PROBLEMS

P3

green set

Problems – green set

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General introduction

The aim of the Nuffield Mathematics Project is to devise a 'contemporary approach for children from 5 to 13'. The guides do not comprise an entirely new syllabus. The stress is on *how to learn*, not on what to teach. Running through all the work is the central notion that the children must be set free to make their own discoveries and think for themselves, and so achieve understanding, instead of learning off mysterious drills. In this way the whole attitude to the subject can be changed and 'Ugh, no, I didn't like maths' will be heard no more.

To achieve understanding young children cannot go straight to abstractions—they need to handle things ('apparatus' is too grand a word for at least some of the equipment concerned—conkers, beads, scales, globes, and so on).

But 'setting the children free' does not mean starting a riot with a roomful of junk for ammunition. The changeover to the new approach brings its own problems. The guide *I do, and I understand* (which is of a different character from the others) faces these problems and attempts to show how they can be overcome.

The other books fall into three categories: Teachers' Guides, Weaving Guides and Check-up Guides. The Teachers' Guides cover three main topics: ● Computation and Structure, ▼ Shape and Size, ■ Graphs Leading to Algebra. In the course of these guides the development of mathematics is seen as a spiral. The same concept is met over and over again and illustrated in a different way at every stage. The books do not cover years, or indeed any specific time; they simply develop themes and therefore show the teacher how to allow one child to progress at a different pace to another. They contain direct teaching suggestions, examples of apparently un-mathematical subjects and situations which can be used to develop a mathematical sense, examples of children's work, and suggestions for class discussions and out-of-school activities. The Weaving Guides are single-concept books which give detailed instructions or information about a particular subject.

The third category of books, as the name implies, will provide 'check-ups' on the children's progress. The traditional tests are difficult to administer in the new atmosphere of individual discovery and so our intention is to replace these by individual check-ups for individual children. These are being prepared by a team from the Institut des Sciences de l'Education in Geneva under the general supervision of Piaget. These check-ups, together with more general commentary, will be issued in the same format as the other guides and, in fact, be an integral part of the scheme.

While the books are a vital part of the Nuffield Mathematics Project, they should not be looked on as guides to the only 'right' way to teach mathematics. We feel very strongly that development from the work in the guides is more important than the guides themselves. They were written against the background of teachers' centres where ideas put forward in the books could be discussed, elaborated and modified. We hope very much that they will continue to be used in this way. A teacher by himself may find it difficult to use them without the reassurance and encouragement which come from discussion with others. Centres for discussion do already exist and we hope that many more will be set up.

The children's work that has been reproduced in these books, like the books themselves, is not supposed to be taken as a model of perfection. Some of it indeed contains errors. It should be looked upon as an example of work that children *might* produce rather than a model of work that they *should* produce.

Foreword

The last few years have been exciting ones for teachers of mathematics; and for those of us who are amateurs in the subject but have a taste for it which was not wholly dulled by the old methods that are so often stigmatised, there has been abundant interest in seeing the new mathematical approach develop into one of the finest elements in the movement towards new curricula.

This is a crucial subject; and, since a child's first years of work at it may powerfully affect his attitude to more advanced mathematics, the age range 5 to 13 is one which needs special attention. The Trustees of the Nuffield Foundation were glad in 1964 to build on the forward-looking ideas of many people and to set up the Nuffield Mathematics Project; they were also fortunate to secure Dr. Geoffrey Matthews and other talented and imaginative teachers for the development team. The ideas of this team have helped in the growth of much lively activity, throughout the country, in new mathematical teaching for children: the Schools Council, the Local Education Authority pilot areas, and many individual teachers and administrators have made a vital contribution to this work, and the Trustees are very grateful for so much readiness to co-operate with the Foundation. The fruits of co-operation are in the books that follow; and many a teacher will enter the classroom with a lively enthusiasm for trying out what is proposed in these pages.

Brian Young
Director of the Nuffield Foundation

Introduction

These problems have been designed for young Secondary-school children, and it is hoped that teachers will find them useful in conjunction with the main work described in the Teachers' Guides. The majority of children should at least be able to 'have a go' at most of the questions, but they should also be encouraged, to the full extent of their individual abilities, to think round a problem and to devise alternatives and generalisations. There should be no question of a race through the collection; the more thoughtful children will go more 'slowly' than those who can only skate over the surface.

The fifty-two problems are available on cards, for issue to individual children. This volume contains not only the problems but solutions and discussion which could lead to creative follow-up work.

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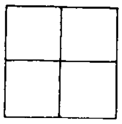
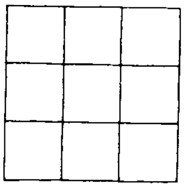
How many squares in this figure?



If we call the figure a 2-by-2 square, then we can draw a 3-by-3 square, like this:
How many squares are there in a 3-by-3 square?

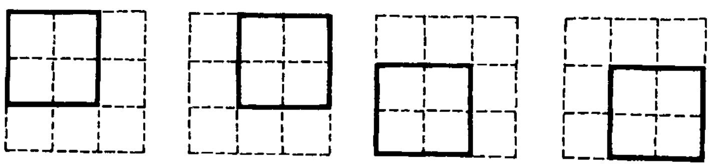


How many squares are there in a 4-by-4 square?

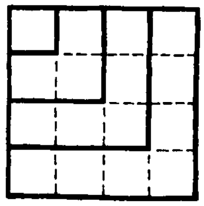
A 2-by-2 square is made up of four small 1-by-1 squares and the outside 2-by-2 square. There are therefore five squares in the 2-by-2 square: four are this size and one is this size:

The whole figure may be built up by using squares cut from cardboard or plywood or any other suitable material, or by using matchsticks (see Problem 8); or it may simply be drawn.

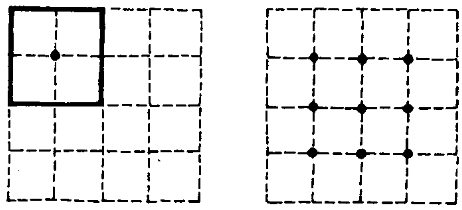
The 3-by-3 square contains nine small 1-by-1 squares and one 3-by-3 square, so we can find at least ten squares in this figure. But the figure also contains some 2-by-2 squares: these can be seen from the heavy lines in the diagram:



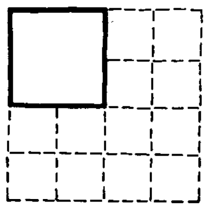
The 4-by-4 square also contains 3-by-3 squares and 2-by-2 squares as well as small 1-by-1 squares; there will be sixteen 1-by-1 squares.



We can best count the 2-by-2 squares by observing that ea intersection of the lines inside the figure is the centre of a 2- square, and that since there are nine such intersections, ther nine such squares in the figure.



Another way of finding the number of squares is to cut out a square of paper the size of a 2-by-2 square, and fit it on to the pattern of lines.



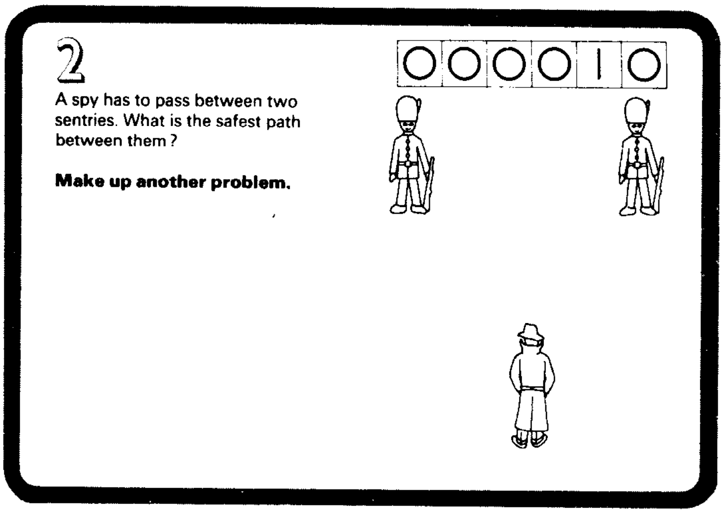
It can be placed in nine positions in the figure, and the boundary of the paper will coincide in turn with each of the nine 2-by-2 squares.

If we tabulate our results, we find:

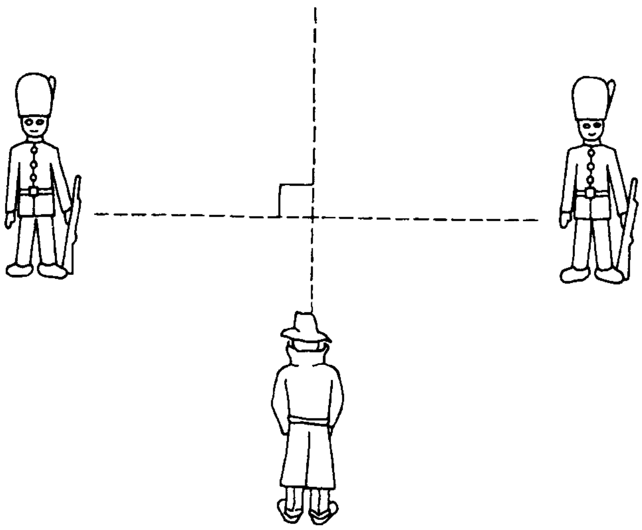
	Number of squares				
Size of square	1-by-1	2-by-2	3-by-3	4-by-4	Total
1-by-1	**1**				**1**
2-by-2	**4**	**1**			**5**
3-by-3	**9**	**4**	**1**		**14**
4-by-4	**16**	**9**	**4**	**1**	**30**

The set of numbers {1, 4, 9, 16, ...} is the set of square numbers. By continuing the pattern which is apparent in this table we can first guess, and then confirm by experiment, that the

5-by-5 square will contain $25 + 16 + 9 + 4 + 1 = 55$ squares and the 6-by-6 square will contain $36 + 25 + 16 + 9 + 4 + 1 = 91$ squares.

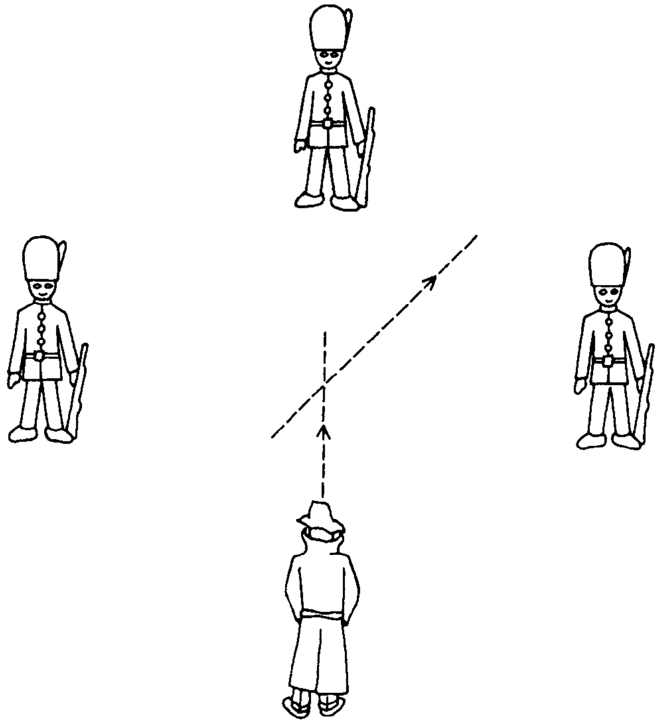


The spy's safest path will be the one which leads him exactly between the two sentries, so that he is no nearer to the one than he is to the other. Technically this will be the line which bisects at right angles the line joining the two sentries, and it will represent the *locus* of a point (or spy) which moves at an equal distance from each sentry.

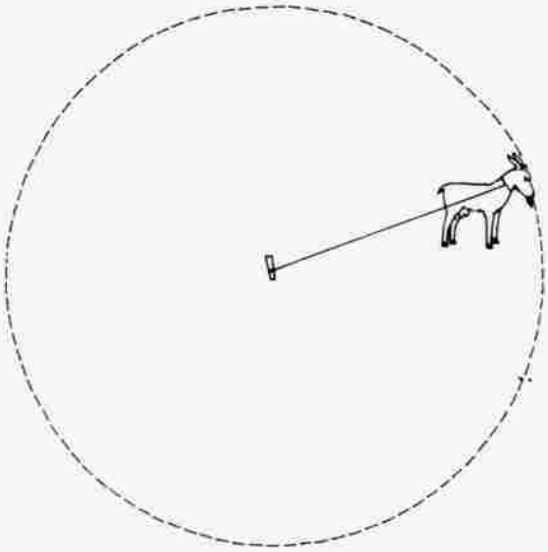


Successive positions for the spy can be marked with the help of a pair of compasses.

If a third sentry is positioned beyond the line which joins the first two, the spy must at some time change the direction of his path if he is not to approach too close to the third sentry. Where should he change direction?



If a goat is tethered to a stake by a chain, he can graze within a distance from the stake which is not much greater than the length of the chain, the limit of his grazing being the circumference of a circle:



This situation can be studied with the help of a pegboard, and a toy goat on the end of a piece of string. So can the situation in which a guard dog is chained to a ring which slides along a bar fixed to a wall.

The boundaries of each of the shaded regions can be described as the loci of points whose movements are subjected to certain restrictions: the goat can move within a given distance from a point; the dog can move within a given distance from a line. No shaded area can be drawn for the spy: his path is not the boundary of a region, but it is the locus of a point moving equidistantly from two points. Suppose he had reason to fear one sentry twice as much as he feared the other, and as a result tried to move so that his distance from the first sentry was twice his distance from the second?

3

When David counted his sweets in fours, he had two left over. When he counted them in fives he had one left over. How many sweets did he have?

Make up another problem like this one and give it to a friend to solve.

If David had two left over when he counted his sweets in fours, he must have had 4 + 2 or 4 + 4 + 2 or 4 + 4 + 4 + 2, etc., assuming that he had more than four to begin with. This means that the number of sweets can be found in this set:
{**6, 10, 14, 18, 22, 26, 30 . . . .**}

Now if he had six sweets, then he would have one left over if he had counted them in fives, so he may have had six sweets. But we can only say that he *may* have had six sweets. Other answers are possible, as we see if we write the set of numbers 5 + 1, 5 + 5 + 1, 5 + 5 + 5 + 1, etc.:
{**6, 11, 16, 21, 26, 31, 36 . . . .**}

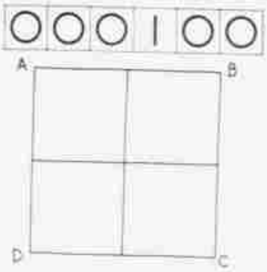
We find that 26 belongs to both sets. Might we expect to find any other numbers in both sets? What would the next number be?

In order to find a unique answer, we must have more information. It would be enough to know that David had more than 10 sweets, but fewer than 30. This would give 26 as the only number which fulfilled all the conditions of the problem.

4

ABCD is a square whose sides are 2 units long. Each of the four small squares has sides one unit long. What is the length of the shortest path from A to C, following the lines? How many 'shortest paths' are there from A to C?

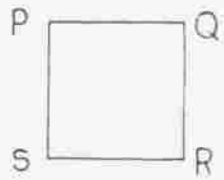
How many shortest paths are there from B to D?



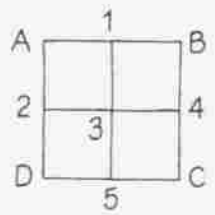
Experiment will show that the shortest path from A to C, following the lines, is 4 units long, and that there are a number of paths which have this minimum length. In fact some shortest paths pass through B and through D, so that there appears to be no advantage in using any of the lines internal to the square and parallel to its sides. (Do these lines have to be equidistant from each other?)

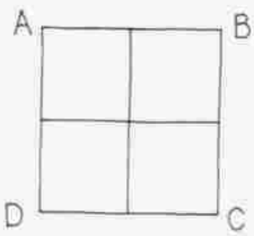
The question of how many paths there are is a little more difficult to decide. One entry into the problem is to evolve a technique using a simpler diagram; this technique can then be employed with the more complicated diagram.

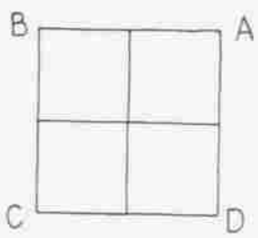
The simplest diagram is a square with no internal lines. The two possible shortest paths are P→Q→R and P→S→R.



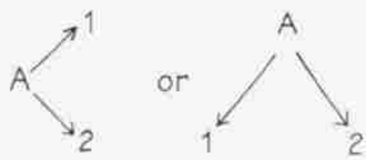
When we add two lines to give us our first diagram, we have five new junctions at which we can change direction. It is necessary to be able to refer to these readily, so we may number them; then we consider what possible routes we can take on the journey.



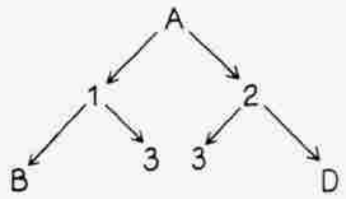




On leaving A, we have to make an immediate choice between two possible paths, one of which leads to 1 and the other of which leads to 2. We may indicate this choice thus:

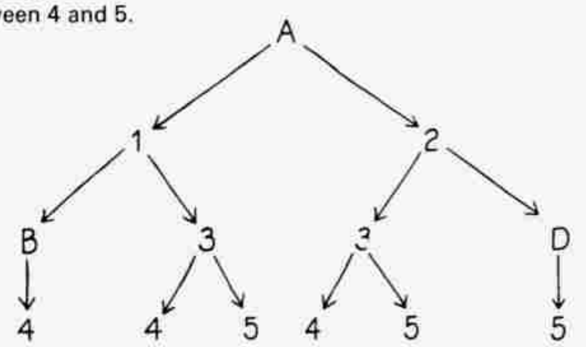


If we choose to go to 1, we have to make a further choice, whether to go to B or to 3; and a similar choice arises if we choose to go to 2. The 'tree' of choices now looks like this:



We note that 3 occurs twice at the end branches of the diagram: this is because we can travel from A to 3 by two separate paths.

There is no need to choose at B and D: we must turn a corner but our path is laid down. At 3 on the other hand we choose between 4 and 5.

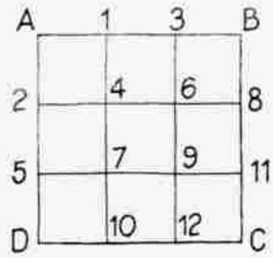


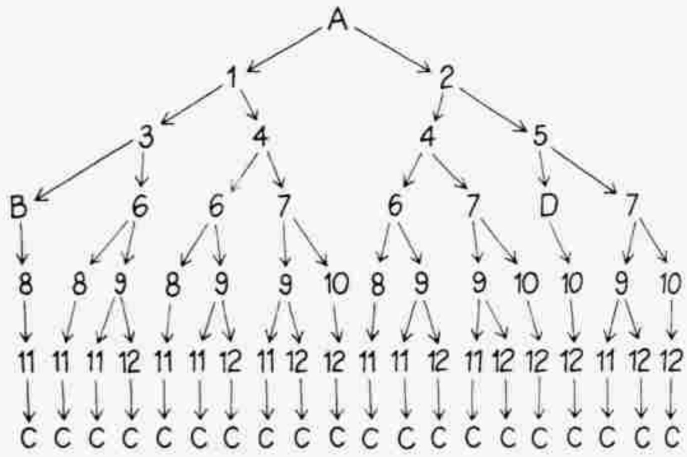
Our choice is now completely exhausted: from either 4 or 5 we can go only to C, if we are to take the shortest route. A route such as A→1→B→4→3→5→C makes use of the choice at 4 between 3 and C, but this is not a shortest path. The shortest path in this diagram is 4 units long, assuming it to comprise four squares with sides of unit length. So in this diagram there are six shortest paths.

There are six shortest paths from B to D, the same number as from A to C. In fact the problems are the same, and are connected by the symmetry of the figure: we can, as it were, flip the square over so that A and B change places and C and D change places:

Similarly the number of paths from C to A, and from D to B is also six.

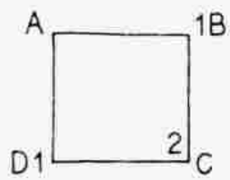
We could set ourselves the problem of finding the number of paths in a nine-square diagram:



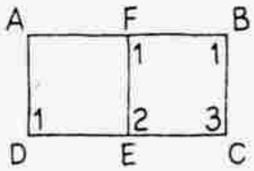


and there are 20 shortest paths from A to C.

Further investigations might seek to discover a general rule for finding the number of shortest paths from one corner to the opposite corner of a square with each side divided into any number of parts: one approach is to calculate the number of shortest paths leading to each intersection, and to see whether a pattern emerges. For instance, on a single square, B can be reached in only one way from A; so can D; and since C may be reached by passing either through B or through D, the number of paths from A to C is 1 + 1 = 2.

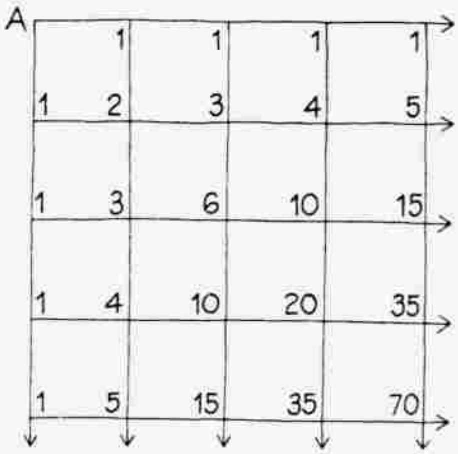


On a 'one-by-two' grid, the number of paths from A to C is 3, which is the sum of the single path to B and of the two possible paths from A to E:



That is, the journey from A to E can be either A→F→E or A→D→E. The journey to C can be the two extensions of these paths, A→F→E→C and A→D→E→C, with the addition of the path through B : A→F→B→C.

Using this additive principle we can write down the number of shortest paths from A to any intersection on a grid of any size.



These numbers may be recognised as being part of Pascal's Triangle (see Problem 9).

5

O	O	O	I	O	I

This is a crossnumber puzzle:

Clues across

1 A square number
3 Number of yards in a furlong
4 A cube number

Clues down

1 Number of inches in 10 feet
2 A multiple of 37

[1]		[2]
[3]		
	[4]	

The principle on which a crossnumber puzzle is solved is analogous to that which underlies the solution of a crossword puzzle: the aid given the solver by the clues is supplemented by the fact that two numbers of the puzzle may have one digit in common, and the solution of one clue gives an additional lead to the solution of one or more other clues.

[1] 1		[2]
[3] 2	2	0
0	[4]	

We now need a square number of two digits, of which the first is 1. The set of square numbers is {1, 4, 9, 16, 25 . . . .}, so we necessarily choose 16 as the answer to 1 across.

For 2 down we need a 3-digit multiple of 37 whose second digit is 0. The search for this set of numbers is one of the tasks for which the desk calculator was designed: the set is {407, 703}. Which do we want?

Since 4 across has to be a two-digit cube, and since the set of cubes is {1, 8, 27, 64, 125 . . . .}, it is clear that we can choose only 7 or 3 as the number in the bottom right-hand square. The full solution to this problem is:

[1] 1	6	[2] 4
[3] 2	2	0
0	[4] 2	7

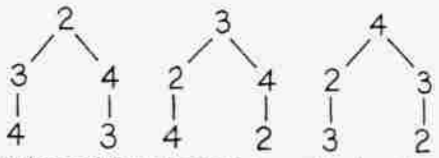
It is worth noting that the search through the multiples of 37 reveals that 37 is a factor of 111, and hence of 222, 333, etc.

In such puzzles no number is allowed to start with zero. In devising a puzzle, it is useful to compile tables of primes, square numbers and of third, fourth and fifth powers of numbers. Many equivalences from tables of quantities can be used for leads into a puzzle, as, for example, the number of half-crowns in £3. A convenient size of grid is 3 × 3 or 3 × 4. The heavy lines separating the light ones need not be placed symmetrically: it is more usual for crossnumber puzzles to have such lines than to have squares blocked out as in a crossword puzzle. The numbering of the clues needs a little care. It is also important that a puzzle should have only one solution.

