# Shape and Size



**Nuffield Mathematics Project** published for the Nuffield Foundation by W & R Chambers and John Murray

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## General introduction

The aim of the Nuffield Mathematics Project is to devise a 'contemporary approach for children from 5 to 13'. The guides do not comprise an entirely new syllabus. The stress is on *how to learn*, not on what to teach. Running through all the work is the central notion that the children must be set free to make their own discoveries and think for themselves, and so achieve understanding, instead of learning off mysterious drills. In this way the whole attitude to the subject can be changed and 'Ugh, no, I didn't like maths' will be heard no more.

To achieve understanding young children cannot go straight to abstractions—they need to handle things ('apparatus' is too grand a word for at least some of the equipment concerned—conkers, beads, scales, globes, and so on).

But 'setting the children free' does not mean starting a riot with a roomful of junk for ammunition. The changeover to the new approach brings its own problems. The guide *I do, and I understand* (which is of a different character from the others) faces these problems and attempts to show how they can be overcome.

The other books fall into three categories: Teachers' Guides, Weaving Guides and Check-up Guides. The Teachers' Guides cover three main topics: ● Computation and Structure, ▼ Shape and Size, ■ Graphs Leading to Algebra. In the course of these guides the development of mathematics is seen as a spiral. The same concept is met over and over again and illustrated in a different way at every stage. The books do not cover years, or indeed any specific time; they simply develop themes and therefore show the teacher how to allow one child to progress at a different pace to another. They contain direct teaching suggestions, examples of apparently un-mathematical subjects and situations which can be used to develop a mathematical sense, examples of children's work, and suggestions for class discussions and out-of-school activities. The Weaving Guides are single-concept books which give detailed instructions or information about a particular subject.

The third category of books, as the name implies, will provide 'check-ups' on the children's progress. The traditional tests are difficult to administer in the new atmosphere of individual discovery and so our intention is to replace these by individual check-ups for individual children. These are being prepared by a team from the Institut des Sciences de l'Education in Geneva under the general supervision of Piaget. These check-ups, together with more general commentary, will be issued in the same format as the other guides and, in fact, be an integral part of the scheme.

While the books are a vital part of the Nuffield Mathematics Project, they should not be looked on as guides to the only 'right' way to teach mathematics. We feel very strongly that development from the work in the guides is more important than the guides themselves. They were written against the background of teachers' centres where ideas put forward in the books could be discussed, elaborated and modified. We hope very much that they will continue to be used in this way. A teacher by himself may find it difficult to use them without the reassurance and encouragement which come from discussion with others. Centres for discussion do already exist and we hope that many more will be set up.

The children's work that has been reproduced in these books, like the books themselves, is not supposed to be taken as a model of perfection. Some of it indeed contains errors. It should be looked upon as an example of work that children *might* produce rather than a model of work that they *should* produce.

## Foreword

The last few years have been exciting ones for teachers of mathematics; and for those of us who are amateurs in the subject but have a taste for it which was not wholly dulled by the old methods that are so often stigmatised, there has been abundant interest in seeing the new mathematical approach develop into one of the finest elements in the movement towards new curricula.

This is a crucial subject; and, since a child's first years of work at it may powerfully affect his attitude to more advanced mathematics, the age range 5 to 13 is one which needs special attention. The Trustees of the Nuffield Foundation were glad in 1964 to build on the forward-looking ideas of many people and to set up the Nuffield Mathematics Project; they were also fortunate to secure Dr. Geoffrey Matthews and other talented and imaginative teachers for the development team. The ideas of this team have helped in the growth of much lively activity, throughout the country, in new mathematical teaching for children: the Schools Council, the Local Education Authority pilot areas, and many individual teachers and administrators have made a vital contribution to this work, and the Trustees are very grateful for so much readiness to co-operate with the Foundation. The fruits of co-operation are in the books that follow; and many a teacher will enter the classroom with a lively enthusiasm for trying out what is proposed in these pages.

**Brian Young**
Director of the Nuffield Foundation

## Contents

## I Introduction

This guide is the first concerned specially with 'Shape and Size' though there is much preliminary material in *Beginnings* ▼.

Many expressions, such as 'spatial relationships', 'learning about shape and size and place', 'geometrical ideas', are used in current discussion on primary school mathematics and sometimes the word geometry is used to embrace all these popular terms.

This guide consists of three sections:
I. A general introduction,
II. A conspectus of the ideas and topics suggested for introduction into the primary school,
III. A more detailed account of how some of these ideas can be put into practice in the classroom, particularly those for young children outlined in the first part of Section II.

A sequel, ▼, will cover further work in the junior school age range, as indicated in the latter part of Section II.

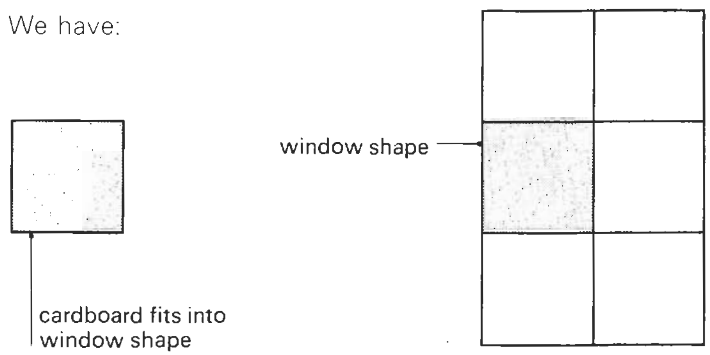
**The most direct applications to the classroom are in Section III, but it is hoped that teachers will first read the introduction. This is important if only to dispel the fear of geometry, and to show that, after all, it does not bring back memories of formal secondary school work – for us, geometry is not theorems.**

**Section II is a panoramic view of the geometrical ideas involved; teachers are advised to refer back to I and II at times, in order to get a picture of the general direction it is hoped the work will follow. In addition, teachers should read *Beginnings* ▼, for many of the activities suggested in it for young children will provide the necessary background experience on which to build the further experience of shape and size as outlined in this guide.**

Geometry is sometimes thought of as investigation or discovery of pattern and relationship in shape, size and place. These are observed in, and derived from, the immediate environment and the much wider world, both natural and man-made. Examples of how discoveries can arise from children's questions and investigations will be the subject of a 'weaving' guide.

As stated in *Mathematics Begins* ❶, mathematics is very much concerned with relationships. A relation makes no sense on its own; examples were given such as *belongs to, was born in the month of.* The idea was then developed that relationships exist between members or 'elements' of sets. Spatial relationships can be considered in a similar way. Suppose Johnny breaks a window at school. As a temporary repair the caretaker fits a piece of cardboard in the window shape.

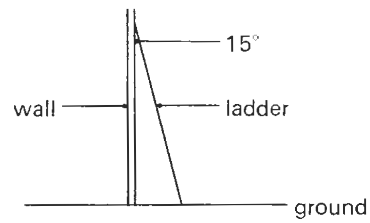
We have:




The cardboard *fits into* the window shape.

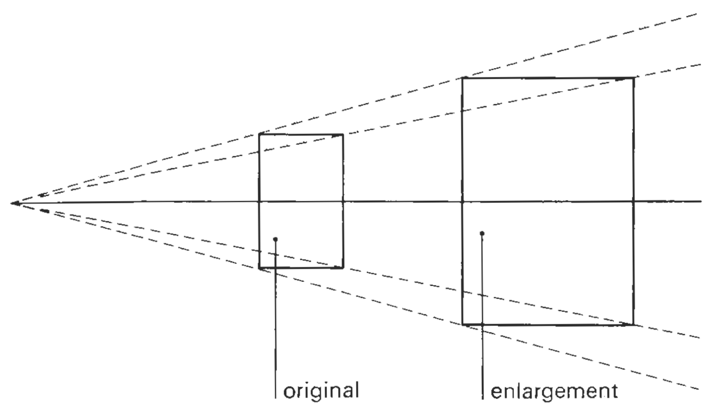
*Fits into* is the relationship between the two shapes. The cardboard fits because the sides and angles correspond to the sides and angles of the window shape.

Later on the glazier uses the same relationship when he fits a new piece of glass. Then the window cleaner arrives. He props his ladder against the wall.




The ladder *is at an angle to* the wall. *Is at an angle to* is the relation. If the angle can be measured the relationship can be stated as 'the ladder makes an angle of 15° with the wall'.

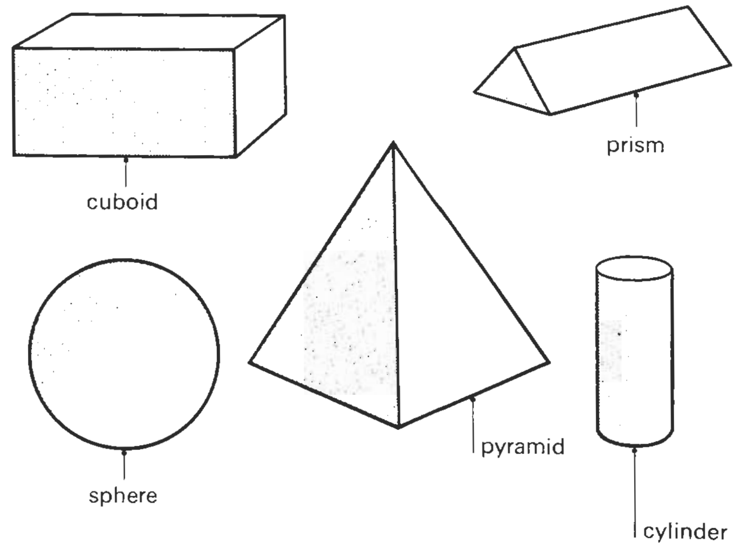
Photographs can be obtained as enlargements of the originals. Later we shall see that there is a relationship in this. The enlargement *is similar to* the original photograph.




This idea also occurs with three-dimensional shapes. Dinky toys and model aircraft, for instance, are scale models of the real things.

These are but a few of the many relationships which will be investigated in work on shapes and size.

We have seen in *Mathematics Begins* ❶ (Chapter 3, 'Early experience of sorting') how a child of two playing with the contents of its mother's handbag is experiencing the idea of a *set*. (A set is a collection of things which are distinguishable from each other and recognisable as belonging to the collection.) This idea of a set can be used to advantage in our study of spatial relationships.




The young child playing with 3D shapes such as cuboids, pyramids, prisms, spheres, cylinders, may find that by building he can stand his cuboid shapes on top of each other in several ways; that his cylinders can be built up one on top of the other but not as easily as cuboids; that pyramids have a 'point', and so on. In doing this he is beginning to sort or classify. Among the set of geometrical solids he is distinguishing the sub-set of cuboids, the sub-set of cylinders and so on. When he looks at 2D shapes, perhaps those which have been noticed as 'faces' or surfaces of the 3D shapes, he finds squares, rectangles, triangles, circles, etc. Some are four-sided shapes (quadrilaterals), some have three sides (triangles), some are circles. If we think of the set of 2D shapes we have the sub-set of all quadrilaterals, all triangles, all circles, etc. Again squares, rectangles, rhombuses, and trapezia, are sub-sets of the set of quadrilaterals.

Having looked at 'relationships' in this general way we can now consider the approach to geometry in the primary school. Perhaps it would be easier to say what our primary school geometry is not. It is *not* concerned with the traditional geometry experienced by so many children at secondary level: and this will lead us to consider why 'our' geometry should be included in the work of the primary school.

Perhaps one of the unfortunate legacies of our own mathematical experiences at school, and of the attitude arising from a traditional approach, is that for many primary school children mathematics is largely concerned with numerical work and little else, while at secondary level it is considered in watertight compartments such as arithmetic, algebra (one), geometry (one), trigonometry, and so on. The very fact that there are aspects other than number relationships shows that mathematics is concerned with other things that are fundamentally important. Primary schools for some time now have realised that the starting point for all their activities and learning springs from, and flourishes best from, experience, and geometrical ideas are part of the experience of all children from their earliest days. Around us, in our everyday life, are shapes of all kinds. Some occur naturally, some are man-made. We see those which are regular and those which are irregular, some display symmetry, others are non-symmetrical. So if the idea of geometry seems too much for some teachers to take because it conjures up visions of a formal secondary school approach, it can just as easily be thought of as concerned with space and shape in the material world, and this world embraces all aspects of a child's life, at home, outside the home and at school. Introducing this idea of geometry will help the child to realise and appreciate that mathematics is something which has grown as man has studied his environment, has tried to describe it and to control it. Geometrical ideas, then, deserve inclusion as just one example of the ubiquity of mathematics.

As … racts the idea of number from his early ex- … of things, so his experience of the …le … bjects will lead to the abstraction … pe and size and space. In addition to this it is a fundamental thesis that children learn most efficiently when discovering for themselves and this aspect of mathematics – the development of spatial concepts, ideas of shapes and size, geometrical ideas, call it what you will – lends itself admirably to discovery, even with the youngest children, and requires little expensive material. For those teachers wishing to make a start at introducing mathematical ideas other than those concerned with number and computation, and to use new approaches to the learning situation, geometrical topics offer a good starting point.

There are three aspects of the 'unity of learning' in which a study of geometry plays its part in the primary school. These can be summarised as the linking of geometry with

a. the rest of the 'mathematics',
b. the rest of the 'class-activity',
c. the rest of the children's lives.

**a. The rest of the mathematics**

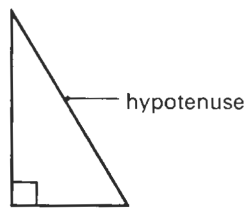
Geometrical ideas can establish a link with other aspects of mathematics, and already it has been noted that spatial relationships are some of the many relationships in mathematics which can overlap and make the subject one whole, not a series of isolated branches. We give a few examples.

i. Patterns with shapes, squares, rectangles, etc., can lead to number relationships and table building and may involve some computation; figurate number such as *square numbers:*

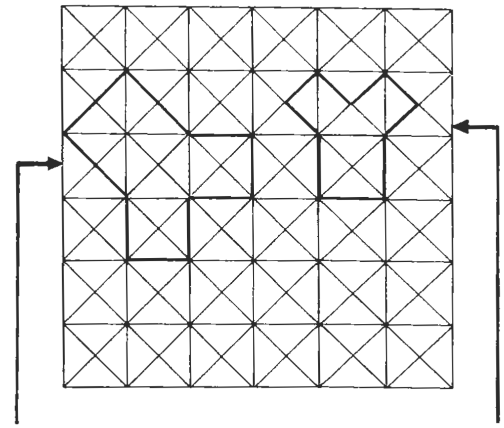
1 4 9 and so on

In order to tell which piece of land belonged to which man the Egyptians had to mark their lands. To do this they needed to use a right angle. To construct a right angle they used the knotted rope to make a triangle with sides 3 units, 4 units and 5 units as shown. There were early attempts at map making, e.g. in the Domesday Book, and Columbus' voyages, which can link history, geography and work on scales and maps.

Earlier it was said that primary school 'geometry' was not concerned with theorems, but with pattern and relationships. But one of the theorems that most teachers will have encountered in their own schooldays, and perhaps remembered, is that of Pythagoras, which states that 'the square on the hypotenuse is equal to the sum of the squares on the other two sides of a right angled triangle'.




History and pattern can come together. For many years before Pythagoras lived, mosaic tile patterns in China illustrated his 'theorem'. This can be linked with discoveries on covering surfaces with tiles and different shapes in our primary school work.



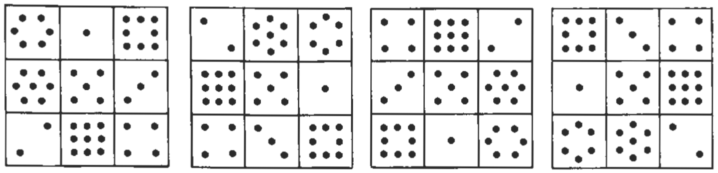
Here the square on the hypotenuse consists of 8 small triangles. On each of the other sides the square consists of 4 small triangles.

For the shaded triangle it can be seen that the square on the hypotenuse is made up of 4 small triangles. The square on each of the other two sides is made up of 2 triangles.

Needlework, too, should not be overlooked, for patterns done by even young children may bring in ideas of symmetry and involve reflection, rotation and translation. (See page 56.)

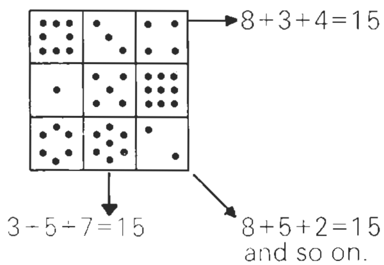
Model making, itself a frequent part of history and geography and other environmental studies, will use ideas of scale and similarity. Wigwams for Indian model villages, for instance, might be made from paper cones with the mathematics arising from the discovery of how to get the height and floor size required. The above are only a few of the links with the rest of the school work, but they serve to remind teachers to be constantly on the lookout for mathematics arising from other work, and just as 'geometry' is a part of the whole mathematics, so it cannot be separated from the whole learning of the children.

The paper is then turned over and these positions are seen:



Starting position. Turning paper through one right angle about centre each time.

For example, the last one could be written:




In fact this is much more than a link with number, for the spirit of elementary mathematics is represented by the fact that any one of the eight positions can always be arrived at from any other by just one of the eight rotations or reflections (see p. 8). The 'eight' doesn't really matter; what matters is the idea that you can always get from some given position to some other given position in just one of a specified series of steps. This will be investigated and discussed in more detail in the sequel.

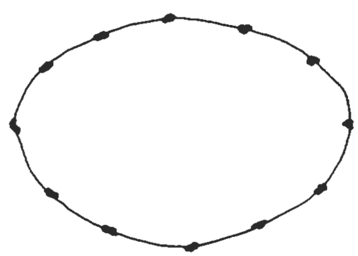
iv. Many of the relationships can be expressed in graphical form, others can lead to generalisation and perhaps an introduction to algebra; symmetry provides some of the preliminary experience of 'motion geometry' and groups for the secondary stage.

The idea of group structure will be discussed in the sequel ▽3 and also in *Computation and Structure* ❸ The basic structural ideas of mathematics are seen even more plainly in a geometrical context than in most others. The point of group structure (with which teachers should not be too concerned at this stage but which is of real importance in ▽3 ) is the fact that it allows us to solve real problems. This is more immediately evident in geometry than in traditional arithmetic, although it is basically present in all elementary mathematics.

**b. The rest of the class-activity**

In the overall pattern of primary school work much of the work on shapes and size can arise from, or be linked with, other branches of learning – something which should always be borne in mind throughout all mathematics at primary school level. In geography and science, for instance, the compass has its connection with the angle, right angle, half right angle and so on. Movement of the earth in relation to the sun leads to shadow variation and height measurement. Maps and scale drawing can start with the immediate environment of the classroom ('maps' of classroom wall as well as floor) and the neighbourhood, and be linked with simple surveying. Distances on the globe will introduce ideas such as the great circle. Latitude and longitude can be considered not as 'angle' but as a special co-ordinate system.

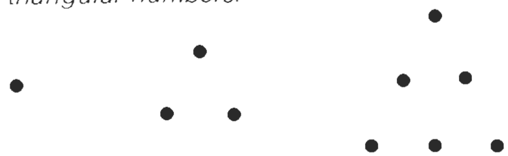
History may bring in the plumb-line and 3, 4, 5 rope used by the Egyptians.



Rope knotted into 12 lengths (made as nearly as possible the same).

Rope stretched to give right-angled triangle.

*triangular numbers:*

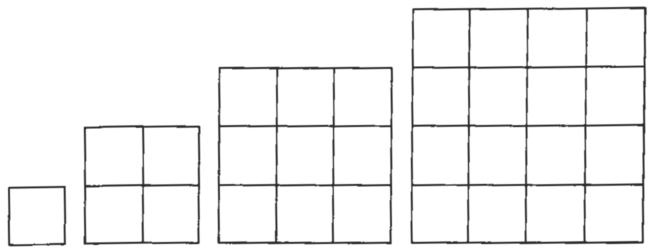


1 3 6 and so on

involve patterns in shape and number.

Examples such as the following:

How many squares can you find in:



| 1 | 5 | 14 | 30 |
|---|---|---|---|
| $1^2$ | $1^2+2^2$ | $1^2+2^2+3^2$ | $1^2+2^2+3^2+4^2$ |

How does the pattern go on?

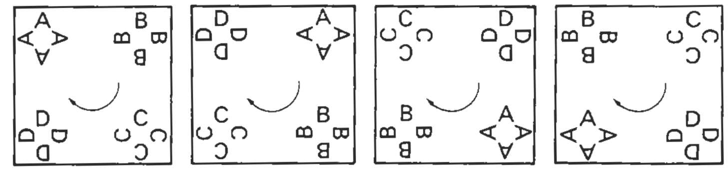
will introduce further number pattern discovery. (For example, the '14' in the above diagram is obtained from 1 square of side three units, 4 squares of side two units and 9 squares of side one unit.)

The way cubes grow can be discovered by building with the shapes, and expressed as

| | | | | |
|---|---|---|---|---|
| 1 | 8 | 27 | 64 | .. |
| or $1^3$ | $2^3$ | $3^3$ | $4^3$ | .. |

Some of these examples will be discussed later in ▼3, and also more fully in *Computation and Structure*.

ii. If a square piece of card is turned about a point at its centre there are four positions where it appears unchanged, i.e. looks like the starting position.

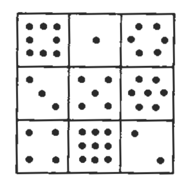


This is an aspect of the *rotational symmetry* of a square. It is a four-fold rotational symmetry. The square card can be turned over and another four positions result. So if we include all the rotations and turning over as well there are eight positions. This is an eight-fold symmetry.

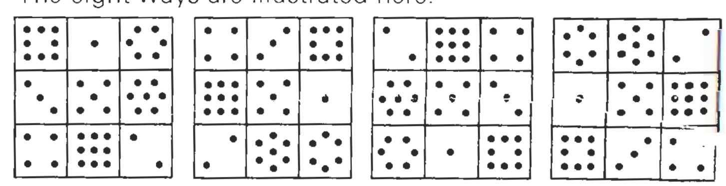
iii. Here we have another link with number.

| | | |
|---|---|---|
| 8 | 1 | 6 |
| 3 | 5 | 7 |
| 4 | 9 | 2 |

This is a '1 to 9 magic square'. The horizontal rows, vertical columns, and the two diagonals give the same total, 15. If this square is printed in dots on tracing paper, or thin writing paper, in the pattern shown and on both sides of the paper so that the dots on each side coincide, eight-fold symmetry can be used to show eight possible ways of transforming this square so that it is still 'magic'.



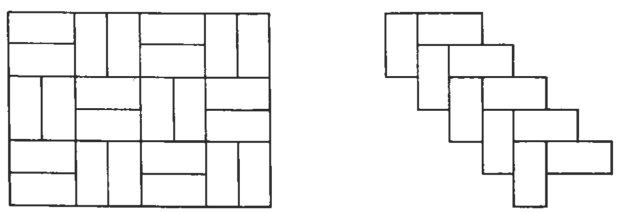
The eight ways are illustrated here:



Starting position. Turning paper through one right angle about centre each time.

### c. The rest of the children's lives

Here an even broader view is taken. Ideas of shape and size are seen as an integral part of mathematics, of the whole school life, and now of the complete environment. In some cases these ideas will arise from things seen and experienced in the child's world and the geometrical ideas drawn from these real things; in others the geometrical ideas may be introduced first and applications of them found outside the classroom. For example, discovery of what shapes occur in a fair-ground, on railways, in the school, in churches, in games such as hopscotch or football, can lead to consideration of why things are a particular shape. Patterns of rectangular tiles seen on floors can be associated with properties of rectangles, symmetry, parallels and so on, and the idea of covering a surface.



These show that such shapes as the rectangle and square used for tiles can occupy the same space in several ways, and can cover a surface without gaps. From this it can be sensed that opposite sides are equal, all angles are equal, and the angles are 'right angles'. Again, patterns of regular hexagons may be seen. These will show that regular hexagons can occupy the same space in several ways and can cover a surface without gaps or overlapping. Their angles, however, are not right angles. Also, regular pentagons can occupy the same space in several ways but cannot cover a surface without gaps. This can tell us something about their angles.

Work on 'strong' shapes, such as the triangle and arch, can lead to discovery of examples of their applications outside the school in such things as bridges and cranes.

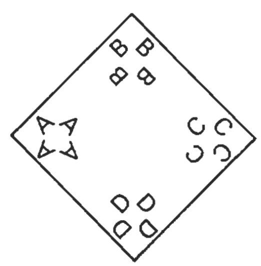
### Summary of aims and approach

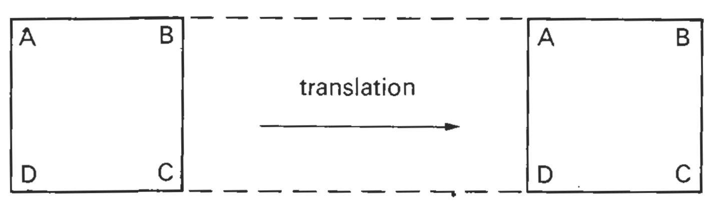
The ideas outlined above are a far cry from the formal approach of Euclid with his 'proofs': the children need a lively, dynamic approach. It is worthwhile going right back to the ages before Euclid and looking at some of the beginnings of geometry. When man began to build he became aware of the need for measuring devices other than those for length. This brought the need for right angles; land had to be marked out, areas calculated, heights of buildings worked out, and so on. These showed the importance of shapes such as the square, rectangle, triangle, circle, and what man could do with them. The properties of these shapes became his tools and these properties were found by using them, by action, by discovery. These experiences inevitably led to a discovery of relationships and rules, to the 'laws' of geometry. Later came the quite different phenomenon of the 'professional', philosophising mathematician looking for proofs in accordance with regulations of his own making. Unfortunately to many teachers the word 'geometry' brings to mind their own experiences at school in what was very much a formal, proof-laden atmosphere, and this in turn leads them to start off with their primary school children with lessons on the 'techniques' of using ruler, compasses, etc., and with constructions of what they call 'geometrical figures'. Our approach should be 'doing', discovering, building up experience; only then should it lead from the particular to the general. The manner of approach of the early practical people gives us a lead.

Having looked back to the practical approach of early man, we must not forget all about the geometry that Euclid was concerned with. But we can look at it in terms of movement or other alteration of shapes rather than through the more static, formal approach of 'proof-seeking'. Let us come nearer to the present day and look at what the mathematician Felix Klein said in 1872. To over-simplify, what he said in effect was that all geometry could be thought about by moving or changing a geometrical figure and seeing which properties did *not* alter. For Euclidean geometry we can consider changes or transformations such as reflection, rotation, translation and find out which properties do not change; e.g. if we look at the square and the properties which do not change when it undergoes various transformations we see:



or through different angles e.g.






Under each of these transformations, length, for instance, is unchanged, so are angle and area. Other transformations give different results. If we look back at the enlargement on page 2 we see that it preserves shape, but the area changes. In the primary school we should provide the experience of moving shapes which bring in elementary ideas of reflection, rotation, translation and enlargement, and thus build up a background for further studies of geometry in this way at a later stage. This is a very different approach from the traditional idea of search for 'proof'. It is certainly no less valid and effective as an approach to geometry: at a later stage we can, if we wish, more readily introduce the necessary axioms and the idea of 'proof'.

The discovery at infant level will be quite informal and will spring from immediate experience, but as it progresses through to the upper stages of the junior school it will need to become more directed. There will always be a place for 'going off on another track', or seizing on some sudden interest of the children, but it is now important that there is progression, that the discovery is purposeful, has a meaning and does not just become random and unrelated activity. This throws an additional burden on the teacher, and, if these activities of the children are to be of real benefit, he must make himself familiar with the mathematical ideas and experience them for himself. Planning and arranging the situations for discovery, taking advantage of interest, are not enough. To see what and how the children are thinking there must be discussion and talking *with* them, and this will often involve asking the vital question which starts off a new line of thought.

This part of the teacher-child relationship is at least as important as the provision of the opportunity for discovery. It will inevitably demand a fresh look at the organisation of classroom work, and this is considered in the guide *I do, and I understand*.

## II Conspectus of ideas

The foregoing has served as general introduction. Let us now consider some geometrical ideas which might be included in primary school mathematics, first in the context of a general guide to the overall work, showing how various topics develop and progress, and then in more detail. This section is in essence a summary of the geometrical ideas which will be covered from the earliest years at school to the final years in the junior school. Little mention is made of where and how these may arise from the environment, or have everyday applications, as this will be developed in the section which gives suggestions for the actual work and activities to be carried out by the children. This summary is followed by a diagram to show how topics are linked together, and in the third section suggestions for children's work follow this 'flow diagram'.

Although not losing sight of the fact that an integrated geometry is part of the whole mathematical experience of the child, we are bound to find that at any one time separate topics are being developed and emphasised. As already stated, in some cases this work will begin with shapes in the environment and from these we shall find the basic geometrical shapes and their properties, in others we shall start with the shapes and see where these occur or are used in the environment. In the course of the discovery of properties of 3D and 2D shapes certain fundamental ideas such as volume, area and angle will be included as these are necessary for a wider study of shapes. The approach to these will begin in a qualitative way, to introduce the basic notions, before considering the quantitative aspect of measurement.

With the discovery approach it is obvious that no set scheme or syllabus of activities or topics can be laid down to fit into a particular age group, but there should be progression, and this really implies that one stage of some particular topic leads on to a further discovery, or the further development of a concept, which in turn may serve as a starting point for the next stage.

With infants most of the activities will be associated with playing, handling and looking at solids, building with bricks, models and toys, etc. This is most valuable experience for the child to familiarise himself with three-dimensional shapes. His immediate environment is largely *three*-dimensional and so we should use this as the starting point for all the work which is to follow throughout his primary school days; this in itself may be a radical reversal of the teacher's own learning experience at school. Three-dimensional space will be experienced through such things as the Wendy House, filling boxes and containers, making houses, ships, aeroplanes, etc., from large boxes and blocks, making them larger or smaller by adding or taking away, building with cuboid shapes, piling bricks, cylinders, on top of each other, fitting boxes inside larger boxes and so on.

There will be no question as yet of measurement in this general experience of 3D space, the experience of filling it and of varying it by increasing or decreasing its dimensions. This may be followed by the first notions of length. Suggestions for activities concerned with this are included in *Beginnings* ▼, to which reference should be made.

Again there will be a pre-measuring stage to widen the vocabulary, by considering *tall* things, *long, wide, deep,* etc., and then by using such things as varying lengths of ribbon to begin comparing lengths by matching. This can be extended with other materials, such as coloured spills and paper streamers, to introduce further comparison and the idea of *shortest, longest* and so on. At the same time two-dimensional surface will be explored by covering. Laying a table-cloth on a table, covering paper with paint, covering surfaces of models and boxes with paper are useful experiences for finding out that surfaces are different sizes. All the above is pre-measuring activity to widen the experience of shapes, and the vocabulary associated with them, and to clarify the use of such words as 'big', 'biggest', etc.

Just as comparison of 3D spaces and shapes is the first stage, leading later to measurement of these, so the same sequence applies for lengths and 2D surfaces. For 3D spaces, containers, boxes, cans are filled with marbles, balls, bricks, cubes; cups with sand or water; large boxes with smaller boxes and so on. Length is measured with ribbon, tape, string, spills, paper; surfaces are covered with dusters, books, sheets of paper, non-rectangular shapes, tiles, etc., and results expressed in these units. 'Home-made' units of measurement will be extensively used before any attempt is made to introduce the standard units.

The first ideas of symmetry can be introduced with young children through patterns such as 'blot patterns', paper-folding and paper-cutting, and further examples of simple 'balance' looked for in nature, e.g. leaves, butterflies; and from activities with bricks and tiles. The notion of symmetry will recur throughout the guides; it will help to classify shapes and suggest geometrical properties.

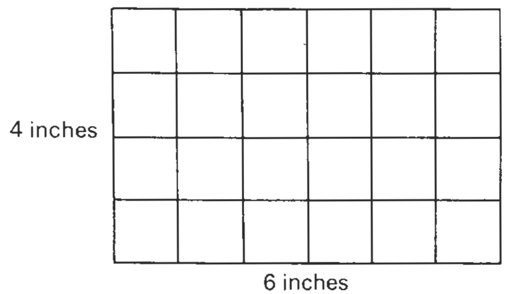
In addition to the usual cube-shaped blocks, other three-dimensional shapes should be available in a 'sorting box', and should include bricks with rectangular cross-section, cylinders, pyramids, prisms and so on. These can be selected into different categories such as those suitable for building, by piling, by shapes of surfaces and so on and the first stages of classification have started. From this and collections of objects brought to school a further vocabulary will be built up to include names of shapes – cube, pyramid, cylinder, sphere, square, rectangle, triangle, circle, hexagon and so on. Young children will also have activities which provide experience leading towards 'conservation' through modelling with clay and plasticine, and putting together two-dimensional shapes; ideas for these are suggested in *Beginnings* ▼.

As stated in *Beginnings* ▼, the idea of 'conservation', or we might say *invariance*, is an important one in concept formation. By this we mean that quantity retains its quantitative value no matter what other changes, e.g. of shape or position, may be imposed on it. In *Beginnings* ▼ mention was made of invariance of length; in *Mathematics Begins* ❶, of invariance of cardinal number. In the particular concepts with which we are concerned here, namely volume and area, the formation and understanding of the concepts depend on the establishment of the idea of invariance, e.g. that a set of solid objects such as bricks can be built into different shapes and the volume occupied by them will remain invariant; that two-dimensional shapes such as a set of tiles can be moved into different positions and the area or surface covered will not be changed.

The ideas of space occupied by a 3D shape, usually experienced by young children by filling 3D spaces, should be developed as they progress into the junior school, in the first place by a continuation of measurement with cupfuls, boxfuls, etc., leading up to the need for standard units, such as the cubic inch for volume and the pint for capacity. Irregular and regular shaped containers will be compared and measured, using these units, but *no attempt ought to be made at this stage to introduce formulae* (e.g. V=LBH for rectangular sectioned containers). Measurement of volume or capacity by counting the number of units used will continue for some time, *discovery* of the generalised way to calculate volume of cuboid shapes being left to the upper stages of the junior school. A similar approach to area will *follow* the absorption of the concept of volume. This is partly because the idea of a 3D space and of filling it seems relatively easy for a child to grasp. It is also possible to come down to 2D by considering the surfaces ('faces') which bound the 3D shapes. This is important because many children at a later stage seem to associate area with 'flat' 2D shapes, and usually only rectangular shapes at that.

