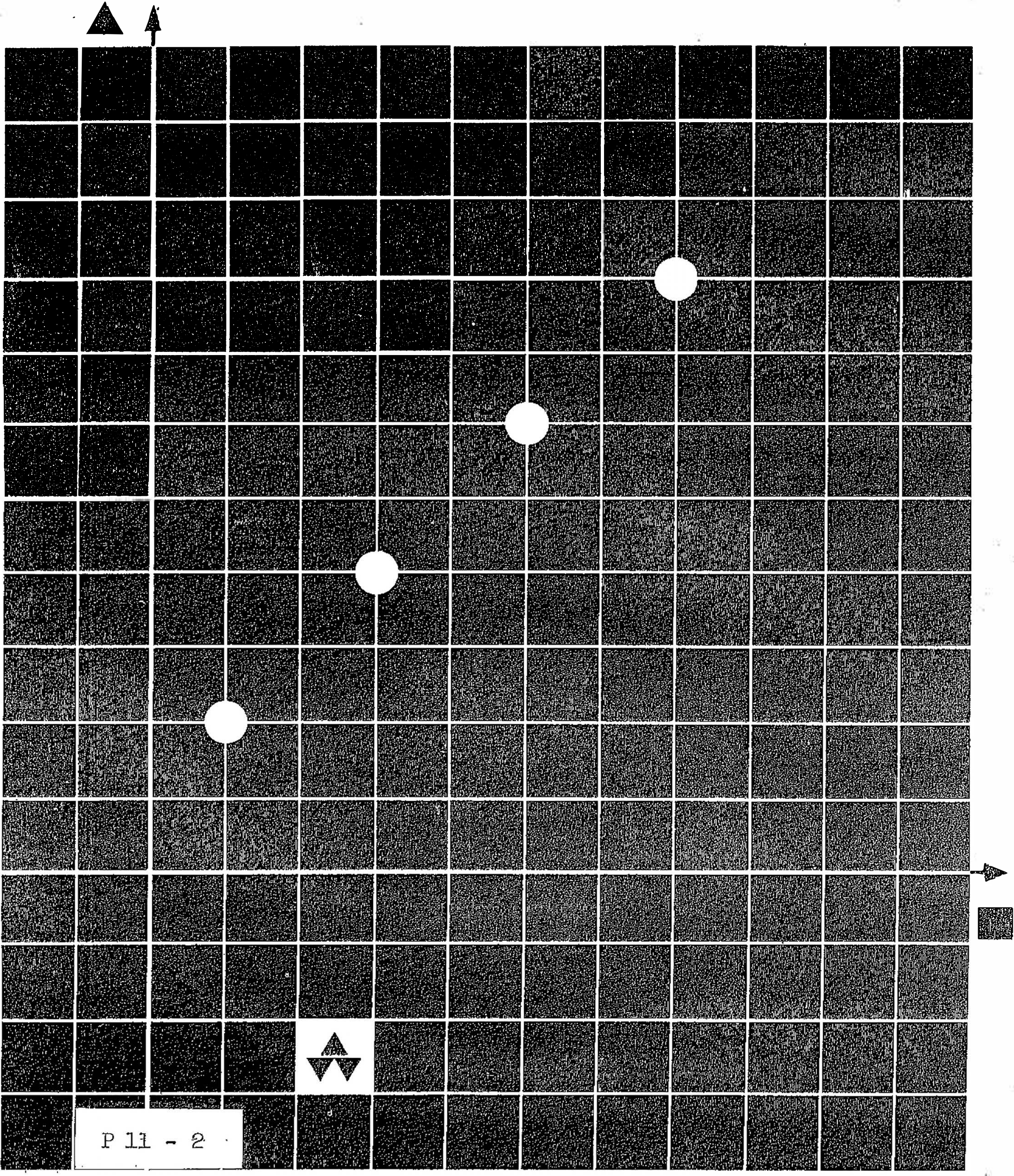

P 11 - 2

ROBERT B. DAVIS

EXPLORATIONS IN MATHEMATICS

THE MADISON PROJECT

STUDENT DISCUSSION GUIDE

EXPLORATIONS IN MATHEMATICS

STUDENT DISCUSSION GUIDE

to accompany

EXPLORATIONS IN MATHEMATICS
A TEXT FOR TEACHERS

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TO THE STUDENT

Some of the ideas in this book were first thought of by great men of the past. Others were first thought of by scientists and mathematicians—usually college professors—who are now alive. Still others were first thought of by teachers who have worked with the Madison Project. But—most important of all—some were first thought of by students just like you.

If **you** get any new mathematical ideas that you think we should know about, please write to us.

Robert B. Davis
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St. Louis, Missouri 63119

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PART ONE ▪ VARIABLES, GRAPHS, AND SIGNED NUMBERS

CHAPTER 1

Variables

One of the major discoveries in mathematics occurred when someone* realized that you could use sentences such as

$$3 + \square = 5.$$

A sentence like this is called an **open sentence** or an **equation.**

(1) Can you write something in the $\square$ in the equation above so as to produce a **true** statement?

(2) Can you write something in the $\square$ in the equation above so as to produce a **false** statement?

(3) Do you know what mathematicians mean when they speak of "the truth set for the open sentence $3 + \square = 5$"? Do you know how to write this truth set?

(4) Do you know what mathematicians mean when they speak of the "rule for substituting"?

(5) Bill claims that the "rule for substituting" says this: if an open sentence has more than one $\square$ in it, you must put **the same number** in every occurrence of the $\square$. Do you agree?

(6) Can you give some examples of using the "rule for substituting"?

(7) Tony says that the "rule for substituting" is not the same as "making a true statement." Do you agree?

(8) For the open sentence

$$\square + \square + \square = 9,$$

can you substitute so as to **obey** the rule for substituting but make a **false** statement?

---

*The original discovery of open sentences (and what we shall call variables) occurred so long ago that its precise origin is not known. An English scholar named A. Henry Rhind found, and bought, what turned out to be a very ancient Egyptian papyrus dating from before 1650 B.C. This papyrus is now the property of the British Museum. It was the work of a very ancient scribe named Ahmes, and it includes the basic idea of **variables**. However, surprising as it may seem, Ahmes and the people of his time did not have a really satisfactory method for writing numbers, and this appears to have prevented them from doing very much with the idea of variables. We, today, can do a great deal with this idea, and will undoubtedly continue to discover more and more ways to use variables.

(9) For the open sentence

$$\square + \square + \square = 9,$$

can you substitute so as to **violate** the rule for substituting but make a **true** statement?

(10) For the open sentence

$$\square + \square + \square = 9,$$

can you substitute so as to **violate** the rule for substituting and make a **false** statement?

(11) For the open sentence

$$\square + \square + \square = 9,$$

can you substitute so as to **obey** the rule for substituting and make a **true** statement?

(12) What is the **truth set** for the open sentence

$$\square + \square + \square = 9\,?$$

Can you find the truth set for each open sentence?

(13) $\square + \square = 12$

(14) $(2 \times \square) + 1 = 7$

(15) $(3 \times \square) + 5 = 38$

(16) $(2 \times \square) + 20 = 25$

(17) $(3 \times \square) + 11 = 22$

(18) $(10 \times \square) + 1 = 28$

(19) What do we mean (in mathematics) when we speak of a **variable**? How can we write a **variable**? How many ways do you know to use a **variable**?

(20) Sometimes, when we want to be very precise, we specify the **replacement set** for the variable. John says that if we specify the replacement set $\boldsymbol{R} = \{3, 4, 5, 6, 7\}$ for the variable $\square$ and if we write the open sentence

$$3 + \square = (2 + \square) + 1,$$

we have really written five mathematical statements that don't have $\square$'s in them! Do you agree?

(21) Do you know what the symbol

$$<$$

means (in mathematics)?

(22) Which statements are **true** and which are **false**?

(a) $3 < 5$

(b) $(2 \times 3) + 1 < 12$

(c) $2 < 1$

(d) $0 < 7$

(e) $1000 < 1{,}000{,}000$

(f) $12 < 7$

(g) $1006 < 1060$

(h) $2000 < 1000$

(i) $5 < 5$

(j) $(3 \times 4) + 1 < 10$

(k) $(3 \times 4) + 1 < 13$

(23) Suppose you wanted to tell somebody these seven statements—**only** these seven, and **no others**:

$$5 + 4 < 21$$

$$5 + 5 < 21$$

$$5 + 6 < 21$$

$$5 + 7 < 21$$

$$5 + 8 < 21$$

$$5 + 9 < 21$$

$$5 + 10 < 21$$

Could you do this by writing only one **open sentence** and by writing the **replacement set** for the variable $\square$?

(24) Paul has written some open sentences and indicated replacement sets for each variable. In each case, can you write the statements Paul means, **without using variables**?

(a) $(3 \times \square) + 1 < 25 \qquad R = \{0, 2, 4, 6\}$

(b) $\square + \square = 2 \times \square \qquad R = \{100, 7, 3, 24\}$

(c) $3 + \square = \square + 3 \qquad R = \{5, 7, 9, 10, 11\}$

(25) Jill wrote these statements:

$$7 + (2 \times 1) < 50$$

$$7 + (2 \times 3) < 50$$

$$7 + (2 \times 4) < 50$$

$$7 + (2 \times 10) < 50$$

Can you indicate these four statements by writing **one open sentence** and by writing the **replacement set** for the **variable**?

(26) Don wrote these statements:

$$(8 + 1) \times (8 - 1) = (8 \times 8) - (1 \times 1)$$

$$(8 + 2) \times (8 - 2) = (8 \times 8) - (2 \times 2)$$

$$(8 + 3) \times (8 - 3) = (8 \times 8) - (3 \times 3)$$

$$(8 + 4) \times (8 - 4) = (8 \times 8) - (4 \times 4)$$

Can you represent Don's four statements by writing **one open sentence** and by indicating the **replacement set** for the **variable**?

(27) Nancy used the letter ***P*** to indicate a variable, and she wrote this for the replacement set for ***P***:

$$R = \{\text{Jill, Eva, Eileen}\}$$

Then she wrote:

I like ***P***.

What did Nancy mean?

(28) Tom used the letter ***P*** to indicate a variable, and he said that the replacement set for the variable ***P*** was to be:

R = {"St. Louis is a city", New York is a city", "Los Angeles is a city", "Miami Beach is a city", "Minot is a city"}

Then Tom wrote:

P is a true statement.

What did Tom mean?

(29) Dick wrote:

P is a false statement.

R = {"Massachusetts is a city", "Connecticut is a city", "Missouri is a city", "Alaska is a city"}

What did Dick mean?

(30) Suppose I write:

I like ***P***.

What would the **truth set** for this open sentence be **for you?**

(31) Kathy uses the symbol "~" to mean "not." Kathy wrote:

$(\sim P)$ is false.

R = {"The violin is a musical instrument", "The trumpet is a musical instrument", "The piano is a musical instrument", "The trombone is a musical instrument"}

What did Kathy mean?

(32) Can **you** make up some examples like these?

(33) Is **replacement set** the same as **truth set**?

CHAPTER 2

The Cartesian Product of Two Sets

When we speak of an **ordered pair** of names (or numerals, or whatever), we mean a pair where order is important. Thus, the ordered pair

(Nancy, violin)

is **not** the same as the ordered pair

(violin, Nancy).

(1) Suppose we write the open sentence

P studies Q,

and we agree to indicate replacements for the variables P and Q by writing:

The ordered pair (P, Q) may be either
(Nancy, violin) or (Bill, clarinet).

What would we mean?

(2) Joe says we would mean

violin studies Nancy

and

Bill studies clarinet.

Did Joe use the **order** correctly?

If A is the set

$$A = \{\text{Al, Bill, Henry}\}$$

and if B is the set

$$B = \{\text{Nancy, Eileen, Eva}\},$$

then the **Cartesian product** $A \times B$ is the set

$$\{(\text{Al, Nancy}), (\text{Al, Eileen}), (\text{Al, Eva}),\\ (\text{Bill, Nancy}), (\text{Bill, Eileen}), (\text{Bill, Eva}),\\ (\text{Henry, Nancy}), (\text{Henry, Eileen}), (\text{Henry, Eva})\}.$$

(3) If

$$A = \{1, 2, 3\}$$

and

$$B = \{5, 6\},$$

can you write the **Cartesian product**

$$A \times B?$$

(4) Using the same sets as in question 3, can you write the Cartesian product

$$B \times A?$$

René Descartes was a French mathematician and philosopher who was born in 1596 and who died in 1650. He made very effective use of the idea of naming points in the plane by using ordered pairs. The adjective "Cartesian" is derived from Descartes' last name. As we shall see, discoveries made by Descartes in the seventeenth century continue to influence our lives today.

(5) Do you know **how** Descartes was able to use ordered pairs of numbers as names for points in the plane?

(6) What was life in the United States like during Descartes' lifetime?

(7) During Descartes' lifetime did they have radios? television? telephone? airplanes? automobiles? bicycles? How did people travel in Descartes' time? Did they have railroads? steam engines? Did they have printed books in those days? Were they able to sail across the Atlantic Ocean?

(8) What kind of music did they have in Descartes' time?

(9) Are there any schools or colleges in the United States today that were in existence during Descartes' lifetime? Are there any in England? in France? in Italy?

(10) Did they have plays in Descartes' day? What books, novels, or plays, if any, might Descartes have read or seen?

(11) Jimmy wanted to give number names to points on a line. First, he named one point "0";

then he named a point "1."

Can you give number names for some other points on Jimmy's line?

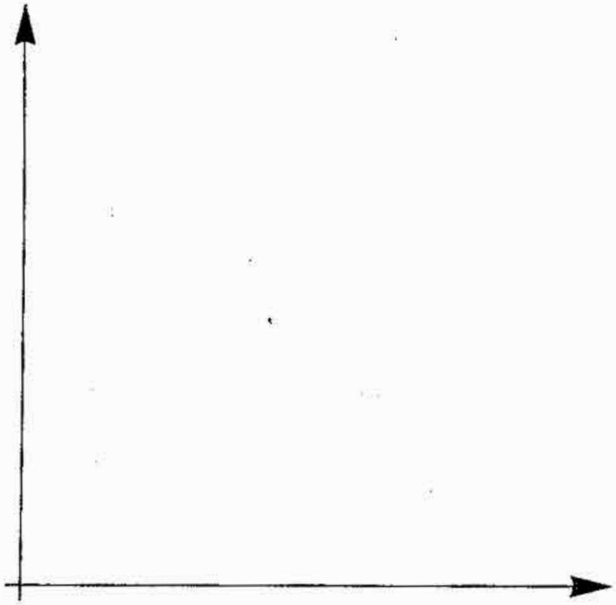
(12) Ellen used a vertical line.

She named one point "0."

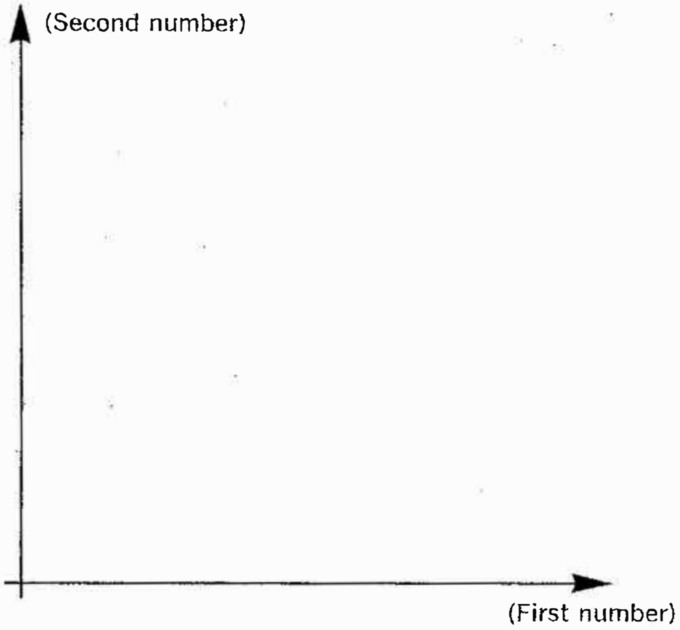
Then she named another point "1."

Can you give number names for some other points on Ellen's line?

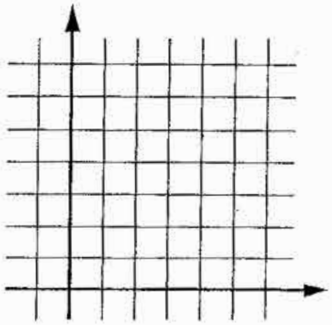
Descartes used a pair of **crossed number lines** in order to name the points in the plane.



Since he was going to use **ordered pairs of numbers**, he had to decide upon an order. He decided to use the **first** number of the ordered pair to refer to the **horizontal** number line and the **second** number of the ordered pair to refer to the **vertical** number line.



(13) If we draw a grid, like this,

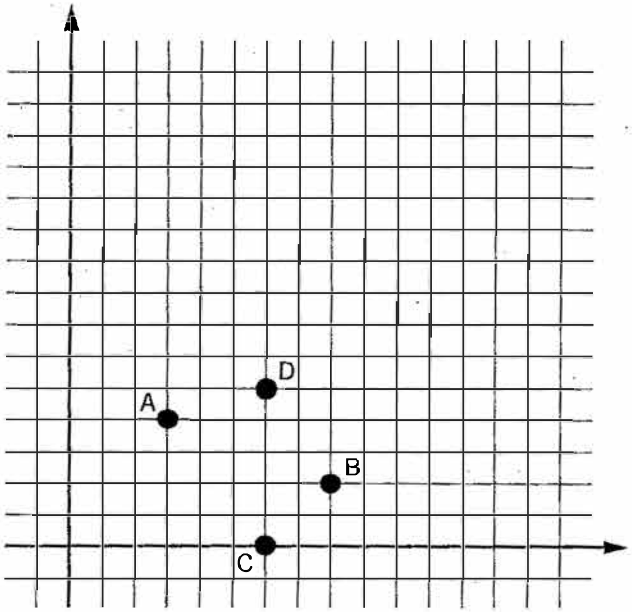


can you find the point Descartes named $(0, 0)$?

(14) Can you find the point Descartes named $(2, 3)$?

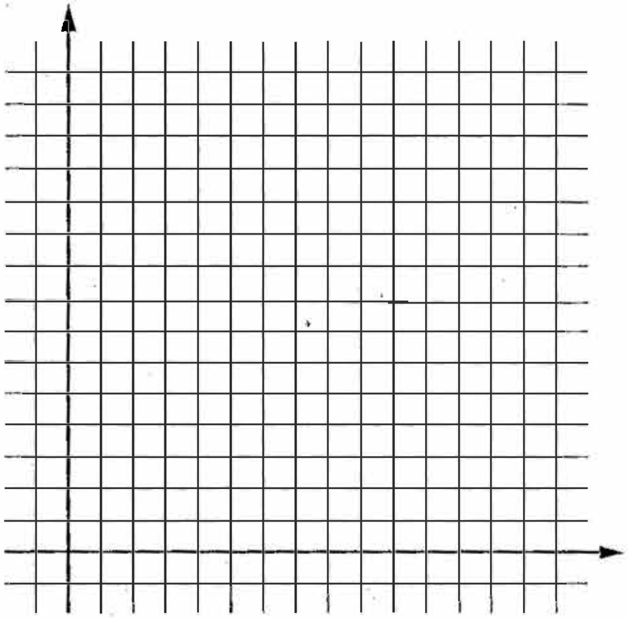
(15) Can you find the point Descartes named $(4, 1)$?

(16) What names (using **ordered pairs of numbers**) would Descartes give to these points?



(17) Descartes could use his "crossed number lines" to make pictures representing Cartesian products. Can you use these "Cartesian coordinates" to make a picture representing

$$A \times B, \quad \text{if } A = \{3, 4\} \text{ and } B = \{0, 1\}?$$



(18) Suppose $A = \{1, 2, 3\}$. Suppose $B = \{2, 3\}$. Can you make a graph (or picture) showing the Cartesian product $A \times B$? How many points will there be in the picture representing $A \times B$?

(19) Suppose that the set M has r elements and the set N has s elements. How many points will there be in the picture representing the Cartesian product

$$M \times N?$$

(20) Suppose that $A = \{\text{Nancy, Jane}\}$ and $B = \{\text{Don, Roy, Louis}\}$. Can you write the Cartesian product

$$A \times B?$$

(21) Suppose that $A = \{\text{red, green, yellow}\}$ and $B = \{\text{hat, scarf}\}$. Can you write the Cartesian product

$$A \times B?$$

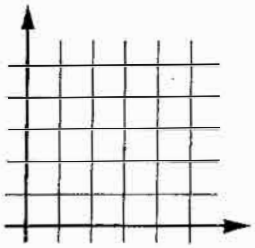
(22) Suppose that $A = \{2, 3, 4\}$ and $B = \{1, 2\}$. Make one graph to show $A \times B$ and another graph to show $B \times A$.

We can play a game using Descartes' method of naming points in the plane by means of ordered pairs. This game is just like tic-tac-toe, only different.

In this game, called "four-in-a-row," if you get four of your marks in an uninterrupted straight line, you win. All of your marks, unlike in tic-tac-toe, will be on the intersection of two lines. (The board allows room for five marks in a line.)

One team's marks are **x**'s, and the other team's marks are **o**'s. The teams take turns naming points they want marked, using Descartes' system of ordered pairs of numbers. The teacher marks the points that each team names. (If you make an illegal move, you lose that turn, and no point is marked.)

The "playing board" looks like this:



(23) Here is a sample game. See if you can keep track of it.

x team: $(3, 2)$
o team: $(2, 2)$
x team: $(3, 3)$
o team: $(3, 1)$
x team: $(4, 5)$
o team: $(0, 0)$
x team: $(1, 3)$
o team: $(2, 1)$

What does the board look like now?

(24) Can you finish the game that we have just started? Which team do you think was ahead at the end of question 23?

CHAPTER 3

Open Sentences With More Than One Variable

(1) Sometimes we want to write an open sentence that has more than one variable in it. Do you know how we can do it? Can you give some examples?

(2) Nancy gave these examples:

$$\square + \triangle = 8$$

$$\square + 3 = \triangle$$

$$\square + \square = \triangle$$

$$\square + \square + \triangle = 9$$

Can you figure out what Nancy meant? How would the "rule for substituting" work for Nancy's open sentences?

(3) Bill says that Nancy forgot to indicate any **replacement sets** for her variables. What do you think?

(4) In Nancy's class, it turned out that some students knew about "negative numbers" and other students did not. In order to be fair, the class agreed to work on Nancy's open sentences, **without using any negative numbers.** Then, in order to make the work easier, the class also agreed **not to use any fractions.** In order to remember this agreement, the class wrote:

$$R_{\square} = \{0, 1, 2, 3, \ldots\}$$

$$R_{\triangle} = \{0, 1, 2, 3, \ldots\}$$

What do you think the class meant?

(5) If

$$R_{\square} = \{0, 1, 2, 3, 4, \ldots\}$$

$$R_{\triangle} = \{0, 1, 2, 3, 4, \ldots\},$$

can you find the **truth set** for each of Nancy's open sentences?

(6) Bruce said he could write the truth set for the open sentence

$$\square + \triangle = 8$$

by means of a table, like this:

$\square$	$\triangle$
0	8
1	7
2	6
3	5

Table for Truth Set

What do you think?

(7) Ellen says that Bruce's table is not complete. Can you finish it?

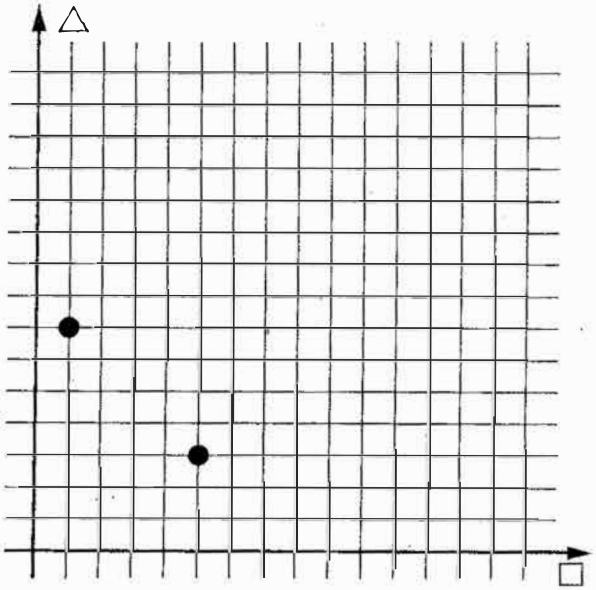
(8) Can you use a table to represent the **truth sets** for Nancy's other open sentences (see question 2)?

(9) Can you make up some open sentences of your own?

(10) Jerry says he can use Descartes' idea of "crossed number lines" to represent the truth set for the open sentence

$$\square + \triangle = 8,$$

by means of a graph. He labeled the horizontal number line with a $\square$ to show that he used it to locate the replacement for $\square$. He labeled the vertical number line with a $\triangle$ to show that he used it to locate the replacements he used for $\triangle$.



Graph for Truth Set

Jerry says the points marked correspond to this table:

$\square$	$\triangle$
1	7
5	3

Can you complete the graph, if

$$R_{\square} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

$$R_{\triangle} = \{0, 1, 2, 3, 4, 5, \ldots\} ?$$

Can you make a graph to show the truth set for each of the following equations? (Use $R_{\square} = \{0, 1, 2, 3, \ldots\}$, $R_{\triangle} = \{0, 1, 2, 3, \ldots\}$.)

(11) $(\square \times 1) + 3 = \triangle$

(12) $(\square \times 2) + 3 = \triangle$

(13) $(\square \times 5) + 3 = \triangle$

(14) $(\square \times 1) + 2 = \triangle$

(15) $(\square \times 1) + 1 = \triangle$

(16) $(\square \times 1) + 0 = \triangle$

(17) $(\square \times 1) + 10 = \triangle$

(18) $\square \times \triangle = 36$

CHAPTER 4

Signed Numbers

The numbers 1, 2, 3, 4, . . . arise whenever we **count** things. Really, these are the **only** numbers that arise from counting, at least the way most people do it.

If we use our imaginations, we might think to add **zero**, which arises in counting how many brothers you have (if you don't have any brothers).

This gives us

0, 1, 2, 3, 4, 5, . . .

(1) Can you mark 0, 1, 2, 3, 4, . . . on a number line?

When we want to **divide things up** (like cakes and pies and candy bars) or when we want to **measure** things, we need more numbers, such as

$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{16}$, $\frac{1}{10}$, $\frac{3}{10}$, $2\frac{1}{2}$, 2.7, and so forth.

(2) Can you show the numbers

$\frac{1}{2}$, $\frac{1}{3}$, $1\frac{1}{4}$, $2\frac{3}{10}$, $4\frac{1}{2}$, 3.7

on a number line? Where would $3\frac{1}{100}$ be?

(3) Do you think there are any **new** kinds of numbers that are different from

counting numbers: 1, 2, 3, 4, . . .

and different from

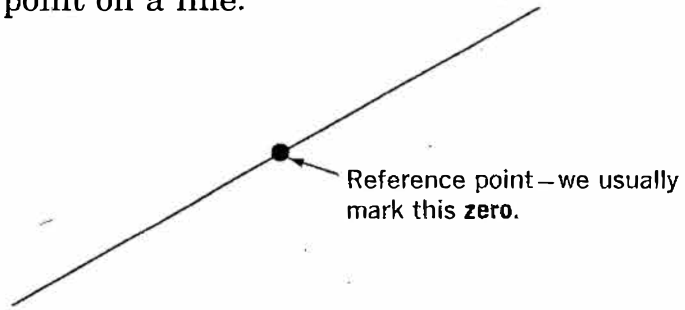
zero: 0

and different from

fractions: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{4}$, $1\frac{1}{10}$, $\frac{99}{100}$, and so forth?

Do you know any other kinds of numbers?

Mathematicians talk about something that they call a **one-dimensional linear vector space.** You meet this one-dimensional linear vector space when you mark a reference point on a line.



Reference point—we usually mark this **zero.**

You can move away from this reference point in **one** direction or in the **opposite** direction.

(4) Can you think of anywhere that you have seen numbers used this way?

(5) Do you know how they determine **zero** on a **centigrade thermometer?**

A certain hotel is built on the side of a steep hill. The result is that the entrance and the lobby are really in the middle of the building. The architect, who was also an amateur mathematician in his spare time, decided to label the lobby floor **zero.** The next floor above the lobby floor he called **positive one.** On the elevator indicator he wrote

$^{+}1.$

The floor just below the lobby level he called **negative one.** On the elevator indicator he wrote

$^{-}1.$

(6) Suppose the elevator starts at

$^{-}3$

and goes to the lobby. Did it go **up** or **down**? How many floors up or down did it go?

(7) Suppose the elevator starts at

$^{+}8$

and goes to the lobby. Did it go **up** or **down**? How many floors up or down did it go?

(8) Suppose the elevator starts at

$^{-}3$

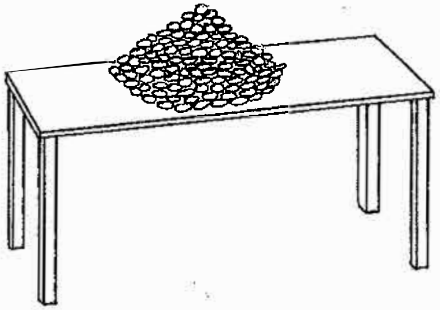
and goes to

$^{+}6.$

How far did it go? Which way, up or down?

(9) Danny has a bag that has lots of pebbles in it.

On the table there is a pile with lots more pebbles.



We need to mark a reference point, so Jerry says "Go!"

Now Danny takes 3 pebbles from the pile on the table, and puts them in the bag.

Are there **more** pebbles in the bag than there were when Jerry said "Go," or are there **less**? How many more or how many less? Do you know how we write this?

(10) Now Danny takes **5** pebbles out of the bag. We can write

$$3 - 5.$$

Are there more pebbles in the bag than there were when Jerry said "Go," or are there less? How many more or how many less? Do you know how to write this

$$3 - 5 = \_\_\_ ?$$

Can you make up a "pebbles-in-the-bag" story for each problem? Can you write the correct **signed number** to describe what happened in each case?

(11) $7 - 2 =$ ____

(12) $5 - 4 =$ ____

(13) $3 - 4 =$ ____

(14) $2 - 10 =$ ____

(15) $6 - 6 =$ ____

(16) $3 - 2 =$ ____

(17) $9 - 10 =$ ____

(18) $2 + 3 - 1 =$ ____

(19) $5 - 4 - 1 =$ ____

(20) $4 - 3 - 2 =$ ____

(21) Can you mark these numbers on a number line?

$$\left\{0,\ {}^{+}1,\ {}^{+}2,\ {}^{+}2\tfrac{1}{2},\ {}^{-}1,\ {}^{-}2\tfrac{1}{2},\ {}^{-}2\right\}$$

(22) Can you mark some more numbers on a number line?

(23) Using Descartes' idea of **crossed number lines**, can you mark these points on a graph?

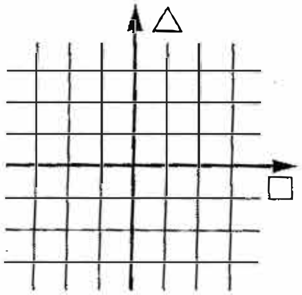
A: $(0,\ 0)$

B: $({}^{+}2,\ {}^{+}3)$

C: $({}^{-}1,\ {}^{-}1)$

D: $({}^{+}2,\ {}^{-}3)$

(24) Can you play our game of "four-in-a-row" on **this** board?



What is the **biggest** number you can put in the ☐?

What is the **smallest** ? What is the biggest number you can put in the △ ? What is the **smallest** ?

(25) Can you mark these numbers on a number line?

$$\{^{+}3,\ ^{-}2,\ ^{-}3,\ ^{-}2\tfrac{1}{4},\ ^{+}4\}$$

(a) Is $^{+}3$ more or less than $^{+}4$?

(b) Is $^{+}3$ to the right or left of $^{+}4$?

(c) Is $^{-}3$ more or less than $^{+}3$?

(d) Is $^{-}3$ to the right or left of $^{+}3$?

(e) Is $^{-}2$ more or less than $^{-}2\frac{1}{4}$?

(f) Is $^{-}2$ to the right or left of $^{-}2\frac{1}{4}$?

(26) Do you know what

$$<$$

means **on the number line?**

(27) Which statements are **true** and which are **false?**

(a) $^{+}5 < {}^{+}7$

(b) $0 < {}^{+}1$

(c) $0 < \frac{1}{2}$

(d) $0 < {}^{+}\frac{1}{10}$

(e) $^{+}\frac{1}{2} < {}^{+}\frac{1}{3}$

(f) $^{+}\frac{1}{10} < {}^{+}\frac{1}{2}$

(g) $^{-}1 < 0$

(h) $^{-}2 < {}^{-}1$

(i) $^{-}1 < {}^{+}1$

(j) $^{+}2 < {}^{-}2$

(k) $^{+}2 < {}^{-}3$

(l) $^{-}5 < 0$

(m) $^{-}5 < {}^{+}\frac{1}{2}$

(n) $^{+}\frac{1}{2} < 0$

(o) $^{-}5 < {}^{+}1$

(p) $^{-}1000 < 0$

(q) $^{-}1000 < {}^{+}1$

(r) $^{+}3 < {}^{-}1000$

(s) $0 < {}^{-}1000$

(t) $^{-}1 < {}^{-}1$

(u) $^{+}7 < {}^{+}7$

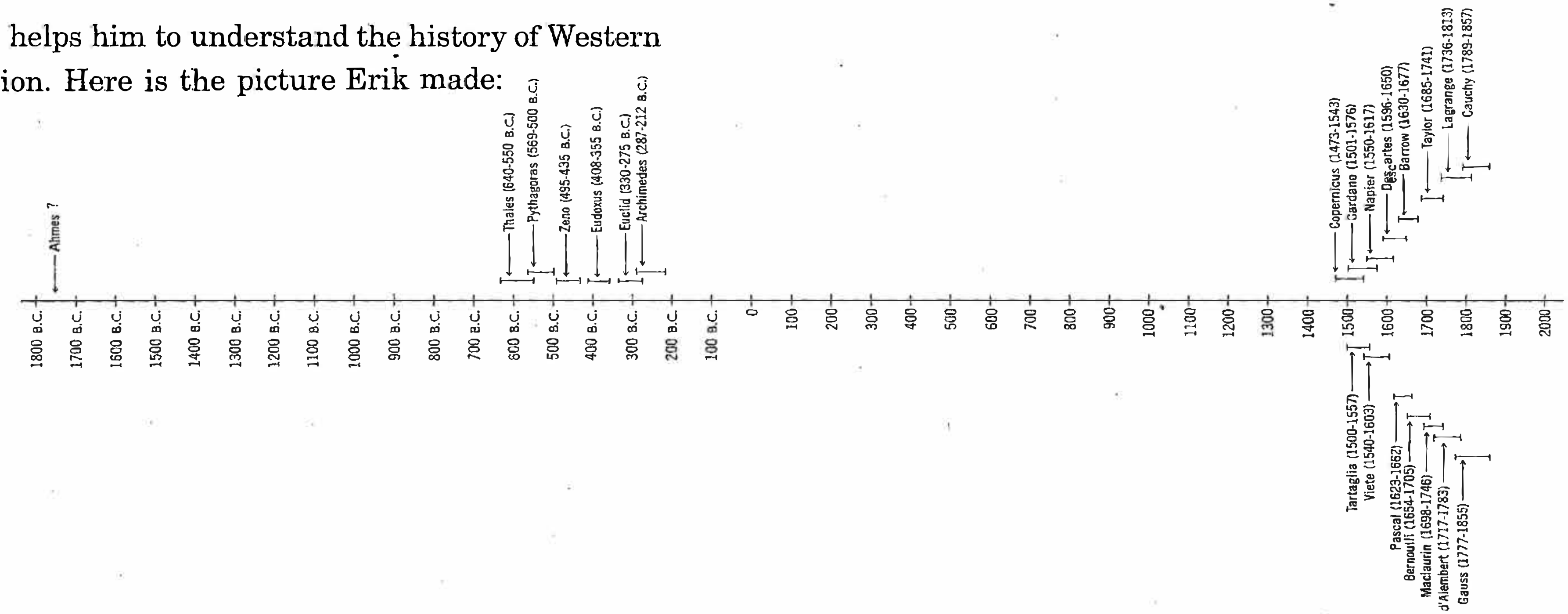
(v) $^{+}7 < {}^{-}10$

(w) $^{-}10 < {}^{+}7$

(28) Erik says that **dates** for B.C. and A.D. **almost** work like the number line, **but not quite**! What do you think ?

(29) Even though the dates are only **almost** like a number line, Erik says that "a sort of number-line

picture" helps him to understand the history of Western civilization. Here is the picture Erik made:



(30) Don says Erik only considered mathematicians, and he left out a great many mathematicians, even at that. Can you add some of the following mathematicians, to Erik's chart?

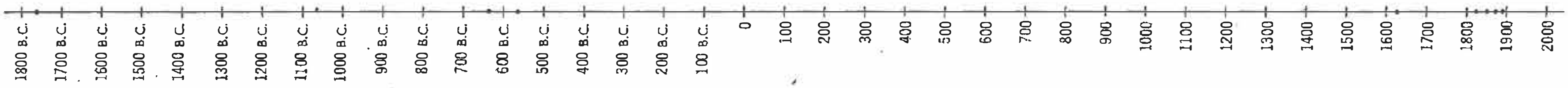
J. W. Alexander	(1856-1915)
P. Alexandroff	(1896-)
Emil Artin	(1898-)
Stefan Bergman	(1898-)
Friedrich Wilhelm Bessel	(1784-1846)
R. H. Bing	(1902-)
George David Birkhoff	(1884-1944)
George Boole	(1815-1864)
George Cantor	(1845-1918)
Constantin Carathéodory	(1873-1950)
Arthur Cayley	(1821-1895)
W. K. Clifford	(1845-1879)
J. W. R. Dedekind	(1831-1916)
Albert Einstein	(1879-1955)
Gottlieb Frege	(1848-1925)
Évariste Galois	(1811-1832)
Kurt Gödel	(1906-)
Herman Grassmann	(1809-1877)
John Graunt	(1620-1674)
Edmond Halley	(1656-1742)
Sir William Rowan Hamilton	(1805-1865)
Felix Hausdorff	(1868-1942)
J. B. van Hemholtz	(1821-1894)
David Hilbert	(1862-1943)
Bela van Kérekjártó	(1898-1946)

Felix Klein	(1849-1925)
Andrei Kolmogorov	(1903-)
Henri Leon Lebesgue	(1875-1941)
Solomon Lefschetz	(1884-)
Deane Montgomery	(1909-)
R. L. Moore	(1882-)
John von Neumann	(1903-1957)
Amalie Emmy Noether	(1882-1935)
G. Peano	(1858-1932)
Jules Poincaré	(1854-1912)
George Polya	(1887-)
L. S. Pontriagin	(1908-)
Srinivasa Ramanujan	(1887-1920)
Georg Friedrich Riemann	(1826-1866)
J. B. Rosser	(1907-)
Bertrand Arthur Russell	(1872-)
Waclaw Sierpinski	(1882-)
T. A. Skolem	(1887-)
M. H. Stone	(1903-)
Alfred Tarski	(1902-)

Oswald Veblen	(1880-)
Karl Theodor Weierstrass	(1815-1897)
Andre Weil	(1906-)
Hermann Weyl	(1885-1955)
Alfred North Whitehead	(1861-1947)
Norbert Wiener	(1894-1964)
R. L. Wilder	(1896-)
E. F. F. Zermelo	(1871-)

(31) Ellen says Erik's chart is not big enough to get all the names in. Can you make a chart that is big enough? (Use the **same scale** from 1800 B.C. through 2000 A.D.)

