The Languages of Strings and Arrows

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THE LANGUAGES OF STRINGS & ARROWS INTRODUCTION

The language of strings is the nonverbal language of sets, that is, collections of objects put together in some way. The language of arrows is the nonverbal language of relations among objects. These languages permeate the *CSMP* strands and are used, separately or together, to present an unending variety of numerical and nonnumerical situations.

Classification: The Language of Strings

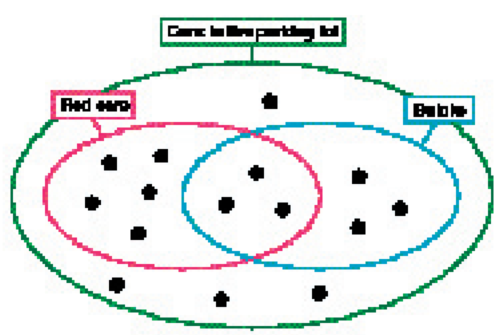
As the word implies, classifying means putting things into classes, or as the mathematician says, sets. The mathematics of sets can help students to understand and use the ideas of classification. The basic idea is simple: Given a set S and any object x, either x belongs to S (x is in S) or x does not belong to S (x is not in S). We represent this simple act of sorting—in or out—by using pictures to illustrate in and out in a dramatic way. Objects to be sorted are represented by dots, and the sets into which they are sorted are represented by drawing strings around dots. A dot inside the region delineated by a set's string is for an object in the set, and a dot outside a set's string is for an object not in the set.

This language of strings and dots provides a precise (and nonverbal) way of recording and communicating thoughts about classification. The ability to classify, to reason about classification, and to extract information from a classification are important skills for everyday life, for intellectual activity in general, and for the pursuit and understanding of mathematics in particular. The unique quality of the language of strings is that it provides a nonverbal language that is particularly suited to the mode of thinking involved in classification. It frees young minds to think logically and creatively about classes and to report their thinking long before they have extensive verbal skills.

In this strand we present situations and ask carefully phrased questions to continue to advance skills in classification, always remembering that the skills grow out of such experiences. To be able to draw strings and dots is not an objective in itself; to develop the mode of thinking involved is the objective. Thus it is important for us to construct the situations carefully. The sets into which we ask students to classify objects must be determined by well-defined attributes; otherwise, there is the added problem of deciding whether a certain object does or does not have a certain attribute. For this reason we make extensive use of A-blocks (attribute blocks) and sets of numbers in designing classification situations. Students can immediately say whether or not a block is red, whether or not it is a square, and so on. They know whether or not a given number is less than 15, whether or not it is odd, and so on.

One reason for classifying objects is to count the objects that have a certain attribute. Suppose, for example, there are eight red cars and six Buicks in a parking lot. If four cars in the lot are neither red nor Buicks, must there be 18 cars in the lot? Suppose we count the cars in the parking lot and find there are exactly 15. Is this possible? A string picture immediately settles this apparent paradox.

There are three red Buicks.



Relations: The Language of Arrows

Relations are interesting and important to us in our everyday lives, in our careers, in school, and in scientific pursuits. We are always trying to establish, explore, and understand relations. In mathematics it is the same; to study mathematics means to study the relations among mathematical objects like numbers or geometric shapes. The tools we use to understand the everyday world are useful to understand the world of mathematics. Conversely, the tools we develop to help us think about mathematical things often serve us in nonmathematical situations.

A serious study of anything requires a language for representing the things under investigation. The language of arrows provides an apt language for studying and talking about relations. Arrow diagrams are a handy graphic representation of a relation, somewhat the same way that a blueprint is a handy graphic representation of a house. By means of arrow diagrams, we can represent important facts about a given relation in a simple, suggestive, pictorial way—usually more conveniently than the same information could be presented in words. The convenience of arrow diagrams has important pedagogical consequences for introducing children to the study of relations in the early grades. A child can read—and also draw—an arrow diagram of a relation long before he or she can read or present the same information in words. The difficulty of presenting certain ideas to children lies not in their intellectual inability to grasp the ideas; rather, the limitations are often mechanical. Arrow diagrams have all the virtues of a good notation: they present information in a clear, natural way; they are attractive, colorful things to look at; they are easy and fun for children to draw. Another educational bonus occurs when arrow diagrams are used imaginatively. A story or problem may be captured by an arrow diagram, and at the same time, the resulting arrow diagram may suggest other, similar situations. This allows students to call upon previous experience or to expand their creativity.

One of the purposes of this strand is to use the power of the language of arrows to help children think logically about relations. Again it must be remembered that our goals concern the thinking process and not the mechanism. The ability to draw prescribed arrows is not the objective in itself, nor is viewing an arrow diagram just another format for drill problems in arithmetic.

The general aim of The Languages of Strings and Arrows strand is to suggest situations that are inherently interesting and thought provoking, and to give children modes of thinking and appropriate languages with which they can organize, classify, and analyze. In addition to a varied assortment of lessons concerning sets and relation, this strand includes lessons involving systematic methods for solving combinatorial (counting) problems; methods for collecting, recording and interpreting data in real life situations; probabilistic thinking; and networks.

Capsule Lesson Summary

Recall the changing seats game and examine an arrow picture for nine people playing the game. Decide how many rounds of the game need to be played before all the players are back in their original chairs. Find that the result is a common multiple of certain numbers which are given by the lengths of cycles in the arrow picture.

Materials

Teacher
- Colored chalk
- Colored pencils, pens, or crayons

Student
- Paper

Description of Lesson

Exercise 1

On the board, draw nine large dots arranged so you can add arrows as shown below.

T: ***Do you remember the game of changing seats? You had to change seats according to an arrow picture. We are not going to play the game today; we will just talk about it. These dots are for chairs in which some children are sitting. How many dots are there?***

S: ***Nine.***

T: ***I'll draw some arrows to show you how the children in these nine chairs change seats.***

Draw red arrows in your picture.

T (pointing to one of the dots)**:** ***Where will the person who starts here go in Round 1?***

Invite a student to point to the appropriate dot. Repeat this question several times, referring to other dots. Then vary the question slightly as follows.

T (pointing to one of the dots)**:** ***Imagine the children have played three rounds of the game. Where will the person from this chair be?***

Let a student point to the appropriate dot. For the benefit of the class, check the student's response by tracing three arrows in succession starting at the original dot. Repeat this question several times, referring to other dots.

T: ***After how many rounds will all the children return to their own chairs?***

Record some students' guesses on the board. For example:

Guesses

0 rounds	6 rounds	10 rounds	30 rounds
15 rounds	8 rounds	20 rounds	1005 rounds

Note: If no one mentions zero rounds, suggest it yourself.

T: ***Let's see if we can answer this question completely.***

Draw a chart on the board and label one of the dots in the three-cycle **b**.

T (pointing to **b**)**:** ***The person who starts here will return after how many rounds?***

S: ***Three.***

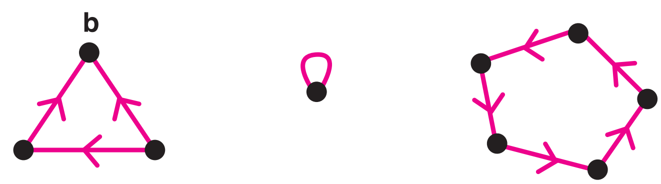
Trace the cycle of three arrows starting at **b** and returning to **b**, and then record 3 in the chart.

T: ***Three rounds is correct. Are there any other times?***

S: ***After six rounds.***

Trace the cycle of three arrows twice, starting at **b** and returning to **b**. Record 6 in the chart. Accept many solutions. (Be sure to include zero rounds.)

a, b, c	
0	
3	
6	
9	
12	
15	
18	
.	
.	
.	



T: ***What do we call these numbers?***

S: ***Multiples of 3.***

In the same manner, consider the other two dots in the cycle with **b**, labeling them **a** and **c**. When it is clear that the situation is the same for **a**, **b**, and **c**, add **a** and **c** to your chart heading.

Label the dot with a loop **d**, and extend your chart with another column.

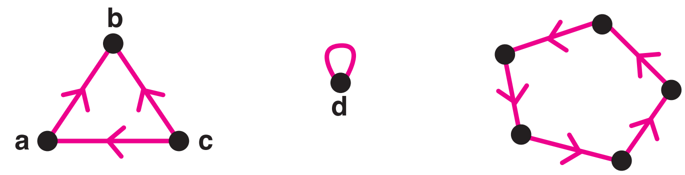
T (pointing to **d**)**:** ***The person who starts here will return after how many rounds?***

S: ***After zero, one, two, three, four, . . . rounds.***

S: ***Any number of rounds.***

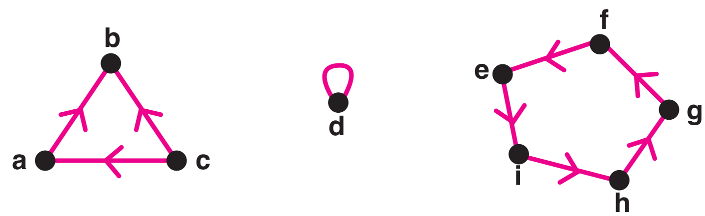
Indicate this in your chart.

a, b, c	d
0	0
3	1
6	2
9	3
12	4
15	5
18	6
.	.
.	.
.	.



Repeat this activity, considering each dot in the five-cycle. Conclude that the children starting in those chairs will return after any round numbered with a multiple of 5.

a, b, c	d	e, f, g, h, i
0	0	0
3	1	5
6	2	10
9	3	15
12	4	20
15	5	25
18	6	30
.	.	.
.	.	.
.	.	.



Consider each guess recorded earlier in the lesson.

Guesses

0 rounds	6 rounds	10 rounds	30 rounds
15 rounds	8 rounds	20 rounds	1005 rounds

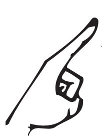
T: ***After how many rounds will all the children have returned to their own chairs? …after zero rounds?***

S: ***Yes.***

T: ***…after 15 rounds?***

Check 15 in this manner.

a, b, c	d	e, f, g, h, i
0	0	0
3	1	5
6	2	10
9	3	15
12	4	20
15	5	25
18	6	30
.	.	.
.	.	.
.	.	.



T: ***Is 15 a multiple of 3?***

a, b, c	d	e, f, g, h, i
0	0	0
3	1	5
6	2	10
9	3	15
12	4	20
15	5	25
18	6	30
.	.	.
.	.	.
.	.	.

T: **d** ***is always in the same chair.***

a, b, c	d	e, f, g, h, i
0	0	0
3	1	5
6	2	10
9	3	15
12		20
15		25
18	6	30
.	.	.
.	.	.
.	.	.

T: ***Is 15 a multiple of 5?***

S: ***Yes.***

T: ***… after six rounds?***

S: ***No, 6 is not a multiple of 5.***

Erase 6 from the list of guesses.

Continue in this manner until each guess has been considered. If a number is not both a multiple of 3 and a multiple of 5, erase it from the list.

T (pointing to the correct guesses)**:** ***Does anyone know another good answer now?***

Check students' answers and record each correct number of rounds in the list. Arrange, and if necessary reorganize, your list so that the numbers can be listed in increasing value. Once a pattern is established, ask students to continue the sequence for several numbers. You should end up with a list similar to this one.

0 rounds
15 rounds
30 rounds
45 rounds
60 rounds
75 rounds
90 rounds
105 rounds
120 rounds

1,005 rounds
⋮
⋮

T: ***What do we call these numbers?***

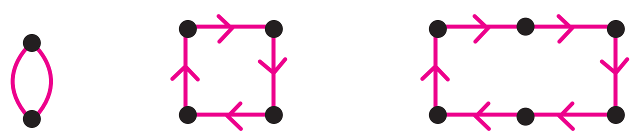
S: ***Multiples of 15.***

Writing Activity

Instruct students to draw their own pictures of a changing seats game. Suggest they include about ten chairs (dots) in their pictures. Suggest they exchange pictures with a partner and study their partner's picture to decide how many rounds it will take before all the children will be back in their own seats.

Extension Activity

Pose other changing seats games and ask students to decide how many rounds it will take for everyone to return to his or her own seat. For example, for the following game, the list would be 0, 12, 24, 36, 48, … (multiples of 12).



Home Activity

Give students a picture for a changing seats game to take home and explain to a family member. The student should decide with a family member which rounds would put everyone back in his or her own seat.

L2 A DETECTIVE STORY

Capsule Lesson Summary

Solve a detective story about a secret number where the clues involve (1) finding all the numbers that can be named using the symbols 3, 5, 4, +, x, (,), each exactly once; (2) locating the secret number in a three-string picture where the strings are for multiples of 3, 4, and 5; and (3) observing that the secret number is on the same +5 arrow road as $\widehat{3}$.

Materials

Teacher	• Colored chalk	**Student**	• Paper • Colored pencils, pens, or crayons • Worksheets L2* and **

Description of Lesson

You may like to allow students to work with a partner or in groups during this lesson. Write 17 on a slip of paper and then fold the paper so that 17 is hidden.

T: ***The name of a secret number is written on this paper. I'll give you some clues and if you are good detectives, you should be able to discover the secret number.***

Clue 1

Write these symbols on the board.

$$3 \qquad 5 \qquad 4$$
$$\times \qquad \times \qquad (\qquad)$$

T: ***The secret number can be written using each of these symbols exactly once.***

Ask students to find possibilities and to write them on a piece of paper.

Observe the students' work and as soon as you see a correct expression using only the above symbols, draw it to the class's attention.

T: ***Nicole and Howard found a number that the secret number could be. They used the symbols in this way*** (write them on the board)**.**

Who knows what number this is?

$$(3 \times 4) \times 5$$

S: ***35, because 3 + 4 = 7 and 7 x 5 = 35.***

Let the students continue to work, periodically drawing the class's attention to a solution found by students and asking the rest of the class what number the expression names. Continue until all six possibilities have been found and are listed on the board.

$$(3 \times 4) \times 5 = 35 \qquad (3 \times 4) \times 5 = 17 \qquad (3 \times 5) \times 4 = 32$$
$$3 \times (4 \times 5) = 23 \qquad 3 \times (4 \times 5) = 27 \qquad (3 \times 5) \times 4 = 19$$

Students may suggest certain other expressions that are for numbers already given. For example, if a student suggests (4 + 3) x 5 or 5 x (3 + 4) when (3 + 4) x 5 is already on the board, use the opportunity to comment on commutativity (you need not use this word). Students should observe 4 + 3 = 3 + 4 and conclude the following:

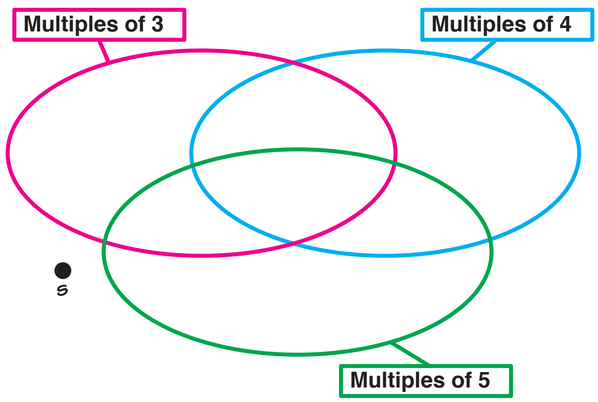
$$(4 \times 3) \times 5 = (3 \times 4) \times 5 = 35$$

T: ***The secret number could be 35, 23, 17, 27, 32, or 19.***

Erase all but a list of these six numbers.

Clue 2

Draw this string picture on the board and ask students to copy it on their papers.



T: ***The second clue is that the secret number is outside all three of these strings. What are some numbers that the secret number could be?***

Instruct student pairs to spend a few minutes trying to locate each of the six numbers in the string picture. Then hold a class discussion of which numbers could be at **s**.

S: ***19.***

T: ***How do you know? Is 19 a multiple of 3*** (trace the red string)***?***

S: ***No, 15 is a multiple of 3, 18 is the next multiple of 3, and 21 is the next multiple of 3. 19 is skipped.***

T (tracing the blue string)**:** ***Is 19 a multiple of 4?***

S: ***No, 20 is a multiple of 4 so 19 cannot be. It is only 1 less than 20.***

T (tracing the green string)**:** ***Is 19 a multiple of 5?***

S: ***No, 19 does not end in 0 or in 5.***

Circle 19 in the list on the board.

If a student suggests a number that cannot be the secret number, very likely many students will disagree immediately. Suppose, for example, someone suggests 27.

T: ***Where does 27 belong in this picture?***

Ask another student to locate 27 in the picture.

T (tracing the red string)**:** ***How do you know that 27 is a multiple of 3?***

S: ***30 is a multiple of 3 and I counted three backwards from 30.***

T (tracing the blue string)**:** ***How do you know that 27 is not a multiple of 4?***

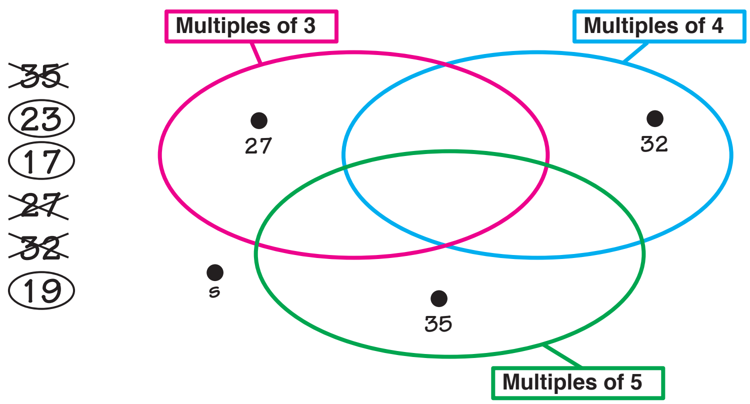
S: ***I counted by fours from 20: 20, 24, 28. 27 is skipped.***

T (tracing the green string)**:** ***How do you know that 27 is not a multiple of 5?***

S: ***27 does not end in 0 or 5.***

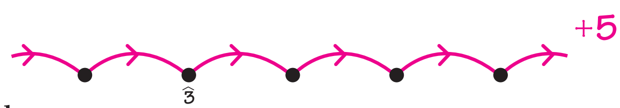
Eliminate 27 from the list of possibilities.

Continue until the class concludes that the secret number could be 19, 17, or 23. Consider any other numbers in the list that have not been circled or crossed out, and ask students to locate them in the string picture.



Clue 3

Draw this arrow picture on the board.

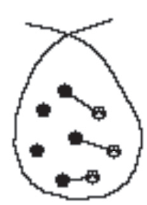


T: ***The secret number is on a +5 arrow road that meets the number $\hat{3}$. I only drew part of the road because it goes on and on in both directions. Which is the secret number? Is it 23, 17, or 19? Write it on your paper.***

Allow a few minutes for student pairs to find the secret number. Some may need to draw the arrow picture. Acknowledge correct answers.

T: ***Who can convince us that one of these numbers (23, 17, 19) is on this arrow road?***

Perhaps a student will ask to label the dots to the right of $\hat{3}$ to show where 17 occurs. Use Eli's bag of peanuts, if necessary, to show that $\hat{3} + 5 = 2$.

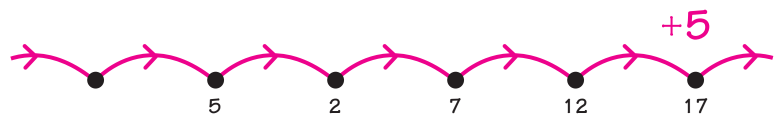


$\hat{3} + 5 = 2$

Continue until the three dots to the right of $\hat{3}$ have been labeled.

S: ***The next number (to the right) is 17.***

Finish drawing the arrow and label its ending dot 17.



