TEACHER'S COMMENTARY UNIT NO. 15

GEOMETRY

PART I

SCHOOL MATHEMATICS STUDY GROUP

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School Mathematics Study Group

Geometry

Unit 15

Geometry

Teacher's Commentary, Part I

Prepared under the supervision of
the Panel on Sample Textbooks
of the School Mathematics Study Group:

Frank B. Allen	Lyons Township High School
Edwin C. Douglas	Taft School
Donald E. Richmond	Williams College
Charles E. Rickart	Yale University
Henry Swain	New Trier Township High School
Robert J. Walker	Cornell University

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Study Group has been provided by the National
Science Foundation.

Below are listed the names of all those who participated in any of the writing sessions at which the following SMSG texts were prepared: First Course in Algebra, Geometry, Intermediate Mathematics, Elementary Functions, and Introduction to Matrix Algebra.

H.W. Alexander, Earlham College
F.B. Allen, Lyons Township High School, La Grange, Illinois
Alexander Beck, Olney High School, Philadelphia, Pennsylvania
E.F. Beckenbach, University of California at Los Angeles
E.G. Begle, School Mathematics Study Group, Yale University
Paul Berg, Stanford University
Emil Berger, Monroe High School, St. Paul, Minnesota
Arthur Bernhart, University of Oklahoma
R.H. Bing, University of Wisconsin
A.L. Blakers, University of Western Australia
A.A. Blank, New York University
Shirley Boselly, Franklin High School, Seattle, Washington
K.E. Brown, Department of Health, Education, and Welfare, Washington, D.C.
J.M. Calloway, Carleton College
Hope Chipman, University High School, Ann Arbor, Michigan
R.R. Christian, University of British Columbia
R.J. Clark, St. Paul's School, Concord, New Hampshire
P.H. Daus, University of California at Los Angeles
R.B. Davis, Syracuse University
Charles DePrima, California Institute of Technology
Mary Dolciani, Hunter College
Edwin C. Douglas, The Taft School, Watertown, Connecticut
Floyd Downs, East High School, Denver, Colorado
E.A. Dudley, North Haven High School, North Haven, Connecticut
Lincoln Durst, The Rice Institute
Florence Elder, West Hempstead High School, West Hempstead, New York
W.E. Ferguson, Newton High School, Newtonville, Massachusetts
N.J. Fine, University of Pennsylvania
Joyce D. Fontaine, North Haven High School, North Haven, Connecticut
F.L. Friedman, Massachusetts Institute of Technology
Esther Gassett, Claremore High School, Claremore, Oklahoma
R.K. Getoor, University of Washington
V.H. Haag, Franklin and Marshall College
R.R. Hartman, Edina-Morningside Senior High School, Edina, Minnesota
M.H. Heins, University of Illinois
Edwin Hewitt, University of Washington
Martha Hildebrandt, Proviso Township High School, Maywood, Illinois
R.C. Jurgensen, Culver Military Academy, Culver, Indiana
Joseph Lehner, Michigan State University
Marguerite Lehr, Bryn Mawr College
Kenneth Leisenring, University of Michigan
Howard Levi, Columbia University
Eunice Lewis, Laboratory High School, University of Oklahoma
M.A. Linton, William Penn Charter School, Philadelphia, Pennsylvania
A.E. Livingston, University of Washington
L.H. Loomis, Harvard University
R.V. Lynch, Phillips Exeter Academy, Exeter, New Hampshire
W.K. McNabb, Hockaday School, Dallas, Texas
K.G. Michaels, North Haven High School, North Haven, Connecticut
E.E. Moise, University of Michigan
E.P. Northrop, University of Chicago
O.J. Peterson, Kansas State Teachers College, Emporia, Kansas
B.J. Pettis, University of North Carolina
R.S. Pieters, Phillips Academy, Andover, Massachusetts
H.O. Pollak, Bell Telephone Laboratories
Walter Prenowitz, Brooklyn College
G.B. Price, University of Kansas
A.L. Putnam, University of Chicago
Persis O. Redgrave, Norwich Free Academy, Norwich, Connecticut
Mina Rees, Hunter College
D.E. Richmond, Williams College
C.E. Rickart, Yale University
Harry Ruderman, Hunter College High School New York City
J.T. Schwartz, New York University
O.E. Stanaitis, St. Olaf College
Robert Starkey, Cubberley High Schools, Palo Alto, California
Phillip Stucky, Roosevelt High School, Seattle, Washington
Henry Swain, New Trier Township High School, Winnetka, Illinois
Henry Syer, Kent School, Kent, Connecticut
G.B. Thomas, Massachusetts Institute of Technology
A.W. Tucker, Princeton University
H.E. Vaughan, University of Illinois
John Wagner, University of Texas
R.J. Walker, Cornell University
A.D. Wallace, Tulane University
E.L. Walters, William Penn Senior High School, York, Pennsylvania
Warren White, North High School, Sheboygan Wisconsin
D.V. Widder, Harvard University
William Wooton, Pierce Junior College, Woodland Hills, California
J.H. Zant, Oklahoma State University

GEOMETRY

Part I

Teachers' Commentary

Contents

INTRODUCTION

The text that you are about to teach from is the result of a collaboration between university mathematicians and experienced high school teachers. The treatment of geometry in this book is very different, especially in the first few chapters, from the treatment that nearly everybody is used to. There is no question that every change in teaching has its price: it calls for a great deal of preparation when a treatment which has become second nature is replaced by a treatment some of whose features are new to the teacher as well as to the student. For this reason, we have made changes only when we became convinced that they were worth the price. It should be remembered also that while any change at all creates some trouble for the teacher, simply because it is a change, this principle does not apply to the student: for him <u>any</u> formal treatment of geometry would be new.

This manual is intended to reduce your troubles to a minimum. It consists of three parts. The main part is a running commentary, referring to particular short passages of the text. In this part, we try to explain what we are driving at, and to warn of possible difficulties. (As of the Fall of 1960, the text has been revised after use in over one hundred classrooms, but it is natural to expect that there will still be difficulties that haven't been recognized and discussed.)

In a very large number of cases, we had trouble deciding what to put into the running commentary and what to leave out. We decided at length that when in doubt we should put things in. Thus we have no doubt included many explanations which are unnecessary. These, however, should be easy to skip.

Obviously, in a tenth-grade textbook many of the discussions have to be logically incomplete. We have cut some corners, expecting the student's intuition to take over, and we believe that this is as it should be. All sorts of questions can come up in class, however, and the chances are that this book will provoke some questions that students don't usually ask in the traditional courses. The running commentary is designed to help you to be one up when this happens. We have also indicated, at some points, the things we think should be emphasized and the general style of presentation that we had in mind.

There are some topics that can't conveniently be dealt with in connection with a particular passage of the text. Some of these topics cut across several chapters. We have therefore added a series of essays, under the general title, Talks to Teachers. These include, in our opinion, some of the most important parts of the commentary. (These will be referred to, hereafter in this manual, simply as the Talks.)

The first of the Talks, entitled Facts and Theories, we believe you will want to read right now and at least once more after you have read well into the text.

Finally, to save you spade-work we have given answers to all problems and solutions to all but the simplest. These are interspersed in the running commentary at the appropriate places. Answers have often been given in simplified radical form or as multiples of π rather than in the form of decimal approximations. We believe this policy should be encouraged, but that the student should be able to supply a decimal approximation on demand.

In addition to the Teacher's Commentary you should have available a copy of Studies in Mathematics, Volume II, Euclidean Geometry Based on Ruler and Protractor Axioms, by C. W. Curtis, P. H. Daus, and R. J. Walker. This contains, especially in the first chapters, much material that could have been put in the Talks to Teachers. It also contains detailed proofs of basic theorems that are not mentioned in the text. The properties stated in these theorems are intuitively obvious and are generally accepted by students without comment. A completely logical development of geometry must, nevertheless, contain proofs of these theorems, and so they are included here for whatever use you wish to make of them. This book will be referred to frequently in this manual. When we do so we will speak of it as "Studies II."

Some teachers may enjoy referring to a lighter presentation of some geometric ideas. To them we suggest Studies in Mathematics, Volume V, Concepts of Informal Geometry.

Although we felt it unwise to make our text logically complete in its proofs we did attempt to give a complete foundation of postulates and definitions. On such a foundation a student can build as elaborate and complete a structure as his capabilities permit, with the help of his teacher and of supplementary reading. The only difficulty apt to be met in laying this foundation is an apparent slowness of the text in coming to grips with really interesting geometric problems. However, you will find that the postulates, definitions and simple theorems in Chapters 2, 3 and 4, although not particularly interesting when you first study them, will be of great value in the later chapters. Moreover, seen from the perspective of the later chapters the basic material of the early chapters takes on a more interesting appearance as its importance to geometry becomes appreciated. If a student is to understand a complicated geometric situation he must first have a clear picture of the fundamentals.

Obviously you are going to like some features of this text better than others. In any case, we ask that you teach each chapter of this book as if you had faith in the presentation. If some features of it don't work, we want to know it, but we can't find out, one way or the other, unless they are given a fair try. A half-hearted experiment in the classroom has some of the disadvantages of a half-hearted back flip in a gymnasium.

USING THE TIME AVAILABLE

This text was written so that very good classes will have enough material to challenge them for a year. It follows, then, that some classes will not be able to cover all the material. You may prefer not to rush through important topics just to cover pages, so this note will suggest the kind of choices that you can make. The choices mentioned are only samples, however, and you will find variations that fit the needs of your own class.

A full course includes all exposition, and a substantial number of problems from each set. Few, if any, students will solve all the problems. An approximation to time allotment for classes which study every topic is given in this table. The names of chapters are topical and are not necessarily the actual chapter titles.

	Chapter	Days		Chapter	Days
1.	Introduction.	3	10.	Parallels in Space	6
2.	Sets, Numbers, Lines.	10	11.	Area, Pythag. Theorem	10
3.	Lines, Planes.	6	12.	Similarity	15
4.	Angles, Triangles	6	13.	Circles, Spheres	13
5.	Congruences.	20	14.	Characterizations of Sets. Constructions.	10
6.	A Closer Look at Proof	6	15.	Area of Circles	5
7.	Inequalities	8	16.	Volumes	8
8.	Perpendiculars in Space	9	17.	Coordinate Geometry	20
9.	Parallels in Plane	17			
	Total	85			87

The list of days must include time used for chapter reviews and tests. Though such work is important, a practical observation is in order: A class that uses two days per chapter for reviewing and testing uses more than one-sixth of the year in that way, and must plan accordingly.

We believe that every course should include careful treatment of the first volume, regardless of the preceding table. This does not mean that proofs of theorems should be memorized or that all problems should be done, however. Selection of material, if necessary, can begin with Volume 2.

The table above shows that if you are not into Chapter 10 by the end of the first semester, and many classes will not be, you will want to plan ahead so that you can study the chapters and topics most important for your students.

For example, you may decide to omit some material in order to devote sufficient time to the chapter on coordinate geometry. A way to do this is to omit Chapter 10 and cover the ideas of Chapters 14, 15 and 16 intuitively while doing selected problems. You may also decide to take up Chapter 17 immediately after Chapter 12.

Or you may decide to teach Chapters 8 and 10 largely on an intuitive basis, using problems to develop major concepts. Similarly for Chapters 14 and 15. Then omit Chapter 16 and treat most of Chapter 17.

Certainly numerous such plans are possible. Ideally, the one basic plan is to cover all material. Realistically, due to factors of time and of individual and group differences, several alternative plans must be considered, evaluated, and reviewed constantly.

We list here what can be omitted, in the order, very roughly, of preference in omission, the last item being the one you should least consider omitting. Chapter 17 is not included in the list, partly because its place in such a list is highly controversial and partly because a reason for omitting other topics is to assure adequate coverage of coordinate geometry.

Proofs in Section 6-5 and in Chapters 16, 14, 10, 8, 15, 7, 13 (after Theorem 13-5), 12.

All text material (except for formulas) in Chapters 16, 14, 15, 13 (after definition of intercepted arc), 10.

We are not proposing that anyone omit anything unnecessarily, for all the material is worthwhile. We are merely proposing that, if pressed for time, you not rush through too much material with your students but instead select the material best suited to their needs.

A WORD ABOUT THE PROBLEM SETS

The problem sets in this book are an extremely important part of the course. Many concepts are developed and expanded there. Careful assignment of the problems is essential so as not to exclude some of the important topics in the development.

Each problem set begins with some simple exercises. Some of the more difficult problems, not necessarily to be found at the end of the set, are starred.

It is hoped that the teacher will read all of the problems in a set before making an assignment. In some cases a sequence of problems builds an important concept, and an assignment should contain all the problems that develop the concept. In some instances a special comment about a problem occurs with the answer to the problem.

We hope that teachers will use their own judgment about the number of problems to assign. It is likely that no student will work all the problems. Certainly most students can be expected to do only some of the large number provided. You have a good chance to allow for individual differences in your assignments.

Proofs, and reasons within proofs, are given in varied form to suggest to the teacher that general understanding of the problems is more important than a rigid form of presentation. (This applies especially to Chapter 5 and the following chapters, in which many of the problems call for proofs of theorems.) The solutions given are not always the only possible solutions, and good original reasoning by students should be encouraged and commended.

The fact that we give a proof, in our solutions, in paragraph form for convenience and brevity does not mean that we believe that every student should give it in this form. The teacher can decide which form has the most educational value for his students at the given time.

On occasion, students should be asked to suggest and solve problems not in the text.

A GUIDE TO THE SELECTION OF PROBLEMS

Following is a tabulation of the problems in this text. It will be noted that the problems are arranged into three sets, I, II, and III. At first glance, one might think that these are in order of difficulty.

THIS IS NOT THE MANNER IN WHICH THE PROBLEMS ARE GROUPED!!!!

Before explaining the grouping, it should be mentioned that it is understood that a teacher will select from all of the problems those which he or she feels are best for a particular class. However, careful attention should be given to the comments on the problems in A Word About the Problem Sets.

Group I contains problems that relate directly to the material presented in the text.

Group II contains two types of problems: (1) some that are similar to those of Group I, and (2) some that are just a little more difficult than those in Group I. A teacher may use this group for two purposes: (1) for additional drill material, if needed, and (2) for problems a bit more challenging than those in Group I, that could be used by a better class.

Group III contains problems that develop an idea, using the information given in the text as a starting point. Many of these problems are easy, interesting and challenging. The student may find them more stimulating than the problems in Groups I or II. However, if time is a factor, a student can very well not do any of them and still completely understand the material in the text. These are enrichment problems.

It is assumed that a teacher will not feel that he or she must assign all of the problems in any set, or all parts of any one problem. It is hoped that this listing will be helpful to you in assigning problems for your students.

We have included in the problem sets results of theorems of the text which are important principles in their own right. In this respect we follow the precedent of most geometry texts. However, all essential and fundamental theorems are in the text proper. The fact that many important and delightful theorems are to be found in the problem sets is very desirable as enrichment.

While no theorem stated in a problem set is used to prove any theorem in the text proper, they are used in solving numerical problems and other theorems in the problem sets. This seems to be a perfectly normal procedure. The difficulty (or danger), as most teachers define it, is in allowing the result of an intuitive type problem, or a problem whose hypothesis assumes too much, to be used as a convincing argument for a theorem. The easiest and surest way to handle the situation is to make a blanket rule forbidding the use of any problem result to prove another. Such a rule, however, tends to overlook the economy of time and, often, the chance to foster the creative spirit of the student. In this text we have tried to establish a flexible pattern which will allow a teacher and class to set their own policy.

GUIDE TO SELECTION OF PROBLEMS

		I	II	III
Chapter 1				
Set	1-1	2,5,6,7,9,10.	3,4,8.	1,11,12.
	1-2	1,3,4.	5,6.	2,7.
Chapter 2				
	2-1	1,3,7,12.	2,4,5,6.	8,9,10,11.
	2-2	1,2,3,5.	6.	4,7.
	2-3	1,3,4.	2.	5.
	2-4	1,2,3.	4,5.	6,7.
	2-6	2,3,4,5,7.	1,8,9.	6.
	2-7a	1,2,3,6.	4,5.	7,8.
	2-7b	1,2,3,7.	5.	4,6.
Chapter 3				
	3-1a	1,2,3,4.		
	3-1b	1,2,3,4.	5.	6,7.
	3-1c	1,2,3,4,6.	5,7,12.	8,9,10,11.
	3-2	1,2,3.	4.	
	3-3	1,2,3,4,7,11,13.	5,6,8,9,12,18.	10,14,15,16,17.
Chapter 4				
	4-1	1,2,3,4,5,6,7,8, 9, 10, 11.	12,13,14,15.	16,17,18,19.

	I	II	III
Chapter 4			
4-3	1,2,3,4,5,6,7,8.	9,10,11.	12.
4-4	1,2,3,4,5,6,9, 10,12.	7,8,11,13.	14,15,16.
Chapter 5			
5-1	1,6,7,9.	2,3,4,8,10, 11,17.	5,12,13,14, 15,16.
5-2	1,2,6,7,8,9.	3,4,5.	
5-4	1,2,3,5,6,7,8.	4,10.	9.
5-5	1,2,3,4,6.	5,7.	
5-6	1,2,3,4,6,9.	5,7,8.	
5-7	1,2,5,6,9,10.	3,4,7,8,11,12, 14,16.	13,15.
5-8	1,3,4,5,6,7.	2,8,11,14,15, 16,17,18,19.	9,10,12,13,20, 21,22,23,24,25,26
Chapter 6			
6-2a	1,2,3,8.	6,7.	4,5.
6-2b	1,2,4,5.	3,6.	
6-3	1,2,3,5,7.	4,6,8,9,10.	
6-4	1,2.	3.	4,5.
6-5	1.	8.	2,3,4,5,6,7.

	I	II	III
Chapter 7			
Set 7-1	1,2,4,5.	3,7.	6,8.
7-3a	1,2,3.		4,5,6.
7-3b	1,2,3,5.	4,6.	
7-3c	1,2,3,4,7,9,11.	5,6,10.	8.
7-3d	1,2,3.	5,6,8.	4,7,9.
7-3e	1,2,3,8.	4,5,6,9.	7.
7-4	1,2,3.	4,5.	
Chapter 8			
8-1	1,2,4,5,6,8,9.	3,7.	10.
8-2a	1,2.		
8-2b	1.	2,3,4,6,7.	5.
8-2c	1,2,3,4.	5.	6.
Chapter 9			
9-1	1,2,3,4,5,7,9.	6,10,12,13.	8.
9-3	1,2,3,4,5.	6.	7,8.
9-4	1,4,5,8.	2,6,7,9.	3,10.
9-6	1,4,6,10. 2	2,3,5,7,8,9,12, 13,16,17,18.	11,14,15,19,20.
9-7	1,3,4,5.	6,8.	2,7.
Chapter 10			
10-1	1,2,4.	3,5.	6,7,8,9.
10-2	1,2,3.	4,5.	6,7.
10-3	1,2,4.	3,6,7.	5,8.

