# Probability and Statistics

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# PROBABILITY AND STATISTICS INTRODUCTION

In today's world, probabilistic and statistical methods have become a part of everyday life. They are an important part of industry, business, and both the physical and social sciences. Just being an intelligent consumer requires an awareness of basic techniques of descriptive statistics, for example, the use of averages and graphs. Probability and statistics are key to the present day modeling of our world in mathematical terms. The lessons in this strand utilize both fictional stories and real data as vehicles for presenting problems and applications. The problems and questions that arise focus attention on key concepts of probability and statistics such as randomness, equally likely events, and prediction.

Probability stories fascinate most students and encourage their personal involvement in the situations. They often relate the probability activities to games they have encountered outside the classroom. This personal involvement builds students' confidence and encourages them to rely on their intuition and logical thinking to analyze the situations. In *IG-VI*, students use dice or other random devices to simulate a situation or to play a game. These activities help students understand the story and also form a basis for predicting the likelihood of particular outcomes. Yet simulations produce only estimates of the probabilities, leaving open the question of a true probability. Pictorial techniques make the analysis of theoretical probabilities accessible. This combination of simulation and analysis of situations demonstrates the strong interdependence between probability and statistics.

Some applications of mathematics are more easily investigated than others. The probability and statistics applications in this strand lean towards the easy end of the spectrum, both because of the small amount of theoretical knowledge required, and because the link between the situation and the mathematical model is easily perceived. In particular, the development of pictorial models for

s the ease of solutions.

## Content Overview

### Probability

There are many methods available for determining probabilities. The simplest techniques, though usually tedious, require listing all possible outcomes. Most powerful techniques rely on formulas involving the multiplication of probabilities. Throughout *CSMP for the Intermediate Grades*, and continuing in *IG-VI*, the lessons in Probability and Statistics review and introduce several efficient pictorial techniques that elementary students can readily apply.

Three lessons this semester provide real-life examples where a knowledge of probabilities contributes to an understanding of the situations. In each case, students will have an opportunity to simulate a process (game or contest) and to use the results of the simulation to predict probabilities and results.  Simulation is a standard technique for estimating probabilities in applied probability and statistics. Its popularity and value have increased greatly because of the advance of computer technology. Every person should be aware of simulation methods as they are so prevalent in today's world.

# PROBABILITY AND STATISTICS INTRODUCTION

There are many advantages to conducting simulation activities with students. First, simulation is a valid technique for estimating probabilities and finding expected values. Second, such activities are pedagogically sound as they promote an understanding of random situations, variability of data, and prediction. Furthermore, in the *CSMP* sequence students have pictorial methods for finding theoretical probabilities with which to compare the results of simulations.

Lessons: P1, 2, and 3

## Statistics

The simulation activities mentioned above have a statistics aspect in that they involve collection of data, graphing of data, interpretation of data, and estimation. These lessons reinforce an important theme: that statistics is estimation and not an exact science. Everyday realization of this fact will contribute to an understanding of statistics in the world.

Internationally, population growth causes a major impact on the policies of many governments. Domestically, the changing age distribution of the United States population affects educational systems and senior citizens' programs. A prerequisite to comprehending the implications of these phenomena is an understanding of the population statistics which reveal changes.

Four lessons this semester introduce methods of graphing and analyzing United States population data for 1920–1990. Since the various graphs are for the same era, students discover relationships among the graphs, and at the same time learn that each graph provides a distinct perspective on recent United States history. The influences of two World Wars and the Great Depression are apparent. A comparison of current population distribution graphs of the United States, Mexico, and Sweden provide an effective conclusion to this series of lessons. The drastic differences among the graphs suggest many social implications as well as demonstrate the strong visual impact of graphs for conveying comparative information.

The problems in these lessons require the application of many mathematical concepts. Reporting growth rates as annual growth rate per 1 000 people employs averages and ratios. The age distribution data is given in percents to allow comparison between countries. Appropriate approximations are made with a rounding operation. And calculators provide the support which allows students to handle real population data.

Lessons: P1, 2, 3, 4, 5, 6, and 7

## Capsule Lesson Summary

If a breakfast cereal company randomly places one of six prizes in each box of cereal, what is the expected number of boxes a person needs to buy to obtain all six prizes? Simulate this problem with a die in order to estimate the answer. Draw a bar graph of the number of times each face of the die occurs during the simulation.

### Materials

**Teacher**
- Grid board
- Calculator
- Die

**Student**
- Die
- Calculator
- Worksheets P1(a), (b), (c), and (d)

## Description of Lesson

### Exercise 1

Refer to the graphs on Worksheet P1(a) as you explain,

**T:** ***Pretend that you roll a die many times and draw a frequency bar graph that shows how often each outcome (1, 2, 3, 4, 5, and 6) occurs. The graph could look like any of the graphs on this worksheet.***

Invite students to describe graphs **A**, **B**, **C**, and **D**. Perhaps they will simply note the order of frequency for each number on the die; for example:

**S:** ***In graph D, 5 occurs most often and then 2, 1, 6, 3 and 4 in that order.***

From these specific descriptions, try to lead the discussion to a more general perspective of each graph; for example:

**S:** ***In graph A, every number occurs with exactly the same frequency. In graph B, all of the numbers occur with close to the same frequency.***

**S:** ***In graph C, 6 seldom occurs but all of the other numbers occur with about the same frequency.***

**S:** ***In graph D, two numbers (2 and 5) occur often while two other numbers (3 and 4) seldom occur.***

**T:** ***If we roll a die about 100 times and draw a frequency bar graph for the outcomes, would you expect our graph to look most like A, B, C, or D?***

Let students express their opinions.

**T:** ***Later in this lesson, you will roll a die many times and graph the data. We will check which type of graph most of you get.***

### Exercise 2

You may like to begin this exercise by asking students if they know of any cereals that have prizes or contests.

**T:** ***What kinds of prizes or contests do breakfast cereals use to attract customers?***

Allow a couple minutes for students to express their ideas.

**T:** ***Nutribest cereal puts a card in each box of cereal. What might these cards feature?***

**S:** ***Athletes or singers or movie stars.***

**S:** ***Animals or cars or airplanes.***

**T:** ***Nutribest offers cards of six different space vehicles. They produce the same number of cards of each spaceship, and they randomly select one card for each box of cereal. If we wanted to collect a complete set of the six spaceships, how many boxes of cereal do you think we would have to purchase?***

Record some predictions on the board.

**T:** ***What is the fewest number of boxes we would have to buy?*** (Six)
***What is the most number of boxes we would have to buy?***

**S:** ***If we were unlucky, we might have to buy hundreds of boxes.***

**Note:** For your information, if there are N of each kind of card available, a person would have to buy at most (5 x N) + 1 boxes.

**T:** ***There are several ways to determine the average number of boxes we would need to buy. We could go to the store and buy many boxes of cereal. Is that practical?***

**S:** ***No, it would be expensive and wasteful.***

**T:** ***We could analyze the situation using a probability tree or the area method, but with six different cards the analysis would become quite complicated.***

***Instead, let's try a new method. Using a die, we can simulate the situation. That is, we assume that the six spaceships are numbered 1, 2, 3, 4, 5, and 6. Each time we roll a die, we pretend to buy a box of cereal and to get a card with the spaceship numbered like the outcome on the die. Then we count how many times we roll the die until all six numbers occur at least once.***

On the board, draw a table, as in the next illustration. Invite two students to perform a simulation for the class. Let one student roll the die and, after each roll, ask the other student to record the result. Stop occasionally to let the class assess the situation. For example, suppose after five rolls the table on the board looks like this.

Number on Die

| | 1 | 2 | 3 | 4 | 5 | 6 | Number of Rolls |
|---|---|---|---|---|---|---|---|
| 1st Trial | \|\|\| | \|\| | 𝍸 | \| | \|\| | \|\|\| | 16 |
| | | | | | | | |

**T:** ***Which spaceship cards have we already collected?***

**S:** ***2, 3, and 5.***

**T:** ***Which spaceships do we still need?***

**S:** ***1, 4, and 6.***

**T:** ***How many times have we rolled the die?*** (five)

Continue only until each number on the die occurs at least once; then enter the number of rolls in the table. For example:

Number on Die

| | 1 | 2 | 3 | 4 | 5 | 6 | Number of Rolls |
|---|---|---|---|---|---|---|---|
| 1st Trial | ||| | || | ~~||||~~ | | | || | ||| | 16 |
| | | | | | | | |

Repeat the activity having two other students do the simulation to assure that all students understand the method and the objective.

**T:** ***We did two simulations of the situation. Would anyone like to change their prediction of the average number of boxes of cereal needed to collect a complete set of six cards?***

Let students change their predictions if they wish.

Pair students with each pair having a die. Refer to Worksheet P1(b) as you explain,

**T:** ***Each pair should perform this simulation several times. One person can roll the die while the other person records the results; take turns. For each simulation trial, continue until all six numbers on the die occur, and then record how many times you rolled the die to get them. There is room on your worksheet to record six simulation trials. You may not have time to complete all six trials.***

Observe the work to confirm that students are performing the simulation correctly.

Once most pairs of students have completed three or four simulation trials, tell everyone to complete the simulation they are working on and then to stop. When you have the class's attention, describe how you will collect the data.

**T:** ***I'd like to record just the lengths (number of rolls) of each of your simulation trials. Did any pair have a simulation trial requiring only six rolls of the die? One person per pair should raise a hand if you did.***

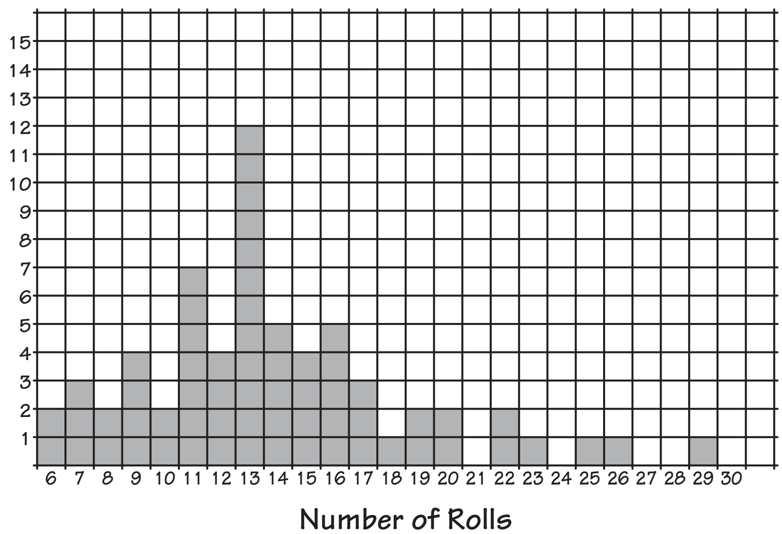
Record a 6 on the board for each hand raised. In a similar manner, continue to collect all of the results. Be sure only one member of a pair raises a hand when appropriate. Your data might look like the following.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 6 | 7 | 7 | 7 | 8 | 8 | 9 | 9 | 9 | 9 | 10 | 10 | |
| 11 | 11 | 11 | 11 | 11 | 11 | 11 | 12 | 12 | 12 | 12 | | | |
| 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | | |
| 14 | 14 | 14 | 14 | 14 | 15 | 15 | 15 | 15 | 16 | 16 | 16 | 16 | 16 |
| 17 | 17 | 17 | 18 | 19 | 19 | 20 | 20 | 22 | 22 | 23 | 25 | 26 | 29 |

Refer to Worksheet P1(c) as you ask students to explain how to draw a bar graph of the data.

