Computation and Structure

Nuffield Mathematics Teaching Project

First draft not for publication

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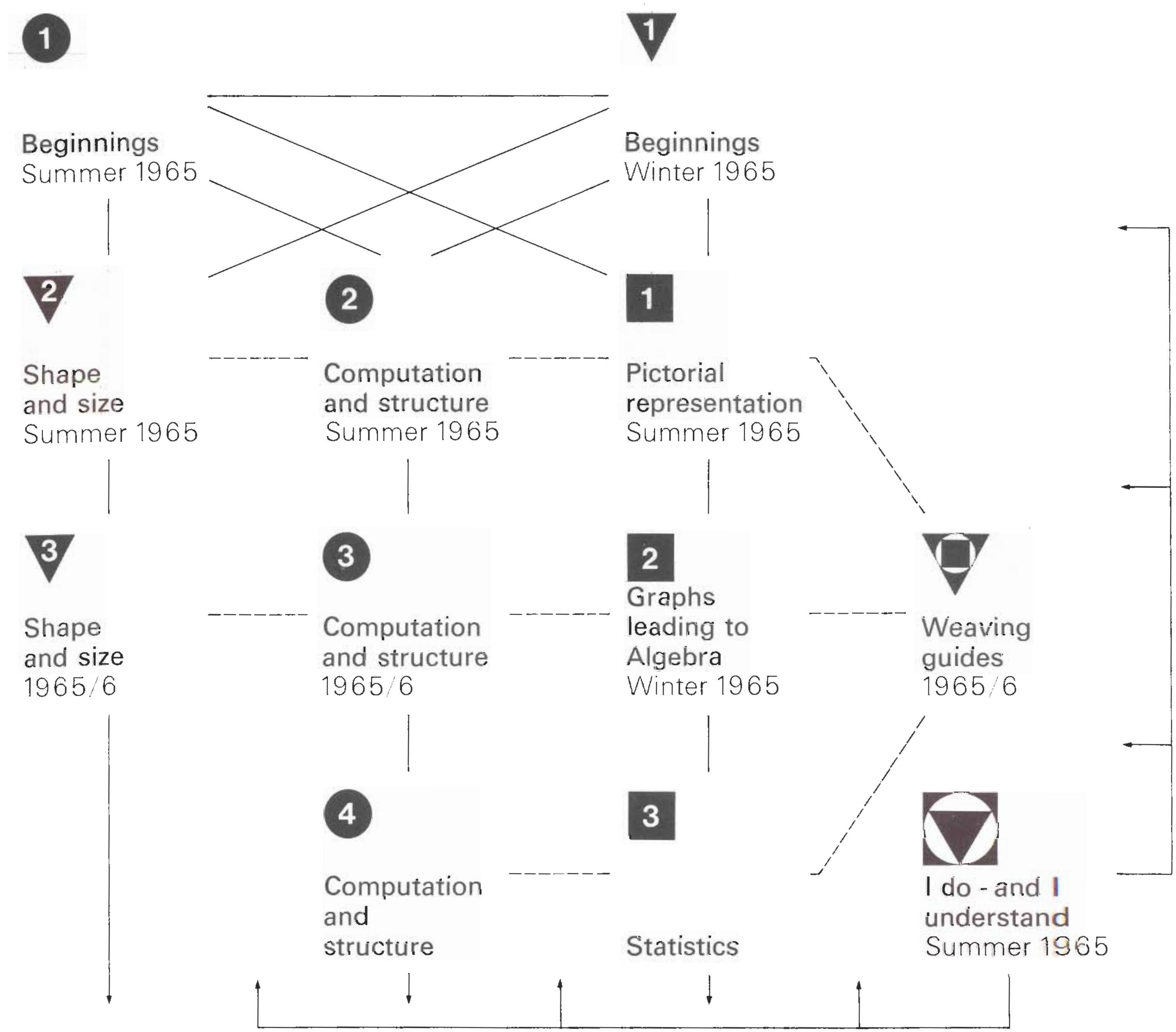
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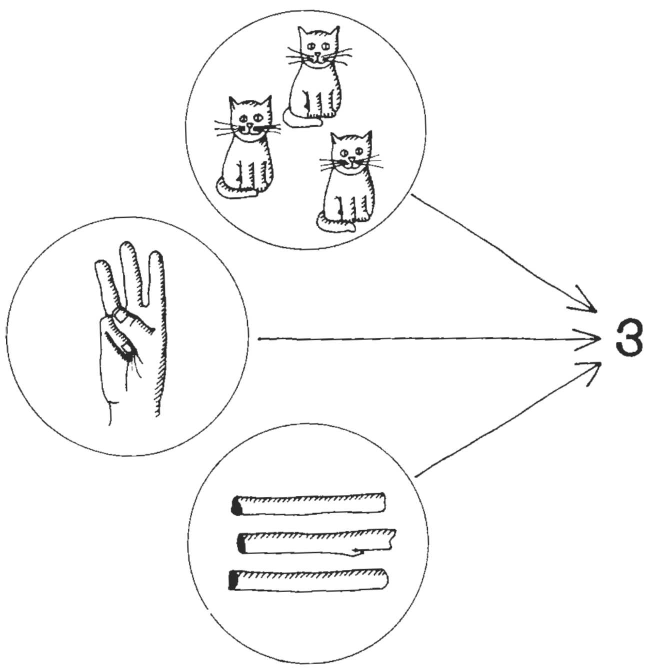
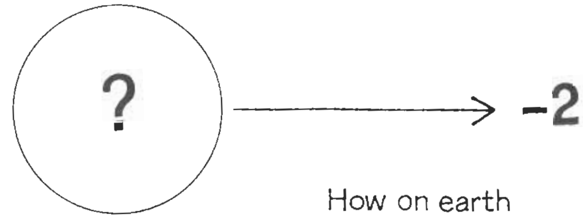
1 Natural Numbers

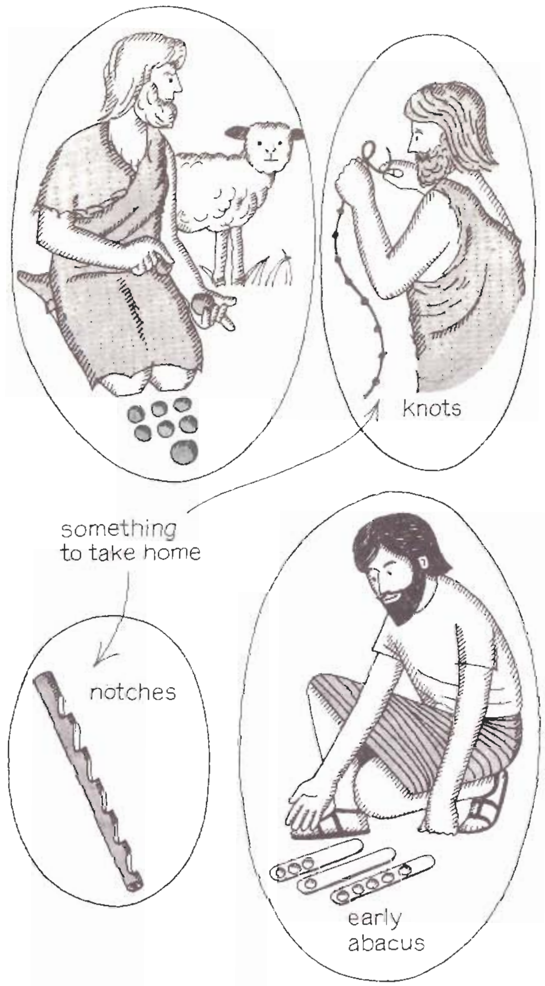
The numbers 1, 2, 3, 4, 5, 6, 7, 8 . . . which we use so often in our everyday lives are called Natural Numbers because, it was thought, they had a natural existence, quite independent of man. They corresponded to something in reality: two eyes, a brace of pheasants, four legs and so on, whereas, as we shall see later, it is necessary to invent other numbers which do not correspond to anything tangible. Consider for example, −1; (no-one has ever seen 'minus one sheep'). All numbers, however, including our Natural Numbers, are an abstract concept and it took man a long time to arrive at a convenient way of representing what he meant by *two, three, four,* etc. While we may agree that the Natural Numbers are easier for us to understand than the other numbers we must, nevertheless, bear in mind that they involve this abstract notion which is still something of a mystery to us. Yet a very large part of mathematics is built upon them.





How on earth could we represent −2 ?

In *Beginnings* ❶ we were concerned with one-to-one correspondence, matching and mapping. Even earlier experiences of the young infant were ones concerned with building a vocabulary such as: *many, few, a lot, more than, less than,* from which developed an awareness of both inequalities and equality. Experiences followed in recognising or becoming aware of 'twoness', 'threeness', etc., through a variety of real situations illustrated by using blocks, bricks, conkers, shells, rods, and *discussion.* The child learned to attach the appropriate number name to a set of objects. All this is very much the same as man experienced himself in the development of this abstract concept of number which he needed to be able to express himself, count and record his possessions: cattle, sheep and crops. From 'word pictures' he hit upon the idea of matching the objects of one set with those of another, perhaps matching *weapons* with *men,* and from this he was able to see if there were too few or too many. Very quickly, no doubt, a standard set was used for matching with a set of real objects and what better at this stage than his fingers or toes? Larger numbers, of course, presented a greater difficulty and it was then that the idea of tallying became a marked step forward: either notches cut into a piece of wood or knots tied in a piece of fibre.



Even though the use of tallying with a model (knots or notches) as a standard, and a one-to-one correspondence between the sheep and the notches, was limited, it was still a big step towards abstract numbers. Man's early concept of fiveness would no doubt be in terms of the number of fingers on the hand and not the abstract *'five'*. This requires a big intellectual leap. The perceptions of matching and one-to-one correspondence, and the actions required in setting up these relationships did not in themselves bring about the idea of numbers, but they greatly increased the chances of their formation.

We must, therefore, be aware of the corresponding development in our children. How does a child recognise the threeness of three? Actions and perceptions are the preliminary steps in the acquisition of the concept of Cardinal Number (number names). Certain logical ideas have to be acquired by the child before a real grasp of numbers is possible; these will come through experience and intuition.

The historical development of natural numbers and how man first began to write them can be fascinating to young children. A teacher aims to gain a child's interest and uses this to create a climate for future learning. It is suggested, therefore, that stories illustrating the development of natural (or, as they are sometimes called, 'counting') numbers be used.

Though no-one is quite clear exactly when people first began to write numbers, we know that in early days the signs representing a numeral were merely strokes which probably represented the fingers of the hand. The word 'digit' which is a word we use for each of the numerals 1 to 9 comes from the Latin word 'digitus': finger. Should we be surprised when children use their fingers to count?

We have often seen children (and adults) using their fingers to perform a calculation. As an example: how many more days of the week are left? If it is now Monday, the child will call off the names of the days of the week and for each day he names he will turn down (or bring up) a finger on his hand:

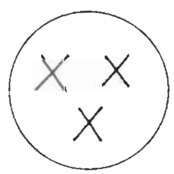
Tuesday → thumb
Wednesday → 1st finger
Thursday → middle finger
Friday → ring finger
Saturday → little finger

He knows, therefore, that there are five days left till the end of the week.

The two sets { Tuesday, Wednesday, Thursday, Friday, Saturday } and { thumb, 1st finger, middle finger, ring finger, little finger } are in one-to-one correspondence; that is, corresponding elements can be 'paired off' (Tuesday, thumb), (Wednesday, 1st finger) and so on.

If two sets are in one-to-one correspondence they have the same associated cardinal number. When we count the number of objects in a set we are using a standard reference set which itself consists of the names of all the (finite) cardinal numbers in order (that is, one, two, three, etc.). When we have matched every object in the set being counted with members of the reference set (in the standard order), then the last named member of the reference set is the name of the cardinal number of the set being counted.

If we think about any particular cardinal number, say 'three', then there are an infinite number of sets of objects which can be put in one-to-one correspondence with the number of crosses in the diagram below



and hence each of these sets is in one-to-one correspondence with each other, since the property of 'one-to-one correspondence' is transitive.

Similarly, to each cardinal number there corresponds an infinite family of sets in one-to-one correspondence with each other. When we ask 'How many?' or 'How many objects are there here?' we are really asking which of these families a given set belongs to.

It is not suggested that children are burdened with these explanations but merely that the teacher should be aware that behind these natural developments mathematical relationships exist.

Our early work in the Junior School concerning computation will be aimed at acquiring an interest in and a real understanding of Natural Numbers, counting natural numbers and also introducing zero, represented by the symbol 0. It is felt that through stories of historical development, with practical illustrations and applications where possible, the children will be helped through interest and their natural curiosity to become aware of our system of notation and place-value. The need for an organised structure should also become apparent.

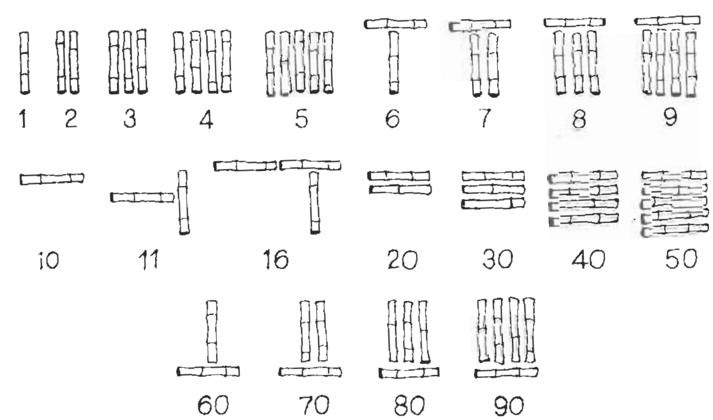
From the Infant School, the young child will be familiar with the numbers 1 to 9 and will also recognise 10 as a number, perhaps even 20, 30, 40, etc., but we must realise that at this stage these are only numbers, and there is little or no appreciation of place value. So, during these early stages we shall confine ourselves to an interest in the numbers 1 to 10, with perhaps 20, 30, 40, . . . 100 for some children.

Other Ways of Writing Numbers

Chinese

Thousands of years ago, the Chinese had a method of representing numbers by using short pieces of bamboo which they placed on the ground or, perhaps, on a piece of board.

This is how they showed their units:

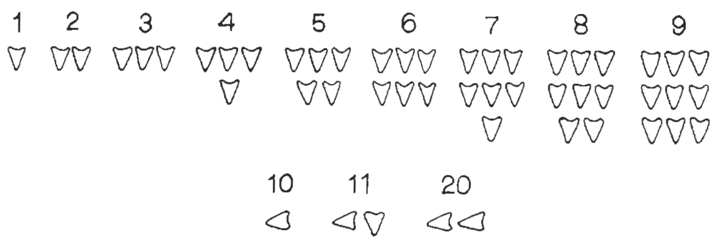


Assignment:

> Get a box of match-sticks or break up some small twigs and make numbers in the old Chinese way.
> Why do you think the Chinese changed the pattern from 6 to 9? Why didn't they go on putting sticks in a row?

Babylonian Numerals (Cuneiform or 'Wedge-shaped')

These were made on slabs of damp clay using a piece of wood to press the wedge-shaped symbols into the clay which was then baked or dried in the sun. Their symbols were even simpler than the Chinese bamboo symbols.

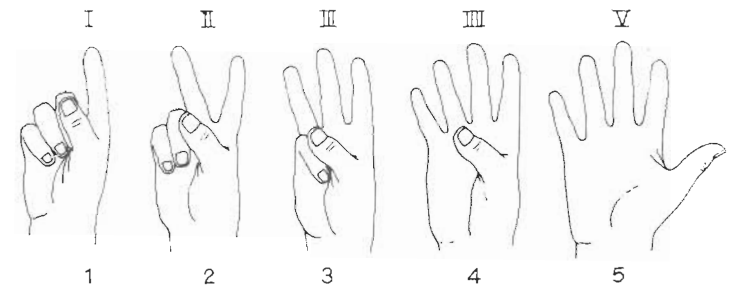


This way of writing numbers was being used during the time of King Nebuchadnezzar, nearly 600 years before the birth of Christ, when Daniel was put into the lions' den.

> Take some plasticine and with the end of a stick or something which will make a mark in the plasticine, set out the Babylonian numbers from 1 to 10.
> Write, in Babylonian, how many years old you are.

Roman numerals

The Romans also used strokes to represent their numbers and their system seems to have been derived from the fingers and hand.



The Romans then proceeded with VI (6), VII (7), VIII (8), VIIII (9), and X (10) as two fives (X)

Extensions of this notation, based on parts of a circle, will be given at a later stage when bigger numbers and place-value are fairly well understood.

Briefly, they are: —
M for 1000 from O, or two semi-circles: — (| D
D for 500 from D, a semi-circle (half of 1000)
C for 100 from curved part of semi-circle
L for 50 from lower part of semi-circle

One small change in the Roman numerals that we still see today is that seldom do we find more than three symbols together. Four (IIII) is now written as IV, one less than five; VIIII (9) is now written as IX (one less than ten) and so on. Note that I in IV has a different significance from the I in VI, determined by its position (this is the beginnings of place-value).

These Roman symbols and the Chinese rod numerals show that at one time five was the basis of their counting systems, almost certainly from the number of fingers on the hand. Thus we have: 1, 2, 3, 4, 5, and '5 and 1', '5 and 2', '5 and 3', two 5's, 'two 5's and 1', etc.

I II III IIII V
VI VII VIII VIIII X

(Ask the children where they might have seen these numerals, e.g. books, clocks, tomb-stones).

It was then easy to use both hands and use ten as a base. *Ten* is used as a base in nearly every part of the world today and we have:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

(11) (12) (13) (14) (15)
10+1, 10+2, 10+3, 10+4, 10+5,

(16) (17) (18) (19) (20)
10+6, 10+7, 10+8, 10+9, Two tens,

(30) (40) (50)
Three tens, Four tens, Five tens....

It can perhaps be seen that, from our stories of the efforts of man in various parts of the world to record number, we are working towards what is a convenient system of notation, the need for this and place-value, also the idea of a number base (see below).

Very little practical or mathematical work can be done as yet in this topic other than of general interest and to help the child's perception of the background to our system of counting numbers.

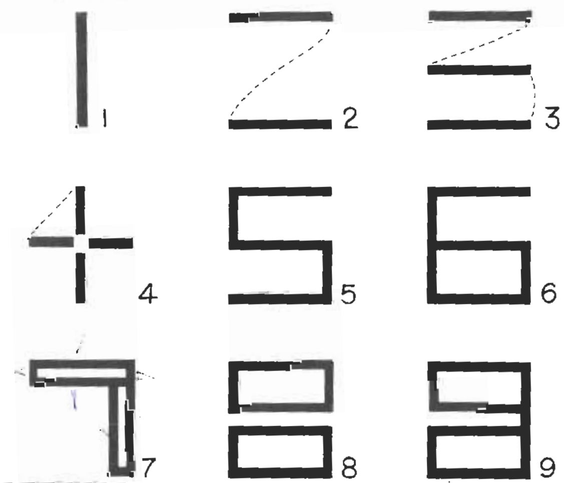
> Make or invent a number system of your own to represent the numbers 1 to 10. It must be easy and simple to use.
> Now see if you can go on to 20.

> Take some sticks and set out the numerals from 1 to 10 in Roman.
>
> Can you take 1 from 4 and leave 5?
>
> Can you make five from two?
>
> Can you do the same sort of thing with the Chinese or Babylonian numerals?

Again, is it really true that *half of 8 is 3*, 'demonstrated' by cutting down the middle?

This sort of activity is not a waste of time, it brings out the difference between the (abstract) numbers themselves and the 'numerals', or actual marks which represent them.

> Our numbers are said to be Arabic. Do you think you can see how their shape might have come about? Make the thick strokes on a piece of paper first.



Summary
We have been concerned with:

1. The historical development of natural numbers for a) interest and b) a visual perception of the abstract notion of a number.

2. The early 'real life' notions of sets, matching and mapping.

3. An awareness of the need for a convenient structure: notation, place-value, and base.

4. Comparisons between other ways of recording numbers and our own Arabic numerals.

2 The development of weights and measures

'Measurement can only be as accurate as the tool used allows and there are many occasions in everyday life when extreme accuracy is unnecessary.'

'Absolute accuracy in measurement, as distinct from counting, is unattainable.'

'Weights and Measures' will provide children with a variety of practical experiences, and work with the system of numbers 1, 2, 3, . . . can be extracted from these. One of the main differences between today's approach in the junior school and the previous more traditional methods is that practical work is now used throughout the course and not in concentrated doses of a few weeks. The previous work on the historical development of number and the ideas in this chapter can be used when the teacher feels it will be most beneficial; sometimes, perhaps, using assignment cards for group practical work and sometimes with the whole class for story and discussion. Practical work, however, will not benefit either teacher or child unless it is purposeful. The work has to be developed and directed by the teacher in a progressive way according to the requirements of the child and the facilities available in the school. Only general guidance can be given here.

There is nothing to recommend the dubious ability to do computational 'sums' in weights and measures; the learning of tables of length and weight (rods, poles, bushels and pecks); or 'reductions up and down'. Visual 'yardsticks' need to be acquired by children first, in order that they may be able to estimate and deal with real-life situations within their experience. Later, we can extend these experiences to longer distances and larger units using contrived situations, perhaps, but we must first ensure that children have a working knowledge of the units with which they are dealing and the instruments they are using for measuring; that they can distinguish between a count (pure number) and a measurement, and can choose the appropriate units for the different ranges of measure. It is also worth noting that, apart from £. s. d., more than two successive units are very rarely used in real situations (e.g. ft, in; st, lb). This tends to make a mockery of some of our text-book arithmetics with their exercises dealing in sums with a heading of:

tons cwt qr st lb oz

Length

Early experiences of length will best be carried out by children using any sort of measure that suitably comes to hand:– lengths of paper, book-lengths, knitting needles, match-boxes, strides, spans (hand-breadths) and finger lengths; something which the child is familiar with and which he understands. We shall, of course, be working towards the need for, and the discovery of, *standard units*.

In the infant school children will have been encouraged to develop their notions of length through vocabulary and comparisons:

longer than

shorter than

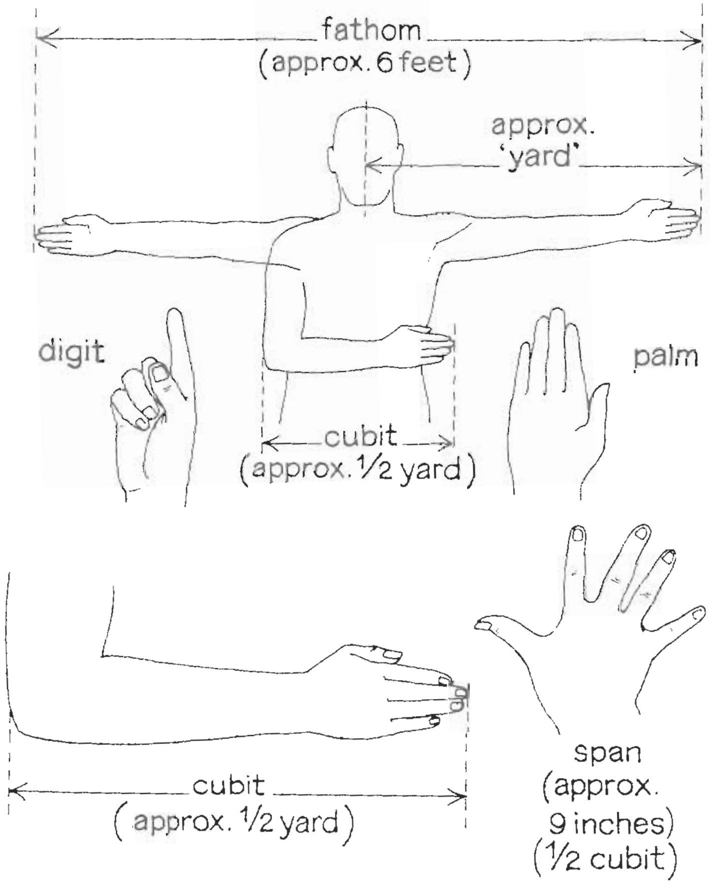
higher or taller than

near and far

Pieces of string, ribbon and wood will clearly have been made shorter if we cut some off and made longer if we join lengths together. Such experiences have given the child an understanding of length: the quality or 'longness' of length. We are now in the position to be able to extend these experiences towards the understanding of the units of measurement, and standard tools involved in this.