T: ***How can we be sure that neither 23 nor 19 can be the secret number?***

S: ***The next number on this arrow road is 22, because 17 + 5 = 22. We skip 19.***

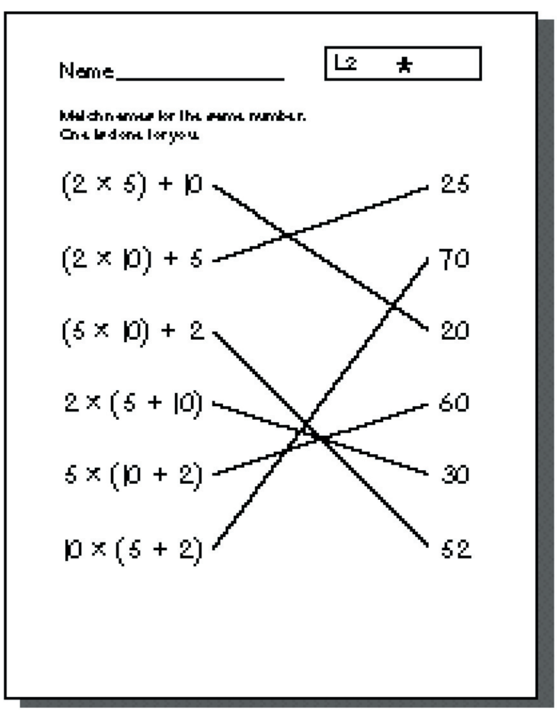
S: ***The next number after 22 is 27, because 22 + 5 = 27. We skip 23.***

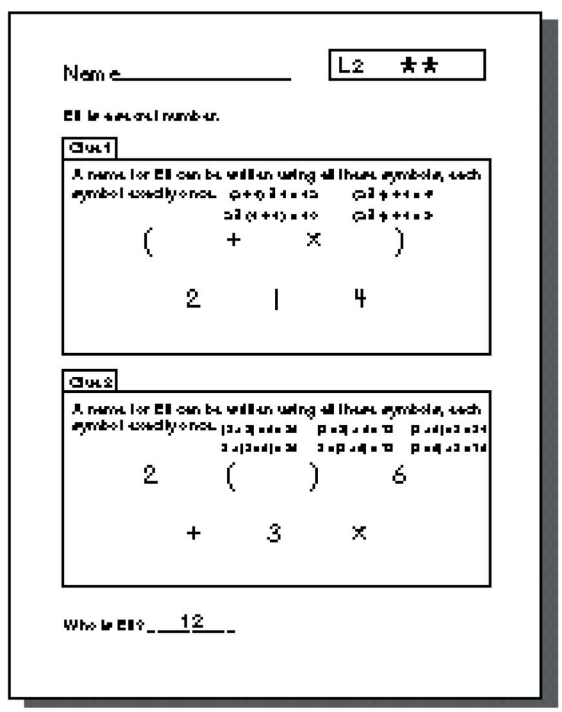
Reveal that 17 is written on the slip of paper.

Worksheets L2* and ** are available for individual work.

Writing Activity

Suggest that students try to write their own detective stories with several clues.





Capsule Lesson Summary

Play The String Game twice, once with two strings and once with three strings.

Materials

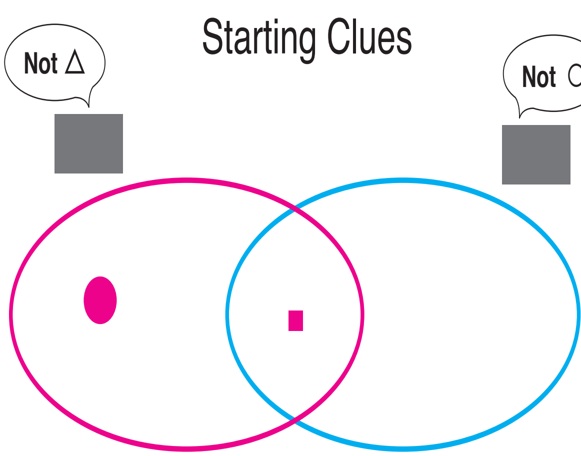
Teacher	• A-Block String Game kit • Colored chalk	**Student**	• None

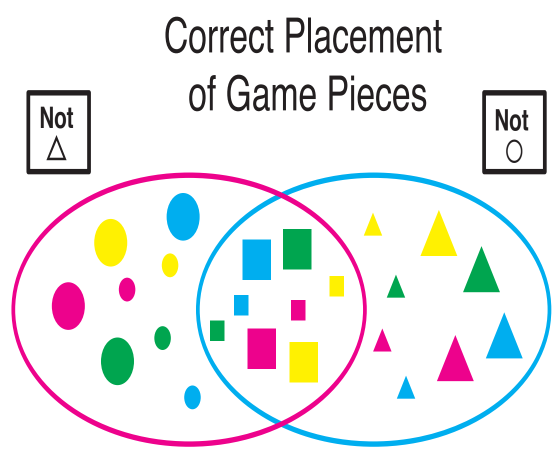
Description of Lesson

Prepare to play the String Game by setting up the team board and taping a list of the 16 possible string labels above it. Divide the class into two or more teams and distribute the game pieces on the team board.

Exercise 1

Play a two-string version of The String Game.† A game is suggested below. The illustration on the left shows two possible starting clues. The illustration on the right shows correct placement of all 24 A-Block pieces and can be used as a crib sheet during the play of the game.

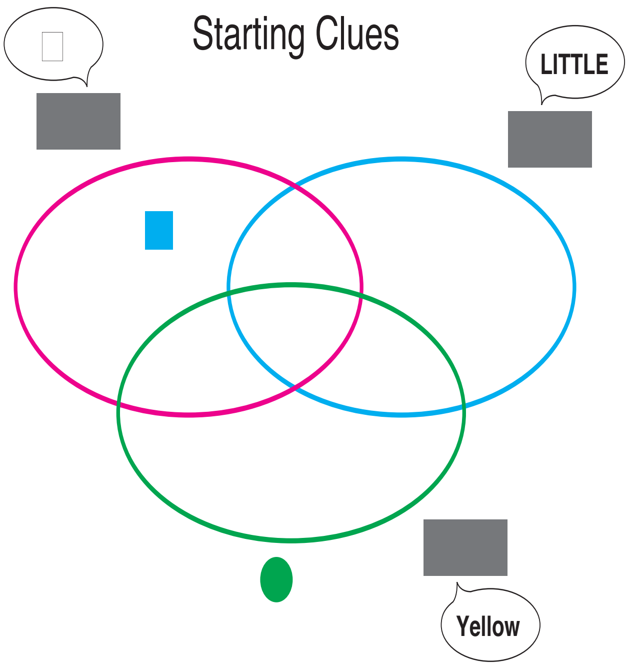


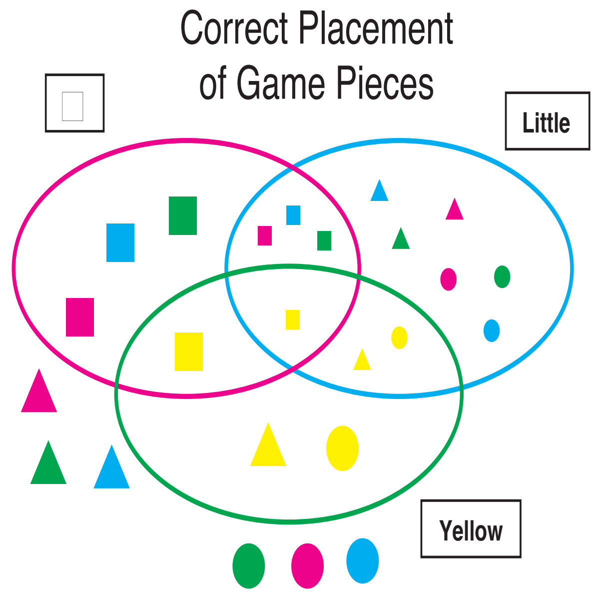


†See the appendix on The String Game for a description of the various versions of the game. This appendix also has an explanation of how the game is played.

Exercise 2

Play a three-string version of The String Game. A game is suggested below. The illustration on the left shows two possible starting clues. The illustration on the right shows correct placement of all the A-Block pieces and can be used as a crib sheet during the play of the game.





Capsule Lesson Summary

Play a composition game with the rule "blue followed by blue is red." Solve a numerical problem involving composition.

Materials

Teacher
- Colored chalk

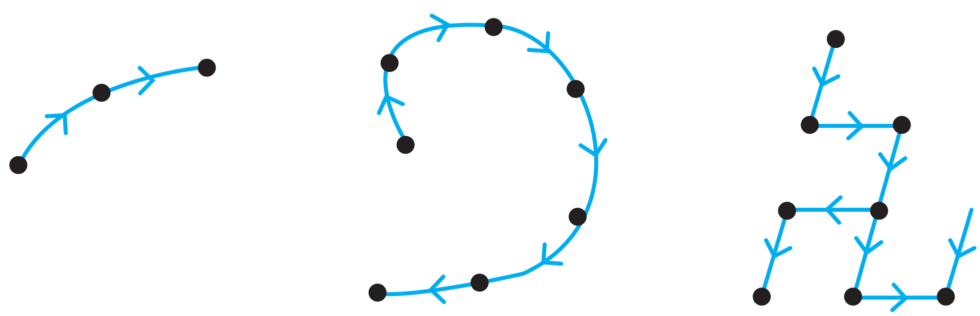
Student
- Worksheets L4* and **
- Colored pencils, pens, or crayons

Advance Preparation: Before the lesson begins, draw the arrow picture from Exercise 1 on the board or prepare it on a transparency. Give the picture plenty of space, and draw arrows that can be easily distinguished.

Description of Lesson

Exercise 1

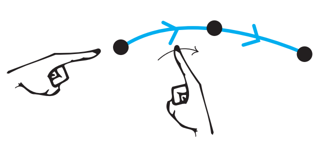
Put this arrow picture on the board.



T: ***We are going to play a game with these blue arrows. The object of the game is to draw red arrows, but there is a rule for drawing red arrows.***

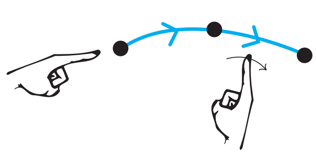
Use the upper left corner of the arrow picture for your explanation. As you explain the rule stated in the left column below, make the motions described in the right column.

T: ***Each time there is a blue arrow …***



(Point and hold your left forefinger on a dot at which the first blue arrow starts. Follow the arrow with your right forefinger in the direction of the arrowhead.)

T: ***… followed by a blue arrow…***



(Stop the motion of your right forefinger at the middle dot; tap the dot; and then follow the next blue arrow. Hold your right forefinger at the ending dot of this blue arrow.)

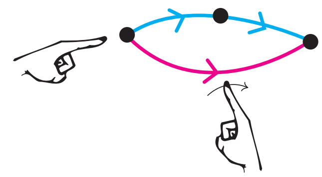
T: ***… then we can draw a red arrow from***

(Tap this dot several times with your left

the dot where the first blue arrow starts… forefinger.)

… to the dot where the second blue arrow ends. (Tap this dot several times with your right forefinger.)

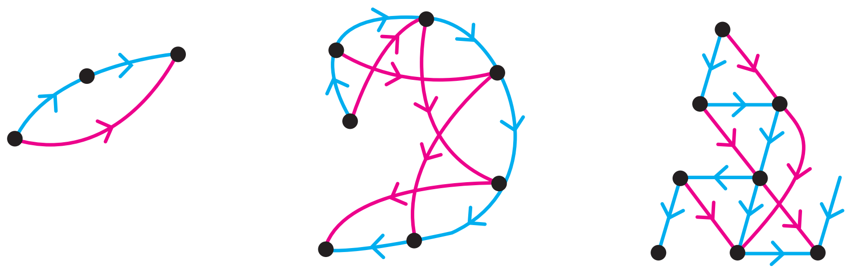
First trace and then draw the red arrow. Emphasize that your left forefinger marks the start of the red arrow and that your right forefinger is at the end.



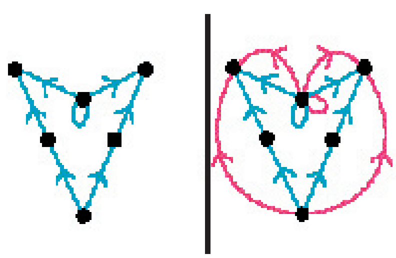
T: ***Where can we draw other red arrows?***

Invite students, one at a time, to show where other red arrows can be drawn. Ask each student who volunteers to first trace a blue arrow followed by a blue arrow, and then to trace how to draw a red arrow. Stop a student who starts to trace against the direction of an arrow and emphasize that the direction of an arrow must be followed. Encourage the class to help you check for mistakes. Let a student draw a red arrow if it has been traced correctly.

When all the red arrows have been found, your picture should look like this one.



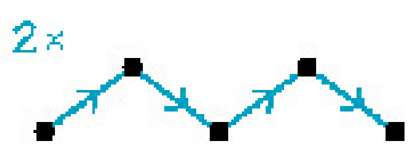
Add this section of an arrow picture to the one on the board and invite students to find places to draw red arrows in it. In this section, the blue loop will contribute to two red arrows and also allow a red loop. A red loop results when you follow the blue loop twice at that dot.



Worksheets L4* and ** are available for students to practice drawing red arrows for the blue followed by blue composition. You may like to allow about five minutes now for individual work. Then continue with Exercise 2.

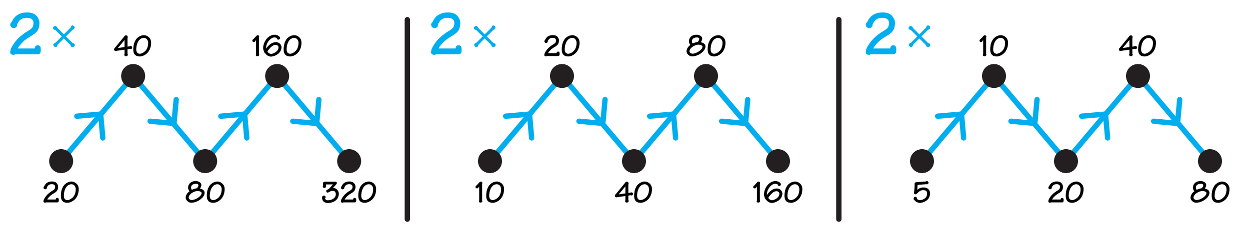
Exercise 2

Erase the board and then draw this arrow picture. Ask students to copy it on their papers.

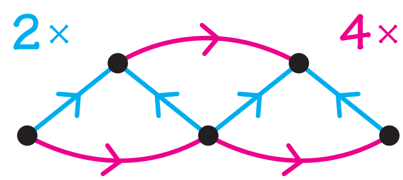


T: ***Right now none of the dots in this picture are labeled. I would like to find places for the numbers 20 and 80. Try to put 20 and 80 in the picture, but remember blue arrows are for 2x. After you locate dots for 20 and 80, label all the dots and check the arrows.***

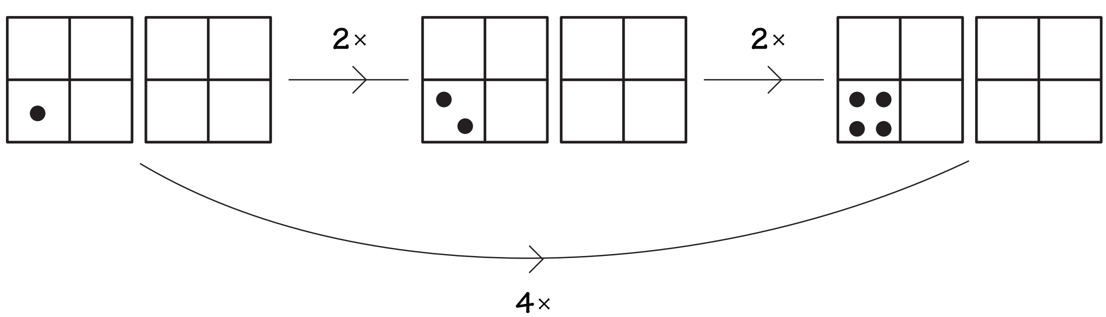
Let students experiment. You may want to remind them to consider the composition of blue followed by blue. There are three different solutions. Encourage students to find more than one solution.



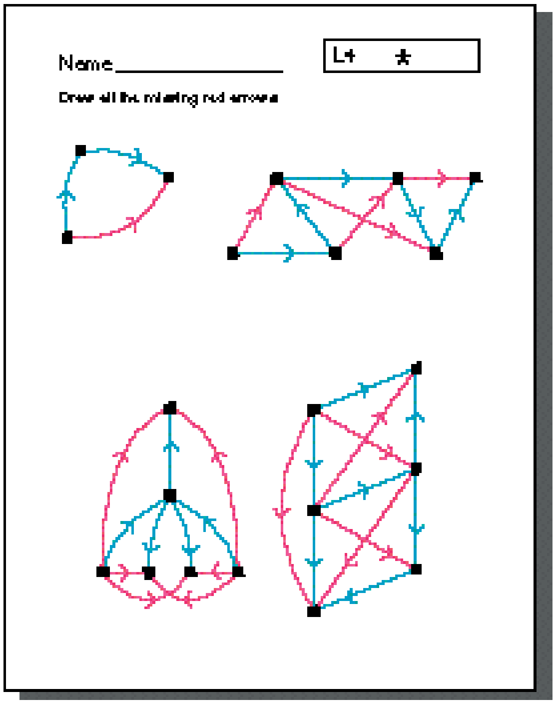
Look for all three solutions and ask students to put them on the board. Encourage students to explain how they found a solution. Discuss the composition 2x followed by 2x is 4x. Draw red arrows for 4x and use red arrows to help explain how to find solutions. 20 and 80 can be located at the beginning and end, respectively, of any red arrow.

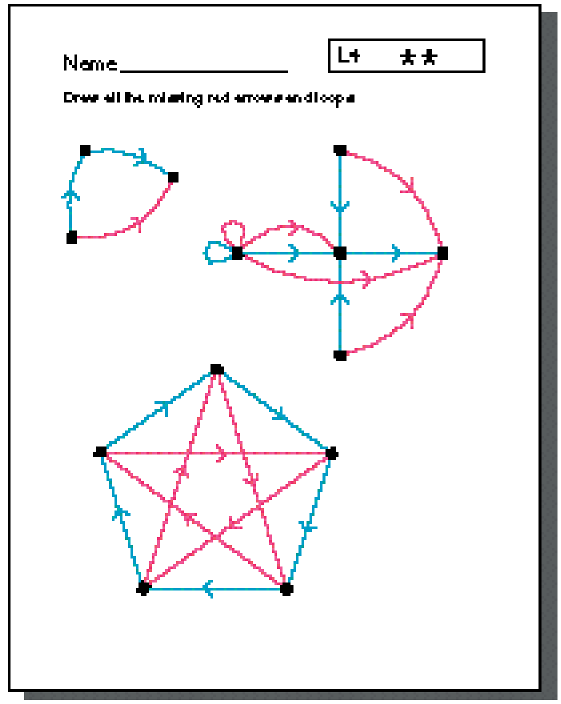


Reinforce this relationship by displaying the Minicomputer with a checker in the 20 position. Then double it and double again. Students should see that 2x followed by 2x is 4x.



L4


Name
L4 ★
Draw all the missing red arrows


Name
L4 ★★
Draw all the missing red arrows and loops

Capsule Lesson Summary

Read the *Dancing Friends* Storybook and explore a new addition operation ⊕ invented for use with just ten whole numbers, 0 through 9.

Materials

Teacher
- *Dancing Friends* Storybook
- Colored chalk
- Numeral cards 0–9 on yarn necklaces

Student
- *Dancing Friends* Storybook
- Paper
- Colored pencils, pens, or crayons

Advance Preparation: Before starting the lesson, punch a hole at the top of each numeral card 0–9, and string each on a piece of yarn to make necklaces.

Description of Lesson

You may like to pair students during this lesson. Distribute copies of the *Dancing Friends* Storybook and instruct students to follow along (without reading ahead) as you read this story. Students will need paper and colored pencils.

Pages 2–3

Read these pages aloud. When you come to the middle of page 3, read the column of addition facts on the left and ask how to extend this list. For example, the next three addition facts might be $8 + 6 = 14$; $9 + 6 = 15$; and $10 + 6 = 16$.

When you finish reading page 3, ask what other games the numbers might play. You may receive a wide variety of suggestions; accept all of them without comment.

Pages 4–5

Read these pages aloud or ask some students to do so. Some students may have difficulty reading $4 \div 8 = \frac{1}{2}$ and will need help.

Record the ten numbers on the board and keep the list on the board for the rest of the lesson.

0 1 2 3 4 5 6 7 8 9

T: ***Can you imagine a game that just these ten numbers could play?***

Pages 6–7

Read these pages aloud or ask some students to do so. Perhaps your students would enjoy reading the number sentences in unison. They may express curious surprise at the number sentences on page 7. Ask the class if they understand 0's game, and let some students try to explain it.