(32) Bill said it would be easier just to mark a dot to show when each mathematician was born. Bill began his chart with dots for Ahmes, Thales, Pythagoras, Polya, Dedekind, Graunt, Peano, and Veblen. At this stage, Bill's chart looked like this:



Can you mark on Bill's chart all the other mathematicians **mentioned** in this chapter?

Harold said these charts seem to say something about history. He made a new chart, marking the dates of birth of the following musicians. What did Harold's chart look like?

Johann Sebastian Bach	(1685-1750)
Karl Philipp Emanual Bach	(1714-1788)
Béla Bartók	(1881-1945)
Ludwig von Beethoven	(1770-1827)
Leonard Bernstein	(1918-)
Johannes Brahms	(1833-1897)
Elliott Cook Carter	(1908-)
François Fréderic Chopin	(1810-1849)
Aaron Copeland	(1900-)
Claude Debussy	(1862-1918)
Giovanni Gabrielli	(1554-1612)
George Gershwin	(1898-1937)
George Friedrick Handel	(1685-1759)
Roy Harris	(1898-)
Franz Joseph Haydn	(1732-1809)
Wolfgang Amadeus Mozart	(1756-1791)
Guiseppe Antonio Paganelli	(1710-1760)
Sergei Sergeevich Prokofiev	(1891-1953)
Sergei Wassilievitch Rachmaninoff	(1873-1943)
Alessandro Scarlatti	(1659-1725)
Domenico Scarlatti	(1683-1757)
Franz Peter Schubert	(1797-1828)
Robert Schumann	(1810-1856)
Dmitri Dmitrievich Shostakovich	(1906-)
Igor Stravinsky	(1882-)
Pëtr Ilich Tchaikovsky	(1840-1893)
Antonio Vivaldi	(1675-1741)
Richard Wagner	(1813-1883)
William Walton	(1902-)
Kurt Weill	(1900-1950)

(34) Ellen made a chart showing birth dates of painters, sculptors, and architects. What did Ellen's chart look like?

(35) Nancy made a chart showing birth dates of playwrights, writers, and poets. What did Nancy's chart look like?

(36) Andy made a chart showing birth dates of explorers. What did Andy's chart look like?

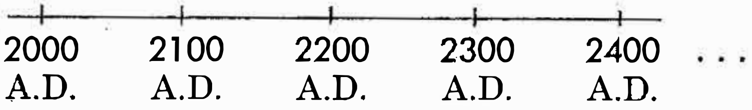
(37) Dick made a chart showing birth dates of scientists. What did Dick's chart look like?

(38) Do these charts suggest anything? How do you explain it?

(39) Do you know the date when cities first began to appear? What sort of chart can you make from the beginning dates of various cities?

(40) Do you know what people mean when they speak of the Renaissance?

(41) What do you think these charts will look like for the following part of the number line?



(42) You may be interested in studying the changes in the population of Europe. A chart can help here, too.

(43) Here is a table for the population of London. You may want to make a chart of this.

Date	Population
1801	1,088,000
1811	1,259,000
1821	1,504,000
1831	1,778,000
1841	2,073,000
1851	2,491,000
1861	2,291,000
1871	3,336,000
1881	3,881,000
1891	4,266,000
1901	4,563,000
1911	4,541,000
1921	4,498,000
1931	4,408,000
1951	3,353,000

Population of London

(44) Do you have any theories on why the graphs look like this? You might try to find the dates of the following events to check against your theories.

Founding of Oxford University
Founding of Cambridge University
Founding of University of Paris
Discovery of the source of the Nile
French Revolution
California gold rush
Discovery of America
First European settlement in South America
First trip around the world
Marco Polo's birth
First universal compulsory education
in the United States
Founding of Hopkins Grammar School
First railroad across United States
Crusades
First European contacts with China
Stradivarius' birth
United States Revolutionary War
United States Civil War
World War I
World War II
Date of discovery of prime numbers, etc.
(Belgian Congo)
Date of Australopithecus Africanus

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CHAPTER 5

Postman Stories

Jerry wrote a story about a very peculiar postman, who behaved like this:

(a) He read all of the mail.

(b) He did not necessarily deliver the mail to the right people. He gave it to anyone he wanted to give it to. (But he **remembered** who **should** have received **it**!)

(c) Later on he would come back and pick up mail he had misdelivered, apologize, and give it to the right person.

Jerry's story also includes a housewife, who also behaves peculiarly:

(a) She tries to keep up-to-date in her estimate of how much **available** money she has.

(b) She never reads the **addresses** on the mail she receives (she figures it doesn't do any good anyhow, because the postman delivers them to whomever he wants), and she never reads the **name** on **bills** and **checks** (but she reads the **amount** and keeps her records **up-to-date**!).

Jerry's story involves **bills**, like

$$^{-}3,\ ^{-}1,\ ^{-}5,\ ^{-}100,\ ^{-}10,$$

and **checks**, like

$$^{+}2,\ ^{+}7,\ ^{+}5,\ ^{+}100,\ ^{+}9.$$

(1) Do you know what Jerry means by a **check**? Who might send you a **check**?

(2) Do you know what Jerry means by a **bill**? Do you like to get **bills**? Who might send you a **bill**?

Jerry's postman sometimes **brings checks**

$$+\ ^{+}3,$$

and he sometimes comes and **takes away** a **check** (that was really for someone else)

$$-\ ^{+}10.$$

The postman sometimes **brings bills**

$$+\ ^{-}7,$$

and he sometimes **takes away** a **bill** (that was really for somebody else).

(3) Does it make you **happy** or **sad** when the postman **brings** a **bill**?

(4) Does it make you **happy** or **sad** when the postman **takes away** a **bill**?

(5) Does it make you **happy** or **sad** when the postman **brings** a **check**?

(6) Does it make you **happy** or **sad** when the postman **takes away** a **check**?

(7) Jerry said, "On Monday morning, the postman brought the housewife a check for \$3 and a check for \$5."

$$+\ {}^{+}3 + {}^{+}5$$

As a result of the postman's visit on Monday morning, did the housewife think she was **richer** or **poorer**? How **much** richer or how **much** poorer?

(8) Can you write a single signed number showing how much **richer** or **poorer** the housewife thought she was?

$$+\ {}^{+}3 + {}^{+}5 =$$

(9) Geoffrey's father says that mathematicians sometimes **leave off** the first "+" sign and write merely

$${}^{+}3 + {}^{+}5.$$

Can you write a single signed number that names the same amount as ${}^{+}3 + {}^{+}5$?

$${}^{+}3 + {}^{+}5 =$$

(10) The housewife thought she had \$120 uncommitted and available **before** the postman came Monday morning. How did she change her records as a result of the postman's visit Monday morning?

$$\cancel{\$120}$$

$$\boxed{} \longleftarrow ?$$

(11) Gloria says the housewife's records should look like this:

$$\cancel{\$120}$$
$$\$130$$

Do you agree?

Can you make up a postman story for each problem? What answer do you get?

(12) ${}^{+}2 + {}^{+}7$

(13) ${}^{+}2 + {}^{-}1$

(14) ${}^{-}5 + {}^{-}2$

(15) ${}^{+}3 + {}^{-}4$

(16) ${}^{+}9 - {}^{+}2$

(17) $-\ {}^{+}5 - {}^{+}2$

(18) $-\ {}^{-}5$

(19) $-\ {}^{-}1 - {}^{-}5$

(20) ${}^{+}10 - {}^{-}100$

(21) $-\ {}^{+}100$

(22) ${}^{+}10 - {}^{+}100$

(23) $-\ {}^{-}7$

(24) ${}^{+}5 - {}^{-}2$

CHAPTER 6

Postman Stories for Products

For products, like

$$^{+}2 \times {}^{+}3,$$

Jerry makes up stories like this:

The postman

Brings or takes away? **brings**

$$\overset{\downarrow}{^{+}}2 \times {}^{+}3$$

How many? **two**

$$^{+}\overset{\downarrow}{2} \times {}^{+}3$$

Bills or checks? **checks**

$$^{+}2 \times \overset{\downarrow}{^{+}}3$$

For how much? **for $3 each.**

$$^{+}2 \times {}^{+}\overset{\downarrow}{3}$$

Can you make up a postman story for each problem? What answer do you get?

(1) $^{+}2 \times {}^{+}3 =$

(2) $^{+}2 \times {}^{+}5 =$

(3) $^{+}2 \times {}^{-}3 =$

(4) $^{+}2 \times {}^{-}5 =$

(5) $^{+}5 \times {}^{+}7 =$

(6) $^{-}2 \times {}^{+}1 =$

(7) $^{-}2 \times {}^{+}5 =$

(8) $^{-}2 \times {}^{-}5 =$

(9) $^{-}2 \times {}^{-}6 =$

(10) $^{+}2 \times {}^{-}6 =$

(11) $^{-}1 \times {}^{-}1 =$

(12) $^{+}5 + {}^{+}3 =$

(13) $-\,{}^{-}5 =$

(14) $^{-}3 \times {}^{-}3 =$

(15) $^{-}2 \times {}^{-}2 =$

(16) $^{+}3 \times {}^{+}3 =$

(17) $^{+}2 \times {}^{+}2 =$

(18) $^{+}2 \times {}^{-}2 =$

CHAPTER 7

Kye's Arithmetic

(1) Miss Parsons was working this **subtraction** problem with her class:

$$\begin{array}{r} 64 \\ -\,28 \\ \hline \end{array}$$

She said, "I can't take eight from four, so I'll regroup the sixty as . . ." At this point a boy named Kye interrupted and said, "Oh, yes! Four minus eight is negative four

$$\begin{array}{r} 64 \\ -\,28 \\ \hline {}^{-}4 \end{array}$$

. . . and twenty from sixty is forty

$$\begin{array}{r} 64 \\ -\,28 \\ \hline {}^{-}4 \\ 40 \end{array}$$

. . . so that you get forty plus negative four, which is thirty-six."

$$\begin{array}{r} 64 \\ -\,28 \\ \hline {}^{-}4 \\ 40 \\ \hline 36 \end{array}$$

What would you say to Kye?

(2) Can you use Kye's method on this problem?

$$\begin{array}{r} 83 \\ -\,25 \\ \hline \end{array}$$

(3) Some other students extended Kye's method. They decided to write "negative signs" **over** the digits to which they apply, so that

$$5\bar{3}$$

means

$$50 - 3$$

or

$$50 + {}^{-}3.$$

If 34 means "thirty plus four," can you say what each of these numerals means?

(a) 72

(b) $7\bar{3}$

(c) $1\bar{3}$

(d) 21

(e) 45

(f) 55

(4) Cynthia wrote:

$$\begin{array}{r} 64 \\ -28 \\ \hline 44 \end{array}$$

What do you think ?

(5) Can you work out these problems by two (or more) different methods?

(a) $\begin{array}{r} 23 \\ +13 \\ \hline \end{array}$ (b) $\begin{array}{r} 59 \\ +37 \\ \hline \end{array}$

(6) Some other students made up a method for subtracting. If we use their method, the problem

$$\begin{array}{r} 64 \\ -28 \\ \hline \end{array}$$

means "how far (on the number line) is it from 28 to 64 ?"

We'll see:

28 plus 2 gets you to 30

$$\begin{array}{r} 64 \\ -28 \\ \hline 2 \end{array}$$

. . . plus 30 gets you to 60

$$\begin{array}{r} 64 \\ -28 \\ \hline 2 \\ 30 \end{array}$$

. . . plus 4 gets you to 64

$$\begin{array}{r} 64 \\ -28 \\ \hline 2 \\ 30 \\ 4 \\ \hline \end{array}$$

. . . and altogether we've added 2 and 30 and 4,

$$\begin{array}{r} 64 \\ -28 \\ \hline 2 \\ 30 \\ 4 \\ \hline 36 \end{array}$$

. . . which is 36.

What do you think about this method ?

CHAPTER 8

Graphs With Signed Numbers

(1) Can you show the **truth set** for

$$(\square \times \square) + (\triangle \times \triangle) = 25,$$

by means of a **table** and a **graph**? (Use both **positive** and **negative** numbers.)

(2) Can you show the **truth set** for

$$\square \times \triangle = 36,$$

by means of a **table** and a **graph**?

(3) Can you show the **truth set** for

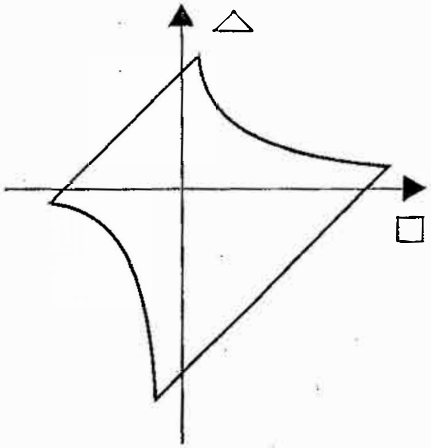
$$\square + \triangle = 10,$$

by means of a **table** and a **graph**? (Use **positive numbers, negative numbers,** and **fractions.**)

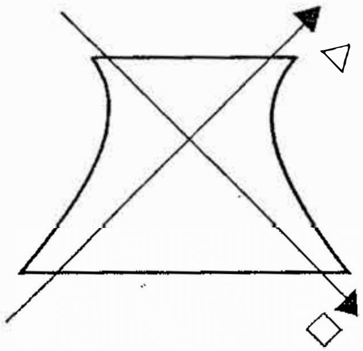
As illustrated at the right, a piece of the graph of the truth set of problem 2,

$$\square \times \triangle = 36,$$

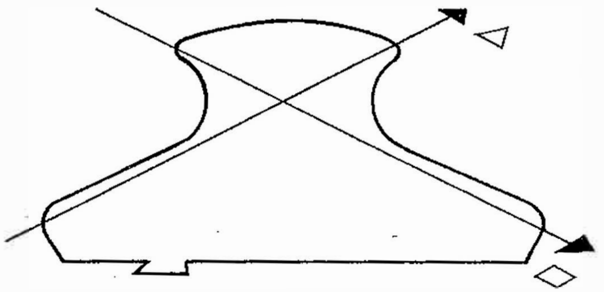
was used by architect Gyo Obata in designing the planetarium in St. Louis, Missouri. A continuous, smooth curve (a hyperbola) was obtained by allowing fractional solutions.



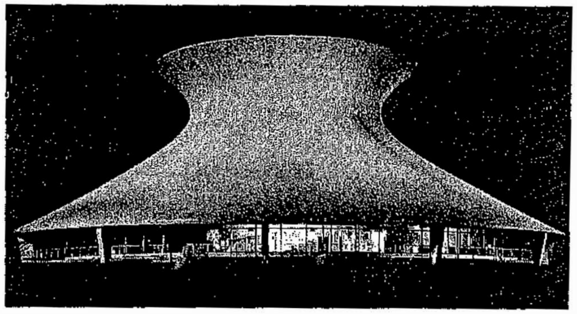
The picture is rotated so that the □ and △ axes appear in their usual positions.



In this diagram the perspective is changed so that the □ and △ axes appear perpendicular.



Because of perspective and changes of scale, the □ and △ axes that fit the profile curve do not seem to be perpendicular.



Actual view of the St. Louis planetarium as seen from the ground.

(4) Can you show the **truth set** for

$$\square \times \triangle = {}^{-}24,$$

by means of a **table** and a **graph**?

Can you make a graph for each truth set?

(5) $\square \times \triangle = {}^{+}24$

(6) $\square \times \triangle = {}^{-}36$

(7) $\square \times \triangle = {}^{+}12$

(8) $\square \times \triangle = {}^{-}12$

(9) $\square - \triangle = 0$

(10) $\square + \triangle = 0$

(11) $\triangle = \square \times \square$

(12) $\triangle + (\square \times \square) = 0$

(13) $\square = \triangle \times \triangle$

(14) $\square + (\triangle \times \triangle) = 0$

(15) $(\square \times \square) + (\triangle \times \triangle) = 169$

(16) $(\square \times \square) + (\triangle \times \triangle) = \frac{25}{4}$

(17) $(\square \times {}^{+}2) + {}^{+}3 = \triangle$

(18) $(\square \times {}^{+}2) + {}^{-}3 = \triangle$

(19) $(\square \times {}^{-}2) + {}^{+}3 = \triangle$

(20) $(\square \times {}^{-}2) + {}^{-}3 = \triangle$

(21) Widge says she uses the symbol "○" like this:

$$^{\circ}({}^{+}3) = {}^{-}3$$

$$^{\circ}({}^{-}3) = {}^{+}3$$

$$^{\circ}({}^{+}10) = {}^{-}10$$

$$^{\circ}({}^{+}\tfrac{1}{2}) = {}^{-}\tfrac{1}{2}$$

$$^{\circ}({}^{+}1.1) = {}^{-}1.1$$

What would this be?

$$^{\circ}({}^{+}5) =$$

(22) Can you find these **"opposites"**?

(a) $^{\circ}({}^{+}1) =$

(b) $^{\circ}({}^{-}4) =$

(c) $^{\circ}(\frac{1}{10}) =$

(d) $^{\circ}(0) =$

(23) Lex says that Widge is finding **additive inverses** and that she finds $^{\circ}(^{+}1)$ by asking, "What can I add to positive one to end up with zero?" What do you think?

(24) Can you find the **truth set** for each open sentence?

(a) $^{+}5 + \square = 0$

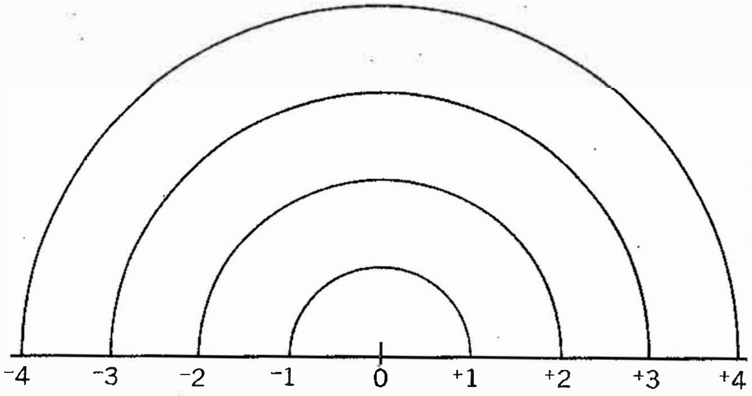
(b) $^{+}7 + \square = 0$

(c) $^{-}3 + \square = 0$

(d) $0 + \square = 0$

Can you give the truth set of each of these open sentences another name?

(25) Cynthia made a **rainbow picture:**



She says that you find the **additive inverse** of a number by "going to the opposite end of the rainbow."

What do you think?

(26) Can you find the **additive inverse** of each number?

(a) $^{+}10$

(b) $^{-}15$

(c) $^{+}3$

(d) 0

CHAPTER 9

Using Names and Variables in Mathematics

THE MEANING OF "EQUAL"

How can "two different numbers" ever be **equal**? This question poses the kind of problem we often encounter when we think carefully about the words we use.

In mathematics it is sometimes important to use words rather carefully. Mathematicians and logicians want to avoid contradictions in what they say. Consequently, they have thought a good deal about the problem of "two different numbers" being "equal," and have decided to resolve the difficulty this way:

(a) Mathematicians distinguish "symbols" or "names" from "things" or "ideas." You have an idea of **two**; but you cannot write the **idea** on the chalkboard. (They don't sell that kind of chalk!)

What you write on the board is a **symbol** or a **name**, such as

$$2$$

or

$$\text{II}$$

or

$$\text{two}.$$

(b) Mathematicians agree that "equality," which they write by using the symbol "=," is a statement about **names**, and not a statement about **things** or **ideas**.

Thus, "two different numbers" never will be "equal." When we write

$$1 + 1 = 2,$$

what we shall mean by this is that

$$1 + 1$$

is a **name** and that

$$2$$

is a **name**, and, in fact, these names **both name the same "thing"** or **"idea."** Both

$$1 + 1$$

and

$$2$$

are names for the **number** two.

(1) In Paul's class, there is a girl named Sandy Davis. Paul claims that we would be using the symbol "=" correctly, as it is used by modern mathematicians, if we wrote

$$\text{Sandy} = \text{Miss Davis}.$$

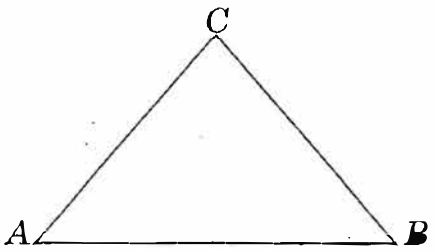
Do you agree?

(2) Jill says we could also write

Hawaii = the only state consisting entirely of islands.

Do you agree?

(3) In the triangle ***ABC***,



can you identify which angle is meant by the notation ∠ ***CAB***? Which angle do we mean when we write

∠ ***CBA*** ?

(4) George says that the triangle ***ABC*** has some kind of symmetry. In fact, he took a protractor and measured ∠ ***CAB*** and also ∠ ***CBA***. He concluded that they were equal, and so he wrote

∠ ***CAB*** = ∠ ***CBA***.

What do you think?

(5) Can you make up some statements where you use the symbol "=" the same way that modern mathematicians do? Can you explain the meaning of your statements?

THE "PRINCIPLE OF NAMES"

Mathematicians have a rule, which we shall call the "principle of names," and will abbreviate "PN." The rule PN says roughly this:

> If you take any **true** statement, take some occurrence of some name in this statement, erase it, and replace it by another name for the same thing, then the new statement you get will also be **true.**
>
> The same thing holds for **false** statements. If you start with a **false** statement and replace some occurrence of some name by another name for the same thing, then the new statement will also be **false.**

To understand PN, let's look at some examples.

(6) Let's use Sandy Davis's name again. If we start with the statement

Sandy was born in St. Louis, Missouri,

(which is true) and if we erase the name "Sandy"

▨ was born in St. Louis, Missouri,

and replace it by "Miss Davis" (which is another name for the same person), we get the statement

Miss Davis was born in St. Louis, Missouri.

Now, according to PN, the statement

Miss Davis was born in St. Louis, Missouri,
should also be true. Is it?

(7) Suppose that we start with the statement

Accra is a city very near the equator,

and suppose that we say

Accra is the capital of Ghana.

Can we write

The capital of Ghana is a city
very near the equator?

(8) Can we start with the statement

$$2 + 4 = 6$$

and use the fact that

$$4$$

names the same number that

$$3 + 1$$

does, to write

$$2 + (3 + 1) = 6?$$

(9) Can we start with the statement

$$5 + 5 + 5 = 15$$

and use the fact that

$$5$$

names the same number that

$$3 + 2$$

does, to write

$$5 + 5 + (3 + 2) = 15?$$

Is this a correct use of PN?

A METHOD FOR SHOWING WHERE (AND HOW) WE HAVE USED PN

Sometimes mathematics looks complicated when you see it written down, and it is useful to have ways of "writing notes to ourselves" so that we can keep track of what is going on. This occasionally happens when we are using PN. In order to keep track of where we use PN, we can use either of two methods. For one method we mark a "gaping hole" for the **name we "erase,"** and into the "gaping hole" we place the **new name for the same thing.** For the other method we agree to underline with a heavy black line **the name which we "erase"** and also **the new name for the same thing.**

Example 1

The "gaping-hole" method of writing:

(i) Sandy was born in St. Louis.

We erase the name "Sandy."

(ii) ▨ was born in St. Louis.

Into the "gaping hole," we insert the name "Miss Davis."

(iii) Miss Davis was born in St. Louis.

The "underlining" method of writing:

(i) $\underline{\text{Sandy}}$ was born in St. Louis.

(ii) Sandy = Miss Davis.

(iii) $\underline{\text{Miss Davis}}$ was born in St. Louis. PN from line (i), using line (ii).

Notice that, in the line above, we have given an "explanation" of what we did, by writing

PN from line (i), using line (ii).

Example 2

The "gaping hole" method of writing:

(i) $\left(3 \times 6\right) + \left(3 \times 1\right) = 21$

(ii) Now, "3×1" names the same thing that "3" names.

(iii) Hence, we can "erase" the name "3×1":

$(3 \times 6) +$ ▨ $= 21.$

(iv) Into the "gaping hole" we can put "3," to get

$(3 \times 6) + 3 = 21.$

The "underlining" method of writing:

(i) $\left(3 \times 6\right) + \underline{\left(3 \times 1\right)} = 21$

(ii) $3 \times 1 = 3$

(iii) $\left(3 \times 6\right) + \underline{3} = 21$ PN from line (i), using line (ii).

(10) Try to rewrite your work on questions 8 and 9, using the "underlining" notation and "explaining" the final step, as in the preceding examples.

THE "USE OF VARIABLES"

There is another rule in mathematics which looks somewhat like PN but is really quite different. We want to be careful not to get the two mixed up.

This other rule is the rule for using variables, which we shall abbreviate "UV."

We have actually learned about UV in Chapter 1, but we did not name it. Let's look at a few examples.

(11) If you start with the open sentence

$$\square + \square = 2 \times \square$$

and if you make a numerical replacement for the variable like this

$$\text{UV: } 3 \longrightarrow \square,$$

what statement do you get?

(12) If you start with the open sentence

$$\square + 1 < 5$$

and if you make a numerical replacement for the variable like this

$$\text{UV: } 8 \longrightarrow \square,$$

what statement do you get?

(13) If you start with the open sentence

$$(\square + \square) + \square = 3 \times \square$$

and if you use the fact that the "open name"

$$\square + \square$$

will always name the same thing that

$$2 \times \square$$

names, can you therefore write

$$(2 \times \square) + \square = 3 \times \square?$$

Have you used UV or PN? Did you use it correctly?

(14) If you start with the open sentence

$$(\square + \square) + \square = 3 \times \square$$

and if you make a numerical replacement for the variable by "putting 4 in every $\square$," what statement do you get? Did you use UV or PN?

(15) Al started with the open sentence

$$(\square + \square) + \square = 3 \times \square,$$

and "put 5 into all the $\square$'s on the left side of the = sign." He got, as a result, the open sentence

$$(5 + 5) + 5 = 3 \times \square.$$

Was Al using UV or PN? Did he use it correctly?

(16) If you start with the open sentence

$$(2 \times \square) + \square = 3 \times \square$$

and use the fact that the open name

$$\square$$

will always name the same thing that

$$1 \times \square$$

names, can you therefore write

$$(2 \times \square) + \underset{\nwarrow}{\xcancel{\square}} = 3 \times \square$$

(Erase this . . .)

$$(2 \times \square) + \underset{\uparrow}{(1 \times \square)} = 3 \times \square?$$

(and put this in its place.)

Did you use UV or PN? Did you use it correctly?

(17) Try to write out your work for question 16, using the "underlining" notation.

(18) How is UV different from PN?

(19) What number is named most often below?

one-half

$^{+}5 - {}^{-}2$

21

seven

X − III

$\frac{1}{2}$

11 - 4

1,000,000,000,003

VII

0.0004

six

35

(1 + 2) + 4

CHAPTER 10

Nora's Secrets

Can you find the truth set for these open sentences?

(1) $\square - 6 = 2$

$\{\quad\}$

(2) $2 \times \square - 6 = 2$

$\{\quad\}$

(3) Can you find the truth set for this open sentence?

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

$\{\ ,\ \}$

Can you find the truth set for each open sentence?

(4) $(\square \times \square) - (12 \times \square) + 35 = 0$

$\{\ ,\ \}$

(5) $(\square \times \square) - (8 \times \square) + 15 = 0$

$\{\ ,\ \}$

(6) $(\square \times \square) - (7 \times \square) + 10 = 0$

$\{\ ,\ \}$

(7) $(\square \times \square) - (6 \times \square) + 5 = 0$

$\{\ ,\ \}$

(8) $(\square \times \square) - (16 \times \square) + 55 = 0$

$\{\ ,\ \}$

(9) $(\square \times \square) - (9 \times \square) + 14 = 0$

$\{\ ,\ \}$

(10) Nora says she knows two secrets about this kind of equation. Do you know what she means?

Can you find the truth set for each open sentence?

(11) $(\square \times \square) - (15 \times \square) + 26 = 0$

(12) $(\square \times \square) - (14 \times \square) + 33 = 0$

(13) $(\square \times \square) - (9 \times \square) + 20 = 0$

(14) $(\square \times \square) - (12 \times \square) + 20 = 0$

(15) $(\square \times \square) - (21 \times \square) + 20 = 0$

(16) $(\square \times \square) - (15 \times \square) + 36 = 0$

(17) $(\square \times \square) - (102 \times \square) + 200 = 0$

(18) Do you know Nora's secrets? If you do, **DON'T TELL!** (It's a SECRET!)

Can you find the truth set for each open sentence?

(19) $(\square \times \square) - ({}^{+}3 \times \square) + {}^{-}10 = 0$

(20) $(\square \times \square) - ({}^{-}5 \times \square) + {}^{-}14 = 0$

(21) $(\square \times \square) - ({}^{-}5 \times \square) + {}^{+}6 = 0$

(22) $(\square \times \square) - ({}^{-}9 \times \square) + {}^{-}22 = 0$

(23) $(\square \times \square) - ({}^{-}3 \times \square) + {}^{-}10 = 0$

(24) $(\square \times \square) - (5 \times \square) + 8 = 2$

(25) $(\square \times \square) - (12 \times \square) + 25$
$= (1 \times \square) + 3$

(26) $(\square \times \square) - (8 \times \square) + 20 = 8$

PART TWO ▪ LOGIC

CHAPTER 11

Logic (by observing how people use words)

(1) Earle wrote

$$\square + \triangle = 8;$$

$$R_{\square} = \{2, 3, 5, 7\};$$

$$R_{\triangle} = \{1, 3, 5\}.$$

$\square$	$\triangle$
3	5
5	3
2	5
7	1
1	7

Table for Truth Set

What do you think?

(2) Joan wrote

P is x;

$R_p = \{$"New Orleans is a city", "New Hampshire is a city", "New Jersey is a state"$\}$;

$R_x = \{\text{true, false}\}$.

Can you make a table to show the **truth set** for Joan's open sentence?

(3) Larry made this table for the truth set of Joan's open sentence.

P	x
New Orleans is a city.	true
New Hampshire is a city.	false
New Hampshire is a city.	true
New Jersey is a state.	false

Do you agree?

(4) John said he was going to keep track of his friends' statements. He was **not** going to see how many

words they used, he was **not** going to worry whether their words were elegant, and he was **not** going to care about what they said. The **only thing** he was going to study was whether their statements were **true** or **false**. John made this table:

□	△	□ and □

He said, "I'm going to study what my friends mean when they use the word '**and**.' I don't know what statement they may put in the □, but it will be either **true** or **false**. I don't know what statement they may put in the △, but it will be either **true** or **false**."

How many possibilities must John allow for in his table?

(5) Eileen said that people might put a **true** statement in the □ and a **true** statement in the △, so she wrote:

□	△	□ and △
T	T	

What do you think ?

(6) Jill said that people might put a **true** statement in the □ and a **false** statement in the △, so she added another line to John's table:

□	△	□ and △
T	T	
T	F	

What do you think ?

(7) Can you add any **more** possibilities to John's table?

(8) John asked one of this friends to make up a sentence,

□ and △,

with

$R_{\square}$ = {"I am ten years old", "I am fifteen years old", "I am seven feet tall "}

$R_{\triangle}$ = {"my name is George", "my name is Albert "} .

How many sentences **could** his friend have written? Can you write them all ?

(9) Henry, who is twelve years old, wrote:

I am ten years old and my name is Albert.

Was Henry's statement **true** or **false**?

(10) Nancy says that if you put a **true** statement in the □ and a **true** statement in the △, then the statement

□ and △

will be **true**; so she wrote:

□	△	□ and △
T	T	T

What do you think?

(11) Can you complete John's table?

(12) Can you make a table of the way **your** friends use the word "**or**"?

(13) Sandy made this table to show how her friends use "**or**":

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

What do you think?

(14) Ann disagreed with Sandy. Ann says **her** friends use "or" this way:

P	Q	P or Q
T	T	F
T	F	T
F	T	T
F	F	F

What do you think?

(15) Alex says sometimes his friends use "or" the way Sandy says and sometimes the way Ann says. What do you think?

(16) Alex gave this example: "I'll either go canoeing all day Saturday or I'll go to the baseball game." Which way is "**or**" used in this sentence, Ann's way or Sandy's way?

(17) Kevin gave this example: "I sure hope I get an A in English or math." Which way is "**or**" used in this sentence?

(18) A waitress said, "You may have potato or spaghetti." Which kind of "or" did she mean?

(19) Do you know what mathematicians mean by the "**inclusive or**"?

(20) Do you know what mathematicians mean by the "**exclusive or**"?

(21) Kathy made a table for her symbol "~," which means "**not**":

P	Q	$\sim P$
T	T	F
T	F	F
F	T	T
F	F	T

Do you agree?

(22) Can you make a table for "$\sim Q$"?

(23) John uses □ and △ for the variables in his tables. Sandy uses P and Q for the variables in her table. In order not to get mixed up, John and Sandy have made a table labeled with both P and Q **and** □ and △:

			"Inclusive or"	"Exclusive or"		
□	△	□ and △	□ or △	□ or △	~□	~△
P	Q	P and Q	P or Q	P or Q	$\sim P$	$\sim Q$
T	T					
T	F					
F	T					
F	F					

Can you fill in the rest of their table?

(24) What do your friends mean by 'if . . . then . . ."? Can you show this by a truth table?

□	△	If □, then △.
P	Q	If P, then Q.
T	T	
T	F	
F	F	

(Mathematicians write "If P, then Q" this way:

$$P \Rightarrow Q$$

or else

$$P \supset Q.)$$

(25) Sandy's father says that mathematicians write

$$P \Leftrightarrow Q$$

to mean

P has the same truth value as Q.

Sandy made a truth table for

$$P \Leftrightarrow Q.$$

Can you figure out how she did it?

P	Q	$P \Leftrightarrow Q$
T	T	
T	F	
F	T	
F	F	

CHAPTER 12

Logic (by making agreements)

(1) Bill says it is very confusing to find people using the same word for different meanings. He says we should agree, **in this class**, that whenever we say "**and**" we will use it according to this table:

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

What do you think?

(2) Paul's father says that mathematicians always use "**or**" to mean the "**inclusive or.**" Let's complete the following truth table for "**or.**" And let's agree that, in this class, we will always use "**or**" according to our table.

P	Q	P or Q

(3) Let's make up a truth table for "**if . . . then . . .**" Let's agree that, in this class, we will always use "**if . . . then . . .**" according to the table we make up.

P	Q	If P, then Q.

(4) Let's make up truth tables for "~," and for "$\Leftrightarrow$," and let's agree that, from now on in this class, we will use "~" and "$\Leftrightarrow$" according to our tables.

P	Q	$\sim P$	$\sim Q$	$P \Leftrightarrow Q$

CHAPTER 13

Some Complicated Formulas in Logic

(1) Larry says he can fill in the column for

$$\sim (P \text{ and } Q)\,.$$

Can you?

(2) Joan says there is really nothing new in this; you just use things you already know, such as "$\sim$" and "**and**." What do you think?

(3) Can you fill in a column for

$$\sim (P \text{ or } Q)\,?$$

(4) Can you fill in a column for

$$\sim (\sim P)\,?$$

(5) Can you fill in a column for

$$\sim (P \Rightarrow Q)\,?$$

(6) Can you fill in a column for

$$\sim (P \Leftrightarrow Q)\,?$$

(7) Can you fill in a column for

$$(\sim P) \text{ and } (\sim Q)\,?$$

(8) Can you fill in a column for

$$(\sim P) \text{ or } (\sim Q)\,?$$

Can you fill in columns with the following headings?

(9) $(\sim P) \text{ or } Q$

(10) $P \text{ or } (\sim Q)$

(11) $Q \Rightarrow P$

(12) $P \text{ and } (\sim Q)$

(13) $(\sim P) \Leftrightarrow Q$

(14) $\sim \left[(\sim P) \Leftrightarrow Q\right]$

(15) $\sim \left\{\sim\left[(\sim P) \Leftrightarrow Q\right]\right\}$

(16) $\left[(\sim P) \Leftrightarrow Q\right] \text{ or } \left[P \Rightarrow Q\right]$

(17) $\left[P \Rightarrow Q\right] \text{ or } \left[Q \Rightarrow P\right]$

(18) $\sim \left\{\left[P \Rightarrow Q\right] \text{ or } \left[Q \Rightarrow P\right]\right\}$

(19) $P \Leftrightarrow (\sim Q)$

(20) $\sim \left[P \Leftrightarrow (\sim Q)\right]$

(21) $P \text{ and } (\sim P)$

(22) $P \text{ or } (\sim P)$

(23) $P \Leftrightarrow (\sim P)$

(24) Michael says he has made an interesting discovery. Have you?

(25) Mark says that the column for

$$\sim (\boldsymbol{P} \text{ and } \boldsymbol{Q})$$

is **exactly the same** as the column for

$$(\sim \boldsymbol{P}) \text{ or } (\sim \boldsymbol{Q})\,.$$

What do you think?

(26) Eileen listened to what Mark said, and wrote

$$\left[\sim (\boldsymbol{P} \text{ and } \boldsymbol{Q})\right] \Leftrightarrow \left[(\sim \boldsymbol{P}) \text{ or } (\sim \boldsymbol{Q})\right].$$

The class named this "Eileen's formula." Can you make up any **other** formulas the way Mark and Eileen made up this one?

CHAPTER 14

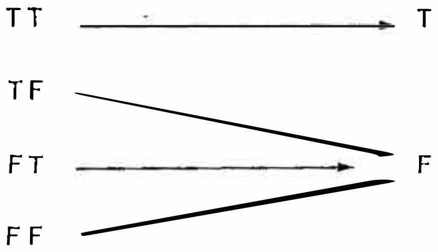
Logic (by thinking like a mathematician)

(1) Randy made up a truth table for "**and**," like this:

□	△	□ and △
P	***Q***	***P*** and ***Q***
T	T	T
T	F	F
F	T	F
F	F	F

Did he use "**and**" the way we have agreed to?

(2) Lex's father says that the relationships in Randy's table can be shown by using what mathematicians call a **mapping**:



He says that TT means that ***P*** is true and ***Q*** is true, and that mathematicians call T the **image** of TT. Do you agree?

(3) Debbie says that, if $V = \{T, F\}$, then

$$\{TT, TF, FT, FF\}$$

is just the **Cartesian product**

$$V \times V.$$

She says Randy has a mapping of $V \times V$ into V.

$$V \times V \longrightarrow V$$

What do you think?

(4) How many **different** ways can you map

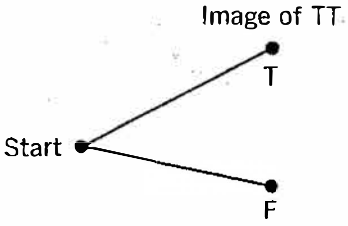
$$V \times V \longrightarrow V?$$

(5) When they want to **count** something, mathematicians sometimes make a special kind of a drawing which is known as a **tree** or a **tree diagram.** Geoffrey tried to count the mappings of $V \times V$ into V by drawing a tree diagram.