It is interesting to note that, historically, the earliest measurements were calculated from parts of the human body: *limb measurements*. Many women measure yards of cloth by stretching the material from the tip of their nose to the end of the middle finger of the outstretched arm. A man's 'pace' is roughly one yard in length, and of course the word 'foot' speaks for itself.

This use of the body as a common measure was an important step forward in the history of measurement and it has its origin in visual perception. Therefore, when a child, too, sees an object for measurement he will match it with something he knows: his hand or arm, creating a model image.



4 digits	1 palm
3 palms	1 span
2 spans	1 cubit
4 cubits	1 fathom

These are relationships between the *limb measurements*, and children could be asked to discover such relationships for themselves.

Here we can begin to train children to estimate and make an approximation of the lengths before actually measuring with them. We can let them discover for themselves just how approximate measurement is anyway, and decide that we can never measure anything *exactly*.

An ability to make reasonable estimations can only come from practical work and from first-hand experience of the measures being used. A child cannot measure successfully with unfamiliar units and so we must ensure that measuring work is a regular part of our programme with gradual progression towards finer limits of accuracy.

> Measure your neck, wrist and waist. What will you use to do this?
>
> How many 'wrist measurements' will go round your neck? (Estimate first)
>
> How many 'neck measurements' will go round your waist? (Estimate first)
>
> See if your measurements are different from your partner's.

These assignments can, of course, be read to the children and directed by the teacher if reading ability is limited. Further questions can also be developed by the teacher e.g. 'How many wrist measurements are there in a neck measurement?'

> Measure the length of the desk using
> a Cubits
> b Spans
> c Palms
> d Digits
>
> First check with the chart that you know what these are.
>
> Make a rough estimate before you measure.
>
> Which did you find gave you the best measurement?
>
> Which took the longest to do?
>
> Which *two* limb measurements would you use to give you the best measurement? Write down this measurement.

Use the length of your span (handbreadth), your pace, arm (from elbow to finger-tip), or the length of your foot to measure:

the length and breadth of books
rooms
halls
playgrounds
boards
desks
papers
window panes
doors.

Think about the best units to use before you measure

Compare your results with those of your partners.

Are they different? Why do you think this is?

The last two 'cards' would probably form the basis for several others. It is not suggested that they be assigned all at once.

Which human units would you use to measure:

the height of a giraffe?
the height of a horse?
a mouse from nose to tail?
the length of your garden?

Write down some more things which you could measure and say which units you would use.

By using human measuring units with various exercises of this kind a teacher can help children to see that, because 'human units' vary in different people, there is a need for a fixed standard unit. Even so, this is not the only value of using hands, feet and arms: the child is using something familiar to him (his body). It can be pointed out to children when we are developing our ideas of measuring length that, again, we are following the historical development of measuring units.

In this preparatory work we are, as previously stated, leading children to recognise the need for standard units; that we make a measurement by finding out how many times a given unit is contained in this required measurement. According to the work of Professor Piaget, some children take a long time to reach the operational stage of thought required to deal with standard units and an increasingly accurate assessment of measure.

There are stages through which a child passes:

1. the use of arbitrary units (whatever is conveniently chosen – matchboxes, etc.)

2. units that are approximately standard – limb measurements

3. the statutory standard units (ft, in).

These stages conform with the historical development of our system of measurement.

Standard Measurements

In the year A.D. 1150 King David of Scotland declared that the inch was to be the average measure of 'the thumbs of three men; that is to say, a medium man, and a man of measurable stature, and a little man. The thumbs are to be measured at the root of the nail.'

King David was well aware that limb measurements depended upon the size of the man, and in order to get some definite length, he decided to take the *average* width of the thumbs of three men of different sizes.

In Edward II's reign a law was passed abolishing all limb measurements and it was declared that an inch would be the length of three barley-corns, or grains.

These measurements were, of course, still far from satisfactory as 'standards' and not until the reign of Queen Victoria was a standard bar of one yard in length decreed to be kept at the Standards Office in London under the Weights and Measures Act.

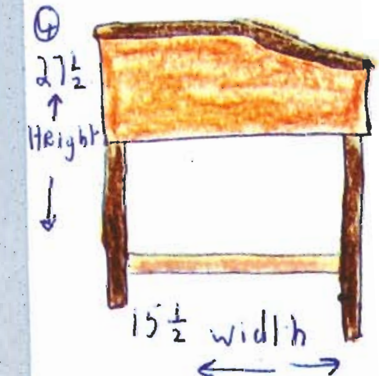
David Pitches

Card 302.

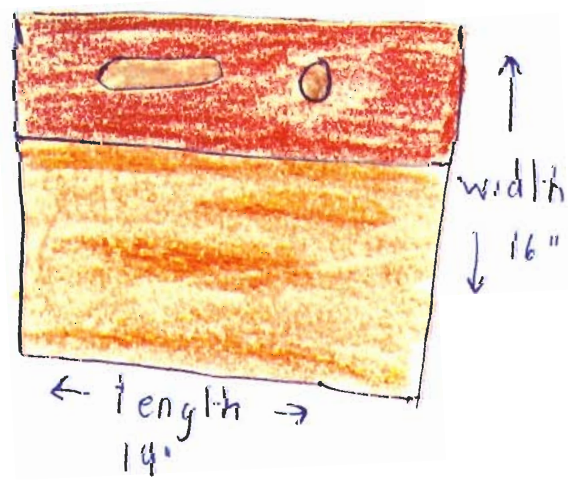
Measuring in spans.

1. My desk is 3 spans long.
2. It's length is $4\frac{1}{2}$ spans.
3. The desk is 3 spans wide.
4. It's width is 4 spans.
5. The desk is 5 spans high

6.



Plan of Desk from above.



Robert Hitchen Age 8 yrs.

Limb measurements.

1. The classroom is 9 paces long and 8 paces wide.

2. The playground is 45 paces long and 25 paces wide.

3. The blackboard is 21 spans long.

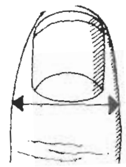
4. The desk is 4 spans and 5 digits long.

5. My book is 13 digits wide and 4 palms long.

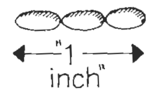
6. The door is 6 spans wide.

7. The cloakroom is 46 foot measurements long and 19 foot measures wide.

My answers are not the same as my partners because he is not the same size as I.



Width of thumb
at root of nail is
King David's 'inch'



of Edward II
3 barley-corns
placed end to end

Further details of the historical development of length will be given at a later stage in the Guide when we extend the system to larger units.

These historical facts can provide enjoyable and stimulating work for junior children in history, English and mathematics. Many children will want to read more of the details of the History of Measurement and reference should be made to the many interesting books contained in the list on p. 67.

When children have been brought to realise how inadequate the human measurements usually are, they will then be ready for work with standard units. It is strongly recommended that lengths of wood or hardboard, or the commercial attractively-produced rulers (without end-pieces or sub-divisions) be purchased for use. The children can be given these to use for measuring tasks to be carried out within the environment of the classroom and school; lengths to be recorded in whole yards and whole feet (after estimation). Measurements of lengths drawn on the blackboard or on the floor, or paper pinned to the wall, and previously measured to an approximate number of yards or feet by the teacher, can be included in the assignments. A collection of pieces of wood, string, ropes and a variety of objects for this kind of measurement can be a very helpful part of the classroom equipment. It is also time to introduce perhaps the most fascinating measuring instrument to young children – the trundle or click-wheel. This has a circumference of one yard and the wheel clicks each time this measurement is made. For measuring playground and halls the trundle-wheel is invaluable.

After practical work using yards and feet children can be asked to complete tables as suggested below:

yd	ft
1	3
2	6
3	9
4	.....
5	.....
6	.....
7	.....
8	.....
9	.....
10	.....
.....	33
.....	36
.....	39
.....	42
.....	45

Complete this table.

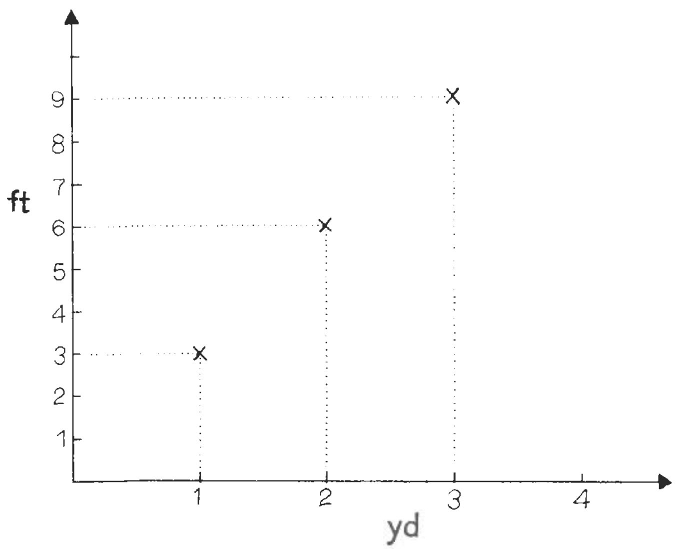
Can you explain what you were doing?

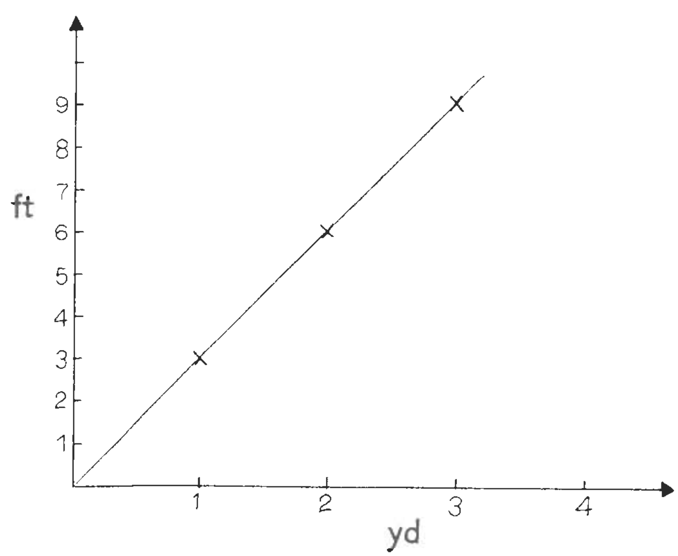
Can you use this table to change yards to feet or feet to yards?

It is at this stage that we have passed from the real world of measurement back to the set of counting numbers. Although the above table is headed 'yd' and 'ft', it is essentially one of ordered pairs of numbers (1, 3), (2, 6), (3, 9) . . . . , the relation being that in each case the second number is *three times* the first. The table can be built up by successive 'counting on' in threes. (Cf. on the following page).

This relationship is abstracted from the experiences of measuring. It must be emphasised that when we are using the natural numbers, even when we are talking about '6 ft', we are using a *mathematical model* of the far more complicated real world. (The 'model' here refers to the use of a simplified system; it is not, of course, a concrete model.) We can *talk* of '6 ft', but no-one can *exactly* measure 6 ft (think of using even a very powerful microscope).

A simple graph can be made illustrating conversion of feet and yards.





With luck the children will notice that the points plotted lie on a straight line, and the temptation to join up should not be resisted. The discovery of zero ('0') can be made. (the count-down '3, 2, 1, zero' is quite possibly part of their vocabulary anyway), and it would be unreasonable if 0 ft did not correspond to 0 yd.

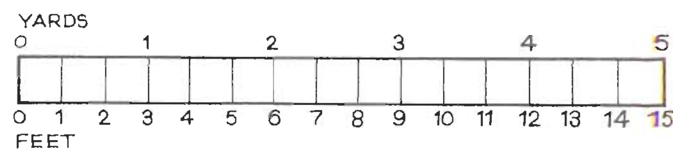
But care is needed. It is dangerous to pretend that children at this stage can properly interpret the intervals in between 0 and 1, 1 and 2, and so on. It is far too early to consider sets of numbers other than {1, 2, 3, . . .}, (with 'zero' sometimes being adjoined to them). Joining up to form the straight line can, however, be interpreted intuitively, e.g.

a. A length *just less than* 6 ft will also surely be *just less than* 2 yd.

b. 'About halfway between 2 yd and 3 yd corresponds to about halfway between 7 ft and 8 ft'.

It would be quite wrong to try to squeeze more out of the graph at this stage, for example, to dabble in decimals. The need for new sets of numbers will arise naturally later, as will the method of introducing them.

A home-made 'converter' is also good experience leading towards the table of 3's.



Following the above experiences children will anyway become dissatisfied with rulers that will only give them measurements in feet and yards. They will want to find the length of the 'bits and pieces' which are left over at the end. This is the time to make the inch rulers available (without other sub-divisions if possible). Some foot and yard rulers, now with inch divisions, will also be helpful at this stage. It is good experience for children if they are now asked to record their error after estimation and also to record measurements in more than one way, e.g.

2yd 1ft or 7ft   1ft 6in or 18in

Write down all the things in the room that you think are about 1 foot in length. Record like this:

Objects estimated as 1 foot long	Measured length (to nearest inch)	Error (to nearest inch)
window pane	9 inches	3 inches

How many feet are there in one yard?

Write down how you found out.

How long is one of your strides? (to the nearest inch).

Mark out one yard on the floor. How much shorter than a yard is one of your strides (to the nearest inch).

Measure the length of the classroom by striding. Now use a ruler (which one?) and work out your error.

From the following table it will be observed that a good deal of computation is necessary to convert from one unit to another.

Object	Estimate	Measurement (to nearest unit) *normally expressed in two ways*	Error
Length of desk	5 ft	5 ft 9 in or 69 in	9 in
Length of teacher's table	4 ft	3 ft 6 in or 42 in	6 in
Width of table	3 ft	2 ft 3 in or 27 in	9 in
Length of corridor	30 yd	45 yd	15 yd
Width of corridor	10 yd	10 ft or 3 yd 1 ft	20 ft
Length of book-shelf	5 ft	5 ft or 1 yd 2 ft	—
Width of book-shelf	1 ft	1 ft 6 in	6 in
Height of book-shelf	3 ft	3 ft 6 in or 1 yd 6 in	6 in

Further practical work with yards, feet and inches

Estimation

Can you say, roughly, without measuring, what are the lengths and the heights of the following?:

a. the classroom door
b. your own height
c. the length of your stride or pace
d. your teacher's height
e. the length and breadth of your exercise book
f. the distance round a tennis ball

Try to find:

1. something that is approximately one yard long
2. something that is approximately one inch long
3. the distance round a tennis ball

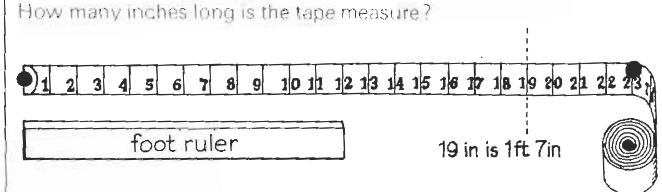
When you have estimated, measure and check, to the nearest unit. What did you use to check the measurement of the tennis ball? Show your teacher how you did this.

Feet and Inches

You will need a tape measure and some foot rulers (without end pieces)

Pin the tape measure out straight on a table or on the wallboard.

How many inches long is the tape measure?



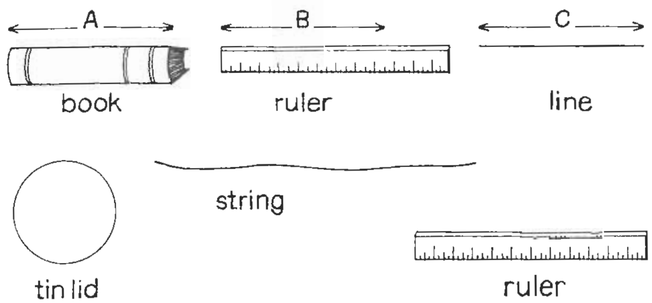
If you choose 18 inches on the tape measure, how many feet and inches is this? Use your ruler to help you.

Now find out how many feet and inches these are:
20 inches, 30 inches, 24 inches, 36 inches, 48 inches, 42 inches, 32 inches, 21 inches, 33 inches, 45 inches.

How much longer than two feet is: 25 inches, 27 inches, 30 inches?

How much shorter than two feet is: 18 inches, 23 inches, 15 inches?

When we measure length or height we are really *matching* the object in question against a standard unit and this enables us to say 'As long as . . . units (yards, feet or inches)'. In fact when a child measures the length of an object with his ruler and then draws a line to represent that length he is really matching twice (ruler to object, length of line to ruler.)

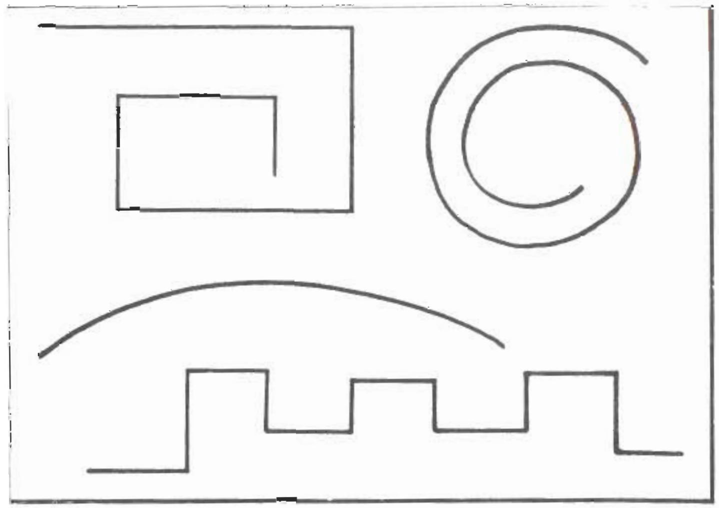


When experiments by Piaget were carried out with young children it was observed that even though they seemed to appreciate the conservation of length when objects to be measured were placed in a straight line, this did not necessarily imply that they had a complete understanding of measurement. When objects were rearranged in curved or zig-zag lines, some children were not so certain. With young juniors, therefore, we give many experiences of curved, round, and zig-zag measurements. Here the child will need an intermediate model of the object he is measuring – in measuring round a tin lid he will take a piece of string or a tape measure to acquire the appropriate length and then match this in turn against his ruler.

Calipers can be introduced to children at this stage for measuring diameters of tin lids, balls and cylinders; alternatively they can be placed between two blocks.

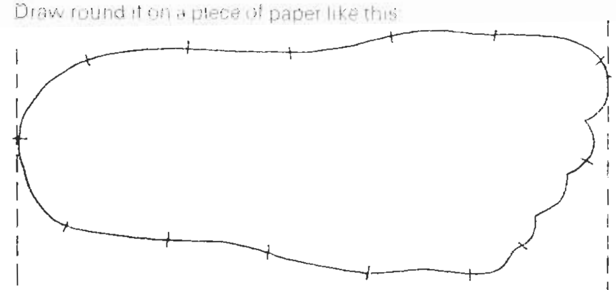
Suggestions for Practical work

Get a large piece of paper and draw lines shaped like these, much bigger of course, using a felt pen. When you have done this, measure them. Think how you are going to do this, and don't forget to estimate first.



What is the length of your foot?

Draw round it on a piece of paper like this:



Draw round some other children's feet. Try to find the longest and shortest in the class.

Now, if a man's foot is a foot long, approximately, how much shorter is yours?

How much shorter than a man's foot is the length of the longest foot in the class?

How much shorter is the length of the shortest foot in the class?

What is the distance all the way round your foot? Compare this with your partner's foot.

Do three of your foot-lengths come to about the same as one of your strides?

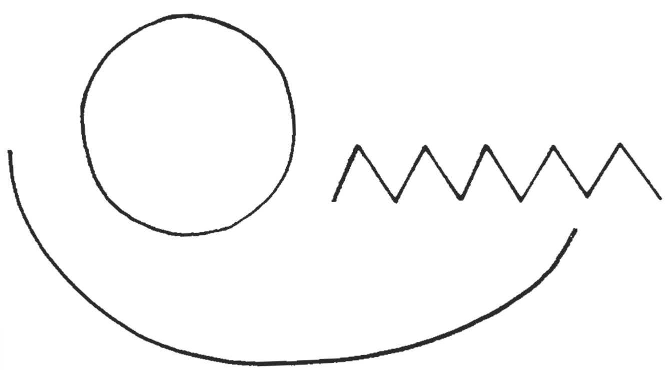
1. Draw a line (without using a ruler) which you think is about an inch long. Now measure it and see how far out you were.

2. Now draw lines (again without using a ruler) which you think are: 2 in long, 3 in long, 6 in long. Now check these with a ruler and record how much too long or too short you were. e.g.

2 in

My line is about $\frac{1}{2}$ inch too long.

3. Now draw curved, round or zig-zag lines which you think are approximately 6 in long.



How are you going to check these?

4. Draw any lines which you think are about a foot in length. What other way can you write 'a foot long'?

Find as many round objects as you can and see if you can measure the distance round them to the nearest inch. (tins and tin-lids, balls and buckets – top and bottom.)

Make a list of these and record your measurements against each. Don't forget to have a good guess first.

Can you measure the longest distance across the circular shapes?

How will you do this?

Get the calipers and see if you can find out how to use them and find this measurement, which is called the *diameter*.

By means of gradual experience of the standard units of measurement required to measure, the children will no doubt ultimately arrive at fractional parts, when they feel they want to be even more precise with their measuring. We can now introduce the foot rulers with sub-divisions of half and quarter inches. Halves and quarters are contained in a child's vocabulary almost as soon as he begins to use other number names: a half of a piece of chocolate, a quarter pound of sweets, half-past four, etc. These may be merely words to the child, so far, but, from his experiences in measurement (and weight and capacity) he can begin to appreciate the meaning of a fractional part. Children are apt to become confused when introduced to fractions through blackboard illustrations and diagrams. We shall therefore, deal with fractions as natural parts of a whole unit and certainly not labour them at this stage.

It must be emphasised that just as no-one has ever *exactly* measured 6 ft (cf. p.9), so no-one has ever measured *exactly* $2\frac{1}{2}$ inches, or indeed $\frac{1}{4}$ lb sweets. We are using fractions here (and certainly only at most $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{3}$ should be introduced in this way) as approximations, as in the rule. 'The fair way to divide a cake into two is for Jack to cut it and then Jill to choose which is the bigger half'.

We probably cannot (and possibly don't want to!) avoid this loose use of fractions in measurement, but we must constantly point out that we are not being precise. For example '$\frac{1}{2}$lb tea' balances against a $\frac{1}{2}$lb weight; it will appear to balance just as well if a few extra leaves are added.

When dealing with *numbers*, $1\frac{1}{2}$ will mean *precisely* that number but this is a much later stage. When dealing with measurements, we shall as soon as possible use decimals, so that we can show clearly the degree of accuracy to which we are working. This will involve an extension of the number system (cf. the sequel ❸).

At the present stage '$\frac{1}{4}$', '$\frac{1}{2}$', '$\frac{3}{4}$' are used colloquially, not as numbers at all but in connection with a rough way of describing measurements. There can, of course, be no question of 'adding' them under these circumstances.

Suggested Practical Work

Measure the cards marked L1 to L6. You will need a ruler with halves and quarters. Find the length, breadth or width, and record your results in your book like this: (Estimate first to the nearest inch.)

