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Teaching project

UNIT XV

ADDITION & LINEAR TRANSLATIONS

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FOR THE

ELEMENTARY SCHOOL

Unit XV

Addition
and
Linear Translations

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MINNESOTA MATHEMATICS AND SCIENCE TEACHING PROJECT

JAMES H. WERNTZ, JR.
Associate Professor of Physics
University of Minnesota
Project Director

PAUL C. ROSENBLOOM
Professor of Mathematics
University of Minnesota
Mathematics Director

ADDITION AND LINEAR TRANSLATIONS

JULIA CLARK — REVISION EDITOR
Fourth grade teacher
St. Paul, Minnesota

DONALD E. MYERS — CONTENT EDITOR
Associate Professor of Mathematics
University of Arizona

HELEN LONGANECKER
Fourth grade teacher
University of Southwestern Louisiana

MARJORY LUCE
Upper elementary teacher
Edina, Minnesota

ROY MUCKEY
Minnemast Center
University of Minnesota

PETER TOMSICH
Minnemast Center
University of Minnesota

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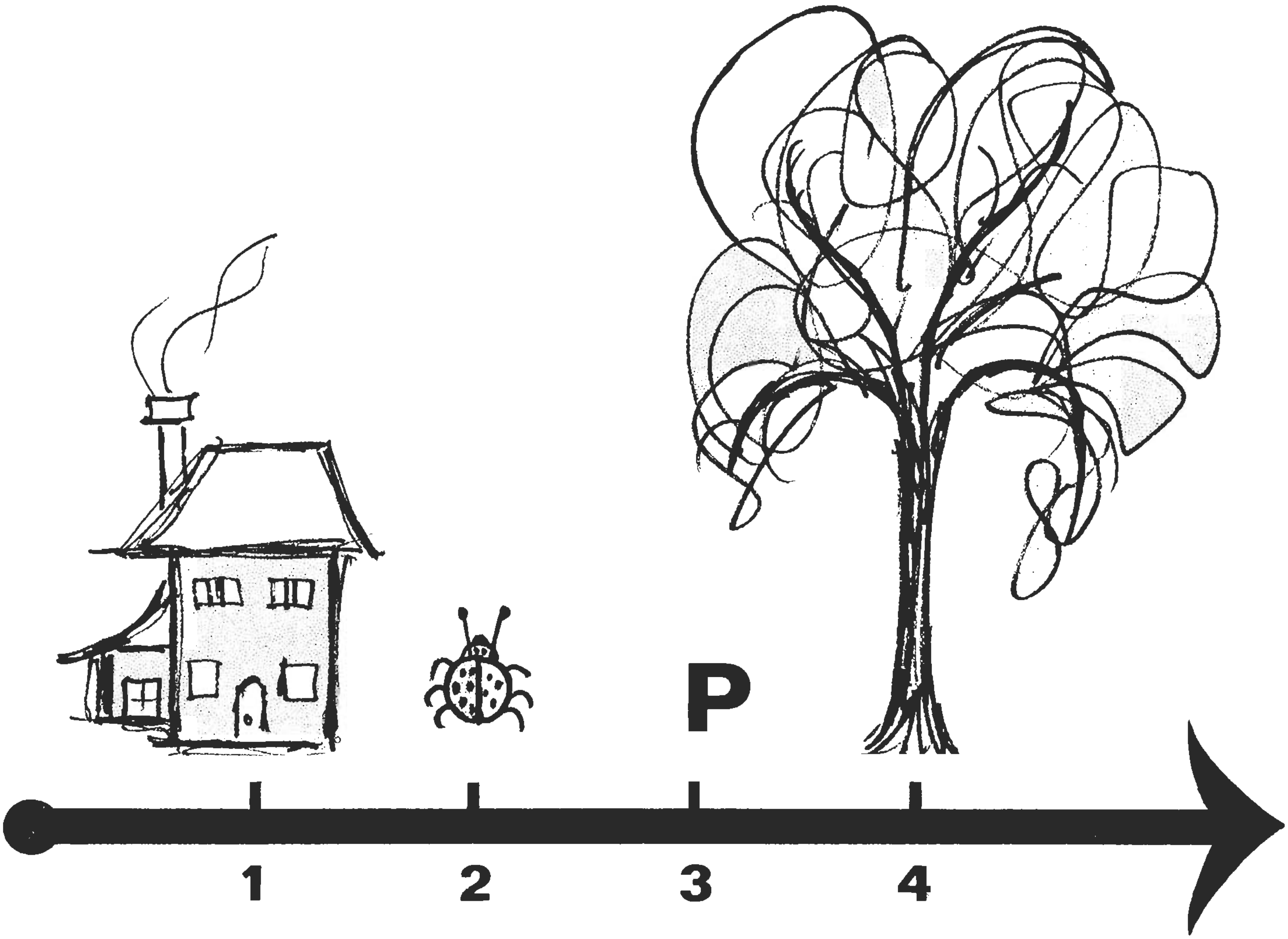
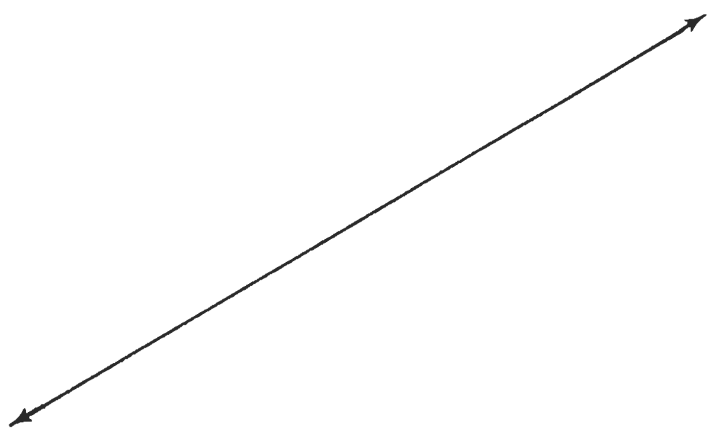
Naming Points On A Ray



NAMING POINTS ON A RAY

Imagine a line placed anywhere. We cannot draw a picture of the whole line because it goes on and on, as far as you like, in both directions. We draw a picture of part of the line and place arrows at each end to show that the line goes on both ways:



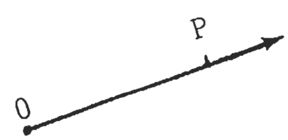
You might think of this line as a long street, or the edge of a ruler.

Suppose you see something on the line, and you want to tell where it is. You need a way to name points on the line.

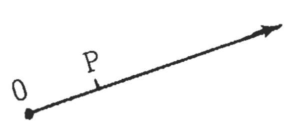
Choose some point on the line; call it "zero", and label it "0". You can tell someone where a point is located on the line by telling them how far it is from zero. You must also tell on which side of zero the point is. The point 0, or zero, together with all points (on the line) on one side of 0 is what we call a ray. We show a ray by drawing a line segment and making a dot at one end point and an arrow at the other:

The arrow shows that the ray goes on and on.

We can tell where the point P is on the ray by telling how far it is from 0. To do this we must choose a unit of length. People in English speaking countries use the inch.



In many other countries the centimeter is used.



NAMING POINTS ON A RAY

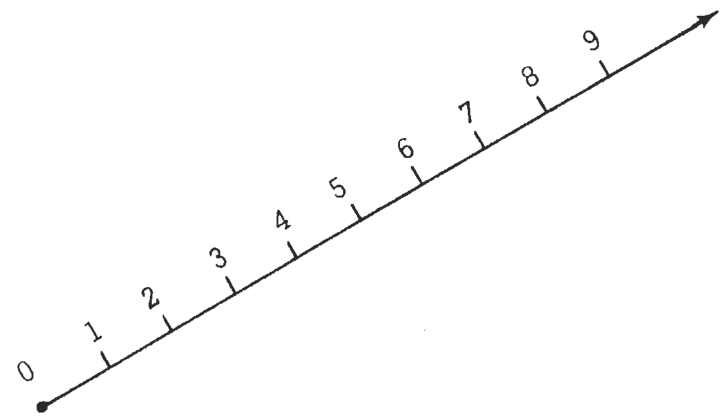
Since scientists often want to study the scientific work of other countries, it is easier for them if all scientists use the same measuring system. So, they usually use centimeters.

We could use any distance we please as a unit of length:

Tommy's unit   Ellen's unit   Ethelbert's unit

Make up names for these units. You may wish to choose a unit of your own. If you use a unit like a "ron," you will have to explain it to other people so that they will be able to understand you.

Choose a unit of length. Mark off this distance from 0 on the ray, and name the point you come to "1." Then mark off this distance from 1 in the same direction and name this new point 2. Keep on in this way as far as you have room on your paper. You will have a scale that looks something like this:



You can think of these numbers as addresses on the line. If the point P, which you were trying to describe, is one of the marked points, then you have a name for P. You may say things like "P is the point 7", or "The red house is at 9", or "The ladybug is at 3". If P is not one of the marked points, you may say "P is between 7 and 8", or "P is a little past 4", or "P is almost at 8".

Later on you will learn names for the other points on the ray. Then you will be able to tell better where P is.

NAMING POINTS ON A RAY

1. On your own paper draw three rays. On one make a scale using a centimeter as the unit of length.
2. On the second make a scale using an inch as the unit of length.
3. On the third make a scale using a unit of length that you invent yourself.
4. Invent a name for your unit.
5. Be ready to explain to the class how long your unit is in terms of centimeters or inches.
6. Johnny called his unit a "dag". He made this table showing how his unit is related to a centimeter:

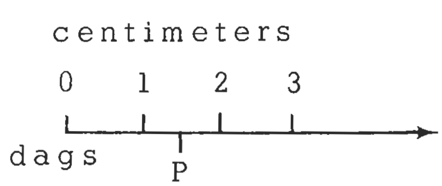
dags	centimeters
0	0
1	between 1 and 2
2	between 2 and 3
3	between 4 and 5
4	between 5 and 6
5	between 7 and 8
6	between 8 and 9

a) Make a table comparing your unit with centimeters and inches.

Your unit of length	centimeters	inches
	between	between
	between	between
	between	between
	between	between
	between	between
	between	between

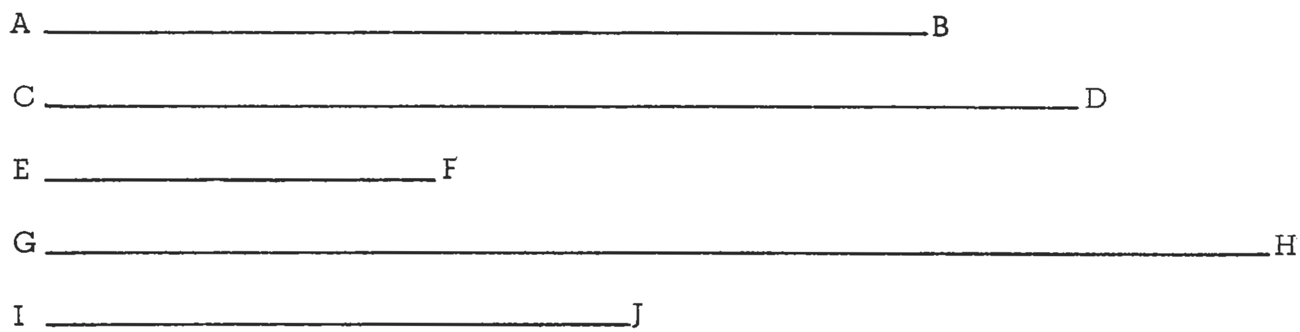
For junior researchers only:

b) About how long is a dag? Guess and check your guess against Johnny's table. Each row of Johnny's table tells you something about the size of a dag. For example, the third row tells you that 2 dags is between 2 and 3 centimeters. Can a dag be longer than the distance from 0 to P on the scale below?



NAMING POINTS ON A RAY

Measure each of the following line segments with all three of your scales. Record the measures of each segment in the chart.

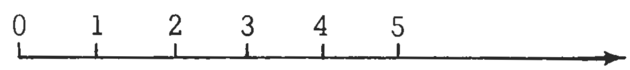


Line segment	Measure in centimeters	Measure in inches	Measure in your own scale
$\overline{AB}$	between 11 and 12 centimeters	between 4 and 5 inches	
$\overline{CD}$	between 13 and 14 cm	between 5 and 6 inches	
$\overline{EF}$	5 cm	2 inches	
$\overline{GH}$	16 cm	between 6 and 7 inches	
$\overline{IJ}$	between 7 and 8 cm	3 inches	

NAMING POINTS ON A RAY

What can you find out about 1 dag from the third row in Johnny's table?

Show on the row below your guess about how long a dag is.



Note: These tables might be posted for several days. The children could then try to discover how long each unit may be.

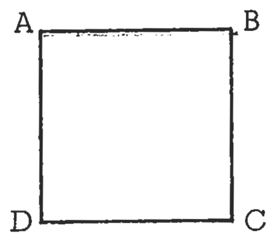
Teacher's Copy

WORKSHEET 1a

Name__________________

Measuring Diagonals

1. The following geometric figure is called a square.



2. Use a red color. Draw a line segment directly from one corner of this square to its opposite corner.

3. The line segment you just drew is called a diagonal of the square.

4. Use a different color and draw in all the other diagonals you can find in this square.

5. How many diagonals are there in square ABCD. 2

6. Find the diagonal that connects points A and C. Try to write the name of this diagonal. $\overline{AC}$ or line segment AC; $\overline{CA}$ or line segment CA

7. Now write the name of the other diagonal. $\overline{BD}$ or line segment BD; $\overline{DB}$ or line segment DB

Teacher's Copy

WORKSHEET 1b   Name________________

8. Look at both diagonals but do not measure them.

Circle your choice of these answers.

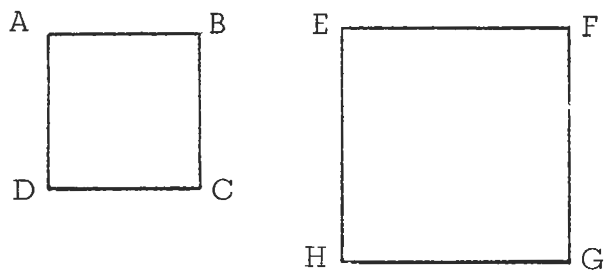
a. AC is longer than BD.

b. AC is shorter than BD.

(c.) AC is the same length as BD.

9. Now measure the two diagonals and check your answer. Were you correct? Yes or no

10. Measure the length of the diagonals in these two squares. Complete the chart. Use your centimeter scale.



Diagonal	Length in centimeters	
$\overline{AC}$		Are $\overline{AC}$ and $\overline{DB}$ the same length? yes no
$\overline{DB}$		
$\overline{EG}$		Are $\overline{EG}$ and $\overline{HF}$ the same length? yes no
$\overline{HF}$		

11. What is always true about the length of both diagonals of a square? They are equal.

Teacher's Copy

WORKSHEET 1c

Name________________

1. Use your centimeter scale to measure the length of one side of each square.

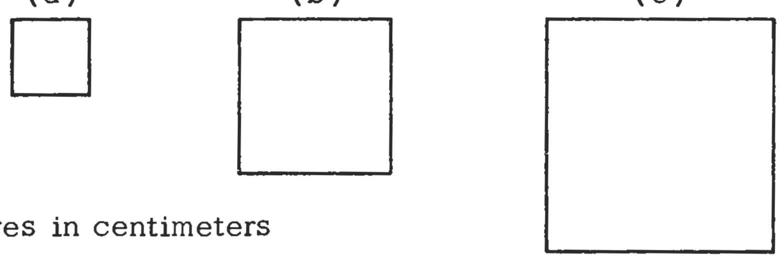
2. Write the length of that side near the square you measure. Also record the length in the chart at the bottom of the page.

3. Now draw one diagonal in each square. Before you measure it, predict how long it is.

4. Write your prediction in the chart.

5. Now measure the length of the diagonal and check your prediction.

(a) (b) (c)



*All measures in centimeters

Square	Length of outside	Predicted length of diagonal	Measured length of diagonal
(a)	1		between 1 and 2
(b)	2		between 2 and 3
(c)	3		between 4 and 5

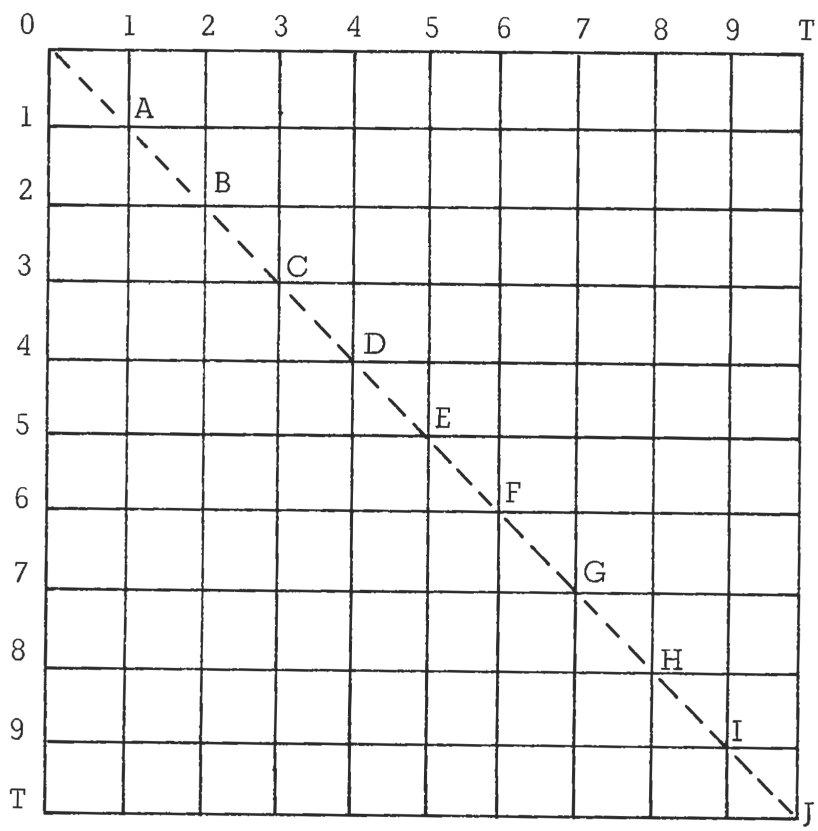
What have you discovered about the length of a diagonal of a square? The length of the diagonal of a square is greater than the length of any one side of the square.