Geoff says you can map the

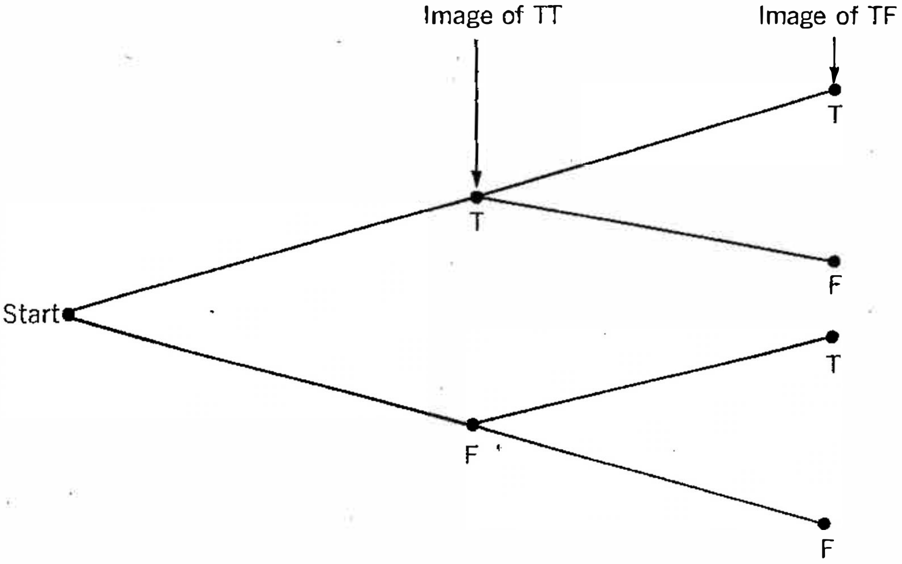
TT

either into T or into F. To show these two choices, he started his tree diagram like this:



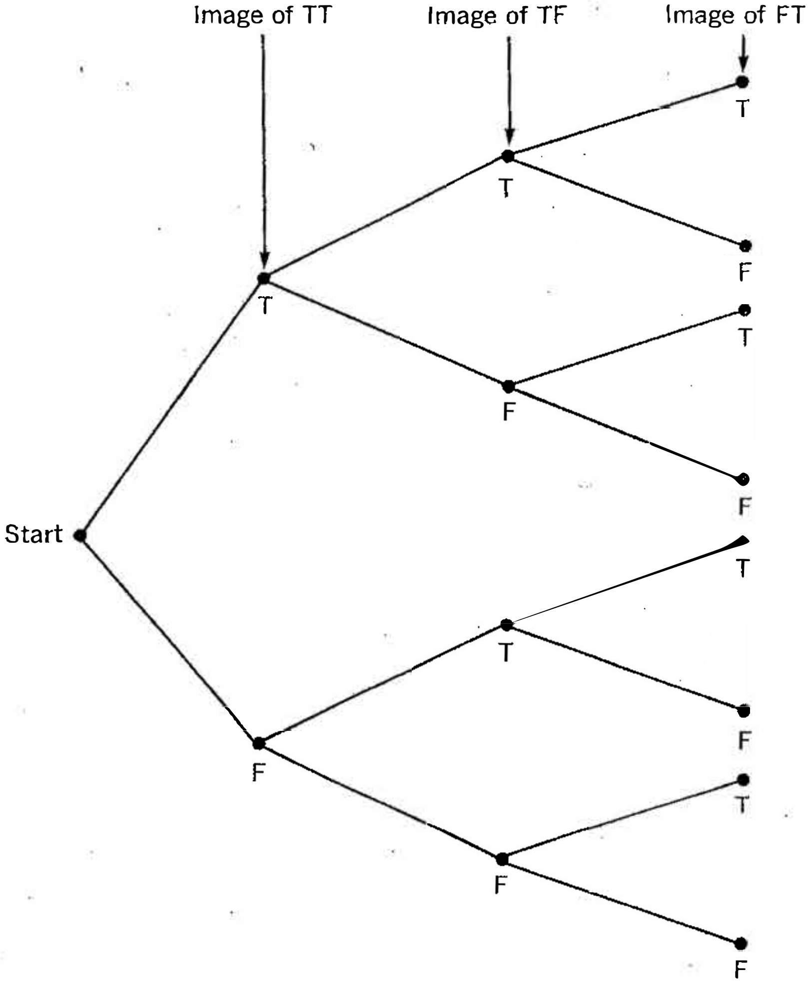
Can you finish Geoff's tree?

(6) After you've mapped TT, Allen says you can map TF either into T or into F:



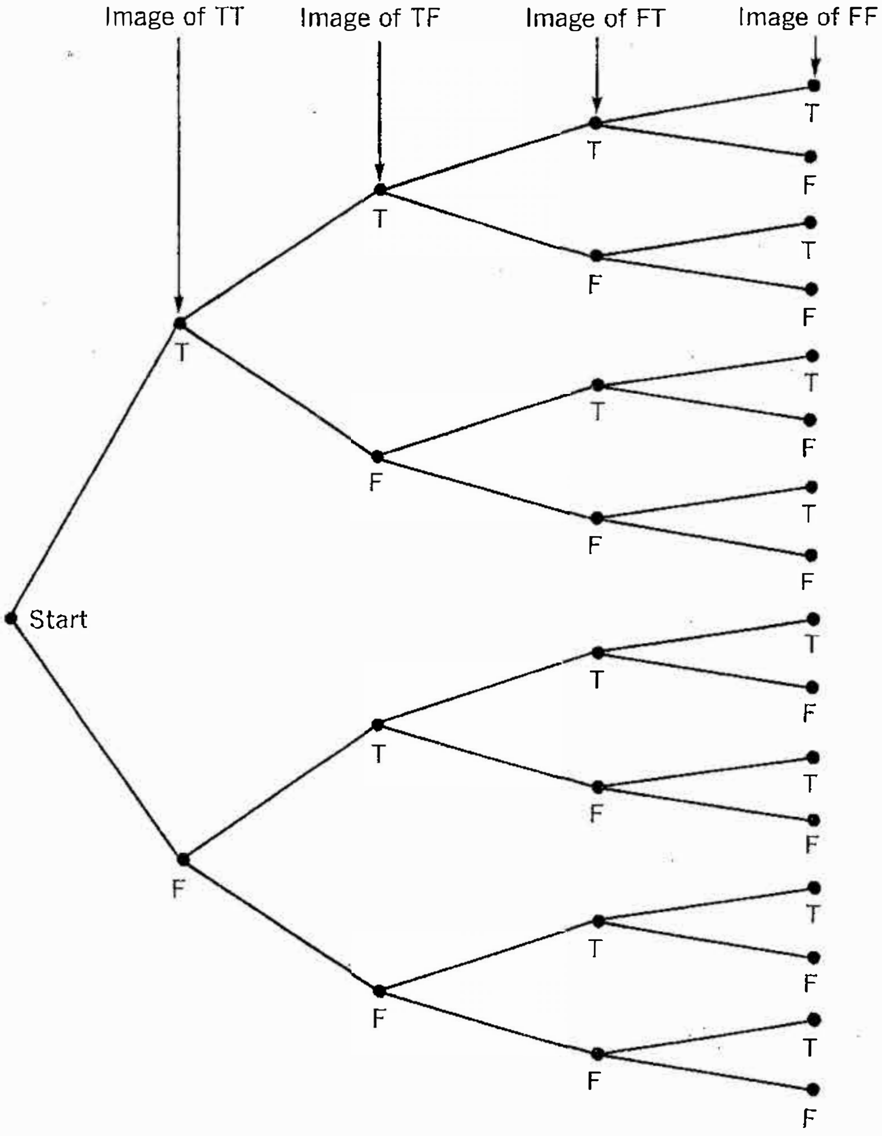
Do you know what Allen means? Can you finish this tree?

(7) Nancy says that after you've mapped TT and TF, you can map FT either into T or into F:



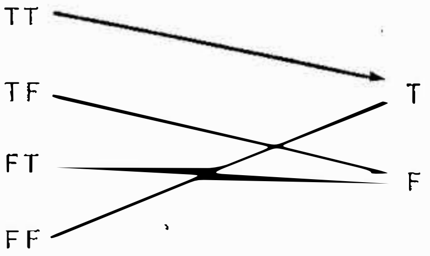
What do you think?

(8) Amy finished the tree like this:



Can you trace a path through Amy's tree that will correspond to the mapping of "**and**"?

(9) Bill represented the mapping of "**or**" (meaning the "**inclusive or**") with an arrow diagram:



Do you agree?

(10) Can you make a diagram of a mapping of

$$\boldsymbol{V} \times \boldsymbol{V} \longrightarrow \boldsymbol{V}$$

to correspond to each of the following mappings?

(a) or

(b) If $\boldsymbol{P}$, then $\boldsymbol{Q}$.

(c) $\boldsymbol{P} \Leftrightarrow \boldsymbol{Q}$

(d) $\sim \boldsymbol{P}$

(e) $\sim \boldsymbol{Q}$

(f) $(\sim \boldsymbol{P})$ or $(\sim \boldsymbol{Q})$

(g) $(\sim \boldsymbol{P})$ and $(\boldsymbol{P})$

(11) How many **possible** mappings of

$$\boldsymbol{V} \times \boldsymbol{V} \longrightarrow \boldsymbol{V}$$

are there? Can you show each of them by an arrow diagram?

(12) Can you find **names** for each mapping of

$$\boldsymbol{V} \times \boldsymbol{V} \longrightarrow \boldsymbol{V}?$$

CHAPTER 15

Inference Schemes

In several preceding chapters, we have looked at logic from the point of view of the logical connectives that commonly occur **within** sentences: connectives such as "and," "or," "not," "if . . . , then . . . ," and so on. We now wish to look at the logical relations that often exist **between** sentences. Here are some examples:

(1) If Mr. Wilson is the guilty person, then he certainly had to be in New York City on July 10, 1967. However, Mr. Wilson was **not** in New York City on July 10, 1967. Therefore, Mr. Wilson **cannot** be the guilty person.

Jerry has tried to take these statements about Mr. Wilson, and represent them as an **inference** scheme:

Jerry lets P stand for "Mr. Wilson is the guilty person."

He lets Q stand for "Mr. Wilson was in New York City on July 10, 1967."

Can you now represent the statements about Mr. Wilson, using Jerry's P and Q?

(2) Marie says the statement, "If Mr. Wilson is the guilty person, then he was in New York City on July 10, 1967," can be represented as

$$P \Rightarrow Q.$$

What do you think?

(3) Nancy says the statement, "Mr. Wilson was **not** in New York City on July 10, 1967," can be represented as

$$\sim Q.$$

What do you think?

(4) Al says the whole discussion about Mr. Wilson can be represented this way:

$$\frac{P \Rightarrow Q, \sim Q}{\sim P}$$

Do you see how Al's notation works?

(5) Consider these statements:

> If Jerry believes that smoking causes cancer, then he would be foolish to smoke. Jerry does believe that smoking causes cancer. Therefore, Jerry would be foolish to smoke.

Can you write out the **inference scheme** that seems to be used here?

(6) Consider these statements:

> If Mr. Harper was in San Diego at 10 A.M., Tuesday, then he must be innocent. If he was

> driving from Los Angeles to San Diego at 10 A.M., Tuesday, then he must be innocent. Now, we know definitely that he was either in San Diego at 10 A.M., Tuesday, or else he was driving from Los Angeles to San Diego at that time. Therefore, Mr. Harper must be innocent.

Can you write out the **inference scheme** that seems to be used here?

(7) Can you make up any **inference schemes** of your own that seem to be valid?

(8) Toby made up this inference scheme:

$$\frac{P \Rightarrow Q,\ Q \Rightarrow R}{P \Rightarrow R}$$

Do you think it is valid? Can you give some examples, using words?

(9) Can you find a way to test the truth of inference schemes by using truth tables?

What good would such a method be?

(10) Sarah made up this inference scheme:

$$\frac{P \Rightarrow Q,\ \sim P}{\sim Q}$$

Do you think it is valid? Can you give some examples, using words? Can you test it by using a truth table?

(11) How many **valid** inference schemes can you list?

(12) Who might be interested in studying inference schemes? Do you think mathematicians would? Do you think logicians would? Do you think lawyers would? Who else might be? What good would it do to study inference schemes?

CHAPTER 16

The Game of Clues

The rules for the game of clues are as follows:

One team (or one person) has a secret. Let's call this team TWS, for "team with secret." The other team seeks to discover this secret. Let's call this team DISC, for "discovery."

1. TWS writes some numbers on a piece of paper which then is sealed in an envelope, or otherwise put where it cannot be read. (For example, someone can fold the paper and sit on it.)

2. DISC seeks to force TWS to disclose the "secret" numbers, and to let everyone read the paper.

3. Only positive integers are allowed. Repetitions are allowed; for example, the secret numbers might be:

$$1, 3, 5, 7, 7, 7, 7.$$

4. In guessing the secret numbers, DISC does **not** have to guess the **order** in which they are written; for example,

$$7, 3, 5, 7, 1, 7, 7$$

would count as the same list as the one given in the rule preceding.

5. TWS writes **clues** on the board, labeling the clues *a*, *b*, *c*, . . . , and so on (it is desirable to omit "F" and "T" as labels, since we have a **different** use for them).

6. The clues may be **true** or they may be **false.**

7. Anytime that DISC believes there is a **contradiction** in a certain set of clues, DISC **lists** the clues in question and tries to show that there **is** a contradiction in these clues.

8. DISC is **right** about the contradiction if the clues they list **do** contain a contradiction, and **if no proper subset** of the clues on the list contains a contradiction.

9. DISC is **wrong** about the contradiction if the clues they list do **not** contain a contradiction or **if a proper subset** of the clues does contain a contradiction.

10. At the start of the game, DISC has 5 points.

11. Anytime DISC is **wrong** about a contradiction, it loses one point.

12. Anytime DISC is **right** about a contradiction, TWS must mark T (for **true**) or F (for **false**) beside each clue that is involved in the contradiction. TWS **must be correct** in marking T's and F's (even though TWS is allowed to make some of the clues themselves false).

13. The game ends in one of two ways: If DISC loses all 5 points, then TWS tears up the secret paper and

never allows it to be read (DISC has "lost"). If, on the other hand, DISC is able to force disclosure of the paper, then everyone on the DISC team is allowed to read it, and DISC has "won."

14. The procedure by which DISC may be able to force disclosure of the secret is this: whenever it believes it is in a position to do so, DISC can list the numbers that it believes must be written on the paper, and can **bet** TWS that **no other collection of numbers** would satisfy all the known truth values of the clues. (That is, no other collection of numbers would make **true** statements of all the clues labeled T and **false** statements of all the statements labeled F.) If TWS **can** find **any other** collection of numbers that will be consistent with the T's and F's, then DISC **loses** the bet, and DISC's points are reduced to zero. (Which, of course, means the secret paper is torn up and the numbers **never** disclosed.)

If TWS **cannot** find any other collection of numbers that will be consistent with the indicated T's and F's, then DISC **wins** the bet, and TWS is forced to disclose the secret.

In order to make the game interesting, TWS must provide a growing collection of interesting clues.

Here is a sample game:

DISC begins, of course, with 5 points.

TWS begins by listing these clues:

a. 5 numbers on paper.
b. All odd numbers.
c. Their sum is 26.
d. The largest number is 7.
e. The smallest number is 8.

DISC says there is a contradiction in clues *a, b,* and *c,* because an **odd** number of **odd** numbers cannot add up to an **even** total.

Since DISC is right about

$$\{a, b, c\},$$

it is necessary for TWS to label *a, b,* and *c* as either T or F; TWS does this as follows:

F *a.* 5 numbers on paper.
T *b.* All odd numbers.
F *c.* Their sum is 26.
d. The largest number is 7.
e. The smallest number is 8.

TWS changes the clues to look like this:

a. 7 numbers on paper.
T *b.* All odd numbers.
c. Their sum is 12.

d. The largest number is 7.

e. The smallest number is 8.

DISC says that

$$\{a, b, c\}$$

still contains a contradiction: an odd number of odd numbers cannot add up to an even sum.

Since DISC is right about this contradiction, TWS must label *a, b,* and *c* as T or F. They do this as follows:

T *a.* 7 numbers on paper.

T *b.* All odd numbers.

F *c.* Their sum is 12.

d. The largest number is 7.

e. The smallest number is 8.

DISC says that

$$\{d, e\}$$

contains a contradiction, because the **largest** number cannot be smaller than the smallest number.

Since DISC is right about this, TWS must mark T's and F's on

$$\{d, e\}.$$

They do this as follows:

T *a.* 7 numbers on paper.

T *b.* All odd numbers.

F *c.* Their sum is 12.

T *d.* The largest number is 7.

F *e.* The smallest number is 8.

TWS changes the clues to read like this:

T *a.* 7 numbers on paper.

T *b.* All odd numbers.

c. Their sum is 13.

T *d.* The largest number is 7.

F *e.* The smallest number is 8.

Although they are not yet forced to do so, TWS labels clue *c* as T, in order to make the game move along faster. The clues now look like this:

T *a.* 7 numbers on paper.

T *b.* All odd numbers.

T *c.* Their sum is 13.

T *d.* The largest number is 7.

F *d.* The smallest number is 8.

(1) Can you finish this game?

(2) Why don't you write your own secret numbers, and make up your own clues?

CHAPTER 17

Measurement Uncertainties

(1) Can you measure how long your classroom is?

(2) Can you measure **exactly** how long the classroom is?

(3) If you are in doubt about your measurement, how doubtful are you? Could you be in error by one yard? by one foot? by one inch? by one-tenth of an inch? by one-hundredth of an inch? by one-millionth of an inch? Would your measurement be **exact**?

CLASS EXPERIMENT 1

(4) Have 10 people, **working independently and in secret, guess** the length of the classroom and write their guesses on a piece of paper. Give these 10 pieces of paper to a trustworthy person. We'll work with these numbers in the next few questions.

(5) How much doubt do you feel about these 10 guesses? Could they be in error by as much as 10 feet? by as much as one yard? by as much as one foot? by as much as one inch?

(6) Let's write all 10 guesses on the chalkboard, converting to the same unit in each case (probably the foot, and its decimal parts, is the best unit to use).

(7) Can you find the **average** of these 10 numbers? What is it?

(8) How much doubt do you feel about this **average**? Could it be in error by as much as 10 feet? by as much as one yard? by as much as one foot? by as much as one inch?

(9) **We want to see how well these 10 people agreed with one another.** (This is why we wanted to work independently, and to write their guesses in secret!) Mathematicians have thought of many different ways of comparing how well different measurements (or guesses) agree.

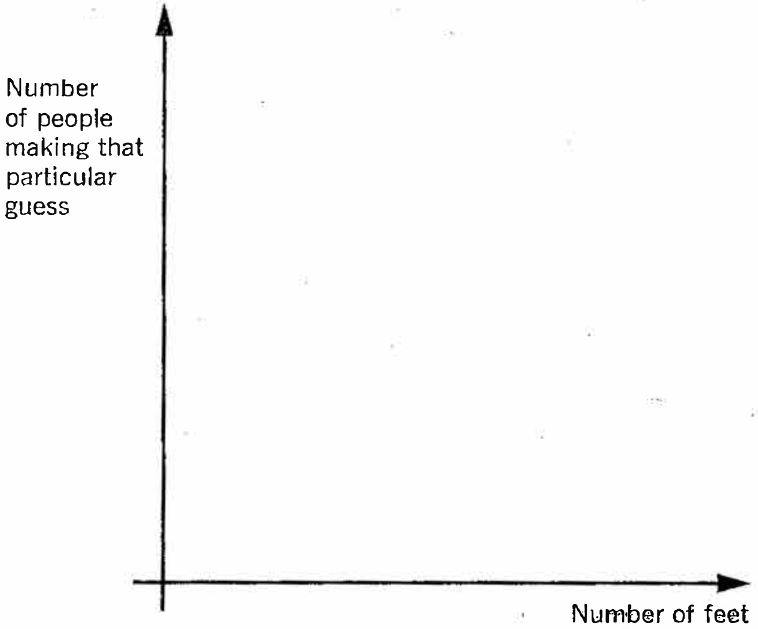
One method is to compute the **range** of the guesses.

For instance, suppose that the guesses were 31 ft, 30 ft, 33 ft, 27 ft, 32 ft, 32 ft, 29 ft, 30 ft, 34 ft, and 28 ft. Then the range (in the sense of statistics) of the 10 guesses would be 7 ft.

Can you see how to find the range of any number of guesses?

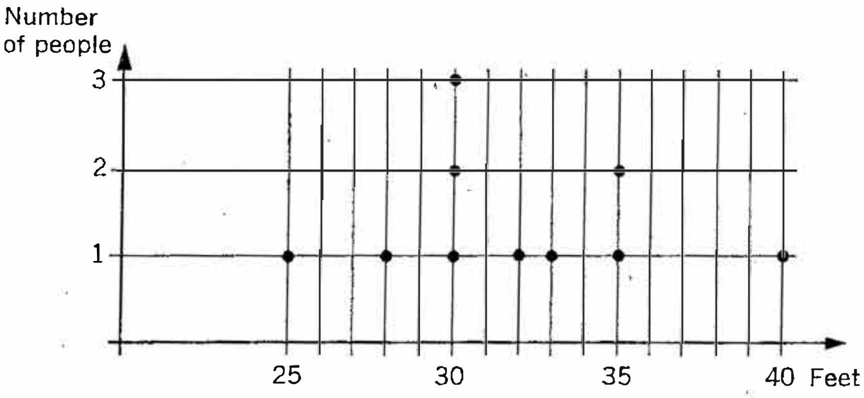
Do you think that range is a good measure of error?

(10) A second method is to plot our points on a **graph,** like this:



Suppose, for example, the guesses were: 30 ft, 33 ft, 35 ft, 30 ft, 25 ft, 32 ft, 30 ft, 28 ft, 35 ft, 40 ft.

For these 10 guesses, our graph might be made to look like this:



This gives a kind of visual picture that suggests how well the different guesses agreed.

Why don't you make a graph using the 10 guesses from **your** class? How well did the people agree?

(11) Another method is **the method of average absolute deviation from the average.** (The name makes this method sound much harder than it really is. After a while this name will make sense to you, if you think about it.) We can illustrate this method, using sample data. If the guesses were 30, 33, 35, 28, 25, 32, 30, 28, 35, and 40, then we can find the **average** like this:

30
33
35
28
25
32
30
28
35
40
316

$$10\overline{)316}^{\,31.6}$$

31.6 is the **average.**

Now, 30 (the first guess) **deviates** from this average by this amount:

$$31.6 - 30 = 1.6.$$

so 1.6 is the **deviation** of the first guess from the average.

The next guess, 33, deviates from the average by this much:

$$33 - 31.6 = 1.4.$$

Similarly, here are the **deviations from the average** for the other guesses:

$$35 - 31.6 = 3.4$$
$$31.6 - 28 = 3.6$$
$$31.6 - 25 = 6.6$$
$$32 - 31.6 = 0.4$$
$$31.6 - 30 = 1.6$$
$$31.6 - 28 = 3.6$$
$$35 - 31.6 = 3.4$$
$$40 - 31.6 = 8.4$$

Consequently, the **deviations** (or deviations from the average) are: 1.6, 1.4, 3.4, 3.6, 6.6, 0.4, 1.6, 3.6, 3.4, 8.4.

What shall we do with **these** 10 numbers? The answer is that we will **average them**!

$$\begin{array}{r} 1.6 \\ 1.4 \\ 3.4 \\ 3.6 \\ 6.6 \\ 0.4 \\ 1.6 \\ 3.6 \\ 3.4 \\ \underline{8.4} \\ 34.0 \end{array} \qquad 10\,\overline{)\,34.0}^{\;3.4}$$

So, the **average** of the deviations from the average is 3.4, using our sample data. Can you compute the **average absolute deviation from the average** using the 10 guesses made in your class? For your 10 guesses, was the average absolute deviation from the average greater or less than that of our sample data? Which 10 guesses are more in agreement, yours or the 10 guesses in the sample data?

(12) Why do you suppose we call this the average **absolute** deviation from the average? Do you know what we mean by **absolute value?**

CLASS EXPERIMENT 2

(13) Have 10 people **measure** the length of the room with 6-inch rulers. As before, the 10 people must work independently and in secret, and each must write his answer before seeing what any of the others have done. Give these 10 pieces of paper to a trustworthy person, who will **keep** them. We want to be able to work with these 10 numbers, and to refer back to them whenever we need to.

(14) How well do these 10 people agree? Could one of them be in error by as much as 10 feet? by as much as one yard? by one foot? by one inch? by one-tenth of one inch? by one-hundredth of one inch?

(15) Compute the **average** of these 10 numbers. Do you think the average could be in error by as much as one foot? by how much?

(16) Compute the **range** of these 10 numbers. Did the "6-inch-ruler" measurements agree **more,** or **less**, than the guesses from Experiment 1?

(17) Use the **method of graphs.** Do the "6-inch-ruler" measurements seem to show **more** agreement, or **less**, than the guesses did?

(18) Use the **method of average absolute deviation from the average**. Do the 6-inch-ruler measurements show **more** agreement, or **less**, than the guesses did?

CLASS EXPERIMENT 3

(19) Have 10 people measure the length of the room, using yardsticks or meter sticks of good quality. How well do the 10 people agree?

CLASS EXPERIMENT 4

(20) Have 10 **teams** of people measure the length of the room, using high-quality tape measures. How well do the 10 teams agree?

(21) How would you find the **exact** length of the room?

CHAPTER 18

Identities

Which are **true**? Which are **false**? Which are **open**?

(1) $29 + 51 = 70$

(2) $\frac{1}{2} + \frac{1}{3} = \frac{1}{5}$

(3) $2 \times 3\frac{1}{2} = 7$

(4) $5 + \square = 6$

(5) $12 + \square = 12$

(6) $\frac{1}{10} + \frac{1}{10} = 3\frac{1}{2}$

Can you find the **truth set** for each open sentence?

(7) ${}^{+}8 + \square = {}^{+}9$

(8) ${}^{+}8 + \square = {}^{+}7$

(9) ${}^{+}8 + \square = 0$

(10) $(\square - 2) \times (\square - 3) = 0$

(11) $\square \times \square = 16$

(12) $\square \times \square = 169$

(13) Can you make up an **open sentence** that will become **true** for **every** legal substitution?

(14) Jerry says this open sentence will become **true** for every substitution:

$$\square \times 0 = 0.$$

Do you agree?

(15) Do you know what we mean by an **identity**?

(16) Sarah says that mathematicians use the symbol "V_x" to mean "for all x," and that they would write Jerry's idea this way:

$$V_x \quad x \cdot 0 = 0.$$

(17) Can you make up any more **identities**?

IMPORTANT: For our future work we will need a long list of identities. The best way to get such a list will be for you to maintain a "cumulative" list of identities as you make them up. Keep this list in a safe place where we can refer to it whenever we may need to.

CHAPTER 19

Making Up Some "Big" Identities by Putting Together "Little" Ones

In making up identities, you probably had a method —whether conscious or unconscious. For example, you may have started an identity something like this

$$[(\Box + 3) \times \triangle] \times (\triangle + \tfrac{1}{2})$$

and then thought to yourself, "Ah! If I now multiply by zero, the result will be zero." Consequently, you write

$$\{[(\Box + 3) \times \triangle] \times (\triangle + \tfrac{1}{2})\} \times 0 = 0.$$

In a simpler case, you may have begun

$$\Box \times \Box$$

and then thought, "Aha! If I now add zero, the sum will by unaffected by the addition of zero," and so you wrote

$$(\Box \times \Box) + 0 = \Box \times \Box.$$

As a third line of reasoning, you may have begun with

$$(\Box + \tfrac{1}{2}) + (\Box + 3)$$

and said to yourself, "Why, all you have to do is to put **exactly** the same thing on the other side of the equals sign, and surely that will give you an identity!" Consequently, you wrote:

$$(\Box + \tfrac{1}{2}) + (\Box + 3) = (\Box + \tfrac{1}{2}) + (\Box + 3).$$

In this chapter we want to investigate these methods for making "fancy" identities out of other, simpler ones.

Probably the best way to carry on our investigation is to look at a few examples.

Example 1

Sometimes you use UV (use of variables).

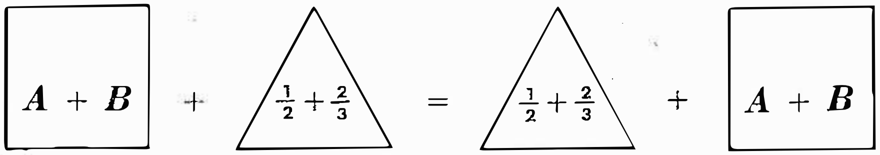
You might start with a simple identity, like

$$\Box + \triangle = \triangle + \Box,$$

and then you use UV to get a more complicated identity. Suppose, for example, we do this:

$$\begin{aligned} \text{UV: } A + B &\longrightarrow \Box \\ \tfrac{1}{2} + \tfrac{2}{3} &\longrightarrow \triangle \end{aligned}$$

The result will be



which we would ordinarily write as

$$(A + B) + \left(\tfrac{1}{2} + \tfrac{2}{3}\right) = \left(\tfrac{1}{2} + \tfrac{2}{3}\right) + (A + B).$$

We have made up this "more complicated" identity by using UV.

If you prefer $\square$'s and $\triangle$'s, instead of A's and B's, to indicate your variables, you can use UV again, like this:

$$\text{UV:}\quad \square \longrightarrow A$$

$$\triangle \longrightarrow B$$

$$(\square + \triangle) + \left(\tfrac{1}{2} + \tfrac{2}{3}\right) = \left(\tfrac{1}{2} + \tfrac{2}{3}\right) + (\square + \triangle)$$

Example 2

Sometimes you use PN (principle of names).

We might, for instance, start with the identity we just got. We could make it still more complicated if we want to. For example,

(i) $\square + \triangle = \triangle + \square$

and we just got the identity

(ii) $(\square + \triangle) + \left(\tfrac{1}{2} + \tfrac{2}{3}\right) = \left(\tfrac{1}{2} + \tfrac{2}{3}\right)$

$$+ \underline{(\square + \triangle)}.$$

We could use PN—the method of "erasing" one name and putting in its place another name for the same thing—to get:

(iii) $(\square + \triangle) + \left(\tfrac{1}{2} + \tfrac{2}{3}\right) = \left(\tfrac{1}{2} + \tfrac{2}{3}\right)$

$+ \underline{(\triangle + \square)}$ PN from line (ii), using line (i).

Q.E.D.*

(Remember, the heavy underlining shows which name was "erased" and replaced by another name for the same thing.)

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*Q.E.D. stands for the Latin term *quod erat demonstrandum*. It means we have now proved what we were asked to prove.

(1) Try to make up some more identities to add to your cumulative list. (Remember to keep your list carefully; we shall need it later.)

(2) Start with the identity

$$\bigtriangledown + 0 = \bigtriangledown,$$

and use UV like this:

$$\text{UV:}\ \boldsymbol{A} + \boldsymbol{B} + \tfrac{1}{2} \longrightarrow \bigtriangledown.$$

What result do you get? Now replace the $\boldsymbol{A}$'s and $\boldsymbol{B}$'s by $\Box$'s and $\triangle$'s. What is your final result?

(3) Start with the identity

$$(\Box + \triangle) + [\Box \times (\triangle + \bigtriangledown)] = (\Box + \triangle) + [\Box \times (\triangle + \bigtriangledown)].$$

Now use PN, and use these identities:

$$\Box + \triangle = \triangle + \Box$$

$$\Box \times (\triangle + \bigtriangledown) = (\Box \times \triangle) + (\Box \times \bigtriangledown).$$

Can you get a more complicated identity as a result? What result did you get? Can you write out each step carefully?

CHAPTER 20

Shortening Lists: "Axioms" and " Theorems"

(1) Jeanne has this list of identities:

$$(\square \times \triangle) + 3 = (\triangle \times \square) + 3$$

$$(\square \times \triangle) + 4 = (\triangle \times \square) + 4$$

$$(\square \times \triangle) + 5 = (\triangle \times \square) + 5$$

$$(\square \times \triangle) + 6 = (\triangle \times \square) + 6$$

$$\vdots$$

What do you suppose the three dots at the bottom mean? Can you make up any more identities that "look like" those on Jeanne's list—that is to say, that have this same pattern?

(2) Albert says he can write **one single identity** to represent Jeanne's entire list. Do you think he can? How?

(3) Suppose you had Albert's single identity. Could you get the identity

$$(\square \times \triangle) + 5 = (\triangle \times \square) + 5$$

from Albert's by using UV? How?

(4) Anne has this list of identities:

$$\square + \triangle = \triangle + \square$$

$$\square \times \triangle = \triangle \times \square$$

$$A + (B \times C) = (C \times B) + A$$

Could you shorten Anne's list, without really losing anything?

(5) Marjory says she could shorten Anne's list to this:

$$\square + \triangle = \triangle + \square$$

$$\square \times \triangle = \triangle \times \square$$

Marjory says that once you know these two, you can always make up

$$A + (B \times C) = (C \times B) + A,$$

by using UV and PN. What do you think?

(6) Take **your** list of identities and shorten it as much as possible, without really losing anything. **What does your final list look like?**

(7) Do you know what mathematicians mean by the word **axiom**?

(8) Do you know what mathematicians mean by the word **theorem**?

CHAPTER 21

How Shall We Write Derivations?

Debbie claimed that she could use

Axiom 1: $\square = \square$

Axiom 2: $\square + \triangle = \triangle + \square$

Axiom 3: $\square \times \triangle = \triangle \times \square$

together with UV and PN and end up with

$$A + (B \times C) = (C \times B) + A.$$

George challenged Debbie to prove it, and so Debbie wrote this:

Statement	**Reason**
(i) $\square = \square$	Axiom 1.
(ii) $A + (B \times C) = A + \underline{(B \times C)}$	UV: $A+(B\times C) \longrightarrow \square$, in line (i).
(iii) $\square \times \triangle = \triangle \times \square$	Axiom 3.
(iv) $B \times C = C \times B$	UV: $B \longrightarrow \square$ $C \longrightarrow \triangle$ in line (iii).
(v) $A + (B \times C) = A + \underline{(C \times B)}$	PN from line (ii), using line (iv).
(vi) $\square + \triangle = \triangle + \square$	Axiom 2.
(vii) $A + (C \times B) = (C \times B) + A$	UV: $A \longrightarrow \square$ $C \times B \longrightarrow \triangle$ in line (vi).
(viii) $A + (B \times C) = \underline{A + (C \times B)}$	Repeat of line (v), in order to avoid confusing the underlining for PN.

(ix) $A + (B \times C) = \underline{(C \times B) + A}$

PN from line (viii),
using line (vii).

Q. E. D.

(1) Who won the argument, George or Debbie?

(2) Andy says he can make an even shorter derivation that will be just as good as Debbie's. Can you? Do you think Andy can?

(3) Study Debbie's derivation very carefully, and then try to make your own derivation that uses the axioms

$$\square = \square$$

$$\square + \triangle = \triangle + \square$$

$$\square \times \triangle = \triangle \times \square$$

and ends up with the theorem

$$(A + B) \times C = C \times (B + A).$$

Try to give reasons, the same way that Debbie did.

CHAPTER 22

Subtraction and Division

(1) Nancy says that we know lots of important identities involving **addition** and **multiplication,** but we do not have any for **subtraction** or **division.** What do you think?

(2) Tony says you can handle **subtraction** by **turning it into addition.** Do you know what Tony means?

(3) Do you know what mathematicians mean by the **additive inverse** or the **opposite** of a number?

(4) Can you find other names for these numbers?

(a) ${}^{\circ}({}^{+}1)$

(b) ${}^{\circ}(2\frac{1}{2})$

(c) ${}^{\circ}({}^{-}3)$

(d) ${}^{\circ}(\frac{{}^{-}1}{2})$

(e) ${}^{\circ}(0)$

(f) ${}^{\circ}(3)$

(g) ${}^{\circ}({}^{\circ}3)$

(h) ${}^{\circ}[{}^{\circ}({}^{\circ}3)]$

(i) ${}^{\circ}[{}^{+}2 + {}^{+}3]$

(5) Jean says Cynthia used to have a "rainbow picture" to show what we mean by **opposites.** Do you know what Jean is talking about?

(6) Debbie says the **official definition** of the **opposite of *A*** is "the number that I can add to ***A*** so that the sum will be zero." What do you think?

(7) Which of these are identities?

(a) $\square + {}^{\circ}\square = 0$

(b) $\square + \square = 0$

(c) $\square + \square = \square$

(d) $\square + \square = 2 \times \square$

(e) ${}^{\circ}(\square + \triangle) = {}^{\circ}\square + {}^{\circ}\triangle$

(f) ${}^{\circ}(\square \times \triangle) = ({}^{\circ}\square) \times ({}^{\circ}\triangle)$

(g) ${}^{\circ}({}^{\circ}\square) = \square$

(h) ${}^{\circ}[{}^{\circ}({}^{\circ}\square)] = \square$

(i) ${}^{\circ}[{}^{\circ}({}^{\circ}\square)] = {}^{\circ}\square$

(8) Dan says you can change subtraction into addition by using the identity

$$\square - \triangle = \square + {}^{o}\triangle .$$

What do you think?

(9) We have seen how "additive inverses" and "subtraction" work. Do you know how "multiplicative inverses" work?

(10) Debbie says that the "multiplicative inverse" of $\square$ is "the number that I can multiply $\square$ by to get 0." What do you think?

(11) Roger thinks Debbie is wrong. He says that the "multiplicative inverse" of $\square$ is "the number you multiply $\square$ by to get 1." Do you agree?

(12) Can you find the truth set for each of these open sentences?

(a) $\quad {}^{+}2 \times \square = 0$

(b) $\quad {}^{+}2 \times \square = 1$

(c) $\quad {}^{+}\frac{2}{3} \times \square = 1$

(d) $\quad {}^{+}\frac{2}{3} \times \square = 0$

(e) $\quad {}^{+}2\frac{1}{4} \times \square = 1$

(f) $\quad 0 \times \square = 1$

(13) Roger says that mathematicians call the "multiplicative inverse" of a number $\square$ the "reciprocal" of $\square$, and that they write ${}^{r}\square$.

What is ${}^{r}7$? What is ${}^{r}\left(\frac{1}{4} + \frac{1}{2}\right)$?

(14) Which of these are identities?

(a) $\quad \square \times {}^{r}\square = 1$

(b) $\quad \square \times {}^{r}\square = {}^{-}1$

(c) $\quad (\square + \triangle) \times {}^{r}(\square + \triangle) = 1$

(d) $\quad (\square + \triangle) \times ({}^{r}\square + {}^{r}\triangle) = 1$

(e) $\quad {}^{r}({}^{r}\square) = \square$

(f) $\quad {}^{r}[{}^{r}({}^{r}\square)] = \square$

(g) $\quad {}^{r}[{}^{r}({}^{r}\square)] = {}^{r}\square$

(15) Can you change division into multiplication by using a "reciprocal"?

(16) Can you write a complete list of the **axioms** that we seem to be using thus far? Do you suppose this list is final? Will we ever want to change it?

CHAPTER 23

Practice in Making Up Your Own Derivations

In this chapter, we want to get some practice in making up our own derivations.