Chapter 1
COMMON SENSE AND ORGANIZED KNOWLEDGE

This chapter should be treated as an introduction. It is not a review of algebra or of the Pythagorean relation. The algebraic problems and the Pythagorean relation are introduced to illustrate mathematical method, not to provide items for forgetful students to relearn during the first week of a new course.

In this chapter it is desired first that the students see the distinction between a problem with an obvious solution and one that requires thought and skill in its solution. Later the need for exact reasoning on the basis of previously defined or accepted information is illustrated. What should be impressed upon the student is the fact that once we establish our basic information we intend to remain within the framework of our system to do the remainder of our work. We have our postulates (which contain undefined terms), and our definitions. On the basis of these (and these alone), we will build up a body of geometrical information by the application of logical reasoning.

As pointed out in the text, it is impossible to define all terms, so we have to begin with some undefined terms. Definitions are just agreements that we make to allow us to substitute a word, phrase or symbol for other phrases that are, in general, longer and more tedious to write out. A definition may be thought of as an abbreviation for a longer phrase or group of phrases. If P and Q represent phrases such that Q is taken as an abbreviation for P, then the abbreviated form Q may be substituted for P in any discussion and the sense of the discussion remains the same. This also works in the reverse order. The expanded form P may replace the abbreviated form Q. For example, consider the definition: A parallelogram is a quadrilateral whose opposite sides are parallel. If we know that the quadrilateral ABCD has $\overline{AB} \| \overline{CD}$ and $\overline{AD} \| \overline{BC}$, then we can abbreviate this by saying that ABCD is a parallelogram.

On the other hand, if we know that ABCD is a parallelogram, then we can assert what this phrase stands for, namely: ABCD is a quadrilateral such that $\overline{AB} \| \overline{CD}$ and $\overline{AD} \| \overline{BC}$,

So we see that the phrase, "ABCD is a parallelogram" and the phrase, "ABCD is a quadrilateral and $\overline{AB} \| \overline{CD}$ and $\overline{AD} \| \overline{BC}$ " can be used interchangeably. Since definitions are agreements that a simple phrase means the same as a more complicated phrase, there is no question about ever trying to prove a definition.

Only a very remarkable student will fully understand the paragraphs about theorems, postulates, proofs and undefined terms, when he first studies this chapter. These ideas will come into sharp focus in the student's mind only when he has had some experience with them. Chapter 1 is designed merely to give the student a sufficiently good idea of what is going on so that he will be better prepared for what follows. For this purpose, short and simple statements to the class are probably best. For example, if a student asks what a proof is, a good answer is that a proof is a complete explanation of why a statement is true. (Later the student will learn, by experience, the way all of us did, what sort of proof is acceptable in mathematics.) In the same spirit, a definition is simply an exact explanation of what a word or phrase means.

The explanation of the meaning of postulates has deliberately been made a little ambiguous. There are two possible viewpoints:

1. Until about 1800, everybody believed that the postulates of geometry were "self-evident truths", and that the theorems proved from them were statements of fact about the outside world, learned by pure reason.

2. Since the discovery of non-Euclidean geometry, it has been plain that the postulates of ordinary geometry are not "self-evident truths". There are many kinds of geometry; all of them are equally valid mathematically; some of the very "peculiar" ones are useful in physics; and _each of them is described by its own set of postulates_. Postulates, therefore, are simply descriptions of the kind of geometrical theory that we propose to investigate at a given time. And when we prove a theorem, we are not showing that the theorem is "true" in the sense that it fits the facts of the outside world. When we prove a theorem, we are merely showing that the theorem holds true in the mathematical system described by our postulates. (See the remarks on non-Euclidian geometry in the chapter on parallels, and the Talks on Miniature Geometries and Non-Euclidean Geometry.)

It does not seem to us that this second viewpoint is suitable for presentation in the second week of the tenth grade. The student would probably be completely bewildered, and he might get the idea that Euclidean geometry is just words, words, words. In Chapter 1 we have therefore been treading a rather fine line, explaining to the student approximately as much as we think that he can understand, and being careful in the process not to make any statements that will have to be corrected later.

What needs to be emphasized, at the start, is that postulates are not just pulled out of the air to satisfy somebody's whim. The space of Euclidean geometry is an extremely good approximation to physical space. This is why it got invented, and this is the most effective way to think about it. We can and we should use our intuition of physical space to help us guess what can be proved and how we can prove it. The proof itself, when we get it, should be logically based on the postulates. A mathematical system, like the geometry we are developing, that consists of postulates and theorems involving undefined and defined terms is called a deductive theory. This theory itself is given meaning and content by exhibiting an interpretation of the undefined terms. When we give the usual interpretation of point, line, and plane from physical space we get our physical geometry, which is an approximate model of our deductive theory. Other interpretations of the undefined terms lead to different models. A further discussion of mathematical models and how they work is given in the *Talks*.

It might be well to return to the latter part of this chapter after the student has had a fair amount of experience with the concepts which we have been trying to explain. After the class has finished Chapter 5, the ideas of *postulate*, *theorem*, *proof* and *undefined* *term* should have become entirely comprehensible. Chapter 6 will clarify some of the more troublesome problems involved in some types of proofs.

The numbers in the left-hand margin refer to the pages in the text that are being commented upon.

Some students may not remember how to solve simultaneous
1 equations. The thing to do here, as far as the class as a whole is concerned, is to provide enough reminders so that the class understands the solution offered by the book.

Notice the manner in which the lengths of the sides of the rectangles are discussed. The sides of the rectangles are merely
2 line segments, and each segment has a length that is a number of inches. Note that we write $x = 8$ and not $x = 8$ inches. There are times when we want to talk about x^2 and we square *numbers*, for example, $(8)^2$, but we do not square 8 inches. The problem here is simply to keep the units of measure out of the mathematical operations and use them in the interpretation of the results of these operations. The lower case letters, x, y, are used to stand only for numbers which are lengths of the sides in some unit, for example: If a rectangle is 8 inches long and x stands for the length, then $x = 8$.

Admittedly, this is a fine point, but we have been very careful about it in the text, and it will be easier on the students if you back us up by being equally careful about it in the classroom.

The usage that we are following is different from that of physics and chemistry courses. Physicists have developed, to a fine point, the art of handling unit signs as if they were algebraic symbols. A simple example of this is

$$6 \text{ ft.} \times 6 \text{ ft.} = 36 \text{ sq. ft.} = 36 \text{ ft.}^2$$

From here they move on to measure accelerations in ft./sec.^2 and
perform cancellations between such expressions according to the
ordinary laws governing fractions. We are not claiming for a
moment that there is anything wrong with this. It is not only
2 very right, but very useful. It is not, however, part of the
natural subject matter of this course, and so we are taking the
more elementary viewpoint that the things we know how to add and
multiply are numbers. This will be quite adequate for our pur-
poses, and the art of handling units algebraically can best be
learned in courses where it is needed.

You may have a student who will enjoy making apparatus to
illustrate the Egyptian method for constructing a right angle.
2 First he needs to tie eleven knots in a piece of cord so tnat
twelve equal lengths result. Then he needs a board and two
tacks. Students can manipulate this simple apparatus to get a
feeling for the operation the Egyptians went through.

Other students may enjoy supplementary reading, for example, an encyclopedia account of the Egyptian pyramids.

Your students may insist that they do not have to try "all"
cases to be sure of getting a right angle when $a^2 + b^2 = c^2$.
3 You will find it hard to argue against the principle of reason-
ing they are using as long as you restrict discussion to this
one case where the mathematical fact is correct in spite of the
reasoning. But try such a thing as the "formula" for primes

$$p = n^2 - n + 41$$

when $n = 1, p = 41$
$n = 2, p = 43$
$n = 3, p = 47$
$n = 4, p = 53$
$n = 5, p = 61$
$n = 6, p = 71$

The first six values for n, and many more, yield prime numbers for p. Your students may believe that this is true for all values of n. If your class does not have anyone who hits upon the revealing number, 41, for which p is not a prime, you can propose this value yourself.

Or, on a different level, mention the rich child who believes -- because of several observations -- that every family has a Cadillac.

Problem Set 1-1

4 2. a. 1.   b. $\begin{array}{r|l} 2 & 135,790 \\ \hline & 67,895 \end{array}$

3. a. 30 mi.

b. Let d be the number of miles between the cities.

$d = \frac{1}{3} d + 7.$

$3d = d + 21.$

$2d = 21.$

$d = 10\frac{1}{2}.$ The distance is $10\frac{1}{2}$ mi.

5 *4. a. 4 in., 1 in.

b. Let n be the number of inches in the shorter piece and $5 - n$ the number of inches in the longer piece. Then $\frac{n}{4}$ is the number of inches in the sides of the smaller square, and $\frac{5 - n}{4}$ is the number of inches in the side of the large square. The problem then requires that $\left(\frac{5 - n}{4}\right)^2 = 4\left(\frac{n}{4}\right)^2.$

$\frac{25 - 10n + n^2}{16} = \frac{4n^2}{16}.$

$0 = 3n^2 + 10n - 25.$

$0 = (3n - 5)(n + 5).$

$3n - 5 = 0$, or $n + 5 = 0$,

$3n = 5$, or $n = -5$ (which is meaningless here).
$n = 1\frac{2}{3}$,
$5 - n = 3\frac{1}{3}$.
The pieces are $1\frac{2}{3}$ and $3\frac{1}{3}$ in. long.

5. This is a right triangle because $(5)^2 + (12)^2 = (13)^2$.
6. Reason (d) is likely. Reason (b) would account for large errors. Reason (a) is unlikely.
7. Since $1^2 - 2\cdot 1 + 2 = 1$ the equation is true if $n = 1$. Yes. No. No.
8. a. The remainder is one.
   b. All of them.
   Comment: Each odd integer can be represented by $2n + 1$ for some integer n. If we expand $(2n + 1)^2$ and divide by 4, the integral part of the quotient is $n^2 + n$ and the remainder is 1. Hence, if 4 is divided into the square of any odd integer, the remainder is 1.

6 9. There are 31 (or in special cases, 30) regions formed, never 32. This problem illustrates the danger of jumping to hasty conclusions.
10. a. Yes. b. Yes. c. The areas are equal. d. The lengths are equal.
11. The area of the rectangle is 63 while the sum of the areas
7 of the pieces is 64. The fallacy is that if the other measurements are correct, the small triangles should have heights of $3\frac{8}{9}$ rather than 4. This can be shown by using similar triangles.

*12. The total time for the trip is the distance, 60, divided by the average speed, 60, and is therefore 1 hour. Since this hour is used up travelling the first 30 miles at 30 miles per hour, our answer must be that the average speed of 60 m.p.h. is then impossible to achieve.

8-10 This is a description of what is involved in setting up a mathematical theory. It took the human race a long time to perfect this idea. You cannot expect your students to grasp it from an abstract description. The understanding of what is involved in logical reasoning will grow throughout the course as students actively engage in logical reasoning. Nobody can learn logical reasoning in a vacuum.

The idea the student needs to get here is that point, line and plane are basic terms in our system and that we define more
10 complex terms like triangle, parallelogram, etc., in terms of point, line and plane.

You can draw dots of different sizes on the blackboard to help get at the idea of point. Or you can mention a star, thousands of times as large as the earth, that is barely visible. Seen up close it is tremendous. Seen from farther and farther away it approximates more and more closely the idea of a point.

10 It may be necessary to point out repeatedly that a line "does not stop".

The plane is the most difficult of the three terms for some students to understand. This is revealed by such incorrect
11 language as "rectangular plane" or "circular plane". A plane is suggested by such convenient objects as the classroom floor, the top of the teacher's desk, and a sheet of paper. Emphasize, whenever you use these objects for illustrative purposes, that a mathematical plane "keeps on going", and move your hand in appropriate directions.

It may help the student if you occasionally, during the first months, suggest that they reread the third paragraph of page 11.

Problem Set 1-2

12 1. a. measurement, size, dimension.

b. dimension, measurement, extent, size.

13 4. plan, houses, churches, schools.

5. plane, bounded by, parallelogram, rectangle, space.

14 6. a. Defining a term usually involves placing it in a class and distinguishing it from other members of the class. The term "something" is an unnecessarily large class into which to place squares. The phrase "that is not round" does not distinguish it from other "somethings". (One satisfactory definition at this point: A square is a rectangle whose sides have the same length.)

b. Only one of the angles has a measure of 90° in a right triangle. (A right triangle is a triangle with one right angle.)

c. "When" refers to time, not to geometric figures. A triangle is not a period of time. (An equilateral triangle is a triangle whose three sides are equal in length.)

d. "Where" denotes a place. "Perimeter" is not a place. (The perimeter of a rectangle is a number equal to the sum of the lengths of its sides.)

e. This is a true statement, but it states a process for computing circumference rather than stating what circumference is. (The circumference of a circle is a number which indicates its length.)

*7. A. False B. True C. False D. True

Chapter 2

SETS, REAL NUMBERS AND LINES

Some of the ways in which the material of this chapter differs from that of a traditional text are: (1) sets are introduced and (2) the real numbers, and thereby arithmetic and algebra, are brought into the course in a fundamental way. The reason for including sets becomes evident when you realize that every geometric figure is most simply studied as a set of points. This book does not treat the theory of sets as an end in itself but introduces its ideas and terminology to the extent that they contribute to the geometry course.

The real numbers are needed in geometry for the measurement of segments, angles, areas and volumes. We introduce them explicitly, rather than use them without any explanation.

The immediate reason for introducing the real numbers in this chapter is that they are needed for the statement of Postulates 2, 3, and 4. These postulates guarantee in effect that lengths of segments are expressible as real numbers, and have the familiar properties that we expect. One important advantage of introducing real numbers so early is that we can use them to define betweenness for points on a line. Then we can define segment, one of the most important geometric figures, in terms of between.

Seeing numbers so strongly emphasized in a geometry course will seem strange at first. At the time when Euclid wrote, algebra hardly existed, except insofar as it was implicit in geometry. In the following two thousand years or so algebra developed to a high degree, but the teaching of elementary geometry has made rather light use of it.

In this book, algebra is used in two important ways. In the first place, it is used in the postulates to make them easier to apply. If we take for granted that the real numbers are known, then it is possible to give a logically complete set of postulates, adequate for proving the theorems, avoiding some of the complications and difficulties involved in, say, Hilbert's Foundations of Geometry. We will see also, as we go along, that a great deal of the traditional material of geometry was really algebraic all along, and is much easier to handle when it is described algebraically. (This is especially true in the chapter on proportion.)

We believe that for your students these simplifications are genuine simplifications, and will make geometry easier for them to understand in the long run. But the algebraic apparatus used in this chapter and later may very well call for more careful preparation than you have ever given before to an early chapter of a textbook.

15 In the form in which we have presented it, the discussion of sets is not really a mathematical theory but simply an explanation of the language in which we propose to talk. As the "homely examples" in this section show, all of the basic ideas about sets -- with the sole exception of the empty set -- are already familiar. Only some of the words in which we talk about them are new.

The standard notation of a set theory is described in Appendix I, entitled A Convenient Shorthand. This is intended to be strictly optional and the title of the appendix is meant to suggest the spirit in which the notation was to be regarded. There is a serious danger in talking too much, and too fancily, about sets, at the high school level: the impression may be conveyed that writing things like $A \cap B \subset C$ is a loftier occupation than proving meaty theorems and solving hard problems in geometry and algebra. This would be sad. We therefore believe that the language of sets should be introduced matter-of-factly without fanfare, and that the notation of set theory should be taught to a given student only if and when the student is prepared to think of it as a matter of convenience.

As a matter of convenience, however, the language of sets is going to be used continually. For example, an angle will be defined as the union of two non-collinear rays. Two lines in the same plane are parallel if they do not intersect, and this means that the lines, considered as sets of points, have no member in common.

16 Notice that we are referring to the rectangles as the union of the four line segments, not the line segments plus the region enclosed by them. Later we shall be concerned with the interior of geometric figures.

17 Such a statement as "...each of the two lines is a set of points." seems to say something specific about "line", which is to be one of our undefined terms. This should not be cause for trouble, however, for the material here is informal and explanatory. It is not part of our formal system of geometry.

Problem Set 2-1

19 1. The intersection is {5, 9, 11}.
The union is {3, 4, 5, 6, 7, 9, 10, 11, 12}.

2. a. S_1 and S_2; S_1 and S_3; S_1 and S_5; S_2 and S_5 if you are a boy, but S_3 and S_5 if you are a girl.
b. S_1.
c. S_1.
d. The set consisting of all members of faculty and students of your school.
e. S_1, S_2, S_3, S_5.

20 3. The set {A}.
The set {B,C}.
The empty set.

4. a. Three committees: {A,B}, {A,C}, {B,C}.
b. {A,B} and {A,C} have A in common. {A,B} and {B,C} have B in common. {A,C} and {B,C} have C in common. "Intersect" means "have a member in common".

5. The set of all positive integers.
6. The empty set. Or, the sets have no common member.
7. The intersection is the segment $\overline{BC}$. The union is the triangle ABC.
8. The set consisting of the one pair (2,1).

21 9. The set consisting of the one pair (4,3).

10. The empty set. Or, there are no common elements.
11. a. The set of all positive integers divisible by 6 (i.e., by both 2 and 3) -- {6, 12, 18, 24,...}.
    b. 6n, where n is a positive integer.
    c. The set of all positive integers divisible by either 2 or 3, {2, 3, 4, 6, 8, 9, 10, 12,...}.
12. a. 1. b. 3. c. 6, 10. d. $\frac{1}{2}n\ (n - 1)$.

---

21 The material in this section, too, is informal. This intuitive development is intended to convince the student that to each point on a line there corresponds a real number, and to each real number there corresponds a point on the line. The feeling for the arrangement of these real numbers on a line is important to the student at this time.

Pages 23 to 28 point out the properties of real numbers concerning inequalities and absolute values, and show their geometric interpretation on a line.

23 Proof of the fact that between any two rational numbers there is a third one is simple, and interesting to some. Intuitively, the "average" seems to be such a number. The following argument justifies this intuitive notion.

1. Let a be the larger and b be the smaller of any two rational numbers. We show that $\frac{a + b}{2}$ is between a and b.
2. $a = \frac{1}{2} a + \frac{1}{2} a < \frac{1}{2} a + \frac{1}{2} b < \frac{1}{2} b + \frac{1}{2} b = b$.
3. $a < \frac{1}{2} a + \frac{1}{2} b < b$.
4. $a < \frac{a + b}{2} < b$.