Teacher's Copy

WORKSHEET 1d Name________________

Squares and Diagonals

1. How many squares can you find that have one corner at point 0. There are ten squares.
2. Record your information in the chart on the next page.
3. Try to complete the chart.



Teacher's Copy

WORKSHEET 1e

Name________________

The best way to check this sheet is as a group activity. The children should take turns identifying squares and then providing the necessary information.

All lengths in centimeters.

Name of Square	Length of one side	Sum of length of two sides	Name of Diagonal	Predicted length of Diagonal (between)	Measured length of Diagonal (between)
01A1	1	2	$\overline{0A}$	1 and 2	1 and 2
02B2	2	4	$\overline{0B}$	2 and 3	2 and 3
03C3	3	6	$\overline{0C}$	4 and 5	4 and 5
04D4	4	8	$\overline{0D}$	5 and 6	5 and 6
05E5	5	10	$\overline{0E}$	7 and 8	7 and 8
06F6	6	12	$\overline{0F}$	8 and 9	8 and 9
07G7	7	14	$\overline{0G}$	9 and 10	9 and 10
08H8	8	16	$\overline{0H}$	11 and 12	11 and 12
09I9	9	18	$\overline{0I}$	12 and 13	12 and 13
010J10	10	20	$\overline{0J}$	14 and 15	14 and 15

What have you found out about the diagonal of any square?

Choose any square. The length of the diagonal of this square is greater than the length of one of its sides but less than the combined length of the two sides connected by the diagonal.

Worksheet 2 provides a review of two scales. As in previous units children should be guided to see that a point on any given line may have a different name. It all depends upon the point of origin as in this drawing. Point 6 on scale A is point 4 on scale B.

WORKSHEET 2a

Teacher's Copy

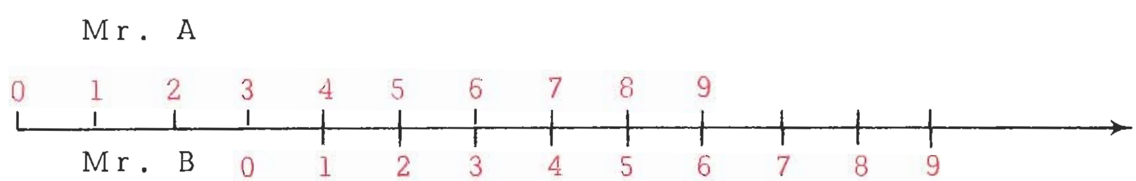
Name____________

The two observers

A little man, Mr. A, stands at the end of a long street. He calls the point where he stands "0".

He describes points on this street by telling their distance from him. Mr. A says, "I am the most important person in the world. I describe things from my point of view".

Mr. A uses a block as his unit of length. Represent a block by a centimeter and mark off Mr. A's scale. Write the numerals above the ray.



Mr. B is another observer. He stands three blocks from Mr. A on that long street. He also uses a block as his unit of length. Mr. B says, "I am the most important person in the world. I describe things from my point of view". So he calls the point where he

WORKSHEET 2b Name Teacher's Copy

stands "0". Write the numerals for Mr. B's scale below the ray.

Use the number ray (Mr. A's and Mr. B's street) that you have just completed to help you finish the following chart and answer the questions.

The address on Mr. A's scale	The address on Mr. B's scale
3	0
4	1
5	2
6	3
7	4
8	5
9	6

When Mr. B's address is 6, Mr. A's address is 9.

When Mr. B's address is 2, Mr. A's address is 5.

When Mr. B's address is 5, Mr. A's address is 8.

At point 6 who has the greater address, Mr. A or Mr. B? Mr. A How much greater? 3

Teacher's Copy

WORKSHEET 2c

Name____________

If you know the address of a point on Mr. B's scale, how do you get the address of that point on Mr. A's scale? You could say, "Address on Mr. A's scale = address on Mr. B's scale + ___3___".

If you know the address of a point on Mr. A's scale what must you do to get the address of that point on Mr. B's scale?

You could say, "Address on Mr. B's scale = address on Mr. A's scale - ___3___".

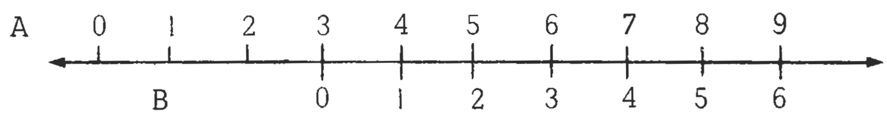
Teacher's Copy

WORKSHEET 3a

Name________________

Mr. A's and Mr. B's street

This is another picture of Mr. A's and Mr. B's street.



Let "a" stand for any point on Mr. A's scale.

Let "b" stand for the same point on Mr. B's scale.

If b = 5, then a = __8__. To find "a" we must add __3__ to b. We can show this briefly by the following:

$$b \xrightarrow{+(\ 3\ )} a$$

If b is 5, then 5 $\xrightarrow{+(\ 3\ )}$ __8__.

If b is 2, then 2 $\xrightarrow{+(\ 3\ )}$ __5__.

If b is 1, then 1 $\xrightarrow{+(\ 3\ )}$ __4__.

If b is 6, then 6 $\xrightarrow{+(\ 3\ )}$ __9__.

Suppose we want to go from an address according to Mr. A to the address of the same point according to Mr. B.

If a is 9, then 9 $\xrightarrow{-(\ 3\ )}$ __6__.

Teacher's Copy

WORKSHEET 3b Name____________

If a is 4, then 4 —-(3)→ 1.

If a is 3, then 3 —-(3)→ 0.

If a is 7, then 7 —-(3)→ 4.

Write over the arrows what you must do to go from a to b.

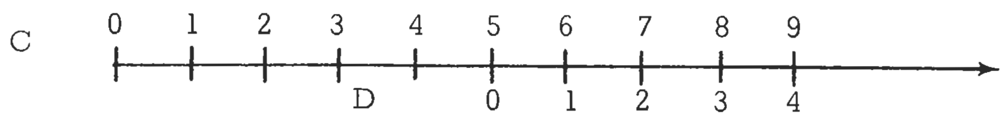
Value of a		Value of b
3	—-(3)→	0
7	—-(3)→	4
9	—-(3)→	6
6	—-(3)→	3

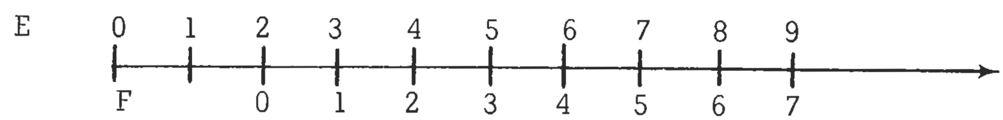
Additional Activity

Present this following Worksheet 3.

The children should notice that the difference between the "0" point on the two scales as they compare them is the amount of variation between the numbers on the scales at any given point.

Try this with other scales:





Ask questions such as:

If c is 6 what is d?

If d is 2 what is c?

If c is 7 what is d?

If e is 6 what is f?

If e is 8 what is f?

If f is 4 what is e?

Teacher's Copy

WORKSHEET 4a

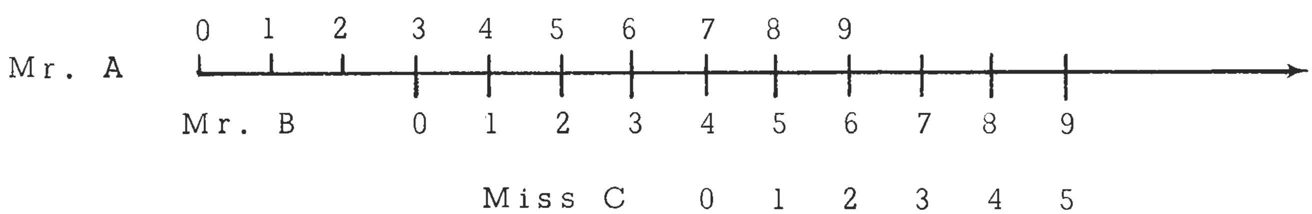
Name______________

What is the address?

Miss C stands at 4 on Mr. B's scale and she describes everything from her point of view.

This picture shows where Mr. A stands, where Mr. B stands, and where Miss C stands.

Use your scale. Set this up.



Use this picture to help you fill in the blanks in the following questions:

Any address on B's scale = the address on C's scale + ___4___.

Any address on C's scale = the address on B's scale - ___4___.

If we use "c" to stand for the address on Miss C's scale, then the step of going from her scale to Mr. B's scale can be described like this:

$$c \xrightarrow{+(\ 4\ )} b.$$

WORKSHEET 4b

Teacher's Copy

Name____________

If we want to show the reverse step going from Mr. B's scale to Miss C's scale, we can describe it like this:

$$b \xrightarrow{-(\ 4\ )} c.$$

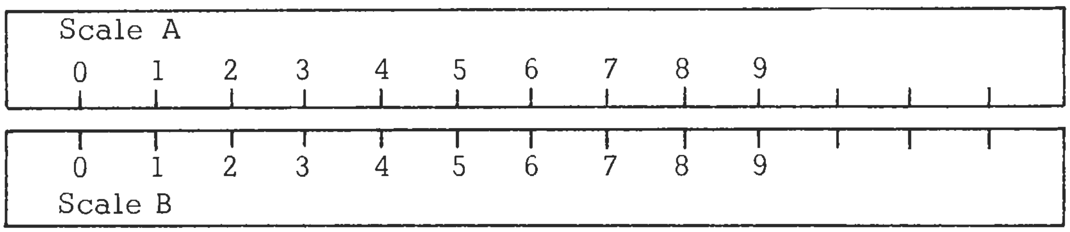
WORKSHEET 5a

Name______________

A simple adding machine

Here are two scales marked with the same scale unit.

Place your scales in this position.



Look at these problems: Do not write in the answers.

1 $\xrightarrow{+4}$	8 $\xrightarrow{-4}$
2 $\xrightarrow{+4}$	7 $\xrightarrow{-4}$
3 $\xrightarrow{+4}$	6 $\xrightarrow{-4}$
5 $\xrightarrow{+4}$	5 $\xrightarrow{-4}$

Now! Read carefully and THINK.

1. You may move one of your scales by sliding it along the edge of the other scale.
2. You are allowed to make only one move.
3. If you make the correct move, your two scales will be able to tell at a glance the answers to the problems above.
4. Try it! Can you do it? If so, fill in the chart and keep your method a secret. Your teacher will discuss it later.

WORKSHEET 5b

Teacher's Copy
Name__________

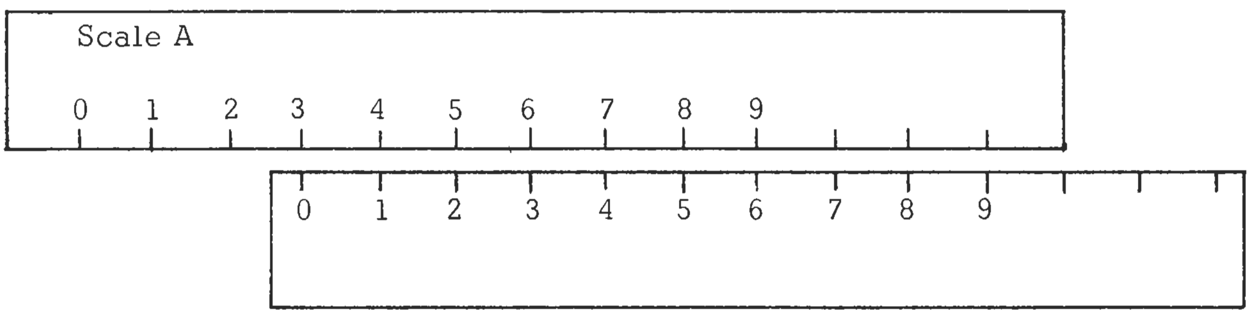
A simple adding machine

1. To begin, both scales must have their zero points together.

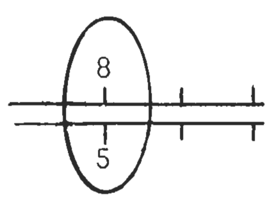
2. To set the scales for adding 3, you must slide one scale 3 units.

3. Then the 3 point of one scale will be at the zero point of the other scale.

4. Here is a picture of two scales. One has been moved to make an adding machine for 3.



5. Choose a numeral from the B scale so that there is a numeral above it on the A scale. Circle your choice.

Sample 

6. Look first at the 5 and then at the 8. Show this by writing 5 ⟶ 8.

7. What would you write above the arrow to complete your picture? 5 —(+3)→ 8

WORKSHEET 5c

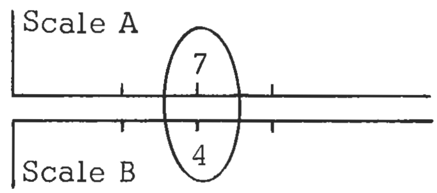
Teacher's Copy

Name_______

8. Fill in this chart. Use the scales shown in problem 4.

Numeral on B scale	Numeral on A scale	What this shows
		——→
		——→
		——→
		——→
		——→

9. Suppose we circle these two numerals.



10. This time we look first at the numeral on the A scale. Then we look at the numeral on the B scale. How would you write this?

7 —(-3)→ 4

11. Fill in this chart. Use the scales shown in problem 4.

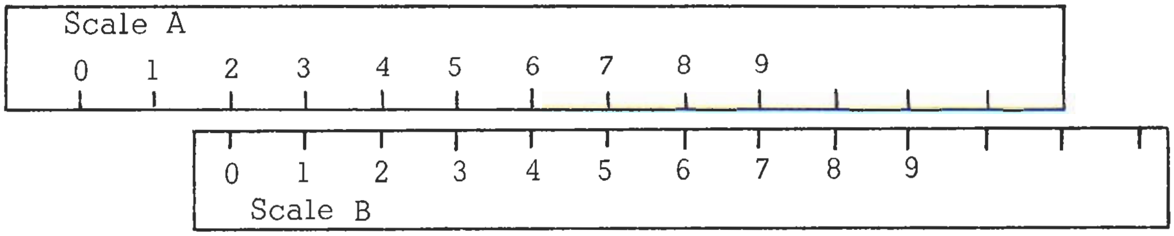
Numeral on A scale	Numeral on B scale	What this shows
		——→
		——→
		——→

WORKSHEET 5d

Teacher's Copy

Name_______________

12.



Scale _A_ was moved _2_ units.

13. The adding machine shows the answers for any

B scale numeral ⟶ A scale numeral

or

A scale numeral ―― B scale numeral

14. Fill in this chart. Use the scales from problem 12.

B scale numeral	A scale numeral	What this shows
		⟶
		⟶
		⟶
		⟶

A scale numeral	B scale numeral	What this shows
		⟶
		⟶
		⟶
		⟶

Teacher's Copy

WORKSHEET 6a

Name________________

Using a simple adding machine

a) $5 \xrightarrow{+4}$ 9

b) 5 $\xrightarrow{+4} 9$

c) $5 \xrightarrow{+(\ 4\ )} 9$

d) $9 \xrightarrow{-4}$ 5

e) $9 \xrightarrow{-(\ 4\ )} 5$

f) 9 $\xrightarrow{-4} 5$

Now, suppose that Miss C is standing at the point marked 2 on Mr. B's scale. Match the zero on Miss C's scale with the point marked 2 on Mr. B's scale. Use the scales to help find the missing information.

g) $3 \xrightarrow{+2}$ 5

h) 3 $\xrightarrow{+2} 5$

i) $3 \xrightarrow{+(\ 2\ )} 5$

j) $5 \xrightarrow{-2}$ 3

k) $5 \xrightarrow{-(\ 2\ )} 3$

l) 5 $\xrightarrow{-2} 3$

Where should the zero on Miss C's scale be placed to solve exercises m, n, o, p, q, and r?

m) $3 \xrightarrow{+6}$ 9

n) 3 $\xrightarrow{+6} 9$

o) $3 \xrightarrow{+(\ 6\ )} 9$

p) $9 \xrightarrow{-6}$ 3

q) $9 \xrightarrow{-(\ 6\ )} 3$

r) 9 $\xrightarrow{-6} 3$

WORKSHEET 6b

Teacher's Copy

Name______________

Use the sliding scales to help in solving the following problems:

s) 0 $\xrightarrow{+3}$ 3

t) 1 $\xrightarrow{+3}$ 4

u) 2 $\xrightarrow{+3}$ 5

v) 3 $\xrightarrow{+3}$ 6

w) 3 $\xrightarrow{-3}$ 0

x) 4 $\xrightarrow{-3}$ 1

y) 5 $\xrightarrow{-(3)}$ 2

z) 0 $\xrightarrow{+3}$ 3

Note: Be sure that children understand that in c), e), i), k), o), and q) there is a blank following the sign.

Teacher's Copy

WORKSHEET 7a

Name______________

Adding 3 numbers on the machine

1. How many numbers can be added at one time?

   Only two numbers

2. Suppose we want to add 3,4,2.

   First we would add __3__ + __4__ to get __7__.

   Then we would add __7__ + __2__ to get __9__

   which is the sum of 3,4,2.

3. At any one time, did you add more than two numerals? Yes (No)

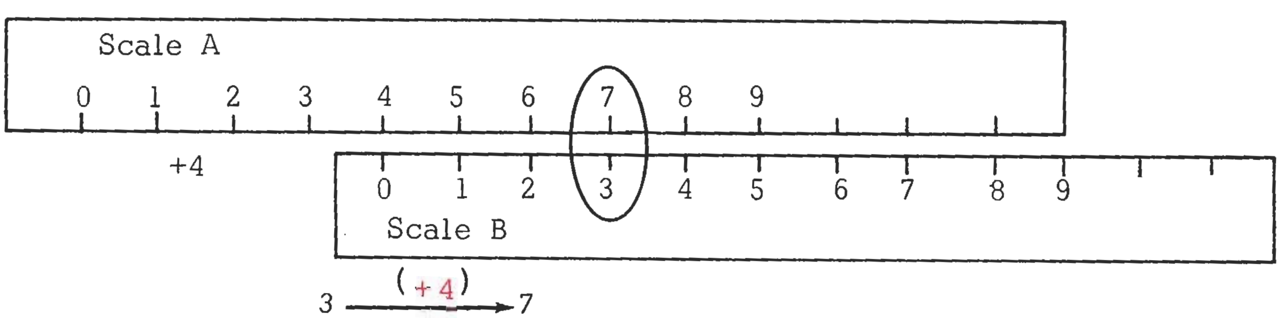
4. How many times did you add? two times

5. If you used your slide rule, how many times would you need to slide one of the scales?

   Slide it two different times.

