed{9}} = 512$ $\quad$ $\boxed{5}^3 = 125$

$7^3 = \boxed{343}$ $\quad$ $5^{\square} = 610$ (No whole number solution) $\quad$ $\boxed{3}^{\boxed{3}} = 27$

The last problem has three solutions.

$\boxed{16}^{\triangle 2} = 256$ $\quad$ $\boxed{4}^{\triangle 4} = 256$ $\quad$ $\boxed{2}^{\triangle 8} = 256$

Exercise 2

Give the class the following task.

T: ***Write 60 as a product of prime numbers; that is, find a multiplication calculation for 60 using only prime numbers. You may use a prime number more than once. Try to use exponents when a prime number occurs more than once in the product.***

Allow several minutes for students to work independently, and then ask for a volunteer to write the prime factorization of 60 on the board.

$$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5$$

T: ***Now, let's make a list of all the positive divisors of 60. Let's also try writing each divisor as a product of prime numbers.***

S: ***15 is a positive divisor of 60, and 15 = 3* x *5.***

Note: You may want students to use the term *prime factorization* and, at the same time, to refer to positive divisors as *factors*.

Continue until your list includes all of the positive divisors of 60, as illustrated below. You may prefer to organize differently, for example, with divisors paired 1 and 60, 2 and 30, and so on.

Positive Divisors of $2^2 \times 3 \times 5$ (60)

$15 = 3 \times 5$	$30 = 2 \times 3 \times 5$
2	$60 = 2^2 \times 3 \times 5$
$4 = 2^2$	5
$20 = 2^2 \times 5$	$6 = 2 \times 3$
3	$12 = 2^2 \times 3$
$10 = 2 \times 5$	1

T: ***Do you notice anything interesting about the positive divisors of 60?***

Accept any correct observations students make.

S: ***60 has three prime divisors: 2, 3, and 5.***

S: ***Each divisor, except 1, is the product of some combination of 2s, 3s, and 5s.***

T: ***Let's write some multiples of 60 as products of prime numbers. Who can suggest one?***

S: ***120 is a multiple of 60; 120 =*** $2^3 \times 3 \times 5$.

Record multiples of 60 in another list on the board. When multiples become very large, or when students begin to discover patterns, accept and record only the prime factorizations of such numbers.

$$\underline{\text{Multiples of } 2^2 \times 3 \times 5\ (60)}$$

$$\begin{aligned}
120 &= 2^3 \times 3 \times 5 \\
240 &= 2^4 \times 3 \times 5 \\
300 &= 2^2 \times 3 \times 5^2 \\
420 &= 2^2 \times 3 \times 5 \times 7 \\
480 &= 2^5 \times 3 \times 5 \\
600 &= 2^3 \times 3 \times 5^2 \\
& \ 2^2 \times 3^3 \times 5^2 \\
& \ 2^8 \times 3 \times 5 \times 7
\end{aligned}$$

T: ***Do you notice anything interesting about our list of multiples of 60?***

Accept any correct observations students make.

S: ***Each multiple has at least*** 2^2 ***in the product of primes.***

S: ***Each multiple of 60 is some whole number times*** $2^2 \times 3 \times 5$.

Exercise 3 ________________

Put this list of numbers on the board.

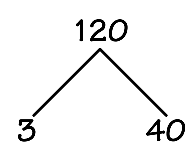
120 75 48 64 540 1056

T: ***On your paper, write each of these numbers as a product of prime numbers. Many of you have devised your own ways to solve such problems. Let me show you one scheme for finding the prime numbers in a product for 120.***

The following dialogue follows just one way students may respond, but the end result should be the same.

T: ***Name any two numbers other than 1 and 120 whose product is 120.***

S: ***3 and 40, 3*** $\times$ ***40 = 120.***



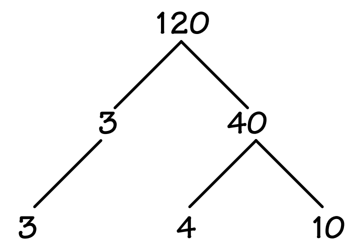
Begin a factorization tree on the board using the two divisors.

T: ***3 is a prime number, but 40 is not. Name two numbers other than 1 and 40 whose product is 40.***

N 18

S: ***4 and 10; 4 x 10 = 40.***

T: ***120 = 3 x 4 x 10. 3 is a prime but 4 and 10 are not. Name two numbers other than 1 and 4 whose product is 4. Name two numbers other than 1 and 10 whose product is 10.***

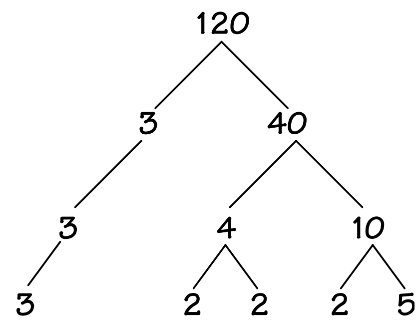


S: ***2 x 2 = 4.***

S: ***2 x 5 = 10.***

T: ***120 = 3 x 2 x 2 x 2 x 5. Since 3, 2, and 5 are all prime numbers, we are done. What is a shorthand way of writing this product?***

S: ***2^3 x 3 x 5.***



Write this in a number sentence on the board.

$$120 = 3 \times 2 \times 2 \times 2 \times 5 = 2^3 \times 3 \times 5$$

Let students work individually or with partners to find the prime factorizations of the other five numbers. Solutions are shown below.

$$75 = 3 \times 5^2$$
$$48 = 2^4 \times 3$$
$$64 = 2^6$$
$$540 = 2^2 \times 3^3 \times 5$$
$$1056 = 2^5 \times 3 \times 11$$

Capsule Lesson Summary

Investigate when 7 x ☐ yields a number less than 7. Put decimal numbers on the Minicomputer using the weighted checkers ②, ③, …, ⑨. Review the rule for counting decimal places when locating the decimal point in the product of decimal numbers. Use a calculator to further investigate this rule. Label arrows in an arrow picture where each arrow is for +, –, x, or ÷ some decimal number.

Materials

Teacher	• Minicomputer set • Weighted checkers • Colored chalk	**Student**	• Calculator • Paper • Colored pencils, pens, or crayons

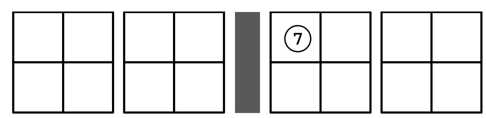
Description of Lesson

Exercise 1

Display four Minicomputer boards and place a ⑦-checker on the 0.8-square.

T: ***What number is on the Minicomputer?***

S: ***7 x 0.8, or 5.6.***



7 × 0.8 = 5.6

Discuss how the students know that 7 x 0.8 = 5.6.

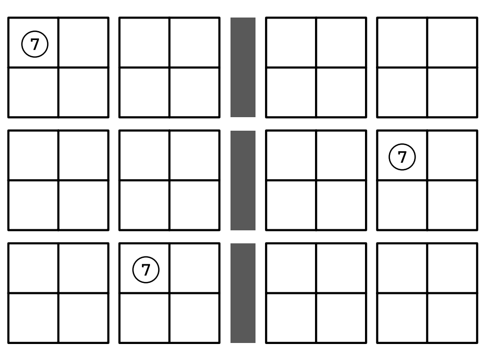
S: ***7 x 8 = 56, so 7 x 0.8 = 5.6.***

S: ***0.8 could represent eight dimes or 80 cents. Seven times 80 cents is $5.60.***

S: ***7 x 8 = 56. 7 x 0.8 must be more than 0.8 and less than 7, so 7 x 0.8 = 5.6.***

Record 7 x 0.8 = 5.6 as the first entry in a list of related number sentences.

Move the ⑦-checker to the squares indicated below. Each time, ask students for the number on the Minicomputer and record the corresponding 7x fact on the board.



= 7 × 80 = 560

= 7 × 0.08 = 0.56

= 7 × 8 = 56

N 19

Ask students to complete four more multiplication facts involving decimals, this time with 0.7x. (Answers are in boxes.)

$7 \times 0.8 = 5.6$	$0.7 \times 0.8 = \boxed{0.56}$
$7 \times 80 = 560$	$0.7 \times 80 = \boxed{56}$
$7 \times 0.08 = 0.56$	$0.7 \times 0.08 = \boxed{0.056}$
$7 \times 8 = 56$	$0.7 \times 8 = \boxed{5.6}$

T: ***What do you notice about the facts on the right compared to those on the left?***

S: ***They are for multiplying by 0.7 instead of 7.***

S: ***The results on the right each have one more decimal place than the corresponding result on the left.***

S: ***Divide each result on the left by 10 to get the corresponding result on the right.***

Write this information on the board, and point to the first 7x fact in the list.

$$7 \times \square < 7$$

T: ***Here 7 x 0.8 is less than 7. In general, what positive numbers can we multiply by 7 to get a number less than 7?***

S: ***Any number between 0 and 1.***

T: ***Why?***

S: ***7 x 0 = 0 and 7 x 1 = 7. Multiplying a number between 0 and 1 by 7 will result in a number between 0 and 7.***

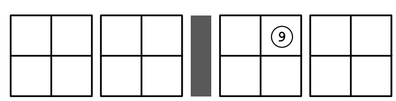
S: ***If we multiply a number less than 1 by a positive number (7), the result is less than the number (7).***

S: ***Think about buying something that costs $7 per pound. Less than a pound costs less than $7.***

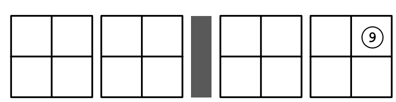
Exercise 2

Display the weighted checkers ②, ③, ④, ⑤, ⑥, ⑦, ⑧ , and ⑨. Present the following problems to the class.

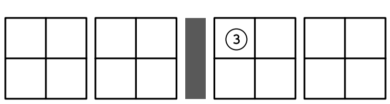
- Put 3.6 on the Minicomputer using exactly one of these checkers.



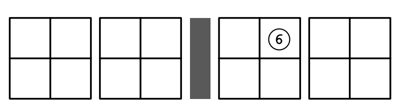
- Put 0.36 on the Minicomputer using exactly one of these checkers.



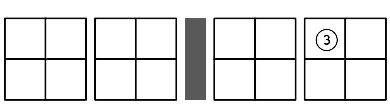
- Put 2.4 on the Minicomputer using exactly one of these checkers.



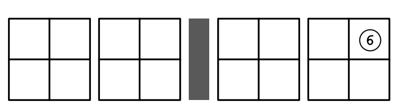
or



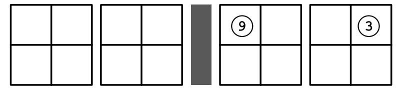
- Put 0.24 on the Minicomputer using exactly one of these checkers.



or



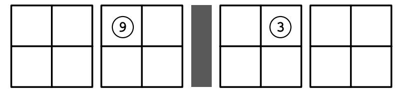
Display this configuration of checkers on the Minicomputer.



T: ***What number is on the Minicomputer?***

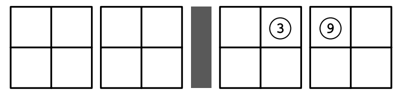
S: ***7.32, 9*** x ***0.8 = 7.2 and 3*** x ***0.04 = 0.12 and 7.2 + 0.12 = 7.32.***

Invite a students to put 73.2 on the Minicomputer using the same two checkers.



= 73.2

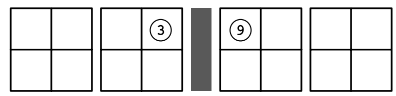
Move the checkers on the Minicomputer to this configuration.



T: ***What number is on the Minicomputer?***

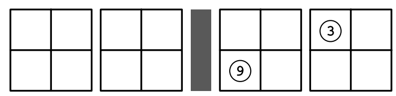
S: ***1.92; 3*** x ***0.4 = 1.2 and 9*** x ***0.08 = 0.72 and 1.2 + 0.72 = 1.92.***

Invite a student to put 19.2 on the Minicomputer using the same two checkers.



= 19.2

Invite a student to put 2.04 on the Minicomputer using these same two checkers.



= 2.04

Exercise 3 _______________

Direct students to copy and solve the following multiplication problems. Mention that they can first multiply the numbers ignoring decimal points and then use estimation, patterns, or a rule to correctly place decimal points in the products. After a while, check the problems collectively.

$$\begin{array}{r} 6.17 \\ \times\ 0.8 \\ \hline 4.936 \end{array} \qquad \begin{array}{r} 4.6 \\ \times\ 9.5 \\ \hline 43.70 \end{array} \qquad \begin{array}{r} 0.73 \\ \times 0.04 \\ \hline 0.0292 \end{array}$$

T: ***How did you determine where to place the decimal points in the products?***

S: ***I used estimation. For example, 4.6*** x ***9.5 is a little less than 5*** x ***10 or 50. Therefore, 4.6*** x ***9.5 = 43.70.***

S: ***I counted decimal places. For example, 6.17 has two digits to the right of the decimal point and 0.8 has one, so the product has three decimal places. 6.17*** x ***0.8 = 4.936.***

Review the rule of counting decimal places and apply it in each of the three problems. Observe that in the third problem zero is in the first decimal place of 0.0292.

Direct students to copy and solve the following multiplication problems. Suggest they first multiply ignoring decimal points and then use the rule about counting decimal places to correctly place decimal points in the products. (Answers are in boxes.)

$0.6 \times 0.3 = \boxed{0.18}$ $\qquad$ $0.2 \times 0.4 = \boxed{0.08}$

$0.9 \times 7 = \boxed{6.3}$ $\qquad$ $0.002 \times 0.04 = \boxed{0.00008}$

$0.09 \times 0.07 = \boxed{0.0063}$ $\qquad$ $0.05 \times 0.6 = \boxed{0.030}$

N 19

Write the following multiplication calculations on the board, and ask students to predict how many decimal places each product will have according to the rule. Write students' responses near the corresponding calculation. For example:

2.44 × 0.15 <4>
126 × 0.45 <2>
3.60 × 0.40 <4>

Invite students to use their calculators to do the three calculations and record the results on the board.

2.44 × 0.15 = 0.366
126 × 0.45 = 56.7
3.60 × 0.40 = 1.44

T: ***Were our predictions of the number of decimal places in each product correct? Why?***

S: ***3.60 = 3.6 and 0.40 = 0.4, so we should look at the third calculation as 3.6 x 0.4. The product should have two decimal places, which 1.44 has.***

S: ***But we could write 1.44 as 1.4400 and have four decimal places.***

S: ***In the other two products, we can add a decimal place by writing another zero on the right. Then the results would still be correct and would agree with our rule. That is, 2.44 x 0.15 = 0.3660 and 126 x 0.45 = 56.70. The calculator does not display unnecessary zeros.***

T: ***Earlier we found that 4.6 x 9.5 = 43.70. Do this calculation on your calculators.***

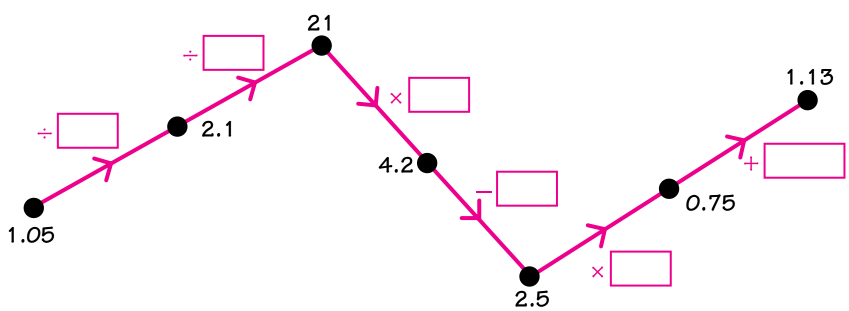
S: ***4.6 x 9.5 = 43.7.***

S: ***The product is the same except that the calculator does not display the unnecessary zero.***

Conclude that the rule about counting decimal places is valid; however, one must sometimes count unnecessary zeros as decimal places.

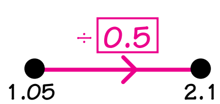
Exercise 4

Draw this arrow picture on the board.

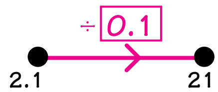


Ask students to use decimal numbers to fill in the boxes for the arrows, allowing individual students to choose which arrow to label.

The following explanations may be given for some of the arrow labels.

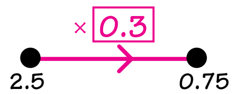


1) 1.05 x 2 = 2.1, and x2 is the same as ÷½ or ÷0.5.
2) 2.1 x 0.5 = 1.05, and the return arrow for x0.5 is ÷0.5.
3) 1.05 ÷ 2.1 = 0.5, so 1.05 ÷ 0.5 = 2.1.



1) 10 x 2.1 = 21, but 10x = $^{10}/_{1}$x = ÷$^{1}/_{10}$ = ÷0.1.

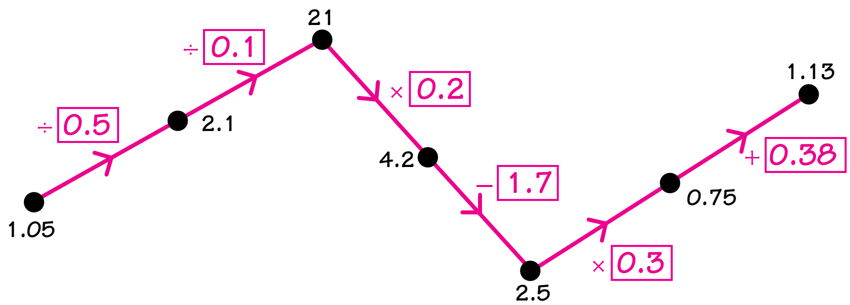
2) 21 x 0.1 = 2.1, and the return arrow for x0.1 is ÷0.1.



1) 25 x 3 = 75 and 2.5 x 3 = 7.5, so 2.5 x 0.3 = 0.75.

2) 2.5 x 0.1 = 0.25 and 0.25 x 3 = 0.75, so 2.5 x 0.3 = 0.75.

The completed arrow picture is shown below.



Capsule Lesson Summary

Introduce the concept of relatively prime numbers. Explore techniques for finding pairs of numbers that are and are not relatively prime, and graph them. Discuss patterns in the graphs.

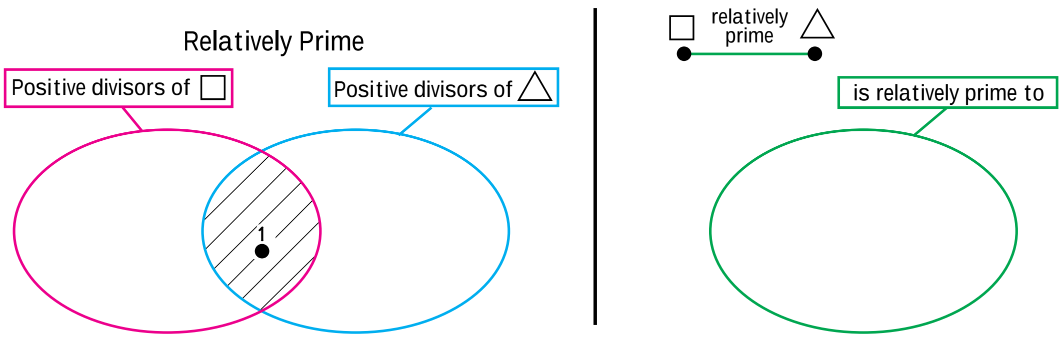
Materials

Teacher	• Colored chalk • Grid board • Blacklines N20(a), (b), and (c)	**Student**	• Grid sheet • Colored pencils, pens, or crayons • Straightedge

Advance Preparation: Use Blackline N20(a) to make copies of a grid sheet for students. Use Blacklines N20(b) and (c) to make copies of the graphs.

Description of Lesson

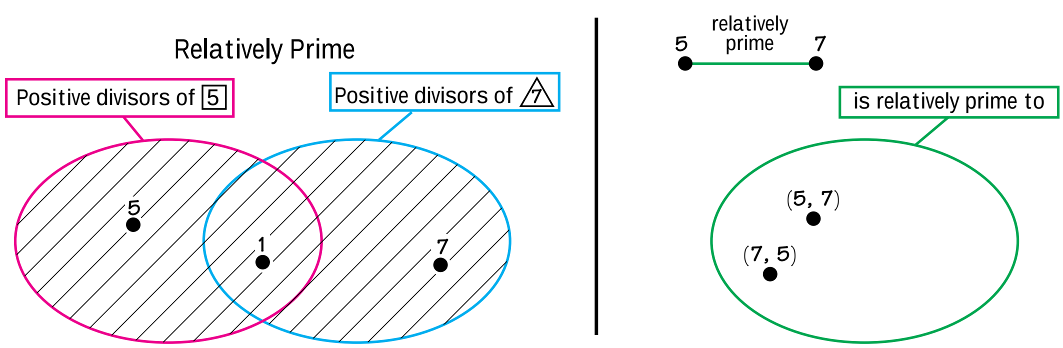
Draw these pictures on the board.



T: ***"Relatively prime" is a relation on positive integers so in this lesson, we will only use the numbers 1, 2, 3, 4, 5 and so on. Numbers are relatively prime if and only if their only common positive divisor is 1. Can you name a pair of numbers that are relatively prime?***

S: *5 and 7.*

Indicate how this pair of numbers fits in both pictures. Note that 5 is relatively prime to 7, and 7 is relatively prime to 5, so both (5, 7) and (7, 5) belong inside the green string.



Let students suggest several more pairs. If only prime numbers are suggested, ask for a pair of relatively prime numbers neither of which is prime, for example, 12 and 35.

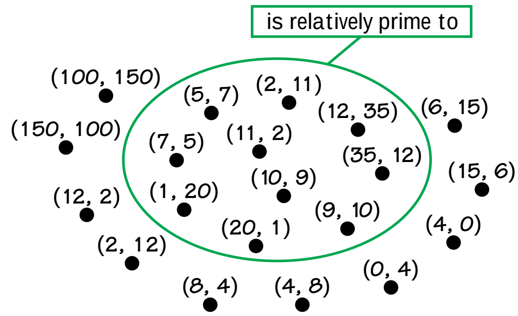
T: ***Name some pairs of numbers that are not relatively prime.***

S: ***6 and 15, because 3 is a positive divisor of both 6 and 15.***

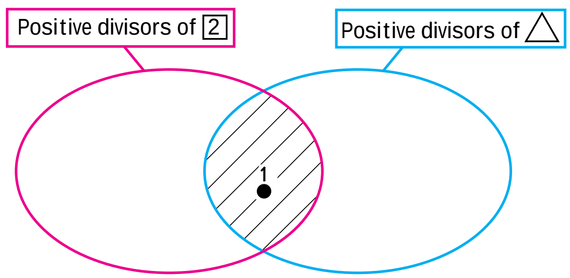
T: ***Where do (6, 15) and (15, 6) belong in the "is relatively prime to" string?***

S: ***Outside the string.***

Allow several more students to suggest pairs of numbers that are not relatively prime. Your "is relatively prime to" string may look similar to this one.



In the two-string picture on the left, put 2 in the box.



T: ***What number could be in the triangle?***

S: ***Any odd number could go in the triangle.***

T: ***What kind of numbers cannot go in the triangle?***

S: ***Even numbers, because 2 is a common divisor of any even number and 2.***

T: ***Can an even number ever be relatively prime to another even number?***

S: ***No, two even numbers always have 2 as a common positive divisor.***

Your class may wish to take the discussion further and make some other conjectures. The following are examples of possible conjectures. Two are not true and one is true. A counter example is given when the conjecture is not true.

- An even number is always relatively prime to an odd number. (Not true; 7 and 28 are not relatively prime)
- 1 is relatively prime to every other integer. (True)
- An odd number is always relatively prime to another odd number. (Not true; 21 and 35 are not relatively prime)

T: ***Do you think that it is more likely that two numbers are relatively prime or are not relatively prime?***

Let students express their opinions.

T: ***This is really not a question we are prepared to investigate thoroughly, but let's look at a sampling of possibilities. Let's graph pairs of numbers that are relatively prime.***

Refer to the copy of the grid on Blackline N20(a) or your grid board, with axes labeled as on the Blackline (to your grid’ s limitation). Call on students to name pairs of numbers that are relatively prime and to draw dots at these points. After collectively graphing several pairs of numbers, organize the class into groups to continue the graphing activity.

In order to better compare relatively prime to not relatively prime, you may suggest that some groups graph (put dots at points for) numbers that are relatively prime and other groups graph numbers that are not relatively prime. Or, within a group of four, two students might do one graph and two students do the other graph.

After a short while, interrupt the group work to ask students to share strategies with the class.

S: ***2 is relatively prime to every odd number; so are 4, 8, and 16.***

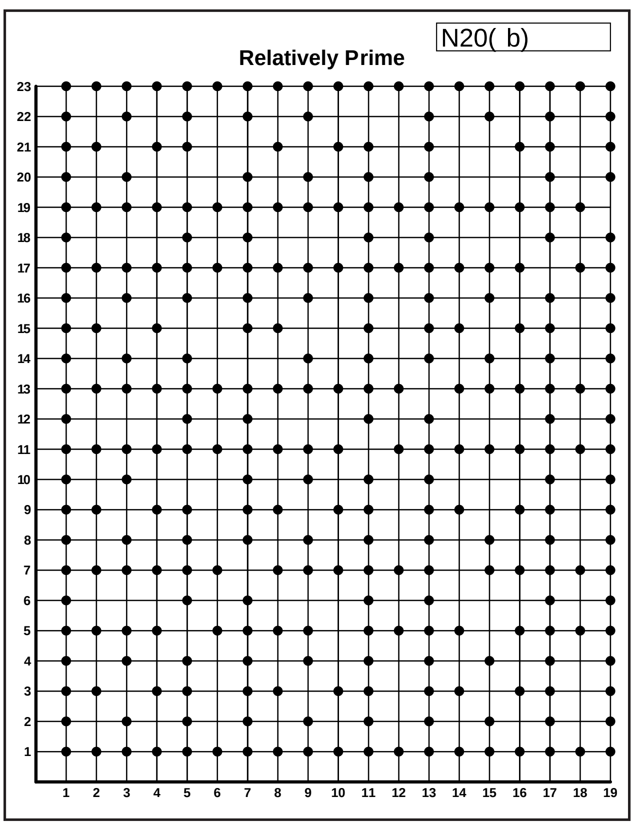
S: ***A number is not relatively prime to itself.***

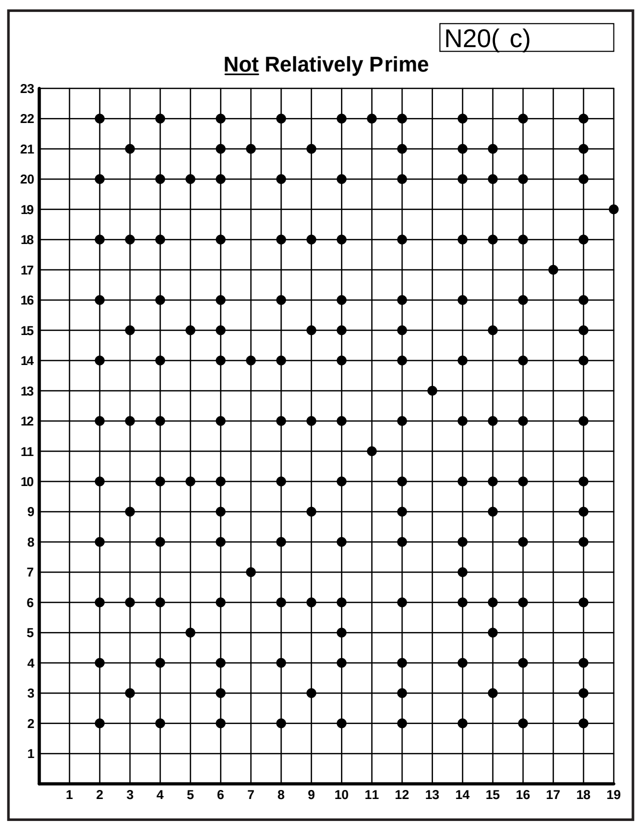
S: ***A prime number is relatively prime to all numbers except its multiples. So 5 is not relatively prime to 5, 10, 15, and 20, but is relatively prime to the other numbers.***

S: ***6 is relatively prime to 1, 5, 7, 11, 13, 17, 19, and 23. In the 6-column the dots for* relatively prime *alternate between being four and two apart.***

S: ***The same is true in the 6-row.***

Let students continue on their graphs until there are some that complete the task. Display students’ complete graphs, or make transparencies of Blacklines N20(b) and (c) to show the class.