Pages 8–9

Read page 8 aloud and then ask students to read page 9 to themselves while you write some problems on the board.

$$6 \oplus 9 = \qquad 5 \oplus 5 \oplus 5 \oplus 5 =$$
$$8 \oplus 8 = \qquad 7 \oplus 7 \oplus 7 =$$
$$7 \oplus 7 = \qquad 9 \oplus 8 \oplus 6 =$$

After most students have finished reading page 9, ask,

T: ***Who can explain 0's game?***

S: ***0 doesn't add in the usual way.***

S: ***8 + 7 = 15, but in 0's game you just keep the last number 5 (the ones digit).***

S: ***In 0's game you add in the usual way and then subtract 10.***

T (pointing to the problems on the board)**: *What would 0's answers be to these problems?***

Ask each student who gives an answer to explain it; for example:

S: ***6 ⊕ 9 = 5.***[†]

T: ***How did you get 5?***

S: ***6 + 9 = 15 and I kept only the 5.***

Continue until all the number sentences are completed.

$$6 \oplus 9 = 5 \qquad 5 \oplus 5 \oplus 5 \oplus 5 = 0$$
$$8 \oplus 8 = 6 \qquad 7 \oplus 7 \oplus 7 = 1$$
$$7 \oplus 7 = 4 \qquad 9 \oplus 8 \oplus 6 = 3$$

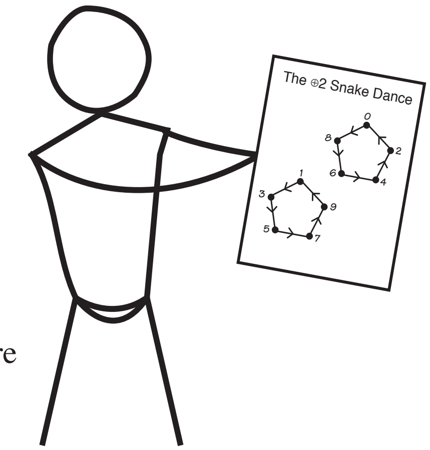
Pages 10–11

Read page 10 aloud or ask some students to do so.

T: ***Look at the ⊕2 dance. Why does it have two pieces?***

S: ***If you add 2 starting at 0, you only get even numbers. If you add 2 starting at 1, you only get odd numbers.***

[†]Read ⊕ as "circle-plus."



Hold the page so that you can trace the arrows in the upper piece of the picture and so that the students can see what you are doing. As you trace each arrow, state the corresponding number sentence.

Point to the lower piece of the arrow picture.

T: ***But suppose we start the ⊕2 snake dance at 1.***

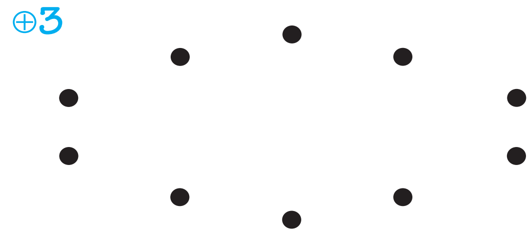
As you did before, trace the arrows in the lower piece of the picture and state the corresponding number sentences.

T: ***What numbers are in this piece of the ⊕2 dance?***

S: ***The odd numbers from 1 to 9.***

Ask the class to think about what a ⊕3 snake dance would look like. If necessary, tell them not to turn the page yet. Allow a minute or two for comments and predictions. You may like to let ten students put on the numeral card necklaces. Ask one of these students to come up to the front and then call for the next number by adding ⊕3 to the first. The student with this number should go to the front of the class and join hands with the first student. Continue calling the next number when ⊕3 is added until all the numbers 0–9 are in the dance.

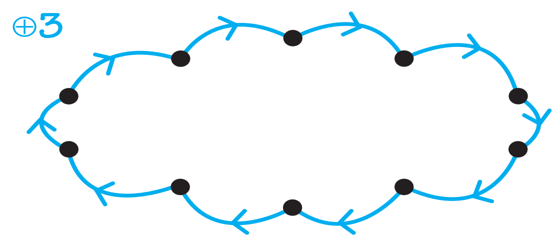
The ⊕3 Snake Dance



Draw this picture of dots for ten numbers on the board and ask the students to copy it at their desks.

T: ***This picture has a dot for each of these ten numbers*** (point to the list of numbers 0–9 on the board or to the ten students) ***because they all can dance in the ⊕3 snake dance.***

The ⊕3 Snake Dance



Instruct students to label the dots and draw arrows for ⊕3 in their pictures as you do so at the board. Use the ten students holding hands to direct how to label dots and draw arrows.

Pages 12–13

Students can compare their pictures to the one shown on page 13. If necessary, remind them that their pictures might look a little different but should still be similar. Suggest that they check their pictures by starting at 0 and following the arrows. They should meet the ten numbers in this order: 0, 3, 6, 9, 2, 5, 8, 1, 4, 7, and return to 0.

Before turning to page 14, ask students to think about what a ⊕5 dance would look like.

Pages 14–15

After you read page 14 aloud, hold up page 15 so you can trace the arrows in the ⊕5 dance and students can see what you are doing. As you trace each arrow, state the corresponding number sentence; for example, while tracing the ⊕5 arrow from 6 to 1 say, “$6 + 5 = 11$, so $6 \oplus 5 = 1$.”

Students may like to observe that in the ⊕5 dance the ten numbers are in pairs. Relate this fact to the ones digit pattern in a +5 arrow road or counting by fives.

T: ***Before you turn the page, try to draw a ⊕7 dance and a ⊕8 dance on your paper.***

When students finish both pictures, let them go on reading the rest of the storybook with their partners. Suggest that they check their pictures against the ones on pages 16 and 18.

Capsule Lesson Summary

Review the story of *Dancing Friends* and the ⊕ (addition with ten number friends) operation. Solve some problems involving the operation ⊕ in this finite system. Draw arrow pictures for the relations ⊕3, ⊕6, and ⊕9, and observe that return arrows for these relations are, respectively, ⊕7, ⊕4, and ⊕1.

Materials

Teacher	• Colored chalk	• Colored pencils, pens, or crayons • Worksheets L6* and **
Student	• Paper	

Description of Lesson

Exercise 1

Briefly review the story of *Dancing Friends*.

T: ***Do you remember the story of* Dancing Friends*?***

S: ***It was about ten numbers and a new game that 0 made up.***

T: ***What happened in the story?***
Why did 0 make up a new game for only ten numbers?

S: ***The boy in the story invited only the numbers 0 through 9, and there weren't enough numbers to play their usual addition, subtraction, multiplication, and division games.***

Write this list of numbers on the board so it can be referred to throughout of the lesson.

0 1 2 3 4 5 6 7 8 9

T: ***Do you remember the*** ⊕ (read as "circle-plus") ***operation?***
What number is 6 ⊕ 7?

A student might add 6 and 7 in the usual way.

S: ***13.***

T: ***But remember only these numbers*** (point to the list) ***are playing the game.***

S: ***3.***

T: ***How did you get 3?***

S: ***6 + 7 = 13 and I just kept the last number 3 (the ones digit).***

Complete a number sentence on the board and write another calculation under it.

$6 \oplus 7 = 3$
$8 \oplus 1$

T: ***What number is 8 ⊕ 1?***

S: ***9.***

T: ***What number is 8 ⊕ 6?***

S: ***4.***

S: ***8 × 6 = 14, so 8 ⊕ 6 = 4.***

$$6 \oplus 7 = 3$$
$$8 \oplus 1 = 9$$
$$8 \oplus 6 = 4$$

Continue this activity, asking your class to complete each of the number sentences below. Answers are in the boxes.

$$4 \oplus 4 \oplus 4 \oplus 4 = \boxed{6} \qquad 7 \oplus 7 \oplus 7 = \boxed{1}$$
$$4 \oplus 4 \oplus 4 \oplus 4 \oplus 4 = \boxed{0} \qquad 7 \oplus 7 \oplus 7 \oplus 7 = \boxed{8}$$

Students will need paper and colored pencils for certain parts of the rest of this lesson.

T: ***I will put some problems on the board. Copy them on your paper, and then figure out which of these numbers*** (point to the list of whole numbers from 0 to 9) ***can be put in the boxes to make true number sentences.***

Write these problems on the board. Answers are in the boxes.

$$7 \oplus \boxed{5} = 2 \qquad 2 \oplus \boxed{1} = 3$$
$$6 \oplus \boxed{9} = 5 \qquad 6 \oplus \boxed{5} = 1$$

Collectively check the above problems and then add this open sentence to the list.

$$\square \oplus \square = 2$$

T: ***The same number goes in both of these boxes. Which of these numbers*** (point to the list of whole numbers from 0 to 9) ***could we put in the boxes to get a true number sentence?***

S: ***6.***

T: ***6 + 6 = 12, so 6 ⊕ 6 = 2. That's right.***

S: ***1.***

T: ***1 + 1 = 2, so 1 ⊕ 1 = 2 also.***
Is there any other number we could put in the boxes?

S: ***No.***

Write this open sentence on the board.

$$\square \oplus \triangle = 5$$

T: ***These frames have different shapes, so we can put different numbers in them.***

S: ***Put 9 in one of them and 6 in the other.***

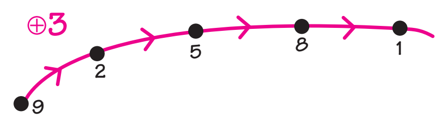
T: ***9 + 6 = 15, so 9 ⊕ 6 = 5.***

Continue until all of the pairs in this chart have been suggested and checked.

□	⊕ △ = 5
9	9
8	8
5	5
4	4
3	3

Exercise 2

Erase the board except for the list of whole numbers from 0 to 9 and then draw this arrow road. Collectively label the dots, starting with 5 at the center dot.



Trace imaginary arrows as you ask the following questions. The first such arrow starts at 1.

T: ***What is the next number we would meet if we drew more ⊕3 arrows?*** **S:** ***4.***

T: ***1 ⊕ 3 = 4. And the next?*** **S:** ***7.***

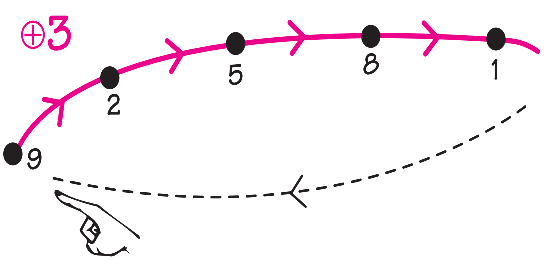
T: ***4 ⊕ 3 = 7. And the next?*** **S:** ***0.***

T: ***7 ⊕ 3 = 0. And the next?*** **S:** ***3.***

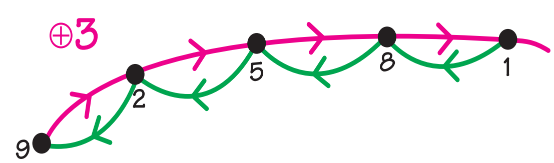
T: ***0 ⊕ 3 = 3. And the next?*** **S:** ***6.***

T: ***3 ⊕ 3 = 6. And the next?*** **S:** ***9.***

With a sweeping motion indicate that the road returns to 9 as you say, "6 ⊕ 3 = 9."



Draw return arrows for each ⊕3 arrow in the picture.



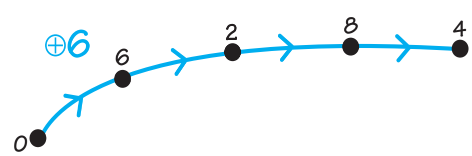
T: ***What could the green arrows be for?***

S: ***⊖3.***

S: ***⊕7.***

If students do not suggest ⊕7 on their own, write ⊕□ in green on the board and ask whether the green arrows could be for ⊕ some number. Check that the green arrows could be for ⊕7 by considering each number. Look at the starting number, follow the arrow saying, "⊕7", and check that the ending number is correct. Then write ⊕7 in green near the arrow road.

Draw this arrow road next to the ⊕3 arrow road. Collectively label the dots, starting with 2 at the center dot.

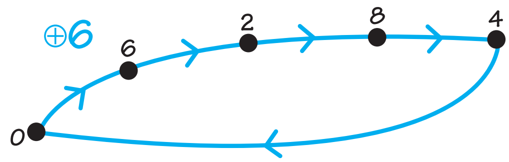


T: ***If we drew another ⊕6 arrow, what***

T: ***is the next number we would meet?***

S: ***0.***

Draw a blue arrow from 4 to 0.



T: ***Where is the number 3 in the ⊕6 snake dance?***

S: ***Only even numbers are in this piece.***

S: ***You need to start a new part of the ⊕6 snake dance with 3 in it.***

Draw arrows starting at 3 as you ask the following questions.

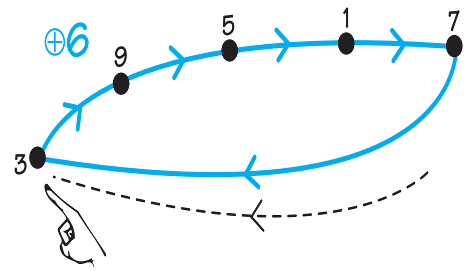
T: ***What is 3 ⊕ 6?*** **S:** ***9.***

T: ***What is 9 ⊕ 6?*** **S:** ***5.***

T: ***What is 5 ⊕ 6?*** **S:** ***1.***

T: ***What is 1 ⊕ 6?*** **S:** ***7.***

T: ***What is 7 ⊕ 6?*** **S:** ***3.***

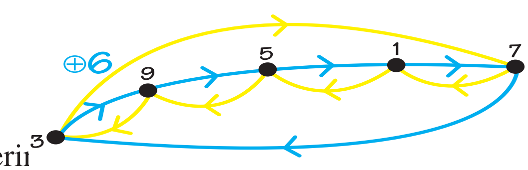


With a sweeping motion indicate that the road returns to 3.

Draw the return arrows in this part of the ⊕6 snake dance.

T: ***What could the yellow arrows be for? Try to label them circle-plus some number.***

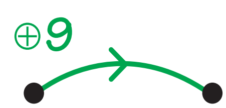
S: ***⊕4.***



Check that the yellow arrows could be for ⊕4 by considering the arrow picture. Check also that the return arrows in the other part of the ⊕6 picture are for ⊕4.

Exercise 3

Erase the board and then draw a ⊕9 arrow.



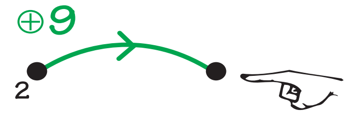
8	7
5	4
4	3
2	1
1	0
7	6
0	9
6	5
3	2
9	8

T: ***How can we label these dots?***

Let students make suggestions. Record correct number pairs in a chart as shown here.

If your students do not find all the possible number pairs, you can help by calling attention to the left column of the list and asking whether any whole number from 0 to 9 is not included. Then ask if a ⊕9 arrow could start at that number. For example, suppose that the pair (2,1) is not in the list.

T: ***2 is not on this side*** (trace down the left column of the chart).



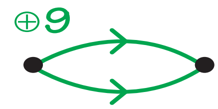
Could this ⊕9 arrow start at 2? (Yes)
If so, what number would be here?

S: ***1.***

T: ***2 + 9 = 11, so 2 ⊕ 9 = 1.***

Repeat this procedure, if necessary, until all ten possible number pairs are listed in the chart.

Draw a return arrow in the picture.



T: ***What could the red arrow be for?***
Could it be for circle-plus some number?

S: ⊕***1.***

Using the entries in the chart, check that the return arrow could be for ⊕1. (You will need to go from right to left in the chart to check the return arrow.) Then write ⊕1 in red near the arrow picture.

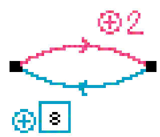
Worksheets L6* and ** are available for individual work.

Home Activity

You may like to let students take home a *Dancing Friends* Storybook and explain the ⊕ operation to other family members.

Name______________ | L6 ★

Fill in the chart with ways to label the dots.

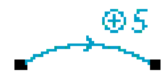


5	7
6	8
9	1
1	3
4	6
7	9
8	0
2	4
3	5
0	2

What could the blue arrow be for? Fill in the blue box.

Name______________ | L6 ★★

Fill in the chart with ways to label the dots.



0	5
1	6
2	7
3	8
7	2
4	9
5	0
6	1
8	3
9	4

Capsule Lesson Summary

Examine numbers that can be named by selecting, in order, one symbol from each of three spinners. The first spinner has two numbers, the second has three operations, and the third has three numbers. Predict how many possibilities there are in this situation and discuss the chances of getting a number less than 10. Make operation tables to find all the possibilities. Discuss how to make a fair game in which a team receives points when the number formed is more (less) than a specified number.

Materials

Teacher		Student	
	• Three spinners • Colored chalk		• Worksheet L7

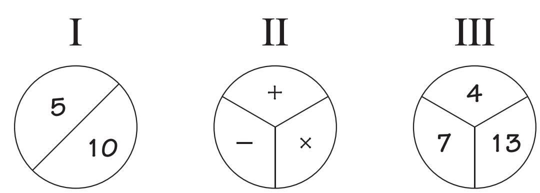
Note: Use Blackline L7 to make three spinners as shown below. You can use a cardboard arrow or a paperclip with a pencil to create the spinner. As an alternative to spinners, you can put folded slips of paper with the numbers or the operations in three boxes labeled I, II, and III. Then let students draw one paper from each box.

Description of Lesson

This would be a good lesson to put students into cooperative groups.

Exercise 1

Display the three spinners and describe them.



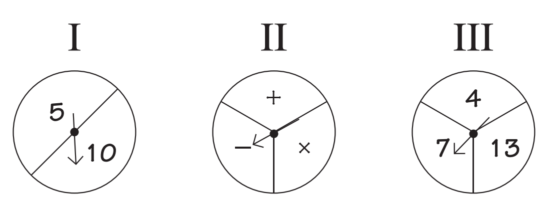
T: ***I have three spinners here. Spinner I is equally divided with the numbers 5 and 10. Spinner II is equally divided with the operations +, –, and*** **x*****. Spinner III is equally divided with the numbers 4, 7, and 13.***

The numbers were chosen so that every combination of something from the first spinner, something from the second spinner, and something from the third spinner names a different number.

Call on a student to spin each spinner once. Ask the class to read aloud the result from each spinner. Write the results in order on the board. For example:

T: ***What number is 10 – 7?***

S: ***3.***



10 − 7 = 3

Repeat this activity, letting other students take a turn at spinning the spinners. Record appropriate number sentences on the board.

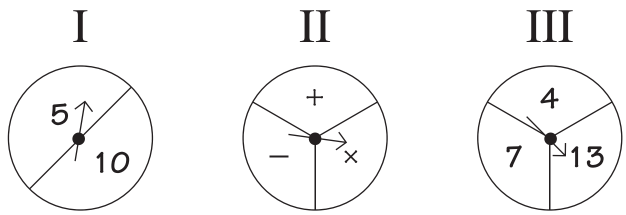
T: ***How many different numbers do you think we can get this way?***

L7

Record the students' predictions on the board. Your chalkboard might look similar to this.

How many different numbers?

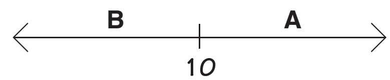
12
8
10
18
22



$$10 - 7 = 3$$
$$5 - 7 = \widehat{2}$$
$$10 + 4 = 14$$
$$5 \times 13 = 65$$

T: ***Suppose we play a game with two teams in which one team (Team A) gets a point if the number formed is greater than 10 while the other team (Team B) gets a point if the number formed is less than 10.***

Draw this number line picture under the spinners.



T: ***How could Team A get a point?***

S: ***The result could be 10 + 4 = 14 and 14 > 10.***

T: ***How could Team B get a point?***

S: ***The result could be 5 + 4 = 9 and 9 < 10.***

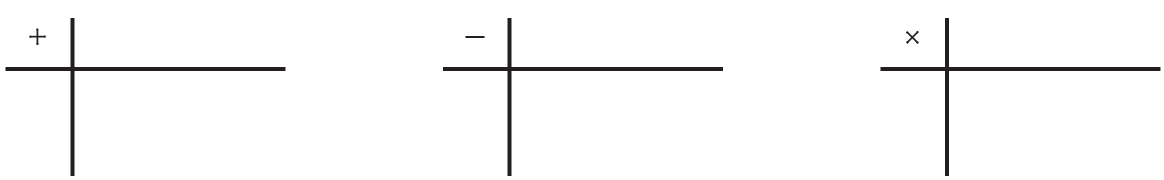
T: ***Do you think this is a fair game?***

Let students discuss the situation in their groups. Some students might want to play many rounds of the game. Accept this as a good idea, but ask if they could answer the question without depending on the results of many rounds of the game. Commend those who notice that combinations with x always give points to Team A and combinations with **–** always give points to Team B, but do not offer this observation yourself.