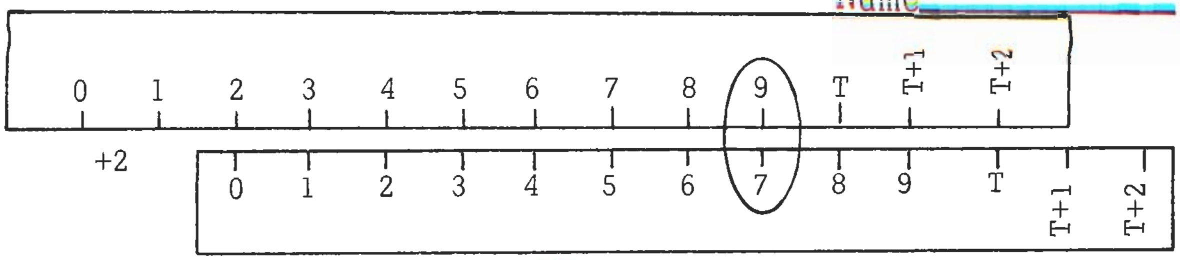
6. Before each slide of the scale, you made sure the __0__ point of one scale was line up with the __0__ point of the other scale.

7. Here are pictures of one way you might add 3,4,2. Complete the sentence below each picture.



WORKSHEET 7b

Teacher's Copy
Name____________



7 $\xrightarrow{+2}$ 9

8. Now combine both sentences for adding 3, 4, 2.

3 $\xrightarrow{+4}$ 7 $\xrightarrow{+2}$ 9

9. Fill in this chart. Use your adding machine to check your answer.

Begin with	Add	Then add	Sentence
3	2	3	3 $\xrightarrow{+2}$ 5 $\xrightarrow{+3}$ 8
1	3	3	1 $\xrightarrow{+3}$ 4 $\xrightarrow{+3}$ 7
5	1	2	5 $\xrightarrow{+1}$ 6 $\xrightarrow{+2}$ 8
4	2	3	4 $\xrightarrow{+2}$ 6 $\xrightarrow{+3}$ 9

10. Finish these sentences.

a) 3 $\xrightarrow{+2}$ 5 $\xrightarrow{-2}$ 3

b) 8 $\xrightarrow{-7}$ 1 $\xrightarrow{+7}$ 8

c) 6 $\xrightarrow{+9}$ 15 $\xrightarrow{-9}$ 6

d) 3 $\xrightarrow{+2}$ 5 $\xrightarrow{+4}$ 9

e) 1 $\xrightarrow{+5}$ 6 $\xrightarrow{+3}$ 9

f) 1 $\xrightarrow{+3}$ 4 $\xrightarrow{+5}$ 9

Teacher's Copy

WORKSHEET 8a

Name_______________

Using Scales to find the Sum

You may make use of the sliding scales (Mr. B's scale and Miss C's scale) to help in solving the following exercises. Remember that Miss C may stand at any place you want her to stand.

1. We write 3 $\xrightarrow{+2}$ $\xrightarrow{+4}$ _____ to show that we start at 3, add 2, then add 4 to the result. You can fill the blank with the final result. If you need to, you may write the result of the first addition between the two arrows, as in a).

a) 3 $\xrightarrow{+2}$ 5 $\xrightarrow{-2}$ 3

b) 8 $\xrightarrow{-7}$ $\xrightarrow{+7}$ 8

c) 6 $\xrightarrow{+9}$ $\xrightarrow{-(\ 9\ )}$ 6

d) 3 $\xrightarrow{+2}$ $\xrightarrow{+(\ 4\ )}$ 9

e) 1 $\xrightarrow{+5}$ $\xrightarrow{+3}$ 9

f) 1 $\xrightarrow{+3}$ $\xrightarrow{+5}$ 9

Teacher's Copy

WORKSHEET 8b

Name________________

g) 7 $\xrightarrow{+8}$ $\xrightarrow{-8}$ __7__

h) 9 $\xrightarrow{-8}$ $\xrightarrow{+8}$ __9__

i) 9 $\xrightarrow{-5}$ $\xrightarrow{+(\ 5\ )}$ 9

j) 3 $\xrightarrow{+\ (\ 6\ )}$ 9

k) 1 $\xrightarrow{+\ (\ 8\ )}$ 9

l) 3 $\xrightarrow{+4}$ $\xrightarrow{+2}$ __9__

Note: The numeral must be supplied in c), d), i), j), and k).

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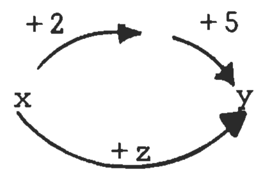
WORKSHEET 9a

Name________________

Making One Move take the Place of Two

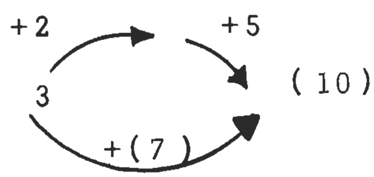
In the diagram



let x be any number you like. Find the value of y by doing what the arrows on top say. Then figure out what number for z will take you from x to y.

<u>Example</u>

Let x be 3. Then use other values of x, and complete the table.



Value of x	Value of y	Value of z
0	7	7
1	8	7
2	9	7
3	10	7

Adding 2, then adding 5, is the same as adding___7___.

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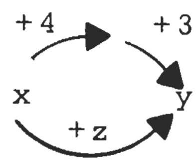
WORKSHEET 9b

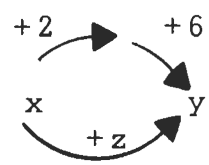
Name________________

Use this diagram and complete the table below:



Value of x	Value of y	Value of z
0	7	7
1	8	7
2	9	7
3	10	7

Adding 4, then adding 3, is the same as adding________.



Value of x	Value of y	Value of z
0	8	8
1	9	8
2	10	8
3	11	8

Adding 2, then adding 6, is the same as adding 8.

Use any numbers you like on the two top arrows. How can you always find the value of the lower arrow?

The sum of the two top arrows equals the lower arrow.

Note: Use as many exercises as necessary until the children grasp the idea.

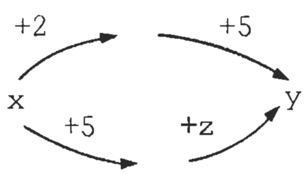
Teacher's Copy

WORKSHEET 10a Name________________

What happens when you change the order of addends?

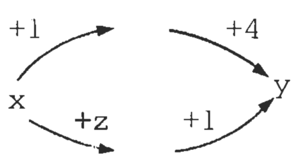
1. In the diagram below, let x be any number you like. Find the value of y, by doing what the arrows on top say. Then figure out what number, z, will help take you from x to y. You may use your sliding scales if you like.

Complete the table below:



Value of x	Value of y	Value of z
0	7	2
1	8	2
2	9	2
3	10	2

2. Study this diagram and complete the table below:



Value of x	Value of y	Value of z
0	5	4
1	6	4
2	7	4
3	8	4

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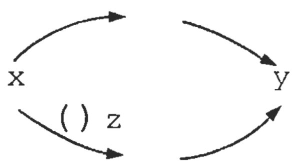
WORKSHEET 10b Name_______________

3. You may choose the numbers to use over the arrows(as in 1. and 2.) in this diagram. Then complete the table below.



Answers will vary.

Value of x	Value of y	Value of z
0		
1		
2		
3		

4. Complete the following sentences:

a) Adding 2 and then 5 to a number gives the same sum as adding 5 and then ___2___.

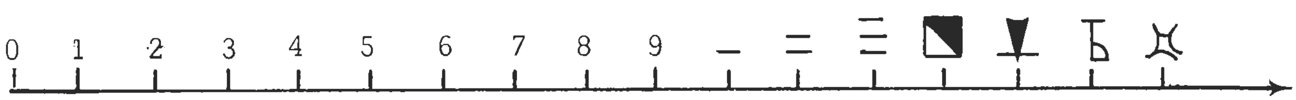
b) Adding 1 and then 4 to a number gives the same sum as adding 4 and then ___1___.

c) Order of addends does not change the _sum_.

WORKSHEET 11a   Name________________

"T" for Ten

So far on our scales, only the points 1, 2, 3, 4, 5, 6, 7, 8, and 9 were labeled. How shall we name the rest of the points on the scale? We could invent a special symbol for each new point.



Do you think this would be a good idea? Explain.

It would be better to have names which have a <u>system</u> to them. A <u>system</u> will help you remember what the names of the points mean.

Once previously we used "T" for ten in our system. On the following scale, print T exactly above the point which is one unit to the right of the point named 9.

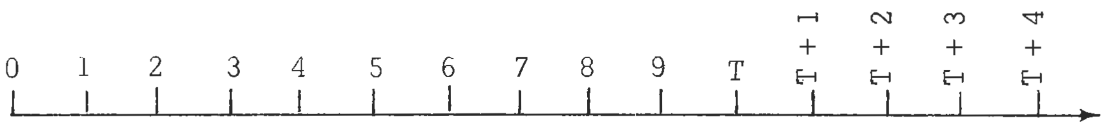
Since adding 1 means moving 1 unit away from 0 in the direction of the ray, we can use T + 1 to name the point one unit beyond T. In the same way, we can

Teacher's Copy

WORKSHEET llb

Name____________________

name the point 2 units beyond T, ___T + 2___; the point 3 units beyond T can be named ___T + 3___.

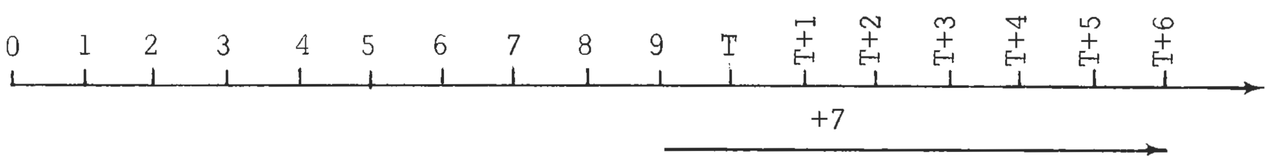


In order to center the name over the point, have children write them as above.

Using only one scale, we can add 7 to any number on the scale by moving 7 units away from the number in direction of the ray. We can find the missing sum in this example,

9 + 7 = ___T + 6___

by starting at 9 and moving 7 units in this way:



Complete the following sentences:

a) 9 + 3 = ___T + 2___

b) ___T + 5___ = 9 + 6

c) ___T + 4___ = 5 + 9

d) 8 + 7 = ___T + 5___

e) 7 + 6 = ___T + 3___

Teacher's Copy

WORKSHEET 11c

Name________________

f) T + 7 = T + 7

g) 4 + 9 = T + 3

h) 7 + 4 = T + 1

i) 9 + 7 = T + 6

j) 9 + 9 = T + 8

k) 9 + 4 = T + 3.

Teacher's Copy

WORKSHEET 12a

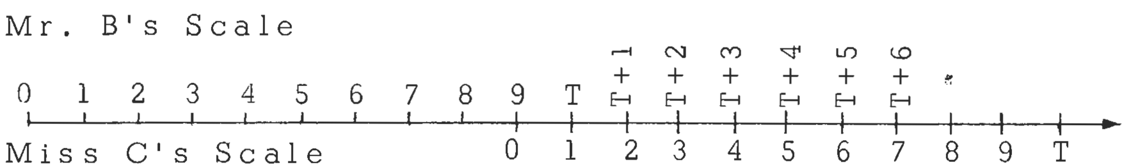
Name__________________

Find the sum or missing addend

Use your private system (T + 1, T + 2, etc.) to mark all the points on Mr. B's scale and Miss C's scale.

Now that you have marked both Mr. B's scale and Miss C's scale, you can let them do the counting for you by sliding Miss C's scale along Mr. B's scale until the 0 is opposite the point named 9 on Mr. B's scale. The point on Mr. B's scale which matched the seven on Miss C's scale is T + 6; so 7 + 9 = <u>T + 6</u>. We could also record this fact in this way:

$$7 \xrightarrow{+9} T + 6$$



Note: It may be difficult for the child to verbalize the method although he may be able to use the scale efficiently. Help him to see that if he wishes to subtract 6 from T + 4 he can place the 6 on Miss C's scale under T + 4 on Mr. B's scale, move 6 units to the left on Miss C's scale, bringing us to 0 on Miss C's scale which is opposite 8 on Mr. B's scale. Therefore, (T + 4) - 6 = 8, or we can say that 8 is the missing addend since 6 + 8 = (T + 4).

Teacher's Copy

WORKSHEET 12b

Name________________

Use your sliding scales to complete the following exercises:

a) 3 + 8 = T + 1

b) $3 \xrightarrow{+9}$ T + 2

c) T + 1 = 5 + 6

d) $7 \xrightarrow{+6}$ T + 3

e) 8 + 7 = T + 5

f) 9 + 8 = T + 7

g) T + 2 = 6 + 6

h) $5 \xrightarrow{+9}$ T + 4

i) 9 + 3 = T + 2

j) T + 2 = 7 + 5.

In the exercises above both <u>addends</u> are given; you found the <u>sum</u>. In the exercises below, the sum and only one addend are given. You are to find the <u>missing addend</u>. How can the sliding scales be used to help you?

k) (T + 7) - 9 = 8

l) (T + 2) - 5 = 7

m) (T + 1) - 8 = 3

n) $(T + 6) \xrightarrow{-9}$ 7

o) (T + 5) - 6 = 9

p) (T + 7) - 4 = T + 3

q) $(T + 6) \xrightarrow{-9}$ 7

r) $(T + 9) \xrightarrow{-T}$ 9

WORKSHEET 13a

Teacher's Copy

Name____________________

Choose a number

1. Choose a number between 0 and T. Add your number to each of the numbers from 0 to T. Make a table of your results.

0 + your number = 8
1 + your number = 9
2 + your number = T
3 + your number = T + 1
4 + your number = T + 2
5 + your number = T + 3
6 + your number = T + 4
7 + your number = T + 5
8 + your number = T + 6
9 + your number = T + 7
T + your number = T + 8

The number I chose was 8.

Answers will vary.

WORKSHEET 13b

Teacher's Copy

Name______________________

2. Record in your notebook an addition chart.

Allow time for discussion.

+	0	1	2	3	4	5	6	7	8	9	T
0	0	1	2	3	4	5	6	7	8	9	T
1	1	2	3	4	5	6	7	8	9	T	T+1
2	2	3	4	5	6	7	8	9	T	T+1	T+2
3	3	4	5	6	7	8	9	T	T+1	T+2	T+3
4	4	5	6	7	8	9	T	T+1	T+2	T+3	T+4
5	5	6	7	8	9	T	T+1	T+2	T+3	T+4	T+5
6	6	7	8	9	T	T+1	T+2	T+3	T+4	T+5	T+6
7	7	8	9	T	T+1	T+2	T+3	T+4	T+5	T+6	T+7
8	8	9	T	T+1	T+2	T+3	T+4	T+5	T+6	T+7	T+8
9	9	T	T+1	T+2	T+3	T+4	T+5	T+6	T+7	T+8	T+9
T	T	T+1	T+2	T+3	T+4	T+5	T+6	T+7	T+8	T+9	2T

What interesting patterns do you see in this addition table? Can this table also help you find the missing addend in a subtraction example?

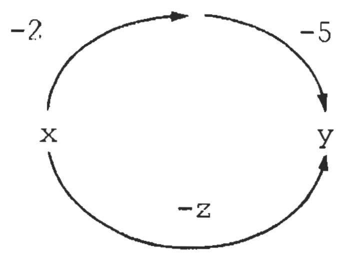
Explain. (For example: (T + 7) - 9 = _____. Find the sum (T + 7) in the row with the known addend 9, and look to the head of the column to find the missing addend, 8.)

Teacher's Copy

WORKSHEET 14a Name_______________

Subtracting two numbers

Let x be any number from 7 to T + T. Compute y from the top arrows. Then work out the value of z. Fill in this table.



Value of x	Value of y	Value of z
7	0	7
T + 3	6	7
T	3	7
T + 5	8	7
T + T	T + 3	7
8	1	7
T + 1	4	7
T + 8	T + 1	7
T + 4	7	7
T + 2	5	7
9	2	7
T + 6	9	7

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WORKSHEET 14b

Name____________________

Subtracting 2 and then subtracting 5 is the same as subtracting ____7____.

Replace 2 and 5 by any two numbers; this will be the same as subtracting __the sum of these two numbers__.

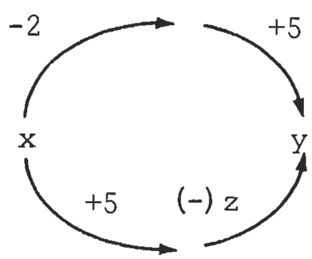
Teacher's Copy

WORKSHEET 15a

Name________________

What rule can you discover?

Let x be any number from 2 to T + T. Compute y from the top arrows. Then work out the value of z. Do you have to add or subtract z? Put the operation sign in the parentheses in front of z.



Value of x	Value of y	Value of z
6	9	2
T + 1	T + 4	2
4	7	2
T + 8	T + T + 1	2
9	T + 2	2
3	6	2
T	T + 3	2
T + 2	T + 5	2

Continued on following page.

WORKSHEET 15b

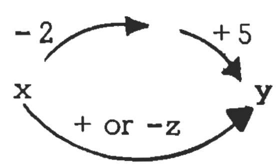
Teacher's Copy
Name______________

Value of x	Value of y	Value of z
5	8	2
T + T	T + T + 3	2
T + 5	T + 8	2
7	T	2

The general rule is: Subtracting 2, then adding 5, is the same as adding 5, then subtracting 2.

Replace 2 and 5 by any numbers. What is the general rule for finding z? Subtracting the first number and adding the second is the same as adding the second and subtracting the first.

In the diagram



try various numbers for x and compute y by the top arrows.

What do you have to do on the bottom arrow to get from x to y? Add. How much? 3

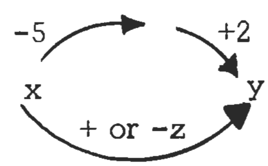
WORKSHEET 15c

Teacher's Copy
Name________________

Subtracting 2, then adding 5, is the same as add-ing 3.

Try other numbers in place of 2 and 5. What is the general rule? If the number to be subtracted is larger than the number to be added, subtract the difference; if the number to be added is larger than the number to be subtracted, add the difference.