5. Hence $\frac{a + b}{2}$ is between a and b.
6. Furthermore, $\frac{a + b}{2}$ is rational.

For a more detailed discussion of irrational numbers see Appendix III, and also Chapter 4 of Studies II.

23 We introduce here symbols that might be new to some students, namely $<$, meaning less than, $>$, meaning greater than, $\leq$, meaning less than or equal to, and $\geq$, meaning greater than or equal to. To say that an inequality can be written in reverse means, for example, that if $7 < 9$, then $9 > 7$. This is a statement in the form if $x < y$, then $y > x$. We also have inequalities of the form $x \leq y$, or $y \geq x$. These could be illustrated in the following manner: To say that $x \leq 8$, means that x can be <u>either</u> less than 8 or equal to 8, for example x can be -12, $-\pi$, 0, 3, 7.999 or 8 itself. For a more detailed treatment of inequalities see Chapter 4, of Studies II. There will also be some discussion of inequalities in Chapter 7 of the text.

24 While the basic algebraic postulates are put in Appendix II for completeness, the postulates (laws) for inequalities are included in the text proper, for many students are not acquainted with them.

25 Some students may be so used to saying "The square root of 9 is plus or minus 3" meaning that 9 has two square roots, 3 and -3, that it will be hard to convince them that the written statement "$\sqrt{9} = \pm 3$" is incorrect. We know of no patent medicine to prescribe. Simply explain, move ahead, and remind later as necessary.

26 <u>Problem Set 2-2</u>

1. All four are true.
2. a. AB is less than CD.
   b. x is greater than y.

c. XY is greater than or equal to YZ.

d. n is less than or equal to 3.

e. 0 is less than 1 and 1 is less than 2.

f. 5 is greater than or equal to x and x is greater than or equal to -5, or x lies between 5 and -5 inclusive.

g. x is positive or x is greater than 0.

3. a. $k > 0$.
   b. $r < 0$.
   c. $t \leq 0$.
   d. $s \geq 0$.
   e. $2 < g < 3$.
   f. $2 \leq w \leq 3$.
   g. $a < w < b$.
   or $b < w < a$.

4. a, c, d, f, h.

5. a. 3.009, 3.05, 3.1.
   b. -3, -2.5, -1.5.
   c. $1\frac{3}{5}$, $1\frac{5}{8}$, $\frac{5}{3}$.
   d. $-1\frac{5}{8}$, $1\frac{3}{5}$, $\frac{5}{3}$.

27 *6. a. T. b. T. c. N. d. S. e. S.
(Note to teacher. Parts (d) and (e) are true for $r > s > 0$ but are not always true for certain negative values.)

*7. a. S. b. T. c. S. d. T. e. T.

---

27 Most students learn what "absolute value" means by looking at several examples. The method of "defining by pointing" helps the student to grasp the concept, but it certainly is not a mathematical method. Assure your students that their notion of absolute value will serve them satisfactorily in geometry.
28 Point out that this particular definition is not intended to be explanatory in the ordinary sense of the word. Awkward though the definition may appear to be, it does pin the idea down and is technically correct, whereas superficially stated "definitions" that sound good often fail to hold up under close inspection.

Problem Set 2-3

29 1. a, c, d.

*2. b, c, d.

3. a. r. b. -r. c. 0.

4. Drawings are omitted.

   a. The set of points to the left of the zero mark.

   b. One point, a unit to the right of 0.

   c. The set of points to the right of 1.

   d. The part of the line to the left of and including 1.

   e. Two points.

   f. The part of the line between 1 and -1 inclusive.

   g. The union of the part of the line to the left of -1 and the part to the right of 1.

   h. The entire line.

30 5. a. The first set includes 0; the second does not.

   b. The first set includes 0 and 1; the second does not.

---

30 Throughout this book, when we speak of "two points", we really mean two. That is, if A and B are two points, then A and B are different. The phrases "three points", "two lines", and so on, are used in the same way. On the other hand, if we say merely that A and B are points of the line L, this allows the possibility that A and B are the same; if we mean that they are different, we either say explicitly that they are different or we say explicitly that there are two of them.

Some usages are matters of convention, and there is not unanimous agreement on them in the mathematical literature. (For example, most algebra textbooks say that every quadratic equation has two roots; and thus the equation $x^2 - 2x + 1 = 0$ has "two roots", which happen to be the "two numbers" 1 and 1.) We have therefore attempted to write this text in such a way that the reader will understand what we mean without having to pay undue attention to the conventions that we are following.

Sometimes we shall use the phrase "two different points" for emphasis -- even when the word "different" is not necessary logically. Postulate 1, for example, uses "different" in this way.

If you want to acquaint yourself in advance with the notations that are adopted in the text, see the index of symbols at the end of the volume.

32

Problem Set 2-4

1. a. $\frac{1}{6}, \frac{1}{18}$.
   b. $54, 1\frac{1}{2}$.
   c. 24, 2.
2. a. 50, 0.5 .
   b. 325, 0.325 .
   c. 7320, 732 .
3. a. The numerical value of the length would be 11 divided by $8\frac{1}{2} = 1\frac{5}{17}$ or approximately 1.3; that of the width would be 1.
   b. The numerical value of the width would be $8\frac{1}{2}$ divided by $11 = \frac{17}{22}$ or approximately 0.77; that of the length would be 1.
4. $36^2 + 48^2 = 60^2 = 3600$.
5. a. $P = 4\cdot 48 = 192$.
   $A = 48^2 = 2304$.
   b. $P = 4\cdot\frac{4}{3} = \frac{16}{3}$.
   $A = (\frac{4}{3})^2 = \frac{16}{9}$.

33 *6.

1. $a^2 + b^2 = c^2$.	1. Given.
2. $\frac{a^2}{n^2} + \frac{b^2}{n^2} = \frac{c^2}{n^2}$.	2. By division.
3. $(\frac{a}{n})^2 + (\frac{b}{n})^2 = (\frac{c}{n})^2$.	3. Another form of Step 2.

*7. If the length of any side of the square is s units, it is given that

$$s^2 = 4s$$

from which $$s^2 - 4s = 0$$

or $$s(s - 4) = 0.$$

The only meaningful solution to this equation is $s = 4$. Area and perimeter will be numerically equal only if a side is 4 units long, whatever the unit may be. Since any change in unit will change the 4 to something else, the area and perimeter will no longer be numerically equal.
(Note to teacher: Be ready to commend other correct proofs students may give. The concept of generalization in mathematics is an important one.)

---

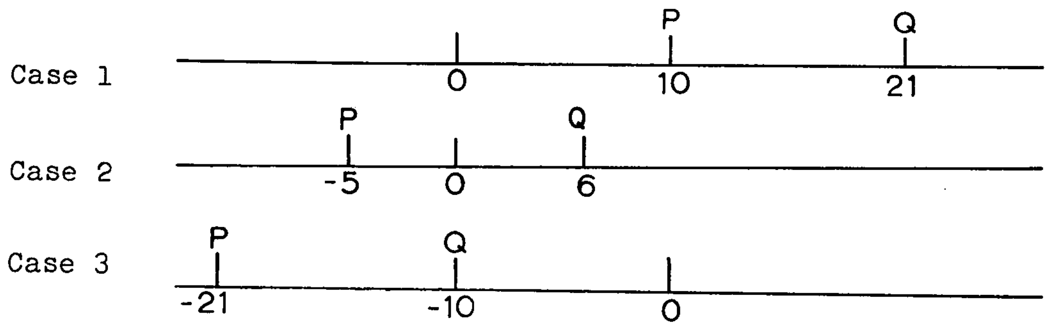
33 Section 2-5 begins by appealing to the student's knowledge. It then "describes this situation" formally, in Postulate 2. The postulate is not a casually chosen group of words to use in playing a game. It is on the contrary a carefully chosen statement that gives one of the basic properties of points; it formalizes something with which the student is already familiar at an informal or intuitive level. Later postulates will continue the process of characterizing point, line, and plane by formalizing properties which are intuitively familiar or which have been suggested by physical experience.

34 Notice how the first strictly geometric definition is set off. This particular definition does not lend itself to a discussion of the nature of mathematical definition as well as some later ones do, so the text postpones such a discussion until a more suitable example appears.

Postulate 2 and the definition of "distance" use some words such as "any", "different", "unique" which have not been defined, and this may bother very dutiful students who are trying to be precise. You can simply say that we are using the

English language in the course, assuming that the meanings of all simple non-geometric terms are known. Such terms are used with their usual meanings. In other words, the language of ordinary speech is assumed. Geometric terms, words with technical meanings, are the ones that are treated carefully within the system of geometry.

35 In Section 2-6 on the infinite ruler, we are trying to prepare the student in an intuitive manner for Postulate 3 (The Ruler Postulate.) When investigating the general rule that the distance between the point that corresponds to x and the point that corresponds to y is $|y - x|$ it might be well to check the rule for some whole numbers first. There are only three cases we have to consider: (1) both points correspond to positive numbers, (2) one point corresponds to a positive number and the other corresponds to a negative number, and (3) both points correspond to negative numbers. The case when one point corresponds to zero has already been considered when discussing absolute values.



It is clear that the distance from P to Q, (which is the same as the distance from Q to P,) is 11 in all three cases above. Now let us check and see if the absolute value of the difference of the corresponding numbers will give the distance between these points regardless of the order in which we take the numbers in the formula, $PQ = |y - x|$.

Case 1. $PQ = |21 - 10| = 11$, and $|10 - 21| = 11$.

Case 2. $PQ = |6 - (-5)| = 11$, and $|-5 - (6)| = 11$.

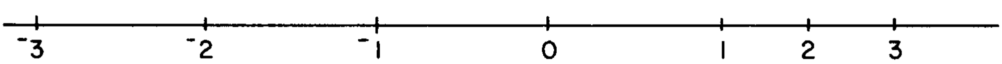
Case 3. $PQ = |-10 -(-21)| = 11$, and $|(-21) - (-10)| = 11$.

Now the Ruler Postulate seems reasonable, because we have seen that it will give us the results that we would expect from
37 the previous discussion. We now have a coordinate system on a line; the number corresponding to a point is the coordinate of that point.

Though the book mentioned previously that a line is a set of points, there was no formal statement about how many points a line contains. Postulate 3 gives us infinitely many points on every line. This is so because we have assumed the real number system and are now postulating a one-to-one correspondence between the set of points on a line and the set of real numbers. (The text will use the phrase "one-to-one correspondence" formally in Chapter 5.)

When we say that the points on a line are in a one-to-one correspondence with the real numbers, we mean: (1) to each point of the line there corresponds exactly one real number and (2) to each real number there corresponds exactly one point of the line. One-to-one correspondences are not unique to mathematics. For instance, how many times have you taken attendance in your class by looking to see if each assigned seat in the classroom is filled? What you have done is to establish a one-to-one correspondence between assigned seats in your classroom and students in your class. If you can match up a seat with each student, you know that all of the students are present.

Postulate 3 is a very powerful tool. Part (3) guarantees that distances on a line behave in a way that we would normally expect them to behave in. It would not be sufficient to postulate just the existence of a one-to-one correspondence. We cannot have anything like this:



because such an "undesirable" ruler does not satisfy Part (3) of Postulate 3.

[page 37]

If you are familiar with the foundations of geometry you may find Postulates 3 and 4, with their strong emphasis on algebra, rather strange. We have introduced real numbers in Postulates 2, 3, 4 as a pedagogical device at the tenth grade level to avoid very subtle and difficult discussions on the theory of measure of segments. (See the Talk on the Concept of Congruence for an indication of this.) One should not infer that we consider this the best treatment at higher levels. In an advanced course in the foundations of geometry we would prefer a treatment of the type given in Hilbert's Foundations of Geometry or Veblen's Monograph on the Foundations of Geometry (Monograph 1 in Monographs on Topics of Modern Mathematics, edited by J. W. A. Young.) In such a treatment the postulates would be more geometric, making no reference to algebraic entities, and our Postulates 2, 3, 4 would appear as theorems -- indeed difficult ones to prove.

Note the contrast with the conventional treatment (and with Euclid) where betweenness is not even mentioned and betweenness relations are taken, when needed, intuitively from pictures. The early introduction of real numbers permits us to define betweenness. The mathematical treatments of Hilbert and Veblen take betweenness as undefined and characterize it by postulates.

Problem Set 2-6

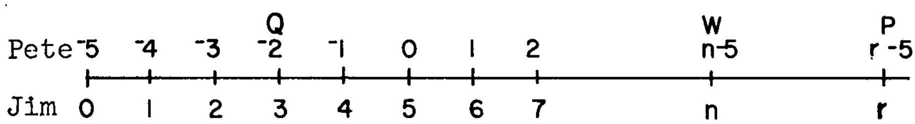
37

1. a. 3.
   b. 3.
   c. 3.
   d. 2.
   e. $|2a|$ or $2|a|$.
   f. 0.

2. a. 12.
   b. 12.
   c. 12.
   d. 12.
   e. $1\frac{1}{2}$.
   f. 10.2
   g. $\sqrt{3} - \sqrt{2}$
   h. $|x_1 - x_2|$ or $|x_2 - x_1|$.
   i. $|4a|$ or $4|a|$.
   j. $|2s|$ or $2|s|$.

(Note to teacher: In (g) point out that $\sqrt{3} - \sqrt{2}$ is exact, while $1.732 - 1.414 = 0.318$ is an approximate result.)

3. a.



b. $|r - 3|$ by Jim's scale.
$|(r - 5) - (-2)| = |r - 3|$ by Pete's scale.

c. $|r - n|$ by Jim's scale.
$|(r - 5) - (n - 5)| = |r - n|$ by Pete's scale.

4. Subtract $\frac{1}{4}$ from the value at Q.

38 5. a. Yes.

b. $p + 2$ and $q + 2$.

c. The distance, by definition, is $|p - q|$. For the new numbering

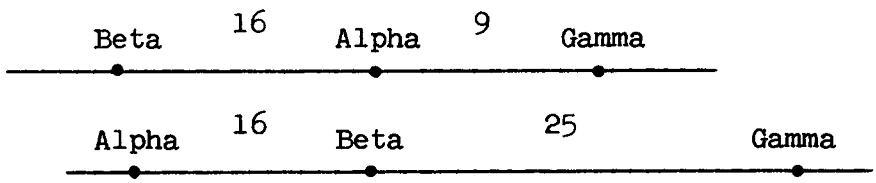
$$|(p + 2) - (q + 2)| = |p - q|.$$

d. Yes.

6. Consider two points with coordinates n and r. After renumbering the original scale, the coordinates will be $(-n)$ and $(-r)$.
The distance between them is $|n - r|$.
It is also true that $|(-n) - (-r)| = |r - n| = |n - r|$.

7. a. No. Gamma.

b. 9 miles or 41 miles



c. Alpha.

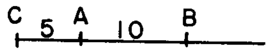
d. Alpha.

(Note to teacher: This problem is leading up to the concept of betweenness.)

39 8. There are 2 possible arrangements.

B can be between A and C.

A can be between C and B.



9. B is between A and C. 14.

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40 The concept of betweenness, though intuitively natural, is one that has rarely been formalized in high school treatments of geometry. From the discussion in the text it can be seen that this can be a very tricky concept if we consider the problem on a closed curve. Fortunately, later discussions and treatments in the text consider betweenness on a line only.

In connection with the idea of betweenness, it might be worthwhile to propose the following problem to the class: In how many ways can four round beads, of different colors, be arranged in a string so as to make a four-bead necklace? The answer is that there are only three different ways. The point is that there is only one way for the first three beads, A, B, C to be arranged in the necklace; the six orders ABC, ACB, BAC, BCA, CAB, CBA all form the same necklace. The only real choice is in the position of the fourth bead, D, and for this there are three possibilities: D can be immediately between A and B, or immediately between B and C, or immediately between A and C.

41 The definition of "between" is followed by a discussion of definitions in mathematics. A mathematical definition must be distinguished from a dictionary definition which often gives only a synonym or description of the term defined. A mathematical definition is, as this manual mentioned in Chapter 1, a formal agreement to use -- when desired -- one phrase as an abbreviation for another phrase.

Notice that a definition is logically very different from a theorem. A typical theorem is in the form, if A, then B, where A and B are statements. It says that statement B is deducible from statement A. For example, let A be the statement "a triangle has two congruent sides" and B the statement "a triangle has two congruent angles." These statements mean different things, and we have learned a geometric fact when we prove that the second statement inevitably follows from the first.

On the other hand a typical definition is of the form: P stands for (or is an abbreviation of) Q, where P and Q are phrases. For example (see Chapter 1, commentary) let P be "parallelogram" and Q be "a quadrilateral whose opposite sides are parallel." No implication is involved here -- P and Q are not even statements. Rather we are making an agreement, motivated by convenience, that the short phrase P shall stand for the long phrase Q. Sometimes, to avoid awkwardness of language, we state a definition in "if--then" form, for example: if the opposite sides of a quadrilateral are parallel, then we call the quadrilateral a parallelogram. Don't be misled by this. No implication is involved. We are not stating a geometric fact, but an agreement about how geometric terminology shall be used, namely that the word "parallelogram" shall stand for the phrase "a quadrilateral whose opposite sides are parallel."

You can discuss definitions in such down-to-earth terms as these: A mathematical definition is a convenient handle for dealing with a mathematical idea just as the set of finger holes in a bowling ball is a convenient handle to use when rolling the ball.

You may want to present the idea of definition to your class like this: Consider the following definition of "honor student." "Students of East High with a deportment grade of A and no academic grade below B are called honor students."

Knowledge that Cindy Marshall is an honor student at East High is also knowledge that she has a deportment mark of A and has no academic mark below B by definition of "honor student". On the other hand, knowledge that Eric Hughes, a student at East High, had A in deportment and no mark below B is knowledge that he is an honor student -- again by definition. "Honor student" is a convenient label that spares people all the words "a student with a deportment mark of A and no academic mark below B".

42 A figure for Theorem 2-1 might lead the students to feel that they can "see" that point B is between A and C. What must be realized is that a figure is not sufficient justification of a proof. To prove this theorem formally we must prove it on the basis of the definition and not the configuration, for the only formal knowledge we have of betweenness is that provided by the definition.

You might wonder why we prove theorems like 2-1 at all; they seem so obvious. Notice that according to our logical program, as outlined in Section 1-2, every statement of our geometry must be either a theorem or a postulate. We could, of course, take as postulates all statements as obvious as Theorem 2-1, and some text-books do this. We choose, rather, to use as few postulates as we feel are pedagogically necessary, and prefer to give proofs of even the "obvious" theorems. This does not mean that either you or your students need spend much time on the proofs. We merely believe that it is good for the students to know that some "obvious" things can be proved, and that mature mathematicians do not regard it a waste of time to devise such proofs (and some of them are unimaginably difficult.)