N 2 0

Let the class discuss the graphs for the rest of the period. The following are observations that students might make. You may wish to prompt the students to look for some of them or suggest some of them yourself during the discussion.

- The diagonal of points except for (1, 1) all have dots in the graph for **not relatively prime**.
- Both graphs are symmetric around the diagonal line passing through (1, 1) (2, 2), and so on. Whether two numbers are relatively prime or not, it does not matter the order in which they occur in an ordered pair.
- Some columns (rows) look the same, for example, the 6-column, the 12-column, and the 18-column. 6, 12, and 18 all have the same prime divisors, namely, 2 and 3.
- There are patterns in diagonals parallel to the one passing through (1, 1), (2, 2), and so on. (Note: You can use a straightedge to better focus on these diagonals.)
- A number and the number one less than it are always relatively prime.

Be open to other patterns students might notice. We have only listed some of many.

T: ***For pairs of numbers within the confines of this grid, is it more common for two integers to be relatively prime or not relatively prime?***

S: ***Relatively prime; there are more dots on the graph for* relatively prime *than for* not relatively prime.**

N21 PERCENT #3 N 21

Capsule Lesson Summary

Use the definition of percents as a composition of relations and a number line model to solve problems such as 28 = □% of 40. Discuss the zoom feature on some copy machines. Given various size copies of a cartoon, determine what zoom percentage was used to make the copies. Draw boxes to fit a 40% reduction and a 150% enlargement of the original cartoon.

Materials

Teacher	• Colored chalk • Blacklines N21(a), (b), and (c)	**Student**	• Cartoon pictures of various sizes • Paper • Metric ruler

Advance Preparation: Use Blacklines N21(a) and (b) to make copies of cartoon pictures of various sizes for students. Use Blackline N21(c) to cut out 40% and 150% copies of the cartoon for your use in checking student work in Exercise 3.

Description of Lesson

Exercise 1

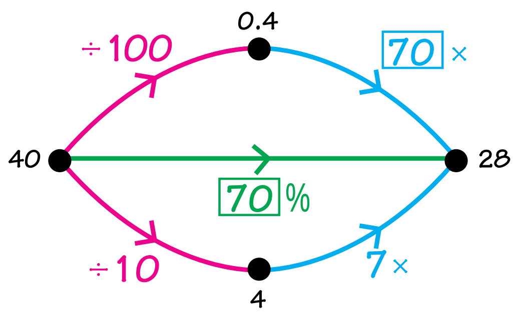
Write this percent problem on the board.

28 = □% of 40

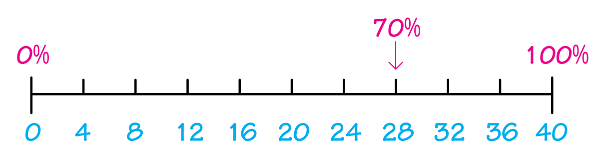
T: ***How can we decide what number goes in the box?***

There are several ways your students may try to solve this problem. Follow your class' s suggestions, but include a description of the following methods.

Arrow Picture



Number Line



Perhaps someone will suggest using equivalent fractions $\frac{28}{40} = \frac{7}{10} = \frac{70}{100}$, and $\frac{70}{100}$ is 70%.

Repeat this exercise with one or two more such problems. For example:

$30 = \boxed{40}\%$ of 75

$65 = \boxed{130}\%$ of 50

N 21

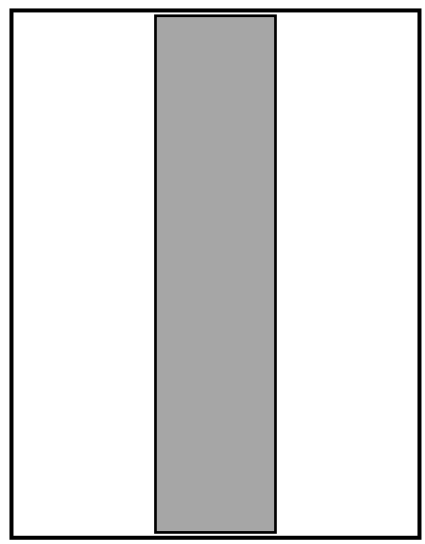
Exercise 2[†]

Begin this exercise with a discussion of copy machines. Include in the discussion the fact that some copy machines have a zoom feature with which one can make enlargements or reductions of pictures.

Distribute copies of Blacklines N21(a) and (b) to students in small groups. Explain that an original cartoon is at the top of N21(a). The other copies of the cartoon were made with the zoom feature of a copy machine. The job of each group is to decide what percent reduction or enlargement was used for each copy. As necessary, suggest that students use rulers to measure the boxes around the original cartoon and the copies.

When most groups have finished, let some explain their methods of solution. Then pose another problem for the groups to consider. Demonstrate with a copy of N21(b) as you observe,

T: ***Notice that the second copy (the 140% copy) on N21(b) almost fills the page from left to right. Suppose we turn the cartoon and want to almost fill the page from top to bottom. What percent enlargement could we make?***



Note: In metric measurement, the length of a standard size paper is about 27 cm (closest whole number of centimeters). The original cartoon box is 15 cm in length. Therefore, an enlargement of approximately 180% (180% of 15 = 27) would almost fill the page from top to bottom.

Exercise 3

On separate pieces of paper, ask groups of students to draw a box that would just fit a 40% reduction of the original cartoon and another box that would just fit an 150% enlargement.

While groups are working, cut out the cartoon copies on Blackline N21(c). One is a 40%-copy and the other is a 150%-copy. Take your copies around to the groups to check their boxes.

Home Activity

Allow students to take home copies of Blacklines N21(a) and (b) to explain the task in Exercise 2 to family members.

[†]The idea for this exercise was suggested by a lesson in *A Collection of Math Lessons* (Grades 6–8) by Marilyn Burns and Cathy Humphreys.

Capsule Lesson Summary

Express some numbers as products of prime numbers. Given the prime factorization of 425, express all of its positive divisors and some of its multiples as products of prime numbers, making use of exponential notation. Draw an arrow picture for the relation "is a multiple of," and include all possible arrows between five given numbers.

Materials

Teacher	• Colored chalk	**Student**	• Paper • Colored pencils, pens, or crayons • Worksheets N22*, **, ***, and ****

Description of Lesson

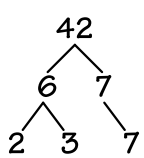
Exercise 1

Write this list of numbers on the board.

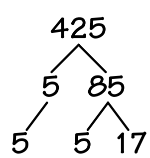
42 425 273 88 135

T: ***On your paper, write each of these numbers as a product of prime numbers. You may use a prime number more than once. Use exponents when a prime number occurs more that once in the product.***

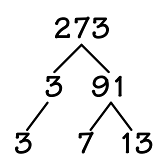
If necessary, briefly review the notion of prime numbers with the class. Let students work on these problems individually or with partners for several minutes. Check the work collectively by inviting students to put their solutions, perhaps using factorization trees, on the board. Solutions and sample trees are shown below.



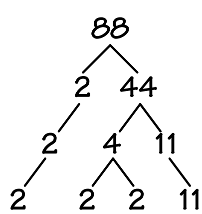
$42 = 2 \times 3 \times 7$



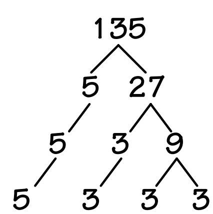
$425 = 5^2 \times 17$



$273 = 3 \times 7 \times 13$



$88 = 2^3 \times 11$



$135 = 3^3 \times 5$

Note: You may want students to use the term *prime factorization* and to refer to positive divisors as factors.

N 2 2

Exercise 2

Choose one of the numbers from Exercise 1 to use in this exercise. The dialogue here uses 425.

T: ***Let's make a list of all of the positive divisors of 425, and let's write each divisor as a product of prime numbers.***

S: ***25 is a positive divisor of 425, and 25 = 5^2.***

Continue until the list includes all of the positive divisors of 425 and their prime factorizations. You may prefer to organize the list with divisors paired, 1 and 425, 5 and 85, and so on.

Positive Divisors of $5^2 \times 17$ (425)
$25 = 5^2$
5
$425 = 5^2 \times 17$
1
17
$85 = 5 \times 17$

T: ***Do you notice anything interesting about the positive divisors of 425?***

S: ***Each divisor, except 1, has some combination of 5s and 17s in its prime factorization.***

T: ***Now let's write some multiples of 425 as products of prime numbers. Who can suggest a multiple of 425?***

S: ***850 is a multiple of 425 because 2 x 425 = 850.***

S: ***850 = 2 x 5^2 x 17.***

S: ***2 125 is a multiple of 425 because 5 x 425 = 2 125.***

T: ***Express 2 125 as a product of prime numbers.***

S: ***2 125 = 5^3 x 17.***

Multiples of $5^2 \times 17$ (425)
$850 = 2 \times 5^2 \times 17$
$1\,700 = 2^2 \times 5^2 \times 17$
$2\,125 = 5^3 \times 17$
$3\,825 = 3^2 \times 5^2 \times 17$
$5^4 \times 17$
$5^3 \times 17 \times 23$
$2^2 \times 5^4 \times 17$

When the multiples become large or when students discover patterns, accept and record only the prime factorizations of numbers.

T: ***Do you notice anything interesting about this list of multiples of 425?***

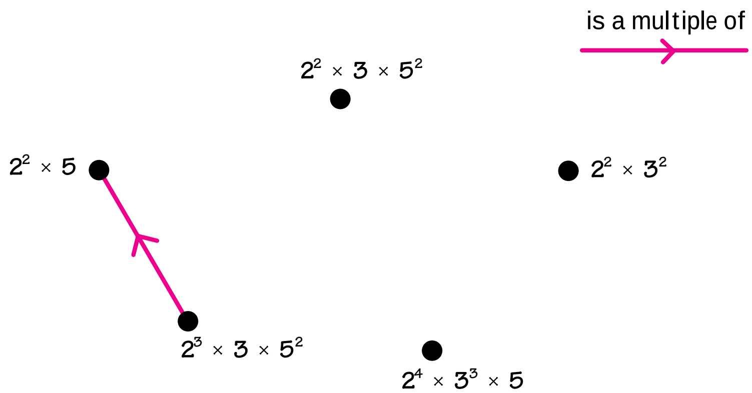
S: ***Each number is 425 (i.e., 5^2 x 17) times some whole number.***

S: ***Each multiple has at least 5^2 and 17 in its prime factorization.***

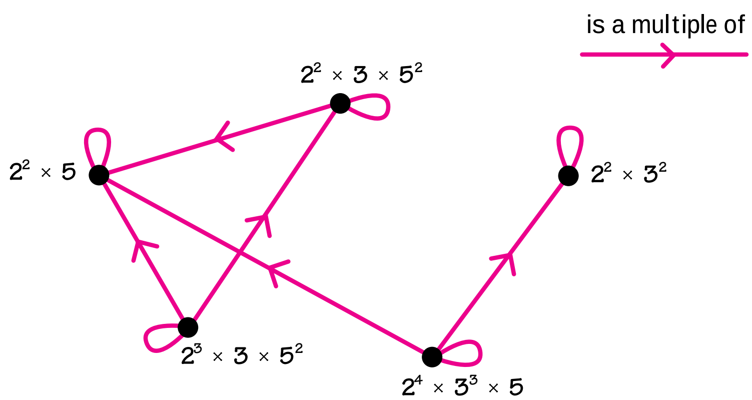
Exercise 3 ________________

On the board, draw and label dots as in the next illustration. Indicate that red arrows will be for "is a multiple of," and ask for a volunteer to draw such an arrow in the picture. Then ask the volunteer to read the information given by the arrow. For example:

S (tracing the red arrow)**:** ***$2^3 \times 3 \times 5^2$ is a multiple of $2^2 \times 5$.***



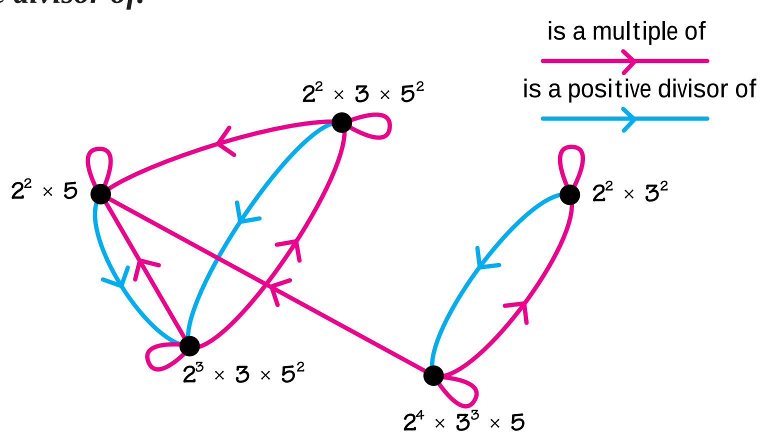
Instruct students to copy the picture and to draw as many red arrows as possible. A completed arrow picture is shown below.



Draw several return arrows in blue and ask,

T: ***What could the blue arrows be for?***

S: ***"Is a positive divisor of."***

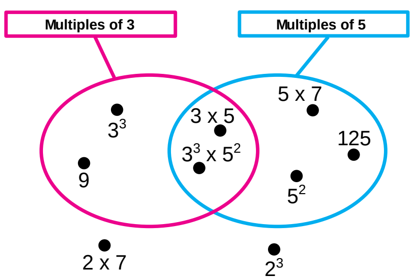


Worksheets N22*, **, ***, and **** are available for individual work.

Name ____________ N22 *

Put these numbers in the string picture.

5×7	3^3	125
9	3×5	5^2
2^3	2×7	$3^3 \times 5^2$



Name ____________ N22 **

Write 70 as a product of prime numbers (prime factoriz ation) . Then list several multiples and all the positive divisors of 70. Write each number in the lists as a product of primes.

Prime Factorization

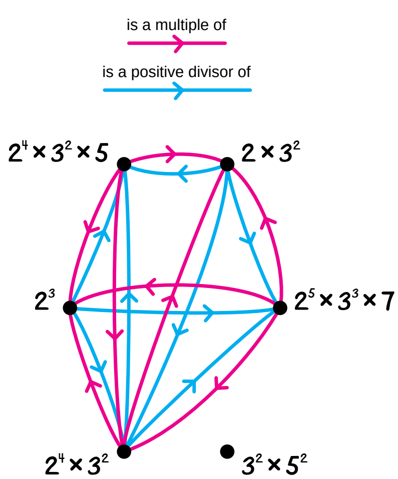
70 = 2 x 5 x 7

Multiples of 70	Positive Divisors of 70
140 = 2^2 x 5 x 7	1
210 = 2 x 3 x 5 x 7	2
350 = 2 x 5^2 x 7	5
700 = 2^2 x 5^2 x 7	7
490 = 2 x 5 x 7^2	10 = 2 x 5
1400 = 2^3 x 5^2 x 7	14 = 2 x 7
	35 = 5 x 7
Many answers are possible.	70 = 2 x 5 x 7

Did you find eight divisors of 70? Yes

Name ____________ N22 ***

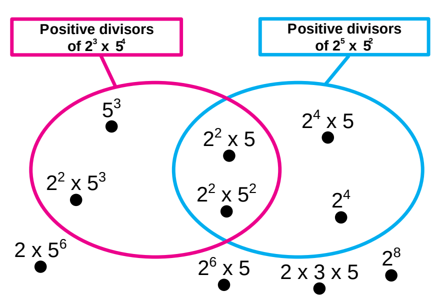
Draw all of the possible red and blue arrows between these dots.



Name ____________ N22 ****

Locate these numbers in the string picture.

$2^2 \times 5$	$2^4 \times 5$	2×5^6	
$2 \times 3 \times 5$	5^3	$2^2 \times 5^2$	
2^8	$2^6 \times 5$	$2^2 \times 5^3$	2^4



Capsule Lesson Summary

Play a Guess My Rule game for an operation $*$ where the rule is $a * b = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{a \times b}$. Discuss the early Egyptians' use of unit fractions. Write some numbers as sums of distinct unit fractions. Discuss problems posed and solved on the Rhind Papyrus.

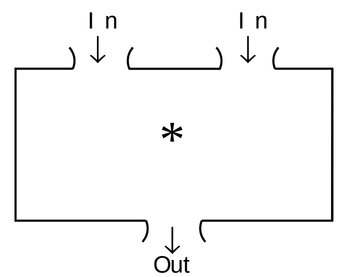
Materials

Teacher	• None	**Student**	• Paper • Worksheet N23

Description of Lesson

Exercise 1: Guess My Rule

This exercise uses an operation rule. You may like to use a "machine" picture to review how an operation rule works.



T: ***I have a secret rule for $*$. I'll give you some clues about my rule. Try to figure out the rule for $*$.***

Write several number sentences on the board as clues. Then write an open sentence and see if anyone can predict which number goes in the box.

$$2 * 5 = \frac{7}{10}$$
$$3 * 1 = \frac{4}{3}$$
$$7 * 3 = \frac{10}{21}$$
$$4 * 5 = \square$$

Note: The rule is $a * b = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{a \times b}$. The number in the box is $\frac{9}{20}$ because $4 * 5 = \frac{1}{4} + \frac{1}{5} = \frac{4+5}{4 \times 5} = \frac{9}{20}$. Do not reveal the rule to the class at this time.

Suggest that students write their guesses on paper for you to check. Acknowledge aloud correct guesses and reject incorrect guesses; for example, "No, $\frac{9}{10}$ is not the number we get using my rule." Let a student who guesses correctly tell the class that $\frac{9}{20}$ is in the box, but do not give away the rule yet. If no one guesses correctly, announce that $\frac{9}{20}$ goes in the box yourself.

Continue the activity with several more examples. (Answers are in boxes.)

$5 * 4 = \boxed{\frac{9}{20}}$	$2 * 9 = \boxed{\frac{11}{18}}$	$3 * 3 = \boxed{\frac{6}{9} = \frac{2}{3}}$
$7 * 4 = \boxed{\frac{11}{28}}$	$4 * 2 = \boxed{\frac{6}{8} = \frac{3}{4}}$	$9 * 9 = \boxed{\frac{18}{81} = \frac{2}{9}}$
$4 * 7 = \boxed{\frac{11}{28}}$	$5 * 10 = \boxed{\frac{15}{50} = \frac{3}{10}}$	$4 * 6 = \boxed{\frac{10}{24} = \frac{5}{12}}$

When many students know the rule, let one explain it to the class. Insist on a clear explanation, and write the rule on the board as described.

S: ***Add the two numbers for the numerator of a fraction, and multiple the two numbers for the denominator.***

$$\boxed{a * b = \frac{a+b}{a \times b}}$$

N 2 3

Check several of the above examples until it is clear that students understand the rule. Let students practice with the rule by solving some more problems. (Answers are in boxes.)

$$8 * 3 = \boxed{\frac{11}{24}} \qquad 9 * 6 = \boxed{\frac{15}{54} = \frac{5}{18}} \qquad 5 * \boxed{7} = \frac{12}{35}$$

$$6 * 12 = \boxed{\frac{18}{72} = \frac{1}{4}} \qquad 7 * 7 = \boxed{\frac{14}{49} = \frac{2}{7}} \qquad \boxed{6} * 7 = \frac{13}{42}$$

T: ***There is another way to explain the rule as the sum of two fractions.***

Invite students to explain how to use equivalent fractions to add fractions. For example:

$$\frac{1}{2} + \frac{1}{5} = \boxed{\frac{5}{10} + \frac{2}{10} = \frac{7}{10}} \qquad \frac{1}{8} + \frac{1}{3} = \boxed{\frac{3}{24} + \frac{8}{24} = \frac{11}{24}}$$

T: ***What do you notice?***

S: ***$2 * 5 = \frac{7}{10}$, and $\frac{1}{2} + \frac{1}{5} = \frac{7}{10}$.***

S: ***$8 * 3 = \frac{11}{24}$, and $\frac{1}{8} + \frac{1}{3} = \frac{11}{24}$.***

Add this information to the statement of the rule on the board.

$$a * b = \frac{a + b}{a \times b} = \frac{1}{a} + \frac{1}{b}$$

T: ***There are two different ways to explain the rule for $*$. Fractions with 1 as denominator are called* unit fractions*. The rule for $*$ suggests an easy way to add two unit fractions.***

Invite students to explain how to use the rule for $*$ to add unit fractions with the following examples. Emphasize that this technique works for unit fractions, but not for other fractions.

$$\frac{1}{9} + \frac{1}{4} = \boxed{9 * 4 = \frac{13}{36}} \qquad \frac{1}{5} + \frac{1}{8} = \boxed{5 * 8 = \frac{13}{40}} \qquad \frac{1}{10} + \frac{1}{7} = \boxed{10 * 7 = \frac{17}{70}}$$

Exercise 2

Tell the following story. Involve the class by interspersing questions and letting them tell what they know about Egypt or about papyrus.

T: ***Let's talk about the history of arithmetic. As far as we know, the first civilization to use fractions was the Egyptian around the year 2000 B.C. About how long ago was that?***

S: ***Almost 4 000 years ago.***

T: ***We know about the Egyptians' use of fractions from many papyrus scrolls discovered by archaeologists.***

Write **papyrus** on the board.

T: ***Papyrus is a type of reed that grows abundantly in shallow water along the Nile River in Egypt. The Egyptians carved utensils from the woody root of the plant and also used the root as fuel. They made mats, sails, and boats from the reed. They ate the pithy inside part of the stem. They also made a type of writing material from the stems of the reeds. Our word* paper *comes from the word* papyrus.**

To make papyrus, the Egyptians split the reeds into long, thin strips and laid them side by side to form a layer. They then laid another layer of strips across the first layer. They glued the two layers together to form one sheet of papyrus. Then many sheets were joined to form a papyrus scroll. After drying, they wrote on the papyrus with ink.

T: ***Papyrus was the most popular type of paper in Egypt for several thousand years and was exported to Europe. Not until 1000 A.D. were papyrus scrolls completely replaced by thin, treated animal skins. Only later did our type of paper come to Europe from China, via Arab traders.***

Thousands of papyrus scrolls have survived to the present day because Egypt is mostly desert and, therefore, very dry. Otherwise, papyrus documents would have rotted with age.

Write **Dr. Rhind** and **1864 A.D.** on the board.

T: ***In 1864, a British archaeologist, Dr. Rhind, bought a papyrus scroll containing many arithmetic problems. This papyrus document was written around 1800 B.C. by a scribe named Ahmes who wrote that he was copying it from a papyrus document written around 2000 B.C. It contains many arithmetic problems that involve fractions.***

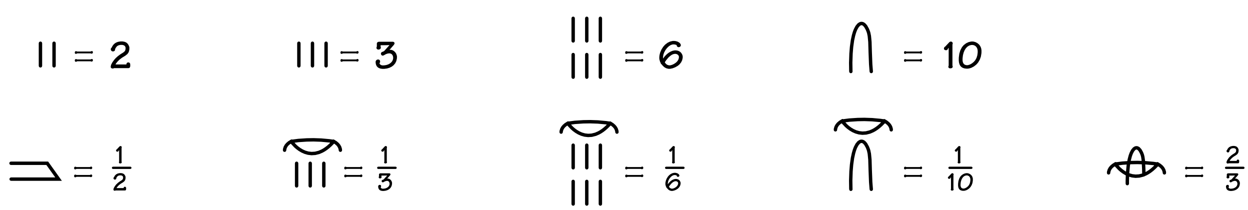
Write **7 loaves among 10 people** on the board.

T: ***For example, one problem was to share fairly seven loaves of bread among ten people. How much bread would each person receive?***

Let students discuss the problem. They should conclude that each person would receive 7 ÷ 10 or $\frac{7}{10}$ or 0.7 loaf of bread.

T: ***However, the Egyptians did not know about decimal numbers like 0.7. Also, they used mostly unit fractions so $\frac{7}{10}$ was unknown.***

Write this information on the board.



T: ***This was the Egyptian way of writing numbers. Notice that they wrote fractions by writing an oval or "eye" above the symbol for a number (indicate $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{10}$). The only exceptions were $\frac{1}{2}$, which had a special symbol ( ), and $\frac{2}{3}$, the only non-unit fraction they used which also had a symbol ( ).***

Instead of the old Egyptian symbols, let's write unit fractions in our usual way.

Egyptians would not have used $\frac{7}{10}$ as the answer to the sharing bread problem since they only knew unit fractions and $\frac{2}{3}$. They would have expressed $\frac{7}{10}$ as a sum of unit fractions. Can you find two or more unit fractions whose sum is $\frac{7}{10}$?

S: ***$\frac{1}{2}$ and $\frac{1}{5}$, because $\frac{7}{10} = \frac{5}{10} + \frac{2}{10} = \frac{1}{2} + \frac{1}{5}$.***

In this problem or in a later problem, students might naturally suggest solutions with repeated unit fractions: $\frac{7}{10} = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10}$ or $\frac{7}{10} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{10}$. If such a solution is given, explain that the Egyptians would accept only solutions in which all of the fractions were different.