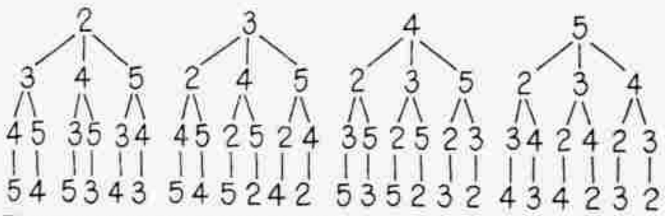
6

Three coloured rods, 2 cm, 3 cm, and 4 cm long, can be placed end to end to form a rod 9 cm long. We can give the arrangement shown here the code number 234. How many arrangements can you find which have numbers different from 234?

By juggling the rods around we find five more arrangements whose code numbers are 243, 324, 342, 423, and 432. Is this a maximum or are there more arrangements which we have not yet found? We may check that if we put 2 first, we can follow this by 34 or by 43; similarly with 3 first, our two arrangements are 324 and 342; and with 4 first we have 423 and 432. We can show this in a tree diagram:

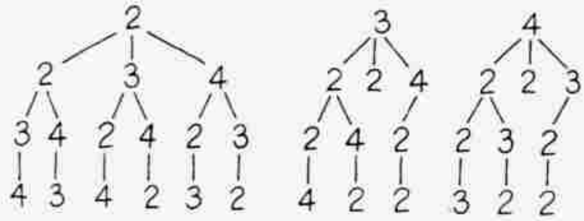


If we add another rod 5 cm long to our collection then our tree will grow very much bigger:



There are twenty-four ways of arranging four rods of different length.

If, instead of adding a 5-cm rod to the collection, we had added a second rod 2 cm long, then the question would be: 'How many arrangements can we make with the numbers 2234?' Again a tree diagram may help:



Our tree has lost some of its branches: we have six numbers beginning with 2, and three each beginning with 3 and 4. We could have found these arrangements by trying to fit a second rod into each of the six arrangements of three rods. We could put the new rod in front of each:

2234, 2243, 2324, 2342, 2423, 2432

This gives us six arrangements which begin with 2 in our tree. If we fit the new 2 in second place, we have:

2234, 2243, 3224, 3242, 4223, 4232

The first two arrangements we have already; so we have four new arrangements.

With 2 in third place we have:

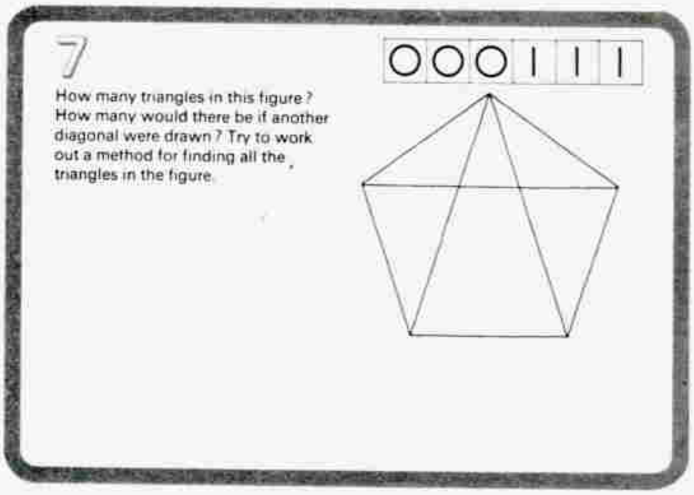
2324, 2423, 3224, 3422, 4223, 4322

but of these only two are new arrangements.

With 2 lying last we have:

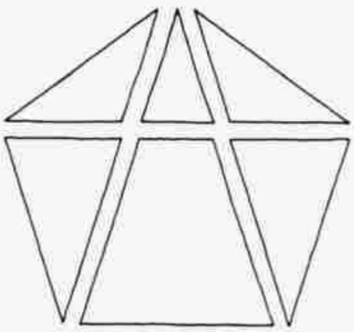
2342, 2432, 3242, 3422, 4232, 4322

but all these arrangements we have already.



Obviously there are several methods. The difficulty lies in locating all of them. It is very easy to miss one or two. Three approaches:

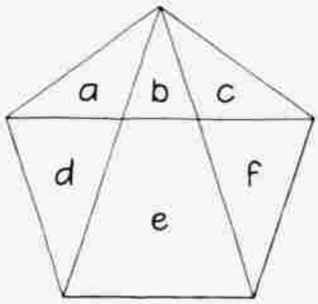
1 A model in card or preferably in plywood is made, with the lines representing cuts which dissect the pentagon into six pieces:



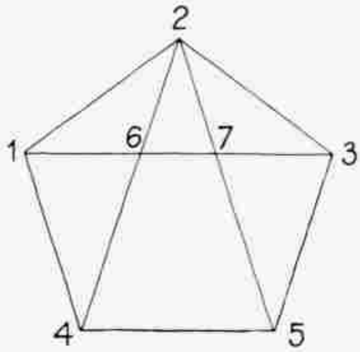
Five of these are triangles already: they can be combined in pairs or in threes to give six more triangles, though they must not change their relative positions or be turned over or rotated.

2 We number the regions of the figure. We then list all possible combinations of regions which border one another and examine each to see whether it is a triangle. This leads to the following list of triangles:

one region	two regions	three regions
a	**ab**	**abc**
b	**ad**	
c	**bc**	
d	**be**	
f	**cf**	



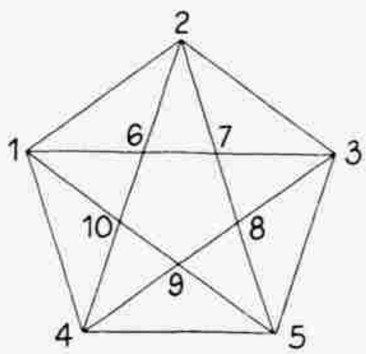
3 We may number the vertices and intersections of the figure, and see which combinations of three numbers mark the corners of a triangle.



Below we have listed all possible combinations of three points. We have then rejected those which all lie on the same line, and those in which two are not connected; and eleven remain. These tally with the eleven we found by using the method in **1**.

123 –	234	345	456	567
124 –	235 –	346	457	
125	236 –	347	467	
126 –	237 –	356		
127 –	245 –	357 –		
134	246	367		
135	247			
136	256			
137	257			
145	267 –			
146 –				
147				
156				
157				
167				

We might use the second method to discover how many triangles there are in the figure when diagonals are drawn from 1 to 5 and from 3 to 4. There will now be 10 vertices or points of intersection . . . .

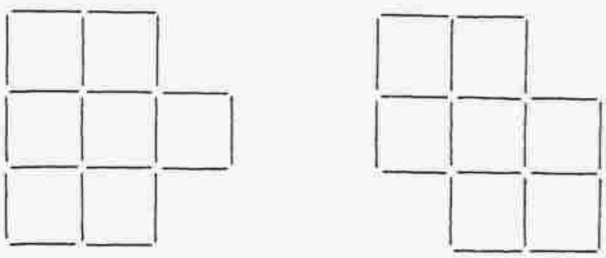


8

This pattern of five squares is made from twelve matches:

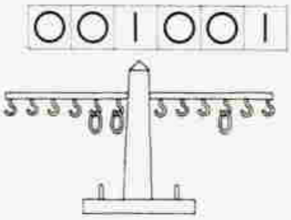
How many squares can you make using twenty matches? (Matches must meet only at their ends, and must not be bent or broken.)

This is a matter for experiment. It seems likely that the answer is 'nine'. Two ways of making these nine squares are:



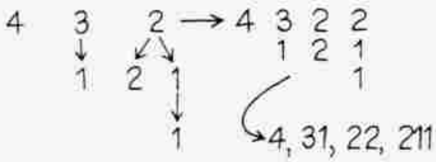
They are pleasing designs. Their compactness suggests that nine is the maximum number of squares which can be made from twenty matches. Some of the matches are part of three different squares. As we increase the number of matches we use, we obtain a relatively greater increase in the number of squares: for instance, from forty matches we can make thirty squares. There may come a point at which we have more squares than matches in the pattern. If so, when does this arise?

9

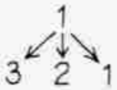


Hang a ring on hook number 4 on the right-hand side of an equaliser. In which different ways can you make the equaliser balance by hanging hooks on the left-hand side? You may count two ways as different if you merely hang rings on the same hooks in a different order, for instance 1 and then 3 can count as different from 3 and then 1.

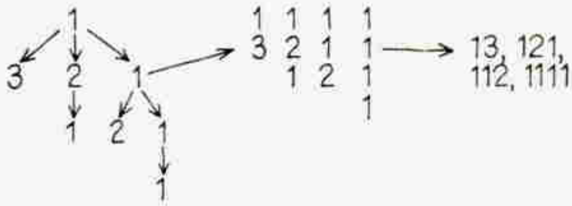
This is essentially the same problem as writing all the numbers whose digits add up to 4, not allowing zeros (4, 31, 13, 22, 112, 121, 211, 1111). We may begin by hanging a ring on hook 4: this is the only single-digit number in the set of our solutions. If we hang a ring on hook 3, then one ring on hook 1 will give the only solution, 31, whose first digit is 3. When a ring is hung on hook 2, we have two possibilities for the next step: another ring on 2, or a hook on 1, followed by a second ring on 1. So far we have four solutions:



If we hang a ring on hook 1, then we may hang a second ring on 3 or on 2 or on 1.



The rings on 1 and 3 balance the ring on 4 on the other side. The rings on 1 and 2 will balance when another ring is placed on hook 1. The two rings on hook 1 can be supplemented either by a ring on hook 2 or by two more rings on hook 1.



Placed in order of size, treating them as numbers, the solutions are 4, 13, 22, 31, 112, 121, 211, and 1111. This set can be partitioned into subsets according to the number of their digits: {4}, {13, 22, 31}, {112, 121, 211}, {1111}. If we write the solutions in columns, we see a pattern more clearly:

4	13	112	1111
	22	121	
	31	211	

The numbers in each column are 1, 3, 3 and 1, and these numbers compose a line of Pascal's Triangle:

1

1 1

1 2 1

<u>1</u> <u>3</u> <u>3</u> <u>1</u>

1 4 6 4 1

More of the lines of this triangle appear in the solutions to the problem if the ring is hung on other numbers besides 4 on the right-hand side of the equaliser.

10 | O O I O I O

My age this year is a multiple of 7 and next year it will be a multiple of 5. If I am not yet 50 can you say how old I am? If I am also over 30, can you say how old I am?

One method of solving this problem is to use a square ten inches by ten inches, divided into 100 one-inch squares, which are numbered from 1 to 100. Coloured cubes are placed on the multiples of 7 and on those of 5, a different colour on each set of multiples.

1	2	3	4		6		8	9	
11	12	13			16	17	18	19	
	22	23	24		26	27		29	
31	32	33	34		36	37	38	39	
41		43	44		46	47	48		
51	52	53	54			57	58	59	
61	62		64		66	67	68	69	
71	72	73	74		76		78	79	
81	82	83			86	87	88	89	
	92	93	94		96	97		99	

This year my age is one of the numbers under a red cube. Next year it will be a number which is under a green cube. These two numbers must be consecutive, and run in the order red →green. A study of the pattern of the square will show that three pairs of numbers satisfy these conditions: they are (14, 15), (49, 50), and (84, 85) and although we cannot say from this exactly how old I am, the possibilities are limited to 14, 49, and 84. We can say that the solution set is {**14, 49, 84**}. Even the knowledge that I am not yet 50 does not help much: it means that we may reject 84 from our set of solutions but we are still left with {**14, 49**}. If we are then told that I am also over 30, this excludes 14, which is less than 30, leaving the unique solution {**49**}.

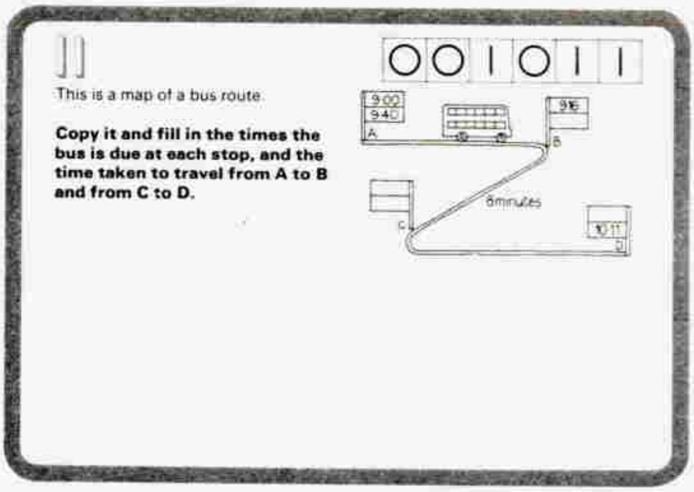
The problem can also be solved by writing down the set of multiples of 7 as far as 100; we refer to this as set A.
A = {**7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98**}.

If one of these numbers is my age this year, my age next year will be one of the numbers in set B:
B = {**8, 15, 22, 29, 36, 43, 50, 57, 64, 71, 78, 85, 92, 99**}.

But my age will also be a multiple of 5, that is a number in set C:
C = {**5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100**}.

We now look for any numbers which are in set B *and* in set C. These will give a set which is the intersection of B and C: {**15, 50, 85**}.

These give my age next year. My age this year is one of the set {**14, 49, 84**}. Having determined this we now impose the restrictions as before.



We have to assume that all buses take the same time over any part of the route: no allowance is made for rush-hour traffic. In this case the journey from A to B will take 16 minutes, and the second bus will be due at B at 9.56. The two buses will be due at C at 9.24 and 10.04 respectively: the journey from C to D takes 7 minutes, and the first bus is due at 9.31.

We can tabulate this information in the form of a timetable. Many children find it much easier to use a timetable when they have constructed one for themselves. If we follow the trend towards the use of the 24-hour clock, then we write 0924 for 9.24 a.m.

Assuming that the times given in the map are both morning times and evening times, we have:

A	**0900**	**0940**	**2100**	**2140**
B	**0916**	**0956**	**2116**	**2156**
C	**0924**	**1004**	**2124**	**2204**
D	**0931**	**1011**	**2131**	**2211**

Local bus or train timetables can now be studied. Children can ask each other questions such as: 'If I wanted to get to the cinema by 3.40 (1540) what is the latest bus I could catch?' or 'If I get to Hill Road bus stop at ten past eight, how long shall I have to wait for the next bus to Outsted?'

Many bus companies also publish fare tables, in this form:

A			
4	B		
8	4	C	
11	8	4	D

Similar tables can be made to show the time taken for a bus to travel from one stop to another:

A			
16	B		
24	8	C	
31	15	7	D

There are many relationships to be discovered here, e.g. the time from A to D is the same as that from A to C together with that from C to D (31 = 24 + 7). Is the same true for fares?

The table for times taken on the return journey can be compared with this table and the reasons for any differences discussed. Average speeds can be worked out. But most important from a practical point of view, problems of travelling from one place to another can be set, the best solution being that which takes the shortest time, or which involves the fewest changes en route . . . .

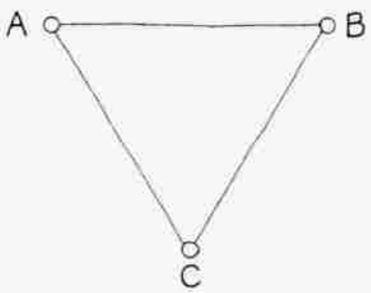
12

If each of four people shakes hands once with each of the other three, how many handshakes will there be? How many if there are five people?

Is there an easy way of working out the number of handshakes for any number of people?

This is a problem to be approached through experiment. A team of four engages in a bout of handshaking, at the end of which everyone has shaken hands three times: the product of four and three is twelve. If this is offered as an answer, it might be useful to make a map or 'graph' of the situation. For two people, only one handshake takes place.

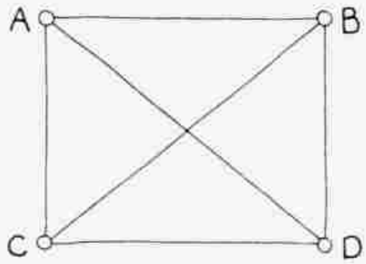
A and B are the people; the line joining them is the handshake. When C meets them, more handshakes occur:



When three people shake hands, there are three handshakes, even though each person shakes hands twice and the product of three and two is six. The key to the solution lies in the fact that one man's handshake is another man's handshake. We have counted each handshake twice, assuming that we have not made any error at all. We halve our product to obtain the number of handshakes. Thus

$$\frac{4 \times 3}{2} = 6$$

The graph for four people shows the six handshakes clearly.

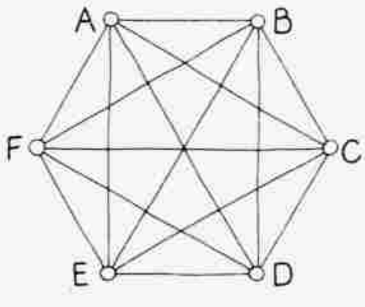


The graph for five people is a pentagon with its diagonals: the graph has ten lines in all.

What are our results so far? We tabulate them and look for a pattern.

No. of people	**1**	**2**	**3**	**4**	**5**
No. of handshakes	**0**	**1**	**3**	**6**	**10**

The numbers in the lower line can be recognised as the triangular numbers together with zero. The next is 15, which should give the number of handshakes which take place among six people.



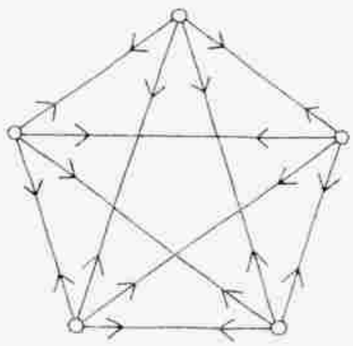
If we take 1 to be the first triangular number, 3 to be the second, then for n people, the number of handshakes will be the (n − 1)th triangular number.

What if there are 1000 people? We can hardly count up as far as the 999th triangular number, which is 499,500. We must if possible establish a relationship between the numbers in the upper and lower lines of our table.

Each of three men will shake hands with two others: the product of three and two is six. But each handshake is counted twice in the product, so we halve the product to obtain three handshakes.

Each of five men shakes hands with four others: the product of five and four is twenty. Half twenty is ten.

The graph for five men shaking hands will be a pentagon with its diagonals. From each vertex of the pentagon, four lines lead to other vertices. But if we show an arrow on every line leaving a vertex, two arrows will share the same line: each line will contain two arrows. So in fact we have not twenty lines but ten.



What have we discovered? We take the number of people and subtract one from that number, since no-one shakes hands with himself. We find the product of the two numbers, and divide this product by two. This can be expressed algebraically as

$\frac{n\,(n-1)}{2}$ which gives the (n − 1)th triangular number.

Check: the 999th triangular number is

$$\frac{1000 \times 999}{2} = \frac{999000}{2} = 499{,}500.$$

If an odd number of people is involved, will each shake hands with the others an odd or an even number of times? Or is there no such general rule? If an even number is involved, . . .

13 O O I I O I

The four numbers given here are part of a sequence with a simple rule for finding other numbers in the sequence. Can you find the numbers which are missing?

☐ ☐ 7, 10, 13, 16, ☐ ☐

Can you find the missing numbers in this sequence?

☐ ☐ 11, 15, 19, 23, ☐ ☐

Make up two problems of your own like these and give them to your friends to solve.

If we make a number strip and block out in some way the numbers 7, 10, 13, 16, we see at once that the numbers form a pattern:

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

and all we need to do to find a solution to the problem is to extend the pattern two places to the right and to the left.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

1, 4, 7, 10, 13, 16, 19, 22

For the second problem, we use precisely the same technique and complete the pattern without difficulty.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31

3, 7, 11, 15, 19, 23, 27, 31

In both these sequences the difference between terms is constant:

Sequence: 1 4 7 10 13 16 19 22
Difference: 3 3 3 3 3 3 3

Sequence: 3 7 11 15 19 23 27 31
Difference: 4 4 4 4 4 4 4

That is to say, once we have found what the difference is, we can simply add this difference to the last term in the sequence to obtain the next term, and by repeating this process, carry the sequence on as far as we wish.

16 + 3 = 19; 19 + 3 = 22; 22 + 3 = 25; 25 + 3 = 28; etc.

If we met a sequence with a difference which was not constant, as in 4, 7, 11, 16, 22 . . . ., we should have to look for a different kind of pattern, and this may very well be brought out by using a grid. We write the next number following 4, that is 5, at the head

	5	8	12	17	23	30
(4)	6	9	13	18	24	31
	7	10	14	19	25	32
		11	15	20	26	33
			16	21	27	34
				22	28	35
					29	36
						37

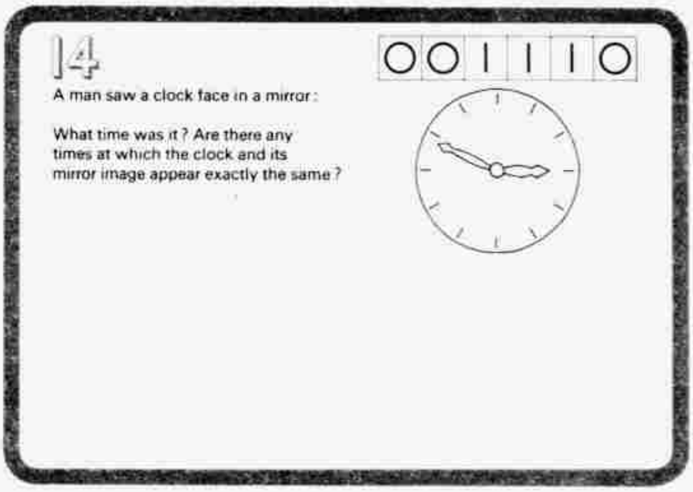
of the *second* column, and then write in the natural numbers in columns, starting a new column immediately after writing one of the numbers in our sequence. The next two terms are 29 and 37. We can of course find these by writing differences:

4 7 11 16 22 29 37 . . . .
3 4 5 6 7 8 . . . .

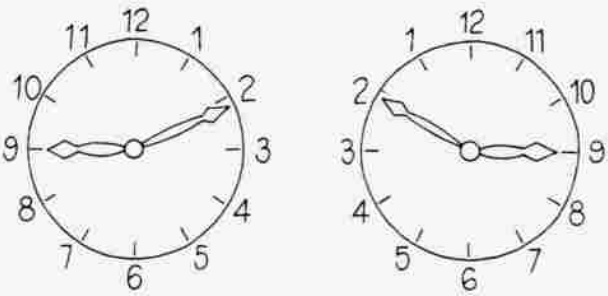
If we plot our two original sequences on the grid we find a different pattern, which reflects the constancy of the differences:

	5	8	11	14	17	20
	6	9	12	15	18	21
4	7	10	13	16	19	22

	4	8	12	16	20	24	28
	5	9	13	17	21	25	29
	6	10	14	18	22	26	30
3	7	11	15	19	23	27	31



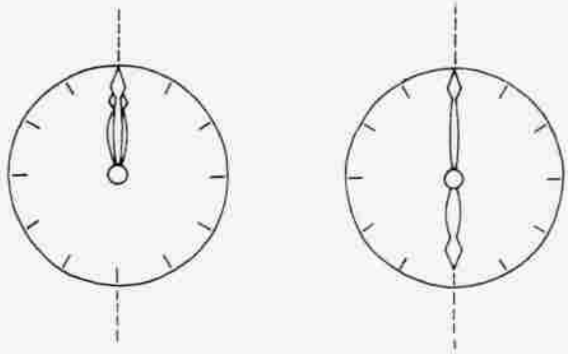
The simplest way to solve this is to find a mirror and a clock, and to adjust the hands of the clock until the mirror image is the one illustrated. Another method is to draw two clocks, making one the mirror image of the other. For simplicity, we may write all numbers the correct way!



It would seem that the time on the clock was ten past nine. A wise check is to put the hands of a clock to this time and to look at it in a mirror.

An old alarm clock whose hands can be turned easily is a useful piece of equipment in the classroom: it can be used to check not only that the time shown on the clock on the card looks like ten to three, but also that ten to three and ten past nine are in fact mirror images of each other. What can we say of times which are mirror images of each other?