S: ***We need to see all the possible numbers you can get in the game. Then we will see how many give Team A a point and how many give Team B a point.***

T: ***One way to do this is to look at the numbers with +, then the numbers with –, then the numbers with x.***

Prepare these three operation tables.



Choose one of the number sentences previously written on the board, for example, 10 + 4 = 14.

T: ***Here we add 10 + 4. Where do I put 10 in the operation table? Where do I put 4?***

Discuss with the class that the first number in an expression comes from the first spinner, and goes along the left in the table for that operation (+, –, or x). The second number in an expression comes from the third spinner and goes across the top. After placing these numbers, locate 14 in the table. Then locate the other results already on the board.

+	4	7	13
5			
10	14		

–	4	7	13
5		$\hat{2}$	
10		3	

×	4	7	13
5			65
10			

Distribute copies of Worksheet L7 to the groups.

T: ***On this worksheet there is a table for +, for –, and for* x*. Fill in the tables to find all of the numbers that are possible to get in the game.***

While students are working, you may need to remind them how to read the subtraction table; that is, which number to subtract from the other. Choose several students to help you fill in the tables on the board.

+	4	7	13
5	9	12	18
10	14	17	23

–	4	7	13
5	1	$\hat{2}$	$\hat{8}$
10	6	3	$\hat{3}$

×	4	7	13
5	20	35	65
10	40	70	130

T: ***How many different numbers can you get in this game?*** (18)

Note: A student who suggests that there are 23 or 33 possible numbers is including the numbers 5, 10, 4, 7, and 13, one or three times, respectively. Remind the class that you are only counting the numbers that you can get in the game.

Compare 18 to the predictions recorded on the board earlier in the lesson. Decide which prediction was closest to the actual number of possibilities.

T (pointing to the three tables)**:** ***For which of these numbers would Team A gain a point? Which of these numbers is more than 10?***

Put checkmarks (✓) by such numbers as they are suggested.

+	4	7	13
5	9	12✓	18✓
10	14✓	17✓	23✓

–	4	7	13
5	1	$\hat{2}$	$\hat{8}$
10	6	3	$\hat{3}$

×	4	7	13
5	20✓	35✓	65✓
10	40✓	70✓	130✓

T: ***How many of the possible numbers are more than 10?***

S: ***11.***

Write the fraction $^{11}/_{18}$ on the board as you say,

T: ***So Team A has 11 chances out of 18 to get a point.***
How many of the possible numbers are less than 10?

S: ***Seven.***

T: ***So Team B has 7 chances out of 18 to get a point.***
Is this a fair game?

A: $^{11}/_{18}$ B: $^{7}/_{18}$

S: ***No.***

T: ***Who has the advantage, Team A or Team B?***

S: ***Team A.***

Erase the checkmarks from the tables.

Tell the class that a toy company has created this game and now wants to make it fair without completely changing the game. Assign each group the job of writing to the toy company to tell them how to make a fair game. Allow time for the groups to discuss and work; then let groups share their ideas.

You might get only a few different suggestions, or you might get a wide variety ranging from simple alterations to very complex ones which may well be correct. Three types of alterations are likely to be suggested:

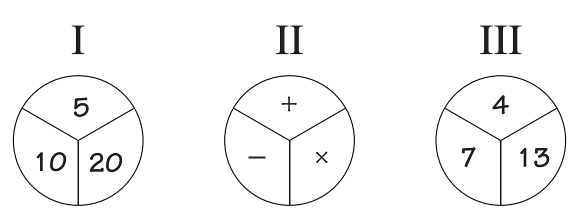
- Change the numbers on one or both of spinners I and III. In this case, groups may experiment with finding numbers to put on the spinners that would make a fair game.
- Make some change on the middle spinner, the operations. Some students may notice that if you eliminate + you will get a fair game. Others might want to try including ÷, in which case the game would favor Team B but only slightly ($^{11}/_{24}$ to $^{13}/_{24}$).
- Change the number that determines which team gets a point (that number was 10 in the example game). To do this, look at all the possibilities in the operation tables and select a number (for example, 15 or 16) for which half the possibilities are more and half are less.

While discussing a change such as the first one above, you may find that you will make the observations presented in the next exercise. If not, proceed with Exercise 2.

Exercise 2

T: ***Suppose we put another number, for example 20, on the first spinner together with the other two numbers already on the spinner.***

Redraw spinner I.



T: ***How many more numbers could we get now in the game?***

S: ***We could get three more with +, three more with –, and three more with* x.**

Students might make erroneous suggestions. For example, someone may say, "Eighteen more; I just doubled what we had before." If so, express doubt and ask for another suggestion.

To show the nine new possibilities, draw a new row in each table as shown below. Instruct students to extend the tables on their worksheets to include 20 as a choice from the first spinner and to complete the tables.

+	4	7	13
5	9	12	18
10	14	17	23
20	24	27	33

–	4	7	13
5	1	$\hat{2}$	$\hat{8}$
10	6	3	$\hat{3}$
20	16	13	7

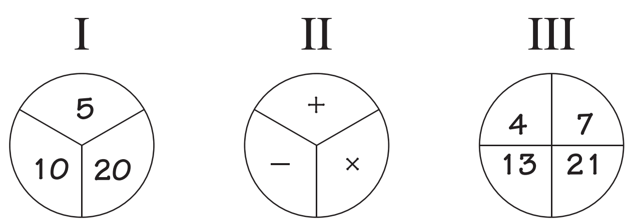
×	4	7	13
5	20	35	65
10	40	70	130
20	80	140	260

T: ***There were 18 possible numbers before. How many are there now?***

S: *27.*

You may want to observe that it is difficult to make a fair game with 27 outcomes because you can not assign half to each team.

Repeat this activity, this time putting 21 on spinner III.



+	4	7	13	21
5	9	12	18	26
10	14	17	23	31
20	24	27	33	41

–	4	7	13	21
5	1	$\hat{2}$	$\hat{8}$	$\widehat{16}$
10	6	3	$\hat{3}$	$\widehat{11}$
20	16	13	7	$\hat{1}$

×	4	7	13	21
5	20	35	65	105
10	40	70	130	210
20	80	140	260	420

T: ***There are 36 possible numbers we could get in this three spinner game. Now how could we decide to give points to Team A and Team B so the game would be fair?***

Let students discuss the situation. There are many solutions, but one that might become evident is to give Team A a point if the number is more than 21 (or 22) and to give Team B a point if the number is less than 21 (or 22).

Writing Activity

Suggest students write a letter to the toy company telling the company how their group was able to make a fair game.

Capsule Lesson Summary

Examine a special kind of graph, a *tree*, and find that in a tree there is one more dot than there are edges. Find that between any two dots there is exactly one path following the edges. Assign numbers (weights) to the edges of a tree and play a game that generates a cycle of arrows going through all the dots.

Materials

Teacher		Student	
	• Colored chalk • Blackline L8		• Tree picture • Worksheets L8* and ** • Colored pencils, pens, or crayons

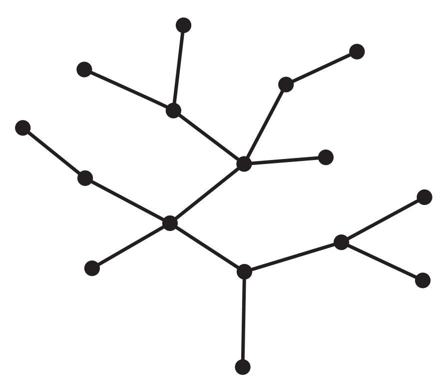
Advance Preparation: Use Blackline L8 to make student copies of the tree used in both Exercises 1 and 2.

Description of Lesson

Note: Trees are a special type of graph that is used often to organize and display information. For example, trees may be used to give probability information in multi-step experiments. This lesson simply introduces the notion of a tree and then uses a weighted tree in a game involving order of numbers.

Exercise 1

Copy the tree picture from Blackline L8 on the board and refer students to their copy of this picture.



T: ***This kind of picture is sometimes called a tree. Do you have any idea why it might be called that?***

S: ***It looks like it has branches.***

T: ***How many dots are there?*** (16)

Direct students to count aloud as you point to each dot.
Trace a line segment connecting two of the dots as you say,

T: ***This is called an edge. How many edges are there in this tree?*** (15)

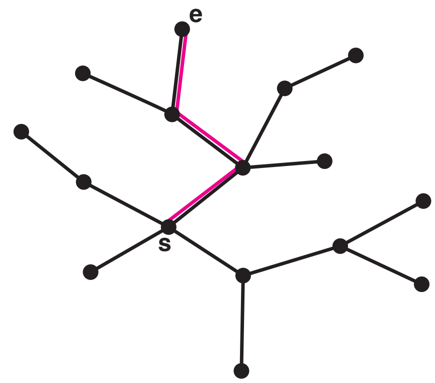
Ask students to count the edges aloud as you point to each edge.
Record the number of edges and the number of dots on the board.

16 dots 15 edges

For the following discussion, chose any two dots in the tree; for example, **s** and **e** in the next illustration.

T: ***Suppose we start here*** (tap one of the dots). ***How many ways are there to go from this point*** (tap the starting dot) ***to this dot*** (tap the other dot) ***following edges? Once we go on an edge we do not want to go back on that edge again.***

Label the starting dot **s** and the ending dot **e** and invite someone to trace a path from **s** to **e**.



T: ***Are there any other paths between these two points (s and e)?***

S: ***No.***

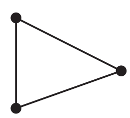
Repeat the questions for other pairs of dots in the tree; you may want to choose one more pair of dots yourself and then let students choose the dots. The class should discover that there is always just one path between two points in the tree. Generalize this observation as you say,

T: ***That is what makes a tree special—there is only one way to get from one point to another following the edges. Let's make another tree. I'll start the tree.***

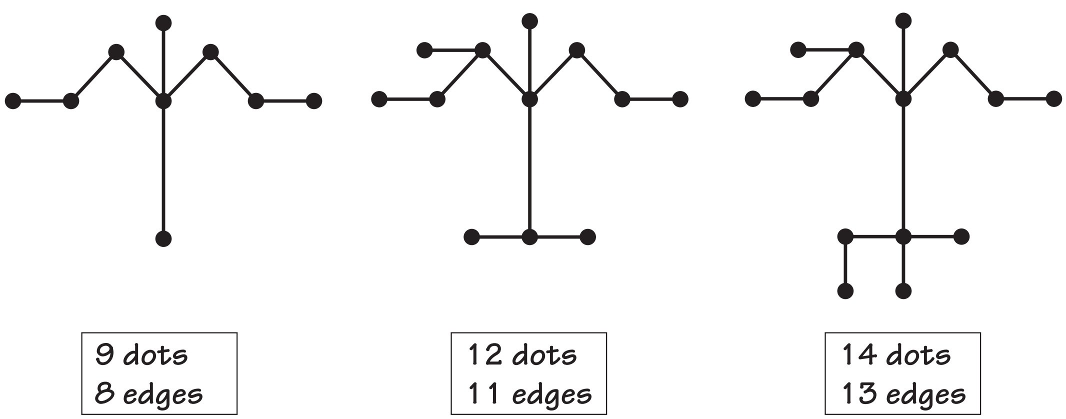
Draw two dots on the board and a line segment connecting them. The size of your dots and the length of your segment will set a size perspective for the tree to be built.

T: ***Who would like to draw another dot and another edge in this tree?***

Continue until 13 or 14 edges have been added to the picture. After each edge is added, ask the students if the picture is still a tree. There should be no cyclic paths, that is, paths that start and end at the same point, as in this example.



Periodically ask how many dots and how many edges are in the tree at that particular stage. The illustrations below show a tree at different stages and the number of dots and edges at that stage.



If no one mentions that the number of edges is always one less than the number of dots (a property of a tree), point this out yourself.

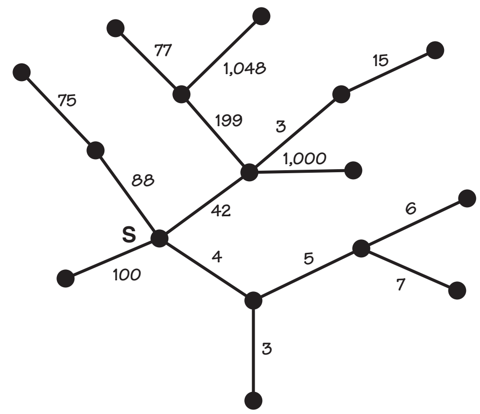
Exercise 2

Refer to the tree on Blackline L8.

T: ***Now I would like you to help me assign numbers to the edges of this tree. The only restriction is that edges meeting at the same dot must be assigned different numbers.***

Point to the edges, one at a time, and let students choose numbers for them. Encourage the class to check that the restriction is obeyed each time they assign a number to an edge.

This picture shows one possible assignment of numbers and will be used to describe a game.

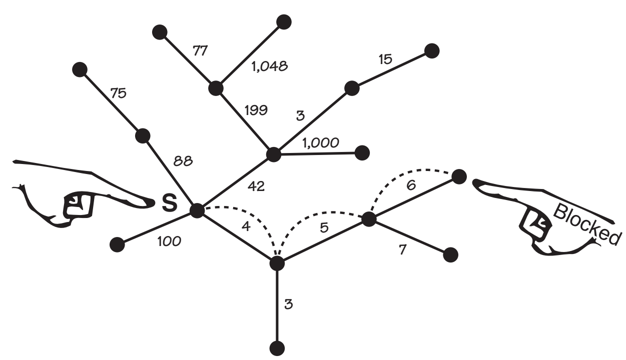


Call on someone to chose a starting dot. Label that dot **s**.

T: ***We are going to play a game called The Red Arrow Game with this tree. This is how you play:***

- ***Begin at the starting dot and follow the edge with the least number.***
- ***When you get to the next dot, try to follow an edge with a greater number but still as small a number as you can.***
- ***You may keep going until you are blocked, that is, you cannot follow an edge with a greater number.***

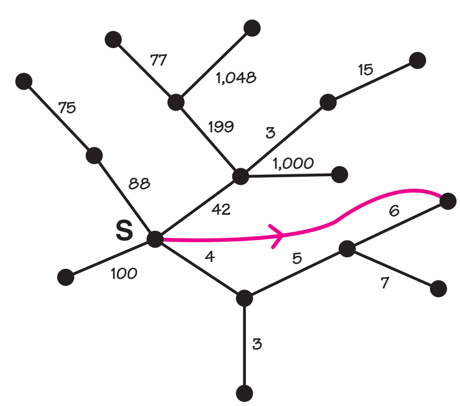
Let's start together and see what happens.



Put your left forefinger on the starting dot and hold it there. Invite a student to begin tracing edges following the rules of the game. First follow the edge labeled 4 since 4 is the least number assigned to an edge at **s**. At the next dot, look at just the edges with greater numbers and chose the edge (in this case, there is only one) with the least number. Follow this edge, labeled 5, to the next dot. Then, again look just at the edges with numbers greater than 5 (in this case, there are two such edges) and choose the edge with the least number (in this case the edge labeled 6) to the next dot. Again look for edges with numbers greater than 6—now there are none so you are blocked.

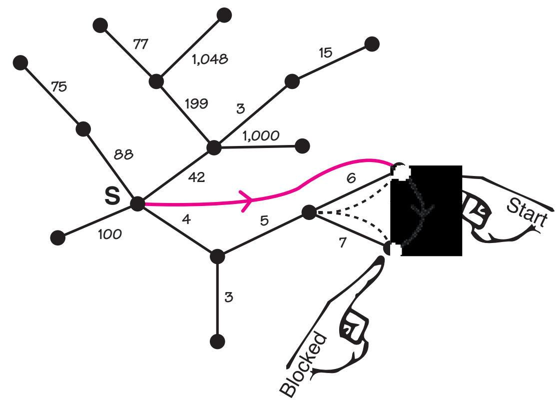
T: ***Now we are blocked because there are no edges here with numbers greater than 6. So, we draw a red arrow from where we started*** (point to **s)** ***to where we ended*** (point to the dot where you are blocked)***.***

Next, we start the game over, but this time we start at the dot where the red arrow ends.



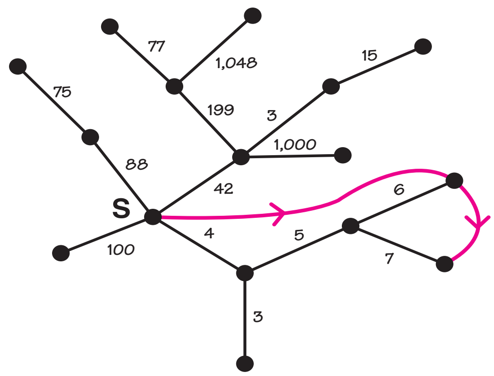
Hold your finger at the new starting dot and invite a student to trace edges following the rules of the game until you are again blocked.

T: ***We are blocked again, so we draw another red arrow from where we started to where we are blocked his time. Then we begin again starting where this new red arrow ends.***

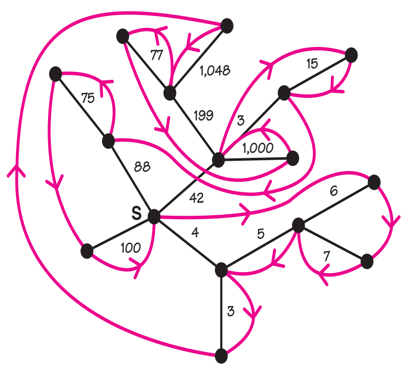


Continue until a red arrow returns to the starting dot **s**.

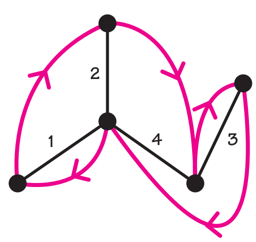
Invite someone to trace the cycle of red arrows starting at **s** and returning to **s**.

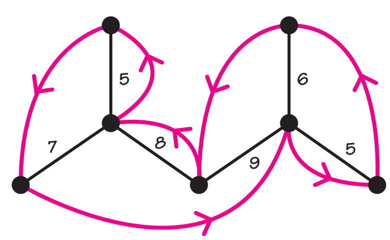


Note: In the preceding picture it does not matter which dot you choose as your starting dot **s**; you will obtain the same cycle of red arrows going through the dots in the same order. What does make a difference, though, is the assignment of numbers to edges. For example, here is the same tree with a different assignment of numbers to edges; a different red cycle results.



If your students understand the rules of The Red Arrow Game fairly well, suggest they play the game with a partner. Direct them to assign numbers to edges on their copy of the tree (Blackline L8) and then use this weighted tree to play the game. If your students need more practice with the game rules, draw one or two very simple trees on the board and play the game collectively with these trees. For example:

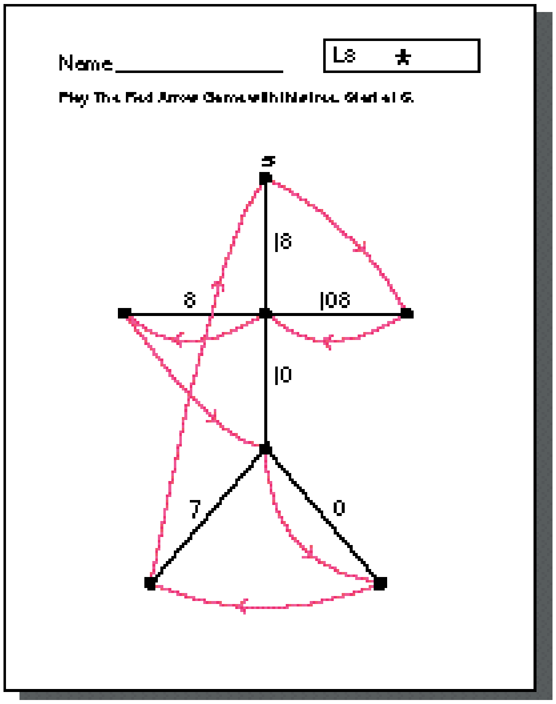


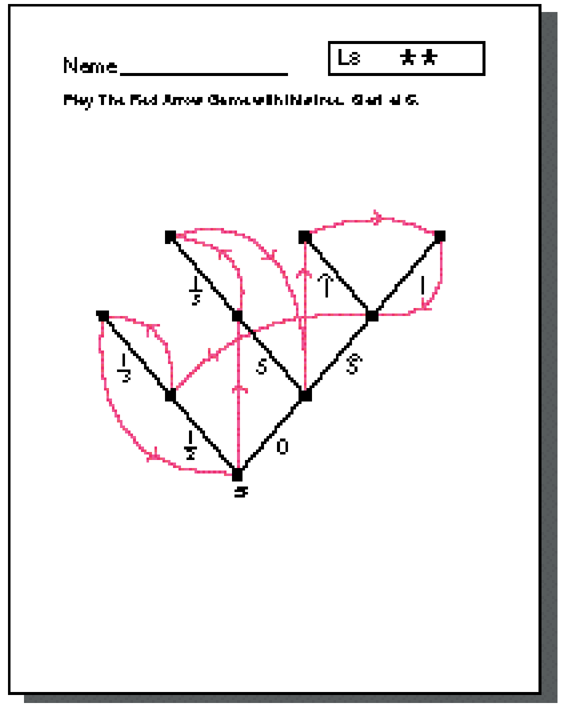


Worksheets L8* and ** are available for individual work.