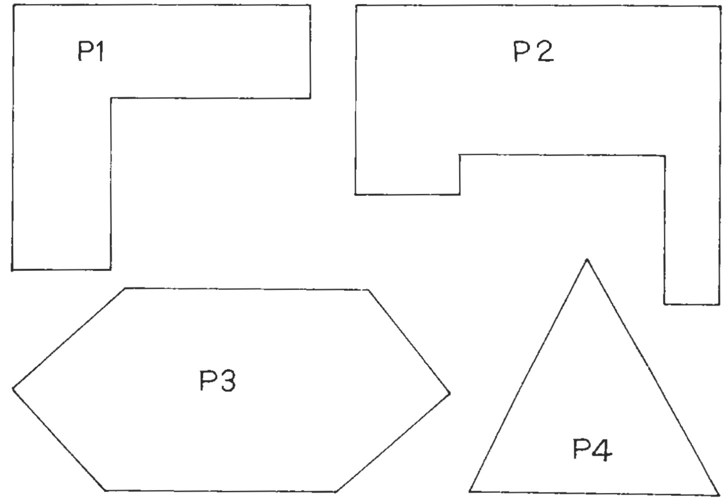
Card	Length			Width		
	Estimated	Measured	Difference	Estimated	Measured	Difference
1						
2						
3						

Which card is the longest?
Which is the shortest?

Get the cards marked P1 to P6. You will notice that these shapes are *irregular*.

Measure the distance all the way round the edges. This is called the *perimeter*. Have a good guess first to the nearest inch. Record your measurements, to the nearest half-inch, and then answer this question:
Which has the longest perimeter?

Some of the 'shapes' in the above assignment might look like these.



Measuring Heights

Measure your own height on the class 'heights ruler'. Get someone to help you to do this.

Now measure your partner's height.

Who is the tallest in the class? Who is the shortest?

What is the difference in heights between each pair of you? (Make a record of all this information.)

How much will you have to grow before you are about 6 ft tall?

If you and your partner stood on each others heads how tall would you both be then?

Goliath's height was said to be 6 cubits and a span. What is this in our units of measurement?

Draw a line on the floor to represent Goliath's height. Then, next to it draw a line to represent your height. Is there much difference?

What is this difference? (Record the measurement in two ways, perhaps three.)

Have you ever seen anyone as tall as Goliath? Can you think of anything about as tall as Goliath?

Could he get through your classroom door without bending down?

Vincent Keirle Age 8 yrs.

Estimation

	Estimate	Measure	Error
The door	8 ft	8 ft 9 ins	9 ins.
My height	4 ft	4 ft ¼ in	¼ in.
My teacher's height	5 ft 11 ins	5 ft 5 ins	6 ins
My stride	2 ft 6 ins	1 ft 9 ins	9 ins
My book	7 ins	8 ins	1 in
A picture	1 ft	1 ft.	0
The blackboard	1 yd	2 ft 11 ins	1 in.
A penny	1 in	1 ¼ ins	¼ in
Distance round a ball	5 in	7 ½ ins	2 ½ ins

I used a piece of string to go round the ball and then measured it against my ruler.

ERTC Gorman

Card 11

All About Feet

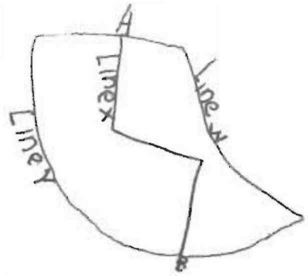
The length of my foot is nearly 8ins
I drew round Lucy Cundliffe's Foot
And round James Blacow's Linda Swans
Lorraine Ellisos, Jacqueline Birtwhistle's,
Julie Carters, Christine Pearson's, Mine,
David Pitches's, Leon Bees Vicki Combells,
Kevin Barrett's, Owen Boguleys, Keith Ushers
and the last was Kevin Goulds.
The boy who has the largest foot is Kevin Barrett
And my foot is 1 inch shorter
If a man's foot is a foot long approximateley
my foot is 4 ins shorter
The longest foot belonging to Kevin Barrett
is 3ins shorter than a man's
The shortest foot in the class belonging to Christine Pearson
is 5¼ ins shorter than a man's
The distance all the way round my foot
is 1ft 6½ ins.
My partner's foot all the way round is 1ft 7ins
My foot all the way round is one inch shorter.
One of my strides measures 2ft
3 of my foot lengths measure 1ft 9¾

Leon Bee.

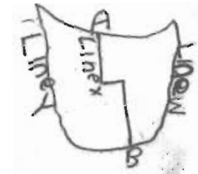
Card 166.

Measuring

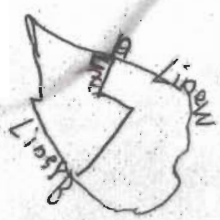
1. I can jump 8 f.t.-1 inch.
2. I can throw a ball 48½ y.d.s.
3. The netball court is 33 y.d.s. I measured it with a trundle wheel.
4. It is 17 y.d.s. wide.
5. The long side of the playground is 76 y.d.s.
6. The short side of the playground is 43 y.d.s. long.



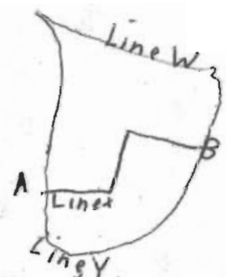
Line x is the quickest way from B to A



Line x is the quickest way from A to B



Line x is the quickest way from A to B



Line X is the quickest way from B to A

Line x is the quickest way from A to B

I measured the straight lines with a ruler and the curvinglines with a piece of cotton

Barry Matthews Chest expansion of Boys in Class 2A Card 8 8

Name	Age	Chest normal	Chest expounded	Chest expansion
Barry Matthews	8	27	28	1"
Leon Bee	9	27½	28½	1"
Trevor Cooper	8	29	30½	1½"
Philip Monoghan	9	32	33	1"
Christopher Bates	9	27	27½	½"
James Blacow	9	26½	28	1½"
David Chapman	9	26	27	1"
David Pitches	9	26	27½	1½"
Eric Gorman	9	27	28	1"
Kevin Barrett	9	29	31	2"
Andrew McClelland	9	25	27	2"
kevin Gould	9	26	27	1"

Table of feet and inches

feet	inches
1	12
2	24
3	.....
4	.....
5	.....
6	.....
7	.....
8	.....
9	.....

Complete this table as you did for yards and feet. What are you doing each time?

The Egyptians used human measurements as you can read in the Bible.

When they made a building they measured in cubits, fathoms, spans, etc.

Imagine that you are an ancient Egyptian.

If a doorway into one of your Pyramids is 4 fathoms high, about how many feet is this?

A statue is 5 cubits high. Approximately how many feet is this?

If one of your ships needs a depth of 30 feet to float and the depth of the water in your harbour is 6 fathoms, can you bring the ship into the harbour?

(Find out from the chart if you cannot remember what cubits, etc. are in our measurements.)

Summary

The approximate nature of measurement.

An indication of how computation can, nevertheless, be derived from practical work.

The historical development of *length*.

The need for standard units.

Vocabulary: length, breadth, width, depth, circumference, diameter, perimeter, height, distance, names of units.

Weight

Weight is the force due to gravity. This is not such an easy concept as might be supposed – though young children will no doubt have heard of 'weightlessness' in space, it is doubtful if they really understand what it is all about.

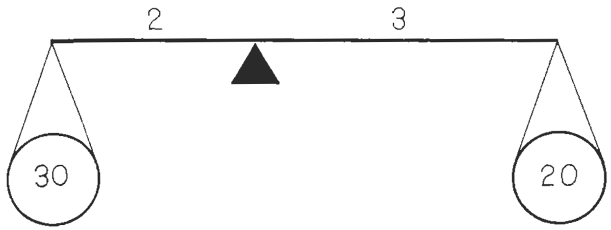
The young child's first experiences of weight have been outlined in *Beginnings* .

We must again point out to children that the measurement of weight is something which cannot be absolutely accurate. We have to work to the most appropriate approximation. Regular insistence upon *'estimate first'* will help this idea. It will also foster real familiarity with units of weight: the ability to choose and work with a sensible unit being an important aspect of measurement.

The following is an experience witnessed in a junior school:

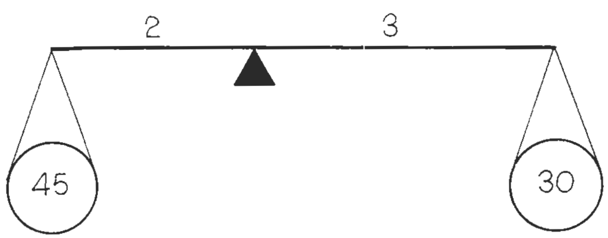
A group of small children were engaged in weighing activities. This particular group were finding how many 1 oz weights balanced a 2 lb weight. The children found that 33 ounce-weights made the balance. They were then asked to change the weights from one pan to another and see what happened. This time 31 ounce-weights balanced the 2 lb weight. This could be embarrassing for the teacher who had too much faith in the balance and the weights. The 'ounce-weights' themselves will not weigh *exactly* the same (even if their markings suggest this), but grosser errors are probably due to the point of balance not being central. The most likely explanation of the paradox just quoted is as follows.

The 'law of the see-saw' is illustrated below.



If the arms are 2 in, 3 in long and the weights in the pans respectively those of 30 oz and 20 oz, they would 'balance', since $30 \times 2 = 20 \times 3$ (this is the principle of the 'equaliser', p.41 below; we are assuming here that the arms themselves and scale-pans are of negligible weight).

Now with the same balances (faulty because the point of balance is not at the centre) if the 30 oz weight were transfered to the other side, it would now balance not with the 20 any longer but with 45, since



$$45 \times 2 = 30 \times 3$$

If a balance is faulty in this way, no amount of tinkering about with plasticine etc. in one pan will make it more reliable.

It is therefore necessary for the point of balance to be at the centre. Of course if the scale pans were then not of the same weight, a piece of plasticine could be put on one to rectify this.

In early civilisations men no doubt picked up objects and used their 'muscle-sense' to estimate weight. It is probable that simple balances and early types of steel-yard were then devised. Stones or pieces of metal of convenient size would no doubt be used as weights. Children will readily see the connection between these early 'weights' and our unit of weight called a 'stone'.

Early 'weights' often had the same name as units of money and capacity (pound, bushel and peck). Some early English records gave weights not only in lbs ozs but in shillings.

There were also two *systems* of weight in most countries: one for goods and one for precious metals. We know this to be true of our own land: Avoirdupois and Troy. These two systems were in use until, almost 100 years ago, it was declared that Troy should be used for precious metals and drugs only. By this time the pound (Avoirdupois) had been fixed at 16 oz; the cwt which had been 100 lb (and still is in America) was fixed at 112 lb and the stone at 14 lb. Additions had been made to allow for discrepancies (an example of the approximation of weight).

Children are often puzzled as to why a *hundredweight* is 112 lb and not 100 lb. The 'make-weight' of 12 lb should, therefore, be explained. From this we can then point out the 'halving' process from which we derived our other 'weights': Otherwise, 28 lb and 14 lb seem impractical.

1 cwt was fixed at 112 lb

$\frac{1}{2}$ cwt was then 56 lb ($\frac{1}{2}$ of 112)

1 quarter was then 28 lb ($\frac{1}{2}$ of 56)

1 stone was then 14 lb ($\frac{1}{2}$ of 28)

Young children acquire their early concepts of weight by picking up objects and using their 'muscle-sense' to feel the gravitational pull. From these experiences they begin to understand what is meant by 'heavy' and 'light', 'heavier' and 'lighter than'.

From experiences through 'muscle-sense' we can then proceed to the use of simple balance-scales. Initially, the child can weigh and find out the heavier or lighter, heaviest and lightest, objects by balancing one against the other.

It is noticeable that, at first, children are concerned with the substance of an object, and its size. They tend to think that the greater amount or the bigger the object, the heavier it is. It is a good idea, therefore, to have a variety of heavy and light materials with which children can experiment:
sand, shells, nails, beads, washers, nuts, conkers, flour, soil – and water.

From experience with these materials and objects we shall be giving children an opportunity to learn that only a small amount of one substance is sometimes sufficient to balance a far greater amount of another. A good example would be: A large packet of corn-flakes and a bag of sugar.

After this kind of experience we can build up the need for standard units of weight as we did for length.

Suggestions for Practical Work

Example 1

> You will need the balance-scales and the boxes of weighing materials. Before weighing, *estimate first.*
>
> 1. How many peas weigh about the same as 6 conkers?
>
> 2. How many nails weigh about the same as a cupful of sand?
>
> 3. How many peas weigh about the same as a cupful of sand?
>
> 4. Which is the heavier: a cupful of peas or a cupful of sand?
>
> 5. Weigh and compare any two materials. Record what you find out in your 'Weight Book'.
>
> 6. When you have weighed and compared all the materials, can you say:
> Which is the heaviest?
> Which is the lightest?
> How did you discover this? Write it down.
>
> 7. Which material would be the best to use as a 'standard' of measurement? (Talk to your teacher about this)

Much benefit can be derived from discussion with children about the various materials and how they can find out the best 'unit' to use for recording their measurement of weight, e.g.
We will use one material to use as a 'weight' so that we can balance the others against it. Which do you think will be the most convenient?

Why have you chosen the . . . ?

Why not use soil?

Why do you need the cup?

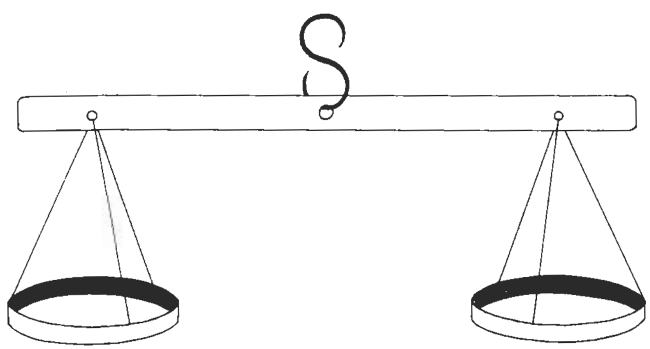
Example 2

Five tins (or parcels) marked by colour, or A, B, C, D, E are made to weigh differently e.g. 4, 8, 6, 12 and 16 oz respectively.

> 1. Take each parcel in turn and *estimate* which you think is the heaviest, next heaviest . . . and so on.
>
> Record like this: —
> Parcel————————is the heaviest.
> Parcel————————is the next heaviest.
> and so on until
> Parcel————————is the lightest.
>
> 2. Now use the balance scales and, without using weights, find which is the heaviest, next heaviest . . . , lightest.
> Write them down in order.
> Show your teacher how you found this out.
>
> 3. Check with your estimations.
> Were you right?
> Can you find two parcels which weigh as much as the heaviest parcel?

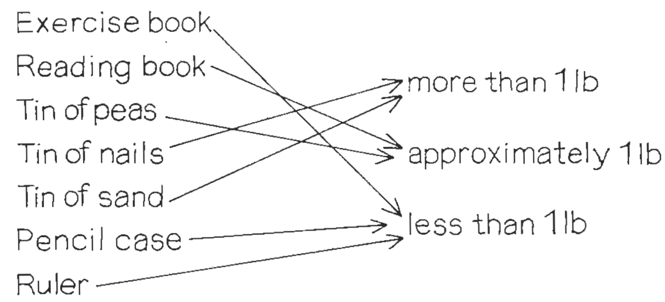
It will be noticed that for nearly every assignment suggested there is an instruction to the children: 'Show your teacher how you found this out' or 'Discuss this question with your teacher'. This is a deliberate invitation to the children to discuss things with the teacher: a most valuable part of the assignment. It then provides an opportunity for the teacher to assess progress and understanding, before the children proceed any further.

Example 3 (For the craft lesson or at home)

> Try to make a weighing machine of your own using a piece of wood (about the size of a ruler), some string and two tin lids. You will also need a hook (a butcher's hook) made from wire. To make a good balance, the 'arms' must be equal.
>
> It should look something like this: —
>
> 
>
> Use plasticine to make it 'balance' if the scale pans are not of the same weight.

When the need has been felt for standard weights it is a good idea to begin with the 1 lb weight since it is the basic standard unit of our system of 'weights'. Children can be given assignments which will familiarise them with this unit:

Example 4.

> Use the scales and a 1 lb weight. Find as many things as you can which you think weigh approximately 1 lb.
>
> Make a list and say if you were about right or not. You can record this as follows:
>
> 
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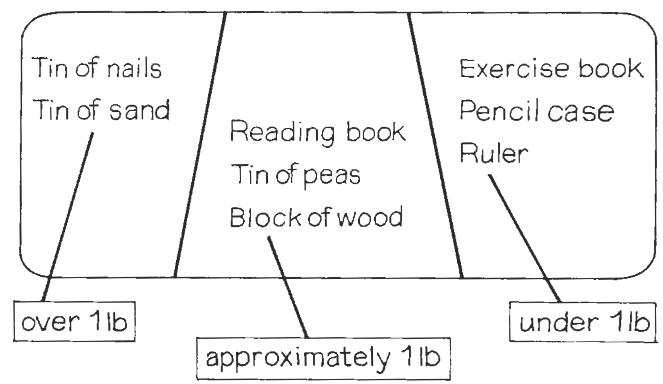
> 
>

Here we have taken a set of objects, Exercise book, reading book, tin of peas, tin of nails, tin of sand, pencil case, ruler and *partitioned* it into three sub-sets (cf. *Beginnings* ❶), those 'over 1 lb', those 'approximately 1 lb in weight' and those 'under 1 lb'.

A partition establishes an equivalence relation, in this case *is in the same category as.* It will be recalled *(Beginnings* ▼ *)* that an equivalence relation has three properties:

i. **R**eflexive

The reading book is in the same category as itself (i.e. 'approximately 1 lb in weight').

ii. **S**ymmetric

If we know that reading book is in the same category as tin of peas, *then* we know that tin of peas is in the same category as reading book.

iii. **T**ransitive

If we know that reading book is in the same category as tin of peas and also that tin of peas is in the same category as block of wood, *then* we know that reading book is in the same category as block of wood.

But we must specify more carefully what is meant by 'approximately 1 lb in weight'. For example, the exercise book is in the same category as the ruler: they are both 'approximately 1 lb in weight', in the sense that they are both within one ounce, say, of one pound; they are not necessarily within one ounce of each other. The partition here is '15 oz or less' 'between 15 and 17 oz' '17 oz or more'.

Hence it is proper to infer from

1. the reading book is in the same weight category as the tin of peas, and

2. the tin of peas is in the same weight category as the block of wood, that

3. the reading book is in the same weight category as the block of wood,

provided that we keep the same meaning and classification for 'is in the same weight category as', i.e. 'they are both within one ounce of 1 lb'.

The phrase 'weighs approximately the same as' is usually used differently: for instance, we could use it to mean that two objects differ in weight by less than one ounce. Now this does *not* establish a partition and hence is *not* an equivalence relation. We must *not* infer from

1. a penny ($\frac{1}{3}$ oz) weighs approximately the same as these sweets (1 oz) and

2. these sweets weigh approximately the same as an ink bottle ($1\frac{3}{4}$ oz) that

3. a penny weighs approximately the same as an ink bottle.

for the difference in weight between the penny and the ink bottle is more than one ounce. If we were silly enough to do this, we might just as well go on and on, choosing objects such that each was less than one ounce heavier than the previous one, and eventually misuse the transitive property to obtain 'a penny weighs approximately the same as an elephant.'

The children could also record their information like this:–

	over 1 lb	about 1 lb	under 1 lb
Exercise book			✓
Reading book		✓	
Tin of peas		✓	
Tin of nails	✓		
Tin of sand	✓		
Pencil case			✓
Ruler			✓

Now write down a list of things which you know are sold in *pounds*, e.g. apples.

From their weighing experiences children need to see for themselves that 16 of their ounces just about balance 1 lb, depending upon the accuracy of the apparatus. Then, having discussed this, we can abstract the mathematical correspondence

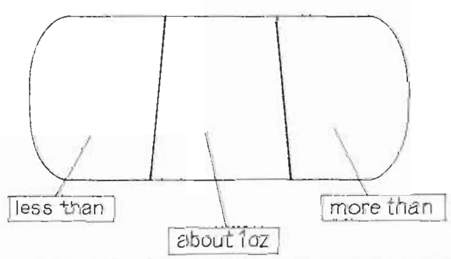
lb	oz
1	16
2	32
....	....

After their experiences with the 1 lb weight the children will feel the need for a smaller unit. We can now introduce the ounce weights. There should be a number of these available for children to weigh in *ounces*. Let them use a variety of objects and materials as before. Recording in lbs and ozs, and in ounces should be encouraged e.g. 1 lb 4 oz or 20 oz.

Example 5

Make a list of things which you think weigh about 1 ounce. Check by weighing and record. Use more than one way to record your discoveries, e.g.

	about 1 oz	more than	less than
Fountain pen		X	
Pencil	X		
Conker			X



When introducing children to the fractional parts of the pound weight it is best that they should be allowed to discover the relationship, between these and ounces for themselves. These facts can be recorded in their 'Book of Weight'. Useful assignments using the 2 oz, $\frac{1}{4}$ lb and $\frac{1}{2}$ lb weights can then follow:

Example 6

You will need a pair of balance scales. On one side you can use any weights you please but on the other side you can only choose from this set —

1 oz, 2 oz, $\frac{1}{4}$ lb, $\frac{1}{2}$ lb, and 1 lb.

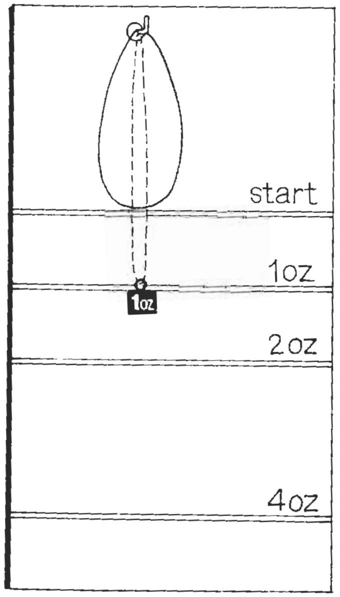
For example: to weigh or balance 1 lb 12 oz in one pan you would need to use the 1 lb, $\frac{1}{2}$ lb, and $\frac{1}{4}$ lb, from the given set, in the other pan.

Now see if you can complete the following:

	balances
9 oz →	½ lb and 1 oz
5 oz →	¼ lb and 1 oz
12 oz →	
13 oz →	
7 oz →	
1 lb 8 oz →	
1 lb 1 oz →	
1 lb 5 oz →	
1 lb 7 oz →	
1 lb 9 oz →	
1 lb 3 oz →	

Example 7

A further useful experience for children is to use the spring-balance. Elastic bands or springs can be calibrated by hanging the spring or elastic band over a nail fixed in the wall. To the lower end can be attached various weights. If a piece of white card is placed behind the spring, its position, when stretched by a particular weight, can be marked.

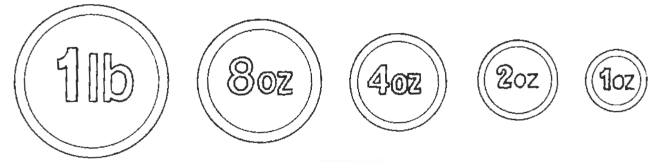


A graph can be plotted from the children's discoveries, using strips of gummed paper to mark the length of stretch. This paper length can then be transferred direct to the graph.

Practical experience which will link up with number bases can be carried out with the set of weights previously given for a weighing assignment. (See Example 6)

Example 8

You will need the following set of weights: — 1 lb, 8 oz, 4 oz, 2 oz, 1 oz. Put these weights in order of size beginning with the heaviest on the left i.e.



Make a drawing like this in your 'Weight Book'.
Underneath each, write the number of ounces in each weight.
Answer these questions.

1. How many 1 oz weights balance the 2 oz weight?
2. How many 2 oz weights balance the 4 oz weight?
3. How many 4 oz weights balance the 8 oz weight?
4. How many 8 oz weights balance the 1 lb weight?

Do you think you could make some ½ oz and ¼ oz 'weights' out of plasticine, using only the scales and 1 oz 'weights' to help you?

Teachers will see that here is an example of the binary scale: (Cf. Number Bases)

16	8	4	2	1
2^4	2^3	2^2	2^1	

(It will be recalled that
2^1 means 2

2^2 means 2×2

2^3 means $2 \times 2 \times 2$

2^4 means $2 \times 2 \times 2 \times 2$)

Some interesting recording using binary notation can be carried out as follows:

Make up some parcels, each weighing a number of ounces or 1 lb and a number of ounces. Ask the children to weigh the parcels and record the number of weights used e.g.