You will probably want to point out to your students that they are not expected to "learn" the proofs of the theorems in this chapter. The theorems may not seem meaty to beginning geometry students, and the proofs are not at all typical of the kind of geometric reasoning they will usually be doing. We do not expect them to know how to write proofs of their own until Chapter 5. The book gives proofs for the sake of completeness.

[page 42]

Go through them once, and then go on. Assure the students that the time for mastering simple geometric proofs will come, and that the book will then help them get a start.

42 The statement that if $x < y$, then $y - x$ is positive, might require some amplification. We can illustrate this with a specific example, letting x and y represent 2 and 7 respectively. If $x < y$, and we subtract the smaller number from the larger, then it is certain that the difference will be a positive number $(y - x > 0)$. If, on the other hand, $x < y$, and we subtract y from x, we would have $x - y < 0$. If we subtract 7 from 2 we get a negative number, which is, of course, less than 0. In the theorem it is given that $x < y$. Then $y - x$ is positive and, by definition of absolute value, $|y - x| = y - x$.

Problem Set 2-7a

42 1. a. 7. d. $8\frac{1}{2}$.
b. 6. e. 0.9.
c. 10. f. $|x_1 - x_2|$.

2. It is only necessary to read a single positive number if one uses the Ruler Placement Postulate. Neither subtraction nor computing an absolute value is necessary.

43 3. $RS + ST = RT$.

4. The coordinate of A is -2; that of B is 14.

5. c. See the Ruler Postulate and definition of between.

6. The point having coordinate x_1. Theorem 2-1.

7. a. By the Ruler Postulate:

$$AE = \left|0 - \frac{1}{3} r\right| = \frac{1}{3} r.$$

$$EF = \left|\frac{1}{3} r - \frac{2}{3} r\right| = \frac{1}{3} r.$$

$$FB = \left|\frac{2}{3} r - r\right| = \frac{1}{3} r.$$

$$AE = EF = FB.$$

b. $AF = |0 - \frac{2}{3}r| = \frac{2}{3}r$.

$AE + EF = AF$ since $\frac{1}{3}r + \frac{1}{3}r = \frac{2}{3}r$.

Therefore, E is between A and F.

*8. The inequality $x > y > z$ can also be written $z < y < x$, in which case $y - z$, $x - z$, and $x - y$ are all positive. Therefore, $CB = y - z$,

$CA = x - z$,

and $AB = x - y$.

From these three equations we observe that

$$CB + BA = x - z = CA.$$

Therefore, B is between A and C. (Note: A brief proof relates $z < y < x$ to Theorem 2-1.)

---

45 The term "ray" might be new to students. The text makes clear the distinction between ray and segment. What should be pointed out to the students is that in the notation for a ray, for instance $\overrightarrow{AB}$, the first letter is the end point and the second is one of the infinitely many points through which the ray passes. It is not correct, therefore, to refer to the ray whose end point is A and which passes through point F as $\overrightarrow{FA}$. The correct notation is $\overrightarrow{AF}$.

Observe that in the figure for Theorem 2-4 the point P need not, in spite of the diagram, lie to the right of point B. P may be the same point as B, or P may be between A and B. However, P cannot be at A, and A cannot be between P and B, since x is a positive number.

Remarks on The Line Separation Theorem. The following theorem is not stated in the text, but is often used tacitly later. It describes the separation of a line by a point, and is closely analogous to the later postulates in Chapter 3 dealing with the separation of a plane by a line and the separation of space by a plane.

The Line Separation Theorem. Let P be a point of the line L. Then L is the union of P and two sets H_1 and H_2 not containing P, such that

(1) No point of L lies in both H_1 and H_2.

(2) If two points Q and R are both in the same set, H_1 or H_2, then P is not between Q and R, and

(3) If Q is in H_1, and R is in H_2, then P is between Q and R.

Proof: Let us set up a coordinate system on L such that P corresponds to 0. Let H_1 be the set of all points of L with negative coordinates and let H_2 be the set of all points of L with positive coordinates. Then L is the union of P, H_1 and H_2, because every real number is positive, negative or zero. P is not in either H_1 or H_2 because 0 is neither positive nor negative. (1) holds because no number is both positive and negative. It remains to verify (2) and (3).

Let Q and R be points with coordinates x and y. Suppose that y is the larger; this is merely a choice of notation. If Q and R are in H_1, then $x < y < 0$; by Theorem 2-1, R is between Q and P; and so P is not between Q and R. If Q and R are in H_2, then $0 < x < y$; Q is between P and R; and so P is not between Q and R. This verifies (2).

Let Q, R, x and y be as before, with $x < y$. If Q is in H_1 and R is in H_2, then $x < 0$ and $y > 0$. Therefore, $x < 0 < y$; and therefore, P is between Q and R. This verifies (3).

This theorem has been deliberately kept out of the text. It is so obvious that students can be expected to use it tacitly and its proof is not very interesting mathematically.

Of course, the half-lines H_1 and H_2 are analogous to the half-planes and half-spaces to be discussed in the next chapter. Notice that a half-line is different from a ray; a ray contains its end-point, but a half-line does not.

Notice that the Line Separation Theorem guarantees that every ray has exactly one opposite ray.

<u>Problem Set 2-7b</u>

47 1. Two.

2. a. Theorem 2-1.
   b. Theorem 2-3.
   c. Definition of between.

48 3. a. Points X and Y and all points of $\overleftrightarrow{XY}$ between X and Y.
   b. Points of $\overline{XY}$ and all points Z of $\overleftrightarrow{XY}$ such that Y is between X and Z.

*4. Case 1. If A is between B and C, then AB + AC = BC. Since AB = BC, this leads to the impossibility AC = 0.

Case 2. If C is between B and A, then BC + CA = BA. This leads to the impossibility CA = 0.

Case 3. B is between A and C, by Theorem 2-2, is the only remaining possibility and must be true.

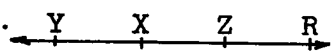
(Note: A proof based on setting up a coordinate system and using Theorem 2-1 is also possible.)

*5. Theorem 2-4.

*6. Proof.

	Statements:	Reasons:
1.	AB + BC = AC.	Definition of between.
2.	AC - AB = BC.	Subtracting AB from each side.
3.	BC > 0.	Distance Postulate.
4.	AC > AB.	If AC - AB > 0, AC > AB.

7. a. $\overleftrightarrow{XZ}$ contains points Y and R but $\overline{XZ}$ contains neither points Y nor R. R belongs to $\overrightarrow{XZ}$ but Y does not. YZ + ZR = YR.

   b. 

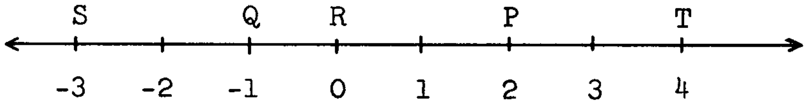
or 

48 <u>Review Problems</u>

1. S_1; S_4; S_3; S_5; the empty set.

49 2. 1; 2; no.

3. a.



b. PQ = 3, RT = 4, TR = 4, PT = 2, QS = 2.

4. a. Positive.

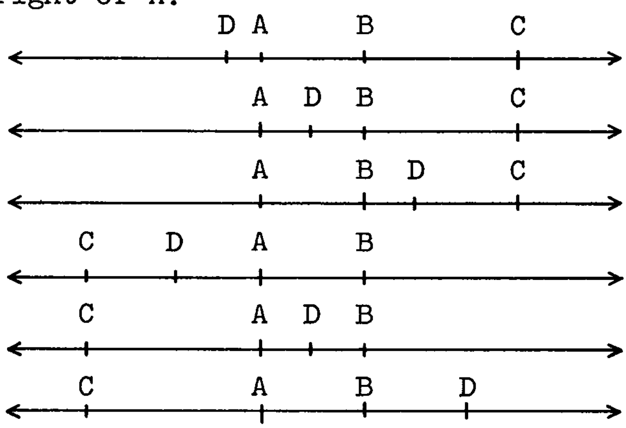
b. Between 0 and 2.

c. Negative.

5. a. AB + BC = AC.

b. AB = BC.

6. There are 12 possible orders. We picture the 6 in which B is to the right of A.



7.

AB + BC = AC. $\overleftrightarrow{DB}$ contains points A and C, but $\overline{DB}$ contains neither point A nor point C. A belongs to $\overrightarrow{DB}$ but C does not.

8. x = 9, y = 4.

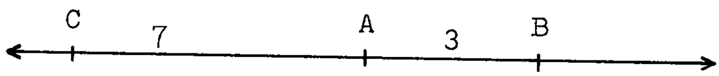
50 9. Perhaps they live in the same house. However, since people are not always precise in every day language usage, it may be that they only live near each other -- as on opposite sides of the street.

10. N - 2.

11. a. $\overline{AF}$ and E.
    b. E and F.
    c. Triangle AFE.
    d. The empty set.
    e. Triangle AEF.

12. a. 5. (ABCD, ABCE, ABDE, ACDE, BCDE.)
    b. 10. (AB, AC, AD, AE, BC, BD, BE, CD, CE, DE.)
    c. 10.

13. No. AC could only be 13 or 7.



14. a. F (Should be 6).
    b. T
    c. T
    d. T
    e. T
    f. T
    g. T
    h. F (Should be 7).

51 15. Yes. Since y is larger than x, the value of y - x will be the same as the value of $|x - y|$.

16. (b) is not a coordinate system because the numbers 4, 3, 2, 1 and 0 each correspond to more than one point. This is not permissible according to Postulate 3.
    (e) is not a coordinate system because the distance between points with coordinates 2 and 1 in the original numbering is $|2 - 1|$ or 1. In the numbering of (e) the distance between the same two points is $|2 - (-1)|$ or 3. By Postulate 2 the same two points can correspond to only one number indicating distance.

17. d; b, e; h; f.

Illustrative Test Items for Chapter 2

A suitable chapter test might be made by selecting problems from the following list. These have been grouped into sets of problems that are similar with the idea that the teacher may wish to make a test by choosing none or more from each set.

In compiling this list and later lists, we generally have omitted items specifically calling for statements of definitions, postulates, theorems, and so on, in the belief that each teacher on his own will draw on this wealth of test material, as well as on his own ingenuity in constructing his own tests.

A. 1. a. Let A be the set of squares of the first eight non-negative integers. List the members of this set.

b. Let B be the set of the first eight even positive integers. List the members of this set.

c. What is the intersection of sets A and B?

d. What is the union of sets A and B?

2. Consider the set of all positive integers divisible by 5. Consider the set of all positive integers divisible by 3. List the first five integers in the intersection of these two sets.

3. The intersection of ray $\overrightarrow{AB}$ and ray $\overrightarrow{BA}$ is ________. The union of ray $\overrightarrow{AB}$ and ray $\overrightarrow{BA}$ is ______________.

B. 1. Arrange the five collinear points E, L. M, S, T in proper order if LM + ME = LE; SE + ET = TS; LS + SM = ML.

2. A number scale is placed on line $\overleftrightarrow{RS}$ with -5 falling at R and 6 at S. If the Ruler Placement Postulate is applied with 0 placed on R and a positive number on S, what will be the coordinate of S?

3. Copy the following sentences and supply the appropriate missing symbols over each letter pair.

a. AB has no end points.

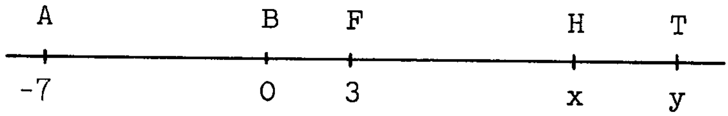
b. The end points of MR are M and R.

c. RQ has one endpoint, R.

4. Three towns Lander, Manton and Amity are collinear but not necessarily in that order. It is 9 miles from Lander to Manton and 25 miles from Manton to Amity.
   a. Is it possible to tell which town is between the other two?
   b. Which town is not between the other two?
   c. What may be the distance from Lander to Amity?
   d. Illustrate with sketches.

C. 1. Given A, B, and C are three collinear points with AB = 8 and CB = 5. If, also, the coordinate of B is -2, and the coordinate of A is less than that of C, what are the coordinates of A and C? Draw two sketches giving different sets of answers.

2.



In the figure:
   a. the length of $\overline{AB}$ is ______________.
   b. the length of $\overline{AH}$ is ______________.
   c. the length of $\overline{BT}$ is ______________.
   d. the length of $\overline{FT}$ is ______________.
   e. the length of $\overline{HT}$ is ______________ or ______________.

3. If A corresponds to 0 and B to 1 on a number line, what set of numbers correspond to the points of the ray $\overrightarrow{AB}$? Of the ray $\overrightarrow{BA}$?

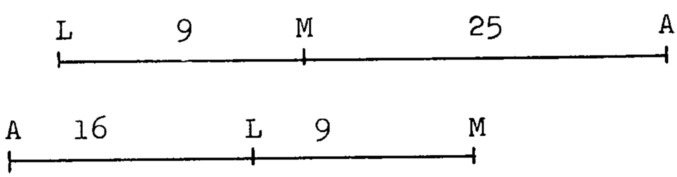
D. 1. a. $|-7| + |3| =$ ______________.
   b. $|-7| - |3| =$ ______________.
   c. $|-7| - |-3| =$ ______________.
   d. $|-7-3| =$ ______________.
   e. $|-7+3| =$ ______________.
   f. $|-7-10| =$ ______________.
   g. $|-7+4| =$ ______________.

2. a. How many square roots does the number 49 have?
   b. $\sqrt{49} =$
3. a. Write as an inequality: K is a negative number greater than -10.
   b. Restate the following in words: $20 > x > 10$.
4. Make a true statement out of each of the following expressions by replacing each question mark by one of the following symbols, $<$, $>$, $=$, $\leq$, $\geq$: $|3-6|$? $|6-3|$; $|-4-2|$? $|-4| - |2|$; $|x + y|$? $|x| + |y|$.

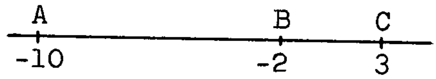
Answers

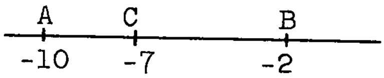
A. 1. a. 0, 1, 4, 9, 16, 25, 36, 49.
   b. 2, 4, 6, 8, 10, 12, 14, 16.
   c. 4, 16
   d. 0, 1, 2, 4, 6, 8, 9, 10, 12, 14, 16, 25, 36, 49.
2. 15, 30, 45, 60, 75, 90.
3. $\overline{AB}$; $\overleftrightarrow{AB}$.

B. 1. 

2. 11.
3. a. $\overleftrightarrow{AB}$.
   b. $\overline{MR}$.
   c. $\overrightarrow{RQ}$.
4. a. No.
   b. Amity.
   c. 34 mi. or 16 mi.
   d. 

C. 1.





2. a. 7

   b. $x + 7$.

   c. y.

   d. $y - 3$.

   e. $y - x$ or $|x - y|$ or $|y - x|$.

3. The set of numbers, x, such that $x \geq 0$. The set of numbers, x, such that $x \leq 1$.

D. 1. a. 10

   b. 4.

   c. 4.

   d. 10.

   e. 4.

   f. 17.

   g. 3.

2. Two; 7.

3. a. $-10 < K < 0$, or $0 > K > -10$.

   b. x is a number between 10 and 20.

4. $=$; $>$; $\leq$.

Chapter 3
LINES, PLANES AND SEPARATION

The material of this chapter differs from that of the traditional text in several ways. First, some elementary solid geometry is introduced, for the authors believe that there should be no undue separation of solid geometry from plane geometry.

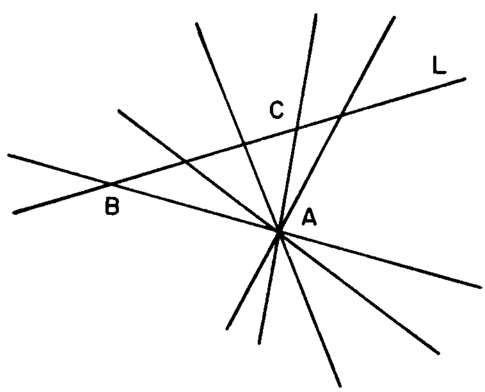
Second, the important idea of convexity is introduced. Most of the familiar geometric figures, such as triangular and rectangular regions, or the interiors of circles and spheres, or rectangular solids and circular cones, are convex sets.

Finally, the separation of a plane by one of its lines and the separation of space by a plane are studied. These ideas are treated purely intuitively in most geometry texts as is indicated by phrases such as "two points are on opposite sides of a line."

54 The text description of the figure on page 54 asserts that points A, B, C and E are coplanar. Actually, F is contained in the same plane as A, B, C and E, and we can say that A, B, C, E and F are coplanar.

54 Most students will not see readily that Postulate 5a really does fill a plane with points. We do not believe that you should press the matter, for most students will not be interested in something so "obvious." You can show inquiring students this by using Postulates 1, 6 and 3 along with Postulate 5a as follows:

A plane has three non-collinear points A, B, C by Postulate 5a. Then by Postulate 1 there is a line L determined by B and C. The plane contains line L by Postulate 6. Line L has infinitely many points by Postulate 3. Point A, in



combination with these points individually, determines infinitely many lines by Postulate 1. All of these lines (and their infinitudes of points) lie in the plane by Postulate 6.

56 Postulate 6 assures us, as the text points out, that a plane is flat. Your students can use a globe in the classroom to see that is is not possible to find two points on a sphere such that the line determined by them lies on the sphere. A sphere as a surface, then, does not satisfy Postulate 6. Other surfaces, for example cylindrical ones, are trickier. Your students can find points on a steam pipe in your room such that the line determined by them lies on the pipe. Pupils should readily see, however, that finding some such pairs of points is not enough. The question remains: do <u>all</u> pairs of points on the pipe satisfy the requirement? Since the answer is no, the cylindrical surface of the pipe does not satisfy Postulate 6.

A triangular region does not satisfy Postulate 6. Although the region contains the segment $\overline{AB}$ joining its points A and B, it does not contain the line $\overleftrightarrow{AB}$ which is determined by the points.

55 Theorem 3-1 could be stated in the if-then form: If two lines intersect, then they intersect in only one point. The two statements are equivalent.

The students should be reminded of the fact that the "if . . . then . . ."relationship is not unique to geometry or mathematics. It is a cause and effect relationship common to science and everyday life, for example: "If I do not sleep for two consecutive nights, then I will be tired." Statements such as this often occur in conversation. Full use of the exercise material in recognizing the hypothesis and conclusion of statements should be made when you reach Section 3-2.