To start with, we'll need a list of axioms. Let's agree to use this list, at least for the time being:

$$\square = \square$$

Reflexive Property of Equality (RPE)

$$\square + \triangle = \triangle + \square$$

Commutative Law for Addition (CLA)

$$\square \times \triangle = \triangle \times \square$$

Commutative Law for Multiplication (CLM)

$$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$$

Distributive Law (DL)

$$\square + (\triangle + \bigtriangledown) = (\square + \triangle) + \bigtriangledown$$

Associative Law for Addition (ALA)

$$\square \times (\triangle \times \bigtriangledown) = (\square \times \triangle) \times \bigtriangledown$$

Associative Law for Multiplication (ALM)

$$\square + 0 = \square$$

Addition Law for Zero (ALZ)

$$\square \times 0 = 0$$

Multiplication Law for Zero (MLZ)

$$\square \times 1 = \square$$

Law for 1 (L1)

$$\left.\begin{array}{l} 1 + 1 = 2 \\ 2 + 1 = 3 \\ 3 + 1 = 4 \\ \quad \vdots \end{array}\right\}$$

Definition of the Numerals 2, 3, 4, . . . (Def. Num.)

$$\left.\begin{array}{l} {}^{+}1 = 1 \\ {}^{+}2 = 2 \\ {}^{+}3 = 3 \\ \quad \vdots \end{array}\right\}$$

Definition of the Numerals $^{+}1$, $^{+}2$, . . .

Every number has an additive inverse. If we use the notation ${}^{\circ}({}^{+}2)$ to mean "the additive inverse of ${}^{+}2$," then ${}^{\circ}({}^{+}2)$ is defined as the number we must add to ${}^{+}2$ in order to get zero.

In general,

$$\square + {}^{\circ}\square = 0$$

Law of Opposites (L. Opp.)

$$\square - \triangle = \square + {}^{\circ}\triangle$$

Definition of Subtraction (Def. Subtr.)

$$\left.\begin{array}{l} {}^{\circ}({}^{+}1) = {}^{-}1 \\ {}^{\circ}({}^{+}2) = {}^{-}2 \\ {}^{\circ}({}^{+}3) = {}^{-}3 \\ \vdots \end{array}\right\}$$

Definition of the Numerals ${}^{-}1, {}^{-}2, {}^{-}3, \ldots$

Every number **except zero** has a "reciprocal," or "multiplicative inverse." If we write

$${}^{r}2$$

to mean the "reciprocal of 2," then

$${}^{r}2 = \tfrac{1}{2}$$

$$2 \times {}^{r}2 = 1,$$

and, in general,

$$\square \times {}^{r}\square = 1, \qquad 0 \not\to \square^{*}$$

Law for Reciprocals (L. Recip.)

$$\square \div \triangle = \square \times {}^{r}\triangle, \qquad 0 \not\to \triangle$$

Definition of Division (Def. Div.)

The list above gives us a reasonable set of axioms for our "algebra." For our "logic," we shall have two rules: PN and UV.

Now let's see if we can make up derivations.

Can you write a derivation for each theorem, using Marjory's method of writing?

(1) $A \times (B + C) = A \times (C + B)$

(2) $A \times (B + C) = (B + C) \times A$

(3) $A \times (B + C) = (C \times A) + (B \times A)$

(4) Lex made up a derivation for the theorem

$$A + (B \times C) = (C \times B) + A.$$

---

*This notation is meant to remind us that, in the axiom $\square \times {}^{r}\square = 1$, zero must never be used as a replacement for the variable $\square$.

Cynthia complained that she couldn't understand Lex's derivation, so Bob tried to describe it.

Lex's derivation **Bob's description**

$A + (B \times C) = A + (B \times C)$

Actually, Lex really began with RPE:

$$\square = \square.$$

Lex knew this was an axiom. Then Lex used UV

$$\text{UV: } A + (B \times C) \longrightarrow \square$$

to get

$$A + (B \times C) = A + (B \times C).$$

$A + (B \times C) = A + (C \times B)$

Now, Lex used PN. He erased the name

$$B \times C,$$

to get

$$A + (B \times C) = A + (\text{▨}).$$

↗
(Gaping hole)

Then, into this gaping hole, he put **another name for the same thing.**

$$A + (B \times C) = A + (C \times B).$$

How did Lex know that $C \times B$ named the same thing that $B \times C$ did? He used CLM

$$\square \times \triangle = \triangle \times \square$$

and used UV

$$\text{UV: } B \longrightarrow \square$$

$$C \longrightarrow \triangle$$

to get

$$B \times C = C \times B,$$

which says that $C \times B$ names the same thing that $B \times C$ does.

$A + (B \times C) = (C \times B) + A$

Here, Lex again used PN. He began with the identity

$$A + (B \times C) = A + (C \times B).$$

He erased the entire right-hand side,

$$A + \left(B \times C\right) = \underset{\substack{\uparrow \\ \text{(Gaping hole)}}}{\rule{3cm}{0.4cm}}$$

and into the gaping hole he put another name for the same thing,

$$\left(C \times B\right) + A,$$

to get

$$A + \left(B \times C\right) = \left(C \times B\right) + A.$$

How did Lex know that

$$\left(C \times B\right) + A$$

named the same thing that

$$A + \left(C \times B\right)$$

named? The answer is that he started with CLA

$$\square + \triangle = \triangle + \square,$$

and used UV

$$\text{UV: } \left(C \times B\right) \rightarrow \square$$

$$A \rightarrow \triangle$$

to get

$$\left(C \times B\right) + A = A + \left(C \times B\right),$$

which says that

$$\left(C \times B\right) + A$$

names the same thing that

$$A + \left(C \times B\right)$$

names.

Do you understand Bob's description of what Lex did?

Can you write out a derivation for each theorem, using Debbie's method?

(5) Theorem: $\square + \square = 2 \times \square$

(6) Theorem: $A + \left(B + C\right) = C + \left(B + A\right)$

(7) Theorem: $3 + 2 = 5$

(8) Theorem: $\left(A + B\right) + \left(C + D\right)$
$= \left[D + \left(C + B\right)\right] + A$

(9) Theorem: $\left(A + B\right) \times \left(A + B\right)$
$= \left\{\left(A \times A\right) + \left[\left(B + B\right) \times A\right]\right\}$
$+ \left(B \times B\right)$

CHAPTER 24

Extending Systems: " Lattices" and Exponents

Frequently, we build up a mathematical system for some reason or other, and are proud of it because it is our own creation and because it seems to "work."

Then, on some black day or other, we discover that our system does not "work" any longer. We reach a point that our system cannot cope with. (This is somewhat like the feeling people had, before Columbus, that if they came to the edge of the world they would fall off. We have come to the "edge" of our beautiful mathematical system, and we seem to be in danger of having nowhere else to go.)

Can we build on to our system? Can we extend it further?

Let's look at this, in two important cases.

I. THE SYSTEM OF "LATICES"

Professor David Page, of the University of Illinois, has introduced an interesting mathematical system, which we might represent in the following way.

To begin with, we write numbers in an array or "lattice" like this:

↑ (and so on)

31	32								
21	22	23	24	25	26	27	28	29	30
11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10

Now, this gives us **a new way to write names for numbers:**

(1) What number do you suppose is meant when we write

3 → ?

(2) What number do you suppose is meant when we write

7 ↑ ?

Can you find simpler names for each of these numbers?

(3) 8 →

(4) 9 ←

(5) 5 ↑ ↑ ↑

(6) 3 ↗

(7) 9 ← ↑ ←

(8) 21 ↓ →

(9) 3 ↑ ↑ ↑ ↑ ↑

(10) 3 ↑ ↑ ↑ ↑ ↓ ↓ ↓

(11) 24 ↙

(12) 26 ↙ →

(13) 27 ↙ → ↑

(14) 27 ↙ → ↑ ↑

(15) 27 ↙ → ↑ →

(16) 27 → → ↑ ↙

(17) 27 ↑ ↙ → →

(18) 27 ↑ → ↙ →

(19) 27 ↑ → ↙ ← ←

(20) 23 ← → ← → ← → ← →

(21) 14 ← → ← → ← → ← → ↑

(22) 14 ← → ← → ← → ← → ←

(23) 14 ← → ← → ← → ← → ↓

(24) 12 ↗ ↗ ↗ ↓ ↓ ↓ ← ←

Which of these open sentences are **identities**?

(25) □ → ← = □

(26) □ ↗ ↓ → = □

(27) □ ↗ ↓ ← = □

(28) □ ↗ ↗ ↗ ↓ ↓ ↓ ← ← = □

(29) □ ↗ ↗ ↗ ↓ ↓ ↓ ← ← ← = □

(30) □ ↑ ↓ ↑ ↓ ↑ ↓ ↑ ↓ → → ↙ ↙ ↑ ↑ = □

Can you find simpler names for each of these numbers?

(31) 15 ↑

(32) 9 → →

(33) 12 ↙ ←

(34) 22 ← ←

(35) Doug's teacher wrote this on the chalkboard:

> In problems 1 through 34 we have been studying the structure of Professor Page's lattice. The structure of a mathematical system is an important idea even if it is a bit hard to say exactly what it is.
>
> We have seen that we "came to the edge" of Professor Page's system and we needed to "extend" it. We found several different extensions which were "legal" since they violated no rules.

Did you find a "natural" extension that did not destroy the structure of Professor Page's system?

(36) Did you finally settle on any definite way, that everyone in class agreed with, to extend Professor Page's lattice?

II. THE SYSTEM OF EXPONENTS

We frequently encounter problems like the following.

$2 \times 2 = 4$

$2 \times 2 \times 2 = 8$

$2 \times 2 \times 2 \times 2 = 16$

$3 \times 3 = 9$

$3 \times 3 \times 3 = 27$

$3 \times 3 \times 3 \times 3 = 81$

$10 \times 10 = 100$

$10 \times 10 \times 10 = 1000$

We can often make these problems simpler, and easier to think about, if we introduce the system that mathematicians call **exponents**. Here is how it works:

$2^2 = 2 \times 2 = 4$

$2^3 = 2 \times 2 \times 2 = 8$

$2^4 = 2 \times 2 \times 2 \times 2 = 16$

$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$

$2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$

$\vdots$

$3^2 = 3 \times 3 = 9$

$3^3 = 3 \times 3 \times 3 = 27$

$3^4 = 3 \times 3 \times 3 \times 3 = 81$

$\vdots$

$5^2 = 5 \times 5 = 25$

$5^3 = 5 \times 5 \times 5 = 125$

$5^4 = 5 \times 5 \times 5 \times 5 = 625$

$$\vdots$$

$10^2 = 10 \times 10 = 100$

$10^3 = 10 \times 10 \times 10 = 1000$

$10^4 = 10 \times 10 \times 10 \times 10 = 10{,}000$

$10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100{,}000$

$$\vdots$$

Can you find simpler names for each of these numbers?

(37) 4^2
(38) 7^2
(39) 1^5
(40) 2^{10}
(41) 100^2
(42) 11^2
(43) $(^-1)^2$
(44) $(^-1)^3$
(45) $(^-1)^4$
(46) $(^-1)^5$
(47) 1^{1965}
(48) $(^-1)^{1965}$
(49) $(^-1)^{1066}$
(50) $(^-1)^{200}$
(51) $(^-1)^{201}$
(52) 10^{100}

Which statements are **true** and which are **false**?

(53) $2^{10} < 10^3$
(54) $2^{10} < 10^2$
(55) $2^{10} < 10^4$
(56) $5^4 < 4^5$
(57) $3^4 < 4^3$

Can you find the truth set for each open sentence? (Let's agree to use only positive integers.)

(58) $2^6 = \square^3$
(59) $3^4 = 9^{\square}$

(60) $5^{10} = 25^{\square}$

(61) $2^3 \times 2^4 = 2^{\square}$

(62) $2^5 \times 2^3 = 2^{\square}$

(63) $3^2 \times 3^3 = 3^{\square}$

(64) $3^5 \times \square^2 = 3^7$

(65) $2^3 \times 5^3 = \square^3$

(66) $2^2 \times 3^2 = \square^2$

(67) $({}^-1)^{301} \times ({}^-1)^{301} = \square$

(68) $2^9 < 10^{\square} < 2^{10}$

(69) $25 < 7^{\square} < 50$

(70) $2^6 < 5^{\square} < 2^7$

(71) $10 < 2^{\square} < 10^2$

(72) $10 < 3^{\square} < 10^2$

(73) $10^2 < 5^{\square} < 10^3$

If we use only **positive integers** as replacements for the variables, which of these are **identities**?

(74) $\square^2 \times \square^3 = \square^6$

(75) $\square^2 \times \square^3 = \square^5$

(76) $(\square^2)^3 = \square^6$

(77) $(\square^2)^3 = \square^6$

(78) $p^a \times q^b = (p \times q)^{a \times b}$

(79) $p^a \times p^b = p^{a \times b}$

(80) $p^a \times p^b = p^{a+b}$

(81) $p^a \times q^a = (p \times q)^{a+a}$

(82) $p^a \times q^a = (p \times q)^a$

(83) $p^a \div p^b = p^{a-b}$

(84) Do you see any need to **extend** our system of exponents?

(85) Thus far, we have been using the set of positive integers as the replacement set for our variables. What other set **might** we use, instead of merely positive integers?

(86) Do you see any reason to extend our system of exponents?

Can your **extended** system cope with these problems? Can you find simpler names for any of these numbers?

(87) 3^1

(88) 3^0

(89) 2^0

(90) 2^{-1}

(91) 3^{-1}

(92) 2^{-5}

(93) 10^0

(94) 10^{-1}

(95) 10^{-2}

(96) 10^{-3}

(97) 10^{-10}

(98) $10^5 \div 10^7 = \square$

(99) $2^5 \div 2^6 = \square$

(100) Mary says that if there really is such a thing as $9^{\frac{1}{2}}$, then

$$9^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 9.$$

Is Mary right? Do you think there really is such a thing as $9^{\frac{1}{2}}$?

Can your extended system cope with these problems? Can you find simpler names for any of these numbers?

(101) $100^{\frac{1}{2}}$

(102) $27^{\frac{1}{3}}$

(103) $1000^{\frac{1}{3}}$

(104) $10{,}000^{\frac{1}{2}}$

(105) $1024^{\frac{1}{5}}$

If we use rational numbers as replacements for the variables a and b, which of the following are identities?

(106) $\square^{a} \times \square^{b} = \square^{a \times b}$

(107) $\square^{a} \times \square^{b} = \square^{a+b}$

(108) $p^a \times q^a = (p \times q)^a$

(109) $\square^{a} \div \square^{b} = \square^{a-b}$

(110) $(\square^{a})^{b} = \square^{a+b}$

(111) $(\square^{a})^{b} = \square^{a \times b}$

(112) Can you describe how you extended the original system of exponents? Why did you do it the way that you did?

CHAPTER 25

Guessing Functions

(1) Ranny made up a rule and Alec tried to guess what it was.

When Alec told Ranny "zero," Ranny answered "three."
When Alec said "one," Ranny answered "five."
When Alec said "two," Ranny said "seven."

□	△
0	3
1	5
2	7

Do you know what rule Ranny made up?

(2) Can you write a formula for Ranny's rule, using □'s and △'s?

(3) Nancy says Ranny's rule is

$$(\square + 3) + 2 = \triangle.$$

Do you agree?

(4) Kathy says Ranny's rule is: "Whatever number they say, double it, and add three." What do you think?

(5) Can you write a formula for Ranny's rule, using □'s and △'s?

(6) Can you make a **graph** for Ranny's rule?

(7) Joan made up a rule. When Alan said "ten," Joan answered "fifty-three." When Alan said "five," Joan said "twenty-eight." Do you know Joan's rule?

(8) Can you write a formula for Joan's rule?

(9) Can you make a graph for Joan's rule?

(10) Why don't you make up your own rule, and see if people can guess it.

(11) Do you know what mathematicians mean by a **function**?

(12) Do you know the difference between a **formula** and a **function**?

CHAPTER 26

Guessing Functions: Form vs. Numbers

The students in a class at Nerinx High School, in Webster Groves, Missouri, have worked out some methods to help them guess functions. If you think these methods would help you, you may want to glance at the following few pages.

I. IS THE FUNCTION LINEAR OR NOT?

Sometimes people make up a rule somewhat like these:

$$(\square \times 3) + 5 = \triangle$$

$$(\square \times 2) + {}^{-}3 = \triangle$$

$$(\square \times 5) + {}^{-}1 = \triangle$$

and so on.

In general, any rule of this sort looks like this:

$$(\square \times \underline{\qquad}) + \underline{\qquad} = \triangle$$

↗ ↖

(Some number here) (Some number here)

Rules (or functions) of this kind are called **linear** functions.

Do you know why?

(1) If the rule we are trying to guess is of this form, we can tell at once by making a graph and looking at it. What will the graph look like?

(2) How can we tell by looking at a **table**?

Which of these functions are **linear**?

(3)

$\square$	$\triangle$
0	4
1	7
2	10
3	13
4	16
⋮	⋮

(4)

$\square$	$\triangle$
0	3
1	5
2	8
3	12
⋮	⋮

(5)

□	△
0	4
1	2
2	4
3	2
4	4
5	2
⋮	⋮

(6)

□	△
0	1
1	2
2	3
3	4
⋮	⋮

(7)

□	△
0	17
1	14
2	11
3	8
4	5
5	2
6	$^{-}1$
⋮	⋮

(8)

□	△
0	1
1	2
2	5
3	10
4	17
5	26
10	101
100	10,001
1000	1,000,001
⋮	⋮

II. IS THE FUNCTION EVEN OR ODD?

Some functions have tables like this:

□	△
1	4
$^{-}1$	4
2	28
$^{-}2$	28
3	108
$^{-}3$	108
⋮	⋮

For these functions, 1 gets the same answer that $^{-}1$ does, 2 gets the same answer that $^{-}2$ does, and so on. Such functions are called **even** functions.

Some functions have tables like this:

□	△
1	2
$^{-}1$	$^{-}2$
2	10
$^{-}2$	$^{-}10$
3	30
$^{-}3$	$^{-}30$
⋮	⋮

In this table, the answer for $^{-}1$ is the **additive inverse** of the answer for 1, the answer for $^{-}2$ is the additive inverse of the answer for 2, and so on. Functions like this are called **odd** functions.

Some functions are **neither even nor odd.** Here is one:

□	△
1	5
$^{-}1$	$^{-}1$
2	8
$^{-}2$	$^{-}4$
⋮	⋮

Which of these functions are **even**, which are **odd**, and which are **neither even nor odd**?

(9)

□	△
1	6
$^{-}1$	6
2	9
$^{-}2$	9
3	14
$^{-}3$	14
⋮	⋮

(10)

□	△
1	2
$^{-}1$	$^{-}4$
2	5
$^{-}2$	$^{-}7$
3	8
$^{-}3$	$^{-}10$
⋮	⋮

(11)

□	△
1	1
$^{-}1$	$^{-}1$
2	14
$^{-}2$	$^{-}14$
3	51
$^{-}3$	$^{-}51$
⋮	⋮

(12) $y = x^2$

(13) $(\square \times \square) = \triangle$

(14) $(\square \times \square) \times \square = \triangle$

(15) $y = x^3$

(16) $y = 5x^2$

(17) $(3 \times \square) + 2 = \triangle$

(18) $y = 3x + 2$

(19) $y = x^4$

(20) $y = x^4 - x^2$

(21) $y = 3x^2 + 2$

(22) $y = x^3 + x$

(23) How can the idea of **even** and **odd** functions help you to guess functions?

III. ELLEN'S METHOD

Sometimes, when she was trying to guess a rule, Ellen would say,

"Use your rule on 10, and tell me the answer."

When she got the answer to this, she would say,

"Use your rule on 100, and tell me the answer."

When she got **this** answer, she would request,

"Use your rule on 1000, and tell me the answer."

(24) Why do you suppose Ellen picked these numbers to ask about?

(25) Using her method, Ellen got this table:

$\square$	$\triangle$
1	2
10	100.1
100	10,000.01
1000	1,000,000.001
$\vdots$	$\vdots$

Can you guess this function?

IV. LOOKING FOR THE RIGHT FORM

Another very good method, which the Nerinx students often use, goes like this:

First try to list several **likely** forms. For example, these forms are often used:

$$y = ax + b$$

$$y = ax + \frac{b}{x}$$

$$y = a + bx + cx^2$$

$$y = ax^2 + \frac{b}{x}$$

In these forms, ***a*, *b*,** and ***c*** are definite numbers, chosen by the people who made up the rule. The letter ***x*** indicates "the number we tell them," and the letter ***y*** represents "the answering number that they tell us."

Once you have written your list of likely forms, try to ask questions which will **eliminate** some of them or **confirm** one of them. Here are some questions the Nerinx students often use:

(26) "Use your rule on 0, and tell me the answer." Suppose the answer is 3.

x	y
0	3

Which forms on the list preceding would be **eliminated** by this answer?

(27) "Use your rule on 0, and tell me the answer." Suppose the answer is: "Our rule doesn't work for 0."

x	y
0	no answer

Which forms on the list would be eliminated by this answer?

(28) Suppose the table was:

x	y
1	9
$^{-}1$	5
2	11
$^{-}2$	3
⋮	⋮

What **forms** does this table suggest?

(29) Suppose the table was:

x	y
1	$^{-}2$
$^{-}1$	$^{-}2$
2	1
$^{-}2$	1

What **forms** does this table suggest?

As you begin to see what **form** the function probably has, you can try to find the actual numbers—that is, the numerical replacements for the variables ***a*, *b*, *c*,** etc., in the form.

(30) Can you find this function?

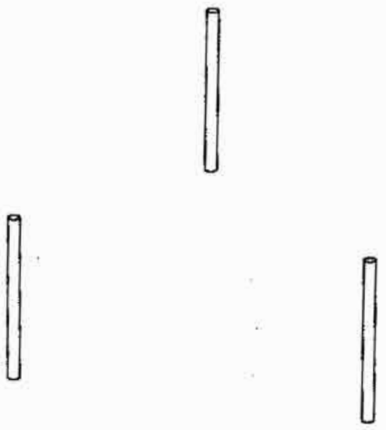
x	y
0	0
1	0
⁻1	0
2	30
⁻2	⁻30
3	240
⁻3	⁻240
10	99,990
100	9,999,999,900
⋮	⋮

CHAPTER 27

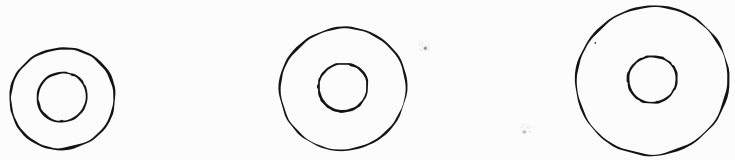
Where Do Functions Come From?

THE EASY THREE-PEG GAME

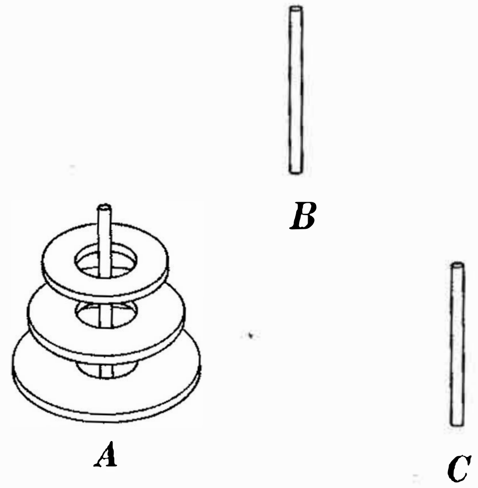
(1) Suppose you have three pegs:



and three washers, of three different sizes:

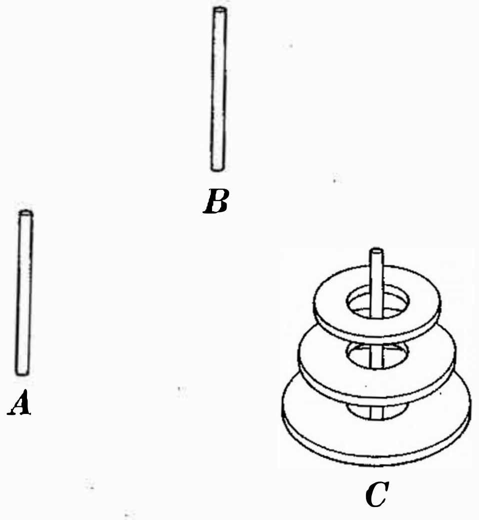


To play the game, you start with all three washers on peg *A*, with the largest washer on the bottom and the smallest washer on the top.



You move one washer at a time, taking it off one peg and putting it on another.

You are **finished** when you have **all three washers on peg *C***, with the largest washer on the bottom and the smallest washer on the top.

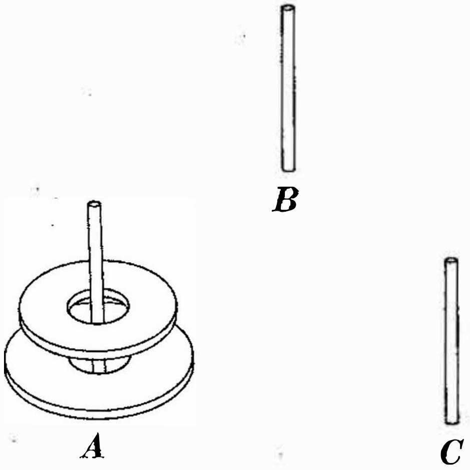


Can you get the washers onto peg ***C*** this way?

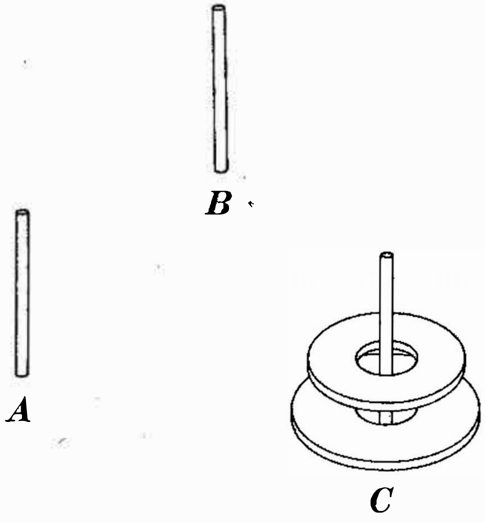
(2) How many moves did you need to get all three washers onto peg ***C***?

(3) Could you have done it in fewer moves?

(4) Suppose we change the rules of the game. Everything else is the same, but we start with only **two** washers:



Can you get both washers onto peg ***C***, with the little one on top?



(5) How many moves did you need?

(6) Jean made this table:

Number of washers ↘

Minimum number ↙ of moves

□	△
2	?
3	?

Can you fill in the two missing numbers in Jean's table?

(7) What would happen if you used **four** washers?

(8) Can you extend Jean's table?

(9) What would happen if you used 100 washers?

(10) Can you write a formula for this function?

(11) Can you make a graph for this function?

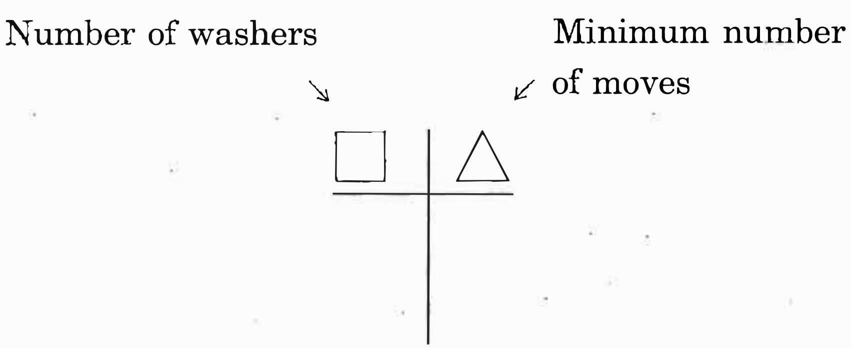
(12) Does it work the same way if you start with only **one** washer?

THE HARD THREE-PEG GAME

(13) You play this game with the same rules as the "easy three-peg game," but you add one **additional** rule: you must **never, at any stage,** put a **larger** washer on top of a **smaller** washer.

With this additional rule, can you move the washers from peg ***A*** to peg ***C***?

(14) Can you make a table for the "hard three-peg game"?



(15) Suppose you used 10 washers, how many moves would you need?

(16) Suppose you used 100 washers, how many moves would you need?

(17) Can you write a formula for this function?

(18) Can you make a graph for this function?

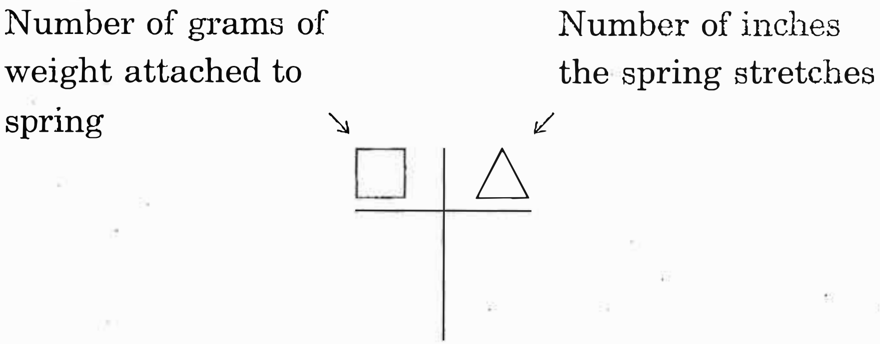
(19) Larry says that if P_n is the number of moves you need for n washers, then

$$P_n = \left(2 \times P_{n-1}\right) + 1.$$

Do you agree?

THE METAL-SPRING FUNCTION

(20) Hang up a coiled metal spring, and hang weights on it. You can get a function like this:



(21) Can you make a graph for this function?

(22) Can you write a formula for this function, using □'s and △'s?

(23) How much would the spring stretch if you hung 27 grams on it?

(24) How much would the spring stretch if you hung 41 grams on it?

(25) How much would the spring stretch if you hung 1,000,000 grams on it?

(26) How much would the spring stretch if you hung 0 grams on it?

(27) How much would the spring stretch if you hung $\frac{1}{100}$ gram on it?

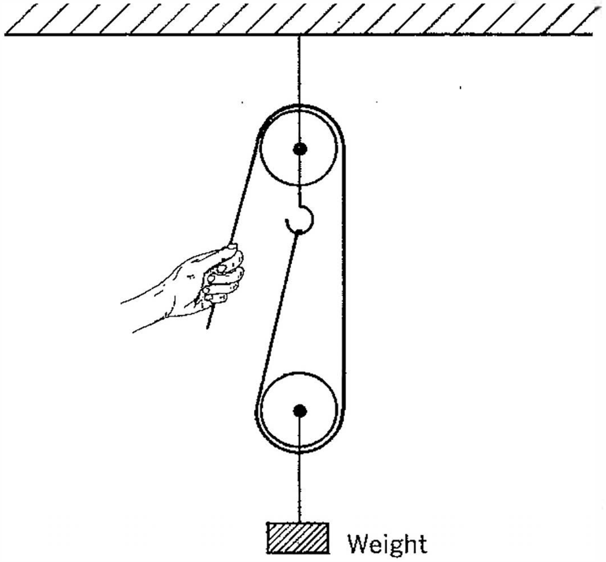
THE RUBBER-BAND FUNCTION

(28) This is exactly like the "metal-spring function," except that you use a "chain" of rubber bands instead

f a metal spring. Can you make a table for this func- ion? Can you make a graph for this function? Can ou write a formula for this function, using □'s nd △'s?

HE PULLEY-DISTANCE FUNCTION

(29) Arrange a pulley like this:



ake a table:

Number of inches hand moves ↘

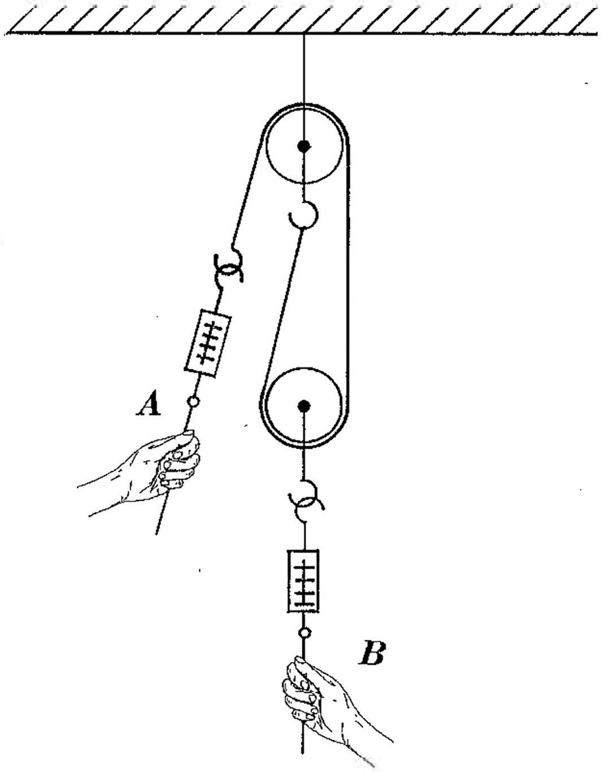
↙ Number of inches weight moves

□	△

(30) Can you make a graph for this function? Can you write a formula for it, using □'s and △'s?

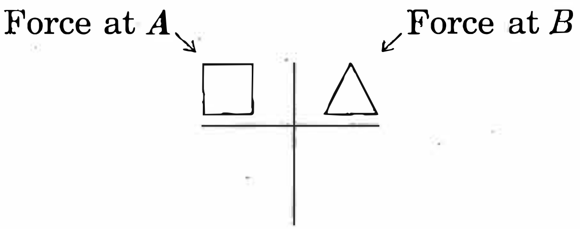
THE PULLEY-FORCE FUNCTION

(31) Use the same pulley as in questions 29 and 30. Use two spring balances (or some other method for measuring **forces**), like this:



Pull hard enough at ***A*** and ***B*** so that the two forces "just balance," and the rope and pulley wheels don't move.

Can you fill in part of this table?



Can you make a graph for this function? Can you write a formula for this function?

(32) Linda's father says that physicists use **conservation laws.** By looking at your work with the pulley, can you find anything that is **unchanged** (or, as the physicists say, **conserved**)?

(33) Jerry says that the hand at A moves farther, but doesn't pull as hard as the force at B. Jerry says that the **sum**

$$F + d$$

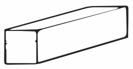
is **conserved**, where F is the **force** and d is the **distance moved.**

That is, if $F(A)$ and $d(A)$ mean the force and distance at A, and $F(B)$ and $d(B)$ mean the force and distance at B, then

$$F(A) + d(A) = F(B) + d(B).$$

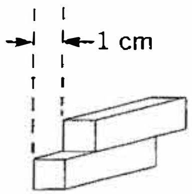
Do you agree?

(34) Toby has some pieces of wood 1 cm by 1 cm by 5 cm, shaped like a block or a brick:



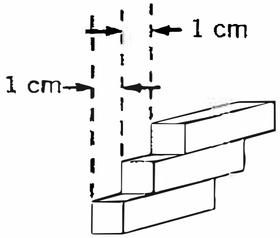
What is the volume of one piece of wood like this? What is the surface area?

(35) Suppose Toby glues together two of the pieces, like this:



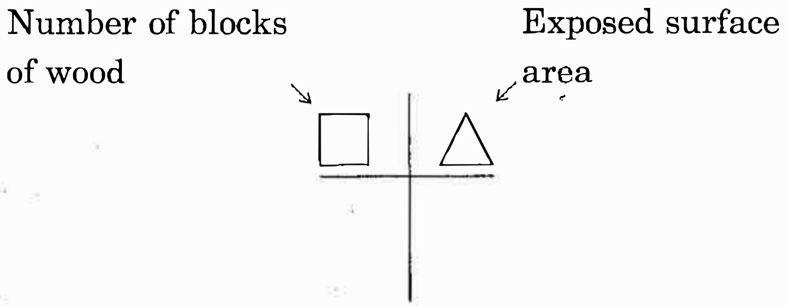
What will the volume be? What will the exposed surface area be?

(36) Suppose Toby glues three pieces together, like this:



What will the volume be? What will the exposed surface area be?

(37) For Toby's "stairs," can you fill in this table?



(38) Can you make a graph for this function?

(39) Can you write a formula for this function?

(40) If Toby used 10 blocks, what would the exposed surface area be?

(41) If Toby used 100 blocks, what would the exposed surface area be?

(42) If Toby used n blocks, what would the exposed surface area be?

CHAPTER 28

The Notation $f(x)$

Can you find the truth set for each open sentence?

(1) $(\square \times \square) - (15 \times \square) + 26 = 0$

(2) $(\square \times \square) - (^{-}5 \times \square) + {}^{+}6 = 0$

(3) $(\square \times \square) - (^{+}2 \times \square) + {}^{-}15 = 0$

(4) $(\square \times \square) - (^{-}3 \times \square) + {}^{-}4 = 0$

(5) $(\square \times \square) - (^{-}3 \times \square) + {}^{-}10 = 0$

(6) $(\square \times \square) - (^{-}9 \times \square) + {}^{+}20 = 0$

(7) Pam made up a rule, which worked like this

If you told her 2, she answered 5.
If you told her 3, she answered 6.
If you told her 4, she answered 7.

Debbie wrote the rule this way:

$P\colon 2 \longrightarrow 5$
$P\colon 3 \longrightarrow 6$
$P\colon 4 \longrightarrow 7$
$P\colon \square \longrightarrow \square + 3$

Can you complete these?

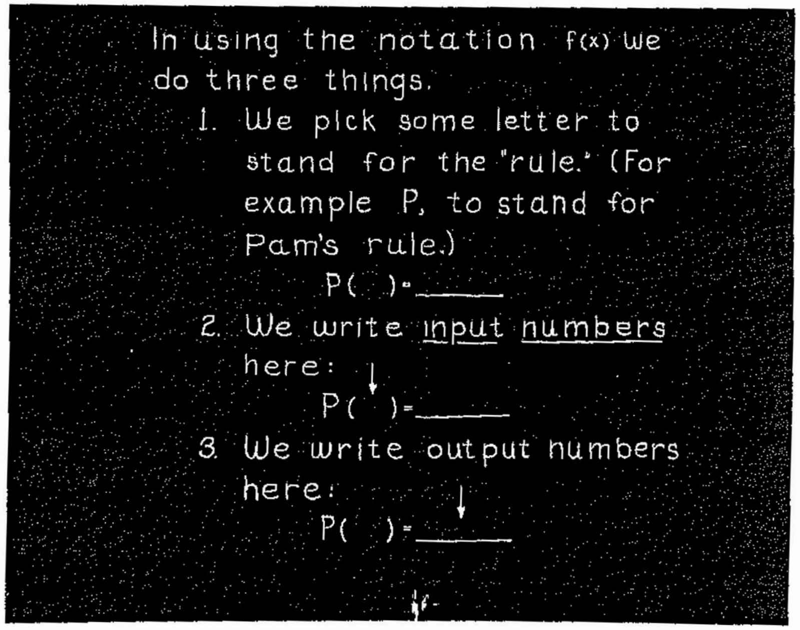
(a) $P\colon 5 \longrightarrow$ ___

(b) $P\colon 6 \longrightarrow$ ___
(c) $P\colon$ ___ $\longrightarrow 15$
(d) $P\colon x \longrightarrow$ ___
(e) $P\colon w \longrightarrow$ ___
(f) $P\colon 1.002 \longrightarrow$ ___
(g) $P\colon x + 3 \longrightarrow$ ___

(8) Frank used Pam's rule this way:

He wrote:	**He read it as:**
$P(2) = 5$	"*P* of 2 equals 5."
$P(3) = 6$	"*P* of 3 equals 6."
$P(\square) = \square + 3$	

Frank wrote this on the chalkboard to explain how he wrote Pam's rule:



In using the notation f(x) we do three things.
1. We pick some letter to stand for the "rule." (For example P, to stand for Pam's rule.)
P( )=____
2. We write input numbers here: ↓
P( )=____
3. We write output numbers here: ↓
P( )=____

n you fill in the missing imput or output numbers?