	1lb	8oz	4oz	2oz	1oz	
Parcel A (2oz)				1	0	(I used one 2oz weight)
Parcel B (4oz)			1	0	0	(I used one 4oz weight)
Parcel C (8oz)		1	0	0	0	and so on
Parcel D (12oz)		1	1	0	0	
Parcel E (20oz)	1	0	1	0	0	

Example 9 Complete the following table.

Weight in lb	Weight in oz
1	16
2	..
3	..
..	64
..	80

When the teacher has checked this with you, make a graph to show the relationship.

Example 10

Can you estimate the following?

1. The weight of a brick.
2. The weight of 3 pennies.
3. The weight of an egg.
4. The weight of a pint of water
5. Your own weight

Which is the heavier: a pound of feathers or a pound of iron? (Careful).
If you only had a pair of scales but no 'weights', how could you weigh 'about a pound' of something?

Example 11

Weigh 6 pennies on a scale then complete the following.
6 pennies weigh approximately ___ oz.

12 pennies weigh approximately ___ oz or ___ lb ___ oz.

24 pennies weigh approximately ___ oz or ___ lb ___ oz.

How many pennies will weigh approximately 1 lb?
Which is heavier, 10 pennies or a $\frac{1}{4}$ lb weight?

Summary

There seems to be little virtue in proceeding often beyond lb and oz at such an early stage. However, an obvious activity for young children is to weigh themselves, thus involving stones and lb. This can be done using bathroom-type scales. The emphasis should still be on pounds and a conversion table for converting stones to lb could be made. It is thought inadvisable to do much more than this.

In this section we have made reference to: —

a. the historical development of weight.
b. the child's concept of weight.
c. the approximation of weight.
d. standard units of weight.
e. application of numbers to weight (number bases and 'tables')

Weight Scale

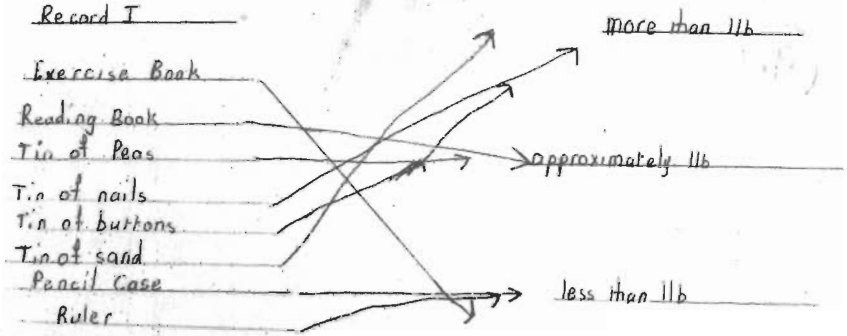
AMOUNT TO BE USED	WEIGHTS USED					
	2lb	1lb	8oz	4oz	2oz	1oz
1 oz						I
2 oz					I	
3 oz					I	I
4 oz				I		
5 oz				I		I
6 oz				I	I	
7 oz				I	I	I
8 oz			I			
9 oz			I			I
10oz			I		I	
11oz			I		I	I
12oz			I	I		
13oz			I	I		I
14oz			I	I	I	
15 oz			I	I	I	I
1lb 0oz		I				
1lb 1oz		I				I
1lb 2 oz		I			I	
1lb 3oz		I			I	I
1lb 4oz		I		I		
1lb 5oz		I		I		I
1lb 6oz		I		I	I	
1lb 7oz		I		I	I	I
1lb 8oz		I	I			
1lb 9oz		I	I			I
1lb 10oz		I	I		I	
1lb 11oz		I	I		I	I
1lb 12oz		I	I	I		
1lb 13oz		I	I	I		I
1lb 14oz		I	I	I	I	
1lb 15oz		I	I	I	I	I
2lb 0 oz	I					

Dawn Jennison

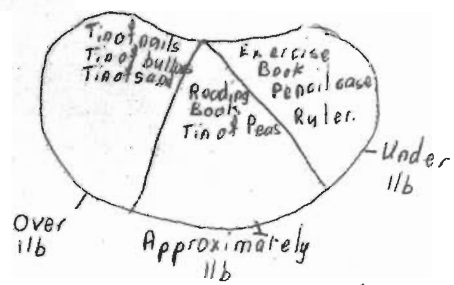
Card 21

Thing's that weigh about 1lb in Weight

Record I



Record II



Record III

	Over 1lb	About 1lb	Under 1lb
Exercise Book			✓
Reading Book		✓	
Tin of Peas		✓	
Tin of Nails	✓		
Tin of Button's	✓		
Tin of sand	✓		
Pencil Case			✓
Ruler			✓

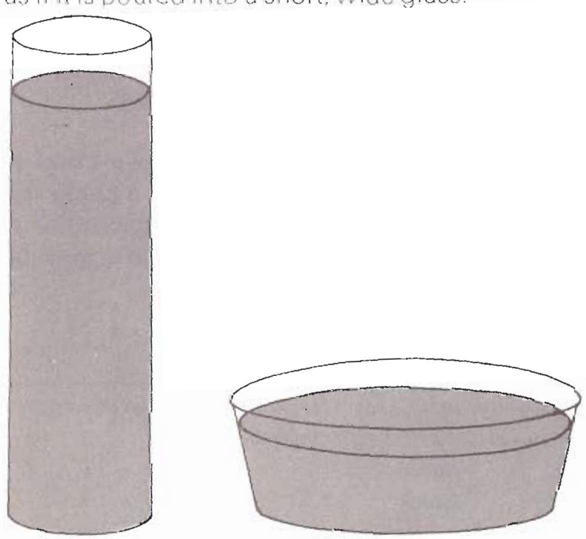
Sets of thing's which are sold in pounds are apples, pears, plums, cabbages, potatoes currants, sugar soap, cherries, carrot's

f. different ways of recording.
g. vocabulary: gravity, heavier, heaviest, lighter, lightest, ounce, pound, stone, balance, scales, names of materials, weight.

Capacity

There are strong links between liquid capacity, other measures of volume, and weight. These *connections* are usually beyond children of the early junior school years but reference should be made to work concerned with cubic measurement and 3D containers in *Shape and Size* 2.

Formal processes ('four rules' of gal, qt, pt.) are out of place. We must provide young children with practical experiences from which they can consolidate their acquisition of the principle of invariance e.g. under certain circumstances a quantity remains unchanged whether close together or spread apart. In the case of liquid measurement (capacity), the amount of liquid in a tall, narrow glass is the same as if it is poured into a short, wide glass.



Unfortunately, a measure of liquid volume is difficult to approximate. The eye is not a trustworthy guide to quantity. Capacity is measured, as in length, by another kind of matching operation: a visual comparison with a standard unit, but it is much more difficult to see. We can help children here if we make use of transparent containers wherever possible. We must provide vessels or containers of all kinds, to be filled with water or sand, and compared in quantity.

We can again point out, though it is almost certain that they will readily find out themselves, that this measurement also, can only be an approximate one. After trying to fill one or two containers to the brim, this should be quite obvious.

The need for standard units can then be built up after practical work of the following nature.

Example 1

You will need the liquid containers (jugs, cans, bucket, etc.) and the funnel. Use the funnel to avoid spilling the water.
Find out and record in your Capacity Book: —

How many cupfuls fill the jug.

How many spoonfuls fill the cup.

How many jugfuls fill the bucket.

How many table-spoonfuls fill the medicine bottle.

How many tea-spoonfuls fill the medicine bottle.

Example 2

You will need the spoons and a jug of water, also the medicine-bottles.

How many tea-spoonfuls of water are approximately the same as a dessert-spoonful?

How many dessert-spoonfuls of water are approximately the same as a table-spoonful?

How many tea-spoonfuls of water are approximately the same as a table-spoonful?

If the doctor ordered '3 teaspoonsful a day', how long would the small medicine-bottle last you?

Example 3

> Fill all the containers with sand or water and discover the capacity of each. Then record their order (which holds the most, next, . . . least)
>
> Now try to make a table for these:
> 2 cupfuls fill 1 jam-jar.
> 3 jam-jars fill 1 jug –
> and so on.
>
> Why can't we use 'tables' like this?

The children have been answering the question 'How much?' in cupfuls, jugfuls and spoonfuls. We can now proceed to obtain real familiarity with vessels holding a pint, $\frac{1}{2}$ pint, quart, and gallon. Cans, bottles, buckets and jugs can now be filled using the standard measures. The only convincing way to show what a bucket holds is to let the child fill it and record for himself. Though quarts are seldom used these days it is valuable to let children discover that a quart is a quarter of 1 gallon.

The word *gallon* was derived from the French 'galon' meaning 'a bowl'. It was used for liquid measure (oil) and dry measure (grain).

The following illustrates the connection between weight and capacity:

'1 gal measure is that which holds 10 lb of water at 62° F'.

'1 gal of corn weighs 8 lb'.

'The Babylonian *Talent* was the weight of 1 cu ft of water'.

Example 4

> You will need the set of liquid measures:
> 1 gallon, $\frac{1}{2}$ gallon, quart, pint, $\frac{1}{2}$ pint, gill.
> Arrange these measures in order of size, starting with the smallest.
> Write them down.
>
> Estimate first, then record:
> How many pints can I pour into 1 quart?
>
> How many pints can I pour into 1 gallon?
>
> How many quarts can I pour into 1 gallon?
>
> How many $\frac{1}{2}$ pints can I pour into 1 quart?
>
> How many $\frac{1}{2}$ pints can I pour into 1 gallon?
>
> How many gills can I pour into 1 pint?
>
> How do you think the quart got its name?
>
> If you pour 3 pints into the gallon measure, how many more pints would you need to fill it?
>
> Can you write out a table for these measures?

NB. In the North of England a gill is often considered to be $\frac{1}{2}$ pint but the statutory measurement is $\frac{1}{4}$ pint.

It must be emphasised that the number of pints in a quart (etc.) is exact in a theoretical table; when actually measuring, there will be inaccuracies, within the measuring instruments themselves, let alone spilling and the impossibility of precise definition of 'full'.

Example 5

> Write down, in your 'Capacity Book' all those things which you know are usually sold in *pints* or *gallons* e.g.
>
> | *Pints* | *Gallons* |
> |---|---|
> | Milk | Petrol |
>
> There are some 'dry' things as well, which are sold in pints.

I filled all the containers with water and discovered the capacity of each.

A bucket held ~~the~~ most water.
A jug held the next most water.
A jam jar held the next most water.
A cupful held the least.

I then made out Capacity Table
2 cupfuls fill 1 jam jar.
3 jam jars fill one jug.
1½ jugs fill 1 bucket.

We can't use tables like this because all jugs and buckets do not hold exactly the same amount.

Young children should be encouraged to experiment and find out something about the approximate capacities of familiar vessels, e.g. kettles, tea-pots, liquid-soap containers, watering-cans, lemonade bottles.

Doubling and halving are apparent in our tables of measures.

When arranging the measures in order children can be asked to record so that this progression becomes more obvious:

gill	$\frac{1}{2}$ pint	1 pint	1 quart	$\frac{1}{2}$ gallon	1 gallon
$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

In the section dealing with 'Weight' an illustration of the binary scale was given:

1 lb	$\frac{1}{2}$ lb	$\frac{1}{4}$ lb	2 oz	1 oz
16	8	4	2	1
2^4	2^3	2^2	2^1	

The doubling or halving process was also given for

112 lb	56 lb	28 lb	14 lb
(1 cwt)	($\frac{1}{2}$ cwt)	(1 qr)	(1 stone)

We can help children's awareness of certain relationships through these examples.

Example 6

You will need a bucket of water and the measuring cans.

1. How many quarts fill the gallon can?
2. How many gills fill the pint can?
3. How many $\frac{1}{2}$ pints fill the quart can?
4. How many quarts does the bucket hold?
5. How many pints is this?
6. How many bucketsful of water would you need to fill a bath holding approximately 20 gallons?

From the last example the teacher can take advantage of the situation to 'build' for tables by counting on in 2's, 4's, and 8's e.g.

gal:	1	2	3	4	5	6	. . .
qt:	4	8	12	16	20	24	. . .

qt:	1	2	3	4	5	6	. . .
pt:	2	4	6	8	10	12	. . .

gal:	1	2	3	4	5	6	. . .
pt:	8	16	24	32	40	48	. . .

Example 7

Take a pint bottle and fill it with water. Carefully pour the water from the pint bottle into a school milk-bottle. Empty the school bottle when full and repeat this until the pint bottle is empty.

How many school milk-bottles can you fill from the pint bottle?

Can you give another word used for the school milk-bottle (part of a pint)? Discuss this with your teacher, then write it down in words and as a number.

How many school milk-bottles could you fill if you had –

a. a pint of milk
b. a quart of milk
c. a gallon of milk?

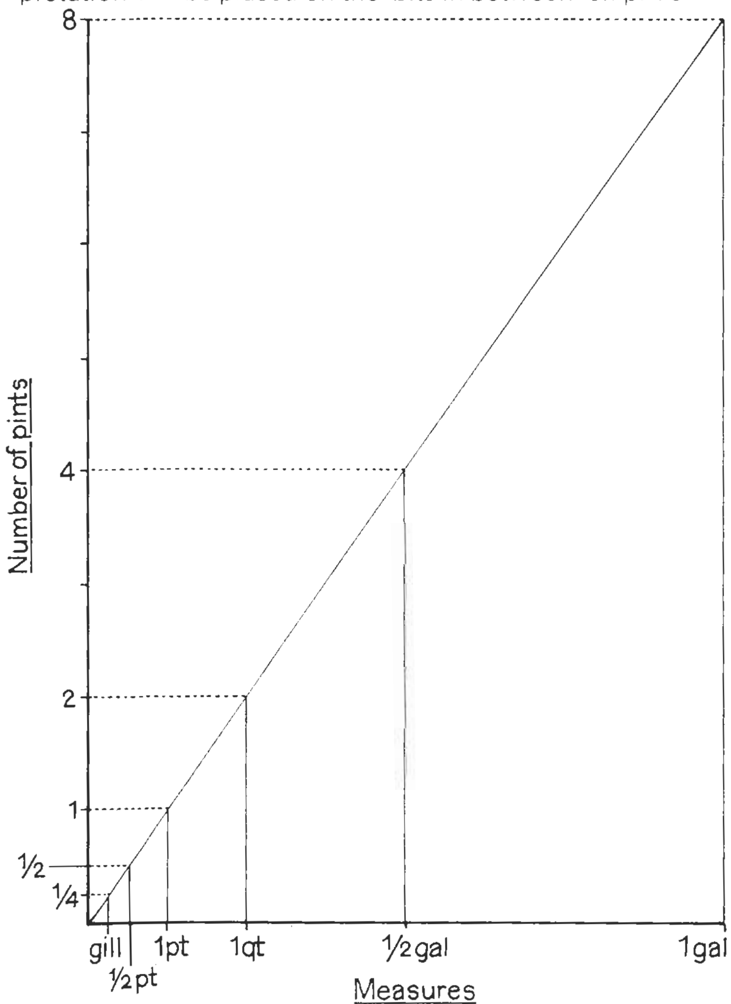
Record your information, then complete this table: —

pints	Number of 1/3 pints
1	3
2	....
3	....
4	12
5	....
6	....
....	21
....	24

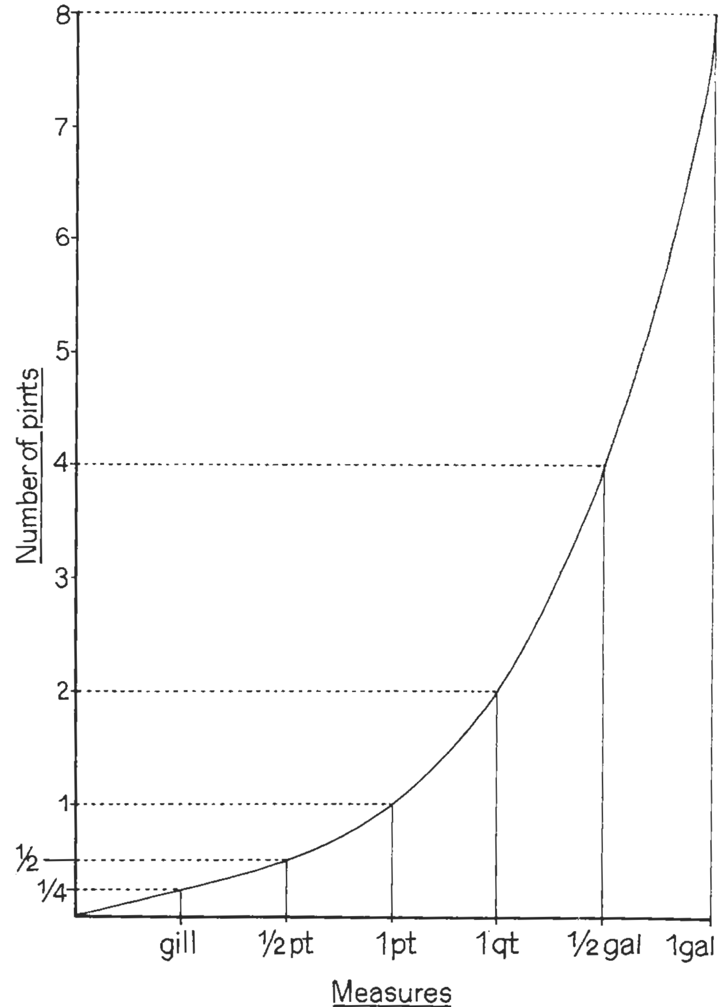
How far can you go?

The school milk-bottle is an obvious 'classroom measure' but has little other practical value than has so far been illustrated: as a fractional part with which the children are so familiar and so it can help in the table of 3's.

The following graph shows the relationship between measures and pints. Once again although the points are joined to show that they lie on a line, only an intuitive interpretation can be placed on the 'bits in between' cf. p. 10.



It may be instructive to look at this second graph (cf. 'Care needed . . .' *Pictorial Representation* **1**).



Here the measures 'gill', '$\frac{1}{2}$ pint', etc., are equally spaced along the horizontal axis although successive measures increase by a *factor* of 2. Joining the points plotted now produces a curve and interpretation of the 'bits in between' becomes even more hazardous.

LINDA
SWAN

Card 54

Using the Right Units for measuring.

I would measure

① My own height in ft and inches

② The length of a fence in yards.

③ A journey from London to Paris, in miles.

④ The cost of a postage stamp in d.

⑤ The price of a new coat in £'s.

⑥ The weight of a box full of groceries in lb's

⑦ The weight of a railway engine in tons.

⑧ The weight of a small twig in ounces.

⑨ The value of a house in pounds. £'s

⑩ How much water a saucepan holds in pints

⑪ The amount of water in a swimming bath in gallons.

⑫ The duration of playtime in minutes

⑬ The life of an elephant in years.

⑭ The circumference of a tennis ball in inches.

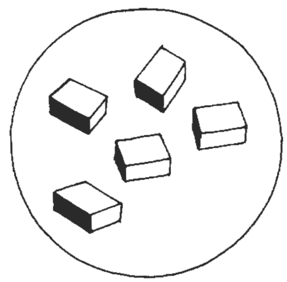
⑮ The span of my hand in inches

Summary

Capacity as a liquid and dry measure.
Its connection with weight and other volume.
Familiarity with containers of different cross-section.
The need for standard units.
Practical experiences using gal, qt, pt, etc.
'Doubling' and 'halving' relationships.
Building up 'tables': 2's, 3's, 4's, 8's.
Vocabulary:
names of standard units
names of containers
capacity.

3 Counting – towards addition

Much early mathematical experience is concerned with counting: ones, shillings, conkers, people . . . , in fact counting the number of elements in a certain set.



The number of bricks in this set is 5. (It would be quite wrong to say 'The set equals 5', as the set is made up of bricks as its elements and of course 5 is an abstract concept).

We shall now be concerned with economy of effort in counting various collections and this will lead on to considering operations such as addition and relationships between numbers.

The earliest mathematical experiences of young children are nearly all based on matching and comparison and they may well meet 'subtraction' and 'division', and certainly inequalities, before the operation of addition traditionally done first. They compare their belongings: 'I've got more than you'. They match objects to discover 'more than' and 'less than' and use the same matching operation when setting the table or sharing things. However, at this stage, it is unlikely that young children would record their early experiences in the symbolism or short-hand of mathematics and it is undesirable that they should. It will be done verbally or in very simple written English.

We must also remember that the four operations are very closely linked: subtraction is the inverse of addition; division the inverse of multiplication. In the early stages children 'subtract' by adding on and before they can multiply they employ successive addition; for division they keep on 'taking away'. This is all part of their experiences and it is artificial to start with one operation and follow with the others in a particular order. However, when we come to developing the structure of mathematics we shall start with the usual 'addition' operation.

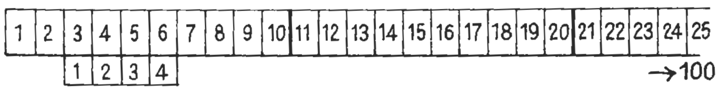
The aim in this part of the guide is to give some ideas on number work: pattern, relationships, operations and properties. The children will, it is hoped be meeting the majority of their number work through practical experiences in real situations, using weights and measures, pictorial representations and shapes. The underlying structure and patterns of the numbers themselves will gradually be abstracted from this experience.

Counting on and back. Use of the Number-Strip

In order to practise computation in an efficient manner, children must be able to 'count on', not just in ones, but in 10's, 2's, 5's, 9's, etc. Many children are made to do 'mechanical' arithmetic with large numbers long before they can count efficiently and this is probably the cause of much of the dislike for mathematics. From their very early work with concrete objects, sorting and matching, recognising the cardinal and ordinal aspects of numbers (cf. *Beginnings* ❶), we can lead children to this next stage of development by representing the set of natural numbers (counting numbers) on a number-strip, very much like the number ladder.

Many teachers fix a 'master-strip' on the class-room wall, at a convenient height for children to use it, usually from 1 to 100. This is made from thin card and given a suitable protective covering of plastic tape or varnish. Smaller strips, representing numbers 1 to 10 are placed in a pocket or hung close to the 'master-strip'.

Children can make their own from strips of one inch graph paper stuck on to thin card. At first, a strip representing the set of numbers 1 to 20 can be used, then 20 to 40, 40 to 60 etc. can be added as required. A smaller strip representing say the numbers 1 to 10 is also needed, of course, for carrying out counting operations; indeed separate strips representing each number from 1 to 10 will probably be used in the initial stages.



Using the strip children can be asked to count in twos, threes, etc. and, of course, tens. This can be useful preparation not only for 'addition facts' but multiplication as well. Successive addition of the starting number will encourage the memorisation of tables when we wish to do this at a later stage, e.g.

2, 4, 6, 8, 10, 12, . . .

3, 6, 9, 12, 15, 18, . . .

4, 8, 12, 16, 20, 24, . . .

but other series should be built up also:

1, 3, 5, 7, 9, . . .

2, 5, 8, 11, 14, . . .

13, 23, 33, 43, 53, . . .

Counting backwards is also useful practice for subtraction and division:

15, 12, 9, 6, 3.

50, 40, 30, 20, 10.

25, 20, 15, 10, 5.

and this same apparatus can be used for multiplication and division:

What number do you reach if you step off five, four times?

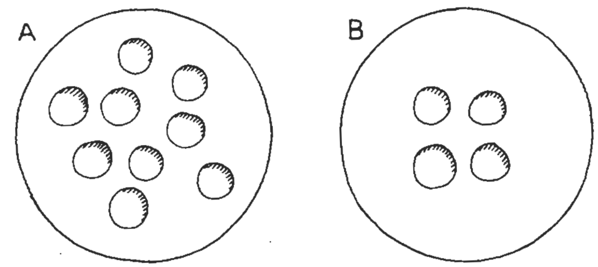
How many 8-strips are needed to reach 24?

(The formulation of such questions must be done carefully. To the question 'How many 5's in 25?', a fair answer would be 'one' – the one after the 2).

Until now, children have not been introduced to equations of the form

$6+4=10$

though they will have met the simpler 'inequality' signs.