Record this solution on the board.

$$\frac{7}{10} = \frac{1}{2} + \frac{1}{5}$$

N 23

T: ***Often our rule for $*$ will suggest a solution. For example, try to use $*$ to express $^{11}/_{30}$ as a sum of unit fractions.***

S: ***$^{11}/_{30} = \frac{5+6}{5 \times 6} = 5 * 6 = ^1/_5 + ^1/_6$.***

Record this solution on the board and present $^8/_{15}$ in a similar manner ($\frac{8}{15} = \frac{5+3}{5 \times 3}$).

$$\frac{11}{30} = \frac{1}{5} + \frac{1}{6} \qquad \frac{8}{15} = \frac{1}{3} + \frac{1}{5}$$

T: ***There is another way to express $^{11}/_{30}$ as a sum of different unit fractions.***

Write a hint on the board.

$$\frac{11}{30} = \frac{10}{30} + \square$$

T: ***Does this suggest another solution?***

S: ***Yes. $^{11}/_{30} = ^{10}/_{30} + ^1/_{30} = ^1/_3 + ^1/_{30}$.***

Record this solution on the board.

$$\frac{11}{30} = \frac{10}{30} + \frac{1}{30} = \frac{1}{3} + \frac{1}{30}$$

T: ***Try to find two ways to express $^{11}/_{12}$ as a sum of different unit fractions.***

Students will observe that you cannot use $*$ because no pair of whole numbers has a sum of 11 and a product of 12. As necessary, give hints leading to the following solutions.

$$\frac{11}{12} = \frac{6}{12} + \frac{3}{12} + \frac{2}{12} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6}$$

$$\frac{11}{12} = \frac{6}{12} + \frac{4}{12} + \frac{1}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{12}$$

Refer to one of the solutions on the board; for example:

T: ***Why did we express $^{11}/_{12}$ as $^6/_{12} + ^3/_{12} + ^2/_{12}$?***

S: ***$^6/_{12}$, $^3/_{12}$, and $^2/_{12}$ each are equivalent to a unit fraction.***

T: ***Can you express $^2/_5$ as a sum of different unit fractions?***

If no productive suggestion comes from the class, write $^2/_5 = \frac{\square}{20}$ on the board as a hint. Suggest that students first find the indicated equivalent fraction for $^2/_5$ and then try to express that fraction as a sum of different unit fractions. Write this solution on the board.

$$\frac{2}{5} = \frac{8}{20} = \frac{5}{20} + \frac{2}{20} + \frac{1}{20} = \frac{1}{4} + \frac{1}{10} + \frac{1}{20}$$

T: ***This problem shows that sometimes we should look at other names for the given fraction.***

Write a list of fractions on the board, and ask students to express them as sums of different unit fractions.

$$\frac{13}{36} \quad \frac{13}{40} \quad \frac{7}{12} \quad \frac{3}{10} \quad \frac{3}{4} \quad \frac{11}{16} \quad \frac{4}{7} \quad \frac{2}{7} \quad \frac{2}{9}$$

Allow students to work with partners on these problems, and then let some students present their solutions. If a problem remains unsolved, you may wish to lead the class to a solution. One or more solutions for each fraction are shown below. Other solutions are possible.

$$\frac{13}{36} = \frac{9+4}{9 \times 4} = 9 * 4 = \frac{1}{9} + \frac{1}{4} \qquad \frac{13}{36} = \frac{12}{36} + \frac{1}{36} = \frac{1}{3} + \frac{1}{36}$$

$$\frac{13}{40} = \frac{8+5}{8 \times 5} = 8 * 5 = \frac{1}{8} + \frac{1}{5} \qquad \frac{13}{40} = \frac{10}{40} + \frac{2}{40} + \frac{1}{40} = \frac{1}{4} + \frac{1}{20} + \frac{1}{40}$$

$$\frac{7}{12} = \frac{6}{12} + \frac{1}{2} = \frac{1}{2} + \frac{1}{12} \qquad \frac{7}{12} = \frac{3+4}{3 \times 4} = 3 * 4 = \frac{1}{3} + \frac{1}{4}$$

$$\frac{3}{10} = \frac{2}{10} + \frac{1}{10} = \frac{1}{5} + \frac{1}{10}$$

$$\frac{3}{4} = \frac{2}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{4}$$

$$\frac{11}{16} = \frac{8}{16} + \frac{2}{16} + \frac{1}{16} = \frac{1}{2} + \frac{1}{8} + \frac{1}{16}$$

$$\frac{4}{7} = \frac{8}{14} = \frac{7}{14} + \frac{1}{14} = \frac{1}{2} + \frac{1}{14}$$

$$\frac{2}{7} = \frac{1}{2} \times \frac{4}{7} = \frac{1}{2} \times \left(\frac{1}{2} + \frac{1}{14}\right) = \frac{1}{4} + \frac{1}{28}$$

$$\frac{2}{9} = \frac{4}{18} = \frac{3}{18} + \frac{1}{18} = \frac{1}{6} + \frac{1}{18}$$

T: ***One set of problems on the Rhind Papyrus involved sharing fairly a number of loaves of bread among ten people. For example, if ten people share one loaf of bread, how much does each person receive?***

S: ***$\frac{1}{10}$ loaf.***

T: ***Similar problems are on your worksheet.***

Distribute Worksheet N23 for individual work. If time allows, check the worksheet collectively.

Home Activity

Suggest that students take home Worksheet N23 and explain to family members the problem of writing a fraction as the sum of unit fractions.

Name ________________ N23

Some problems on an Egyptian papyrus involved sharing loaves of bread among ten people. For each problem, write the solution as a unit fraction or as a sum of different unit fractions. Two problems are done for you.

Number of loaves	Number of people	Solution
2	10	$\frac{2}{10} = \frac{1}{5}$
3	10	$\frac{3}{10} = \frac{2}{10} + \frac{1}{10} = \frac{1}{5} + \frac{1}{10}$
4	10	$\frac{4}{10} = \frac{8}{20} = \frac{5}{20} + \frac{2}{20} + \frac{1}{20} = \frac{1}{4} + \frac{1}{10} + \frac{1}{20}$
5	10	$\frac{5}{10} = \frac{1}{2}$
6	10	$\frac{6}{10} = \frac{5}{10} + \frac{1}{10} = \frac{1}{2} + \frac{1}{10}$
7	10	$\frac{7}{10} = \frac{5}{10} + \frac{2}{10} = \frac{1}{2} + \frac{1}{5}$
8	10	$\frac{8}{10} = \frac{5}{10} + \frac{3}{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}$
9	10	$\frac{9}{10} = \frac{5}{10} + \frac{4}{10} = \frac{1}{2} + \frac{1}{4} + \frac{1}{10} + \frac{1}{20}$

Other solutions are possible.

N24 REVIEW OF POSITIONAL SYSTEMS — N 24

Capsule Lesson Summary

Review several different base abaci and writing in different bases. Present a calculation and decide in which base system the calculation was done. Discover a rule for distinguishing even and odd numbers written in base three.

Materials

Teacher
- Minicomputer checkers (optional)

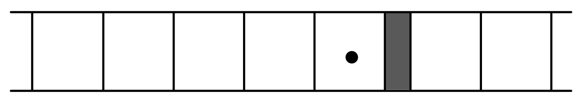
Student
- Paper
- Worksheets N24*, **, and ***

Description of Lesson

Note: Because there are several different bases referred to in this lesson, you must be careful how you read numerals. Read numerals written in bases other than base ten digit by digit; for example, read “31” written in base four as “three, one, base four.”

Exercise 1

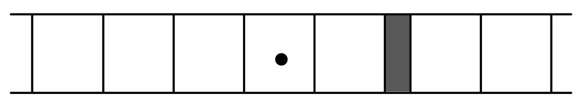
Draw this part of an abacus on the chalkboard and put a checker on the ones board.



T: ***What number is on the abacus?***

S: ***1.***

Move the checker one board to the left.



T: ***Now what number is on the abacus?***

Allow students to respond without commenting yourself. Someone should observe that the question cannot be answered until we know more about the abacus.

S: ***We don’t know what number it is until we know the rule for this abacus.***

T: ***That’s right. Which rules are you familiar with?***

S: ***Two checkers on a board trade for one checker on the next board to the left.***

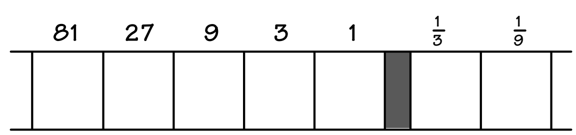
S: ***That would be the binary or base two abacus. We also know about base three and base twelve.***

T: ***Suppose this were the base three abacus. What would the rule be?***

S: ***Three checkers on a board trade for one checker on the next board to the left.***

T: ***How would we label the boards of the base three abacus?***

Label the boards with student assistance.



N 2 4

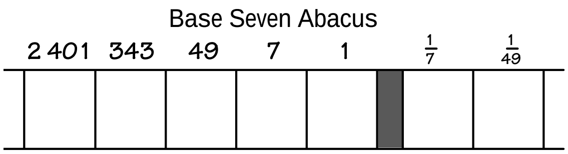
T: ***If we were conserving checkers, what would be the most we would ever need to put on one board of this base three abacus?***

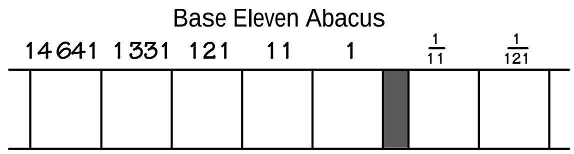
S: ***Two checkers. If there were three or more checkers we could make a trade, exchanging three checkers for one checker on the next board to the left.***

T: ***Yes. How many digits do we need for writing in base three?***

S: ***Three digits: 0, 1, and 2.***

Repeat this exercise for base seven and for base eleven.





The class should conclude that at most six checkers are needed on one board of the base seven abacus and that seven digits (0, 1, 2, 3, 4, 5, 6) are needed for writing in base seven. At most ten checkers are needed on one board of the base eleven abacus and eleven digits are needed for writing in base eleven.

Exercise 2

Write this information on the board.

Calculation	Base
12 + 13 = 30	

T: ***Here is an addition problem that is correct. Can you tell in what base system this problem was done?***

Allow a few minutes for students to study the situation and to experiment independently before leading a collective discussion.

S: ***It cannot be in the decimal system because 12 + 13 ≠ 30.***

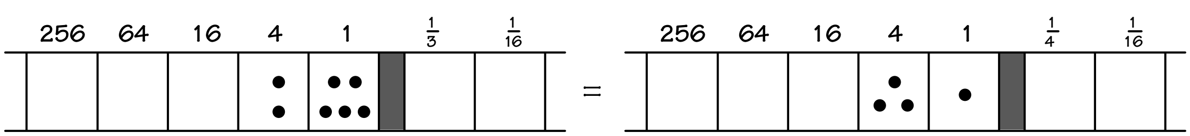
S: ***It cannot be in base two because 0 and 1 are the only two digits used in base two.***

S: ***Also, it cannot be in base three because the digit 3 is not used in base three.***

Some students may rely strictly on the abacus as an aid to finding a solution.

S: ***Maybe it was done in base four. Let's try using the base four abacus to do the problem.***

Relabel the abacus so that it is a base four abacus, and ask students to put on checkers for 12 and for 13. Invite students to make trades until at most three checkers are on any board.

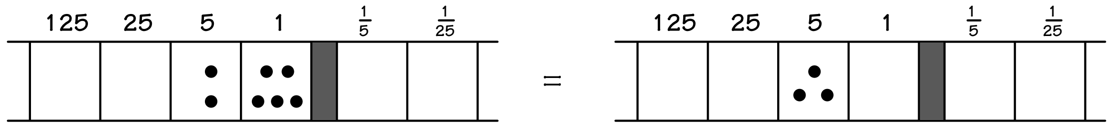


T: ***In base four 12 + 13 = 31, not 30.***

S: ***I tried something different. I looked just at the ones place and asked in which base would 2 + 3 leave 0 in the ones place. It could be base five because 2 + 3 = 10 in base five.***

$$\begin{array}{r} 12 \\ +13 \\ \hline 30 \end{array}$$

Once base five is suggested, use the base five abacus to do the calculation and verify that it is indeed the correct base.



Calculation	Base
12 + 13 = 30	5

Continue in a similar manner with these problems (Answers are in boxes.) The last problem may require some discussion as many answers are possible.

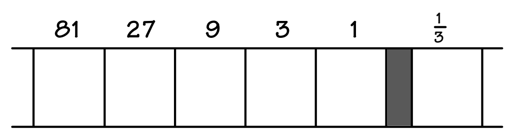
Calculation	Base
$3 \times 4 = 13$	$\boxed{9}$
$44 + 33 = 121$	$\boxed{6}$
$7 \times 8 = 56$	$\boxed{10}$
$2 \times 2 = 4$	$\boxed{\text{Any base greater than 4}}$

Exercise 3

T: ***In the decimal (base ten) system, how do we recognize odd and even numbers?***

S: ***Look at the ones digit. It is even if the ones digit is 0, 2, 4, 6, or 8. It is odd if the ones digit is 1, 3, 5, 7, or 9.***

T: ***Yes. Let's examine the situation for base three writing and see if we can find a rule.***



On the board, write the list of decimal numbers below, and ask students to write each number in the base three system on their papers. When many students have finished, check the work collectively before continuing. (Answers are in boxes.)

Decimal Writing	Base Three Writing	Decimal Writing	Base Three Writing
12	$\boxed{110}$	5	$\boxed{12}$
10	$\boxed{101}$	7	$\boxed{21}$
32	$\boxed{1012}$	13	$\boxed{111}$
40	$\boxed{1111}$	27	$\boxed{1000}$

T: ***Do you notice anything interesting about the base three writing for odd numbers and for even numbers?***

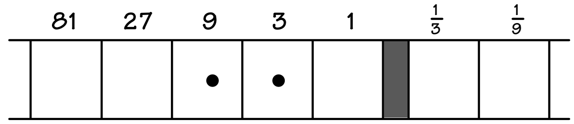
S: ***We cannot use the rule of looking at the ones digit.***

T: ***Yes, but can we find a new rule? I'll give you a hint. Think about putting each number on the base three abacus.***

N 24

Allow a few minutes for examining the situation. If no good idea is forthcoming, make the following suggestion.

Ask a student to put 12 on the abacus.



T: ***How many checkers did we use?***

S: ***Two checkers.***

Record this information in a third column. (See the illustration below.) Most students will not need to actually put checkers on the base three abacus to determine the number of checkers. Continue filling in this column until it is complete.

Decimal Writing	Base Three Writing	Number of Checkers	Decimal Writing	Base Three Writing	Number of Checkers
12	110	2	5	12	3
10	101	2	7	21	3
32	1012	4	13	111	3
40	1111	4	27	1000	1

T: ***What do you notice?***

S: ***An even number requires an even number of checkers, and an odd number requires an odd number of checkers.***

You may like to let students pick another even and another odd number to check.

T: ***How can we explain this?***

S: ***Each time you put a checker on the base three abacus (to the left of the bar), you add an odd number. If there are an odd number of checkers, then the number is odd because it is the sum of an odd number of odd numbers. The sum of an even number of odd numbers is even, so an even number of checkers results in an even number.***

S: ***When a number is given in base three writing, we can tell whether it is odd or even by the number of checkers on the abacus.***

Let students describe the rule in their own ways; then summarize the situation.

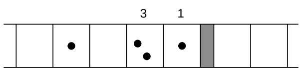
T: ***So, in the base three system, a number is odd if and only if the sum of its digits is odd. The number is even if and only if the sum of its digits is even.***

Worksheets N24*, **, and *** are available for individual work.

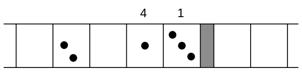
Name ______________ N24 *

Put each number on the abacus and fill in the boxes.

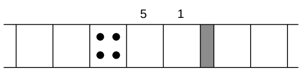
Decimal Writing | Base Three Writing

34  1021

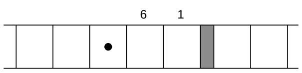
Base Four Writing

135  2013

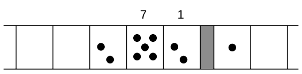
Base Five Writing

100  400

Base Six Writing

36  100

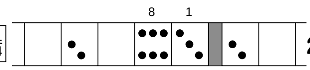
Base Seven Writing

$135\frac{1}{7}$  252.1

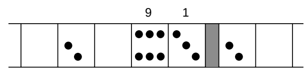
Name ______________ N24 **

Put each number on the abacus and fill in the boxes.

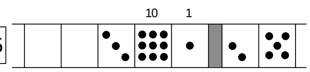
Decimal Writing | Base Eight Writing

$1075\frac{1}{4}$  2063.2

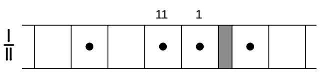
Base Nine Writing

$1515\frac{2}{9}$  2063.2

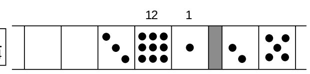
Base Ten Writing

391.25  391.25

Base Eleven Writing

$1343\frac{1}{11}$  1011.1

Base Twelve Writing

$541\frac{29}{144}$  391.25

Name ______________ N24 ***

Fill in the boxes showing the base for each calculation.

Calculation	**Base**
15 + 12 = 30	7
2 × 2 = 10	4
22 + 20 = 112	3
6 × 8 = 48	10
100 + 10 = 110	2
3 × 24 = 132	5

Any whole number larger than 2 is also a solution.

Capsule Lesson Summary

Include division by decimal numbers in a mental arithmetic activity. Use the result of a division calculation to do other related division calculations. Use multiplication functions to produce several division calculations with the same solution and, in particular, to transform a division problem such as $7.5\overline{)271.8}$ into an easier problem such as $15\overline{)543.6}$ or $75\overline{)2718}$. Use calculators in an exercise involving estimation and division of decimal numbers.

Materials

Teacher	• Calculator	**Student**	• Paper • Calculator

Description of Lesson

Exercise 1

Conduct a mental arithmetic activity as suggested below. Write the corresponding division facts on the board so that students can more easily make use of patterns and previous results.

60 ÷ 1 (60)	60 ÷ 10 (6)	60 ÷ 0.1 (600)
60 ÷ 2 (30)	60 ÷ 20 (3)	60 ÷ 0.2 (300)
60 ÷ 3 (20)	60 ÷ 30 (2)	60 ÷ 0.3 (200)
60 ÷ 4 (15)	60 ÷ 40 (1.5)	60 ÷ 0.4 (150)
60 ÷ 5 (12)	60 ÷ 50 (1.2)	60 ÷ 0.5 (120)
60 ÷ 6 (10)	60 ÷ 60 (1)	60 ÷ 0.6 (100)

Refer to a problem where the divisor is a decimal to ask,

T: ***What number is 60 ÷ 0.1?*** (600) ***How did you do the calculation?***

S: ***Think about 60 ÷ 0.1 as asking, "How many dimes are there in $60?" There are ten dimes in one dollar, so there are 10 x 60 or 600 dimes in $60.***

S: ***60 ÷ 10 = 6, and 60 ÷ 1 = 60, so 60 ÷ 0.1 = 600. Each time, the second number (10, 1, 0.1) is divided by 10 and the quotient (6, 60, 600) is multiplied by 10.***

S: ***÷ 0.1 is the same as ÷ $\frac{1}{10}$, and ÷ $\frac{1}{10}$ is the same as 10x. 10 x 60 = 600.***

Encourage students to make use of earlier results as you continue the mental arithmetic activity.

N 2 5

Exercise 2

On the board, write this list of division problems headed by a division fact. Suggest that students use the given information to solve the other problems. Solutions are in boxes.

$$434 \div 7 = 62$$

$$4\,340 \div \boxed{70} = 62 \qquad 217 \div 7 = \boxed{31}$$

$$\boxed{4\,340} \div 7 = 620 \qquad 43.4 \div \boxed{0.7} = 62$$

$$868 \div 7 = \boxed{124} \qquad \boxed{217} \div 3.5 = 62$$

$$868 \div \boxed{14} = 62 \qquad 21.7 \div \boxed{0.35} = 62$$

Erase the calculations that do not have 62 as the quotient.

T: ***Why do each of these division calculations have the same solution: 62?***

S: ***In each case, you can multiply or divide 434 and 7 by the same number to get the new calculation. For example, 2 x 434 = 868 and 2 x 7 = 14, so 868 ÷ 14 = 62. Also, 434 ÷ 10 = 43.4, and 7 ÷ 10 = 0.7, so 43.4 ÷ 0.7 = 62.***

T: ***Can we use this observation to create other division facts for 62?***

Add students' suggestions to the list on the board. For example:

$$86.8 \div 1.4 = 62 \qquad 1\,736 \div 28 = 62$$

$$8.68 \div 0.14 = 62 \qquad 2\,170 \div 35 = 62$$

T: ***This multiplication technique provides a way to simplify some division problems involving decimal numbers.***

Write this problem on the board.

$$7.5\overline{)271.8}$$

T: ***Can you think of another division problem with the same solution that might be easier to solve?***

Write some problems that students suggest on the board, for example:

S: ***Multiply each number by 2.***
2 x 7.5 = 15 and 2 x 271.8 = 543.6.

$$15\overline{)543.6}$$

S: ***Multiply each number by 10.***
10 x 7.5 = 75 and 10 x 271.8 = 2 718.

$$75\overline{)2\,718}$$

S: ***Multiply each number by 4.***
4 x 7.5 = 30 and 4 x 271.8 = 1 087.2.

$$30\overline{)1\,087.2}$$

Instruct students to solve the problem of their choice independently. Insist that students continue the division process until the remainder is 0.

After a while, call on volunteers to explain their methods. Two samples are shown below. The class should find that indeed the problems do have the same solution and that 271.8 ÷ 7.5 = 36.24.

$75\overline{)2718}$ = 36.24

75)2 718	
−2 250	30
468	
−450	6
18	
−15	0.2
3	
−3	0.04
0	

$15\overline{)543.6}$ = 36.24

15)543.6	
−450.0	30
93.6	
−90.0	6
3.6	
−3.0	0.2
0.6	
−0.6	0.04
0	

T: ***Here we continued the division process until the remainder was 0. Is that always possible?***

Let students express their opinions.

T: ***Let's try another problem. What division calculation does the fraction ⅚ suggest?***

S: ***5 ÷ 6, since ⅚ = 5 ÷ 6.***

$$\frac{5}{6} = 5 \div 6$$

T: ***Let's use long division to find a decimal name for ⅚.***

Let students tell you how to proceed in the division process. For example:

6)5.0000	
−4.8	0.8
0.2000	
−0.1800	0.03
0.0200	
−0.0180	0.003
0.0020	
−0.0018	0.0003
0.0002	

T: ***There is still a remainder, so we could continue by adding more zeros. But would the process ever end?***

S: ***No, the pattern among the remainders, 0.2, 0.02, 0.002, and 0.0002, suggests that the process would continue.***

T: ***If we could continue forever, what would the result be?***

S: ***0.83333…or*** $0.8\overset{*}{3}$.

T: ***So a decimal "nickname" for ⅚ is*** $0.8\overset{*}{3}$.

Write these one-star problems on the board, asking students to copy and solve them independently. Remind students that they can change a problem involving decimals into one that is easier to solve. Insist that the division process be continued until the remainder is 0 or until a pattern emerges. (Answers are in boxes. Each multiplication function in parentheses yields a division problem with a whole number divisor.)

* $0.8\overline{)18.88}$ $\boxed{23.6}$ (10×, 5×)

$3.5\overline{)210.7}$ $\boxed{60.2}$ (2×, 4×, 10×)

N 25

As students complete the one-star problems, write these two- and three-star problems on the board.

** $0.62\overline{)25.172}$ $\boxed{40.6}$ (100×) $0.085\overline{)0.60605}$ $\boxed{7.13}$ (200× / 1000×)

*** $\frac{5}{9} = 5 \div 9 = \boxed{0.\overset{*}{5}}$ $\frac{22}{7} = 22 \div 7 = \boxed{3.\overline{142857}^{*}}$

Exercise 3 (optional)

This calculator activity involving estimation and division of decimal numbers may be used during this lesson or at another time if you wish.

Write this information on the board. $7 \div \square = 60.$ ______

T: ***Using your calculators, find a number to put in the box so that the quotient is between 60 and 61.***

If students protest that the problem is impossible because 60 is more than 7, assure them that there are solutions and suggest they try decimal numbers.

Record some attempts on the board even though the quotients may not be between 60 and 61. These may serve as hints. For example:

$7 \div 0.1 = 70$ $7 \div 0.13 \approx 53.846153$

$7 \div 0.2 = 35$ $7 \div 0.15 \approx 46.666666$

$7 \div 0.11 \approx 63.636363$

Occasionally call attention to the information on the board. For example:

T: ***7 ÷ 0.1 = 70 and 7 ÷ 0.2 = 35. What does that tell us?***

S: ***A solution for the number in the box is between 0.1 and 0.2, since 60 is between 35 and 70.***

T: ***What are some numbers between 0.1 and 0.2?***

S: ***0.18, 0.15, and 0.12.***

T: ***What happens to the quotient when we increase the divisor (for example, from 0.1 to 0.2)?***

S: ***As we increase the divisor, the quotient decreases. For example, 7 ÷ 0.1 = 70 and 7 ÷ 0.2 = 35. 0.1 is less than 0.2, while 70 is more than 35.***

After a while students should find some solutions; for example:

$7 \div 0.115 \approx 60.869565$

$7 \div 0.116 \approx 60.344827$

If time allows, present the following problems in a similar manner.

$13 \div \square = 60.$ ______ $100 \div \square = 15.$ ______

Feel free to invent your own problems and to use this exercise at other times in order to improve students' knowledge of decimal numbers.

Capsule Lesson Summary

Review several methods of doing percent calculations. Solve percent problems involving savings, loans, and interest. Observe that a% of b = b% of a. Solve percent problems involving sales, discounts, and taxes.

Materials

Teacher		Student	
	• Colored chalk		• Calculator • Paper • Worksheets N26*, **, ***, and ****

Description of Lesson

Exercise 1

Write the following problems on the board. Invite students to solve the problems without the use of calculators and to explain their methods. Accept explanations based on patterns, intuition, or graphics. (Answers are in boxes.)

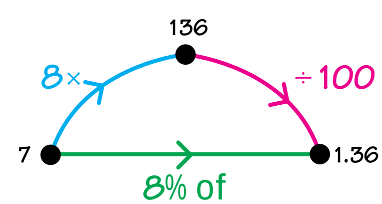
50% of 46 = $\boxed{23}$	50% of $\boxed{92}$ = 46
25% of 20 = $\boxed{5}$	25% of $\boxed{80}$ = 20
10% of 62 = $\boxed{6.2}$	10% of $\boxed{620}$ = 62
30% of 60 = $\boxed{18}$	30% of $\boxed{200}$ = 60
8% of 17 = $\boxed{1.36}$	8% of $\boxed{212.5}$ = 17

The following dialogue gives possible solutions for a few of the above problems.

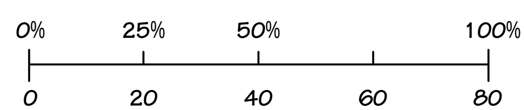
S: ***25% of 20 = 5, since 50% of 20 = 10 and ½*** x ***10 = 5.***

S: ***10% of 62 = 6.2, since "10% of" is the same as ÷10 and 62 ÷ 10 = 6.2.***

S: ***8% of 17 = 1.36, because 8*** x ***17 = 136 and 136 ÷ 100 = 1.36.***



S: ***25% of*** $\boxed{80}$ ***= 20 because 50% of 80 = 40 or 100% of 80 = 80.***



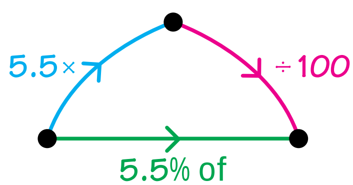
S: ***30% of 200 = 60. 60 = 3*** x ***20 and 10% of 200 = 20, so 30% of 200 = 60.***

Exercise 2

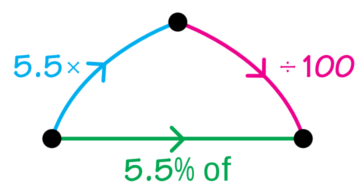
Allow students to use calculators to do the calculations in this exercise.