**S:** ***Count how many times each number appears in the list on the board. Draw an appropriate vertical bar for that number of rolls. For example, there are two 6s so draw a bar two units high above 6.***

Direct students to graph the collective data on their worksheets. During the individual or partner work, sketch the bar graph for your data on the grid board.




**T:** ***What information does the graph give us about the number of boxes of cereal we are likely to buy in order to collect a full set of the six cards?***

Encourage comments such as the following.

**S:** ***It usually requires buying between 11 and 16 boxes.***

**S:** ***It is unlikely that we would have to buy more than 20 boxes.***

**S:** ***We would be very lucky to complete the set after buying only 6, 7, or 8 boxes of cereal.***

**T:** ***On the average, how many boxes does it appear we would have to buy to obtain all six cards?***

When students answer this question, insert the terms *mean*, *median*, and *mode* as appropriate. For example:

**S:** ***I think it would take 13 boxes because in our class 13 occurs more often than any other result.***

**T:** ***13 is the most frequent result; 13 is called the mode of our data.***

After a while, direct the discussion to calculate the mean, medium, and mode of your data.

**T:** ***We have learned about three different types of averages: mode, median, and mean. Do you remember how to calculate each of these averages?***

**S:** ***The mode is the most frequent result. For our data, the mode is 13.***

**S:** ***To find the median, first list all of the data from fewest boxes (rolls) to most boxes (rolls) as we have already done on the board. Then select the result exactly in the middle of the list.***†

**S:** ***To find the mean, add all of the results and divide by the number of trials.***

Suggest that students use calculators to do this calculation. Encourage the use of shortcuts, for example:

**S:** ***There are two 6s, so add 12.***

**S:** ***There are twelve 13s. 10 x 13 = 130 and 2 x 13 = 26, so add 156.***

The mean of the sample data in this lesson is 890 ÷ 64, or approximately 13.9.

You may need to remind students of the rounding operation in order to round the mean to the nearest one-tenth (0.1). That is,

**T:** ***The calculator displays 13.90625 for 890 ÷ 64. What number is 13.90625 ® 0.1?***

Record the three averages on the board:

Mode: 13 Median: 13 Mean: 13.9

**T:** ***Each of these three averages***†† ***is a good estimate of the number of boxes of cereal we would need to buy to collect a complete set of six cards. The simulation provides a fairly easy way to calculate a good estimate. To obtain an even better estimate, we could program a computer to repeat the simulation thousands of times and then have the computer calculate the mean, median, and mode.***

Compare the averages on the board to earlier predictions.

## Exercise 3

†If there is an odd number of entries, the median is the value of the middle entry. If there is an even number of entries, the median is the mean of the two middle entries. For the data in this lesson description there are 64 entries so the median is the mean of the 32nd and the 33rd entries, namely 13 and 13. So the median is 13.

††You may wish to observe that many people use the term *average* to refer only to the mean, ignoring the other types of average.

Refer students to their simulation trials recorded on Worksheet P1(b).

**T:** ***Now let's look at how often in your simulations each number on the die occurred.***

Direct each pair of students to total the number of times each number on the die occurred during the several simulation trials they performed. Students can then draw a bar graph of their totals on Worksheet P1(d).

When most students have finished their bar graphs on Worksheet P1(d), compare the results to graphs on Worksheet P1(a).

**T:** ***Compare your bar graph on Worksheet P1(d) to the four graphs on Worksheet P1(a). Which, if any, of the four graphs is most similar to your graph?***

Let students display their graphs to the class as they express their opinions. Most of their graphs are likely to be similar to graph **B** on Worksheet P1(a). Conclude that all six numbers on a die tend to occur with approximately the same frequency.

## Writing Activity

You may like students to take lesson notes on some, most, or even all their math lessons. The "Lesson Notes" section in Notes to the Teacher gives some suggestions and refers to forms in the Blacklines you may provide to students for this purpose. In this lesson, for example, students can note how a simulation activity is used to estimate the answer to a problem involving probabilities.

## Capsule Lesson Summary

Tell a story about an archery contest between two people. Given each archer's probability of hitting the target, use a die to simulate the contest and to estimate each archer's probability of winning. Assess the accuracy of the estimates by determining the probabilities with an analysis based on area.

### Materials

**Teacher**
- Meter stick
- Calculator
- Colored chalk

**Student**
- Calculator
- Die
- Metric ruler
- Colored pencils, pens, or crayons
- Worksheet P2(a), (b), and (c), P2* and **

## Description of Lesson

### Exercise 1

Describe an archery contest as follows:

**T:** ***Two archers are practicing and they devise a simple contest. They take turns shooting at a small target; the first one to hit the target wins. They repeat this contest many times.***

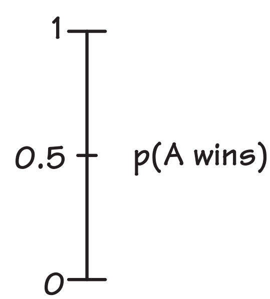
Write this information on the board.

Archers A: p(A hits) = $\frac{1}{2}$
B: p(B hits) = $\frac{1}{2}$
A always shoots first.

**T:** ***Suppose the two archers, A and B, are evenly matched. Each hits the target one-half of the time. B always lets A shoot first. Remember, the first archer to hit the target is the winner. Who do you think is favored?***

**S:** ***Archer A, because they are of equal ability but A always shoots first.***

Draw a 0 to 1 line segment on the board as you ask,

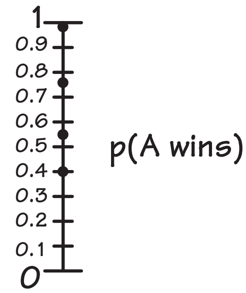



**T:** ***Do you recall how to use a "probability stick" to display probability estimates?***

**S:** ***Draw a dot on the line segment. A dot at 1 means that A always wins and a dot at 0 means that A never wins. Estimate the probability of A winning on a scale from "never" to "always" with a dot at 0.5 for A winning about one-half of the time.***

Invite students to draw dots on the 0 to 1 line segment for their estimates of **A**'s probability of winning. You may need to distinguish the probability that **A** hits the target from the probability that **A** wins the contest. Encourage students to explain and discuss their estimates.

Then divide the line segment into ten equal parts.




**T:** ***We can assign decimal numbers between 0 and 1 to your estimates of the probability that A wins.***

Invite students to read the decimal labels for the dots already on the segment; for example, 0.4, 0.57, 0.75, and 0.98.

**T:** ***There are several ways to determine A's probability of winning. We can obtain an estimate of the probability by simulating the contest with a die. How might we do this?***

Lead the class to consider the following method:

- Select two students to play the archers **A** and **B**.
- Let **A** and **B** take turns rolling a die with **A** always going first.
- **A** hits the target if **A** rolls 4, 5, or 6; similarly, **B** hits the target if **B** rolls 4, 5, or 6.
- The game continues until one player hits the target and wins.

**T:** ***Why do we say A or B hits the target with a roll of 4, 5, or 6? Why not just 6?***

**S:** ***There are three out of six chances to get 4, 5, or 6 on a roll. This agrees with A's and B's probability of hitting the target on each shot; that is, $\frac{3}{6} = \frac{1}{2}$.***

**S:** ***If they hit the target only by rolling a 6, their probability of hitting the target on each roll would be $\frac{1}{6}$, not $\frac{1}{2}$.***

**T:** ***Could we choose numbers other than 4, 5, or 6 to have A or B hit the target?***

**S:** ***Yes, any three numbers from 1 to 6 would work equally well. For example, A (and B) could hit the target by rolling an odd number.***

**Note:** It is convenient, but not necessary, that **A** and **B** hit the target with the same set of numbers.

Select two students to play the roles of archers **A** and **B**. Let the rest of the class give directions to the two players as they use the die to simulate the archery contest several times.

**S:** ***Archer A rolls the die first. If the result is 4, 5, of 6, then A wins. Otherwise it is B's turn. They alternate until one of them wins by rolling a 4, 5, or 6.***

Let the two students play about four games, and record each winner on the board. A sample record is shown here. Emphasize that archer **A** always rolls first even if **A** just won the previous game.

| Game | Winner (A or B) |
|---|---|
| 1 | A |
| 2 | A |
| 3 | B |
| 4 | A |

Pair students so that each pair has a die and a copy of Worksheet P2(a). Suggest that each pair of students play ten games, always letting **A** roll first. Check to see that students are performing the simulation correctly and that they are recording the winners.

After most pairs have played the game ten times, total the results. An easy way to do this is to ask each pair of students how many games were won by **A** and by **B**, and keep a running total. For example:

**S:** ***A** won eight times, **B** won twice.*

**S:** ***A** won six times, **B** won four times.*

| | | |
|---|---|---|
| A | ~~8~~ | 14 |
| B | ~~2~~ | 6 |

Continue until all pairs have reported; for example:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | ~~8~~ | ~~14~~ | ~~21~~ | ~~29~~ | ~~36~~ | ~~43~~ | ~~49~~ | ~~58~~ | ~~64~~ | ~~71~~ | ~~77~~ | 85 |
| B | ~~2~~ | ~~6~~ | ~~9~~ | ~~11~~ | ~~14~~ | ~~17~~ | ~~21~~ | ~~22~~ | ~~26~~ | ~~29~~ | ~~33~~ | 35 |

**T:** ***According to our simulation, what is an estimate of A's probability of winning?***

**S:** ***$^{85}/_{120}$, since A won 85 out of 120 games (85 + 35 = 120).***

**T:** ***How can we find a decimal name for $^{85}/_{120}$?***

**S:** ***Calculate 85 ÷ 120.***

Let students use a calculator to do the division calculation. Similarly, use the simulation results to estimate **B**'s probability of winning.