Home Activity

If students are comfortable with the game, they may like to take Worksheets L8* and ** home to show a family member how to play The Red Arrow Game.


Name
L8 ★
5
18
8
108
10
7
0


Name
L8 ★★
1/3
1/3
5
5
1
1/2
0
5

Capsule Lesson Summary

Review not-cards as string labels. Analyze the information obtained from correctly placed pieces in a string game until the string labels are determined. Play a three-string version of The String Game.

Optional: Play a three-string game involving multiples and order.

Materials

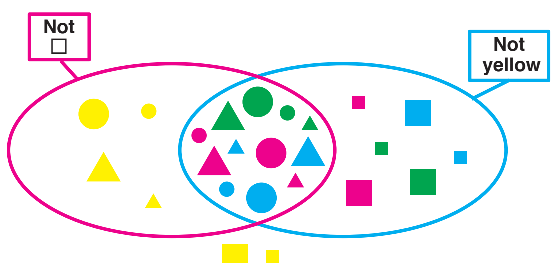
Teacher
- A-Block String Game kit
- Colored chalk
- Colored markers

Student
- None

Description of Lesson

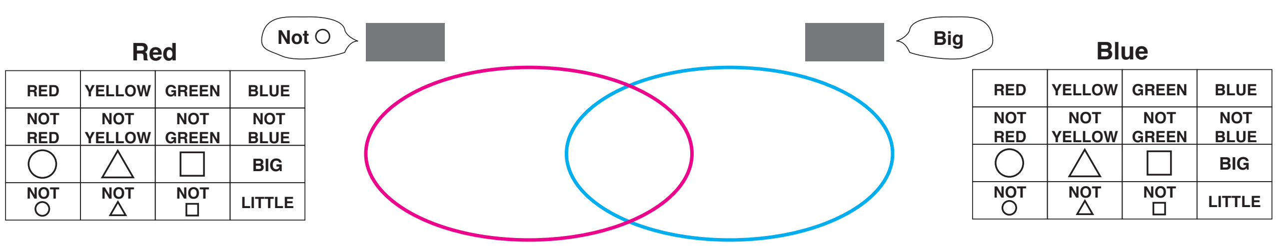
Exercise 1

Begin the lesson with a brief review of not-cards. First, place several pieces in one string labeled **NOT YELLOW**. Then add a second string labeled **NOT** ☐ and place several pieces in the resulting two-string picture. The correct placement of all A-Block pieces is shown here for your reference.



Exercise 2

Set up your board for The String Game as illustrated below. The bubbles show what is on the hidden string labels. Tape two A-Block String Game Posters (Version B) to the board, one for the red string and one for the blue string.

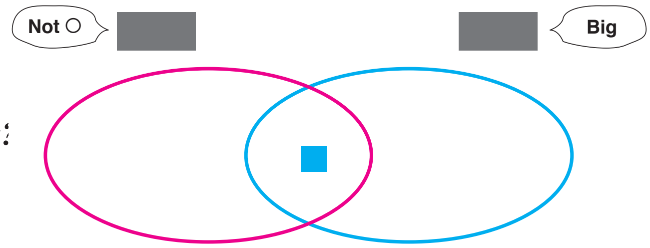


T: *We are going to play The String Game today, but first we are going to look at what information we get from knowing where some of the A-block pieces belong in the picture. I will place some pieces correctly and you can use these as clues to give information about what labels the strings can and cannot have.*

Clue 1

T: ***The big blue square is in the middle region, in both the red and the blue strings.***

What does this tell us about the strings? Are there some labels (point to one of the lists) ***that the strings cannot have?***



S: **NOT BLUE.**

T: ***We can cross off* NOT BLUE *for which string?***

S: ***Both.***

T: ***If a string were labeled* NOT BLUE*, where would this piece belong?***

S: ***Outside that string.***

Use red and blue crayons or markers to cross off **NOT BLUE** from both lists. In the same manner, let students eliminate as many labels as they can from both lists. Each time they cross off a label on one list because the corresponding string cannot have that label, they should observe that the same label should be crossed off on the other list, also. A piece in the center region gives the same information about both strings.

A student may suggest incorrectly that a particular label be crossed off the lists; for example:

S: ***Cross off* NOT RED.**

T: ***But this piece*** (pointing to the big blue square) ***is not red.***

When your class has exhausted the information given by this clue, they should find that there are eight remaining possibilities for each string.

Red

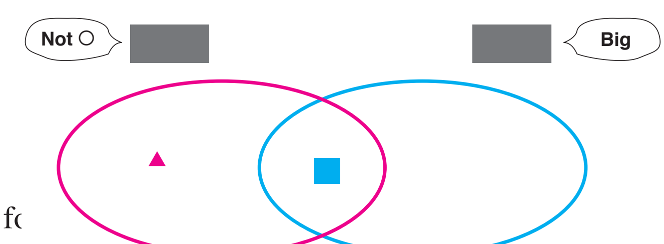
~~RED~~	~~YELLOW~~	~~GREEN~~	~~BLUE~~
NOT RED	NOT YELLOW	NOT GREEN	~~NOT BLUE~~
~~○~~	~~△~~	~~□~~	~~BIG~~
~~NOT ○~~	~~NOT △~~	NOT □	~~LITTLE~~

Blue

~~RED~~	~~YELLOW~~	~~GREEN~~	BLUE
NOT RED	NOT YELLOW	NOT GREEN	~~NOT BLUE~~
~~○~~	~~△~~	□	BIG
NOT ○	NOT △	~~NOT □~~	~~LITTLE~~

Clue 2

T: ***I'll give you another clue. The little red triangle is in the red string but not in the blue string.***



Consider the labels remaining as possibilities fc string could be for any of them. The following dialogue gives a discussion of two of the eight remaining labels.

Point to **BLUE** on the Red list and then trace the red string as you say,

T: ***Could the red string be for* BLUE*?***

S: ***No, the triangle is red!***

Cross off **BLUE** from the Red list. Point to **NOT ○** on the Red list and then trace the red string as you ask,

T: ***Could the red string be for* NOT ○*?***

S: ***Yes, both pieces in it are not circles.***

Continue until all of the eight labels have been discussed for the red string. The illustration below shows which are eliminated by this clue.

Consider the blue string. The analysis involved is slightly different for this string because the small red triangle is outside the blue string. Because of the position of this piece in the picture,

- whenever a label is eliminated as a possibility for the red string, it remains as a possibility for the blue string; and
- whenever a label remains as a possibility for the red string, it is eliminated as a possibility for the blue string.

The following dialogue gives a discussion of two of the eight labels remaining after the first clue.

T: ***Could the blue string be for* BLUE** (point to **BLUE** on the Blue list)***?***

S: ***Yes, the small red triangle is outside the blue string; only blue pieces are inside the blue string.***

Point to **NOT ○** on the Blue list and then trace the blue string as you ask,

T: ***Could the blue string be for* NOT ○*?***

S: ***No, because the little red triangle is not a circle and it is outside the blue string.***

If no one responds, ask where the little red triangle would belong in the picture if the blue string had the label **NOT ○**.

After all of the eight possibilities for the blue string have been discussed, your lists should look like these.

Red

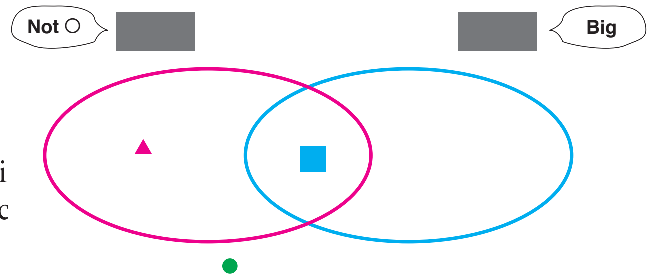
~~RED~~	~~YELLOW~~	~~GREEN~~	~~BLUE~~
~~NOT RED~~	NOT YELLOW	NOT GREEN	~~NOT BLUE~~
~~○~~	~~△~~	~~□~~	~~BIG~~
NOT ○	~~NOT △~~	NOT □	~~LITTLE~~

Blue

~~RED~~	~~YELLOW~~	~~GREEN~~	BLUE
NOT RED	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	~~△~~	□	BIG
~~NOT ○~~	NOT △	~~NOT □~~	~~LITTLE~~

Clue 3

T: ***The next clue is that the little green circle is outside both strings.***



As before, cross out appropriate labels on the two lists. Since the little green circle is outside both strings, the analysis for both strings is simi to that used to consider the blue string in the sec clue (small red triangle outside the blue string).

After considering all the remaining labels (from the second clue) on the lists, you should have two possibilities for the red string and three possibilities for the blue string.

Red

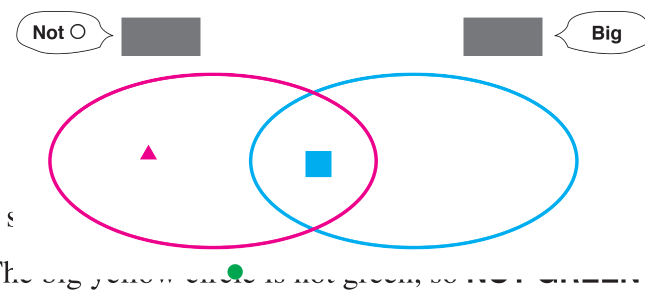
~~RED~~	~~YELLOW~~	~~GREEN~~	~~BLUE~~
~~NOT RED~~	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	~~△~~	~~□~~	~~BIG~~
NOT ○	~~NOT △~~	~~NOT □~~	~~LITTLE~~

Blue

~~RED~~	~~YELLOW~~	~~GREEN~~	~~BLUE~~
~~NOT RED~~	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	~~△~~	~~□~~	BIG
~~NOT ○~~	~~NOT △~~	~~NOT □~~	~~LITTLE~~

Clue 4

T: ***The big yellow circle is inside the blue string but outside the red string.***



From this clue your class should determine the s

- The red string must be for **NOT** ○. (The big yellow circle is not green, so **NOT GREEN** can be crossed off the Red list.)
- The blue string must be for **BIG**. (The big yellow circle is not blue and is not a square, so **BLUE** and □ can be crossed off the Blue list.)

Red

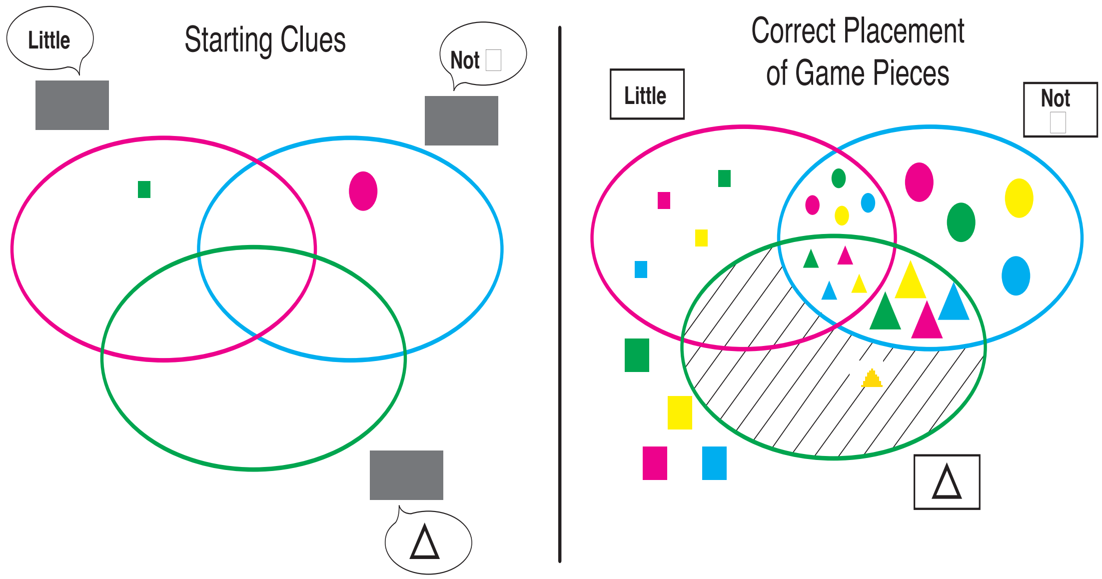
~~RED~~	~~YELLOW~~	~~GREEN~~	~~BLUE~~
~~NOT RED~~	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	~~△~~	~~□~~	~~BIG~~
NOT ○	~~NOT △~~	~~NOT □~~	~~LITTLE~~

Blue

~~RED~~	~~YELLOW~~	~~GREEN~~	~~BLUE~~
~~NOT RED~~	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	~~△~~	~~□~~	BIG
~~NOT ○~~	~~NOT △~~	~~NOT □~~	~~LITTLE~~

Exercise 3

Play The String Game with three strings in the usual way. Possible string labels and appropriate starting clues for a game are suggested here. The picture on the right may be used as a crib sheet.



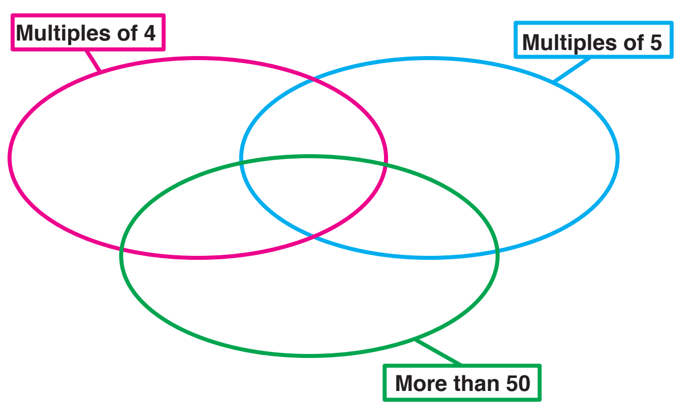
At the end of the game, decide with your class that two of the regions in the picture are empty and hatch them.

Exercise 4 (optional)

Do this exercise if there is time and student interest.

Draw this string picture and put these lists of numbers on the board. If you prefer to play with more than two teams divide the numbers accordingly.

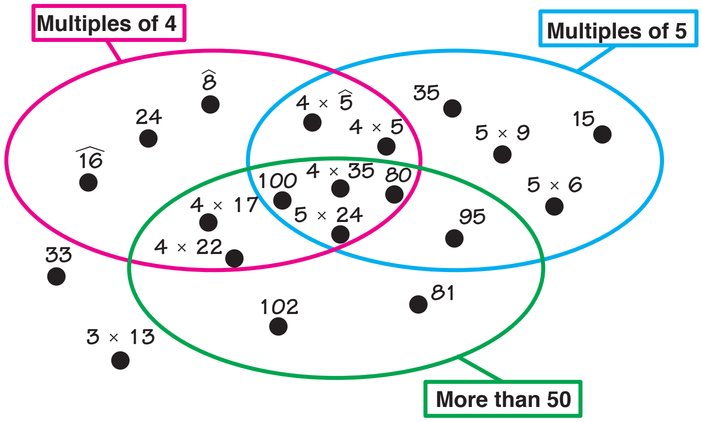
Team A	Team B
$\widehat{16}$	5×24
81	80
4×22	24
15	4×17
5×6	35
100	5×9
$4 \times \widehat{5}$	$\widehat{8}$
33	102
4×35	4×5
95	3×13



T: ***We are going to play a game with numbers. During a turn, you will have only one chance to put a number in the string picture. The first team to put all the numbers from their list where they belong in the string picture wins.***

Alternating teams, let students place numbers in the string picture. If a number is placed correctly, leave the number in the string picture and erase it from the list in which it occurs; if placed incorrectly, erase it from the picture but leave it on the list.

The next illustration shows the correct placement of the numbers in both lists.



Center Activity

Students can set up analysis games for one another in a center using the A-Block String Game kit.

Capsule Lesson Summary

Read and discuss the storybook *I Am Not My Name*. Find many different names for 0 and for $\frac{1}{2}$, and perhaps for other numbers such as 1.

Materials

Teacher
- *I Am Not My Name* Storybook
- Colored chalk

Student
- *I Am Not My Name* Storybook

Description of Lesson

Distribute copies of the storybook *I Am Not My Name* and let students look at them until you are ready to begin the lesson.

On the board, draw a line segment and place two dots on it, one for the number 0 and one for the number 4.

Pages 2–7

Read these pages aloud or ask some students to do so.

T: ***Can you imagine a dance that the numbers 16, 8, 4, 2, 1, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, and so on could do?***

S: ***An "is greater than" dance.***

S: ***A 2x dance.***

S: ***A $\frac{1}{2}$x dance.***

Pages 8–9

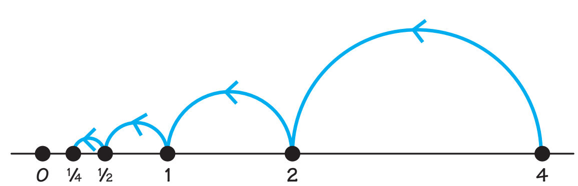
Read these pages aloud or ask some students to do so.

T (pointing to the number line on the board)**:** ***Where is 2 on this number line?***

When a student indicates the point halfway between 0 and 4, draw a dot there and label it 2. Emphasize that 2 is halfway between 0 and 4 on a number line. Draw a blue arrow from 4 to 2.

Continue in the same manner asking these questions.

- Where is 1 on this number line?
- Where is $\frac{1}{2}$ on this number line?
- Where is $\frac{1}{4}$ on this number line?



Put your finger halfway between 0 and $\frac{1}{4}$ on the number line as you ask,

T: ***What number is here?***

S: ***$\frac{1}{8}$.***

T: ***On the number line, what number is halfway between 0 and $\frac{1}{8}$?***

S: *$\frac{1}{16}$.*

T: ***On the number line, what number is halfway between 0 and $\frac{1}{16}$?***

S: *$\frac{1}{32}$.*

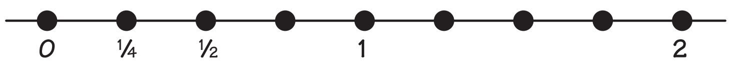
T: ***What could the blue arrows be for?***

S: ***$\frac{1}{2}x$.***

Pages 10–13

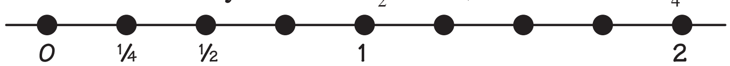
Read these pages aloud or ask some students to do so.

Erase the arrows in the picture. Rescale the number line on the board by repositioning the numbers $\frac{1}{4}$, $\frac{1}{2}$, 1, and 2, each one dot to the right. (4 is excluded from the part of the number line drawn.) Erase the unlabeled dot next to 0. Draw additional dots so that there are seven equally spaced dots between 0 and 2.



T: ***Where is $\frac{3}{4}$ on this number line?***

When a student suggests the dot halfway between $\frac{1}{2}$ and 1, label that dot $\frac{3}{4}$.



Pages 14–15

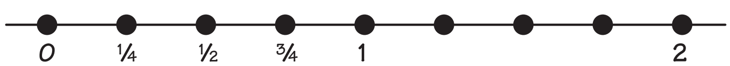
Read these pages aloud.