55 Teachers will recognize the proof of Theorem 3-1 as being indirect. The text does not wish to describe indirect proof at this point, or even to describe proof at all. The thing to do, we believe, is to go through the proof once with emphasis on

understanding and then go on without asking students to learn the proof. Theorem 3-1 and the method of indirect proof are discussed in Chapter 6.

One of the problems in the teaching of geometry is that of keeping emphasis on the ideas of proofs rather than on rote memory. Teachers have their own ways of doing this, such as changing the labels on figures, encouraging students to come up with different proofs, going from paragraph form to two-column form and vice-versa. In other words, discourage mere memorization of proofs. (Be careful not to discourage mental effort, however.)

56 The discussion in the text of a way in which Theorem 3-2 could be proved suggests that you avoid a proof now--or at least avoid emphasizing one. The proof goes: It is impossible for a line and a plane not containing the line to intersect in two different points because then the line, by Postulate 6, would lie in the plane.

56-58 The text proves Theorems 3-2, 3-3, and 3-4 in Chapter 6.

Some time spent on the drawing of planes and lines in three-space is recommended. Some very simple demonstrations with a piece of cardboard (representing part of a plane) and a pencil (representing part of a line) might be performed to illustrate and clarify those postulates and theorems that make reference to three-space.

57 You might ask questions designed to clarify some of the postulates of this chapter: for example, for Postulate 7, "Why does a stool with three legs tend to be more stable than a chair with four legs?"

Problem Set 3-1a

53 1. One.

Infinitely many lines can be drawn.

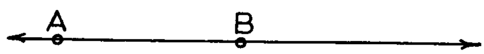
2. No.

Three.

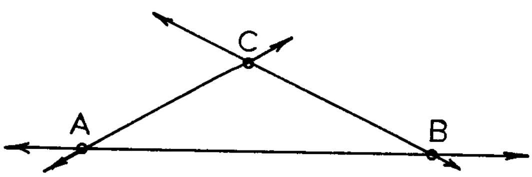
3. Three
4. No end-points. Two end-points.

<u>Problem Set 3-1b</u>

55 1. P and Q are the same point by Theorem 3-1.
2. Infinitely many.
   One.
   None, if the points are non-collinear; one, if collinear.
3. Postulate 1.
4. a. One line, by Postulate 1.



   b. Three lines if the points are non-collinear. There are three pairs of points, and each pair determines a line, by Postulate 1.



   One line if the points are collinear.

56 5. a. Six: $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{BD}$, $\overleftrightarrow{CD}$.
   b. One if D is collinear with A, B, C.
   Four otherwise: $\overleftrightarrow{AB}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{BD}$, $\overleftrightarrow{CD}$.
6. a. A set of points is collinear if there is a line such that each point of the set lies on the line.

A set of points is coplanar if there is a plane such that each point of the set lies in the plane.

b. For each plane there are at least three non-collinear points which lie in this plane.

7. "Contains" form.
Given any two different points, they lie on exactly one line.

Problem Set 3-1c

58 1. Infinitely many.
Infinitely many.
One, if the points are non-collinear; infinitely many, if the points are collinear.

59 2. The ends of the three legs are always co-planar. The ends of the four legs may not be coplanar.

3. Point.
Line.

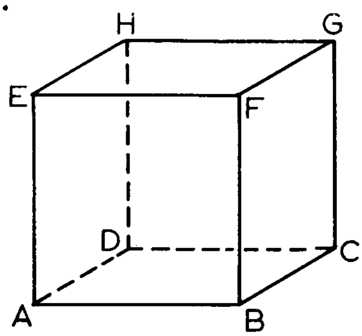
4. No. Yes. Yes. Yes, if $n > 2$.

5. A set of three or more points is non-collinear if there is no line which contains them all.

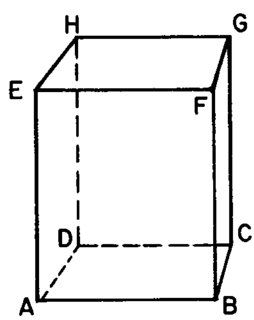
6. Yes, if A, B, C are non-collinear.
No, if A, B, C are collinear.

7. a. A.
b. C.
c. E.
d. Non-collinear, or coplanar.

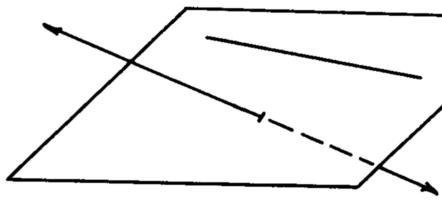
9.



OR



10.



60 11. a. An edge of the tetrahedron is the line segment determined by two vertices.

b. Six: $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{BD}$, $\overline{CD}$.

c. Yes, for example, the edges $\overline{AB}$ and $\overline{CD}$ have no point in common.

d. No. The faces can be paired six ways; each pair has an edge in common.

12. Seven: ABC, ABE, BCE, CDE, ADE, ACE, BDE.

Problem Set 3-2

61 1. a. Hyp: John is ill.
Concl: He should see a doctor.

b. Hyp: A person has red hair.
Concl: The person is nice to know.

c. Hyp: Four points lie on one line.
Concl: They are collinear.

d. Hyp: I do my homework well.
Concl: I will get a good grade.

e. Hyp: A set of points lies in one plane.
Concl: The points are coplanar.

f. Hyp: Two lines intersect.
Concl: They determine a plane.

2. a. If two lines are different, then they have at most one point in common.

b. If a student is a geometry student, then he knows how to add integers.

c. If it rains, then it pours.

d. If a point is not on a line, then the point and the line are contained in exactly one plane.

e. If a practice is dishonest, then it is unethical.

f. If two lines are parallel, then they determine a plane.

62 3. Postulate 1: If points P and Q are different, then there is exactly one line which contains them.

Theorem 3-1: If lines L_1 and L_2 are different, they intersect in at most one point.

4. a. No. The theorem places the intersection of two lines as a condition for the conclusion while not asserting that any two lines must intersect. The statement in this problem asserts that two lines must always intersect.

b. If two lines intersect in a point, then there is exactly one plane containing them.

Before introducing the postulates on separation it may be well to look back and re-examine the postulates we already have. Postulates 1, 5, 6, 7, 8 are similar in that they are purely geometric and describe how points, lines and planes lie on or are "incident with" each other. They are called *incidence* postulates. On the other hand, Postulates 2, 3, 4 involve algebra; they are concerned with properties of measurement, and so are called *metrical* postulates.

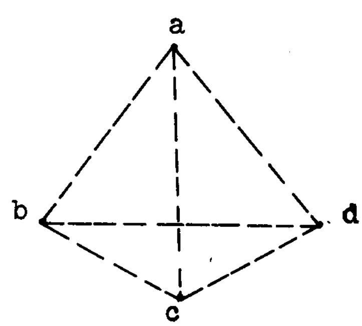
The incidence postulates are simple ones that *logically* form a natural unit for beginning the course. But *pedagogically* this does not seem attractive, for two reasons. First, the incidence postulates would confront the student with solid geometry in his first approach to a new subject. Second, the proofs of the basic incidence theorems (for example Theorems 3-1, . . . , 3-4) involve the indirect method, which causes difficulty for many students.

To avoid these difficulties we have split off Postulate 1 from the incidence postulates and joined it to Postulates 2, 3, and 4 to form the basis of a beginning unit on measurement in Chapter 2. This makes use of the student's knowledge of algebra, and involves, geometrically, only sets of points contained in a line. Then Chapter 3 discusses the incidence properties of points, lines and planes and separation properties. These are non-metrical in character.

The discussion of the preceding three paragraphs suggests a rather basic theoretical point, namely, the effect on a mathematical theory of introducing new postulates. The next few paragraphs use a miniature geometry to illustrate this basic point of theory. We intend this as interesting background material for its broad effect, rather than for any immediate application to the text.

Examine Postulates 1, 5, 6, 7, 8. You see that they include familiar determination and intersection properties of points, lines and planes in Euclidean solid geometry, and also, in Postulate 5, a minimal indication of how numerous points are. You probably have in mind, in any case, that a line and a plane contain infinitely many points. But this can not be proved on the basis of Postulates 1, 5, 6, 7, 8. We show this by exhibiting an appropriate "model" for Postulates 1, 5, 6, 7, 8. The model is a concrete system of objects which satisfy these postulates. Expressed differently, we get a model of our mathematical theory by assigning specific meaning to the undefined terms "point," "line" and "plane," in such a way that the postulates become true statements.

To construct our model, consider a set of four distinct objects, a, b, c, d. For example, we can take four dots on a piece of paper as indicated in the diagram. We can think of them if we wish as the vertices of a triangular pyramid. Interpret "point" to mean any one of the



objects a, b, c, d; "line" to mean any pair of these objects; "plane" to mean any triple of them. Then our postulates are no longer statements involving undefined or uninterpreted terms, but become definite statements (true or false) about the objects a, b, c, d. Thus Postulate 1 now says: any two of the objects a, b, c, d are contained in a unique pair of them. This is trivially true. Similarly, Postulate 6 says that if a triple of the objects contains two of them, then it contains the pair composed of these two. This is also a trivial truth. Similarly it can be shown that each of the Postulates 1, 5, 6, 7, 8 is satisfied when point, line and plane are interpreted in the given way. In virtue of this the system composed of the four "points" a, b, c, d, the six "lines" (a, b), (a, c), (a, d), (b, c), (b, d), (c, d) and the four planes (a, b, c), (a, b, d), (a, c, d), (b, c, d) is called a model for postulates 1, 5, 6, 7, 8.

Since the model satisfies Postulates 1, 5, 6, 7, 8 it must satisfy the theorems which are deduced from these postulates (using no others), for example, Theorems 3-1, 3-2, 3-3, 3-4. This is easily verified. Now you can see that the principle that a line contains infinitely many points can't be deduced as a theorem from Postulates 1, 5, 6, 7, 8. For if this could be done, our model would have to satisfy this principle -- and it doesn't, since each of its lines contains exactly two points.

Now you can see the effect of introducing the metrical postulates, in particular Postulate 4, the Ruler Postulate. This guarantees that a line is rich in points, and that its infinitude of points are arranged on the line and determine distances in just the way we want for the kind of geometric theory we are constructing. The introduction of the metric postulates excludes finite models, of the type we have discussed, which do satisfy the incidence postulates. This illustrates the basic theoretical point we mentioned earlier: in general, as new postulates are added in a mathematical theory, the scope of its application, that is the family of models which satisfy the postulates, is reduced. See the Talks: <u>The Concept of</u>

Congruence and Miniature Geometries.

62 Notice that in sets D, E, F there are infinitely many pairs of points such that the segments joining them are contained in the set. The existence of a single pair of points P, Q such that $\overline{PQ}$ does not lie in the set is sufficient to eliminate the possibility of convexity. Thus the union of the set of points in the interior of a circle and one point outside the circle is not a convex set.

Separation properties are not explicitly mentioned or explained in Euclid or in conventional texts. They appear in geometry in statements such as, "Consider two triangles which have the same base and a pair of vertices on opposite sides of the base." They appear in everyday life when we say, for example, that the town hall and the school are on the same side
63 of the main highway. Notice how the text uses the basic idea of segment to give a precise statement of what is involved in the separation of a plane by one of its lines. The intuitive idea of two points being on the "same side" of line L is expressed precisely by the condition that the segment joining them does not intersect L. Notice how the precise formulation of the separation postulate agrees with our intuitive ideas about separation.

66 Postulate 10, the Space Separation Postulate, is entirely similar to Postulate 9, the Plane Separation Postulate. The corresponding result for a line can be proved from the Ruler Placement Postulate, and was given at the end of Chapter 2 of the Commentary.

Problem Set 3-3

66 1. a. Yes. The line segment joining any two points of the line lies entirely in the line.

b. No. There is one segment joining the two points and it does not lie in the two points.

c. Yes.

d. No. Any segment containing the removed point would not lie entirely within the set even if its end-points were within the set.

e. No. For any two points, R and S, of the set the segment $\overline{RS}$ does not lie in the surface. (Ordinary 3-space is considered here.)

f. Yes.

g. No. No. Yes.

h. No. Yes. No.

i. No. Yes. Yes. No.

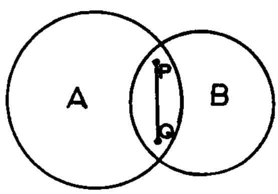
j. Two. Half-spaces.

67 2. No. It is necessary that for <u>every</u> two points, the entire segment joining them lies in the set.

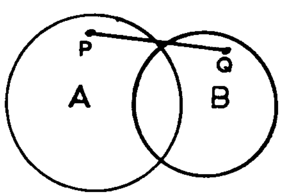
3. V only. V is the only set in which the segment between any two points is contained in the set.

4. Yes. Take any two points P and Q in the plane. By Postulate 6, we know that the line containing these points lies in the plane. Hence $\overline{PQ}$ is contained in the plane, making the set convex.

5. a. Yes. For any points P and Q in the intersection:



b. No. Points P and Q may be selected as follows:



6. No. Any segment containing the removed point would not lie entirely in the set even if its end-points were in the set.

7. Yes.

8. Any figures of the following nature:

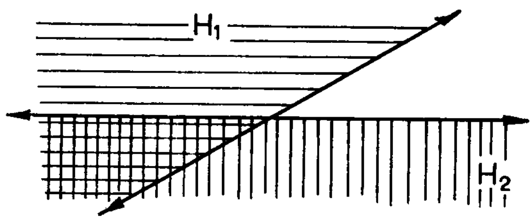
CONVEX NOT CONVEX

9. Yes.

10. No.

11. a. No. The line separating the half-planes is not contained in the union.

    b. No. A large region of the plane is still not covered, as in the diagram.

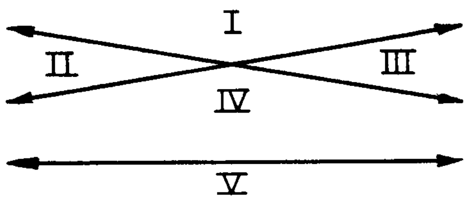


12. a. Two. Half lines.

    b. The Line Separation Statement: Given a point and a line containing it. The points of the line different from the given point form two sets such that (1) each of the sets is convex and (2) if P is in one set and Q is in the other, then the segment $\overline{PQ}$ contains the given point.

68 13. A ray has an end-point, but a half-line has no end-point.

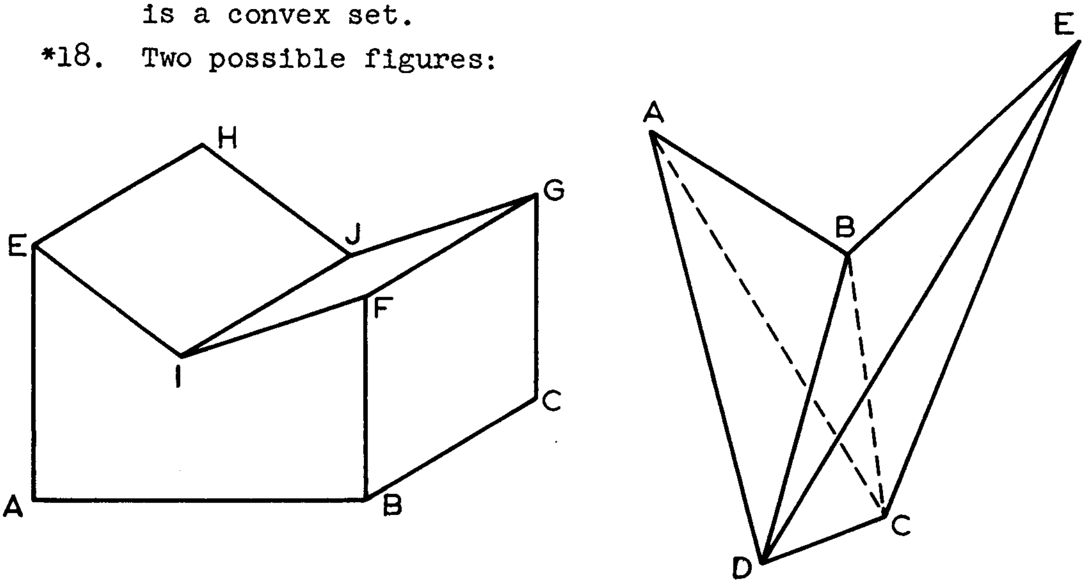
14. No. Yes. No.* Yes. Yes.
(*Three lines can separate a plane into five regions if we allow two parallels through a point to a line. This would give:



However, if we should assume only one parallel through a point to a line, we could not get five regions.

Note that within our postulational system so far developed we do not know which choice, if either, we will accept, or which will be excluded.)

15. Four. Three.
16. Eight. Four.
*17. Consider the segment $\overline{PQ}$ joining any two points P and Q of the intersection. $\overline{PQ}$ is contained in the first set, since it is convex. $\overline{PQ}$ is contained in the second set, since it is convex. By the definition of intersection, the intersection contains all points common to the two sets. Therefore, the intersection contains $\overline{PQ}$, and the intersection is a convex set.
*18. Two possible figures:



[page 68]

Review Problems

68 1. Yes. No. They may intersect in a point (as the corner of a room where two walls and the floor meet). Also, there may be no point common to all three if there are three lines each of which is the intersection of two of the planes.

2. One plane.

3. a. If a zebra has polka dots, then it is dangerous.
   b. If a rectangle has sides of equal lengths, then it is a square.
   c. If Oklahoma wins, then there will be a celebration.
   d. If two straight lines intersect, then they determine a plane.
   e. If a dog is a cocker spaniel, then it is sweet-tempered.

4. Each half-plane is convex. Yes.

69 5. From this statement one gets the impression that a plane has boundaries. To have said, "The top of the table, if it were absolutely flat and smooth, would give a good idea of a small part of a plane," would have been a better statement.

6. Three non-collinear points.
   A line and a point not on the line.
   Two intersecting lines.

7. In the set.

8. Yes.

9. No. Since L_2 lies entirely in plane E, if the two lines were to intersect, L_1 would have to contain some other point of plane E. This is impossible by Theorem 3-2.

10. a. One line contains all points of the set.
    b. One plane contains all points of the set.
    c. Yes.
    d. Yes.
    e. Yes.
    f. No.
    g. Yes.

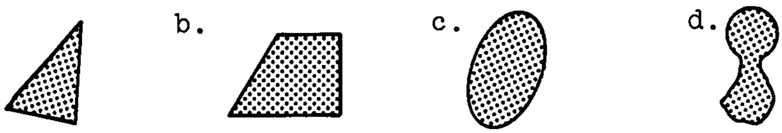
11. Yes, by Postulate 7.
12. Yes.