The times at which the clock is its own mirror image are twelve o'clock and six o'clock. At these times the clock has **bilateral** symmetry through a line joining the 12 to the 6:



A further question to ask is: how often in the course of a day are the hands exactly together? Obviously they are exactly together at twelve o'clock. They will also be together just after five past one.

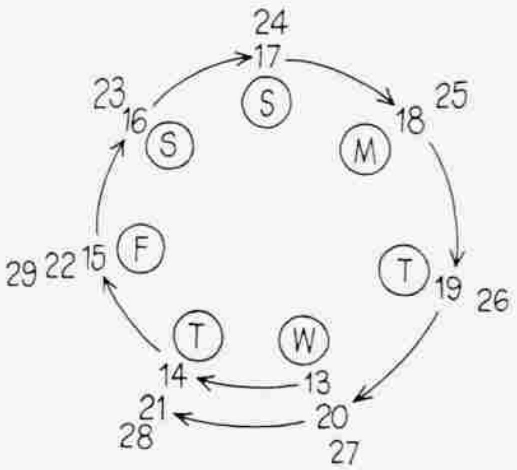
The question, 'At what times are they exactly together?' is a little more difficult to answer. The usual way to do this is by algebra, but the fact that this position occurs ten times between twelve o'clock and the next twelve o'clock, means that it occurs at equal intervals of $\frac{60 \times 12}{11}$ minutes, or every 65 minutes $27\frac{3}{11}$ seconds . . . .

15 OO||||

If 13th January falls on a Wednesday, on what day of the week will 29th January fall? On what day of the week will 13th January fall next year?

Make up a similar problem.

The 29th January falls on a Friday. One way of calculating this is to write the days of the week as if they were 'hours' on a clock, and then to count round: if the 13th falls on a Wednesday, then the 14th will fall on a Thursday, and so on.



If we count backwards we find that we should arrive at 1 on a Friday, this being New Year's Day, 1st January. At the other end the 30th will fall on a Saturday and the 31st on a Sunday, so that we have now partitioned the thirty-one days of January into seven subsets:

Friday	{**1, 8, 15, 22, 29**}
Saturday	{**2, 9, 16, 23, 30**}
Sunday	{**3, 10, 17, 24, 31**}
Monday	{**4, 11, 18, 25**}
Tuesday	{**5, 12, 19, 26**}
Wednesday	{**6, 13, 20, 27**}
Thursday	{**7, 14, 21, 28**}

The second number in each subset differs from the first by 7. The third differs from the first by fourteen, and the third differs from the first by 21. These numbers, 7, 14, 21, are multiples of seven. It ought not to surprise us that, if we know the date of a Friday, then we may find the date of the following Friday by adding 7; and the date of the Friday after that by adding 14, and so on.

If we write the dates of the month in the form of a calendar, we can see that we are using the numbers in the subsets which we have just found.

January

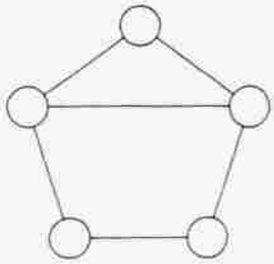
Sunday		**3**	**10**	**17**	**24**	**31**
Monday		**4**	**11**	**18**	**25**	
Tuesday		**5**	**12**	**19**	**26**	
Wednesday		**6**	**13**	**20**	**27**	
Thursday		**7**	**14**	**21**	**28**	
Friday	**1**	**8**	**15**	**22**	**29**	
Saturday	**2**	**9**	**16**	**23**	**30**	

We can check from this table (calendar) that if the 13th falls on a Wednesday, then so also will the 20th ($13 + 7 = 20$) and the 27th ($13 + 14 = 27$). The 29th, falling two days later than the 27th, will be a Friday, two days after the Wednesday.

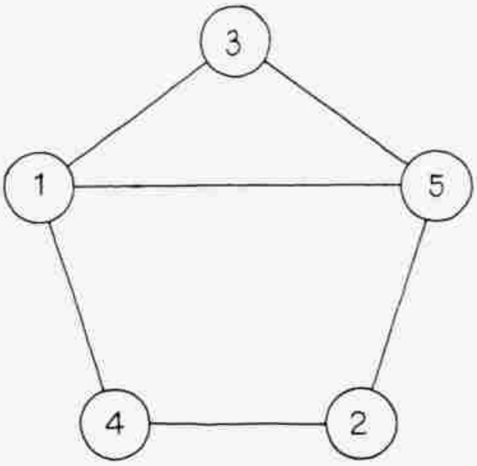
The problem of finding on what day of the week 14th November will fall is more difficult: it is easier to calculate on what day 13th January will fall in a year's time. The 13th January will recur after 365 days, unless this year is a leap year. Since Wednesday recurs every seven days, we can forecast that after 52 weeks of seven days, which is $52 \times 7 = 364$ days, it will again be a Wednesday. The 365th day will be a Thursday. Next year 13th January will fall, not on a Wednesday, but on a Thursday. If this year is a leap year, with 366 days, since $366 = 364 + 2$, 13th January will fall two days later in the week, on a Friday.

16

Place each of the numbers 1, 2, 3, 4, 5 in one of the circles in this figure so that no number is connected directly to a number next door to it. (For example, 3 must not be connected to 2 or 4.)



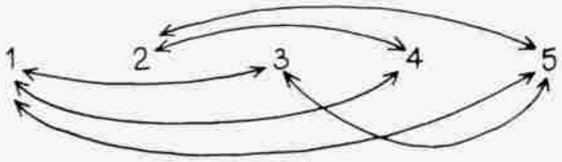
The solution to this problem can be found by trial and error: it is very helpful to draw the pentagon on a large piece of card and slide numbered discs around until a solution is found. There are two solutions, each of which is the reflection of the other. Here is one of them:

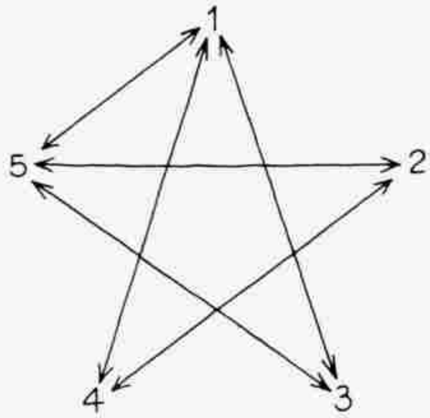


We can start to find a solution by considering the relation: 'is next door to'. If we write the numbers in order, we can insert arrows to show this relation.

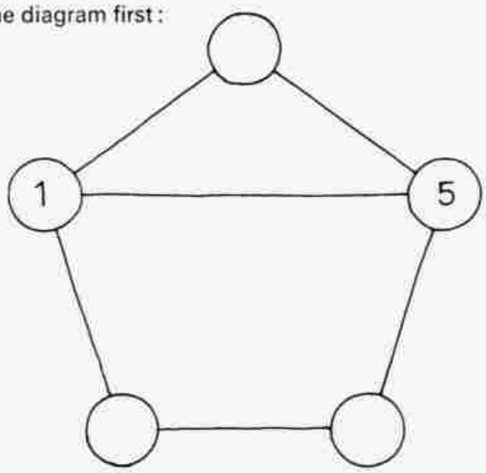
$1 \longleftrightarrow 2 \longleftrightarrow 3 \longleftrightarrow 4 \longleftrightarrow 5$

What we are concerned with is the relation: 'is not next door to'. This can be shown more clearly by writing the numbers in a circular formation:



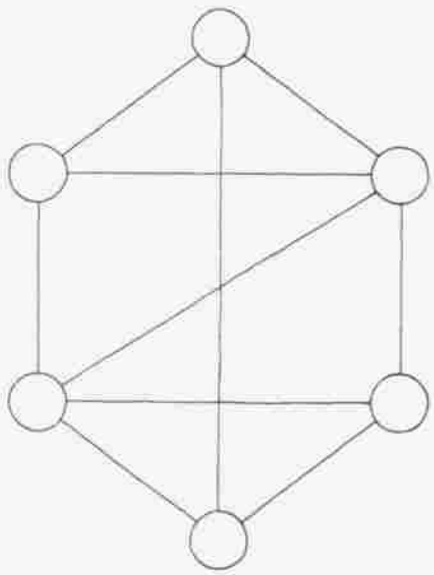


In the diagram two circles are connected to *three* other circles. The lines which connect the circles can only join numbers which are *not* next door to each other; and the only numbers which are *not* next door to *three* others are 1 and 5. We put these in the diagram first:



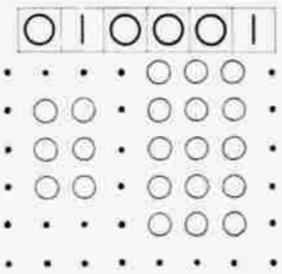
Now 1 is also not next door to 3 and 4, so these must be entered in the circles joined to 1. But 4 *is* next door to 5, so 3 must go in the top circle: the last remaining circle is the home for 2.

Can we use the same method for placing the numbers from 1 to 6 in this pattern, so that no adjacent numbers are joined directly by a line?



17

We may call 6 a 'rectangular' number because six pegs on a pegboard can be arranged in a rectangular pattern or 'array'. Another rectangular number is 15. We make it a rule that no holes must be left empty within the rectangle. What other rectangular numbers can you find which are not greater than 30?



There is only one rectangular number less than 6, and this is **4.** The set of rectangular numbers up to 30 is: {**4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30**}.

The other numbers below 30 are: **1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29,** and if we exclude 1, these are the prime numbers below 30. They are the numbers which cannot be expressed as the product of any two numbers in the set: {**2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,** . . . .}.

Thus whereas 22 can be written as the product of 2 × 11, it is not possible to write 23 in a similar way. (If we include 1 in the set we could write 23 = 1 × 23, and thus the distinction between the prime and the rectangular or **composite** numbers, would disappear.)

The prime numbers appear in other situations. If we make a list of natural numbers as far as 30 and then write 1 against every number, 2 against every second number, 3 against every third number, and so on, we shall have this pattern:

1	1	**11**	1 11
2	1 2	**12**	1 2 3 4 6 12
3	1 3	**13**	1 13
4	1 2 4	**14**	1 2 7 14
5	1 5	**15**	1 3 5 15
6	1 2 3 6	**16**	1 2 4 8 16
7	1 7	**17**	1 17
8	1 2 4 8	**18**	1 2 3 6 9 18
9	1 3 9	**19**	1 19
10	1 2 5 10	**20**	1 2 4 5 10 20

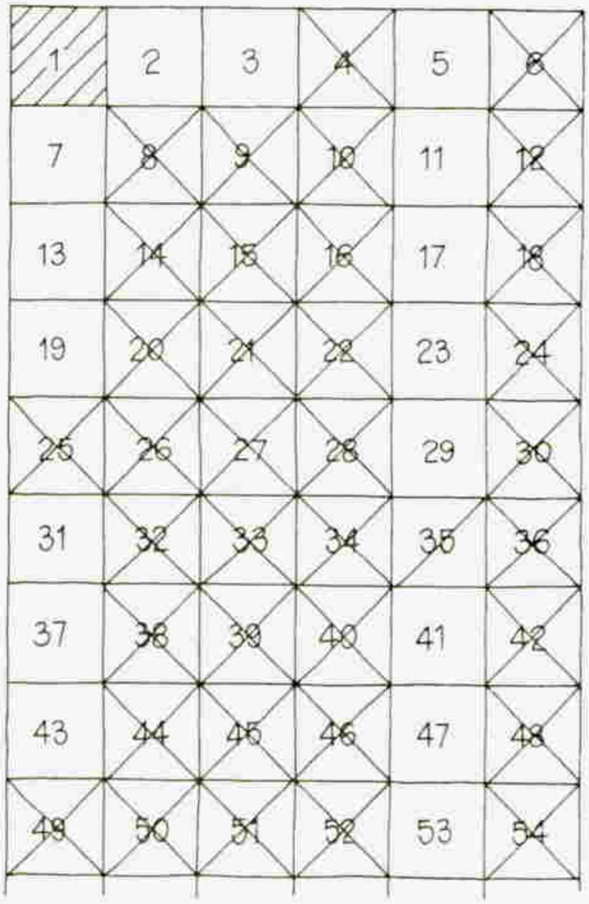
21	1 3 7 21
22	1 2 11 22
23	1 23
24	1 2 3 4 6 8 12 24
25	1 5 25
26	1 2 13 26
27	1 3 9 27
28	1 2 4 7 14 28
29	1 29
30	1 2 3 5 6 10 15 30

We call the numbers written after each of the numbers we first wrote down, the 'factors' of that number. Thus 6 has as factors the set {1, 2, 3, 6}; while 23 has as factors the set {1, 23}. We notice that some numbers have more factors than others. We may tabulate the number of factors:

Number of factors	Numbers
1	**1**
2	**2, 3, 5, 7, 11, 13, 17, 19, 23, 29**
3	**4, 9, 25**
4	**6, 8, 10, 14, 15, 21, 22, 26, 27**
5	**16**
6	**12, 18, 20, 28**
7	—
8	**24, 30**

The prime numbers appear as the set of numbers which have only two factors. What is the distinction between the numbers which have an odd number of factors and those which have an even number of factors? Those with an odd number appear to be the set {1, 4, 9, 16, 25 . . . .}.

To find the prime numbers quickly we use the Sieve of Eratosthenes. We write the numbers in a grid, and block in those which have the factor 2, or the factor 3, or the factor 4 and so on, leaving the 2 itself, 3 itself and so on ; 4 is blocked in as having a factor 2. If we write the numbers in a grid six squares wide, then all the prime numbers except 2 and 3 appear in just two columns of the grid :



18 O I O O I O

Ask a friend to think of a number between 1 and 20; and by asking not more than five questions, to which he may answer only 'yes' or 'no', find out what the number is.

How many questions would you need to ask in order to find the number he had chosen if it was between 1 and 100? Between 1 and 1000?

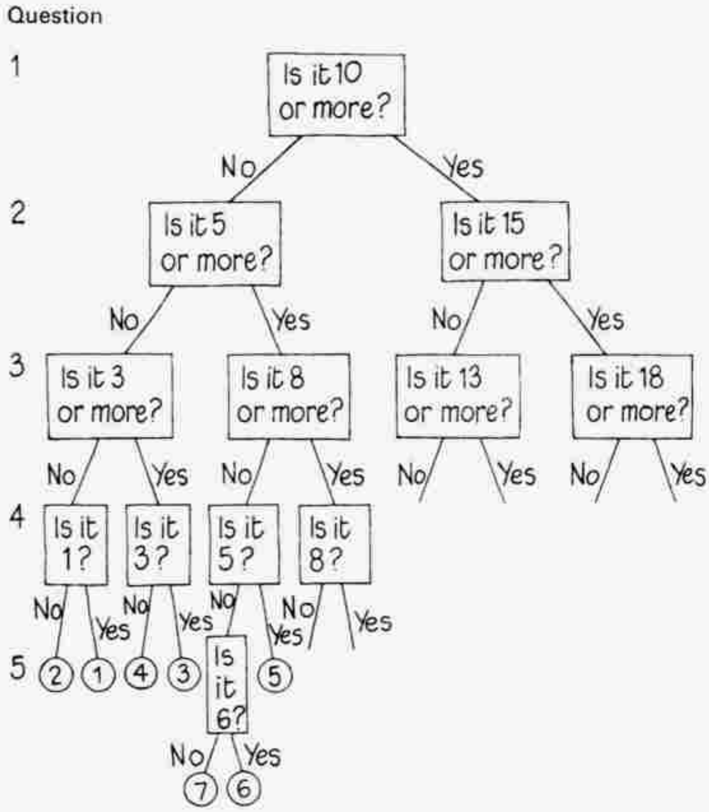
This is not impossible. A technique could be worked out by considering simple cases first, but a better lead may be given by considering how we might solve a much harder task.

Suppose we think of a number between 1 and 2,000,000. Finding a number within this range is a massive task, which calls for a really heavy and blunt instrument for its completion. A little thought and prompting might lead to a simplification by asking the question: 'Is it greater than one million?'. Whichever answer is given, we can at once eliminate half the possibilities, and if we so wish we can continue in this manner.

If we now apply this instrument to the more tractable problem of locating a number between 1 and 20, we can see quickly that as an instrument it is still highly effective. The first question: 'Is it greater than 10?' will eliminate half the possibilities. Suppose the number is 4, then we can proceed by trying to eliminate about half the remaining possibilities at any time.

Is it greater than 10?	No:	**1 2 3 4 5 6 7 8 9 10**	**11 12 13 14 15 16 17 18 19 20**
Is it greater than 5?	No:	**1 2 3 4 5**	**6 7 8 9 10**
Is it greater than 3?	Yes:	**1 2 3**	**4 5**
Is it 4?	Yes:	**4**	**5**

We can illustrate our technique diagrammatically. The flow diagram is appropriate: we shall change the wording of the question slightly and significantly.



The diagram is incomplete, but if it were completed, we should see that five questions would be enough to enable us to discover which number between 1 and 20 had been thought of. If on the other hand we apply the rules of 'Twenty Questions', and count the statement 'Then it must be 4' as a last question, we shall need to ask up to six questions.

If the number is known to be between 1 and 50, then we follow the same technique, choosing a number which is halfway along the range, say 25.

Question 1 Is it 25 or more? Answer, No.
Question 2 Is it 12 or more? Answer, No.

We now know that the number is below 12. But all we need to ask to find a number below 20 are six questions, so that we may be able to find a number which is less than 12 by asking five questions at most, giving us the seven questions which we are allowed to ask. That this can be done can be checked by a trial.

Let us now try to find a number between 1 and 100. Opposite each question and its answer we shall write the range of numbers in which the one we are looking for is to be found.

Is it 50 or more ?	**Yes**	**(50–100)**
Is it 75 or more ?	**No**	**(50–74)**
Is it 62 or more ?	**Yes**	**(62–74)**
Is it 68 or more ?	**No**	**(62–67)**
Is it 65 or more ?	**Yes**	**(65–67)**
Is it 65 ?	**No**	**(66, 67)**
Is it 66 ?	**No**	**(67)**
Is it 67 ?	**Yes**	

So we need 8 questions for a number below 100.

Suppose the number is less than 1000.

Is it 500 or more ?	**No**	**(1–499)**
Is it 250 or more ?	**Yes**	**(250–499)**
Is it 375 or more ?	**No**	**(250–374)**
Is it 312 or more ?	**Yes**	**(312–374)**
Is it 343 or more ?	**No**	**(312–343)**
Is it 327 or more ?	**Yes**	**(327–343)**
Is it 335 or more ?	**No**	**(327–334)**
Is it 330 or more ?	**Yes**	**(330–334)**
Is it 332 or more ?	**No**	**(330, 331)**
Is it 330 ?	**No**	**(331)**
Is it 331 ?	**Yes**	

So, in eleven questions, we have found a number less than 1000.

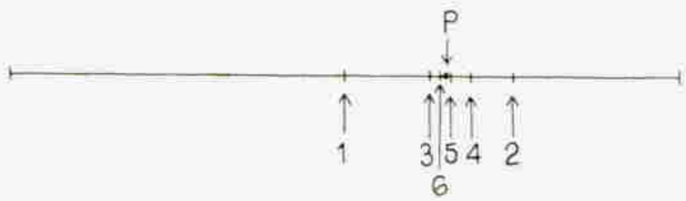
This same technique can be used for locating any word in the English language (or in any other) with the help of a dictionary, if one is allowed to ask twenty questions. This would bear investigation . . . .

This whole process has a close connection with the binary notation and especially with the recording of binary fractions. If we choose to use powers of two: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, in our questioning, this would go as follows:

Is it greater than 512 ?	**No**	
Is it greater than 256 ?	**Yes**	
Is it greater than 384 ?	**(= 256 + 128)**	**No**
Is it greater than 320 ?	**(= 256 + 64)**	**Yes**
Is it greater than 352 ?	**(= 320 + 32)**	**No**
Is it greater than 336 ?	**(= 320 + 16)**	**No**
Is it greater than 328 ?	**(= 320 + 8)**	**Yes**
Is it greater than 332 ?	**(= 328 + 4)**	**No**
Is it greater than 330 ?	**(= 328 + 2)**	**Yes**
Is it 331 ?		**Yes**

If we replace our sequence of answers, beginning with the first 'yes' – 'yes, no, yes, no, no, yes, no, yes, yes' with figures, using one for yes and 0 for no, then the resulting number, 101001011, is the binary representation of denary 331.

A point on a unit line can be given a value as a binary fraction by a process of continued halving. We halve the unit line: if our point lies in the left-hand segment, we record 0; if in the right-hand segment we record 1. We then halve the segment in which the point lies, and record as before; if the point lies exactly on the dividing line, we record 1 and the process is finished.



P has the binary value: 0·101001 . . . . to six places of 'bicimals'. The next division will be close enough to P as far as the eye can discern. If we give the value as 0·1010011, then we are working to a degree of accuracy corresponding to the nearest $\frac{1}{2^7} = \frac{1}{128}$; three more divisions would fix the position of the point to the nearest $\frac{1}{1000}$, which reflects the fact that a number below 1000 may be found by asking ten questions: the eleventh question is needed only to confirm what we have already deduced.

19

O I O O I I

Fill in the missing figures in these additions:

```
  4 *
+ * 8
  7 4

  3 *
+ * 7
 1 2 0
```

Make up two more problems like these and give them to your friends to solve.

It might be as well here to begin with a problem simpler even than either of these, such as **4***, in order to remind ourselves

```
  4*
+ *3
  78
```

that we are looking for a number of units and a number of tens, and that if we write the problem as (4 × 10) + (* × 1) + (* × 10) + (3 × 1) = (7 × 10) + (8 × 1) we can see that it is the same problem as **43**, and that if we solve separately the

```
  43
+ **
  78
```

two problems 3 + * = 8 and 40 + * = 70 we shall have our solution, which is of course

```
  45
+ 33
  78
```

We can see our first problem as either

```
  4*        48
+ *8   or + **
  74        74
```

Now the problem which we find in the units column, **8 + * = 4**, has no solution in the set of natural numbers. If on the other hand we try all the possibilities, we do find one which will give us 4 units:

. . . 8 + 3 = 11; 8 + 4 = 12; 8 + 5 = 13; 8 + 6 = 14.

The problem of what to do with the 1 in 4 is solved by recognising that 14 is 10 + 4, and that we may write 10 as 1 in the 'tens column'.

```
  46
+ *8
  74
  1
```

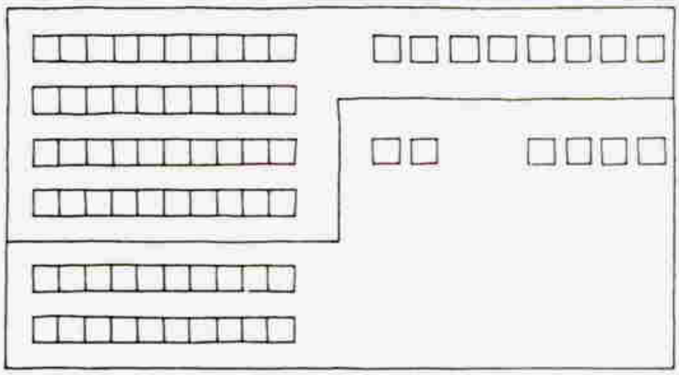
We now have to solve 4 + 1 + * = 7, to obtain a complete solution:

```
  46
+ 28
  74
```

The second problem is solved in the same way, with an additional comment that we can treat 120 either as twelve tens or as one hundred and two tens:

This is the philosophy behind our habit of calling 1234 either one thousand two hundred and thirty-four, or twelve hundred and thirty-four, the latter especially in historical time-telling.

The problems can be solved by using Dienes' base ten blocks. We set up 4 long tens and 8 unit cubes, and add to these enough of either in order to get seven long tens, or their equivalent in unit cubes, and four unit cubes. In the second problem the desirability of introducing a third column of figures is reinforced by the recognition that 12 long tens can be exchanged for 2 long tens and one flat hundred.