For example $9>4$ (9 is greater than 4). The diagram shows two sets, one with 9 elements and the other with 4.

It is important to communicate as accurately as possible. The sign '>' should be used only in comparing *numbers*. It would be correct to write
'The number of elements in Set A is greater than the number of elements in Set B'

but nonsense to write
'Set A > Set B'
for we have not defined the meaning (if any) of a set itself being 'greater' than another one. This *might* mean for example 'more important' or 'takes up more space' and it is wrong to use such a private shorthand which could lead to muddled thinking.

The children should understand what they are really doing with numbers rather than beginning to use the symbols too hurriedly. At this stage, therefore, we can continue to record as suggested in *Beginnings* ❶, e.g.

i 2 ⟶ 4 ⟶ 6 ⟶ 8 ⟶ 10

or

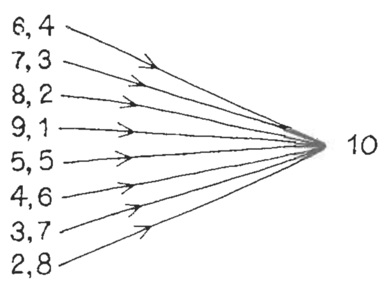
2, 4, 6, 8, 10, 12, ...

ii

add

2, 4 ⟶ 6

3, 8 ⟶ 11



Suggestions for more work with the Number-Strip

A. Count in twos from:

1. 2 to 10
2. 18 to 22
3. 14 to 20
4. 10 to 20
5. 20 to 40
6. 65 to 81
7. 7 to 19
8. 12 to 30
9. 42 to 54
10. 60 to 70

B. Count back in twos from:

1. 9 to 1
2. 19 to 1
3. 50 to 30
4. 35 to 21
5. 70 to 50
6. 99 to 85
7. 26 to 12
8. 20 to 2

C. Using the set of numbers from 1 to 20, write down
a) the even numbers
b) the odd numbers

What is the number of your house?
What are the numbers of the houses each side of you?
Is the number of your house odd or even?

The same kind of practice can be carried out with different 'counting on' or 'counting back' numbers.

In many ways the number-strip serves the same purpose as the 'number-track' of the Stern apparatus. It can give children a visual representation of moves and their patterns so that they achieve an awareness of relationships.

It is not necessary, of course, for the children to record all the moves on the number-strip, in fact they should have times when they are left alone to make their own discoveries at will. It must be remembered that our aim is to eliminate the need for 'counting on' in ones and to familiarise the child with number series as a basis for future operational experiences.

Karen Girling
Class 1.

My Magic Ready Reckoner.

My Mummy has a Washing Machine.
It save's Mummy time and does not make her tired.
This morning we all made a Magic Ready Reckoner.
We had had two pieces of squared paper.
We put a number in each square up to 40.
On the small paper we put a number in every square up to 20.
This is useful for adding up and taking away.
I can do my work more quickly with this apparatus.

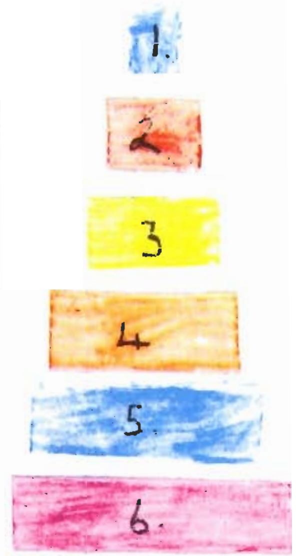
Stephen Bell I.Q 98

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	2	3	4	5	6	7	8	9	10	11	12	13	
2	3	4	5	6	7	8	9	10		12	13		15
3	4	5	6	7	8	9	10		12	13		15	16
4	5	6	7	8	9	10		12	13		15	16	17
5	6	7	8	9	10		12	13		15	16	17	
6	7	8	9	10		12	13		15	16	17		19
7	8	9	10		12	13	14	15	16	17		19	20
8	9	10		12	13		15	16	17		19	20	21
9	10		12	13		15	16	17		19	20	21	22
10	11	12	13		15	16	17	18	19	20	21	22	23
11	12	13		15	16	17	18	19	20	21	22	23	24
12	13		15	16	17	18	19	20	21	22	23	24	25
13		15	16	17	18	19	20	21	22	23	24	25	26

Linda Tyson

I.Q. 91, age 9

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	13
2	3	4	5	6	7	8	9	10	11	12	13	14
3	4	5	6	7	8	9	10	11	12	13	14	15
4	5	6	7	8	9	10	11	12	13	14	15	16
5	6	7	8	9	10	11	12	13	14	15	16	17
6	7	8	9	10	11	12	13	14	15	16	17	18
7	8	9	10	11	12	13	14	15	16	17	18	19
8	9	10	11	12	13	14	15	16	17	18	19	20
9	10	11	12	13	14	15	16	17	18	19	20	21
10	11	12	13	14	15	16	17	18	19	20	21	22
11	12	13	14	15	16	17	18	19	20	21	22	23
12	13	14	15	16	17	18	19	20	21	22	23	24



using these strips
I made some sums.

3 + 6 = 9. 5 + 17 = 22 1 + 8 = 9.

6 + 15 = 21. 4 + 13 = 17. 6 + 13 = 19

5 + 11 = 16. 2 + 14 = 16. 4 + 13 = 17.

The following example provides a variety of number sequence work:

D. What number is missing from the following:

1. 3, 5, 7, 9, [ ]
2. 5, 8, 11, 14, [ ]
3. 30, 40, [ ], 60, 70
4. 6, [ ], 16, 21, 26
5. 26, 24, [ ], 20.

E. What numbers are a) 6 more than

1. 6	2. 12	3. 15	4. 10
5. 21	6. 29	7. 18	8. 28
9. 56	10. 35		

b) 6 less than

1. 18	2. 12	3. 24	4. 15
5. 36	6. 21	7. 69	8. 81
9. 42	10. 30		

F. Time yourself with an egg-timer and see how many numbers you can write down following a pattern, e.g. 6, 10, 14, 18, 22, . . .

Try the following:

1. Up in tens from 2
2. Down in tens from 75
3. Down in threes from 78
4. Up in fours from 4
5. Up in nines from 9

G. House Numbers

In a street the odd numbers are on one side of the road and the even numbers on the other side. Here are some of the house numbers:

12, 28, 67, 10, 27, 58, 42, 47, 19, 101, 1, 99.

Write down, in columns, the numbers of the houses on each side of the road, in order, and if any are missing, put them in.

So far, we have not referred to Place Value. This will be dealt with a little later on. Children, at this stage, are using numbers as names but getting an insight into our economical system of notation through their many experiences in number-work of all kinds. They recognise that 10 is a bigger number than 9 and so 20 one more than 19. They will have had experience of collecting together sets of 10 so that 20, 30, 40, 50, etc. does mean something to them even though they may not, at this stage, understand the significance of the position of each numeral.

Below are some ideas to help to provide children with a mastery of numbers.

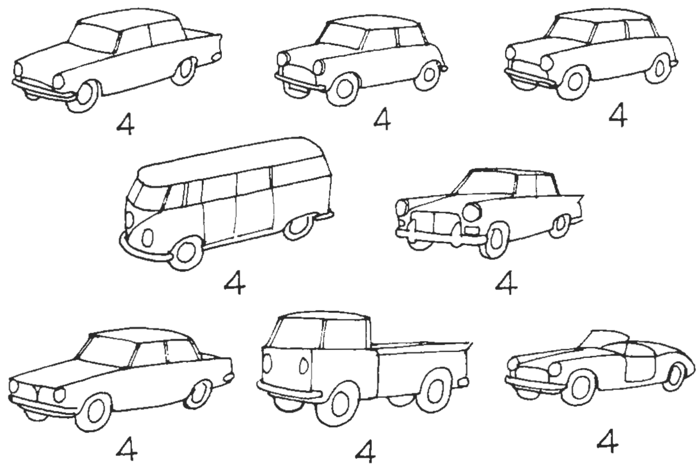
How many wheels on the 8 'dinky' cars?

Record the number of feet corresponding to the number of yards

ft	yd
3	1
6	2
..	3
..	4
..	5
..	..
..	..

> See how many pints of water will fill the large tin, using the quart (2 pints) measure.

If we take the problem of 'How many wheels on the eight dinky cars?' we should probably find that the child would count in ones, but, we must, in this situation, grasp the opportunity to lead the child to count in fours:



leading to

4, 8,16, 20, 24, 28, 32

These kinds of experience are beneficial long before we ask the child to memorise (learn tables). By the time we arrive at this stage, if adequate experiences have been encountered and number series have been met in all kinds of situations, most children will have built a good deal into their memory-store which will improve the chances of 'learning tables' considerably.

Practical work related to 'table-facts' can be obtained from the following examples:

2's quarts and pints, pairs of objects

3's length; sides of triangles (kept apart)

4's Perimeters of squares increasing sides by one unit at a time; capacity (qts and gals); chairs and legs; animals and legs

5's clocks (5 min intervals); fingers and toes

6's money, length

7's calendar (weeks, days)

8's capacity (pints and gallons)
length (furlongs, miles)
money (half-crowns)
weight (stones, cwts)

9's square measure; pattern of nines, one less than 10

10's metric measures

12 and 20 as dozens and scores will also figure frequently in tables of measures. Doubling and halving will also be experienced by young children quite early. They should be encouraged to make comparisons and in weights and measures we can show the doubling and halving process at work as:

112lb	56lb	28lb	14lb
8pt	4pt	2pt	1pt

The '100' Square

At all times we should try to provide a variety of experiences in our number work so that patterns and relationships are seen in many situations. We do not want children always to be thinking of number in terms of length or on a horizontal strip.

The '100' square can be a useful aid to children for observing the patterns which are made by numbers. They enjoy colouring certain numbers and seeing a pattern emerge. This all helps their understanding and gives a visual image of what could be most abstract to a young mind.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

100 square

These squares can be duplicated and children asked to cross off or colour numbers in a particular sequence. Patterns and useful information can be obtained in this way.

Children can be asked 'What is the interval between numbers in the vertical columns or diagonal columns?'

'100' squares can be numbered in a different way and numbers omitted for children to fill in. There are many variations of this on the same theme.

100	90	80	70	60	50	40	30	20	10
	89	79	69	59	49	39	29	19	9
98		78	68	58	48	38	28	18	8
97	87		67	57	47	37	27	17	7
96	86	76		56	46	36	26	16	6
95	85	75	65	55	45	35	25	15	5
94	84	74	64	54		34	24	14	4
93	83	73	63	53	43		23	13	3
92	82	72	62	52	42	32		12	2
91	81	71	61	51	41	31	21		1

Many teachers are already finding various structural appparatus also a useful way of building up patterns of number.

Other useful practical work can be obtained from:

Numbers on tickets

Pages of a book

Numbering a plan of seats for a concert

Another useful device for displaying patterns of numbers on the '100 square' is to make the square on a larger sheet of thin card, (10 inches square is a useful size) and cut windows in other 10 inch squares of card so that for each card, when placed on the '100 square' a pattern of twos, or threes, etc. shows through. Interesting here is that when the cards for twos and threes are placed on the square, the pattern of sixes shows through, illustrating that 6 is a multiple of 2 and 3.

'100 Square' with 'window-cards'

'Window-card' for two

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

'Window-card' for three

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

6

1	2	3	4	5	**6**	7	8	9	10
11	**12**	13	14	15	16	17	**18**	19	20
21	22	23	**24**	25	26	27	28	29	**30**
31	32	33	34	35	**36**	37	38	39	40
41	**42**	43	44	45	46	47	**48**	49	50
51	52	53	**54**	55	56	57	58	59	**60**
61	62	63	64	65	**66**	67	68	69	70
71	**72**	73	74	75	76	77	**78**	79	80
81	82	83	**84**	85	86	87	88	89	**90**
91	92	93	94	95	**96**	97	98	99	100

1

9

1	2	3	4	5	6	7	8	**9**	10
11	12	13	14	15	16	17	**18**	19	20
21	22	23	24	25	26	**27**	28	29	30
31	32	33	34	35	**36**	37	38	39	40
41	42	43	44	**45**	46	47	48	49	50
51	52	53	**54**	55	56	57	58	59	60
61	62	**63**	64	65	66	67	68	69	70
71	**72**	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	**90**
91	92	93	94	95	96	97	98	**99**	100

12

1	2	3	4	5	6	7	8	9	10
11	**12**	13	14	15	16	17	18	19	20
21	22	23	**24**	25	26	27	28	29	30
31	32	33	34	35	**36**	37	38	39	40
41	42	43	44	45	46	47	**48**	49	50
51	52	53	54	55	56	57	58	59	**60**
61	62	63	64	65	66	67	68	69	70
71	**72**	73	74	75	76	77	78	79	80
81	82	83	**84**	85	86	87	88	89	90
91	92	93	94	95	**96**	97	98	99	100

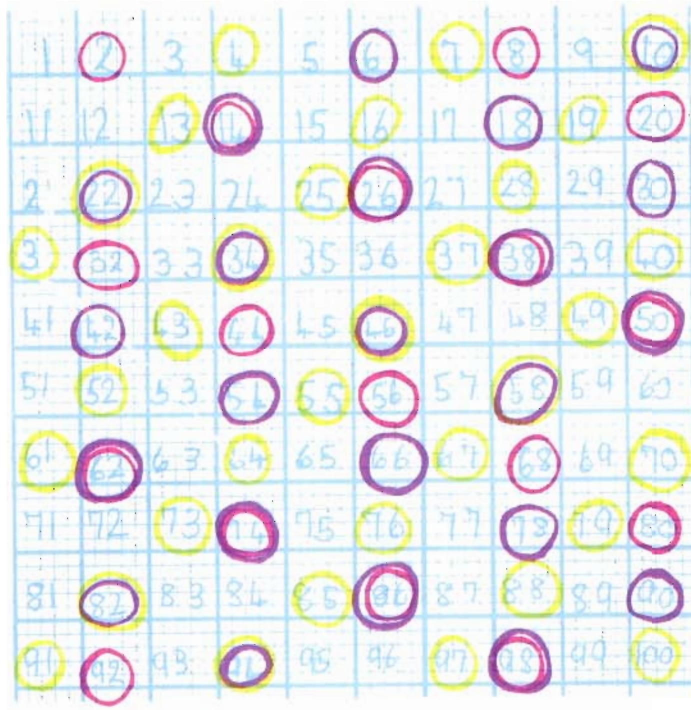
1

11

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	**22**	23	24	25	26	27	28	29	30
31	32	**33**	34	35	36	37	38	39	40
41	42	43	**44**	45	46	47	48	49	50
51	52	53	54	**55**	56	57	58	59	60
61	62	63	64	65	**66**	67	68	69	70
71	72	73	74	75	76	**77**	78	79	80
81	82	83	84	85	86	87	**88**	89	90
91	92	93	94	95	96	97	98	**99**	100

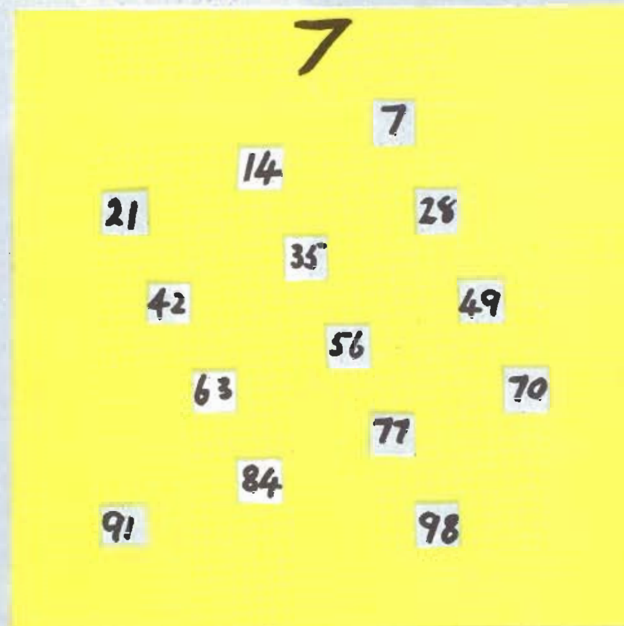
Kathleen Harrison Age 9

Adding Square



This square is very interesting. It shows that every number which has a red ring round it is every sixth number starting from two instead of nothing. Aswell as that the numbers which have purple rings round them are every fourth number starting from six. Also each number that has a yellow ring round it is counting in three's starting from number four.

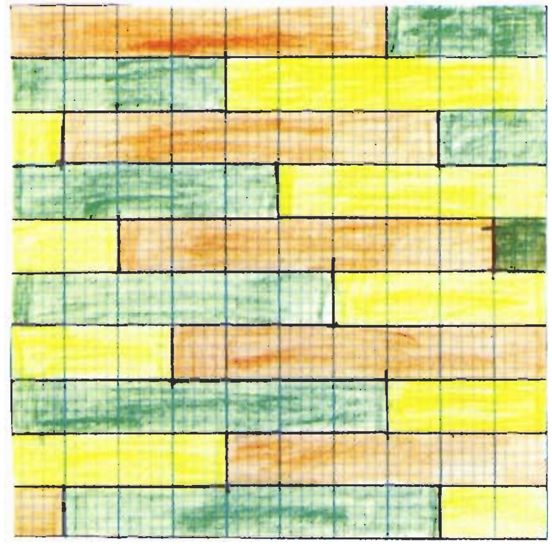

1 2 3 4 5 6 7 8 9 10
11 12 13 14 15 16 17 18 19 20
21 22 23 24 25 26 27 28 29 30
31 32 33 34 35 36 37 38 39 40
41 42 43 44 45 46 47 48 49 50
51 52 53 54 55 56 57 58 59 60
61 62 63 64 65 66 67 68 69 70
71 72 73 74 75 76 77 78 79 80
81 82 83 84 85 86 87 88 89 90
91 92 93 94 95 96 97 98 99 100


7
7
14
21
28
35
42
49
56
63
70
77
84
91
98


7
7
14
21
28
35
42
49
56
63
70
77
84
91
98

Kim Harrison Age 9

The pattern of seven and ten



This is a simple way of adding up. First it shows how to add up to ten, on the first line it shows seven and three, on the second six and four, on the third seven, two, and one, on the fourth five and five, on the sixth two, seven, and one, on the seventh seven and three and so on. Secondly in sevens each block is seven, and you can add up very easily with the help of it

4 + 3 = 7	3 + 4 = 7.	7 + 3 = 10	3 + 7 = 10
6 + 1 = 7	6 + [illegible] = 7	6 + 4 = 10	4 + 6 = 10
2 + 5 = 7	5 + 2 = 7	5 + 5 = 10	5 + 5 = 10
0 + 7 = 7	7 + 0 = 7	8 + 2 = 10	2 + 8 = 10
		9 + 1 = 10	1 + 9 = 10
		10 + 0 = 10	0 + 10 = 10

'Window-cards' for both two and three

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

From the diagram can be seen the pattern of sixes emerging when a card with windows for twos and a card with windows for threes has been placed over the '100 Square'.

It is a useful aid for young children to keep a tabulated reference in their record books with particular reference to multiplication facts. An example of this is given below:

Table of Twos

6d	10s	pt	1	2	3	4	5	6	7	8	9
1s	£1	qt	2	4	6	8	10	12	14	16	18

Eights

2s 6d	fur	pt	st	1	2	3	4	5	6	7	8	9
£1	ml	gal	cwt	8	16	24	32	40	48	56	64	72

Summary of what we have done so far

Use of the number-strip to present visual representation of numbers.

Experience in counting on and back.

Preparation for addition, subtraction and multiplication.

Number series: building up addition and multiplication facts.

100 square and number patterns.

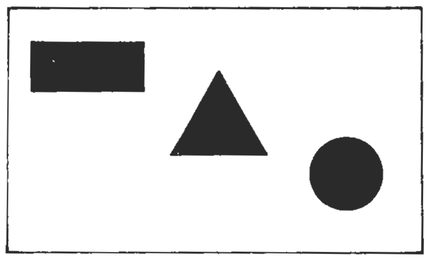
Supplementing practical work.

Vocabulary: diagonal, sequences, odd, even, doubling, halving.

4 The operation of addition

In *Beginnings* ❶ we saw that children sort and classify and partition sets of objects into sub-sets. They then also learn to count the number of elements in a set.

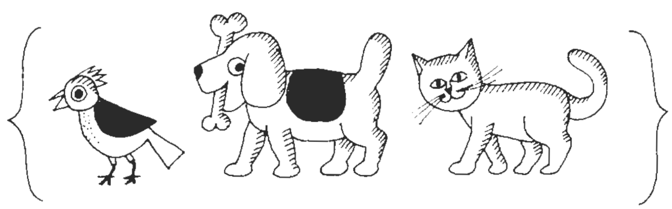
A number tells us how many elements a set has. For instance, all sets of objects whose elements can be matched with this set:



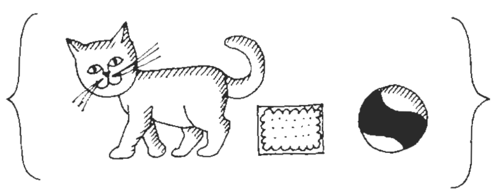
have the common property of *threeness*.

The *union* of two sets is the set of elements which belong to at least one of them.

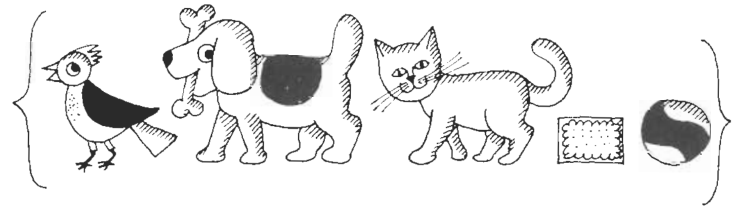
For example, let set A consist of a bird, the cat Copper and the dog Bonzo:



and let set B consist of a biscuit, a ball and the cat Copper:

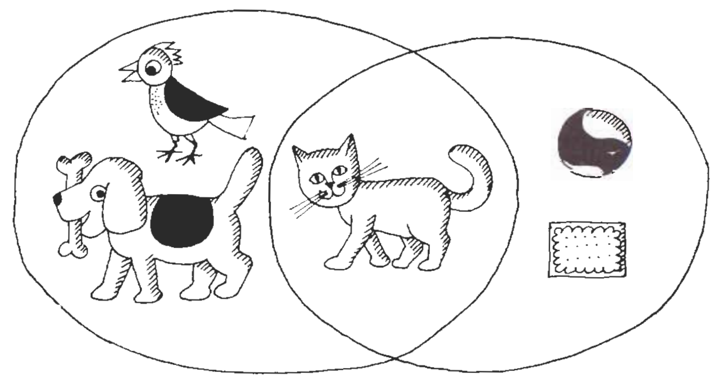


Then the union, written A U B, consists of



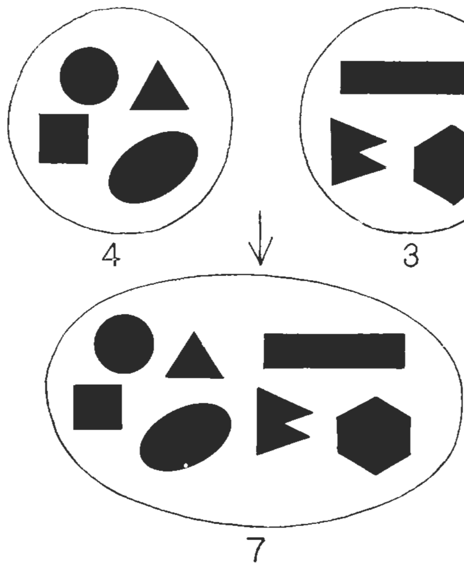
This is the set of elements which belongs to A or B or both.

The *intersection* of two sets consists of the set of elements which belong to both.



In this example, the intersection of A and B (written A ∩ B) consists of the cat Copper who is the only 'element' belonging to both A and B.

We now consider only sets which are *disjoint*, i.e. have no elements in common, and we define addition in terms of their union.



The *sum* of the numbers of elements in two disjoint sets is the number of elements in their union.