(a) $P(4) = \_\_\_$

(b) $P(1) = \_\_\_$

(c) $P(\frac{1}{2}) = \_\_\_$

(d) $P(\_\_\_) = 22$

(e) $P(x) = \_\_\_$

(f) $P(t) = \_\_\_$

(g) $P(w) = \_\_\_$

(h) $P(\_\_\_) = s + 5$

(i) $P(y + 4) = \_\_\_$

(9) John made the rule

$$J\colon x \longrightarrow x^2 - 2x + 3.$$

an you complete these?

(a) $J\colon 3 \longrightarrow \_\_\_$

(b) $J\colon 4 \longrightarrow \_\_\_$

(c) $J\colon \_\_\_ \longrightarrow 18$

(d) $J\colon w \longrightarrow \_\_\_$

(e) $J\colon s + 3 \longrightarrow \_\_\_$

(10) We can also use Frank's method to write John's le:

$$J(x) = x^2 - 2x + 3.$$

Can you complete these?

(a) $J(0) = \_\_\_$

(b) $J(1) = \_\_\_$

(c) $J(2) = \_\_\_$

(d) $J(100) = \_\_\_$

(e) $J(\_\_\_) = 83$

(f) $J(t + 2) = \_\_\_$

(g) $J(2 \times w) = \_\_\_$

(h) $J(\square) = \_\_\_$

(i) $J(N) = \_\_\_$

(11) Ruth made up the rule

$$R(\square) = (\square \times \square) - (7 \times \square) + 10.$$

Can you find the truth set for the open sentence

$$R(\square) = 0\,?$$

(12) For Ruth's rule

$$R\colon \square \longrightarrow [(\square \times \square) - (7 \times \square) + 10],$$

Alex made up the open sentence

$$R(\square) = {}^{-}2.$$

Can you find its truth set?

(13) Al used Ruth's rule and made up the open sentence

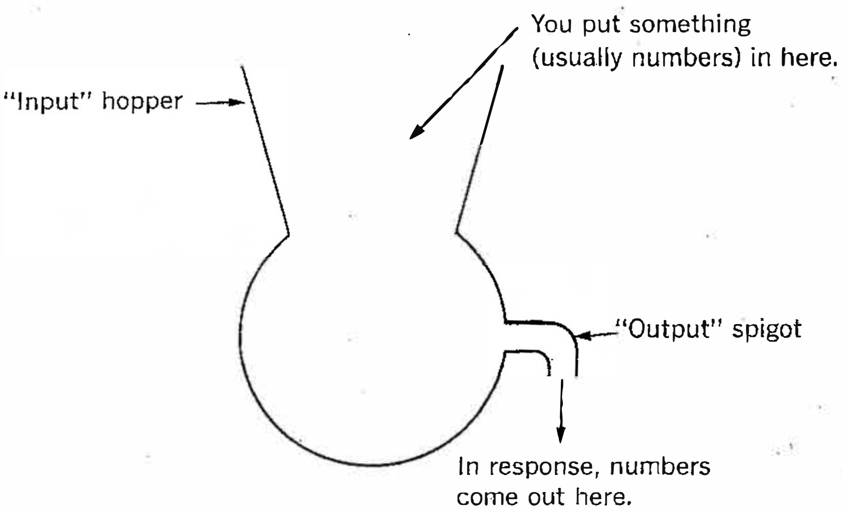
$$R(\square) = 4.$$

Can you find the truth set for Al's open sentence?

(14) Can you find $R(7)$?

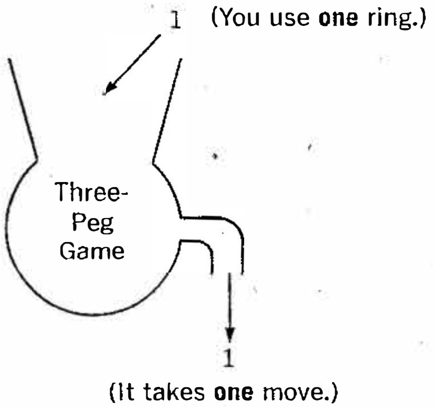
(15) Can you find $R(^-1)$?

(16) Ruth's father says the idea of a "function" or a "rule" is always something like this:

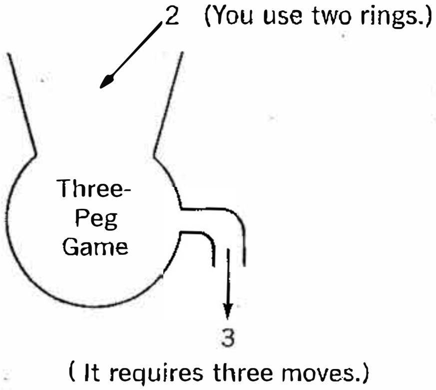


What do you think?

(17) Charles says the three-peg game is a good example of a function. If we use one ring, it takes one move to complete the puzzle:



If, instead, we use two rings, the puzzle requires three moves:



Suppose we construct a "function machine" that will match up numbers the same way that the three-peg game does. What number would come out of the spigot of this machine if you tossed "three" into the hopper?

(18) How many different ways of looking at functions do you think there are?

CHAPTER 29

Some Operations on Equations

(1) Jerry says that two different equations may have the same truth set. What do you suppose he means?

(2) Do these equations have the same truth set?

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

$$(\square - 2) \times (\square - 3) = 0$$

(3) Do these equations have the same truth set?

$$(\square \times \square) - (4 \times \square) + 4 = 0$$

$$3 + \square = 5$$

(4) Do these open sentences have the same truth set? (Use only integers as replacements for the variable $\square$.)

$$3 < \square + 1 < 6$$

$$(\square \times \square) - (7 \times \square) + 12 = 0$$

(5) Do these open sentences have the same truth set?

$$2 < \square + 1 < 5$$

$$(\square \times \square) - (7 \times \square) + 10 = 0$$

(6) Eileen says that two equations which have the same truth set are called **equivalent** equations. What do you think?

For each of the following pairs of equations, can you decide whether or not the two equations are **equivalent**?

(7)
$$\begin{cases} (2 \times \square) + 1 = 177 \\ (2 \times \square) + 2 = 178 \end{cases}$$

(8)
$$\begin{cases} \square + 3 = 10 \\ \square + 5 = 12 \end{cases}$$

(9)
$$\begin{cases} (2 \times \square) + 11 = 34 \\ (2 \times \square) + 12 = 35 \end{cases}$$

(10)
$$\begin{cases} (3 \times \square) + 191 = 273 \\ (3 \times \square) + 196 = 278 \end{cases}$$

(11)
$$\begin{cases} \square + 3 = 7 \\ (2 \times \square) + 6 = 14 \end{cases}$$

(12) $$\begin{cases} \square + 5 = 21 \\ \square + 10 = 42 \end{cases}$$

(13) $$\begin{cases} (3 \times \square) + 11 = 51 \\ (3 \times \square) + 22 = 102 \end{cases}$$

(14) $$\begin{cases} (3 \times \square) + 2 = 100 \\ (5 \times \square) + 4 = 200 \end{cases}$$

(15) $$\begin{cases} (3 \times \square) + 2 = 100 \\ (6 \times \square) + 4 = 200 \end{cases}$$

(16) $$\begin{cases} (3 \times \square) + 2 = 100 \\ (6 \times \square) + 2 = 200 \end{cases}$$

(17) $$\begin{cases} (3 \times \square) + 2 = 100 \\ (3 \times \square) + 102 = 200 \end{cases}$$

(18) $$\begin{cases} \square + 3 = 10 \\ (2 \times \square) + 6 = 20 \end{cases}$$

(19) $$\begin{cases} (13 \times \square) + 1791 = 2564 \\ (15 \times \square) + 1792 = 2565 \end{cases}$$

(20) There are certain things you can do to an equation to produce a new equation. For example, you might add 3 to the "left-hand side" of the equation.

If you start with the equation

$$\square + 10 = 100$$

and add 3 to the left-hand side, you get the new equation

$$\square + 13 = 100.$$

If you add 3 to the left-hand side of one equation, in order to get a new equation, do you suppose the new equation will have the same truth set as the original equation?

(21) Do **something** to the equation

$$(3 \times \square) + 2 = 35$$

so as to produce a new equation. Did you change the truth set?

(22) Do something to the equation

$$\square + 3 = 7$$

so as to produce a **new** equation which will have a **different** truth set.

(23) Do something to the equation

$$\Box + 3 = 7$$

so as to produce a **new** but **equivalent** equation.

(24) Beryl says that things you do to equations that produce new equations with the **same** truth set are called **transform operations.** What do you think?

(25) Can you use a **transform operation** on the equation

$$(3 \times \Box) + 25 = 85\ ?$$

What new equation did you get?

(26) Lex says that he knows five different kinds of transform operations. How many do you know?

(27) Jerry said he thought Lex was wrong. Lex said, "Let me give you a clue!" Then Lex wrote:

$$\begin{cases} \Box + 3 = 5 \\ \Box + 103 = 105 \end{cases}$$

$$\begin{cases} \Box + 3 = 5 \\ (2 \times \Box) + 6 = 10 \end{cases}$$

$$\begin{cases} \Box + 3 = 5 \\ \Box + 2 = 4 \end{cases}$$

$$\begin{cases} (2 \times \Box) + 4 = 16 \\ (1 \times \Box) + 2 = 8 \end{cases}$$

$$\begin{cases} 2 \times \Box = 102 \\ \Box + \Box = 102 \end{cases}$$

Do you know what Lex meant?

(28) Jeannie started with nine equations and did "something" to each one, to produce a new equation in each case. Can you describe what Jeannie did in each case? Was it a **transform operation** or not?

(a) Jeannie's original equation: $3 + \Box = 21$

Jeannie's new equation: $4 + \Box = 21$

(b) Jeannie's original equation: $3 + \Box = 50$

Jeannie's new equation: $4 + \Box = 55$

(c) Jeannie's original equation:

$$(2 \times \square) + 5 = 10$$

Jeannie's new equation:

$$(4 \times \square) + 10 = 20$$

(d) Jeannie's original equation:

$$(\square + \square) + 5 = \square + 8$$

Jeannie's new equation:

$$(2 \times \square) + 5 = \square + 8$$

(e) Jeannie's original equation:

$$(\square + \square) + 10 = 15$$

Jeannie's new equation:

$$\square + \square = 5$$

(f) Jeannie's original equation:

$$(\square + \square) + 9 = 49 + \square$$

Jeannie's new equation:

$$\square + 9 = 49$$

(g) Jeannie's original equation:

$$(2 \times \square) + 7 = 31$$

Jeannie's new equation:

$$(2 \times \square) + 14 = 62$$

(h) Jeannie's original equation:

$$(\square + 3) = 10$$

Jeannie's new equation:

$$(\square + 3) + 5 = 10 + 5$$

(i) Jeannie's original equation:

$$(\square \times \square) - (9 \times \square) + 14 = 0$$

Jeannie's new equation:

$$(\square - 2) \times (\square - 7) = 0$$

(29) What do we mean by a **transform operation**?

(30) Are these **equivalent equations**?

$$(\square \times \square) - 16 = 0$$

$$[(\square \times \square) - 16] + 25 = 25$$

(31) John says that Lex's "five kinds of transform operations" are not really all different. What do you think?

(32) John wrote:

$$7 - 5 = 7 + {}^{-}5$$

What do you suppose he meant?

(33) John also wrote:

$$(3 \times \square) + 10 = 25$$

$$[(3 \times \square) + 10] - 10 = 25 - 10$$

$$[(3 \times \square) + 10] + {}^{-}10 = 25 + {}^{-}10$$

What do you suppose he meant?

(34) How many **different** kinds of transform operations do you know?

Can you find the truth set for each equation?

(35) $(\square \times \square) - (5 \times \square) + 10 = 4$

(36) $(\square \times \square) - (15 \times \square) + 5 = {}^{-}31$

(37) $(\square \times \square) - (12 \times \square) + 20$
$= {}^{-}2 + (1 \times \square)$

(38) $[(\square + \square) \times (\square + 13)] + \square$
$= [(\square + \square) \times (\square + 13)] + 7$

(39) $(3 \times \square) + 1951 = (2 \times \square) + 1500$

(40) Are these equivalent equations?

$$(\square - 3) = 5$$

$$(\square - 3)^2 = 25$$

CHAPTER 30

Some Operations on Inequalities

When mathematicians say that two inequalities are "equivalent," they mean that both inequalities have the same truth set.

In this chapter, let's agree to use only positive integers as replacements for the variables.

(1) David made up the inequality

$$(2 \times \square) + 1 < 10.$$

Can you make up an **equivalent** inequality?

(2) Tom made up the inequality

$$\square + 5 < 8.$$

Can you make up an **equivalent** inequality?

(3) Are these inequalities equivalent?

$$\square + 3 < 10$$

$$(2 \times \square) + 6 < 20$$

(4) Are these inequalities equivalent?

$$\square - 2 < 4$$

$$^{-}1 \times (\square - 2) < {^{-}4}$$

(5) Are these inequalities equivalent?

$$\square + 3 < 7$$

$$\square + 4 < 10$$

(6) What **transform operations** can you find for **inequalities**?

CHAPTER 31

"Variables" vs. "Constants"

In the seventeenth century René Descartes, whose work we have encountered earlier, decided to distinguish between different ways that we use □, △, *x, y, A, B,* and so on. In some cases, Descartes would say that we were dealing with **constants.** In other cases, he would say we were dealing with **variables.***

In order not to get mixed up, Descartes decided to use **letters near the beginning of the alphabet** to represent **constants.** (We shall do the same thing, but we shall add also letters near the beginning of the Greek alphabet; so we shall use *a*, *b*, *c*, α, β, γ, and so forth, to refer to **constants.**)

When he wanted to refer to what he called **variables,** Descartes used **letters near the end of the alphabet.** (We shall too, but we shall also use □, △, etc. Hence, when we want to refer to what Descartes called variables, we can write: *x, y, z, w, u, v,* □, △, ▽, ▭, etc.)

Modern mathematicians and logicians do not always use the words "variable" and "constant" in quite the same way that Descartes did, but nonetheless Descartes' idea is really still valuable and is still used in one form or another.

Now, what was it that Descartes meant, anyhow?

Let's give some examples first.

Example 1

Some airplanes have two engines, some have three, and some have four. Perhaps, then, to the engineer or designer who sets out to design a new airplane, the number of engines is a "variable."

However, to the pilot who flies the plane after it is built, the number of engines is a "constant"—if he takes off with a two-engined plane, he cannot simply decide in midflight to change to three engines.

Example 2

Suppose a teacher is making up a test. If he wants to put in one question about quadratic equations, he can write:

Find the truth set for

$$(\square \times \square) - (5 \times \square) + 6 = 0.$$

---

*In fact, Descartes was not the earliest mathematician to decide to distinguish "variables," or "unknowns," from "constants." This distinction was used earlier by the great mathematician François Viète (1540-1603), and also by the Englishman Thomas Harriot (1560-1621). [Incidentally, Sir Walter Raleigh sent Harriot to (what is now) the United States to survey (what is now) North Carolina.] Viète and Harriot wrote the distinction differently, however; they used vowels for "unknowns," and used consonants to represent "constants."

Or, if he prefers, he can write:

Find the truth set for

$$(\Box \times \Box) - (16 \times \Box) + 55 = 0.$$

Or he can, instead, write:

Find the truth set for

$$(\Box \times \Box) - (20 \times \Box) + 96 = 0.$$

Hence, to the teacher who is making up the problem, the numbers to go here

$$(\Box \times \Box) - (\underset{\uparrow}{\_\_\_} \times \Box) + \_\_\_ = 0$$

and here

$$(\Box \times \Box) - (\_\_\_ \times \Box) + \underset{\uparrow}{\_\_\_} = 0$$

are "variables."

For the student taking the test, however, these numbers are **constants**. The student is supposed to answer the question that was actually asked, and not some other question that might have been asked.

Perhaps the important idea is that, while **all** these letters (a, b, x, y, etc.) and frames ($\Box$, $\triangle$, etc.) really name **variables** (and that is what the modern logician would say), there may be a definite point in time when we choose to make numerical replacements for these variables, and that time may come sooner for some variables than it will for others.

Thus the pilot is still free to determine the direction of the airplane—for him, that is still a variable—but the choice of how many engines to put on it was made long before he took off.

To emphasize this distinction, we could write our quadratic equation problem like this

$$(\Box \times \Box) - (a \times \Box) + b = 0$$

or else like this

$$x^2 - ax + b = 0.$$

To the modern logician, $\Box$, x, a, and b **all denote variables.** But the teacher will make a numerical replacement for the variables a and b, so that—if we are students—by the time the problem gets to us, **a and b will be constants.**

(1) Suppose you are the teacher. Make numerical replacements for the variables a and b in the equation

$$(\Box \times \Box) - (a \times \Box) + b = 0,$$

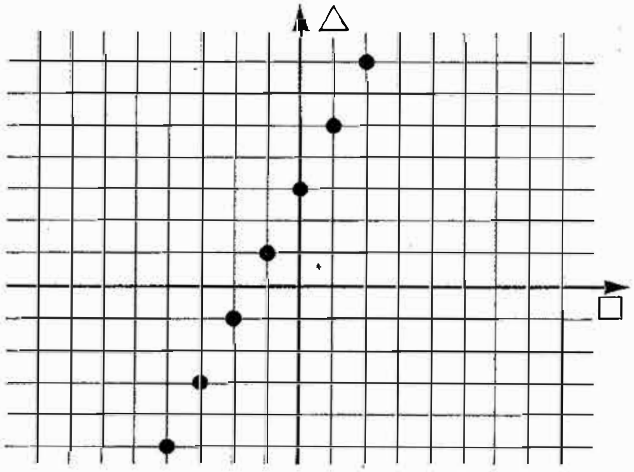
so as to get a reasonably easy examination question. (Make sure you can solve it yourself!)

(2) Now suppose you are the student. Exchange your paper with someone else, and see if you can solve the quadratic equation he made up. (Also, see if he can solve yours!)

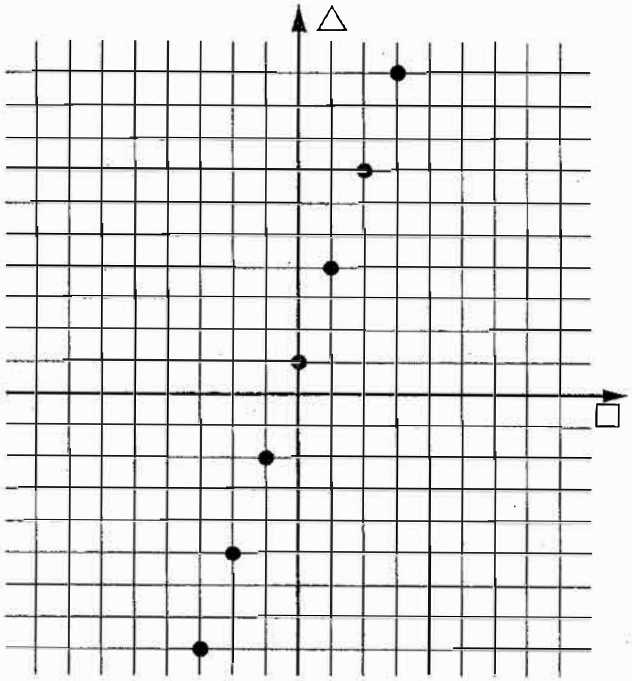
(3) In modern language, we would call the $\square$, *a*, and *b* of question 1 **variables**. Descartes, however, would have called some of them variables and some of them constants.

Which would Descartes call constants? Which would he call variables?

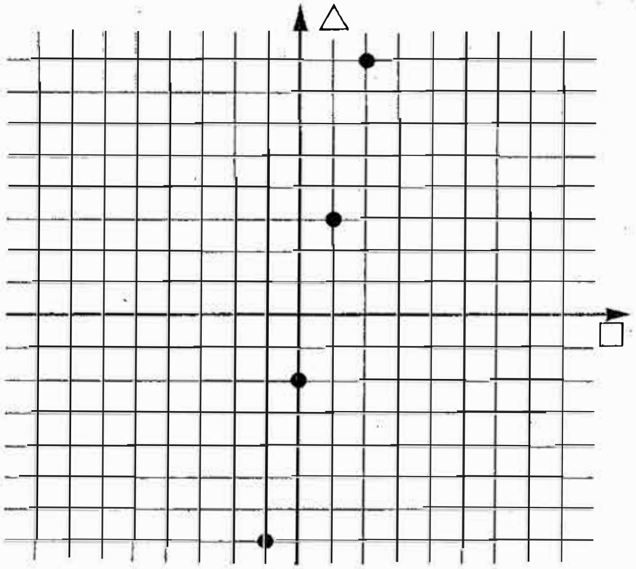
(4) If we study these graphs, we notice two important patterns.



$(2 \times \square) + 3 = \triangle$



$(3 \times \square) + 1 = \triangle$



$(5 \times \square) + {}^{-}2 = \triangle$

We can say that if the equation is

$$(\boldsymbol{a} \times \square) + \boldsymbol{b} = \triangle ,$$

then the "slope" or "steepness" pattern is: "over one to the right, and up ***a***." The number ***b*** also has a geometric significance. Do you know what it is?

(5) In question 4, which would Descartes call **variables** and which would he call **constants**?

(6) Geoff says that before you start in on working a problem, you choose definite numbers for the constants. While you are working that one problem, you don't change these constant numbers. But once you finish that one problem, if you want to go on to another problem, you may choose new constants. What do you think?

(7) Can you give some examples of variables and constants?

What could you write on a piece of paper so that your assistant will be able to help you, if he uses only UV and simple arithmetic?

(3) Suppose you want your assistant to help you solve problems like these:

$$(3 \times \square) + 5 = 11$$

$$(2 \times \square) + 15 = 29$$

$$(5 \times \square) + 6 = 17$$

What could you write on a piece of paper so that your assistant will be able to help you, if all he knows is simple arithmetic and how to use UV correctly?

(4) Do you know what mathematicians mean when they talk about "**the general form of a problem**"? What do they mean when they say "**the general solution of a problem**"?

(5) Mathematicians call the equation $3x + 5 = 0$ **linear** (or "of degree one"), they call the equation $x^2 - 5x + 6 = 0$ **quadratic** (or "of degree two"), and they call the equation $x^3 - 2x^2 + 3x - 5 = 0$ **cubic** (or "of degree three").

What do you suppose they would call each of the following?

(a) $x^2 - 13x + 22 = 0$
(b) $x^3 + 2x^2 = 16$
(c) $x - 2 = 0$
(d) $3 + x = 7$
(e) $x^2 = 4$
(f) $x^4 + x^2 - x = 1$
(g) $x^7 + x^6 - x^3 + x + 9 = 0$
(h) $x^3 - x^2 + x - 1 = 0$

(6) How could you write the **general form** for a linear equation?

(7) Could you write the **general solution** for the general linear equation?

(8) How could you write the **general form** for a quadratic equation?

(9) How could you write the **general solution** for the general quadratic equation?

(10) How could you write the **general cubic equation**? the **general fourth-degree equation**? the general **fifth-degree equation?**

(11) How could you write the **general solution** for each type of equation in question 10?

CHAPTER 33

Hints on How To Solve Problems

One of the very important mathematicians of this century is George Polya, of Stanford University. Professor Polya has succeeded in solving many problems that no one else had ever been able to solve. He has reflected on this experience, and written a number of suggestions on how to go about solving difficult or puzzling problems.*

The kind of suggestions Professor Polya has made go somewhat like this:

(a) If the problem is too hard, can you see **which part** of the problem is making it hard?

(b) Is there any way to eliminate (or ignore) the "hard part"? Could you solve the **easy** part of the problem?

(c) Could you find a **different** problem that might be easier? (**After** you've solved this easier problem you may have learned enough so that **then** you can solve the hard problem. Obviously, you want the easy problem to be at least a little bit similar to the original hard problem.)

(d) Can you change this **new** problem around, so that it will turn into some kind of problem **that you already know about**? This is often called "reducing it to a problem that has already been solved."

(e) If you succeed in solving one kind of problem, you may want to ask yourself if what you have just done might let you go forward and solve some other harder (or more general) problems.

In the next chapter, we want to work on a famous mathematical problem, namely, the task of **finding the general solution of the general quadratic equation**. Professor Polya's suggestions can help us.

First, however, it may be wise to practice using some of his methods.

(1) Solving **the general quadratic equation** is a fairly hard problem. Perhaps we should start with some easier ones. Are there any quadratic equations that you already know how to solve? Can you make up some "easy" quadratic equations that you can easily solve right now?

(2) Solving **the general cubic equation** is also a hard problem. Are there any cubic equations that you can solve right now?

---

*George Polya, *How to Solve It* (Doubleday Anchor Books, New York, 1957).
George Polya, *Mathematics and Plausible Reasoning*, Two volumes (Princeton University Press, Princeton, N.J., 1954).
George Polya, *Mathematical Discovery. On Understanding, Learning, and Teaching Problem Solving*, Two volumes (John Wiley and Sons, New York, 1965).

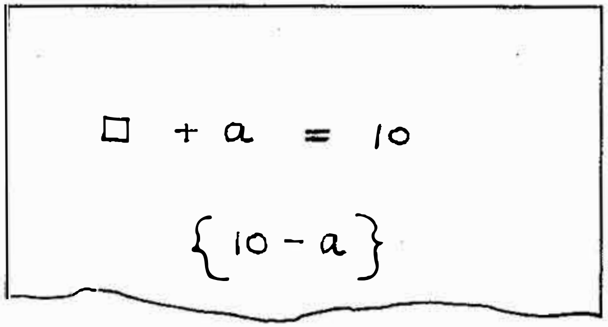
(3) Let's try reducing a new problem to an old one that you already know how to solve. Suppose that your assistant has a piece of paper that tells him how to solve equations like this:

$$\square + 3 = 10$$

$$\square + 5 = 10$$

$$\square + 1 = 10$$

Here, in fact, is what the paper says:



$$\square + a = 10$$

$$\{10 - a\}$$

Now, one day your assistant is confronted by the problem

$$\square + 18 = 25.$$

He can't do this by merely using UV and the formula on his piece of paper. Why not? What can he do?

(4) Can you extend your solution to harder or more general problems? You already know how to solve the equations

$$x^2 = 9$$

$$x^2 = 16$$

$$x^2 = 121$$

and so on. Could you use this to help you solve

$$(x - 2)^2 = 49?$$

(5) Again, try reducing a new problem to an old one that you already know how to solve. If you can solve the equation

$$(x - 3)^2 = 121,$$

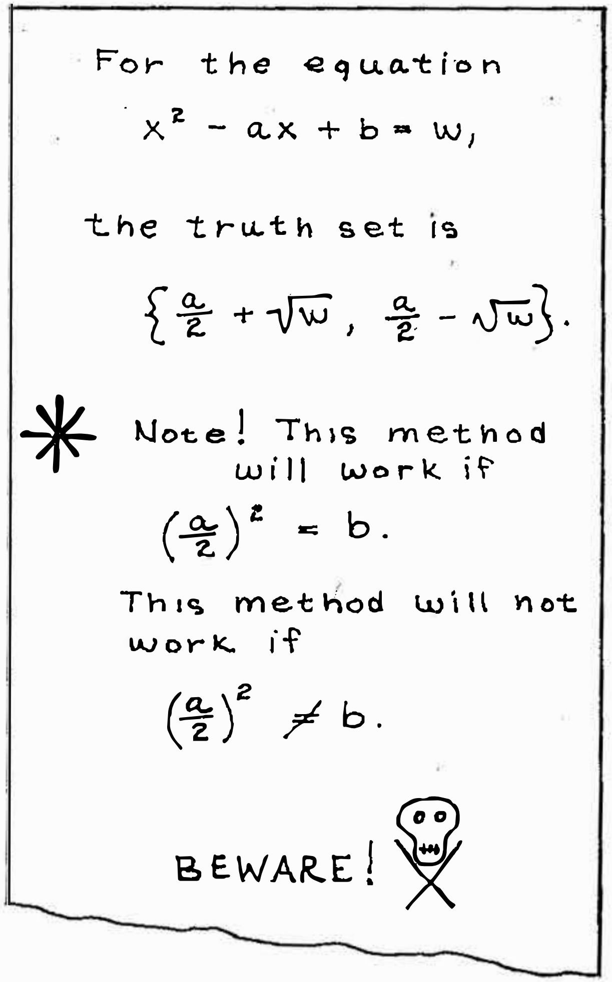
can you use that to help you solve

$$x^2 - 6x + 9 = 144?$$

(6) Can you solve the equation

$$x^2 - 6x + 7 = 79?$$

(7) Your assistant has a piece of paper which says:



For the equation

$$x^2 - ax + b = w,$$

the truth set is

$$\left\{\frac{a}{2} + \sqrt{w},\ \frac{a}{2} - \sqrt{w}\right\}.$$

✱ Note! This method will work if

$$\left(\frac{a}{2}\right)^2 = b.$$

This method will not work if

$$\left(\frac{a}{2}\right)^2 \neq b.$$

BEWARE!

Now, your assistant runs into this problem:

$$x^2 - 8x + 10 = 19$$

What should he do?

CHAPTER 34

All the Quadratic Equations in the World

In this chapter, we want to solve the general quadratic equation. We'll try to proceed by small steps, and make use of Professor Polya's suggestions.

(1) How can you **write** the general quadratic equation?

(2) Can you think of **any** quadratic equations that are so easy that you can solve them just by looking at them?

(3) Can you find the truth set for the open sentence

$$x^2 = 16?$$

(4) Can you find the truth set for the open sentence

$$(x - 1)^2 = 49?$$

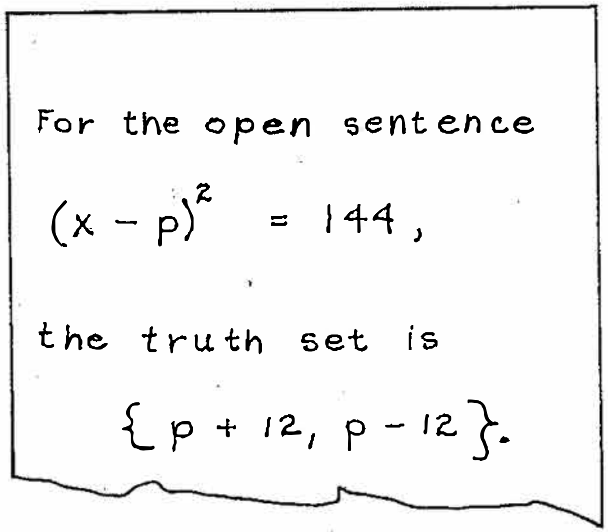
(5) Can you find the truth set for the open sentence

$$(x - 3)^2 = 144?$$

(6) Can you find the truth set for the open sentence

$$(x - 2)^2 = 81?$$

(7) Paul wrote this on a piece of paper:



For the open sentence

$(x - p)^2 = 144$,

the truth set is

$\{p + 12, p - 12\}$.

Then Paul gave the paper to his assistant. Suppose his assistant needs to solve the equation.

$$(x - 2)^2 = 144.$$

Can he do it?

(8) Jerry says that Paul was foolish to write "144" on the paper. What do you think?

(9) What do you suppose Jerry wrote on the paper that he gave to his assistant?

(10) Suppose the next problem was

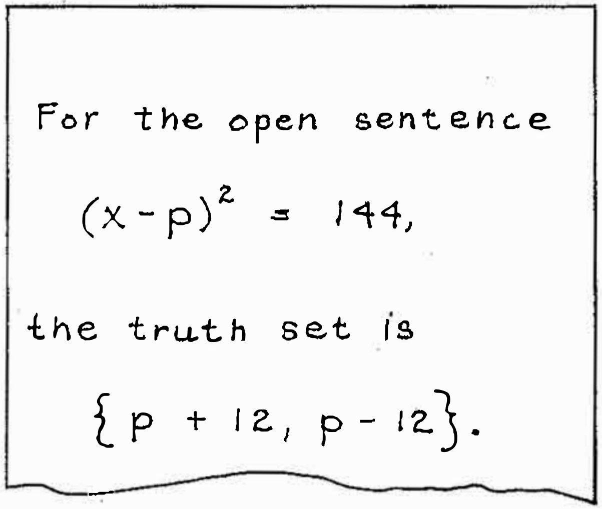
$$(x - 5)^2 = 81.$$

Can Paul's assistant solve this? Can Jerry's assistant solve it?

(11) For the problem

$$(x - 1)^2 = 144,$$

Paul's assistant looked at his paper



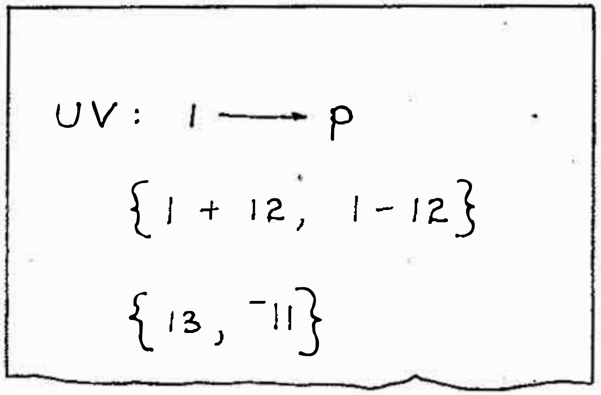
For the open sentence

$(x-p)^2 = 144,$

the truth set is

$\{p + 12, p - 12\}.$

and wrote:



UV: $1 \longrightarrow p$

$\{1 + 12, 1 - 12\}$

$\{13, {}^-11\}$

Did the assistant do the right thing?

(12) Pretend you are Paul's assistant. (That means you know about UV and simple arithmetic, but you don't know anything about equations unless you can read it on a slip of paper!) How would you solve

$$(x - 11)^2 = 144?$$

(13) Stop pretending you are Paul's assistant. Now you are a very clever scientist. Can you find the truth set for the open sentence

$$(x - h)^2 = k?$$

(Remember, in Descartes' words h and k are "constants." That means that somebody else will put numbers in for h and k, **before they give you the problem.**)

What will you write on the slip of paper you give to your assistant?

(14) Pretend you are Jerry's assistant. How will you solve each of the following equations? (Whenever you use UV, write it down as UV: $3 \longrightarrow h$, substituting for 3 whatever number you do use, and so on.)

(a) $(x - 1)^2 = 9$

(b) $(x - 4)^2 = 169$

(c) $(x - 15)^2 = 225$

(d) $(x - 10)^2 = 9$

Before we can go any further, we'll need to use some identities. It might be a good idea to practice a few identities right now.

(15) What do we mean by an **identity**?

(16) Is this an identity?

$$(\square - 3) \times (\square - 3) = (\square \times \square) - 9$$

(17) Is this an identity?

$$(R + S)^2 = R^2 + 2RS + S^2$$

(18) This is the beginning of a triangular array of numbers that mathematicians call "Pascal's triangle":

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Can you fill in the next line across? Can you fill in the line after that? and the next?

(19) Can you see how Pascal's triangle can help you remember these identities?

$$(R + S)^0 = R^0 = 1$$

$$(R + S)^1 = R^1 + S^1$$

$$(R + S)^2 = R^2 + 2RS + S^2$$

$$(R + S)^3 = R^3 + 3R^2S + 3RS^2 + S^3$$

$$(R + S)^4 = R^4 + 4R^3S + 6R^2S^2 + 4RS^3 + S^4$$

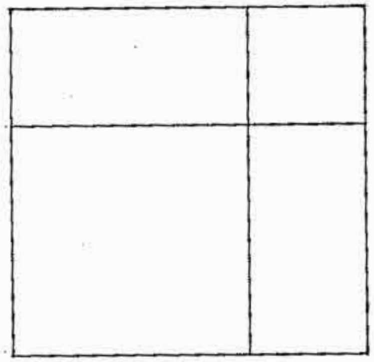
(20) Can you write down the missing part for each of these identities?

(a) $$(R + S)^5 = R^5 + 5R^4S + 10R^3S^2 + \underset{\uparrow}{\underline{\qquad}} + 5RS^4 + S^5$$

(b) $$(R + S)^6 = R^6 + \underset{\uparrow}{\underline{\qquad}} + 15R^4S^2 + 20R^3S^3 + \underset{\uparrow}{\underline{\quad}} + 6RS^5 + \underset{\uparrow}{\underline{\quad}}$$

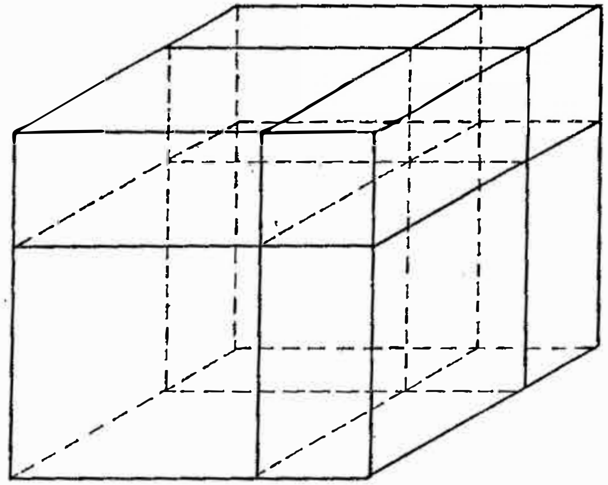
(c) $$(R + S)^7 = R^7 + \underset{\uparrow}{\underline{\qquad\qquad\qquad\qquad}}$$

(21) Which identity in question 19 does this picture uggest?

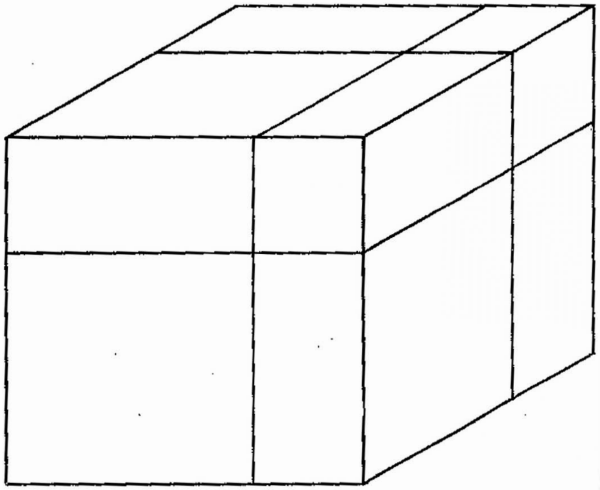


(22) Here you get your choice of two versions of this picture. Both versions show a cube sliced up into pieces, some of which are themselves cubes.

(a) The cluttered one



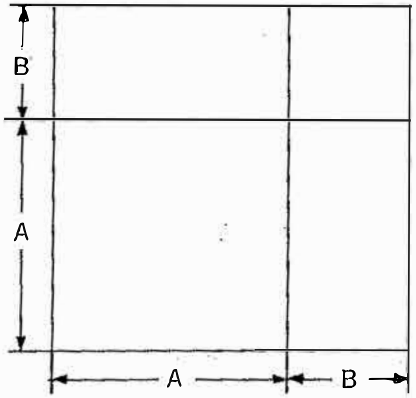
(b) The uncluttered one



Whichever picture you choose, they are both supposed to represent the same big block of wood, which has been sliced into smaller pieces.