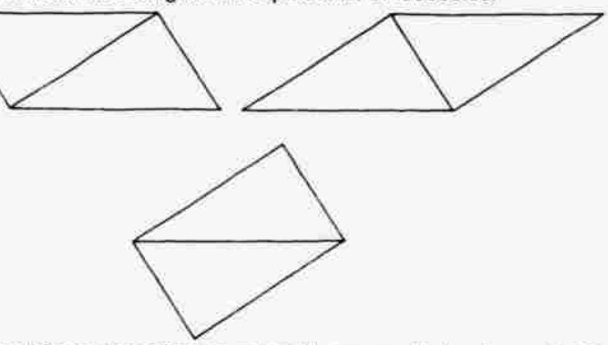
20 O I O I O O

Cut two triangles of the same shape and size from a piece of card. Can you assemble them to make a parallelogram?

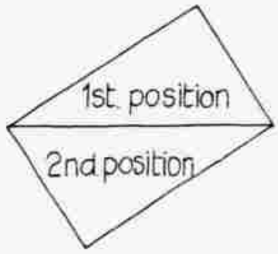
Can you cut this parallelogram with a single cut, and reassemble the pieces to make a rectangle?

Experiment will show that any two congruent triangles will combine to form a parallelogram. Congruence is best seen first as a property which will allow one triangle cut from card to fit exactly on another. Then no matter how the two triangles are placed in relation to each other, rotated or turned over, they remain congruent because at any moment they can be picked up and made to fit one on the other.

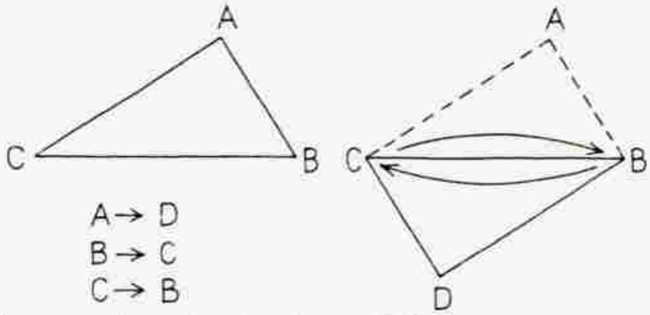
The triangles can be combined to form a parallelogram in three distinct ways, each resulting in a parallelogram of a distinct shape, unless the triangles are equilateral or isosceles.



In doing this we have first to rotate the second triangle so that it will fit the first. It can then be moved without rotation into three positions to fit the first. It is possible to draw the parallelograms by taking a single triangle cut from card. We mark the positions of its vertices on the paper; then we rotate it, fit it in a new position and mark the fourth vertex of the parallelogram.

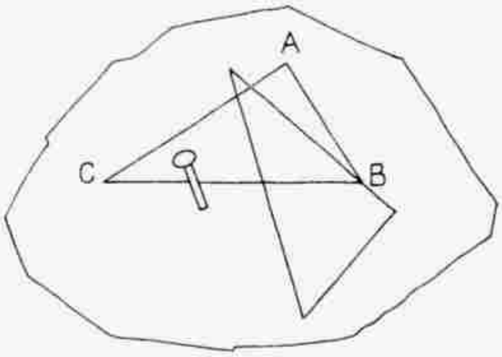


We can trace the movements of various parts of the triangle. If we label the vertices A, B, and C, then we discover that after the triangle has been rotated, B and C have changed places, and A has moved across to D, the fourth vertex of the parallelogram.

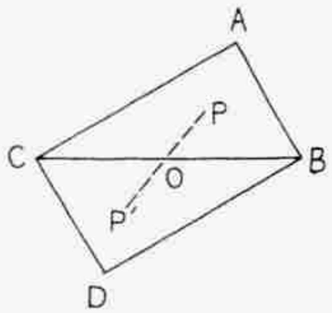


Has every point on the triangle moved? If so, is there a rule for determining exactly where any point will go? One way of finding out is to simulate this movement of the triangle by means of tracing paper. We draw the triangle on paper – ABC. We then copy this on to the tracing paper.

Then we take a pin and push it through the tracing paper, so that it acts as a centre about which we can rotate the tracing paper. Can we place the pin so that the image rotates and fits with the original triangle to give our parallelogram?

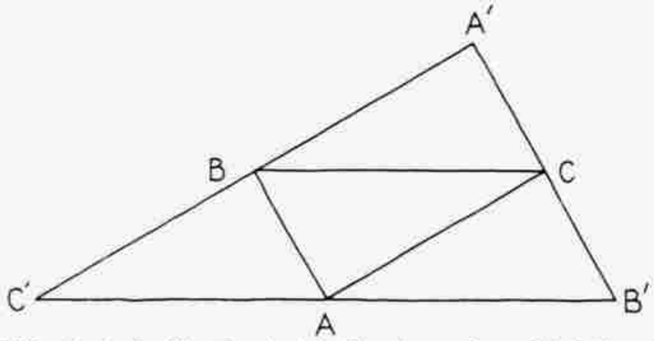


Repeated experiment together with a little observation will locate the desired position of the pin at the mid-point of CB; and a rotation about the mid-point of AB and of AC in turn will give the other two parallelograms. So after any of these rotations has taken place, one point, which is the centre of rotation, has not moved. All other points in the triangle do move: in particular B moves to C and C moves to B. What happens to a point P, say, in the interior of the triangle?



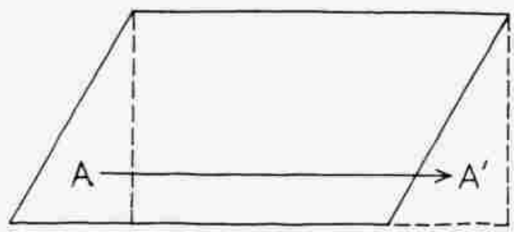
Experiment will show that any point P will map on to a point P′, such that POP′ is a straight line, O being the centre of rotation, and OP = OP′. For this reason in particular AOD is a straight line: OA = OD, so that O is the point at which the diagonals CB and AD of the parallelogram bisect each other.

If we start with our triangle with CB horizontal but with A below CB, and then rotate the triangle about the mid-point of each side in turn, we obtain a larger triangle composed of four smaller ones.

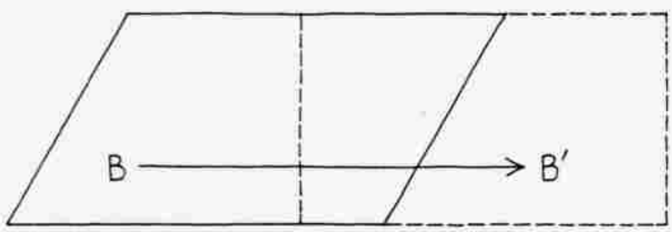


What is the justification for labelling the vertices of this triangle A′, B′ and C′? Is C′AB′ a straight line? How many parallelograms does the figure contain? Which angles in the figure are the same? Which lines are the same length?

To cut a parallelogram in two parts which will fit to make a rectangle is easy enough:



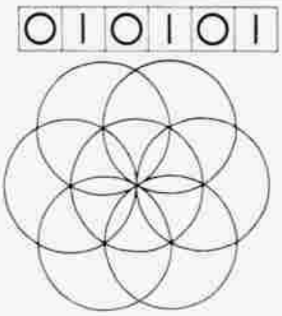
The triangle A is cut off by a cut perpendicular to the horizontal sides: it is then 'translated' or slid along to the position A′. Suppose the cut was made in another position. Could we still make a rectangle? The answer is yes.



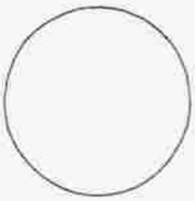
How far does each point of B move? What measurements give the area of the rectangle? What does this tell you of the area of a parallelogram? What does it tell you of the area of each of the two triangles which compose the parallelogram?

21

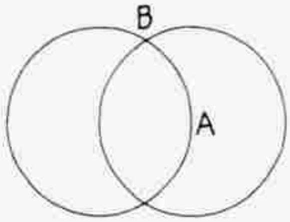
How many circles are there in this figure? What is the least number of colours needed to colour the figure so that no two neighbouring regions are the same colour? Draw the figure and find out. What shapes can you make by joining with straight lines the points where the circles intersect?



One way to answer the first question is to construct the figure for oneself. We begin by drawing the central circle:

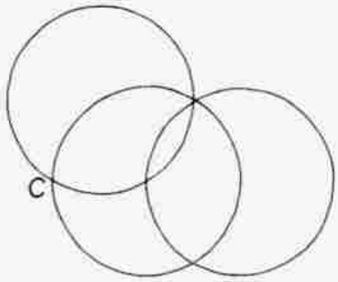


The other circles each pass through the centre of this circle. Placing the pencil of our compass on this point, and twirling the instrument round this point, we observe that the compass point must be placed on the circumference of the first circle when we draw each of the other circles. We draw one of these, with its centre at A:

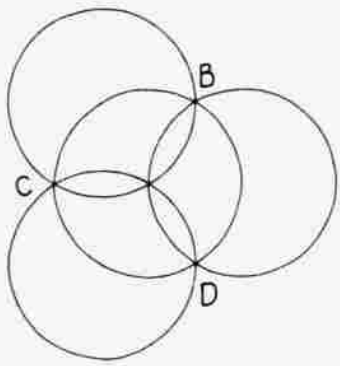


We observe that the circumferences of two other circles intersect on the circumference of the first circle at or near A. Where will their centres be? They must both lie on the circumference of the first circle: where exactly must they be?

We can postpone answering this question and transfer our attention to the fact that a third circle passes through B. We may draw this by placing the compass point on B and making a mark on the circumference of the first circle with the compass pencil: this mark will be the centre of the next circle to be drawn.



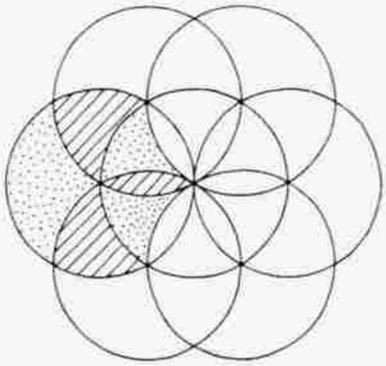
We may repeat this process to draw the circle passing through C.



The figure can be completed by a study of the symmetry of the original diagram. This diagram can be obtained by reflecting the four circles we have so far in an axis which passes vertically through the centre of our first circle: the positions of the centres of the remaining circles to be drawn can be guessed as the points of intersection at B, C, and D. The complete figure will contain seven circles.

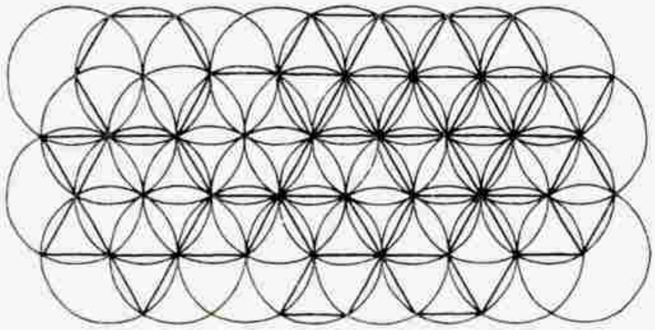
The figure can now be coloured. It will be found that two colours are sufficient for colouring each region distinctly from its neighbours. (We allow two regions whose boundaries meet at only one point to have the same colour.) If we begin at point C and colour the regions which meet at C, alternately red and green, we find that we can colour the six regions in accordance with the regulation, and that this pattern of colour can be extended to cover all regions in the figure.

How many regions will be coloured red and how many green? How many regions in the whole figure? How many regions at each stage in its drawing?



The figure as it stands contains the framework for constructing such important geometrical shapes as the equilateral triangle and the regular hexagon. If the pattern of circles is extended, it provides a framework for covering the plane with a tessellation of equilateral triangles, or of regular hexagons, or of rhombuses, or of other figures formed by combining two or more equilateral triangles.

A word of warning: this exercise often attracts criticism as being a time-wasting one, and there is a danger that it becomes just a pretty picture. It is important that the mathematical significance of the exercise is made clear to the child, and the suggestions made above, if followed, should ensure this.

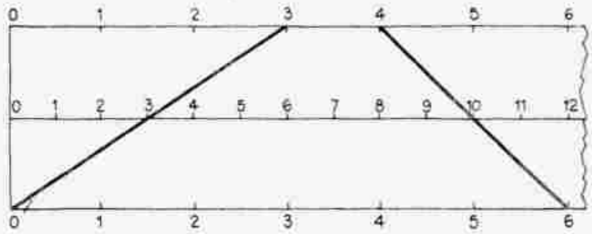


22

O | O | | O

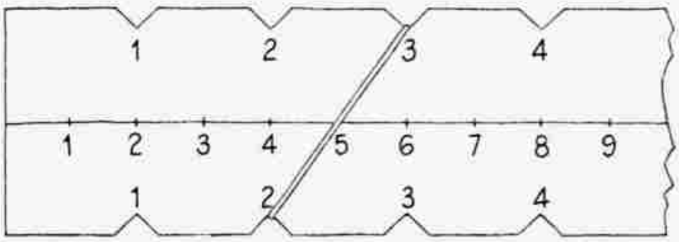
Can you discover the purpose of this figure?

Copy this on a piece of card, cut it out, and use it to add and subtract numbers.



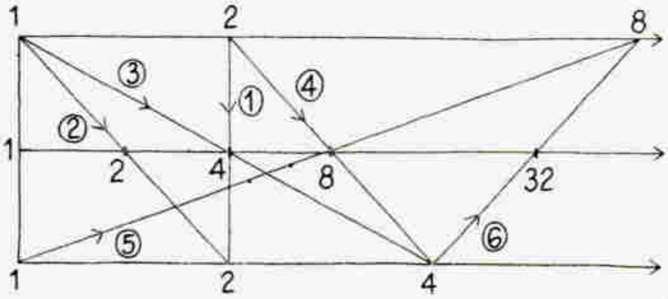
This nomogram can be used to add two numbers. If a line is drawn through two points on the top and bottom scales, say 4 and 6, it will pass through the point numbered 10, which is the sum of these numbers. It can obviously be used also for subtraction, the inverse of addition.

If the diagram is drawn on a piece of card, notches can be cut at unit intervals along the edge of the card, and a rubber band slipped over the card will give the line from which the sum can be read.

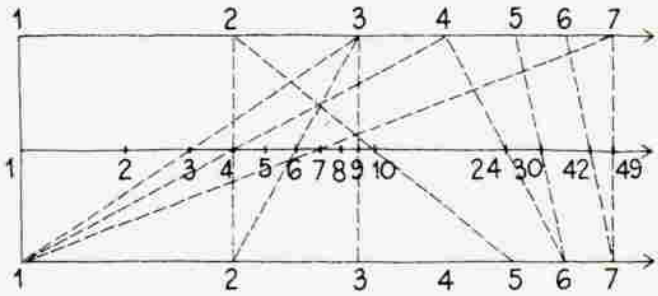


The nomogram can be extended not only to the right but to the left, into the 'negative' segment of the number line. On this nomogram the idea of an additive inverse can be demonstrated: the line joining $^{+}2$ and $^{-}2$ passes through 0, as does the line joining ^{+}a and ^{-}a, whatever the value of a.

A nomogram for multiplication would clearly be useful, but its construction is only possible if logarithmic scales can be drawn in place of the linear scales for addition. These scales are to be found marked on the multiplication slide rule which is one of the pieces of equipment included in the Multiboard. These scales are marked on the upper and lower lines: the central scale will show the product.



If a slide-rule scale is not available we can begin by choosing any point to mark as 2, and build up our numbers as we go.



The ringed numbers show the order in which we might work. First we decide where to mark 2 on the top line.

Then we proceed:

1 We mark 2 the same distance along the bottom line. The beginning of each line is marked 1. Where the line joining the 2 on the top line to 2 on the bottom line crosses the centre, we mark the product $2 \times 2 = 4$.

2 Where the line from top 1 to bottom 2 crosses the centre, we mark $1 \times 2 = 2$.

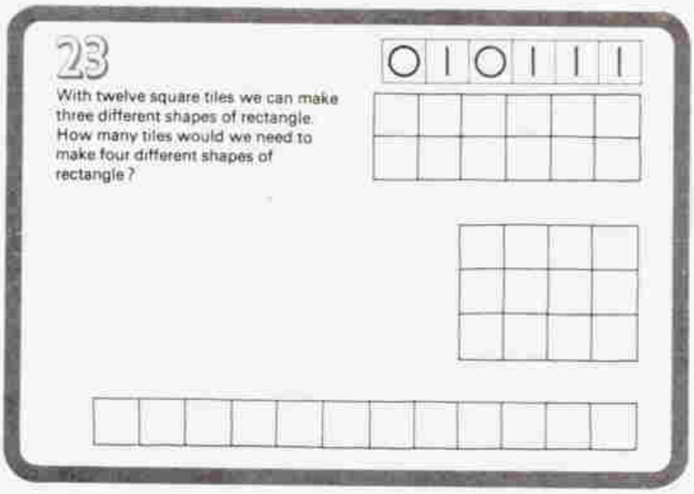
3 The line joining top 1 to centre 4 will cut the bottom line at 4.

4 The line joining top 2 to bottom 4 will cut the centre line at 8.

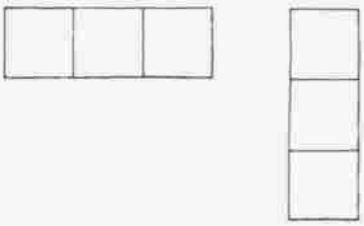
5 The line from bottom 1 through centre 8 will cut the top line at 8.

6 The line joining bottom 4 to top 8 will cut the centre line at 32.

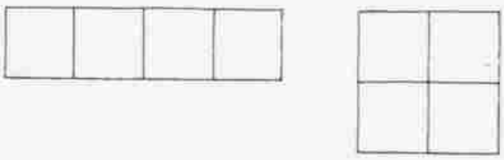
So far all the numbers we have are taken from the set {1, 2, 4, 8, ...}. Other numbers can be placed by trial and error.



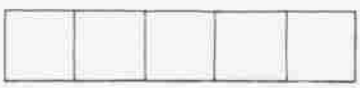
First, we must allow that a square may count as a rectangle in this problem; and that 'different' means 'different no matter which way we turn the rectangle', so that these two rectangles do not have different shapes:



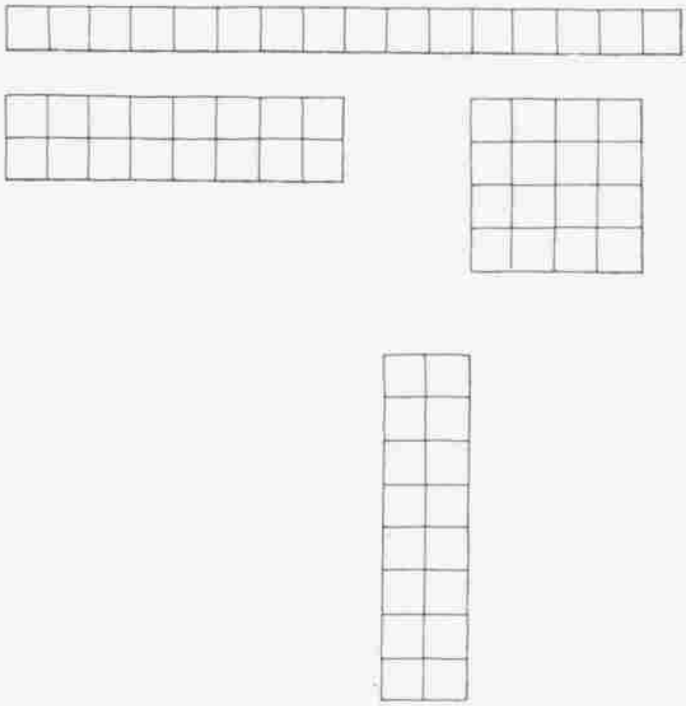
If we had four tiles, we could make these rectangles:



With five tiles we could make one rectangle only:

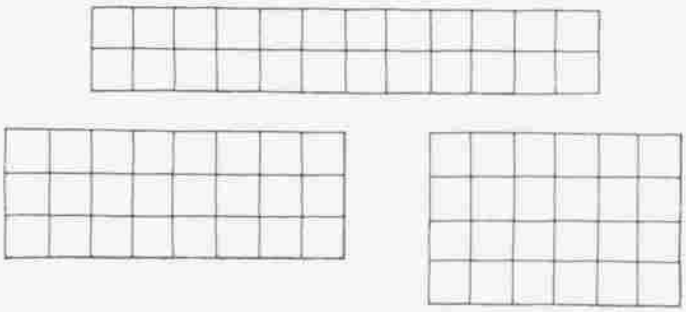


By experimenting with various numbers we find that with sixteen tiles we can make:



but the second and fourth rectangles illustrated here are the 'same': they represent two positions of the same rectangle.

In fact we need twenty-four tiles to make our distinct rectangles. One will be twenty-four tiles long and one wide. The others will be:



How many tiles will we need to make five different rectangles?
Is there any pattern here?

Number	Factors (including the number itself)
2	1, 2
3	1, 3
4	1, 2, 4
5	1, 5
6	1, 2, 3, 6
7	1, 7
8	1, 2, 4, 8
9	1, 3, 9
10	1, 2, 5, 10
....	....
16	1, 2, 4, 8, 16
....	....
24	1, 2, 3, 4, 6, 8, 12, 24

Does this table help you towards a solution?

24

O I I O O O

Complete these tables: the first is an addition table.

+		2		
	8		9	
8				
3		5	8	
	9			
6				9

		4	9	
		8	18	
3		12		
	35			14
				2

Make up similar problems.

It must be understood that these tables are of the nature of 'composition' tables, whereby the numbers along the top and in the left-hand column combine to give the numbers in the main body of the table, each of these numbers being at the intersection of a row and column. First we see that adding maps (3, 2) on to 5; the second operation *could* be multiplication, since $3 \times 4 = 12$.

The first square will contain numbers found by adding the numbers in the top row and the left-hand column. We can immediately write:

+		2	a	
	8		9	
8		10		
3		5	8	
	9			
6		8		9

We have written in the results: $2 + 8 = 10$, $2 + 6 = 8$. Now from the addition $3 + a = 8$, we can derive $a = 5$, and the rest of the table can be filled in rapidly:

+	4	2	5	3
4	8	6	9	7
8	12	10	13	11
3	7	5	8	6
5	9	7	10	8
6	10	8	11	9

No pattern is intended to emerge. A pattern does occur if the numbers which 'generate' the square are written in order:

	2	3	4	5
3	5	6	7	8
4	6	7	8	9
5	7	8	9	10
6	8	9	10	11
8	10	11	12	13

The second square, which we have decided to complete by multiplying the generating numbers, presents a little more difficulty. We can immediately write in 2 in the top place in the left-hand column, since $2 \times 4 = 8$ and $2 \times 9 = 18$. We can then write in 27 as 3×9.

X		4	9	
2		8	18	
3		12	27	
	35			14
				2

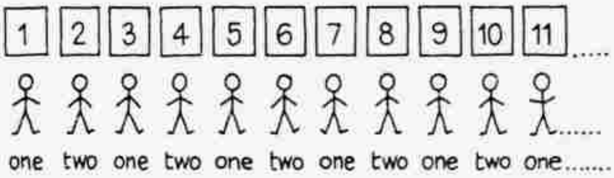
Now we come to what appears to be an impasse. This can be overcome by considering what factor is common to 14 and 35. The answer is 7, so this must be entered beneath the 3 in the left-hand column ; and the square is completed easily.

X	5	4	9	2
2	10	8	18	4
3	15	12	27	6
7	35	28	63	14
1	5	4	9	2

25

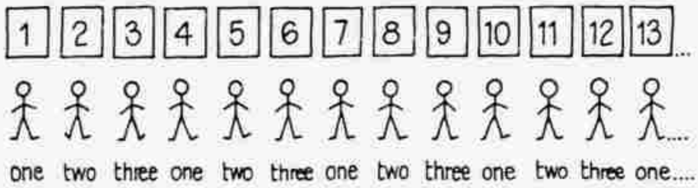
A group of children stood in a line. They then numbered themselves in twos, the first saying, 'one', the next, 'two', the next, 'one', and so on. If the last child said 'one', what does this tell you about the number in the group?