In the diagram



try various numbers for x and compute y by the top arrows. What do you have to do on the bottom arrow to get from x to y? Do you add or subtract? Subtract.

How much? 3

Subtracting 5, then adding 2, is the same as subtract-ing 3.

Try other numbers in place of 5 and 2. What is the general rule? Answers will be the same as the previous general rule.

WORKSHEET 16a

Teacher's Copy

Name________________

This worksheet should be completed with the class and teacher working together.

Some laws of the number system

You have discovered that adding 2, then adding 3 is the same as adding 5. Why? 5 = (2) + (3) _______________.

We can use this knowledge to add twelve and three:

Twelve = T + 2

Twelve and three = (T + 2) + 3.

We have placed parentheses around the "T + 2" to show that we are thinking of this combination as a single number.

According to what you have discovered, you can see that (T + 2) + 3 = T + (2 + 3) and you know that 2 + 3 = 5. Therefore, you find twelve + three = fifteen (the English number word for T + 5).

You have used the general rule that when you add three or more numbers, you get the same sum no matter how you associate the numbers.

WORKSHEET 16b

Teacher's Copy
Name________________

For example, if you add 2, 3, 4 and 5, you can associate the numbers in any of these ways:

1) $[(2 + 3) + 4] + 5 =$ 14

2) $(2 + 3) + (4 + 5) =$ 14

3) $2 + [3 + (4 + 5)] =$ 14.

1. The first string of symbols tells you to add 3 to 2. This equals 5.
   Add 4 to the result 5. This equals 9.
   Add 5 to the last result 9. This equals 14.

2. The second string of symbols tells you to add 3 to 2. This equals 5.
   Add 5 to 4. This equals 9.
   Add these two sums: 5 + 9 = 14.

3. What does the third string tell you?
   Add 4 + 5. This equals 9.
   Add 3 to the result. This equals 12.
   Add 2 to the result 12. This equals 14.

Teacher's Copy

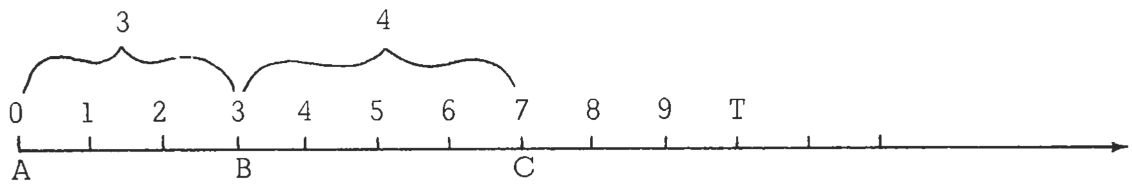
WORKSHEET 16c

Name_______________

Compare the three answers. How do they compare?

They are the same: 14 in each case.

Here is a number line which shows a trip of 3 and a trip of 4.



If you go 3 blocks from A to B and 4 blocks from B to C, you go ___7___ blocks altogether.

On your return trip you go 4 blocks from C to B and 3 blocks from B to A.

Is your return trip longer or shorter than your first trip? No, the return trip is neither longer or shorter.

When you add two numbers, does their order make any difference? No.

This is part of a general rule: When you add two or more numbers, you can commute them by changing the order in any way you please without changing the sum.

Teacher's Copy

WORKSHEET 17a

Name________________

Use the laws

Use what you have learned about associativity and commutativity to help you add the following. For example:

$$13 + 14 = \quad 27$$

Think, $13 = (T + 3)$

$14 = (T + 4)$

so, $(T + 3) + (T + 4) = (T + T) + (3 + 4) =$

$2T + 7 = 27.$

a. $15 + 14 =$

$15 = (\underline{T + 5})$

$14 = (\underline{T + 4})$

$(\underline{T + 5}) + (\underline{T + 4}) = (\underline{T + T}) + (\underline{5 + 4})$

$= \underline{2T + 9} = \underline{29}$

WORKSHEET 17b

Teacher's Copy

Name________________

b. 11 + 17 =

11 = (T + 1)

17 = (T + 7)

(T + 1) + (T + 7) = (T + T) + (1 + 7)

= 2T + 8 = 28

c. 15 + 13 =

15 = (T + 5)

13 = (T + 3)

(T + 5) + (T + 3) = (T + T) + (5 + 3)

= 2T + 8 = 28

d. 14 + 14 =

14 = (T + 4)

(T + 4) + (T + 4) = (T + T) + (4 + 4)

= 2T + 8 = 28

WORKSHEET 17c

Teacher's Copy
Name ______________

e. 12 + 15 =

12 = (T + 2)

15 = (T + 5)

(T + 2) + (T + 5) = (T + T) + (2 + 5)

= 2T + 7 = 27

f. 18 + 11 =

(T + 8) + (T + 1) = (T + T) + (8 + 1)

= 2T + 9 = 29

g. 17 + 12 =

(T + 7) + (T + 2) = (T + T) + (7 + 2)

= 2T + 9 = 29

h. 12 + 13 =

(T + 2) + (T + 3) = (T + T) + (2 + 3)

= 2T + 5 = 25

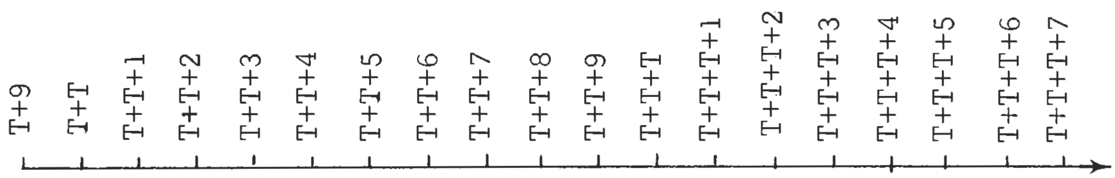
WORKSHEET 18a

Teacher's Copy

Name__________________

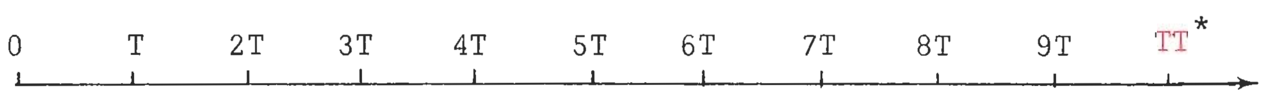
BEYOND "T + T"

As we lay off points beyond T + T on the number line, we name the points T + T + 1, T + T + 2, T + T + 3, etc.



When we come to T + T + T + T + T + T + T, it becomes tiresome to write all this and it is also hard to read because you cannot see easily how many T's there are. A natural abbreviation is: 7T.

So we mark off our scale like this:



*One hundred is written as TT. The sum of 9T + 9 + 1 is split into 9 + 1 = T. Then we see 9T + T = 10T or TT.

and mark the points between with symbols like "3T + 2," "5T + 8," etc. To add such numbers as 3T + 4T we use our general law about association in addition.

Teacher's Copy

WORKSHEET 18b

Name________________

$3T + 4T = (T + T + T) + (T + T + T + T)$

$= T + T + T + T + T + T + T$

$= \underline{\quad 7 \quad} T$

We left out the parentheses in the second line because it makes no difference how we associate these numbers.

To add $(3T + 8)$ and $(4T + 1)$ we change the order and reassociate:

$(3T + 8) + (4T + 1) = (3T + 4T) + (8 + 1)$

$= (\underline{\ 7\ })T + (\underline{\ 9\ })$

$= \underline{\quad 79 \quad}$

To add $(3T + 8)$ and 7, we reassociate several times:

$(3T + 8) + 7 = 3T + (8 + 7)$

$= 3T + (T + (\underline{\ 5\ })$

$= (3T + T) + 5$

$= (\underline{\ 4\ })\ T + 5$

$= \underline{\quad 45 \quad}$

WORKSHEET 19a

Teacher's Copy

Name________________

Addition beyond "T + T"

Use what you have learned about associativity and commutativity in adding the following:

Example: 38 + 41 = (3T + 8) + (4T + 1)

= (3T + 4T) + (8 + 1)

= 7T + 9

= 79.

1. 24 + 35 = (2T + 4) + (3T + 5)

= (2T + 3T) + (4 + 5)

= 5 T + 9

= 59

2. 52 + 46 = (5T + 2) + (4T + 6)

= (5T + 4T) + (2 + 6)

= 9 T + 8

= 98

3. 65 + 12 = (6T + 5) + (T + 2)

= (6T + T) + (5 + 2)

= 7 T + 7

= 77

WORKSHEET 19b

Teacher's Copy
Name______________

4. 37 + 62 = (3 T + 7) + (6 T + 2)

= (3 T + 6 T) + (7 + 2)

= 9 T + 9

= 99

Example: 45 + 38 = (4T + 5) + (3T + 8)

= (4T + 3T) + (5 + 8)

= (4T + 3T) + (T + 3)

= (4T + 3T + T) + 3

= 8T + 3

= 83

5. 64 + 29 = (6 T + 4) + (2 T + 9)

= (6 T + 2 T) + (4 + 9)

= (6 T + 2 T) + (T + 3)

= (6 T + 2T + T) + 3

= 9 T + 3

= 93

WORKSHEET 19c

Teacher's Copy
Name________________

6. 12 + 79 = (T + 2) + (7 T + 9)

= (T + 7 T) + (2 + 9)

= (T + 7 T + T) + 1

= 9 T + 1

= 91

For Junior Researchers only:

7. 54 + 29 = 83

8. 72 + 18 = 90

9. 49 + 30 = 79

10. 32 + 58 = 90

The partitioning of a sum

We will investigate how many different ways a number can be partitioned into two addends. Let's take the number 5 as an example. We can partition 5 in three different ways: 5 + 0, 4 + 1, and 2 + 3.

We do not consider 3 + 2 as another different way of partitioning, since 3 + 2 and 2 + 3 both involve the same addends. In other words this is just a different way of writing the same sum. For the same reasons we could have said 1 + 4 instead of 4 + 1 or 0 + 5 instead of 5 + 0.

Since the order of the addends makes no difference, one may arbitarily decide to write the larger addend first.

Teacher's Copy

WORKSHEET 20a

Name________________

How many parts are there?

Work out this partition chart. Use counters to help you, if necessary.

n	Partition of n	Number of partitions
0	0+0	1
1	1+0	1
2	2+0, 1+1	2
3	3+0, 2+1	2
4	4+0, 3+1, 2+2	3
5	5+0, 4+1, 3+2	3
6	6+0, 5+1, 4+2, 3+3	4
7	7+0, 6+1, 5+2, 4+3	4
8	8+0, 7+1, 6+2, 5+3, 4+4	5
9	9+0, 8+1, 7+2, 6+3, 5+4	5
T	T+0, 9+1, 8+2, 7+3, 6+4, 5+5	6

WORKSHEET 20b

Teacher's Copy
Name________________

What is the pattern in the number of different partitions? Beginning with 2, each two consecutive numbers have the same number of partitions which increase by one with each pair. This will be difficult for the children to verbalize; do not insist on verbalization.

____

Predict the number of partitions of T + 1 into two parts. ____6____ Work them out and check your prediction.

Teacher's Copy

WORKSHEET 20c

Name________________

Predict the number of partitions of 2T + 4 into 2 parts. 13 Check your partitions of 2T + 4. (2T+4)+0, (2T+3)+1, (2T+2)+2, (2T+1)+3, (2T)+4, (T+9)+5, (T+8)+6, (T+7)+7, (T+6)+8, (T+5)+9, (T+4)+T, (T+3)+(T+1), (T+2)+(T+2)

n	Partition of n into 2 parts	Number of partitions of n into two parts
T+1	(T+1)+0, (T)+1, 9+2, 8+3, 7+4, 6+5	6
T+2	(T+2)+0, (T+1)+1, (T)+2, 9+3, 8+4, 7+5, 6+6	7
T+3	(T+3)+0, (T+2)+1, (T+1)+2, (T)+3, 9+4, 8+5, 7+6	7
T+4	(T+4)+0, (T+3)+1, (T+2)+2, (T+1)+3, (T)+4, 9+5, 8+6, 7+7	8
T+5	(T+5)+0, (T+4)+1, (T+3)+2, (T+2)+3, (T+1)+4, (T)+5, 9+6, 8+8	8

Teacher's Copy

WORKSHEET 20d

Name________________

n	Partition of n into 2 parts	Number of partitions of n
T+6	(T+6)+0, (T+5)+1, (T+4)+2, (T+3)+3, (T+2)+4, (T+1)+5, (T)+6, 9+7, 8+8	9
T+7	(T+7)+0, (T+6)+1, (T+5)+2, (T+4)+3, (T+3)+4, (T+2)+5, (T+1)+6, (T)+7, 9+8	9
T+8	(T+8)+0, (T+7)+1, (T+6)+2, (T+5)+3, (T+4)+4, (T+3)+5, (T+2)+6, (T+1)+7, (T)+8, 9+9	10
T+9	(T+9)+0, (T+8)+1, (T+7)+2, (T+6)+3, (T+5)+4, (T+4)+5, (T+3)+6, (T+2)+7, (T+1)+8, (T)+9	10
2T	(2T)+0, (T+9)+1, (T+8)+2, (T+7)+3, (T+6)+4, (T+5)+5, (T+4)+6, (T+3)+7, (T+2)+8, (T+1)+9, T+T	11
2T+1	(2T+1)+0, (2T)+1, (T+9)+2, (T+8)+3, (T+7)+4, (T+6)+5, (T+5)+6, (T+4)+7, (T+3)+8, (T+2)+9, (T+1)+T	11

WORKSHEET 21a

Teacher's Copy

Name________________

Ways an El can have n cents

Note: The red and green counters could be used to represent different coins. Use a magic marker to label red counters as 2-cent pieces and green as 1¢ or penny coins.

2¢	1¢ or penny
red	green

If you prefer, let each child mark his own with a pencil as he needs them.

In Lower Moneyville where the Els live there are only pennies and 2-cent pieces. If an El has 5 cents, he can have:

		2-CENT PIECES		PENNIES
	a)	0	and	5
<u>or</u>	b)	1	and	3
<u>or</u>	c)	2	and	1

a) may be abbreviated as (0, 5).

b) may be abbreviated as (1, 3).

c) may be abbreviated as (2, 1).

The first number is the number of 2-cent pieces and the second number is the number of pennies.

Teacher's Copy

WORKSHEET 21b

Name________________

Work out the following table which will show the different ways the Els can have n cents.

How many 2¢ pieces can he have? Write it first.
How many 1¢ or pennies? Write it next.

n cents	Ways an El can have n cents	Number of ways
1	(0,1)	1
2	(0,2), (1,0)	2
3	(0,3), (1,1)	2
4	(0,4), (1,2), (2,0)	3
5	(0,5), (1,3), (2,1)	3
6	(0,6), (1,4), (2,2), (3,0)	4
7	(0,7), (1,5), (2,3), (3,1)	4
8	(0,8), (1,6), (2,4), (3,2), (4,0)	5
9	(0,9), (1,7), (2,5), (3,3), (4,1)	5
T	(0,T), (1,8), (2,6), (3,4), (4,2), (5,0)	6

WORKSHEET 21c

Teacher's Copy
Name________________

What is the pattern? Beginning with 2 each two consecutive numbers have the same number of ways; these ways increase by one.

Predict the number of ways an **El** can have T + 1 cents.

There will be six different ways.

Check your prediction. (0, T+ 1), (1,9), (2,7), (3,5), (4,3), (5,1)

Predict the number of ways an **El** can have 2T + 4 cents.

thirteen ways Check your prediction.

(0, 2T+ 4), (1, 2T+ 2), (2, 2T), (3, T+ 8), (4, T+ 6), (5, T+ 4), (6, T+ 2), (7, T) (8,8), (9,6), (T, 4), (T+ 1, 2), (T+ 2, 0)

Note: Children should be encouraged to discover as many possibilities as they can. They may not discover all possible combinations. It is suggested that the chart be checked with the class after the children have been given a specified time to work.

WORKSHEET 22a Name________________

Ways an Og can have n cents

In Upper Moneyville where the Ogs live, there are pennies, 2-cent pieces, and 3-cent pieces. Work out the following table to show the different ways an Og can have various amounts of money. For example, an Og can have 4 cents in 4 different ways:

	3-CENT PIECES	2-CENT PIECES	PENNIES
a)	0 3-cent pieces,	0 2-cent pieces,	4 pennies;
or b)	0 3-cent pieces,	1 2-cent pieces,	2 pennies;
or c)	0 3-cent pieces,	2 2-cent pieces,	0 pennies;
or d)	1 3-cent piece,	0 2-cent pieces,	1 penny,

which we may abbreviate by simply writing (0, 0, 4), (0, 1, 2), (0, 2, 0), (1, 0, 1).

WORKSHEET 22b

Teacher's Copy
Name________________

n cents	Ways an Og can have n cents	Number of ways
0	(0, 0, 0)	1
1	(0, 0, 1)	1
2	(0, 0, 2), (0, 1, 0)	2
3	(0, 0, 3), (0, 1, 1) (1, 0, 0)	3
4	(0, 0, 4), (0, 1, 2), (0, 2, 0) (1, 0, 1)	4
5	(0, 0, 5), (0, 1, 3), (0, 2, 1) (1, 0, 2), (1, 1, 0)	5
6	(0, 0, 6), (0, 1, 4), (0, 2, 2), (0, 3, 0) (1, 0, 3), (1, 1, 1) (2, 0, 0)	7
7	(0, 0, 7), (0, 1, 5), (0, 2, 3), (0, 3, 1) (1, 0, 4), (1, 1, 2), (1, 2, 0) (2, 0, 1)	8

WORKSHEET 22c

Teacher's Copy

Name________________

n cents	Ways an Og can have n cents	Number of Ways
8	(0, 0, 8), (0, 1, 6), (0, 2, 4), (0, 3, 2), (0, 4, 0)	T
	(1, 0, 5), (1, 1, 3), (1, 2, 1)	
	(2, 0, 2), (2, 1, 0)	
9	(0, 0, 9), (0, 1, 7), (0, 2, 5), (0, 3, 3), (0, 4, 1)	T + 2
	(1, 0, 6), (1, 1, 4), (1, 2, 2), (1, 3, 0)	
	(2, 0, 3), (2, 1, 1)	
	(3, 0, 0)	
T	(0, 0, T), (0, 1, 8), (0, 2, 6), (0, 3, 4), (0, 4, 2), (0, 5, 0	T + 4
	(1, 0, 7), (1, 1, 5), (1, 2, 3), (1, 3, 1)	
	(2, 0, 4), (2, 1, 2), (2, 2, 0)	
	(3, 0, 1)	

Teacher's Copy

WORKSHEET 22d

Name________________

Use the information from Worksheet 22 and from the table you have just finished to complete the following table:

n cents	Number of ways an El can have n cents	Number of ways an Og can have n cents	Og minus El
0	(1)	(1)	(0)
1	(1)	(1)	(0)
2	(2)	(2)	(0)
3	(2)	(3)	(1)
4	(3)	(4)	(1)
5	(3)	(5)	(2)
6	(4)	(7)	(3)
7	(4)	(8)	(4)
8	(5)	(10)	(5)
9	(5)	(12)	(7)
T	(6)	(14)	(8)

American Money

The aim of this lesson is for the children to discover the various sums which could be formed using different coins. The situations should be so structured that the possibilities are limited by the value of the coins available.