Write both probabilities on the board.

$$p(\text{A wins}): \frac{85}{120} = 85 \div 120 \approx 0.71$$
$$p(\text{B wins}): \frac{35}{120} = 35 \div 120 \approx 0.29$$

**Note:** You may need to remind the class of the rounding operation in order to round each probability to the nearest one-hundredth (0.01). "≈" means "approximately equal to." Explain this symbol if you use it.

Compare the estimate of **A**'s probability of winning with earlier predictions.

## Exercise 2

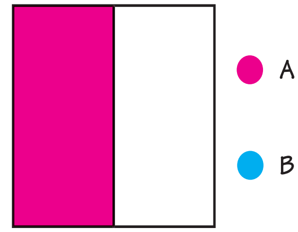
**T:** ***Simulation is a good way to estimate the probability of an event. The estimates are very accurate if we can simulate the event a lot, say hundreds or thousands of times. Often people use computers to perform simulations and to calculate probability estimates.***

***But we can never be sure that a simulation gives an accurate estimate of a probability. Maybe the die is bad or maybe we had a biased way of rolling the die. Maybe many more (or fewer) 4s, 5s, and 6s appeared than usual. To find the theoretical probability, we use other methods.***

Draw a square on the board. 60 cm by 60 cm is a convenient size.

**T:** ***Let's study this game of archery and use a square to show the theoretical probability. How does the game start?***

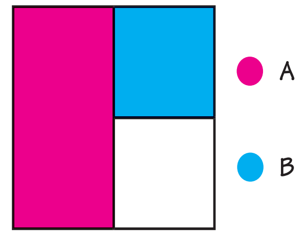
**S:** ***Archer A shoots and has probability ½ of hitting the target. A wins immediately by hitting the target.***




**T:** *How could we represent that on the square?*

**S:** *Divide the square in two equal parts and color one-half red.*

**T** (pointing to the unshaded region)**:** *What happens if* **A** *misses the target on the first shot?*



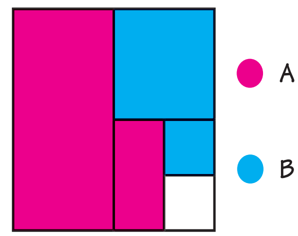

**S:** **B** *shoots with probability $\frac{1}{2}$ of hitting the target.* **B** *wins by hitting the target. Divide the unshaded region into two equal parts and color one-half blue.*

**T:** *So far, how does the red area compare to the blue area?*

**S:** *There is twice as much red as blue.*

**T:** *What happens next in the archery game?*

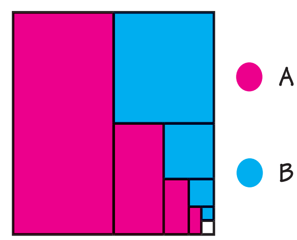
Let students explain the next two steps that lead to this picture.




**T:** *So far, how does the red area compare to the blue area?*

**S:** *There is twice as much red as blue, because the large red piece is twice the size of the large blue piece, and the small red piece is twice the size of the small blue piece.*

Let students continue the analysis for a few more steps.




**S:** *This could go on forever!*

**S:** *If we keep on going, how will the red area compare to the blue area?*

**S:** *There is always twice as much red as blue.*

**T:** *Therefore, what fraction of the square will be shaded red? Blue?*

**S:** *$\frac{2}{3}$ red and $\frac{1}{3}$ blue.*

Write this information on the board.

$$p(\text{A wins}) = \frac{2}{3}$$
$$p(\text{B wins}) = \frac{1}{3}$$

**T:** *What is a decimal name for $\frac{2}{3}$?*

**S:** *0.666 … or $0.\dot{6}$.*

**T:** *We found that the probability that archer* **A** *wins this game is $\frac{2}{3}$ or $0.\dot{6}$. How does this compare to the estimate we got from our simulation?*

**S:** *Our estimate, 0.71, is close to $0.\dot{6}$.*

## Exercise 3

Describe the archery game with this information.

A: p(A hits) = $\frac{1}{3}$

B: p(B hits) = $\frac{1}{2}$

A always shoots first.

**T:** ***Suppose the two archers are not of equal ability. For example, maybe archer* A *hits the target only one-third of the time while* B *hits the target one-half of the time. Now, if the archers play the same game with* A *always shooting first, who do you think is favored?***

**S:** **B** ***is a better shot, but* A *has the advantage of shooting first. Maybe* B *is favored now; maybe* A *is still favored.***

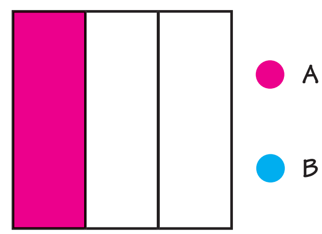
**S:** ***It could be a fair game.***

**T:** ***Let's analyze this game to determine who is favored.***

Refer students to Worksheet P2(b), while you draw a square on the board.

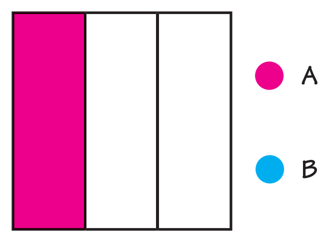
Do the following analysis immediately as a class or, if you prefer, allow students to work in partners or independently for a few minutes, and then lead a collective discussion of the analysis.

**T:** ***How does the contest start?***




**S:** ***Archer* A *shoots first and has probability ⅓ of hitting the target. Color one-third of the square red.***

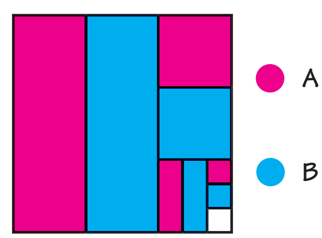
**S:** ***If* A *misses on the first shot, archer* B *shoots and has probability ½ of hitting the target. Color one-half of the unshaded region blue.***




**T:** ***So far, how do the red area and the blue area compare?***

**S:** ***They are equal.***

Let students continue the analysis as indicated in this illustration.




**T:** ***If we continue, what fraction of the square will be red? Blue?***

**S:** ***½ red and ½ blue.***

**T:** ***Who is favored in this game?***

**S:** ***It is a fair game.***

# P2

Write this information on the board.

p(A wins) = $\frac{1}{2}$

p(B wins) = $\frac{1}{2}$

**T:** ***Therefore, in this game,* A*'s advantage of always shooting first exactly balances* B*'s advantage of being the better shot.***

The following optional exercise is a simulation of the previous activity. If you choose to skip it, Worksheets P2* and ** are available for individual work.

## Exercise 4 (optional)

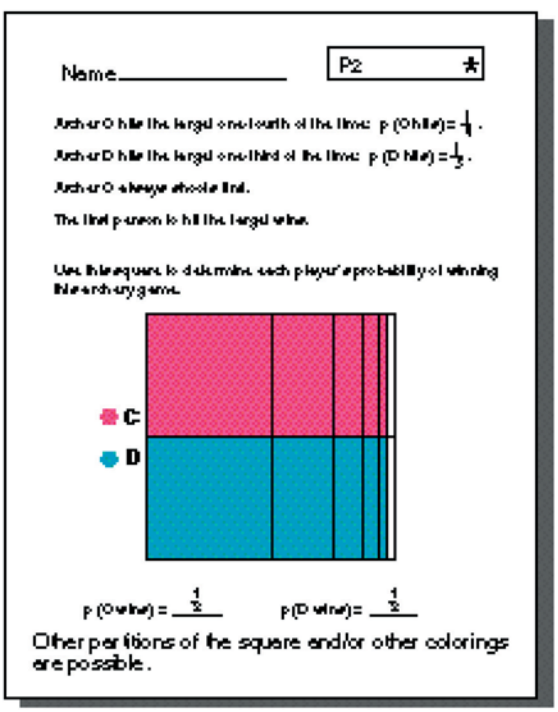
**T:** ***In the game we just analyzed, archer* A *has probability* ⅓ *of hitting the target and archer* B *has probability* ½ *of hitting the target. According to our analysis, if* A *always starts, the game is fair. How could we use a die to simulate this game to see if we arrive at the same conclusion?***

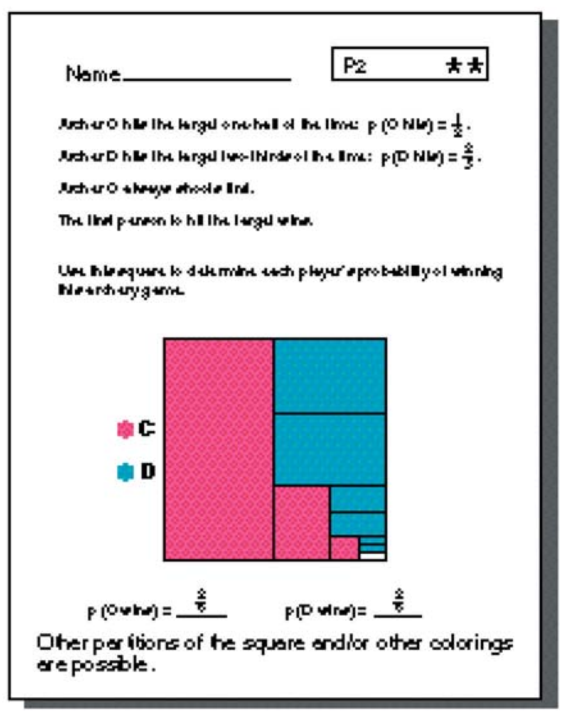
Lead the class to consider the following method:

- **A** and **B** take turns rolling a die with **A** always going first.
- **A** hits the target and wins if **A** rolls 5 or 6; **B** hits the target and wins if **B** rolls 4, 5, or 6.
- Continue until one player wins.

**Note:** Of course, students could pick any two numbers on the die for **A** and any three numbers for **B**.

Pair students with a die and a copy of Worksheet P2(c). Let them conduct the simulation for this situation. After most pairs of students complete ten games, total the results at the board. Note whether **A** and **B** win approximately the same number of times as predicted by the analysis.



## Capsule Lesson Summary

Introduce a carnival-type game that involves tossing a coin onto a grid of squares. Simulate the game many times to estimate the probability of winning both with a dime and with a quarter. find the theoretical probability of winning each game by calculating the areas of regions representing success to compare to the area of a grid square.

## Materials

**Teacher**
- Meter stick
- Chalkboard compass
- Colored chalk
- Blackline P3

**Student**
- Calculator
- Metric ruler
- One dime and one quarter
- 3½ cm grid sheet
- Worksheets P3(a) and (b)

**Advance Preparation:** Use Blackline P3 to make copies of a 3½ cm grid sheet for students.

## Description of Lesson

### Exercise 1

Describe the following carnival-type game.