Illustrative Test Items for Chapter 3

1. If two different lines intersect, their intersection is ________. If two different planes intersect, their intersection is ________. If a plane and a line not contained in the plane intersect, their intersection is ________.

2. Which of these regions, if any, is not convex?

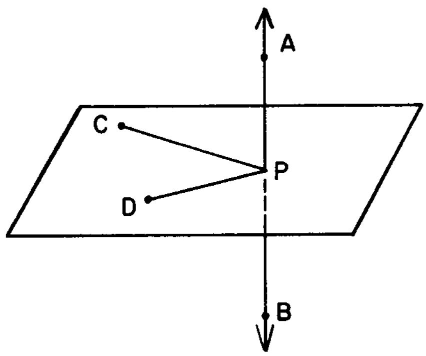
a. b. c. d.



3. Which, if any, of the following can separate a plane?

a. Segment b. Point c. Line d. Ray

4. Fill in the blanks in the statements below on the basis of the figure shown. IMPORTANT: If none of the points given satisfies the condition, write NONE in the blank space.



Points A, P, and ________ are collinear.
Points D, P, and ________ are collinear.
Points P, D, B, and ________ are coplanar.
Points C, A, B, and ________ are coplanar.

5. Write each of the following statements in "if-then" form:
   a. Two different lines have at most one point in common.
   b. Any three non-collinear points lie in exactly one plane.
6. Complete:
   a. The set of all points in a plane which lie on one side of a given line of the plane is a ________ .
   b. The two sets of points into which a ________ separates space are each called half spaces.
7. How many planes can contain one given point? Two given points? Three non-collinear points?
8. Indicate whether True or False:
   a. ______ A line and a plane always have at most one point in common.
   b. ______ Two lines always lie in the same plane.
   c. ______ There are lines which do not intersect each other.
   d. ______ If three points are collinear they are coplanar.
   e. ______ A point and a line always lie in one and only one plane.
   f. ______ Given two different points A and B. There are at least two different lines that contain both A and B.
   g. ______ Every two points are collinear.
   h. ______ A line has two end-points.
   i. ______ There is a set of four points which lie in no plane.
   j. ______ Given two points, there is more than one plane containing them.
9. State the Plane Separation Postulate in your own words.

Answers

1. A point. A line. A point.
2. d.
3. c.
4. B.

   None.

   A.

   P.
5. a. If L_1 and L_2 are two different lines, then they have at most one point in common.

   b. If A, B and C are three non-collinear points, then they lie in exactly one plane.
6. a. Half-plane.

   b. Plane.
7. Infinitely many. Infinitely many. One.
8. a. F; b. F; c. T; d. T; e. F; f. F; g. T; h. F; i. T; j. T.

Chapter 4
ANGLES AND TRIANGLES

High school geometries usually take the notion of interior for granted. A person is supposed to know from looking at a figure when a point lies in the interior of an angle, for example. Most things move along without undue difficulty unless somebody raises such a question as: But what reason can you give to support your claim that point B lies in the interior of angle AOC? Such a question can hardly be answered when there is no formal knowledge from which to reason. This book provides such formal knowledge by treating notions of betweenness, order and interior.

Another way in which this book differs from almost every other text is in its careful treatment of angles: their definition, their separation properties and their measure. This last is done in a way to suggest an analogy with the measure of distance presented in Chapter 2.

There is a clear-cut distinction in this text between an angle and the measure of an angle. An angle is a set of points; its measure is a number. Such a distinction between the point set and the number is usually not made in text books, the word "angle" being used for both.

At the end of this chapter you will see the beginning of something that may strike you as very peculiar. The use of the words <u>equal</u> and <u>congruent</u> in this book is different from the common usage, and you should have early advance warning of this, so as to be ready for it. Near the end of this chapter, it is explained that if $m\angle A = m\angle B$, then the angles are called <u>congruent</u>, and we write $\angle A \cong \angle B$. In Chapter 5 we will give a similar definition of congruence for segments. That is, if $AB = CD$, then the segments $\overline{AB}$ and $\overline{CD}$ are called <u>congruent</u>, and we write $\overline{AB} \cong \overline{CD}$.

(Many texts also say that two triangles are equal, as an abbreviation of the statement that the areas of the triangles are the same. In this book, this abbreviation will be avoided; we shall simply say that the triangles have the same area.)

There is nothing complicated about our terminology, but you may find it hard to get used to. To avoid trouble which might otherwise start soon, we recommend that at the earliest opportunity you read the talk on <u>Equality, Congruence, and Equivalence</u> in which we explain what we have on our minds, and how and why we have departed from the traditional terminology.

In this chapter we have omitted -- rightly, we believe -- the proofs and even the statements of various simple and obvious theorems of a foundational character. Some of these will be discussed in Chapter 6, but for a thorough logical treatment of the material of this chapter, see Chapter 5 of Studies II.

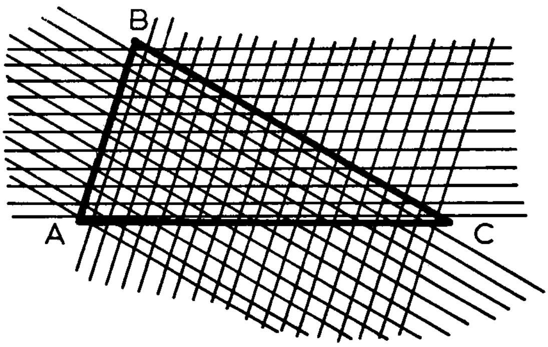
71 No matter what text is used, students must be cautioned that when using three letters to denote an angle, they must write the letter corresponding to the vertex between the other two letters.

72 The three vertices of a triangle are the vertices of the three angles of the triangle. To verify the statement that the angles of $\triangle ABC$ are not contained in the triangle, check to see if the set of points in $\angle ABC$ is contained in the set of points of $\triangle ABC$. If we remember that the set of points in $\angle ABC$ is the union of two rays, each of which extends infinitely far in one direction, and the set of points of $\triangle ABC$ is the union of three segments, then we see that the triangle cannot possibly contain its angles.

73 We could define the interior of $\angle BAC$ as the intersection of the set of points on that side of $\overleftrightarrow{AC}$ containing B with the set of points on that side of $\overleftrightarrow{AB}$ containing C. This intersection is diagrammed on page 74.

74 The interior of $\triangle ABC$ may also be defined as the intersection of three half-planes: (1) the side of $\overleftrightarrow{AC}$ that contains B, (2) the side of $\overleftrightarrow{BC}$ that contains A, and (3) the

side of $\overleftrightarrow{AB}$ that contains C. A cross hatching of the intersection of these half-planes will graphically illustrate that this region is the same set of points as indicated in the text.



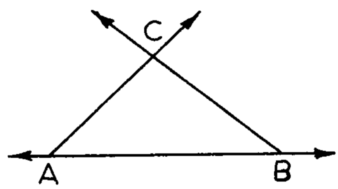
Problem Set 4-1

75 1. union, rays, line.

2. union, segments, non-collinear.

3. No. $\overline{AC}$ and $\overline{AB}$ are line segments, but the sides of $\angle A$ are rays.

4. No. Although the union contains the triangle, the union also contains the rest of the sides of the angles.



Union of $\angle$ A and $\angle$ B

5. Seven.

6. $\angle$NPR, $\angle$ NPT, $\angle$ MPS, $\angle$ MPT.

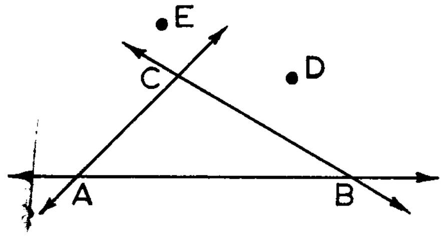
7. $\angle$AEC, $\angle$ CEB, $\angle$ BED, $\angle$ DEA.

8. Eight. $\angle$ A, $\angle$C, $\angle$ABC, $\angle$ABD, $\angle$CBD, $\angle$ ADC, $\angle$ ADB, $\angle$CDB.
Two.

76 9. $\angle$AMB, $\angle$ BMC, $\angle$ CMD, $\angle$ DME, $\angle$ EMF, $\angle$ FMA, $\angle$ AMC, $\angle$ BMD, $\angle$CME, $\angle$ DMF, $\angle$ EMA, $\angle$ FMB.

10. $\triangle$ ABC, $\triangle$ ABF, $\triangle$ BCF, $\triangle$ ACD, $\triangle$ FCD, $\triangle$ AFD, $\triangle$ AGD, $\triangle$ GFD, $\triangle$ AED, $\triangle$ AEG, $\triangle$ EBD, $\triangle$ ABD, $\triangle$ BCD, $\triangle$ GDC.

11. a. D, F, M.
b. E, G, H.



12. No. No. It lies on the angle itself.
13. Yes. No.
14. No.
15. Yes. No.
16. a. Yes. D is such a point.
    b. Yes. E is such a point.
17. P is in the interior of $\triangle ABC$.
18. a. Yes.
    b. Not necessarily. P and C could be on opposite sides of $\overleftrightarrow{AB}$.
19. A and C are in opposite half-planes determined by line m.

---

77 Section 4-2 is not an integral part of the course, and the information presented in it will not be referred to again in the text. The material is made available to those classes whose teachers consider it appropriate in the local curriculum.

79 You may wonder, after seeing the $m\angle A$ notation, why the text uses AB instead of $m\overline{AB}$ in Chapter 2. Actually $m\overline{AB}$ does have the advantage of consistency but we do not feel that this is to offset the advantages AB has: of convenience and of common usage. $\overline{AB}$ and AB are different symbols for different entities. So are $\angle A$ and $m\angle A$.

79 It will be noted that in this treatment of measurement of
angles, it is understood from the start that the unit of measure
80 is the degree. This is implicit in Postulate 11, and in this
respect the Angle Measurement Postulate may seem more satisfying than Postulate 2 concerning distance, where a unit of measure was chosen but left unspecified. There is nothing especially logical, however, about the choice of degree measure for angles: it merely happens to be customary and familiar.

81 You may notice a similarity between the Angle Construction Postulate and the Ruler Postulate. We again have a one-to-one correspondence, this time between rays in a half-plane from a point on the edge of the half-plane and the numbers between 0 and 180.

Some additional mention of the use of the degree sign may be necessary. When we label figures, as in the figure at the top of page 80, the degree sign is used only to indicate that the number appearing to the left of it is the degree measure of the angle, to distinguish from the use of a lower case letter to identify the angle. For example, we may have an angle of a°, and we must distinguish this from the angle that could be identified by the letter "a". We may speak of $\angle QAB$ as "a 40 degree angle" or we may say that $\angle QAB$ is an "angle whose measure (now understood to be degree measure) is 40." One may ask, "Why even mention the degree once we have established it as our unit of measure?" The reason is that the degree is not the only unit by which we can measure angles. There is, of course, the radian, which is fundamental to trigonometry, and we must be absolutely certain with what unit we are working.

80 One difference in this treatment of geometry is that under our definition of an angle there is no angle whose measure is 0, nor is there one whose measure is 180. Since the idea of a "180° angle" or "a straight angle" has been used in geometry for so long, it might be a little hard for us as teachers to become accustomed to this usage. In thinking of angles as point-sets it is apparent that an angle whose measure is 0 is indistinguishable from a ray, and an angle whose measure is 180 cannot be distinguished from a line. Hence, no such "angles" appear in this treatment. Another reason for not allowing these special angles is that it is impossible to determine the interior of an angle of zero measure or of one whose measure is 180. Incidentally, Euclid never used "straight angles."

79,80 Note carefully how the ray $\overrightarrow{AC}$ in the figure on page 79 corresponds to the number 180 and how this <u>can</u> be used to determine the measures of other angles as illustrated on page 80. Note also that the ray $\overrightarrow{AB}$ corresponds to 0. Although we do not allow the possibility of an angle of 180°, this does not eliminate the possibility of <u>two</u> angles having the sum of their

82 measures equal to 180, and thus we do have supplementary angles. (See Postulate 14.)

The phrase "linear pair" will probably be new to you. It is an easily remembered name that simplifies the statement of Postulate 14 and some of the subsequent definitions and proofs. On the other hand, we have not found it necessary to use the phrase "adjacent angles". Linear pair is easily defined, for it involves only the notion of opposite rays. The idea of adjacent angles is more complicated, for it involves the idea of separation in a plane. Two angles are adjacent if they have a common side and their other two sides are contained in the opposite half-planes determined by the line containing the common side.

Problem Set 4-3

83 1. a. 60. g. 25.
b. 30. h. 70.
c. 30. i. 70.
d. 30. j. 90.
e. 70. k. 125.
f. 15. l. 100.

2. a. p; b. m; c. q; d. n;

84 4. The remaining angle has a measure of 50.

5. a. BHG or GHB.
b. BFG or GFB.

6. a. XZY or YZX.
b. XZK or KZX.
c. KZY or YZK.
d. 180.

85 7. $a = 52$, $b = 128$, $c = 52$.

8. 70°; 90°; 144°; 164.5°; $(180 - n)^\circ$, for $0 < n < 180$, n°, for $0 < n < 180$, $(90 + n)^\circ$, for $0 < |n| < 90$.

9. 75, 105.
10. 120.
11. 36, 144.
12. a. One way by the Angle Construction Postulate.
    b. Two ways. There are two half-planes in E whose edges contain $\overrightarrow{AC}$.

---

5 Notice that the definition of right angle precedes any mention of perpendicularity. Various approaches would have been possible; the one used seems to be simplest logically, for it permits lines, rays and segments to be included in one definition of perpendicular.

36 The text points out that a ray or a segment <u>determines</u> a unique line which contains it. When two lines intersect, four rays are determined. These rays in turn determine four angles. Sometimes we refer to the angles as angles formed by the lines. (A mathematical purist might want to replace the phrase "if the two lines containing the two sets determine a right angle" by "if the union of the two lines containing the two sets has a right angle as a subset".)

87 Theorem 4-4 could, with proper restatement, be taken as the definition of right angles. In that case the definition of right angle actually used in the text would be replaced by a theorem.

88 Alternate proof for Theorem 4-7:

Given that $\overrightarrow{AC}$ and $\overrightarrow{AE}$ are opposite rays, and $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are opposite rays so that $\angle 1$ and $\angle 2$ are vertical angles. Let $m\angle 3 = r$. Then by Postulate 14, $m\angle 1$ must be 180-r, and $m\angle 2$ must also be 180-r. Therefore, $m\angle 1 = m\angle 2$, and $\angle 1 \cong \angle 2$, which was to be proved.

Problem Set 4-4

89 1. a. Only one.

b. Infinitely many.

2. $\overleftrightarrow{ON} \perp \overleftrightarrow{RS}$. $\angle RON$ and $\angle SON$ are supplementary and have equal measures. Therefore, each has a measure of 90, making $\overleftrightarrow{ON} \perp \overleftrightarrow{RS}$.

3. a. $\overrightarrow{XR}$ and $\overrightarrow{XS}$.

b. $\angle RXB$ and $\angle SXA$.

c. None occur.

d. $\angle RXB$ and $\angle RXA$.

$\angle SXA$ and $\angle SXB$.

90 4. a. 80°.

b. 10°.

c. 45.5°.

d. $(90 - x)^\circ$, for $0 < x < 90$.

e. x°, for $0 < x < 90$.

f. $(x - 90)^\circ$, for $90 < x < 180$.

5. a. 90.

b. 45.

6. a. Two pairs.

b. 70, 110, 110.

c. 90.

7. r, $(180 - r)$, $(180 - r)$.

8. $m\angle BGD = 90$.

Proof: $m\angle AGC + m\angle CGE = 180$.

$\frac{1}{2}\, m\angle AGC + \frac{1}{2}\, m\angle CGE = 90$.

$m\angle BGC + m\angle DGC = 90$.

$m\angle BGD = 90$.

90 9. If either angle were not acute its measure would be greater than or equal to 90. Then the sum of the two angles would not be 90 so that they would not be complementary as given. Hence, both angles must have measures less than 90 and by definition be acute.

10. Let the measure of each of the congruent angles be m. Since they are also supplementary, $m + m = 180$, $2m = 180$ and $m = 90$. Hence, each angle is a right angle.

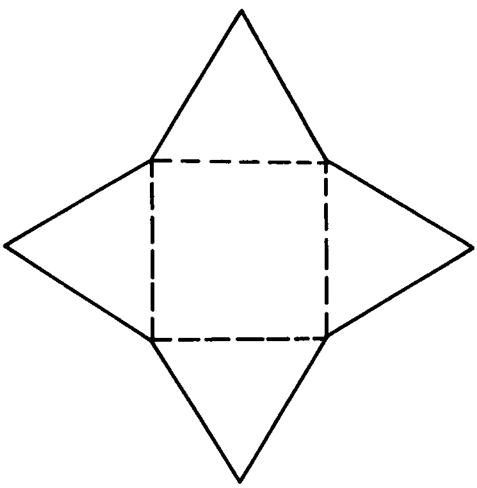
11. m ∠BGD = 90. (Definition of perpendicular.)
m ∠AGB + m ∠BGD + m ∠DGE = 180. (The Angle Addition Postulate and the Supplement Postulate.)
m ∠AGB + m ∠DGE = 90. (Subtraction.)
Therefore, ∠AGB and ∠DGE are complementary. (Definition of complementary.)

91 12. g = c. (Vertical angles have equal measures.)
b + c + d = 90. (Perpendicular lines form right angles.)
Therefore, b + g + d = 90. (Algebraic substitution of g for c.)
a = 90. (Perpendicular lines form right angles.)
Hence, b + g + d = a. (Algebraic substitution.)

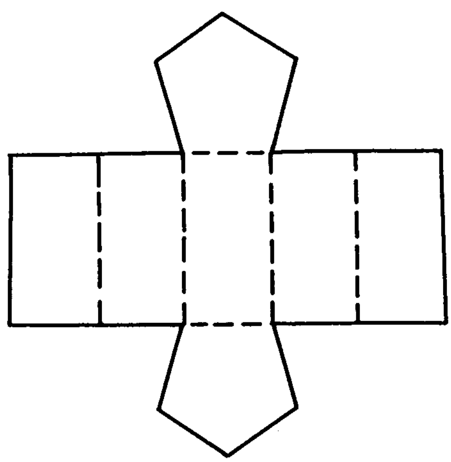
13. a. False. An exception occurs if $\overrightarrow{OB}$ lies in the exterior of ∠AOC.
b. False. An exception occurs if $\overrightarrow{OB}$ lies in the interior of ∠AOC.
(Note to teacher: Point out that one exception is sufficient to prove a statement false.)