**Children of twelve or thirteen, and even older, frequently confuse perimeter and area, and have ill-conceived notions about area and volume in general. This may be largely due to thinking in terms of linear measurement, confusion about units, and above all to a *too early* introduction to area of rectangular shapes incorrectly depicted as 'Length × Breadth', and the volume of cuboid shapes introduced first by way of a formula, such as V=L × B × H. *The most important thing is that all this comes later than the developments described in the first part of Section III.* Statements such as '6 inches × 4 inches = 24 square inches' are frequently used by teachers and hence by children as a way of finding area**

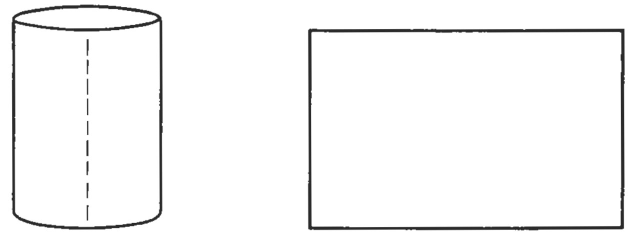
of rectangular shapes. In fact what we really do is to multiply two *numbers* to arrive at a quick way of finding the answer instead of counting squares. The *number* of inches in the length is multiplied by the *number* of inches in the breadth.



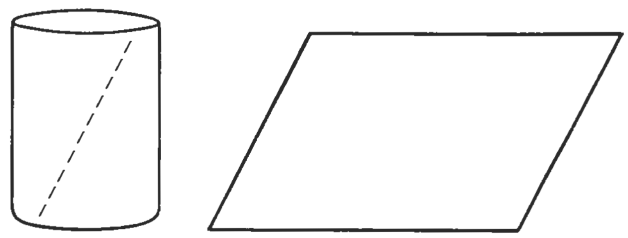

In the case shown in the diagram we know that the real length, for practical purposes, is 6 inches, and the breadth is 4 inches. So we create a 'mathematical model' of the situation and say $6 \times 4 = 24$. We then return to the real and use this model to say that, for practical purposes, the area is 24 square inches. $A = L \times B$ should only be used as an aid to memory when the idea of the generalised way of calculating area has been discovered and understood.

Teachers should emphasise at all stages that the measure of area is that of surface covered or alternatively the amount of surface an object has, and that any measurement actually made is an approximation.

Area can be developed by extending the earlier work on covering surfaces such as table tops, floors, curved surface of cylindrical tins, faces of boxes, using 'home-made' units for comparison and measurement. This will lead to discussion on the need for standard units and then these will be used to find approximations to the areas of irregular shapes such as leaves, soles of shoes, and so on, by counting. (Indeed, the irregular shapes could well come first.) Much later will come the emphasis on rectangular shapes and the *discovery* of the generalised way of calculating such areas. A useful extension of this work is for surfaces on cylinders to be 'opened out' and for the faces of boxes to be considered in the same way.



Cylinder cut vertically and 'opened out' produces a rectangle.



Cylinder cut at an angle not perpendicular to base, and 'opened out' gives a parallelogram.

This cut is difficult to make on a rigid cylinder. The easier way is to make a cylinder of thin card or stiff paper and then to squash it flat before drawing the line and cutting.

This can be used to 'discover' that under certain conditions ('on the same base and between the same parallels') a rectangle and a parallelogram can have the same area.

Symmetry will be developed through more pattern work, such as paper-folding and cutting, potato prints, and everyday occurrences such as laying the table. (See pages 48 to 56.)

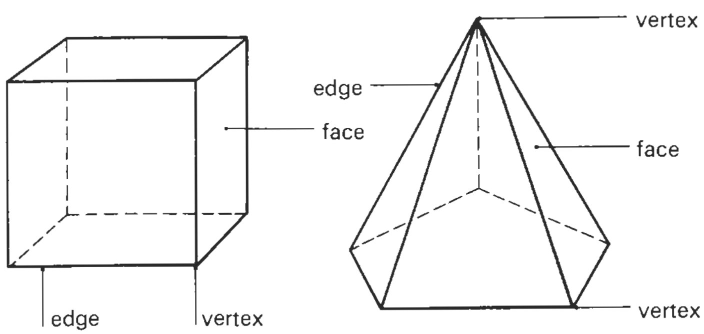
These activities will involve shapes being moved by such methods as folding them and turning them over, turning them round into new positions, moving them along equal distances in the same direction. These movements have already been referred to on page 8, and will be developed more fully in 3. But at this early stage the work will involve a great deal of observation of symmetry and pattern, and on this more precise mathematical ideas will be built later on.

The symmetry involved in fitting together regular 3D cuboid shapes, and 2D rectangular shapes will also introduce the 'square corner' or right angle, and the first 'static' notion of angle.

Soon children will see actual rotation through an angle and find a measure for the amount of turning.

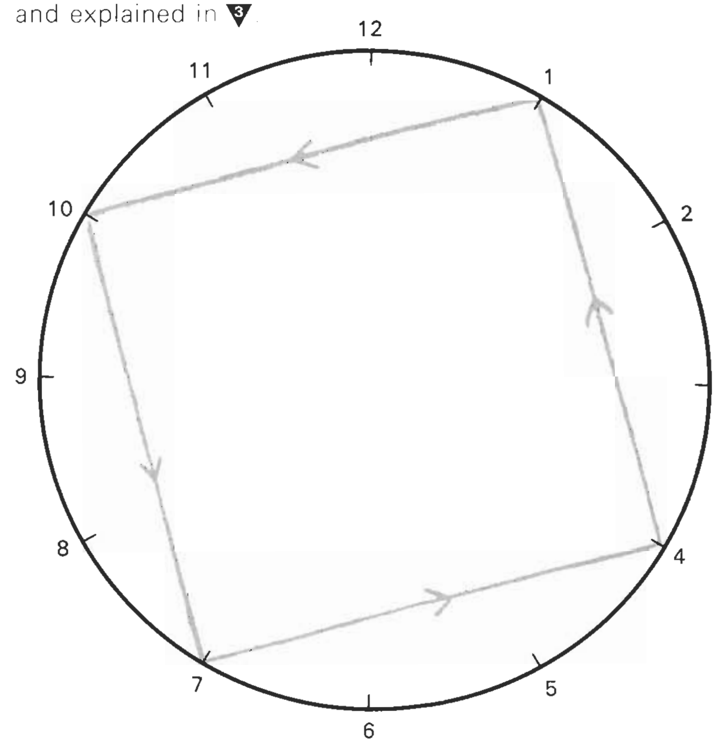
*Horizontal, vertical* and *perpendicular* will develop further aspects of the right angle. The shortest distance from a point to a straight line (again involving a right angle) can be discovered through experiment.

So far the work in the infant school and lower junior school has been concerned with general ideas about 3D and 2D shapes, some sorting and classification, increasing vocabulary, associating these shapes with everyday things, the first notions of volume, area and angle. In some cases we consider activities with three-dimensional shapes first, e.g. filling containers which follows on naturally from the earlier activities, and fitting bricks together. In others it is easier to see something in a two-dimensional setting, e.g. the right angle. This experience and background can now be used at about middle junior school age to look at 3D and 2D shapes in a more detailed way. Vocabulary associated with solids (such as cubes, pyramids) will be introduced. The words *face, edge, vertex* will all be used and there will be investigation of solids with faces of different shape. It will be seen that these shapes are related to the number of edges, e.g. triangle, quadrilateral, polygon.




Further study will show that different sorts of quadrilaterals appear as quadrilateral faces of solids, e.g. square, rhombus, rectangle. This is a refinement of the earlier sorting experience of young children and involves the discovery of more shapes, increasing vocabulary, and the finding of ways in which shapes differ or are alike.

Having looked at 2D shapes by way of discovery from 3D shapes we shall further investigate these as frameworks using metal or card strips, straws, wooden rods. Properties of triangles will arise and also further work on the comparison of different quadrilaterals. Discovery of the rigidity or otherwise of 2D shapes will occur and so will the idea of parallel lines, both ideas being linked with examples in the everyday world. Pattern making on paper, and with stitching or coloured lines on circles, will help to bring out characteristics and properties of some of the shapes studied so far, and can be associated with work on 'modular' arithmetic (cf. ❷, Chapter 6). These ideas will be amplified and explained in ▼3.



As a preview, the square in the clock diagram is obtained by starting from 1 and successively counting on 9 clockwise. In turn we reach 1, 10, 7, 4, 1, 10, 7, 4, 1, . . . . , and joining up we obtain the square. More interesting shapes are obtained by counting on, for example, 7 each time.

Blot patterns, paper folding and cutting, leaves, etc. will have introduced the idea of simple balance or reflection (one axis of symmetry), and activities with squares and rectangles have given experience of more than one axis of symmetry. These ideas of symmetry can now be extended to include patterns, shapes, letters and so on, with more than one axis of symmetry, and the fitting of shapes on to each other by turning them over. Some simple discoveries will be made about shapes which can be turned round as a first introduction to rotation. The basic idea of rotation was illustrated in the Introduction to the guide, and will be explained in more detail in ▼3.

The discoveries about axes of symmetry can be brought into the children's work on angles. Paper folding can produce compass cards with half right angles, and the idea of the arms of an angle being moved to show half right angles and multiples of half right angles will be associated with this. The pattern work with 2D shapes and the idea of covering surfaces that has been associated with area will lead to discovery about the most suitable shapes for covering a surface, to work on tile patterns, to angles associated with parallel lines; experience will be gained that can be used later when considering the sums of angles of triangles, quadrilaterals, and regular polygons.

All this previous experience of 3D and 2D shapes will be used for further discoveries more suitable in general for older juniors, but in some cases only suitable for the more able children at this stage of the school. Three-dimensional shapes will be made as solids by paper folding, or as frameworks by using wooden dowel rods, or straws joined with pipe cleaners. This work should be based on looking at such solid 3D shapes and endeavouring to evolve ways of making them from the children's own ideas, not by copying nets involving standard constructions with compasses and ruler. The discussion of relationships between numbers of faces, edges and vertices can arise from this work. The observation of regular solids will lead to further work on faces, pointing out ideas of congruence and similarity. Moving congruent shapes can introduce first ideas of translation. Similarity may appear from the study of photographic enlargement and model making, and can eventually be extended to include work on height finding.

Previous work on triangles, using axes of symmetry, tile patterns, paper folding, can be used to investigate the sum of the angles of the triangle, proceeding from particular cases to a general conclusion, and to discovery about area of a triangle, and of the sum of angles of polygons, expressed in right angles. Further work on angles will be introduced to show measurement in degrees, using the clock face, and introducing paper folding to divide the circumference of a circle into 6, 10 and 12 parts. This will be associated with turning involving change in direction, and angles as fractions of whole turns. A simple protractor may be introduced at this stage and used for surveying and for plotting bearings. This could to advantage be a circular protractor with degree markings in 10-degree (or possibly 5-degree) intervals with no numerals printed on it.

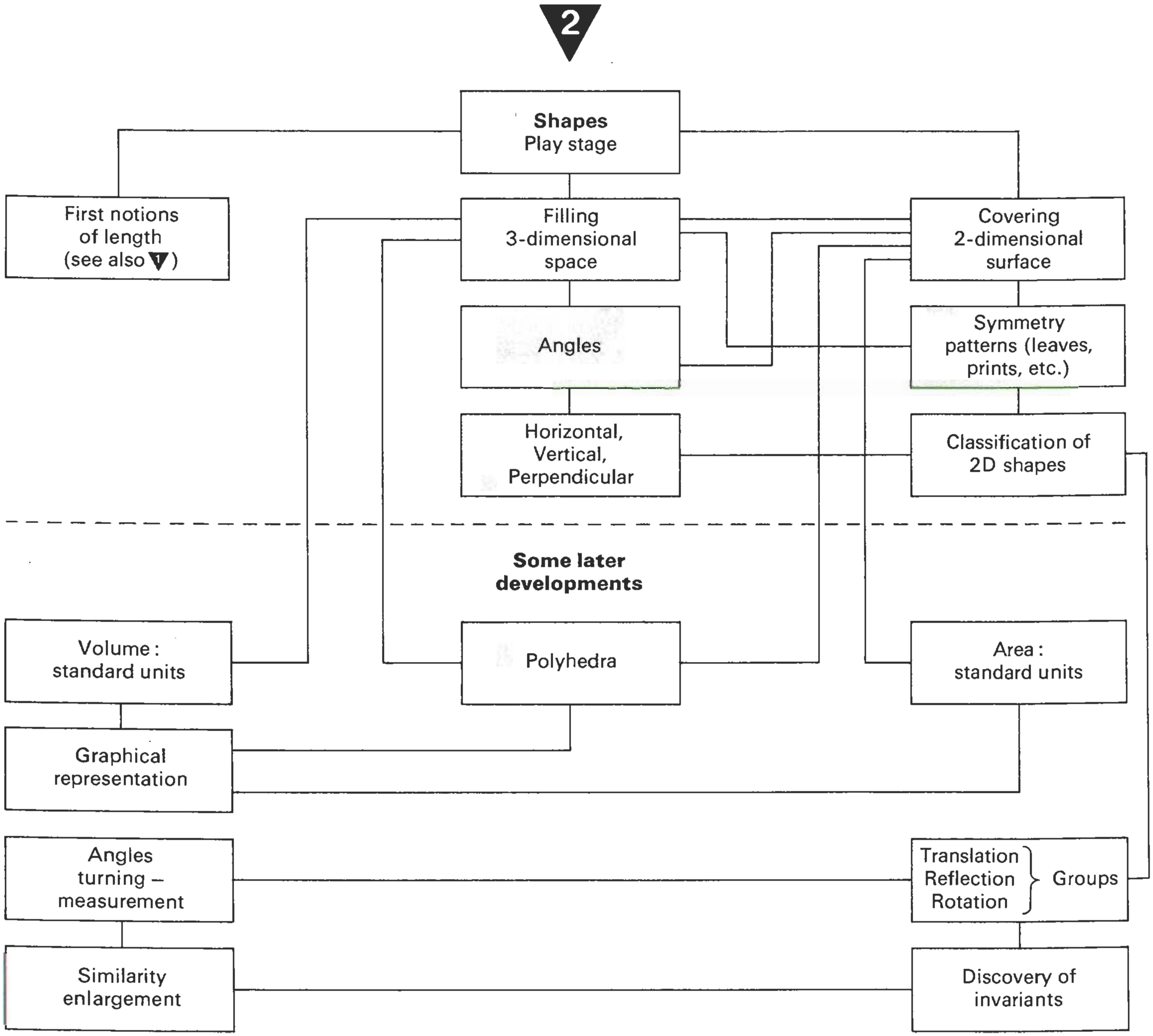
Cuboid shapes can be used for more work on measurement of volumes and discovery of the generalised way of calculating such volumes, and for investigating the growth of a cube. This can be linked with number work and graphical representation. Surface area of cubes can be found and the surface area per unit volume investigated and related to applications in everyday life. Area of rectangular 2D shapes, and the discovery of the generalised way of calculating area will develop in a similar way. The growth of a square can also be associated with number work and graphical representation (see *Computation and Structure*). Similar investigations can be carried out by fitting together triangles to make larger triangles, and rhombuses to make larger similar rhombuses.

Further study of 3D and 2D shapes, and their dimensions will lead to discovery of such relationships as maximum area for constant perimeter, circumference and diameter of circles, area and diameter of circles, perimeters of rectangles of constant area, all of which will provide material for graphical work going on at this stage of the junior school.

At this level it is now possible to develop the idea of symmetry further and link it with ideas going on in number work. Rotational symmetry will be investigated first through everyday objects and then by fitting a square into a square framework without turning it over. This can be associated with work in modular arithmetic, and the first ideas of group structure introduced which, as already mentioned, will be explained in ▼3. This can be extended to turning the square over as well, and to similar work with rectangle and equilateral triangle. Some children may be able to progress to the first stages in work on the three transformations of reflection, rotation and translation, and to discover which properties do not change under these.

At all times indications will be given to show where any particular piece of work or discovery fits into the overall pattern, and advantage will be taken of mathematics which can arise from other school work.

On the next page is a diagram showing the main topics which will arise in this guide, together with some indication of later developments.


2
Shapes
Play stage
First notions
of length
(see also 1)
Filling
3-dimensional
space
Covering
2-dimensional
surface
Angles
Symmetry
patterns (leaves,
prints, etc.)
Horizontal,
Vertical,
Perpendicular
Classification of
2D shapes
Some later
developments
Volume:
standard units
Polyhedra
Area:
standard units
Graphical
representation
Angles
turning –
measurement
Translation
Reflection
Rotation
Groups
Similarity
enlargement
Discovery of
invariants

## III Children's activities

The suggestions outlined in this section are largely intended for the younger children in the junior school, and in some cases overlap and follow suggestions given in *Beginnings* ▼1 A sequel, *Shape and Size* ▼3, will contain suggestions for some of the older children in the junior school.

The activities suggested here are set out in such a way as to show progression in each topic, and are intended for working by small groups of children. A suitable group is three or four children. In this way there willl be co-operation, with children helping each other through discussion among themselves. Recording will vary; in some cases each child will make his own attempt, in others a joint piece of recording may serve best to bring out the discoveries. Frequently there is a 'spiral' treatment of a topic, returning to the subject at different levels of development. The reasons for this are (i) to build on previous experience and thus to amplify and provide progression in understanding, and (ii) it is sometimes of value to look at something in retrospect. Some point which may not be fully grasped initially will frequently become simplified when tackled at the next stage. And, of course, the children will be older and more mature in appreciation.

There is one final aspect to the suggestions for activities which is of great value in assessing understanding and that is for children to make up their own assignments and problems for each other to try after having worked through the ideas suggested for their group. It has the added advantage of confirming and extending the learning of the child who has to think out the problem for another to do. Exercises which are purely personal may not sustain interest. Thinking out a problem for someone else involves the imagination and thought not only of the child attempting to solve the problem, but also of the one who has to compile it.

Some of the topics will involve looking at things in the real world outside the school and associating these with the discoveries in the classroom. At times the reverse may apply. For instance, one of the assignments involves fitting blocks and wooden bricks together in the classroom and then finding how this occurs in buildings, floors, and so on. But this could arise the other way. A general study of a building site might lead to discussion on the ways bricks fit together, and in this case the teacher can refer to the assignments in this guide to see where mathematics can be used. Although suggestions for this approach are not given in detail teachers should always be on the lookout for such instances arising.

### 1 Filling three-dimensional space

#### Volume and capacity

Suggestions for activities leading to the ideas of volume and capacity are put first because they afford a natural overlap with activities suggested in *Beginnings* ▼1 (e.g. filling containers), and in fact may be a duplication of some of these. This will assist teachers in making the progression from one guide to the other.

We have seen in *Beginnings* ▼1 that many of the activities of young children are associated with three-dimensional objects: bricks, blocks, toys, models, containers, and so on. The first ideas of volume and capacity can develop from this and so will be considered before area. It is also stated in *Beginnings* ▼1 that, before children were given activities which involved quantitative comparison of length, volume, capacity and area leading up to measurement, they should have achieved a grasp of the idea of 'conservation'. Conservation of volume, it will be remembered, is the term used in connexion with acceptance that the space filled remains unchanged despite the rearrangement of the parts, or of the shape. Activities to give experience from which this understanding will develop are given in *Beginnings* ▼1 and reference should be made to this.

As with measurement of length where children use a variety of material for measuring in different ways before meeting standard units, so with 3D shapes they should have plenty of activities involving the two aspects of volume (as the amount of space a solid occupies, and as the amount of space in a hollow container, alternatively referred to as capacity) without thinking about measurement in standard units.

Invariance of volume is discussed in *Beginnings* ▼ and, as a check that children have understood this, here are three tests designed for individuals rather than groups.

The first refers to the volume of water being unchanged when poured into different jars, and the others to the volume of a solid shape made of cubes being unchanged when the cubes are rearranged.

1. For this you will need: a jug of coloured liquid, two glass beakers of the same size and shape; one jar taller and narrower than the beakers; one jar shorter and wider than the beakers.

Place the two beakers of same shape and size in front of the child. Pour some coloured liquid from a jug into one beaker until it is about half full. Ask the child to pour the same amount from the jug into the other beaker. Ask, 'Have the two beakers got the same amount of liquid in them?' 'How do you know?' Then ask him to pour the liquid from his beaker into the tall jar (or shallow dish). Put this by the side of your beaker containing liquid. Ask whether your beaker and the tall jar have the same amount of liquid in them.

If he says yes, ask him why he is sure.

If he says no, he needs more experience with pouring and filling various containers with liquid, at frequent intervals. Suggestions for these activities are given in *Beginnings* ▼.

2. You will need 24 one-inch cubes.

Ask the child to make two blocks which look the same, each to contain the same number of cubes. How does he know?

Move one set of cubes to make a new shape. Do the two sets still take up the same amount of space?

If he says yes, ask him why he is sure. Answers which indicate the blocks were the same to start with, with same amount of wood in each (or similar ideas), are satisfactory and show the child has grasped the concept.

If he says no, then he requires more activities with bricks and blocks, building with the same number and rearranging a number of bricks to make different shapes and so on before he is tested again. Suggestions for a sequence of such activities are outlined in *Beginnings* ▼.

### Third Test

Material required:

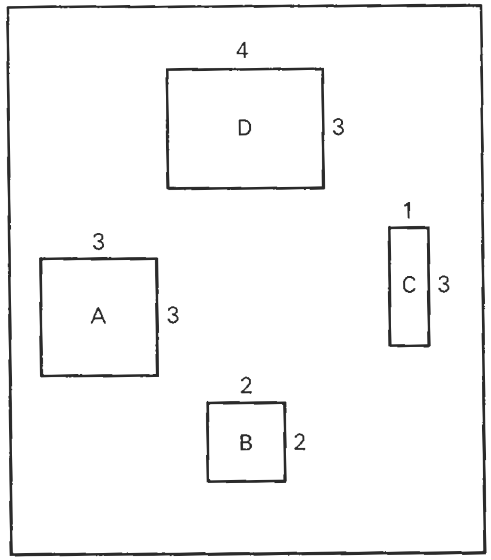
a square of card or lino painted blue to represent water such as a lake (about 13 inches by 15 inches);

four 'islands' placed as in the diagram overleaf—use plain card for these;

a 'block' of wood made up of 36 inch cubes to be placed on 'island' A;

144 inch cubes made of wood.

What to do:
Place the board in front of the child with A on the left as in the diagram. Put the 36 inch cubes as a 'block' on position A. Say 'Here is a house on this island. It is built right up to the edge of the water, but it is not in the water.'




Test:
'Use the other cubes. Can you make a house on this island (point to B) so that it takes up the same space as the first (on A)? It must fill all the island.' If necessary put the instructions as a story, 'The people who live here (point to A) move and go to live here (point to B). They must have just the same space in which to live, no more or no less.' When the child has finished, do *not* ask him to explain what he has done, but go on to what follows keeping the building intact on B.

Note any spontaneous corrections. Allow the child to move or change the 'block' A but *avoid suggesting* that he counts the cubes. If the child spontaneously arrives at a solution by counting, say 'Could you do it another way?'

Leave the building on B and use more cubes. Repeat the procedure, this time for building on C a 'house' with the same space as that on A. Tell the child to use more cubes and repeat for building on D as on A. As previously the building must fill all the surface of the island.

When the three buildings B, C, D are finished say 'Explain (or tell me) what you have done. Are you sure that this house (point to B) contains the same amount of space as that (point to A)? Why?' Ask the same for C and D.

Finally, 'How do you know that you are right?' Allow the child to make any alterations or movements that he wishes. Ask him for his reasons for these. Ask all the questions necessary to find out the procedures used by the child.

If the three buildings are equal in volume to A, stop the activity. If not go on to the following; ask each of these questions:

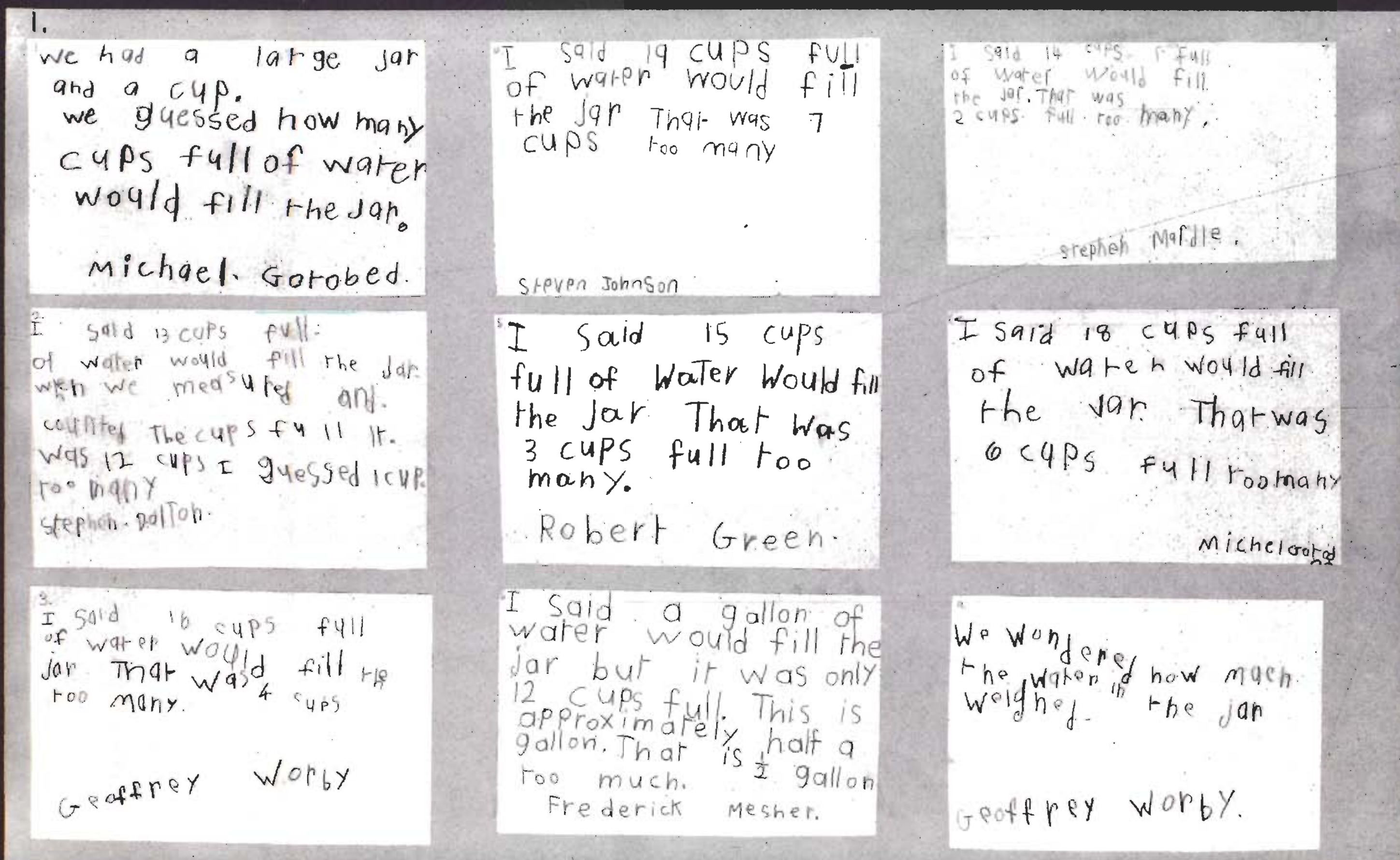
'This house (B) – does it occupy (have) the same amount of space or volume as that (C)?' Repeat for B and D. In the same way compare C and B, C and D and finally D and B, D and C.

When children seem to have satisfactorily grasped the idea of invariance of volume under the above conditions, activities such as the following can be introduced:

> You will need a large jar (e.g. an empty 40-oz Gloy jar) and a cup.
> Estimate how many cupfuls of water you would have to use to fill the jar.
> Then pour water into the jar a cupful at a time until it is full.
> How many did you use?
> How near were you?
> Did you estimate too many or not enough?

(Introduce the word 'estimate' rather than 'guess'. 'Guess' can lead to some light-hearted answers – a boy when asked to 'guess' the length of the school corridor thought it a joke to reply, 'Two miles'.)

The following sequence of children's work shows how an activity which began with filling a jar with water developed into an interesting discovery about weight. This was done by a group of seven-year-old boys in an infants' school. These young children talk more easily than they can write—in some cases they have tried to make a record of their discussion with the teacher, in others the teacher herself has tried to write the main points which came out in discussion.



2. Stephen Said get Some scales that go like this ∪ and Put the water in and weigh it. We havent any scales like this.

Steven Johnson

"I said find something that doesnt weigh anything and Put the water in it. We couldnt find something that didnt weigh anything.

Frederick Mesher

Miss Branney Said If We find out how much the jar weighs....

We weighed the jar. It Weighed 4 Pounds and 4 ounces

Michael said We know how much the jar and water weigh together. 9 pounds and 3 ounces (2lb) (2lb) (1lb) (1lb) (1lb) (1lb) (1oz) (1oz) (1oz) (1oz) We didnt have enough weights. We had to borrow a one pound weight from Miss Wildman and a one pound weight from Mrs Bannerjee.

To find out how much the Water weighed We took away the weight of the jar from the Weight of the jar and Water together.

Robert Green

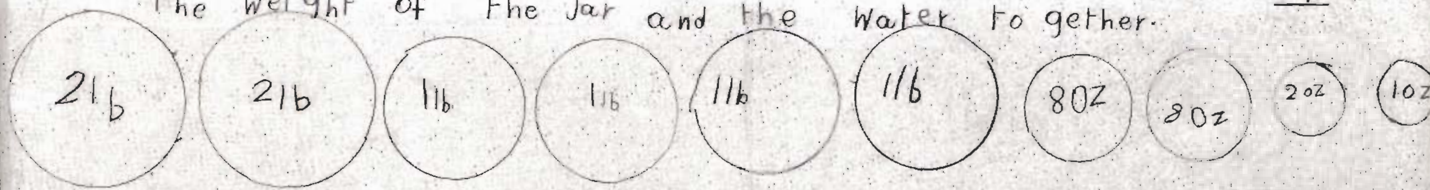
We took two (2lb) (2lb) weights away for the 4 pounds (2lb) (2lb). We couldnt take away 4 ounces be cause there were only 3 ounces there (2oz) (1oz).

Stephen took one (8oz) weight and changed it into two (4oz) weights Then he took away one (4oz) weight and said that makes 15 ounces left The answer is 4lbs 15 ounces

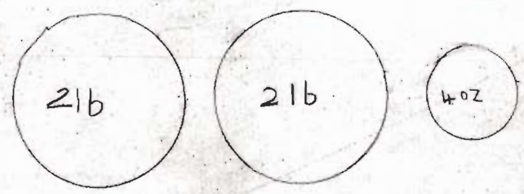
3.

Boys.

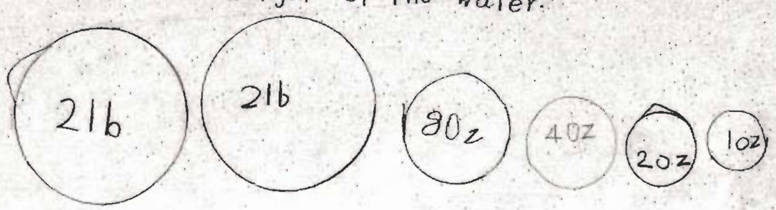
The weight of the jar and the water together.




The weight of the jar without the water.




The weight of the water.



You will need the chalk box, pen-nib box and some sand.
Estimate how many small boxfuls of sand will fill the large box.
Fill the large box with sand by using the small box.
Count how many small boxes full of sand you used.
How good was your estimate?
Was it too many or not enough?

Similar activities can be given using wooden cubes to fill a box, packing small boxes of about the same size into a larger box, and so on.

Other activities:

(Marbles and a jam jar)
How many marbles do you think you will need to fill this jar?
Try it and see.
Do you think marbles are very good for finding how much space there is in the jar?
Why?

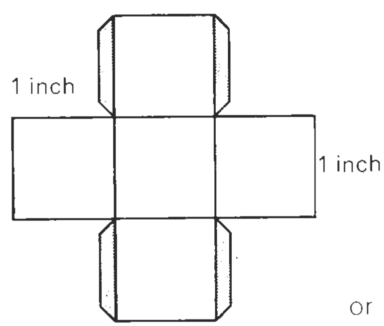
(Discuss this with children, and ask for suggestions for better material,etc.)

(Small balls and a large box)
How many balls do you think you will need to fill this box?
Try it and see how near you were.

Marbles, cubes, and so on can also be used for packing (or trying to fill) containers such as cylindrical tins.

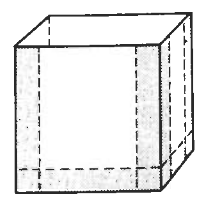
From these activities teachers can discuss the various things used, e.g. marbles, sand, 1-inch cubes for filling boxes and other containers; pen-nib boxes or match-boxes to measure the sand needed to fill a container. Comparisons can be made – for some 3D shapes, e.g. some of the boxes, 1-inch cubes provide an easy way of measuring the amount of space: for others, it is easier to use a small box to measure sand to fill the container. Further discussion can bring out the difficulty of comparing the volume (capacity) of containers using these small boxes, etc., as match-boxes, pen-nib boxes, cups, will vary. Can we have something which everyone can use? (Cf. inches, pints, as introduced in *Computation and Structure* ❷.)

For packing into some boxes the 1-inch cube is useful. When cubes fit without leaving gaps we can count the number used. The amount of space a 1-inch cube occupies is *1 cubic inch*. So in such cases we can measure the amount of space in cubic inches by using these cubes. But what about the gaps when cubes do not fit, or containers such as jars and cylindrical tins? We have seen that these cubes are not suitable. Small boxes were used to measure the amount of sand to fill containers; now we can introduce a 'standard' box. Some children may be able to make an open 1-inch cube of stiff cartridge paper or strawboard to use as a 'box' for this purpose. But where difficulty is experienced by young children it should be made by the teacher.




Paper folded and fastened with an adhesive at the flaps.

or



5 separate 1-inch squares of strawboard fastened with Sellotape.

This 'box' can then be introduced as a measure for filling containers with sand, etc., and the fact that it holds 1 cubic inch discussed. Practice in using this can be given, and children can count the number of cubic inches a container holds. We now have two 'measures', both of which can be used to find volume or capacity: solid 1-inch cubes can be used when they fit suitably, and a 1-inch cube 'box' can be used for filling containers.