We now come more formally to operations on numbers, and we shall be considering the set { 0, 1, 2, 3, . . . } i.e. zero as well as the natural numbers.

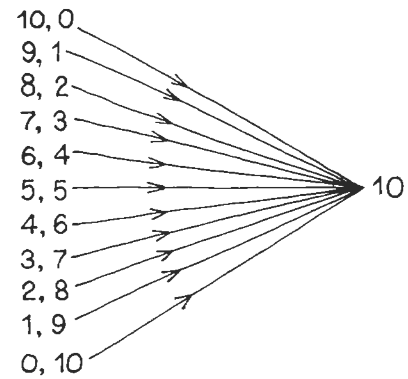
Perhaps we should begin by asking 'What is an operation?' In the case of addition, and this is the first operation which we shall require our children to perform, we have an operation on *two* numbers and it is therefore called a binary operation. The binary operation of addition on two of the numbers of the set results in another number also belonging to the set.

In *Beginnings* ❶ we recorded in the form

$$6, 4 \longrightarrow 10$$

We say that 10 is 'the sum of' the numbers 6 and 4.

Other pairs of numbers also are mapped on to 10: —



The teacher will recognise these very important number pairs as 'number bonds' for 10.

Children should already have had practice in mapping pairs of numbers on to their 'image' (in this case 10) in this way.

The set of ordered pairs whose image under this mapping 'add' is a given number (here 10) form an *equivalence class*.

i. *Reflexive* For example 8, 2 is in the same class ('has 10 as image') as itself.

ii. *Symmetric If* we know that 8, 2 is in the same class as 3, 7, *then* we know that 3, 7 is in the same class as 8, 2.

iii. *Transitive If* we know that 8, 2 is in the same class as 3, 7 (i.e. both have image 10) *and* that 3, 7 is in the same class as 1, 9, *then* we know that 8, 2 is in the same class as 1, 9.

The mapping therefore defines an *equivalence relation* between the ordered pairs and instead of writing

'(9, 1) is equivalent to (7, 3)'

or '(2, 8) is equivalent to (10, 0)'

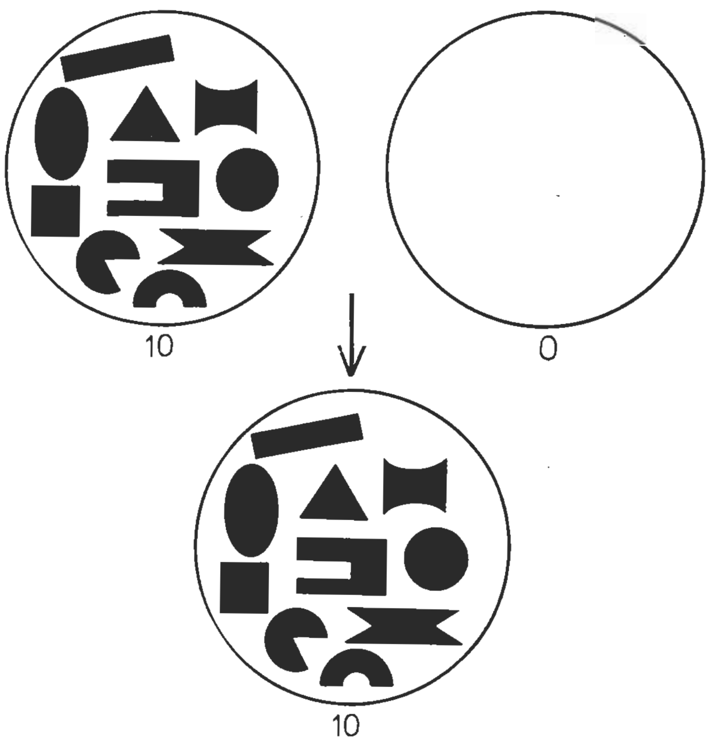
we shall as shorthand write

9+1=7+3

2+8=10

6+4=10 is a shorthand form for 'the image of the number pair 6, 4 under the addition mapping (add) is the same as that of the number pair 10, 0.'

10+0 *can be written as* 10, since 0 is the 'neutral element' for addition, that is it leaves any number unchanged when added to it.



We shall for the present use the '=' sign only to mean *represents the same number as* (i.e. has the same image under the addition mapping). The following are abuses of the sign:

1 pink doll	=2s 6d
Parking all day	=4s 6d (Seen at the entrance to a London car-park)
Ans	=25
22yd	=1 cricket pitch
long	=length (often used for English grammar exercises)

When using the addition operation, especially in the very early stages, it is advisable to concentrate on the *operation* itself, and extra time spent with numbers which the child can add easily will be repaid later on. Use of the number-strip and other aids: bricks, cubes, structural apparatus, are all helpful and basic to the child's understanding of the operation he is to perform.

Young children receive first-hand experience and useful practice when they are engaged in activities of the classroom shop.

As witnessed in one school, the children were given a coin of particular value, namely a shilling, and told to go and see how many different things they could buy at the shop. This led of course to:

6d 6d (2 sixpenny articles)

8d 4d

9d 3d

2d 3d 7d

3d 3d 3d 3d (4 threepenny articles)

and so on.

This was valuable in exposing the children to 'the story of 12'.

6+6=12

8+4=12

9+3=12

7+5=12

When dealing with three amounts, it will be discovered for example that

2+3=5

so (2+3)+7=5+7=12

the brackets being used to show that we first find the image of (2+3), namely 5, and then in turn the image of 5+7, namely 12. but also

3+7=10

so 2+(3+7)=2+10=12

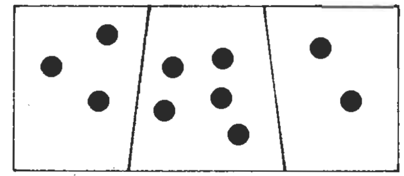
and so (2+3)+7=2+(3+7)

This illustrates the *associative property* of the numbers we are considering:

(a+b)+c=a+(b+c).

Having made the discovery that (2+3)+7=2+(3+7), it is permissible to drop the brackets and write 2+3+7, the understanding being that it doesn't matter in which order the *operations* are carried out.

A set of objects can be partitioned (cf. *Beginnings* ❶) in different ways, and the children can be asked to record their various re-arrangements in terms of the numbers of elements of the sub-sets, e.g.



3 + 5 + 2 = 10

Partitioning in other ways. 8+2=10, etc.

4 + 6

Handling sets of objects in this way children can also see that addition has the 'commutative property', i.e. the order of the *numbers* doesn't matter. For example,

6+4=4+6

8+3=3+8

Using both the commutative and the associative properties allows us to add in any convenient order we wish. For example to add

6+7+2+3+4

we can write:

(6+4)+(7+3)+2

= 10 + 10 +2

= 22.

The number strip can be used with great effect also and all kinds of useful and important combinations can be discovered e.g.

6+9=15

16+9=25

26+9=35

36+9=45

46+9=55

Important still are the 'addition facts' or 'bonds' but these should not be approached in a dreary or monotonous way (similar to the learning of 'tables') but for example as patterns:

6+0=6

5+1=6

4+2=6

3+3=6

2+4=6

1+5=6

0+6=6

Children can be led to observe symmetrical patterns, and the role of zero can again be pointed out, though it should not be laboured as it has been in the past. The symmetrical patterns will also emphasise the commutative property. This can be shown by means of structural apparatus, but also tables or squares such as the following can be most useful for observing:—

+	0	1	2	3	4
0	0	1	2	3	4
1	1	(2)	3	4	5
2	2	3	(4)	5	6
3	3	4	5	(6)	7
4	4	5	6	7	(8)

i. The commutative property, e.g. 2+3=3+2.
ii. The 'doubled' numbers which appear on the diagonal from upper left to lower right

1+1 (2)
2+2 (4)
3+3 (6)
4+4 (8)

Children can be asked to find a number which when added to another number gives a sum in the body of the table square. e.g.
What number must be added to 3 to give 7?
This question can be asked in the form
3+□=7. What number must be put in the box to make this a true statement?

From their 'counting on' experiences, children have little difficulty with examples such as 6+4=□. The real problems are contained in the following situations:

6+□=10

□+5=8

□+3=△+2

Each of these is an 'open sentence'. The problem is to find numbers to put in the □ or △ in order to make the sentence *true*. In the first example, the required number is 4; in the last, we can find many pairs of numbers, for example 5 in □ and 6 in △. In fact a table could be made

■	▲
5	6
6	7
2	3
0	1
7	8

etc.

Alison Wood Age 8 years.

4	+	3	=	5	+	2
7	+	9	=	8	+	8
7	+	4	=	5	+	6
8	+	10	=	9	+	9
4	+	6	=	5	+	5
3	+	12	=	9	+	6
1	+	7	=	4	+	4
6	+	4	=	5	+	5
7	+	9	=	8	+	8
12	+	6	=	3	+	15
2	+	9	=	6	+	5
20	+	9	=	23	+	6

Beverley Burgess Age 7years

15 + 9 > 2 + 4 + 3

9 + 3 + 4 > 2 + 4

2 + 6 > 1 + 2 + 2

9 + 2 + 8 > 4 + 2

9 + 21 > 1 + 8 + 2

2 + 8 + 11 > 3 + 1

8 + 9 + 11 > 2 + 2

11 + 13 > 2 + 3 + 2

11 + 12 + 19 > 9 + 3

12 + 10 > 2 + 3 + 6

19 + 20 + 16 > 2 + 7

19 + 2 > 1 + 7 + 6

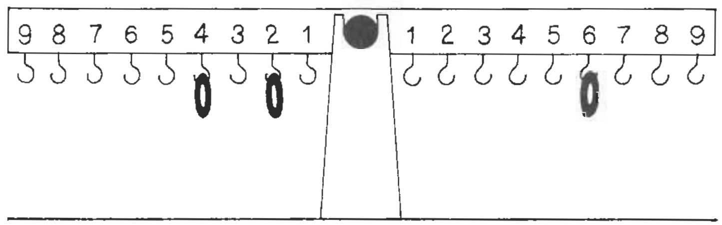
Before we become too pre-occupied with equalities, it is, perhaps, a good idea to allow children to discover and record those combinations of numbers which are not equal, and introduce them to the sign of inequality ≠.

Children could then record

6+5≠10

6+4≠6+5

A useful piece of apparatus for children to discover equalities and inequalities is the 'Equaliser'.



Children hang washers on the numbered hooks and discover and record such facts as: A washer on the 6 and a washer on the 4 on one side, balance a washer on the 7 and a washer on the 3, on the other side. They can see that this corresponds to the fact that

6+4=7+3

and then use the equaliser for discovering similar relationships.

They can also be guided to discover the important generalisation that if two numbers are equal, then adding the same number to each results again in two equal numbers, e.g.

if 3+4=7

then (3+4)+5=7+5.

From their experiences in placing washers on each side of the equaliser, children will probably discover that there are more instances when the two sides do not 'balance' than when they do. We can take advantage of this and encourage them to record these facts not only in the form 6+4≠7+2

but, for example,

6+4 > 7+2

9 < 7+6

> means 'is greater than'

< means 'is less than'

The following types of discovery can be made: —

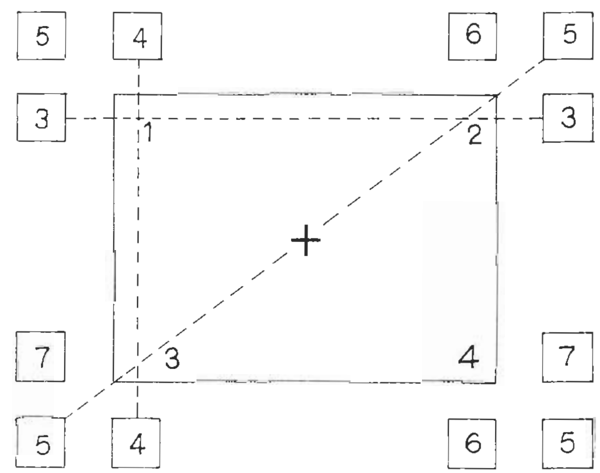
i. *If* 9>7,

then 9+2>7+2

ii. If one washer on the 9 on one side outbalances one on the 7 on the other side, then '3 washers on the 9' will still outbalance '3 washers on the 7'.

Some activities

i.



One large card and twelve smaller cards can be made on which the appropriate numbers are written. The children can be asked to put the cards in their correct positions, after adding up, down, across and back, also diagonally.

Some teachers may prefer to duplicate the central square and leave the small ones blank for the children to complete. An assignment could be:

> Start with the large square in the middle.
>
> Add the numbers up, then down e.g.
>
> 3+1=4
>
> 1+3=4
>
> Now add the numbers across and back.
>
> Add the numbers in the opposite corners, both ways.
>
> Record what you do.

Magic Squares

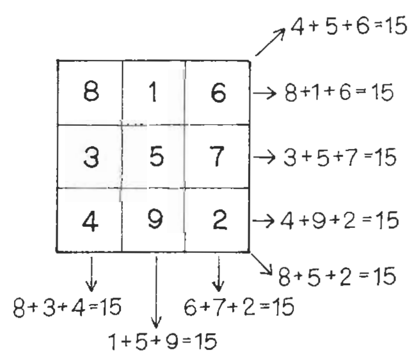
A 'magic square' consists of a square array of numbers, with the rules

i. No number may appear more than once.

ii. The sums of the numbers in the various rows and columns must be the same.

iii. So must the sum of the numbers along a diagonal.

Examples:



16	3	2	13
5	10	11	8
9	6	7	12
4	5	14	1

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Given these 'basic' squares, others can be made for example by adding a given number to each entry, e.g.

8	1	6
3	5	7
4	9	2

add 2 →

10	3	8
5	7	9
6	11	4

8		
	5	
		2

> Can you complete the 'magic square'?
> Use only the numerals 1 to 9.
> Some have been put in for you.
> What is the total of each row or column?

Class 2A by Anne Urwin, Ruth Duncan and Graham Kay

Magic Squares.

2	9	4
7	5	3
6	1	8

12	5	10
7	9	11
8	13	6

4	9	5	16
14	7	11	2
15	6	10	3
1	12	8	13

15	10	8	1
6	3	13	12
9	16	2	7
4	5	11	14

17	8	1	24	15
10	21	19	12	3
4	20	13	6	22
23	14	7	5	16
11	2	25	18	9

23	6	19	2	15
10	18	1	14	22
17	5	13	21	9
4	12	25	8	16
11	24	7	20	3

A Magic Square is a square which has all sides, across, down or diagonal adding up to the same number

A 3^2 Magic Square consists of 9 small boxes and 4^2 has 16 small boxes

Magic Squares

Cheryl Brayne Age 9

A

8	1	6
3	5	7
4	9	2

B

6	1	8
7	5	3
2	9	4

2	7	6
9	5	1
4	3	8

C

6	7	2
1	5	9
8	3	4

D

Magic Squares

All of these Magic Squares add up to 16

9, 5, 1, is the same as 6, 5, 4,

and 8, 5, 2, is the same as 3, 5, 7,

If you put A by a mirror you will see B. because B is a reflexion of A.

I put C by a mirror instead of seeing 2, 7, 6 I saw 6, 7, 2 which is D.

		10
	9	
8		6

Complete this 'square'.
Look at the diagonal row first.

13		
	10	12

The sum of the numbers in each row or column is 30

24		
	15	
	27	6

Look at the diagonal row.

Other useful activities are 'number games', with which most teachers are fully familiar, e.g. dominoes, snap.

'Dominoes' can be made from pieces of card e.g.

8	4+5
9	4+4

and this well-known game played using 'addition facts' instead of the usual dot patterns: —

8	4+5	9	3+2	5	6+3

Magic Squares and games all provide a variety of experiences in the addition and recognition of numbers.

5 Place value

From their experiences with number strips and other equipment and the study of other notations children have been receiving an informal introduction to our system of place value. There is, however, still a danger of underestimating the difficulties which this presents to children. Teachers know what pitfalls are created by 0; in the past even older children have been confused over the values of each numeral contained in a given number (e.g. what does the '5' represent in 256?). This is probably due to the fact that many of them had never seen a practical representation of the position and consequent value of a numeral in our system of notation.

From the beginning we need a realistic approach to the problem of introducing children to place value. Suggestions for an early introduction through the historical development of early systems of notation (e.g. Babylonian, Egyptian and Roman) have already been given; this can be a very rewarding approach if teachers will take the time and give children the opportunity to construct these early counting aids with sand, pebbles, twigs and plasticene.

Various structural apparatus is helpful at this stage to establish the notion of place-value. It must be emphasised that the account which follows is not a sequence which must be followed blindly. Some children will take more readily to one piece of apparatus than another (and so a variety should be available) and some will quickly get the successive ideas and not need so much of these experiences.

Use of different 'number bases' focuses on place value in its own right without being tied to the customary 'base ten'. When we write 37 this means (by convention)

3 tens and 7 units

which could be written

3 (10) + 7

but reference should be made to p. 22 where the weight of Parcel A is recorded as 10 since it balanced with 1.2-oz weight and 0.1-oz weights.

Instead of 10 meaning

1 *ten* and 0 units

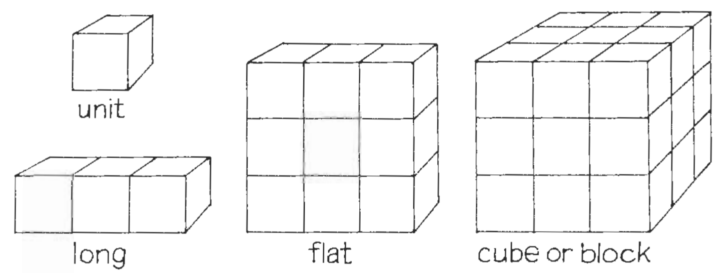
in the 'scale of two' (or 'binary' scale), 10 means

1 *two* and 0 units.

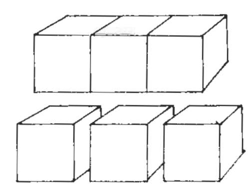
The *base* is 2 instead of 10.

The Dienes multi-base arithmetic blocks give structural experience with different bases (3, 4, 5, 6, 10). The material is so designed that the children build according to the base that they are using. At first, it is recommended that children play with the material until they discover the relationships between the pieces.

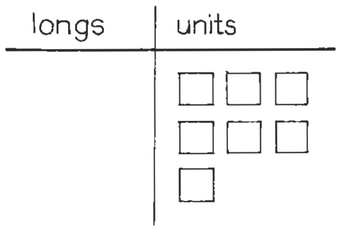
As an example, these are the standard pieces in base 3.



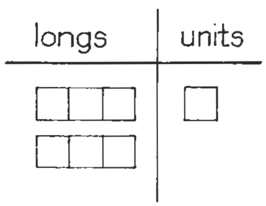
Given a miscellaneous collection of these, children will probably start just playing with them, making towers, houses, aeroplanes . . . and this phase must not be hurried – in one class the period concerned varied from one day to three weeks. They will later discover that three 'units' match against a 'long'



and then start making exchanges (3 units for a long, 3 longs for a flat and so on). They can actually *see* that on collecting 3 units they can be traded for 1 long, or given, for example, 7 units



6 can be traded for 2 longs

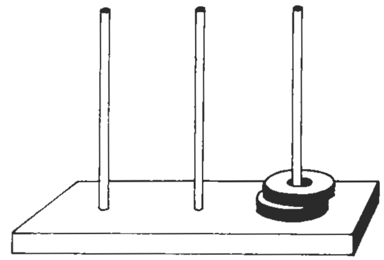


and this is a first step towards realising that for example seventeen units (in conventional base 10) can be replaced by

1 ten and 7 units

and that this is what we mean when we write '17'. Children should get used to working in different bases on different occasions, and the teacher should provide the opportunity for them to change over frequently, the '10-box' (leading to our 'usual' notation) happening to be one of them.

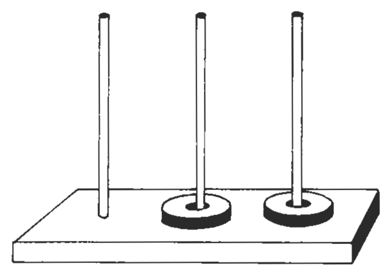
Various forms of abacus are useful in the same connection, though it is more sophisticated to exchange say 3 washers on one peg for a similar one *of the same size* on the next.



Abacus consisting of wooden pegs (or meat skewers mounted on plasticine) and plastic washers.

Here, for example, having agreed to be using 'base 3', a washer on the right counts as a unit, one in the middle as 3 and a left-hand one as 3^2 (i.e. 9).

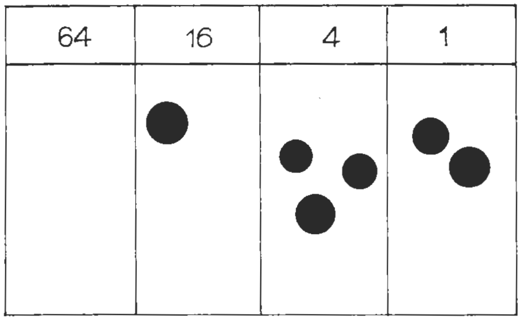
A game can be played by two children each having such an abacus, using a 'spinner' with the numbers 0, 1, 2. The first might score 2 (washers placed as above) and on the next turn a further 2 – altogether 4 'units' of which 3 would then be traded for one on the middle peg.



This represents a score of 4.

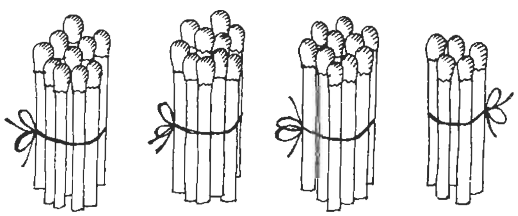
The game could end, for example, when the first child had a sufficient score to place a washer on the left-hand peg.

Another variation is the 'counting board' here shown with base 4 but the set of numbers at the top can be changed according to the base used.



This could be home-made from strawboard using small plastic tablets, buttons, shells, etc., as counters.

All sorts of other aids can be used. One of the simplest is bundles of sticks, matchsticks, spills, straws or pen-handles. These can be tied in bundles of ten.



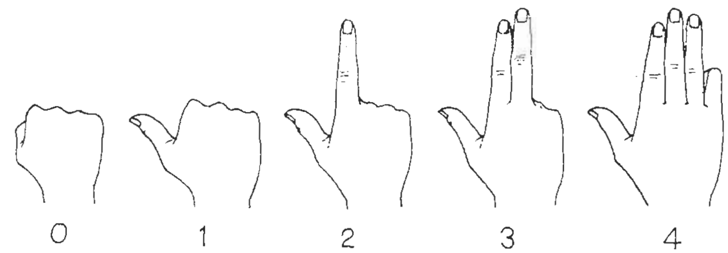
3 bundles of ten   and 5   total 35

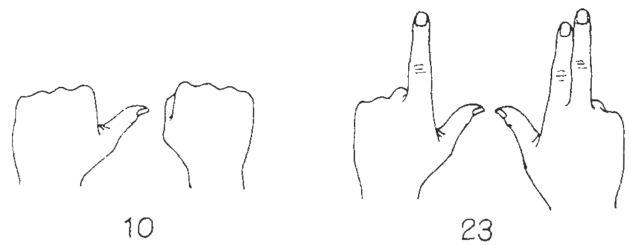
Ancient man used to think in 5's because he had five fingers on his hand (the Roman system is also based on fives). Children could therefore be asked to count in base 5 using their fingers and hands. 5 in this system would be written as 10 (i.e. **1** (5) +**0**), 6 as 11 (i.e. **1** (5) +**1**) and so on.

Base 5

Number in base 10	5^2	5	1
11		2	1
4		0	4
5		1	0
6		1	1

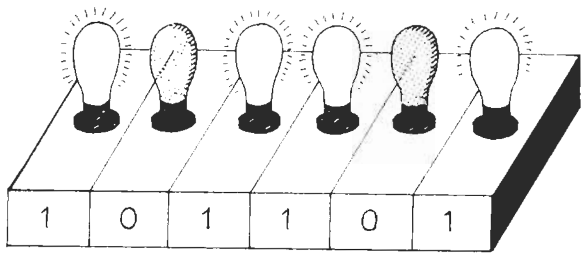
Using the hands for Base 5. (Each finger on the right hand represents a unit and each finger of the left hand represents five)





that is 2(5) +3, i.e. 13 in base 10

The 'binary' system (base 2) is particularly useful because the only symbols used are 0 and 1, and these can be represented by *on* and *off* in an electrical circuit.