It is recommended that the class do the following lesson together. Use counters. Mark them on the white side.

You have been given the following coins.

25¢ 10¢ 10¢ 5¢

1¢ 1¢ 1¢ 1¢ 1¢ = 55¢

You wish to purchase a bottle of milk which costs 20¢. What coins <u>could</u> you give the storekeeper?

Possible answers are: 10¢ 10¢

<u>or</u> 10¢ 5¢ 1¢ 1¢ 1¢ 1¢ 1¢

<u>or</u> 25¢ if storekeeper has 5¢ change.

The storekeeper does not have any change. (It is too early in the morning.) He asks you to give him the coins which have the smallest value. What will you give him? <u>a dime, a nickel and 5 pennies</u>

Draw them.

10¢ 5¢ 1¢ 1¢ 1¢ 1¢ 1¢

You will now have a quarter and a dime left.

Now, you decide you must purchase a bar of soap for 8¢. What coins could you give the storekeeper?

Answer: a dime (2¢ change)

or a quarter (17¢ change)

If you give him a dime (10¢), and he gives you 2¢ change, you now have a quarter and two pennies left.

You have just remembered that mother also wants some bread. This costs 22¢. What coins would you give the storekeeper?

Answer: a quarter (3¢ change)

or

a quarter plus two pennies (5¢ or a nickel in change)

After this initial lesson with the class, this kind of activity could be worked out whereby half the class is assigned as storekeepers and half as customers.

It is significant to note that the illustration used did not require that the storekeeper have any change at all at the time of the first purchase.

The possibilities of the situation might be explored on the basis of the initial purchase being made with different coins. What would the exchanges be in the various purchases.

WORKSHEET 23

Teacher's Copy
Name________________

What change must you have?

Label counters for coins. You are the storekeeper. You do not like to keep too much change on hand. What is the least amount of change you must have to begin the day? 1 dime, 9 pennies

These are the customers.	They buy:	They have:
John	tablet 10¢ then pencil 7¢	quarter
Bob	airplane 17¢ then comic book 10¢	quarter dime
Dad	ballons 10¢ then paints 22¢	quarter dime

Now you make up a problem like this for the class.

Can the storekeeper start with no change? No.

Addition and subtraction patterns

Worksheets 24-31 provide further practice in addition and subtraction. At the same time these worksheets develop the idea of representing numbers by letters in equations.

Let the letters x, y, a and b represent the numbers in the boxes below them. These numbers were chosen to have special addition and subtraction relationships, namely: From

x	a	b	y
T + 2	9	3	6

T + 2	9	3	6

we see

$$T + 2 = 9 + 3 \quad \text{and} \quad 9 - 3 = 6.$$

In general, we express these relationships with the letters as

$$x = a + b \quad \text{and} \quad a - b = y.$$

Tell the children that there is a "Hidden Secret" or "Magic Formula" for solving Worksheets 24-30. Then let them discover the secret without any help from the teacher.

WORKSHEET 24

Teacher's Copy
Name________________

Can you find the secret?

Fill in the blanks.

		a	b	
1.	T + 9	T	9	1
2.	T + 3	7	6	1
3.	T+3	7	6	1
4.	2T + 4	T + 9	5	T+4
5.	9	7	2	5
6.	T + 5	T	5	5

If you find the secret, make a puzzle like this for the class.

WORKSHEET 25

Teacher's Copy
Name________________

How can you tell?

Fill in one box at a time.

	8	2T + 1	T+4	T + 7	T+2	T
a	5	T+3	9	T	6	8
b	3	8	5	7	6	2
	2	5	4	3	0	6

	5T + 4	7T+3	9T+1	8T+8	TT	5T + 7
a	3T + 4	4T + 5	5T+2	6T + 6	7T+7	2T+9
b	2T	2T + 8	3T + 9	2T+2	2T+3	2T+8
	T + 4	T+7	T + 3	4T + 4	5T + 4	1

WORKSHEET 26

Teacher's Copy
Name______________

Keep the pattern

Fill the blanks but keep the pattern.

2T + 8	2T	8	T + 2
T + 6	T+2	4	8
T+8	T + 3	5	8
2T	T + 2	8	4
2T	T + 4	6	8
T + 3	T + 2	1	T+1
T + 5	T+2	3	9
2T+1	T + 5	6	9

WORKSHEET 27

Teacher's Copy
Name______________

Who's missing?

Find the value of x.

x			y
2T + 2	T + 4	8	6

Find the value of y.

x			y
T + 7	T + 5	2	T + 3

Find the value of a.

	a	b	
T + 8	T	8	2

Find the value of b.

	a	b	
2T + 1	T + 7	4	T + 3

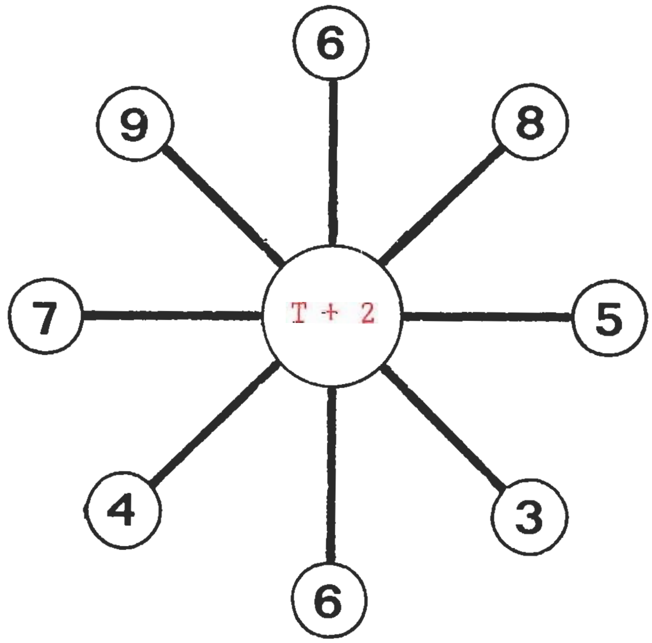
WORKSHEET 28a

Teacher's Copy

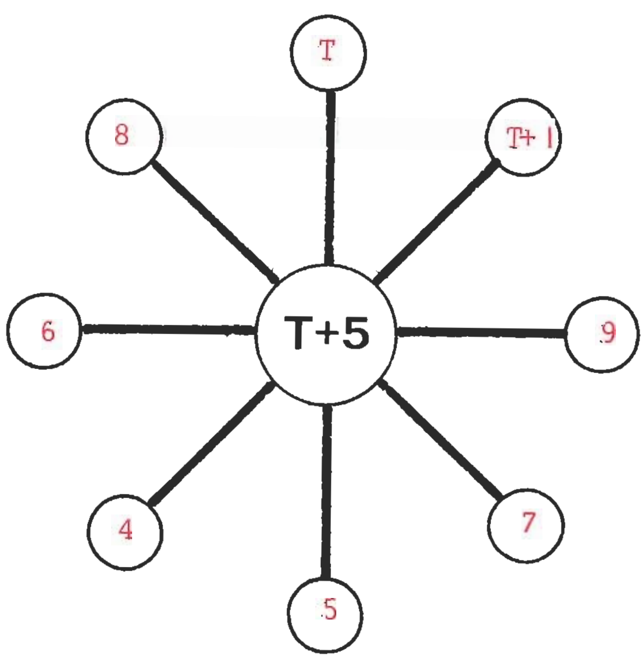
Name________________

What number belongs?

What number belongs in the empty circle?



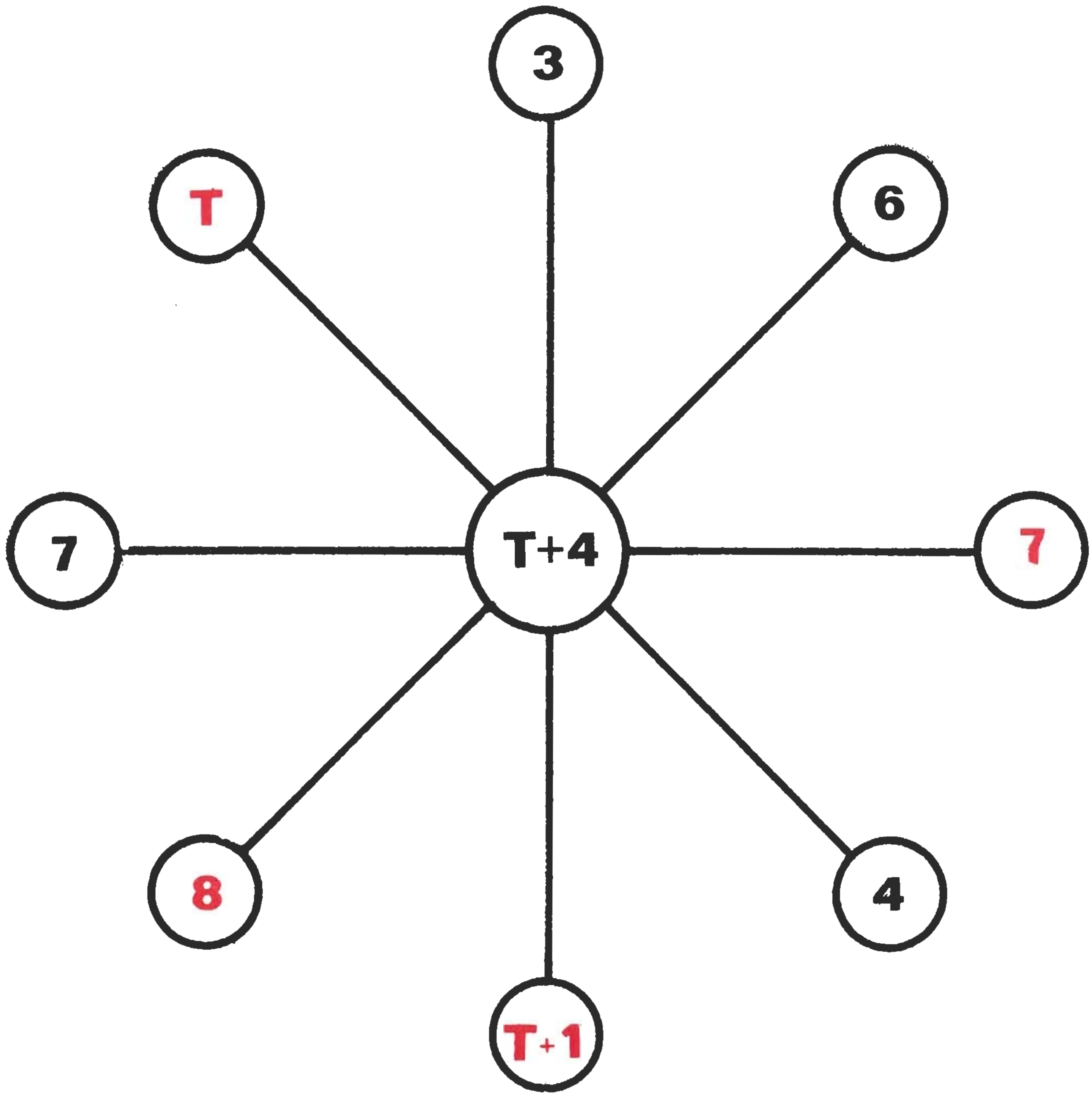
Select the numbers which <u>could</u> be placed in the circles.



Teacher's Copy

WORKSHEET 28b

Name____________



WORKSHEET 29

Teacher's Copy

Name________________

Write the secret

Write the secret by using the letters instead of the numbers.

x	a	b	y
T + 4	T	4	6

a + b = x, a - b = y

x	a	b	y
T + 2	8	4	4

a + b = x, a - b = y

x	a	b	y
T + 7	9	8	1

a + b = x, a - b = y

Now prepare a secret for the class.

WORKSHEET 30

Teacher's Copy

Name______________

Write a number sentence

Now that you know the secret write the number sentence for each.

T + 6	T + 1	5	6

T + 1 + 5 = T + 6

T + 1 - 5 = 6

2T	T + 7	3	T + 4

T + 7 + 3 = 2T

T + 7 - 3 = T + 4

2T + 3	T + 5	8	7

T + 5 + 8 = 2T + 3

T + 5 - 8 = 7

Teacher's Copy

WORKSHEET 31 Name________________

What number stands for the letter?

T + 2	9	2T + 1	3

If, a + b = x

and

a - b = y

Then: a = T + 2

b = 9

x = 2T + 1

y = 3

Choose your own letters instead of a, b, x, y and write the secret in those letters.

l	m	n	p
T	6	4	2

m + n = l, m - n = p

Number charts

Some children experience difficulty in mastering addition. A device that can be helpful and fascinating is a number chart. The following story could be used to introduce number chart activities and to arouse interest in them.

> Ethelbert said, "I have a foolproof way to add 3 to any number. I never make a mistake if I use my method."
>
> Zasu said, "I sometimes make mistakes, and I don't like to. Please, Ethelbert, show me how you do it."
>
> Ethelbert showed the chart he had made.

Put the chart on the chalkboard and have children fill in their copy of the blank 3-chart.

> Zasu said, "My, all you need to know to make this chart is how to count. You could go on and on with this chart until you were worn out or until you didn't know any more numerals. But, I still don't see how it can help you add 3."
>
> Ethelbert had to explain how it worked. "It is really easy," he said. "Suppose, to see how it works, we take a problem where we know the answer. Take 5 + 3 = _______. Find 5 on the chart. It is in the 2 column. Now, can you find where 5 + 3 is?"
>
> Zasu said, "Oh, I see that the number directly below 5 in the number 2 column is the answer. Then, if I wanted to know what (T + 9) + 3 is, I'd look in the 1 column just below T + 9. There I would find my answer which is 2T + 2."

At this point have some 3-charts to add more difficult numbers with 3. Let one child ask the question, "What is (3T + 9) + 3?" Suggest that the child who answers says something like this, "I find 3T + 9 in the 0 column. I look at the number just below. It is 4T + 2. So, 3T + 9 + 3 = 4T + 2." This activity could continue until all children have had a turn calling out a problem. See if the class can see other implications in the three table, such as going down two numbers in the same column to add 6. Let the class have an addition activity in adding 6. Be sure to call each column the number of the first numeral at the top of that column. The first column would then be the 0 column; the second column, the 1 column; and the third column, the 2 column.

> Adolph said, "Anyway, I never make mistakes in adding 3. If you could show me a foolproof way of adding 7, or 8, or even T + 2, then I would be interested."
>
> Ethelbert said, "Those are just as easy to do."

Have the class suggest an easy way to add numbers. Try to lead the class to see that a number chart for any number would supply an easy way to add that number. Teacher may say, "What do you suppose Ethelbert had done that made him think that adding seven was very easy?" We hope someone will suggest that Ethelbert had made a 7 chart. Assign the class number charts 3 through 12 to make. Help the class plan how to use a straight edge to make an attractive chart. These number charts would make a very nice display for a mathematics bulletin board.

Read the following story to the class:

Marie went to the bulletin board and examined a number chart. There she looked at the 3-charts that the other children were using to make the (x + 6)-charts. She sat down and studied her own chart awhile. She began writing and working hard. Soon her hand was waving like a flag in a strong wind. She almost shouted, "Look, what I found! I made up a new kind of table!"

Marie's table helped her a lot. It helped by showing her which column to look down to find the sum of 3 and any other number in the 3-chart. We have already noticed and discussed that. Then she noticed which column to look down to find the sum of 4 and 5. See if you can use your 3-chart to discover what Marie discovered."

Teacher's Copy

WORKSHEET 32

Name______________

A 3-chart

Finish the chart.

0	1	2
3	4	5
6	7	8
9	T	T + 1
T + 2	T + 3	T + 4
T + 5	T + 6	T + 7
T + 8	T + 9	2T
2T + 1	2T + 2	2T + 3
2T + 4	2T + 5	2T + 6
2T + 7	2T + 8	2T + 9
3T	3T + 1	3T + 2
3T + 3	3T + 4	3T + 5
3T + 6	3T + 7	3T + 8
3T + 9	4T	4T + 1
4T + 2	4T + 3	4T + 4
4T + 5	4T + 6	4T + 7
4T + 8	4T + 9	5T
5T + 1	5T + 2	5T + 3
5T + 4	5T + 5	5T + 6
5T + 7	5T + 8	5T + 9

Return to the 3-chart. Have the children use it to complete Worksheet 33. After completion of the table, stimulate discussion to indicate that the children have discovered that when we add 6 to a number, x, the sum is always located in the same column as x and that you move down two rows. You might, for example, ask, "Where in the 3-chart do we find the sum when we add 6 to number x?"

If the only answer that emerges is that the sum is in the same column as x, suggest that they check this rule to see if it works for all the numbers in the table. Yes. Ask if it would work for numbers that aren't in the table. Yes. Then continue discussion until children discover and state the other part of the answer.

Teacher's Copy

WORKSHEET 33

Name________________

Can you find X and X + 6?