**T:** ***Toss-a-Coin is a common carnival game. A person tosses a coin, usually a dime, onto a large grid board laying on the ground. If the coin lands touching a grid line, the person loses. If the coin lands completely within a square, the person wins money or a prize. Has anyone ever played a game like this?***

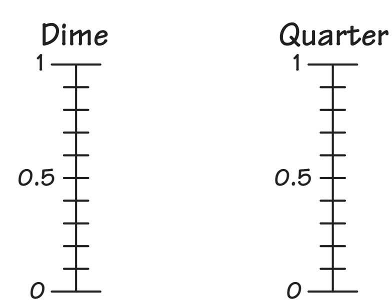
Let students describe their experiences, if any. Then refer to the 3½ cm grid sheet as you explain,

**T:** ***You will play the game on your table with this grid sheet. You can toss a penny, a nickel, a dime, a quarter, a half-dollar, or a silver dollar. With which coin would you be most likely to win? Lose?***

**S:** ***The dime is the smallest coin so it would give the highest probability of winning.***

**S:** ***The silver dollar would yield the lowest probability of winning because it is the largest coin.***

**T:** ***Let's play this game with dimes and quarters.***




Draw probability sticks for each coin on the board. 50 cm is a convenient length for the 0 to 1 line segments.

Recall the use of probability sticks to display probability estimates.

**T:** ***Think about tossing a dime (or a quarter) onto the grid of squares on this sheet. Predict the probability that the coin will land completely within a square, not touching any grid line.***

Let students predict each probability by drawing a dot on the appropriate probability stick. Encourage students to give decimal names for the probabilities they predict.

Pair students and give each pair a dime and a quarter. Instruct the pairs to place some paper or cardboard under the grid sheet on a table. You may also suggest they frame the sheet with books or use the sheet to line the bottom of a shallow box. Direct each student to toss each coin (the dime and the quarter) ten times and to record the number of wins on Worksheet P3(a). A pair of students will record the wins for 40 tosses, ten per student with each coin. Insist that students toss the coin a short distance (20 to 30 cm), not simply drop it.

After a while, collect the data on the board by asking each pair of students for their total number of wins in the 20 tosses with each coin. For example:

| | |
|---|---|
| Dime: | 7, 5, 6, 7, 5, 2, 4, 3, 10, 8, 3, 4, 6 |
| Quarter: | 3, 4, 1, 3, 2, 0, 1, 6, 3, 1, 2, 4, 4 |

**T:** ***How do we calculate the mean (average) of the number of wins in 20 tosses of each coin?***

**S:** ***Add the total number of wins for each coin and divide by the number of pairs of students.***

As a class, mentally sum each set of numbers. Then let students use a calculator to do the division calculation, and record the results on the board. For example:

Winning Tosses

| | Total | Mean | |
|---|---|---|---|
| Dime | 70 | $70 \div 13 \approx 5.4$ | |
| Quarter | 34 | $34 \div 13 \approx 2.6$ | |

You may need to remind students of the rounding operation in order to round each mean to the nearest one-tenth (0.1).

**T:** ***What might be a good estimate for the probability of winning with the dime? With the quarter?***

**S:** ***On the average, we won 5.4 out of 20 times tossing the dime, so the probability of winning with the dime is about*** $5.4/20$ ***or 0.27.***

**S:** ***We averaged 2.6 wins out of every 20 tosses with the quarter, so the probability of winning with the quarter is about*** $2.6/20$ ***or 0.13.***

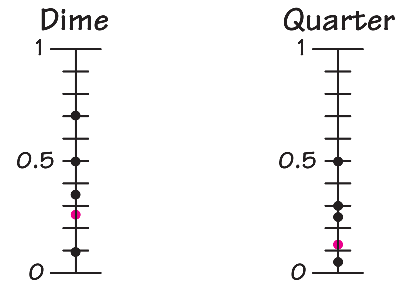
Add this information to the record on the board.

Winning Tosses

| | Total | Mean | Estimate of Probability |
|---|---|---|---|
| Dime | 70 | $70 \div 13 \approx 5.4$ | $\frac{5.4}{20} = 0.27$ |
| Quarter | 34 | $34 \div 13 \approx 2.6$ | $\frac{2.6}{20} = 0.13$ |

**T:** ***These are estimates for the probability of winning in two Toss-a-Coin games. How do these estimates compare to our predictions?***

Invite students to draw red dots for these estimates on the appropriate probability sticks. Compare the estimates to the earlier predictions. For example:



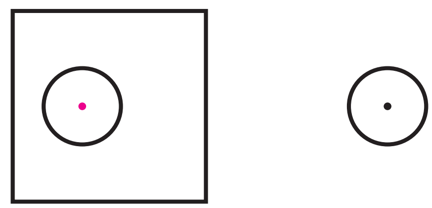

**T:** ***A simulation like this gives us good estimates of the probability of winning each game and allows us to decide which predictions are fairly accurate and which are unreasonable. Of our original predictions, which were close?***

## Exercise 2

Draw a 50 cm by 50 cm square on the board. Draw a circle of radius 10 cm next to the square. Clearly mark the center of the circle.

**T:** ***Let's pretend that this square is one square of the grid and that this circle is a coin. Consider the center of the circle. Show me a point inside the square where the center of the circle could be so that the circle is entirely within the square.***

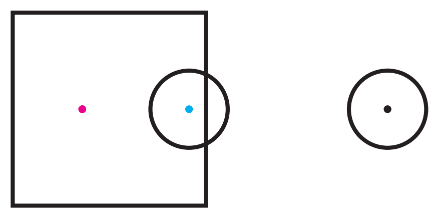
Invite a student to indicate a point within the square. Draw a circle with its center at that point and with a radius of 10 cm. If the circle lies entirely within the square, draw a red dot at its center, as illustrated here.



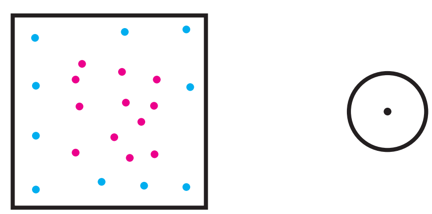
Erase the circle, leaving the red dot.

**T:** ***Where inside the square could the center of the circle be so that the circle overlaps the square?***

In a similar manner as above, invite a student to select a point within the square. Draw the appropriate circle; if it overlaps the square, draw a blue dot at its center.



Invite students to locate many red dots and blue dots within the square, red dots representing centers of 10 cm circles (coins) that lie completely within the square and blue dots representing centers of 10 cm circles that overlap the square.



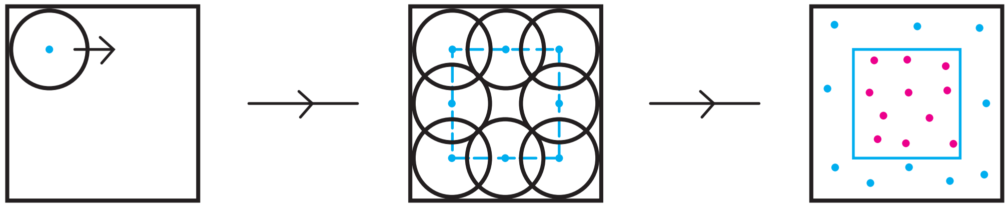
**T:** ***If we continued drawing red dots, what shape would they eventually fill?***

**S:** ***A square.***

**T:** ***How could I draw this interior square?***

# P3

Let students discuss methods of finding this interior square. One method is to imagine placing the circle (coin) inside the grid square, near one corner, touching two sides of the square. Then imagine rolling the circle along the four sides of the square, marking the center of the circle as it rolls. This determines a blue square, all of whose interior points are red. Other methods are possible.



**T:** ***So if the center of the coin lands inside this interior square when we toss it, we win. We lose if the center of our coin lands on the interior square or outside of it. Now let's try to calculate the probability of winning.***

Write this information on the board.

$$p(\text{win}) = \frac{\text{Number of ways to win}}{\text{Number of ways to win or lose}}$$

**T:** ***Usually we calculate the probability of winning by determining the number of ways to win and the total number of ways to win or lose. But in this problem, there are infinitely many ways to win (red dots) and infinitely many ways to lose (blue dots).***

Erase the right side of the above equation on the board.

**T:** ***How should we calculate the probability of winning in this situation?***

**S:** ***Find the area of the interior square and the area of the grid square.***

Add this information to the equation on the board.

$$p(\text{win}) = \frac{\text{Area of interior square}}{\text{Area of grid square}}$$

Let students direct you in measuring the sides of the two squares and in calculating their areas in the picture on the board.

| | Length of side | Area |
|---|---|---|
| Interior square | 30 cm | 900 $cm^2$ |
| Grid square | 50 cm | 2 500 $cm^2$ |

**T:** ***What is the probability of winning?***

**S:** $\frac{900}{2\,500}$ ***or*** $\frac{9}{25}$.

**T:** ***What is a decimal name for*** $\frac{9}{25}$***?***

**S:** ***0.36.***

**T:** ***So if our grid squares and coins were the size of the square and circle on the board, our probability of winning would be 0.36. But both our grid squares and coins are much smaller. How can we find our probability of winning?***

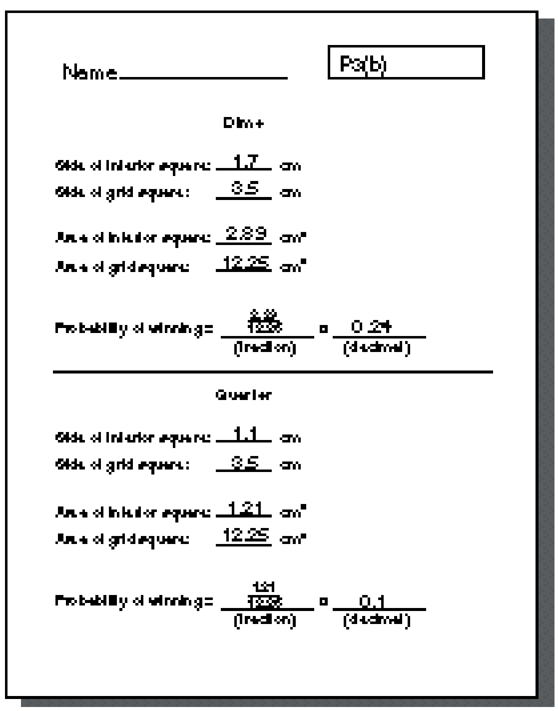
Let students discuss how to adapt the method just demonstrated on the board to their situation. They should mention the following steps.

- Move a dime inside one of the grid squares on the grid sheet to determine an interior square indicating a safe region within which the center of the dime can land.
- Measure the sides of the interior square and of the grid square.
- Calculate the areas of the interior square and the grid square.
- Divide the area of the interior square by the area of the grid square to find the probability of winning.