T: ***Why did $\frac{1}{2}$ begin to cry?***

S: ***The dot for $\frac{1}{2}$ is also the dot for $\frac{2}{4}$.***

Point to the appropriate dots as you say,

T: ***This dot is for $\frac{1}{4}$ and this dot is for $\frac{3}{4}$; so this dot*** (point to the dot labeled $\frac{1}{2}$) ***must be for $\frac{2}{4}$.***



Pages 16–21

Read these pages aloud.

Pages 22–23

Read these pages aloud. Discuss why 175 x 45 x 0 x 45 x 720 x 75 easily can be seen to be a name for 0.

T: ***What are some other names for 0?***

Here are some examples of other names for 0 that may be offered.

1,000,000 – 1,000,000	10 – 10 + 2 – 2
(5 x 50) – 250	100 – 1 – 99
16 + 16 – 32	$50 + \widehat{25} + \widehat{25}$
$1 + \widehat{1}$	0 ÷ 2[†]

T: ***How could the boy in the story have more than one name?***

S: ***Maybe he has a nickname as well as his real name.***

You might ask several students to give their full names.

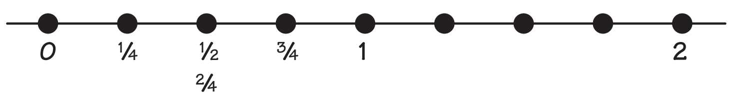
Pages 24–25

Read these pages aloud.

Rescale the number line on the board by repositioning the numbers $\frac{1}{4}$, $\frac{1}{2}$ ($\frac{2}{4}$), $\frac{3}{4}$, and 1 as in the next illustration. (2 is excluded from the part of the number line drawn.)

Ask someone to locate and label a dot for $\frac{1}{8}$.

Label some of the other dots in terms of eighths as long as there is student interest. It may help the students to consider $\frac{1}{2}$ first.



Pages 26–27

Read these pages aloud. With class assistance, draw and label dots for $\frac{9}{8}$ and $\frac{10}{8}$ on the number line.

Pages 28–31

Read these pages aloud. Ask questions such as the following:

T: ***What are some other names for $\frac{1}{2}$?***
What is a name for $\frac{1}{2}$ that is 2x some other number? (2 x $\frac{1}{4}$)
What is a name for $\frac{1}{2}$ that is 4x some number? (4 x $\frac{1}{8}$)
Do you remember a name for $\frac{1}{2}$ we used with money? (0.50 or 0.5)

Practice Activity

With your class, count by fourths, for example, $\frac{1}{4}$, $\frac{2}{4}$ ($\frac{1}{2}$), $\frac{3}{4}$, $\frac{4}{4}$ (1), $\frac{5}{4}$, $\frac{6}{4}$ ($1\frac{1}{2}$), …, and by eighths.

[†] 0 = 0 ÷ 2, but 0 ≠ 2 ÷ 0.

L11 MULTIPLICATION MODULO 10 #1

Capsule Lesson Summary

Review the addition with ten friends operation, ⊕, introduced in *Dancing Friends,* and introduce a multiplication operation, ⊗, for the ten friends. Draw an arrow picture for ⊗2 with all ten numbers —0, 1, 2, 3, 4, 5, 6, 7, 8, and 9—in it.

Materials

Teacher
- Colored chalk

Student
- Paper
- Worksheets L11* and **

Description of Lesson

Exercise 1

During this exercise you may prefer to ask students to copy and solve the problems on their papers, before doing a class check. List the whole numbers from 0 to 9 on the board. Keep the list on the board for quick reference during the lesson.

0 1 2 3 4 5 6 7 8 9

Briefly review the operation ⊕ (read as "circle-plus") invented by the number 0 in the story *Dancing Friends*. Write these problems on the board and do them collectively; answers are in boxes.

$9 \oplus 9 = \boxed{8}$	$7 \oplus 5 = \boxed{2}$	$6 \oplus 6 = \boxed{2}$
$3 \oplus 6 = \boxed{9}$	$4 \oplus 7 = \boxed{1}$	$5 \oplus 8 = \boxed{3}$

Ask which numbers can be put in the frames in each of the following number sentences. A table of solutions for the problem on the right is given below it. (The □ and the △ numbers can, of course, be reversed.)

$\boxed{7} \oplus 3 = 0$

$8 \oplus \boxed{4} = 2$

$\boxed{9} \oplus 5 = 4$

$\square \oplus \triangle = 1$

□	△
1	0
2	9
3	8
4	7
5	6

T: ***The ten number friends 0 to 9 did ⊕ dances. Do you think they could invent a way to multiply, say ⊗, and do ⊗ dances?***

What number do you think 2 ⊗ 4 (read as "two circle-times four") ***is?***

S: ***8.***

T: ***What number do you think 3 ⊗ 4 is?***

S: ***2.***

T: ***How did you get 2?***

S: ***3 x 4 = 12, and we keep only the ones digit.***

Make sure the class understands this last example and then continue with other number sentences. Answers are in boxes.

$2 \otimes 4 = \boxed{8}$	$4 \otimes 5 = \boxed{0}$	$5 \otimes 3 = \boxed{5}$
$3 \otimes 4 = \boxed{2}$	$2 \otimes 9 = \boxed{8}$	$4 \otimes 7 = \boxed{8}$
$4 \otimes 4 = \boxed{6}$	$3 \otimes 7 = \boxed{1}$	$6 \otimes 4 = \boxed{4}$

Write this open sentence on the board.

$$3 \otimes \square = 4$$

T: ***3 ⊗ one of these numbers*** (point to the list 0 to 9) ***is 4. Which number could be in the box?***

S: ***8, because 3 x 8 = 24 and 4 is in the ones place.***

$$3 \otimes \boxed{8} = 4$$

T: ***If I change this number (4) to 7, which number could be in the box?***

S: ***9, because 3 x 9 = 27 which ends in 7.***

$$3 \otimes \boxed{9} = 7$$

T: ***If I change this number (7) to 1, which number could be in the box?***

S: ***7, because 3 x 7 = 21 which ends in 1.***

$$3 \otimes \boxed{7} = 1$$

Write this open sentence on the board.

$$\square \otimes 2 = 8$$

T: ***Which number could be in the box?***

S: ***4, because 4 x 2 = 8.***

S: ***9, because 9 x 2 = 18 which ends in 8.***

T: ***If I change this number (8) to 6,***
which number could be in the box?

$$\square \otimes 2 = 6$$

S: ***3, because 3 x 2 = 6.***

S: ***8, because 8 x 2 = 16 which ends in 6.***

Write this open sentence on the board.

$$5 \otimes \square = 0$$

T: ***Which number could be in the box?***

Students should suggest that 0, 2, 4, 6, or 8 could be in the box. Observe that these are all of the even numbers in the list 0 to 9.

T: ***When we take 5 ⊗ some number, we can get 0 or we can get…?***

S: ***5.***

Change the open sentence to reflect this possibility.

T: *Now which number could be in the box?*

$$5 \otimes \square = 5$$

Students should suggest that 1, 3, 5, 7, or 9 could be in the box. Observe that these are all the odd numbers in the list 0 to 9.

Write this open sentence on the board.

$$\square \otimes \triangle = 4$$

T: ***The number in the box may be different from or the same as the number in the triangle.***

Students should be able to find all the solutions. List them in a chart as they are suggested. (The $\square$ and the $\triangle$ numbers can, of course, be reversed.)

1	4
2	2
2	7
3	8
4	6
6	9
8	8

Exercise 2

Erase the board and then write $\otimes 2$ in a color.

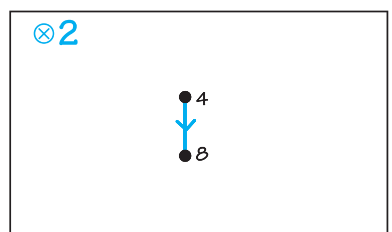
T: ***Let's put the ten number friends in a ⊗2 arrow picture.***
What number would you like to start with?
Remember, the ten dancing friends are the numbers 0 to 9.

The following is a sample dialogue, starting with the number 4. The illustrations on the right show how the picture on the board progressively develops.

S: *4.*

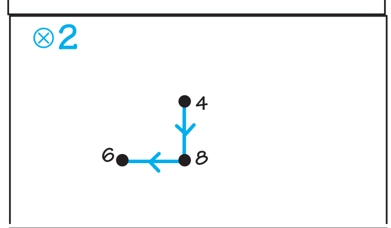
T: *4 ⊗ 2 = …?*

S: *8.*



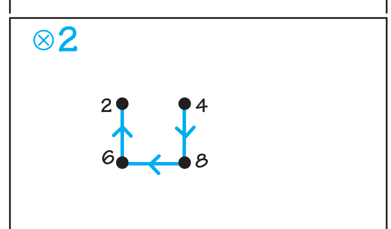
T: *8 ⊗ 2 = …?*

S: *6.*



T: *6 ⊗ 2 = …?*

S: *2.*

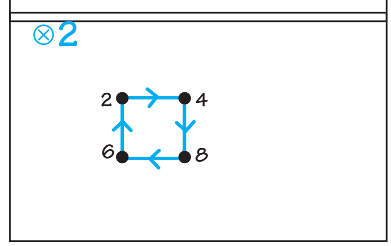


T: *2 ⊗ 2 = …?*

S: *4.*

T: ***We already have a dot for 4.***

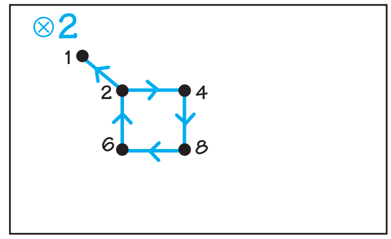
T: ***Where should we put 1 in this picture?***



S: ***Near 2.***

T: ***Why there?***

S: ***Because 1 ⊗ 2 = 2.***

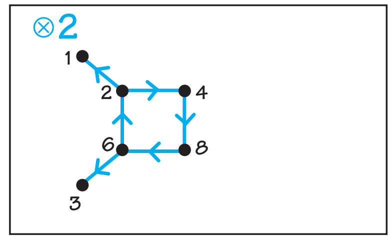


T: ***What number friends are we missing?***

S: ***3.***

T: ***Where should we put 3?***

S: ***Near 6, because 2 ⊗ 3 = 6.***

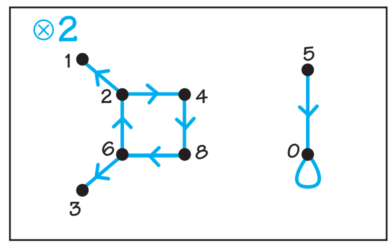


T: ***Where should we put 5?***

S: ***Near 0, because 5 ⊗ 2 = 0.***

S: ***0 is not in the picture yet.***

S: ***Draw a loop at 0 because 0 ⊗ 2 = 0.***



T: ***What number friends are missing?***

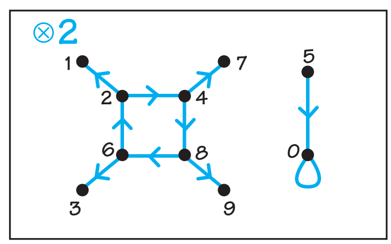
S: ***7 and 9.***

T: ***Where should we put 7?***

S: ***Near 4, because 7 ⊗ 2 = 4.***

T: ***Where should we put 9?***

S: ***Near 8, because 9 ⊗ 2 = 8.***



Worksheets L11* and ** are available for individual work.

Home Activity

Students may like to take Worksheets L11* and ** home and explain the new multiplication game that the numbers 0–9 are playing. If earlier you sent home a copy of the storybook *Dancing Friends*, remind parents/guardians about that story.

Name________________ L11 ★

Complete these number sentences about multiplication with ten number friends.

$1 \otimes 6 = \boxed{6}$	$9 \otimes 1 = \boxed{9}$
$2 \otimes 6 = \boxed{2}$	$2 \otimes 8 = \boxed{6}$
$3 \otimes 6 = \boxed{8}$	$3 \otimes 8 = \boxed{4}$
$4 \otimes 6 = \boxed{4}$	$5 \otimes 7 = \boxed{5}$
$5 \otimes 6 = \boxed{0}$	$4 \otimes 9 = \boxed{6}$
$6 \otimes 6 = \boxed{6}$	$7 \otimes 3 = \boxed{1}$

Name________________ L11 ★★

Find several solutions to this number sentence.
One is done for you.

$\square \otimes \triangle = 2$

1	2	$1 \otimes 2 = 2$
3	4	$3 \otimes 4 = 2$
6	2	$6 \otimes 2 = 2$
7	6	$7 \otimes 6 = 2$
4	8	$4 \otimes 8 = 2$
8	9	$8 \otimes 9 = 2$

The $\square$ and $\triangle$ numbers can be reversed.

L12 MULTIPLICATION MODULO 10 #2

Capsule Lesson Summary

Review the story of *Dancing Friends* and the operations ⊕ and ⊗. Also recreate the ⊗2 arrow picture with the ten numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Draw an arrow picture for ⊗3. Provide individual work on other similar arrow pictures.

Materials

Teacher
- Colored chalk

Student
- Worksheets L12* and **
- Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

Begin the lesson by briefly recalling the story of *Dancing Friends*.

T: ***How many and which number friends were invited to play in the dancing games?***

S: ***Ten friends: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.***

List the whole numbers from 0 to 9 on the board.

0 1 2 3 4 5 6 7 8 9

Review the operation ⊕ by doing these problems as a collective activity. (Answers are in the boxes.)

$5 \oplus 3 = \boxed{8}$ $\quad 5 \oplus 8 = \boxed{3}$ $\quad \boxed{7} \oplus 4 = 1$

$7 \oplus 5 = \boxed{2}$ $\quad 6 \oplus \boxed{9} = 5$ $\quad 1 \oplus \boxed{9} = 0$

Continue by asking for pairs of numbers to go in the box and the triangle of this open sentence. A table of possible answers is given here. (Also, the □ and △ numbers can be reversed.)

□ ⊕ △ = 6

□	△
0	6
1	5
2	4
3	3
7	9
8	8

Review the operation ⊗ by doing these problems as a collective activity. (Answers are in the boxes.)

$3 \otimes 4 = \boxed{2}$ $\quad 3 \otimes 4 = \boxed{2}$ $\quad$ (Answers: 3 and 8)

$5 \otimes 4 = \boxed{0}$ $\quad 5 \otimes 4 = \boxed{0}$ $\quad$ (Answers: 1 and 6)

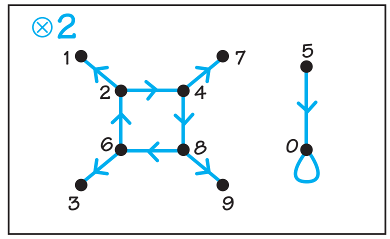
$4 \otimes 7 = \boxed{8}$ $\quad 4 \otimes 7 = \boxed{8}$

Continue by asking for pairs of numbers to go in the box and the triangle of this open sentence. A table of possible answers is given here. (Also, the □ and △ numbers can be reversed.)

□	⊕ △ = 0
0	1, □, 9
2	5
4	5
6	5
9	5

Exercise 2

T: ***Do ~~you remember how~~ we put the ten number friends in a ⊗2 picture?***



With students help, recreate the ⊗2 picture from Lesson L11.

Keep this picture on the board for the remainder of the lesson.

T: ***Let's put the ten number friends in a ⊗3 picture. What number would you like to put in first?***

The following is a sample dialogue starting with the number 5. The illustrations on the right show how the picture on the board progressively develops.

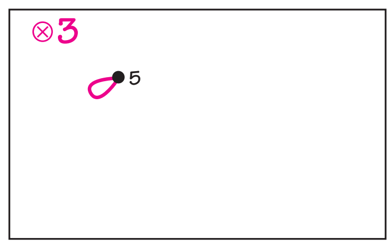
Note: Careful location of the numbers in the picture will yield a more attractive final drawing (see the last illustration in the sequence).

S: ***5.***

T: ***5 ⊗ 3 = ?***

S: ***5.***

S: ***There's a loop at 5.***

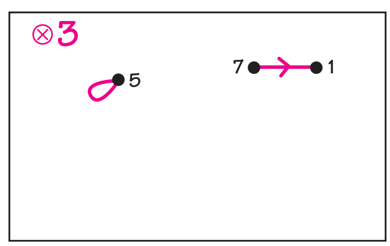


T: ***What number would you like to put in the picture next?***

S: ***7.***

T: ***7 ⊗ 3 = …?***

S: ***1.***



T: ***1 ⊗ 3 = …?***

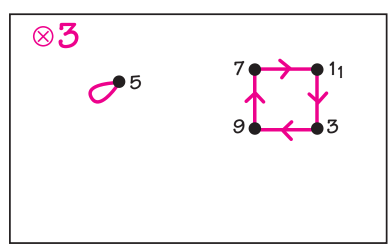
S: ***3.***

T: ***3 ⊗ 3 = ?***

S: ***9.***

T: ***9 ⊗ 3 = …?***

S: ***7.***



T: ***We have all the odd numbers in our picture; let's put in an even number next.***

S: ***2.***

T: *2 ⊗ 3 = …?*

S: *6.*

T: *6 ⊗ 3 = …?*

S: *8.*

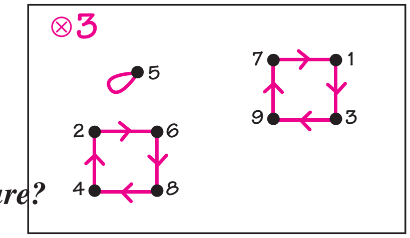
T: *8 ⊗ 3 = …?*

S: *4.*

T: *4 ⊗ 3 = …?*

S: *2.*

T: *Are any of the ten number friends missing from our picture?*

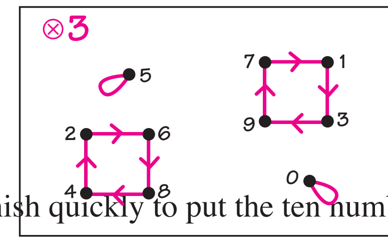


S: *0.*

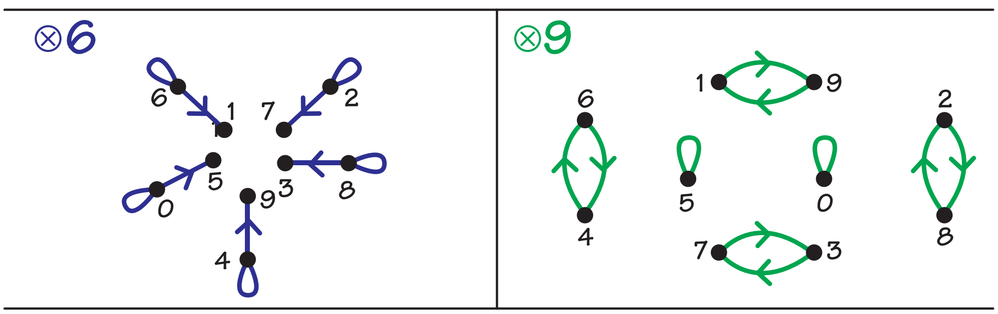
T: *0 ⊗ 3 = …?*

S: *0.*

S: *There's a loop at 0.*



Distribute copies of Worksheets L12* and **. Ask students who finish quickly to put the ten number friends in a ⊗6 picture and in a ⊗9 picture.

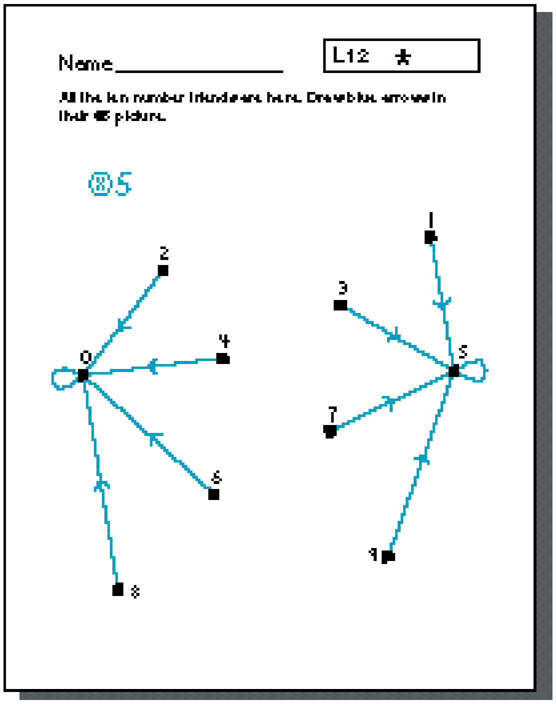


Extension Activity

Invite students to prepare class posters or a booklet with all the different ⊗ arrow pictures.