Write the following problems on the board. Let students discuss and compare the two problems. After a while, draw an arrow picture beneath each problem and let students tell you how to label the arrows.

Melanie puts \$88 in a bank and earns $5\frac{1}{2}$% interest for one year. How much interest does she earn in one year?



Howard puts money in the bank and earns $5\frac{1}{2}$% interest for one year. At the end of the year, he earns \$88 interest. How much money did Howard put in the bank?



T: ***How could we label the dots to solve these two problems?***

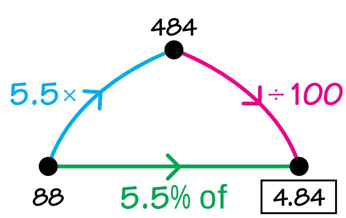
S: ***Melanie puts \$88 in the bank. To compute her interest, we must find 5½% of \$88. Put 88 at the starting dot of the 5.5% arrow. Calculate 5.5 x 88 and divide that number by 100.***

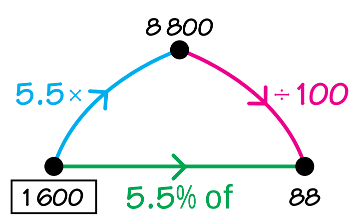
S: ***Since 5.5 x 88 = 484 and 484 ÷ 100 = 4.84, Melanie earns \$4.84 interest in one year.***

Label the dots in the arrow picture on the left.

S: ***Howard earns \$88 interest. So the problem is "5.5% of what number is 88?" Put 88 at the ending dot of the 5.5% arrow.***

With student input, complete the arrow picture.





S: ***Howard put \$1 600 in the bank at the beginning of the year.***

T: ***In each arrow picture, which dot is for the amount of interest earned?***

S: ***The ending dot of the 5.5% arrow.***

T: ***Which dot is for the amount of money put in the bank?***

S: ***The starting dot of the 5.5% arrow.***

Review the similarities and differences between the two problems. Especially observe how the class decided where to put 88 in each case.

T: ***Why is a bank willing to pay you interest when you give them money to save for you? How does the bank make a profit?***

After students respond, select two students to help you act out the following skit.

T: ***Kevin has $2 000 and goes to a bank. When offered 7% yearly interest, he agrees to deposit his money in the bank. A day later, Tracy needs to borrow $2 000 to buy a used car and she goes to the same bank. What do you think the bank does?***

S: ***It lends her money, but at an interest rate higher than 7%.***

T: ***Yes, the bank lends Tracy $2 000 at a 16% yearly interest rate.***

T: ***A year later, Kevin goes to the bank to collect his money. How much money does he receive?***

S: ***$2 140. 7% of $2 000 = $140.***

T: ***The next day Tracy goes to the bank to pay off her loan. How much does she have to pay the bank?***

S: ***$2 320. 16% of $2 000 = 320.***

T: ***How much profit has the bank made?***

S: ***$180 (320 – 140 = 180).***

T: ***Why didn't Kevin lend the money directly to Tracy?***

S: ***They might not have known each other.***

S: ***Kevin might not have trusted Tracy to return the money.***

S: ***Tracy might not have been able to pay Kevin back exactly when he wanted the money.***

Exercise 3

Write the following problems on the board. Instruct students to copy and complete the calculations on their papers. (Answers are in boxes.)

16% of 50 = $\boxed{8}$ 25% of 12 = $\boxed{3}$
50% of 16 = $\boxed{8}$ 12% of 25 = $\boxed{3}$

28% of 75 = $\boxed{21}$ 150% of 24 = $\boxed{36}$
75% of 28 = $\boxed{21}$ 24% of 150 = $\boxed{36}$

After a while, check the problems collectively. Discuss any problems students have difficulty with.

T: ***What do you notice?***

S: ***The problems are in pairs with the same result for each pair of problems.***

S: ***It appears that you can switch the two numbers in a percent calculation and not change the result.***

N 26

You may like to observe for which of the four operations +, –, x, and ÷ you can switch the numbers without changing the result. That is,

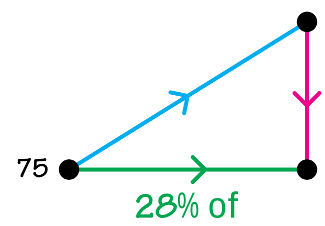
$$\text{(Yes)} \quad 12 + 4 = 4 + 12 \qquad 12 \times 4 = 4 \times 12$$

$$\text{(No)} \quad 12 - 4 \neq 4 - 12 \qquad 12 \div 4 \neq 4 \div 12$$

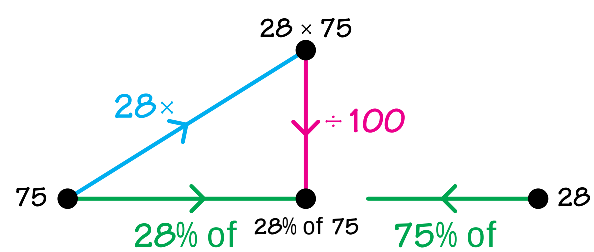
T: ***With percent calculations, as with addition and multiplication, switching the two numbers does not change the result. Do you understand why this is true?***

Let students express their opinions while you begin an arrow picture on the board.

T: ***Let's show that 28% of 75 is the same as 75% of 28 in an arrow picture without separating the two calculations.***



Invite students to label the arrows and put **28 x 75** and **28% of 75** at the appropriate dots. Then add the following information to the arrow picture.



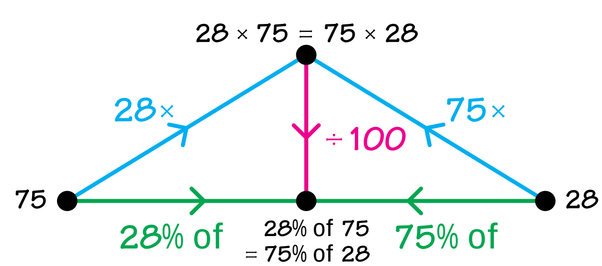
T: ***We would like to show that 28% of 75 and 75% of 28 are the same. Can anyone convince us that the 75% arrow should end here (at* 28% of 75*).***

S: ***75 x 28 = 28 x 75, so a 75x arrow starting at 28 ends at the top dot.***

Draw an appropriate arrow.

S: ***75% of is the same as 75x followed by ÷100. Therefore, we can draw the 75% arrow from 28 to the dot for* 28% of 75.**

S: ***28% of 75 is the same as 75% of 28.***



Illustrate that 28 and 75 could be replaced by another pair of numbers throughout the picture. Then write "17% of 10" and "10% of 17" on the board.

T: ***Knowing that you can switch the two numbers often makes a percent problem easier. Which do you find easier, 17% of 10 or 10% of 17?***

S: ***10% of 17, because “10% of” is the same as ÷10 and 17 ÷ 10 = 1.7.***

T: ***So what number is 17% of 10?***

S: ***Also 1.7.***

Present the following two problems in a similar manner. Students may decide that 200% of 46 is easier to calculate mentally. (Answers are in boxes.)

$$46\% \text{ of } 200 = \boxed{92} \qquad 200\% \text{ of } 46 = \boxed{92}$$

Ask students to copy and solve the following problems. (Answers are in boxes.) Encourage students to do the calculations mentally, not with a calculator. Your students may use methods other than those indicated below.

$$13\% \text{ of } 50 = \boxed{50\% \text{ of } 13 = 6.5} \qquad 62.3\% \text{ of } 100 = \boxed{100\% \text{ of } 62.3 = 62.3}$$

$$35\% \text{ of } 20 = \boxed{20\% \text{ of } 35 = 7} \qquad 26\% \text{ of } 150 = \boxed{150\% \text{ of } 26 = 26 + 13 = 39}$$

Worksheets N26*, **, ***, and **** are available for individual work.

Home Activity

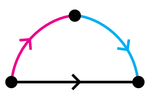
Suggest that students explain the percent property a% of b = b% of a to family members.

Name ________________ N26 *

Complete.

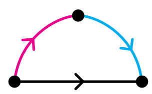
50% of 132 = $\boxed{66}$	50% of $\boxed{264}$ = 132
10% of 84 = $\boxed{8.4}$	10% of $\boxed{840}$ = 84
20% of 115 = $\boxed{23}$	20% of $\boxed{575}$ = 115
1% of 410 = $\boxed{4.1}$	1% of $\boxed{41000}$ = 410

You may use the arrow pictures to help solve these problems.



30% of

All 90 students in sixth grade at Hopkins School voted for their favorite movie. *Space Cadets* won with 30% of the votes. How many students voted for *Space Cadets*? 27



30% of

Sheila received 90 votes for student body president which is 30% of the total votes. How many students voted? 300

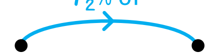
Name ________________ N26 **

15% of

Lou has $54.50 and wants to buy an anniversary present for his parents. He considers a silver tray costing $48. However, Lou must remember that there is a 15% combined luxury and sales tax.
How much tax is there on the $48 silver tray? $7.20

Does Lou have enough money to buy the silver tray? no
If yes, how much change will he receive? ____
If no, how much more money does he need? 70¢

$7\frac{1}{2}$% of



Lou also considers a large gardenia plant costing $50. There is only a $7\frac{1}{2}$% sales tax on the plant.
How much tax is there on the gardenia plant? $3.75

Does Lou have enough money to buy the gardenia plant? yes
If yes, how much change will he receive? 75¢
If no, how much more money does he need? ____

Name ________________ N26 ***

A	B	C
Spring Sale! 25% off all winter wear. Additional 10% off on items originally priced $50 or more.	*Spring Sale!* $\frac{1}{3}$ off all winter clothing. Make room for Spring!	*Spring Sale!* 30% off every winter clothing item in stock.

Three nearby stores are having winter clearance sales. Some items are available at all three stores; some only at two of the three stores. Fill in this table to show where to pay the least amount for individual items.

Item	Original Price	Available at	Best Discounted Price (at which store)	$7\frac{1}{2}$% Sales Tax	Best Total Cost
Sweater	$44.40	A, B, & C	29.60 (**B**)	2.22	$31.82
Boots	$39.75	A & C	27.83 (**C**)	2.09	$29.92
Coat	$78	A, B, & C	50.70 (**A**)	3.80	$54.50
Jacket	$62	A & B	40.30 (**A**)	3.02	$43.32
Suit	$210.90	B & C	140.60 (**B**)	10.55	$151.15

Name ________________ N26 ****

TIRE SALE!
All prices reduced 15%

1. Ms. Thomas saved $12 by buying two tires at the tire sale.
   What was the original price of the two tires? $80
   How much did Ms. Thomas pay for the tires? $68

2. You can buy a set of luggage for $160 cash. If you buy it on the the installment plan, you must make 12 monthly payments of $15.25 each. What is the total cost to buy the luggage on the installment plan? $183
   What percent over the cash price is the installment plan price? $14\frac{3}{8}$%

3. Drew prepares a contest box with tickets like these:
   SORRY, TRY AGAIN WINNER
   He puts 500 tickets in the box of which 1% are WINNER tickets. Now Drew wants to take out some of the SORRY, TRY AGAIN tickets so that 2% of the tickets will be WINNER tickets. How many SORRY, TRY AGAIN tickets should he remove? 250

4. Suppose a customer buys an item from Store **C** (see Worksheet N26***). Would the total cost be different if the store:
   - discounted the original price first and then added sales tax, or
   - added sales tax to the original price and then discounted that amount? Either way, the total cost would be the same.

N27 COUNTING POSITIVE DIVISORS #1 N 2 7

Capsule Lesson Summary

Express some numbers as products of prime numbers. Given the prime factorization of 637, express all of its positive divisors and some of its multiples as products of prime numbers, making use of exponential notation. Use a string picture as an aid for counting the number of positive divisors of $3^2 \times 7^3$.

Materials

Teacher	• Colored chalk	**Student**	• Paper • Colored pencils, pens, or crayons • Worksheets N27*, **, and ***

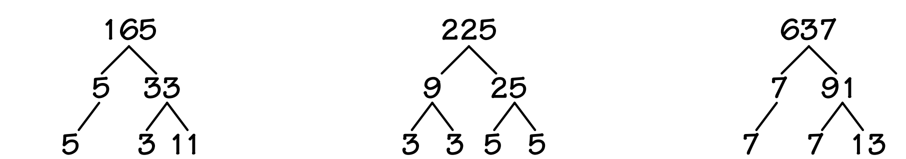
Description of Lesson

Exercise 1

Write this list of numbers on the board. 165 225 637

T: ***On a piece of paper, write each of these numbers as a product of prime numbers. You may use a prime number more than once. Use exponents when a prime number occurs more than once in the product.***

Allow several minutes for individual or partner work on these problems. Check the work collectively by inviting students to put their solutions, perhaps using factorization trees, on the board. Solutions and sample trees are shown below.



$165 = 3 \times 5 \times 11$ $225 = 3^2 \times 5^2$ $637 = 7^2 \times 13$

Some students may need help with 637. You can use the following discussion to find a prime factorization for 637.

T: ***63 is a multiple of 7. Why?***

S: ***9 x 7 = 63.***

T: ***Is 630 a multiple of 7?***

S: ***630 is also a multiple of 7; 90 x 7 = 630.***

T: ***How much more than 630 is 637?***

S: ***637 is 7 more than 630.***

Record this information in a factorization tree for 637. (See above.)

N 27

T: ***7 is a prime number. Is 91 prime?***

S: ***No, 7 x 13 = 91.***

S: ***13 is a prime number.***

S: ***So, 637 = 7 x 7 x 13 = 7^2 x 13.***

Whether or not you need to lead the above discussion, continue the lesson with an examination of positive divisors and multiples of 637.

T: ***Name a positive divisor of 637 and, if possible, express it as a product of prime numbers.***

S: ***49 is a positive divisor of 637 and 49 = 7^2.***

Record the positive divisors of 637 in a list on the board.

Positive Divisors of $7^2 \times 13$

$49 = 7^2$
7
$91 = 7 \times 13$
$637 = 7^2 \times 13$
1
13

T: ***Do you notice anything interesting about the positive divisors of 7^2 x 13?***

S: ***Each divisor, except 1, has only 7s and/or 13s in its prime factorization.***

T: ***Yes. Now let's write some multiples of 637 as products of prime numbers. Who can suggest a multiple of 637?***

Record multiples of 637 in a list on the board as students suggest them. You may encourage observation of patterns by recording only prime factorizations.

Multiples of $7^2 \times 13$

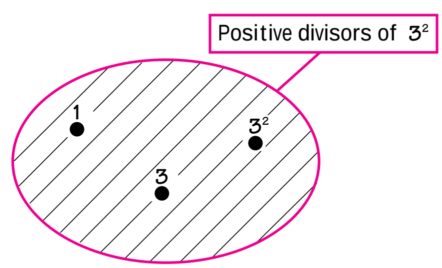
$2 \times 7^2 \times 13$	$2 \times 5 \times 7^2 \times 13$
$2^2 \times 7^2 \times 13$	$7^5 \times 13$
$2^3 \times 7^2 \times 13$	$3 \times 7^7 \times 13$
$5^8 \times 7^2 \times 13$	$7^3 \times 13^2$

Exercise 2

T: ***How many positive divisors does 3^2 have?***

S: ***Three; 1, 3, and 3^2 are positive divisors of 3^2.***

Show the three divisors in a string picture.



T: ***What are some other numbers that have exactly three positive divisors?***

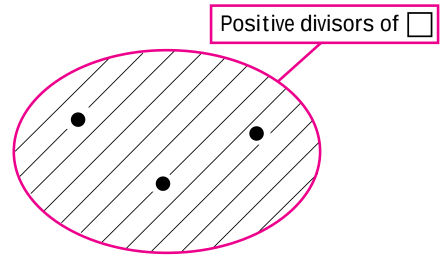
Allow a few minutes for students to think about the question. Encourage students to express the numbers as products of prime numbers.

S: ***4 has three positive divisors.***

T: ***Yes. How do you express 4 as a product of prime numbers?***

S: ***4 = 2 x 2 = 2^2.***

Alter the string picture (as shown here) by erasing the numerals, and start a list of the numbers with exactly three positive divisors.



Continue to let students suggest numbers that have exactly three positive divisors. Any disputes may be settled by listing and then counting the positive divisors of a number.

Numbers with exactly three positive divisors

$9 = 3^2$
$4 = 2^2$
$25 = 5^2$
$49 = 7^2$
$121 = 11^2$

T: ***Do you notice anything interesting about numbers with exactly three positive divisors?***

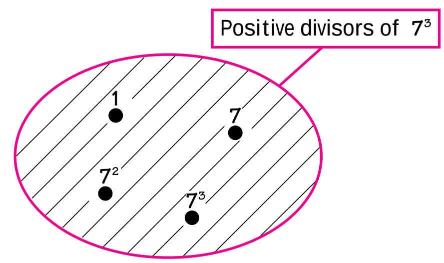
S: ***Each number with exactly three positive divisors is the square of a prime number.***

A few students may incorrectly suggest that any number squared has exactly three positive divisors. In that case, choose a non-prime number such as 6 to show that $6^2 = 36$ and 36 has more than three positive divisors.

T: ***How many positive divisors does 7^3 have?***

S: ***Four; 1, 7, 7^2, and 7^3 are positive divisors of 7^3.***

Show the four divisors of 7^3 in a string picture.



T: ***Name some other numbers with exactly four positive divisors.***

List numbers on the board as students suggest them.

Note: Any positive prime number to the third power has exactly four positive divisors. Also, any number that is the product of two distinct prime numbers has exactly four positive divisors. Accept both types of numbers, but be certain that students recognize numbers of the former type.

Numbers with exactly four positive divisors

$343 = 7^3$	$6 = 2 \times 3$
$8 = 2^3$	$10 = 2 \times 5$
$125 = 5^3$	$15 = 3 \times 5$
$27 = 3^3$	$35 = 5 \times 7$
$1\,331 = 11^3$	$77 = 7 \times 11$

S: ***Any prime number to the third power has exactly four positive divisors.***

T: ***So 3^2 has exactly three positive divisors and 7^3 has exactly four positive divisors. How many positive divisors does $3^2 \times 7^3$ have?***

Record estimates on the board. If a student says that $3^2 \times 7^3$ has twelve divisors, ask for an explanation but do not insist on the one that follows. Otherwise, continue the lesson by drawing the string picture shown in the next illustration (but with no labels on the dots).

S: ***Twelve.***

T: ***How did you conclude that $3^2 \times 7^3$ has exactly 12 positive divisors?***

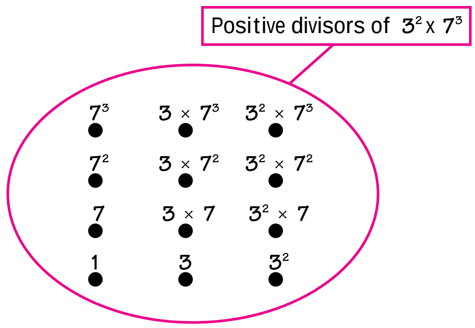
N 27

S: ***3^2 has three positive divisors and 7^3 has four positive divisors. $3 \times 4 = 12$.***

S: ***Look at the exponents. Increase each exponent by 1 and then multiply. $3 \times 4 = 12$.***

Arrange twelve dots in a string picture on the board.

Ask students to name positive divisors of $3^2 \times 7^3$, and put the divisors in the string in the arrangement shown below, no matter in which order they are given.



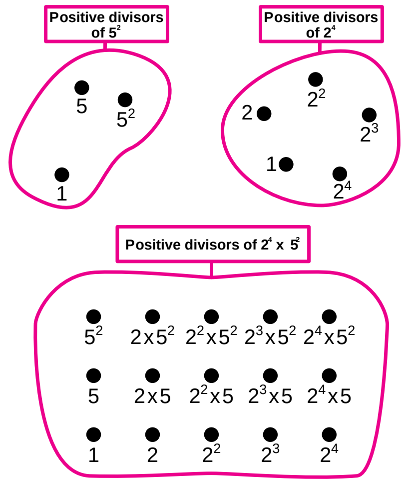
T: ***Is there any positive divisor of $3^2 \times 7^3$ that we left out?*** (No)
How many positive divisors does $3^2 \times 7^3$ have? (Twelve)

If appropriate, comment on how the string picture supports multiplication methods of counting the number of positive divisors.

Worksheets N27*, **, and *** are available for individual work.

Name ____________ N27 *

Label the dots.

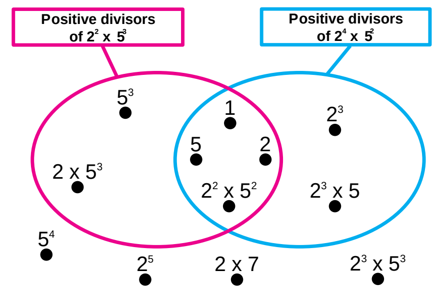


Other arrangements of the numbers within each string are possible. Our choice for the lower string suggests a counting method and allows a quick check that all the positive divisors of $2^4 \times 5^2$ are included.

Name ____________ N27 **

Put these numbers in the string picture.

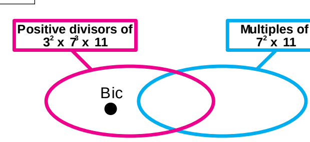
1	2	$2^2 \times 5^2$
5	5^3	2×5^3
2^3	5^4	$2^3 \times 5$
2^5	2×7	$2^3 \times 5^3$



Name ____________ N27 ***

Bic is a secret number.

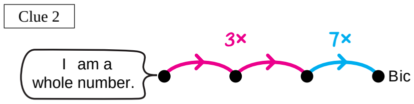
Clue 1



Bic could be $1, 3, 3^2, 7, 7^2, 7^3, 11, 3 \times 7, 3 \times 7^2, 3 \times 7^3, 3 \times 11,$
$3^2 \times 7, 3^2 \times 7^2, 3^2 \times 7^3, 3^2 \times 11, 7 \times 11, 3 \times 7 \times 11, 3^2 \times 7 \times 11$

(Hint: There are between 15 and 20 possibilities for Bic.)

Clue 2



Bic could be $3^2 \times 7$, $3^2 \times 7^2$, $3^2 \times 7^3$, or $3^2 \times 7 \times 11$

Clue 3

$$\text{Bic} \sqcap (5 \times 7^3) = 7^2$$

Who is Bic? $3^2 \times 7^2$

N28 BASE $\hat{2}$ ABACUS #1 N 28

Capsule Lesson Summary

Introduce the rule of the base $\hat{2}$ abacus. Practice putting on numbers, reading them, and making trades. Observe the effect of moving checkers to the left on the abacus. Starting with 2 000 checkers on the ones board, make trades to put 2 000 in standard configuration on the base $\hat{2}$ abacus.

Materials

Teacher
- Colored chalk
- Minicomputer checkers (optional)
- Weighted checker set (optional)
- Blackline N28

Student
- Paper
- Base $\hat{2}$ abacus

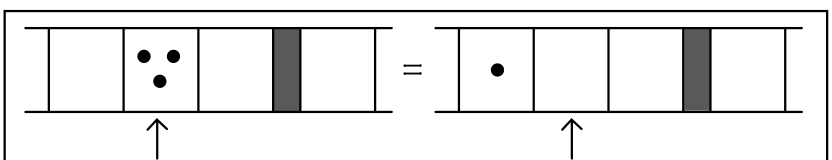
Advance Preparation: Use Blackline N28 to make copies of an abacus for students.

Description of Lesson

Exercise 1

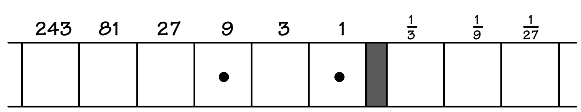
Draw part of an abacus on the chalkboard, and let the class choose which base abacus it will be.

If they choose base three, for example, briefly review its rule:



Three checkers on a board trade for one checker on the next board to the left.

Then, with student assistance, label the boards of the abacus. Put on a number with two or three checkers. For example:

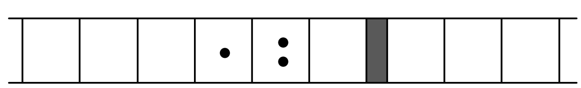


T: ***What number is on the abacus?*** (10) ***What happens if we move all these checkers one board to the left?***

S: ***We get 30 on the abacus; we multiply the number by 3.***

Demonstrate this fact several times by moving all the checkers one board to the left, each time noting the number on the abacus: 10, 30, 90, 270.

Clear the abacus and erase the board labels. Then put on this configuration of checkers.

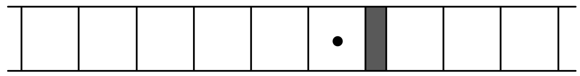


T: ***Today we are going to consider an abacus that uses a new kind of rule. The number on this abacus is 0. In fact, anytime there is one checker on a board and two checkers on the next board to the right, the number is 0.***

N 28

Remove the checkers. Put one checker on the first board left of the bar.

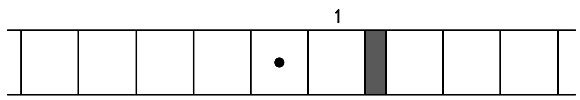
T: ***What number is this?***



S: ***1.***

Label the board and put the checker on the next board to the left.

T: ***Do you know what number this is?***



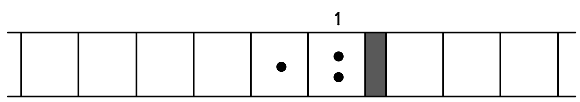
Students will need time to study the situation.

S: ***$\hat{2}$.***

T: ***How did you get $\hat{2}$?***

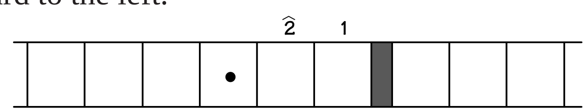
Invite someone to give an explanation. You may need to do some prompting with such a new situation. Perhaps a student will put this configuration on the abacus.

S: ***This is 0; that's the rule of this abacus. Two checkers on the ones board is 2, and $\hat{2} + 2 = 0$. So the other checker must be for $\hat{2}$? it is on the $\hat{2}$s board.***



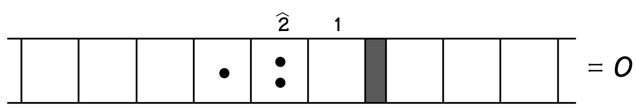
Label the board and put the checker on the next board to the left.

T: ***What number is this?***



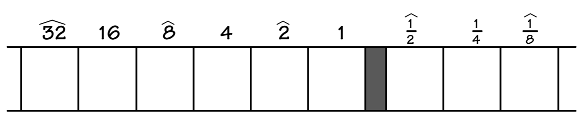
S: ***4.***

S: ***This is 0. Two checkers on the $\hat{2}$s board is $\hat{4}$ and $4 + \hat{4} = 0$.***



Continue in this manner until each board is labeled. Instruct students to label the boards on their copies of the abacus.