Allow students to work on Worksheet P3(b) with partners. Encourage the use of a calculator to do the area calculations. Just before the end of the class period, let some students present their final results to the class. Compare their results to the predictions the students made and to the results of the simulations in Exercise 1.

## Home Activity

Students may enjoy taking home copies of the grid to tell family members about the carnival game with coins.


Name________________ P3(b)

Dime

Side of interior square: 1.7 cm
Side of grid square: 3.5 cm

Area of interior square: 2.89 $cm^2$
Area of grid square: 12.25 $cm^2$

Probability of winning = 2.89/12.25 (fraction) = 0.24 (decimal)

Quarter

Side of interior square: 1.1 cm
Side of grid square: 3.5 cm

Area of interior square: 1.21 $cm^2$
Area of grid square: 12.25 $cm^2$

Probability of winning = 1.21/12.25 (fraction) = 0.1 (decimal)

# P4 POPULATION GROWTH RATE #1

## Capsule Lesson Summary

Review the rounding operation. Define population growth rate. Using population data, determine the net population gain or annual growth rate of several cities.

### Materials

| Teacher | | Student | |
|---|---|---|---|
| | • None | | • Calculator<br>• Paper<br>• Worksheets P4* and ** |

**Advance Preparation:** The inclusion of population data for your city is highly recommended for Exercise 3. Recent almanacs are a reliable source of population data. With Internet access, you may find data for your city at the U.S. Census Bureau web site <www.census.gov>.

## Description of Lesson

### Exercise 1

Write these two problems on the board.

387 ® 10 = [ ]     387 ® 100 = [ ]

**T:** ***Who can explain the rounding operation?***

**S:** ***The rounding operation calls for the multiple of the second number closest to the first number. For example, 387 ® 10 = 390 since 390 is the closest multiple of 10 to 387.***

**S:** ***387 ® 10 = 390, because when you count by 10 the closest you get to 387 is 390.***

**S:** ***387 ® 100 = 400, because 400 is the closest multiple of 100 to 387.***

Write the following problems on the board and solve them in a similar manner. (Answers are in boxes.)

| | |
|---|---|
| 73 ® 10 = [70] | 671 ® 100 = [700] |
| 129 ® 10 = [130] | 8 438 ® 100 = [8 400] |
| 58 711 ® 1000 = [59 000] | 5,555,444 ® 1000 = [5,555,000] |
| 16.84 ® 0.1 = [16.8] | 9.36 ® 0.1 = [9.4] |
| 0.372 ® 0.1 = [0.4] | 0.372 ® 0.01 = [0.37] |

For problems involving decimals, you may remind students of calculator relations such as [+][0.1][=] … . For example, to solve 16.84 ® 0.1, consider counting by 0.1 on the calculator by pressing [+] [0] [.] [1] [=] [=] [=] and so on.

### Exercise 2

Present the following U.S. population information.

**T:** ***In 1790, the United States' population was only about 4 million. That is about half the present population of New York City. Today we are going to learn about the United States' population growth.***

***Births can increase a country's population. What are other ways for a country's population to increase or decrease?***

**S:** ***Deaths.***

**S:** ***Immigration and emigration.***

You may need to explain that an *emigrant* is a person who leaves his or her country to live elsewhere.

**T:** ***When we put all four of these factors together, we get a country's net growth.***

Write the following statements on the board.

**In 1990, the United States showed a net gain in population. Its rate was an additional 9.8 people for every 1 000 people.**

Invite students to discuss what this statement means. They may suggest some of the following ideas:

- Consider the United States' population being divided into groups of 1 000. For each group of 1 000 people at the beginning of the year, there were 1 009 or 1 010 people at the end of the year.
- We say "9.8 people" because 9.8 is an average calculated from many groups of 1 000 people.
- 9.8 people for every 1 000 means the same as 98 people for every 10 000.

On the board, begin a table with this line of information.

**T:** ***A city has a population of 5 000 and its annual growth rate is 6.4 per 1 000. What is that city's net population gain in one year?***

| Population | Annual Growth Rate per 1000 people | Annual Net Gain |
|---|---|---|
| 5 000 | 6.4 | |
| | | |

**S:** ***32, since 5 x 6.4 = 32.***

**T:** ***Why did you calculate 5 x 6.4?***

**S:** ***You can imagine that the city's population is divided into five groups of 1 000 people. Each group of 1000 gains, on the average, 6.4 people. Therefore, we multiply 5 x 6.4 to find the net gain.***

Record this answer and extend the table.

**T:** ***If a city with 17 000 people has a growth rate of 10.7 per 1000, what is its annual net gain?***

| Population | Annual Growth Rate per 1000 people | Annual Net Gain |
|---|---|---|
| 5 000 | 6.4 | 32 |
| 17 000 | 10.7 | |
| | | |

**S:** ***181.9 since 17 x 10.7 = 181.9.***

**T:** ***Let's use the rounding operation to round each net gain to the nearest person. What does 181.9 ® 1 equal?***

**S:** ***182.***

Record 182 in the table.

Proceed similarly for two or three more lines of data. (Answers are in boxes.) When considering the population of 1,086,000, you may need to discuss that 1,000,000 people could gather in 1000 groups of 1000 people. Thus, there are 1 086 thousands in 1,086,000. Students may notice that this can be determined by covering the last three zeros in 1,086,000.

Add a line of data for Chicago.

**T:** ***What do you think a negative growth rate means?***

**S:** ***The city lost population in that year.***

**T:** ***What was Chicago's net gain that year?***

**S:** ***2 784 x ⁻7.4 = ⁻20 601.6, so the annual net gain is ⁻20 602.***

| Population | Annual Growth Rate per 1000 people | Annual Net Gain |
|---|---|---|
| 62 000 | 10.7 | 663 |
| 965 000 | 2.3 | 2 220 |
| 1,086,000 | 2.3 | 2 498 |
| 2,784,000 (Chicago) | ⁻7.4 | |

Record this result in your table.

Then add this line of data to the table.

**T:** ***How should we calculate net gain when a city's population is 7 800?***

| Population | Annual Growth Rate per 1,000 people | Annual Net Gain |
|---|---|---|
| 7 800 | 15.3 | |
| | | |

Accept suggestions from students. Make sure that both of the following methods are discussed.

- Use 7 800 ® 1 000 = 8 000 to round 7 800 to the nearest 1 000. Then calculate 8 x 15.3 = 122.4, and 122.4 ® 1 = 122.
- Calculate 7 800 ÷ 1 000 = 7.8 (i.e., there are 7.8 thousands in 7 800). Then calculate 7.8 x 15.3 = 119.34, and 119.34 ® 1 = 119.

Discuss the effect of rounding by comparing the two results, 122 and 119. Note that the two results are quite close and that either result would be acceptable for most uses of this data.

Proceed similarly for these lines of data. Let students choose whether or not to first round the population to the nearest thousand before multiplying by the annual growth rate. (Both answers are provided in the boxes.)

| Population | Annual Growth Rate per 1,000 people | Annual Net Gain |
|---|---|---|
| 12 360 | 3.8 | 46 or 47 |
| 606 900 (Washington, D.C.) | ⁻4.9 | ⁻2 974 |
| | | |

Erase all of the data from the board before proceeding.

## Exercise 3

Put these lines of information in the table.

| Population | Annual Growth Rate per 1000 people | Annual Net Gain | Students' Predictions |
|---|---|---|---|
| 63 000 | 16.4 | 1033 | X |
| 63 000 | 8.6 | | |
| 93 720 | 16.4 | | |
| 93 720 | 10.4 | | |

Instruct students not to use calculators in answering the next questions.

**T:** ***A city of 63 000 people with a growth rate of 16.4 per 1 000 has an annual net gain of 1033 people. Is the net gain for a city of 63 000 people and growth rate of 8.6 per 1000 more or less than 1033?***

**S:** ***Less, since that city's population is the same as the first city's but its growth rate is lower.***

**S:** ***Less, since we would calculate 63 x 8.6 instead of 63 x 16.4.***

Write "less than 1 033" in the column for students' predictions. Consider the next two lines of data similarly, asking students to predict whether the net gain will be more or less than 1033. Students should predict that the net gain in a city with 93 720 people and a growth rate of 16.4 will be more than 1033. The class may be undecided about the last line of data since 93 720 is more than 63 000 but 10.4 is less than 16.4.

If you wish, ask students to estimate the net gain for the three lines of data. Then invite students to use their calculators to calculate the three unknown net gains.

Note whether the students' predictions were accurate.

| Population | Annual Growth Rate per 1000 people | Annual Net Gain | Students' Predictions |
|---|---|---|---|
| 63 000 | 16.4 | 1033 | X |
| 63 000 | 8.6 | 542 | less than 1033 |
| 93 720 | 16.4 | 1542 or 1537 | more than 1033 |
| 93 720 | 10.4 | 978 or 975 | undecided |

Add this line of data to the table.

| Population | Annual Growth Rate per 1000 people | Annual Net Gain |
|---|---|---|
| 5 000 | | 42 |

**T:** ***If a city of 5 000 gains 42 people in one year, how can we calculate its growth rate?***

Use one or both of the following two methods of solution, whichever your students suggest.

- Consider 5 x ☐ = 42 and fill in the box by trial and error. For example, students may try 5 x 8 = 40, 5 x 9 = 45, 5 x 8.5 = 42.5, and so on until they determine that 5 x 8.4 = 42.
- Calculate 42 ÷ 5 = 8.4.

Proceed similarly with two or three more such problems. (Answers are in boxes, using ® 0.1 to round each growth rate to the nearest tenth.)

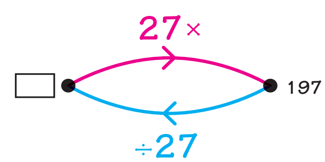
| Population | Annual Growth Rate per 1000 people | Annual Net Gain |
|---|---|---|
| 23 000 | 1.8 | 42 |
| 27 000 | 7.3 | 197 |
| 164 812 | 2.2 | 366 |
| | | |

You may suggest the division method to solve the above problems with the following discussion.

Refer to one line of data and write an open sentence on the board.
For example: 27 × □ = 197 □

**T:** ***Is there another number sentence we could write using these numbers?***

An arrow picture might help students identify 197 ÷ 27 = □.




**T:** ***What is the growth rate for that city?***

**S:** ***About 7.3, since 197 ÷ 27 is about 7.296 and 7.296 ® 0.1 = 7.3.***

197 ÷ 27 = □

Invite students to solve a couple more problems using the division method. (Answers are in boxes.)

| Population | Annual Growth Rate per 1000 people | Annual Net Gain |
|---|---|---|
| 1,216,000 | 6.8 | 8 269 |
| 82 281 | ⁻13.6 | ⁻1118 |
| | | |

If you can find population data for two consecutive (recent) years for your city, determine the annual net gain in the one year, and then ask the class to find the annual growth rate for your city.