Use the 3-chart to help in completing the following chart. Let x be the number to which 6 will be added.

x	x + 6	Column in which x is found	Column in which x + 6 is found
1	7	1	1
3	9	0	0
0	6	0	0
8	T + 4	2	2
2T + 1	2T + 7	0	0
4T + 3	4T + 9	1	1
5	T + 1	2	2
T + 9	2T + 5	1	1
4T	4T + 6	1	1
6T + 6	7T + 2	0	0
T + 3	T + 9	1	1
8T + 2	8T + 8	1	1
T + 7	2T + 3	2	2
3T + 5	4T + 1	2	2
5T + 2	5T + 8	1	1
8T	8T + 6	2	2

Teacher's Copy

WORKSHEET 34a

Name________________

Using the 3-chart

Let x and y be any number you want. Complete the chart below by using the 3-chart.

x	y	x + y	Column of x	Column of y	Column of x + y
3	T + 8	2T + 1	0	0	0
8	3T + 2	4T	2	2	1
T + 1	5T + 6	6T + 7	2	2	1
T + 4	2T + 9	4T + 3	2	2	1
2T + 3	4T + 4	6T + 7	2	2	1
2T + 7	T + 7	4T + 4	0	2	2
3T + 3	3T + 1	6T + 4	0	1	1
4T + 5	5T + 3	9T + 8	0	2	2
4T + 9	2T + 5	7T + 4	1	1	2
5T + 2	4T + 2	9T + 4	1	0	1
5T + 8	3T + 9	9T + 7	1	0	1

Teacher's C..y

WORKSHEET 34b

Name____________

x	y	x + y	Column of x	Column of y	Column of x + y
3	3T + 3	3T + 6	0	0	0
T + 3	5T + 7	7T	1	0	1
2T + 3	T + 9	4T + 2	2	1	0
3T + 4	T + 5	4T + 9	1	0	1
5T + 4	T	6T + 4	0	1	1
T + 9	2T + 4	4T + 3	1	0	1
4T + 1	4T + 6	8T + 7	2	1	0
3T + 6	6T + 9	TT + 5	0	0	0
5T + 2	5T + 2	TT + 4	1	1	2
T + 5	T + 8	3T + 3	0	0	0
2T + 7	3T + 1	5T + 8	0	1	1
4T + 8	4T + 5	9T + 3	0	0	0

WORKSHEET 35a

Teacher's Copy
Name____________

Marie's Table

In which column is the sum found when 3 is added to a number? When 4 is added? When 5 is added? When 6 is added? Can you predict in which column the sum will be found when 6 is added to a number? 7? 8? 9?

0 Column	$\xrightarrow{+3}$	0	Column
1 Column	$\xrightarrow{+3}$	1	Column
2 Column	$\xrightarrow{+3}$	2	Column
0 Column	$\xrightarrow{+4}$	1	Column
1 Column	$\xrightarrow{+4}$	2	Column
2 Column	$\xrightarrow{+4}$	0	Column
0 Column	$\xrightarrow{+5}$	2	Column
1 Column	$\xrightarrow{+5}$	0	Column
2 Column	$\xrightarrow{+5}$	1	Column
0 Column	$\xrightarrow{+6}$	0	Column
1 Column	$\xrightarrow{+6}$	1	Column
2 Column	$\xrightarrow{+6}$	2	Column

Teacher's Copy

WORKSHEET 35b

Name____________

Agnes looked at Marie's table and said, "I can make Marie's table shorter." Using the table from Worksheet 35a see if you can fill in Agnes' table.

0 Column + 0 Column = __0__ Column

0 Column + 1 Column = __1__ Column

0 Column + 2 Column = __2__ Column

1 Column + 0 Column = __1__ Column

1 Column + 1 Column = __2__ Column

1 Column + 2 Column = __0__ Column

2 Column + 0 Column = __2__ Column

2 Column + 1 Column = __0__ Column

2 Column + 2 Column = __1__ Column

WORKSHEET 36a

Teacher's Copy
Name________________

A 12-chart

Finish the 12-chart below. Choose any number for x and complete the table on the following page.

0	1	2	3	4	5	6	7	8	9	T	T + 1
T + 2	T + 3	T + 4	T + 5	T + 6	T + 7	T + 8	T + 9	2T	2T + 1	2T + 2	2T + 3
2T + 4	2T + 5	2T + 6	2T + 7	2T + 8	2T + 9	3T	3T + 1	3T + 2	3T + 3	3T + 4	3T + 5
3T + 6	3T + 7	3T + 8	3T + 9	4T	4T + 1	4T + 2	4T + 3	4T + 4	4T + 5	4T + 6	4T + 7
4T + 8	4T + 9	5T	5T + 1	5T + 2	5T + 3	5T + 4	5T + 5	5T + 6	5T + 7	5T + 8	5T + 9

Teacher's Copy

WORKSHEET 36b

Name__________

x	y	x + y	Column in which x is found	Column in which y is found	Column in which x + y is found
5	4	9	5	4	9
T	4	T + 4	T	4	2
2T	4	2T + 4	8	4	0
T + 5	4	T + 9	3	4	7
3T + 4	4	3T + 8	T	4	2
4T + 8	4	5T + 2	0	4	4

Look at the second chart. What do you notice about the numbers in the last three columns? See next page.

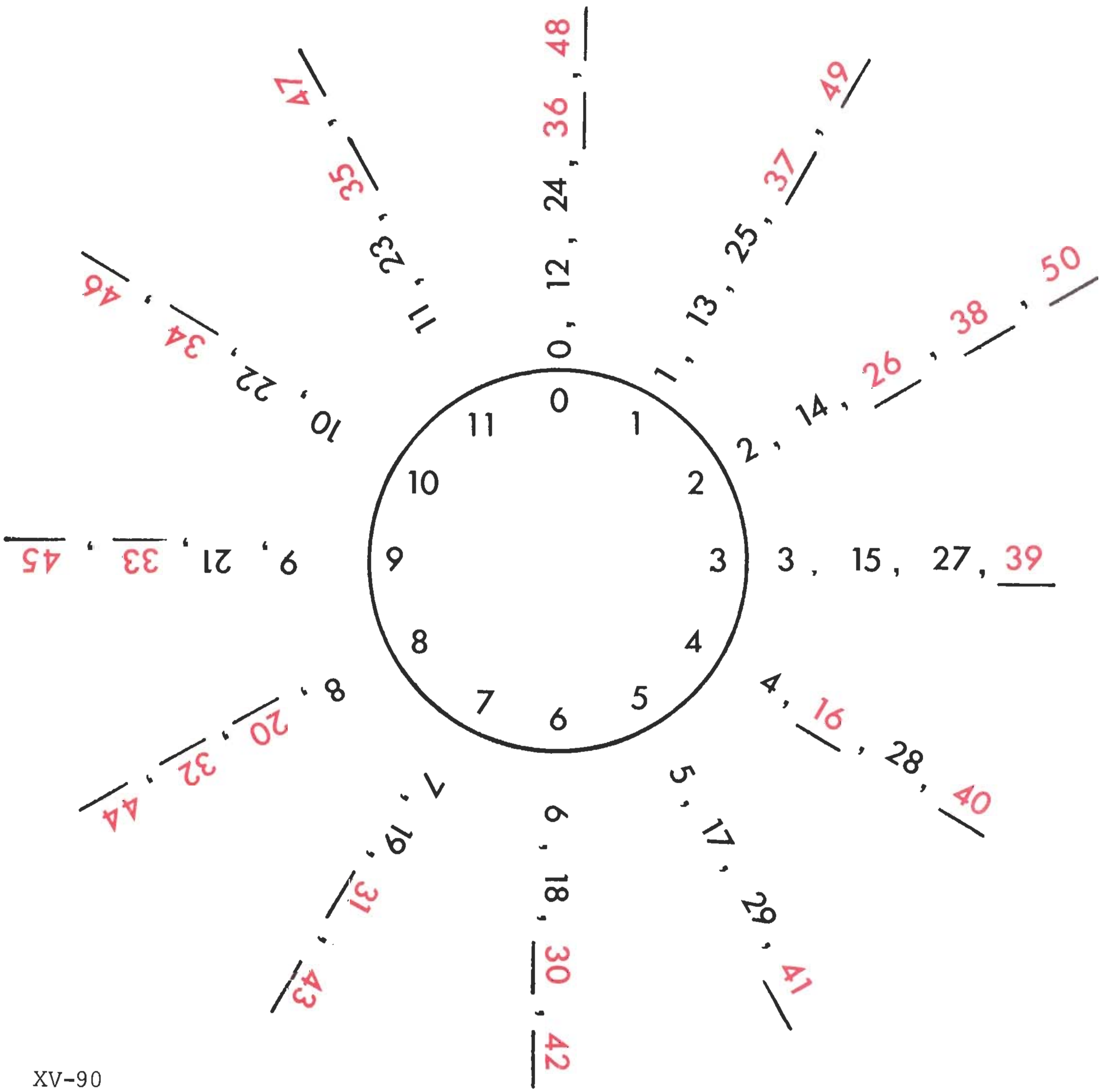
Probably the first response that will be given is that "the number in the last column is the sum of the two just before it." Have the children check to see if this rule works for all the rows. They should discover that it doesn't work in all instances. Ask them if they can figure out how to modify the rule to account for these discrepancies. Explain, if need be, that the rule works until the sum of the two is 12 or more. Then the number in the last column is the sum of these two reduced by, or minus, 12.

WORKSHEET 37

Teacher's Copy
Name______________

The clock chart

Don made a 12-chart in a different way. He was thinking of a clock. Can you finish his chart?



WORKSHEET 38a Name______________

Don's 12-Table

Don began a table for his 12-chart. The "+" in the upper left hand corner shows the operation to be done. Since the operation is addition, choose two numbers to be added, for example, 3 and 4. At the left side of the table, locate the first number, 3. The line of blank spaces directly to the right of 3 is called the 3-row. At the top of the table, locate the second number, 4. The line of blank spaces directly below 4 is called the 4-column. Find the space that is in both the 3-row and the 4-column. In this way, any space can be named by a row number and a column number.

Add the two chosen numbers and referring to Don's 12-chart find the column that contains their sum. Put this column number in the space which we found by using the 3-row and 4-column of our table. Follow these steps to add other pairs of numbers and complete the table.

Be sure to notice interesting patterns as you write.

WORKSHEET 38b

Teacher's Copy

Name________________

+	0	1	2	3	4	5	6	7	8	9	T	T + 1
0	0	1	2	3	4	5	6	7	8	9	T	T + 1
1	1	2	3	4	5	6	7	8	9	T	T + 1	0
2	2	3	4	5	6	7	8	9	T	T + 1	0	1
3	3	4	5	6	7	8	9	T	T + 1	0	1	2
4	4	5	6	7	8	9	T	T + 1	0	1	2	3
5	5	6	7	8	9	T	T + 1	0	1	2	3	4
6	6	7	8	9	T	T + 1	0	1	2	3	4	5
7	7	8	9	T	T + 1	0	1	2	3	4	5	6
8	8	9	T	T + 1	0	1	2	3	4	5	6	7
9	9	T	T + 1	0	1	2	3	4	5	6	7	8
T	T	T + 1	0	1	2	3	4	5	6	7	8	9
T + 1	T + 1	0	1	2	3	4	5	6	7	8	9	T

WORKSHEET 39a   Name_______________

Read a 7-chart

Adolph was pleased with himself. He said, "I've worked out a chart which is something like our 3-chart." (See the 7-chart below.)

Rachel answered, "The 7-chart is easy to understand. It is just like a calendar. Look!" What do you suppose Rachel meant?

This is the way Rachel made her 7-chart.

Rachel's 7-CHART

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
0	1	2	3	4	5	6
7	8	9	T	T + 1	T + 2	T + 3
T + 4	T + 5	T + 6	T + 7	T + 8	T + 9	2T
2T + 1	2T + 2	2T + 3	2T + 4	2T + 5	2T + 6	2T + 7
2T + 8	2T + 9	3T	3T + 1	3T + 2	3T + 3	3T + 4
3T + 5	3T + 6	3T + 7	3T + 8	3T + 9	4T	4T + 1
4T + 2	4T + 3	4T + 4	4T + 5	4T + 6	4T + 7	4T + 8

WORKSHEET 39b

Teacher's Copy

Name____________

Use the 7-chart, and a piece of scratch paper, if necessary, to help in completing the following chart.

Choose a number from the 0-Column	Choose a number from the 1-Column	Column in which the sum of the numbers is found
2T + 1	T + 5	1
2T + 8	2T + 9	1
7	3T + 6	1
Choose a number from the 3-Column	**Choose a number from the 6-Column**	**Column in which the sum of the numbers is found**
T + 7	3T + 4	2
3T + 8	2T	2
T	2T + 7	2
Choose a number from the 5-Column	**Choose a number from the 2-Column**	**Column in which the sum of the numbers is found**
3T + 3	9	0
T + 2	3T	0
T + 9	T + 6	0

Teacher's Copy

WORKSHEET 39c

Name_____________

Use your chart and a sheet of scratch paper to determine the following questions:

a. What column will the sum be in when you add a number from the 0-Column and a number from the 1-Column? 1 column

b. How does the column for the sum change when you add a number from Column 1 to another column?

The number of the column becomes one larger.

c. Can you predict which column the sum of two numbers will be in? Fill out the chart below:

A number from Column			A number from Column			A number from Column
	0	+		2	=	2
	4	+		2	=	6
	5	+		2	=	0
	4	+		4	=	1
	6	+		0	=	6
	0	+		1	=	1
	1	+		1	=	2

Teacher's Copy

WORKSHEET 39d

Name______________

A number from Column	2	+	A number from Column	1	=	A number from Column 3
	3	+		1	=	4
	4	+		1	=	5
	5	+		1	=	6
	6	+		1	=	0

Sample Charts for Teachers

4- CHART

0	1	2	3
4	5	6	7
8	9	T	T + 1
T + 2	T + 3	T + 4	T + 5
T + 6	T + 7	T + 8	T + 9
2T	2T + 1	2T + 2	2T + 3
2T + 4	2T + 5	2T + 6	2T + 7
2T + 8	2T + 9	3T	3T + 1
3T + 2	3T + 3	3T + 4	3T + 5
3T + 6	3T + 7	3T + 8	3T + 9
4T	4T + 1	4T + 2	4T + 3
4T + 4	4T + 5	4T + 6	4T + 7
4T + 8	4T + 9	5T	5T + 1
5T + 2	5T + 3	5T + 4	5T + 5
5T + 6	5T + 7	5T + 8	5T + 9
6T	6T + 1	6T + 2	6T + 3
6T + 4	6T + 5	6T + 6	6T + 7
6T + 8	6T + 9	7T	7T + 1
7T + 2	7T + 3	7T + 4	7T + 5
7T + 6	7T + 7	7T + 8,	7T + 9
8T	8T + 1	8T + 2	8T + 3
8T +4	8T +5	8T + 6	8T + 7
8T + 8	8T + 9	9T	9T + 1
9T + 2	9T + 3	9T + 4	9T + 5

Sample Charts for Teachers

6-CHART

0	1	2	3	4	5
6	7	8	9	T	T+ 1
T+ 2	T+ 3	T+ 4	T+ 5	T+ 6	T+ 7
T+ 8	T+ 9	2T	2T+ 1	2T+ 2	2T+ 3
2T+ 4	2T+ 5	2T+ 6	2T+ 7	2T+ 8	2T+ 9
3T	3T+ 1	3T+ 2	3T+ 3	3T+ 4	3T+ 5
3T+ 6	3T+ 7	3T+ 8	3T+ 9	4T	4T+ 1
4T+ 2	4T+ 3	4T+ 4	4T+ 5	4T+ 6	4T+ 7
4T+ 8	4T+ 9	5T	5T+ 1	5T+ 2	5T+ 3

8-CHART

0	1	2	3	4	5	6	7
8	9	T	T+ 1	T+ 2	T+ 3	T+ 4	T+ 5
T+ 6	T+ 7	T+ 8	T+ 9	2T	2T+ 1	2T+ 2	2T+ 3
2T+ 4	2T+ 5	2T+ 6	2T+ 7	2T+ 8	2T+ 9	3T	3T+ 1
3T+ 2	3T+ 3	3T+ 4	3T+ 5	3T+ 6	3T+ 7	3T+ 8	3T+ 9
4T	4T+ 1	4T+2	4T+ 3	4T+ 4	4T+ 5	4T+ 6	4T+ 7
4T+ 8	4T+ 9	5T	5T+ 1	5T+ 2	5T+ 3	5T+ 4	5T+ 5
5T+ 6	5T+ 7	5T+ 8	5T+ 9	6T	6T+ 1	6T+ 2	6T+ 3
6T+ 4	6T+ 5	6T+ 6	6T+ 7	6T+ 8	6T+ 9	7T	7T+ 1
7T+ 2	7T+ 3	7T+ 4	7T+ 5	7T+ 6	7T+ 7	7T+ 8	7T+ 9
8T	8T+ 1	8T+ 2	8T+ 3	8T+ 4	8T+ 5	8T+ 6	8T+ 7
8T+ 8	8T+ 9	9T	9T+ 1	9T+ 2	9T+ 3	9T+ 4	9T+ 5

Numbers and their squares

Materials

oak tag

Draw "square" number charts as indicated below. These charts are called "square" because they have the same number of rows and columns. The construction of these squares differs from the earlier charts in that here both of the rows and columns are labeled with numerals, the 1st row and the 1st column with numeral 1; the second row and the second column with numeral 2, etc. The difference that is important to the child is that the entries of the first row begin with 1 instead of 0. Do not fill in the squares until class work is begun.

Procedure

When we studied the numeration unit, we made 2-charts, 3-charts, and 5-charts. We saw many relationships and patterns that numbers made. Today we are going to make a 1-square, 2-square, 3-square, and 4-square. Look for a different relationship from those we talked about before. As you help me make these number squares for the classroom, keep this problem in mind.

1-Square

	1
1	1

2-Square

	1	2
1	1	2
2	3	4

3-Square

	1	2	3
1	1	2	3
2	4	5	6
3	7	8	9

4-Square

	1	2	3	4
1	1	2	3	4
2	5	6	7	8
3	9	T	T+1	T+2
4	T+3	T+4	T+5	T+6

Does anyone see a pattern? Look at the 4-chart. Find column 4 and row 4.

Column ↓

Row →

	1	2	3	4
1	1	2	3	4
2	5	6	7	8
3	9	T	T+1	T+2
4	T+3	T+4	T+5	T+6

What numeral is in the box which is located by a move down in column 4 to row 4? Notice that this box is in both the last row and the last column of the 4-Square. In the same way, for each square we will ask which number is found in the box which is in both the last row and last column.

No. of Square	Answer
1	1
2	4
3	9
4	T + 6
5	2T + 5
6	3T + 6
7	4T + 9
8	6T + 4
9	8T + 1
T	TT

This table may take several days to complete.

Note: The children are not being asked to learn multiplication facts even though the charts look like it. Be very careful not to inject this into the discussion.