We can draw the line of children, giving each one a number label in order, and writing underneath what each child said.



We are not told how many children were in the group: there may not have been as many as eleven, and the dots show that there may have been more than eleven. What we can see is that number 2, number 4, number 6, and so on, would each have said 'Two'. Number 1, number 3, number 5, and so on, would each have said 'One'. If the last child said 'One', his number must have been one of this set: {1, 3, 5, 7, 9, 11 . . . .}, and the number in the group would be an odd number.

If we are given more information, we can narrow down the range of possible numbers in the group. Suppose the group had counted in threes, and the last person had said 'Two'.



The last person must have had one of these numbers: {**2, 5, 8, 11,** . . . .}.

If we extend our two sets:
{**1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29,** . . . .}
{**2, 5, 8, 11, 14, 17, 20, 23, 26, 29,** . . . .}
we can say that the number we are looking for is in both these sets, and is a member of the set:
{**5, 11, 17, 23, 29** . . . .}.

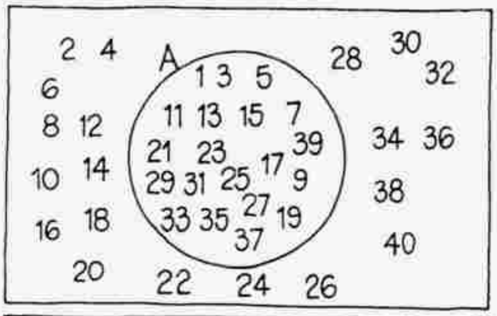
The difference between successive numbers in this set is 6, so that the set will continue:
{. . . . **29, 35, 41, 47** . . . .}.

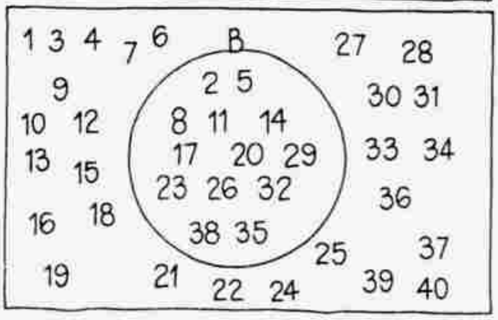
If now we are told that when the children in the group counted in fives, the last child said 'Four', we know that our number must also belong to the set:
{**4, 9, 14, 19, 24, 29, 34, 39** . . . .}.

The numbers which appear in all three sets:
A = {**1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39** . . . .}
B = {**2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38**}
C = {**4, 9, 14, 19, 24, 29, 34, 39** . . . .}
are now very few: the only one we see is 29, and we can write the solution set as {29 . . . .}.

If we are told that the number of children in the group was less than 32, we now know that there were just 29 in the group.

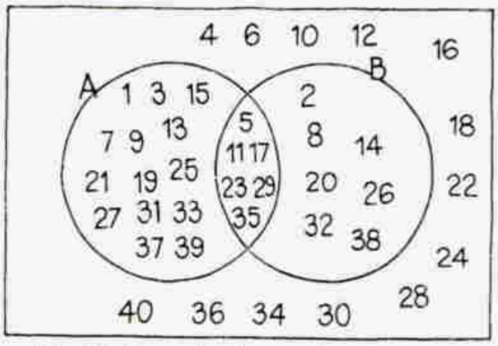
If we call our three sets A, B, and C, we can put all our information into a diagram. Taking all the natural numbers as far as 40, and writing them inside a rectangle, we can write Set A in a loop inside the rectangle:



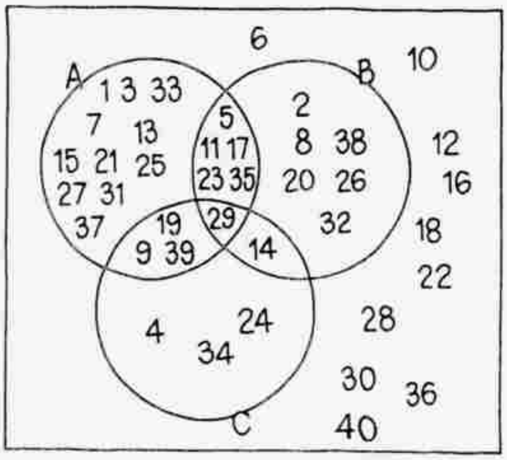


We can do the same for Set B.

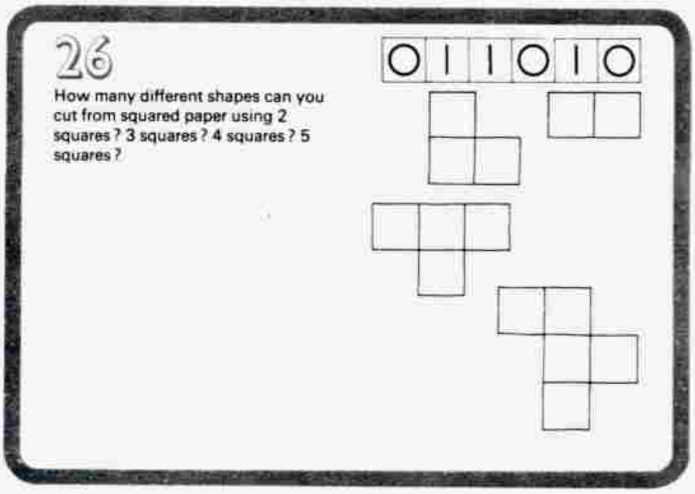
We can combine these two loops in one diagram, making the loops overlap to accommodate the numbers which must appear in both:



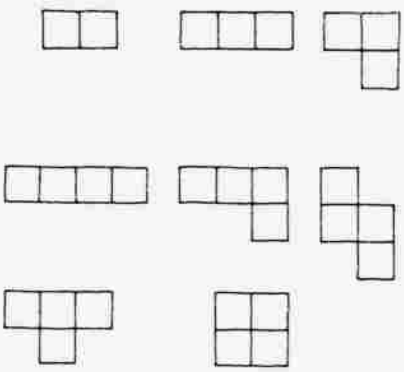
Finally we add a third loop for the numbers in Set C:



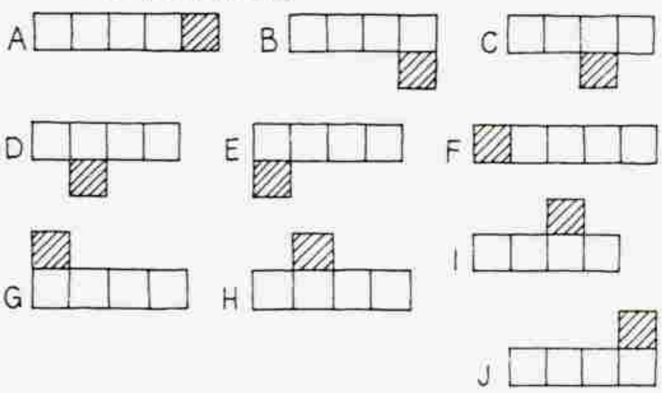
The only number in the intersection of all three loops is 29. What would be the next number to appear in this region of the diagram?



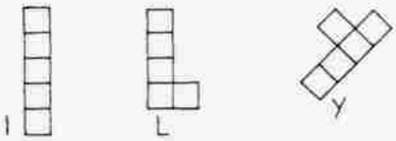
If we allow only combinations of squares which are distinct from each other, no matter how they are rotated or reflected (so that we may turn them over or twist them round as we will, and never find two to fit each other), then we can draw shapes with one square and with two squares one way only: three squares can be made into two distinct shapes; four squares into 5 distinct shapes, and 5 squares into 12 distinct shapes. These combinations are sometimes given the name domino, tromino, tetromino, and pentomino, respectively.



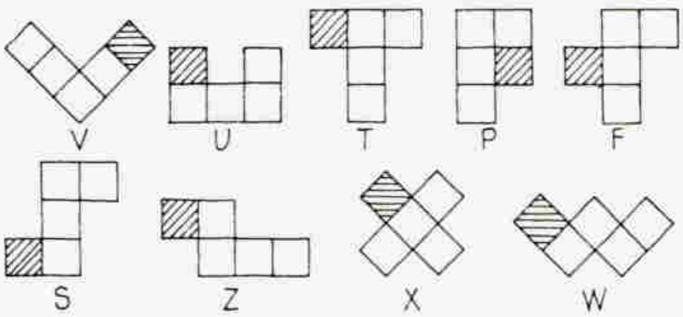
The twelve pentominoes can be built up from the tetrominoes by adding a fifth square to a tetromino and checking to see whether a pentomino congruent to the new one has already been found. For instance, by adding one square to the straight tetromino, we obtain these pentominoes:



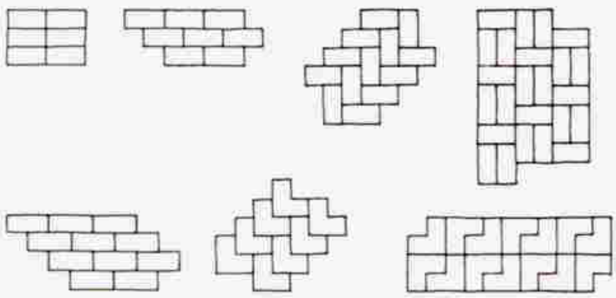
Of these, A and F are obviously congruent. B, E, G, and J can all be made to fit one another, by turning them round or by turning them over; so can C, D, H, and I. So we have only three distinct pentominoes so far. We can call each by the name of a letter which it might be said to resemble:



From the remaining tetrominoes we can make:



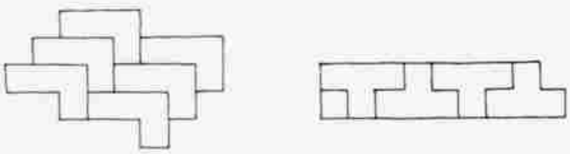
It is clear that the domino and trominoes will all tessellate in the plane, that is, a regular and repeating tile pattern can be made with each of these shapes: some will be simple, some complicated.



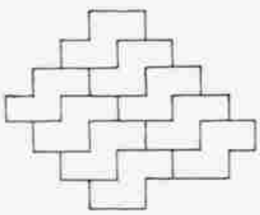
Furthermore, all these shapes are either rectangular, or can be combined into a rectangle, so all will tessellate to give a rectangle.

Whether or not this is true of the tetrominoes, we cannot immediately be sure. The straight tetromino will obviously tessellate to form a rectangle; so will the square tetromino. The L tetromino and the T tetromino can each be tessellated to form a 4-by-4 square; or alternatively a 4-by-4 square can be dissected either into four L tetrominoes or into four T tetrominoes.

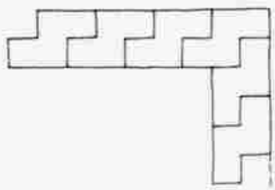
Other tessellations can be made with either shape:



The 'skew' tetromino can be tessellated in the plane without difficulty:



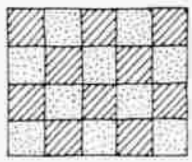
We may try to fit it into a rectangle:

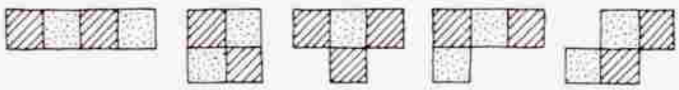


But we cannot completely fill another corner of the rectangle, so we are forced to conclude that this shape will not tessellate to fill a rectangle of finite dimensions.

Can we take the five distinct tetrominoes and with them make a rectangle (which must be either a 2×10 rectangle or a 5×4 rectangle, since each of the five contains four squares)? The answer is no, and the proof is simple.

Let us assume that we can construct a 5-by-4 rectangle from the five tetrominoes and that the rectangle is marked in unit squares. We could then colour alternate squares in the rectangle, as if it were part of a chess board. There would be ten squares of each colour. We then dismantle the rectangle and study the tetrominoes, which are now all coloured.





We now see a discrepancy. The tetrominoes can only be coloured as they are here, if they had originally been assembled in a coloured rectangle; but instead of there being ten squares of each colour, there are nine of one and eleven of the other. The villain of the set is the T tetromino which must have three squares of one colour, whereas each of the other pieces has two of each colour. So the original assembly as a rectangle is impossible.

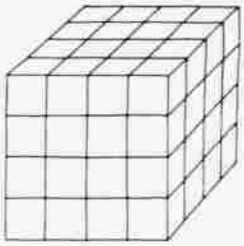
Similar work can be done with the twelve pentominoes.

27

O I I O I I

A block of wood is a cube with edges 4 cm long. It is made by putting together a number of smaller cubes each with edges 1 cm long. How many of these were needed to make the large cube? How many of them are on the outside of the large cube?

How many are hidden inside?



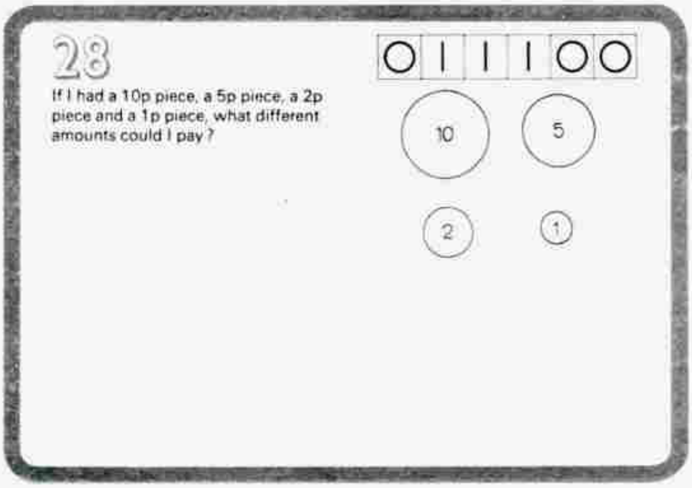
If possible this question should be approached with a large quantity of Dienes M.A.B. apparatus to hand. The large cube may then be built up from smaller cubes or flats, and the number of small cubes which goes into its composition may be directly counted or at least readily calculated without any formulae such as V = lbh. The answer is of course 64 : some of these small cubes are visible, others are not.

How many are visible? If we count the number of cubes on each face, we have exactly 16; and there are six faces. But it is not true that 96 cubes are visible – why not? Some cubes in fact appear on two faces of the large cube and some indeed appear on three faces. A count must be taken.

Number of cubes appearing on:	
3 faces	**8 (one at each vertex)**
2 faces	**2 × 12 (two along each edge)**
1 face	**4 × 6 (four on each face)**
Total	**8 + 24 + 24 = 56**

So nearly all the 64 cubes are visible. It is worth noting that if the outside of the cube were painted, the eight cubes which appear on three faces of the large cube would themselves be painted on three faces. The 24 which appear on two faces would themselves be painted on two faces; and the 24 which appear on only one face of the large cube would be painted on one face only. The cubes in the interior would remain unpainted; and there are 64 − 56 = 8 of these.

Investigation can be continued into the 5 × 5 × 5 block, or the 10 × 10 × 10 block.



There are 15 different amounts which can be paid with four differently valued coins, no two or three of which are together equivalent in value to any one or two of the others. One way of finding these is to make a list:

1p → **1p**
1p, 2p → **3p**
1p, 2p, 5p → **8p**

A more concise way is to make a table and to indicate by using zeros and ones in each row what coins are to be included in the total for that row:

10p	5p	2p	1p	Total
0	**0**	**0**	**1**	**1p**
0	**0**	**1**	**1**	**3p**
0	**1**	**1**	**1**	**8p**

etc.

Each of the rows contains four digits and each row can be considered as a binary number. The greatest binary number which can be made with four digits is 1111, or 15 in base ten, so that we may pay 15 different sums of money with four coins of different values.

All that we need to do to find these 15 is to list each binary number, and against each to write the corresponding amount. If we write the column headings as:

10p 5p 2p 1p

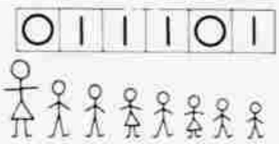
and give these values to the digits in each number, we shall discover the amounts in order of size.

10p	5p	2p	1p	Total
0	**0**	**0**	**1**	**1p**
0	**0**	**1**	**0**	**2p**
0	**0**	**1**	**1**	**3p**
0	**1**	**0**	**0**	**5p**
0	**1**	**0**	**1**	**6p**
0	**1**	**1**	**0**	**7p**
0	**1**	**1**	**1**	**8p**
1	**0**	**0**	**0**	**10p**
1	**0**	**0**	**1**	**11p**
1	**0**	**1**	**0**	**12p**
1	**0**	**1**	**1**	**13p**
1	**1**	**0**	**0**	**15p**
1	**1**	**0**	**1**	**16p**
1	**1**	**1**	**0**	**17p**
1	**1**	**1**	**1**	**18p**

Which values are missing in the totals? Why should these particular values be missing?

29

Mrs Chalmers is buying Christmas presents for her seven children to give one another. Each child gives a present to each of the others. How many presents must she buy?



Make up similar problems.

Each child must give six presents, one to each of his brothers and sisters, so Mrs Chalmers must buy $7 \times 6 = 42$ presents.

We can construct a table showing the number she would have bought at successive stages of the family's growth.

No. of children	No. of presents	Difference
1	$1 \times 0 = 0$	
2	$2 \times 1 = 2$	**2**
3	$3 \times 2 = 6$	**4**
4	$4 \times 3 = 12$	**6**
5	$5 \times 4 = 20$	**8**
6	$6 \times 5 = 30$	**10**
7	$7 \times 6 = 42$	**12**

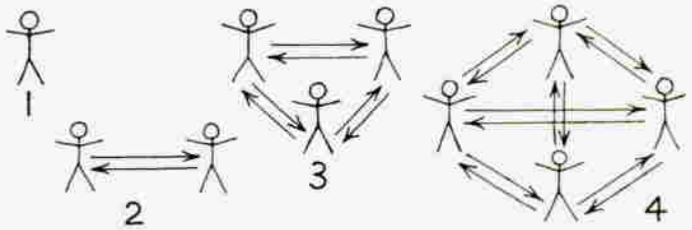
From this table we see that:

a) There is a progressive increase in the number she must buy. The difference column displays the set of even numbers: why?

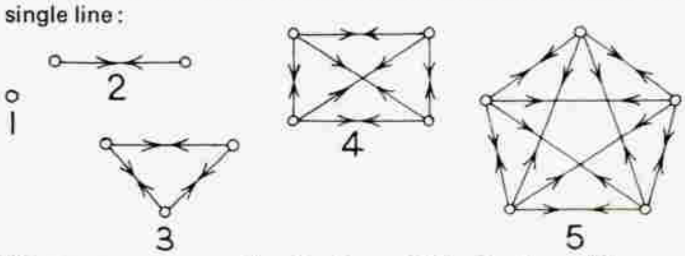
b) The product of two consecutive numbers is always even: why?

c) The number of presents she must buy is in each case twice a triangular number: why?

The answers to these questions may perhaps be found by considering a diagrammatic representation of the situation as it develops. We can show the relationship 'gives a present to' by an arrow.

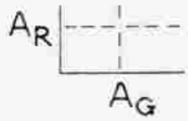


The situation gets too complicated for us to represent it clearly in this manner with more than 4 people. We can simplify our diagrams by replacing people by points and our twin arrows by a single line:

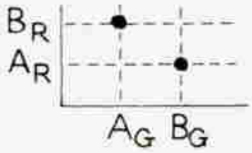


We may now compare the situation with that in the problem where people shake hands with one another. We may represent each situation by a polygon with its diagonals: the total number of lines needed to join n points to one another is $\frac{n(n-1)}{2}$. In the case of giving presents, each line shows the path of two presents, so $n(n-1)$ presents will pass. When one person is involved, no lines are needed: a second person will be joined to the first by one line; a third will be joined to the first two by two lines, and so on. If each line represents two presents, then 1, 2, 3 . . . . lines represent 2, 4, 6 . . . . presents.

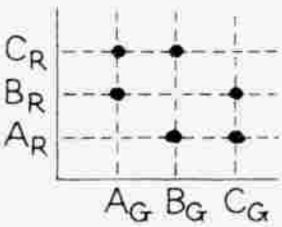
We may also represent the number of presents by points on a lattice. To do this we have to represent each person in two roles: one as a giver, one as a receiver. We can name the children Andrew, Betty, Charles, Deborah, Eileen, Francis, and Gertrude. A_G is Andrew giving, A_R is Andrew receiving, and so on. We write names along two axes and show a present passing between two children by marking a circle at the intersection of two lines.

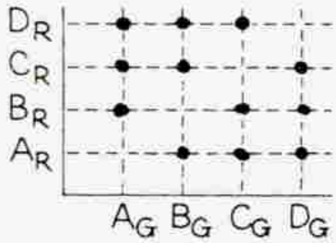


Andrew does not give a present to himself.

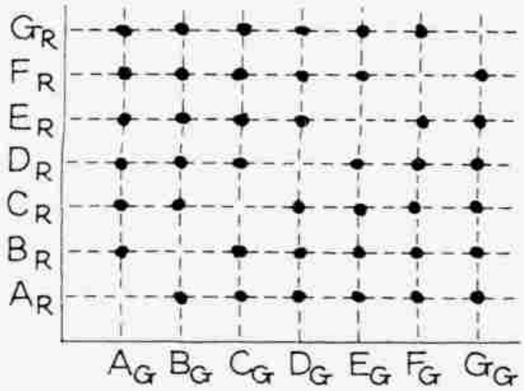


Andrew and Betty give each other a present, but they do not give themselves presents.





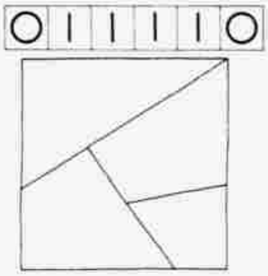
The final diagram will be:



The pattern shows clearly that the number of presents given is twice a *triangular number*: it also shows incidentally that $n^2 - n$ is also twice a triangular number. There are in the final diagram seven rows and columns, each with six points marked with circles: the total number of circles is thus the product of seven and six, or 42.

30

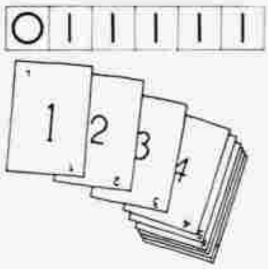
Cut a square into four pieces of any shape, jumble the pieces up and ask a friend to reassemble them to form the square.



No comment, except that you can start again with other shapes and different numbers of pieces . . . .

3]

Make a set of cards showing the numbers 1 to 12. Put them in order and then deal them one at a time, like playing cards, into two piles. What numbers will you find in each pile?



If the cards are dealt alternately into two piles, one pile will contain the even numbers {2, 4, 6, 8, 10, 12}, while the other will contain the odd numbers {1, 3, 5, 7, 9, 11}.

If the cards are now put in order again and are dealt into three piles, we find that the numbers in the three piles are:
{1, 4, 7, 10}
{2, 5, 8, 11}
{3, 6, 9, 12}

The numbers in the third pile are recognisable as the multiples of 3, but at first sight the numbers in the other two piles appear to fit no pattern. However, a study of these numbers will reveal that there is a difference of 3 between successive numbers; thus $1 + 3 = 4$, $4 + 3 = 7$, and so on. This is also true of the multiples of three: $3 + 3 = 6$, $6 + 3 = 9$, etc.

If we deal the cards into groups of 4, then the fourth pile will contain the multiples of 4. The fact that these divide by 4 exactly may help us to classify the numbers in the other sets:
{1, 5, 9}
{2, 6, 10}
{3, 7, 11}

If we divide the numbers in the first set by 4, we obtain each time a remainder of 1. The numbers in the second set give a remainder of 2 when they are divided by 4, and those in the third set a remainder of 3.

This pattern appears also in the numbers in the three piles:
{1, 4, 7, 10}
{2, 5, 8, 11}
{3, 6, 9, 12}

We find that 1, 4, 7, 10 all give a remainder of 1 when they are divided by 3, and so on.