Containers such as boxes, jars, tins, etc. can then be compared for size (volume). Methods already experienced by the children can be used: e.g.

i. filling several jars, cans, with water using something such as a cup as a measure,

ii. filling chalk boxes, shoe boxes, chocolate boxes with sand using a small box such as a pen-nib box, match-box, inch cube 'box',

iii. packing boxes with wooden cubes.

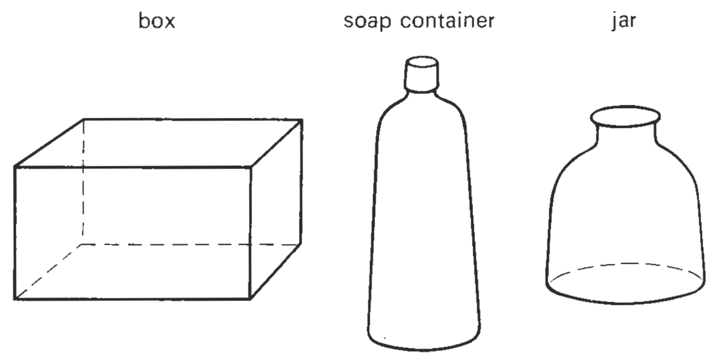
In all cases the children should be asked to use the apparatus or material available to find their own way of arranging the containers in order of size (volume), biggest first, . . . smallest last. And a final question, 'Can you think of another way of doing it?'
(e.g. Weighing, as suggested by some seven-year-olds.)

A development of this is to provide several different-shaped containers, e.g.
a jar, a box, a liquid soap container.

> Find which has the largest volume (or which holds the most) and which is the smallest, and then arrange them in order of size.
> (Children may use any method they choose.
> Discuss this with them afterwards.)
> Could you think of another way? Try it.

The above assignments on comparing the volume and capacity of containers, and arranging them in order of size by volume may be compared with Chapter 6 of *Mathematics Begins* ❶. In fact, children arranging the nesting boxes in order of size as suggested there were doing so by volume. In the activities outlined above children were asked to arrange various containers in *order* of size by volume (capacity).

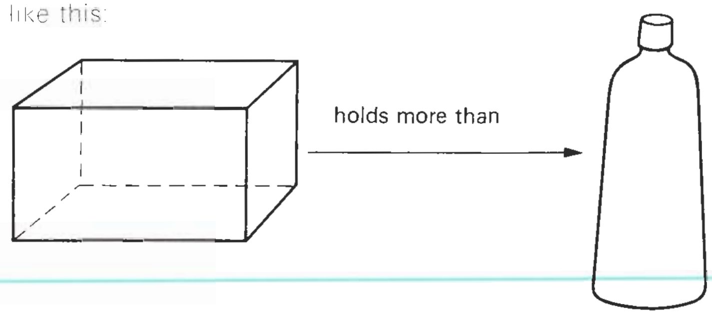
In the diagram we see that given any pair of containers there is a relationship – *that one holds more than the other.*



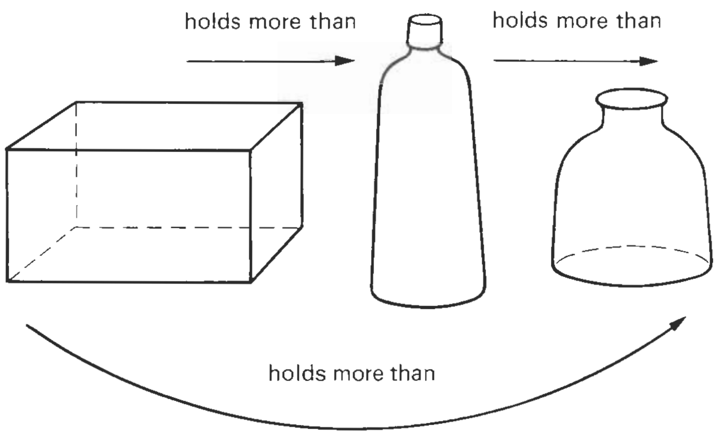

(largest capacity on left – smallest on right)

For instance, the box bolds more than the soap container, and again the soap container holds more than the jar.

The method of recording shown in *Mathematics Begins* ❶ can be used to show the relationship between each pair like this:




These can be ordered, and the relationship shown thus:




In *Mathematics Begins* ❶ reference was made to *transitive* relations and here we have another example. From discovery that the box holds more than the soap container, and the soap container holds more than the jar, we can see that the box holds more than the jar.

If children use the same unit for measuring the amount of sand each container holds, for instance, a small pen-nib box, one way of recording the result might be:

'I used about 15 small boxes full of sand to fill the large box. The liquid soap container took about 13 small boxes to fill it, and the jar about 10; I can see that the large box has the biggest volume, the soap container next, and the jar has the smallest volume. $15 > 13 > 10$.'

Provide boxes with lids on (e.g. chalk box, shoe box) and plenty of wooden cubes.

> See if you can make a shape just like the box, using the cubes.
> Show your teacher when you have finished.
> Do the box and your model take up the same amount of space?
> Are they the same size (volume)?
> How many cubes did you use?
> Now make a different shape using all these cubes.

Discuss examples of *conservation* with the children. The teacher can use these as a check on the understanding of conservation in a way similar to that suggested in the tests given earlier in this guide. Such questions as the following can be asked:
'Did all your models take up the same amount of space? How do you know?'

Reference to pages 16 and 17 will remind teachers about conservation.

> When you have worked through these cards try to make up some for others to do, using the ideas you have discovered.

### Looking back over what has been done

The main points covered have been: the idea of a three-dimensional object or shape occupying a space; the amount of space inside a container; conservation of volume, and activities leading to measurement and comparison of volume and capacity. There was preliminary discussion on the need for a standard unit. *Note that no mention has been made of calculating volumes of cuboids, etc. by using linear dimensions.*

But before considering the further development of the use of standard units another aspect of filling a three-dimensional space will be considered.

### Which shapes fit together best?

This will be considered from several aspects – both to provide early experience on which to build ideas of symmetry, experience and discovery of properties of 'regular' shapes 3D and 2D (such as the square and rectangle) and square corners or right angles, and as preparation for discovery at a much later stage of the generalised way of calculating the volume of cuboids, etc.

For these activities a supply of solid shapes of different kinds is required and there should be at least two dozen of each type and size for a group of four children. The shapes needed are cubes, small rectangular bricks, pyramids, cylinders, prisms, spheres. Some of these are easily obtainable, such as 1-inch cubes, others such as small bricks with rectangular section can be cut from lengths of wood (e.g. a 6-foot length of 2-inch by 1-inch), sandpapered and painted. Small tins of the same shape as each other, or pieces cut from broom handles, will serve as cylinders. In many cases the work can be done with the larger bricks, blocks, etc., found in most infant schools. Assignments are given to children from which they may discover which shapes fit together best without leaving a gap, and may find some of the properties of these shapes.

Examples:
A group of children will have, say, twenty-four of each shape of brick or block of a given size (cube, cuboid, prism, cylinder, etc.)

> Use all the blocks of one kind.
> Try to build a wall which has at least two thicknesses of brick.
> Do this with all the shapes. You may put the bricks in any pattern you wish.
> Which shapes did you find were the easiest for building walls?
> Why do you think this is?

The children should have found that the cube and rectangular shaped bricks were the best. This is because these bricks can occupy the same place in several ways. They have 'square corners' and parallel faces which fit together, and opposite faces of the same shape and size. House bricks, toy building bricks, constructional games use these shapes for this reason. Children will come across many examples, particularly in their toys and games, and so these 'regular' shapes are considered before irregular shapes.

The development of this discovery, that cuboid shapes fit together best, will be to find out more about these shapes, and some assignments are given below.

> Build a wall two bricks thick using the cubes.
> Can you take out any brick, turn it round and replace it in its hole?
> See if you can find out how many ways you can replace a brick.
> You may mark the faces of the brick if this will help you.

> Build a wall two bricks thick using the rectangular bricks.
> Find some walls made of bricks – at school or outside the school, and see if you can find out how the bricks are fitted together. (A garden wall may be useful.)
> Which do you think gives the strongest wall, one made from cubes or one made from rectangular bricks?
> Why?
> Can you take out any brick from your wall and turn it round and replace it?
> In how many ways can you put it back?
> You may mark the face to help you.

The following five illustrations are reproductions of part of a large display mounted by a group of eight- to nine-year-old children in connexion with activities on volume and capacity similar to those outlined above on pages 16 to 25


WHICH IS THE LARGEST BOX?
How many balls do you think you will need to fill this box?
Try it and see how near you were.
Do you think balls are good things to use to find how much space there is in the jar? Why?
RED OXO
CHALK BOX
Our Team
Stephen Goodwin - Aged 8
Fiona Adams - Aged 8
Michael Glennon - Aged 8
Andrew Dale - Aged 8
CHOCOLATE BOX
FLAT CHOCOLATE BOX

# MARBLES AND A JAR

## Filling a jar with marbles

We took a one lb jam jar and a box of marbles. Then we guessed how many marbles it would take to fill the jar.

### Our Guesses

| | Name | Number of Marbles |
|---|---|---|
| | Fiona said | 79 |
| (Susan) | I said | 51 |
| | Gillian said | 80 |
| | Peter said | 90 |

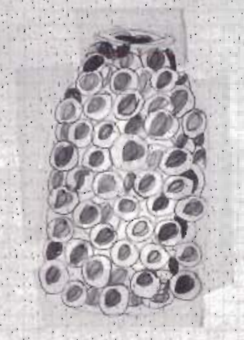
1) We wondered how many marbles there really were in the jar so we filled it full to the top, and then counted them. There were 80.

2) We recounted them to check and found that our first answer was correct.

3) Therefore, Gillian's guess was quite correct, Fiona's was only 1 out, Peter guessed 10 too many while I guessed 29 too few. I must try to do better next time.

4) We all think that marbles are not very good for finding how much space there is in the jar because
(Peter) They are round, and leave spaces in between.
(Fiona) They don't go right to the edge of the jar.
(Gillian) They don't fill the jar completely.

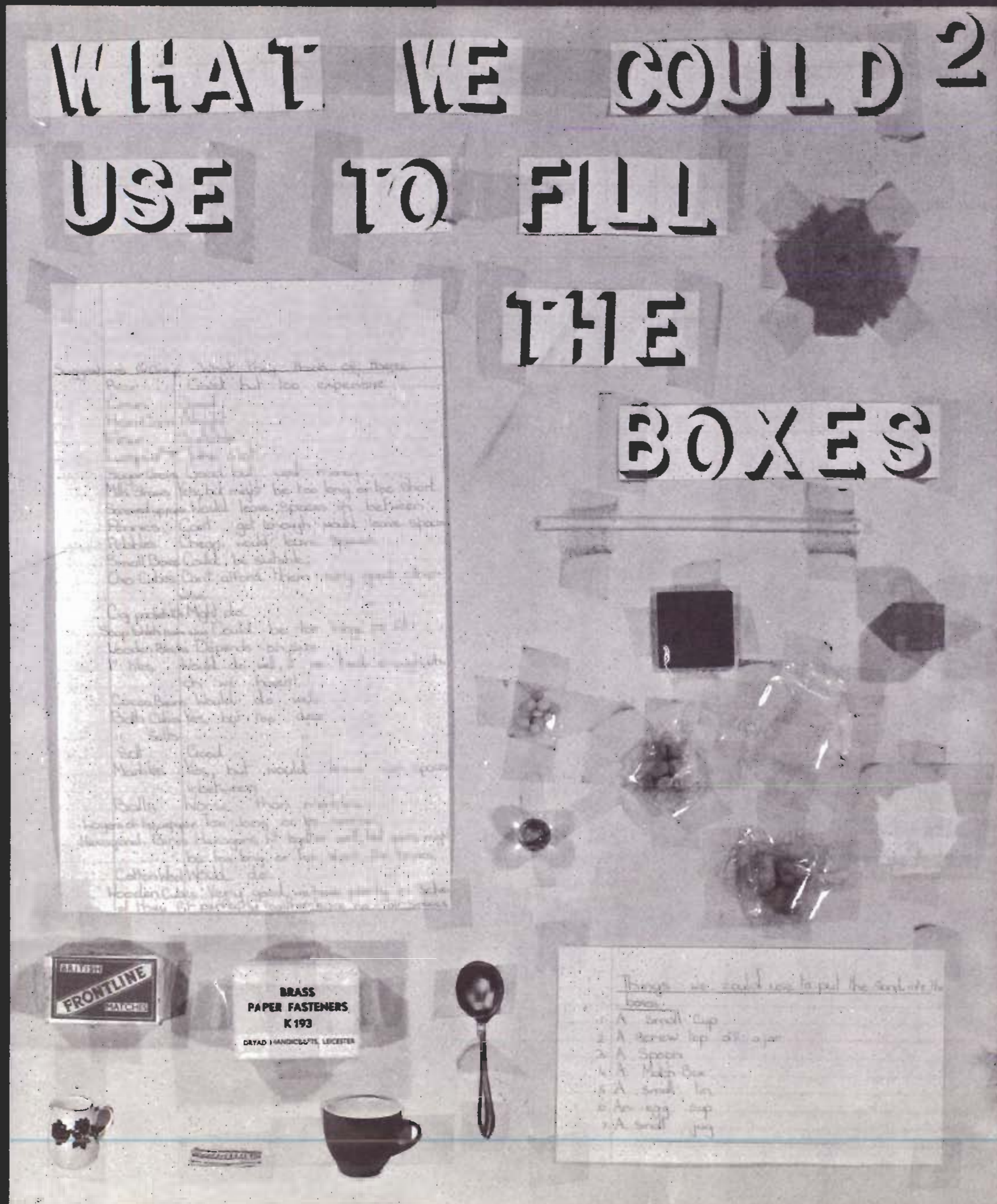
I, Susan Binns, have recorded this.

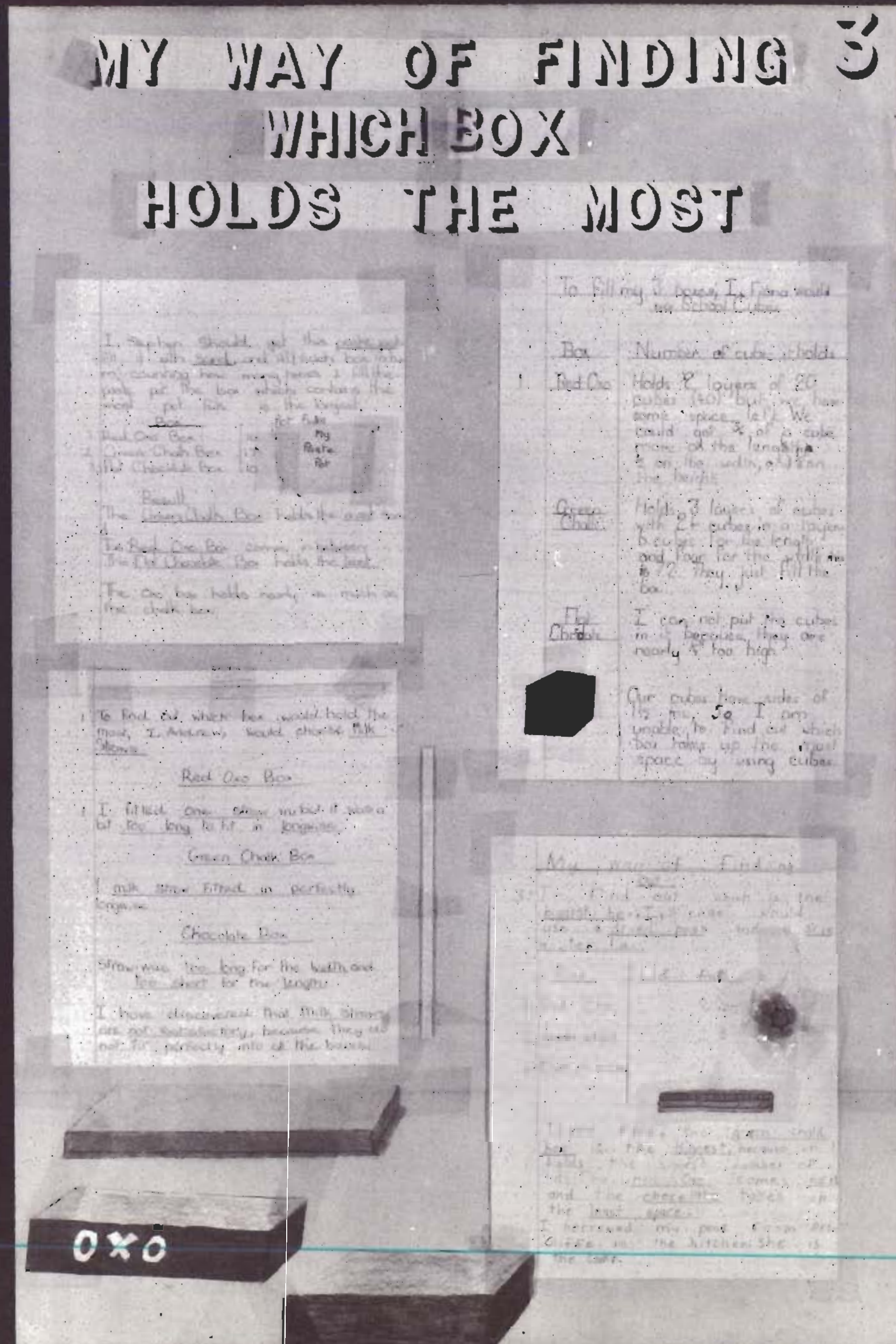


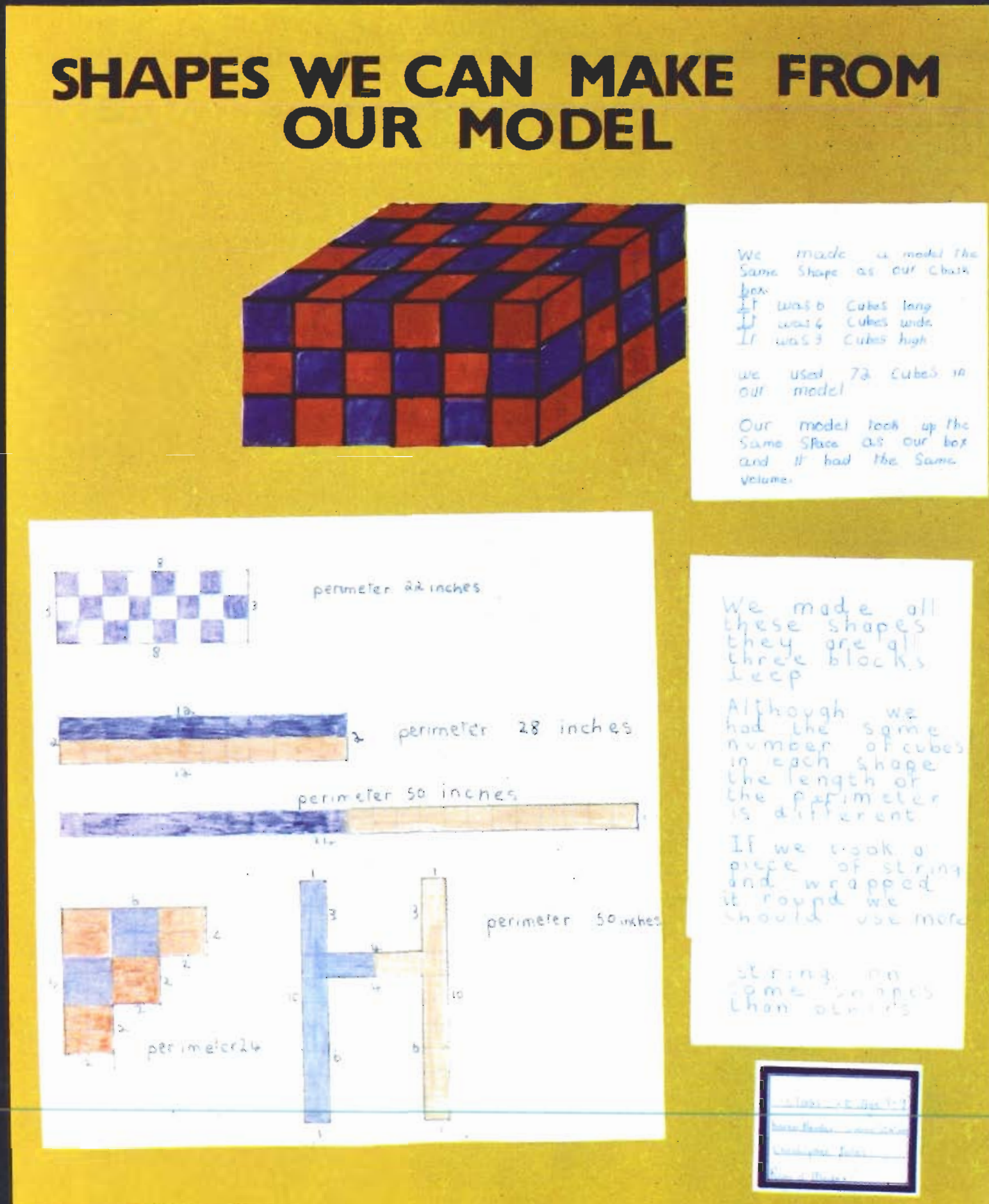
I, Peter, drew the Marbles.

### Our Team

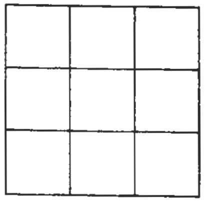
1. Vanda Spencer — Aged 9
2. Jillian Homer — 8
3. Susan Binns — 9
4. Peter Cartlidge — 8


WHAT WE COULD USE TO FILL THE BOXES
2
BRITISH FRONTLINE MATCHES
BRASS PAPER FASTENERS K 193
DRYAD HANDICRAFTS, LEICESTER
Things we could use to put the sand into the boxes.
1. A Small Cup
2. A Screw top off a jar
3. A Spoon
4. A Match Box
5. A Small tin
6. An egg cup
7. A Small jug

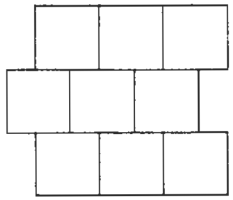

3
MY WAY OF FINDING
WHICH BOX
HOLDS THE MOST
OXO


SHAPES WE CAN MAKE FROM OUR MODEL
We made a model the Same Shape as our Chalk box.
It was 6 Cubes long
It was 6 Cubes wide
It was 3 Cubes high
we used 72 Cubes in our model
Our model took up the Same Space as our box and it had the Same volume.
perimeter 22 inches
perimeter 28 inches
perimeter 50 inches
perimeter 50 inches
perimeter 24
We made all these shapes they are all three blocks deep
Although we had the same number of cubes in each shape the length of the perimeter is different
If we took a piece of string and wrapped it round we should use more
string on some shapes than others

Build a wall using all your cubes.



Build it like this so that the cubes do not overlap.



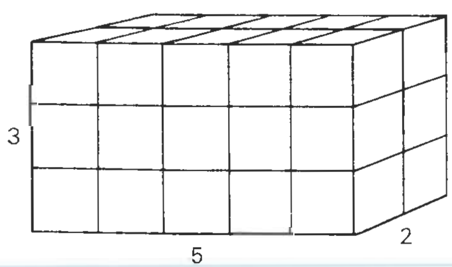
Not like this

How many did you use?
Then make two or three different walls using all the cubes.
Did each wall take up the same amount of space (volume)?
(Now do the same with the rectangular bricks.)

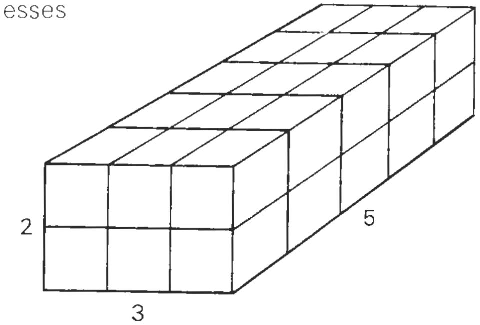
At a later stage children may use this work as a link with computation. For instance if 30 bricks are made into a wall they could be arranged as:

5 bricks in a layer of one thickness
3 layers
2 thicknesses

leading to the expression $(5 \times 3) \times 2$ for the number of bricks, which indicates a sequence of multiplication.




*or*
3 bricks in a layer of one thickness
2 layers
5 thicknesses



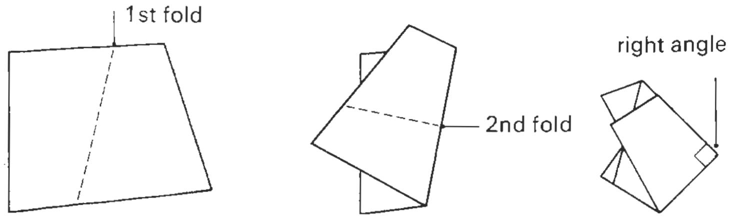

leading to the expression $(3 \times 2) \times 5$ for the number of bricks, and so on.

Look at your walls. Look at the corners of the bricks.
Why do you think cubes and rectangular bricks fit together well?
Look round the school and see if you can find any other examples of these shapes fitted together.
Now try to write down as many reasons as you can why these shapes are used a lot in building.

Suggest that the children make up some small assignments for each other using their discoveries.

The ideas brought out will probably include the facts that there are 'square corners' which fit together well, that the bricks can be replaced easily, that it is easy to make the top of the wall level (linked with parallel lines of mortar, etc.)

These ideas can be investigated further in the classroom. The teacher will have to introduce the term 'right angle' for the 'square corner' of rectangular faces, and show the children how to make a 'right angle tester' by folding a piece of paper twice.

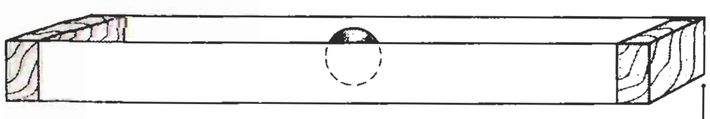



(new edge folded over on itself)

It is interesting to ask the children why we do not use the corner of a sheet of exercise paper. Can they suggest a reason?

Can you see any more right angles in the classroom? (Or about the school?)
Test them by fitting your folded paper right angle on to the shapes.
Open out your folded paper. How many right angles can you find where the creases cross?

Do you know how a bricklayer (builder) makes sure the bricks (floor) are level? (Have any children fathers who are bricklayers?)
What would happen to this marble if the desk (floor) was not level?
Try it.
Make a tester from two rulers and a marble.
Check the level of surfaces.



A 'level' tester

Two upright rulers with small blocks or cubes at ends. Bind on with Sellotape

Introduce the word 'horizontal' (not only of 'plane' but lines, e.g. mortar lines between bricks, window frame bars, etc.)
'Can you see any more horizontals in the classroom?
Test with spirit level.'

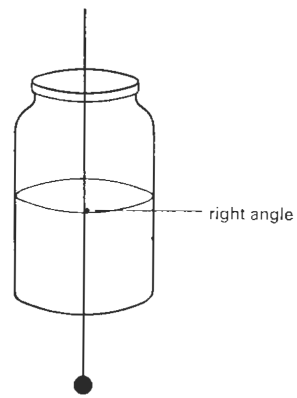
What else must a builder check in his building?
('Vertical' may have to be brought out by questioning. Does he want it to lean over?)
Have any children watched Father hanging wallpaper?
How does he get the first strip 'upright'?
Introduce the word 'vertical'.
How do the bricklayer and Father check this? (Plumb-line.)
Make a plumb-line from thin string and a small heavy weight.
Find some examples of what look like verticals and test them with the plumb-line.

Children should make up a few of their own assignments on these lines for each other to try.

### Vertical and Horizontal

A jam jar about half full of water is required.

'Stand jar on desk. Does the water surface look horizontal?
(Test with spirit level held in front of jar.)
Carefully tilt jar one way. Is the surface still horizontal?




Some teachers may prefer to use a 'jar' with a rectangular base, such as a small fish tank.

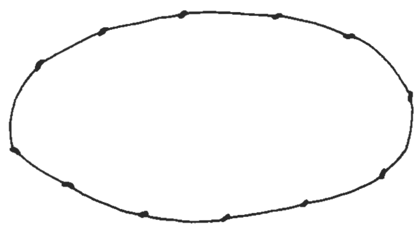
Tilt in some other directions and check horizontal.
Hold plumb-line in front of the jar. Put your eyes so that the surface of the top of the water looks like a straight line.
What angles do you see where the plumb-line crosses the line of the water surface?
Can you say what angles you see when a vertical line crosses a horizontal line?'

### More right angles

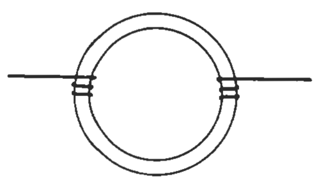
'How many right angles can you find between pairs of edges of a brick?
Does it matter which way you tilt the brick?'

### Link with history: Story of Egyptian rope-stretchers

Prepare a piece of rope in loop form knotted to show 12 intervals as nearly as possible equal. The rope can be of any length, but if several groups are doing this it is interesting to have different-sized loops.



An alternative arrangement is to use 12 curtain rings or large washers and 12 lengths of stiff wire bent round the rings.



Stiff but will slip along ring. A stake can be put through the ring.

> In ancient Egypt, surveyors and builders found a way to make a right angle by stretching a long rope. They used this to make 'square corners' for their buildings and when marking out their land. The rope was divided into 12 spaces as nearly as possible equal. One man held the two ends of the rope together to make a loop. Two other men stretched the rope by holding knots, and this made a 'square corner' or right angle. Try and do this yourself.

> Take the loop of rope. Three of you hold knots at different places on the rope and stretch it tightly to make a right angle like this:
> You may have to try several different ways before you find the right places.
> What shape does the rope make?
> When you have found the right places to hold the rope count the spaces on each side of the shape. What do you notice?

Teachers should be aware that 3, 4, 5 is not the only combination which will give a right-angled triangle, and they may wish to consider whether some very able children might find others. For example, a triangle with sides (5, 12, 13) will be right angled. Other 'Pythagorean triples' giving right-angled triangles are:
(8, 15, 17)
(7, 24, 25)
(9, 40, 41)
and, of course, multiples of any of these, e.g. (6, 8, 10) (30, 40, 50).

### What has been covered in these assignments

Cuboids or cubes fit together best. They can be turned over and replaced, so opposite faces are the same shape and size; they have 'square corners' to faces (right angles). They can be placed in rows (parallel) and are easy to build level (horizontal and vertical). This led to further discovery on right angles as 'static' angle shapes.

Later this experience will be used when we come to consider the discovery of the generalised way of finding the volume of cuboid shapes, when children can be given sets of regular solids and asked to find which solids can be used to fill space completely.

## 2 From three dimensions to two

### Fitting shapes together – Symmetry

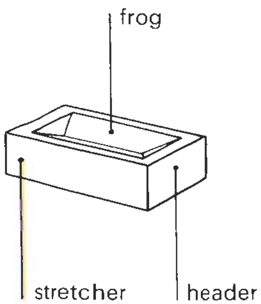
The patterns shown by the faces of bricks can now be used to develop the idea of covering a two-dimensional surface. Most of the tiles they see on walls, floors they walk on, paving stones they run and jump on to and from school, use rectangular shapes, so the work in the early stages will largely be concerned with these shapes. And in any case it is a natural development from the three-dimensional bricks.

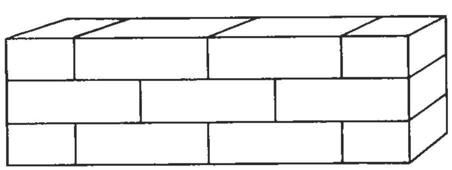
In this section when activities will be on fitting shapes together (and not on area), we are concerned with discovering that some shapes will fit together without leaving a gap or overlapping. We are not concerned with boundaries, e.g. what happens at the end of the wall or on reaching the edges of the floor.

Refer to patterns in brickwork, tiles and floors again, the children to find such patterns.

Most brickwork in housebuilding and other load-bearing walls is arranged in one of the following ways so that joins between bricks are not directly one above the other. This is called bonding.