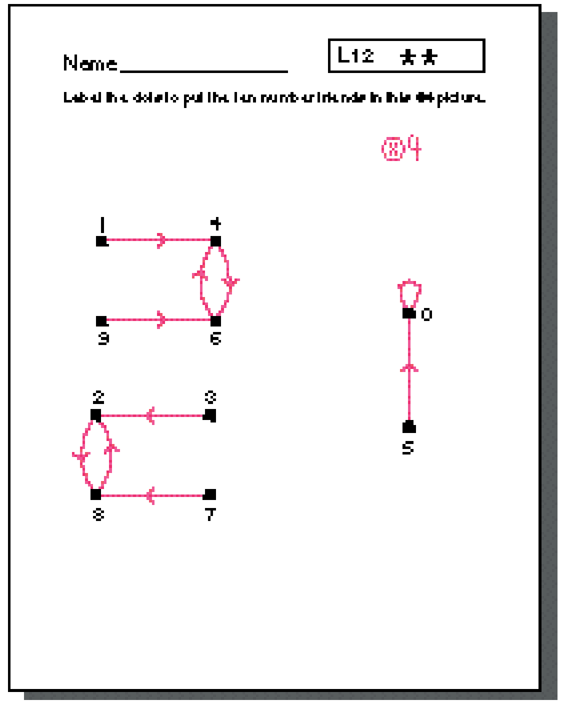
Writing Activity

Suggest students write to a friend describing one of the ⊗ arrow pictures and how to construct it.


Name________________

L12 ★

All the ten number friends are here. Draw blue arrows in their ⊗5 picture.


Name________________

L12 ★★

Label the dots to put the ten number friends in this ⊗4 picture.

L13 BASKETBALL DATA

Capsule Lesson Summary

Introduce stem-and-leaf charts as a way to record the season record data for a basketball team. These charts emphasize place value and display data in a concise format.

Materials

Teacher
- Season record for a basketball team

Student
- Basketball data
- Paper

Advance Preparation: Before teaching this lesson, obtain the season record for a local basketball team. You may do this by checking the newspaper regularly; or perhaps you can ask for the information from the athletic department of a local high school, college, or professional team. Alternatively, use the season record for St. Louis University on Blackline L13.

Description of Lesson

This lesson would best fit in the schedule shortly after the basketball season is over. Display the season record for a team of your choice. This lesson is presented using the record on Blackline L13. Arrange that students have a copy of the record.

T: ***What can you tell me about the basketball season for St. Louis University?***

S: ***They lost a lot.***

S: ***They usually scored in the 60's.***

Exercise 1

T: ***We are going to make some special charts for this basketball team's season record. First we'll just look at what our team scored each game.***

Draw this picture on the board and refer to the first game in the record.

Tens	Ones
4	
5	
6	
7	
8	3
9	

T: ***What did our team (St. Louis University) score in its first game?*** (83) ***I'll put 83 on the chart as 8 tens and 3 ones.***

In a stem-and-leaf chart, you only put the stem number in one time, in this case 8, for 8 tens. Then, the next time the team scores in the 80's (for example, 86) you already have the stem (8) and you add another leaf (in this case, 6). Call on students, one at a time, to record the score for one game of your team or St. Louis University. You may need to help the first few students until the idea of the chart becomes clearer. A complete stem-and-leaf chart for the season record of St. Louis University is shown here. A chart for your team may need to have an extended stem (tens column) to include scores in the 100's.

When the chart is complete, ask the class some questions that are easily answered by looking at the stem-and-leaf chart. For example,

Tens	Ones
4	9
5	5 4 8 9 7
6	8 1 6 5 8 5 1 2
7	4 8 9 5 5
8	3 8 0 3 1 5 3 2
9	

T: ***What was the least number of points the team scored in a game this season? What was the greatest number?***
How many times did the team score in the 60's?
How many times did the team score more than 75 points?
Where are most of the scores?What score did they have most often?

You may like to rearrange the data to better see the range and distribution of scores.

Tens	Ones
4	9
5	4 5 7 8 9
6	1 1 2 5 5 6 8 8
7	4 5 5 8 9
8	0 1 2 3 3 3 5 8
9	

Exercise 2

T: ***Now let's look at when our team won and when they lost. In a chart we'll show by how much they won or lost.***

Draw this picture on the board and refer to the first game in the record.

Lost By		Won By
7	0	
	1	
	2	

T: ***Did our team win or lose the first game?*** (Lost)
By how much? (7)

Point out to the class that on one side of this chart you record "won by" amounts and on the other side you record "lost by" amounts. Again, the stem is for tens.

Direct students to complete the Won By or Lost By column of the season record. Then call on students, one at a time, to record the results in the chart. Again, you may need to help the first few students until the idea of this double stem-and-leaf chart becomes clearer. A complete chart for the season record of St. Louis University is shown here.

When the chart is complete, direct the class to make some observations about the data.

Lost By		Won By
2 2 4 1 8 2 8 1 7	0	3 1 4 2
0 8 3 9 3 1 6	1	2
2 0 3	2	0 0
5	3	

T: ***What kind of a season did St. Louis University have?***
How many wins? How many losses?
Did they have many close games?
What was their best win?
What was their worst loss?
How many times did they lose by less than 10 poi

Again, you may like to rearrange the data to better view i

Lost By		Won By
8 8 7 4 2 2 2 1 1	0	1 2 3 4
9 8 6 3 3 1 0	1	2
3 2 0	2	0 0
5	3	

Additional Practice

Select some other data that would be interesting to view in a stem-and-leaf chart. For example, invite students to record the day of the month of their birthdays.

L14 TALKATIVE NUMBERS #1

Capsule Lesson Summary

Tell a story to introduce a multiples relation: two whole numbers are related if and only if one number is a multiple of the other. Construct pictures involving this relation and label the dots.

Materials

Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons
- Worksheets L14* and **

Description of Lesson

Start the lesson by telling this story to your class. If necessary, recall that the whole numbers are 0, 1, 2, 3, 4, and so on.

T: ***In the world of whole numbers, it has been raining continuously for an entire month. At the school, the students (numbers) are not able to play outside because of the rain so they are getting noisier and noisier. Each day the talking at the school is getting worse.***

The number 0 is the school principal and the number 1 is the assistant principal. They have a meeting to try to figure out what to do to make the situation better. They decide upon a rule and then call an assembly of the entire school. 0 tells the numbers that a rule is going to be imposed during this rainy period. "From now on," 0 announces, "two numbers may talk to each other if and only if one of them is a multiple of the other."

Write the rule on the board.

Two numbers may talk to each other
if and only if
one of them is a multiple of the other.

T: ***Do you think this rule will restrict the talking? Are there numbers who can no longer talk to each other?***

Allow students to suggest several examples of number pairs who may not talk to each other according to the rule; for example, 7 and 8; or 10 and 15; or 4 and 6.

T: ***The numbers are not sure that they understand this rule.***

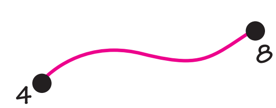
Do you think that very many numbers may talk to one another with this rule in effect? Can you think of an example of two numbers who may talk to one another?

Suppose a student suggests 4 and 8.

T: ***Why can 4 and 8 talk to one another?***

S: ***Because 8 is a multiple of 4.***

Draw dots for 4 and 8 and connect them with a red cord.



T: ***8 can talk to 4 and 4 can talk to 8. Instead of drawing two red arrows, I'll draw one red cord between them.***

Suppose a student suggests an incorrect pair of numbers, for example 9 and 12

T: ***Is 9 a multiple of 12?***

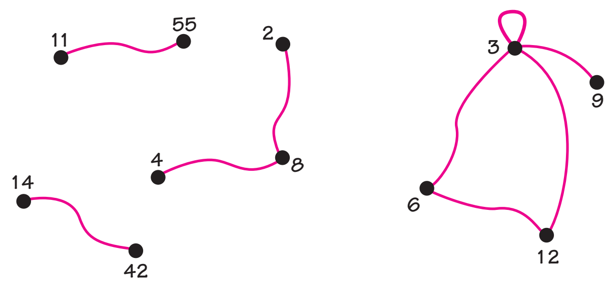
S: ***No.***

T: ***Is 12 a multiple of 9?***

S: ***No.***

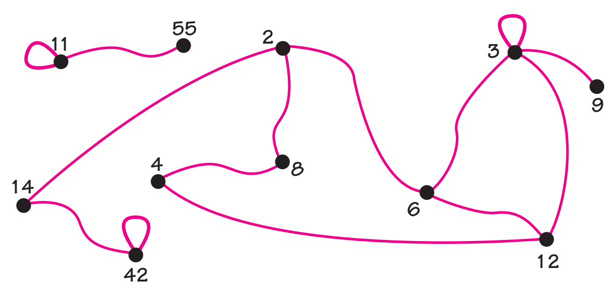
T: ***So 9 and 12 may not talk to one another.***

Continue until several pairs of numbers have been suggested. Your class may develop an illustration similar to the one below.



T: ***Looking at the picture, do you see where we could draw other cords between these numbers?***

Very likely there will be many cords that could be added to your picture; accept a few suggestions such as in the illustration below. Students may observe that there could be a loop at every dot.



Erase the board.

T: ***Since 0 and 1 made up this rule, do you think they will follow the rule? What will happen if they do?***

S: ***0 can talk with every whole number, because 0 is a multiple of every whole number.***

S: ***The multiples of 1 are 0, 1, 2, 3, 4, and so on. So 1 can talk with all the whole numbers.***

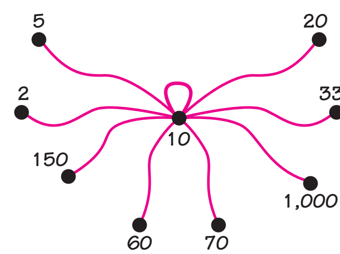
Do not expect your students to make these observations immediately; it is likely they will need to check to see that 0 and 1 can both talk to many numbers.

T: ***Let's talk about one of the students, say the number 10.***

Draw a dot on the board and label it 10.

Ask students for some of the numbers that 10 may talk to and record them in the picture. If no one suggests numbers less than 10, ask the students whether or not 10 may talk to any whole numbers less than itself. (Yes, 10 may talk to 0, 1, 2, and 5.) Do not include 0 and 1 in your picture, but note that we know 10 can talk with 0 and 1 because every whole number can.

Your picture might look similar to this one.



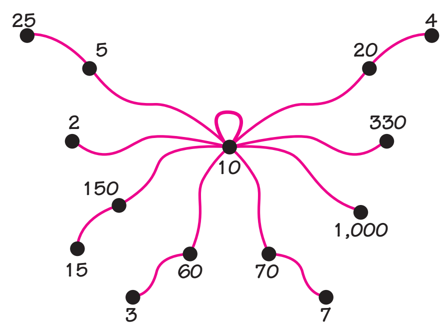
Point to one of the numbers in the picture other than 10; for example, 5.

T: ***Can you think of a number 5 may talk to that is not already in our picture?***

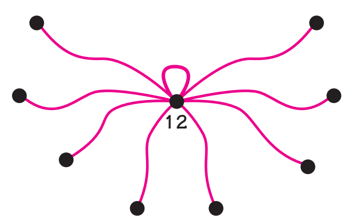
S: ***25.***

Draw a dot for the number suggested and the appropriate cord.

In a similar manner extend your picture several times as illustrated here.



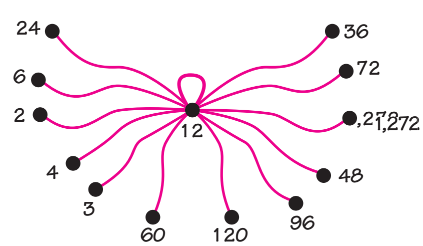
Erase the board and then draw this picture.



T: ***Draw a picture on your paper showing some numbers 12 may talk with. You do not need to include 0 and 1, because we know every whole number can talk to 0 and 1.***

As students are working, walk around the room and observe which numbers they are putting in their pictures. Ask students, one at a time, to put particular numbers in the picture on the board; choose a wide range of numbers from their papers. This illustration is an example of numbers that could be put in the picture.

Check that each number in the picture has permission to talk with 12; for example:



T: ***Why may 12 and 6 talk to one another?***

S: ***12 is a multiple of 6.***

T: ***Why may 12 and 24 talk to one another?***

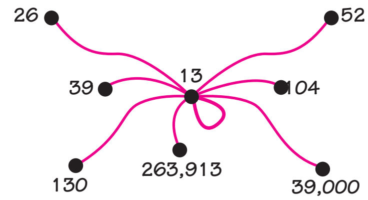
S: ***24 is a multiple of 12.***

Worksheets L14* and ** are available for individual work.

Extension Activity

L14

Give students the problem of finding many numbers, other than 0 and 1, that 13 may talk with, as illustrated here.



Observe that 13 has only student friends greater than itself to talk with, whereas 12 had friends both less and greater than itself to talk with.

Look for other numbers that, like 13, can only talk with student friends greater than themselves. (Prime numbers)

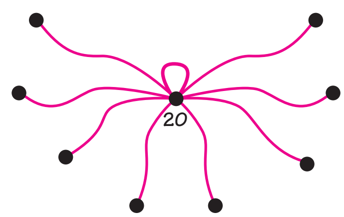
Look for other numbers that, like 12, have friends both less and greater than themselves to talk with. (Composite numbers)

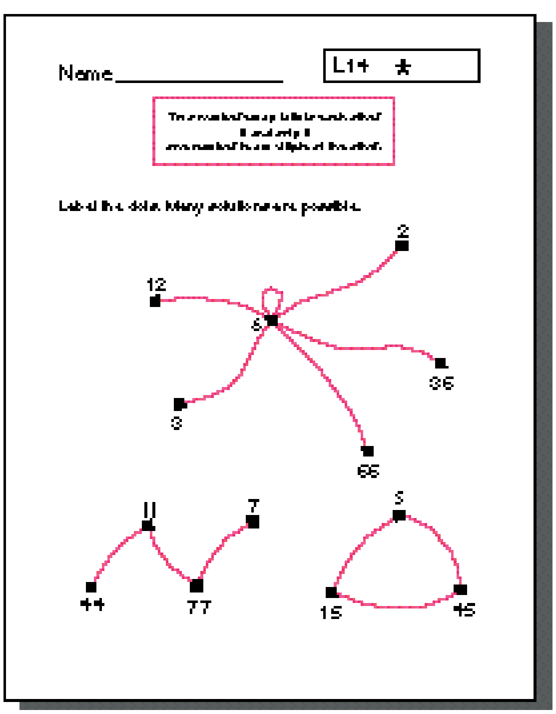
Writing Activity

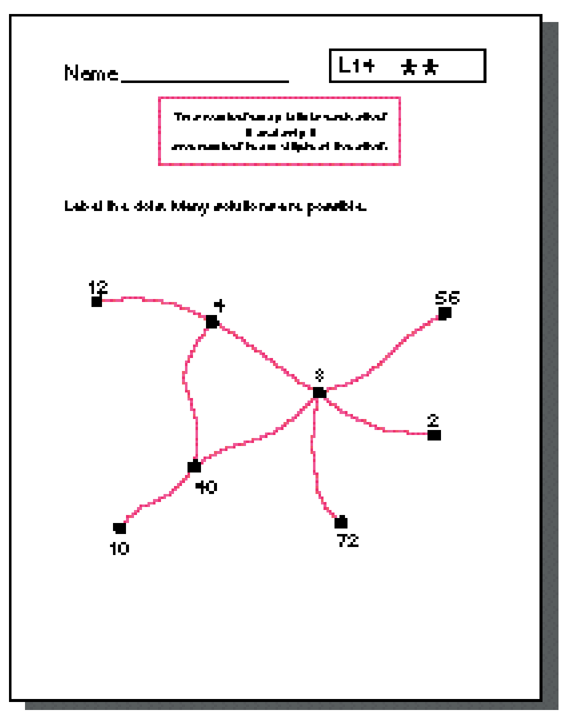
Suggest that students write an article about what happens in the whole numbers' school when this rule is in effect.

Home Activity

Send home a problem like one of those in the lesson for students to do with a family member. For example, using the talkative numbers rule, students and family members can put numbers in the picture illustrated here.







L15 TALKATIVE NUMBERS #2

Capsule Lesson Summary

Review the story from Lesson L14 and the multiples relation. Create pictures and label dots in pictures involving this relation. Find numbers that can be in given pictures about the story.

Materials

Teacher
- Colored chalk

Student
- Paper
- Colored pencils, pens, or crayons

Description of Lesson

Exercise 1

Briefly review the story from Lesson L14.

T: ***Do you remember the story I told you about a school in the world of whole numbers?***

S: ***0 is the principal and 1 is the assistant principal.***

S: ***It was raining for a long time, and the numbers were getting too noisy.***

S: ***0 and 1 made a rule that two numbers may talk to one another if and only if one of them is a multiple of the other.***

Write the rule on the board.

Two numbers may talk to each other
if and only if
one of them is a multiple of the other.

T: ***Can you name two numbers that may talk to one another?***

S: ***5 and 10; 10 is a multiple of 5.***

S: ***60 and 20. 60 is a multiple of 20.***

S: ***0 can talk to every whole number.***

T: ***Why is that?***

S: ***0 is a multiple of every number.***

S: ***1 can talk to every whole number, because every whole number is a multiple of 1.***

Continue until 10 to 12 examples have been suggested.

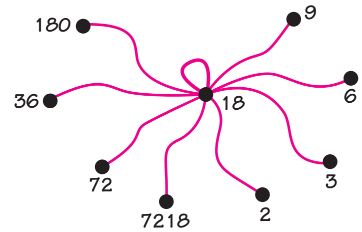
T: ***How did we show in a picture that two numbers may talk to each other?***

S: ***We drew a red line (cord) between their dots.***

T: ***A red cord is like an arrow that goes in both directions.***

L15

Draw a dot for 18. Then ask students for some of the numbers that 18 may talk with and record them in the picture. Students should include numbers greater than, less than, and equal to 18. This picture shows some of the many possibilities.



Point to one of the numbers in the picture other than 18 (for example, 6), and draw two cords starting at the dot for that number.

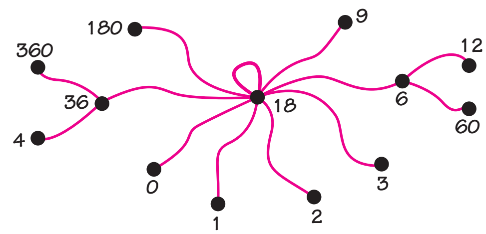
T: ***What are some numbers 6 may talk to that are not already in our picture?***

S: ***12.***

S: ***60.***

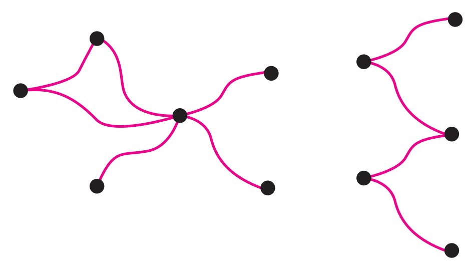
Note: It is possible that 6 may talk to other numbers already in the picture. If one of them is suggested, simply draw the cord and ask your question again.

Label the dots being considered with the numbers suggested. Repeat this activity using one of the other numbers in the picture. For example:



Exercise 2

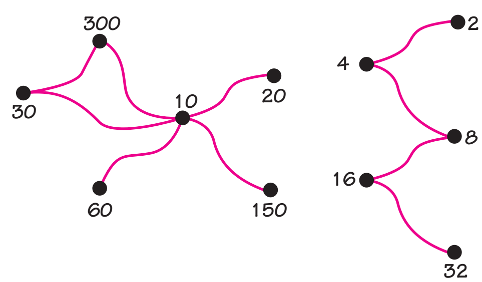
Erase the board and then draw this picture.



T: ***This picture shows some of the cords that can be drawn between 11 numbers. What numbers could the dots be for?***

Invite students to label dots in the picture, and check that whenever two numbers are connected by a red cord, the numbers may talk according to the rule.

The illustration here shows one way to label the dots. There are, of course, many possibilities.



Exercise 3

Draw two dots on the board, and label them 12 and 18.