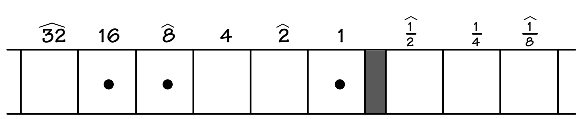
T: ***Does this abacus remind you of any other abacus?***



S: ***It is like the binary abacus, except every other board is for a negative number. Instead of 1, 2, 4, 8, 16, 32, and so on, here it is 1, $\hat{2}$, 4, $\hat{8}$, 16, $\widehat{32}$, and so on.***

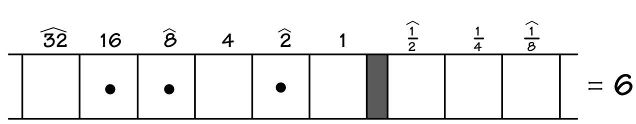
T: ***Just like the binary abacus, we can always put numbers on this abacus with at most one checker on a board.*** (Point to the $\hat{2}$s board.) ***Because this board is for $\hat{2}$, we call this abacus the*** **base** ***$\hat{2}$ abacus.***

Display this configuration and ask students to write the decimal name for this number.

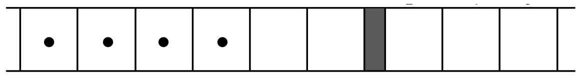


S: ***$1 + \hat{8} + 16 = 9$.***

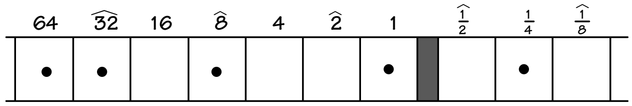
T: ***Now put 6 on your abacus. Remember, try to do it with at most one checker on a board.***



Ask students to put on one or two more numbers such as $\widehat{20}$ and 25.25.

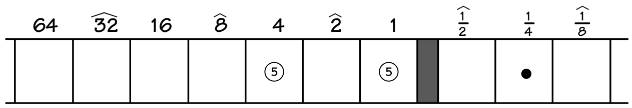


= $\widehat{20}$



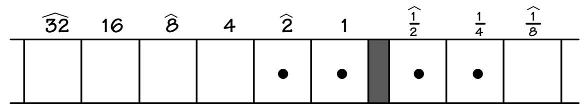
= 25.25

Note: Previous experience with weighted checkers might suggest other correct configurations, as illustrated below. Accept these as correct but encourage configurations with at most one checker on a board.



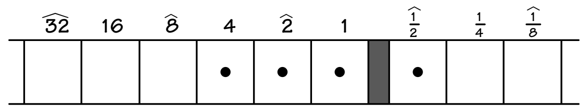
= 25.25

Put this configuration of checkers on the abacus.



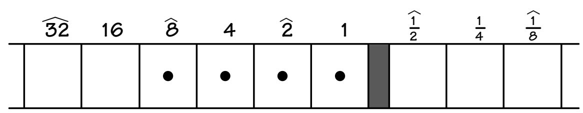
T: ***What number is this?***

S: ***$\widehat{1.25}$, because $\widehat{2} + 1 = \widehat{1}$ and $\widehat{\frac{1}{2}} + \frac{1}{4} = \widehat{\frac{1}{4}}$.***

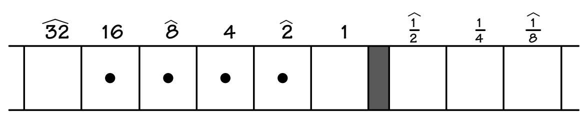


Move each checker one board to the left and ask for the new number.

S: ***2.5***



Continue to move each checker one board to the left a couple more times, each time asking for the new number.



S: ***$\widehat{5}$.***

S: ***10.***

T: ***What happens when we move all of the checkers one board to the left?***

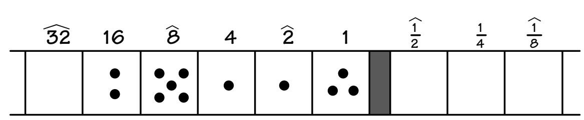
S: ***Moving the checkers one board to the left multiplies the number by $\widehat{2}$.***

S: ***On the binary abacus, moving checkers one board to the left doubles the number. Here it is almost the same except it also changes from positive to negative or from negative to positive.***

You may wish to compare this activity as well to a similar one on the base three abacus.

Exercise 2 ______________

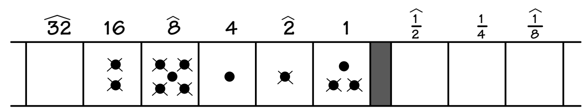
Display this configuration of checkers.



T: ***Let's display the same number with fewer checkers.***

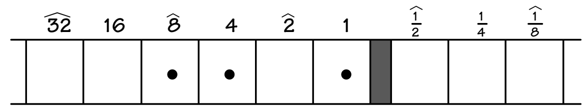
N 2 8

Invite students to take off checkers that represent 0, using the rule of the abacus.



S: ***One checker on the $\widehat{2}$s board wipes out two checkers on the ones board.***

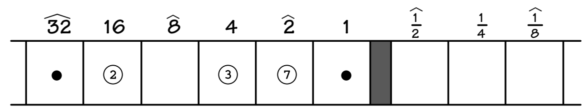
S: ***Two checkers on the 16s board wipe out four checkers on the $\widehat{8}$s board.***



T: ***What number is on the abacus?***

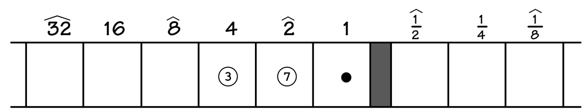
S: $\widehat{3}$.

Display this configuration of checkers, and ask students to take off checkers that represent 0 until there is at most one checker on a board.

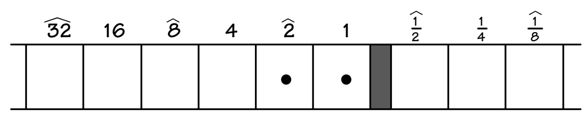


The following is one possible sequence of steps.

S: ***One checker on the $\widehat{32}$s board wipes out two checkers on the 16s board.***

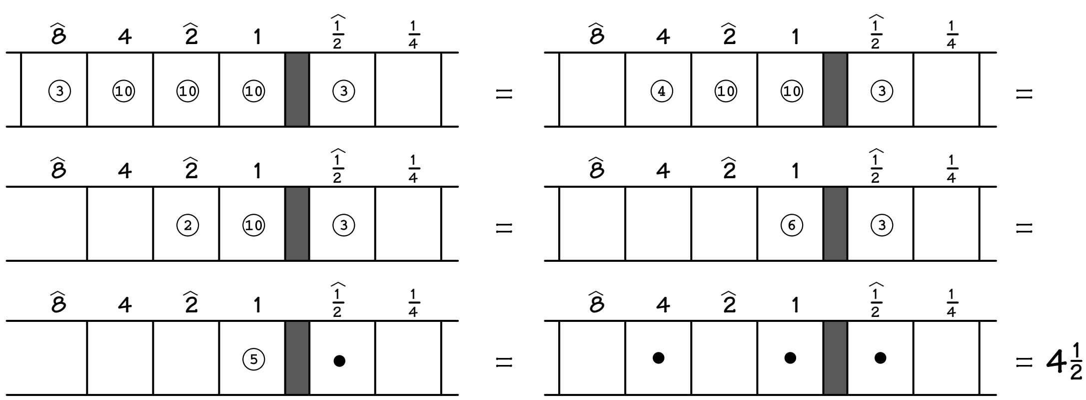


S: ***Three checkers on the fours board wipe out six checkers on the $\widehat{2}$s board.***



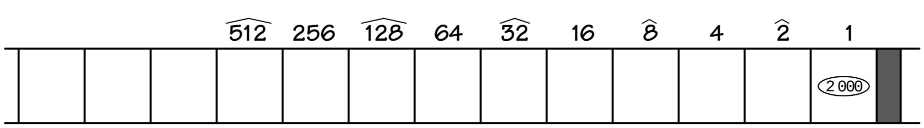
T: ***Which number is on the abacus?*** ($\widehat{1}$)

Repeat this activity with another starting configuration, such as the one shown below.



Exercise 3

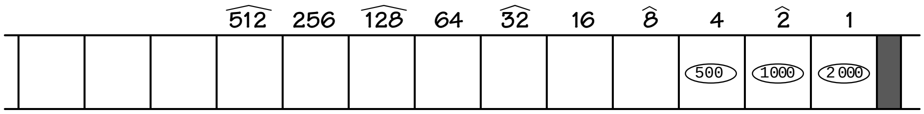
Redraw the abacus as shown below, and put on 2 000.



T: ***Lets try to display 2 000 with fewer checkers.***

Students may want to make suggestions like putting 30 checkers on the 64s board and leaving 80 checkers on the ones board. There are many good ideas, but indicate that you will look at a method that puts on more checkers in order to use the rule of the abacus to remove checkers.

Put on these checkers as you ask,

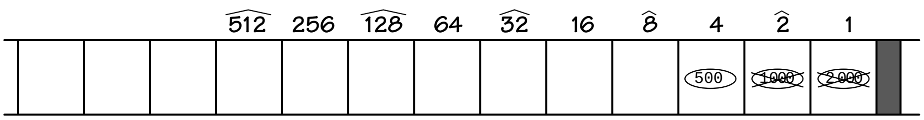


T: ***Which number is on the abacus?***

S: ***It is still 2 000 because 500 checkers on the fours board together with 1 000 checkers on the $\widehat{2}$s board represent 0.***

T: ***Do you see how we could get a lot fewer checkers on the abacus without changing the number?***

S: ***1 000 checkers on the $\widehat{2}$s board wipe out 2 000 checkers on the ones board.***

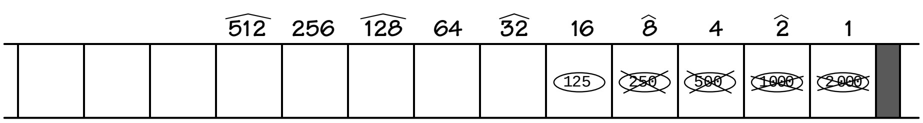


T: ***Could we use this method again to get 2 000 with still fewer checkers?***

S: ***Put 250 checkers on the $\widehat{8}$s board. Then 250 checkers on the $\widehat{8}$s board wipe out 500 checkers on the fours board.***

T: ***Remember we want to always keep 2 000 on the abacus.***

S: ***125 checkers on the 16s board together with 250 checkers on the $\widehat{8}$s board represent 0, so put 250 checkers on the $\widehat{8}$s board and 125 checkers on the 16s board. The number is still 2 000. The 250 checkers on the $\widehat{8}$s board wipe out 500 checkers on the fours board.***

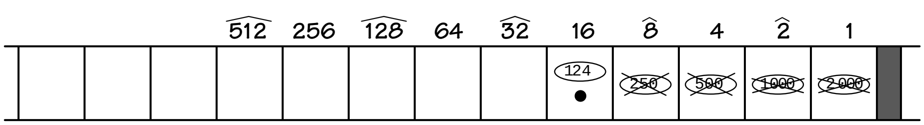


T: ***How should we continue?***

S: ***There are an odd number of checkers on the 16s board; the same method will not work.***

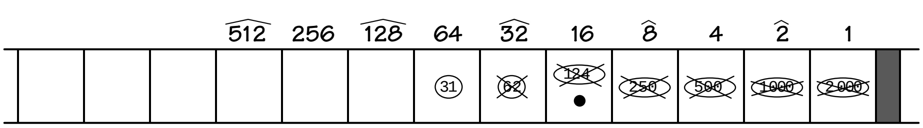
At this point, you may need to give another hint.

T: ***Think of 125 as 124 + 1.***



S: ***31 checkers on the 64s board together with 62 checkers on the $\widehat{32}$s board represent 0.***

S: ***Then, 62 checkers on the $\widehat{32}$s board wipe out 124 checkers on the 16s board.***

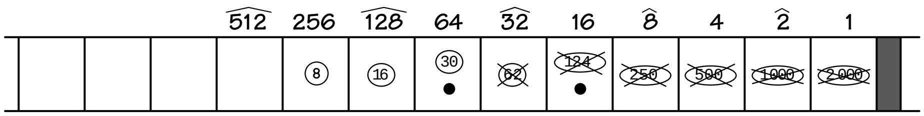


N 2 8

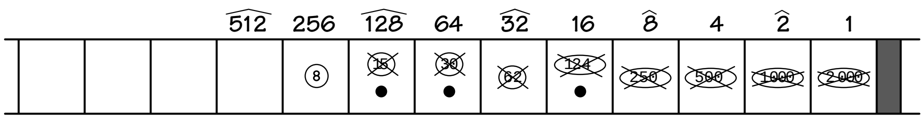
S: ***Since 31 = 30 + 1, we need 15 checkers on the $\widehat{128}$s board. But then we would need $7\frac{1}{2}$ checkers on the 256s board.***

S: ***We could put 8 checkers on the 256s board and 16 checkers on the $\widehat{128}$s board.***

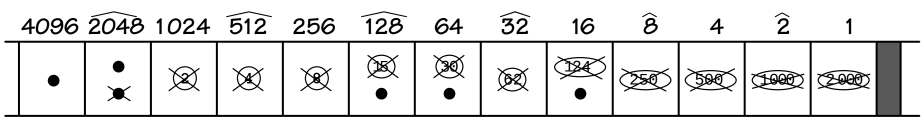
If necessary, make this suggestion yourself.



S: ***Now, 15 checkers on the $\widehat{128}$s board wipe out 30 checkers on the 64s board, leaving one checker on the $\widehat{128}$s board and one on the 64s board.***



Continue in this manner until each board has at most one checker left on it.



T: ***On your paper, write the base $\widehat{2}$ name for 2 000.***

After a couple minutes, invite a student to put the result on the chalkboard.

Decimal Writing		Base $\widehat{2}$ Writing
2 000	=	1100011010000

Writing / Home Activity

Invite students to write a letter to a family member describing the base $\widehat{2}$ abacus.

N29 COUNTING POSITIVE DIVISORS #2 N 29

Capsule Lesson Summary

Express a number as a product of prime numbers, and count all of its positive divisors. Suggest a method for counting the positive divisors of a number given its prime factorization. Use string pictures to count the number of positive divisors of $3 \times 5^2 \times 7^4$. Find the probability that a positive divisor of $2^3 \times 3 \times 5^3$ is a multiple of 2.

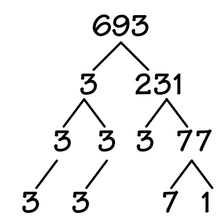
Materials

Teacher • Colored chalk **Student** • Paper

Description of Lesson

Exercise 1

Instruct students to work individually or with partners to find the prime factorization of 693. Encourage the use of a factorization tree.



$$693 = 3^2 \times 7 \times 11$$

T: ***Name some multiples of 693 and write each of those multiples as a product of prime numbers.***

As students suggest multiples of 693, record them in a list on the board. For example:

Multiples of $3^2 \times 7 \times 11$	
$2 \times 3^2 \times 7 \times 11$	$2^2 \times 3^3 \times 7 \times 11$
$3^3 \times 7 \times 11$	$3^2 \times 7^3 \times 11^2$
$2^2 \times 3^2 \times 7 \times 11$	$3^2 \times 5 \times 7 \times 11$

T: ***What do you notice about this list of multiples of 693?***

S: ***Each multiple is a whole number times $3^2 \times 7 \times 11$.***

T: ***Now let's examine the positive divisors of 693. First, predict how many positive divisors $3^2 \times 7 \times 11$ has.***

Record students' predictions on the board. If 12 is given, ask for an explanation.

S: ***3^2 has three divisors. Both 7 and 11 have two divisors. Multiply $3 \times 2 \times 2 = 12$.***

T: ***If we make a list of the positive divisors of $3^2 \times 7 \times 11$, perhaps we will see a way to count them.***

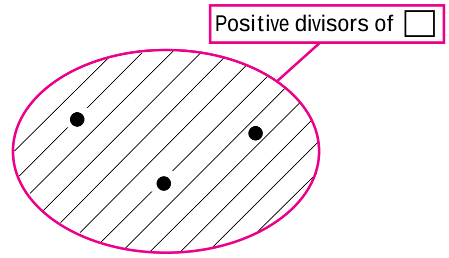
List the positive divisors of $3^2 \times 7 \times 11$ on the board as students suggest them. Build the list using the pattern array illustrated here.

Positive Divisors of $3^2 \times 7 \times 11$		
1	3	3^2
7	3×7	$3^2 \times 7$
11	3×11	$3^2 \times 11$
7×11	$3 \times 7 \times 11$	$3^2 \times 7 \times 11$

T: ***How many positive divisors does $3^2 \times 7 \times 11$ have?*** (12)

N 2 9

Exercise 2



Draw this string picture on the board.

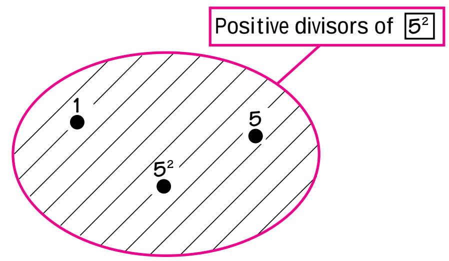
T: ***What are some numbers with exactly three positive divisors?***

S: ***9.***

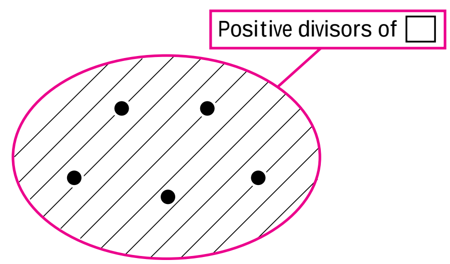
S: ***25.***

S: ***4.***

S: ***Any square of a prime number has exactly three positive divisors.***



Let students check a few examples by labeling the dots when the square of a prime is put in the box. Leave the labeling with 5^2 in the box on the board.



Draw this string picture next to the first one.

T: ***What are some numbers that have exactly five positive divisors? Try to express such numbers as products of prime numbers.***

S: ***81 = 3^4 and 81 has exactly five positive divisors: 1, 3, 3^2, 3^3, and 3^4.***

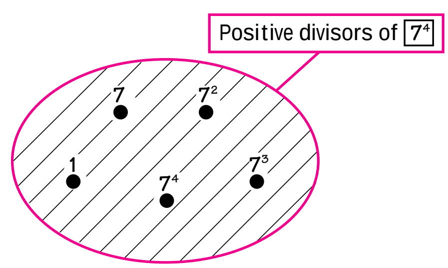
S: ***16 or 2^4.***

S: ***5^4.***

S: ***11^4.***

S: ***7^4.***

S: ***Any prime number to the fourth power has exactly five positive divisors.***



Put 7^4 in the box and ask students to label the dots.

T: ***5^2 has three positive divisors, and 7^4 has five positive divisors. How many positive divisors does $5^2 \times 7^4$ have?***

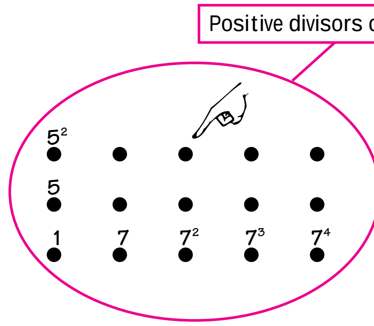
While students are considering the question, draw and label a third string for the positive divisors of $5^2 \times 7^4$. Put this picture below the other two string pictures already on the board.

S: ***Fifteen.***

Some students may have discovered a method for calculating the number of positive divisors; however, for the benefit of those who have not, continue as follows.

T: ***How might we arrange the dots in this string picture for the positive divisors of $5^2 \times 7^4$?***

S: ***Three by five. Three rows starting with the three divisors of 5^2, and five columns starting with the five divisors of 7^4 (or vice versa).***

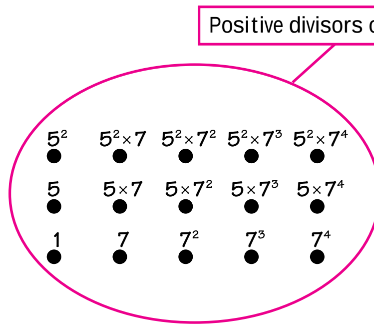


Inside the string, draw three rows of five dots each. Invite a student to label the dots in the first column and another to label the dots in the bottom row. Then point to a dot as you ask,

T: ***How could we label this dot?***

S: ***$5^2 \times 7^2$.***

Of course, other answers would be correct, but this one makes the best use of the array. Continue in this manner until all of the dots are labeled, as shown here.



T: ***How many positive divisors does $5^2 \times 7^4$ have?***

S: ***Fifteen.***

S: ***Add 1 to each exponent and multiply: $(2 + 1) \times (4 + 1) = 3 \times 5 = 15$.***

T: ***Can you predict the number of positive divisors of $3 \times 5^2 \times 7^4$?***

Allow a couple minutes for students to consider this situation.

S: ***$3 \times 5^2 \times 7^4$ has twice as many positive divisors as $5^2 \times 7^4$. All of the positive divisors of $5^2 \times 7^4$ are also positive divisors of $3 \times 5^2 \times 7^4$, and $3\times$ any positive divisor of $5^2 \times 7^4$ is a positive divisor of $3 \times 5^2 \times 7^4$.***

S: ***3 has two positive divisors, 5^2 has three positive divisors, and 7^4 has five positive divisors; $2 \times 3 \times 5 = 30$.***

S: ***Add 1 to each exponent and then multiply. $3 = 3^1$ so $(1 + 1) \times (2 + 1) \times (4 + 1) = 2 \times 3 \times 5 = 30$.***

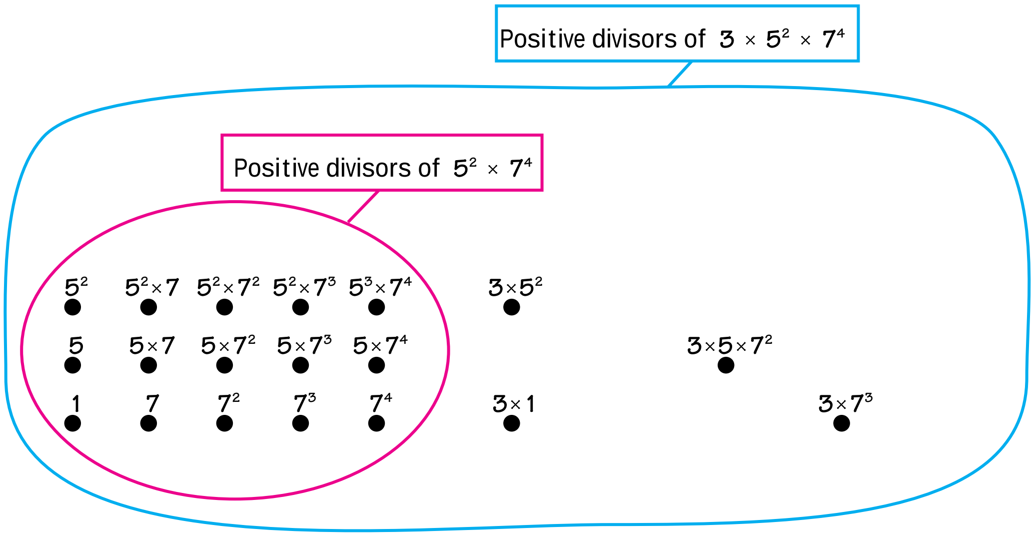
Draw a large blue string for the positive divisors of $3 \times 5^2 \times 7^4$ around the string for the positive divisors of $5^2 \times 7^4$ (see the next illustration).

T: ***All of the positive divisors of $5^2 \times 7^4$ are in this string. Which divisors of $3 \times 5^2 \times 7^4$ are outside the red string*** (point to the region between the strings)***?***

S: ***Any divisor that is $3\times$ a positive divisor of $5^2 \times 7^4$. For example, 5×7^2 is a divisor of $5^2 \times 7^4$ and $3 \times 5 \times 7^2$ is a divisor of $3 \times 5^2 \times 7^4$.***

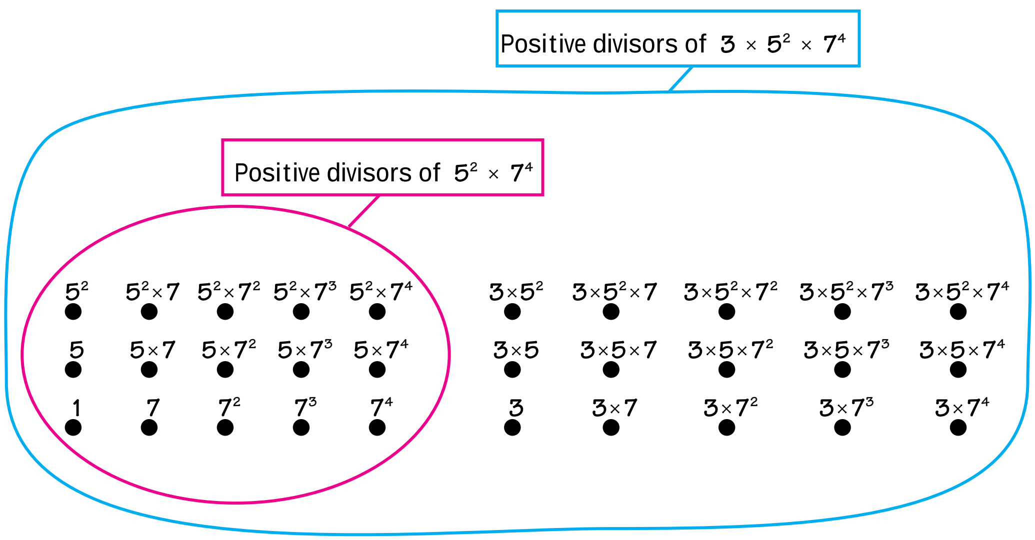
N 29

Put $3 \times 5 \times 7^2$ in the string picture together with a few other numbers of the form 3x a positive divisor of $5^2 \times 7^4$.



T: ***How many positive divisors does $3 \times 5^2 \times 7^4$ have?***

S: ***Thirty; there are 15 that are also positive divisors of $5^2 \times 7^4$ and 15 that are not.***



Exercise 3 (optional)

Write this number on the board. $2^3 \times 3 \times 5^3$

T: ***How many positive divisors does $2^3 \times 3 \times 5^3$ have?***

S: ***32, because $(3 + 1) \times (1 + 1) \times (3 + 1) = 4 \times 2 \times 4 = 32$.***

T: ***Imagine that each positive divisor of $2^3 \times 3 \times 5^3$ is written on a separate slip of paper and that we place those slips of paper in a box. If we randomly choose one slip of paper from the box, what is the probability that we will get a multiple of 2?***

There are essentially two ways to approach this problem.

- Students might suggest counting the divisors of $2^3 \times 3 \times 5^3$ that are also multiples of 2 and comparing that number to 32. In this case, help the class to be systematic by first counting divisors with 2^1 in their prime factorizations, then divisors with 2^2 in their prime factorizations, and finally divisors with 2^3 in their prime factorizations. A list of these divisors is given below.

  2, 2×3, 2×5, 2×5^2, 2×5^3, $2 \times 3 \times 5$, $2 \times 3 \times 5^2$, $2 \times 3 \times 5^3$
  2^2, $2^2 \times 3$, $2^2 \times 5$, $2^2 \times 5^2$, $2^2 \times 5^3$, $2^2 \times 3 \times 5$, $2^2 \times 3 \times 5^2$, $2^2 \times 3 \times 5^3$
  2^3, $2^3 \times 3$, $2^3 \times 5$, $2^3 \times 5^2$, $2^3 \times 5^3$, $2^3 \times 3 \times 5$, $2^3 \times 3 \times 5^2$, $2^3 \times 3 \times 5^3$

  Note that there are eight numbers in each list, so altogether there are 24 multiples of 2 that are divisors of $2^3 \times 3 \times 5^3$.