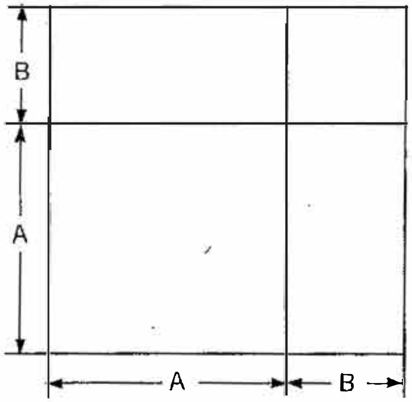
How many small pieces are there?

Can you find the volume of each of the small pieces if the faces have the dimensions shown below?



(23) Which identity in question 19 do these volumes suggest? (In fact, if you really "believe" this picture, it virtually gives you a **proof** of one of the identities. How come?)

(24) If you were not sure of your answer to question 21, suppose the dimensions were written in like this:



Compute the area of the large square (of side $A + B$) by two different methods.

Now, which identity in question 19 does this suggest? (In fact, if you "believe in" this picture, you have almost "proved" the identity. How come?)

(25) Anne says that she remembers

$$(R + S)^2$$

by saying:

"The square of the first term	R^2
plus twice the product of the two terms	$R^2 + 2RS$
plus the square of the second term."	$R^2 + 2RS + S^2$

Can you complete this identity?

$$(A + B)^2 = A^2 + \_\_\_\_$$

(26) We've spent enough time looking at identities. Let's get back to work trying to solve

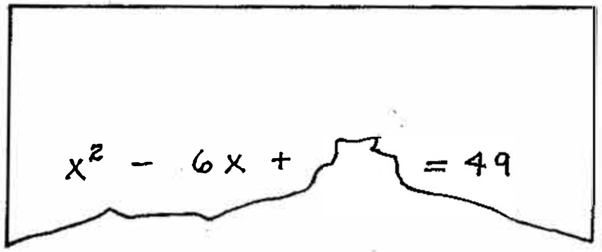
$$x^2 + Ax + B = W.$$

Where were we? How far had we gone?

(27) Solve

$$(x - 3)^2 = 169.$$

(28) Suppose you saw this problem:



When you find the missing piece of paper, what number do you **hope** will be written on it?

Solve these equations.

(29) $(x - 10)^2 = 9$

(30) $(x - 1)^2 = 196$

(31) $(x - 7)^2 = 36$

(32) $x^2 - 16x + 64 = 81$

(33) $x^2 - 2x + 1 = 4$

(34) $x^2 - 14x + 49 = 9$

(35) $x^2 - 20x + 100 = 121$

(36) $x^2 - 6x + 9 = 16$

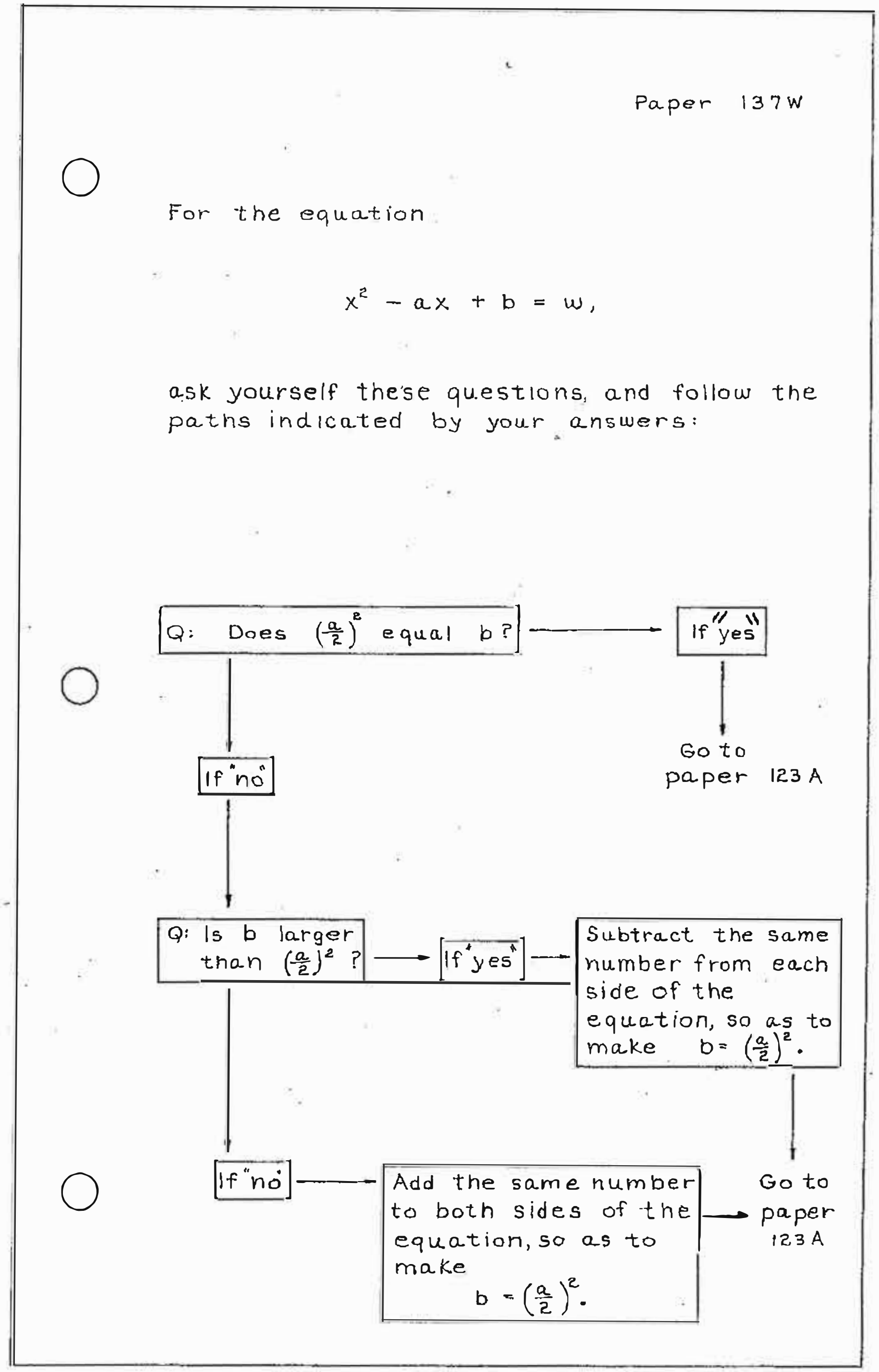
(37) Can you find the truth set for the open sentence

$$x^2 - ax + \left(\frac{a}{2}\right)^2 = w\,?$$

(38) Can you find the truth set for the open sentence

$$x^2 - ax + b = w\,?$$

(39) Ellen gave her assistant a piece of paper which said:



Paper 137W

For the equation

$$x^2 - ax + b = w,$$

ask yourself these questions, and follow the paths indicated by your answers:

Q: Does $\left(\frac{a}{2}\right)^2$ equal b?

If "yes"

Go to paper 123 A

If "no"

Q: Is b larger than $\left(\frac{a}{2}\right)^2$?

If "yes"

Subtract the same number from each side of the equation, so as to make $b = \left(\frac{a}{2}\right)^2$.

Go to paper 123 A

If "no"

Add the same number to both sides of the equation, so as to make $b = \left(\frac{a}{2}\right)^2$.

Go to paper 123 A

What do you suppose it says on Paper 123A?

(40) Pretend that you are Ellen's assistant. How would you solve the equation

$$x^2 - 10x + 15 = 39?$$

Trace out your path on Ellen's "map" (which mathematicians call a "flow chart").

(41) Pretend that you are Ellen's assistant. How would you solve the equation

$$x^2 - 12x + 45 = 18?$$

Again, trace out your path on Ellen's flow chart.

(42) Pretend that you are Ellen's assistant. How would you solve the equation

$$x^2 - 22x + 121 = 196?$$

Trace out your path on the flow chart on Paper 137W.

(43) Can Ellen's assistant solve **any quadratic equation in the world?**

(44) Anne and Jeanne worked out a general solution for the general quadratic equation this way:

$$\begin{aligned} x^2 - Ax + B &= W \\ x^2 - Ax &= W - B \\ x^2 - Ax + \left(\frac{A}{2}\right)^2 &= W - B + \left(\frac{A}{2}\right)^2 \\ \left(x - \frac{A}{2}\right)^2 &= W - B + \left(\frac{A}{2}\right)^2 \end{aligned}$$

What is the truth set for this open sentence? (Remember, in Descartes' words, x is the "variable" or the "unknown," whereas A, B, and W are "constants.") Mathematicians call this method "completing the square."

CHAPTER 35

Some History

There have been two advanced periods of civilization in European or Western history. The first was the civilization of the ancient Greeks (and their neighbors, such as Sumerians, Persians, and so forth). If we try to identify the early beginnings of this ancient civilization by using mathematics as our criterion, we might decide that it was well under way, in Babylonia, by 2000 B.C.*

Just before 1000 B.C., the site of the "most advanced" civilization began to shift from Egypt and Babylonia to the areas inhabited by the Hebrews, Assyrians, Phoenicians, Greeks, and at least as far west as Sicily. Of this newly developing society, Howard Eves writes, "The static outlook of the ancient orient became impossible and in a developing atmosphere of rationalism men began to ask why as well as how."+

Some great mathematical thinkers of this period were Thales (around 600 B.C.), Pythagoras (born about 575 B.C.; died about 500 B.C.), Zeno (495-435 B.C.), Eudoxus (408-355 B.C.), Diophantus (born about 400 B.C.), Euclid (330-275 B.C.), Archimedes (287-212 B.C.), and Hipparchus (born about 160 B.C.). Using these men as guides, we might say that the "golden age" of Greek mathematics began roughly at 600 B.C., and began to vanish around 200 B.C. Some Greek mathematicians were still at work as late as 250 A.D. (e.g., Pappus), but by then the "golden age" was well over. Indeed, all of ancient civilization was gradually destroyed as a living society, although fragments of it remain, in various forms, even today. Human life in the Western world entered the period known as the "Dark Ages."*

(Mathematical activity was not confined solely to Europe. Indeed, very important mathematical discoveries were made by the Hindus, particularly with regard to better methods for writing mathematics. These Hindu discoveries were later to play a very important role in Western mathematical progress, but the Hindu results were largely unknown to the ancient Greeks.)

The dates for the Dark Ages may be taken as roughly 450 A.D. until 1000 A.D. Of this period, Eves writes:

> The period starting with the fall of the Roman Empire in the middle of the fifth century and extending into the eleventh century is known as Europe's Dark Ages, for during this period civilization in western Europe reached a very low ebb. Schooling became almost nonexistent, Greek learning all but disappeared, and many of the arts and crafts bequeathed by the ancient world were forgotten. Only the monks of the Catholic monasteries, and a few cultured laymen, preserved a slender thread of Greek and

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*Compare the interesting accounts given in: Howard Eves, *An Introduction to the History of Mathematics*, Revised Edition, pp. 30-31, and elsewhere (Holt, Rinehart, and Winston, New York, 1964).

+Ibid., p. 52

*Compare Eves, *op. cit.*, p. 165 and Chapter 8.

> Latin learning. The period was marked by much physical violence and intense religious faith. The old social order gave way and society became feudal and ecclesiastical.
>
> The Romans had never taken to abstract mathematics, but contented themselves with merely practical aspects of the subject associated with commerce and civil engeering. With the fall of the Roman Empire and the subsequent closing down of much of east-west trade and the abandonment of state engineering projects, even these interests waned and it is no exaggeration to say that very little in mathematics, beyond the development of the Christian calendar, was accomplished in the West during the whole of the half millennium covered by the Dark Ages.*

Around 1000 A.D. Western society "started moving again"—started building along a "new frontier" of civilization, as we today might describe it—although for the 500 years from 1000 A.D. until 1500 A.D. the pace was fairly slow. This rebirth occurred in part as a result of Asian culture reaching Europe, particularly by way of Spain. The case of the French mathematical scholar and churchman, Gerbert, is interesting and points in the direction of what was to come. Gerbert was born in Auvergne, France, about 950 A.D. He traveled to Spain, where he studied in a Moslem school, and he may have been the person who introduced into Europe some of the Hindu methods for writing mathematics. He may also have designed clocks and musical instruments. In the year 999 he became the leader of the Roman Catholic Church, assuming the title of Pope Sylvester II.

As we have seen earlier, in the early part of the thirteenth century (just after 1200 A.D.) something new was added to European life: the great universities were started, particularly at Paris, Oxford, Cambridge, Padua, and Naples. A new civilization was beginning to appear—the one of which we, today, are the most recent part.

As history goes, our civilization is surprisingly new: even being generous, we would say that it is less than 1000 years old; looked at more narrowly, we might date it from about 1453, in which case it is more like 500 years old. Many parts of the United States are of the order of 100 years old (e.g., the state of Colorado), and others (such as the city of St. Louis, Missouri) are about 200 years old. Of course, all of our "civilization" in the United States has been built upon the civilization of Europe, and in surprisingly many ways we have even built upon what we have learned from the civilizations of the ancient Greeks and their neighbors. We are nonetheless in many important respects a surprisingly new society, and we clearly have the feeling that we are headed for new frontiers—although, as always when civilization is moving forward to new and unprecedented heights, we cannot see where we are going.

In all of the approximately 4000 years from the earliest beginnings of ancient civilization, through the Dark Ages, and up until the present time, at what point did the study of quadratic equations appear?

Clearly, were you able to show a problem in quadratic equations to a "typical" man of the Dark Ages,

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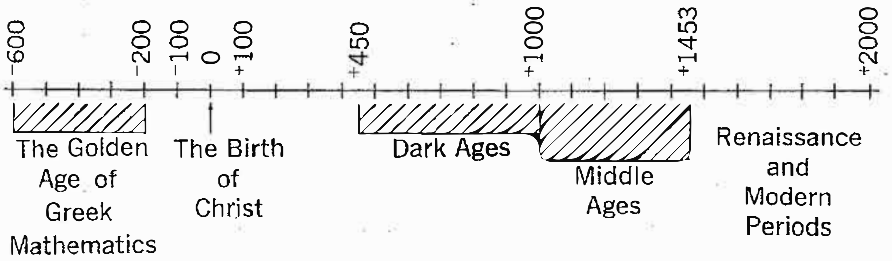
*Eves, *op. cit,* Chapter 8.

he would have been unable to solve it. Presumably he would even have been unable to understand what it was that you were trying to do. Quite likely he would not have cared, anyhow.

But the question is, was this ignorance of something that the ancients had worked on and finally came to understand, or was it ignorance of new mathematical discoveries that were not made until after the Renaissance? Had this knowledge been "lost," or had it not yet been discovered?

When did men first learn to work with quadratic equations, and to understand them?

A Broad Perspective of History Shown on a Number Line



The answer is surprising, if not nearly incredible. The study and understanding of quadratic equations is very old indeed. It dates from the early beginnings of ancient civilization. Here is what Eves writes about it:

"By 2000 B.C. Babylonian arithmetic had evolved into a well-developed rhetorical algebra. Not only were quadratic equations solved, both by the equivalent of substituting in a general formula and by completing the square, but some cubic (third degree) and biquadratic (fourth degree) equations were discussed."*

We need to clarify one point. The way that we today **write** quadratic equations is due to Descartes, who lived in the seventeenth century. Hence, the ancients clearly did not have our modern method of writing quadratic equations. (But, then, they also wrote numbers differently, so that does not necessarily put them out of the running for **understanding** quadratic equations and being able to work with them.)

It might be well for us to recapitulate the process of "completing the square." Here it is, using numbers:

(a) $x^2 - 6x + 4 = 11$

(b) We see that **this number** is not what we wish it were. Why? In fact,

$$\tfrac{1}{2} \times 6 = 3$$

$$3^2 = 9,$$

and so we wish that this number were 9.

(c) Consequently, we decide to add 5 to each side of the equation, so that we have

$$x^2 - 6x + 9 = 16.$$

---

*Eves, *op. cit.*, p. 33.

(d) Now, we can rewrite this as

$$(x - 3)^2 = 16.$$

(e) Evidently, the truth set for the equation in line (d) is:

$$\{7, {}^{-}1\}.$$

(f) Unfortunately, **we have solved the wrong problem.** The set $\{7, {}^{-}1\}$ is the truth set for the equation

$$(x - 3)^2 = 16.$$

However, we were asked to find the truth set for the open sentence

$$x^2 - 6x + 4 = 11.$$

(g) Fortunately, the two changes in the equation which we have made were both "transform operations." Hence, the equation

$$x^2 - 6x + 4 = 11$$

has exactly the same truth set as the equation

$$(x - 3)^2 = 16.$$

(h) Hence, the truth set for the equation

$$x^2 - 6x + 4 = 11$$

is

$$\{7, {}^{-}1\}.$$

If we use Descartes notion of "constants" and "variables," we can use this same method of "completing the square" to derive the general solution of the general quadratic equation.

(a) The general quadratic equation can be written in the form

$$x^2 - Ax + B = W.$$

(b) Now, we want to make sure that

$$\left(\tfrac{1}{2} \times A\right)^2 = B.$$

Since we do now know, in general, whether or not this is true, we can avoid the matter entirely by subtracting B from each side of the equation:

$$x^2 - Ax \qquad = W - B.$$

(c) Now, since we want to see

$$\left(\frac{A}{2}\right)^2$$

inserted on the left-hand side, our simplest procedure will be to put it there. How can we do this in a legal fashion (that is, by using a "transform operation")? The answer is simple: we shall add

$$\left(\frac{A}{2}\right)^2$$

to each side of the equation, which gives us:

$$x^2 - Ax + \left(\frac{A}{2}\right)^2 = W - B + \left(\frac{A}{2}\right)^2.$$

(d) Since we have now made the left-hand side into a "perfect square," we can write:

$$\left(x - \frac{A}{2}\right)^2 = W - B + \left(\frac{A}{2}\right)^2.$$

(e) For this equation, the truth set is

$$\left\{\frac{A}{2} + \sqrt{W - B + \left(\frac{A}{2}\right)^2},\ \frac{A}{2} - \sqrt{W - B + \left(\frac{A}{2}\right)^2}\right\}.$$

We can now hand our assistant a piece of paper that says:

For the open sentence

$$x^2 - Ax + B = W,$$

the truth set is

$$\left\{\frac{A}{2} + \sqrt{W - B + \left(\frac{A}{2}\right)^2},\ \frac{A}{2} - \sqrt{W - B + \left(\frac{A}{2}\right)^2}\right\}.$$

This is a secret from antiquity. Use it in good health!

(1) Many books in the first half of the twentieth century wrote the general quadratic equation in the form

$$ax^2 + bx + c = 0.$$

Use the formula which we just obtained to find the truth set for this equation.

Although the ancients solved the general quadratic equation, and also some cubic and quartic equations, they were never able to solve the general cubic equation nor the general quartic equation. In large part they must have been handicapped by their lack of our modern methods for **writing** mathematics. (Imagine doing mathematics, for example, with your eyes closed!)

When did man first come to understand the general cubic and quartic equations? Here is what Eves writes, describing the event:

> Probably the most spectacular mathematical achievement of the sixteenth century was the discovery, by Italian mathematicians, of the algebraic solution of cubic and quartic equations. The story of this discovery, when told in its most colorful version, rivals any page written by Benvenuto Cellini. Briefly told the facts seem to be these. About 1515, Scipione del Ferro (1465-1526), a professor of mathematics at the University of Bologna, solved algebraically the cubic equation $x^3 + mx = n$, probably basing his work on earlier Arabic sources. He did not publish his result but revealed the secret to his pupil Antonio Fior. Now about 1535, Nicolo of Brescia, commonly referred to as Tartaglia (the stammerer) because of a childhood injury which affected his speech, claimed to have discovered an algebraic solution of the cubic equation $x^3 + px = n$. Believing this claim was a bluff, Fior challenged Tartaglia to a public contest of solving cubic equations, whereupon the latter exerted himself and only a few days before the contest found an algebraic solution for cubics lacking a quadratic term. Entering the contest equipped to solve two types of cubic equations, whereas Fior could solve but one type, Tartaglia triumphed completely. Later Girolamo Cardano, an unprincipled genius who taught mathematics and practiced medicine in Milan, upon giving a solemn pledge of secrecy wheedled the key to the cubic from Tartaglia. In 1545, Cardano published his *Ars magna*, a great Latin treatise on algebra, at Nuremberg, Germany, and in it appeared Tartaglia's solution of the cubic. Tartaglia's vehement protests were met by Lodovico Ferrari, Cardano's most capable pupil, who argued that Cardano had received his information from del Ferro through a third party and accused Tartaglia of plagiarism from the same source. There ensued an acrimonious dispute from which Tartaglia was perhaps lucky to escape alive.
>
> Since the actors in the above drama seem not always to have had the highest regard for truth, one finds a number of variations in the details of the plot.
>
> The solution of the cubic equation $x^3 + mx = n$ given by Cardano in his *Ars magna* is essentially the following. Consider the identity
>
> $$(a - b)^3 + 3ab(a - b) = a^3 - b^3.$$
>
> If we choose a and b so that
>
> $$3ab = m, \qquad a^3 - b^3 = n,$$
>
> then x is given by $a - b$. Solving the last two equations simultaneously for a and b we find
>
> $$a = \sqrt[3]{\left(\frac{n}{2}\right) + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}$$
>
> $$b = \sqrt[3]{-\left(\frac{n}{2}\right) + \sqrt{\left(\frac{n}{2}\right)^2 \quad \left(\frac{m}{3}\right)^3}}$$
>
> and x is thus determined.
>
> It was not long after the cubic had been solved that an algebraic solution was discovered for the general quartic (or biquadratic) eqution. In 1540, the Italian mathematician Zuanne de Tonini da Coi proposed a problem to Cardano which led to a quartic equation. . . Although Cardano was unable to solve the equation, his pupil Ferrari succeeded, and Cardano had the pleasure of publishing this solution also in his *Ars magna.**

(2) Explain Cardano's solution of the equation

$$x^3 + mx = n.$$

(3) Cardano, in effect, had written this note:

For the equation

$x^3 + mx = n,$

$x = a - b$, where

$$a = \sqrt[3]{\left(\frac{n}{2}\right) + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}$$

$$b = \sqrt[3]{-\left(\frac{n}{2}\right) + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}$$

---

*Eves, *op. cit.*, pp. 220-221.

(Today we know that there may be other solutions for this equation, but Cardano's work is correct as far as it goes.)

Suppose Cardano met the equation

$$x^3 + ax^2 + bx + c = 0.$$

What would he do?

[Hint: He would use Polya's idea of "reducing it to some other problem that he already knew how to solve." One way to do this is to write

$$x = t + \alpha.]$$

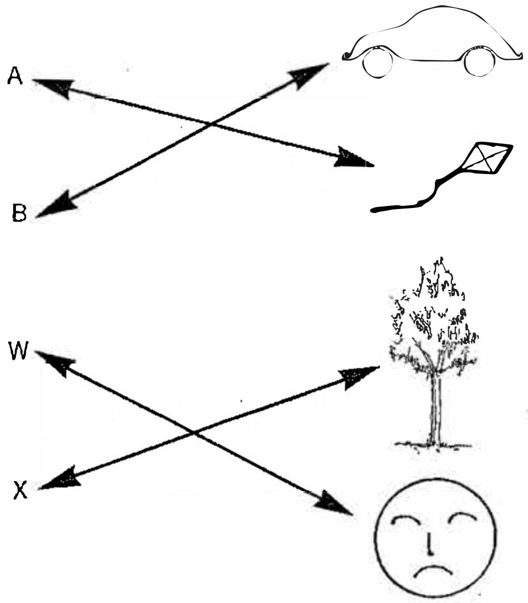
Can you finish this problem?

CHAPTER 36

The Idea of "Mappings" or "Correspondences"

Mathematicians nowadays think that the idea of correspondences is very important. Let's see if we can figure out what they mean when they use the word **correspondence.**

Jerry made this correspondence:



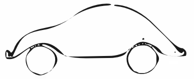
(1) In Jerry's scheme, what **corresponds to** A?

$A \longleftrightarrow$ ____

(2) In Jerry's scheme, what **corresponds to** W?

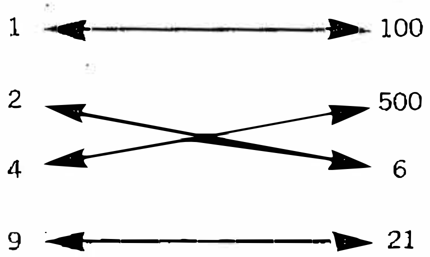
$W \longleftrightarrow$ ____

(3) In Jerry's scheme, what **corresponds to**



?

(4) Andy made this correspondence:

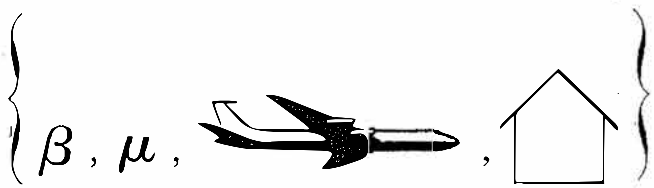


In Andy's scheme, what corresponds to 1?

(5) In Andy's scheme, what corresponds to 2?

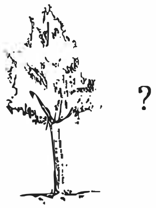
(6) Suppose that A is this set:

And suppose that B is this set:



Can you make a **correspondence** between the elements of A and the elements of B?

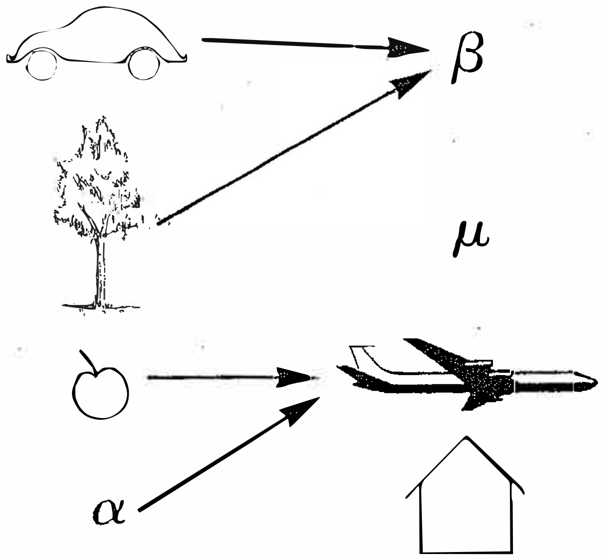
In your scheme, which element of B corresponds to



Will others in your class have a correspondence different from yours?

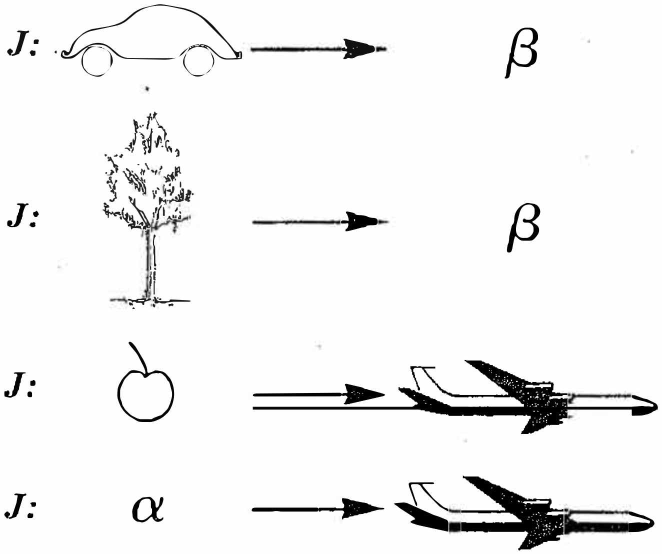
Which correspondence in your class is correct?

(7) Joe says this is a **mapping** of the set A into the set B:



What do you think?

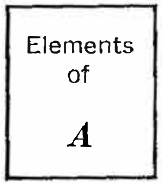
(8) Tom wrote Joe's mapping like this:



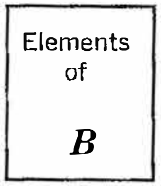
Can you make up another mapping of the set A into the set B? Can you write it the way Joe wrote his? Can you write it using Tom's method?

(9) How many **different** mappings of set A into set B can you find? How many different mappings of A into B do you suppose there are altogether?

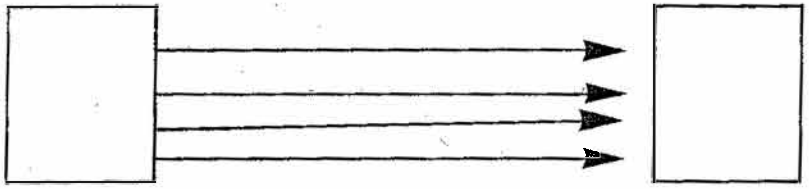
(10) Al says one way to show a mapping of set A into set B is to list the elements of A in a column here



and to list the elements of set B in a column here

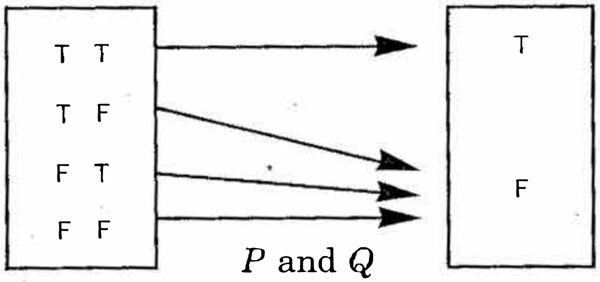


and then to draw arrows from each element of A to some of the elements of B, sort of like this:



Can you make up a mapping and write it this way?

(11) Sam used Al's method to write "P and Q" as a mapping of $\{TT, TF, FT, FF\}$ into $\{T, F\}$ like this:



What do you think?

(12) Can you use Al's method to write "P or Q" as a **mapping** of $\{TT, TF, FT, FF\}$ into $\{T, F\}$?

(13) Can you use Al's method to write

$$P \Rightarrow Q$$

as a **mapping** of $\{TT, TF, FT, FF\}$ into $\{T, F\}$?

(14) Can you use the operation

$$U : \square \longrightarrow {}^{\circ}\square$$

to map the set $\{{}^{+}1, {}^{+}3, {}^{-}4, {}^{-}10\}$ into the set $\{{}^{-}1, {}^{-}3, {}^{+}10, {}^{+}4\}$?

Can you write the **mapping** by Al's method?

(15) Can you use the operation

$$U : \square \longrightarrow {}^{\circ}\square$$

to map the set $\{{}^{+}1, {}^{+}2, {}^{+}3, {}^{-}5, 0\}$ into the set $\{{}^{-}1, {}^{-}2, {}^{-}3, {}^{-}4, {}^{-}5, {}^{+}1, {}^{+}2, {}^{+}5, 0\}$?

Can you use Al's method to write this mapping?

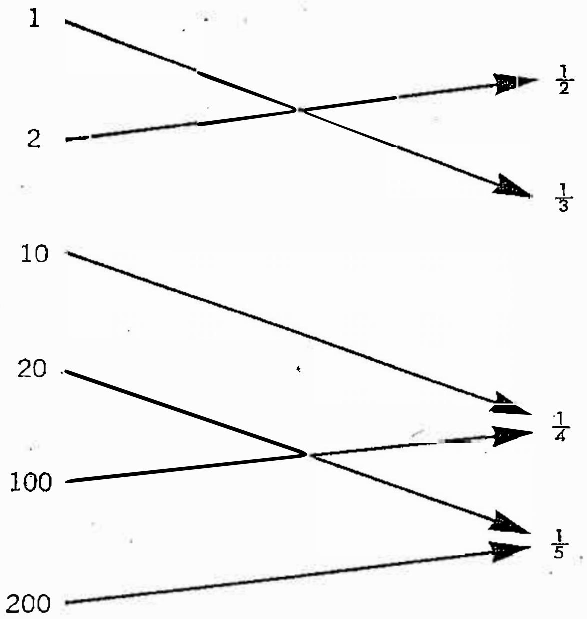
(16) Can you use the operation

$$U : \square \longrightarrow {}^{\circ}\square$$

to map the set $\{{}^{-}1, {}^{+}2, {}^{+}3, {}^{+}5\}$ into the set $\{{}^{+}1, 0, {}^{-}2, {}^{+}4\}$?

Can you use Al's method to write this mapping?

(17) Can you take the mapping



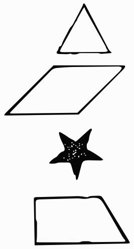
and write it, using Tom's method?

Mappings appear in many different disguises. Mathematicians try to see through these disguises and recognize the mapping, whenever they can.

(18) Tony says that the "Guessing Functions" game is really a mapping in disguise. What do you think? Can you make up a rule and show how it can be written using Al's method?

(19) Elizabeth says that hanging weights on a spring gives you a mapping. What do you think?

(20) Toby says that using a magnifying glass gives you a mapping of "pictures" into "large pictures." Suppose you used a magnifying glass on these pictures:



What would you get? Can you write this, using Al's method?

(21) Ellen says that you can use equations like

$$(\square \times \square) - (\underset{\uparrow}{\underline{\qquad}} \times \square) + \underset{\uparrow}{\underline{\qquad}} = 0$$

to map the ordered pair (5, 6) into the nonordered pair (2, 3). Can you figure out how she does it?

(22) Using Ellen's method, the ordered pair (8, 15) would map into _______.

Since Ellen maps (5, 6) into (2, 3),

$$(5, 6) \longrightarrow (2, 3),$$

mathematicians say that

"(2, 3) is the **image** of (5, 6)."

(23) When you use Ellen's mapping, what is the **image** of (8, 15)?

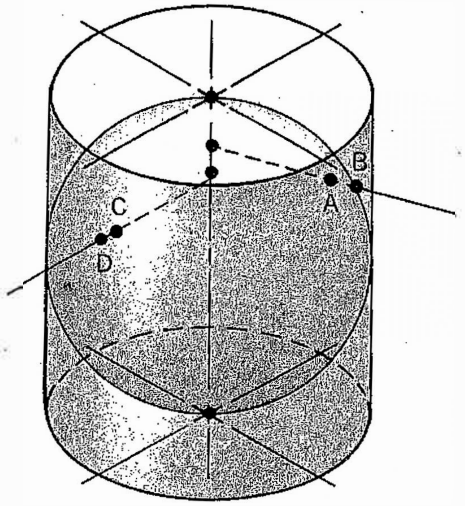
(24) When you use Ellen's mapping, what is the **image** of (9, 14)?

(25) When you use Ellen's mapping, what is the **image** of (13, 22)?

(26) Can you write Ellen's mapping, using Al's method?

(27) The world globe is (approximately) a sphere. How could you make a flat map of the world?

(28) Lex used this method:



He mapped point ***A*** (on the sphere) onto point ***B*** (on the cylinder). He mapped point ***C*** (on the sphere) onto point ***D*** (on the cylinder).

What will Lex's flat map look like? Will a country be the same size on the flat map as it is on the sphere? If not, will it be larger or smaller on the cylinder than on the sphere?

(29) John says that when we use **exponents** we are using a **mapping** in disguise.

John wrote:

$$E : \square \longrightarrow 10^{\square}$$

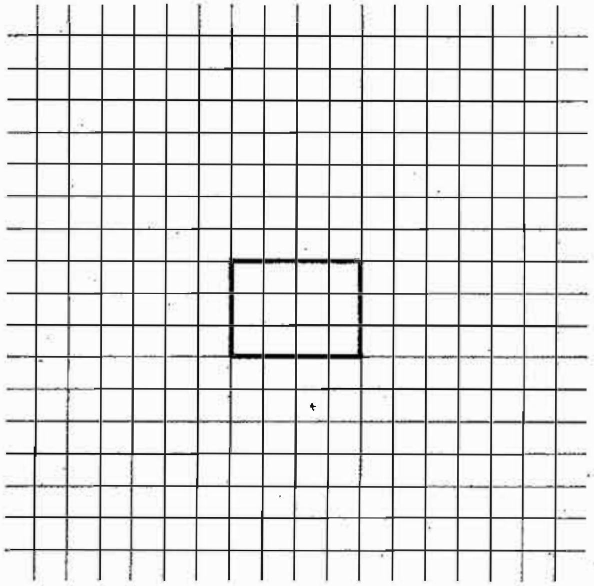
Can you find the image of 2, using John's mapping?

(30) Use John's mapping on the set $\{1, 2, 3, 4\}$. What is the image set?

(31) Can you use Al's method to write the mapping in question 30?

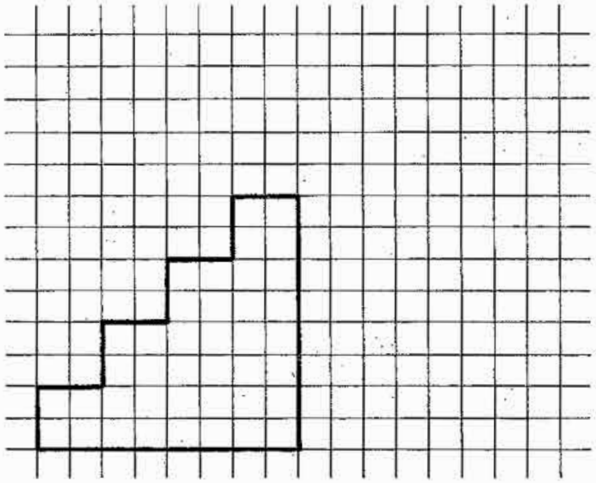
(32) Using John's mapping, what is the **image** of 5?

(33) Bill mapped the figure

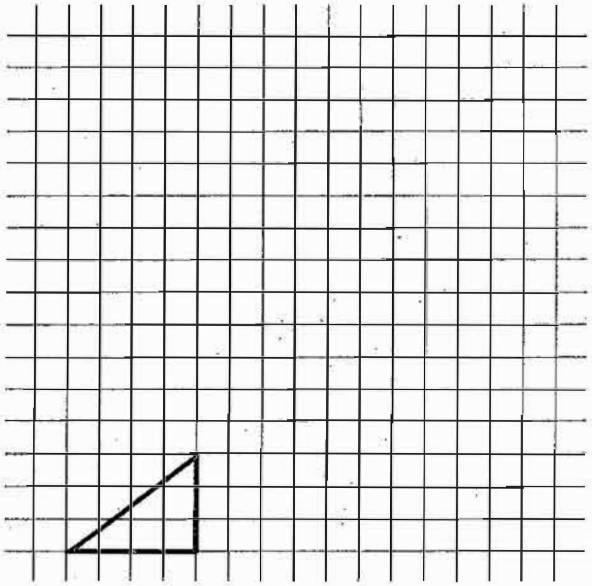


into the number 12.

Using Bill's mapping, can you find the **image** of the following figure?



(34) Using Bill's mapping, can you find the **image** of this figure?



(35) Can you write Bill's mapping, using Al's method?

(36) Debbie says that Bill mapped **plane figures** into **numbers**, by using the idea of **area.** She says she will map **plane figures** into **segments**, by drawing the figure on a grid, pretending the sun is directly overhead, and mapping the figure into its **shadow** on the x-axis. Debbie is pretending that the x-axis is the ground.