T: ***The numbers 12 and 18 are good friends, but under this new rule 12 and 18 may not talk to one another. What do you suppose they will try to do?***

S: ***Find another number that they both may talk with.***

T: ***Yes, they have the idea that they can relay messages through such a common friend. Can you help them find such numbers?***

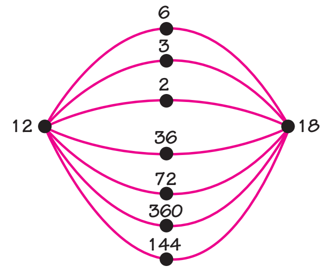
S: ***6.***

S: ***3.***

S: ***0.***

S: ***1.***

T: ***They decide to look only for student friends, and not to relay messages through the principal (0) or the assistant principal (1).***

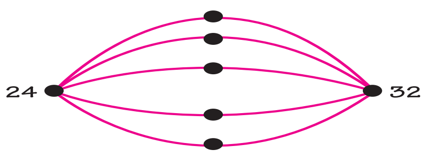


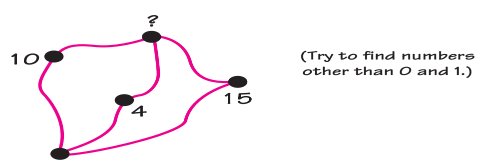
As students suggest numbers, record them in the picture.

As an individual or cooperative group activity, ask students

- to find some numbers 24 and 32 may both talk with; and
- to find some numbers 4, 10, and 15 may all talk with.

Indicate the preceding problems by drawing these pictures on the board. Ask students to copy the pictures, label the dots, and extend the pictures if they can show more possibilities.

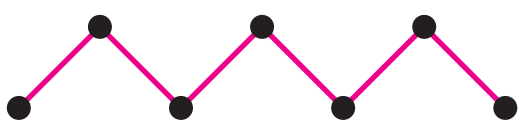




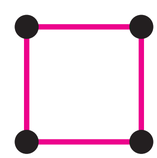
Solutions: Whole numbers that 24 and 32 may talk with are 2, 4, 8; and 96, 192, …, 96 x n, ….
Whole numbers that 4, 10, and 15 may talk with are 60, 120, 180, …, 60 x n ….

Extension Activity

Present problems such as the following:



Find seven numbers that may talk in this way, but so that no additional talking can take place.



Find four numbers that may talk in this way, but so that the diagonal numbers are not allowed to talk.

Home Activity

Using the talkative numbers rule, pose one or two problems such as the following for students to work on with a family member.

- Find some numbers that 11 (or 20) may talk with.
- Find numbers other than 0 and 1 that both 4 and 6 may talk with.

L16 WHO IS MAX?

Capsule Lesson Summary

Solve a detective story about a secret number called Max. One clue involves finding all the numbers that can be named using, in order, one symbol from each of three given sets. A second clue involves putting numbers on the Minicomputer by moving exactly one checker of a given configuration on the Minicomputer. The final clue shows Max's location in a three-string picture. Play a game with strings for multiples of 10 and multiples of 4.

Materials

Teacher
- Minicomputer set
- Colored chalk

Student
- Paper

Description of Lesson

Exercise 1

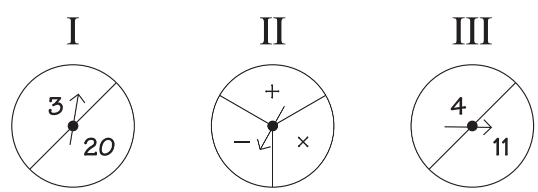
You may like to let students work in pairs or groups of four during this exercise.

T: ***I have a secret number that I'll call Max.***
You will be detectives and try to discover Max.

Clue 1

Briefly recall the game from Lesson L7 *Three Spinners*. Draw three imaginary spinners on the board as you describe them.

T: ***Pretend that I have three spinners.***
Spinner I is equally divided with the numbers 3 and 20.
Spinner II is equally divided with the operations +, −, and x.
Spinner III is equally divided with the numbers 4 and 11.



You can get a name for Max by spinning the three spinners and writing the results in order. What are some numbers Max could be?

Instruct students to write possibilities on their papers. When most students have found several possibilities, record a few on the board.

S: ***12, because 3 x 4 = 12.***

S: ***14; 3 + 11 = 14.***

S: ***80; 20 x 4 = 80.***

T: ***How many numbers for Max do you think we could get in this way?***

Let students predict how many and, if you like, record a few predictions on the board.

T: ***How can we find all the possibilities for Max? Do you remember what we did before?***

If no one suggests making tables for each of the three operations (+, –, and x) suggest this yourself. Begin an addition table on the board.

T: ***This table is just for addition.***
I'll put the numbers from the first spinner here
(put 3 and 20 along the left side) ***…***

…and the numbers from the third spinner here
(put 4 and 11 across the top)**.**

+	4	11
3		
20		

Invite students to complete the table on the board.

T: ***So far we have four possibilities for Max.***
What other tables do we need?

S: ***One for – and one for* x.**

Draw tables for each of these operations and invite students to complete them. At some point, students should realize that this clue gives 12 possibilities for Max. Compare 12 to the predictions made earlier.

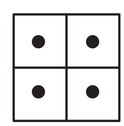
+	4	11
3	7	14
20	24	31

–	4	11
5	$\hat{1}$	$\hat{8}$
10	16	9

×	4	11
3	12	33
20	80	220

Clue 2

Display one Minicomputer board and put on this configuration.



T: ***What number is on the Minicomputer?***

S: ***15.***

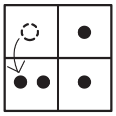
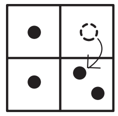
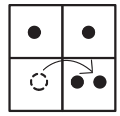
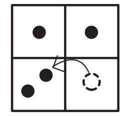
T: ***Could Max be 15?***

S: ***No, 15 is not one of the numbers in the tables.***

T: ***You can get Max on the Minicomputer by moving exactly one of these checkers. What numbers could Max be?*** (Point to the numbers in the three tables.)

Whenever a student suggests a number from the tables, ask the student to convince the class that the number is a possibility for Max by moving one of the checkers in the configuration on the Minicomputer.

Circle numbers in the tables as they are offered and checked.

 = 9  = 12  = 14  = 16

T: ***What is the greatest number we can get by moving one of these checkers?***

S: ***22.***

Ask a stu
is on the

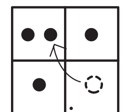
+	4	11
3	7	14
20	24	31

−	4	11
5	$\hat{1}$	$\hat{8}$
10	16	9

×	4	11
3	12	33
20	80	220

T (pointing to the multiplication table)**:** ***So which of these numbers do we need to check?***

S: ***12.***



Invite a student to show how to get 12 on the Minicomputer by moving one checker.

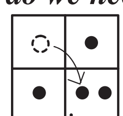
T: ***What is the least number we can get by moving exactly one of these checkers?***

S: **8.**

Ask a student to move one of the checkers so that 8
is on the Minicomputer.

T (pointing to the addition table)**:** ***So which of these numbers do we need to check?***

S: ***14.***



Invite a student to show how to get 14 on the Minicomputer by moving one checker.

T (pointing to the subtraction table)**:** ***Which of these numbers do we need to check?***

S: ***9 and 16.***

Invite students to show how to get 9 and 16 by moving one checker.

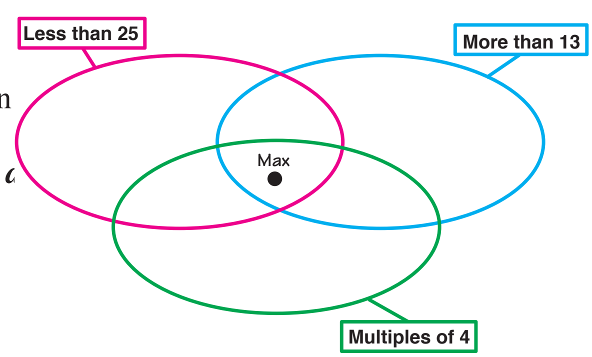
Clue 3

Draw this string picture on the board.

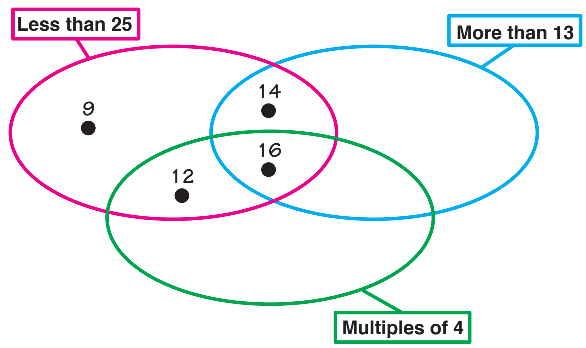
T: ***Max is one of these four numbers*** (point to the circled numbers in the tables)***, and I've shown you where Max is in this string picture. Which number is Max?***

Instruct students to write their answers on paper, and then

S: ***16 is less than 25; 16 is more than 13; and 16 is a***



Ask students to locate the other three numbers
(9, 12, and 14) in the picture.

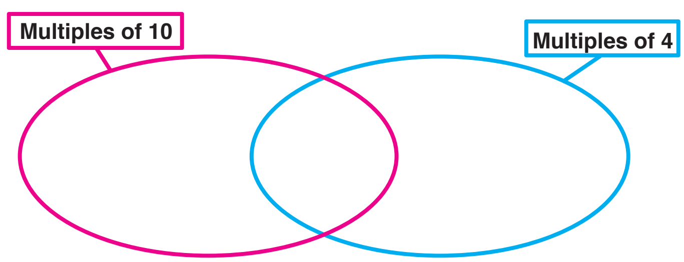


Exercise 2

Note: This exercise need not be done as a team game; it can be done as well as a cooperative class activity where students put all the numbers correctly in the string picture.

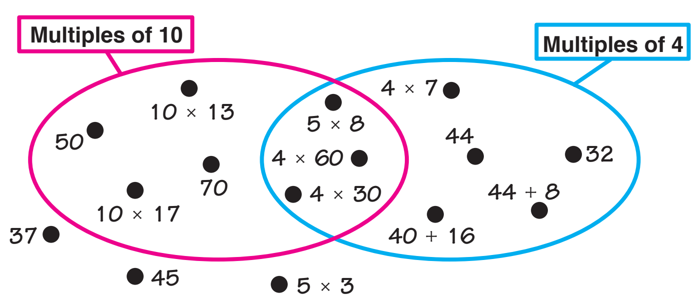
Erase the board and draw a string picture as shown below. Make two lists of numbers near the strings.

Team A	Team B
37	70
44 + 8	4 × 7
4 × 6	32
44	4 × 60
4 × 30	45
50	40 + 16
5 × 8	5 × 3
10 × 13	10 × 17



Organize the class to play a game placing the numbers in the picture. If a number is placed correctly, erase it from the list; if a number is placed incorrectly, erase it from the string picture. Continue until one team wins by placing all of its numbers in the picture.

Correct placement of all the numbers is given below for your convenience. You may wish to use this picture as a crib sheet during the play of the game.



Capsule Lesson Summary

Analyze the information obtained from three starting clues (correctly placed pieces) in a string game and find that one string is determined. Determine the other string from one additional clue. Play a three-string version of The String Game.

Optional: Play a three-string game involving multiples and order.

Materials

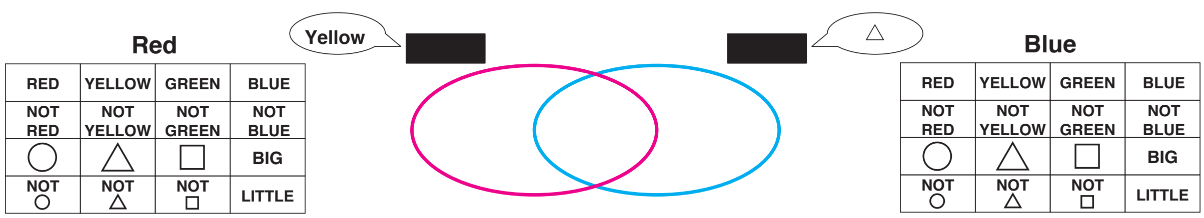
Teacher • A-Block String Game kit
• Colored chalk

Student • None

Description of Lesson

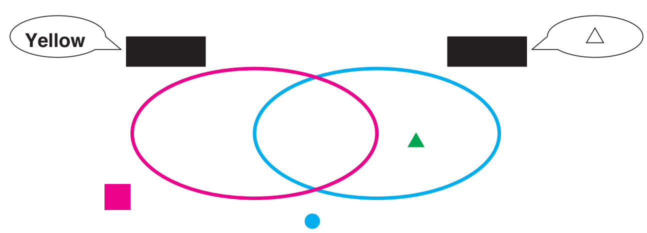
Exercise 1

Before the lesson begins, set up your board for The String Game as illustrated below. The bubbles show what is on the hidden tags. Tape two A-Block String Game Posters (Version B) to the board.



T: ***We are going to play The String Game today, but first let's look at what information we get from knowing where some of the A-block pieces belong in the picture.***

Put the little green triangle, the little blue circle, and the big red square in the picture.



T: ***What does this information tell us about the strings? Are there some labels*** (point to one of the lists) ***that the strings can not have?***

When explaining why a label can be crossed off, a student should identify which of the three pieces gives that information and why. If no one responds, choose a label on one of the lists and ask if the string could have that label. The following sample dialogue indicates how some labels are eliminated from the lists.

S: ***The red string cannot be for* GREEN*, because the little green triangle is outside the red string.***

T: ***Could the blue string be for* GREEN*?***

S: **Yes.**

Cross off **GREEN** from the Red list.

S: ***Neither string can be for* LITTLE*, because the little blue circle is outside both strings.***

Cross off **LITTLE** from both lists.

S: ***The blue string cannot be for* NOT △*, because there is a triangle in it.***

T: ***Could the red string be for* NOT △*?***

S: ***No, because the big red square and the little blue circle are outside of it, and they are not triangles.***

Cross off **NOT △** from both lists.

Continue in this way until the red string is determined (**YELLOW**) and there are only two remaining possibilities for the blue string (**GREEN** and △).

Red

~~RED~~	YELLOW	~~GREEN~~	~~BLUE~~
~~NOT RED~~	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	~~△~~	~~□~~	~~BIG~~
~~NOT ○~~	~~NOT △~~	~~NOT □~~	~~LITTLE~~

Blue

~~RED~~	~~YELLOW~~	~~GREEN~~	~~BLUE~~
~~NOT RED~~	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	△	~~□~~	~~BIG~~
~~NOT ○~~	~~NOT △~~	~~NOT □~~	~~LITTLE~~

Your students might be surprised that the red string is determined even though no piece is inside the red string as a clue.

T: ***The red string is for* YELLOW*, but we don't know yet whether the blue string is for* GREEN *or for* △*. I'll give you another clue.***

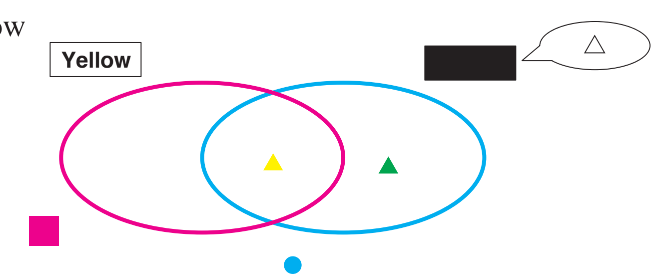
Hold up the little yellow triangle as you ask,

T: ***Do we know anything about where this piece belongs in the picture?***

S: ***It must be in the red string because it is yellow.***

S: ***It could be in the middle or in just the red string.***

If no one suggests this, ask whether the little yellow triangle belongs inside or outside the red string.



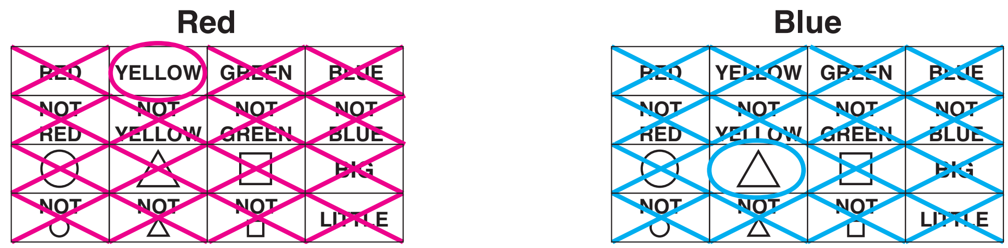
T: ***I'll show you where it belongs.***

Now do we know what the blue string is for?

S: *Triangles.*

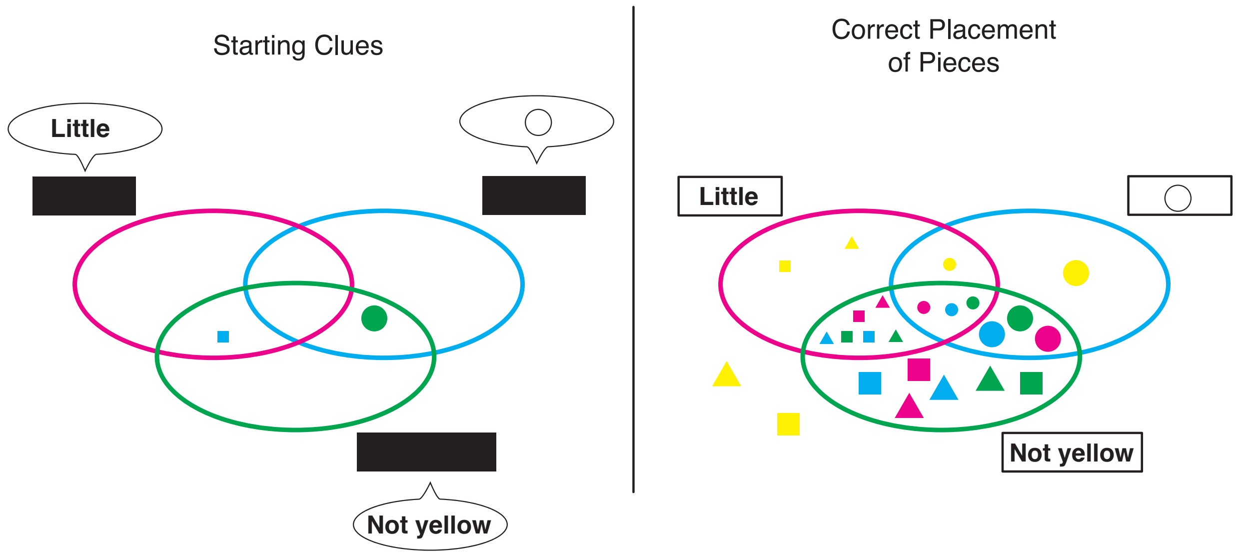
T: *Why can we cross off* **GREEN** *from the Blue list?*

S: *Because there is a yellow piece in the blue string.*



Exercise 2

Play The String Game with three strings in the usual way. A possible game is suggested below.

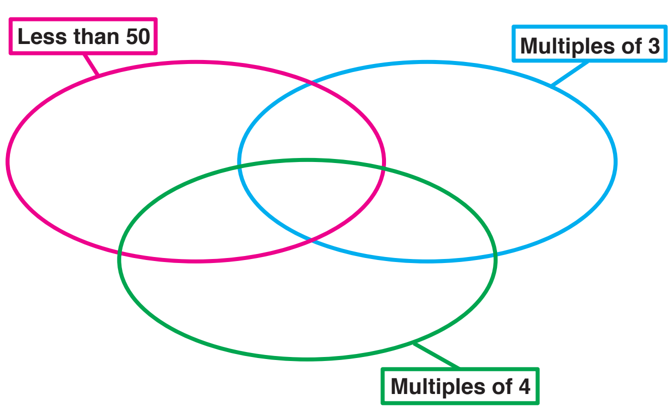


Exercise 3 (optional)

Do this exercise if there is sufficient time and student interest.

Draw this string picture and put these lists of numbers on the board. If you use more than two teams, divide the numbers accordingly.

Team A	Team B
$\widehat{12}$	4 × 15
3 × 25	$\widehat{16}$
49	3 × 23
$\widehat{9}$	$\widehat{25}$
4 × 7	36
24	50
52	51
3 × 32	40 × 7
5 × 11	7 × 10



T: ***We are going to play a game with numbers. During a turn, you will have only one chance to put a number in the string picture. The first team to put all its numbers where they belong wins.***

Alternating teams, let students place numbers in the picture. If a number is placed correctly, leave the number in the string picture and erase it from the list in which it occurs; if placed incorrectly, erase it from the picture but leave it on the list.

The next illustration shows the correct placement of all the numbers in both lists.