If the wall is only half a brick thick i.e. $4\frac{1}{2}$ inches, the method is known as *Stretcher Bond.*

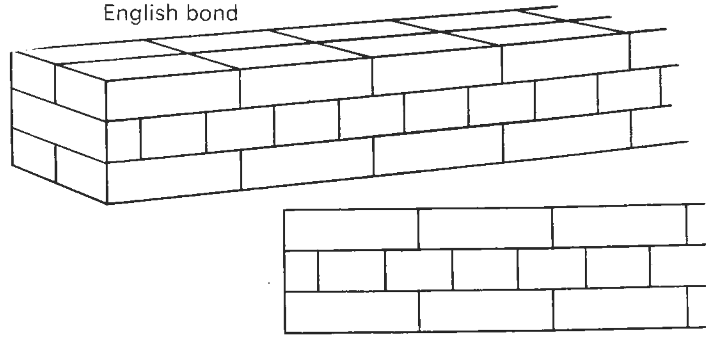


stretcher bond

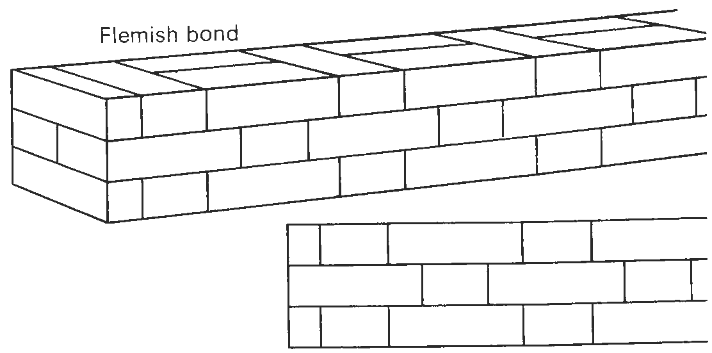
The commonest ways of fitting bricks together for a wall a brick thick, i.e. 9 inches, are *English Bond* and *Flemish Bond.*

English bond



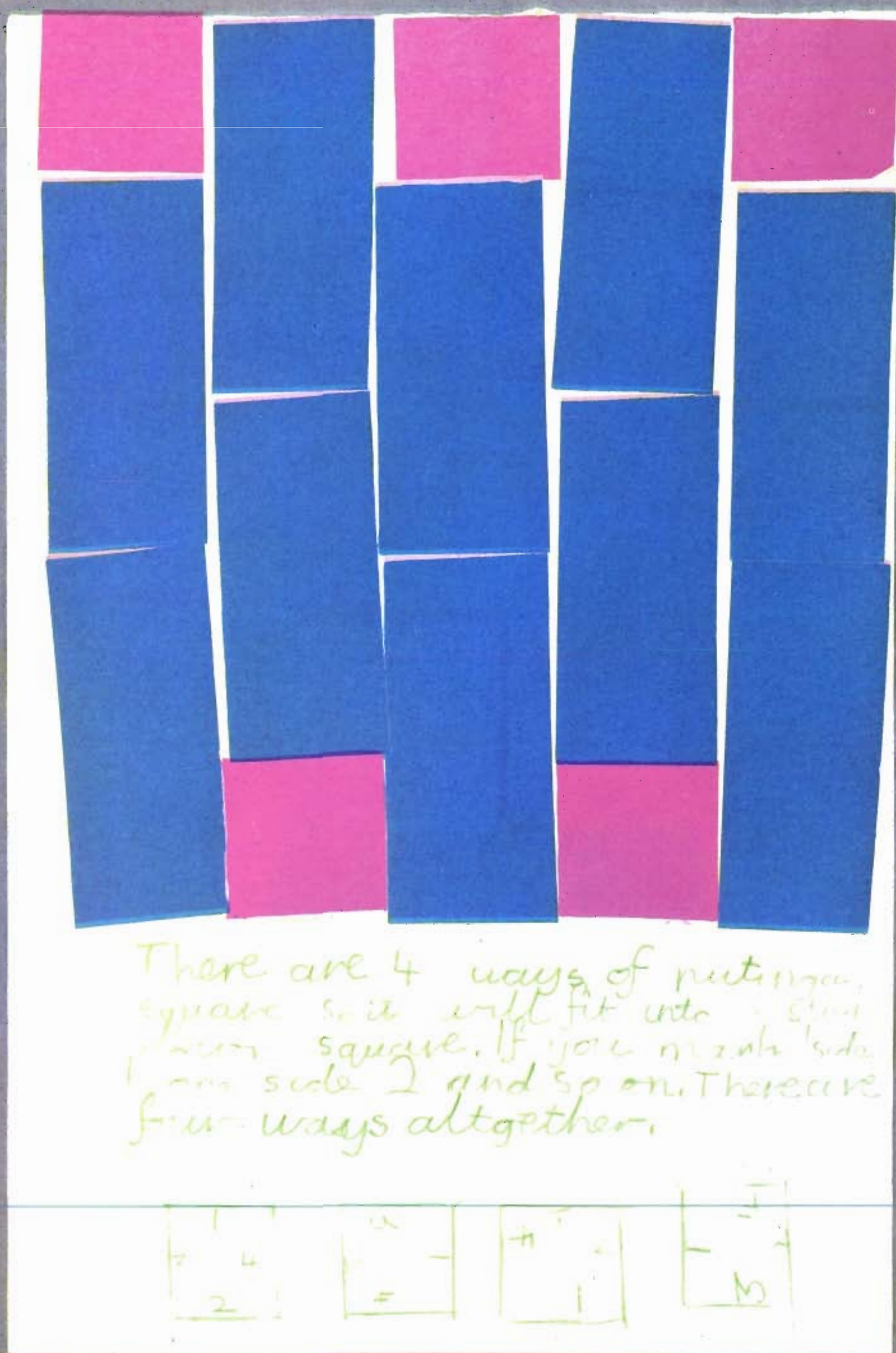
front view

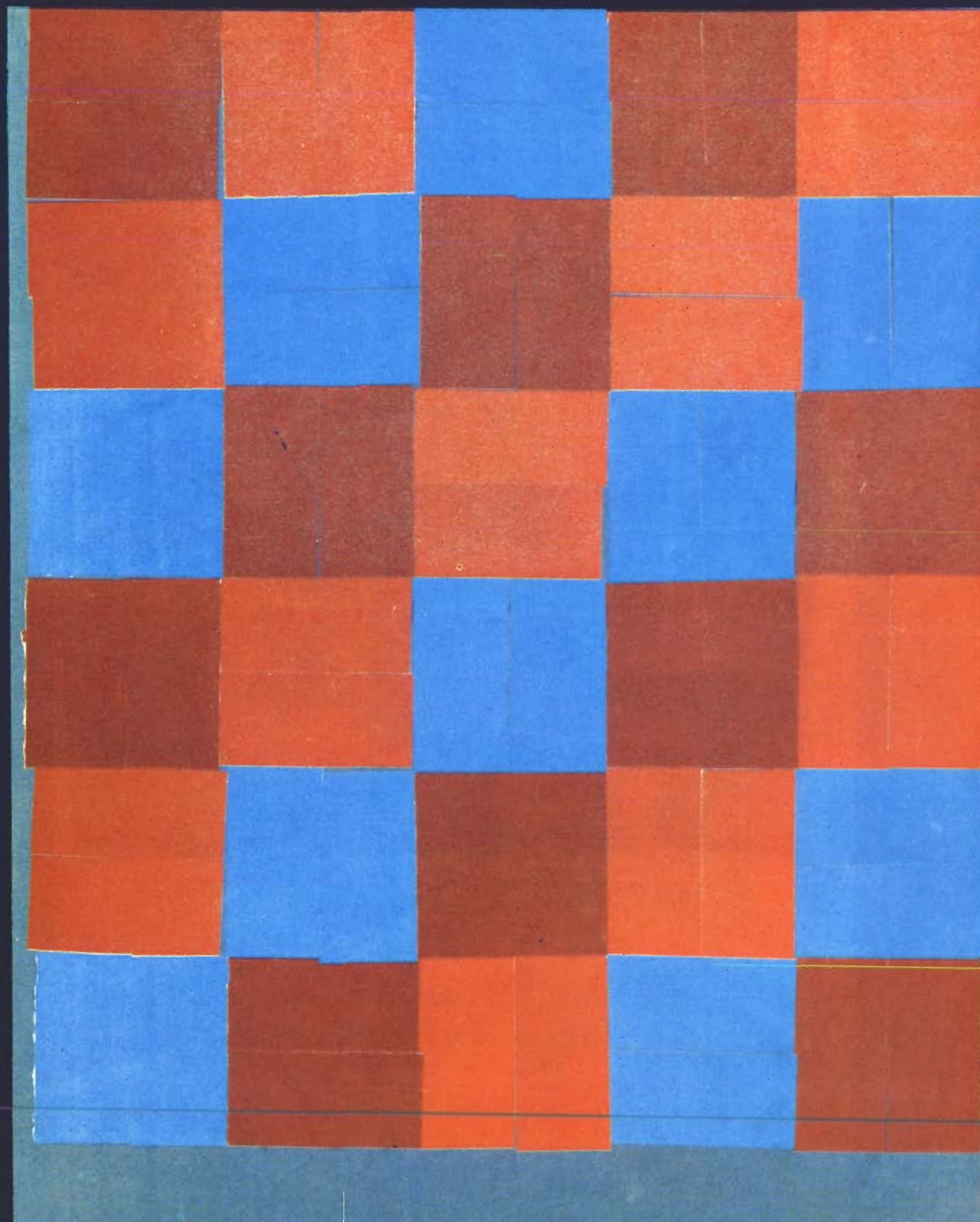
Flemish bond



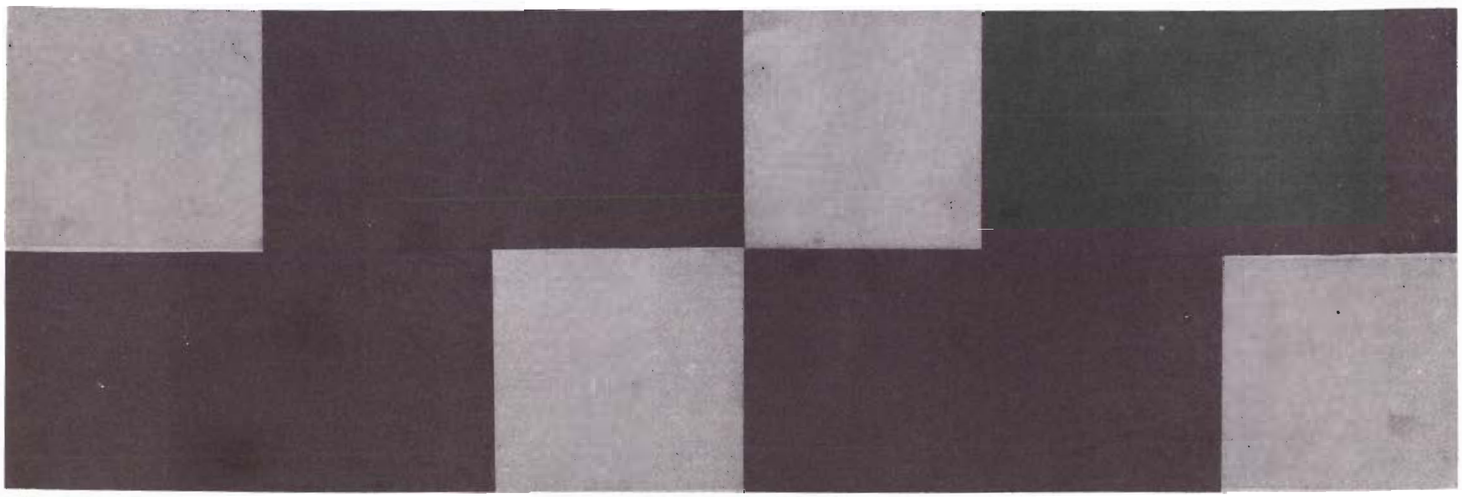
front view

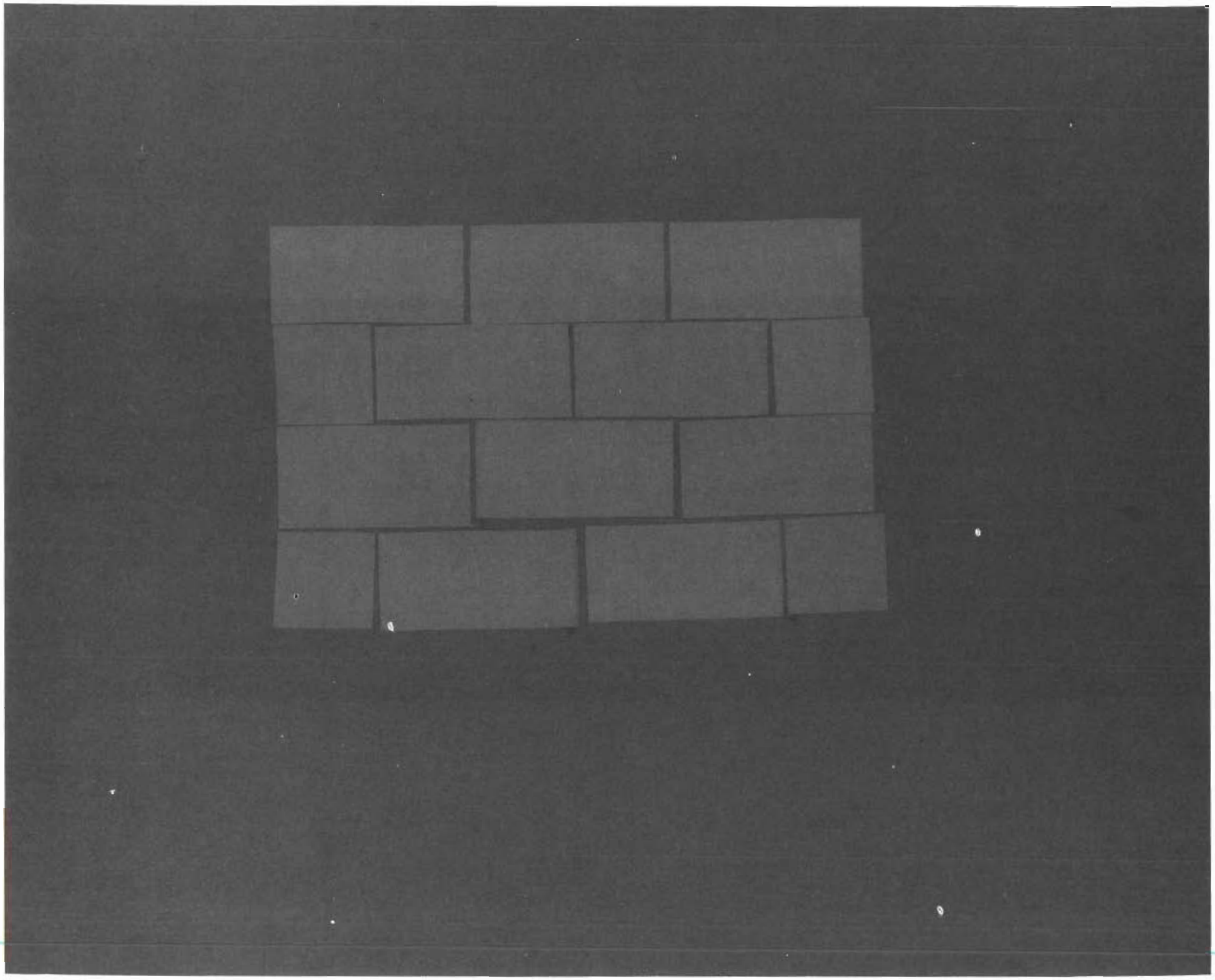
This page is by a below-average eight-year-old. His friend wrote, 'When you put the rectangle down you can put it down two ways, one one way and one the other way. But if you could turn it over you could turn it four ways.'





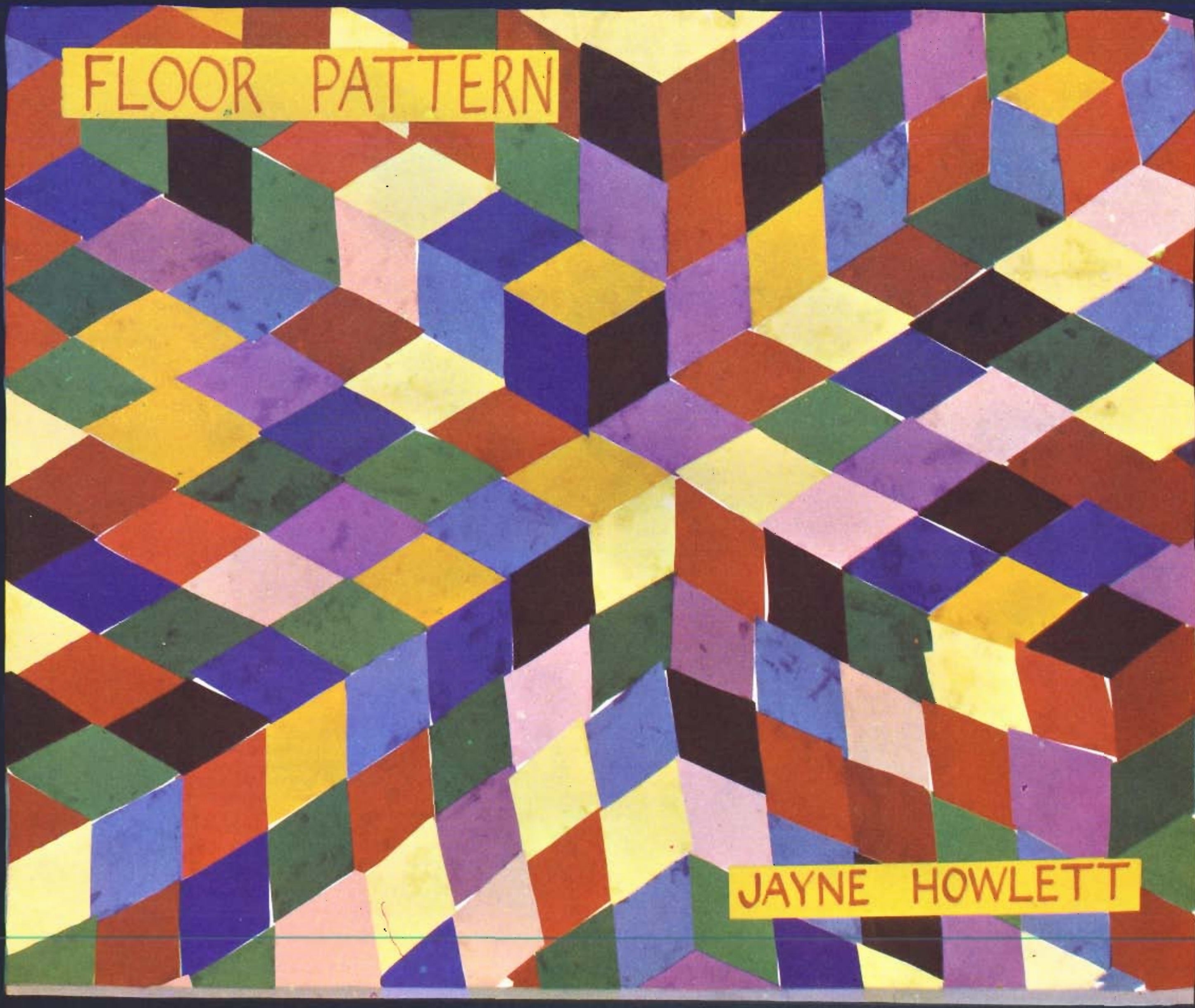
Tile patterns using squares and rectangles made by eight-year-old children.





Reproduction of classroom floor pattern by a nine-year-old girl.

An imaginary floor tile pattern using rhombuses made by a group of nine-year-old girls.



Jayne Howlett

We made a picture of a floor made from diamond shapes in all colours.

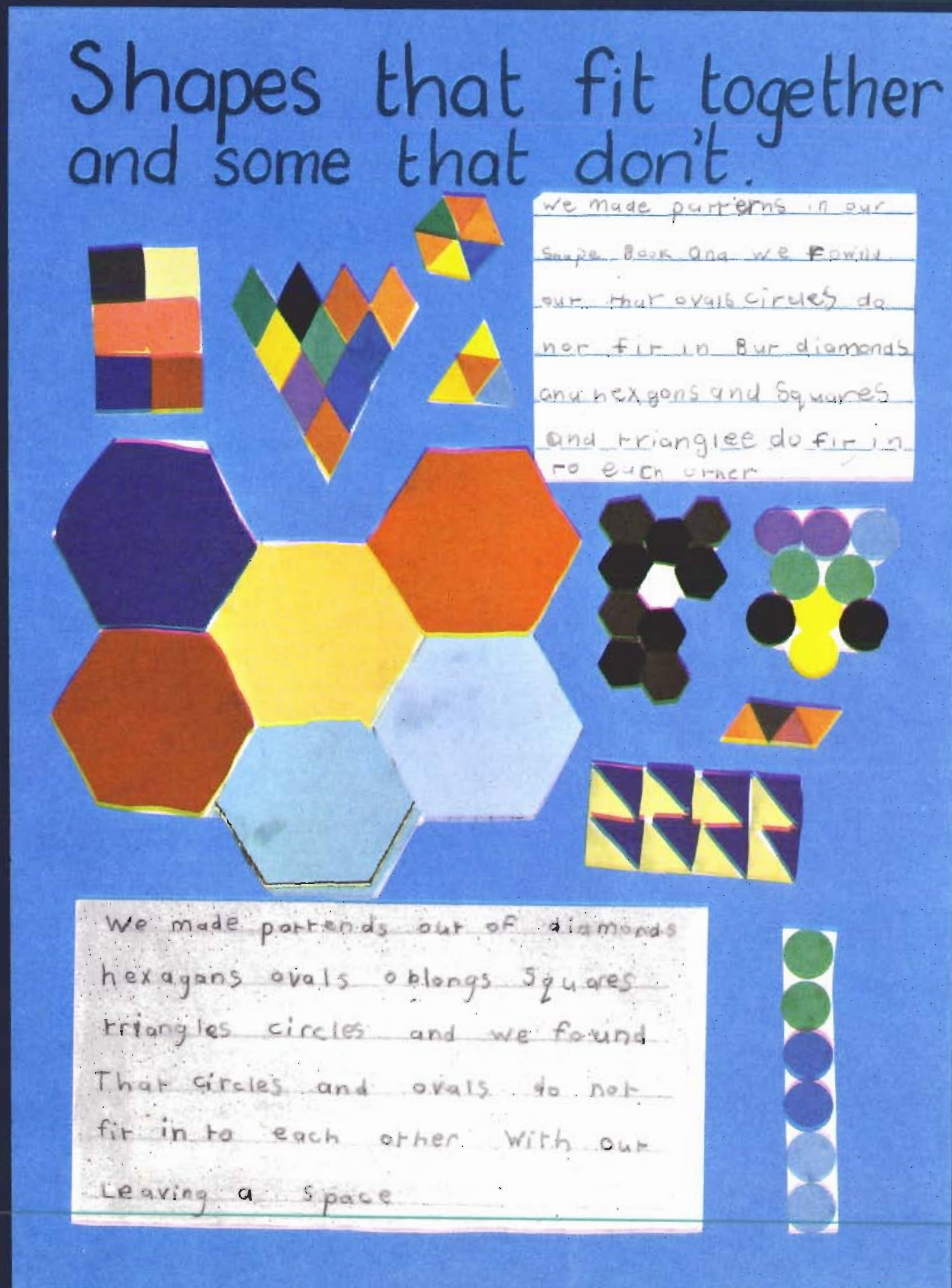
If you fold any diamond shapes in half you will find that they are symmetrical. A diamond shape can be put back in four different ways so that the whole is mapped on to the whole.

When fitted on the paper they look like little boxes piled up. Sometimes a diamond looks like a square when you are looking down on it but from the side

Geraldine Beak

Jayne Howlett & I got a big piece of card and a lot of sticky paper. I also got the equila-
-teral triangle and made some diamond shapes I stuck them on to the card and made a pattern. If you put them on top of each other they look like a lot of boxes. When we started we thought it looked like a star shape If you fold any diam-
-ond in half you can see that it is symmetr-
-ical. You can put any diamond shape back in four ways

A page from a book on 'Shapes and how we use them' made by some six- and seven-year-old children in an infants' school.



When the children are looking at a wall to study the brick pattern they can be asked what part of the bricks they are looking at (the 'ends', 'sides', etc.) This will be an opportunity to introduce the word 'face'. They can then be asked such questions as, 'What shape do you see?' (Rectangles) 'How could you make a pattern like this without using bricks?' (Draw round faces of bricks, use squared paper, cut out shapes from card or coloured paper and arrange in patterns, and so on.) If there is a parquet or wood block floor, or tiles, in the school they can look for other patterns of shapes fitting together. Paving stones on pavements and paths will also provide useful patterns. Later on in this section an account is given of the interest aroused by this kind of investigation in one school, and the many different places in which they found patterns.

### Other suggestions for work

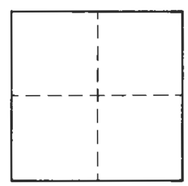
Groups of children can be given tiles made from coloured card, or cut from lino or vinyl floor covering. These can be in squares or rectangles.

> Make tile patterns by fitting these shapes together. See how many ways you can make a pattern with the rectangles. (Repeat with squares and rectangles used together.)
>
> Find how many ways you can turn a square round so that it still fits into the pattern. What about a rectangle?
>
> How much do you have to turn a rectangle so that it fits into its hole in a new position? More than for a square? How much more?

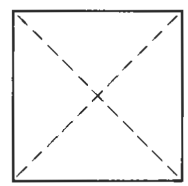
It is advisable in general to keep to these shapes, and not introduce irregular quadrilaterals, or other regular shapes at this stage, for in fact the bricks and tiles which children see all about them are mostly square or rectangular. Fitting other shapes such as triangles, quadrilaterals, regular polygons and so on, to try to cover a surface, could well come later, but an illustration is given opposite of some work done in this connexion by 6-and 7-year-olds. This work has introduced the first ideas of 'rotational symmetry', purely as experience on which to build mathematical ideas associated with it later on. Having discovered how squares and rectangles can be turned round to fit into a space as tiles, other ideas of symmetry can now be looked at.

> Trace round the edges of rectangular and square bricks, or tiles or other rectangular or square shapes, on a sheet of paper.
> Cut out paper shapes.
> How many ways can you fold these shapes so that one half matches the other half?

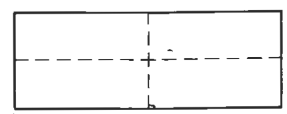
This introduces the idea of symmetry by 'reflection' and the children should be asked to describe what the shape on each side of the fold looks like.



Most will find this way.



Some will find this way of folding.

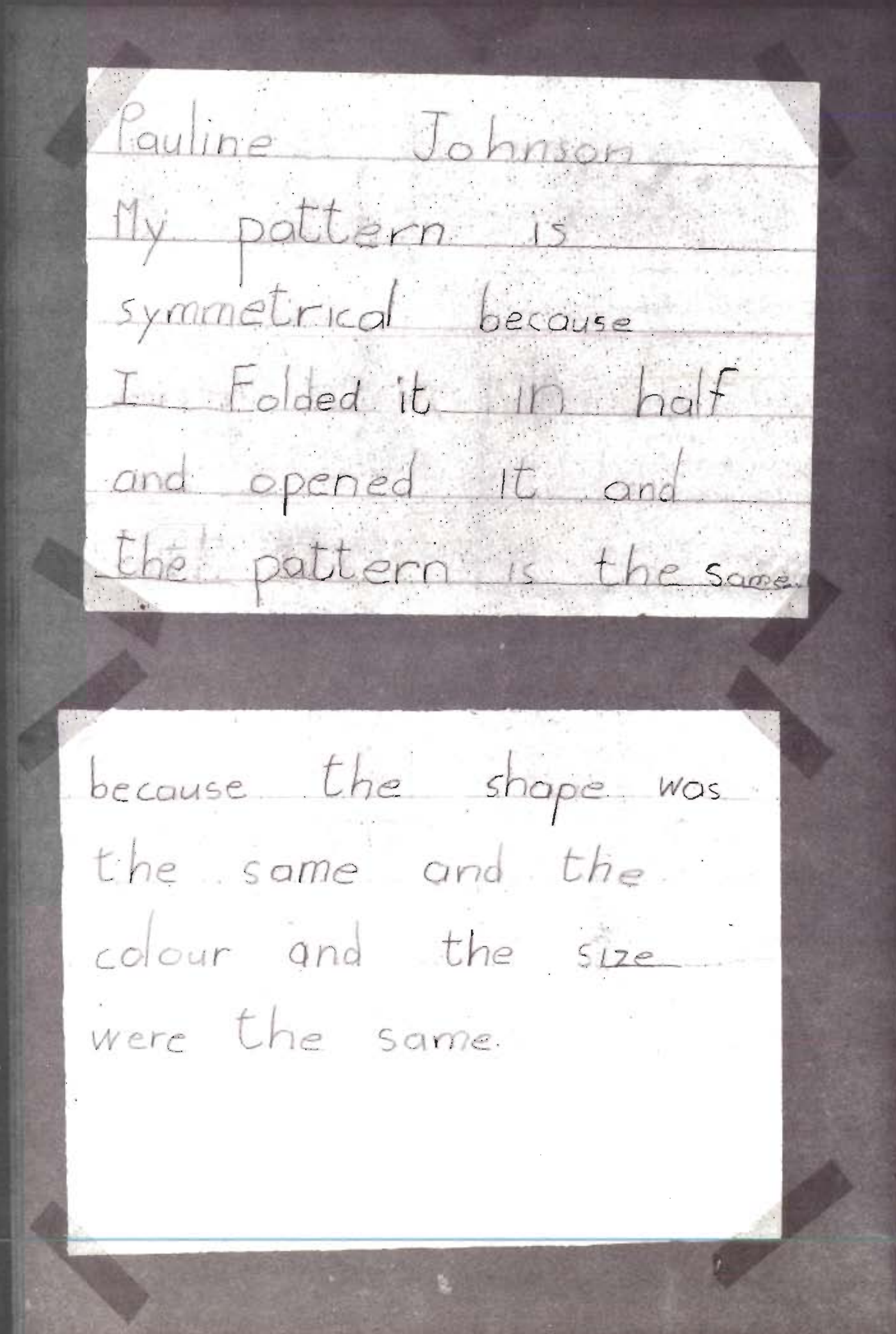


This seems sufficient introduction with rectangular shapes for young juniors. A further development of this with regular shapes and irregular shapes linked with axes of symmetry will come later in the junior school. Of course if any child presses the discovery further on his own initiative this should be encouraged and the teacher can refer to the later section in this guide to see possible lines of development.

The idea of symmetry, or balance, in shapes can now be investigated in other ways. One avenue for exploration is pattern work in art and craft activities. Blot patterns may

The following work is by a group of 'below-average' nine-year-olds.

Pauline Johnson (aged seven) did an 'ink-blot' pattern and wrote the following comment:



have been made already in the infants' school and these can be recalled or developed again. Others can be made by paper folding and cutting, pricking, drawing. These may arise from work in other lessons, in which case they provide material for mathematical experience, or they may be introduced as an assignment in the time allocated to mathematical activities. In the latter case assignment instructions might be given thus:

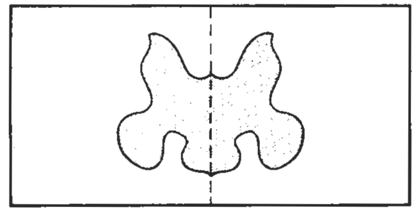
> Take a plain sheet of paper about the size of a page in an exercise book, or slightly larger. Fold it in two.
> Open out the paper and drop a 'blob' of paint (or ink) on the fold. Now fold your paper again and carefully press it all over. Open it and lay it flat to dry.
> What do you notice about the shape you have made?
> Tell me about the shapes you can see on each side of the fold. How has this happened?
> Does it look like anything you know?

(It may resemble a leaf, or a butterfly, for instance.)

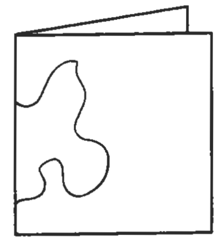
It is probably better to give the above instructions orally to a group of children doing this and to word these in a manner suitable for presentation to the age and ability of the children concerned.

Although the interest lies mainly in the discovery of the shapes produced, it is useful to take a closer look, and the children's description of what was done and the result, is valuable. It will first be in speech, and this may be sufficient for most children, but those who wish should be encouraged to write about their activity and discovery.

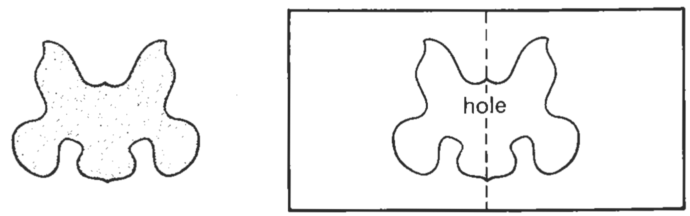
Similar ideas can be suggested orally, or on cards, for the examples illustrated below. If this occurs in other work, such as art, it should be discussed as illustrations of symmetry and not just left as patterns.