Worksheets P4* and ** are available for individual work.

Name________________ | P4 ★

Complete.

33 Ⓡ 5 = 35 | 16 837 Ⓡ 10 = 16 840
33 Ⓡ 8 = 32 | 16 837 Ⓡ 100 = 16 800
82 Ⓡ 10 = 80 | 16 837 Ⓡ 1000 = 17 000
637 Ⓡ 10 = 640 | 5.862 Ⓡ 1 = 6
73.27 Ⓡ 1 = 73 | 5.862 Ⓡ 0.1 = 5.9
58.18 Ⓡ 0.1 = 58.2 | 5.862 Ⓡ 0.01 = 5.86

Name________________ | P4 ★★

Fill in the boxes.

| City | Population | Annual Population Growth Rate per 1000 People | Annual Net Gain |
|---|---|---|---|
| Hondo, Texas | 5 000 | 9.8 | 49 |
| Harrisburg, Pennsylvania | 79 697 | −14.7 | −1 172 |
| Colo. Springs, Colorado | 281,140 | 30.7 | 8 631 |
| Chicago, Illinois | 2,783,726 | −7.4 | −20 600 |
| Springfield, Oregon | 6 208 | 8.7 | 54 |
| Bogalusa, Louisiana | 21 823 | −13.8 | −301 |

A city with a population of 32 000 on January 1, 1994 has a population growth rate per 1 000 people of 13.6 in 1994 and 12.9 in 1995. What is the city's population on:
January 1, 1995? 32 435
January 1, 1996? 32 853

## Capsule Lesson Summary

Review calculations dealing with population growth data. Organize population data into a table and a graph. Ask quantitative and social questions about the data and the graph.

### Materials

**Teacher**
- *IG-VI Probability Poster* #1
- Blacklines P5(a) and (b)

**Student**
- Calculator
- *News* paragraph
- Growth rate graph

**Advance Preparation:** Use Blackline P5(a) to make copies of the *News* paragraph for students. Use Blackline P5(b) to make copies of the growth rate graph for students.

## Description of Lesson

### Exercise 1

Begin this table on the board.

| Year | Population Jan. 1 | Annual Growth Rate per 1000 | Annual Net Gain |
|---|---|---|---|
| 1987 | 696,000 | 2.9 | |
| 1988 | | | |

**T:** ***This population data is for South Dakota. We are going to study South Dakota's population changes over a five-year period. What was South Dakota's net population gain in 1987?***

**S:** ***2 018, since 696 x 2.9 = 2 018.4 and 2 018.4 ® 1 = 2 018.***

**T:** ***Therefore, what was South Dakota's population on January 1, 1988?***

**S:** ***696 000 + 2 018 = 698 018.***

Record these answers in the table.

Continue in a similar manner, adding one line of data at a time, until the following table is complete. Let students choose whether or not to round off the January 1 population to the nearest thousand before multiplying by the annual growth rate, but ask that they be consistent throughout the activity.

| Year | Population Jan. 1 | Annual Growth Rate per 1000 | Annual Net Gain |
|---|---|---|---|
| 1987 | 696,000 | 2.9 | 2018 |
| 1988 | 698,018 | ⁻1.3 | ⁻907 |
| 1989 | 697,111 | ⁻1.6 | ⁻1115 |
| 1990 | 695,996 | 8.6 | 5990 |
| 1991 | 701,986 | 10.1 | 7100 |
| 1992 | 709,086 | 9.7 | 6877 or 6878 |

Write this data on the board.

| | Population | Growth Rate per 1000 |
|---|---|---|
| U.S. 1990: | 249,000,000 | 10.4 |

**T:** ***In 1990, the United States' population was about 249,000,000 and our growth rate was 10.4 per 1 000. About what was the net population gain for the United States in 1990?***

**S:** ***2,589,600; 249 000*** x ***10.4 = 2,589,600.***

You may need to remind students that one million people make up 1 000 groups of 1 000 people each. Therefore, there are 249 000 groups of 1000 in 249,000,000.

**T:** ***The annual net gain was 2,589,600. What was the daily net gain?***

**S:** ***About 7 095 per day, since there are 365 days in a year and 2,589,600 ÷ 365 ≈ 7 095 (rounded to the nearest 1).***

**T:** ***Yes, the United States was gaining about 7 095 people every day. How many is that per hour?***

**S:** ***About 296. There are 24 hours in a day and 7 095 ÷ 24 = 295.625.***

**T:** ***What was the net gain per minute?***

**S:** ***About 5 new people every minute, since there are 60 minutes in an hour and 296 ÷ 60 = 4.93.***

## Exercise 2

Distribute copies of the *News* paragraph on Blackline P5(a).

**T:** ***Pretend that this is a newspaper article. I'll ask you some questions about it. When was the population growth rate 11.9?***

**S:** ***It was 11.9 in 1965 and 1980.***

**T:** ***What was the growth rate in 1935?***

**S:** ***6.3.***

**T:** ***In what year mentioned in the article was the growth rate the greatest?***

**S:** ***In 1920, when the rate was 18.8.***

**T:** ***In which five-year periods did the growth rate decline?***

Accept any of the following answers: 1920–1925, 1925–1930, 1930–1935, 1955–1960, 1960–1965, 1970–1975, and 1980–1985.

**T:** ***What do you think of the way this data is presented? Can you suggest better ways to organize the data?***

**S:** ***Put the information into a table.***

Following students' suggestions, draw a table on the board. It is not necessary to complete the table. For example, you might draw something like this:

| Year | Growth Rate per 1000 |
|---|---|
| 1920 | 18.8 |
| 1925 | 13.8 |
| 1930 | 7.3 |

**T:** ***Are there other ways to organize the data besides in a table?***

**S:** ***We could graph the data.***

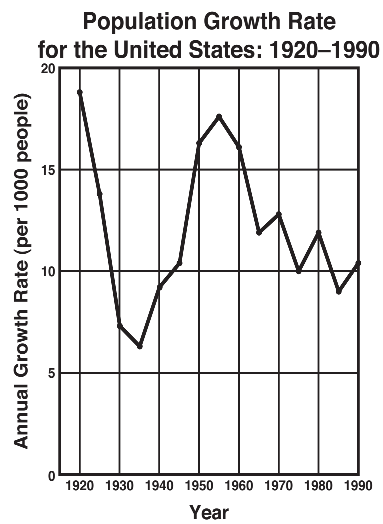
**T:** ***How would you suggest drawing a graph?***

Following students' directions, draw a rough sketch on the board of the graph they describe. They may suggest a format similar to the one on Blackline P5(b).

Once students describe how to label and mark the axes, distribute copies of the growth rate graph on Blackline P5(b) and display *IG-VI Probability Poster* #1. Instruct students to graph the growth rate data from the *News* paragraph. While students work individually or with partners, invite students to plot points in the graph on the poster. Encourage accurate placement of dots.

**T:** ***Let's connect the dots on the graph so it is easier to see when the growth rate increased or decreased.***

***Now I have a few more questions about this data. In what year between 1920 and 1990 was the growth rate the lowest?***




**S:** ***1935.***

**T:** ***When was the growth rate less than 10 per 1 000 people?***

**S:** ***In 1930, 1935, 1940, and 1985.***

**T:** ***In which periods did the growth rate increase?***

**S:** ***From 1935 to 1955, from 1965 to 1970, from 1975 to 1980, and from 1980 to 1990.***

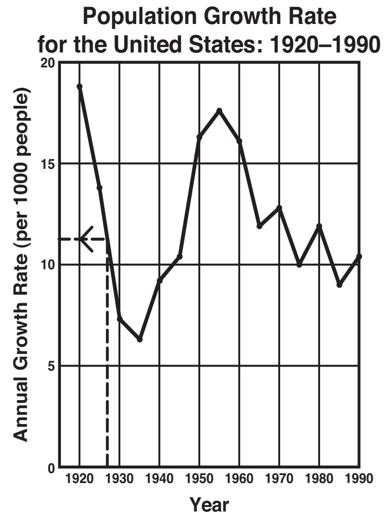
Discuss how the graph made these questions easier to answer as compared to using the original disorganized data.

Point to the dots on the graph for 1925 and 1930.

**T:** ***Since we know the growth rates in 1925 and 1930, we can draw these two dots. Then we can join the dots with a line segment. Do we know the growth rate for 1927? Could we estimate it?***

**S:** ***We could use the graph to estimate it, but our estimate may not be exact.***

Your students might use the following method (see the dotted arrows in the next illustration) to estimate the growth rate for 1927 to be about 11 per 1000.




**T:** ***This is probably a good estimate for 1927, but we cannot be sure. If 1927 had a very large or very small growth rate, it would not be shown since our graph includes data for 1925 and for 1930, but not for 1927.***

***The growth rates were quite low from 1930 to 1940. What historical events might have caused low growth rates during that period in the United States?***

**S:** ***The Great Depression in the early 1930s.***

**S:** ***World War II in the late 1930s and early 1940s.***

**T:** ***The growth rate in 1985 was also quite low. Do you think it was low for the same reasons as from 1930 to 1940?***

Students should realize that other reasons besides depression and war could have caused a low growth rate in 1985. For example, due to social reasons people may have been marrying later or choosing to have fewer children.

**T:** ***From 1920 to 1990, did the total United States' population ever decrease?***

**S:** ***No, the growth rate was always above zero, so the population was always increasing.***

Some students might observe that the growth rate decreased often (1920–1935, 1955–1965, 1970–1975, and 1980–1985) and conclude that the total population also decreased in those years. For these students, point out that the graph is of growth rate, not of total population. Even in 1935 and 1985, the growth rate was positive. Therefore, the population was always increasing, although more slowly at times. You might use this analogy: a child's growth rate can be slow or fast, but the height of a child increases even when the growth rate is very slow.

**T:** ***If the United States' population were to ever decrease, how would that be shown on the growth rate graph?***

**S:** ***The growth rate would be negative. You would have to extend the graph below 0 to allow negative numbers.***

The following question is optional and need not be considered if too little time remains in the class period.