This puts the odd numbers in a new light: they form the set of numbers which give a remainder of 1 when they are divided by 2, while the even numbers give no remainder when they are divided by 2.

If we study the numbers in these sets, we can discover certain relationships. If we add any two numbers in the set {1, 4, 7, 10}, we obtain a number in the set {2, 5, 8, 11, 14, 17, 20}. This set contains the four numbers {2, 5, 8, 11}, which are the second set in our deal into three piles. Similarly, any two members of the set {2, 5, 8, 11} have a sum in the set {1, 4, 7, 10, 13, 16, 19, 22}.

We can see a similar relationship in the sets of odd and even numbers, although whereas two odd numbers add to give an even number, two even numbers also add to give an even number. We may express this fact in an addition table:

+	E	O
E	E	O
O	O	E

This tabulates the relations: E + E = E, E + O = O, O + E = O, O + O = E.

If we now substitute *remainders* for O and E, we can write 0 (zero) for E, and 1 (one) for O, since these are the remainders left on dividing the even or the odd numbers by 2.

+	0	1
0	0	1
1	1	0

If we take the sets of numbers obtained by dealing the cards into three piles, and build a similar addition table, we shall have:

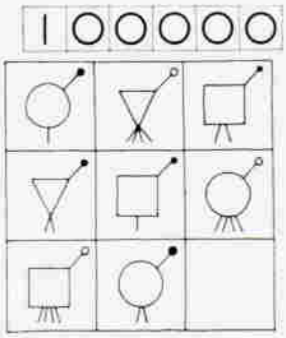
+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

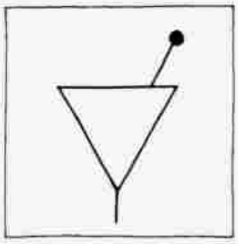
This table is reproduced in the commentary on Card 32.

32

Draw the missing figure.

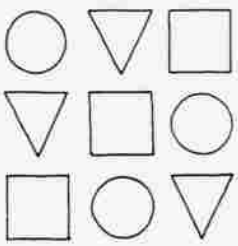
Invent more problems like this.





The missing figure has a triangular body, a black head and one leg. The total in each row and column is then 2 black heads, 7 legs altogether and one of each shape.

There is a balance about the whole thing. We see that the bodies are arranged thus:



Here each shape occurs once in each row and once in each column.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

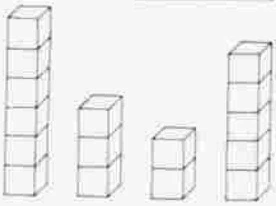
What is this table? Any connection with the above discussion?

The legs are arranged thus:

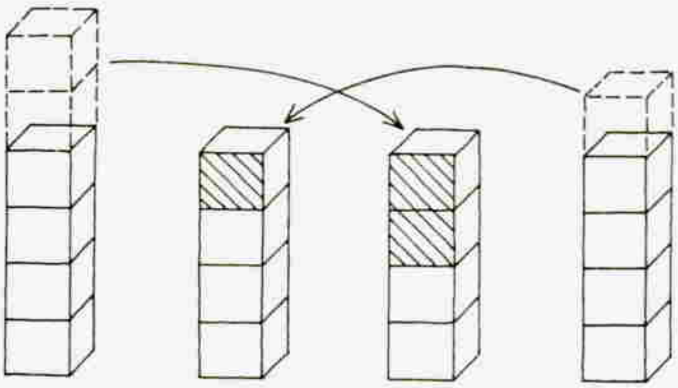
1	4	2
2	1	4
4	2	1

33

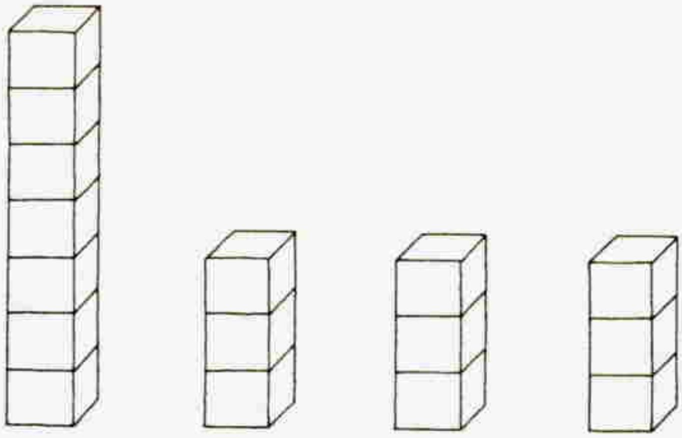
Pile 16 one-inch cubes in 4 piles, containing 6, 3, 2, and 5 cubes. A 'move' consists in moving one or more cubes from one pile to another. Can you make all the piles the same height in two moves? Can you find an arrangement of 16 cubes in 4 piles which needs 3 moves to get the same number in each pile?



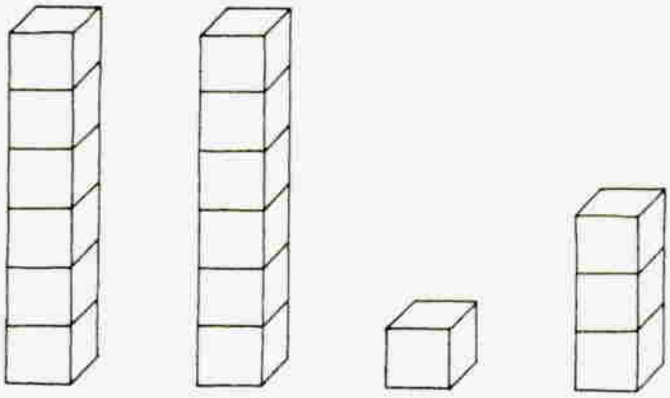
We can represent the solution to the first problem in this diagram:



The second problem has several solutions, of which this is the simplest:

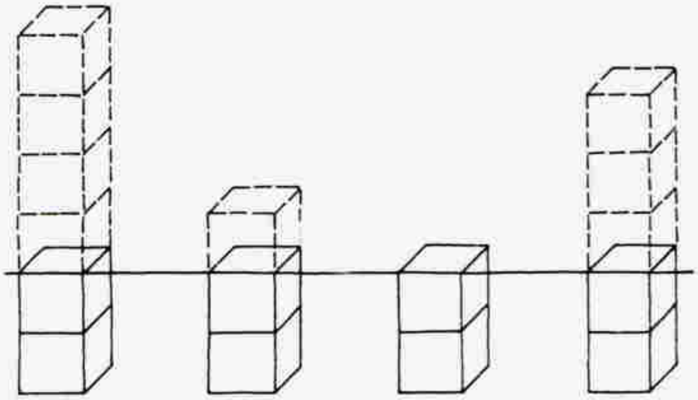


and this is another:

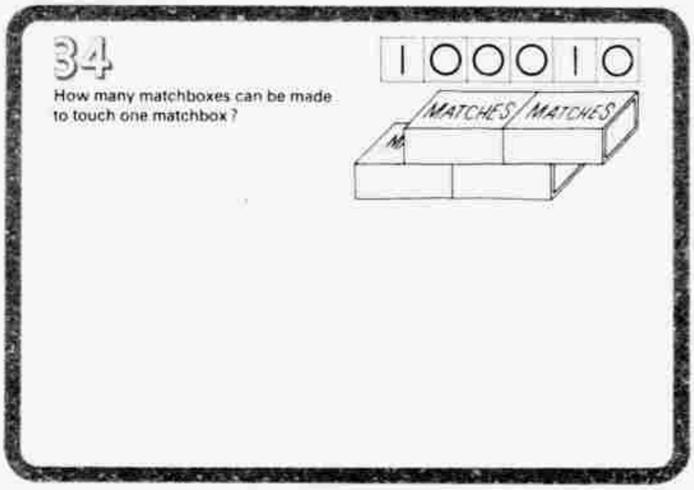


Is there any situation with four piles which needs four moves for its solution?

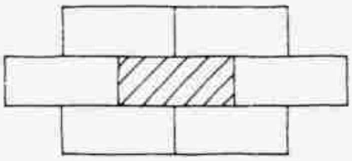
The process of making each pile the same height is analagous to that of finding the arithmetic mean of four numbers. The familiar method of adding the numbers and then dividing by the number of numbers to find a mean can be represented by piling all the cubes in a single pile and then distributing them one by one into four piles. An improved method is to remove the cubes in each pile which stood above the level of the top of the smallest pile and to redistribute these: since there are 8 of them, we place 2 of them on each pile.



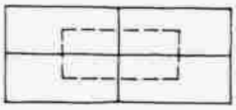
A still more sophisticated method depends on our recognising that the mean height of the piles will be greater than the height of the lowest pile, but less than the height of the tallest pile. We may then guess at what the mean might be, and try to make all piles that height. If we guess the mean height to be 3, we should find ourselves with 4 piles of 3 and 4 cubes over: we then add one of these to each pile. If we were to guess the mean height to be 5 we should have 3 piles of 5 and one pile of 1 : we should be 4 cubes short, and by removing one cube from each pile, we may then make a fourth pile of four.



If we lay a matchbox down on a flat surface, we can arrange for six other matchboxes to touch its sides, provided all the boxes lie flat on the surface, and are all arranged lengthwise :

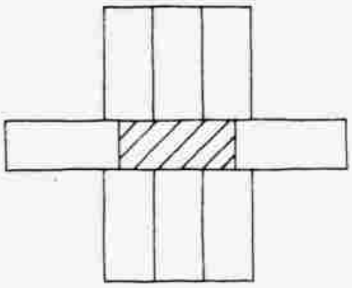


Four more matchboxes can be arranged on top of this array so as to touch the first box (shown here by the line of dashes) :

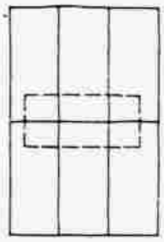


The box may rest on four other boxes, so that the total number of boxes touching the original box is $6 + 4 + 4 = 14$.

If we alter the position of the boxes on the same plane as the first box, we may increase the number touching it on this plane from six to eight :



and we may increase the number touching it above and below from four to six :

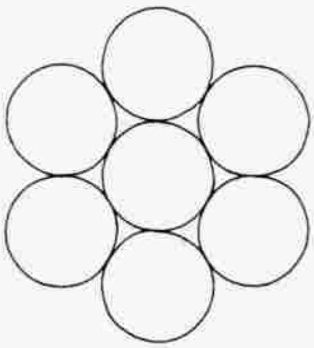


So the total is now $8 + 6 + 6 = 20$.

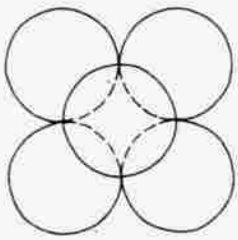
Can this number be increased if we are allowed to stand the boxes on end ?

It is worth checking that fourteen is the maximum number of *cubes* which can touch another cube, all cubes being of the same dimensions. Fourteen is also the maximum number of cuboids which can touch a single cuboid of the same dimensions if all are closely packed in space, and if all are arranged in the same way, that is, labels uppermost and all the same way up.

Six pennies can be made to touch a penny on the same plane :

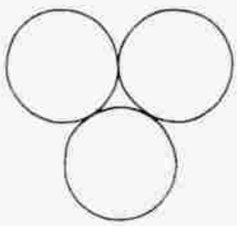


The central penny can rest on four others:

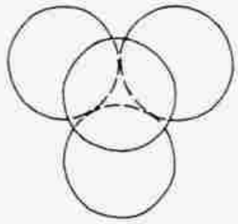


Can it rest on five others? On six others?

Three pennies can be arranged so that each touches the other two:



A fourth penny can be added so that each touches the other three:



Can five pennies be arranged so that each touches the other four?

Six matches can be arranged so that each touches the other five:

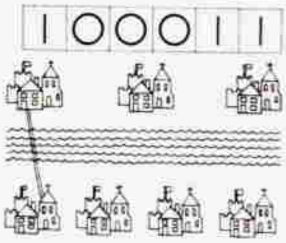
Can seven matches, or pencils, be so arranged?

Three people can touch one another by holding hands: what is the maximum number of people who can touch each other? (!!)

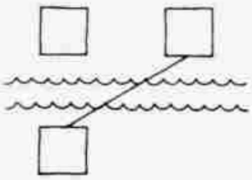
35

A telephone engineer has to lay cables across a river, connecting each of three villages on the north bank to each of four villages on the south bank. How many cables must he lay?

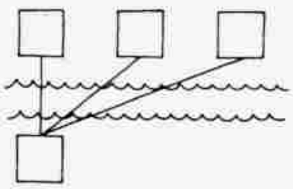
Make a table showing how many cables would be needed to connect up to six villages on either bank.

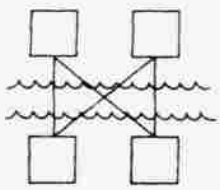


We need a single cable to join two villages, one on either bank, assuming that we are concerned only with submarine cables. A second village on either bank will require a second cable.

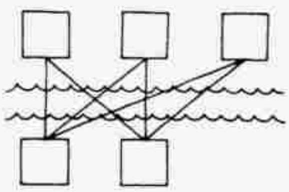


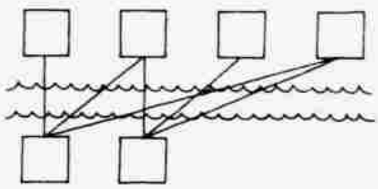
If a new village springs up on the north bank, a new cable will be needed to join it to the village on the south bank. But if the new village is on the south bank, it must be joined to both villages on the north bank, and two new cables will be needed.





Adding further villages will increase the requirements for cables.





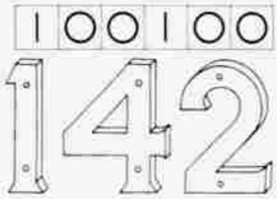
We are now in a position to begin entering items in our table. The entry for the number of cables needed to join three villages on the north bank to two on the south bank will be made in the intersection of the third column and the second row, and so on.

South \ North	1	2	3	4
1	1	2	3	
2		4	6	8
3				

As we fill in more of these spaces we see familiar patterns emerging. It might now be of interest to vary the 'rules', e.g. by also connecting neighbouring villages on the same bank or by introducing an island . . . .

36

A man went into a shop and bought some big numbers to put on his front gate. What do you think the number of his house was if he bought a 1, a 2, and a 4?



There are six possibilities if we assume that he will use all three, and only three:

1 2 4
1 4 2
2 1 4
2 4 1
4 1 2
4 2 1

These can be found by trial and error. It is best actually to have three such numbers and shuffle them around: if we put down the 1 first, then the others can follow in the order 24 or 42. For any number we place first, the other two can be arranged in two ways. Since we can place any of the three numbers first, the total arrangements are $3 \times 2 = 6$.

Suppose we had an extra number, say 7. Then we have four numbers, each of which we could lay down first: the other three can be laid down in six different ways, as we have just seen. The total arrangements of four figures are therefore $4 \times 6 = 24$, or since the 6 was obtained from 3×2, we must write $4 \times 3 \times 2 = 24$. This number can be checked by writing out all the arrangements of four figures. What if there are five figures? And what if two figures are the same, as in 122 or in 1447?

These arrangements can be illustrated by using people. Two people can stand side by side in two distinct ways; three can stand in a row in six distinct ways; four can stand in a line in twenty-four ways. If we rely on the argument or on the pattern we were building up in the previous paragraph, we can continue the calculations:

Number of objects	Number of arrangements
1	**1**
2	**2**
3	$3 \times 2 = 6$
4	$4 \times 3 \times 2 = 24$
5	$5 \times 4 \times 3 \times 2 = 120$
6	$6 \times 5 \times 4 \times 3 \times 2 = 720$
....	....

The objects must all be distinguishable if all arrangements are to appear different. We can use the desk calculator to discover in how many ways ten children can be arranged differently: it is in fact 3,628,800.

If our man had been buying one of the numbers for his next-door neighbour, to replace one that had worn away, and if the houses were all numbered consecutively even or odd according to the side of the street, can we say what the numbers of their houses were?

37 I O O I O I

Mrs Brown breeds dogs. At present she has eleven of which seven are spaniels and eight are puppies. How many spaniel puppies has she?

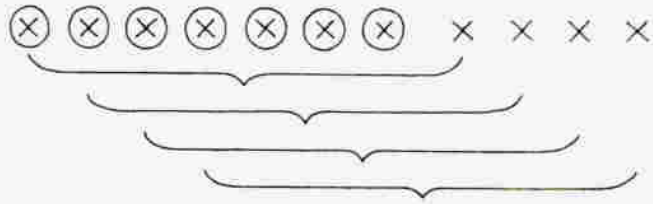
Make up a similar problem.

At first sight there does not seem to be enough information in this question to allow us to give a definite answer, and in the accepted sense there is no single 'right' answer. But we may give an answer to within limits. We know that Mrs Brown cannot have more than a certain number: there are only seven spaniels in all, and if all of these were puppies, then she would have seven spaniel puppies. This would be possible since she has eight puppies in all. Must any of her spaniels be puppies; or must any of her puppies be spaniels?

If none of her spaniels were puppies, then she would have seven spaniels and eight puppies of other breeds, or fifteen dogs in all. Since she has only eleven dogs, some of them at least must be both spaniels and puppies. All these relationships and possibilities can be shown graphically in various ways.

First we might line up all her dogs. Each dog is shown as a cross, the spaniels by ringed crosses.

Now eight of these must be puppies. We may arrange our dogs so that all the puppies are grouped together, and we can then show, by a bracket, which are the puppies:

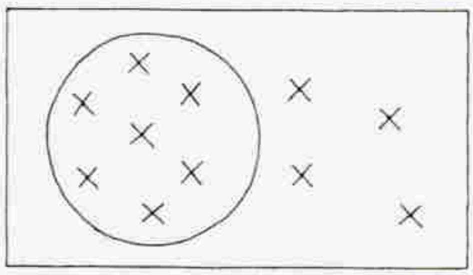


This bracket may be placed in four positions. In its first position it includes all seven spaniels and one other breed; in its last position it includes four spaniels and four other breeds. We may show the results in a table:

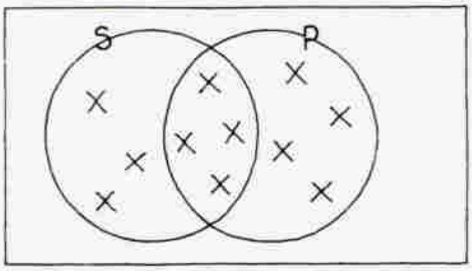
Spaniels	Other breeds
7	**1**
6	**2**
5	**3**
4	**4**

Since there are only four 'other breeds', there are no other possibilities; and we conclude that at least four spaniels must be puppies, though any number up to all seven may be.

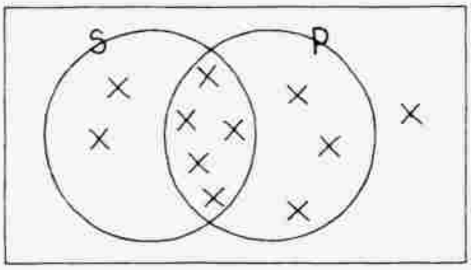
Another means of illustrating all the possible situations is the Venn diagram. In this, crosses representing the dogs are drawn inside two loops. The one contains spaniels, the other puppies. The two loops are drawn inside an enclosing rectangle, which contains eleven crosses, each representing a dog. In this rectangle we first draw the loop which encloses the spaniels.



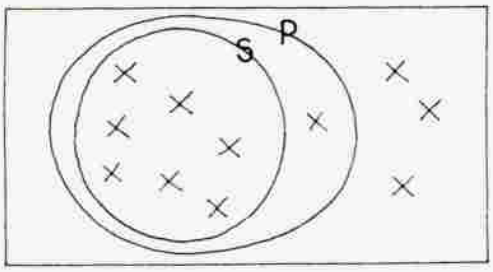
The second loop must enclose eight crosses. Let us first draw it so that it encloses all the crosses which are not enclosed by the first loop.



The loop S encloses seven spaniels, the loop P encloses eight puppies. At least four dogs must lie within the *intersection* of the two loops: these are both spaniels and puppies, that is, they are spaniel puppies. If we allow one dog to escape from the loop P so that it lies outside both loops, this will indicate that this dog is neither a spaniel nor a puppy: it is an adult of another breed. But to compensate for the loss of this dog, the loop P must take in another dog, and the only supply of other dogs is inside the loop S. The intersection of P and S will now contain five spaniel puppies.



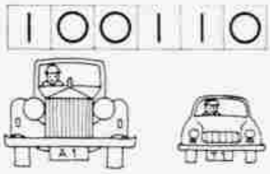
If this process is continued it reaches its limit when all loop S is included in loop P:



All the spaniels in this case are puppies. One puppy is not a spaniel; and three dogs are neither puppies nor spaniels. Notice that in each case four dogs are not included in S, that is, they are not spaniels; and three dogs are not included in P, that is, they are not puppies.

38

When the people of Perm Island were given independence, they gave each of the motor vehicles on the island a new registration number, consisting of one letter and a number from the set {1, 2, 3 . . . . 8, 9}. How many different registrations can be obtained in this way?



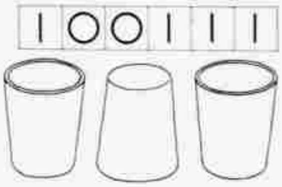
We can make a list of all possible letter-number pairs: this would be tedious. Or we may argue that we should have 9 registrations containing A, 9 containing B, 9 containing C, and so on, so that in all we should have $26 \times 9 = 234$ different registrations. If we allowed reversals so that T2 and 2T were both permitted, then we could have $2 \times 234 = 468$ registrations. We might exclude 1I as being too easily confused with I1: it would depend on how we chose to print our Is and ones on the number-plates.

If the number of vehicles exceeded 468 (or 467, excluding 1I), we could allow single letters in combination with numbers between 1 and 99: this would give us $99 \times 26 = 2574$ registrations or double this number with reversals permitted.

39

Three tumblers are placed on a table, with the centre one upside down, and the others the right way up.

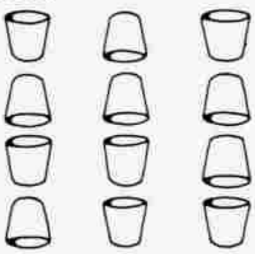
You are allowed to turn any of the tumblers over, but you must turn just two at a time. Can you do this so that all three finish the right way up?



Extended experimentation will point to a high probability that the answer to the question is 'no'. We may arrive at a position in which all three tumblers are upside down, but not when all three are the right way up. Can we prove that it is impossible to have them all the right way up?

It is easy to see intuitively that it cannot be done. If we start with the tumblers in the position shown, and turn the two outside ones, all three will then be upside down. If on the other hand we turn the centre one, we must at the same time turn one of the others, and we shall again have one upside down.

Let us now give a value to each tumbler. If it is the right way up, we give it the value 1; if it is upside down, we give it the value 0. The total value of the tumblers in each of several positions is shown below:



$1 + 0 + 1 = 2$

$0 + 0 + 0 = 0$

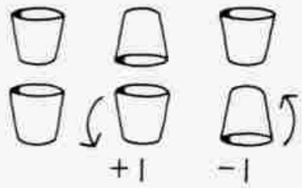
$1 + 1 + 0 = 2$

$0 + 1 + 1 = 2$

These are the only positions we can reach with our tumblers, if we abide by the rules. When all the tumblers are the right way up, their total value is 3:

$1 + 1 + 1 = 3$

Now all the values of the positions we have been able to obtain have been even: 2 or 0, counting 0 as even. The value of the starting position was an even number, 2. When we turned two tumblers, we added 1 to this total when we turned the tumbler which was already upside down, but we subtracted 1 when we turned another tumbler which was already the right way up.



$1 + 0 + 1 = 2$

$1 + 1 + 0 = 2$

The final value remained unaltered: $2 + 1 - 1 = 2$. By turning the two tumblers which are the right way up, we subtract 1 twice from the value: $2 - 1 - 1 = 0$. In either case we finish with an even number: we cannot change the value to an odd number. Only if the value of our starting position is an odd number can we achieve the value 3, which is the total value when all tumblers are the right way up.