- Students might suggest counting the divisors of $2^3 \times 3 \times 5^3$ that are not multiples of 2. This approach requires noting that the divisors of $2^3 \times 3 \times 5^3$ that are not multiples of 2 are exactly the divisors of 3×5^3. Since there are eight ($2 \times 4 = 8$) divisors of 3×5^3, there are 24 ($32 - 8 = 24$) divisors of $2^3 \times 3 \times 5^3$ that are multiples of 2.

Comparing 24 (divisors of $2^3 \times 3 \times 5^3$ that are multiples of 2) to 32 (divisors of $2^3 \times 3 \times 5^3$), the students should conclude that the probability that a divisor of $2^3 \times 3 \times 5^3$ is a multiple of 2 is $^{24}/_{32}$, or $^3/_4$.

You might wish to pose other questions of this nature, such as the following:

What is the probability that a positive divisor of $2^3 \times 3 \times 5^3$ is:
- a multiple of 3? ($^{16}/_{32}$, or $^1/_2$)
- a multiple of 6? ($^{12}/_{32}$, or $^3/_8$)
- a multiple of 10? ($^{18}/_{32}$, or $^9/_{16}$)

Writing Activity

Students may enjoy writing other probability problems similar to the one in Exercise 3.

Capsule Lesson Summary

Review the rule of the base $\widehat{2}$ abacus. Practice putting on and decoding numbers. Observe the effect of moving checkers to the left on the abacus. Discover the base $\widehat{2}$ name for ⅓.

Materials

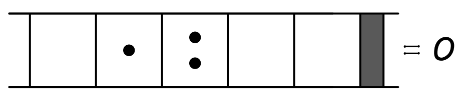
Teacher
- Colored chalk
- Minicomputer checkers (optional)

Student
- Paper
- Worksheets N30*, **, ***, and ****

Description of Lesson

Exercise 1

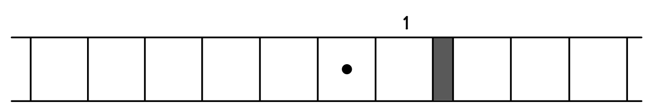
Draw part of an abacus on the chalkboard and review the rule for the base $\widehat{2}$ abacus.



One checker on a board wipes out two checkers on the next board to the right.

Observe that on every abacus the first board to the left of the bar is the ones board, and label it.

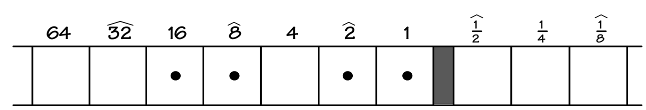
Put the checker on the next board to the left.



T: ***What number is this?***

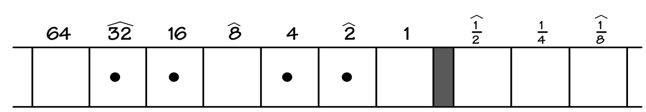
S: $\widehat{2}$.

Label the $\widehat{2}$s board and continue in this manner until each board is labeled. Display this configuration of checkers.



T: ***What number is on the abacus?*** (7)

Move each checker one board to the left.



T: ***Now what number is on the abacus?***

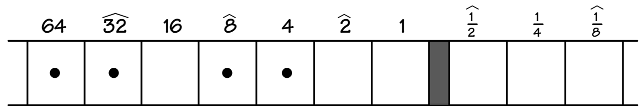
S: ***$\widehat{14}$; just double and change from positive to negative.***

S: $\widehat{2} \times 7 = \widehat{14}$.

Remind the class that moving checkers to the left on the base $\widehat{2}$ abacus has the same effect as on any other abacus, namely multiplying by the base ($\widehat{2}$ in this case).

N 30

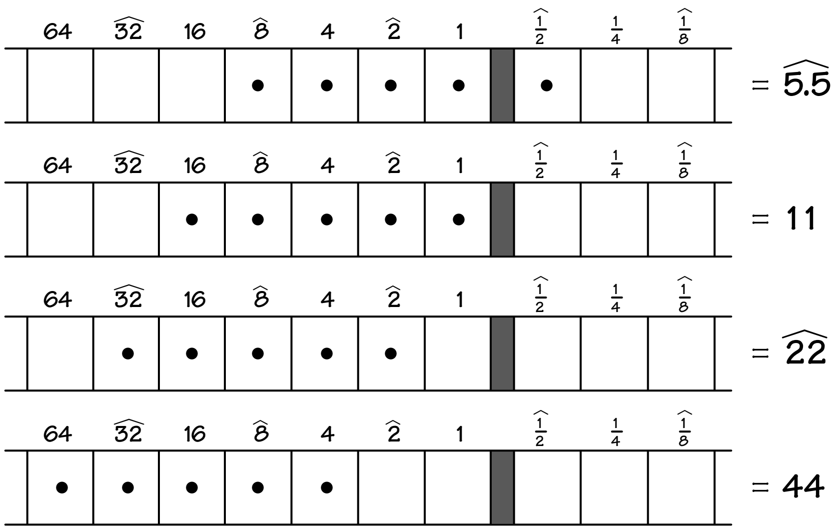
Again move the checkers one board to the left and ask for the number.



S: ***28.***

Write this multiplication fact on the board.

Repeat this activity starting with another number. For example, ask a student to put $\widehat{5.5}$ on the abacus.



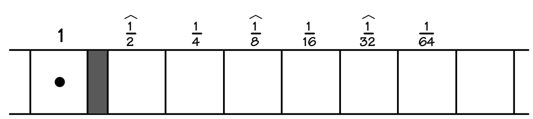
Exercise 2

Briefly review the place value of boards to the right of the bar by placing a checker successively on the $\widehat{\frac{1}{2}}$s board, the $\frac{1}{4}$s board, and the $\widehat{\frac{1}{8}}$s board. Observe that, like the binary abacus, it is easy to put on fractions whose denominators are powers of 2.

T: ***What could we do to get ⅓ on this abacus?***

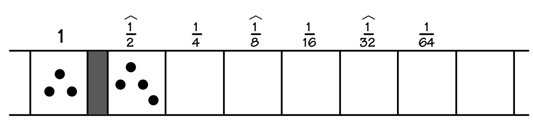
S: ***Put 1 on the abacus and then try to get all of the checkers in groups of three. Then if we look at just one checker from each group of three checkers, we will have ⅓.***

Put 1 on the abacus.

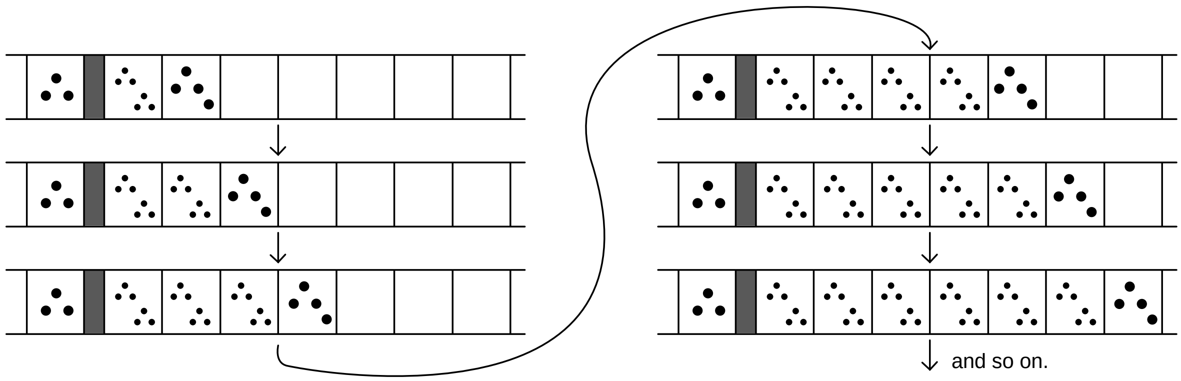


T: ***Now, what should we do?***

If necessary, suggest the following move yourself. Two checkers on the ones board wipe out four checkers on the $\widehat{\frac{1}{2}}$s board. Notice that the choice of two checkers on the ones board was to get a group of three.



Continue adding checkers and forming groups of three checkers until someone comments that this procedure will never end.

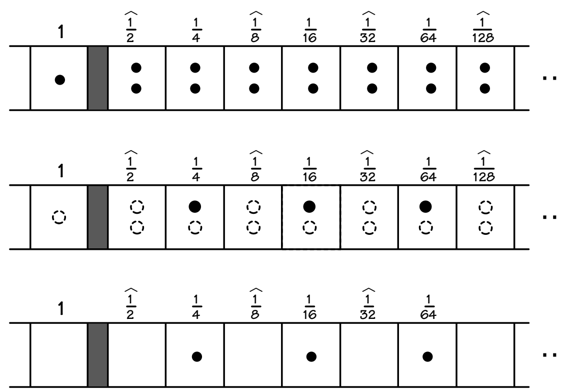


T: ***What number is on the abacus?***

S: ***1.***

T: ***Now let's show ⅓.***

Let a student remove two out of every three checkers, leaving this configuration on the abacus.



T: ***What can we do so that there is at most one checker on any board?***

S: ***Remove checkers using the rule of the base $\widehat{2}$ abacus: one checker wipes out two checkers on the next board to the right.***

T: ***What number is this?***

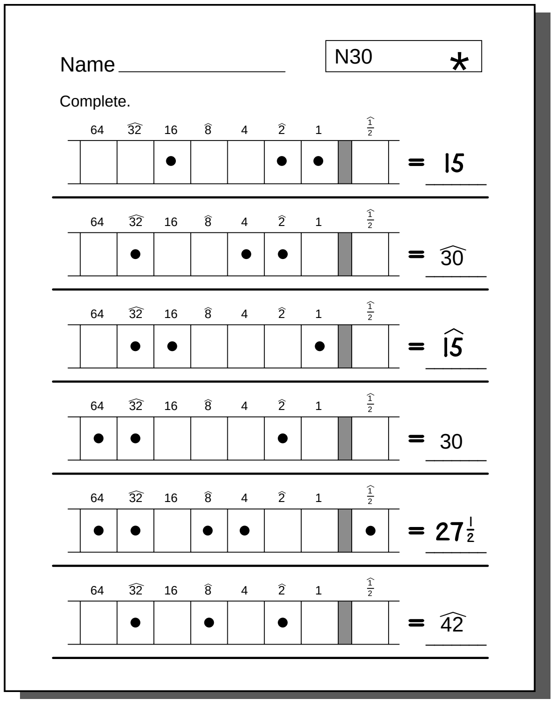
S: ***⅓.***

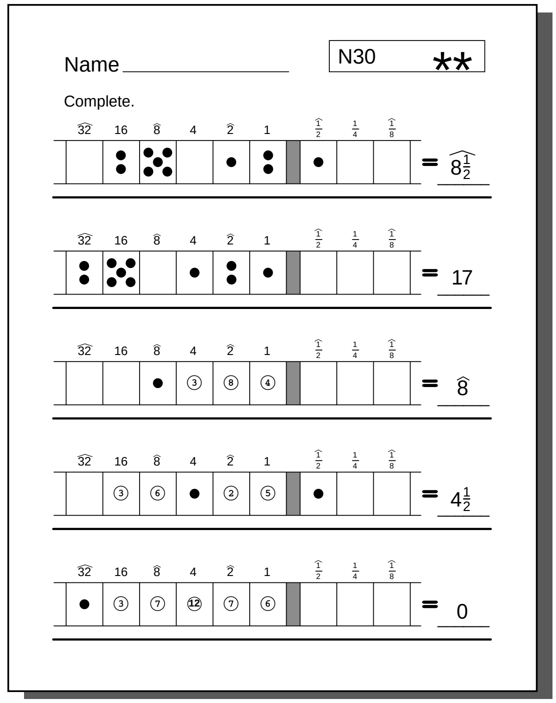
T: ***On your paper, write the base $\widehat{2}$ name for ⅓.***

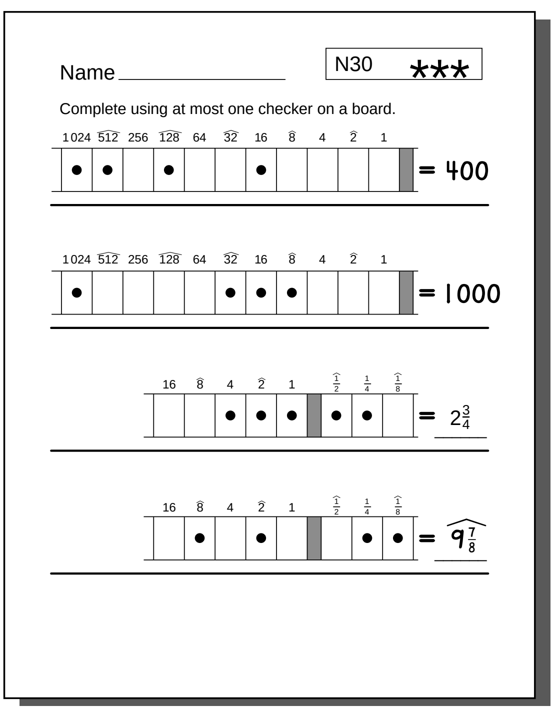
$$\frac{1}{3} = \underset{\text{Base } \widehat{2} \text{ Writing}}{0.010101\ \ldots} = 0.\overset{*}{\overline{01}}$$

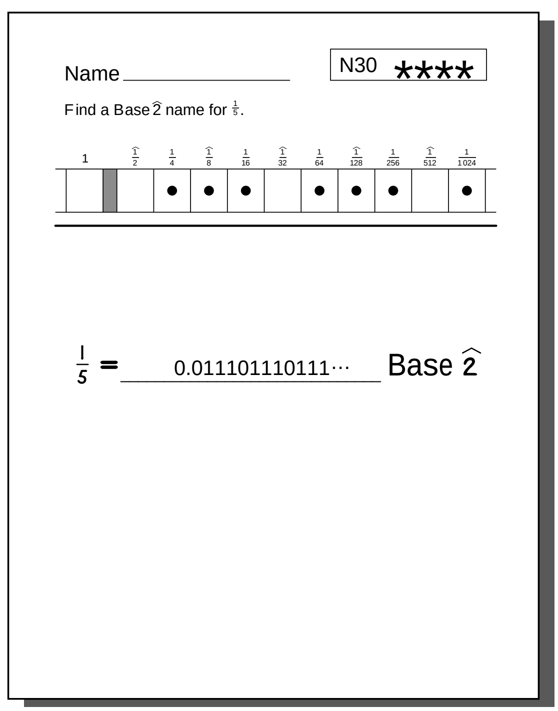
Some students might notice that the base $\widehat{2}$ and binary names for ⅓ are the same.

Worksheets N30*, **, ***, and **** are available for individual work.


Name
N30 *
Complete.
64 32 16 8 4 2 1 1/2
= 15
= 30
= 15
= 30
= 27 1/2
= 42


Name
N30 **
Complete.
32 16 8 4 2 1 1/2 1/4 1/8
= 8 1/2
= 17
= 8
= 4 1/2
= 0


Name
N30 ***
Complete using at most one checker on a board.
1 024 512 256 128 64 32 16 8 4 2 1
= 400
= 1000
16 8 4 2 1 1/2 1/4 1/8
= 2 3/4
= 9 7/8


Name
N30 ****
Find a Base 2 name for 1/5.
1 1/2 1/4 1/8 1/16 1/32 1/64 1/128 1/256 1/512 1/1 024
1/5 = 0.011101110111··· Base 2

Capsule Lesson Summary

Use compositions to label arrows in an arrow picture that includes addition and multiplication functions. From the arrow picture, conclude that $+\frac{7}{12}$ followed by 12x is the same as 12x followed by +7, and that $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$. Play a game where the object is to label the dots in a two-string picture using a set of non-standard names for numbers.

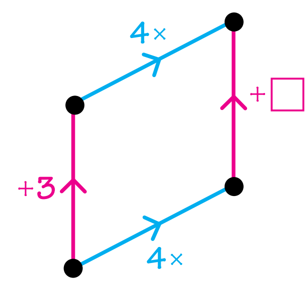
Materials

Teacher	• Colored chalk	**Student**	• Paper • Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

Draw this arrow picture on the board.
Allow room for later extension.



T: ***What could this arrow on the right be for?*** (+12)

Allow students to check their answers by labeling the dots, but also encourage analysis of the situation.

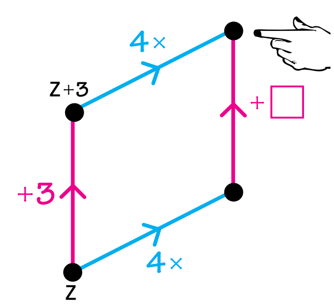
S: ***+12, since 4* x *3 = 12.***

T: ***It appears that +12 is correct, but let's try to convince ourselves.***

Label the lower left dot **z** and refer to it as you say,

T: ***Suppose we put some number here; call it* z*.*** (Trace the +3 arrow and point to its ending dot.) ***What do we know about this number?***

S: ***It is* z *+ 3.***

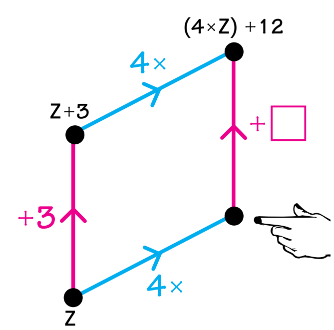


Label the dot **z + 3**. Trace the 4x arrow starting at **z + 3** and point to its ending dot.

T: ***What do we know about this number?***

S: ***It is 4* x *(*z *+ 3). You multiply both* z *and 3 by 4, so the number is (4* x z*) + (4* x *3) or (4* x z*) + 12.***

Label the upper right dot; then point to the lower right dot.

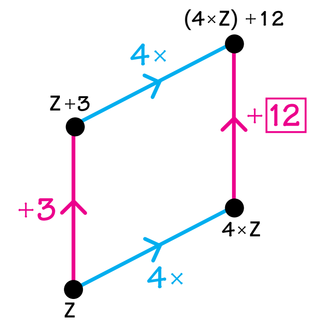


T: ***What do we know about this number?***

S: ***It is 4* x z*.***

N 31

Trace the arrow from **4** x **z** to (**4** x **z**) + **12**.

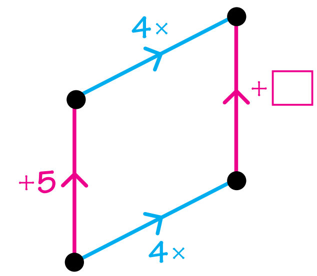


T: ***What could this arrow be for?***

S: ***+12, since the arrow goes from (4 x z) to (4 x z) + 12.***

T: ***Since z could be any number, we know that this arrow could be for +12 regardless of how we label the dots.***

Erase the labels for the dots and relabel the arrows, as shown here.

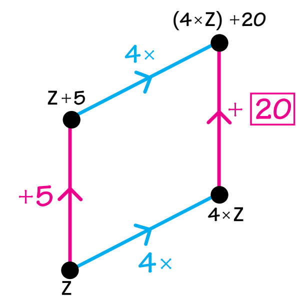


T: ***If we change +3 to +5, how would this change the arrow on the right?***

S: ***Now it would be for +20, because 4 x 5 = 20.***

S: ***If we label the dots, it appears that it would be for +20.***

Proceed as before to confirm that the unlabeled arrow is for +20.

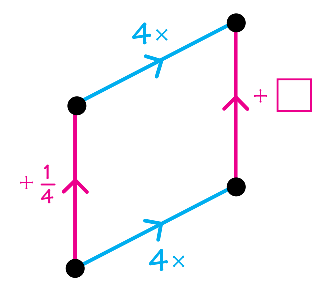


T: ***What do you notice about the two addition arrows?***

S: ***When one is for +5, the other is for +20 and 4 x 5 = 20.***

S: ***In the previous problem, one was for +3 and the other was for +12, and 4 x 3 = 12.***

You may want to repeat this activity several times, each time changing the addition functions. For example, you could use +7 (+35), –5 (–20), and +20 (+100). Once students recognize the 4x pattern, present the following problem.



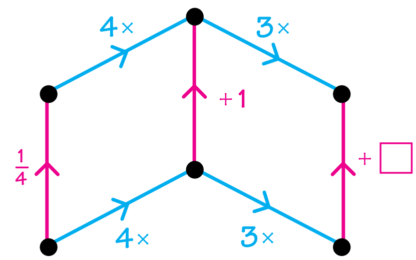
T: ***What could the arrow on the right be for?***

S: ***+1, since 4 x $\frac{1}{4}$ = 1.***

Again, you may like to repeat this activity, changing the addition functions to:

$+\frac{7}{4}$ (+7) $+\frac{3}{2}$ (+6) $+\frac{1}{8}$ ($+\frac{4}{8}$ or $+\frac{1}{2}$)

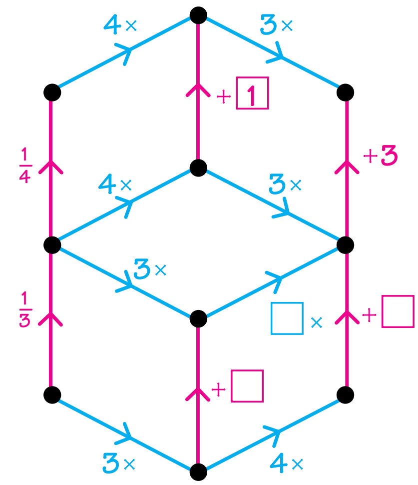
Relabel the arrows and extend the arrow picture, as shown here.



T: ***What could this arrow on the right be for?***

S: ***+3, because 3*** x ***1 = 3.***

Extend your picture again to include seven new arrows, as shown here.



Trace and then label the appropriate arrows as you ask,

T: ***$+\frac{1}{3}$ followed by 3x is the same as 3x followed by plus what number?***

S: ***+1, because 3 x $\frac{1}{3}$ = 1.***

T: ***4x followed by 3x is the same as 3x followed by times what number?***

S: ***4x, since 4 x 3 = 12 and 3 x 4 = 12.***

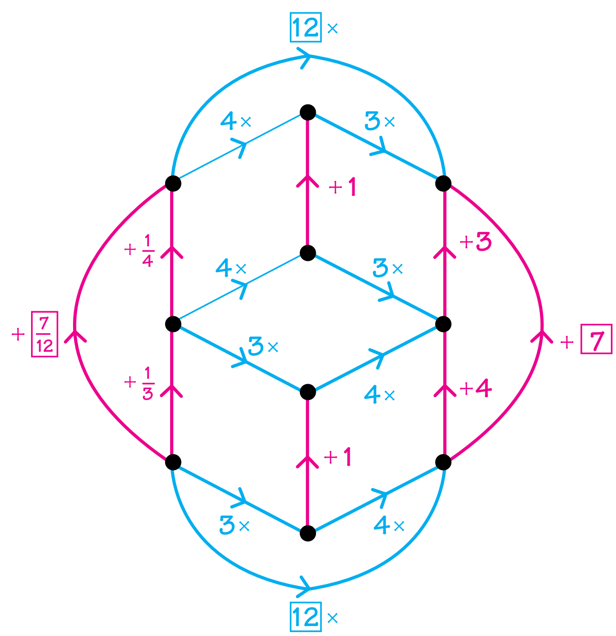
S: ***4x. Both 4x followed by 3x and 3x followed by 4x are the same as 12x.***

Point to the last unlabeled arrow on the right.

T: ***What could this arrow be for?***

S: ***+4, since 4 x 1 = 4.***

Add four composition arrows as shown below. Working from the right side to the left side of the picture, ask students to label the arrows. (Answers are in boxes.)



S: ***On the right +4 followed by +3 is the same as +7.***

S: ***On the top and bottom the arrows are both for 12x. 4x followed by 3x is 12x, and also 3x followed by 4x is 12x.***

S: ***On the left we have $+\frac{1}{3}$ followed by $+\frac{1}{4}$. Since $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$, that is $+\frac{7}{12}$.***

S: ***The arrow on the left is for $+\frac{7}{12}$. We know that $+\frac{7}{12}$ followed by 12x is the same as 12x followed by +7 because 12 x $\frac{7}{12}$ = 7.***

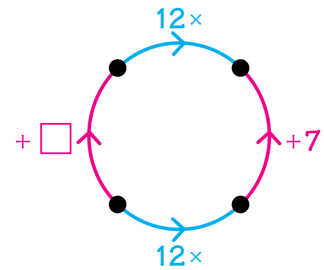
N 31

To emphasize the latter fact, draw these arrows in a separate picture.

T: ***This situation is similar to the arrow pictures we studied at the beginning of the lesson. How could we fill in the box?***

S: ***12 x $^7/_{12}$ = 7, so the arrow on the left could be for +$^7/_{12}$.***

T: ***Do you notice anything interesting about the big arrow picture?***



Let students comment freely about the situation.

S: ***Look at the +$^7/_{12}$ arrow. 7 = 4 + 3 and 12 = 4 x 3.***

S: $\frac{1}{3} + \frac{1}{4} = \frac{4+3}{3 \times 4} = \frac{7}{12}$.

If this last observation is made, summarize the fact in a number sentence on the board.

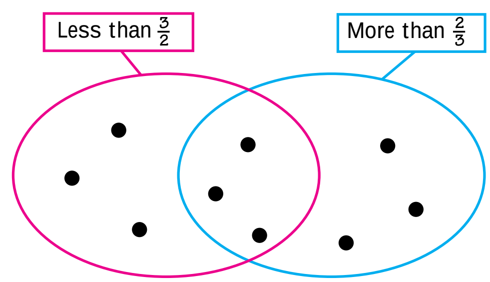
$$\frac{1}{3} + \frac{1}{4} = \frac{4+3}{3 \times 4} = \frac{7}{12}$$

Exercise 2

For this exercise you may like to suggest that students draw and label dots in a string picture on their papers.

Divide the class into two or more teams. Draw two strings on the board and list numbers in two or more sections of a team board similar to that pictured below.

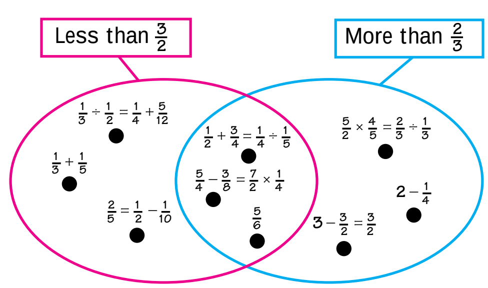
Team A	Team B	Team C
$3 - \frac{3}{2}$	$\frac{1}{4} + \frac{5}{12}$	$\frac{2}{3} \div \frac{1}{3}$
$\frac{1}{3} \div \frac{1}{2}$	$\frac{1}{2} - \frac{1}{10}$	$\frac{3}{2}$
$\frac{2}{5}$	$\frac{5}{2} \times \frac{4}{5}$	$\frac{5}{4} - \frac{3}{8}$
$\frac{7}{2} \times \frac{1}{4}$	$\frac{1}{4} \div \frac{1}{5}$	$\frac{1}{2} + \frac{3}{4}$
$\frac{1}{3} + \frac{1}{5}$	$\frac{5}{6}$	$2 - \frac{1}{4}$



T: ***We are going to play a game. The teams will take turns trying to label correctly the dots in the string picture. You may only use the labels in your section of the team board. The team that correctly uses all of its labels first is the winner. There may be different names for the same number here, so a dot may have more than one label.***

Alternating teams and taking turns among the members of a team, call on students to label the dots. When a team member correctly places a number in the string picture, erase it from the team board.