**T:** ***When was the net gain of the United States' population greater, in 1920 or in 1985?***

**S:** ***It is hard to tell. The growth rate was greater in 1920 but the population was greater in 1985.***

Write this information in the board.

| | Population | Growth Rate per 1000 |
|---|---|---|
| 1920: | 106,000,000 | 18.8 |
| 1985: | 238,000,000 | 9.0 |

**T:** ***These are the growth rates and population for 1920 and for 1985. Let's calculate the net gain of population in each year.***

**S:** ***The net gain in 1920 was 18.8*** x ***106 000 or 1,992,800 people.***

**S:** ***The net gain in 1985 was 9.0*** x ***238 000 or 2,142,000 people.***

**S:** ***So the net gain of population was greater in 1985 than in 1920 even though the growth rate was much lower (less than half) in 1985.***

Save Poster #1 with the completed graph for use in Lesson P6.

## Capsule Lesson Summary

Compare graphs of the population growth rate and the total population of the United States from 1920 to 1990. Discuss and graph the 1990 age group distribution data for the United States.

### Materials

**Teacher**
- *IG-VI Probability Posters* #1 and #2
- Colored chalk

**Student**
- Calculator
- Colored pencils, pens, or crayons
- Worksheets P6 (no star), P6* and **

**Advance Preparation:** *IG-VI Probability Poster* #1 should have a graph of the U.S. population growth rate that was drawn in Lesson P5. You may like to use Blackline P6 to make a form for starting a population pyramid graph at the end of the lesson.

## Description of Lesson

### Exercise 1

Display *IG-VI Probability Posters* #1 and #2.

**T** (pointing to Poster #1)**:** ***Who recognizes this graph?***

**S:** ***It is the graph that we drew of the United States' population growth rate from 1920 to 1990.***

**T** (pointing to Poster #2)**:** ***This is a graph of the total United States' population from 1920 to 1990. What was the United States' population in 1950?***

**S:** ***About 150 million.***

**T:** ***In 1925?***

**S:** ***About 117 million.***

**T:** ***When did the United States' population reach 200 million?***

**S:** ***About 1968.***

**T:** ***Last time we agreed that the United States' population has always increased. How does each of these graphs show that fact?***

**S:** ***The graph of total population is always rising.***

**S:** ***The growth rate shows that the growth rate was always positive.***

**T:** ***Both of these graphs concern the United States' population from 1920 to 1990. Do you see any similarities in the shapes of these two graphs?***

**S:** ***When the growth rate was very high in the 1950s, the total population grew very quickly.***

**S:** ***The total population graph seems to level off some between 1930 and 1940 due to low growth rates from 1930 to 1940.***

## Exercise 2

Write this information on the board.

United States: K-8
1970: 36,629,000 students
1980: 30,625,000 students
1990: 33,164,000 students

**T:** ***In 1970, there were about 36,629,000 students in kindergarten through the eighth grade in the United States. In 1980, there were only 30,625,000 students in the same grades, even though the total population increased from 1970 to 1980. Can you explain this?***

Invite students to express their opinions.

**T:** ***What do you suppose were some effects of this large decrease in the number of elementary school students? Who might be very concerned?***

Students may suggest effects such as school closings and teacher lay-offs.

Notice that the population of K–8 students then increased substantially by 1990. With this observation students may discuss the need for more schools again and a demand for teachers.

## Exercise 3

Refer to the table on Worksheet P6 (no star).

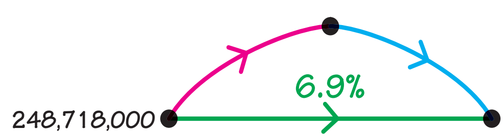
**T:** ***This table shows the percent of Americans in various groups in 1990. What percent of Americans were in the 10–14 age group?***

**S:** ***6.9%.***

**T:** ***How can we calculate the number of Americans in that age group?***

**S:** ***Calculate 6.9% of the total U.S. population, 248,718,000.***

Draw a composition arrow picture for the calculation on the board.

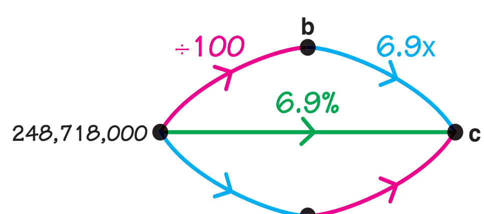



**T:** ***How could we calculate 6.9% of 248,718,000?***

**S:** ***Divide 248,718,000 by 100 and multiply the result by 6.9.***

**S:** ***Multiply 248,718,000 by 6.9 and divide the result by 100.***

Encourage students to mention both orders for doing the operation. Label the arrows and draw the reverse order arrows.




Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

**T:** ***Can you do these calculations on your calculator?***

**S:** ***No. 248,718,000 does not fit on the calculator display; the calculator displays only eight digits.***

**T** (pointing to **b**): ***Can you calculate 248,718,000 ÷ 100 without a calculator?***

**S:** ***Yes. 248,718,000 ÷ 100 = 2,487,180. When you divide by 100 you can drop the last two zeros.***

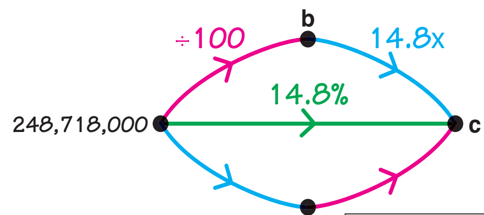
Put 2,487,180 at **b**.

**T** (pointing to **c**): ***Use your calculator to do the calculation 6.9 x 2,487,180.***

**T:** ***What percent of Americans in 1990 were less than ten years old?***

**S:** ***14.8%, because 7.5% were in the 0–4 age group and 7.3% were in the 5–9 age group and 7.5 + 7.3 = 14.8.***

Relabel the arrow picture.




Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

**T:** ***How many Americans in 1990 were less than ten years old?***

**S:** ***About 36,810,264 since 14.8 x 2,487,180 = 36,810,264.***

Label **b** and **c** in the arrow picture to explain this method of calculation.

**T:** ***What do you expect is the sum of all of the numbers in the percent column of the table.***

Some students may be able to explain why the total should be 100.

**T:** ***Use your calculator to add those numbers. What sum do you get?***

**S:** ***100.***

**T:** ***Why should the sum be 100?***

**S:** ***100% of the total population means all Americans. Every American must be in one of the age groups.***

**T:** ***We calculated that about 36,810,264 Americans were less than ten years old. Of these 51% were boys. Is that more or less than one-half of the Americans under ten years old?***

**S:** ***More, since 50% is ½.***

**T:** ***Why do you suppose that there were more boys than girls under ten years old?***

After a brief discussion, explain that slightly more than one-half of all births are boys.

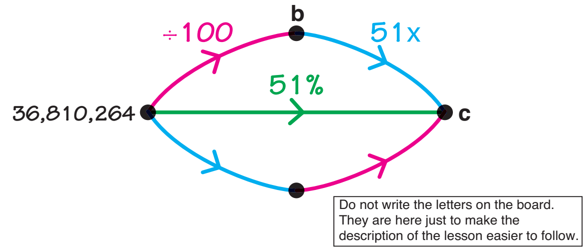
**T:** ***If 51% of Americans under ten years old were boys, what percent were girls?***

**S:** ***49%, since 51 + 49 = 100.***

**T:** ***We know that in 1990 there were about 36,810,264 Americans under ten years old. If 51% of them were boys, how do I calculate the number of boys under ten years old?***

**S:** ***Calculate 51% of 36,810,264; divide 36,810,264 by 100 and then multiply that number by 51.***

Ask students to relabel the arrow picture to indicate these calculations.




**T** (pointing to **b**)**:** ***What number is 36,810,264 ÷ 100? Do not use your calculator.***

**S:** ***368,102.64***

**T** (pointing to **c** and tracing the green arrow)**:** ***So what number is 51% of 36,810,264?***

**S:** ***51*** x ***368,102.64 or about 18,773,235.***

**T:** ***How can we calculate the number of girls less than ten years old?***

**S:** ***Subtract the number of boys under ten from the total number of children under ten; 36,810,264 – 18,773,235 = 18,037,029.***

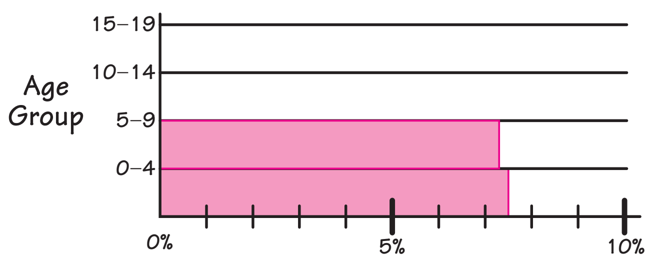
**S:** ***Calculate 49% of 36,810,264; 49*** x ***368,102.64 ≈ 18,037,029.***

Using either method determines that there were about 18,037,029 girls less than ten years old in 1990.

Refer to Worksheet P6* as you explain,

**T:** ***Now let's graph the data in the table. You will find a graph already started on Worksheet P6*.***

Explain what you are doing as you draw this part of a graph on the board. You may like to do this on a transparency made from Blackline P6.

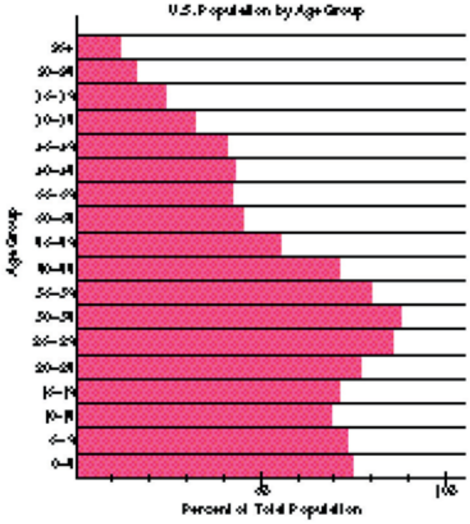



**T:** ***Here the bars indicate that 7.5% of the population was in the 0–4 age group and 7.3% was in the 5–9 age group. How should we continue this graph?***

Invite students to continue the graph on their copies of the worksheet. You may first want to collectively add bars for the data for 10–14 and 15–19 age groups. As students complete the graph on Worksheet P6*, direct them to answer the questions on Worksheet P6**. Discuss some of the questions on P6** as a group.

Name________________ | P6 ★

Complete the graph.




Name________________ | P6 ★★

1. What percent of Americans in 1990 were 15–19 years old? 7.1%

   How many Americans in 1990 were 15–19 years old? 17,658,978

2. What percent of Americans in 1990 were 65 years or older? 12.5%

   How many Americans in 1990 were 65 years old or older? 31,089,750

   What percent of Americans were less than 65 years old in 1990? 87.5%

3. Many Americans still remember World War II. What percent of the Americans in 1990 were born before 1940 (near the beginning of World War II)? 25.5%
   (Hint: Think about their ages.)