14. 162.

91 15. a.



b.

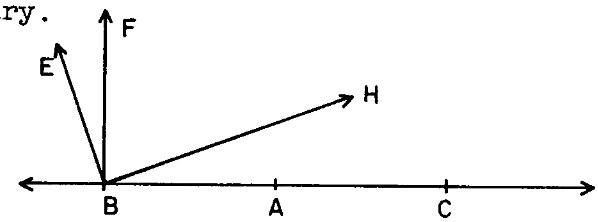


Review Problems

1. Protractor.
2. 0, 180.
3. Acute.
4. Linear pair.
5. Complement.
6. Obtuse.
7. Congruent.
8. Right angle.
9. Congruent.
10. Acute.
11. Union; rays.
12. Non-collinear; triangle.
13. X, T, $\overleftrightarrow{RS}$.
14. 90, 180, supplementary.

93 15. Vertical.

16. a. 110.
    b. 70.
    c. 110.



17. a. 130. b. 65. c. 50. d. 130.
18. 65, 115.
19. 15, 75.
20. If both are right angles.
21. Yes, any vertex of the triangle.
22. Not necessarily. The statement would not be true if the sum were 180 or larger.

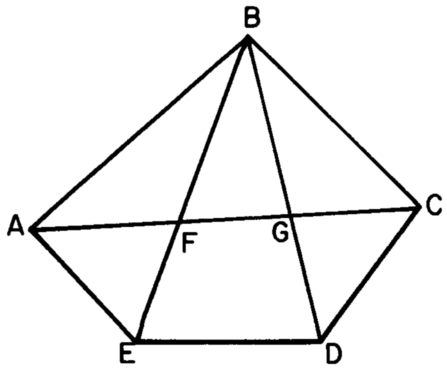
94 23. Yes. See figure on page 57.

24. 5.
25. Yes.
26. Yes.
27. No.
28. 12.
29. S and T are on opposite sides of $\overleftrightarrow{VU}$.
R and T are on opposite sides of $\overleftrightarrow{VU}$.
R and S are therefore on the same side of $\overleftrightarrow{UV}$, so that they are in the same half-plane. Since a half-plane is convex $\overline{RS}$ does not intersect $\overleftrightarrow{UV}$.
30. By the Supplement Postulate, $\angle 2$ is a supplement of $\angle x$ and $\angle s$ is a supplement of $\angle y$. $\angle z \cong \angle s$ because supplements of congruent angles are congruent.
31. Supplements of congruent angles are congruent.
32. The measure must be between 0 and 180.
33. No. The point P must be limited to a half-plane with the ray $\overrightarrow{XY}$ on its edge.
95 34. a. Angle Addition Postulate. b. Supplement Postulate.
35. No. O may not be between C and D.

Illustrative Test Items for Chapter 4

1. Indicate whether each statement is true or false.
   a. A point on the edge of a half-plane belongs to that half-plane.
   b. If two complementary angles are congruent, then each is a right angle.
   c. For every positive number r, there is an angle, $\angle A$, such that $m\angle A = r$.
   d. If a point is in the exterior of any one of the angles of a triangle, then it is in the exterior of the triangle.
   e. If D is in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$.
   f. If D is in the exterior of $\angle ABC$, then $m\angle DBA + m\angle ABC = m\angle DBC$.
   g. If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O, then $\angle AOC \cong \angle BOD$.
   h. If $m\angle Q = 100$, then $\angle Q$ has no complement.
   i. If a point is in the interior of an angle of a triangle, it is in the interior of the triangle.
   j. The intersection of two half-planes whose edges have only one point in common is the interior of an angle.
   k. The interior of an angle is a convex set.
   l. If two angles have the same measure, then they are vertical angles.
   m. The supplement of $(90 - x)^\circ$ is $(x + 90)^\circ$.
   n. Every angle is congruent to itself.
   o. Vertical angles are never supplementary.

2. a. In the figure below, there are a number of triangles. Five of these triangles have been listed below. Use the remaining space to list all of the other triangles you can find in the figure.



△ BAF.

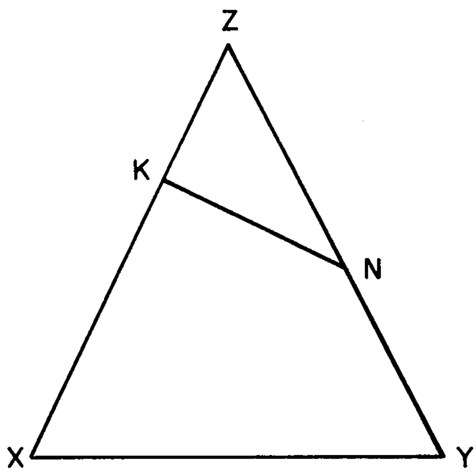
△ BFG.

△ BCG.

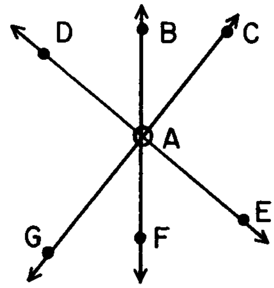
△ AEF.

△ GCD.

b. List all of the angles in the figure below.



c. State the number of different angles in the given planar figure.
How many different angles are there if the three lines are not coplanar?
How many linear pairs of angles are in the figure?



3. Multiple Choice. Select the one correct answer.

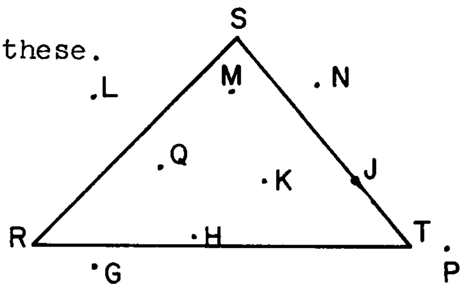
a. Which of these points is not in the interior of any angle?

L, P, H, M, none of these.

b. Which of these is determined by $\overrightarrow{RS}$ and $\overrightarrow{RT}$?
∠RST, ∠TRS, △SRT, ∠RTS, none of these.

c. Which point is in the exterior of △RST?

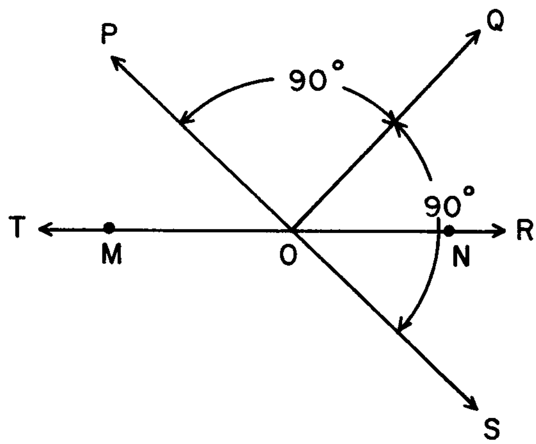
G, R, H, J, none of these.



Probs. a - c.

d. ∠TOP and ∠ROS are:
supplementary angles,
perpendicular,
complementary angles,
vertical angles,
none of these.

e. ∠QOR and ∠ROS are:
supplementary angles,
perpendicular,
complementary angles,
vertical angles,
none of these.



Probs. d - g.

f. $\angle QOS$ is:

a right angle, an acute angle,
a vertical angle, none of these.

g. $\overrightarrow{OQ}$ is perpendicular to:

$\overrightarrow{RT}$, $\overleftrightarrow{OQ}$, $\overleftrightarrow{PS}$, $\overleftrightarrow{MN}$, none of these.

h. If $\overrightarrow{AB} \perp \overleftrightarrow{NS}$, then:

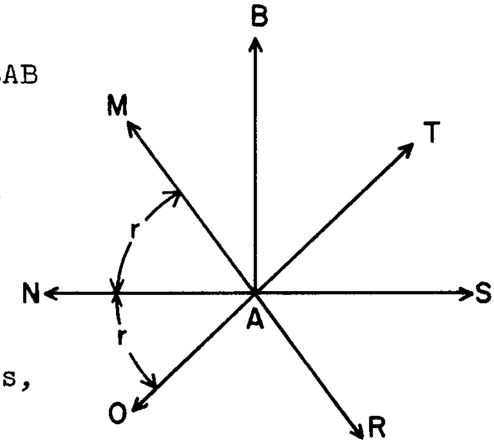
$m\angle MAN = m\angle BAT$,
$m\angle MAN = m\angle TAS$,
$m\angle MAN = m\angle BAM$,
$m\angle MAN = m\angle BAN$,
none of these.

i. If $\overrightarrow{AB} \perp \overleftrightarrow{NS}$, then $\angle NAB \cong \angle SAB$ because:

they are both acute,
they are complements of congruent angles,
they both have the same measure,
they are vertical angles,
none of these.

j. $m\angle MAT$ equals:

180 - 2r, 2r,
180 - r, 180,
none of these.



Probs. h - j.

4. MATCHING. Below are a number of statements or phrases in one column and a list of words or expressions in the other. Complete each statement by selecting the proper word or expression from the right-hand column.

a. An angle with measure less than 90 is ____________.

b. The supplement of a 60° angle has measure ___________.

c. The number of degrees in a right angle is __________.

d. If $\angle ABC$ is a right angle, then rays $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are _________.

e. Angles with the same measure are _____________.

f. The complement of a 60° angle has measure ___________.

g. If the sum of the measures of two angles is 90, the angles are _______.

h. An angle with a measure of more than 90 is ___________.

i. The supplement of a right angle has measure _________.

j. Complements of congruent angles are ______.

k. If $m\angle ABC + m\angle RST = 90$, then $\angle ABC$ is the _________ of $\angle RST$.

l. The supplement of an acute angle is _______.

m. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays. Ray $\overrightarrow{AE}$ is situated so that $m\angle CAE = m\angle BAE$. $\angle CAE$ is a ________ angle.

n. The measure of an angle that is twice its supplement is ___________.

o. The measure of an angle whose measure is half that of its complement is _________.

perpendicular
obtuse
right
90
acute
120
triangle
complement
congruent
30
n
complementary
supplementary

5. $\overrightarrow{XA}$ and $\overrightarrow{XB}$ are opposite rays on the edge of half-plane H. S and R are points of H such that $m\angle RXB = 35$, $m\angle RXS = 90$. Make a sketch and answer the following:
   a. Name a pair of perpendicular lines in H, if any occur.
   b. Name a pair of complementary angles in the sketch, if any occur.
   c. Name a pair of vertical angles in H, if any occur.
   d. Name two pairs of supplementary angles in the sketch, if two pairs occur.
   e. Name two acute angles in the sketch if any occur.
   f. Name two obtuse angles in the sketch if any occur.
6. Find $m\angle B$ in each of the following, where $\angle B$ is the supplement of $\angle A$.
   a. $m\angle A = 30$. b. $m\angle A = n$. c. $m\angle A = 45-n$.
   d. $m\angle A = 120$.
7. Find $m\angle B$ in each of the following, where $\angle B$ is the complement of $\angle A$.
   a. $m\angle A = 38$. b. $m\angle A = 49$. c. $m\angle A = n$
   d. $m\angle A = n+25$.
8. a. If one of a pair of vertical angles has a measure of x, write the formulas for the measures of the other three angles formed.
   b. If three rays have a common endpoint and two of them are opposite rays, what is the sum of the measures of the angles in the resulting figure?
   c. H is a point in the interior of $\angle RST$. $m\angle HST = 10$ and $m\angle RST = 30$. What is the value of $m\angle HSR$?
   d. If two congruent angles are supplementary, what kind of angles are they?
   e. If each of two vertical angles has measure 1, what is the measure of each of the other vertical angles in the figure?
   f. If the difference between the measures of two complementary angles is 8, what is the measure of each angle?

9. a. Sketch two angles such that their intersection is a set of three points.

   b. Is every point in the interior of an angle a point of the angle?

   c. Given Δ RST and a point P. P and R are on the same side of $\overleftrightarrow{ST}$. P and S are on the same side of $\overleftrightarrow{RT}$.

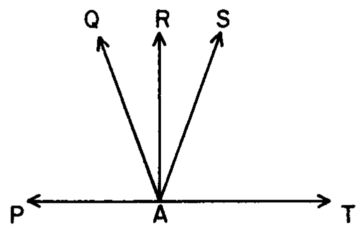
      Is P in the interior of $\angle$ RTS?

      Is P in the interior of Δ RST?

10. a. If the ray $\overrightarrow{AC}$ lies in a plane, how many rays $\overrightarrow{AB}$ are there in the plane such that $m\angle BAC = 110$? Draw a sketch.

   b. In the planar figure it is given that $\overrightarrow{AR} \perp \overleftrightarrow{PT}$ and that $m\angle QAR = m\angle SAR$.

      Prove: $\angle PAQ \cong \angle SAT$.

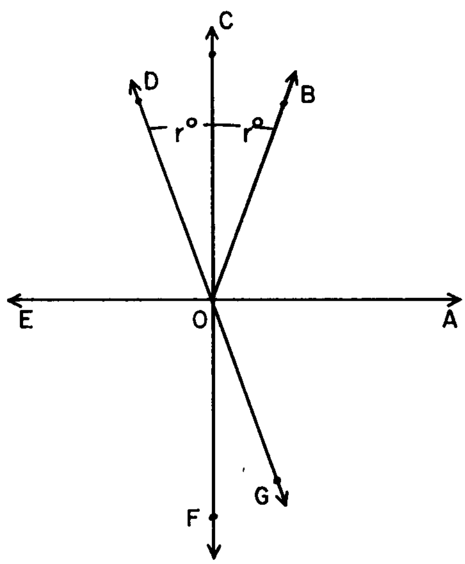


11. In the figure $\overleftrightarrow{AE} \perp \overleftrightarrow{CF}$. For each of the congruences below state the theorem which justifies it.

   a. $\angle AOB \cong \angle DOE$.

   b. $\angle DOF \cong \angle BOF$.

   c. $\angle DOC \cong \angle FOG$.



Answers to Illustrative Test Items for Chapter 4

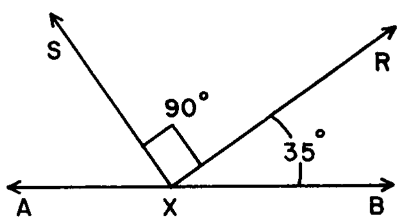
1. a. False.
   b. False.
   c. False.
   d. True.
   e. True.
   f. False.
   g. False. (O may not be between C and D.)
   h. True.
   i. False.
   j. True.
   k. True.
   l. False.
   m. True.
   n. True.
   o. False.

2. a. ΔABE, ΔBED, ΔBCD, ΔABC, ΔBAG, ΔBFC.
   b. $\angle$X, $\angle$Y, $\angle$Z, $\angle$ZKN, $\angle$ZNK, $\angle$XKN, $\angle$YNK.
   c. 12, 12, 12.

3. a. P.
   b. $\angle$ TRS.
   c. G.
   d. Vertical angles.
   e. Complementary angles.
   f. a right angle.
   g. $\overleftrightarrow{PS}$.
   h. None of these.
   i. They both have the same measure.
   j. 180-2r.

4. a. Acute.
   b. 120.
   c. 90.
   d. Perpendicular.
   e. Congruent.
   f. 30.
   g. Complementary.
   h. Obtuse.
   i. 90.
   j. Congruent (or acute).
   k. Complement.
   l. Obtuse.
   m. Right.
   n. 120.
   o. 30.

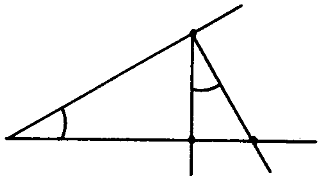
5.



   a. None occur in H.
   b. $\angle$BXR and $\angle$SXA.
   c. None occur in H.
   d. $\angle$RXA and $\angle$RXB, $\angle$BXS and $\angle$AXS.
   e. $\angle$SAX and $\angle$RXB.
   f. $\angle$SXB and $\angle$RXA.

6. a. 150. b. 180 - n. c. (135 + n). d. 60.
7. a. 52. b. 41. c. (90 - n). d. (65 - n).
8. a. x, 180 - x, 180 - x. b. 180. c. 20.
   d. Right. e. 179. f. 41,49.

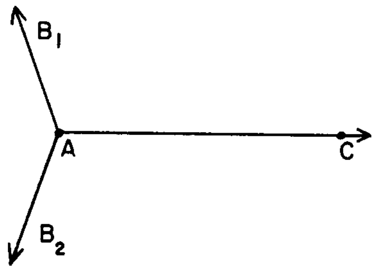
9. a.



b. No; no point in the interior is a point of the angle.

c. Yes, Not necessarily.

10. a. Two.



b. ∠PAR and ∠TAR are right angles, by the definition of perpendicular.

∠PAQ and ∠SAT are complements of congruent angles, ∠QAR and ∠SAR.

∠PAQ ≅ ∠SAT, because complements of congruent angles are congruent.

11. a. Complements of congruent angles are congruent.

b. Supplements of congruent angles are congruent.

c. Vertical angles are congruent.

Chapter 5

CONGRUENCES

The treatment of congruence in this chapter will seem unfamiliar to many teachers, but the two Talks, Equality, Congruence, and Equivalence and The Concept of Congruence, should be helpful to them. The difference in treatment lies chiefly in the fact that congruence is regarded here as a special kind of one-to-one correspondence. Our notation was chosen to show how the corresponding parts of two triangles are paired without referring to a diagram. Correct use of this symbolism should eliminate confusion about what the corresponding parts are in any particular problem.

We have included problems to familiarize students with the new terminology; the rest of the problems in the chapter are familiar in type. In this book, as in most books, the students are expected to develop a working knowledge of proof by working with congruence of triangles.

Students should show progress, while studying this chapter, in their ability to recognize different proofs of a theorem. The tendency for them to think that a mathematical problem has only one method of solution should be replaced gradually by the practice of examining each proof as an example of correct logical reasoning.

The extent to which a proof is detailed is mainly a matter between the teacher and student. We believe it desirable to develop flexibility of methods dependent upon the problem at hand and the mathematical maturity of the students involved. As the student progresses he should be encouraged to omit minor steps where understanding is not impaired and convenience results. For example, if the hypothesis of a theorem says that M is the midpoint of $\overline{AB}$, the teacher may require in the first proofs the student does that $AM = MB$ be justified in two steps:

1. M is the midpoint of $\overline{AB}$.	1. Hypothesis.
2. AM = MB.	2. Definition of midpoint.

As he learns, the student should be permitted to telescope this into one step by saying AM = MB, by definition of midpoint (or even, by hypothesis). The important thing is to advance the student's growth in the direction of appreciating and understanding proof.

A$\leftrightarrow$D can be read: Points A and D correspond to
98 each other, or A corresponds to D.