A final picture is shown here for your reference.



Capsule Lesson Summary

Play Guess My Rule for an operation $*$ where the rule is $a * b = (a \div b) - 1 = (a - b) \div b$. Find pairs of numbers satisfying equations such as $\square * \triangle = 3$. Observe patterns in the possible number pairs.

Materials

Teacher
- None

Student
- Paper
- Worksheets N32*, **, ***, and ****

Description of Lesson

Exercise 1

Announce to the class that you have a secret rule for an operation $*$. You may want to remind students about operations using an operation machine. Then write several clues for $*$ on the board.

T: ***I have a secret rule for $*$. Here are four examples of how to use $*$. With these clues, try to figure out my rule. What number do you think goes in the box?***

$$8 * 2 = 3$$
$$20 * 4 = 4$$
$$20 * 2 = 9$$
$$10 * 2 = 4$$
$$40 * 5 = \square$$

Note: The rule is $a * b = (a \div b) - 1 = (a - b) \div b$. The number in the box is 7 because $40 * 5 = (40 \div 5) - 1 = 7$, or $40 * 5 = (40 - 5) \div 5 = 7$. Do not explain the rule to the class at this time.

Suggest that students write their guesses on paper for you to see. Acknowledge correct guesses and reject incorrect guesses; for example, "No, we do not get 8 using my rule." Let a student who guesses correctly tell the class that 7 is in the box, but do not permit the student to give away the rule. If no one guesses correctly, put 7 in the box yourself.

Continue the activity with several more examples.

$$15 * 2 = \boxed{6.5 \text{ or } 6\tfrac{1}{2}} \qquad 56 * 8 = \boxed{6}$$
$$36 * 10 = \boxed{2.6} \qquad 10 * 4 = \boxed{1.5 \text{ or } 1\tfrac{1}{2}}$$
$$8 * 3 = \boxed{\tfrac{5}{3} \text{ or } 1\tfrac{2}{3}} \qquad 9 * 4 = \boxed{1.25 \text{ or } 1\tfrac{1}{4}}$$

T: ***Who can explain my rule?***

S: ***Divide the first number by the second number, then subtract 1 from the result. For example, $8 * 2 = 3$ because $8 \div 2 = 4$ and $4 - 1 = 3$.***

N 3 2

S: ***Subtract the second number from the first number, then divide the result by the second number. For example, 8 * 2 = 3 because 8 – 2 = 6 and 6 ÷ 2 = 3.***

Accept any reasonable and correct explanation your students offer. Write an equation for the rule on the board.

$$(a \div b) - 1 = a * b = (a - b) \div b$$

Let students use either description of the rule to explain their solutions for the following problems. (Answers are in boxes.)

$15 * 3 = \boxed{4}$ $\qquad$ $12 * \boxed{4} = 2$

$56 * 7 = \boxed{7}$ $\qquad$ $11 * \boxed{2} = 4.5$

$7 * 4 = \boxed{\tfrac{3}{4} \text{ or } 0.75}$ $\qquad$ $\boxed{72} * 9 = 7$

You may want to give students an opportunity to practice with the rule for * before going on to Exercise 2. Use Worksheet N32* for this purpose.

Exercise 2

Write this open number sentence on the board.

$\square * \triangle = 3$

$\square$	$\triangle$
8	2
16	4
12	3
32	8
4	1
20	5
40	10

T: ***On your paper, find pairs of numbers for the box and the triangle so that when we use the operation * on the numbers, the result is 3.***

Allow several minutes for independent work. When most students have found a few pairs of numbers, record some possibilities in a table on the board. For example:

T: ***Do you notice anything interesting about these pairs of numbers?***

S: ***If we know one solution, for example, 8 for $\square$ and 2 for $\triangle$, then we can multiply both numbers by the same number (except 0) and get another solution.***

S: ***Each number for $\square$ is 4x the corresponding number for $\triangle$. (8 = 4 x 2, 16 = 4 x 4, 12 = 4 x 3, and so on.)***

T: ***Why do you think there is a 4x pattern?***

S: ***According to the rule for *, ($\square \div \triangle$) – 1 = 3 so $\square \div \triangle$ = 4. That means $\square = 4 \times \triangle$.***

Let students express their ideas no matter how awkward the explanations are.

Repeat this activity with another open number sentence such as this one.

T: ***Can you find a pattern here?***

S: ***Yes, a number for △ is ⅔x the corresponding number for □.***

S: ***□ = 1.5 x △, and 1.5 = 0.5 + 1.***

□ * △ = 0.5

□	△
3	2
384	256
6	4
12	8
1	$\frac{2}{3}$
1.5	1
0.75	0.5

Worksheets N32*, **, ***, and **** are available for individual work. It may be necessary to call attention to the fact that the rule for * changes on Worksheet 32****.

Home Activity

Students may like to invent secret rules for an operation * and try letting family members guess their rules.

Name ______________ N32 ∗

$a \ast b = (a \div b) - 1$
or
$a \ast b = (a - b) \div b$

Complete.

$25 \ast 5 = \boxed{4}$

$28 \ast 4 = \boxed{6}$

$42 \ast \boxed{7} = 5$

$\boxed{6} \ast 6 = 0$

$19 \ast 2 = \boxed{8.5}$

$\boxed{0} \ast 8 = \hat{1}$

$7 \ast 5 = \boxed{\frac{2}{5}}$

$5 \ast \boxed{3} = \frac{2}{3}$

$\frac{2}{3} \ast \frac{2}{7} = \boxed{1\frac{1}{3}}$

$6 \ast 0.1 = \boxed{59}$

$8 \ast 0.2 = \boxed{39}$

$\frac{1}{5} \ast \frac{1}{4} = \boxed{\widehat{\frac{1}{5}}}$

Name ______________ N32 ∗∗

$a \ast b = (a \div b) - 1$
or
$a \ast b = (a - b) \div b$

Complete this table.

$\square \ast \triangle = 6$

35	5
21	3
7	1
70	10
3.5	0.5
2	$\frac{2}{7}$
42	6
4.2	0.6

Other solutions are possible.

Name ______________ N32 ∗∗∗

$a \ast b = (a \div b) - 1$
or
$a \ast b = (a - b) \div b$

Complete.

$51 \ast 17 = \boxed{2}$

$4.5 \ast 0.5 = \boxed{8}$

$2.8 \ast 0.1 = \boxed{27}$

$\frac{3}{2} \ast \frac{1}{2} = \boxed{2}$

$\frac{5}{4} \ast \frac{2}{3} = \boxed{\frac{7}{8}}$

$\frac{2}{3} \ast \frac{5}{9} = \boxed{\frac{1}{5}}$

$\boxed{91} \ast 13 = 6$

$3.5 \ast \boxed{0.5} = 6$

$\boxed{0.8} \ast 0.1 = 7$

$\frac{2}{5} \ast \boxed{\frac{1}{5}} = 1$

$\boxed{8} \ast \frac{1}{2} = 15$

$\frac{3}{5} \ast \boxed{\frac{1}{3}} = \frac{4}{5}$

Name ______________ N32 ∗∗∗∗

$a \ast b = (a \div b) + \frac{1}{2}$

Complete.

$256 \ast 64 = \boxed{4.5}$

$4.8 \ast 4 = \boxed{1.7}$

$12.4 \ast 0.4 = \boxed{31.5}$

$\frac{2}{3} \ast \frac{1}{6} = \boxed{4\frac{1}{2}}$

$\frac{3}{8} \ast \frac{3}{5} = \boxed{1\frac{1}{8}}$

$\frac{2}{7} \ast \frac{3}{7} = \boxed{1\frac{1}{6}}$

$108 \ast \boxed{12} = 9.5$

$\boxed{30} \ast 4 = 8$

$17 \ast \boxed{4} = 4.75$

$\boxed{\frac{7}{8}} \ast 7 = \frac{5}{8}$

$\frac{6}{5} \ast \boxed{\frac{4}{5}} = 2$

$\boxed{\frac{1}{14}} \ast \frac{5}{7} = \frac{3}{5}$

Capsule Lesson Summary

Put some decimal numbers on the Minicomputer, first with one and then with two of the weighted checkers ②, ③, ④, …, ⑨. Change numbers on the Minicomputer by moving exactly one checker. Play Minicomputer Golf cooperatively.

Materials

Teacher
- Minicomputer set
- Weighted checkers
- Colored chalk
- Blackline N33

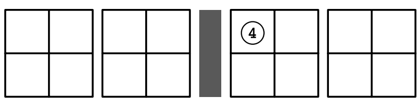
Student
- Minicomputer sheet
- Pencil
- Worksheets N33*, **, ***, and ****

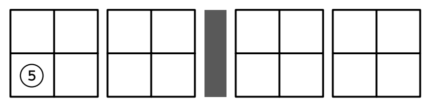
Advance Preparation: Use Blackline N33 to make a Minicomputer sheet for student use in Exercise 2.

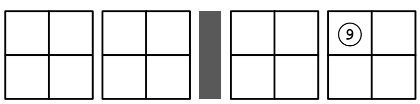
Description of Lesson

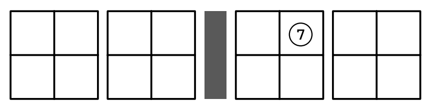
Exercise 1

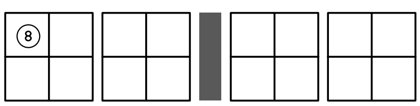
Display four Minicomputer boards and the weighted checkers ②, ③, ④, ⑤, ⑥, ⑥, ⑧, and ⑨. Ask students to use exactly one of these weighted checkers to put on each of the following numbers: 3.2, 100, 0.72, 2.8, 640, and 1.2. (Answers are given below.)

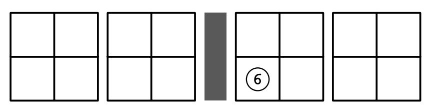
3.2 =

Another solution is to put the ⑧-checker on the 0.4-square.

100 =


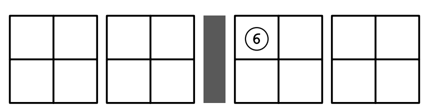
0.72 =


2.8 =


640 =


1.2 =

Another solution is to put the ③-checker on the 0.4-square.

Clear the Minicomputer and put a ⑥-checker on the 0.8-square.

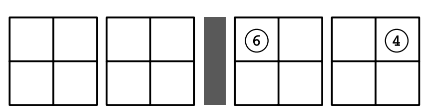


T: ***What number is this?***

S: ***4.8 (6* x *0.8).***

T: ***Can you put 4.96 on the Minicomputer by adding exactly one of these checkers?***

Invite a student to put on a checker. For example:


= 4.96

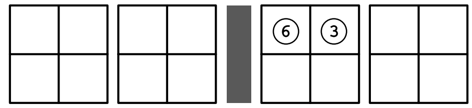
S: ***4* x *0.04 = 0.16 and 4.8 + 0.16 = 4.96.***

N 33

Remove just the ④-checker.

T: ***Can you put 6 on the Minicomputer by adding exactly one of these checkers?***

Let a student put on a checker. For example:



= 6

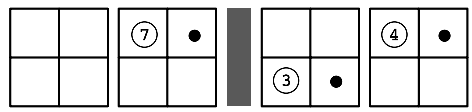
S: ***3* x *0.4 = 1.2 and 4.8 + 1.2 = 6.***

At this point you may want to let students practice on other such problems individually or with partners. Use Worksheets N33* and ** for this purpose. Give students a short time to work on the worksheets before starting Exercise 2.

Exercise 2

Display this configuration of checkers on the Minicomputer.

T: ***Let's change this number by moving one checker at a time. Each time, tell us what increase or decrease we make to the number on the Minicomputer.***



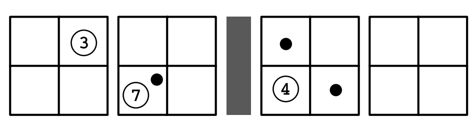
Move the regular checker from the 0.04-square to the 2-square.

S: ***An increase of 1.96, because 0.04 + 1.96 = 2.***

Continue in this manner, making the following moves:

- Move the ⑦-checker from the 8-square to the 2-square.
  (A decrease of 42; 56 – 42 = 14)
- Move the regular checker from the 4-square to the 0.8-square.
  (A decrease of 3.2; 4 – 3.2 = 0.8)
- Move the ③-checker from the 0.2-square to the 40-square.
  (An increase of 119.4; 0.6 + 119.4 = 120)
- Move the ④-checker from the 0.08-square to the 0.2-square.
  (An increase of 0.48; 0.32 + 0.48 = 0.8)

After making the above moves, this configuration will be on the Minicomputer. You may want to check that your configuration of checkers agrees in order that the remainder of this exercise is easier to follow.



Write this information on the board.

+2.4	–0.08	–13.72
+9.2	–108	–1.2

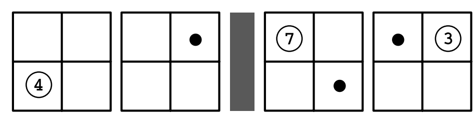
T: ***Each of these changes can occur by moving exactly one checker from the square it is on to another square. Each of the six checkers on the Minicomputer is involved in exactly one of these six changes.***

Let students work individually or with partners for a while. Then begin asking students to announce solutions. After a move is demonstrated, return the checkers to the previous configuration. Continue until the class finds all six moves.

- +2.4: Move the ④-checker from the 0.2-square to the 0.8-square. (0.8 + 2.4 = 3.2)
- +9.2: Move the regular checker from the 0.8-square to the 10-square. (0.8 + 9.2 = 10)
- –0.08: Move the regular checker from the 0.1-square to the 0.02-square. (0.1 – 0.08 = 0.02)
- –108: Move the ③-checker from the 40-square to the 4-square. (120 – 108 = 12)
- –13.72: Move the ⑦-checker from the 2-square to the 0.04-square. (14 – 13.72 = 0.28)
- –1.2: Move the regular checker from the 2-square to the 0.8-square. (2 – 1.2 = 0.8)

Exercise 3

Display this configuration of checkers on the Minicomputer and allow sufficient time for students to compute the number.



Goal: 32.32

T: ***What number is this?*** (89.9)

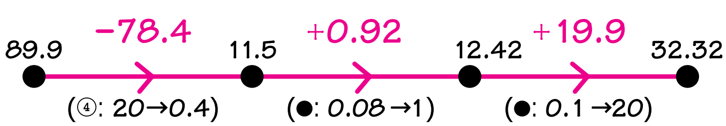
We are going to play Minicomputer Golf with 89.9 as the starting number and 32.32 as the goal. Let's work together and try to reach the goal in as few moves as possible.

Call on students to make moves, but do not be concerned that they contribute to a minimal solution. Suppose the first student moves the ④-checker from the 20-square to the 0.4-square.

T: ***What is the effect of this move?***

S: ***That move decreases the number on the Minicomputer by 78.4, because 80 – 78.4 = 1.6. The new number is 11.5 (89.9 – 78.4 = 11.5).***

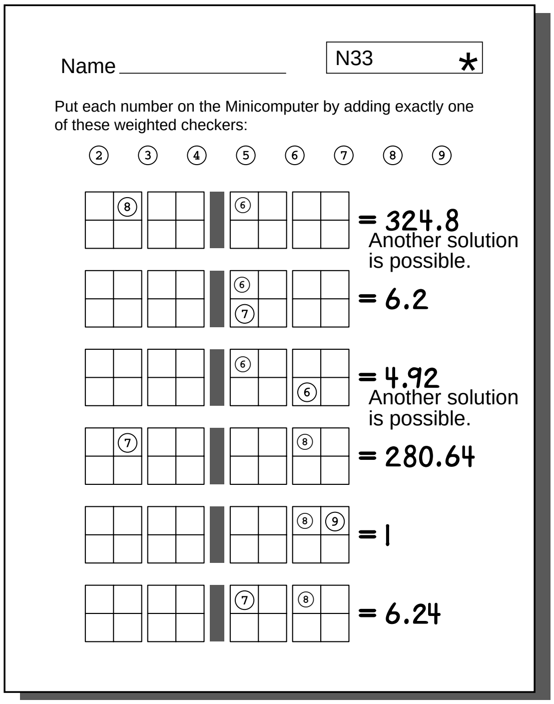
Record this information in an arrow road and continue until the class reaches the goal. A shortest solution involves three moves, but do not expect your students to find such a short solution.

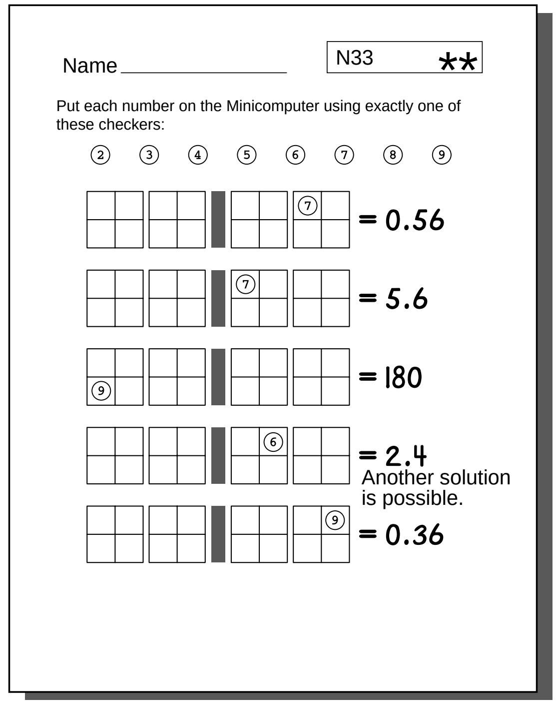


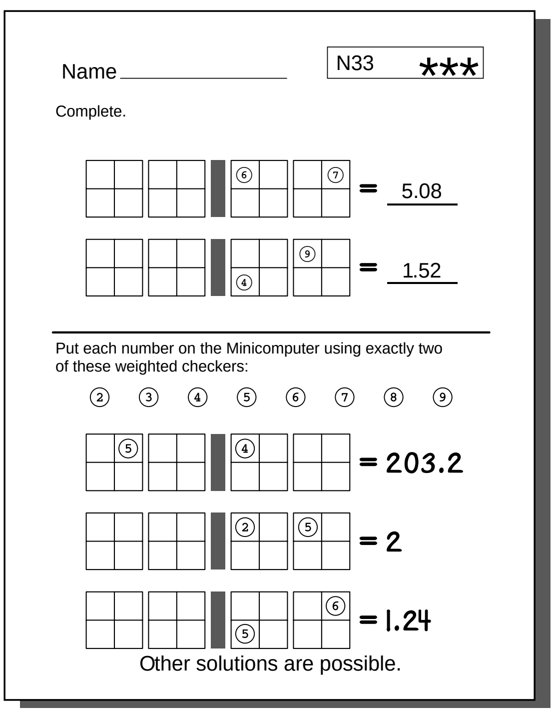
Goal: 32.32

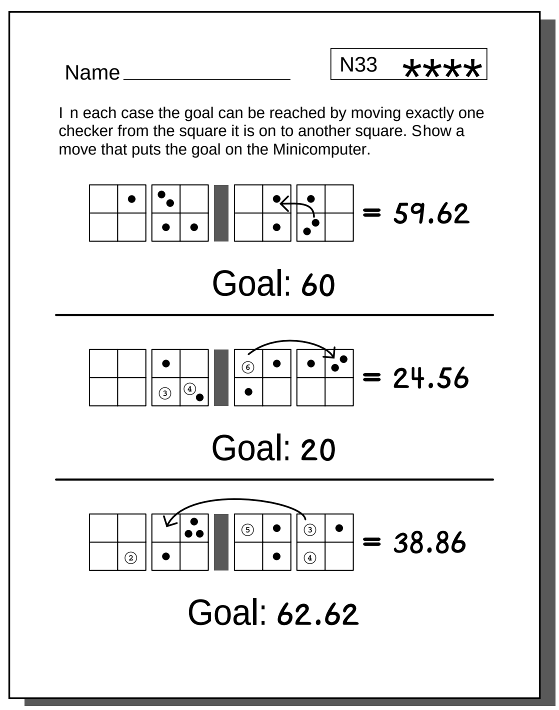
Repeat this activity with the same starting configuration but with a goal of 44.44. You may prefer to let students work in small groups to solve the problem.

Worksheets N33*, **, ***, and **** are available for individual work.


Name
N33 *
Put each number on the Minicomputer by adding exactly one of these weighted checkers:
2 3 4 5 6 7 8 9
= 324.8
Another solution is possible.
= 6.2
= 4.92
Another solution is possible.
= 280.64
= 1
= 6.24


Name
N33 **
Put each number on the Minicomputer using exactly one of these checkers:
2 3 4 5 6 7 8 9
= 0.56
= 5.6
= 180
= 2.4
Another solution is possible.
= 0.36


Name
N33 ***
Complete.
= 5.08
= 1.52
Put each number on the Minicomputer using exactly two of these weighted checkers:
2 3 4 5 6 7 8 9
= 203.2
= 2
= 1.24
Other solutions are possible.


Name
N33 ****
In each case the goal can be reached by moving exactly one checker from the square it is on to another square. Show a move that puts the goal on the Minicomputer.
= 59.62
Goal: 60
= 24.56
Goal: 20
= 38.86
Goal: 62.62

Capsule Lesson Summary

Introduce a story posing a linear programming problem in which cost must be minimized while certain other requirements are fulfilled. Investigate solutions which satisfy the requirements and examine their costs. Look at cost lines on a grid. This preliminary work is background for Lesson N36 in which the problem is solved.

Materials

Teacher
- Grid sheet transparency or *IG-VI World of Numbers Poster* # 1
- Markers
- Blackline N34

Student
- Grid sheet

Advance Preparation: Use Blackline N34 to prepare a grid sheet transparency and to make copies of the grid sheet for students. You may want to draw the first table on the board before starting the lesson.

Description of Lesson

Present the following situation to the class.

T: ***Dr. Dantzig is a veterinarian. She was hired by the Wolfe Kennel to help provide a good, healthy diet for the dogs. The Wolfe Kennel specializes in Great Danes, and today Dr. Dantzig is concerned about what diet to recommend for a Great Dane.***

The kennel stocks two kinds of dog food: Brand X and Brand Y. Here is what Dr. Dantzig has found out about one scoop of each kind of dog food.

Draw this table on the board.

	Units of Carbohydrates	Units of Vitamins	Units of Protein
Brand X	1	4	5
Brand Y	3	4	1

T: ***This table tells the number of units of carbohydrates, vitamins, and protein supplied per scoop of dog food. How many units of each nutrient does two scoops of Brand X provide?***

S: ***2 units of carbohydrates, 8 units of vitamins, and 10 units of protein.***

T: ***What about three scoops of Brand Y?***

S: ***9 units of carbohydrates, 12 units of vitamins, and 3 units of protein.***

T: ***How many units of each nutrient does three scoops of Brand X and five scoops of Brand Y mixed together provide?***

Allow a minute for students to make the necessary calculations.

S: ***Three scoops of Brand X provide 3 units of carbohydrates and five scoops of Brand Y provide 15 units of carbohydrates. Altogether that is 18 units of carbohydrates.***

S: ***There are 12 units of vitamins from three scoops of Brand X and 20 units from five scoops of Brand Y, so altogether that is 32 units of vitamins.***

S: ***There are 15 units of protein from three scoops for Brand X and 5 units from five scoops of Brand Y, so altogether that is 20 units of protein.***

T: ***Being a veterinarian, Dr. Dantzig knows that Great Danes require a minimum of 8 units of carbohydrates, 20 units of vitamins, and 7 units of protein each day to remain healthy. These amounts are the minimum daily requirements (MDR) of these nutrients for Great Danes.***

Add this information to your table.

	Units of Carbohydrates	Units of Vitamins	Units of Protein
Brand X	1	4	5
Brand Y	3	4	1
MDR	8	20	7

T: ***Dr. Dantzig must recommend a diet that* at least *meets the minimum daily requirement for Great Danes. Would three scoops of Brand X and five scoops of Brand Y meet the MDR for all of the nutrients?***

S: ***Yes, it would provide more than the MDR for all three nutrients.***

T: ***Could you meet the MDR with only Brand X? How many scoops would it take?***

S: ***Eight scoops of Brand X would provide 8 units of carbohydrates, 32 units of vitamins, and 40 units of protein. That is exactly the MDR for carbohydrates and more than the MDR for vitamins and protein.***

T: ***Could you meet the MDR with only Brand Y? How many scoops would it take?***

S: ***Seven scoops of Brand Y would provide 21 units of carbohydrates, 28 units of vitamins, and 7 units of protein. That is more than the MDR for carbohydrates and vitamins and exactly the MDR for protein.***

T: ***Let's record this information using ordered pairs of numbers. In each ordered pair, the first number is for the scoops of Brand X and the second number is for the scoops of Brand Y.***

Record this information on the board.

Meets the MDR

(3, 5)
(8, 0)
(0, 7)

T: ***Can you think of some other mixtures of the dog foods that meet the MDR for the Great Danes?***

Allow a couple of minutes for students to find such mixtures. As different mixtures are suggested, collectively check how many units of each nutrient they provide. Then record them on the board. For example:

Meets the MDR

(3, 5) (3, 3)
(8, 0) (4, 2)
(0, 7) (2, 4)
(5, 1) (3, 4)

T: ***In addition to concern for a Great Dane's nutritional requirements, the kennel owner is also concerned about the cost of feeding an animal. He tells Dr. Dantzig that one scoop of Brand X costs about \$0.40 and that one scoop of Brand Y costs about \$0.30.***

Add the cost information to the table.

	Units of Carbohydrates	Units of Vitamins	Units of Protein	Cost
Brand X	1	4	5	$0.40
Brand Y	3	4	1	$0.30
MDR	8	20	7	

Indicate ordered pairs in the list starting with (8, 0) to check the cost.

T: ***How much would eight scoops of Brand X cost?***

S: ***$3.20, because 8 x 0.40 = 3.20.***

Record the cost on the board as you repeat,

T: ***The cost of eight scoops of Brand X is $3.20 What is the cost of seven scoops of Brand Y?***

c(8, 0) = $3.20

S: ***7 x 0.30 = 2.10; so the cost of seven scoops of Brand Y is $2.10.***

c(0, 7) = $2.10

T: ***What is the cost of three scoops of Brand X and five scoops of Brand Y?***

S: ***(3 x 0.40) + (5 x 0.30) = 2.70; so the cost is $2.70.***

Ask students to calculate the cost of each mixture remaining in your list.

c(3, 5) = $2.70
c(8, 0) = $3.20
c(0, 7) = $2.10
c(5, 1) = $2.30

c(3, 3) = $2.10
c(4, 2) = $2.20
c(2, 4) = $2.00
c(3, 4) = $2.40

T: ***Dr. Dantzig must find the least expensive diet for a Great Dane which meets the minimum daily requirement for carbohydrates, vitamins, and protein. She decides to look at the costs for various mixtures.***

Project a transparency of the grid sheet or display *IG-VI World of Numbers Poster* # 1. Refer students to their copies of the grid. Modify the dialogue in this section to fit your data.