4. What percent of Americans were 5–19 years old in 1990? 21.3%

   How many Americans were 5–19 years old in 1990? 52,976,934

   In 1990, 88% of Americans 5–19 years old were in elementary or secondary schools. How many students is that? 46,619,702

## Capsule Lesson Summary

Discuss historical and social questions related to a graph of the United States' population by age group. Find the median age of the 1990 U.S. population. Compare age group population graphs of the United States, Sweden, and Mexico.

### Materials

| Teacher | | Student | |
|---|---|---|---|
| | • *IG-VI Probability Poster #3*<br>• Markers<br>• Colored chalk | | • Calculator<br>• Worksheets P7 (a), (b), and (c), P7* and ** |

## Description of Lesson

### Exercise 1

Display *IG-VI Probability Poster #3*.

**T:** ***From the graph, we see that in 1990 there were fewer Americans in the 55–59 age group than in the 60–64 or the 50–54 age groups. What could have caused this?***

**S:** ***The Depression.***

**S:** ***World War II.***

**T:** ***Let's determine which is more reasonable. When did the Depression occur?***

**S:** ***It started in 1929 and continued through the 1930s.***

**T:** ***When was the United States involved in World War II?***

**S:** ***From 1940 to 1945.***

If your students do not know these facts, provide them yourself.

**T:** ***What are the possible years of birth of people who were in the 55–59 age group in 1990?***

**S:** ***1931 to 1935; 1990 – 59 = 1931 and 1990 – 55 = 1935.***

**S:** ***The Depression must be the reason for fewer people in that age group.***

**T:** ***Another unusual feature of the graph is that the percent of Americans in each of the age groups 5–9, 10–14 and 15–19 is less than that in the age groups 25–29 and 30–34. What could explain this?***

**S:** ***People might be having children later or deciding to have fewer children.***

**T:** ***In the last lesson on statistics, we saw that there were about six million fewer students in elementary school in 1980 than in 1970. Does the graph help explain how this was possible, even though the total population was increasing?***

**S:** ***The number of children in the school age groups was decreasing between 1970 and 1985.***

## Exercise 2

**Note:** Conduct this activity using either mass in kilograms or weight in pounds. If you prefer you can replace weight with height.

Write **Median Weight** on the board.

**T:** *I want to find the* **median weight** *of students in this class. The median weight is a weight such that the number of students heavier than that weight is equal to the number of students lighter than that weight.*[†] *What might be a good estimate of the median weight of this class?*

Accept any reasonable estimate, for example, 90 pounds.

**T:** *How many weigh less than 90 lbs.? More?*

If this estimate is not the median weight, invite a student to suggest another weight for the median and ask the same questions. Continue until the median weight, for example, 82 lbs., is found.

**T:** *The number of you who weigh less than 82 lbs. is the same as the number who weight more than 82 lbs. The median weight of this class is 82 lbs.*

*What do you think is meant by the* **median age** *of the United States' population?*

**S:** *The age that one-half of the people are younger than.*

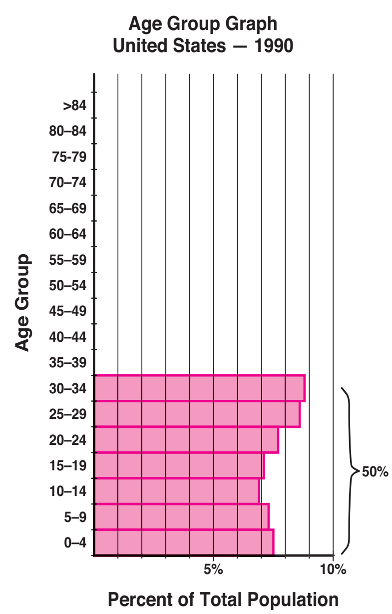
**T:** *What do you think the median age is in the United States?*

Let students make estimates of the median age.

**T:** *How can we use the age group graph to determine the median age of Americans in 1990?*

**S:** *One-half of the population is the same as 50% of the population. Starting at the 0–4 age group, add the percent of people in each age group until we reach 50%.*

From the graph, estimate the percent in each age group and keep a running total. Your estimates may differ slightly from the estimates in this picture. You may refer to percents given on Worksheet P6 to get more accurate percents in each age group.




**T:** *What was the median age in 1990?*

**S:** *32 years old.*

Compare this answer to students' estimates.

**T:** *Do you think the median ages in other countries are higher, lower, or about the same?*

Let students express their opinions.

[†]The following is a more precise definition of *median*: If there are an odd number of measurements, the median equals the middle measurement when all the measurements are ordered. If there are an even number of measurements, the median is the mean average of the two middle measurements when all the measurements are ordered.

## Exercise 3

Refer to Worksheets P7(a), (b), and (c).

**T:** ***These are the 1990 age group graphs for the United States, Mexico, and Sweden.***

Lead a discussion of the similarities and differences among the three graphs. Encourage comments about what each graph reveals about that country. Many points may be brought up, for example:

- Mexico has the largest percentage of young people; Sweden has the smallest.
- Sweden has the largest percentage of old people; Mexico has the smallest.
- Sweden and the United States both have a comparatively small percentage of their populations in the 55–59 age group.
- Mexico has problems related to a rapidly increasing population.
- Sweden has the most uniform size age groups.

**T:** ***Which country do you think has the fewest children in the 0–4 age group?***

**S:** ***Sweden, because less than 7% of their people are in that group.***

**S:** ***Sweden, because they have the smallest population.***

**T:** ***Which country do you think has the most children in the 0–4 age group?***

**S:** ***Maybe Mexico does since almost 13% of their people are in that age group.***

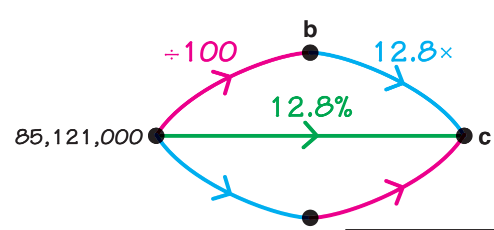
**S:** ***Maybe the United States does since the United States' population is much larger than Mexico's.***

On the board, draw a composition arrow picture, as illustrated below.

**T:** ***Let's calculate the number of children in the 0–4 age group in Mexico, in the United States, and in Sweden. Suppose we do Mexico first. How should we label the arrow picture?***

**S:** ***According to the graph, about 12.8% of all Mexicans are in the 0–4 age group. Label the green arrow 12.8%. Then one arrow could be for 12.8x and the other arrow could be for ÷ 100.***

**S:** ***Put the population of Mexico at the dot on the left.***




Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

**T** (pointing to **b**)**:** ***What number is 85,121,000 ÷ 100? Do not use your calculator.***

**S:** ***851,210.***

**T:** ***Now, how can we calculate the number of Mexicans in the 0–4 age group?***

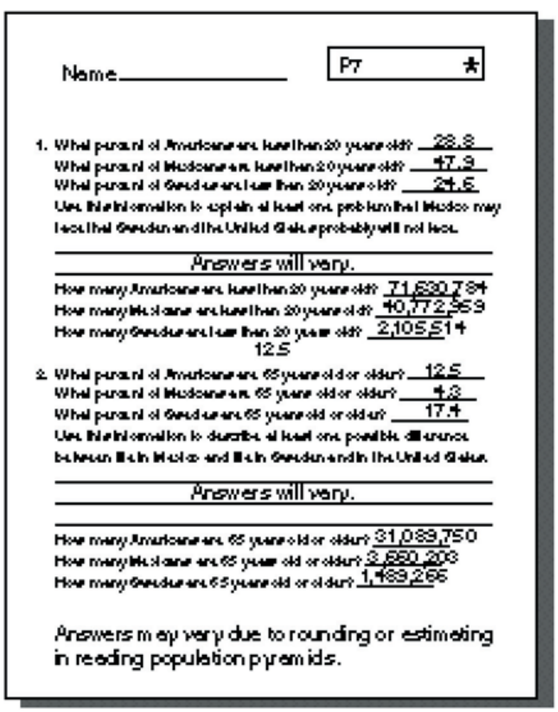
**S:** ***Multiply 12.8 x 851,210. That is 10,895,488.***

Write the result on the board. In a similar manner, use the arrow picture to calculate that there are about 18,653,850 (7.5 x 2,487,180) Americans and about 573,453 (6.7 x 85 590) Swedes in the 0–4 age group. Note that there are more Americans than Mexicans in the 0–4 age group even though the percentage of Mexicans is greater.

# P7

Instruct students to complete Worksheets P7* and **. Caution students to use the correct national population and/or correct graph when answering each question. Allow time at the end of the period to check some of the answers and to discuss different problems the United States, Sweden, and Mexico may have.   For example:

- The United States was closing schools in 1980, but now must be re-opening them and building more. Mexico must need lots of schools.
- The United States and especially Sweden have a large percentage of people over 65 years old who may require financial assistance.
- Mexico's population is rapidly increasing which could cause shortages of food, schools, and health facilities.
- Some large cities have problems due to declining populations, while Mexico City has problems due to a rapidly increasing population.


Name__________

P7 *

1. 28.8

47.9

24.6

Answers will vary.

71,630,784

40,772,969

2,105,514

12.5

2. 12.5

4.3

17.4

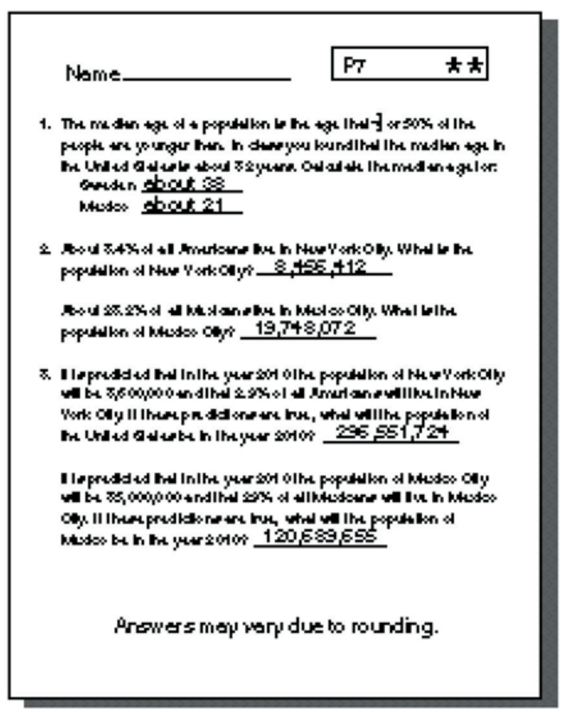
Answers will vary.

31,089,750

3,680,203

1,489,266

Answers may vary due to rounding or estimating in reading population pyramids.


Name__________

P7 **

1. about 38

about 21

2. 8,455,412

19,748,072

3. 296,551,724

120,689,655

Answers may vary due to rounding.