Have the children make a booklet of their own squares and their own table.

WORKSHEET 40

Teacher's Copy
Name________________

Making a Booklet on Squares

Complete these charts.

1-SQUARE

	1
1	1

2-SQUARE

	1	2
1	1	2
2	3	4

3-SQUARE

	1	2	3
1	1	2	3
2	4	5	6
3	7	8	9

MY RECORD

NUMBER OF SQUARE	ANSWER
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
T	100

WORKSHEET 41

Teacher's Copy
Name__________

Fill in the square and record its answer.

4-SQUARE

	1	2	3	4
1	1	2	3	4
2	5	6	7	8
3	9	T	T + 1	T + 2
4	T+ 3	T+ 4	T + 5	T + 6

WORKSHEET 42

Teacher's Copy

Name________________

Complete this square and record its answer.

5-SQUARE

	1	2	3	4	5
1	1	2	3	4	5
2	6	7	8	9	T
3	T + 1	T + 2	T + 3	T + 4	T + 5
4	T + 6	T + 7	T + 8	T + 9	2T
5	2T + 1	2T + 2	2T + 3	2T + 4	2T + 5

WORKSHEET 43

Teacher's Copy
Name____________

Fill in the square and record its answer.

6-SQUARE

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	7	8	9	T	T + 1	T + 2
3	T + 3	T + 4	T + 5	T + 6	T + 7	T + 8
4	T + 9	2T	2T + 1	2T + 2	2T + 3	2T + 4
5	2T + 5	2T + 6	2T + 7	2T + 8	2T + 9	3T
6	3T + 1	3T + 2	3T + 3	3T + 4	3T + 5	3T + 6

WORKSHEET 44

Teacher's Copy
Name____________________

7-SQUARE

Fill in the square and record its answer.

	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	8	9	T	T + 1	T + 2	T + 3	T + 4
3	T + 5	T + 6	T + 7	T + 8	T + 9	2T	2T + 1
4	2T + 2	2T + 3	2T + 4	2T + 5	2T + 6	2T + 7	2T + 8
5	2T + 9	3T	3T + 1	3T + 2	3T + 3	3T + 4	3T + 5
6	3T + 6	3T + 7	3T + 8	3T + 9	4T	4T + 1	4T + 2
7	4T + 3	4T + 4	4T + 5	4T + 6	4T + 7	4T + 8	4T + 9

Teacher's Copy
Name________________

WORKSHEET 45

Fill in the square and record its answer.

8-SQUARE

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	9	T	T + 1	T + 2	T + 3	T + 4	T + 5	T + 6
3	T + 7	T + 8	T + 9	2T	2T + 1	2T + 2	2T + 3	2T + 4
4	2T + 5	2T + 6	2T + 7	2T + 8	2T + 9	3T	3T + 1	3T + 2
5	3T + 3	3T + 4	3T + 5	3T + 6	3T + 7	3T + 8	3T + 9	4T
6	4T + 1	4T + 2	4T + 3	4T + 4	4T + 5	4T + 6	4T + 7	4T + 8
7	4T + 9	5T	5T + 1	5T + 2	5T + 3	5T + 4	5T + 5	5T + 6
8	5T + 7	5T + 8	5T + 9	6T	6T + 1	6T + 2	6T + 3	6T + 4

Teacher's Copy
Name________________

WORKSHEET 46

Fill in the square and record its answer.

9-SQUARE

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	T	T + 1	T + 2	T + 3	T + 4	T + 5	T + 6	T + 7	T + 8
3	T + 9	2T	2T + 1	2T + 2	2T + 3	2T + 4	2T + 5	2T + 6	2T + 7
4	2T + 8	2T + 9	3T	3T + 1	3T + 2	3T + 3	3T + 4	3T + 5	3T + 6
5	3T + 7	3T + 8	3T + 9	4T	4T + 1	4T + 2	4T + 3	4T + 4	4T + 5
6	4T + 6	4T + 7	4T + 8	4T + 9	5T	5T + 1	5T + 2	5T + 3	5T + 4
7	5T + 5	5T + 6	5T + 7	5T + 8	5T + 9	6T	6T + 1	6T + 2	6T + 3
8	6T + 4	6T + 5	6T + 6	6T + 7	6T + 8	6T + 9	7T	7T + 1	7T + 2
9	7T + 3	7T + 4	7T + 5	7T + 6	7T + 7	7T + 8	7T + 9	8T	8T + 1

WORKSHEET 47

Teacher's Copy
Name________________

Fill in the square and record the answer.

THE T-SQUARE

	1	2	3	4	5	6	7	8	9	T
1	1	2	3	4	5	6	7	8	9	T
2	T + 1	T + 2	T + 3	T + 4	T + 5	T + 6	T + 7	T + 8	T + 9	2T
3	2T + 1	2T + 2	2T + 3	2T + 4	2T + 5	2T + 6	2T + 7	2T + 8	2T + 9	3T
4	3T + 1	3T + 2	3T + 3	3T + 4	3T + 5	3T + 6	3T + 7	3T + 8	3T + 9	4T
5	4T + 1	4T + 2	4T + 3	4T + 4	4T + 5	4T + 6	4T + 7	4T + 8	4T + 9	5T
6	5T + 1	5T + 2	5T + 3	5T + 4	5T + 5	5T + 6	5T + 7	5T + 8	5T + 9	6T
7	6T + 1	6T + 2	6T + 3	6T + 4	6T + 5	6T + 6	6T + 7	6T + 8	6T + 9	7T
8	7T + 1	7T + 2	7T + 3	7T + 4	7T + 5	7T + 6	7T + 7	7T + 8	7T + 9	8T
9	8T + 1	8T + 2	8T + 3	8T + 4	8T + 5	8T + 6	8T + 7	8T + 8	8T + 9	9T
T	9T + 1	9T + 2	9T + 3	9T + 4	9T + 5	9T + 6	9T + 7	9T + 8	9T + 9	TT

Patterns in charts and squares

Have the children recall that on the 2-chart in Unit XI, Numeration, we discovered that:

1	2
3	4
5	6
7	8
9	T
T+1	T+2
T+3	T+4
T+5	T+6

(1) the sum of any two numbers in the right-hand column will be found in the right-hand column (6 + T = T + 6);

(2) the sum of two numbers in the left-hand column will be found in the right-hand column (5 + 7 = T + 2);

(3) the sum of a number in the left-hand column and a number in the right-hand column will be in the left-hand column (2 + 3 = 5).

<u>Then proceed to inspect the chart containing the answers of all the squares.</u> Are there any pairs of numbers in the right-hand column whose sum is in the right-hand column?

<u>Example</u>

9 + (T + 6) = 2T + 5

(3T + 6) + (6T + 4) = 9T + T

Now we can look further. What rows are these numbers in?

(3)		(4)		(5)
9	+	T + 6	=	2T + 5
(6)		(8)		(T)
3T + 6	+	6T + 4	=	TT

Decomposition of integers into special addends

No. of Square	Answer
1	1
2	4
3	9
4	T+6
5	2T+5
6	3T+6
7	4T+9
8	6T+4
9	8T+1
T	TT

Decomposition of integers into special addends provides a review of addition and extends the understanding of the composition of sums. It is especially important in that it is fundamentally very similar to part of the process of decomposing an integer in order to write it in a different base.

Notice, as the integers are decomposed, only the numbers from the answer column of the record chart on this page can be used. Each integer to be decomposed is written in a circle to the left of the chart. Space has been provided for answers up to 6 decompositions.

Questions to consider:

1. How many addends did it take to make 9? the most? the least? (Do this with other numbers.)

2. Is there any pattern? Could you tell ahead how many addends there will be?

WORKSHEET 48a

Name________________

(1)	1
(2)	1 + 1
(3)	1 + 1 + 1
(4)	1 + 1 + 1 + 1 4
(5)	1 + 1 + 1 + 1 + 1 4 + 1
(6)	1 + 1 + 1 + 1 + 1 + 1 4 + 1 + 1
(7)	1 + 1 + 1 + 1 + 1 + 1 + 1 4 + 1 + 1 + 1

WORKSHEET 48b

Teacher's Copy
Name______________

(8)

1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
4 + 1 + 1 + 1 + 1
4 + 4

(9)

1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
4 + 1 + 1 + 1 + 1 + 1
4 + 4 + 1
9

(T)

1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
4 + 1 + 1 + 1 + 1 + 1 + 1
4 + 4 + 1 + 1
9 + 1

Teacher's Copy

WORKSHEET 48c

Name________________

T+1

1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
4 + 1 + 1 + 1 + 1 + 1 + 1 + 1
4 + 4 + 1 + 1 + 1
9 + 1 + 1

T+2

1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
4 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
4 + 4 + 1 + 1 + 1 + 1
4 + 4 + 4
9 + 1 + 1 + 1

T+3

1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
4 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
4 + 4 + 1 + 1 + 1 + 1 + 1
4 + 4 + 4 + 1
9 + 1 + 1 + 1 + 1
9 + 4

Teacher's Copy

WORKSHEET 48d

Name_______________

T+4

1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
4 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
4 + 4 + 1 + 1 + 1 + 1 + 1 + 1
4 + 4 + 4 + 1 + 1
9 + 1 + 1 + 1 + 1 + 1
9 + 4 + 1

T+5

1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
4 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
4 + 4 + 1 + 1 + 1 + 1 + 1 + 1 + 1
4 + 4 + 4 + 1 + 1 + 1
9 + 1 + 1 + 1 + 1 + 1 + 1
9 + 4 + 1 + 1

WORKSHEET 49

Teacher's Copy
Name________

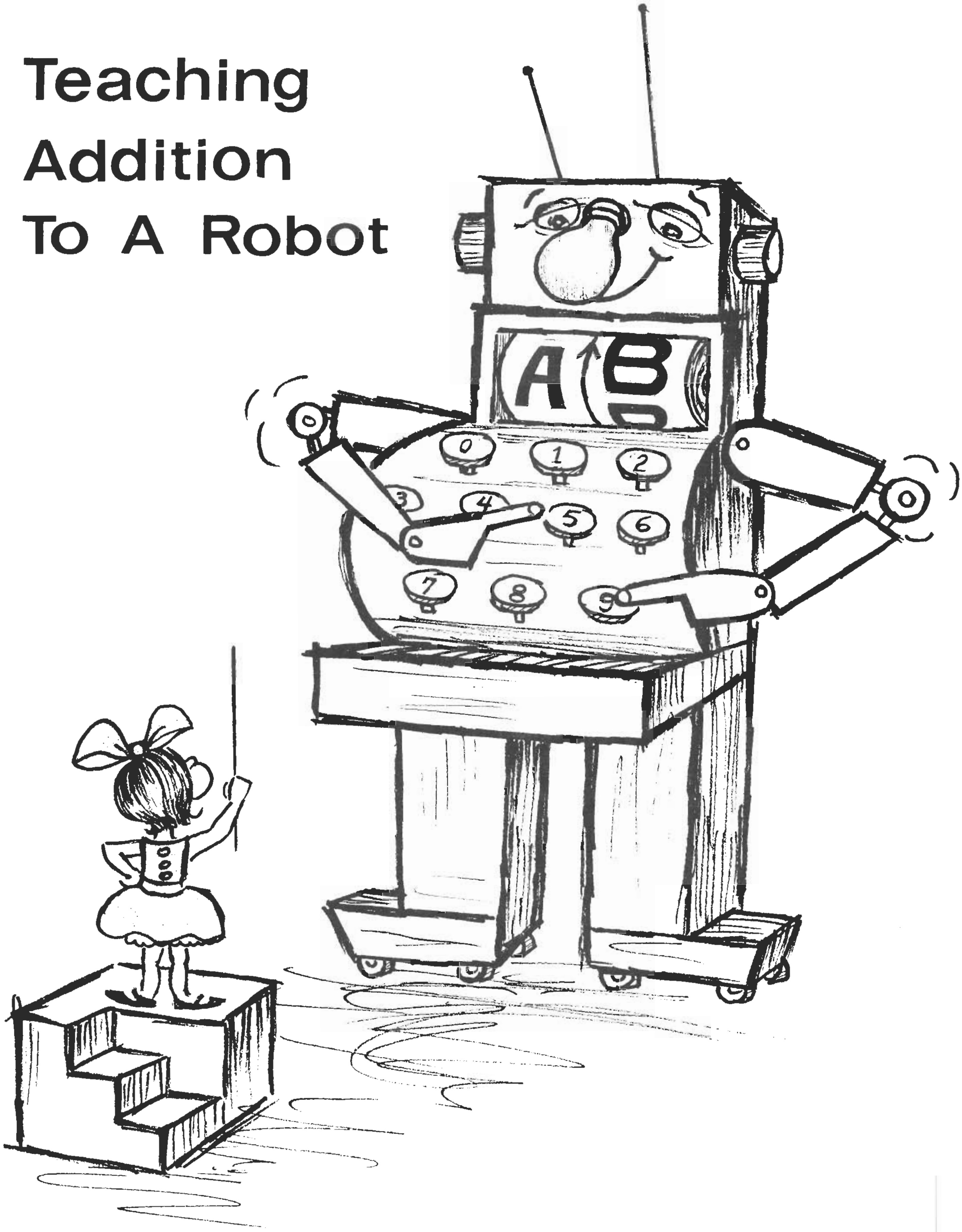
Break Through the Barrier

These integers are sums.

Use numbers from the answer column of your record chart on Worksheet 40. Write the least number of addends for each sum.

(7T + 2)	(6T + 4) + 4 + 4
(3T + 7)	(3T + 6) + 1
(5T + 5)	(4T + 9) + 4 + 1 + 1
(8T + 7)	(8T + 1) + 4 + 1 + 1
(9T + 3)	(8T + 1) + 9 + 1 + 1 + 1
(6T + 9)	(6T + 4) + 4 + 1

Teaching Addition To A Robot



TEACHING ADDITION TO A ROBOT

A computing machine is a giant mechanical slave. It does exactly what you tell it to do, neither more nor less. It does it very fast and it never gets tired. It has no brains, so if you make a mistake in telling it what to do, it will follow your instructions blindly.

The machine has a register, like the top of a cash register, to hold the number it is working on. This register is called the <u>accumulator</u>. If you are using only numbers from 0 to 9T + 9, you need two places in the accumulator:

A	B

To hold 2T + 5, you simply register 2 at A, and 5 at B:

2	5	(2T + 5)

To hold 3T, you register 3 at A. What should you register at B? _______ Of course, you put 0 there, since 3T + 0 = 3T. Adding zero to a number does not change the number.

3	0	(3T or 3T + 0)

TEACHING ADDITION TO A ROBOT

We can easily build into the robot's circuits the part of our addition table that has sums less than T in it.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	
2	2	3	4	5	6	7	8	9		
3	3	4	5	6	7	8	9			
4	4	5	6	7	8	9				
5	5	6	7	8	9					
6	6	7	8	9						
7	7	8	9							
8	8	9								
9	9									

To tell the machine how to add 3 to the number in the accumulator is easy if the number at B is less than 7. Give the following instructions:

a. Add 3 to the number at B. The machine already knows how to do this since the sum will be less than T. (See table above.)

b. Do not change the number at A.

Teacher's Copy

2	5	+3	becomes	2	8
4	3	+3	becomes	4	6
8	1	+3	becomes	8	4
7	6	+3	becomes	7	9
9	2	+3	becomes	9	5
5	4	+3	becomes	5	7
3	0	+3	becomes	3	3

If the number at B is 7, your instructions to the machine are:

a. Change the number at B to 0.

b. Add 1 to the number at A.

For example: | 4 | 7 | +3

$$(4T + 7) + 3 = 4T + (7 + 3)$$
$$= 4T + T$$
$$= 5T$$
$$= 5T + 0$$

or $47 + 3 = 50$ | 5 | 0 |

Teacher's Copy

Write out instructions for adding 3 to the number in the accumulator if the number at B is 8.

a. Change the number at B to 1.

b. Add 1 to the number at A.

Give an example:

6	8

+3

$(6T + 8) + 3 = \underline{6}\,T + (8 + 3)$

$= \underline{6}\,T + (T + 1)$

$= (\underline{6}\,T + T) + 1$

$= \underline{7}\,T + 1$

or $\underline{68} + 3 = \underline{71}$.

7	1

1	8	+3	becomes	2	1
4	8	+3	becomes	5	1
8	8	+3	becomes	9	1
5	8	+3	becomes	6	1
2	8	+3	becomes	3	1

Teacher's Copy

Write out instructions for adding 3 to the number in the accumulator if the number at B is 9.

a. Change the number at B to 2.

b. Add 1 to the number at A.

Give an example:

4	9

+3

$(\underline{4}\,T + 9) + 3 = \underline{4}\,T + (9 + 3)$

$= \underline{4}\,T + (T + 2)$

$= (\underline{4}\,T + T) + 2$

$= \underline{5}\,T + 2$

or $\underline{49} + 3 = \underline{52}$.

5	2

6	9	+3	becomes	7	2
3	9	+3	becomes	4	2
8	9	+3	becomes	9	2
5	9	+3	becomes	6	2

To add 2T + 3 to the number in the accumulator, if the

Teacher's Copy

number at B is less than 7, you give the instructions:

a. Add three to the number at B.

b. Add 2 to the number at A.

For example: | 4 | 5 | + (2T + 3)

$$(4T + 5) + (2T + 3) = (4T + 2T) + (5 + 3)$$

$$= 6T + 8$$

or 45 + 23 = 68 | 6 | 8 |

Complete the following:

3	6	+	(2T + 3)	becomes	5	9
6	2	+	(2T + 3)	becomes	8	5
7	1	+	(2T + 3)	becomes	9	4
5	4	+	(2T + 3)	becomes	7	7
1	3	+	(2T + 3)	becomes	3	6

If the number at B is 7, you instructions are:

a. Change the 7 at B to 0.

b. Add __3__ to the number at A.

For example: | 6 | 7 | + (2T + 3)

$$(6T + 7) + (2T + 3) = (6T + 2T) + (7 + 3)$$
$$= (6T + 2T) + (T)$$
$$= 6T + 2T + T$$
$$= 9T$$
$$= 9T + 0$$

or 67 + 23 = 90 | 9 | 0 |

Complete the following:

3	7	+	(2T + 3)	becomes	6	0
1	7	+	(2T + 3)	becomes	4	0
5	7	+	(2T + 3)	becomes	8	0
0	7	+	(2T + 3)	becomes	3	0
2	7	+	(2T + 3)	becomes	5	0

Of course, if your sum is greater than 9T + 9, then you don't have room in your accumulator to register it. We shall discuss later units how to deal with larger numbers.