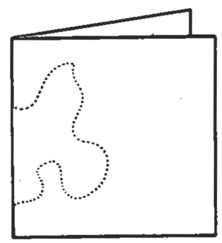
blot pattern one fold



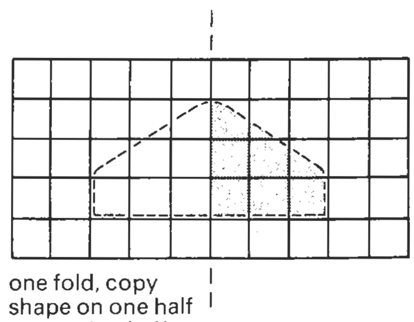
one fold, shape cut out



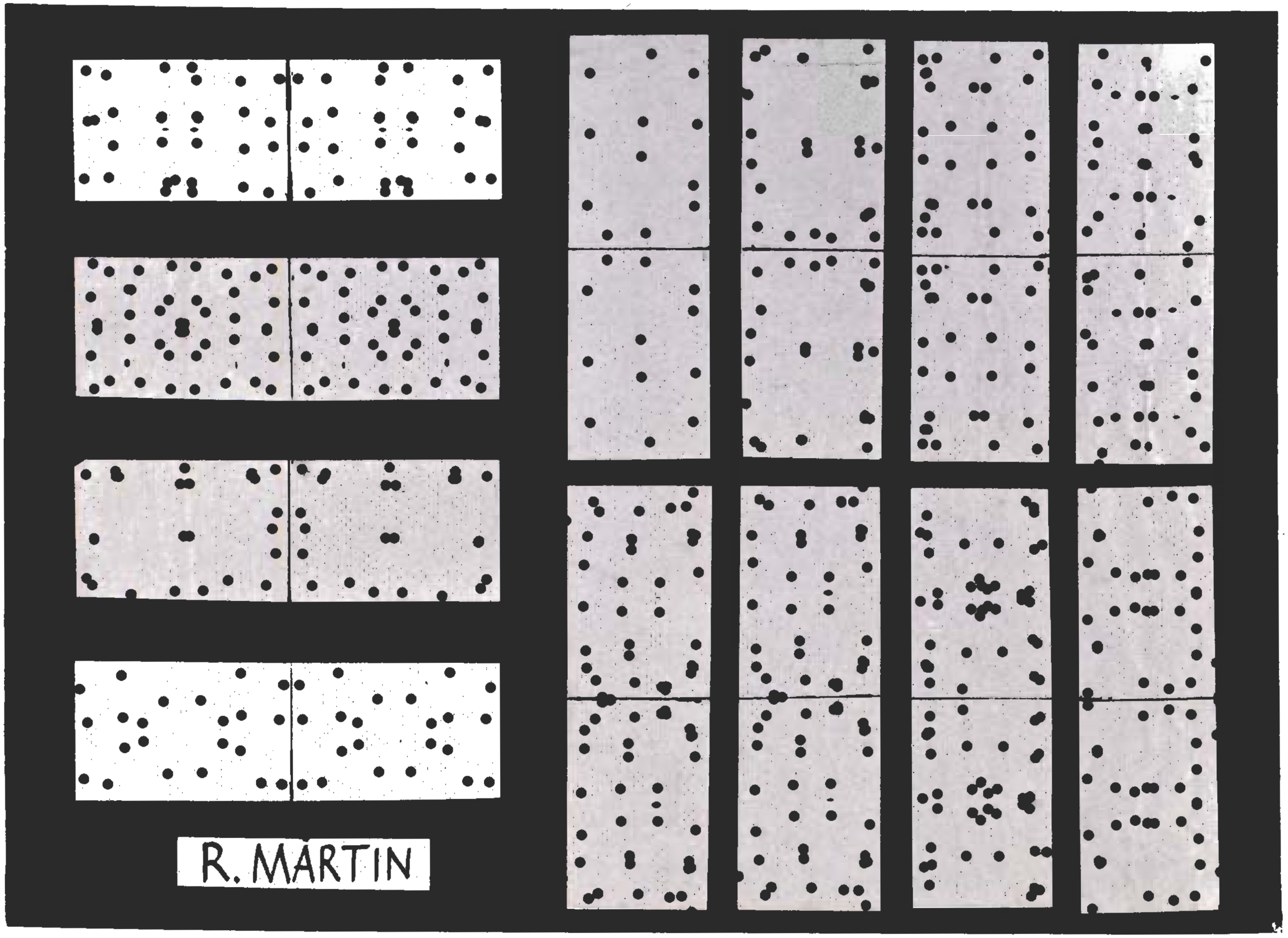

two resultant shapes from cut-out

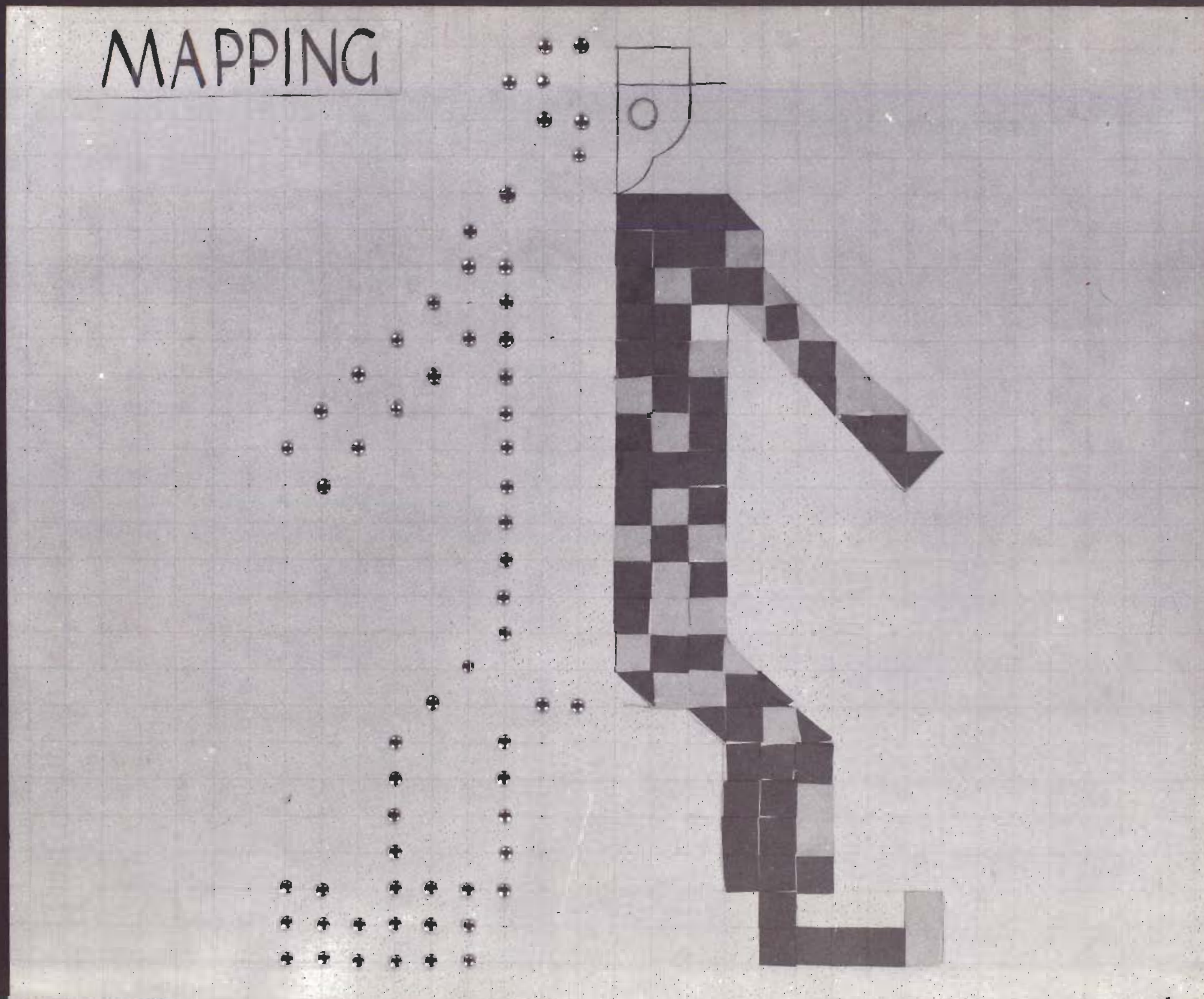


one fold, prick through

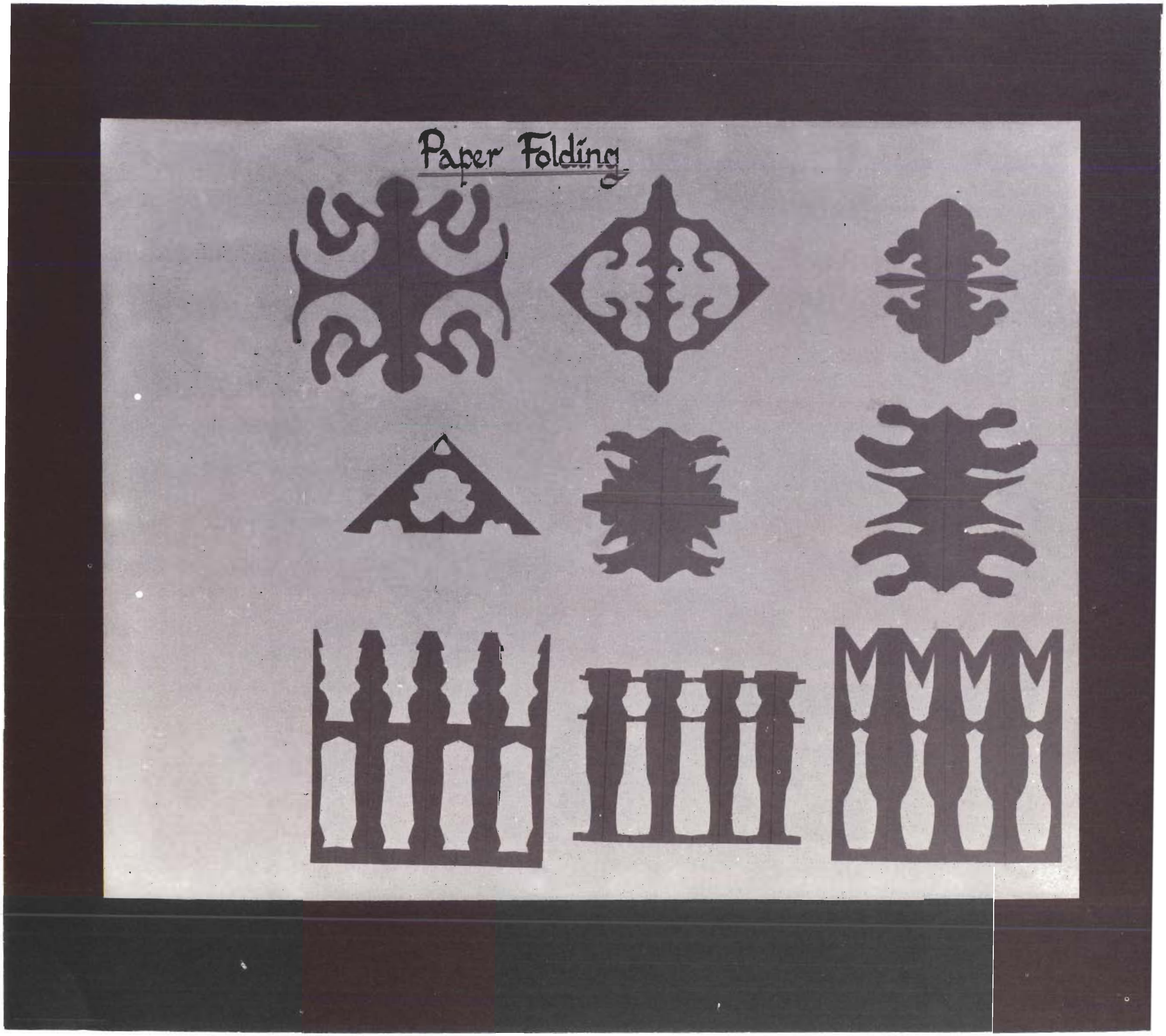


one fold, copy shape on one half on to other half

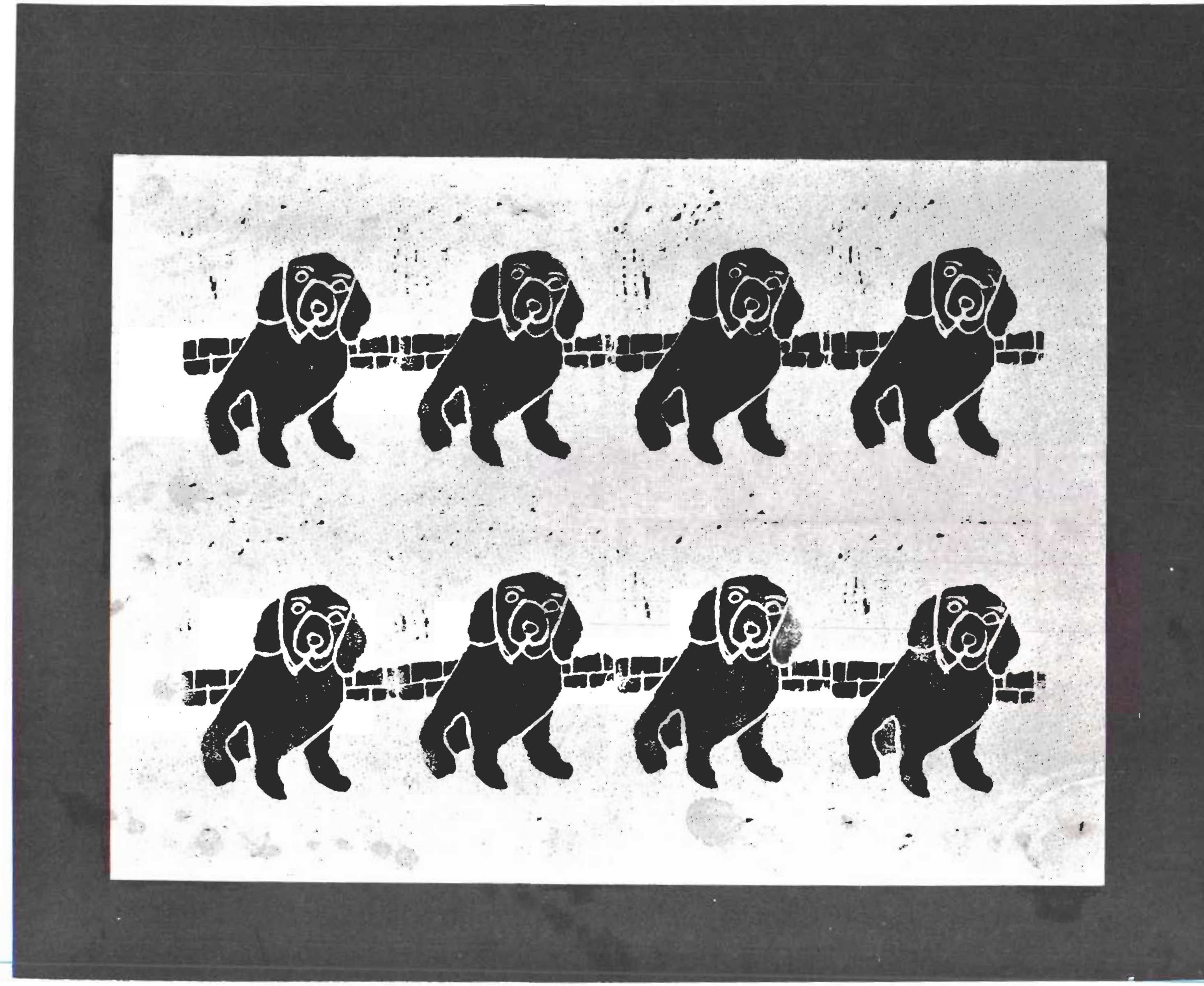

R. MARTIN


MAPPING

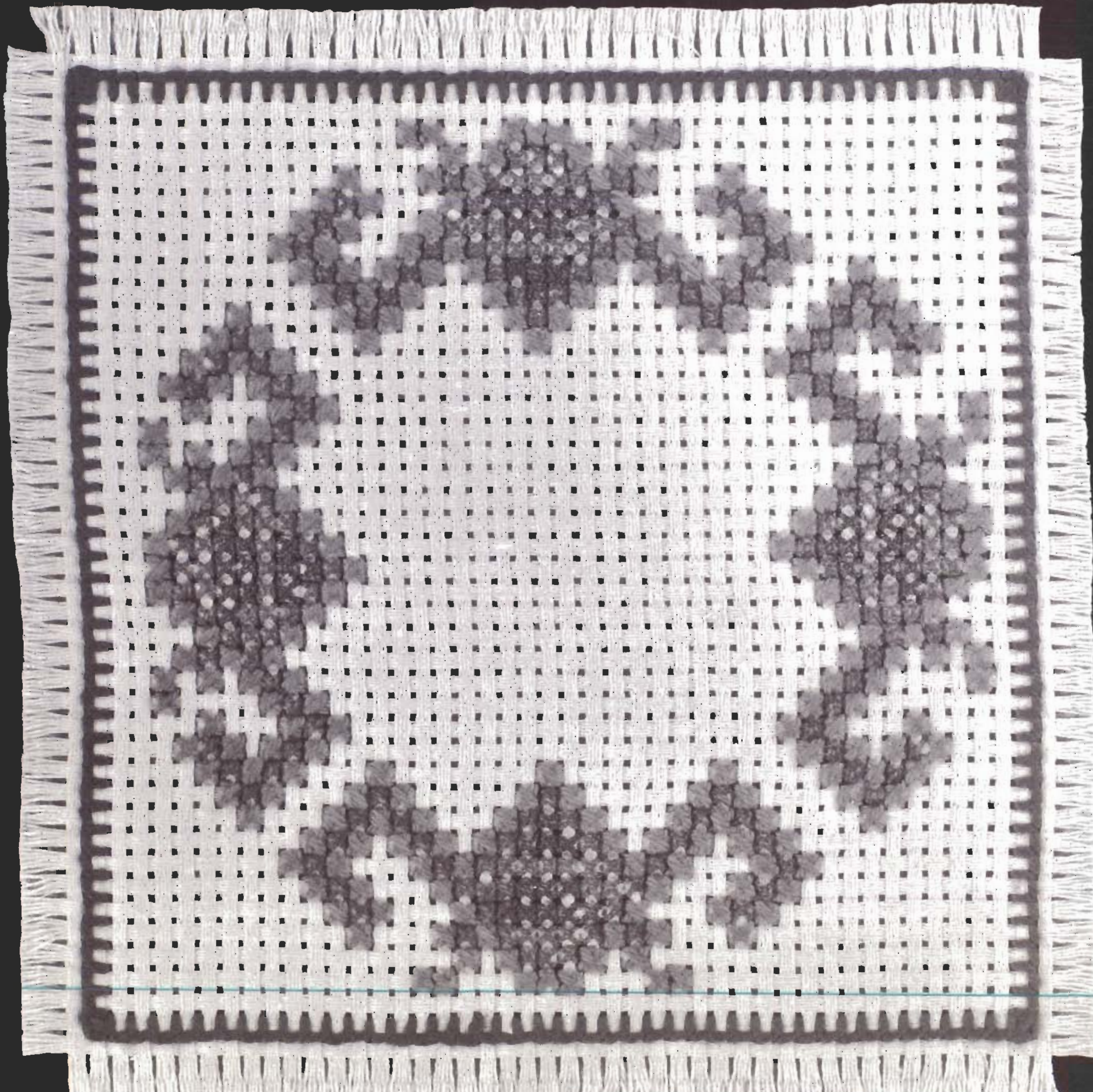
Two pages from a large book on 'Symmetry' produced by some nine-year-old children.


Paper Folding

These lino-prints done by nine-year-old children show 'translation'.



Needlework by eight-year-olds. The blue mat illustrates 'reflection'. 'Rotational symmetry' can be seen in the yellow mat.



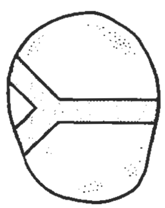
## More activities with patterns

The patterns involving symmetry which have been discussed so far have been mainly concerned with folding or cutting one shape and looking at the resultant symmetrical shape. But many young children enjoy making repeating patterns, such as potato prints or simple needlework patterns, and these may introduce ideas of rotation, reflection, and translation (see page 8). As with the other activities on symmetry this will largely be observation and experience of translations and pattern on which more precise mathematical ideas will be built later on. Much of the work will arise naturally from pattern making in art activities and teachers should discuss the ways in which the patterns have occurred.

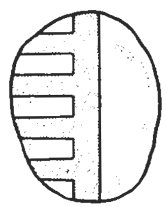
At other times, activities involving reflection, rotation and translation in patterns can be suggested.

For instance, when making potato prints it might be suggested that the potato is cut to give a pattern

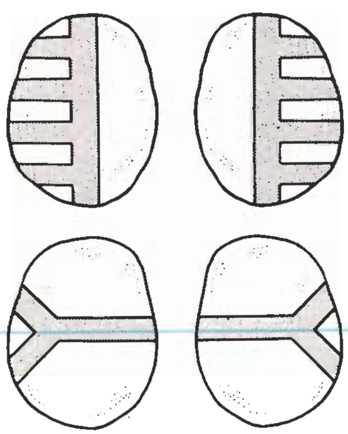
such as this



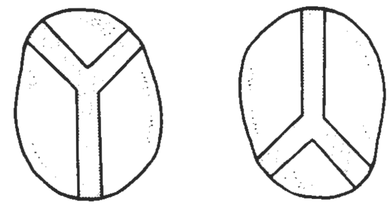
or this



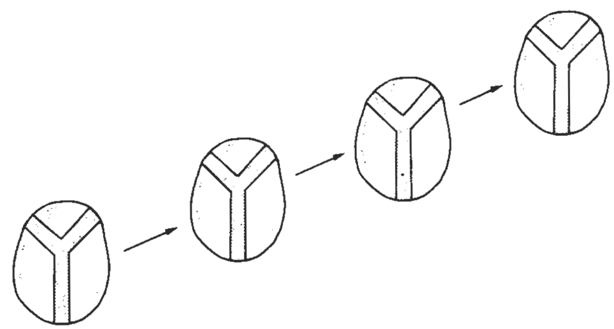
This could be used for patterns showing 'reflection'.



Rotation can be shown thus:

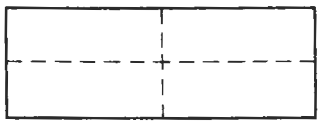


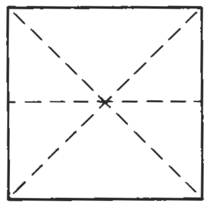
and repeating a pattern can introduce translation:



These are but a few of the ways in which transformations can be observed in patterns which are well within the scope of young children, and teachers will be able to think of many other examples. The value at this stage is in the observation of pattern and symmetry, and the discussion on how the pattern was made, the ways in which the shapes were moved to produce the result. Later, in 3, there will be further examples of stitching and pattern making when 'curve-stitching' to produce envelopes and linking with modular arithmetic will be introduced.

The teacher can now introduce the idea of an 'axis of symmetry' by saying that for the shapes and patterns made we can use this expression for the 'fold'. Interesting shapes can be produced following this by using the experience gained by the children in folding squares and rectangles of paper. These can be folded once, or twice, or more, and shapes cut out after folding. The children can guess what the resultant shape will look like before opening the paper out.





There are many examples of natural shapes which are symmetrical and this is the time to make a collection of these. Children should be encouraged to look for such things as leaves, butterflies, and other shapes which could be 'folded' or cut, in one way, in two ways, (for some children even three ways, four ways . . . ) so that symmetry can be seen, and to show the 'axes of symmetry'.

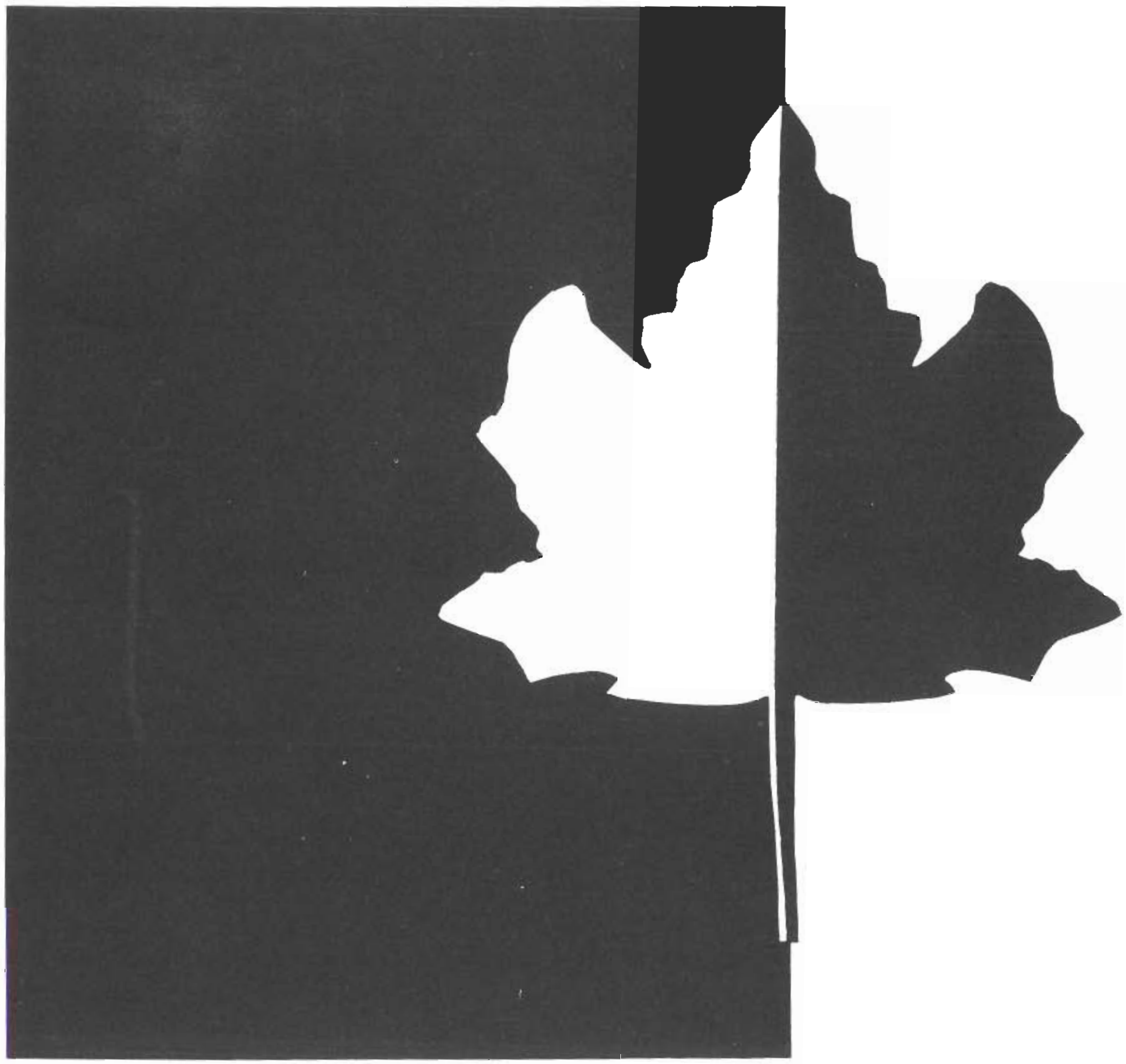
The illustrations on pages 51 to 53 are some examples of children's work on symmetry:

At this point it may be interesting to read the comments of a headmistress of a school where some of the activities so far suggested were tried with some eight-year-old children. It is particularly interesting to note that the children and teacher did not stop at the activities suggested by the assignments, but discovered and developed further ideas and interests of their own:

Symmetry—Although we started with only a few children, this age group became extremely interested and we were obliged to extend all the activities to the whole year group (complete range of ability). In fact the idea of symmetry so captured their imaginations that they collected, cut, drew and examined for a whole week, all day and every day. We limited them to a week but felt that we had rushed through and could easily have spent a good deal more time consolidating ideas, etc. We started by showing them an assortment of objects and getting from them whether they were symmetrical and from which angle. They were able to see this easily, even the less able, and could give us the names of many objects – not around them – which were symmetrical from some views but not others. They could see that a circle could be folded in an infinite number of ways (and didn't they enjoy the word infinite!) We have found that they have associated the idea with their other practical work. As we find up to this age in all their activities, they express themselves so well in speech, but become so much more abbreviated in writing that the excitement of the discoveries does not come through so well in writing. We wished very much that we had had a hidden tape recorder.

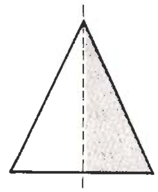
Fitting together 3D shapes and fitting together 2D shapes – Once again we've had to let all the age group spend all day for all this week on this project, although, as with symmetry, we started with a small group. They were absolutely fascinated and brought in odd bits of paper on which they had sketched patterns of brickwork and flooring. They have been all over the school 'rubbing' not only bricks, but gratings, milk crates, windows, wallpaper, wooden planks. They have seen patterns everywhere, and again words over-spilling, so that you need to be on the spot. This led to 'play' with wooden shapes, squares, rectangles, hexagons, trapezia, rhombuses, triangles. This they loved doing and spent many dinner times and playtimes making all-over patterns, etc. Some of the children have worked through playtimes and dinner times! An interesting time was had by two boys who were sure they could fill a space with circles. This arose after the first introduction to shapes and solids, when I showed them a variety of everyday objects. They got hoops, tins, quoits, rolls of gummed paper of various sizes and tried to cover the corridor – this during playtime, but found they were still left with some small spaces. ('We'll have to fill those with bits of paper or putty.') They loved all the names; some of them found the names of as many regular-sided shapes as possible. This seems the right time to 'catch' the enthusiasm. They were able to see patterns in pictures, too.'

This pattern was made by an intelligent eight-year-old by tracing round an actual leaf.

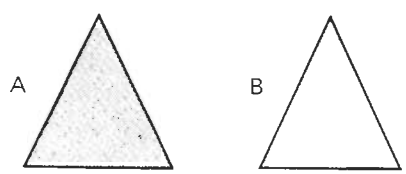


## The mathematics from the assignment suggestions

From fitting bricks and blocks together patterns of faces were seen. These introduced shapes such as the square and rectangle, which were then used as tiles for covering a surface. From this it could be discovered that these plane shapes can be turned round or over to fit the hole from which they were taken. They were also shapes which could be folded symmetrically. This led to further discovery about symmetry and the idea of an axis of symmetry. Plane shapes which have an axis of symmetry (or more than one) so that one half can be folded over to match the other half, can also be turned completely over so that the whole shape matches the original. This shows two ideas of symmetry: we could say that *folding* 'maps' one half on to the other, that *lifting and turning over* 'maps' the whole on to the whole.



In the case of folding along an axis of symmetry it can be seen that the shaded half will match the other half. Each point on the shaded half has its image in the other half, so folding maps one half on to the other half.




If we turn the shaded triangle over so it fits the new position each point in the shaded shape A has a corresponding image point in shape B. So in this case turning completely over maps the whole on to the whole.

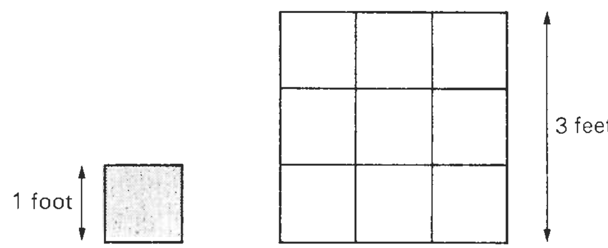
Other transformations are to be observed from pattern work and serve as background experience for later development.

## Covering a surface – Area

It was said above that the ideas which many children at secondary level have about area are confused and ill-conceived. In fact a group of adults when asked what they understood by 'area' answered almost unanimously, 'Length times breadth' and in fact one or two argued that irregular shapes such as leaves had 'no area' because they could not 'work it out'. Frequent confusion about units also occurs. An intelligent 11-year-old girl whose standard in computation was very high regularly stated area measurement in inches, and frequently lapsed into misinterpreting perimeter for area.

W. W. Sawyer in his excellent book, *Vision in Elementary Mathematics*, has this to say:

'One idea needs to be cleared up a little before we start, the idea of area. It seems to take children a long time to distinguish between *area* and the *distance round* some region. In the figure, children when asked for area of the larger square will repeatedly answer, "12 feet". This is, of course, the distance round the boundary; it is the length of the fence that would be required to enclose this region.




When we ask for the area, we want to know how many black squares would be needed to cover the square on the right. The answer, of course, is 9, which is 3×3. But time and again children seem to relapse into giving the distance round when the area is asked for. It seems necessary to emphasise this distinction, perhaps by having children actually pasting square pieces of paper over the region for which the area has been asked. Every time the word or idea of area is involved, it is well to make some reference to pasting paper squares or to covering the floor, or to some similar illustration that will make clear which concept we have in mind. Anyone without classroom experience may be surprised how long it takes some children (including clever ones) to fix this distinction.'

This confusion is nearly always due to inadequate preparation in the early stages. For many children area is presented by way of textbook diagrams of rectangular outlines and the immediate jump to a formula, $A = L \times B$. It is completely divorced from the idea of covering a real surface, and children are taught that '6 inches × 4 inches = 24 square inches'. The young child's experience inside and outside school is largely with three-dimensional objects and understanding how to fill a three-dimensional space or playing with a solid object is much more natural and easy to comprehend than the idea of area. It is also essential that there should be preliminary work on measuring the distance round objects (perimeter) and covering surfaces in many ways before any attempt is made to arrive at a way of measuring area.

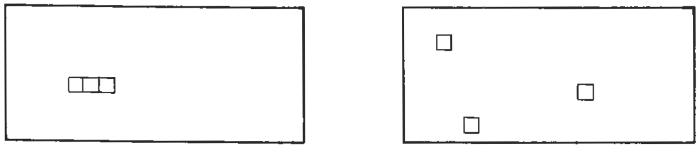
In *Beginnings* ▼, activities involving measuring the distance round objects using a variety of materials have been suggested when considering the notion of length and these will introduce children to the idea of perimeter. Reference should also be made to *Computation and Structure* where this is amplified. When activities on covering a surface begin teachers should remind children of this distinction between perimeter and covering a surface, and frequently refer to the preliminary work on length. Some activities on covering have already been mentioned in *Beginnings* ▼. Laying a cloth on a table, covering surfaces with paint, putting paper on desks and tables when painting and pasting, activities in the Wendy House such as floor covering, and making patterns with large mosaic tiles, are valuable experience which infants can have and prepare the way for a more detailed investigation of covering a surface at junior school stage. Already in this guide one aspect of covering a surface has been discussed. This was fitting tiles together to introduce the idea of symmetry, and it can now be extended to bring in the first ideas of area as measurement of surface covered, or the amount of surface which a shape has, and that this is an approximate measurement. Before any activities on measurement of volume and capacity were introduced a check on the children's understanding of the fact that volume and capacity were unchanged under certain conditions was suggested. Experiences which will help towards an understanding of area also being unchanged have been suggested in *Beginnings* ▼, and before activities on measurement of area are attempted teachers should make a test or check on the children's understanding of this idea. Some suggestions for this are given below:

1. You will need two postcards (or card of about the same size as a postcard). Place these on the table. Ask the child if the cards cover the same amount of the table. If he says yes: tear the card in two, replace it on the table with the pieces apart and ask whether the two pieces cover the same amount of table as the whole card. If yes: tear the two pieces again each in two, replace the four pieces on the table separated, and repeat the question. Do the four pieces of card cover the same amount of table as the whole card? If yes: tear into eight pieces and repeat the question.

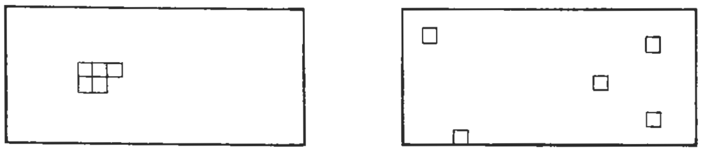
If at any stage the child says no, he requires further experiences at frequent intervals, with activities involving covering a surface as outlined in *Beginnings* ▼.

2. This may be given instead of, or in addition to, the above test, and may be compared with the corresponding one for volume on page 17. You will need two identical sheets of paper, and a dozen 1-inch wooden cubes. Place the two sheets of paper on the table a few inches apart. Tell the child the sheets represent two fields of grass. There are two farmers, and each has one field. Each farmer begins to build farm buildings on his field. Place one cube on each paper, then a second and a third. On one sheet place the cubes close together, on the other sheet scatter the cubes over the paper.

For example:



Tell the child each farmer has a cow. Ask the child whether the two farmers have the same amount of grass for their cows to eat. If the child says yes, continue to put buildings on each farm, always putting the same number of cubes on each sheet. Keep the cubes close together on one farm and scattered on the other, e.g.



Again ask whether the farmers have the same amount of grass for their cows or not. If he still says yes, ask him why he is sure. If he replies that the farmers started with the same grass, that they have the same buildings and therefore the same grass left (or words to convey this idea) he has grasped the concept..

Unsatisfactory answers at any stage indicate that the child needs more activities with putting 2D shapes together and covering surfaces as outlined in *Beginnings* ▼, before he is tested again.

When children reach the stage where they give satisfactory answers to these tests activities can be introduced which give experience in comparing the area of surfaces by the amount of surface covered. Home-made units and a variety of ways will be used for this. From this experience the need for standard units can be introduced at a later stage, and *much* later the discovery of the easy way to calculate the area of rectangular shapes.

Assignments

Provide a set of books of the same shape and size, (e.g. exercise books).

> Estimate the number of books you will need to cover the top of the table, so that the books do not overlap.
> Then fit the books on to the table top to cover it.
> How many did you use? How near was your estimate?