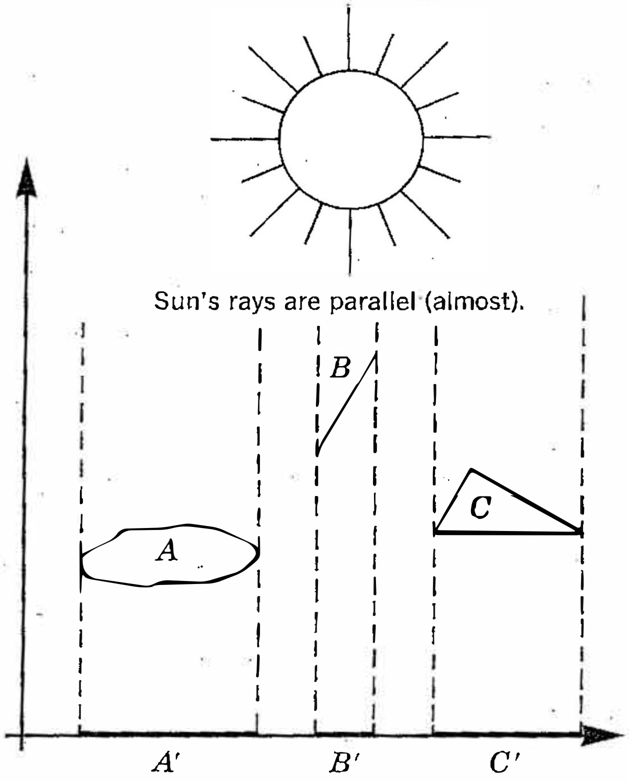
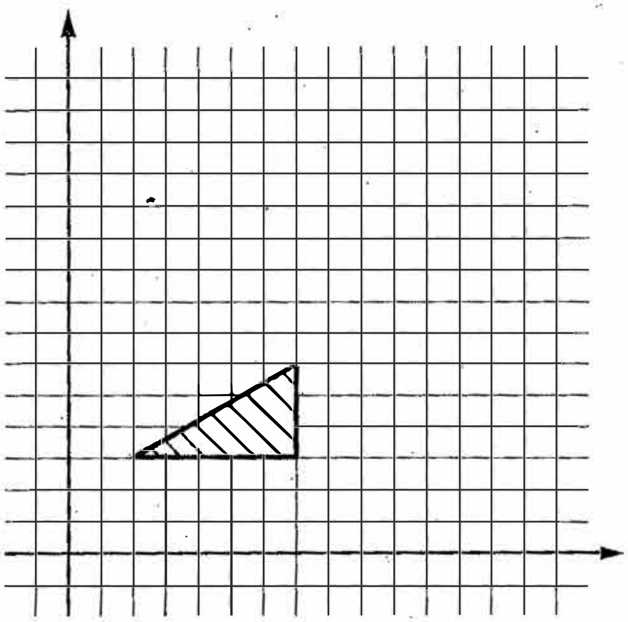


Figure A is mapped into its shadow A'.
Figure B is mapped into its shadow B'.
Figure C is mapped into its shadow C'.

Using Debbie's mapping, can you find the image of this triangle?



(37) Using Debbie's mapping, choose some figures of your own, and see if you can find the "shadows."

(38) Using Debbie's mapping, what is the image of the point (3, 4)?

(39) Suppose $0 < b$. Using Debbie's mapping, what is the image of the point (a, b)?

(40) Suppose $b < 0$. Using Debbie's mapping, what is the image of the point (a, b)?

(41) Can you use a light bulb or a flashlight to map some physical objects into their shadows?

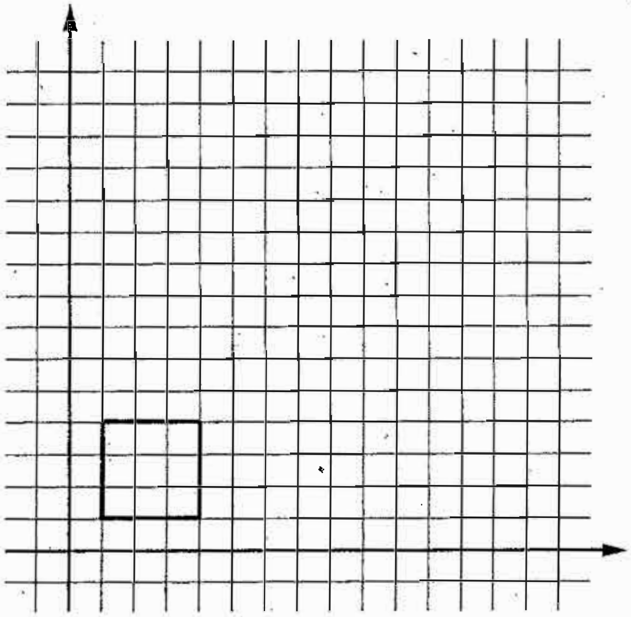
(42) Earle says you can map the plane (which mathematicians write "E_2") into itself.

Earle made up this mapping:

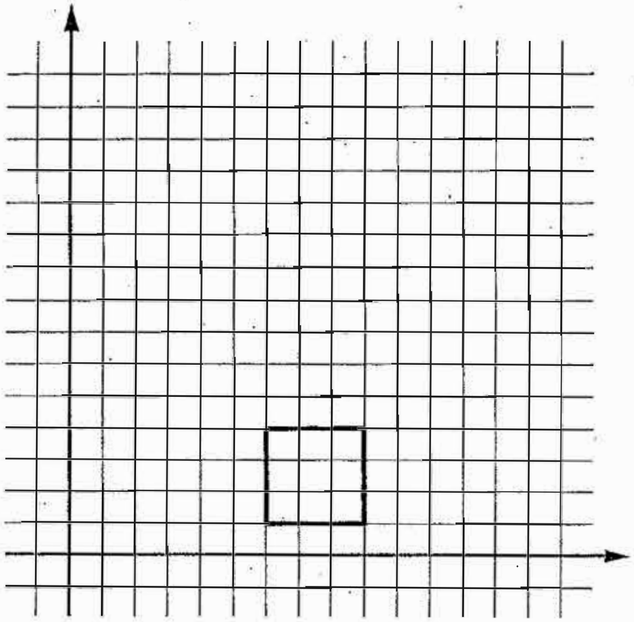
$$X_{\text{new}} = X_{\text{old}} + 5$$

$$Y_{\text{new}} = Y_{\text{old}}$$

He used his mapping to transform the figure

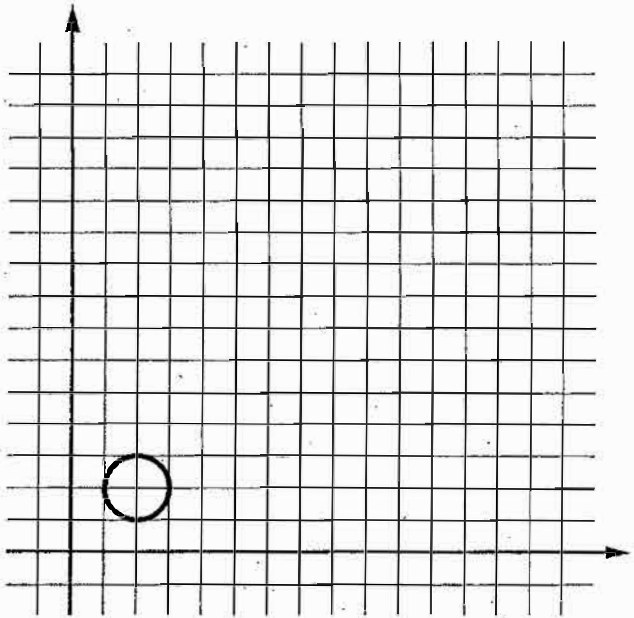


into this figure:

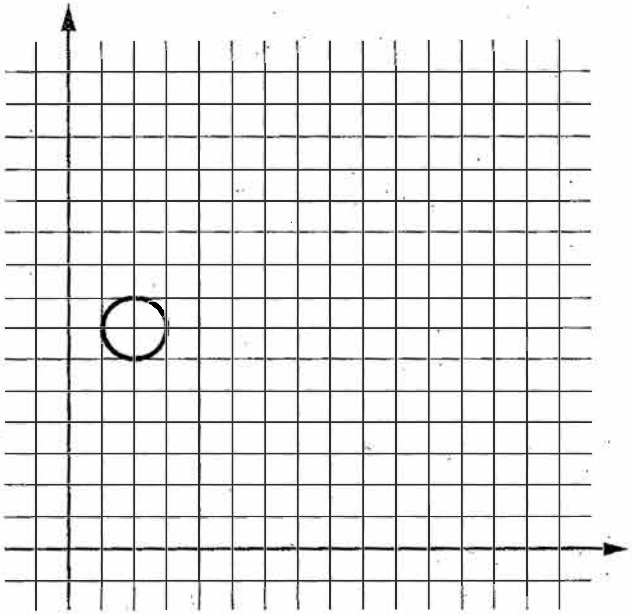


What do you think?

(43) Dexter used Earle's mapping to find the image of this figure.



He got:



Do you agree? Can you draw a figure in E_2 and then find its image, using Earle's mapping?

(44) Bernie mapped E_2 into E_2, using this mapping:

$$X_{\text{new}} = Y_{\text{old}}$$

$$Y_{\text{new}} = {}^{\circ}X_{\text{old}}$$

Using Bernie's mapping, what is the image of $(1, 0)$? What is the image of $(0, 0)$? What is the image of $(0, 1)$? Can you show this by Al's method (as Al did in question 10, earlier in this chapter)? If you **start** with the set $\{(1, 0), (0, 0), (0, 1,\}$, what is the image set?

(45) Nancy mapped E_2 into E_2 by this mapping:

$$X_{\text{new}} = {}^{\circ}Y_{\text{old}}$$

$$Y_{\text{new}} = X_{\text{old}}$$

Can you draw a figure in E_2 and then find its image, using Nancy's mapping?

(46) Draw some figure in E_2. Can you find its image using Bernie's mapping?

(47) Ted mapped E_2 into E_2 like this:

$$X_{\text{new}} = {}^{\circ}X_{\text{old}}$$

$$Y_{\text{new}} = Y_{\text{old}}$$

Can you find the image of $(0, 0)$, using Ted's mapping? Can you find the image of $(1, 0)$? Can you find the image of $(0, 1)$? Can you show the mapping of $\{(1, 0), (0, 0), (0, 1)\}$, using Al's method?

(48) Draw some figure in E_2. Can you find its image, using Ted's mapping?

CHAPTER 37

Candy-Store Arithmetic

Mathematicians have often built important and elaborate mathematical systems by starting with a very "commonplace" idea, which they have been able to extend in some significant way. Let's see if we can do it.

(1) Andy goes to a candy store that sells peppermints (for 2¢ each), chocolate almond bars (for 10¢ each), and chocolate-covered ants (for 50¢ a box). We can write this as:

$$\begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix}$$

Suppose that today Andy buys three peppermints, one chocolate almond bar, and zero boxes of chocolate-covered ants (as a matter of fact, Andy **always** buys zero boxes of chocolate-covered ants). We can write this as:

$$\begin{pmatrix} 3 & 1 & 0 \end{pmatrix}$$

The numbers

$$\begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix}$$

are called a **price matrix**, and the numbers

$$\begin{pmatrix} 3 & 1 & 0 \end{pmatrix}$$

are called a **demand matrix.***

Can you multiply the demand matrix

$$\begin{pmatrix} 3 & 1 & 0 \end{pmatrix}$$

by the price matrix

$$\begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix}$$

to get the **amount of money** that Andy spent?

$$\begin{pmatrix} 3 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix} = ?$$

(2) Joan says that you write:

$$\begin{aligned} \begin{pmatrix} 3 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix} &= (3+2) \times (1+10) \times (0+50) \\ &= 5 \times 11 \times 50 \\ &= 2750¢ \end{aligned}$$

Do you agree?

---

*With the introduction of **matrices** we are turning to the mathematics of quite recent times. Indeed, the algebra of matrices was introduced in the year 1857 by the English mathematician Arthur Cayley (1821-1895). (The singular form is *matrix*. The plural form of the word is *matrices*.)

(3) Nancy says that you write:

$$\begin{pmatrix}3 & 1 & 0\end{pmatrix} \times \begin{pmatrix}2\\10\\50\end{pmatrix} = (3 \times 2) + (1 \times 10) + (0 \times 50)$$

$$= 6 + 10 + 0$$

$$= 16\text{¢}$$

Do you agree?

(4) Jill says that you write:

$$\begin{pmatrix}3 & 1 & 0\end{pmatrix} \times \begin{pmatrix}2\\10\\50\end{pmatrix} = (3 \times 50) + (1 \times 10) + (0 \times 2)$$

$$= 150 + 10 + 0$$

$$= 160\text{¢}$$

Do you agree?

(5) Suppose that Andy goes to the store on Thursday and buys

$$\begin{pmatrix}4 & 2 & 0\end{pmatrix}.$$

What did he buy? How much money did he spend?

(6) Toby went to the store and bought

$$\begin{pmatrix}1 & 3 & 0\end{pmatrix}.$$

What did he buy? How much money did he spend?

(7) One day the store had a special sale. For that day only their prices were

$$\begin{pmatrix}1\\5\\25\end{pmatrix}$$

How much did each item cost at the sale price?

(8) On the day of the sale, Nancy bought

$$\begin{pmatrix}4 & 3 & 0\end{pmatrix}.$$

What did Nancy buy? How much money did she spend?

(9) Up until now we have been dealing with a candy store that sells peppermints and chocolate almond bars, and tries to sell boxes of chocolate-covered ants.

Suppose we now try to build an **abstract** system. We will forget all about stores and prices and quantities. All we will remember is the **pattern** of what we have been doing. **Using this same pattern**, let's multiply these two matrices:

$$\begin{pmatrix}5 & 2\end{pmatrix} \times \begin{pmatrix}4\\3\end{pmatrix} = ?$$

(10) Can you use this same **pattern** to multiply the following two matrices?

$$\begin{pmatrix}1 & 3 & 0 & 2\end{pmatrix} \times \begin{pmatrix}5\\0\\19\\8\end{pmatrix} = ?$$

(11) Suppose that someone knew all about □, △, *a*, *b*, *c*, *x*, *y*, . . . notation, but **did not** know matrix notation. Could you write something that would show him immediately how matrix notation works?

(12) Jane says she could show him matrix notation by writing:

$$\begin{pmatrix} A & B & C \end{pmatrix} \times \begin{pmatrix} D \\ E \\ F \end{pmatrix} = (A + D) \times (B + E) \times (C + F)$$

Do you agree?

(13) Hal says he could explain matrix notation by writing:

$$\begin{pmatrix} A & B & C \end{pmatrix} \times \begin{pmatrix} A \\ B \\ C \end{pmatrix} = (A \times A) + (B \times B) + (C \times C)$$

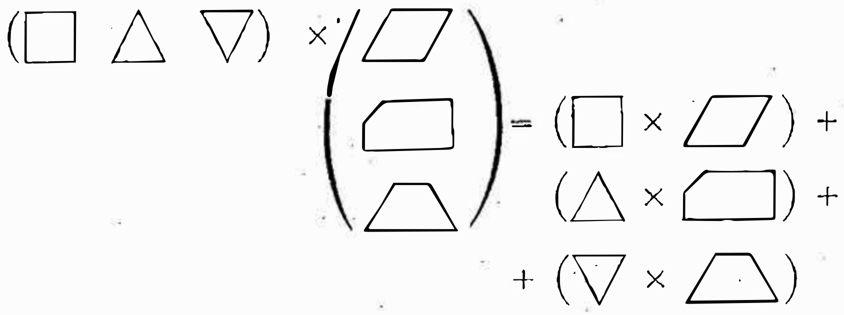
Do you agree?

(14) Ellen says she could explain matrix notation by writing:

$$\begin{pmatrix} A & B & C \end{pmatrix} \times \begin{pmatrix} D \\ E \\ F \end{pmatrix} = (A \times D) + (B \times E) + (C \times F)$$

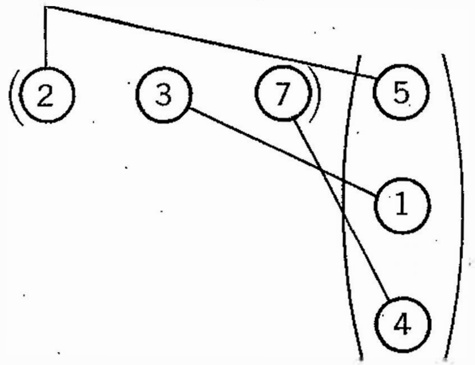
What do you think?

(15) Courtney says he would write:

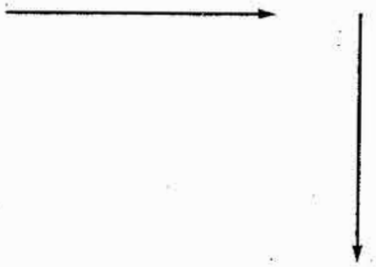


What do you think?

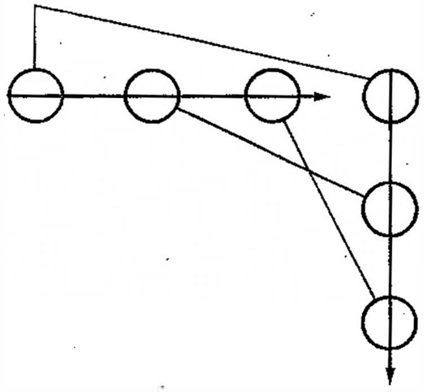
(16) Joe says that the way to multiply matrices is **by pairs**



and "from left to right on the left and from top to bottom on the right,"



which we might also draw like this:



Do you think this is a good description?

(17) Can you multiply these matrices?

$$\begin{pmatrix} 3 & 7 & 10 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} = ?$$

(18) Can you multiply these matrices?

$$\begin{pmatrix} 8 & 12 & 15 \end{pmatrix} \times \begin{pmatrix} 10 \\ 5 \\ 2 \end{pmatrix} = ?$$

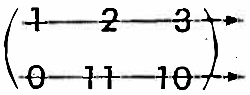
(19) Can you multiply these matrices?

$$\begin{pmatrix} A & B & C \end{pmatrix} \times \begin{pmatrix} W \\ X \\ Y \end{pmatrix} = ?$$

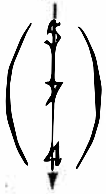
(20) Can you extend the idea of matrix multiplication, still using the same pattern, so that you can multiply these matrices?

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 11 & 10 \end{pmatrix} \times \begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix} = ?$$

(21) Lex says the idea is to use **rows** from the left-hand matrix



and to use **columns** from the right-hand matrix.



What do you think?

(22) Ellen says that the answer has this form: A matrix with some number **here** . . .

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 11 & 10 \end{pmatrix} \times \begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} \_\_\_ \\ \_\_\_ \end{pmatrix}$$

and with some number **here**.

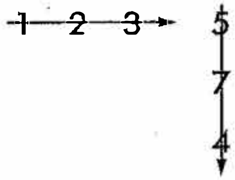
If we write

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 11 & 10 \end{pmatrix} \times \begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} \Box \\ \triangle \end{pmatrix},$$

what number should go in the $\square$ in order to make a true statement?

(23) What number should go in the $\triangle$?

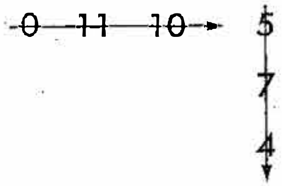
(24) To find the $\square$ number, Eva wrote this:



$$(1 \times 5) + (2 \times 7) + (3 \times 4) = 5 + 14 + 12$$
$$= 31$$

Do you agree?

(25) To find the $\triangle$ number, Marilyn wrote this:



$$(0 \times 5) + (11 \times 7) + (10 \times 4) = 77 + 40$$
$$= 117$$

Do you agree?

(26) Can you multiply these two matrices?

$$\begin{pmatrix} 1 & 0 & 2 \\ 5 & 0 & 11 \end{pmatrix} \times \begin{pmatrix} 3 \\ 14 \\ 10 \end{pmatrix} = \begin{pmatrix} \_\_\_ \\ \_\_\_ \end{pmatrix}$$

(Nancy says the answer should be a matrix with two numbers in it. Do you agree?)

(27) Can you multiply these two matrices?

$$\begin{pmatrix} 1 & 5 & 7 \\ 10 & 3 & 0 \\ 4 & 6 & 8 \end{pmatrix} \times \begin{pmatrix} 2 \\ 9 \\ 11 \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}$$

(28) Hal says the answer should be a matrix like this:

$$\begin{pmatrix} A \\ B \end{pmatrix}$$

Do you agree?

(29) Tom says the answer to question 27 should be a matrix like this:

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

Do you agree?

(30) Jane says the answer to question 27 should be a matrix like this:

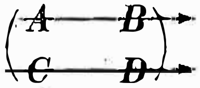
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

What do you think?

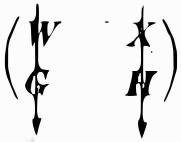
(31) Can you extend the idea of matrix multiplication, so that you can multiply "2 - by - 2" matrices?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} W & X \\ G & H \end{pmatrix} = \begin{pmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{pmatrix}$$

(32) Lex says that the idea is to use **rows** from the left-hand factor



and **columns** from the right-hand factor.



What do you think?

(33) Ellen says that

$$\begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 4 & 7 \\ 6 & 11 \end{pmatrix} = \begin{pmatrix} 42 & 76 \\ 14 & 25 \end{pmatrix}.$$

What do you think?

(34) Jerry says that Ellen got the "42" by saying

$$(3 \times 4) + (5 \times 6) = 42.$$

What do you think?

(35) How did Ellen get the "25"?

(36) How did Ellen get the "76"?

(37) How did Ellen get the "14"?

(38) Can you multiply these two matrices?

$$\begin{pmatrix} 7 & 1 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 5 \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{pmatrix}$$

(39) Amy knows all about $\square$, $\triangle$, A, B, C, . . . , and so on, but she **does not** know how to multiply 2-by-2 matrices. Can you write something that will show Amy how to multiply 2-by-2 matrices?

(40) Jerry wrote:

$$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 12 & 7 \end{pmatrix}$$

Does this show Amy how to multiply 2-by-2 matrices?

(41) Steve wrote:

$$\begin{pmatrix} \square & \triangle \\ \triangledown & \text{⬠} \end{pmatrix} \times \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

$$= \begin{pmatrix} (\square \times W) + (\triangle \times Y) & (\square \times X) + (\triangle \times Z) \\ (\triangledown \times W) + (\text{⬠} \times Y) & (\triangledown \times X) + (\text{⬠} \times Z) \end{pmatrix}$$

Does this show Amy how to multiply 2-by-2 matrices?

(42) Mary wrote:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \begin{pmatrix} (A \times W) + (B \times Y) & (A \times X) + (B \times Z) \\ (C \times W) + (D \times Y) & (C \times X) + (D \times Z) \end{pmatrix}$$

Does this show Amy how to multiply 2-by-2 matrices?

(43) Who is right, Steve or Mary?

Can you multiply these 2-by-2 matrices?

(44) $\begin{pmatrix} 1 & 7 \\ 2 & 0 \end{pmatrix} \times \begin{pmatrix} 3 & 4 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$

(45) $\begin{pmatrix} 5 & 1 \\ 3 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$

(46) $\begin{pmatrix} 5 & 7 \\ 3 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$

(47) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$

(48) Jill says that we **add** matrices like this:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \begin{pmatrix} A + W & B + X \\ C + Y & D + Z \end{pmatrix}$$

What do you think?

(49) Can you add these two matrices?

$$\begin{pmatrix} 1 & 3 \\ 7 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 3 & 10 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

(50) Can you write some axioms for arithmetic and algebra?

(51) Do you know an identity that is called the "commutative law for addition"?

(52) In arithmetic and algebra, what is special about the number 1?

(53) Can you write the identity that is known as the "law for 1"?

(54) In arithmetic and algebra, what is special about the number 0?

(55) Can you write the identity that is known as the "addition law for zero"?

(56) Can you make up an "addition law for zero" that will apply to 2-by-2 matrices?

(57) What 2-by-2 matrix corresponds to the number 0?

(58) Is there a "multiplication law for zero" that works for matrices?

(59) Can you write what might be called a "law for 1" for **matrices**, instead of **numbers**?

(60) Jerry says the "law for 1" for matrices would look like this:

$$\begin{pmatrix}\text{Any 2-by-2}\\ \text{matrix}\end{pmatrix} \times \underset{\substack{\uparrow\\ \text{The ``1'' ma-}\\ \text{trix, what-}\\ \text{ever that}\\ \text{may be}}}{\begin{pmatrix}\quad & \quad\\ & \end{pmatrix}} = \begin{pmatrix}\text{The same 2-by-2}\\ \text{matrix that you}\\ \text{started with}\end{pmatrix}$$

Can you use the ***A***, ***B***, ***C***, ***D***, . . . notation to write Jerry's law?

(61) Mary wrote this:

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix} \times \underset{\substack{\uparrow\\ \text{The ``1''}\\ \text{matrix, if}\\ \text{there really}\\ \text{is any}}}{\begin{pmatrix}\quad & \quad\\ & \end{pmatrix}} = \begin{pmatrix} A & B\\ C & D\end{pmatrix}$$

What do you think?

(62) Can you find the "1" matrix to put into "Mary's law"?

(63) Don says the "1" matrix should be

$$\begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}.$$

(64) Can you write the "law for 1" for matrices? What is the "1" matrix?

(65) George says that, for numbers, there is an axiom that says that every number except zero has a multiplicative inverse, so that

$$A \times {}^{r}\!A = 1.$$

Do you think this axiom applies to the system of matrices?

(66) Can you multiply these two matrices?

$$\begin{pmatrix} 3 & 5\\ 2 & 1\end{pmatrix} \times \begin{pmatrix} \frac{^-1}{7} & \frac{5}{7}\\ \frac{2}{7} & \frac{^-3}{7}\end{pmatrix} = \begin{pmatrix}\quad & \quad\\ & \end{pmatrix}$$

(67) George says that if we use ***A*** to mean

$$\begin{pmatrix} 3 & 5\\ 2 & 1\end{pmatrix},$$

then the multiplicative inverse ${}^{r}\!A$ would be

$$\begin{pmatrix} \frac{^-1}{7} & \frac{5}{7}\\ \frac{2}{7} & \frac{^-3}{7}\end{pmatrix}.$$

What do you think?

(68) Can you find the multiplicative inverse of this matrix?

$$\begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix}$$

(69) Can you find the multiplicative inverse of this matrix?

$$\begin{pmatrix} \frac{{}^{-}1}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{{}^{-}3}{7} \end{pmatrix}$$

(70) Do the matrices

$$\begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} \frac{{}^{-}1}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{{}^{-}3}{7} \end{pmatrix}$$

satisfy the commutative law for multiplication?

(71) Do **all** matrices satisfy the commutative law for multiplication?

(72) Can you multiply these two matrices?

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

(73) Can you multiply these two matrices?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

(74) Can you find the multiplicative inverse of this matrix?

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

(75) Can you find the multiplicative inverse of this matrix?

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Can you find the multiplicative inverses for these matrices?

(76) $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

(77) $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ where p is a number, and $p \neq 0$.

(78) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(79) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(80) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

(81) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

(82) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

(83) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

(84) $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

(85) $\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$

(86) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(87) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(88) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

(89) Can you add these two matrices?

$$\begin{pmatrix} 1 & 9 \\ 11 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 5 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

(90) Can you find an additive inverse for the number $^{+}7$?

(91) Can you find an additive inverse for the number $^{-}3$?

(92) Can you find an additive inverse for the number 0?

(93) Can you find an additive inverse for this matrix?

$$\begin{pmatrix} ^{+}2 & ^{+}3 \\ 0 & ^{+}1 \end{pmatrix}$$

(94) Can you find an additive inverse for this matrix?

$$\begin{pmatrix} ^{-}5 & ^{+}2 \\ ^{-}1 & ^{+}7 \end{pmatrix}$$

(95) Can you find an additive inverse for every matrix?

(96) Can you find an additive inverse for this matrix?

$$\begin{pmatrix} ^{+}7 & ^{+}3 \\ ^{-}1 & 0 \end{pmatrix}$$

(97) Can you find an additive inverse for this matrix?

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

CHAPTER 38

Ricky's Special Matrix

(1) Can you multiply these two matrices?

$$\begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix} \times \begin{pmatrix} 4 & 10 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

(2) Can you multiply these two matrices?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} R & S \\ T & U \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

(3) Ricky says that he has found a special matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Can you tell what is "special" about Ricky's matrix?

(4) Can you find any other "special" matrices? What is "special" about them?

(5) Mary says that she has found a special matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Can you tell what is "special" about Mary's matrix?

(6) Can you multiply these two matrices?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

(7) Can you multiply these two matrices?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

(8) Jeff says that

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is a "special" matrix. What does it do?

(9) Can you multiply these two matrices?

$$\begin{pmatrix} 7 & 11 \\ 13 & 19 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

(10) What "special" thing did the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

do to the matrix

$$\begin{pmatrix} 7 & 11 \\ 13 & 19 \end{pmatrix} ?$$

Would it do the same thing to **every** 2-by-2 matrix? Can you prove it?

(11) Can you multiply these two matrices?

$$\begin{pmatrix} 7 & 11 \\ 13 & 19 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

(12) What "special" thing did

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ do to } \begin{pmatrix} 7 & 11 \\ 13 & 19 \end{pmatrix} ?$$

Would it do the same thing to **every** 2-by-2 matrix? Can you prove it?

(13) Nora says that

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

is a "special" matrix. She says she found out by multiplying

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

What "special" thing does

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

do to

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} ?$$

Would it do the same thing to **every** 2-by-2 matrix?

(14) Can you find any other "special" matrices?

(15) Can you find a matrix that would just **double** each element of any 2-by-2 matrix?

(16) Dexter says that "double each element" means: You start with any 2-by-2 matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and multiply by your "special" matrix to get

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \underset{\uparrow}{\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}} = \begin{pmatrix} 2A & 2B \\ 2C & 2D \end{pmatrix}$$

Dexter's special matrix

What do you think?

(17) Can you multiply these two matrices?

$$\begin{pmatrix} 3 & 5 \\ 7 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

(18) Can you multiply these two matrices?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

(19) Can you multiply these two matrices?

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

(20) Can you multiply these two matrices?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

(21) What does the matrix

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

do to the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} ?$$

(22) Can you find a "special" matrix that will turn

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

into

$$\begin{pmatrix} C & D \\ A & B \end{pmatrix} ?$$

What is the special trick in doing this?

CHAPTER 39

Matrices: A New Mathematical System

We have now built up a new mathematical system—namely, the system of 2-by-2 matrices. Although, like much modern mathematics, this system was made up "just for fun," **it turns out to be a very valuable system.** If you continue the study of mathematics, you will find yourself using this system again and again.

But this is not our concern right now.

We have created a new mathematical system. Let us now explore it! See if you can think of any interesting questions to ask.

Here are some that other people have asked.

(1) Jean wants to know: Do 2-by-2 matrices satisfy the commutative law for addition?

(2) Hal wants to know: Do 2-by-2 matrices satisfy the commutative law for multiplication?

(3) Jerry wants to know: Do 2-by-2 matrices satisfy the addition law for zero?

(4) Ellen wants to know: Do 2-by-2 matrices satisfy the multiplication law for zero?

(5) Andy says that every integer or rational number has an additive inverse. Does every 2-by-2 matrix have an additive inverse?

(6) Does every number have a multiplicative inverse?

(7) Does every 2-by-2 matrix have a multiplicative inverse?

(8) Do you know what mathematicians mean by **algebraic closure**?

(9) Is the set of positive integers closed under addition?

(10) Is the set of positive integers closed under subtraction?

(11) Is the set of positive integers closed under multiplication?

(12) Is the set of positive integers closed under division?

(13) Is the set of 2-by-2 matrices closed under addition?

(14) Is the set of 2-by-2 matrices closed under multiplication?

(15) Is the set of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

closed under addition?

(16) Jerrold says that you can match up numbers and matrices like this:

$$0 \longleftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$1 \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$2 \longleftrightarrow \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$3 \longleftrightarrow \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

What matrix would Jerrold match up with the number 7? with the number $^-1$?

(17) Debbie says there is something special about Jerrold's matching:

> If you add two numbers,
>
> $$2 + 3 = 5,$$
>
> you get a result that corresponds to adding the "matched" matrices:
>
> $$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

Does Jerrold's "matching" also work out like this for **multiplication**?

(13) Nancy says that Jerrold's matching is what mathematicians call an **isomorphism.**

Do you agree?

(19) Debbie says you can match numbers like this to get an isomorphism with respect to addition:

$$1 \longleftrightarrow 2$$
$$2 \longleftrightarrow 4$$
$$3 \longleftrightarrow 6$$
$$4 \longleftrightarrow 8$$

What do you think?

(20) Nancy says Debbie's matching is also an isomorphism with respect to multiplication. Do you think Nancy is right?

(21) Charles made this up and claims it is a strange kind of isomorphism:

+	×
101	2
202	4
303	8
404	16

Do you see how it works?

CHAPTER 40

Matrices and Transformations

(1) Ted has a method for using matrices to map a plane into a plane. Suppose he is using the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

He would start with a point—say, (1, 2)—and write it as a **column matrix**

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and multiply like this:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

He would say: The **image** of (1, 2) is (2, 4)

Can you find the image of (3, 1) using

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}?$$

(2) Can you give a **geometric** description of Ted's mapping?

(3) If you use Ted's idea, but use the matrix

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

instead of

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

can you describe the mapping geometrically?

(4) If you use Ted's idea, but use the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

can you describe the mapping geometrically?

(5) If you use Ted's idea, but use the matrix

$$\begin{pmatrix} 0 & {}^{-}1 \\ 1 & 0 \end{pmatrix}$$

can you describe the mapping that you get?

(6) What mapping do you get from this matrix?

$$\begin{pmatrix} {}^{-}1 & 0 \\ 0 & {}^{+}1 \end{pmatrix}$$

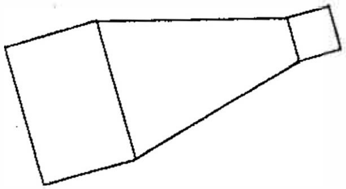
(7) Make up some 2-by-2 matrices yourself, and see if you can find what kinds of geometric mappings your matrices produce?

CHAPTER 41

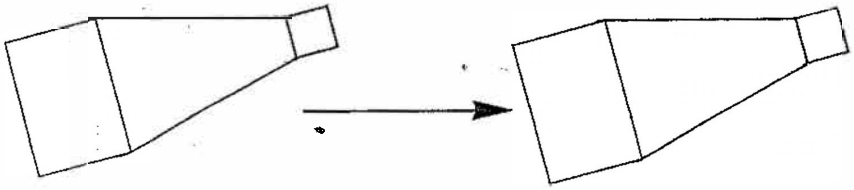
Matrices and Space Capsules

You may find it easier to think about **matrices** and **transformations** if you know something about where they are used.

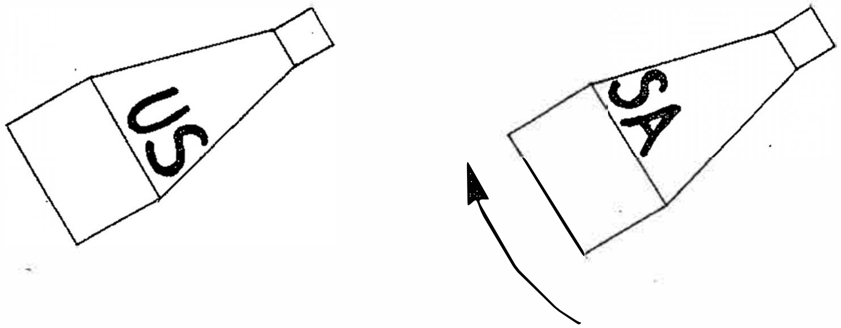
One example, very much in the spirit of the preceding chapter, comes from space science. Suppose we have a rocket or a space capsule (or, for that matter, an airplane)



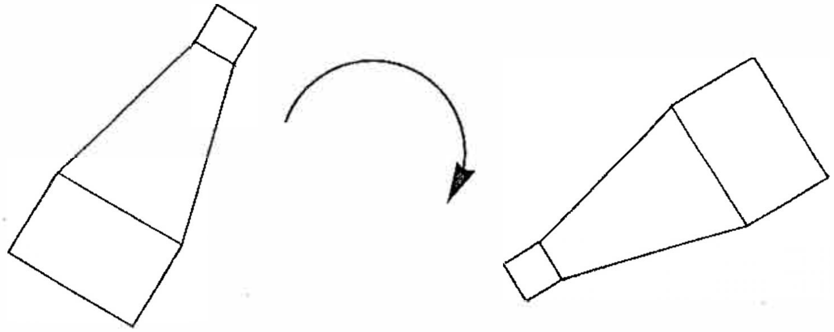
which is moving in space. Its motion can be very complicated. It can "move along a path from one spot to another,"



but—really at the same time as the motion above—it can also change its **orientation** (or, as it is known in space science, its **attitude**). For example, it can rotate like this:



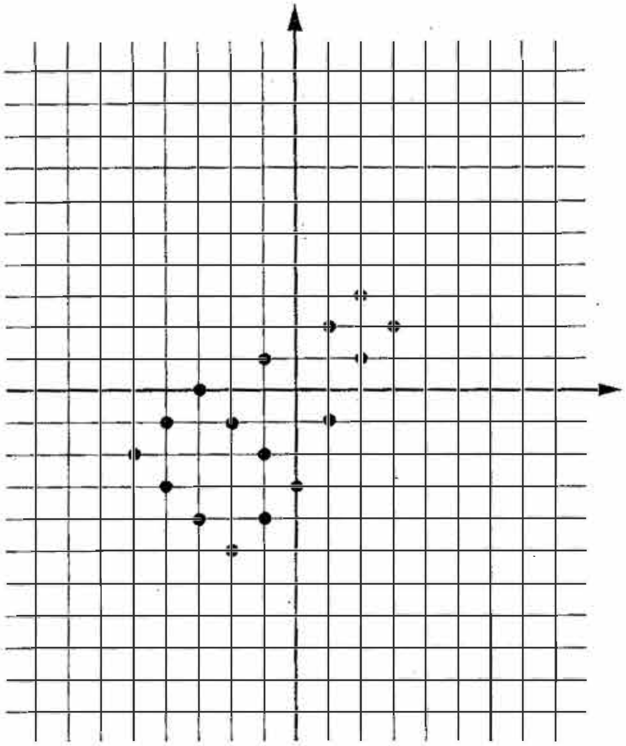
Or it can "flop over" like this:



Now, it is essential to predict, and to observe, the motion of space capsules very precisely, using appropriate mathematics and high-speed digital computers.

The "flopping" kinds of motions are observed using "before" and "after" pictures of the kind we have just been studying. The "flopping" itself is regarded as a **transformation**, and is studied by means of its corresponding matrix.