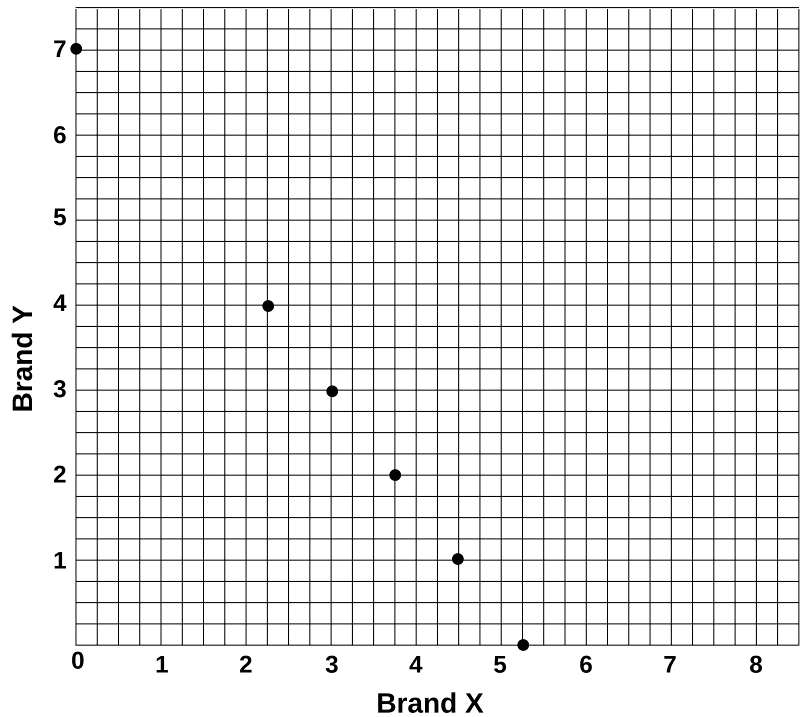
T: ***Notice that there are two mixtures that cost $2.10. On the grid, locate the point for (0, 7) and the point for (3, 3).***

Invite a student to locate these points and to draw the dots in black. If you use the poster, work neatly, as it will be needed again in Lesson N36.

T: ***Find some other mixtures that cost $2.10, and locate the points for those mixtures on your grid.***

N 3 4

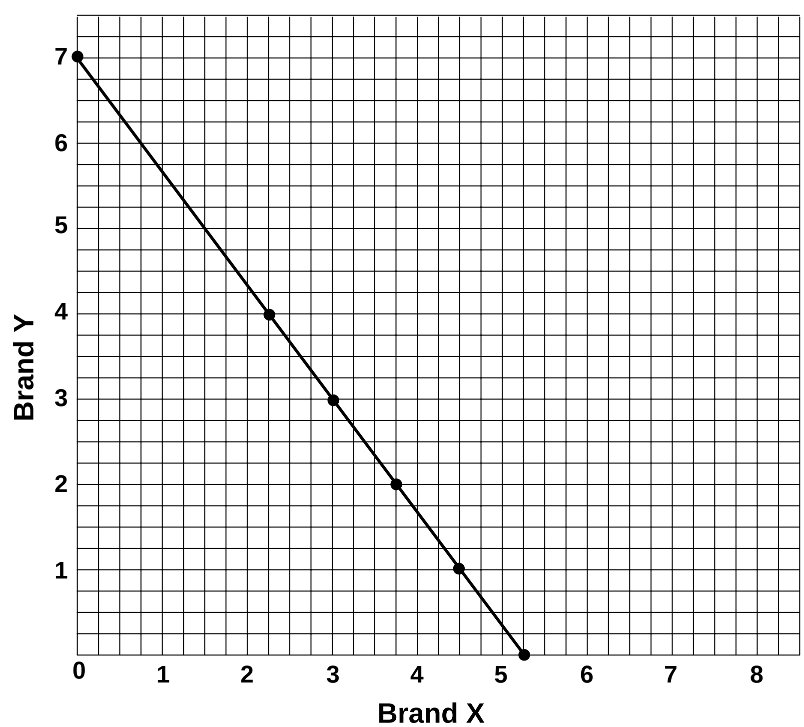
Let students work independently or in groups on this problem. Call on students to indicate points they find on the grid at the front of the room. Your graph should look similar to this one.



T: ***What do you notice about these points?***

S: ***They seem to be in a line.***

Draw the line using a straightedge.



T: ***Let's pick a point on the line and see if that point corresponds to a mixture that costs $2.10.***

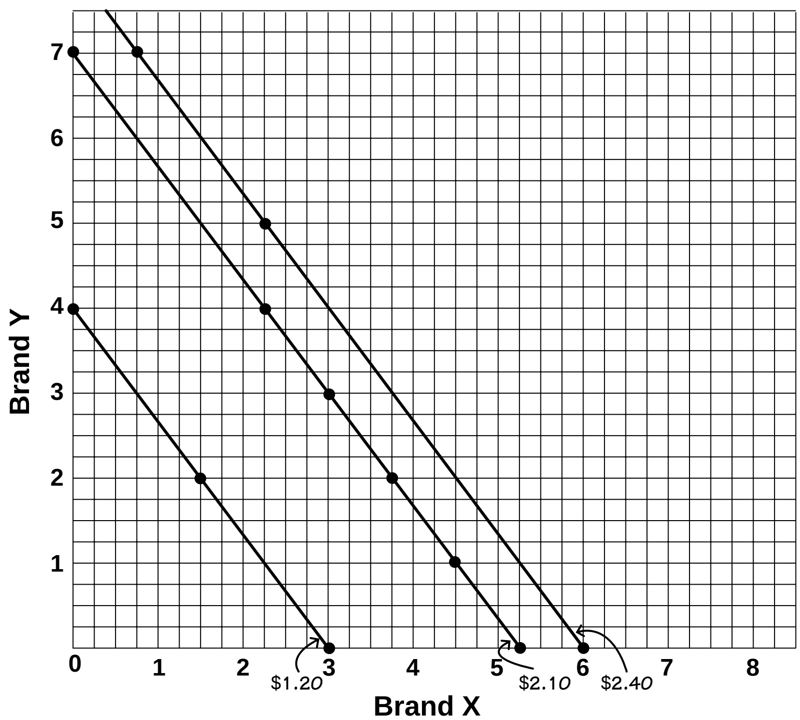
Invite a student to pick a grid intersection point on the line, and ask the class to calculate the cost for the mixture represented by that point. In this case there are only a couple of grid intersection points, such as (1.5, 5).

T: ***How much does one and one-half scoops of Brand X and five scoops of Brand Y cost?***

S: ***The cost is \$2.10; 1.5 x 0.40 = 0.60, 5 x 0.30 = 1.50, and 0.60 + 1.50 = 2.10.***

T: ***It seems that each point on this line represents a mixture that costs \$2.10. For this reason, we call it the* \$2.10 line.**

Label the line with the cost, and ask students to find the \$ 2.40 line and the \$ 1.20 line. As students complete this task on their grid sheets, call on students to locate points and draw the lines on the poster.



T: ***What do you notice about these cost lines?***

S: ***They are parallel.***

S: ***The price increases as we look at lines further to the right.***

Save the grid sheet transparency or the poster for use in Lesson N36. Also, save students' copies of the grid sheet.

Capsule Lesson Summary

In cooperative groups, solve a problem involving patterns and powers of 2. Present or write explanations for methods of solution.

Materials

Teacher	• Blackline N35	**Student**	• Paper • Colored pencils, pens, or crayons • Calculator • Counters or other props • Round table problem

Advance Preparation: Use Blackline N35 to make copies of the statement of the round table problem for students.

Description of Lesson

Organize the class into small cooperative groups for problem solving. Each group should have supplies such as paper, colored pencils, calculator, and props or manipulatives. Provide each group with copies of the problem (Blackline N35) and read it together as a class. At this time, check that everyone understands the problem but leave further discussion of a solution to the groups.

Direct students to work on the problem cooperatively in their groups and to try to find methods of solving the problem that everyone in their group can explain. You may want students to write explanations and answers on their papers individually, but suggest that each group prepare to present their solution to the class.

As you observe group work, look for different techniques so that you can arrange for different approaches to be presented to the whole class.

For your information, the table following the statement of the problem gives chair numbers to take for various numbers of people in class. Students may observe some patterns to explain how to decide which chair to take.

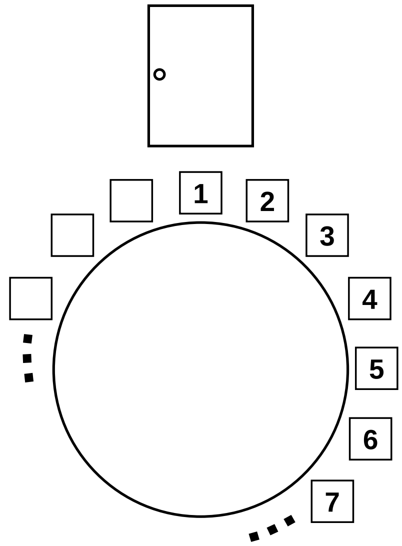
Ms. Bell has one very nice prize to give to just one person in the class. She decides on the following method of selecting who gets the prize.

> Everyone in the class takes a seat around a big round table. Ms. Bell starts with the person closest to the door and goes around the table clockwise saying to students in order, "In–Out–In–Out…." When she says "out" to a student, the student must leave the table. She continues around and around the table until just one student remains. That person gets the prize.

Suppose you really would like to get the prize, but you don't know how many people will be in the class until you enter the room. How do you decide which chair at the table to take?

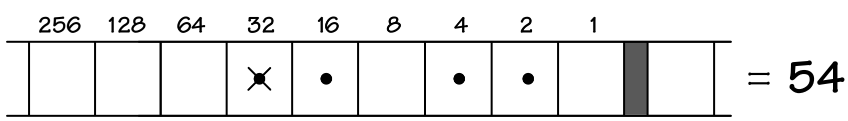
N 35

# People	Chair #
1	1
2	1
3	3
4	1
5	3
6	5
7	7
8	1
9	3
10	5
11	7
12	9
13	11
14	13
15	15
16	1
17	3
18	5
19	7
20	9
21	11
⋮	⋮

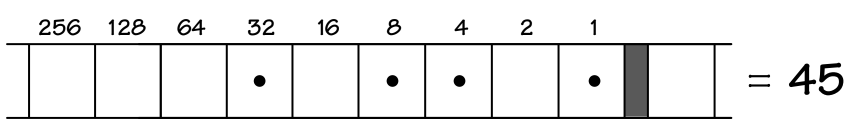


Observations:

- Never take an even numbered chair.
- If the number of people in class is a power of 2, take chair # 1. For example, $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, and so on.
- Observe how many people are in class and subtract as great a power of 2 as possible (to remain positive). Double the result and add 1. Take that chair number.
- Put the number of people in class on a binary abacus. Here, for example, suppose the number is 54. Remove the checker furthest to the left (subtract a power of 2).



Move the remaining checkers one board to the left (double), and add 1.



The result is the chair number to take.

Capsule Lesson Summary

Review the story and problem of Lesson N34. Determine mixtures that meet the minimum daily requirement (MDR) for each of the various nutrients and find corresponding lines on the grid. Locate a region for mixtures which satisfy the MDR for all of the nutrients. Use the fact that cost lines are parallel to locate the least expensive mixture satisfying the MDR.

Materials

Teacher
- *IG-VI World of Numbers Posters* # 1 and # 2
- Colored markers
- Demonstration translator

Student
- Colored pencils, pens, or crayons
- Grid sheet (with cost lines from Lesson N34)

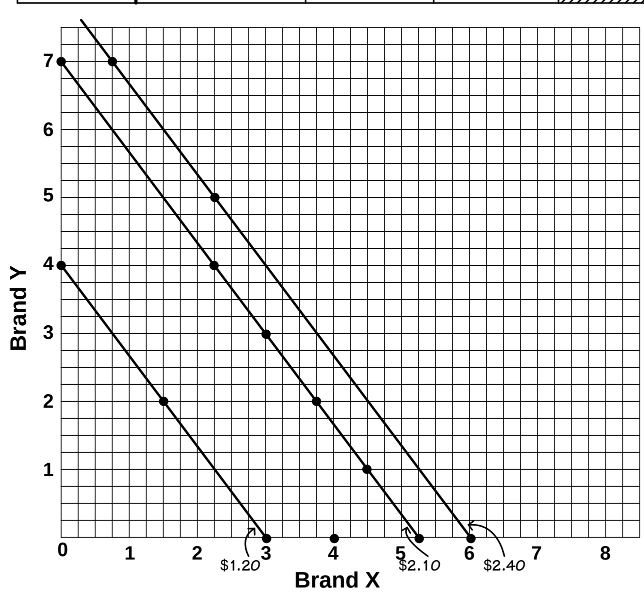
Advance Preparation: The grid transparency or *IG-VI World of Numbers Poster* # 1 used in Lesson N34 should now have three cost lines drawn on it. The same is true for students' copies of the grid sheet. You may want to prepare the table as in the first illustration on the board before starting the lesson.

Description of Lesson

Exercise 1

Ask the class to recall the story of Dr. Dantzig at Wolfe Kennel. Be sure to review the problem of providing a nutritionally adequate, low cost diet for Great Danes. Summarize the pertinent information in a table on the board (see the illustration below), and display the graph (on the grid transparency or Poster # 1) with the three cost lines drawn on it from Lesson N34. Students should have their own copies of the graph so that they can follow the collective lesson by doing similar work on the graph.

	Units of Carbohydrates	Units of Vitamins	Units of Protein	Cost
Brand X	1	4	5	\$0.40
Brand Y	3	4	1	\$0.30
MDR	8	20	7	



T: ***Dr. Dantzig must find the least expensive diet for a Great Dane which meets the minimum daily requirements (MDR) for carbohydrates, vitamins, and protein.***

Earlier we looked at the costs of various mixtures of dog food. (Point to the graph.) ***What do these lines represent?***

S: ***They are cost lines. Each mixture has a cost, and mixtures that cost the same are represented by points on the same line.***

S: ***All of the points on a cost line represent mixtures costing the same.***

T: ***What did we notice about these three cost lines?***

S: ***They are parallel.***

T: ***Yes; if the lines were not parallel, what would happen?***

S: ***The lines would cross.***

T: ***What would it mean if two cost lines crossed?***

S: ***If two of the lines crossed, then the mixture represented by the point where they cross would have two different costs.***

Students may find it difficult to verbalize this idea, so direct the discussion as necessary.

T: ***About where would we find the $2.50 line?***

S: ***A little to the right of the $2.40 line.***

Invite a student to trace approximately where the line would be on the poster.

T: ***How could we locate it exactly?***

S: ***Find some mixtures that cost $2.50. The points which represent such mixtures are on the $2.50 line.***

S: ***One scoop of Brand X and seven scoops of Brand Y cost $2.50, so the point (1, 7) is on the $2.50 line. Use a translator to draw a line parallel to the other lines and through the point (1, 7).***

Alternatively, locate two points for mixtures costing $ 2.50 (for example, (1, 7) and (4, 3)) and draw the line determined by those points. Observe that this line is parallel to the other cost lines.

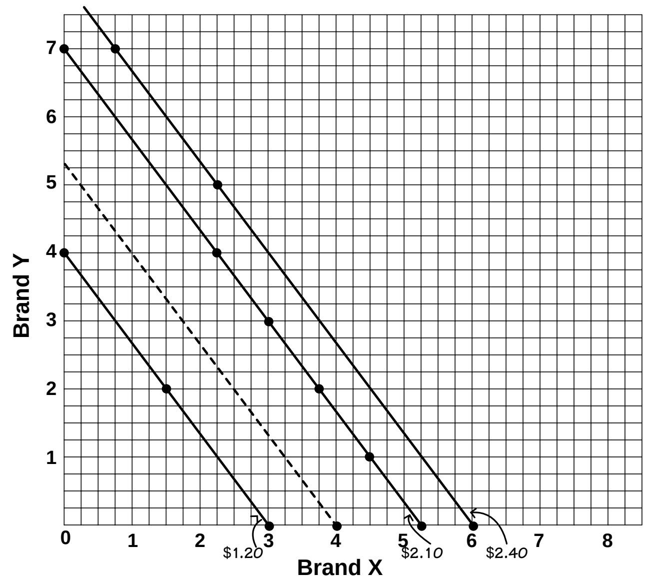
T: ***About where would we find the $1.80 line?***

S: ***Between the $1.20 line and the $2.10 line, but closer to the $2.10 line.***

Invite a student to trace approximately where the line would be on the poster.

T: ***I will draw a cost line on the grid and you tell me which one it is.***

Draw a dotted line, as shown here.



Allow a few minutes for students to study the line.

S: ***\$1.60; because it goes through the point (4, 0) and four scoops of Brand X cost \$1.60 (4 x 0.40 = 1.60).***

T: ***Dr. Dantzig is still looking for a mixture of dog food that will satisfy the minimum daily requirements and also be the lowest in cost. Which mixture do you think she should recommend?***

Record some suggestions on the board for future reference.

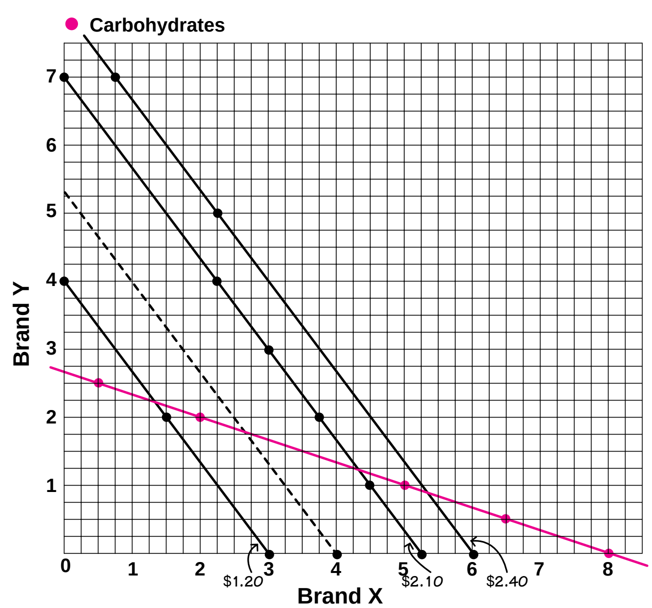
T: ***Now that we know something about the cost, let's look at the mixtures that satisfy the MDR for each of the nutrients (*** refer to the table ***). We can begin with carbohydrates. What are some points representing mixtures that exactly satisfy the MDR for carbohydrates?***

S: ***(5, 1) because (5 x 1) + 3 = 8.***

S: ***(2, 2) because (2 x 1) + (2 x 3) = 8.***

T: ***We'll mark those points on the grid in red.***

Continue this activity until there are four or five red dots on the poster.



S: ***The red dots are in a line.***

Draw the carbohydrate line in red.

T: ***What can we say about points on this red line?***

S: ***They are for mixtures that provide exactly 8 units of carbohydrates.***

T (indicating the appropriate region)**:** ***What can we say about points above this red line?***

S: ***They are for mixtures that provide more than 8 units of carbohydrates.***

T: ***What about the points below the red line?***

S: ***They are for mixtures that provide less than 8 units of carbohydrates.***

T: ***What information do we get from the red line crossing the cost lines?***

S: ***The point where the red line crosses a cost line is a mixture with cost given by the cost line and with 8 units of carbohydrates.***

N 3 6

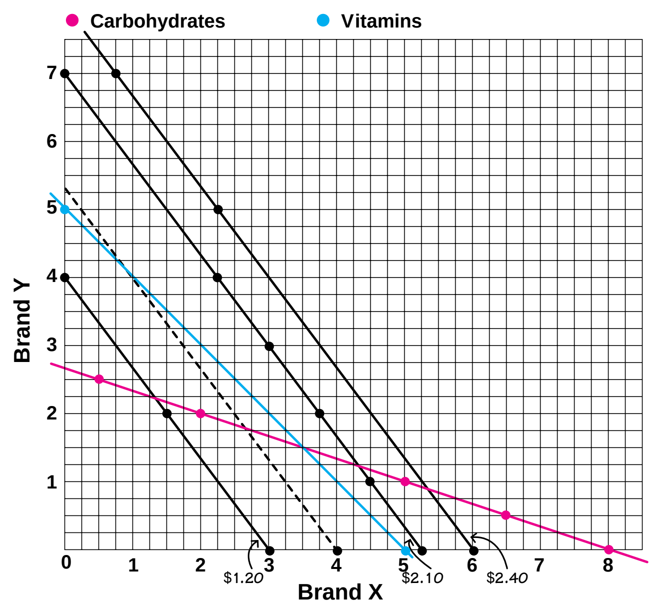
T: ***Now let's consider the MDR for vitamins. What are some points for mixtures which provide exactly 20 units of vitamins?***

If necessary, remind the class that you are now considering vitamins only with no restriction on carbohydrates or protein.

S: ***(5, 0) or (0, 5), because either five scoops of Brand X or five scoops of Brand Y alone provide 20 units of vitamins.***

S: ***You can draw a line through those two points.***

Draw the vitamin line in blue.



S: ***The red line and the blue line cross.***

T: ***Yes, what does this mean?***

S: ***The crossing point, (3.5, 1.5), is for a mixture that provides exactly 8 units of carbohydrates and 20 units of vitamins.***

T: ***What about the points that are above the red line and above the blue line?***

S: ***They are for mixtures which provide more than the MDR for both carbohydrates and vitamins.***

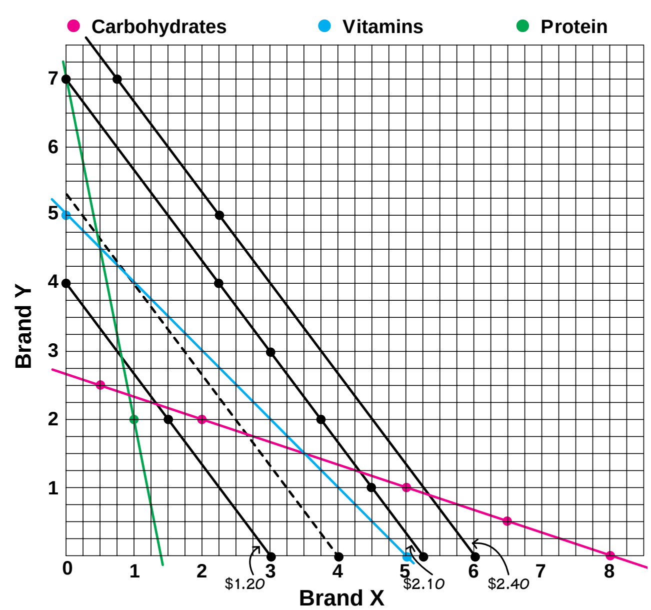
T: ***What should we consider next?***

S: ***A protein line.***

T: ***Name some points for mixtures which provide exactly 7 units of protein.***

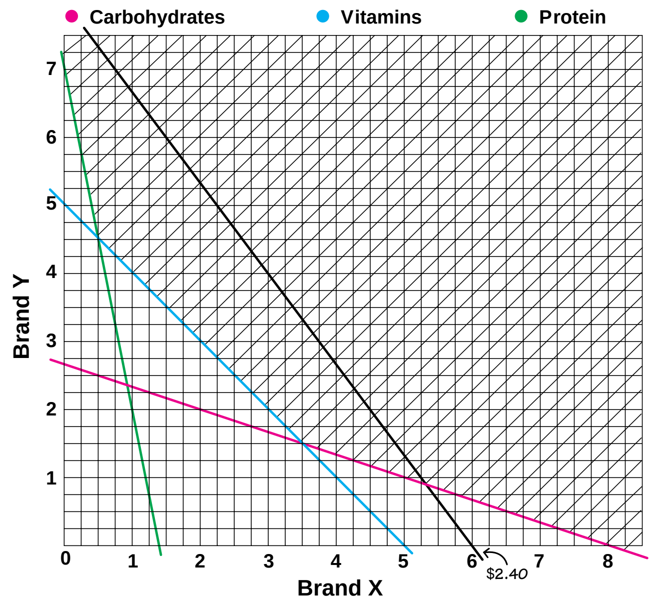
Locate some points as suggested by students, for example, (0, 7) and (1, 2), and draw the protein line in green.

Note: If in the process of getting the three nutrient lines on the grid the picture has become messy or hard to read, display Poster # 2. This poster has the carbohydrate line, the vitamin line, and the protein line on it. In addition, the $ 2.40 line is on it for reference.



T: ***Where are the points for mixtures which satisfy (possibly exceed) all of the minimum daily requirements?***

Invite a student to indicate where such points are on the poster.



S: ***All of the points that are on or above the red line and on or above the blue line and on or above the green line.***

T: ***Where is the point for a mixture that satisfies the MDR and is least expensive?***

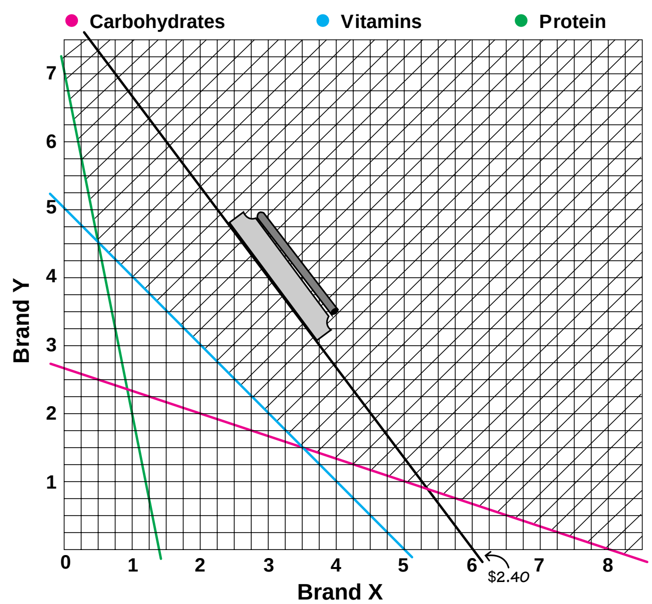
Check any suggestion that is made.

S: ***(3.5, 1.5); where the blue line and the red line cross.***

S: ***That mixture costs $1.85 and it meets all the requirements.***

T: ***Is this mixture the least expensive? Let's check.***

Place the demonstration translator on the $ 2.40 line as shown below.



T: ***What happens when I roll the translator down toward 0?***

S: ***It indicates lines for smaller costs.***

S: ***We must find the lowest point which still is in the region for mixtures satisfying the MDR.***

Roll the translator to go through any of the points that have been suggested.

S: ***The lowest point is where the green line crosses the blue line.***

T: ***Yes; what is the cost of that mixture?***

S: ***½ scoop Brand X and 4 ½ scoops Brand Y; that mixture costs $1.55.***

Writing Activity

Suggest that students write a letter to Dr. Dantzig to explain how she might solve the problem of providing a nutritionally adequate, low cost diet for Great Danes. They can also explain how a solution would change if the units of nutrients in one brand of dog food changed.