(Similar activities with dusters, sheets of newspaper, used postage stamps on surfaces of various sizes. Let children suggest which they should use, e.g. stamps for small surfaces such as bookcovers, newspaper for large surfaces such as corridors, or floors in classroom.)

Provide sets of geometrical shapes made of plastic material, or card. These should include squares, rectangles, regular triangles, regular hexagons, regular pentagons, circles, with a sufficient number of each of the same size to try to cover the surfaces to be investigated.

> Take all the triangles. Estimate how many you will have to use to cover the front of the large reading book.
> Then use the triangles to cover your book. How many did you use? Was your estimate too big, or too small?

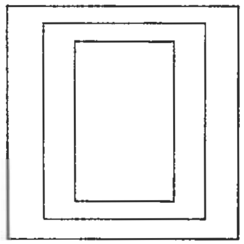
Repeat with the other shapes, i.e. use only circles; hexagons; rectangles; and so on.

> With which of these shapes did you find you could cover a surface?
> Which were not very good for this purpose?
> Write about this in your own way.

Give children several books (or similar shapes) with covers of different area.

> Arrange these books in order of size by the surface of front cover.
> Which is biggest, which is smallest?
> Write about how you did it.

Note whether children compare by using units to cover each, or by placing books on each other thus:



Compare the area (amount of surface) of various shapes using a grid, e.g. chicken wire. This has spaces roughly hexagonal in shape. Table mats, postcards, books, any flat objects with an interesting irregular shape are useful for this work.

Children might be set this task:

> Take three of these shapes. Use the wire netting to find which has the biggest surface on one side. Tell your teacher how you did it and write and draw about it.

Some eight-year-olds in one school were moving from one classroom to another. The desks were being carried by eleven-year-olds. A discussion arose as to whether they would have enough room to arrange the desks in groups of three, or whether they would have to be arranged in rows. One eight-year-old boy said, 'They will still take up the same space, because it doesn't matter which way you put them; each desk will still cover the same space on the floor.'

From this remark they started work on comparing areas of the corridor. The teacher provided sheets of newspaper and the children found their own way of comparing areas.

This is the time to discuss with the children various ways of covering a surface they have used so far.

E.g. 'Let's look at all the things we've used for finding which of these has the biggest area.' (Triangles, squares, circles, wire netting, etc.) 'Which do you think were useful?'

Discussion should lead to the idea of using squares, triangles, regular hexagons; and why pentagons, circles are not as useful. By analogy with the use of the cube for 3D space the square is probably the most useful. At this stage areas will be measured by the number of standard squares required to cover them and estimates must be made of the odd bits at the boundary of the surface covered. Children may well get on to the idea of taking two odd bits which together would seem to have just about the same area as the standard square; alternatively, they could get a reasonable approximation by ignoring the 'small' odd bits and counting the 'large' ones in as if they were each covered by a standard square. Much useful discussion could arise over various methods of approximating if these are suggested by the children.

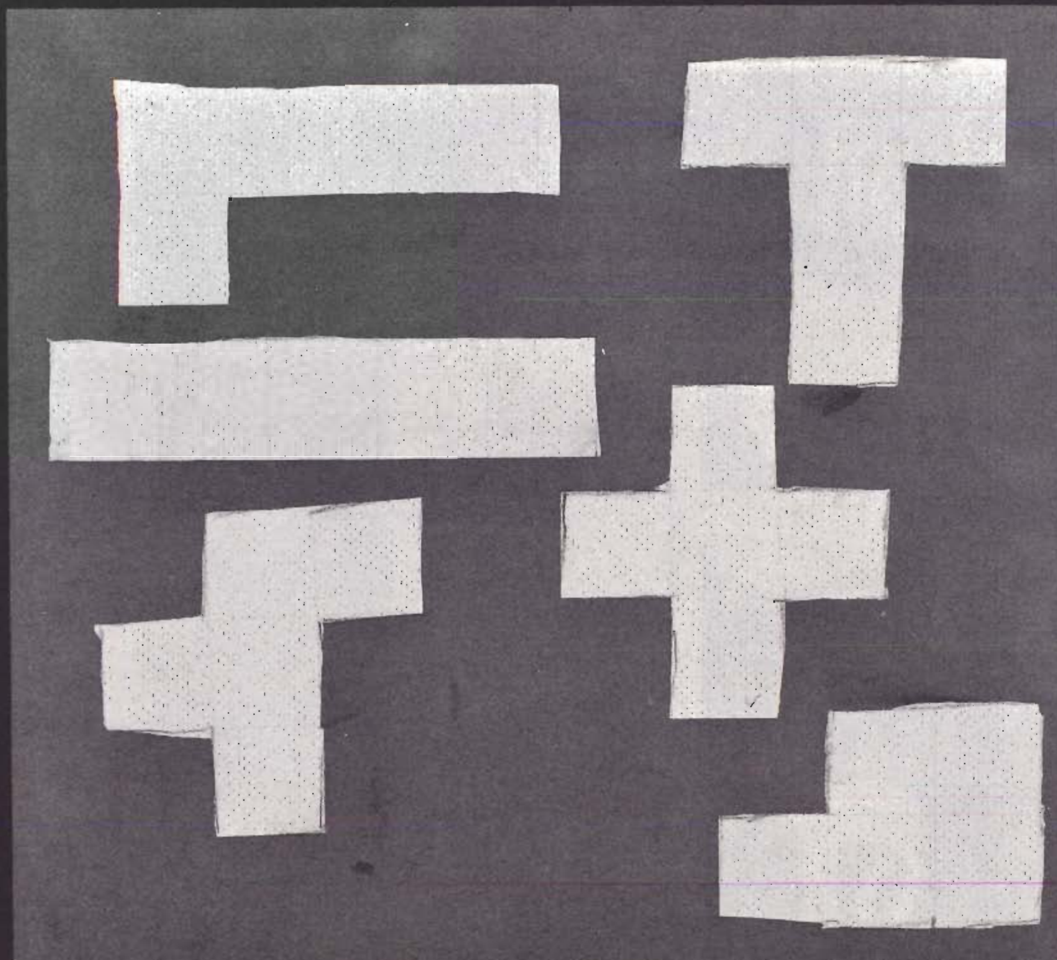
Some assignments using squares can now be given.

Provide some tracing paper marked in squares ($\frac{1}{2}$ in, $\frac{3}{4}$ in, or 1 in will do).

> Cover the front of your book with the squared paper and find how big it is by counting squares. Now do the same on a different book cover. Which was bigger? How do you know?

> Use graph paper marked in $\frac{1}{2}$ in, $\frac{3}{4}$ in or 1 in squares in a similar way. Trace round regular shapes and compare areas.
>
> Then try it on irregular shapes such as leaves using the square grid on the tracing paper.

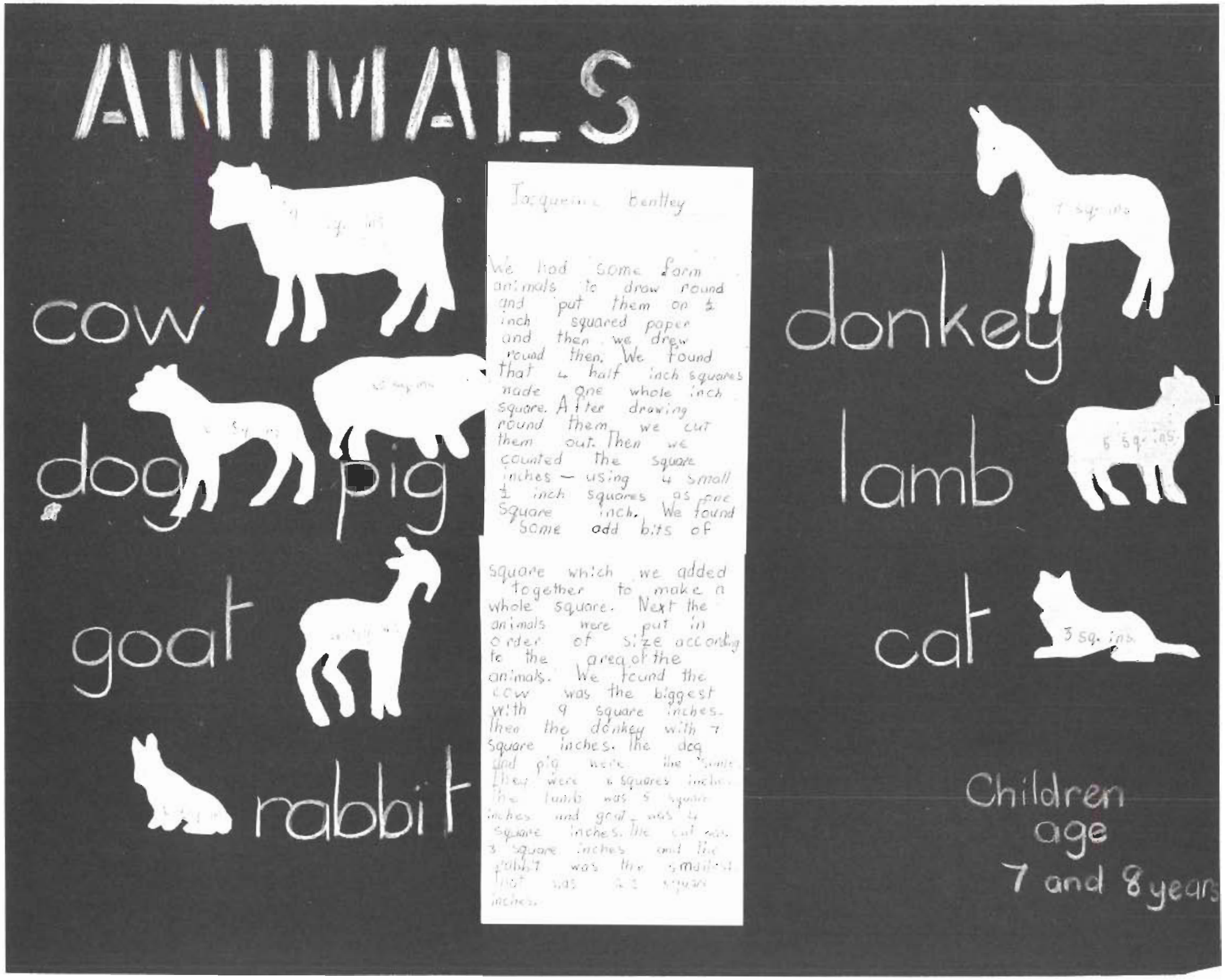
When a group has done this discuss with them what to do with the bits of squares round the edge of the shapes.



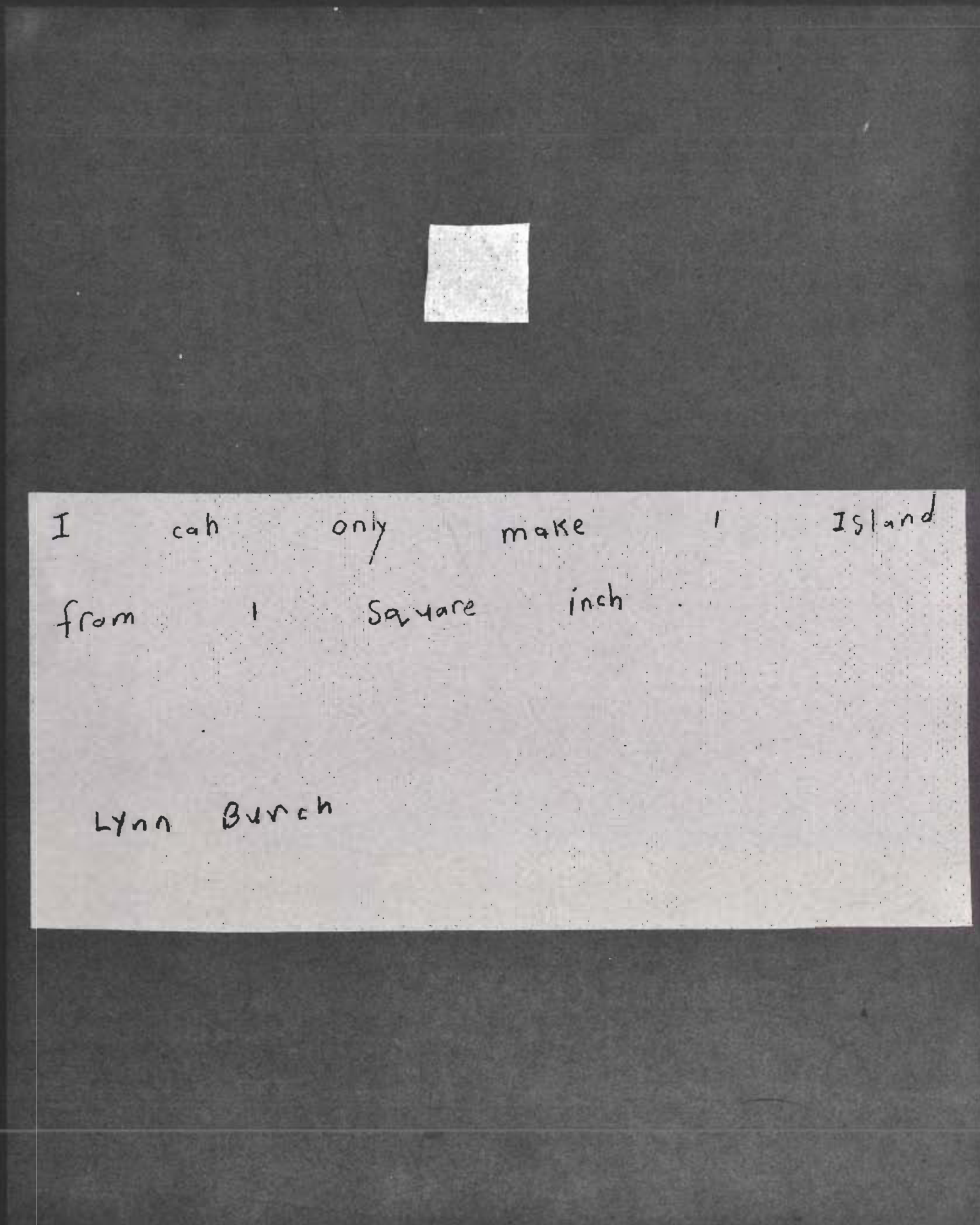
I made 6 different islands from 5 square inches

Peter REDWOOD

Further children's work on area

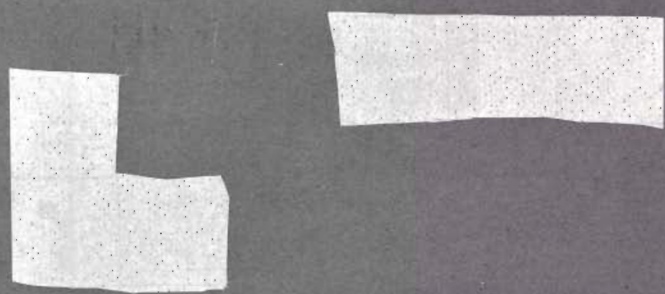


The following sequence was done by six-year-olds. Formulating the rules for 'islands' and a more systematic treatment would make a good project for older children.



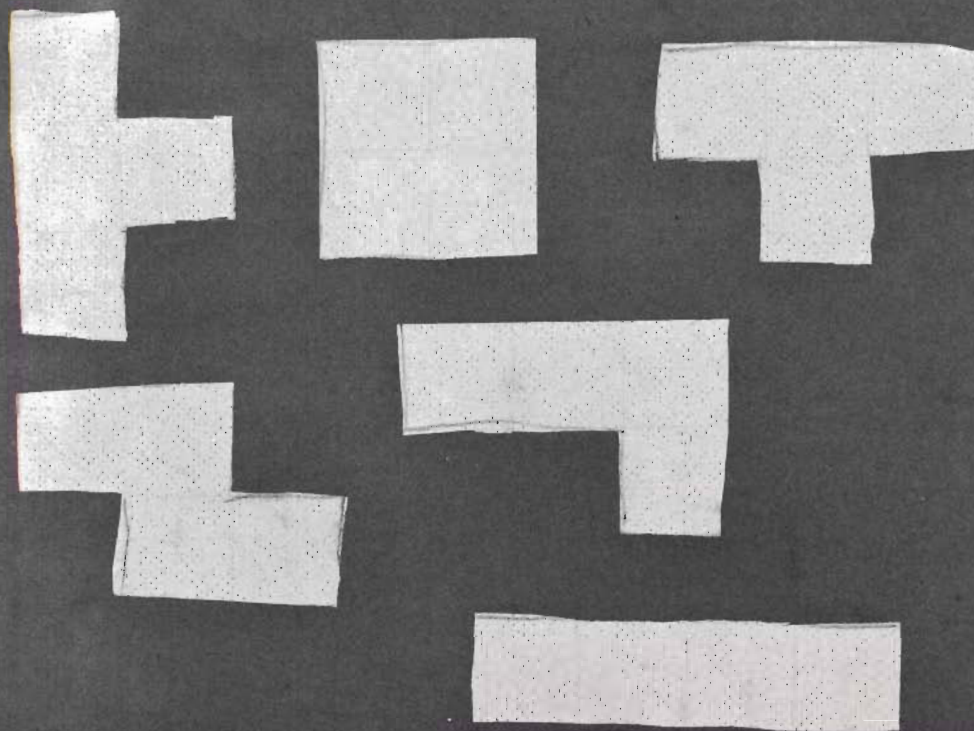
I can only make 1 island from 2 square inches

David



I can make 2 different Islands with 3 square inches

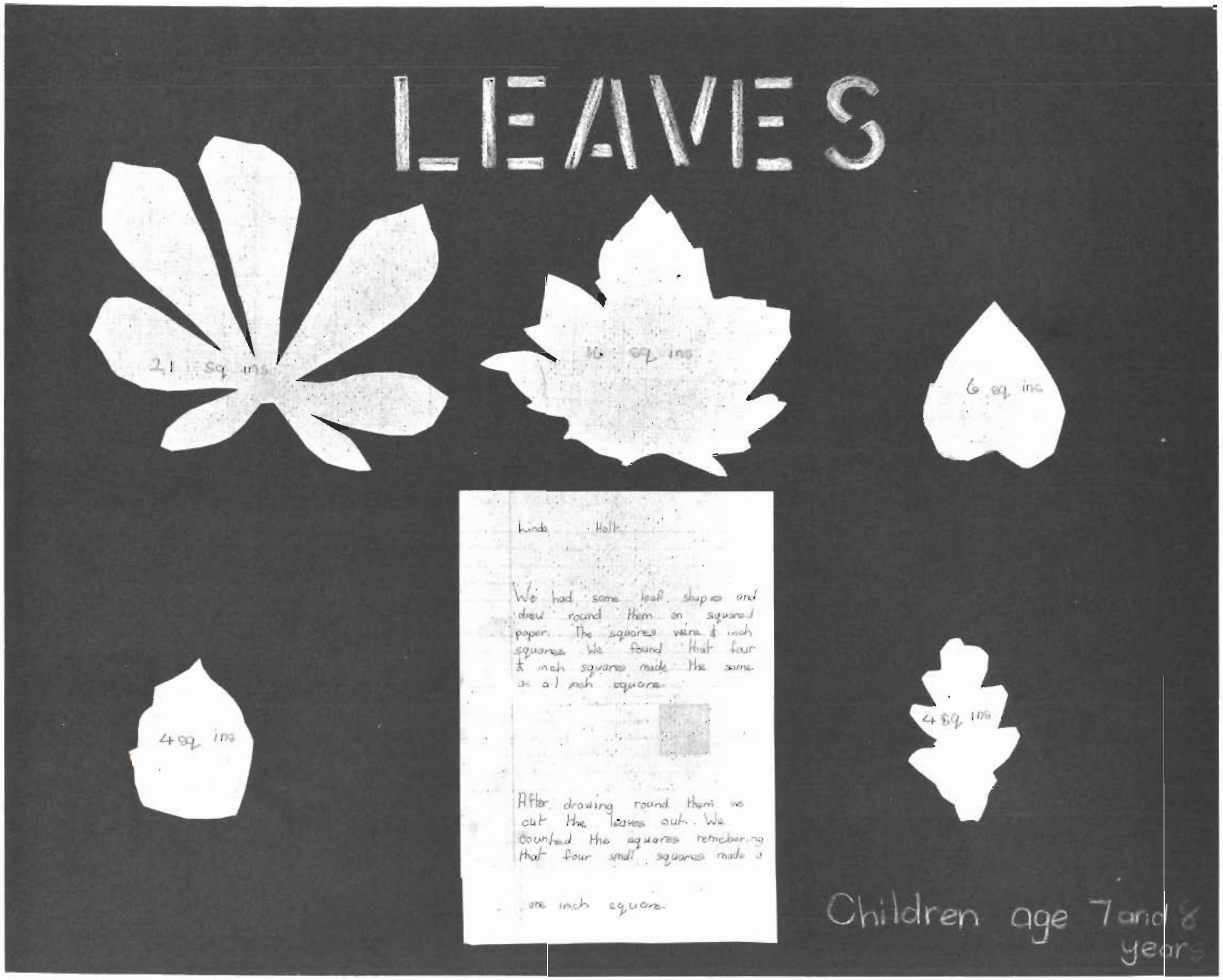
Kim Browring

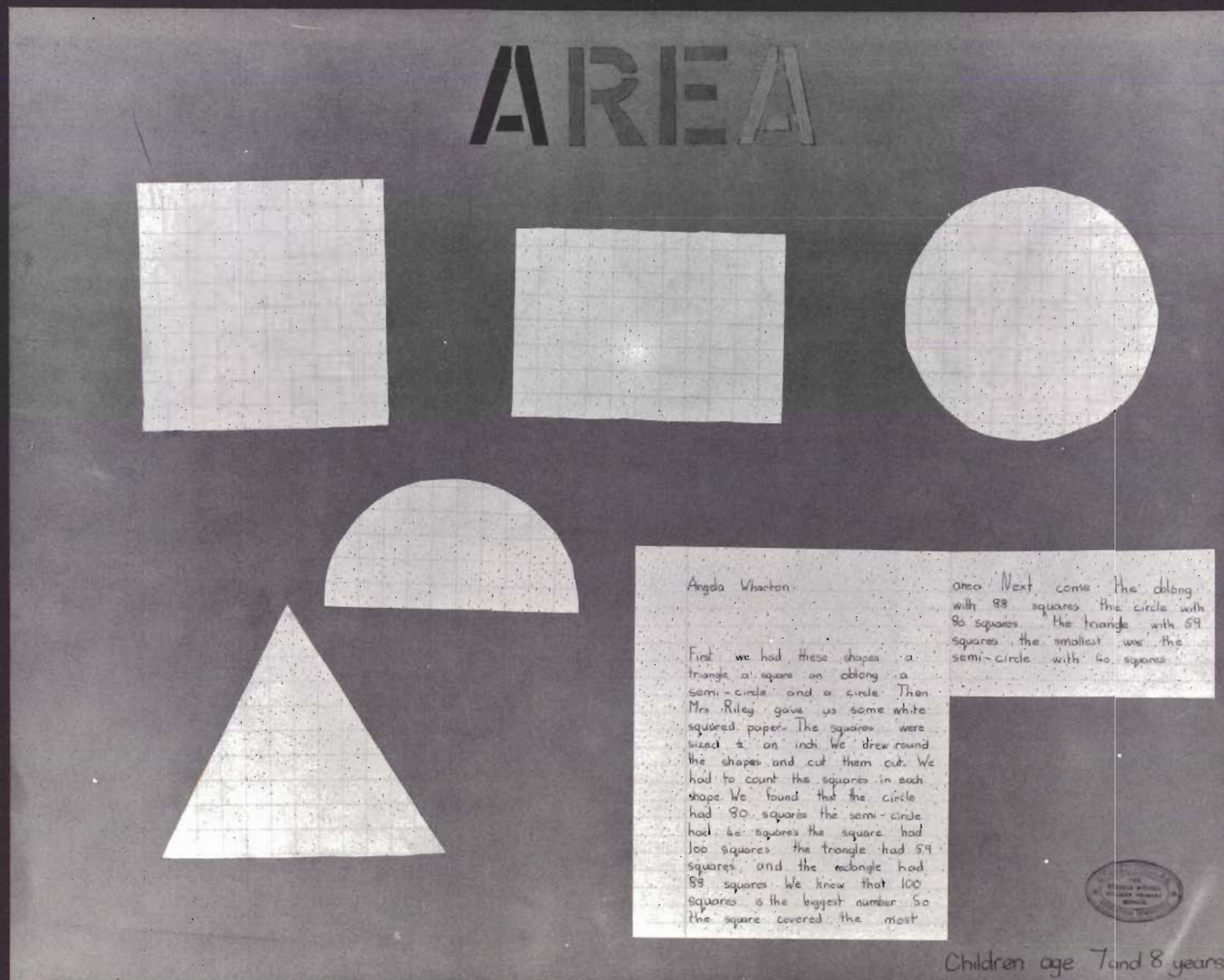


I made 6 aifferenr islonds Froom

4 sqvare inches .

ALLAN PICTON


LEAVES
21 sq ins
16 sq ins
6 sq ins
4 sq ins
4 sq ins
Linda Holt
We had some leaf shapes and drew round them on squared paper. The squares were ½ inch squares. We found that four ½ inch squares made the same as a 1 inch square.
After drawing round them we cut the leaves out. We counted the squares remembering that four small squares made a one inch square.
Children age 7 and 8 years


AREA
Angela Wharton
First we had these shapes a triangle a square an oblong a semi-circle and a circle Then Mrs Riley gave us some white squared paper. The squares were sized ¼ an inch We drew round the shapes and cut them out. We had to count the squares in each shape We found that the circle had 80 squares the semi-circle had 40 squares the square had 100 squares the triangle had 59 squares and the rectangle had 88 squares We know that 100 squares is the biggest number So the square covered the most
area Next come the oblong with 88 squares the circle with 80 squares the triangle with 59 squares the smallest was the semi-circle with 40 squares
Children age 7 and 8 years

Other activities:

> Draw round the sole of your shoe on the squared paper. When your group have all done this find out whose shoe covers up the biggest surface on the paper.

Provide paper with squares on one side only.

> Draw an animal on the plain side of the paper. Cut it out. Turn it over and compare it with the animals of the others in your group. Find out who has the animal with the biggest area. Arrange all of the cut-outs in order of area, biggest first, smallest last. Write about what you did in your group.

A frieze of these, or leaves, could be made for a wall display.

> Can you use the squared paper to find the surface area of this tin? (or other cylindrical object.)

Provide a box with lid, e.g. a chalk box, and a cylindrical tin or container, e.g. a salt drum.

> See if you can find which of these has the larger outside surface area. You may use the squared paper to help you.

Larger squares of cardboard, or hardboard, can be used for comparing large surfaces such as desk tops, the tops of low cupboards and so on. Classroom floor areas can be measured and compared by counting tiles or wood floor blocks.

One other interesting assignment can now be given:

> Use the nailboard and rubber bands to make as many shapes as you can with an area of 6 squares. Copy some of those which you think are interesting in colour on the squared paper.

(No mention here of the kind of shapes which might be investigated – non-rectangular for instance; this and comparison of perimeter will come at a later stage.)

Finally, as with previous topics, children should make up simple tasks for each other to try.

### Looking back over what has been done

The main object of these last assignments has been to introduce the idea of area as covering a surface, or the amount of surface a 2D shape has. From covering surfaces in various ways it can be found that some shapes such as the square, triangle and regular hexagon can be used to cover a surface without leaving a gap or without overlapping, and that the square is probably the most suitable to use as a unit for measuring area. An approximate way of finding areas of irregular shapes, and surfaces of three-dimensional objects was also introduced. At this stage no mention was made of calculating areas by using dimensions such as length and width, and this will come later. The discovery that squares, rectangles, triangles and hexagons fit without a gap while pentagons, for instance, do not, will not be developed further at this stage. Later, as part of the 'spiral' treatment, we shall return to this to discover something about the angles of these shapes.

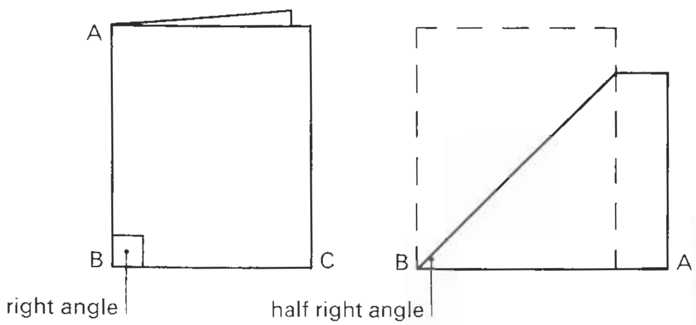
## 3 More about angles

### Right angles and half right angles

In their activities involving fitting square and rectangular tiles children made discoveries about right angles. This was further developed by folding paper to make a right angle, and linked with discoveries about vertical and horizontal. From folding squares and rectangles in the work on symmetry we can now look at 'half right angles'. This will be useful experience for later work on angles when children use the compass with a pointer to consider the idea of an angle by turning.

Suggestions for activities:

Fold a piece of paper to make a right angle (see page 34). Look at the edges of the paper which make the right angle. Fold the paper so that the edges fit one on top of the other like this:

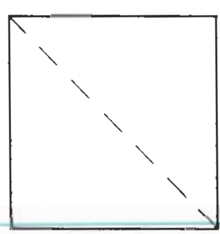



What part of a right angle is the new shape you have made?

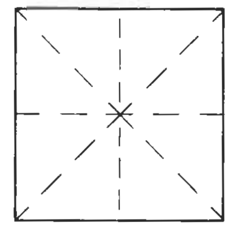
Teachers may have to show children how to fold the paper as in the diagram.

Use the new angle shape (half right angle) to check examples:

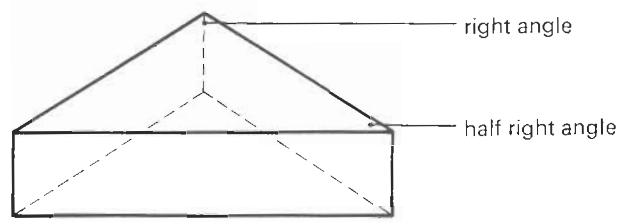
Fold a square of paper along a diagonal (from corner to corner). Use your half right angle to check the angles of the triangle you have made. What do you discover?



Fold a square of paper in half in as many ways as you can. You may remember how you did this when making 'symmetry patterns'. Open out the paper. Check the angles between the creases with your paper 'half right angle'. What do you notice?