(1) Suppose a space capsule is represented by this set of points:



"Before"

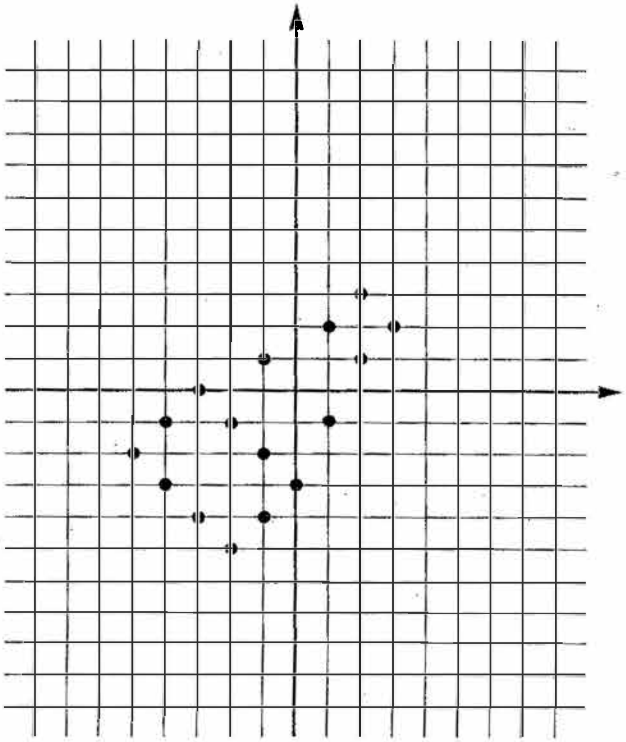
The set of points plotted on the graph above is the set $\{(2,1), (3,2), (2,3), (1,2), (^-1,1), (^-3,0), (^-4,^-1), (^-5,^-2), (^-4,^-3), (^-3,^-4), (^-2,^-5), (^-1,^-4), (0,^-3), (1,^-1), (^-2,^-1), (^-1,^-2)\}$.

Now, at this instant, a computer down on earth sends up a signal which causes the capsule to fire some small "flipping" rockets and "flop over." The computer on earth made a transformation using the matrix

$$\begin{pmatrix} 0 & 1 \\ ^-1 & 0 \end{pmatrix}.$$

Assuming that the rockets all worked correctly, and the capsule did what the computer ordered, what is the new "position" or "attitude" of the space capsule?

(2) Suppose that the capsule started in this position

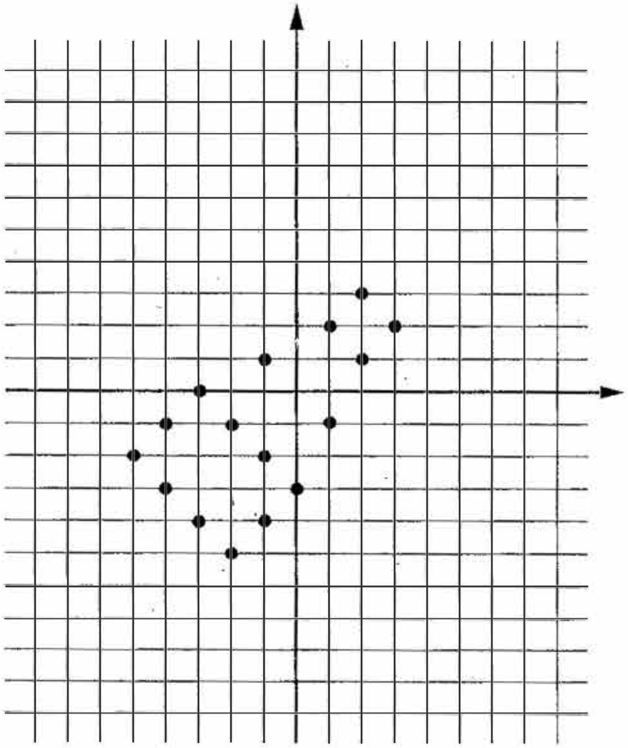


and the computer used the matrix

$$\begin{pmatrix} 0 & ^-1 \\ 1 & 0 \end{pmatrix}.$$

What would the capsule's position be after the maneuver was completed?

(3) Suppose the capsule started in this position

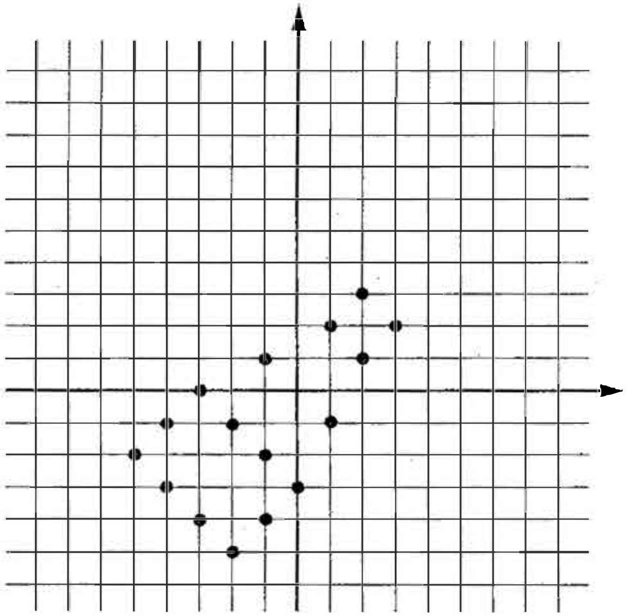


and the computer called for a "flipping" movement according to the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

How would the capsule look after this maneuver was completed?

(4) Suppose the capsule started in this position



and the computer called for a shift in attitude based on the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

But . . . **as soon as this maneuver was completed**, someone discovered that the computer had made a mistake! Instead of calling for the matrix

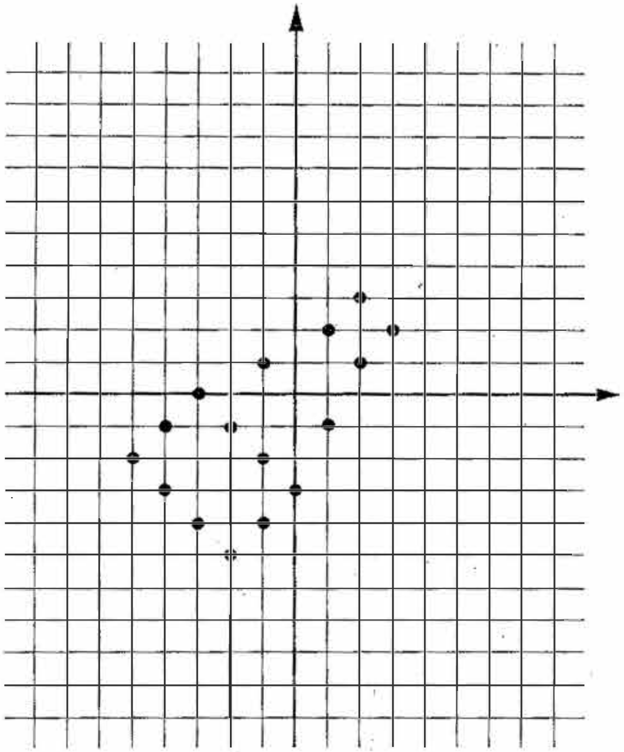
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the computer ought to have called for a shift based on the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

What matrix will get us back to where we **ought** to be?

(5) Suppose the space capsule started like this



and the computer called for a shift based on the matrix

$$\begin{pmatrix} 0 & 1 \\ {}^{-}1 & 0 \end{pmatrix}.$$

After this maneuver was completed, the computer called for a shift based on the matrix

$$\begin{pmatrix} {}^{-}1 & 0 \\ 0 & {}^{-}1 \end{pmatrix}.$$

After this second maneuver was completed, the computer called for a third, using the matrix

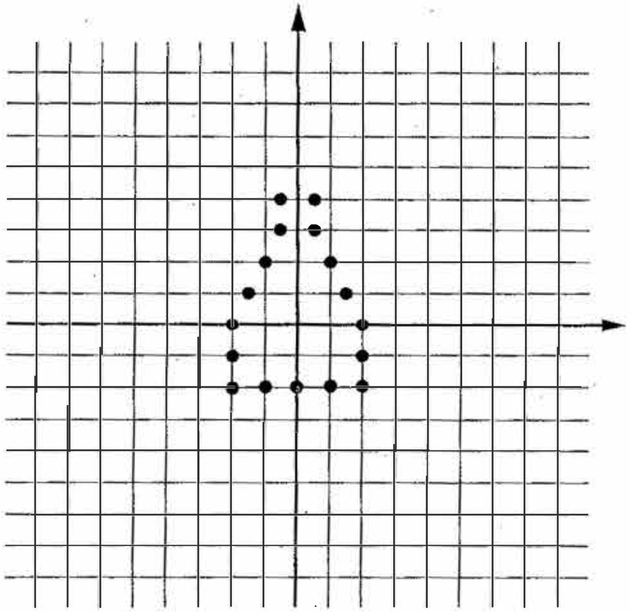
$$\begin{pmatrix} 0 & 1 \\ {}^{-}1 & 0 \end{pmatrix}.$$

After the third maneuver was completed, the computer called for a fourth, using the matrix

$$\begin{pmatrix} 0 & {}^{-}1 \\ 1 & 0 \end{pmatrix}.$$

What was the space capsule's position after the fourth maneuver had been completed?

(6) A space capsule went on a long flight, lasting seven months. At the beginning of the flight, the capsule's attitude was like this:



During the flight, the computer called for 260 shifts, according to this list:

31 shifts, each based on the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

26 shifts, based on the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

203 shifts, based on the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

What was the attitude of the space capsule after **all** of these maneuvers had been completed?

A TABLE OF MATRIX INVERSES

You may find it convenient to have this table of matrix inverses available in case you ever need to use it.

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \times \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \times \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix} \times \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 12 \\ 2 & 2 \end{pmatrix} \times \begin{pmatrix} -\frac{1}{2} & 3 \\ \frac{1}{2} & -\frac{5}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 3 \\ \frac{1}{2} & -\frac{5}{2} \end{pmatrix} \times \begin{pmatrix} 10 & 12 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 10 \\ 5 & 8 \end{pmatrix} \times \begin{pmatrix} -4 & 5 \\ \frac{5}{2} & -3 \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ \frac{5}{2} & -3 \end{pmatrix} \times \begin{pmatrix} 6 & 10 \\ 5 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix}7 & 10\\5 & 7\end{pmatrix}\times\begin{pmatrix}{}^{-}7 & 10\\5 & {}^{-}7\end{pmatrix}=\begin{pmatrix}{}^{-}7 & 10\\5 & {}^{-}7\end{pmatrix}\times\begin{pmatrix}7 & 10\\5 & 7\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\begin{pmatrix}3 & 7\\5 & 12\end{pmatrix}\times\begin{pmatrix}12 & {}^{-}7\\{}^{-}5 & 3\end{pmatrix}=\begin{pmatrix}12 & {}^{-}7\\{}^{-}5 & 3\end{pmatrix}\times\begin{pmatrix}3 & 7\\5 & 12\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}\times\begin{pmatrix}1 & {}^{-}1 & 0\\0 & 1 & {}^{-}1\\0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & {}^{-}1 & 0\\0 & 1 & {}^{-}1\\0 & 0 & 1\end{pmatrix}\times\begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}2 & 3\\5 & 5\end{pmatrix}\times\begin{pmatrix}{}^{-}1 & \frac{3}{5}\\1 & {}^{-}\frac{2}{5}\end{pmatrix}=\begin{pmatrix}{}^{-}1 & \frac{3}{5}\\1 & {}^{-}\frac{2}{5}\end{pmatrix}\times\begin{pmatrix}2 & 3\\5 & 5\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 1 & 1\\0 & 1 & 1 & 1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 1\end{pmatrix}\times\begin{pmatrix}1 & {}^{-}1 & 0 & 0\\0 & 1 & {}^{-}1 & 0\\0 & 0 & 1 & {}^{-}1\\0 & 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & {}^{-}1 & 0 & 0\\0 & 1 & {}^{-}1 & 0\\0 & 0 & 1 & {}^{-}1\\0 & 0 & 0 & 1\end{pmatrix}$$

$$\times\begin{pmatrix}1 & 1 & 1 & 1\\0 & 1 & 1 & 1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 4\\4 & 5 & 6\\1 & 1 & 1\end{pmatrix}\times\begin{pmatrix}1 & {}^{-}2 & 8\\{}^{-}2 & 3 & {}^{-}10\\1 & {}^{-}1 & 3\end{pmatrix}=\begin{pmatrix}1 & {}^{-}2 & 8\\{}^{-}2 & 3 & {}^{-}10\\1 & {}^{-}1 & 3\end{pmatrix}\times\begin{pmatrix}1 & 2 & 4\\4 & 5 & 6\\1 & 1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

CHAPTER 42

Simultaneous Equations

(1) These are known as "simultaneous equations":

$$\left\{\begin{aligned} \square + \triangle &= 10 \\ \square - \triangle &= 8 \end{aligned}\right.$$

The same number must go in both □'s, and the same number in both △'s. Can you find the □ number and the △ number to make **both** statements true?

(2) Can you find the □ number and the △ number to make **both** statements true?

$$\left\{\begin{aligned} \square + \triangle &= 25 \\ \square - \triangle &= 23 \end{aligned}\right.$$

(3) Can you make a numerical replacement for the variable A and a numerical replacement for the variable B, so that **both** statements will be true?

$$\left\{\begin{aligned} A + B &= 16 \\ A - B &= 12 \end{aligned}\right.$$

(4) Can you make a numerical replacement for the variable x and a numerical replacement for the variable y, so that **both** statements will be true?

$$\left\{\begin{aligned} x + y &= 101 \\ x - y &= 99 \end{aligned}\right.$$

(5) Can you find the truth set for this pair of simultaneous equations?

$$\left\{\begin{aligned} A + (2 \times B) &= 104 \\ A - (2 \times B) &= 96 \end{aligned}\right.$$

(6) Can you find the truth set?

$$\left\{\begin{aligned} (2 \times A) + (3 \times B) &= 103 \\ (5 \times A) + (5 \times B) &= 255 \end{aligned}\right.$$

(7) Debbie has a secret method for solving simultaneous equations. She used her secret method on the pair of simultaneous equations

$$\left\{\begin{aligned} (2 \times A) + (3 \times B) &= 14 \\ (4 \times A) + (5 \times B) &= 26 \end{aligned}\right.$$

and she says the correct replacements are

$$\begin{aligned} 4 &\longrightarrow A \\ 2 &\longrightarrow B. \end{aligned}$$

Is Debbie right?

(8) Debbie explained her secret method like this: First, she took the equations

$$\begin{cases} (2 \times A) + (3 \times B) = 14 \\ (4 \times A) + (5 \times B) = 26 \end{cases}$$

and rewrote them as a problem in matrix multiplication:

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 14 \\ 26 \end{pmatrix}.$$

Then, she looked in a table to find the **multiplicative inverse** of the matrix

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}.$$

What she found was

$$\begin{pmatrix} {}^{-}\frac{5}{2} & \frac{3}{2} \\ 2 & {}^{-}1 \end{pmatrix}.$$

Then, she took this inverse, and wrote

$$\begin{pmatrix} {}^{-}\frac{5}{2} & \frac{3}{2} \\ 2 & {}^{-}1 \end{pmatrix} + \left[\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} \right] = \begin{pmatrix} {}^{-}\frac{5}{2} & \frac{3}{2} \\ 2 & {}^{-}1 \end{pmatrix} \times \begin{pmatrix} 14 \\ 26 \end{pmatrix}.$$

Then she used the **associative law for multiplication** (ALM), and got

$$\left[\begin{pmatrix} {}^{-}\frac{5}{2} & \frac{3}{2} \\ 2 & {}^{-}1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \right] \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} {}^{-}\frac{5}{2} & \frac{3}{2} \\ 2 & {}^{-}1 \end{pmatrix} \times \begin{pmatrix} 14 \\ 26 \end{pmatrix}.$$

Then, she said

$$\begin{pmatrix} {}^{-}\frac{5}{2} & \frac{3}{2} \\ 2 & {}^{-}1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and so she wrote

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} {}^{-}\frac{5}{2} & \frac{3}{2} \\ 2 & {}^{-}1 \end{pmatrix} \times \begin{pmatrix} 14 \\ 26 \end{pmatrix}.$$

Finally, she carried out both of these matrix multiplications and got

$$\begin{pmatrix} A \\ \\ B \end{pmatrix} = \begin{pmatrix} ({}^{-}\frac{5}{2} \times 14) + (\frac{3}{2} \times 26) \\ \\ (2 \times 14) + ({}^{-}1 \times 26) \end{pmatrix},$$

which is the same as

$$A = ({}^{-}\tfrac{5}{2} \times 14) + (\tfrac{3}{2} \times 26) = {}^{-}35 + 39 = 4$$

$$B = (2 \times 14) + ({}^{-}1 \times 26) = 28 - 26 = 2.$$

Can you understand Debbie's "secret method"? Do you think you can use it to solve simultaneous equations?

(9) Try Debbie's "secret method" on this pair of simultaneous equations:

$$\begin{cases} (2 \times A) + (3 \times B) = 32 \\ (4 \times A) + (5 \times B) = 60 \end{cases}$$

Were you able to make it work?

Can you solve these pairs of simultaneous equations?

(10) $$\begin{cases}(5 \times A) + (6 \times B) = 183 \\ (3 \times A) + (4 \times B) = 115\end{cases}$$

(11) $$\begin{cases}(7 \times A) + (10 \times B) = 440 \\ (5 \times A) + (7 \times B) = 310\end{cases}$$

(12) $$\begin{cases}(3 \times A) + (7 \times B) = 12 \\ (5 \times A) + (12 \times B) = 20\frac{1}{2}\end{cases}$$

Can you solve these systems of simultaneous equations?

(13) $$\begin{cases}(1 \times A) + (2 \times B) + (4 \times C) = 33 \\ (4 \times A) + (5 \times B) + (6 \times C) = 76 \\ (1 \times A) + (1 \times B) + (1 \times C) = 15\end{cases}$$

(14) $$\begin{cases}A + (2 \times B) + (4 \times C) = 35 \\ (4 \times A) + (5 \times B) + (6 \times C) = 46 \\ A + B + C = 7\end{cases}$$

(15) Mary Frances says that Debbie's method for simultaneous equations is really like a method for solving "one equation in one unknown." To convince her friends, Mary Frances wrote this:

	Debbie's "Secret" Method	Mary Frances' "Shorthand" Way of Writing Debbie's Method	An example of "One Equation in One Unknown"
The original problem:	$\begin{pmatrix}7 & 10\\ 5 & 7\end{pmatrix} \times \begin{pmatrix}A\\ B\end{pmatrix} = \begin{pmatrix}27\\ 19\end{pmatrix}$	Let M stand for the matrix $\begin{pmatrix}7 & 10\\ 5 & 7\end{pmatrix}$; let $\vec{u}$ (for "unknown") stand for the column matrix* $\begin{pmatrix}A\\ B\end{pmatrix}$; $\vec{K}$ (for "known") stand for the column matrix $\begin{pmatrix}27\\ 19\end{pmatrix}$; then, the matrix problem can be written as: $M \times \vec{u} = \vec{K}$.	$3 \times N = 12$
Identify the coefficients of the "unknowns":	$\underset{\substack{\uparrow\\ \text{here}}}{\begin{pmatrix}7 & 10\\ 5 & 7\end{pmatrix}} \times \begin{pmatrix}A\\ B\end{pmatrix} = \begin{pmatrix}27\\ 19\end{pmatrix}$	$\underset{\substack{\uparrow\\ \text{here}}}{M} \times \vec{u} = \vec{K}$	$\underset{\substack{\uparrow\\ \text{here}}}{3} \times N = 12$
Find the multiplicative inverse for this coefficient:	$\begin{pmatrix}-7 & 10\\ 5 & -7\end{pmatrix} \times \begin{pmatrix}7 & 10\\ 5 & 7\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	${}^{r}M \times M = I$, where I stands for the matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\frac{1}{3} \times 3 = 1$

---

*We are using arrows over letters (as with $\vec{u}$) to indicate column matrices.

"Left-multiply" the original equation by this inverse:	$\begin{pmatrix} {}^{-}7 & 10 \\ 5 & {}^{-}7 \end{pmatrix} \times \left[\begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} \right] = \begin{pmatrix} {}^{-}7 & 10 \\ 5 & {}^{-}7 \end{pmatrix} \times \begin{pmatrix} 27 \\ 19 \end{pmatrix}$	${}^{r}M \times (M \times \vec{u}) = {}^{r}M \times \vec{K}$	$\frac{1}{3} \times (3 \times N) = \frac{1}{3} \times 12$
Use ALM:	$\left[\begin{pmatrix} {}^{-}7 & 10 \\ 5 & {}^{-}7 \end{pmatrix} \times \begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \right] \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} {}^{-}7 & 10 \\ 5 & {}^{-}7 \end{pmatrix} \times \begin{pmatrix} 27 \\ 19 \end{pmatrix}$	$({}^{r}M \times M) \times \vec{u} = {}^{r}M \times \vec{K}$	$\left(\frac{1}{3} \times 3\right) \times N = \frac{1}{3} \times 12$
Use the "inverse" property:	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} {}^{-}7 & 10 \\ 5 & {}^{-}7 \end{pmatrix} \times \begin{pmatrix} 27 \\ 19 \end{pmatrix}$	$1 \times \vec{u} = {}^{r}M \times \vec{K}$	$1 \times N = \frac{1}{3} \times 12$
Use the Law for 1, in appropriate form:	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$	$1 \times \vec{u} = \vec{u}$	$1 \times N = N$
And get:	$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} {}^{-}7 & 10 \\ 5 & {}^{-}7 \end{pmatrix} \times \begin{pmatrix} 27 \\ 19 \end{pmatrix}$	$\vec{u} = {}^{r}M \times \vec{K}$	$N = \frac{1}{3} \times 12$
Complete any unfinished multiplications:	$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} ({}^{-}7 \times 27) + (10 \times 19) \\ (5 \times 27) + ({}^{-}7 \times 19) \end{pmatrix}$	$\vec{u} = {}^{r}M \times K$ is as far as you can go in the "shorthand" notation.	$N = 4$
	$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} {}^{-}189 + 190 \\ 135 + {}^{-}133 \end{pmatrix}$		4 → N
	$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$		(or, the truth set is {4})
	1 → A 2 → B (or the truth set is {1, 2})		

Do you think Mary Frances has a good idea here, or not? What is wrong with her idea? Is there anything good about it?

CHAPTER 43

Taxis, Widgets, and Alpha-Beta-Gamma Mix

(1) The Hurry-Up Taxi Company owns some sedans and some station wagons. In 1965 they owned 3 station wagons and 7 sedans. In 1966 they owned 5 station wagons and 12 sedans. In 1965 the company earned a profit of $53,560, after all expenses were paid. In 1966 the company earned a profit of $91,000, after all expenses were paid.

For 1967 they can buy a few additional cars. Should they buy sedans or station wagons?

(2) The Acme Widget Company sells widgets. However, they also sell wigglyups. On the day before Christmas they were hurrying to get all their orders packed into cartons, sealed, addressed, and shipped off.

Somebody found two cartons which had been packed and sealed, but not labeled! What was in them?

Well, they were **either** the order for Smith's Department Store or the order for Edward's Emporium. But which were they?

The Smith's order called for 2 cartons, one containing 7 widgets and 10 wigglyups and the other containing 5 widgets and 7 wigglyups.

The Edward's order called for 2 cartons, one containing 6 widgets and 10 wigglyups, the other containing 5 widgets and 8 wigglyups.

One man suggested weighing the cartons. They did and found that the first carton weighed 59.4 pounds. The second carton weighed 41.8 pounds.

Larry, one of the men in the shipping room, said, "Now I know; this must be the order for Smith's!"

Bill, another shipping room man, said, "I'm sorry, Larry, old fellow, but you're wrong! That must be the order for Edward's!"

Was either man right? What do you think?

(3) The Cochran Chemical Company had a large supply of three chemicals, which we'll call alphathane, betathane, and gammathane.

They also had a District Manager who wrote all of his records on old envelopes and then usually lost the envelopes.

The First Vice-President telephoned the District Manager and asked how much alphathane, betathane, and gammathane he had in his storage spaces.

But the District Manager had written the amounts on an envelope, and he couldn't find the envelope. Then the District Manager spotted an old envelope where he had worked out some calculations, and he read this to the First Vice-President:

"If I wanted to make 370 tons of **soft** alpha-beta-gamma mix, I would need all of the alphathane I have, plus twice as much betathane as I have, plus 4 times as much gammathane as I have. If I wanted to make 880 tons of **ordinary** alpha-beta-gamma mix, I would need 4 times as much alphathane as I have, 5 times as much betathane as I have, and 6 times as much gammathane as I have. But, on the other hand, I could mix together all the alphathane, betathane, and gammathane that I have, and I would get 180 tons of **hard** alpha-beta-gamma mix. Does that answer your question?"

Does it?

CHAPTER 44

New Ways of Writing Old Numbers

(1) Dan made up a number system with numbers like $\alpha, \beta, \gamma, \delta, \epsilon, \lambda, \upsilon, \mu, \cdots$. Dan's system worked like this:

$$\square \times \triangle = \triangle \times \square$$

$$\square + \triangle = \triangle + \square$$

$$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$$

$$\square \times (\triangle \times \bigtriangledown) = (\square \times \triangle) \times \bigtriangledown$$

$$\square + (\triangle + \bigtriangledown) = (\square + \triangle) + \bigtriangledown$$

$$\square \times \alpha = \alpha$$

$$\square + \alpha = \square$$

$$\square \times \beta = \square$$

$$\left.\begin{aligned} \beta + \beta &= \gamma \\ \gamma + \beta &= \delta \\ \delta + \beta &= \epsilon \\ &\vdots \end{aligned}\right\}$$

Is Dan's system really a new system or is it just a new way of writing an old system?

(2) Sarah says Dan's system is really an old system. Dan just writes it a new way. Sarah says she knows what α really is. Do you?

(3) In Dan's system, what is β? How do you know?

(4) In Dan's system, what is γ?

(5) Is Dan's system really a new system, or is it really a new way of writing an old system?

(6) Ellen made up a number system, with numbers like $\phi, \xi, \mu, \zeta, \theta, \kappa, \tau, \psi, \cdots$. Ellen's system worked like this:

$$\square \times \triangle = \triangle \times \square$$

$$\square + \triangle = \triangle + \square$$

$$\square + (\triangle + \bigtriangledown) = (\square + \triangle) + \bigtriangledown$$

$$\square \times (\triangle \times \bigtriangledown) = (\square \times \triangle) \times \bigtriangledown$$

$$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$$

$$\square + \phi = \square$$

$$\square \times \xi = \xi$$

$$\square \times \phi = \square$$

$$\phi \neq \xi$$

$$\left.\begin{array}{lcl} \phi + \xi & = & \xi \\ \xi + \xi & = & \mu \\ \mu + \xi & = & \zeta \\ \zeta + \xi & = & \theta \\ & \vdots & \end{array}\right\}$$

Is Ellen's system really a new system or is it a new way of writing an old system?

(7) Jerry says that Ellen's system is really an old system. Jerry says that ϕ is really 0, ξ is really 1, μ is really 2, and so on. Do you agree?

(8) Martha says that Ellen's system is really a new system, because ϕ is somewhat like 0, but not entirely like 0. What do you think?

(9) Louis made up a method of writing numbers using 2-by-2 matrices. How do you suppose Louis wrote 0?

(10) How do you suppose Louis wrote 1?

(11) How do you suppose Louis wrote 2?

(12) How would $2 \times 3 = 6$ be written, in Louis's system?

(13) How would $2 + 3 = 5$ be written, in Louis's system?

(14) How would Louis write $^-4$?

(15) How would Louis write the integer A?

(16) Bernice says that Louis has set up an **isomorphism** between our usual numbers and a subset of the set of 2-by-2 matrices. What do you think?

(17) What subset of the set of 2-by-2 matrices did Louis use?

(18) Under Louis's isomorphism, what "old-fashioned" **number** corresponds to the following matrix?

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

CHAPTER 45

The Hesitant Search for New Numbers

The origin of the basic ideas of counting surely dates back to quite early prehistoric times. Of course, as we have seen, our present method of writing the numbers that we use in counting (that is to say, 1, 2, 3, . . .) comes from the Hindus. Our method may have been introduced to France and Italy, by way of Spain, by Pope Sylvester II, who, as a young man, had studied in Spanish schools run by the Moslems. The date for this is about 1000 A.D.

However, it appears that the method of writing

$$1,\ 2,\ 3,\ 4,\ \ldots$$

was perhaps the only number idea that Pope Sylvester II brought back from Spain. He appears not to have brought back the important idea of zero, although the Hindus had conceived the idea of zero at least as early as 800 A.D.

There is a strange theme of searching and rejecting that threads through the history of mathematics, from the ancients down until nearly the present day: this is the hesitant search for new kinds of numbers.

If, in fact, you know only the "counting" numbers,

$$1,\ 2,\ 3,\ 4,\ \ldots,$$

then what do you do when you want to cut a pie, or divide up a candy bar, or give number names to all of the points on the number line?

This, and similar problems, led men to invent numbers such as

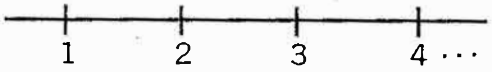
$$\tfrac{1}{2},\ \tfrac{1}{3},\ 2\tfrac{1}{7},\ \tfrac{2}{3},$$

and so on.

In 1484 the French mathematician Chuquet worked out many ideas about **exponents**. He came to recognize the role of negative integers as exponents, which we have seen in such problems as

$$\frac{x^5}{x^7} = x^{-2}.$$

Over a century earlier, Nicole Oresme (born in Normandy about 1323; died in 1382) had worked with the **number line**:



Both of these notions create a natural role for negative numbers, but their introduction and acceptance were gradual. People needed them, but they "didn't really believe in them."

For example, in 1544 the German mathematician Michael Stifel (1486-1567) published a volume entitled *Arithmetica Integra*. In this book, Stifel recorded the lines of "Pascal's triangle" as far as the line for

$$(R + S)^{17} = R^{17} + 17R^{16}S + \cdots + S^{17}.$$

He used letters to represent "unknowns," and used the modern symbols + (for addition), – (for subtraction), and $\sqrt{\ }$ (for square root). However, when Stifel encountered negative numbers as elements of the truth set for an open sentence, he rejected them, apparently not considering them "really appropriate," or something of the sort.*

In 1572 Rafael Bombelli worked effectively with negative numbers, and appeared to have a considerable understanding of them.

But, even after you have the counting numbers†

$$1,\ 2,\ 3,\ 4,\ \ldots$$

and also zero

$$0,\ 1,\ 2,\ 3,\ 4,\ \ldots$$

and also fractions and "mixed numbers"

$$0,\ 1,\ 2,\ \frac{1}{2},\ \frac{1}{3},\ \frac{3}{4},\ 2\frac{1}{7},\ \ldots$$

and also negative numbers,

$$0,\ 1,\ 2,\ \frac{1}{2},\ \frac{1}{3},\ \frac{3}{4},\ 2\frac{1}{7},\ {}^{-}1,\ {}^{-}2,\ \frac{{}^{-}1}{2}$$

you still encounter the need for new kinds of numbers. Why?

We have said earlier that the ancients, and the early Renaissance mathematicians, had worked out the general solution of the general quadratic equation. That was so, in a purely "procedural" sense that paralleled our own derivation of the solution of the general quadratic equation. They knew what to do—provided it worked out satisfactorily!

Now, there is one step in the procedure that will sometimes **fail** to work out satisfactorily: this is the process of taking the square root.

This can be written either as

$$x^2 - Ax + B = W$$

$$\left\{\frac{A}{2} + \sqrt{W - B + \left(\frac{A}{2}\right)^2},\ \frac{A}{2} - \sqrt{W - B + \left(\frac{A}{2}\right)^2}\right\}$$

or else as

$$ax^2 + bx + c = 0$$

$$x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

If

$$W - B + \left(\frac{A}{2}\right)^2$$

is a perfect square, such as 16 or 49 or 121, there is no difficulty. But there are two cases in which there

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*Compare Eves, *op. cit.*, p. 219.

†Nowadays, in the twentieth century, you can find different books by different authors that use the words "counting numbers" to have different meanings. This, of course, is inevitable. Not all authors agree, no matter what the topic under discussion may be. In particular, one will nowadays sometimes see the words "counting numbers" used to refer to the elements of the set {1, 2, 3, 4, . . .} and in other books these words will refer to the elements of the set {0, 1, 2, 3, 4, . . .}. If we construe that "counting" refers to the process most of us use when we "count on our fingers," then it seems reasonable to say that the counting numbers are 1, 2, 3, 4, . . .; on the other hand, as is sometimes convenient, if we choose to regard the counting numbers as the ordinary answers to questions of the form "how many?" then it is reasonable to assume that we are talking about the set {0, 1, 2, 3, 4, . . .}, since it may well happen that when someone says "How many brothers do you have?" the answer will turn out to be 0. Some people wish that all books were in complete agreement, but there is reason to feel that as long as life goes on this will not occur, and perhaps is is a good thing that it won't.

are difficulties—difficulties which early Renaissance mathematicians found quite serious. One case occurs if we encounter square roots such as

$$\sqrt{2} \quad \text{or} \quad \sqrt{45} \quad \text{or} \quad \sqrt{11},$$

and so on.

The other case occurs if we encounter square roots such as

$$\sqrt{^-1} \quad \text{or} \quad \sqrt{^-4} \quad \text{or} \quad \sqrt{^-49},$$

and so on.

(1) What can you say about these two kinds of square roots (for example, $\sqrt{2}$ versus $\sqrt{^-4}$)?

(2) What can you say about the truth set for the open sentence

$$x^2 = {}^-4?$$

(3) Here is a research problem. Using the isomorphism

$$A \longleftrightarrow \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

to guide you, develop some new numbers, so that you will be able to solve the equation

$$x^2 = {}^-4.$$

If you check for algebraic closure under addition and under multiplication, you can work out an entirely new mathematical system. (Incidentally, the system you make up here is one which Descartes encountered, but he rejected it as not making sense. Today it is one of the most important mathematical systems that we know about.)

CHAPTER 46

Determinants

Don made up a mathematical system, like a game, by making up some rules.

(a) Don would begin by writing 4 numbers, like these:

$$\begin{matrix} 7 & 8 \\ 1 & 2 \end{matrix}$$

(b) Don **did not** want these to be the same thing as a matrix, because he already knew about matrices, and he wanted this to be a **new** system. So Don **did not** write:

$$\begin{pmatrix} 7 & 8 \\ 1 & 2 \end{pmatrix}$$

Instead he used **straight** lines and wrote:

$$\begin{vmatrix} 7 & 8 \\ 1 & 2 \end{vmatrix}$$

(c) Don said, "Whenever I write

$$\begin{vmatrix} 7 & 8 \\ 1 & 2 \end{vmatrix}$$

what I really mean is

$$(7 \times 2) - (1 \times 8)."$$

(1) What number does Don mean when he writes

$$\begin{vmatrix} 7 & 8 \\ 1 & 2 \end{vmatrix} \quad ?$$

(2) What number does Don mean when he writes

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \quad ?$$

(3) What number does Don mean when he writes

$$\begin{vmatrix} 3 & 2 \\ 15 & 10 \end{vmatrix} \quad ?$$

(4) What number does Don mean when he writes

$$\begin{vmatrix} 4 & 2 \\ 3 & 8 \end{vmatrix} \quad ?$$

(5) Jane doesn't know Don's system, but she does know all about how **variables** work. Can you use variables to show Jane exactly how Don's system works?

(6) What number does Don mean when he writes

$$\begin{vmatrix} 1 & 1 \\ 3 & 7 \end{vmatrix} \quad ?$$

(7) What number does Don mean when he writes

$$\begin{vmatrix} 1 & 1 \\ 7 & 3 \end{vmatrix} \quad ?$$

(8) What number does Don mean when he writes

$$\begin{vmatrix} 3 & 7 \\ 1 & 1 \end{vmatrix} \quad ?$$

(9) What number does Don mean when he writes

$$\begin{vmatrix} 1796 & 301 \\ 255 & 186 \end{vmatrix} ?$$

(10) What number does Don mean when he writes

$$\begin{vmatrix} 301 & 1796 \\ 186 & 255 \end{vmatrix} ?$$

(11) Sandy's father says that somebody else, who called these things **determinants**, already invented Don's system before Don did.

(12) Alice says that Don's system doesn't look like it will ever be good for anything. What do you think?

CHAPTER 47

Matrix Inverses: A Research Problem

As we saw in Chapter 33, Professor George Polya of Stanford University has tried to describe some of the methods that scientists and mathematicians use in research.

We can try to practice some of these methods ourselves. First, we need a problem to work on. Here is one:

Problem: If you are given any 2-by-2 matrix, say

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

can you find a matrix

$$\begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}?$$

That is to say, if you are given a 2-by-2 matrix, can you find its multiplicative inverse?

Suggestion: Can you read about this problem somewhere?

Answer: Actually, you could read quite a bit about this problem. However, that may not be necessary just yet. Here is one problem that somebody else has already figured out:

If you are given the matrix

$$\begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix},$$

its inverse is

$$\begin{pmatrix} 12 & {}^-7 \\ {}^-5 & 3 \end{pmatrix}.$$

That is to say,

$$\begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} \times \begin{pmatrix} 12 & {}^-7 \\ {}^-5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Can you see a pattern?

Let's see if we can use this idea to help us to find the inverse of each of the following matrices.

(1) $\begin{pmatrix} 12 & {}^-7 \\ {}^-5 & 3 \end{pmatrix}$ (5) $\begin{pmatrix} 9 & 10 \\ 8 & 9 \end{pmatrix}$

(2) $\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$ (6) $\begin{pmatrix} 0 & 1 \\ {}^-1 & 0 \end{pmatrix}$

(3) $\begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}$ (7) $\begin{pmatrix} 2 & 2 \\ 5 & 6 \end{pmatrix}$

(4) $\begin{pmatrix} 13 & 2 \\ 6 & 1 \end{pmatrix}$ (8) $\begin{pmatrix} 4 & 6 \\ 1 & 2 \end{pmatrix}$

(9) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

(10) $\begin{pmatrix} 5 & 9 \\ 2 & 4 \end{pmatrix}$

(11) $\begin{pmatrix} 5 & 6 \\ 8 & 10 \end{pmatrix}$

(12) $\begin{pmatrix} 25 & 23 \\ 1 & 1 \end{pmatrix}$

(13) $\begin{pmatrix} 8 & 5 \\ 1 & 1 \end{pmatrix}$

(14) $\begin{pmatrix} 9 & 5 \\ 1 & 1 \end{pmatrix}$

(15) $\begin{pmatrix} 7 & 5 \\ 1 & 1 \end{pmatrix}$

(16) $\begin{pmatrix} 7 & 4 \\ 1 & 1 \end{pmatrix}$

(17) $\begin{pmatrix} A & B \\ 1 & 1 \end{pmatrix}$

Some of these problems seem to be harder than others. If we can see which problems are "easier" and which are "harder," that may give us a clue as to how to proceed.

If any of your answers thusfar have been **wrong**, those problems deserve special attention! Professor Jerrold Zacharias, a physicist at Massachusetts Institute of Technology, has suggested that the "exploitation of error" is a powerful tool in scientific research. What can we learn from looking carefully at the problems that were wrong? **How** were they different from those we got right? **In what way** were the wrong answers "wrong"? Were they **completely** wrong or **almost** correct?

If you want, make up some matrices yourself and try to find their inverses. If you have trouble, see what you can learn by "exploiting error."

Some people claim that the idea of determinants, from Chapter 46, can be helpful to us. If you wish, see if determinants really can be helpful.

(18) Can you find the inverse for **any** matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} ?$$

(19) **When** can you find the inverse for the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} ?$$