These lamps represent the binary number 101101.

Two in our decimal notation (Denary Scale) would be written 10 in the Binary Scale since 2=**1** (2)+**0** and six in the Denary Scale would be written 110 in the Binary Scale, since 6=**1** (2^2)+**1** (2)+**0**.

Denary	1	2	3	4	5	6	7	8	9
Binary	1	10	11	100	101	110	111	1000	1001

The binary number 101101 is
1 (2^5)+**0** (2^4)+**1** (2^3)+**1** (2^2)+**0** (2)+**1**
i.e. 45, in base 10.

An introduction to binary notation can be given through the use of kitchen weights. (See p. 21.)

Another scale which children enjoy using is the scale of 8, sometimes called 'Spider Arithmetic'.

64	8	1
8^2	8^1	1

Children enjoy counting in different bases and could be asked to construct a counting sequence using different number scales, e.g.

Base 10:— 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 . . .

Base 8:— 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14 . . .

Base 5:— 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22 . . .

Base 3:— 1, 2, 10, 11, 12, 20, 21, 22, 100, 101, 102 . . .

When children get used to the idea of their particular abacus they can be asked to set up numbers and then record the numbers they have made. Teachers could then say what numbers are to be set up on the abacus and the empty column should be dealt with frequently. The following specimen assignments use base 10, but of course others should be used on the same footing.

Show these numbers on your counting board

1 ten and 5 ones

1 ten and 2 ones

2 tens and 7 ones

4 tens

3 tens and 5 ones

1 hundred

1 hundred 2 tens and 5 ones etc.

Record in numerals and then in words.

Show these numbers on your abacus:

11, 10, 19, 24, 50, 23, 27, 29, 19, 41 etc.

Add 2 more to the above numbers on your board and then record the new number.

Write the numbers in words also.

Make numbers on your board and then record like this:

15 is ten and ones

98 is tens and ones.

Now make numbers with 'empty spaces' and record these, e.g.

20

105

Show these numbers on your board:

9, 13, 18, 51, 60, 90, 100,

106, 160, 350, 530, 503

Now add 10 to these numbers.

100 is 10 tens. 140 is 14 tens.

How many tens are there in the following?

150, 170, 210, 350, 270, 700, 200, 80+60, 40+50

What is the '3' worth in these numbers?

130, 350, 13, 23, 301, 231.

Using your abacus and 3 counters only, what is

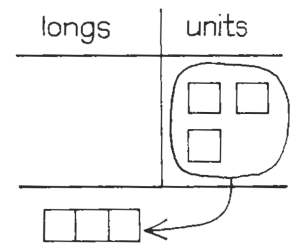
a. the biggest number you can make?
b. the smallest number you can make?

What other numbers can you make?

Addition

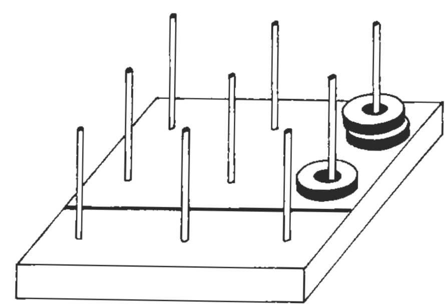
Concrete materials are again useful when adding numbers involving 'carrying'.

For example, using Dienes' apparatus (again with base 3 for illustration)

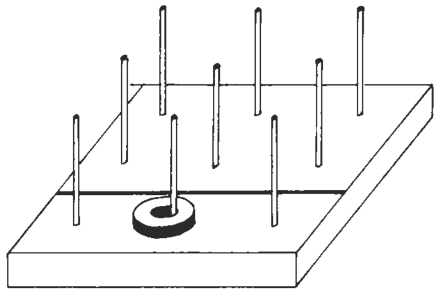


the addition of 2 'units' and 1 'unit' gives three units which can be traded for one 'long'. Much progressive practice (using different bases and eventually units, long, flats and blocks) leads to real understanding of 'place value' in action.

Addition can also be practised with the various forms of abacus, e.g. one which has three rows.



Setting up the same problem as before (still in base 3), the addition of 2 and 1 would now be represented as follows



the three 'unit' washers being replaced by one 'three' washer.

At about this time the children will be ready to record e.g.

Base 4

longs	units
1	1
1	3
3	0

and eventually discard the apparatus altogether. Problems can be posed such as

2	8
1	7
4	3

This represents an addition. What base is being used?

A quick reply might be 'The biggest number I can see is 8, so it's base 9', but of course a closer inspection will reveal base 12.

Addition tables for various bases can be made, e.g.

Base 5

+	1	2	3	4	10
1	2				
2		4			
3		10	11	12	
4				13	
10					20

Children by now will enjoy checking their work by using different bases, e.g. building up a table such as

Base 10	Base 3	
1	1	
2	2	
3	10	i.e. $1(3)+0$
4	11	$1(3)+1$
5	12	$1(3)+2$
6	20	$2(3)+0$
7	21	$2(3)+1$
8	22	$2(3)+2$
9	100	$1(3^2)+0(3)+0$
10	101	$1(3^2)+0(3)+1$
11	102	$1(3^2)+0(3)+2$
12	110	$1(3^2)+1(3)+0$

and working etc.

Base 3

1	2	1	$1(3^2)+2(3)+1(1)=16$
	1	2	$1(3)+2(1)=5$
2	1	0	**21**

$2(3)+1(3)+0(1)=$**21**

Before proceeding to deal with the operation of addition with numbers up to 100 and then 1000 it is, perhaps, a sound idea to let children see for themselves just what a hundred or a thousand looks like. This can be done quite effectively with dots on squared paper or other suitable objects, e.g. coins. It is time well spent to allow pupils to do this and make the necessary comparisons. Illustrations of 100 and 1000 can be displayed on the classroom wall. Children will most likely realise that 'grouping' in tens helps them in their counting and confirms the need for a convenient system of place-value.

To further an awareness of place-values, children can be encouraged to write down their calculations in expanded notation form, e.g.

for 42+10+23, we have

4 tens and 2 units

1 ten and 0 units

2 tens and 3 units

Total 7 tens and 5 units or 75.

An extension of earlier work with the Roman method of representing numbers could also be given effectively here. For large numbers, the Romans used the circle. 1000 was represented by the full circle ◯, five hundred by half the circle (a semi-circle) ◖ or ◗, one hundred by the curved part of the semi-circle (or) and fifty by a special symbol ◟ which was probably the lower half of ((100).

Later, 1000, was represented by two semi-circles (twice 500) ◖◗ or ↀ or M; 500 by the letter D; 100 by the letter C; and 50 by the letter L.

Thus we have: —

L	C	D	M	LX	CVI	DCX
50	100	500	1000	60	106	610
				(50+10)	(100+5+1)	(500+100+10)

XL

40

(10 less than 50)

Useful experience can be obtained from books with chapters and pages numbered in Roman notation; and of course there are monuments and tomb-stones . . .

Children can also delve into books and find how larger numbers (100, 1000, etc.) are written in other systems, such as the Chinese and Babylonian, cf. p. 3.

Further work with the number-strip can be done after place-value has been established, to extend children's number knowledge, especially up to 100.
Counting on and back is still useful practice to lead to the discovery of patterns, e.g.

9+7=16	6+9=15	18+9=27	(perhaps calculated by adding 10 and taking away 1)
19+7=26	16+9=25	25+9=34	
29+7=36	26+9=35	16+9=25	
39+7=46	36+9=45		

58+3=

68+3=

78+3=

69+3=

etc.

Once children have acquired this facility with numbers, they are ready to benefit from written practice in addition, first with tens and units and then with larger numbers, as they come to extend their number knowledge to numbers greater than 100, now understanding and appreciating place-value.

These points, however, should be kept in mind:

1. Written computation practice should supplement and not replace oral practice.

2. To give children more practice than they require to maintain efficiency is a waste of time. In brief, we must consider carefully the time when we do 'sums' and, also, the time spent in doing them.

The need is to concentrate on the operation rather than the numbers in this activity e.g.

75+125=200 125+ 75=☐

Practice in adding three numbers can also be given and brackets used to associate the pairs e.g.

(75+25)+30=130

15+(27+33)=15+60=75

From counting and adding experiences teachers should exploit the properties of associativity and commutativity whenever possible, e.g.

27+53=53+27

=(53+7)+20

=60+20

=80

or

53+27=(53+20)+7

=73+7

=80

We shall not introduce subtraction as an operation until the next volume, ❸ . At this stage, many problems classified as 'subtraction' can be treated instead in terms of addition. For example, the traditional type of question *'Take 39 from 47'* was tackled with mumbo-jumbo something like '9 from 7 I can't, borrow a 10, . . .' But the very wording of the question is jargon. 'How many more is 47 than 39?' makes more sense, and this problem can be tackled by asking 'How much do I have to add to 39 to reach 47?' By counting on, the child uses 'complementary addition' (cf. shopkeeper's addition, p.65). From 39, 1 on to reach 40 and a further 7 to reach 47: total 8'.

'How much greater is 160 *than* 57?'

We seek the number to put in the box to make

57+☐=160

a true statement.

The 'working' might go like this:

57+3=60

57+3+100=60+100

57+*103*=160.

Mathematics and English

It should be the main aim in the teaching of mathematics to make the pupil translate into mathematics, readily and naturally, the questions of the common world, and ultimately to translate his answer back into common usage.

J. Bronowski

It cannot be emphasised too strongly that discussion and writing about their experiences is important to children if we are to help them to think and to solve problems. We frequently hear 'He can do his mechanical sums but not his problems.' Children learn from each other when talking about a problem which confronts them. A teacher, too, can pose a leading question and find out how the child's mind is working towards solving a particular problem. It is hoped that the material in this guide will suggest a number of such problems requiring translation 'from English into mathematics'. Others will be found in text-books, which must however be used with discretion. It is *not* suggested that slabs of 'problems' should solemnly be worked through. But if some suitable realistic ones can be found, the children could be asked to analyse what computation is required without necessarily carrying it through to the answer.

Perhaps a better source of problems is the children themselves. They would not dream of devising conundrums on bath-filling or hole-digging: a class of 7 and 8-year-olds for example made up questions on pop records, marbles, ages, football matches, and cowboys and Indians.

As well as 'translating from English to mathematics' (words to symbols), experience in the opposite process is also valuable. For example, given a mathematical statement such as

$7 + 5 = 12$

or $7 + \square = 12$,

children could be asked to make up a sensible story to illustrate the sentence, which would include 'translating' from mathematical symbolism into words.

When asked to write a sentence about '60+40=100' a nine-year-old boy wrote:

'About the only thing I can write of here is what I read about crashes in cars. If two cars hit each other head on and one is going at 40 and the other at 60 there will be a total crash of 100 m.p.h. because 60+40=100. I didn't know that you could work out the crash speed so easy as that.'

Perhaps the reason why 'Johnny can't do his problems' is that we have taught him to use the short-hand of mathematics too soon, without constant reference to the spoken and written word. From the reference books listed below, (p.67) teachers will find plenty of suggestions for topics of interest or projects concerned with the history and development of numbers, measurement and geometry. This kind of work combines English with mathematics and encourages reading, discovering and recording facts: graphical representation as well as computation can be extracted from experiences of this kind.

Closure and checks of accuracy

We have been concerned with the set of numbers $\{0, 1, 2, 3, \ldots\}$ i.e. the natural numbers together with zero. This set is called *closed* under addition, as the operation of addition performed on any two elements results in another element of the same set (e.g. $2+4=6$ and 6 is also a member of the set concerned).

If we take the set of *even* numbers $\{2, 4, 6, 8 \ldots\}$ we find that when we add any two *even* numbers we still obtain another even number e.g.

$$2+4=6$$

$$4+4=8$$

$$16+16=32$$

$$14+16=30 \quad \text{etc.}$$

so the set of *even* numbers is also closed under addition (the sum of two even numbers is another even number).

But if we take the set of odd numbers: $\{1, 3, 5, 7 \ldots\}$ the closure property no longer holds e.g.

$$3+5=8$$

$$5+13=18$$

The sum of two *odd* numbers is an *even* number.

We can construct a table of even and odd numbers to show this: —

+	E	O
E	E	O
O	O	E

In a dining hall 22 seats were empty. 51 people were sitting in seats so altogether there were seventy three seats.

51 + 22 = 73

The total number of people killed in car crashes in 1963 and 1964 was 197. Three more people were killed in 1964 than 1963. How many people were killed in 1963?

100 + 97 = 197

I did a hundred and eighty stitches altogether when I made a sewing bag. 100 stitches were blue. How many stitches did I do in yellow?

100 + 80 = 180

This table is interpreted as follows.

+	E	O ↓
E		
→ O		E

E stands for 'even', O stands for 'odd'.

When we add an odd number (indicated by the horizontal arrow at O) and another odd number (indicated by the vertical arrow) then the result is even, and this is recorded by inserting an E (for even) where the row of the first arrow meets the column of the second.

+	E ↓	O
E		
→ O	O	

This entry shows that the result of adding an odd number and an even one is an *odd* number, and similarly for the other entries.

These results can be useful to children when adding numbers. If they are adding even numbers e.g. 2+10+6, they should then know that the sum of these will be even, and this provides a check for their work.

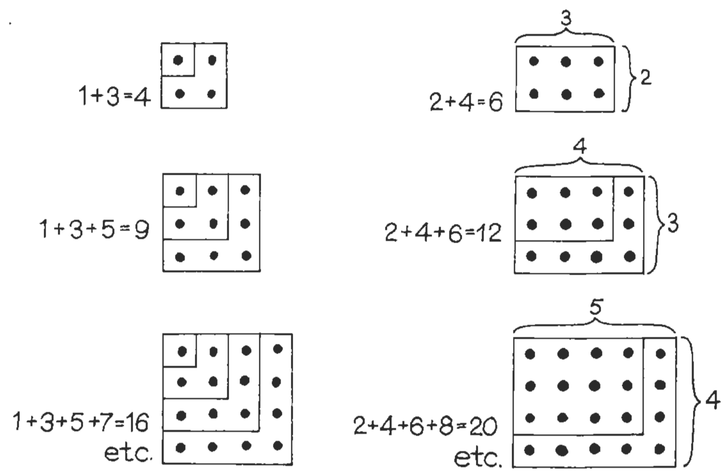
An odd number of *odd* numbers in addition will result in an odd number, e.g.

$$6 + \textcircled{7} + 4 + \textcircled{5} + \textcircled{9} = 31$$

(3 odd numbers)

The children can elaborate on these patterns of behaviour of 'odd' and 'even', and this again encourages them to keep checks on their calculations.

Patterns of odd and even numbers are a good source of investigation and discovery, e.g.



Summary of this section: —

Number knowledge through counting on and back.

Using the union of disjoint sets to define the sum of two numbers.

Addition: a binary operation.

Equalities and Inequalities.

Place Value: different number bases.

Extension of number notation to 100 and beyond.

Odd and even numbers.

6 Time

Dictionary definitions of the word 'time' read as follows: —

'A limited stretch of continued existence'.

'The interval between two events'.

From these two definitions we can appreciate why it is that the concept of time is not an easy one to establish.

'Telling the time' and 'timing' (bouncing a ball, skipping, etc.) are two quite different notions. They come at a fairly early age but the one is just 'dial-reading' and the other counting at regular short intervals. Both, however, eventually help towards understanding the passage and measurement of time.

A short history of time-keeping will give a reasonable indication of the development of understanding in young children.

Early men noticed sunsets and sunrises and so measured time by the interval between these:— the day. They watched the shadows lengthen and shorten, eventually recognising that the sun was responsible for night and day.

At first, the day could be reckoned from sunset to sunset, or sunrise to sunrise. Others would no doubt reckon from noon to noon when the sun was highest in the sky.

From the shadows cast by the sun, man found a way of using observations to reckon times during the day: he noticed their lengthening and shortening.

Using the same observations we can ask children to discover time periods in the same way, using a shadow stick and measuring the shadow at regular times during the day. Cleopatra's Needle, standing on the bank of the Thames, was probably used for this same purpose hundreds of years ago.

The moon is also a source for time discoveries. New moons and full moons were found to recur every 28 days. These 'moon periods' of 28 days give us the lunar month from which we derive our calendar month. Realising that similar things (growth of crops, harvest-time, lambing) happened after every 12 or 13 'moon periods', man became aware of the yearly cycle, so that he now had some fairly good measures of time: the year, the month, and the day. Once again, we observe a doubling and halving process: 1 month of 28 days, if halved, gives us a fortnight, 14 days, and half of this is 7 days or 1 week.

In early days main events of a personal nature were used to describe the time of day (meal-times, milking time, bed-time) and the main events of the year would also mark periods of time (e.g. from seed-time to harvest-time).

It has been suggested that in very young children an appreciation of time is acquired in a somewhat similar way:

1. Personal experiences related to child's age: bed-time, school-time, meal-time, morning and afternoon.
2. Telling the time in hours and half-hours.
3. Conventional time-words: calendar, days of the week, months and, eventually, years.
4. Time since last birthday; time at other places in the world; time since holidays.

We can help children's appreciation of periods of time if we provide them with experiences which have a definite beginning and ending e.g. watching and using egg-timers, using a stop-watch, counting aloud. Guessing how long children are reading, skipping, running, can also help.

Encouraging an awareness of the succession of events in their lives can also be of great value:

Time between school and home-time

Time allowed for play-time, dinner-time

How long do we sleep?

Home-made timing devices:

pendulums

sand and water clocks (tins with holes)

candle-clocks

shadow sticks

From guessing the time and estimating intervals of time we can proceed towards the necessity for instruments which give us an accurate measurement of time, and standard units of time.

Apparatus which can be of great benefit in the classroom:

Real clocks with clear faces and Arabic numerals.

Plastic or wooden clocks with hour and minute hands which can be 'set' by children

Egg-timers

Photographic clocks (second-timers)

Pulsometers (small 'egg-timers': 15, 30 and 60 seconds)

Metronomes

Stop-watches

Large clearly-printed calendars

Small calendars for individual use

Other valuable items are time-tables (school, rail, coach).

Regular recordings of nature observations (bulb and plant-growth; seasonal changes etc.) can help the awareness of periods of time.

Telling the time by reading a clock-face is still, perhaps, the most important part of 'Time' to children and this should be done first in hours, then half-hours, quarter-hours, and, lastly, 5 minute periods, using real clocks as much as possible.

As well as telling the time, a child has to read and record it in two or three different ways: from the verbal 'quarter to three' there follows the writing of these words which are then converted to the conventional 2.45 and ultimately a.m. or p.m. Later, we shall use the relationship between the continental 24 hour clock and our own 12 hour clock to introduce children to 'other clocks' and remainder or modulo arithmetic.

Use the clock. (o'clock means 'of the clock')

1. Write down the time 3 hours *later* than each of these times: —

1 o'clock 6 o'clock 3 o'clock 8 o'clock

2. Write down the time 3 hours *earlier* than these times: —

6 o'clock 4 o'clock 12 o'clock 1 o'clock

You can record like this:

3 hours later
7 o'clock ——→ 10 o'clock

Can you discover:

1. How many minutes there are in 1 hour?

2. How many minutes there are in $\frac{1}{2}$ hour?

3. How many minutes there are in $\frac{1}{4}$ hour?

When you have found out, ask the teacher to check them for you.
When the minute-hand passes from 2 to 3, how many minutes have gone by?

You can write down the time in two ways:

3 o'clock is also written as 3.00

Half-past one is also written as 1.30

A quarter past seven is also written as 7.15

A quarter to two is also written as 1.45

Now write down these times in two ways: —

8 o'clock, half-past six, quarter to five.

Set some times on the clock (use only hours, halves and quarters) and record them in two ways.

When does the minute hand point straight up?

When does the minute hand point straight down?

When does the minute hand point to your left?

When does the minute hand point to your right?

The calendar provides many interesting activities for young children and they usually find brief historical details fascinating.

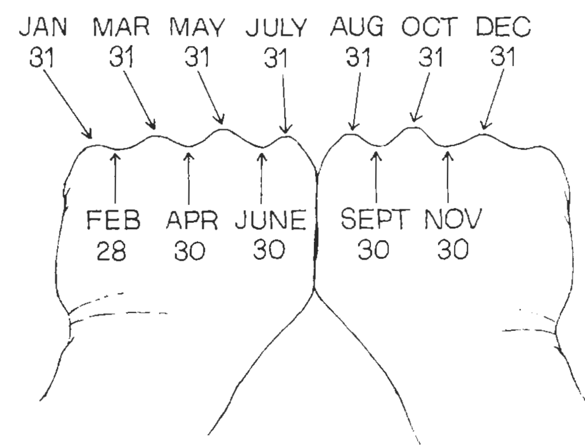
The child soon discovers that there are 12 months in a year and gradually learns the names of these. Julius Caesar can figure in their stories of the months. When he became Emperor of Rome, his astronomer discovered that the calendar was 80 days wrong. Previously each month contained 30 or 29 days, alternately. 46 B.C. was known as the 'year of confusion'. Caesar's astronomer estimated that the true average number of days in a year was $365\frac{1}{4}$. To arrange this on the 'Julian Calendar' the twelve months had 31 and 30 days alternately, except for February which became the odd month and had 29 days for 3 years but 30 days every 4th year. This made up the $\frac{1}{4}$ day.

The month of July was named after Julius Caesar because this was his birthday month. When the next emperor ruled he wanted a month after his name. As this was Augustus we know how August got its name. However, because July had 31 days, Augustus wanted his month to have 31 days also. To do this he decreed that a day should be taken from February and added to August. This is why July and August are the only two months together having 31 days and why February has 28 days except for a leap-year when it has 29 days.

To remember the months of the year and the number of days in each, the teacher and children will be familiar with the rhyme: —

30 days hath bright September etc.

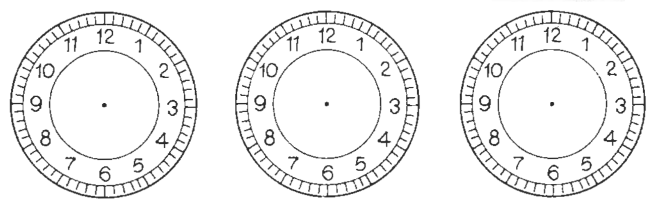
The following 'ready-reckoner' may interest children even more: Using the knuckles of the hands:



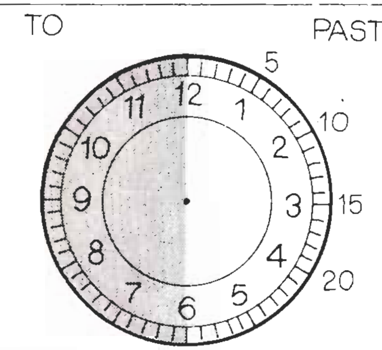
For the children

Use the calendar and the month of February to count:

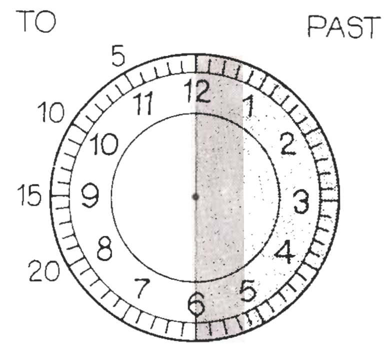
1. In 7's from 1 to 28
2. In 7's from 2 to 26
3. In 7's from 5 to 27
4. In 7's from 7 to 28
5. How many days are there in February?
6. What is the date 7 days after February 14th?
7. What is the date 14 days after February 14th?
8. What day will it be 3 weeks after the first Monday in the month?



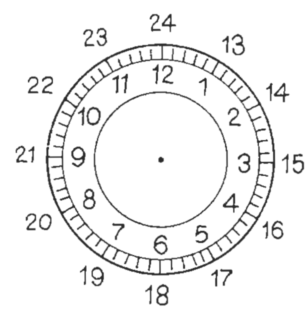
Draw in the hands of the clock for 12 o'clock, 3.45, quarter to 5



These numbers on the outside show us the number of minutes *past* the hour. Continue with these.