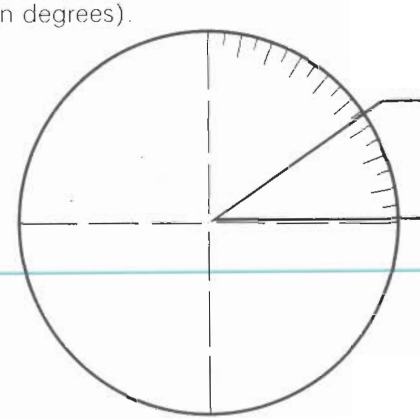


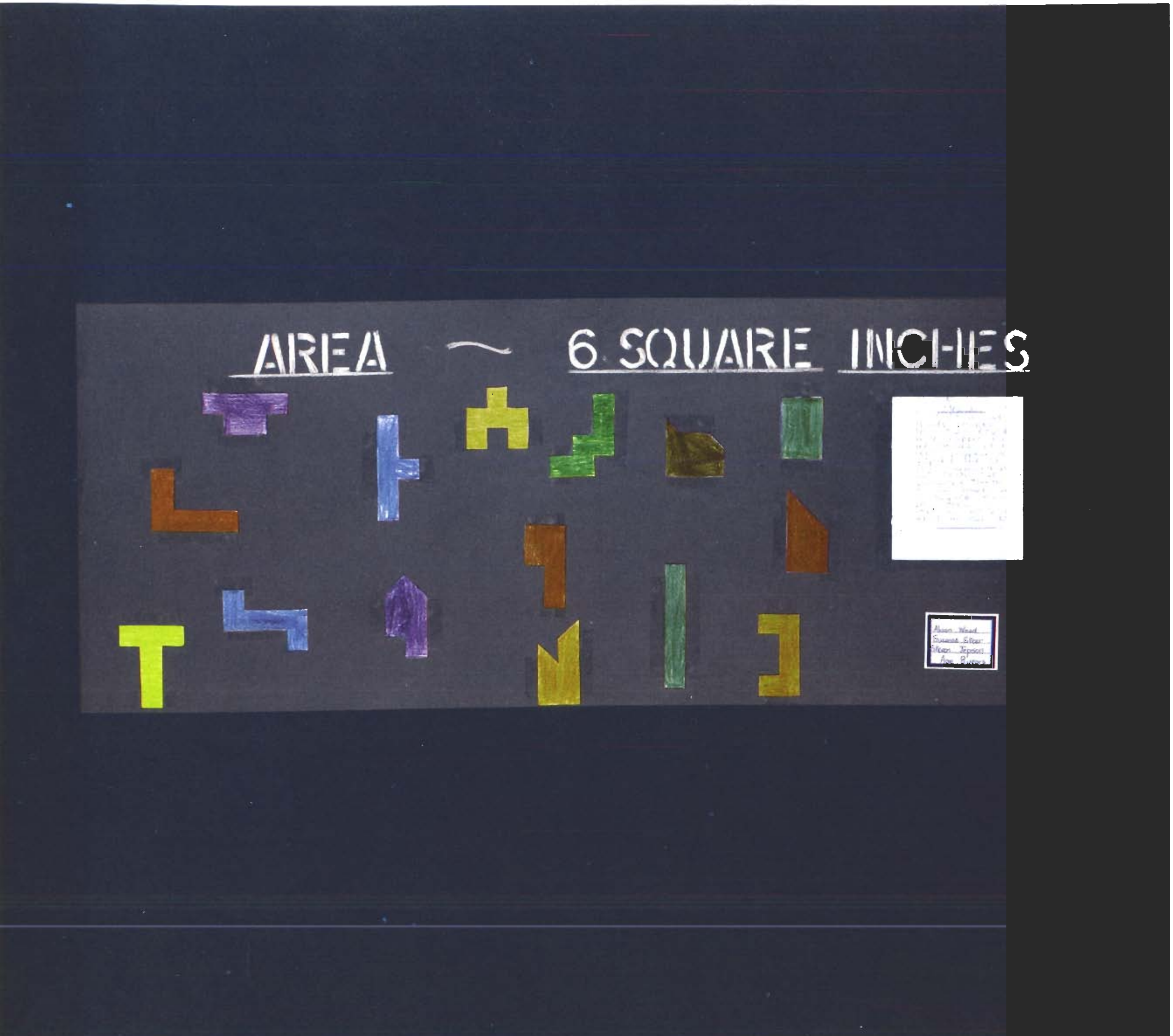
Some of the 'bricks' which young children use are prisms like this:

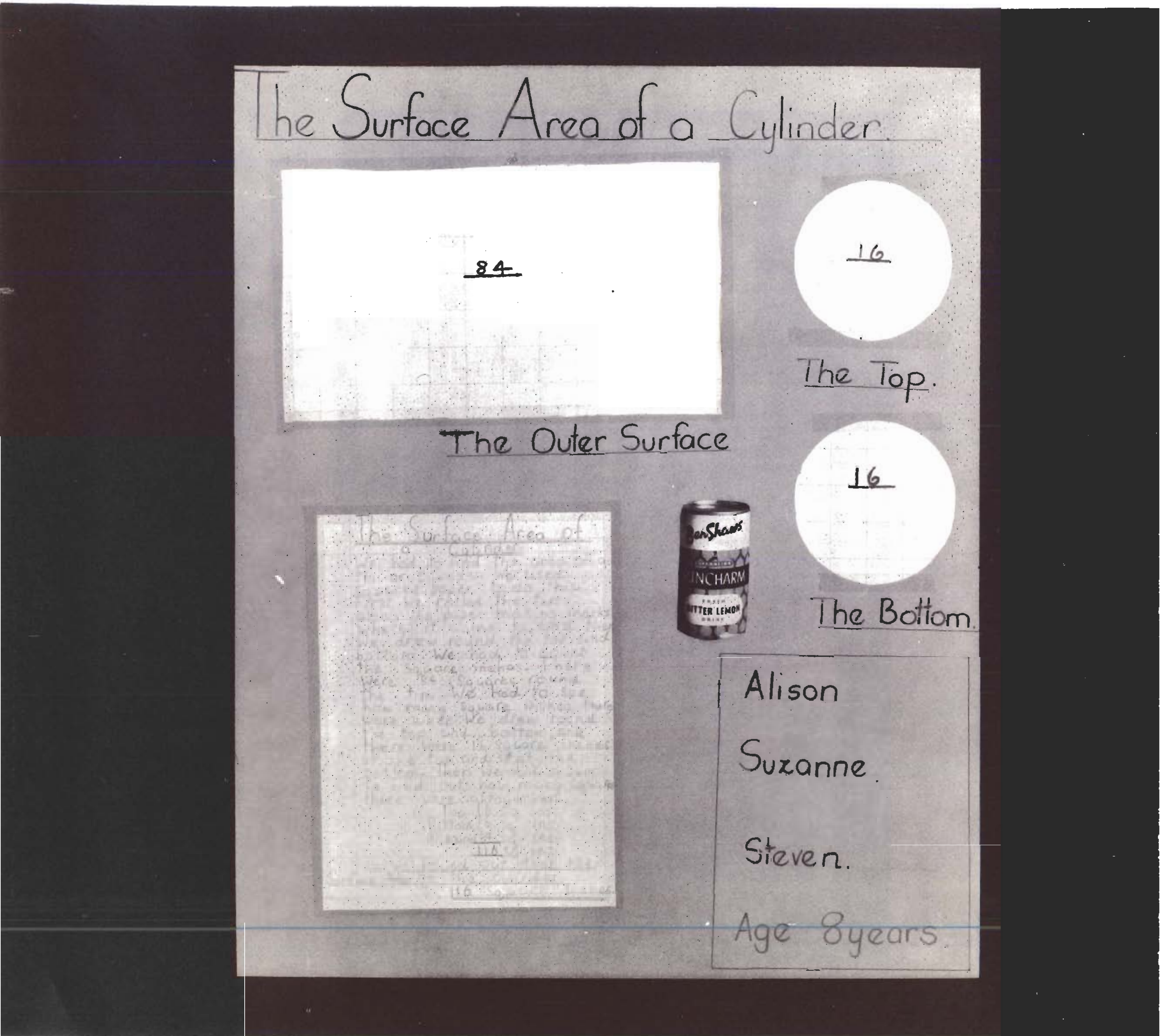



The angles between the edges of a face can be checked with the folded paper 'right angles' and 'half right angles'.

The simplified protractor suggested on page 13 can also be used as a 'carrier' to check angles (not as means of measuring in degrees).




AREA ~ 6 SQUARE INCHES


The Surface Area of a Cylinder.
84
The Outer Surface
16
The Top.
16
The Bottom.
Alison
Suzanne.
Steven.
Age 8years.

Estimate the number of exercise books you will need to cover the top of the table so that the books do not overlap. Then fit the books on to the table top to cover it.

How many did you use?

How near was your estimate?

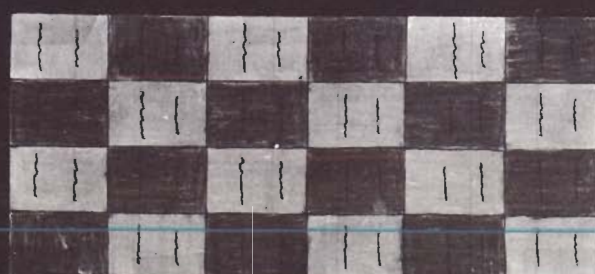
We took a pile of exercise books, and estimated how many it would take to cover Mrs Cotton's table.

I guessed 33;
Diane " 20,
Peter " 29
and Paul " 32

Then we covered the table with books, so that they just touched.

We had 6 books just touching each other along the long side and 4 books along the short side.
∴ 24 books were required altogether.

My estimate was 9 out.
Peter's estimate was 5 out.
Diane's estimate was 4 out.
and Paul's estimate was 8 out.
Diane's estimate was the nearest.



Looking down on the books.

The protractor is placed over the folded paper 'half right angle' as shown and the arms of the angle marked on the protractor in chinagraph pencil.

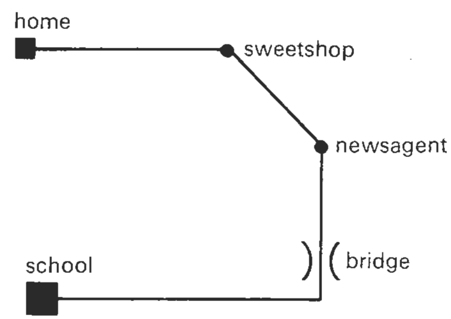
The protractor is then lifted and the 'copy' of the angle 'carried' and placed over any other angle it is desired to compare with the original angle.

At this stage the suggestions have been limited to right angles and half right angles associated with previous work on fitting shapes and paper folding. However, the comparison of angles should not be limited to these if it arises spontaneously from children's own observation and discovery.

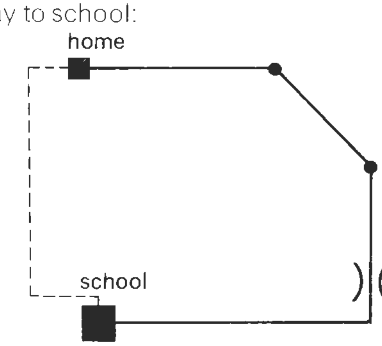
**First stages in drawing plans**

At this stage young children are frequently introduced to the first ideas of drawing simple plans and maps of the classroom and finding their own position, or the position of desks and other furniture in the room. One possible line of development from other class activities is outlined below. No 'assignments' are suggested as most teachers will be able to develop the ideas within their own school and local environment, and may have other ideas of the way plans can be introduced. The suggestions given here are meant only to indicate one possible way.

For instance, in local or environmental studies, or geography, a start can be made by children observing things on the way to school – houses, shops, roads and road junctions, factories, bridges and so on. A visual record can be made for these local studies and this will involve drawing maps. At first these will be no more than the children's own imaginative interpretation of the way to school, or a walk round the district. This can lead to some interesting discussion with the children on the way they come to school, and on giving directions about the route they take, without considering any idea of distance or scale. As adults we very rarely give an indication of distance when asked directions to reach some specified place. Rather we use such expressions as, 'Take the first turning on the left and then turn right by the cinema'. With children discussion can arise from questions such as, 'How many main roads do you have to cross on the way to school?' Parents are concerned about the safety of their children on the way to school and may suggest alternative routes which do not involve crossing busy roads, and these can be discussed. Definite points on the way to school can be located and routes planned to include these, e.g. 'Which way would you come to school if you wanted to buy some sweets at a sweet shop, and go to the newsagent's to buy a comic?' This could be represented by a simple 'network' map:



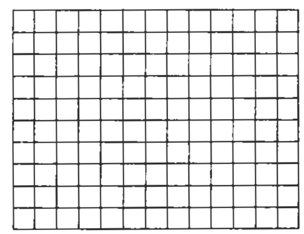

A shorter alternative route might be added to this to show the usual way to school:




Children can be asked to describe another way they could use if some of the roads on their usual route were closed for road works. They can give directions to a friend to indicate the way to get from his house to theirs for a visit or a party. After discussion about directions and routes from the above and similar suggestions, children can attempt simple 'rough' sketches of these as 'networks', without any indication of distance. With some children this may have to be introduced first by the teacher making 'network' maps on the blackboard as examples, as they arise from the discussion, and then leaving the children to attempt their own. Note that no mention is made of measurement or distance at this stage. The above are only a few suggestions and teachers will be able to think of others suitable to their school environment.

A more representative plan can be made from a model of a small part of the district round the school. The teacher will have to outline the roads or streets on a base of thick card or hardboard and the children can then make models to represent houses, shops, the school, and other buildings, and place these in position. These are not made to scale, but can show some idea of relative size. The next stage can be to produce a map from this model. The idea of scale involves some skill in measurement and is not an easy idea for young children to grasp. It should not be introduced here. Rather, the teacher can produce another copy of the roads drawn to the same scale and the children can 'map' the houses and other buildings from the model to the new plan by looking at the model from above and representing them by rectangles or squares. They can see the relationship between the buildings as seen from above and the drawings of these on the map. Further examples of looking at objects from above and drawing plans can be introduced in the classroom to develop the idea of a plan without the added complication, at this stage, of introducing scale. Children can stand on top of desks, for instance, and look down on a flat rectangular surface which may have a circular hole for an inkwell. They can make their own attempts at drawing what this looks like without any measuring. As long as the children have some idea of relative size, scale should not be introduced too early.

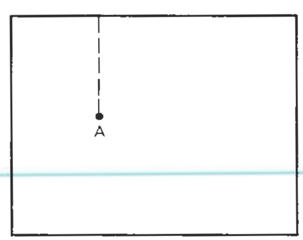
But later, having progressed through the first ideas of making a 'map' they can be given activities involving simple scale drawing and plans. This can be built on previous experience such as work on 'square corner' right angles, and also serve as introduction to the idea of perpendicular. But what may appear a simple activity such as drawing a scale plan of a desk top may well produce results such as: 'length about 3 ft 7 in, width about 1 ft 5 in', which are too difficult to convert into an easy scale such as 1 inch to 1 foot. It may be advisable to start with larger plans such as that of a room and in the first instance to use approximate measures such as paces to find the length of walls. For example, in the diagram below, children could measure the length and width of the room by pacing, and record the result as: length about 13 paces, width about 10 paces. This can easily be transferred to 1-inch squared paper to produce a simple plan with the first idea of scale: 1-inch represents 1 pace.




Using squared paper simplifies these first attempts at making a plan, and it is advisable to keep to rectangular shapes. When plans are made on plain paper the 'square corners' can be drawn by using the folded paper right angles and the sides drawn to the same scale as on the squared paper (1-inch represents 1 pace) by measuring with a ruler marked in inches.

## 'Perpendicular'

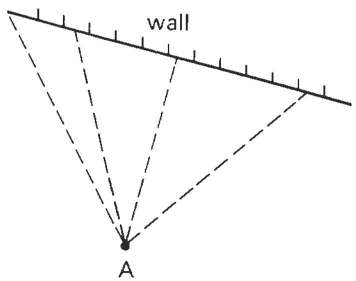
At this stage this is probably sufficient introduction to making plans using a simple scale. The foregoing activities can be associated with the work on right angles already done and then used to introduce the idea of *perpendicular*.




If this is the plan of the classroom and the child (or desk) is at A, the real distance from a wall is the shortest distance between the child (or desk) and the wall.

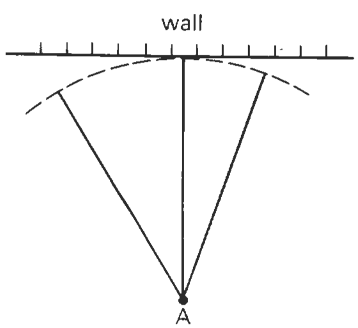
**Suggestions for activities to establish this idea**

The following may be given orally to a small group of children. The first two can be done initially in the playground if desired.




A child to stand at A, say about 10 yards from the wall. He has to run and touch the wall as quickly as he can. What part of the wall should he run to? Why? Try running to different parts of the wall (see dotted lines). Discover that he must take the shortest path.

Find out more about the shortest distance. e.g. One child to stand at A holding one end of a rope or long piece of string, on the floor. His partner stretches the rope until he reaches what he estimates is the shortest distance from the first child to the wall.




Then by holding the rope taut and moving it in an arc, as shown by the dotted line, he can find the position for the shortest distance. Discuss with children why this shows the shortest distance.

When the position is found, the line of shortest distance can be marked in chalk on the floor.

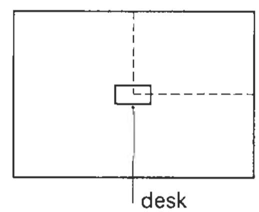
Give each pair of children a large piece of paper (at least imperial size).

> Fold the large piece of paper twice to make a 'square corner' (right angle). Use this to check the angle between the line of shortest distance and the wall. What do you discover about this angle?

Do several of these.

'Does each of the shortest lines make a right angle with the wall?'

Similar assignments can be given inside the classroom to find two lines to fix the position of a desk as in the diagram:

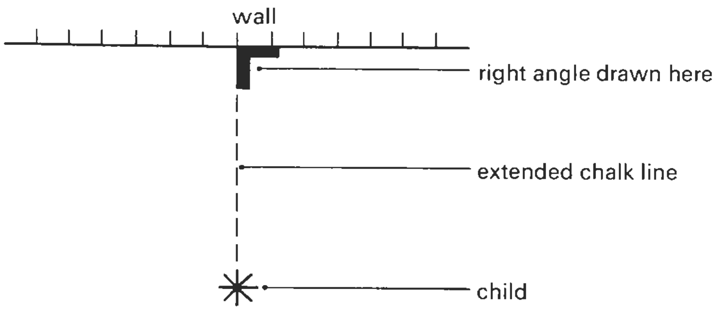



As an additional part of the work children could be asked to estimate the distance each time and then to measure it.

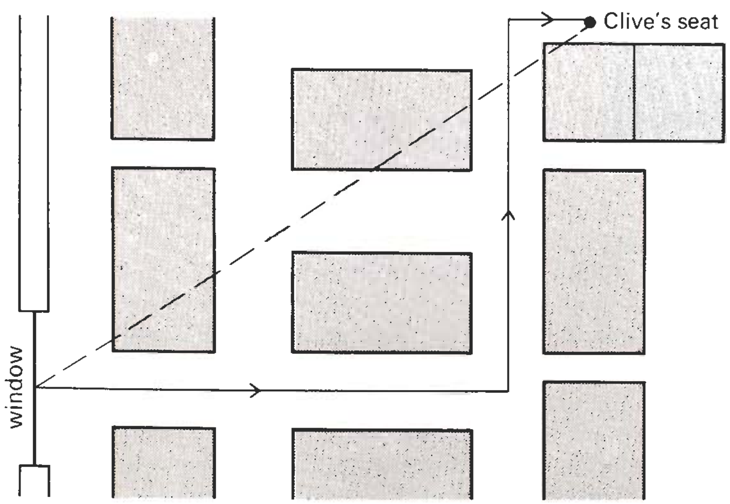
Having discovered something about the angle which the line of shortest distance makes with the wall it is interesting to suggest that children try this in another way.

> Use your 'square corner' right angle to make a right angle with the wall. Draw a chalk line to show this.
> Then use a long piece of wood with a straight edge to draw this line about 6 feet long starting from the wall.
> Get your partner to stand on the line at about 6 feet from the wall.
> Is his shortest route to the wall shown by the chalk line? Why?
> Use the rope to check this in the same way as you did before.

The diagram will help teachers to see the idea for this assignment:




There was an interesting development of this in a group of eight-year-old children. Clive suddenly said, 'When I walk from the window to my desk I go along paths which are perpendicular to each other.' His teacher asked him to explain this and he said he could only do this by showing her what he meant. The route he took is shown arrowed on the diagram.

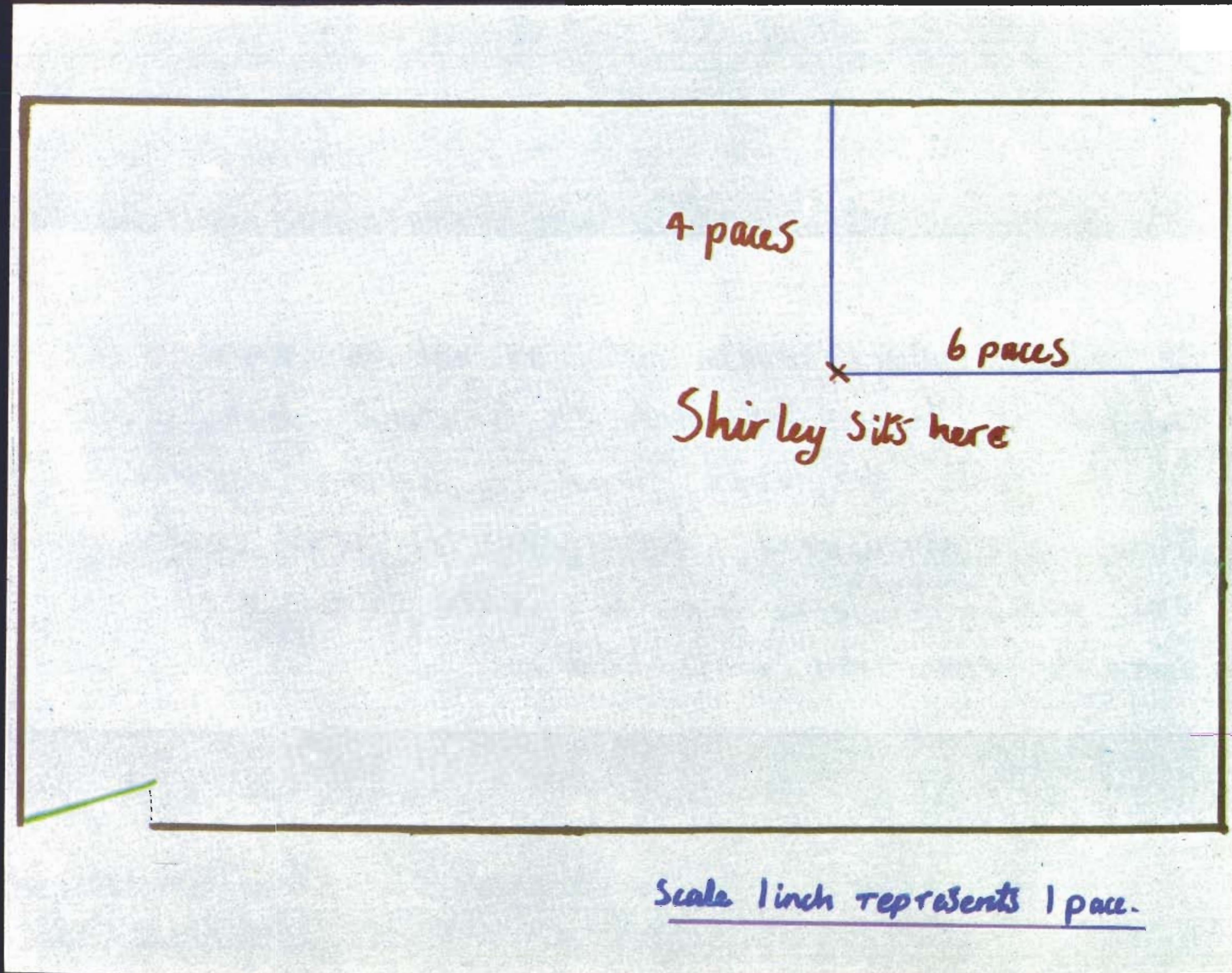



His description was, 'First I go along this line and that is perpendicular to the wall. Then I turn a right angle and go along here, then I turn another right angle and go to my desk.' The teacher then asked him, 'Do you think that is the shortest distance for you to walk to your desk?' When he replied, 'Yes,' she asked, 'How do you know?' There was a pause while he was obviously trying to put his words together. Then he said, 'Each time I turn I go at right angles, so they are perpendicular to each other, so each bit is the shortest distance, so the whole lot must be the shortest distance.' The teacher thought that this was the end of the discussion, and was satisfied as the boy had obviously reasoned from his practical experience about perpendiculars – but Clive was not satisfied. A few minutes later he came to his teacher and said, 'If the desks weren't in the way, I think it would be shorter to go straight to my desk than along those three lines.' The discussion continued. When asked if he could find out if this was so, he and the boy who sat next to him obtained a long tape measure and measured the alternative 'routes'. The teacher went on, 'Could you find out without actually measuring in feet and inches with a tape measure?' This defeated them until she said, 'Well, could you use a piece of string?' They soon saw the way to do this by finding that they needed more string to stretch along the three perpendicular paths than by stretching it direct from the window to the desk (as shown by the dotted line on the diagram).

### Looking at what has been done

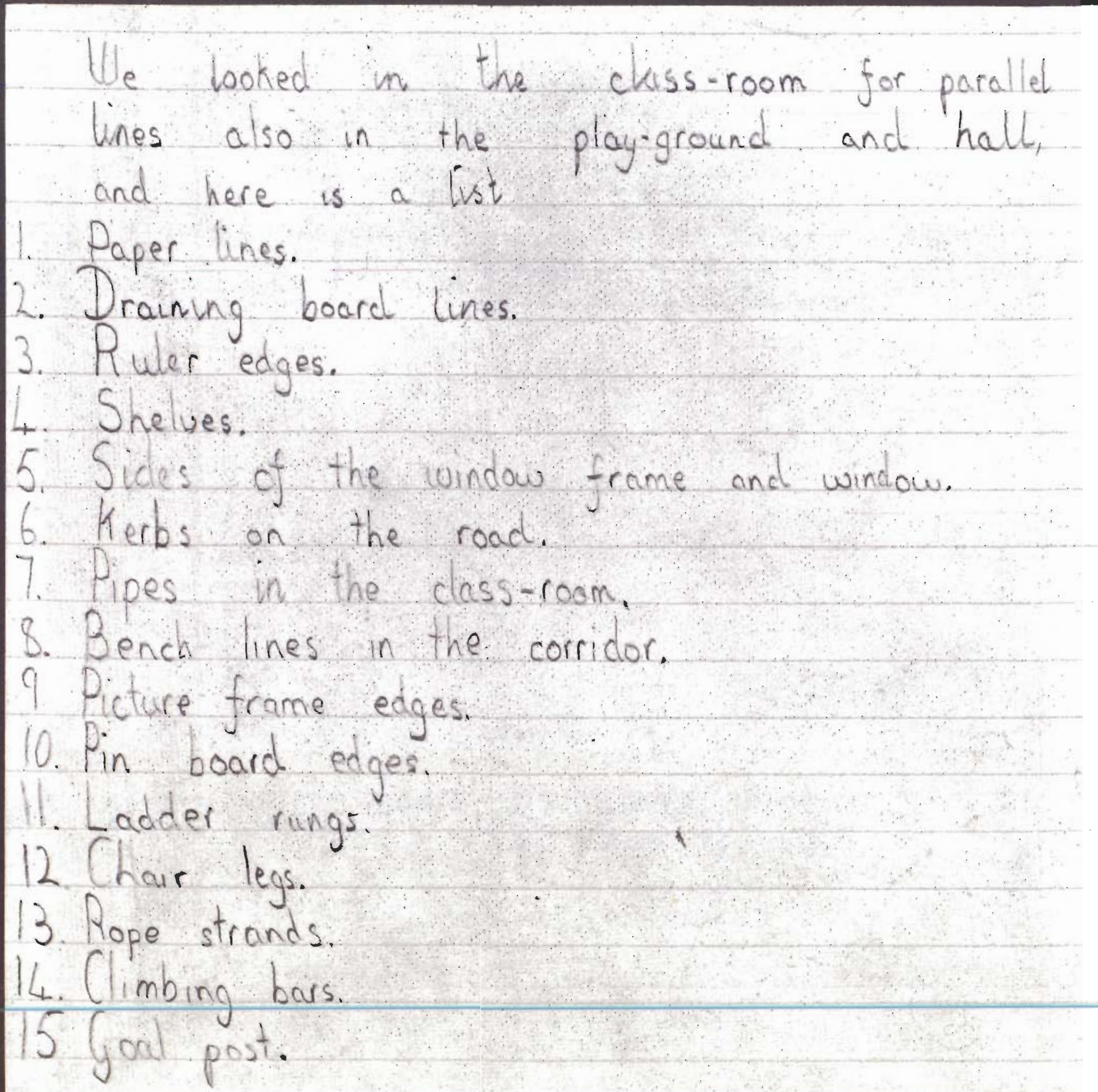
Teachers could now introduce the word *perpendicular*. Up to now the right angle has been seen as a 'square corner' in shapes, or between edges of surfaces and faces, and also where vertical and horizontal lines meet or cross. We have now seen that when one straight line meets another at right angles we use the term *perpendicular* – one line *'is perpendicular to'* the other. This was introduced as the shortest distance from a point to a straight line. The shortest distance from a given point to a given line is along the perpendicular from the point to the line.

Here is a 'seat plan' made by two eight-year-olds.



To find out where Shirley sat, we got a rope and put it from the wall to Shirley's desk. At the wall the rope made a square corner. Then we measured how far it was with our paces. It was 6 paces from one wall and 4 from the other wall.

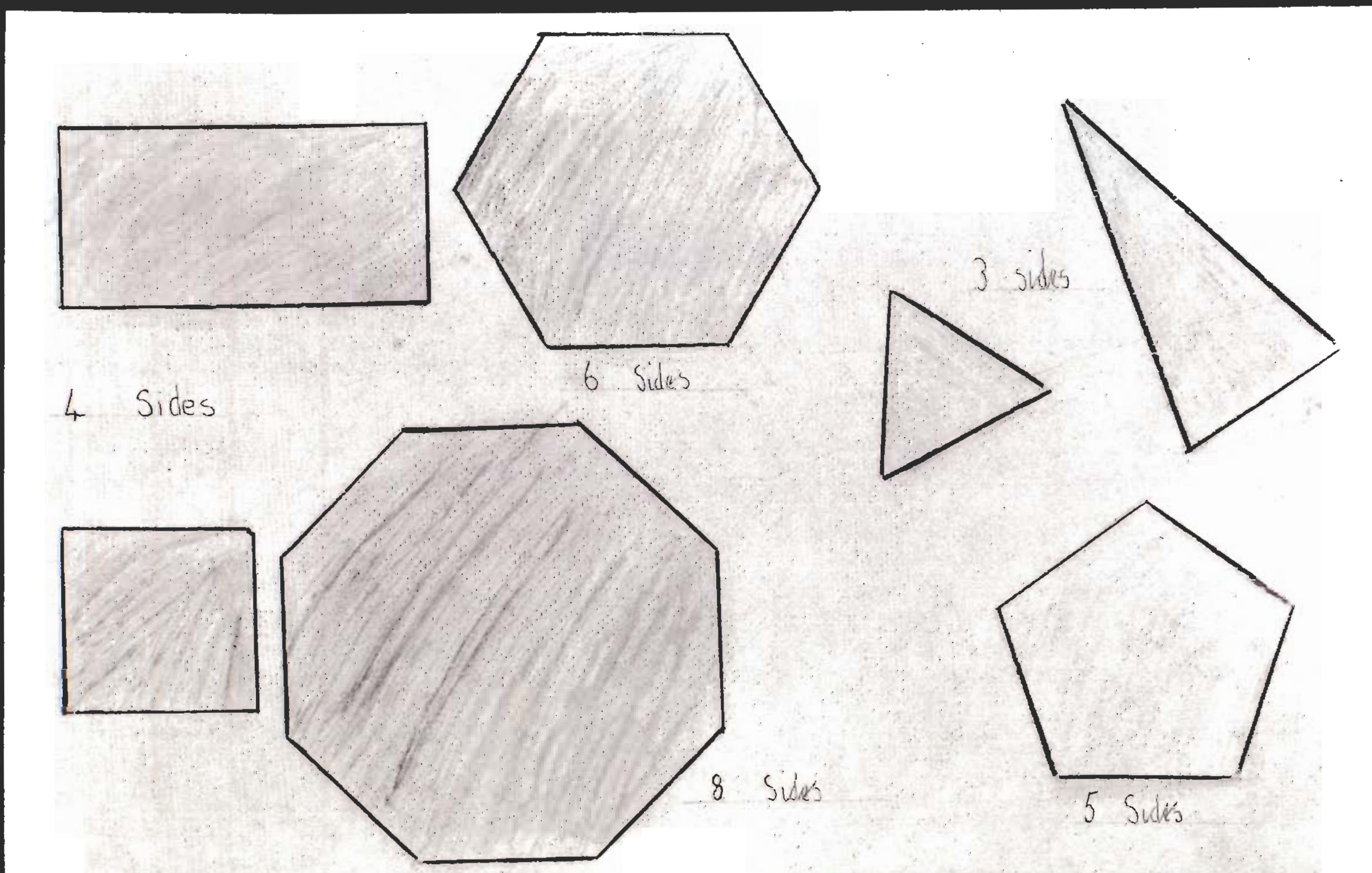
The following is part of the work of one eight-year-old girl on 'parallel and perpendicular'.



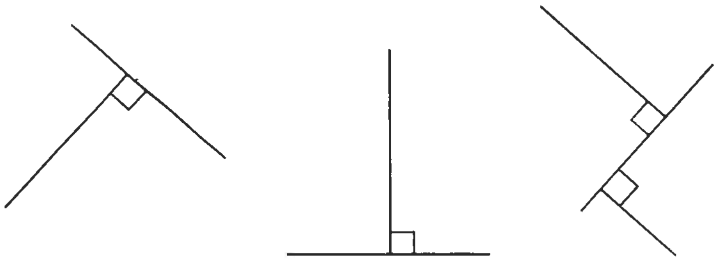
We looked in the class-room for parallel lines also in the play-ground and hall, and here is a list

1. Paper lines.
2. Draining board lines.
3. Ruler edges.
4. Shelves.
5. Sides of the window frame and window.
6. Kerbs on the road.
7. Pipes in the class-room.
8. Bench lines in the corridor.
9. Picture frame edges.
10. Pin board edges.
11. Ladder rungs.
12. Chair legs.
13. Rope strands.
14. Climbing bars.
15. Goal post.

Shapes which have an even number of sides have their opposite sides parallel. None of the sides of shapes with an odd number of sides are parallel. The rectangle and square have sides which are perpendicular.


4 Sides
6 Sides
3 sides
8 Sides
5 Sides

To fix the notion of perpendicular, children can be asked to find examples in the school. They can also use their folded paper 'right angles' to draw some lines where one is perpendicular to the other, e.g.



They can look at shapes they know (squares, rectangles, etc.) such as geometrical shapes they have used for covering surfaces, and also in everyday objects, and find which of these have one side perpendicular to another, and so on.

## Parallel lines

We shall now consider *parallel* lines and then return to *perpendiculars* to discuss both in connexion with *relations* (*Mathematics Begins* ❶).
'Parallel' should not be defined formally but examples and counter examples given until the idea is clear.

> You have found some shapes which have edges which meet at a right angle, and other lines in the classroom which cross at right angles. Now can you find some which are parallel?
> Make a list and think of some examples outside the school too. You may make drawings if you wish.

> Look at two lines in a writing book, or on a sheet of writing paper. Do the lines look parallel?
>
> Does it matter which way you turn the paper to look at the lines?
>
> Get some squared paper. Do the lines which are drawn up and down the paper look parallel?

> Work with a partner. You will need one sheet of lined writing paper between the two of you and a ruler each. Put one ruler along the top line, and the other ruler along the third line from the top. Do the lines look parallel? What would happen if you could draw these lines longer and longer and longer? Do you *think* they would ever meet?

Repeat with other lines, e.g. first and sixth, third and sixth, top and bottom, and so on.

Children can be given cardboard or plastic shapes, or shapes cut from lino, (squares, rectangles, rhombuses, and so on) which they used in earlier work and asked to find which shapes have right angles, and which have opposite edges parallel. They can also look for similar shapes about the classroom and elsewhere. These can be tabulated in lists.

## A look at equivalent relations

In *Mathematics Begins* ❶ the use, and misuse, of the equality sign was considered in some detail by looking at *equivalence relations*. It will be recalled in particular that the relation *is equal to* for numbers (i.e. 'is the same number as') is an example of an *equivalence relation*. There are three requirements for this. The relation should be:

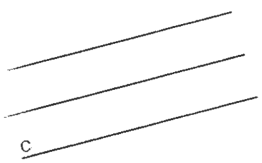
i. **R**eflexive
a=a (where 'a' stands for 'any number')
For example, 4=4, 7=7, 12=12
ii. **S**ymmetric
If a=b then b=a
For example, if we have learned that 3+4=7 then we know that 7=3+4
iii. **T**ransitive
If a=b and b=c, then a=c
For example, if we have learned that 3+4=7 and that 7=6+1, then we know that 3+4=6+1

Relations in general were the subject of Chapter 2 in *Mathematics Begins* ❶, and when considered in the light of the above 'test' it can be seen that some relations are equivalence relations and others are not. Another example of an equivalence relation given in *Mathematics Begins* ❶ was the relation *is parallel to*. But is the relation 'is perpendicular to' also an equivalence relation?

The discovery and experience about parallel lines and perpendicularity can be used to see if, in fact, both are equivalence relations. The 'test' is to see if they satisfy the requirements given above (Reflexive, Symmetric, Transitive).

In the diagrams below, *a, b, c* are the names of straight lines in a *plane*.