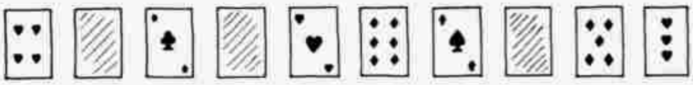
Once we realise this fact, we can say whether one position can be reached from another, no matter how many tumblers we have, provided we must turn just two at a time. For instance, we can operate on these nine tumblers so that they all finish the right way up:

$0+1+1+0+1+0+1+1+0 = 5$

$1+1+1+1+1+1+1+1+1 = 9$

Both positions have a value which is an odd number; by turning two tumblers at a time we can obtain one position from another, even though we make additional rules, such as that two neighbouring tumblers must be moved each time.

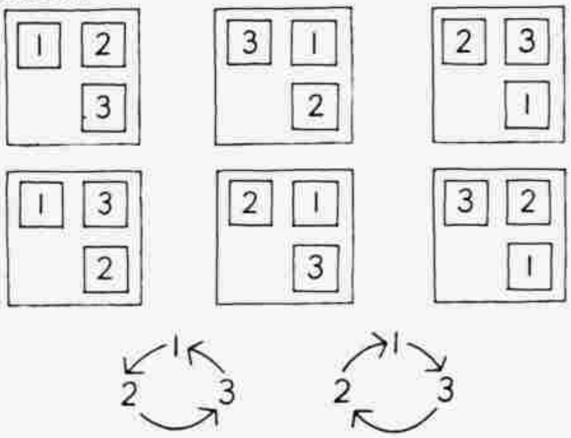
We could work with playing cards. Is it possible to turn two neighbouring cards in this line so that all cards finish face up?



The answer is 'no'. The value of the line as shown is 7. When all ten cards are face uppermost the value will be 10, and we cannot change the value from an odd number to an even number by turning two cards at a time.

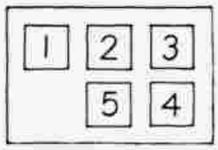
We may express this formally by saying that we cannot change the *parity* of the system. A system with an odd parity retains an odd parity: one with an even parity retains an even parity.

Another problem which involves the idea of parity is that of sliding numbered blocks around in a limited space, so that they finish in a given order. For instance, if we have three cards numbered 1, 2, and 3, which may slide around in a square, we can obtain three distinct arrangements: these we will regard as being the arrangements 123, 312, and 231. The mechanics of the apparatus, which allows a card to slide sideways or up or down but not diagonally, forbid us to obtain any of the other possible arrangements, 132, 213, and 321, although any one of these may be obtained from any other by sliding. These six arrangements of 1, 2, and 3 are the only ones possible: each set happens to contain the cyclic arrangements of 1, 2, 3 either in that order or in reverse order.



What is pertinent to the original problem is that each of the arrangements can be obtained from the arrangement 123 by a series of interchanges. For example, 132 may be obtained by interchanging 2 and 3: 312 may be obtained by interchanging first 1 and 2 and then 2 and 3. By interchanging any two at a time, all six arrangements can be obtained:
123→132→312→213→231→321.

Another order might be:
123→213→231→321→312→132.
Now the first, third and fifth interchange in each case produced one of the set 132, 213, 321, while the second and fourth interchange produced 312 and 231. No interchange was necessary to produce 123. What we have demonstrated is that an odd number of interchanges appears to produce one of the arrangements 132, 213, 321, while an even number appears to produce one of the arrangements 123, 312, 231. On this basis we can classify the six arrangements in two subsets, the members of the same subset having the same *parity*.



If we extend the problem to five numbers sliding in a rectangle, then we shall find that of the 120 possible arrangements, only 60 can be made from the starting position shown, and each of these sixty will require an even number of interchanges to restore them to the order 12345, which is merely the reversal of the procedure we were discussing above. For instance, 41532 is one of the 60. It can be obtained from 12345, or restored to 12345 in 4 steps: 41532→14532→12534→12354→12345. 14253 can be restored in three steps: 14253→12453→12354→12345; but an odd number of steps is required here, and this re-arrangement could not be achieved by 'sliding' in the rectangular box.

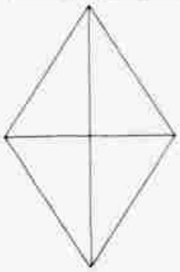
Further investigations might be of interest involving changing the 'rules' with the three tumblers, e.g. from a given arrangement the problem might be to get one tumbler upside down and the others the right way up *in a definite order* and a 'move' might consist of turning one tumbler over and interchanging the other two . . . .

40

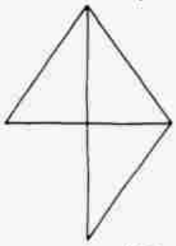
I O I O O O

The outside lines of this figure are all equal in length: they form a rhombus. What other shapes can you see in the figure?

Draw the shape on paper, cut it out, and cut along all the lines. What shapes can you make from the four pieces?

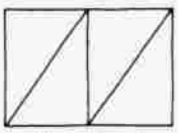


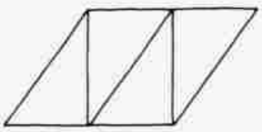
The other shapes in this figure are either triangular or pentagonal:



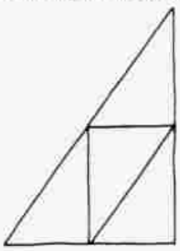
In all, the figure contains eight triangles: four of these remain when the figure is cut up.

A variety of shapes can be made from the four triangles: a rectangle, a parallelogram, and a

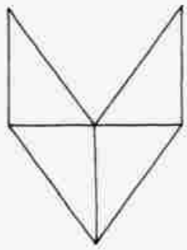




right-angled triangle are some of the simpler shapes.

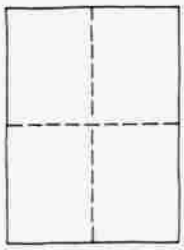


Others such as the fox will be found as children let

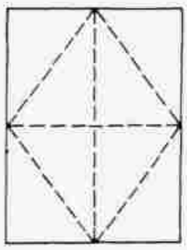


their imagination roam.

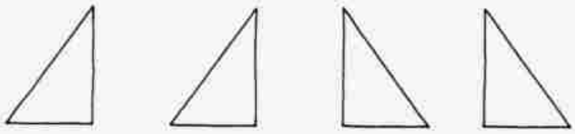
A rhombus may be folded from a rectangle, first by folding edge to edge, and then by folding along the lines joining



the mid points of the edges.



The pieces cut off can be arranged to form a rhombus congruent to the one which is left: each rhombus has an area equal to half that of the original rectangle, whose length and height are retained in the diagonals of the rhombus. If the rhombus is cut from coloured sticky paper, and is then cut into four triangles, two of these will be found to match exactly. The other two will be congruent to these but 'back-to-front' as it were. They may be regarded as mirror images:



Does this affect the number or nature of the shapes which can be made by sticking these four triangles on to a sheet of paper?

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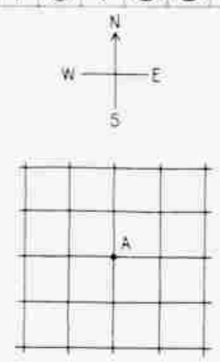
The lines on the right are part of a very large grid. Starting from A a man may move along the lines in four directions, north N, east E, south S, west W. A journey of one square east, two squares south and one square west can be recorded as ESSW. Find five different journeys which will bring him back to his starting point.



It is not difficult to find five journeys which lead back to the starting point. If we restrict ourselves to two moves only we can find four such journeys: EW, WE, NS, SN. If we allow ourselves four moves, then such journeys as EEWW, or NEWS, or NWSE, will serve. Longer journeys can be found and on an infinite lattice there will be no limit to the number of journeys which can be made, and which will return a traveller to the starting point.

What do we notice about the four journeys of two moves? They can be paired. One pair will be (NS, SN) and the other (EW, WE): the members of the first pair both contain N and S, and the members of the second pair both contain E and W. In so far as all these journeys return us to the same point, it does not matter in which order we choose to move. If we are moving a unit east and a unit west, it does not matter if we move east first or west first. If we are moving one unit north and one unit south, then we shall return to the starting point irrespective of whether we first move north or south.

The four journeys, EW, WE, NS, SN, all lead back to the starting point. It is a temptation to write EW = WE = NS = SN since all these journeys have the same outcome; but we must be clear about what we are saying here.

The journeys are **equivalent**, in that they all lead eventually to the same point, which in this case is the starting point. The two journeys EN and NE are also equivalent since they both lead to the same point, though this is not the starting point.

Journeys which return to the starting point are a special case, in that they are equivalent to no journey at all. At the end of the journey, the traveller is no better off in terms of position than the person who stayed at home and made no move. His journey is no journey at all: his instructions were not 'EW' or 'SENW' but 'stay still'. We can give a letter to such a 'journey': we might call it I (since S for 'stay still' is obviously not available for use) and say it stands for 'idle', though in fact this move is an 'identity' move, which leaves the traveller in a position identical to his previous one.

Now we have a whole class of journeys equivalent to one another: EW = WE = NS = SN = NEWS = NESW = ... = I. We can find other classes of journey. For instance, there is the class which contains all journeys equivalent to EN: we have EN = NE = NNES = EENW . . . . , and so on.

A little thought will show that for each point on the lattice there exists a whole class of journeys, any member of which will take us to that point from the starting point.

How can we tell whether two journeys are equivalent? We use two principles here. The first is that the order in which we write down the steps of the journey has no effect on its outcome. The journey EW was equivalent to the journey WE. We can check that the six journeys, ENS, ESN, SEN, SNE, NSE, and NES, all lead to the same point, one unit to the east of the starting point.

The second principle is that WE, EW, NS, and SN can all be replaced by I wherever they occur; and if I appears in the recording of any journey, it can be ignored. This means that EIS represents the same journey as ES: the journey IE is the journey E. The instruction to 'stay still' at one point in the journey may add to the length of time the journey takes, but it does not affect its outcome.

We can now simplify a journey such as SENWSNWES by substituting I for EW, WE, NS or SN wherever they occur, if necessary altering the order of the letters so that the pairs we are after do appear together.

SENWSNWES
= **SENW I I S**
= **SNEWS**
= **I I S**
= **S**

So SENWSNWES is equivalent to a journey of one unit south, and this can be checked on a grid.

Such 'simplifications' might form the basis of a game (treasure-hunt ?).

We could restrict ourselves to one dimension, e.g. NNNNSSSNNN. What about a shorter way of writing this ?

What about 3 dimensions ? (e.g. travelling up and down in a lift as well).

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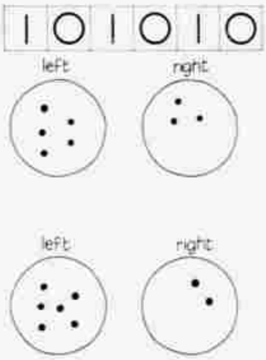


The first picture shows two loops: the left-hand loop contains five counters, and the right-hand loop contains three counters. If we move a counter from the right-hand loop to the left-hand loop, there will be six counters in the left-hand loop and two in the right-hand loop.

How many distinct ways can you find of distributing eight counters between two loops?

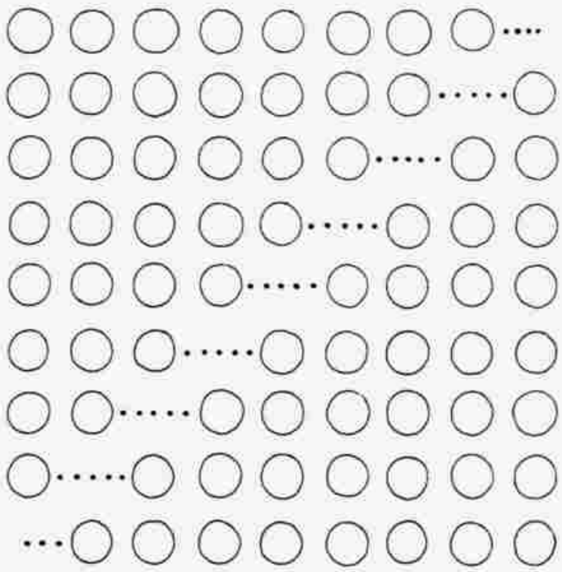
This problem can easily be solved by trial and error. It is best if we begin by putting all the counters in one loop, let us say the left-hand loop. There will be none in the right-hand loop, and we can describe this situation by the pair of numbers (8, 0). We can check that $8 + 0 = 8$. We now move one counter from the left-hand loop to the right-hand loop, and we now have seven in one and one in the other. This situation can be described by the pair of numbers (7, 1), and once again $7 + 1 = 8$. Eventually we shall have moved all the counters into the right-hand loop, and we shall have found nine distinct situations.

By 'distinct situations' we mean, for example, that five in the left-hand loop and three in the right-hand loop is a different situation from three in the left-hand loop and five in the right-hand loop. When we describe these situations by writing pairs of numbers, we must agree that (5, 3) and (3, 5) are distinct pairs, even though $5 + 3 = 3 + 5$. We must say that the order in which the numbers are written matters.

We can summarise our results in a table:

Left-hand loop	Right-hand loop	Number pair	Check
8	**0**	**8, 0**	$8 + 0 = 8$
7	**1**	**7, 1**	$7 + 1 = 8$
6	**2**	**6, 2**	$6 + 2 = 8$
5	**3**	**5, 3**	$5 + 3 = 8$
4	**4**	**4, 4**	$4 + 4 = 8$
3	**5**	**3, 5**	$3 + 5 = 8$
2	**6**	**2, 6**	$2 + 6 = 8$
1	**7**	**1, 7**	$1 + 7 = 8$
0	**8**	**0, 8**	$0 + 8 = 8$

If we have enough counters we can set up all these situations in separate lines:



If we have eight counters, we can find nine distinct ways of dividing them into two *subsets*. How many ways could you find of dividing seven counters into two subsets?

The [illegible]
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can [illegible]
oth[illegible]
in [illegible]
on [illegible]
to [illegible]
alu[illegible]
we sp[illegible]

a [illegible]
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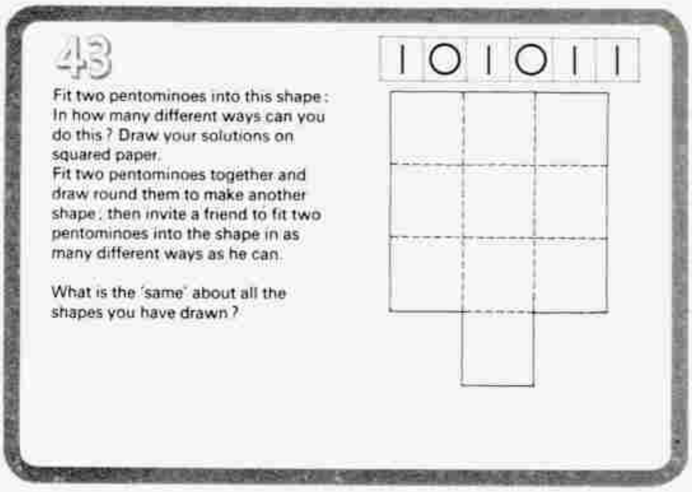
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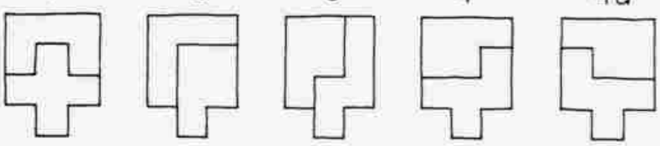
Fit two pentominoes into this shape: In how many different ways can you do this? Draw your solutions on squared paper.
Fit two pentominoes together and draw round them to make another shape; then invite a friend to fit two pentominoes into the shape in as many different ways as he can.

What is the 'same' about all the shapes you have drawn?

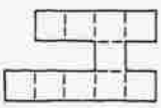
For solving the problems on this Card, we need a set of pentominoes which can be made or bought: there are twelve distinct pentominoes (see Problem 26).

Solutions are:

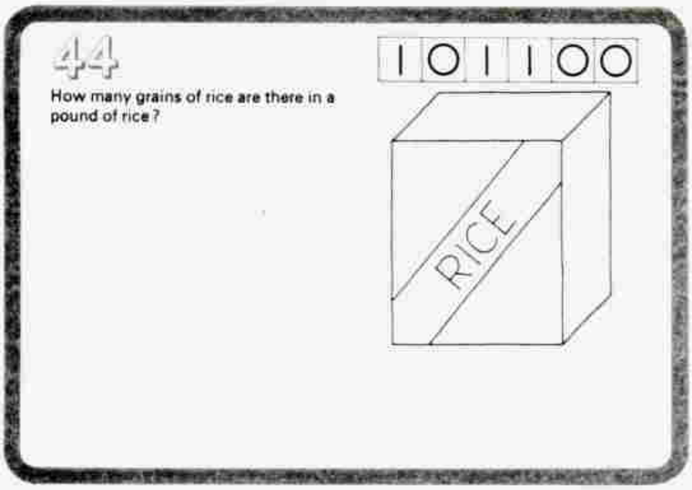
1 2 3 4 4a

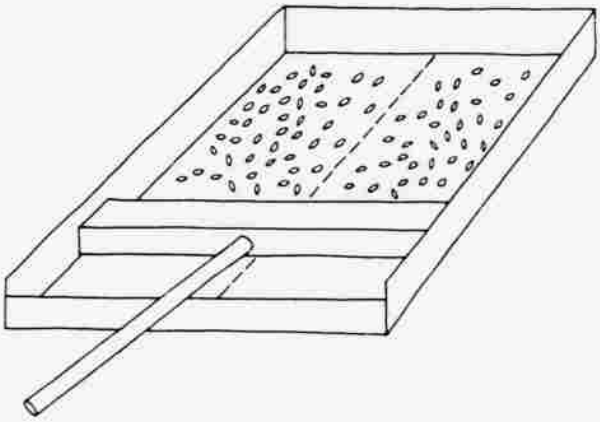


All other solutions are mirror images of one of these, just as 4a is a mirror image of 4. Can we find a shape into which only two particular pentominoes will fit? Here is one such shape:



Three things may be said about these shapes: they are all rectilinear, that is they are all composed of straight lines; these lines all meet at right angles; and each shape occupies ten squares, assuming that a pentomino is made from five squares. The last statement is equivalent to saying that all pentominoes have the same area, and so do the shapes into which two pentominoes will fit. If the pentominoes are made from one-inch squares, each pentomino has an area of 5 square inches, and each of our drawn shapes has an area of 10 square inches. All the shapes do not have the same perimeter, nor do all the pentominoes; it seems then that shapes with different perimeters may yet have equal areas.





There are a lot of grains of rice in a pound, and the task of counting them appears Herculean. Given scales which will weigh an ounce, we may reduce this task to one of finding how many grains there are in an ounce. Then by successive doublings of this number, we can obtain the number in a pound. For example:

Weight	Number of grains
1 oz	**983**
2 oz	**1966**
4 oz	**3932**
8 oz	**7864**
16 oz	**15,728**

This exercise is useful mainly in that it gives experience in counting large numbers. The physical task of counting 983 grains is made easier by grouping the grains in tens, and then in tens of tens; a better appreciation of place value may be gained (incidentally) in the process of counting. A subsidiary exercise would be to count the same number in nines or in eights, emphasising that ten is an arbitrary number to use as a base for counting.

If the task of counting the number of grains in an ounce seems too great, then it can be reduced by using a halving tray. This has three raised sides, and a pusher is used to make sure that the grains of rice are evenly spread over the surface: this is done by shaking the tray gently and adjusting the area over which the rice is spread by moving the pusher backwards or forwards. A line down the centre of the tray is used as a guide for separating the rice into two equal parts, only one of which need then be counted. If this is done carefully, the difference between the two parts can be reduced to four or five grains: any such error will accumulate as the number is doubled, but the proportionate error will remain constant. Any results must be approximate. An exact answer can be found by sharing the task of counting a pound of rice among the whole class, but even then a recount might well produce a different result, and the variation in the size of grains will mean that no two separate pounds are likely to contain the same number of grains, although there is a limit to the amount of variation possible.

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of this
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Weight
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2 oz
4 oz
8 oz
16 oz

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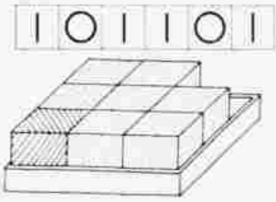
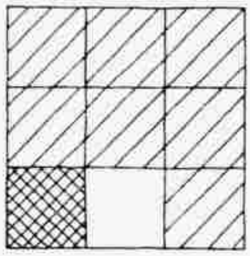
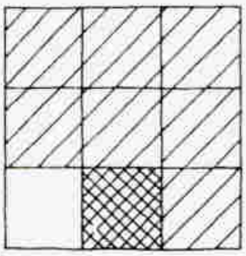
Place seven yellow inch cubes and one green one in a 3-inch square tray, with the green cube in one corner. Leave a space in the opposite corner. If you slide one cube at a time, how many moves will it take to get the green cube into the opposite corner?



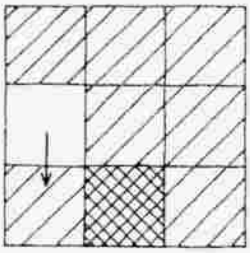
Thirteen moves will be needed. Three moves will be needed to create a space beside the green cube:

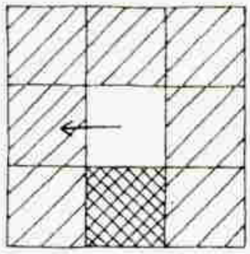


The green cube is slid into this space:

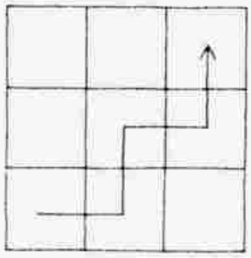


Two moves are needed to clear a new space for the green cube to move into:



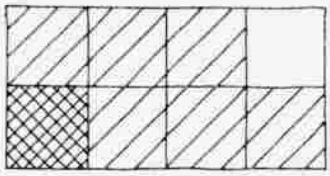


and from now on the green cube will move on every third move. Its path may be this:

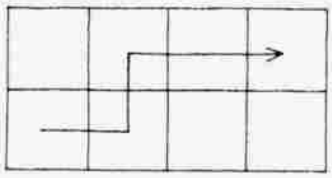


and the total number of moves will be 4 + 3 + 3 + 3 = 13.

If we use a rectangular tray 4 inches by 2 inches, we find it rather more difficult to plan a route. The simplest route for this situation

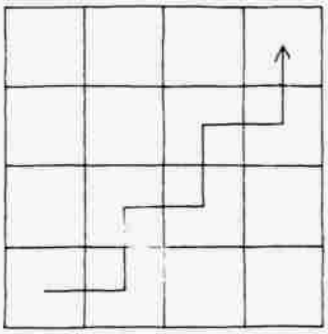


seems to be:



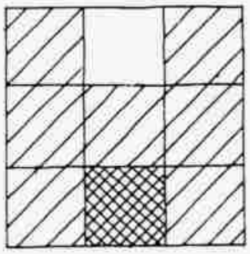
for which route 15 moves are necessary, this number being the sum of 4 + 3 + 3 + 5.

In a square 4 inches by 4 inches, the path is more readily seen as



and the number of moves needed is $6 + 3 + 3 + 3 + 3 + 3 = 21$.

The game can be played by two players, each trying to solve the problem in fewer moves than his opponent. Variants on the problem can easily be invented, e.g. the least number of moves required to move the block from one edge of the tray to the opposite edge:



or the least number of moves needed to change from this pattern:

R	R	G	G
R	R	G	G
B	B	Y	Y
B	B	Y	

to this:

B	B	B	B
G	G	G	G
R	R	R	R
Y	Y	Y	

A little ingenuity will lead to the discovery of many others.