These numbers on the outside show us the number of minutes *to* the hour. Continue with these.



On the continent they use the 24 hour clock. The hours are from 1 to 12 in the morning but, after midday, they continue with 13 hours, 14 hours and so on, whereas we say 1 p.m., 2 p.m., etc.

Here are the hours on the continental clock.

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
		3					8				

From 13 to 24, write underneath the hours on our clock.
If it is 4 p.m. in this country what would they give for the time in France?

Look at these times given below. Use a clock and give the time for 2 hours *later*:

1. 2 o'clock
2. 5 o'clock
3. 8 o'clock
4. 9 o'clock

What will be the time 3 hours *earlier* than these:

1. 6 o'clock
2. 8 o'clock
3. 11 o'clock

If your clock is at 9p.m. write down what time it will be in:

1. 2 hours time
2. 4 hours time
3. 6 hours time
4. 5 hours time
5. 3 hours time

We can see from using the 12-hour clock that we are only concerned with numbers from 1 to 12. Unlike the continentals we stop at 12 and go back to 1, 2, 3, etc. If we can persuade children to accept that '12' on the clock could be re-christened zero (as a starting point) we can introduce remainder or 'modulo' arithmetic and construct tables and patterns from this.

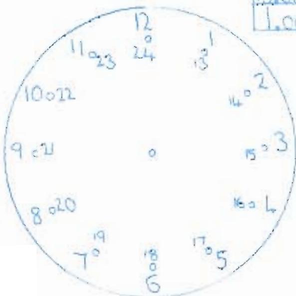
mod 12

	+	0	1	2	3	4	5	6	7	8 ↓	9	10	11
	0	0	1	2	3	4	5	6	7	8	9	10	11
	1	1	2	3	4	5	6	7	8	9	10	11	0
	2	2	3	4	5	6	7	8	9	10	11	0	1
	3	3	4	5	6	7	8	9	10	11	0	1	2
	4	4	5	6	7	8	9	10	11	0	1	2	3
	5	5	6	7	8	9	10	11	0	1	2	3	4
→	6	6	7	8	9	10	11	0	1	(2)	3	4	5
	7	7	8	9	10	11	0	1	2	3	4	5	6
	8	8	9	10	11	0	1	2	3	4	5	6	7
	9	9	10	11	0	1	2	3	4	5	6	7	8
	10	10	11	0	1	2	3	4	5	6	7	8	9
	11	11	0	1	2	3	4	5	6	7	8	9	10

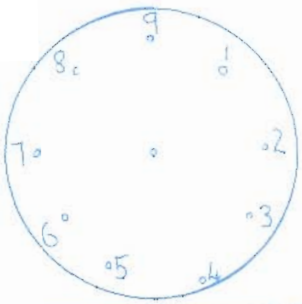
We have replaced '12' on the clock by '0', and we now have the addition table 'modulo 12'. We start at 0 and when we pass 11 we count on 0, 1, 2, 3, . . . instead of 12, 13, 14, 15, . . . , so that 13 is replaced by 1, 14 by 2, etc. just as on the clock. As an example, the arrows indicate 6 and 8 and where the '6' row and '8' column intersect shows 2. This means that the result of 'adding modulo 12' the numbers 6 and 8 is 2. A model for this is, of course, to set a clock at 6 o'clock and move it on by 8 hours. It will then read 2.

Page 1

13.00	20.00	16.00	23.00	18.00
1.00	8.00	4.00	11.00	16.00



A 24 hour Clock

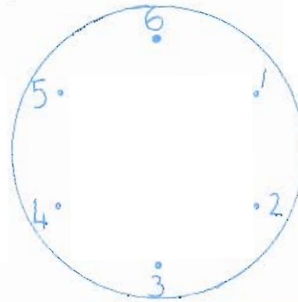


A Nine Dial or clock

1 Start At	7	6	5	7	8	8
2 Add on	5	8	6	7	3	8
3 Time on 9 clock	3	5	2	5	2	7
4 Total of 1 and 2	12	14	11	14	11	16

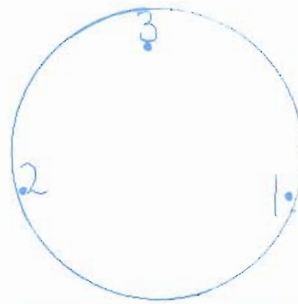
This is really the Remainder of the 9 x table

Page 2



A Six Dial or clock

NUMBER	8	10	12	13	16	17	20	23	24	26
SHOWN AS ON 6 CLOCK	2	4	6	1	4	5	2	5	6	2
AS REMAINDER	2	4	0	1	4	5	2	5	0	2



This is a 3 dial or clock

NUMBER	7	4	2	3	8	17	5	6	10	9
SHOWN AS ON 3 CLOCK	1	1	2	3	2	2	2	3	1	3
AS REMAINDER	1	1	2	0	2	2	2	0	1	0

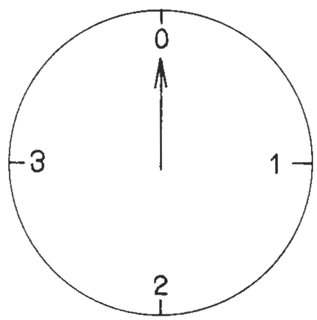
A number strip could be made for arithmetic 'modulo 12',

1	2	3	4	5	6	7	8	9	10	11	0	1	2	3	4	5	6	7	8	9	etc.

1	2	3	4	5	6	1	2	3	4	5	6	7	8

but the 'clock' (with zero instead of 12) perhaps affords a better illustration.

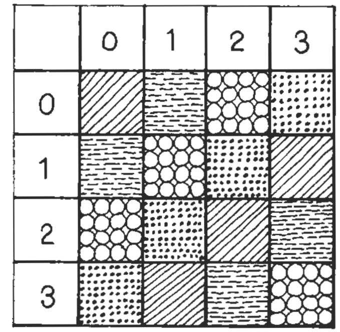
If instead, we had a 'four-hour' clock (starting from 0 again), we should have an arithmetic 'modulo 4' and be concerned only with the numbers 0, 1, 2, 3.



Children could be asked to use this clock and, say, starting at 2, give the time 2 hours later. They could construct a table from their experiences with this clock and discover that under addition the properties of closure, associativity and commutativity all hold, as we found for the set of natural numbers.

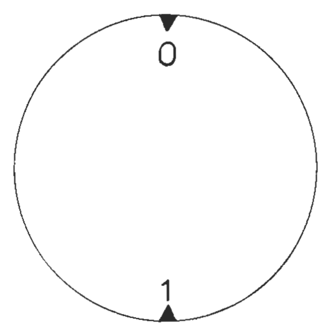
⊕	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Patterns can be seen from such tables by colouring the squares containing each given number in a distinctive way (e.g. 0 red, 1 orange, 2 green, 3 blue).



How this 'modulo' arithmetic can illuminate the process of subtraction is described in the next volume, ❸ .

One of the developments in mathematics over the past century has been the introduction of different 'arithmetics' – we are not always concerned with the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, . . . but, perhaps with a different set, as for example with the clock which only uses twelve distinct numbers. A digital computer can only in effect use two numerals 0 ('off') and 1 ('on') and we end here with an 'arithmetic' associated with it, namely 'modulo 2'. Our 'clock' now is particularly simple.



Starting from 0, one hour (say) is recorded as 1 and two hours by returning to 0 again, so that counting would be recorded by this crude 'clock' as follows:

	0	1	2	3	4	5	6	7	8	9 . . .
'mod 2'	0	1	0	1	0	1	0	1	0	1 . . .
	even	odd	even	odd	even	odd	even	odd	even	odd

The 'addition' table is particularly simple.

+ (mod2)	0	1
0	0	1
1	1	0

It is no coincidence that this pattern is like that for 'even' and 'odd' (p. 52). In fact those numbers represented by 0 (modulo 2) are *even* and those represented by 1 (modulo 2) are *odd*.

The significance of such patterns will be explored in both ❸ and ▼3.

7 Money

Today's children are more used to handling money than their predecessors and most infant schools have their 'shops' as a main activity. There is a danger, therefore, of wasting time doing too many money 'sums', especially if there is emphasis on amounts of money which do not really come within the child's sense of values. It is doubtful if amounts of more than £10 really mean anything to a child up to the age of 8 or 9. Concentration on sums of money up to £1 will probably be far more rewarding.

There is no really satisfactory substitute for handling real money. If, however, teachers feel that money in the classroom is undesirable then cardboard coins are the next best thing. These facsimiles of the real coins are much preferable to some plastic types of 'coin' which are not true representations.

It is through coins and bank-notes that money transactions are made and we should continue to give familiarity with the common coins used; copper and silver; the 10s and £1 note at a later stage. The $\frac{1}{2}$d can be used quite naturally and it is perhaps a good thing to remember that, unlike the 'halves' of measurements (length and weight) there is nothing approximate about the value of money: that of a halfpenny is exactly half that of a penny. Counting coins, as a bank-clerk or cashier does, is essential practice. Coins of the same kind can be dealt with first. Heaps of pennies can be stacked in piles of 12; threepenny bits in piles of 4; sixpences in piles of 2, and so on. When there are insufficient coins to form a complete pile, the child can include these remainders in the final amount, e.g. 6 piles of 12 pennies and 4 over

6s 4d.

Here the child is working in 'base 12' arithmetic and this stacking of pennies will help to eliminate the common error of giving 17 pence as 1s 7d.

Stacking pennies

Count these amounts of copper (pennies) like a cashier and record like this:

pence	s	d
15	1	3
14		
20		

14d 20d 30d 17d 25d 36d 34d 28d 33d 37d

Now find the total value of these heaps of pennies. Put them together and count them like a cashier.

Record like this:

pence or like this: 9+4=13

9 *Total* 1s 1d

4

—

13 *1s 1d*

a. 6d, 7d	b. 9d, 5d	c. 8d, 4d	d. 10d, 10d
e. 11d, 11d	f. 10d, 11d, 3d	g. 9d, 8d	h. 5d, 10d, 3d

Make up some questions like this on your own.

We can proceed from coins of the same value to heaps of mixed coins. 20 pennies, 5 threepenny-bits, and 6 sixpences will be stacked in piles of 12, 4 and 2:

12 pennies (1s) and 8d over.,

4 threepences (1s) and 3d over.,

6 sixpences (3s) and nothing over.

The child counts up the number of shillings and finds that he has five. Adding the remainders (spread out on the desk as distinct from piles of shillings) he finds he has 11d which is not another complete shilling. He knows, therefore, that he has altogether 5s 11d.

How many *sixpences* are worth one shilling?

How many *threepences* are worth one shilling?

How many threepences are worth *sixpence?*

Make piles of the following numbers of threepenny bits.
6 7 4 9 10 11 12 16

Find the value of each pile by sorting as far as possible into shillingsworths. Record like this:

12 threepenny-bits are worth 3s 0d.

Do the same replacing the threepenny-bits by sixpences.

Change these amounts of money into pennies and record your answers like this: —

s	d	pence
1	6	18
1	4	

1s 4d 2s 3d 2s 6d 1s 11d 3s 0d 2s 5d

Do some of your own.

Set out the following coins and give the value of the total amount: —

a. 4 sixpences, 4 threepences, 6 pennies

b. 5 threepences, 6 shillings, 9 pennies

c. 12 sixpences, 13 pennies, 6 threepences.

Make up some of your own.

How many shillings are worth a florin?

How much more is a half-crown?

How many shillings do you think there were in a crown?

How much are the following amounts worth?

a. A florin, a half-crown and 3 sixpences?
b. 2 half-crowns and 2 florins?
c. 4 half-crowns and 4 florins?
d. 5 florins and 4 half-crowns?

For values up to £1, a coin-tray can be very helpful. This helps the child to discover how many of each coin, from penny to half-crown, are needed to reach the value of £1. Real coins have to be used and then, only the correct amount, e.g. 20 shillings, will fit into each compartment.

The photograph shows a coin-tray which is particularly useful for the activities suggested above. The tray holds £6 in real money made up of £1 in each of the following:

half-crowns in eights
florins in tens
shillings in tens
sixpences in tens
threepenny-bits in fours
pennies in twelves

It can be obtained from
Clearex Products Ltd., Heather Park Drive, Wembley, Middx.,
at a cost of 7s 6d.

Derek Taylor Age 8

Price List

Sugar		8d	per lb.
Butter	4s	2d	per lb.
Eggs	4s	a	doz.
Cheese	2s	10d	per lb.
Lard	1s	10d	per lb
Tea	1s	9d	per ¼ lb
Bacon	4s	0d	per lb

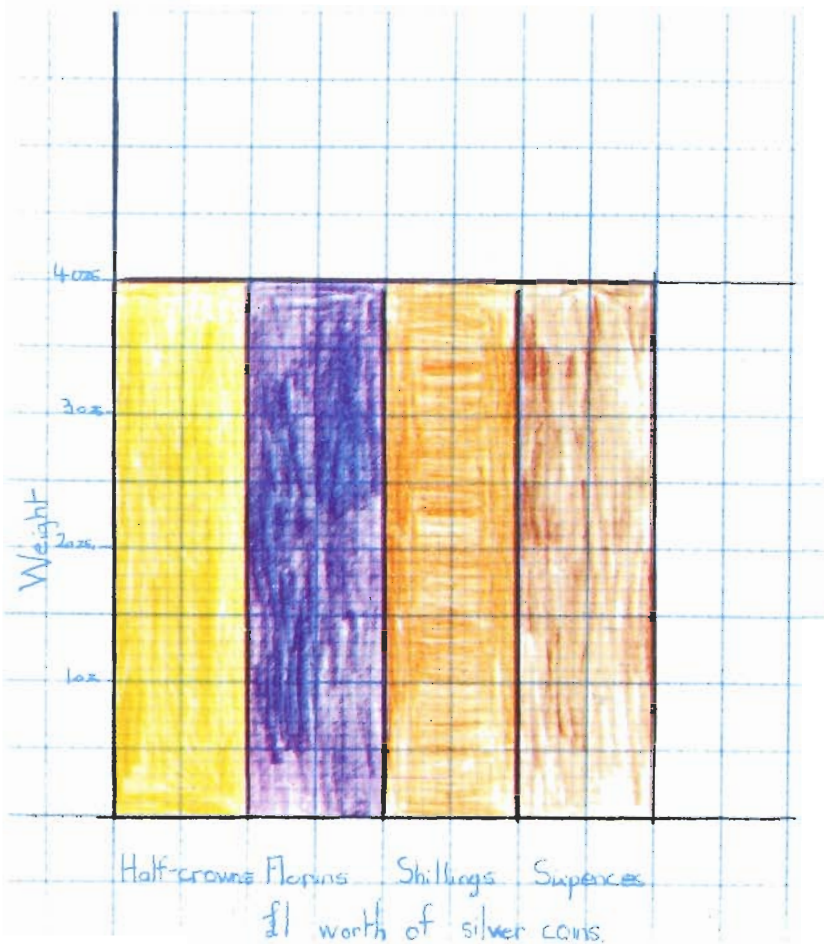
Mrs Taylor

	s	d.
1 doz eggs	4	8
1 lb bacon	4	0
½ lb Cheese	1	5
¼ lb Tea	1	9
2 lb Sugar	1	4
1 lb Lard	1	10
1 lb Butter	4	2
	18	6

Coin Recognition

Alan Clark Age 9years 7months

Silver Graph



This block graph shows that all silver coins that are equal to £1 in value always weigh 4 ozs.

Money Stories

When we use the letter L in our money we are using a Latin word libra which meant a pound. So that is why we use the letter L as pound.

Vincent Keirle Age 8

When we use d for pennies in our money it is the first letter of a Roman coin. It was called a denarius. It was a small silver coin. 240 of them made a pound. Pennies used to be small silver coins. 240 pennies make a pound in money now.

Beverley Fallon Age 8

Hazel Cross. 8 yrs.

Silver Coins.

Long ago the silver coins were made with smooth edges out of pure silver. Some people used to scrape off bits of silver from the coin and it grew smaller and smaller.
At last when so many people had scraped bits of silver off a coin it was very tiny.
It was not worth its value.
The minters started to make silver coins with milled edges.
That is how they know if someone has scraped any bits off the coins.
If they see any silver coins with smooth edges they will not take them

Practical work with money can take many imaginative forms, mainly connected with the idea of shopping, e.g.

Current shop prices at different kinds of shops: greengrocer, fishmonger etc. (A check could be kept on varying prices and simple block graphs made).

Cafe, market, fair

Post office

Train and bus fares

Cost of clothes

Checking bills

Making up bills for partners to check

Costing with quantities, e.g.
4 at 4d each
4 yd at 1s 3d a yard

Measuring amounts of money by weight

Using ready-reckoners.

Tabulating the numbers of pence and shillings, and later shillings and pounds, can be included in children's record books.

Number of pence	Number of shillings
12	1
. .	2
36	.
48	.
60	.
. .	6

Number of shillings	20	40	. .	80	100
Number of pounds	1		3	.	.

Similar tabulations can be made for (half-crowns, pounds), (sixpences, shillings) etc.

Other useful experiences include
a. Choosing suitable coins with which to purchase an article and (b) making up a sum of money in different ways.

Which coin or coins would you offer the shopkeeper if you wanted to pay for:
1 A quarter lb of sweets at 10d per lb?
2 A book costing five shillings?
3 An orange costing 5d?
4 A toy costing 4s 6d?
5 Two comics at 5d each?

How many comics can you buy with 2s?

How many ice-lollies can you buy with 2s?

Write down what you think you could buy with various coins or amounts of money.

What 3 coins are together equal in value to 10d?

What 5 coins are together equal in value to 11d?

Can you think of more than one way of doing this?

Write down the ways you can make up a shillings worth of coins e.g. 6d and 6d.

See how many different things you can buy from the shop with one shilling. You must spend all your shilling each time. e.g.

2 fireworks at 6d each.

Possible shop items displayed with prices:

lollipops 1d each

chocolate bars 5d each

oranges 3d each

fireworks 6d each

ruler 9d

notebook 7d

biscuits 2d each

comic 4d

1s 6d	Use 6 coins
2s 6d	Use 5 coins
5s 0d	Use 8 coins
10s 0d	Use 9 coins
£1	Use 9 coins

Appropriate coins can be placed here

There seems little point in 'reduction exercises' such as converting £3 12s 5d to pence, or 16s 8d to halfpence. However real situations arise in school such as the costing of

savings stamps 22 at 6d

admission fees 100 at 3d

school magazines 30 at 6d

Such problems could be dealt with by class discussion as to the most suitable way of calculating the totals, and if kept within sensible limits there is benefit to be derived. Perhaps a simple graph could be drawn, or a ready reckoner made.

Coins can be interesting to children and other discoveries can be made apart from their values. For example: —

What is the date on the penny?

What is the date on the halfpenny?

What is the date on the sixpence?

What is the date on the threepenny-bit?

How many edges has the threepenny bit?

Which coin measures about 1" across its diameter?

Another interesting experiment for young children concerns the weight of coins.

Use the scales to find the weight of:

£1 worth of half-crowns

florins

shillings

sixpences

mixed silver coins.

Use the scales to find the weight of five shillingsworth of:

a. pennies
b. halfpennies
c. pennies and halfpennies

What have you discovered?
Is this information useful?
Who could use it?
Can you use this information?

A brief history of the development of coins and our monetary system can be interesting. Starting with early ideas of exchange and barter, children can be made aware of the need for money and a convenient range of coinage. The types of metal used can be discussed and why, for instance, certain coins have a milled edge. Bank notes can also be discussed, e.g. why elaborate etchings, water-marks and signatures are found on these.

Giving change can be a very useful mathematical experience for young children. 'Shopkeeper's Addition' or complementary addition should be encouraged. It will help the child's understanding of this operation in ordinary number work, e.g.

6+ ☐ =10

What must be added to 6 to reach 10? (cf. p. 51).

Shopkeeper's Addition

Giving change

If you don't know, ask your teacher how the shopkeeper gives change. Then pair off with your partner and practise giving change like the shopkeeper. Use the real coins or the cardboard ones.

1. Spend 1s 8d and give shopkeeper 2s 0d.
2. Spend 1s 8d and give shopkeeper 2s 6d.
3. Spend 3s 6d and give shopkeeper 2 half-crowns.
4. Spend 2s 4d and give shopkeeper 1 ten-shilling note.
5. Spend 8s 6d and give shopkeeper 4 half-crowns.
6. Spend 12s 6d and give shopkeeper 1 pound note.

It may be necessary, of course, to start with change from a shilling before proceeding to examples such as the above, especially with less able children. Recording is not necessary at the above stage. This type of experience can, however, be followed by such examples as: —

What change shall I get from 2s 6d if I spend: —

6d	9d	11d	4d
1s 2d	1s 6d	10d	3d

How much change?

Work with your partner and use real coins.

From 6d spend

4d	1d	3d	5d	2d
1½d	2½d	4½d	5½d	3½d

From 1 shilling spend:

3d	6d	9d	1d	8d
9d	2d	7d	10d	5d

From 2s 6d spend:

1s 6d	2s 3d	1s 0d	2s 0d
3d	9d	1s10d	1s 3d

Do these the shopkeeper's way.

Recording can now be in the form

Money given	Spent	Change
2s 6d	1s 3d	1s 3d
2s	9d	1s 3d

Change this Paper-money into shillings, recording like this: —

£	s	shillings
1	10	30

1. £1
2. £2 10s
3. £3
4. £1 15s
5. £3 10s
6. £5
7. £1 5s
8. £10 0s
9. £8
10. £6

Now change the above amounts into a. half-crowns
b. florins.

At this stage we are mainly concerned with recognising the value of money and mental agility in handling sums of money, giving change etc. No doubt some of the more able children will be able to deal with more than two amounts and check numerous items on bills etc. but until children are very confident and can add two amounts of money mentally, there does not seem much point in proceeding to larger amounts and more than two items.

Learning 'tables of pence' (20d for 1s 8d, 24d for 2s 0d, etc.) hardly seems worth while either. If we keep to small amounts and children receive plenty of experience in stacking pennies in piles of 12, they should then have no difficulty in converting from pennies to 's d'.

Through their experiences with money children can gain useful vocabulary: —

names of coins

cost, fare, article, cash, bank, cashier,

treasurer, shopkeeper, amount, spent, bill, total, change

as well as the names of numerous items from the various shopping activities.

Summary

Coin recognition

Counting in appropriate base (12 for shillings and pence, etc.)

Giving change (shopkeeper's addition)

Counting money by weight

Brief historical development

Tabulation: ready reckoner and tables

Topics of interest: shops, prices and fares

Addition of small amounts of money

Value of bank-notes.

8 Book List

This list is by no means exhaustive. It is simply a selection of books which have been found helpful at the level of this guide.

Romance in Arithmetic	Bowman	U.L.P.
Number Stories of Long Ago	Smith D. E.	Ginn
The Story of Measurement (Series)	Smith T.	Blackwell
The Story of Numbers	Smith T.	Blackwell
Man must Measure	Hogben	Rathbone
How Time is Measured	Hood	O.U.P.
The Golden Book of Mathematics	Adler	Golden Library
The Giant Colour Book of Mathematics	Adler	Hamlyn
Stories of Mathematics	Williams S.	Evans
How Money has Developed	Groom	Routledge
Building up Mathematics	Dienes	Hutchinson
Think of a Number (1, 2 and Teacher's Book)	Moss	Blackwell
Riddles in Mathematics	Northrop	Penguin,
Magic House of Numbers	Adler	Dobson
Number Patterns	James	Oxford
Exploring Mathematics on Your Own (Series of Books)	Johnson and Glenn	Murray
Number in the Primary School	Brearley	Froebel Foundation
Counting and Measuring	Churchill	Routledge and Kegan Paul
A Background to Primary School Mathematics	Adams	O.U.P.
First Steps in Practical Number Work	Mott	Evans
Primary Mathematics	Flavell and Wakelam	Methuen