Most people write numbers in the short way we have used on the machine:

69 for 6T + 9

50 for 5T + 0, etc.

Notice that in everyday life we do not write "0" in "0T + 7." We simply write, "7," whereas our brainless machine writes

0	7

If you want to read what other people write, or want to tell other people what you have done, you must use the same system of numerals as they do.

Our notation helps you understand better what you are doing. For example, you see right away that

"6T + 9" means 6 tens and 9

"5T" means 5 tens, etc.

Whenever you have trouble in understanding arithmetic, it will be easier if you write numbers in our private notation.

You should practice your arithmetic in the ordinary notation, too, so that you will be able to communicate with others better.

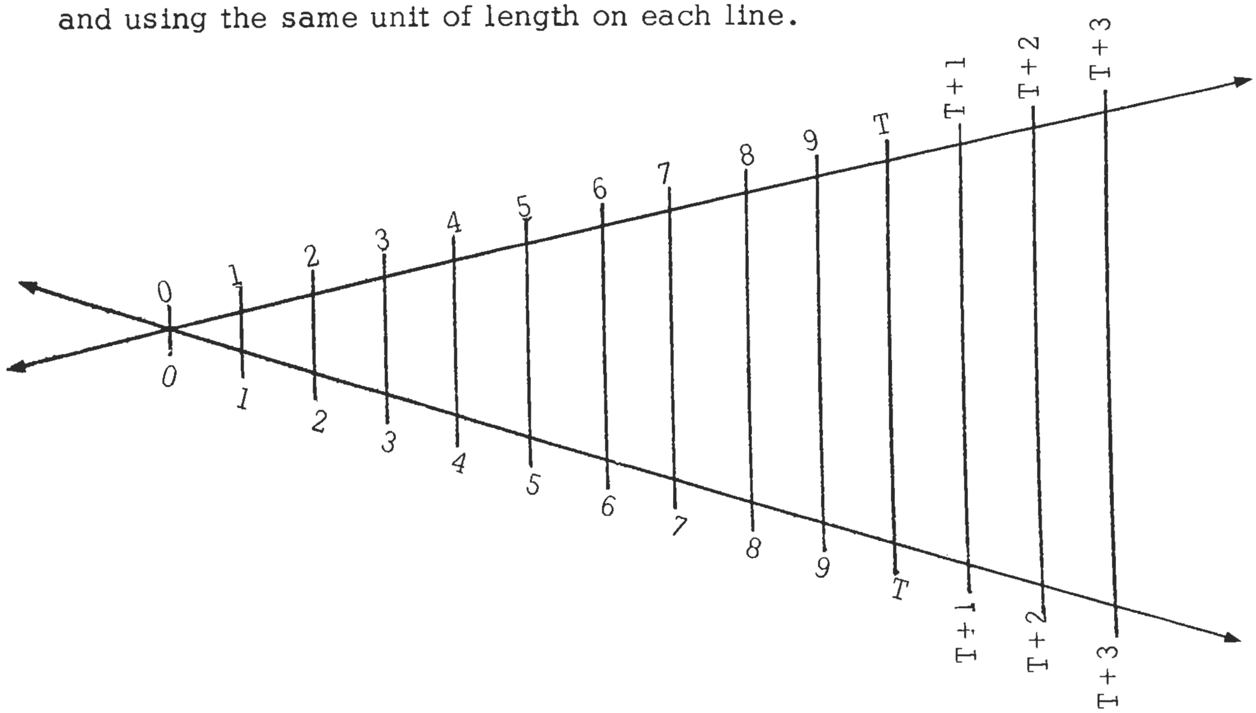
Scales on intersecting and parallel lines

It is suggested that the children not be kept working on the following exercises too long at a time. As soon as they are able to work independently, go on to the next experiment. Provide Minnemast rulers and the plastic straight edge.

Perhaps time (and a place on a bulletin board) could be set aside each week for several weeks to allow children a chance to report orally on their investigations. Their work might be exhibited on the bulletin board as they explain their discoveries.

Experiment 1

Ask the children to draw two intersecting lines on a sheet of plain paper. (Teacher may demonstrate on the chalkboard or the peg board.) Make scales on both lines using the point of intersection as 0 on each scale and using the same unit of length on each line.



Give the following directions to the children: (Do not demonstrate for them.)

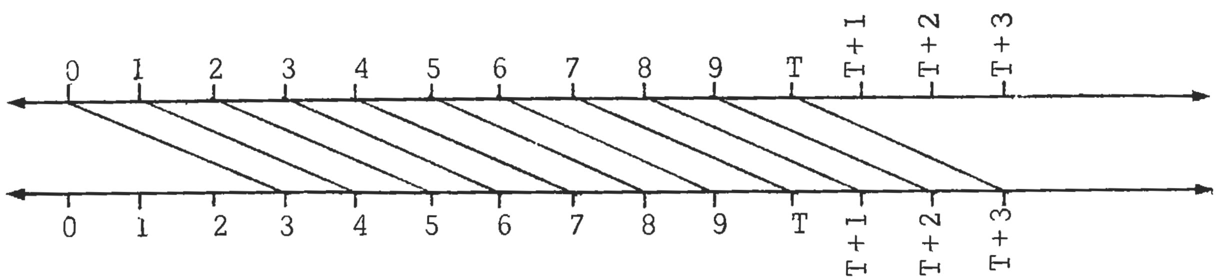
a. Join the 1-points on the two scales by a line.

b. Draw the line joining the "2-points" on the two scales; the line joining the "3-points", etc.

c. Where does the line joining the 1-points meet the line joining the two 2-points? (Where do any two lines in this family meet?) Lines may be extended as far as size of paper permits.

Experiment 3.

a. Make scales on two parallel lines as in Experiment 2.



b. Draw a line joining the 0-point of one scale to the 3-point of the other scale.

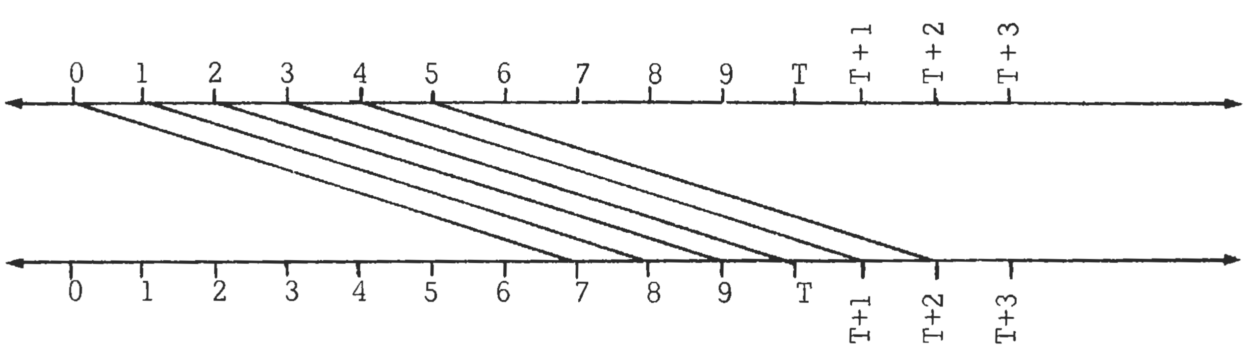
c. Draw a line joining the 1-point of one scale to the 4-point of the other scale; a line joining the 2-point with the 5-point, and so on.

d. What do you notice about the intersection of the lines of this family? <u>There is no intersection.</u>

<u>Note</u>: A family of parallel lines is merely a set of parallel lines of the scales which are its foundation.

Experiment 4.

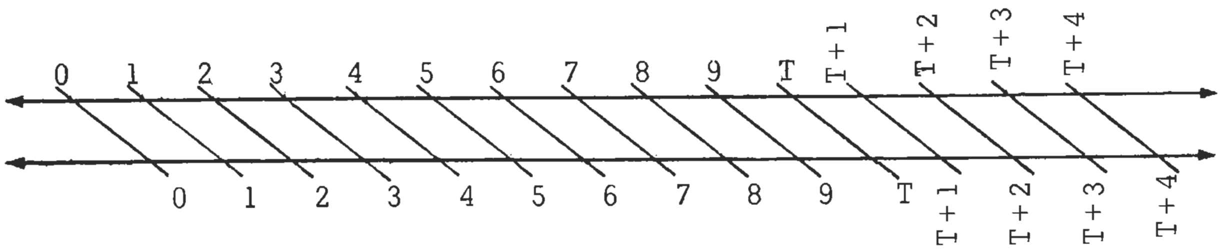
Proceed as in Experiment 3. Perhaps children might do this independently.



Experiment 2.

In this experiment scales will be drawn on parallel lines. Parallel lines on a flat sheet of paper are lines which do <u>not</u> intersect. Remind the children that a line extends on and on in both of its directions. When we say that parallel lines do not intersect, we mean that they do not meet no matter how far they are extended. Two horizontal lines on a page of notebook paper are parallel.

a. Choose a zero point on each line. Make scales on each line, using the same unit of length on both. Make both scales go in the same direction.



Note: 1. Let children select a unit of length.

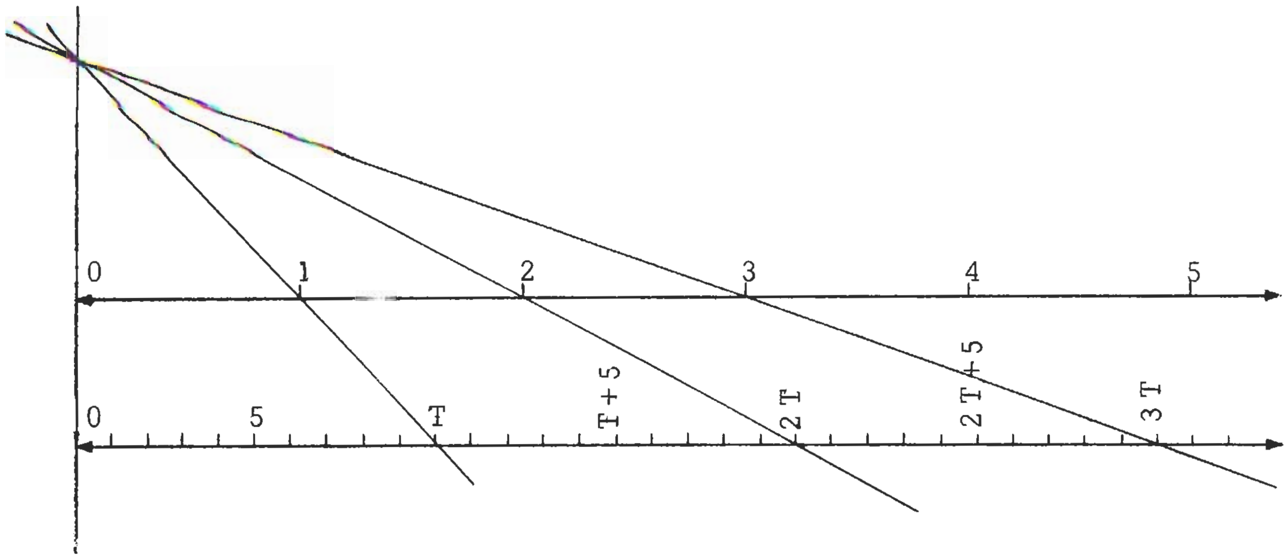
2. Let them label two 0 points anywhere they wish.

3. Papers should differ.

b. Complete as in Experiment 1. Draw the line joining the two 0-points; the two 1-points; the two 2-points, etc.

c. Where does the line joining the two 0-points cross the line joining the two 1-points? Where do any two lines of this family intersect?

Experiment 5.



a. This experiment should be done on a large sheet of paper or on the chalkboard.

b. Draw two parallel lines.

c. Make the scale on one line much smaller than the scale on the other line.

d. Draw the line joining the two 0-points.

e. Draw the line joining the 1-point on the first line to the T-point on the second line.

f. Draw the line joining the 2-point on the first line to the 2T-point on the second line.

g. Continue as above.

h. Where do the first two lines of this family intersect?

i. Where do the second and third lines of this family intersect?

j. What is the general pattern? Can you predict where the line joining the 8-point of the first line to the 8T-point of the second meets the line joining the two 0-points?

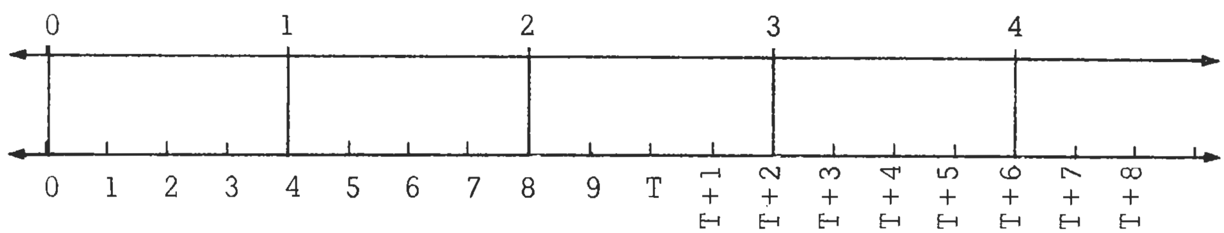
Experiment 6.

Try a similar experiment joining:

the first line		second line
0-point	to	0-point
1-point	to	4-point
2-point	to	8-point
3-point	to	(T + 2)-point

Continue in this manner. Do the results of this experiment resemble the results of Experiment 5.

The child obtains a family of concurrent lines (lines through one point). By accident a child may choose the units on the two lines in such a way as to obtain a family of parallel lines.



If this doesn't occur in your class, do not raise the question at this time. If it does occur in your class, ask the children what would happen if the length of the unit on the second line were changed. Is there any other choice of unit which would also give a family of parallel lines? Do not dwell on this point too long.

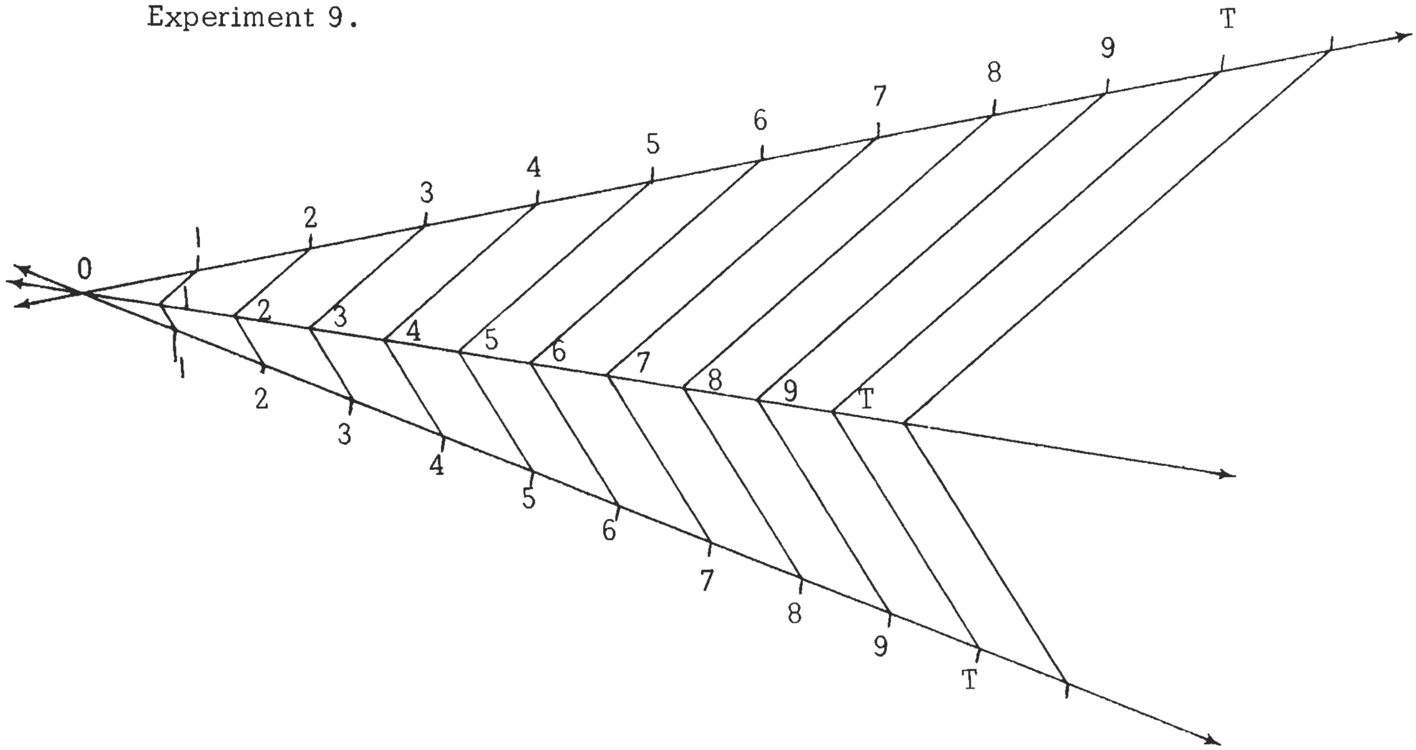
Experiment 7.

If you do a similar experiment joining the 1-point of the first scale to the 3-point of the second, how would you pair up the other points?

Experiment 8.

How would this experiment work if the scales were made on intersecting lines using the intersection point as the 0-point on both scales?

Experiment 9.



In this experiment we have scales on three lines which all intersect. The 0-point of each scale is the point of intersection. Begin with the bottom two scales. Join the 1-points, the 2-points, the 3-points, and so on of these two scales. Do any of the lines in this family meet? This tells us that these form a family of parallel lines.

Then do the same for the two top scales in the diagram. Join the 1-points, the 2-points, the 3-points, and so on. Do any of these lines meet? So we can call them a family of parallel lines.

Now draw the lines joining the 1-points, 2-points, 3-points, and so on of the bottom and top scales. Can we say anything about this family?

We can see that in this diagram there are two ways of going from the bottom scale to the top scale. If we start at the 2-point on the bottom scale, we can go to the 2-point of the middle scale and then to the 2-point of the top scale. Or we can go from the 2-point of the bottom scale directly to the 2-point of the top scale. Does this work for any point on the bottom scale?

Experiment 10.

Follow the instructions from Experiment 9 using these 3 parallel lines.