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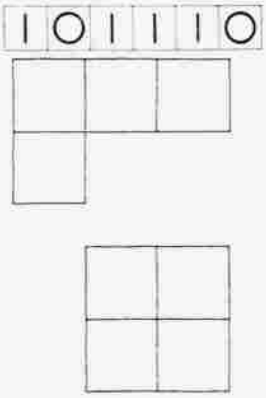
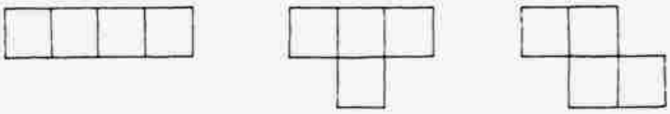
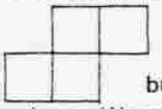
These two shapes are each made from four squares joined along their edges; they are known as 'tetrominoes'. Can you find any other tetrominoes? Cut a set of tetrominoes from squared paper. Can you partition them into subsets according to their symmetry?



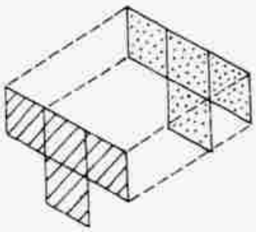
The five distinct tetrominoes are the two illustrated above, with the addition of these three: the I tetromino, the T tetromino, and the Z tetromino.



We may call the ones illustrated on the Card the L tetromino and the square tetromino.

We could have an S tetromino  but this would fit on to the Z tetromino if it were turned over. We may call these two tetrominoes 'congruent' and treat them as if they were the same tetromino.

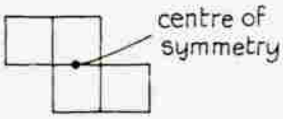
If we make two T tetrominoes and colour the upper side of each one blue, and the under side red, then the two will fit if we put the two red sides together. The reason this happens is that the



T tetromino is symmetrical about an axis of symmetry, and if it is turned over about this axis, it still appears the same shape.

Which other tetrominoes have this property?

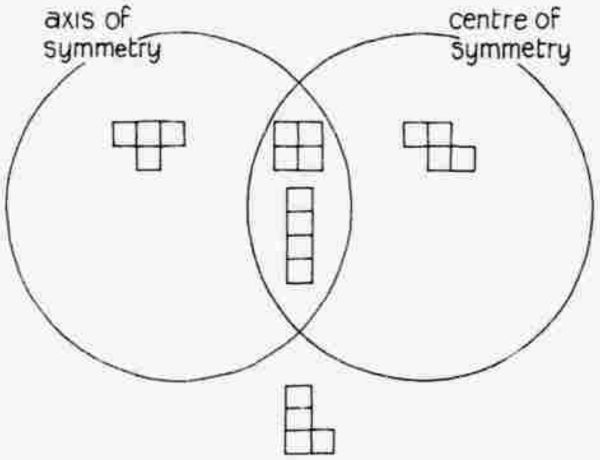
If we place one T tetromino on top of the other and rotate the top one, it will not fit on the other until it has turned completely round and is back in its original position. If, however, we place one Z tetromino on top of another and rotate the top one, then it will fit on the other when it has turned half way. This happens because the Z tetromino has a *centre* of symmetry. (Where is it?)



We can build up a table to show which tetrominoes have either or both of these properties:

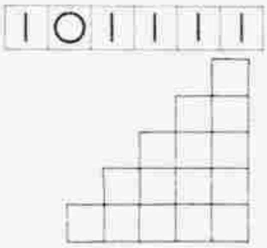
	Axis of symmetry	Centre of symmetry
Square	√	√
I	√	√
T	√	×
L	×	×
Z	×	√

We may show these results in a Venn diagram (see overleaf).


axis of symmetry
centre of symmetry

47

The staircase shown here is made from cubes and goes up five steps. How many cubes will be needed to make a staircase with ten steps up?

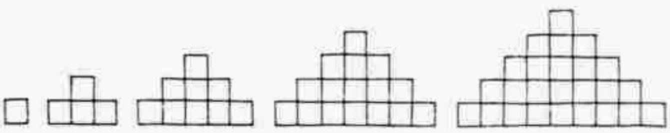


For a staircase ten steps up we should need 55 cubes. This is the sum of 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10. If we build the staircase in stages we shall need one brick for the first step, three bricks for the first two steps, six bricks for the first three steps, and so on. This table gives sums for parts of the staircase.

Number of steps	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Number of cubes	**1**	**3**	**6**	**10**	**15**	**21**	**28**	**36**	**45**	**55**

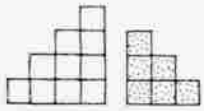
The set {1, 3, 6, 10, 15, 21, 28, 36, 45, 55 . . . .} is known as the set of triangular numbers. Our staircase is triangular in shape.

If our staircase goes down again on the far side, then we find that we shall need a different set of numbers of cubes.

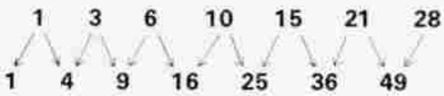


1 4 9 16 25

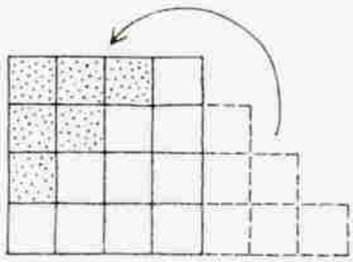
This set {1, 4, 9, 16, 25 . . . .} contains numbers which are formed from the set of triangular numbers. Each staircase can be split into two: for example, the staircase with 16 cubes is built from one of 10 and another of 6:



This relation can be shown by arrows:



The two parts of each staircase will in fact fit to form a square:



and the set {1, 4, 9, 16, 25, 36, 49 . . . .} is known as the set of square numbers.

We can continue the set of square numbers by using the pattern made by the differences between each number and the next.

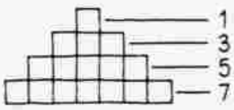
1 4 9 16 25 36 49 . . . .
3 5 7 9 11 13 . . . .

If we did not recognise these differences as the set of odd numbers, we could still decide what came next by taking the differences between numbers in the lower row:

3 5 7 9 11 13 . . . .
2 2 2 2 2 . . . .

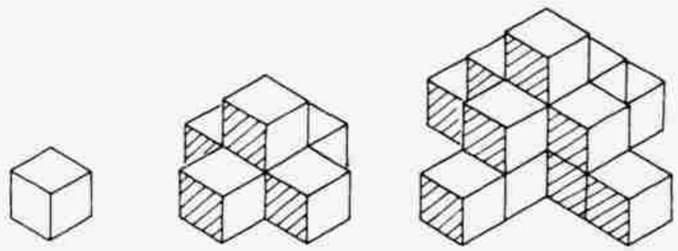
The set of odd numbers continues with 15, 17, 19 . . . . and the set of square numbers can now be continued: 49 + 15 = 64, 64 + 17 = 81, 81 + 19 = 100.

The staircase is formed from layers, each of which contains an odd number of cubes:



etc.

If we build our staircase with four different staircases going up to a central column, then we get another pattern of numbers.



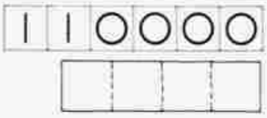
1 6 15

We can calculate that the next staircase is going to use 4 lots of 6 cubes, plus a column of 4 in the centre, which gives a total of $(4 \times 6) + 4 = 28$. In order to continue the pattern past the next number, which is $(4 \times 10) + 5 = 45$, we can use the technique of looking at the pattern of differences:

1 6 15 28 45
5 9 13 17
4 4 4

By adding 4 to 17, we get 21, and by adding 21 to 45 we get 66, which is the number of cubes we should need for a staircase going 6 steps up. We can continue these calculations for as long as we wish.

48



Take a strip of paper and draw three lines across it as shown. Cut along the lines. How many pieces of paper have you got now?

Can you cut a strip of paper into four pieces with only two cuts? How many pieces can you cut the strip into with three cuts?

With one cut we cut the paper into two pieces; with two cuts we cut it into three pieces; with three cuts we cut it into four pieces. We can tabulate our results.

Number of cuts	**1**	**2**	**3**	**4**	**5**	....
Number of pieces	**2**	**3**	**4**	**5**	**6**	

The number of pieces will always be one more than the number of cuts.

If, after cutting the strip of paper into two pieces with one cut, we then place one strip on top of the other and cut again, we shall have four pieces. If we repeat this action, a third cut will give us eight pieces.

Number of cuts	**1**	**2**	**3**	**4**	**5**
Number of pieces	**2**	**4**	**8**	**?**	**?**

If we now fold the piece of paper before we cut it, we find that one cut will produce three pieces. If we fold the strip over and over again, and then cut it, how many pieces do we expect to have?

We shall have five pieces. The table of results gives:

Number of folds	**0**	**1**	**2**	**3**	**4**
Number of pieces	**2**	**3**	**5**	**9**	**17**

It may be difficult to fold five times and then cut: how many pieces should we expect to have if we could do this?

If, after folding the paper once, then cutting it and obtaining three pieces, we next merely put these pieces together and then cut them, we should have seven pieces after the second cut.

Number of cuts	**0**	**1**	**2**
Number of pieces	**1**	**3**	**7**

How many pieces should we expect to have after the third cut?

49

| I | I | O | O | O | I |

How many numbers greater than 0 and less than 1000 can you put into the counter register of a desk calculator by turning the handle twice?

By turning the handle twice, we can put 2 into any of the last three places of the counter register. This will give 2, or 20, or 200.

If we turn the handle once, move the carriage, and then turn the handle a second time, we will get numbers containing two ones: 11, 101, and 110.

This exhausts the possibilities for two turns of the handle unless we adopt a new stratagem, that of turning the handle backwards on its second turn. If the first turn registers 10, then a turn backwards in the units place will give 10 − 1 = 9. We may write a backwards turn by showing a 1 with a bar across it, to indicate that we are subtracting this figure from the total:

11 = 10 − 1 = 9

We can now write a bar over the second 1 of 11, 101, and 110:

11 = 9
101 = 99
110 = 90

Now we are in a position to make the first turn in the fourth, or thousands, place in the calculator, since although this gives 1000, a second turn will subtract 1 or 10 or 100 from this total:

1001 = 999
1010 = 990
1100 = 900

The full solution set of our problem is:
{**2, 9, 11, 20, 90, 99, 101, 110, 200, 900, 990, 999**}
and this set contains twelve numbers.

If we are allowed 3 turns of the handle, we can first write down all the numbers whose digits add up to 3:
{**3, 12, 21, 30, 102, 111, 120, 201, 210, 300, 1002** . . . .}.

We may select numbers over 1000 but below 2000 provided we are going to turn the handle backwards to reduce these to below 1000.
{. . . . **1002, 1011, 1020, 1101, 1110, 1200**}.

We can check that we have all the numbers over 1000 that we want: the last 3 digits form numbers in the solution set for two turns of the handle:

2 → 1002
11 → 1011
20 → 1020
101 → 1101
110 → 1110
200 → 1200

The full solution set for three turns contains 28 numbers:

3			
12 = 8	**102**	**201**	**1002 = 998**
12	**102 = 98**	**201 = 199**	**1011 = 991**
21	**111**	**210**	**1011 = 989**
21 = 19	**111 = 109**	**210 = 190**	**1020 = 980**
30	**111 = 91**	**300**	**1101 = 901**
	111 = 89		**1101 = 899**
	120		**1110 = 910**
	120 = 80		**1110 = 890**
			1200 = 800

{**3, 8, 12, 19, 21, 30, 80, 89, 91, 98, 102, 109, 111, 120, 190, 199, 201, 210, 300, 800, 890, 899, 901, 910, 980, 989, 991, 998**}.

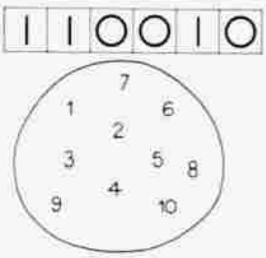
We can pair these 28 numbers each as the reversal of another. We must treat 3 as if it were 003, the reversal of 300; 80 is its own reversal: 080.

3 → 300	**80 → 80**	**111 → 111**
8 → 800	**89 → 980**	**199 → 991**
12 → 210	**91 → 190**	**899 → 998**
19 → 910	**98 → 890**	**989 → 989**
21 → 120	**102 → 201**	
30 → 30	**109 → 901**	

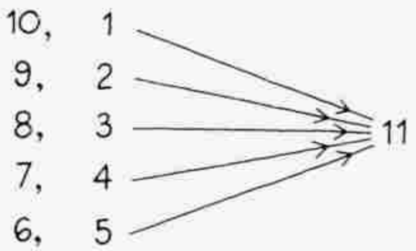
There are 16 of these pairs. Is this something we could have calculated? How many pairs in the set for two turns?

50

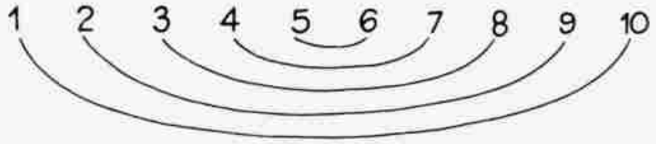
Can you pair off the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 so that each pair adds up to the same number?
Can you do the same with all the even numbers from 2 to 20?
And with all the odd numbers from 3 to 21?
Does this help us to find the sum of all the numbers from one to ten? Or of all the even numbers from 2 to 20?



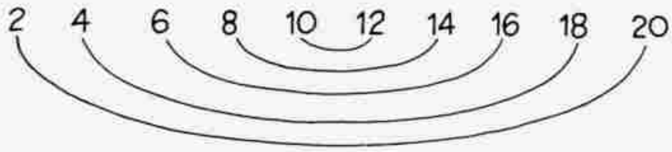
Trial and error will give an answer to the first part of this problem:



and another pattern can be made by joining the members of each pair when the numbers are written in order:



If we now have to work with the even numbers from 2 to 20, we can pair off the 2 with the 20, and so on, and obtain similar patterns to those we made with the numbers 1 to 10.



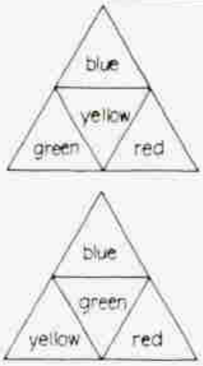
Each pair sums to 22. Since there are 5 pairs, we are presented with a quick method of finding the sum of all the even numbers from 2 to 20: it will be $22 \times 5 = 110$.

We may do the same with the odd numbers from 3 to 21. We can pair them off in such a way that the sum of each pair is 24. Again we shall have 5 pairs, whose total sum will be $5 \times 24 = 120$.

Is there a quick way of finding the sum of the following numbers?
4 8 12 16 20 24 28 32 36

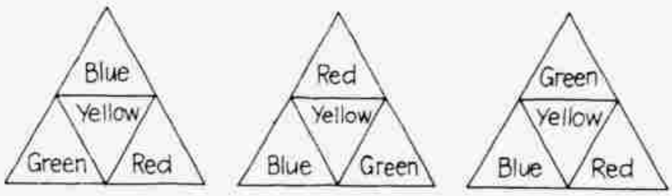
51

Cut out some equilateral triangles from paper, draw lines on each triangle which divide it into four similar triangles, and colour each of these four triangles with a different colour. If you are allowed to use only these same four colours each time, how many distinct triangles can you produce? Here are two distinct triangles to start with:



If these are now folded to give tetrahedra, with each face coloured differently, how many would be distinct?

This is a practical problem: the triangles must be drawn, coloured and cut out, and any two must be compared to ensure that they are distinct from one another. The two triangles on the left below are identical, whereas that on the right is distinct from either:



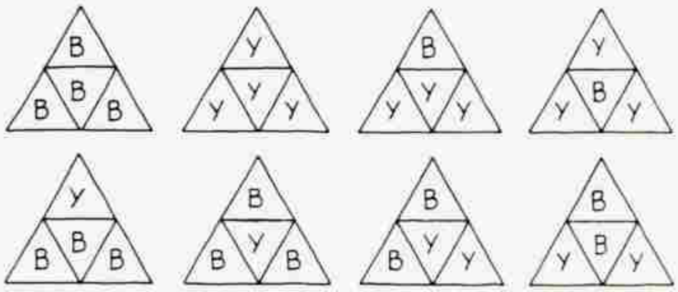
(By rotating the centre triangle clockwise through 120°, it will present exactly the same appearance as the triangle on the left. This is what we mean when we say that the triangles are not distinct.)

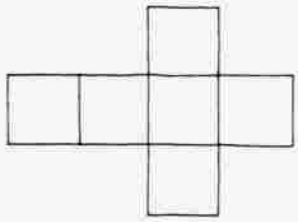
Now, no matter in what order we colour the red, blue and green triangles, we cannot produce a triangle distinct from those we have already found, as long as we colour the centre triangle yellow. Every triangle we colour will be found to be identical in appearance to one of these when we have rotated it through either 120° or 240°. Only by changing the colour of the centre triangle can we find new distinct patterns of colour.

We may choose any of four colours for the **centre triangle**: given the colour of the centre triangle, we can colour two distinct triangles, so the total number of distinct triangles obtainable is eight.

It is now a simple task to fold each up into a tetrahedron, and see how many distinct tetrahedra we have made.

Suppose we were limited to two colours only. We may use one or both: in this case we can colour a tetrahedron in five different ways. These can be folded from the triangles shown below:

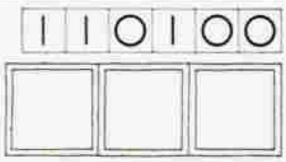




How many distinct cubes can be coloured using six colours, one for each face?

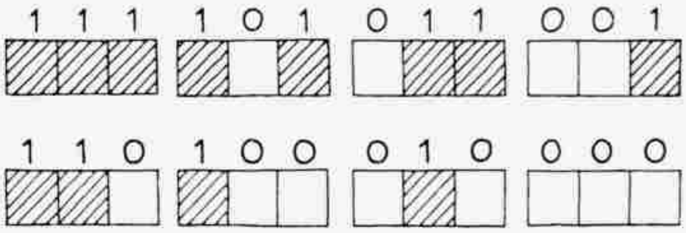
52

A man is fitting coloured glass into a window which is in three sections. In each section he can fit a red pane or a yellow pane. How many different ways are there of glazing the window using these colours?



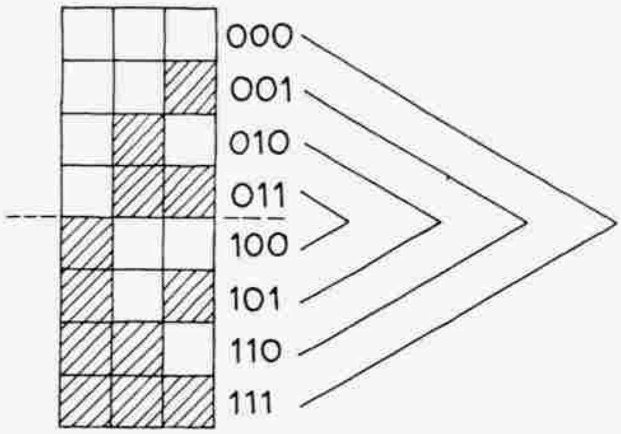
By experiment we can find eight different ways.

If we give numerical values to each pane, 1 if it is glazed in red and 0 if it is glazed in yellow, then we may record the possibilities as follows (shading represents red, non-shading yellow):



These 'numbers' 111, 110, 101, 100, 011, 010, 001, 000 are all different, and in fact, as we should expect, these are the eight possible arrangements of three figures, each of which must be either 1 or 0. Such 'numbers' will be familiar; the binary notation employs 0 and 1 only, and each of these numbers is a binary number. Translated into denary notation, they are the numbers 7, 6, 5, 4, 3, 2, 1, and 0.

If we reverse the order of these numbers and arrange them in a column, we have this pattern of windows:



If we coloured a large rectangle in this way, cut it out and folded it in two along the dotted line, a red section would meet a yellow section and vice versa. If we add two numbers which are equal distances from the dotted line, shown as joined by the lines on the right of the diagram, we find the sum of each is 111:

000	**001**	**010**	**011**
111	**110**	**101**	**100**
111	**111**	**111**	**111**

The two numbers are *complementary*, with respect to 111, or 7: in denary notation we should have $0 + 7 = 7$, $1 + 6 = 7$ and so on. Using binary notation, the complement is easy to discover: we simply supply a 1 wherever the first number was zero, and zero where the first was 1.

The same kind of pattern can be devised for four panes . . . .

The following publications of the Nuffield Mathematics Project appeared in 1967-8:

Introductory Guide

I do, and I understand ●■▼ (1967)
This Guide explains the intentions of the Project, gives detailed descriptions of the ways in which a changeover from conventional teaching can be made and faces many of the problems that will be met.

Teachers' Guides

Pictorial Representation [1] (1967)
Designed to help teachers of children between the ages of 5 and 10, this Guide deals with graphical representation in its many aspects.

Beginnings ▼ (1967)
This Guide deals with the early awareness of both the meaning of number and the relationships which can emerge from everyday experiences of measuring length, capacity, area, time, etc.

Mathematics Begins ❶ (1967)
A parallel Guide to *Beginnings* ▼, but more concerned with 'counting numbers' than with measurement. It contains a considerable amount of background information for the teacher.

Shape and Size ▼ (1967)
The first Guide concerned principally with geometrical ideas. It shows how geometrical concepts can be developed from the play stage in *Beginnings* ▼ to a clearer idea of what volume, area, horizontal and symmetrical really mean.

Computation and Structure ❷ (1967)
Here the concept of number is further developed. A section on the history of natural numbers and weights and measures leads on to the operation of addition, place value, different number bases, odd and even numbers, the application of number strips and number squares.

Shape and Size ▼ (1968)
Continues the geometrical work of ▼. Examination of two-dimensional shapes leads on to angles, symmetry and patterns, and links up with the more arithmetical work of ❷.

Computation and Structure ❸ (1968)
Suggests an abundance of ways of introducing children to multiplication so that they will understand what they are doing rather than simply follow rules.

Weaving Guides

Desk Calculators ●■▼ (1967)
Points out a number of ways in which calculators can be used constructively in teaching children number patterns, place value and multiplication and division in terms of repeated addition and subtraction.

How to Build a Pond ●■▼ (1967)
A facsimile reproduction of a class project.

The Teachers' Guides, together with *Graphs Leading to Algebra* [2] (1969: see opposite) and *Desk Calculators*, are summarised in *The Story So Far*.

Nuffield Mathematics Project publications appearing May, 1969:

Teachers' Guides

Graphs Leading to Algebra 2

This Teachers' Guide develops the use of coordinates and introduces open sentences and truth sets. It goes on to deal with the graphical aspect of these mathematical statements, introducing graphs of inequalities, intersection of two graphs and graphs using integers.

Computation and Structure 4

The main concern of this Teachers' Guide is with the introduction of the integers $\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, {}^{+}3 \ldots\}$. In the past children have been introduced to positive and negative numbers through the application of these and have been taught 'tricks' for using them in mathematical operations. This Guide builds up the idea of the integers in terms of ordered pairs of numbers before introducing the number line and other applications: this lays a sound foundation for operations on integers. The Guide ends with a short section on large numbers and indices.

Weaving Guides

Environmental Geometry

One of the 'Weaving Guides', this book concentrates on making children more critically aware of shapes in their environment and the interrelationships of them. It deals with relative size and position and with recurring shapes and their properties. It is intended mainly for Infants and lower Juniors.

Probability and Statistics

A 'Weaving Guide' designed to build up, in a very practical way, a critical approach to statistical information and assertions of probability. It demonstrates the many ways in which data can be collected and organised and it attempts to define the criteria for selecting the 'best' way for any given situation. Probability is introduced largely through games, but ways of predicting probable outcomes are investigated in detail.

Other publications

Problems – Green Set

This publication consists of a Teachers' Book accompanied by a set of fifty-two cards for distribution to the children. Two further sets of Problems are in preparation.

The first set of Problems is intended for use with young Secondary pupils. The problems on the cards are reprinted in the Teachers' Book, with solutions and a considerable amount of background material and suggestions for follow-up work. All the topics covered by these cards are included in the Teachers' Guides already published, but they are presented in such a way that children who have not followed a 'Nuffield-type' course can do the problems and enjoy them.

The Story So Far

This booklet is an outline of, and index to, the ground covered by the first nine Teachers' Guides of the Project. Its purpose is twofold: to provide easy references to topics in these Guides for those using them day by day (making a straight index proved an impossible task); and to save teachers of older children having to read through all the early Guides to find out 'what had happened previously'.

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