**Parallel**



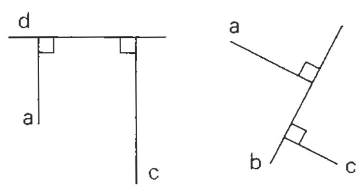

Let // stand for 'is parallel to'

i. **R**eflexive? a//a
The line a 'is parallel to' itself is 'obvious'. So the relation 'is parallel to' is reflexive. **Yes**

ii. **S**ymmetric? If a//b, then b//a
If the line a is parallel to line b, then we know that line b is parallel to line a. The relation 'is parallel to' is symmetric. **Yes**

iii. **T**ransitive? If a//b and b//c, then a//c
We can see that given line a is parallel to line b, and line b is parallel to line c, then line a is parallel to line c. The relation 'is parallel to' is transitive. **Yes**

**Perpendicular**




Let ⊥ stand for 'is perpendicular to'

i. **R**eflexive?
It is obvious that line a is not at right angles to itself. So the relation 'is perpendicular to' is not reflexive. **No**

ii. **S**ymmetric? If a ⊥ b, then b ⊥ a
If the line a is perpendicular to line b (i.e. if a is at right angles to b) we know that b must be perpendicular to a. The relation satisfies the requirement. 'Is perpendicular to' is a symmetric relation. **Yes**

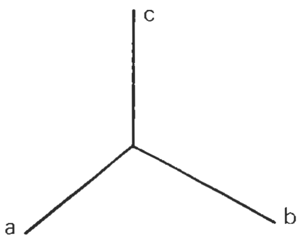
iii. **T**ransitive?
If line a is perpendicular to line b, and line b is perpendicular to line c, then line a is *not* perpendicular to line c. Line a is, in fact, parallel to line c. So the relation 'is perpendicular to' is *not* transitive. **No**

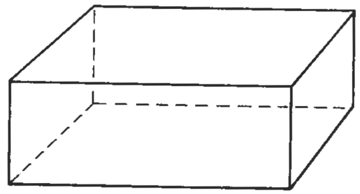
From the above it can be seen that 'is parallel to' and 'is perpendicular to' behave differently. The relation 'is parallel to' satisfied the three requirements, R, S, T and is an *equivalence relation*, just as 'is equal to' was seen to be an equivalence relation in the example for numbers. But the relation 'is perpendicular to' does not satisfy the three requirements, and so it is not an equivalence relation.

It is not suggested that the above should necessarily be presented to children in this way, but from their discoveries about parallel lines, and lines which are perpendicular to (at right angles to) each other, teachers can discuss the fact that the two relations behave differently.

As a postscript, if the lines were not restricted to lying in a plane, it is *possible* that lines a, b, c could all be perpendicular to each other (think of the corner of a brick), but if a ⊥ b and b ⊥ c we cannot *infer* that a ⊥ c (think of the steps of a spiral staircase and the central shaft).

For the corner of a brick we have:

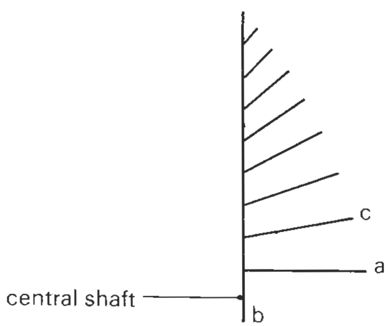


$a \perp b$, $b \perp c$, $c \perp a$

At any corner the edges are perpendicular to each other.

But for a spiral staircase, each 'step' is perpendicular to the central shaft, but the steps are not perpendicular to each other.



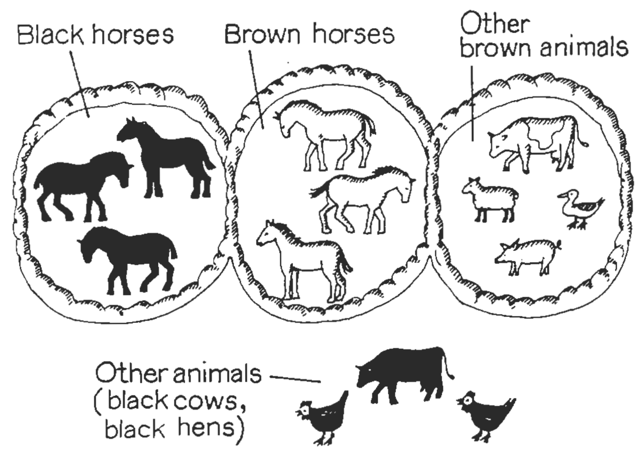

$a \perp b$, $b \perp c$. but a is <u>not</u> perpendicular to c

### 4 Classification of 2D shapes

**This final section of the guide describes only a possible line of development. It is looking some way ahead, and such work should not, of course, be rushed.**

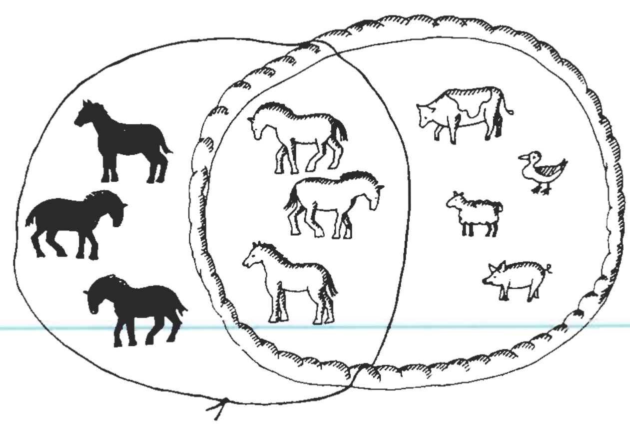
In *Mathematics Begins* ❶ a good deal of sorting activity was suggested. This experience can now be turned to the classification of geometrical shapes, but first we recall the earlier work by giving an example from the classroom.

Alan and Stephen (aged seven) were playing with model animals and pieces of papier mâché hedgerow on a large tray. They were told the story that one farmer was interested in all the horses and another in all the brown animals, and the problem was presented to arrange the hedges so that each farmer could be told where to look for the animals he wanted. Stephen assembled the horses and Alan the brown animals, and there was a tussle when the brown horses were discovered. The first plan was as follows:

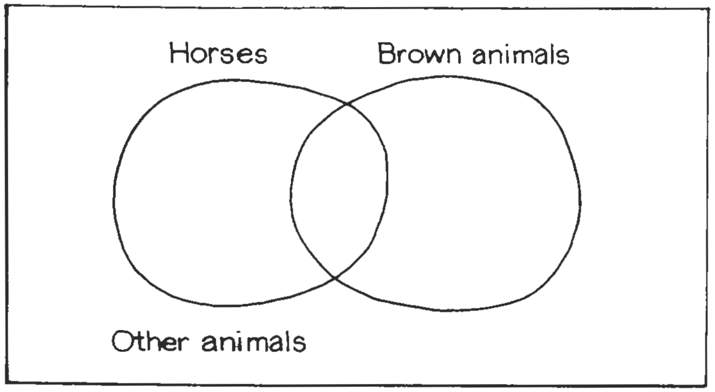



The first farmer was to be told to 'look in the first field and the second field' and the second farmer to 'look in the second field and the third field'.

Later a piece of string was used as a 'wire fence'.



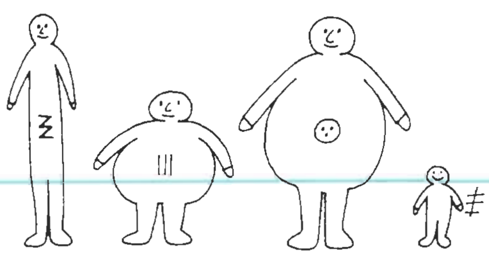
All horses were inside the fence, all brown animals inside the hedge and so the brown horses were in the common enclosure, the intersection of the set of horses, and the set of brown animals.



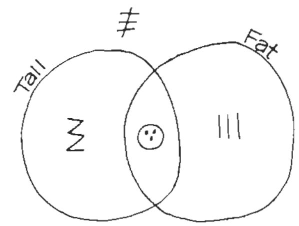

The 'universe' being considered is the given set of farm animals.

The animals were then mixed up again. Alan was asked to pick out the horses and from these the brown ones so that he had the three brown horses. Starting again, Stephen picked out first the brown animals and from these the horses, again ending up with the three brown horses. They readily agreed that if you wanted the brown horses it didn't matter in which order the sorting was done – 'brown' first or 'horses' first.

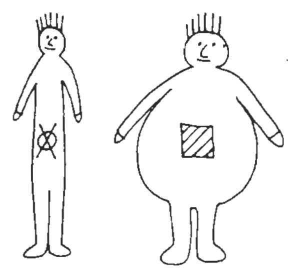
Alan and Stephen were then ready to go over to drawing. Their teacher started drawing grotesque people and asked if they should be called 'tall' or 'fat'. Each was given a squiggle to identify him, which Alan and Stephen found amusing.



They agreed that the first was tall but not fat, the second fat but not tall, the third both and the fourth neither, and they put in the corresponding squiggles in a diagram.

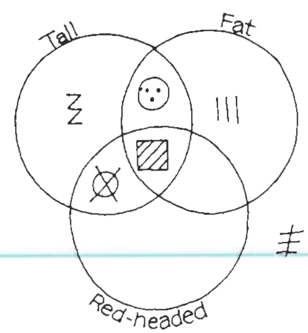



It happened that a red ball-pen was on the table, so the teacher introduced a new category, 'red-headed' (indicated by  here).

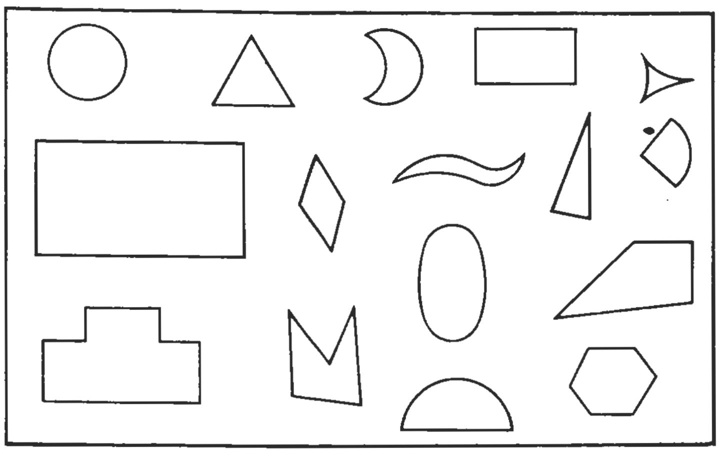


etc.

The children added to the diagram, by this time making their own drawings and having much discussion about the correct categories and where the squiggles should be placed.

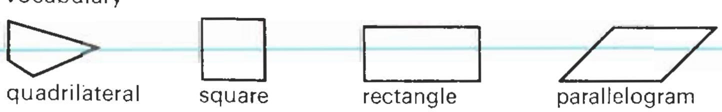


We can use this successful attempt at classification and the approach adopted by these children, to return to the classification of some two-dimensional shapes (cf. p. 2), and then see how this has uses in fostering the beginnings of logical argument. Children can be given a collection of shapes cut from thin card to represent two-dimensional plane shapes, some bounded by straight lines (squares, rectangles, pentagons, hexagons, parallelograms, and so on) and others bounded by curves (circles, ellipses, and so on) and asked to partition the set into two sub-sets. They should be allowed to sort in their own way and then to discuss the reasons for their particular ways of sorting. The discussion can lead to sorting into the particular two sub-sets, (i) those shapes bounded by straight lines and (ii) those bounded by curves. The actual objects can be put in appropriate 'pens' made with loops of string, or the children may suggest other ways of recording the sorting. Later, instead of using actual shapes a duplicated 'puzzle sheet', as shown below, can be given, e.g.



We now concentrate on the shapes bounded by straight lines (and call them *polygons*) and see what can be done with these in the way of sorting.

vocabulary

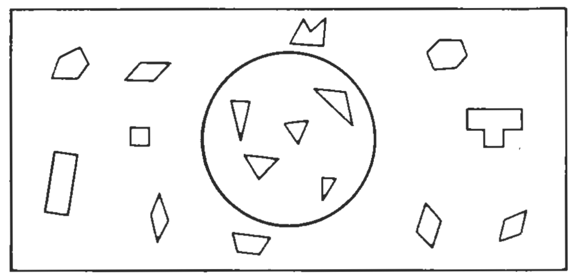




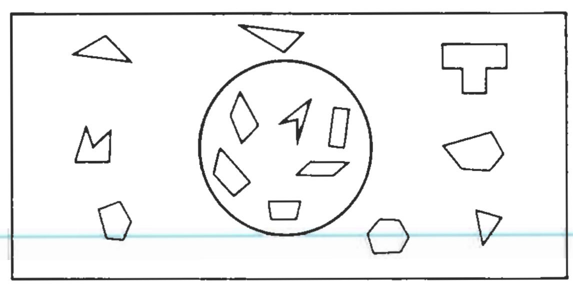
For this they will need a collection of polygons (several of each type). As a suggestion they could be asked:

> Take the set of shapes and sort them into sub-sets like this: those with 3 sides, those with 4 sides, those with 5 sides, and so on. Make a loop of string or coloured wool round each of your sub-sets, or draw chalk rings round them on the floor.
>
> What do you call the sub-set of shapes with 3 sides? Do you know the name for the sub-set of shapes with 4 sides?

Children can discover that, for example, the set of triangles is *included* in the set of polygons.



The children can then sort again to show that the set of quadrilaterals is included in the set of polygons.



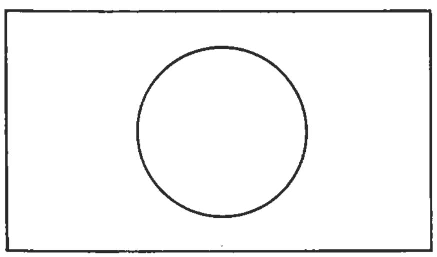
In these activities the 'universe' is the given set of polygons (i.e. the set of polygons they are using), not all the polygons in the world. Teachers will recall that the 'universe' in the farm animal activity was the given set of animals.

The statement
'the quadrilaterals are all polygons' (true)

can be contrasted with its *converse*
'the polygons are all quadrilaterals' (false).

We may now be able to discuss other examples leading to logical thinking which are not concerned with shapes, e.g. 'Think about all the children who come to your school (the set of children attending _____ school), and the children in your own class (the set of children in class 2).'

Discuss representing this in the diagram form.



If a child attends the school he belongs inside the large 'pen'. Can we tell whether he is inside the small one?

Which set of children should we put inside the small 'pen'?
If a child is in Class 2 is he also in the school?
Is it true to say that if John is a pupil attending _____ school he is necessarily in Class 2?

Other examples might be discussed, and perhaps diagrams drawn, e.g.

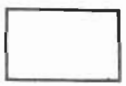
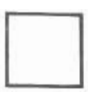
All children in the school.
Those who wear school uniform:

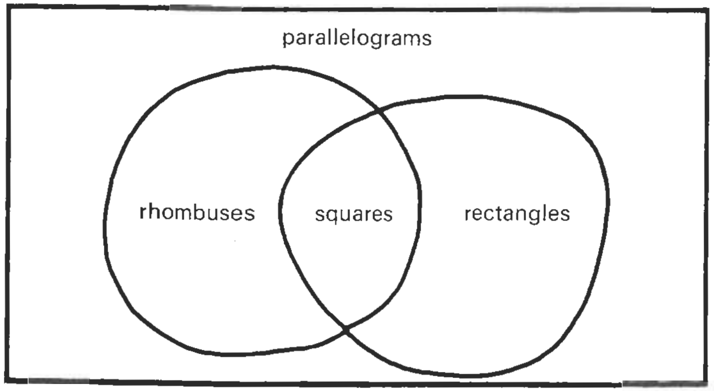
If a child wears school uniform does he attend this school? If he attends this school, does he necessarily wear school uniform?

Teachers can think of other situations which lend themselves to this treatment and which can be represented by diagrams. Children can then be asked to decide which statements are true or false. ('*If* he attends the school, *then* he is in Class 2' and so on.)

We can use the experience about shapes which has been built up earlier in the guide to find out more about the quadrilaterals. The discoveries about right angles, parallels, perpendicular, about shapes which fit together without a gap or without overlapping, and about symmetry, can all be used for further activities.

In this paragraph the reader is reminded of certain properties and subsequently we see how the children might discover them. Let us look at some quadrilaterals, for instance, parallelograms, rectangles, rhombuses and squares. A parallelogram has opposite sides parallel and equal, and opposite angles equal. A rectangle possesses all these properties, but also more. Similarly a square has all the properties of a rectangle and some additional ones. In everyday language we look at things in an *exclusive* way, and might therefore be tempted to say that a square is not a rectangle. But in mathematics we use an *inclusive* definition, and hence think of a rectangle as a special kind of parallelogram, and a square as a special kind of rectangle. So a square is an even more special kind of parallelogram. A rhombus has all the properties of a parallelogram, and more, and a square has all the properties of a rhombus with some additional ones. So again, we can say that a rhombus is a special kind of parallelogram, and a square is a special kind of rhombus – and hence, as before, an even more special kind of parallelogram.

| | opposite sides equal? | opposite sides parallel? | opposite angles equal? | diagonals bisect each other? | diagonals at right angles? | all sides equal? | one angle a right angle? | diagonals equal? |
|---|---|---|---|---|---|---|---|---|
|  parallelogram | Yes | Yes | Yes | Yes | not necessarily (if so it's a rhombus) | not necessarily (if so it's a rhombus) | not necessarily (if so it's a rectangle) | not necessarily (if so it's a rectangle) |
|  rectangle | Yes | Yes | Yes | Yes | not necessarily (if so it's a square) | not necessarily (if so it's a square) | Yes | Yes |
|  rhombus | Yes | Yes | Yes | Yes | Yes | Yes | not necessarily (if so it's a square) | not necessarily (if so it's a square) |
|  square | Yes | Yes | Yes | Yes | Yes | Yes | Yes | Yes |




From this it can be seen that squares have the properties of rectangles, for instance. Squares are a special kind of rectangle, but this does not mean that all rectangles are squares.

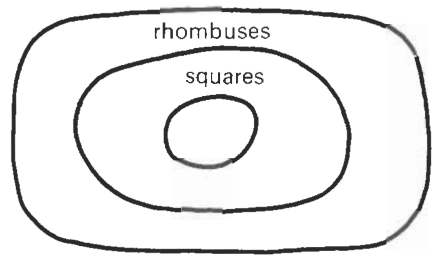
### Children's activities

Strips of metal, plastic, cardboard or other suitable material, with holes punched about a quarter of an inch from each end will be required. A group of children working together will need at least four strips of each of these lengths:
2 inches, 3 inches, 4 inches, 5 inches, 6 inches.

> Fasten four strips of the same length, using paper fasteners, to make a square. What do you know about the angles?
> How can you check this? (Use folded paper right angle.)
> Now can you move the strips to make another shape?
> Is it still a square? How can you tell?
> What name would you give the new shape that you have made? (The name *rhombus* should be given and the unofficial 'diamond' avoided.)
> Make other squares using different lengths and move them to make new shapes.

Discuss with the children what they can see that both the squares and the new shapes have in common (4 equal sides). They can then be asked such questions as 'In what way are squares and rhombuses alike?' 'Is the square a special kind of rhombus?' 'In what way is it special?'

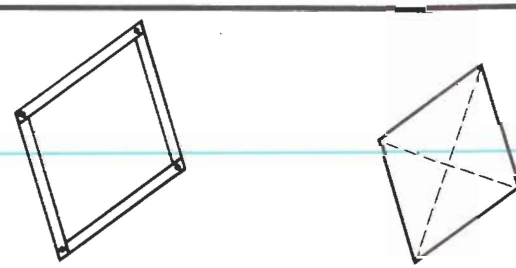
The point to bring out is that quadrilaterals having four equal sides are rhombuses. A square has four equal sides (and also four right angles) so a square is a special kind of rhombus.




This shows diagrammatically that the set of all squares is included in the set of all rhombuses; the set of all rhombuses is included in the set of all quadrilaterals; and the set of all squares is included in the set of all quadrilaterals.

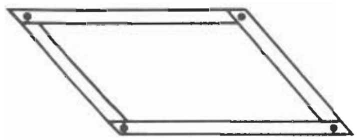
The properties of a rhombus can also be investigated through symmetry.

> Make a rhombus from cardboard strips as before. Place this on a sheet of paper and draw round the inside of the framework. Cut out this rhombus shape.
> How many axes of symmetry has this shape?
> Fold it along an axis of symmetry so that one half matches the other.
> What does this tell you about the lengths of the sides which fit on to each other? What can you discover about the angles?
>
> Now fold it along the other axis of symmetry. What can you discover about the sides and angles this time?



For the next activity four punched strips will be required – one long pair of the same length and one short pair of the same length.

Fasten the strips as before using paper fasteners, with strips of the same length opposite each other like this:



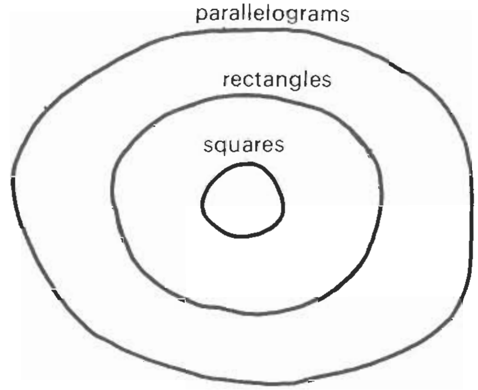
Move the strips to make different shapes. Can you make a shape with two axes of symmetry? With only one? With none? You know the opposite sides are the same length because you used strips of the same length. Can you find out anything else about one pair of opposite sides? Think of the distance between them, and their direction. What word can you use to describe this? What about the other pair of opposite sides?

The teacher can discuss this and introduce the name for these shapes – *parallelogram*.

Make one of the angles a right angle. What happens to the other angles? Use your folded paper right angle to check all four angles. Are the opposite sides still parallel? What name would you give to the shape with four right angles? (rectangle) Can you think of another shape which has four right angles?

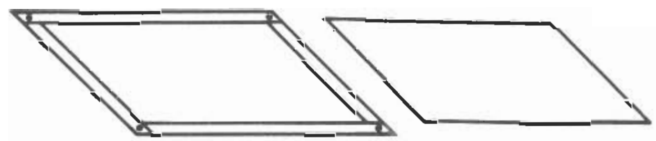
From these activities we see that a rectangle is a special kind of parallelogram for which the angles are right angles. We can use the word *rectangle* for a plane shape bounded by four straight lines which has four right angles. So the square is a rectangle, but it has something more – four equal sides.

Diagrammatically we have:



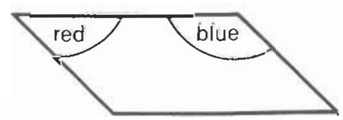

Another activity with parallelograms:

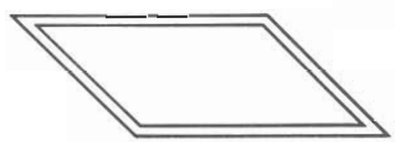
Use the strips as before to make different parallelogram frameworks. Draw some of these shapes on thin card by drawing along the insides of the strips like this:



Cut out the parallelograms drawn on card. Make several of these. Here is one example:

Colour the angles as shown:



Now place the shape on a sheet of plain paper and draw a frame round it like this.

Find how many ways you can fit the shape into its frame without turning it over.

The point to bring out in discussion is that 'half a complete turn' or a complete turn will do this. Actually it can be done with multiples of a half turn, e.g. $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, and so on, but we are only concerned with a half turn and a complete turn back to its original position.

Ask the children to make a half turn with the shape so that it fits into the frame. What can they discover about the opposite sides of the shape from the fact that it fits into the frame in this way? They can then look at the angle coloured red. Does it fit into the new position? What about the angle coloured blue? Does it fit into its new position? 'What can you discover about the angles of a parallelogram from this?' Can you think what would happen if one of the angles was a right angle? What about the other angles?

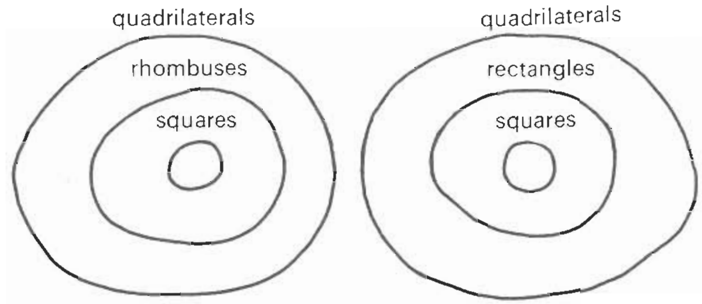
The above activity can be repeated with a rhombus. Use four equal strips for this – does this make any difference to the number of ways it will fit its frame? (cf. parallelogram)

Try again with a square.

It will be recalled that the set of quadrilaterals having four equal sides were called rhombuses, and the set of quadrilaterals which contained four right angles were rectangles. We then discovered that a square had the properties of a rhombus (so the set of squares was included in the set of rhombuses), and also some additional property (four right angles).

A square had the properties of a rectangle (so the set of squares was included in the set of rectangles) and also some additional property (four equal sides). So if we look at the properties of these shapes we see that squares are included in both the set of rhombuses and the set of rectangles.

This can be represented in diagrams:




The next stage is to see how we can combine these. The classification of farm animals, and men with various characteristics (tall, fat, red-headed, etc.) experienced by the children which was described in the first part of this section, is a similar situation and indicates that this approach and possible way of recording it is not restricted to one specific classification. We can now see how it can be applied to the shapes.
It might be presented in the following way. The children will need a collection of cardboard or plastic shapes – rhombuses, rectangles and squares (several of each, of different sizes).

Sort these shapes into two sets. In one put all the shapes which have four sides of equal length. In the other put those which have four right angles. Look at the two sets. Are there some shapes which you can put in either set? Which shapes are these? (squares)

One way of showing this is by a table:

| | Must have 4 equal sides | Must have 4 right angles |
|---|---|---|
| Rhombuses | ✓ | |
| Rectangles | | ✓ |
| Squares | ✓ | ✓ |

Put a tick in the appropriate columns.

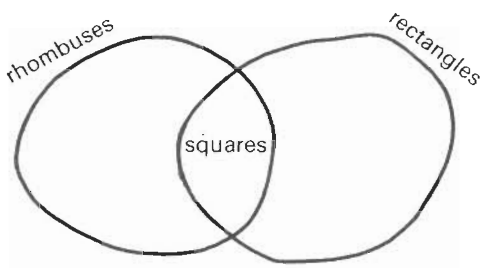
Which shapes are included in both columns?

'Can you show this another way with loops round the shapes?' (Cf. the work of Alan and Stephen on page 92.)

This may well be suggested through discussion in a similar manner to the discussion about the animals. It may be necessary to lead up to this by such questions as:

'In which loop can you put the squares?'
'Can you put them in the other loop too? Why?'
'Can you arrange your loops so that it looks as if the squares are in one loop and in the other loop too?'

The result we wish to arrive at is:

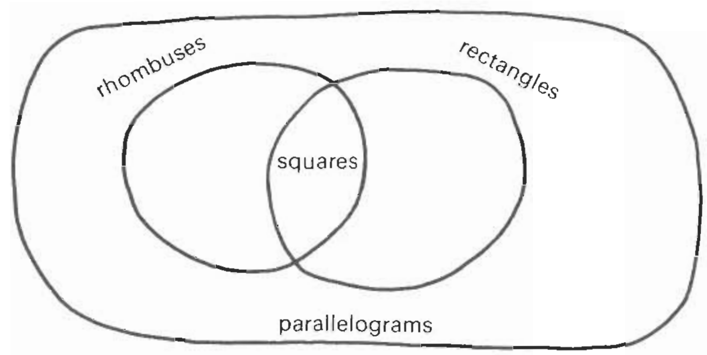



When this is discovered the idea can be fixed by asking such questions as:

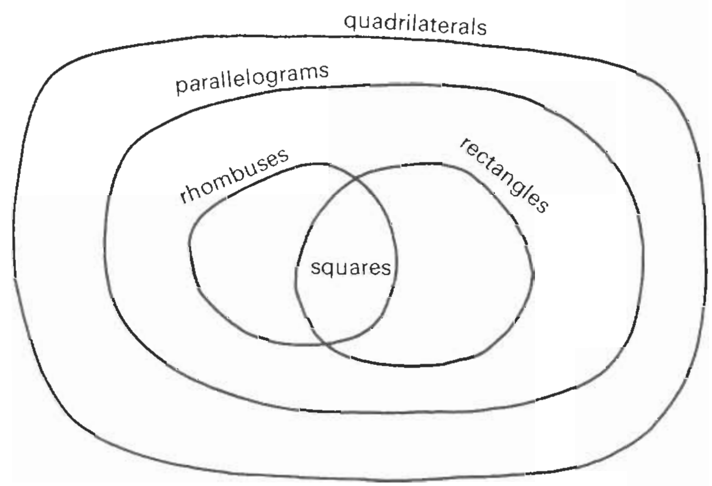
'Do the squares look as if they are inside the loop which surrounds the rhombuses?'
'Do the squares look as if they are inside the loop which surrounds the rectangles?'
Trace round the loops with your finger to show this if you wish. 'Are the squares included in both loops?'

Earlier it was stated that when children are recording their classifications the 'universe' was the given set of shapes, e.g. the actual set of quadrilaterals they had in front of them to sort.

Through discussion we may now extend the diagram showing the inclusion of squares in both the set of rhombuses and the set of rectangles, to this:




and then to this:




and to lead to the idea that this same diagram can also represent the 'universe' as the set of all the quadrilaterals in the world· that it can represent the set of *all* parallelograms, and so on.

Instead of using ticks in the diagram on p. 98 we could record as follows:

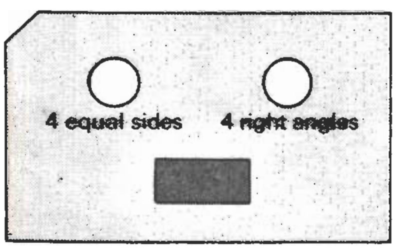
| | Must have 4 equal sides | Must have 4 right angles |
|---|---|---|
| Rhombus | Yes | No |
| Rectangle | No | Yes |
| Square | Yes | Yes |

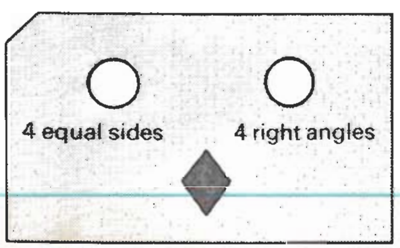
or instead of using Yes and No, we could write 1 and 0:

| | Must have 4 equal sides | Must have 4 right angles |
|---|---|---|
| Rhombus | 1 | 0 |
| Rectangle | 0 | 1 |
| Square | 1 | 1 |

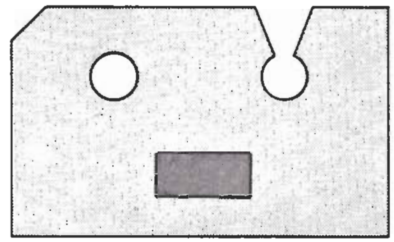
This would enable the information to be fed into a computer which can record 1's and 0's, in the form of two possible states (electric circuits being 'on' or 'off').

The following activity illustrates this idea (instead of 'on' and 'off' we use 'cut' or 'not cut').



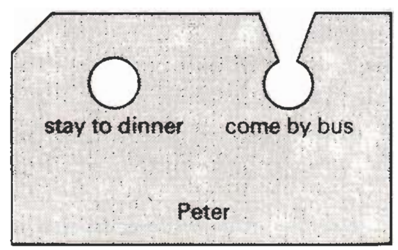




A number of cards are made, each with two holes punched as shown and a picture of a rectangle, rhombus or square underneath. The children cut from the hole to the edge of the card as appropriate, e.g. for a 'rectangle' card, they will cut for '4 right angles' but not for '4 equal sides'.

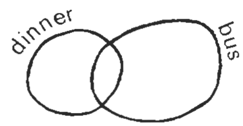


The cards are then stacked so that the pairs of holes are all in line (you can see through). How can the 'rectangle' cards be picked out without looking through all the pack? A child can put a knitting needle through the 'four right angles' hole and the 'rectangle' cards will drop out (but also of course the 'squares' which are included in the set of rectangles). Starting again with the cards thus removed, a needle through the other hole will release the 'four equal sides' cards, i.e. those with a square on them, leaving those which are 'rectangles' but not 'squares' on the needle.

The children can now transfer their new method of sorting to other situations.

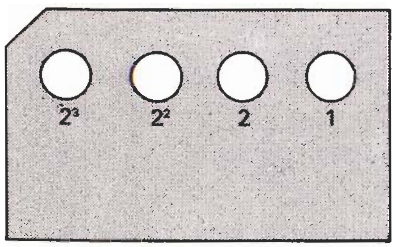



For example, the cards could have their names on them and the holes represent 'stay to dinner' and 'come to school by bus'. Peter comes by bus but doesn't stay to dinner. Again the cards can be stacked and sorted into the various categories (stay to dinner but not come by bus, etc.). This information could then be recorded in other ways already familiar, e.g.



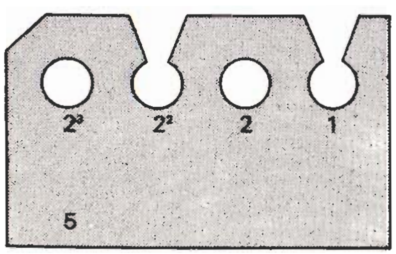

Later, further holes could be punched in the cards to show other categories, e.g. those who have satchels.

Finally, punched cards can also be used to record numbers using the binary scale (cf. ❷).

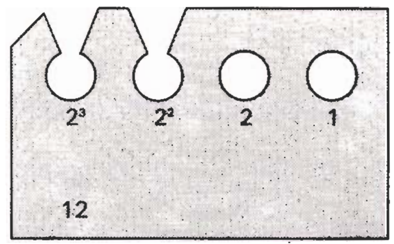



Holes represent (from the right) 1, 2, $2^2$(=4) and $2^3$(=8).

For example: $5=2^2+1$ and so the '$2^2$' and the '1' holes would be cut.




This illustrates that 5 is written in binary notation as 101.




12 in binary notation is 1100, being $\underline{1}\ (2^3)+\underline{1}\ (2^2)+\underline{0}\ (2)+\underline{0}\ (1)$.

Using knitting needles, it is possible to pick out, for example, the even numbers, and ingenious children will find ways of picking out those greater than 5, etc.

This last section illustrates that mathematics cannot be kept in separate compartments – geometrical classification has led to methods of storing and sorting numbers.

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