ABC$\leftrightarrow$DEF can be read: The points A, B, and C and the points D, E and F correspond to each other in the order named, or briefly, A, B, C correspond to D, E, F.

97 In this introduction we first develop the intuitive idea of a <u>congruence</u> between two geometric figures. A congruence means intuitively that there is a particular way of moving one figure so that it coincides with another. We proceed, as quickly as possible, to the idea that a congruence can be described by explaining where each point in a certain finite set of points is going to go. The idea behind this treatment is to get the student accustomed to writing down the sets of matching pairs, so as to prepare the way for the formal mathematical treatment of congruences between triangles.

Two figures are congruent if there is a congruence between them; that is, speaking informally, if one of them can be moved so as to coincide with the other. In this chapter, however, heavy stress is given to the idea of <u>a</u> congruence between two
99 figures, for there may be more than one congruence possible between the two. This stress should begin at the very beginning of the chapter. In this spirit, it should be made plain that a problem based on this section is not to be considered solved if the student has merely determined that two figures are congruent. The problem is solved only when a <u>particular</u> congruence between the two figures is exhibited.

100 For some pairs of triangles there is a unique one-to-one correspondence between vertices that is a congruence. However,

in the case of a pair of isosceles or equilateral triangles, if there exists a congruence between them, then there is more than one congruence between them.

Problem Set 5-1

100 1. ABC ⟷ QPR.
DEF ⟷ SUT.
DFE ⟷ TSU.
EDF ⟷ EFD.
UST ⟷ UTS.
KLNO ⟷ IJGH.

101 2. RFH ⟷ ACB.
MXPQ ⟷ LEKW.
DZG ⟷ TYL.

3. ABC ⟷ PNQ.
KXY ⟷ IHJ.
GDEF ⟷ WRLM.

102 4. AFEG ⟷ WTSX.
HIJK ⟷ NRPQ.
CLM ⟷ CML.
UZY ⟷ UYZ.
CLM ⟷ UYZ.
CLM ⟷ UZY.

5. a, d.

103 6. b, c, e, g, h.

104 7. ABC ⟷ ABC. ABC ⟷ ACB.
ABC ⟷ BAC. ABC ⟷ BCA.
ABC ⟷ CAB. ABC ⟷ CBA.

105 8. ABCD ⟷ ABCD. ABCD ⟷ ADCB.
ABCD ⟷ BCDA. ABCD ⟷ DCBA.
ABCD ⟷ CDAB. ABCD ⟷ CBAD.
ABCD ⟷ DABC. ABCD ⟷ BADC.

9. a. Yes. b. Yes. c. No. d. Yes. e. Yes. f. Yes.
g. Not always.

10. (a,d), (c,e).

106 11. ABCD $\leftrightarrow$ ABCD.
ABCD $\leftrightarrow$ BADC.
ABCD $\leftrightarrow$ DCBA.
ABCD $\leftrightarrow$ CDAB.

12. a. Slide the line to the right or rotate about the point halfway between A and B. The first of these motions takes B to C but the second does not.
b. Rotate the line in the plane (or in space) about B.

13. a. If they have the same length.
b. If they have the same measure.
c. Always.
d. If they have the same radius.
e. If their edges have the same length.
f. Always.
g. Always.

107 14. a. Rotate the circle about its center.
b. Turn the circle over in space, leaving the diameter containing B fixed.

15. a. Slide the frieze horizontally. There are infinitely many translations of this type that result in congruences.

Using the line of the frieze as an axis, rotate the frieze a half-turn about this axis and then translate the frieze horizontally. There are infinitely many motions of this type that result in congruences.

b. Translate horizontally. Infinitely many. Rotate in the plane through 180° about a point on the line halfway between two successive intersections.
Infinitely many.

108 16. (a) and (e). A turn-over is needed.
(b) and (c). No turn-over is necessary.
(d) and (f). No turn-over is necessary.

17. ABCDE $\leftrightarrow$ ABCDE.
ABCDE $\leftrightarrow$ BCDEA.
ABCDE $\leftrightarrow$ CDEAB.
ABCDE $\leftrightarrow$ DEABC.
ABCDE $\leftrightarrow$ EABCD.
ABCDE $\leftrightarrow$ AEDCB.
ABCDE $\leftrightarrow$ EDCBA.
ABCDE $\leftrightarrow$ DCBAE.
ABCDE $\leftrightarrow$ CBAED.
ABCDE $\leftrightarrow$ BAEDC.

---

109 We now begin to talk about congruence in a careful way in terms of distance and angular measure. It may be helpful to restate the definition on this page using symbols:

Definition: Consider angles $\angle A$ and $\angle B$,
$\angle A \cong \angle B$ if $m\angle A = m\angle B$.
Consider segments $\overline{AB}$ and $\overline{CD}$,
$\overline{AB} \cong \overline{CD}$ if $AB = CD$.

Since any definition is an agreement that one expression is an abbreviation for another, the sentence "$\angle A \cong \angle B$" may be replaced by the sentence "$m\angle A = m\angle B$" and the sentence "$m\angle A = m\angle B$" may be replaced by the sentence "$\angle A \cong \angle B$". A related thing holds for segments. The sentence "$\overline{AB} \cong \overline{CD}$" may be replaced by the sentence "$AB = CD$" and the sentence "$AB = CD$" may be replaced by the sentence "$\overline{AB} \cong \overline{CD}$".

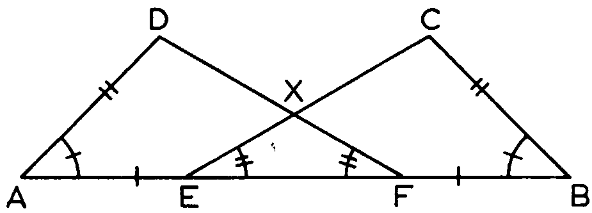
The question may very well arise as to why we have two different ways of writing exactly the same thing. If $\overline{AB} \cong \overline{CD}$ means that $AB = CD$, why bother to introduce the notation $\overline{AB} \cong \overline{CD}$? This would be a valid objection if we were talking about congruence of segments only. But we will be talking about congruence of segments, angles and triangles; and while the technical definitions of congruence are different for these three cases, the basic intuitive idea is the same. The basic intuitive idea is that two figures (of any sort whatever) are congruent if one can be moved so as to coincide with the other. In the Appendix on Rigid Notion (in volume II) this basic unity of the idea of congruence is described in an exact mathematical form. In the meantime, it seems worthwhile to emphasize this unity by using the same word, congruence, and the same symbol, $\cong$, whenever the idea occurs. Notice that in the definition of congruent angles and segments the idea of a one-to-one correspondence does not occur, as it does in the development of the basic idea of a congruence between two triangles. The idea does appear, however, in the general definition of congruence given in the Appendix on Rigid Motion.

In the table on Page 109 of the text note that the expressions on the left and right in each line are interchangeable, but this does not say that we can use the symbols " $\cong$ " and " = " interchangeably.

To help make this clear let us skip ahead and examine Postulate 15 (The S. A. S. Postulate). "Given a correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence." Let us consider the word "congruent" that is underlined above. This may not be replaced by "equals", since "equals" means "is the same as", and we would not be able to talk about two different triangles being congruent. Using "equals" we would be able to talk only about the identity congruence, which is rather uninteresting. In the statement of the above postulate it is possible to replace the phrase, "are congruent to" by the phrase, "have the same measure as."

111 In the definition of a congruence between two triangles we see that we must have a one-to-one correspondence between the vertices of the triangles such that (1) each pair of corresponding sides are congruent and (2) each pair of corresponding angles are congruent. Conditions (1) and (2) might be stated in this alternate manner: (1') each pair of corresponding sides have the same length and (2') each pair of corresponding angles have the same measure.

112 The text shows how to mark diagrams to indicate which parts of figures that are known to be congruent in the statement of a problem. Students should be encouraged to mark the figures they draw for themselves when this practice is not continued in the text. They will soon see that this is a very convenient method of translating the written information to their figures. As a student's analysis of a particular problem develops, he may wish to mark additional elements, the congruence of which he has established by using the given data. For example, suppose that it is given for the following figure that $\overline{AE} \cong \overline{FB}$, $\overline{AD} \cong \overline{BC}$, and $m\angle A = m\angle B$. The figure is marked accordingly:



Suppose it is required to prove that $\triangle EXF$ is isosceles. After the student has proved that $\triangle ADF \cong \triangle BCE$ and that $\angle CEB \cong \angle DFA$, he can put a pair of appropriate marks on these angles and show visually how much he has accomplished.

112 In answer to the question in the text, "Would it be correct to write $AB \cong DE$ or $\angle A = \angle D$? Why or why not?" (Refer to the figure above the question in the text.) $AB \cong DE$ is incorrect because AB and DE are numbers and we should speak of them as being equal rather than congruent. $AB = DE$ is correct. If we wish to emphasize the idea of a congruence, we can write a different correct statement, $\overline{AB} \cong \overline{DE}$. $\angle A = \angle D$ is incorrect in this case because $\angle A$ is not the same angle as $\angle D$, but $\angle A$ is congruent to $\angle D$ and we should write $\angle A \cong \angle D$ or else $m\angle A = m\angle D$.

[page 112]

The text emphasizes the fact that we may use the expressions "AB = DE" and "$\overline{AB} \cong \overline{DE}$"; "$\angle A \cong \angle D$" and "$m\angle A = m\angle D$", interchangeably. You may decide for yourself which notation is easier for you to use in a particular problem.

Let us once again, before reaching the S.A.S. Postulate, remind the teacher of the careful use of the correspondence idea in making statements about congruence in this text. You often hear people say that two triangles are congruent without indicating the particular correspondence between the vertices needed to prove the triangles congruent. Thus the statement that $\triangle ABC$ and $\triangle DEF$ are congruent is abbreviated -- without regard to the order in which letters are written -- as $\triangle ABC \cong \triangle DEF$, or $\triangle ABC \cong \triangle FED$, or $\triangle ABC \cong \triangle DFE$, and so on. These statements about congruence are treated in some courses as different correct ways of saying the same thing.

This is the idea of congruence that is explained in some conventional texts, but it is not the idea that gets used. Every time we seem to be using the idea that two triangles are congruent, it soon becomes clear that what we are really using is the fact that they are congruent _in a particular way_; that is, under a particular correspondence. For example, if we go on to infer that "corresponding sides have the same length", then we are claiming to know which side corresponds to which side. That is, what is being used is a correspondence between the triangles. The treatment in this text is based on the idea that we should talk explicitly about the ideas that we are really using. The unfamiliarity of this treatment may make it hard for us as teachers to get used to it. But the student, at this point, is not used to any formal mathematical treatment of congruence, and it ought to be easier to teach him to read what is written _on_ the lines than to teach him to read _between_ them. As a practical matter, the conventions of this chapter for the expression

$$\triangle ABC \cong \triangle DEF$$

[page 112]

seem to be efficient. It is very easy to read off which sides and angles are congruent, instead of having to remember the correspondence without benefit of concise memoranda. (Refer to the discussion on page 111 in the text.)

Problem Set 5-2

113 1. $\overline{MQ}$. F. R.

$\angle R$ $\overline{MR}$.

$\angle Q$. $\overline{QR}$.

2. $\overline{BA} \cong \overline{BF}$. $\angle A \cong \angle F$.

$\overline{RA} \cong \overline{RF}$. $\angle ABR \cong \angle FBR$.

$\overline{RB} \cong \overline{RB}$. $\angle ARB \cong \angle FRB$.

3. $\angle M \cong \angle F$. $\overline{MR} \cong \overline{FH}$.

$\angle R \cong \angle H$. $\overline{MK} \cong \overline{FW}$.

$\angle K \cong \angle W$. $\overline{RK} \cong \overline{HW}$.

4. $\angle R \cong \angle A$. $\overline{RQ} \cong \overline{AB}$.

$\angle Q \cong \angle B$. $\overline{RF} \cong \overline{AX}$.

$\angle F \cong \angle X$. $\overline{QF} \cong \overline{BX}$.

114 5. $\angle A \cong \angle B$. $\overline{AZ} \cong \overline{BZ}$.

$\angle AWZ \cong \angle BWZ$. $\overline{AW} \cong \overline{BW}$

$\angle AZW \cong \angle BZW$. $\overline{WZ} \cong \overline{ZW}$.

6. $\triangle ABW \cong \triangle MKF$.

7. $\triangle ABC \cong \triangle DEF$.

Two triangles congruent to the same triangle are congruent to each other.

(The student may be permitted to generalize the situation still more by substituting "figure" for "triangle" in this statement.)

8. a. The triangles are the same size and shape.

b. The triangles are the same size and shape.

c. The triangles vary in size and shape.

d. A possible idea is the statement of Postulate 15.

115 9. a. $\cong$, $=$, $\cong$, $=$, $\cong$, $\cong$ or $=$, $=$.
b. The sixth.
c. The third.

---

115 From the pictures and intuitive development, it seems very likely that $\triangle ABC \cong \triangle DEF$ under the stated conditions, and we make this intuitively reasonable idea our Postulate 15. The usual proof of this statement (S.A.S.) involves the superimposing of one triangle upon the other. This method of proof is not valid under our postulates. It is a fact that the S.A.S. Postulate cannot be proved on the basis of the preceding postulates.

117 Here we give the student an example of an "original" theorem, and explain how one might think of a proof and write it out. It is well known to mathematicians that proofs must not depend on information taken from figures. It may seem odd, therefore, that the examples of proof in Section 5-4 appear to depend on the figures that are given. This is not really true; the use of the figures is merely a matter of convenience, and they have been used because at this rather difficult stage of his development the student badly needs all the help he can get.

All valid geometric proofs are independent of figures in precisely this way. In Studies II, this fact is dramatized by the total omission of all figures. But such a treatment in the tenth grade would be more than flesh and blood could stand. And over and above this fact, the use of figures to aid intuition and stimulate the imagination is one of the most important things that we are trying to teach. Not even the best and most mature mathematicians have found a way to live by logic alone.

118 In the proof of Example 1 the reason column contains three definitions, one theorem, and one postulate. There is an implied use in Step 1 of the fact that $\overline{BH}$ is given

bisected by $\overline{AR}$. Actually some people would write "Given" as the reason for Step 1. Others, wishing to avoid any telescoping of steps early in the year, might prefer two steps:

$\overline{BH}$ bisects $\overline{AR}$ at F.	Given.
AF = RF	Definition of bisect.

A list of acceptable reasons for two-column proofs follows:

Given.
Definitions.
Postulates already set down.
Previously proved theorems or corollaries.
Principles of algebra or elementary logic.

119 The blanks in the proof of Example 2 can be filled in with:

2.	$\angle AHB \cong \angle FHB$.	.
4.	$\triangle AHB \cong \triangle FHB$.	By the S.A.S. Postulate.
5.		By the definition of congruence between triangles.

Problem Set 5-4

120 1. a, c, e, f, g, h.

121 2.

1. AC = DC.	1. Given.
2. BC = EC.	2. Given.
3. $\angle ACB \cong \angle DCE$.	3. Vertical angles are congruent.
4. $\triangle ACB \cong \triangle DCE$.	4. S.A.S. [The teacher may prefer a full statement of the postulate at this stage.]
5. $\angle B \cong \angle E$.	5. Definition of a congruence between triangles.

3.

1. $\overline{RB} \cong \overline{HB}$.	1. Given.
2. $\angle x \cong \angle y$.	2. Given.
3. AB = FB.	3. From the definition of midpoint.
4. $\triangle ABR \cong \triangle FBH$.	4. S.A.S.
5. $\angle R \cong \angle H$.	5. Corresponding parts of congruent triangles are congruent.

4.

a. 1. AD = BC.	1. Sides of a square have the same length.
2. AR = BR.	2. Definition of a midpoint.
3. $\angle A \cong \angle B$.	3. Each angle of a square is a right angle. All right angles are congruent.
4. $\triangle ARD \cong \triangle BRC$.	4. S.A.S.
5. RD = RC.	5. Definition of congruent triangles.

b. $\angle ADR \cong \angle BCR$, $\angle ARD \cong \angle BRC$ (corresponding parts of congruent triangles) and $\angle RDC \cong \angle RCD$ (complements of congruent angles are congruent).

5.

1. AB = FH.	1. Given.
2. $m\angle x = m\angle g$.	2. Given.
3. BH = HB.	3. Identity.
4. $\triangle ABH \cong \triangle FHB$.	4. S.A.S.
5. $m\angle A = m\angle F$.	5. Corresponding parts of congruent triangles are congruent.

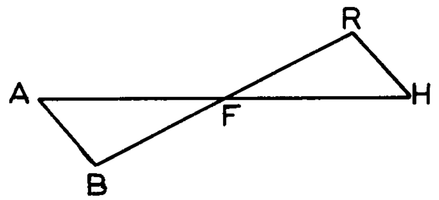
6.

1. AB = FB.	1. Given.
2. $m\angle ABH = m\angle FBH$.	2. Given.
3. BH = BH.	3. Identity.
4. $\triangle ABH \cong \triangle FBH$.	4. S.A.S.
5. AH = FH.	5. Definition of congruent triangles.

7.

Given: $\overline{AH}$ and $\overline{RB}$ bisect each other at point F.

To prove: $\triangle FAB \cong \triangle FHR$.



1. $\overline{AF} \cong \overline{HF}$. $\overline{FB} \cong \overline{FR}$.	1. Definition of bisect.
2. $\angle AFB \cong \angle HFR$.	2. Vertical angles are congruent.
3. $\triangle AFB \cong \triangle HFR$.	3. S.A.S.

8.

1. AE = DE. CE = BE.	1. Definition of bisect.
2. $\angle CED \cong \angle BEA$.	2. Vertical angles are congruent.
3. $\triangle CED \cong \triangle BEA$.	3. S.A.S.
4. CD = BA.	4. Definition of a congruence between triangles.

Similar proof for AC = DB.

9.

a. 1. AD = BC.	1. Sides of a square are congruent.
2. DF = CQ.	2. Given.
3. AD - DF = BC - CQ.	3. Subtraction.
4. AF + FD = AD.	4. Definition of between.
5. AF = AD - FD.	5. Subtracting FD from both sides of Step 4.
6. BQ + QC = BC.	6. Definition of between.
7. BQ = BC - QC.	7. Subtracting QC from both sides of Step 6.
8. AF = BQ.	8. From Steps 3, 5 and 7.
9. AR = BR.	9. Definition of midpoint.
10. $\angle A \cong \angle B$.	10. All angles of a square are right angles and all right angles are congruent.
11. $\triangle ARF \cong \triangle BRQ$.	11. S.A.S.
12. RF = RQ.	12. Corresponding parts.

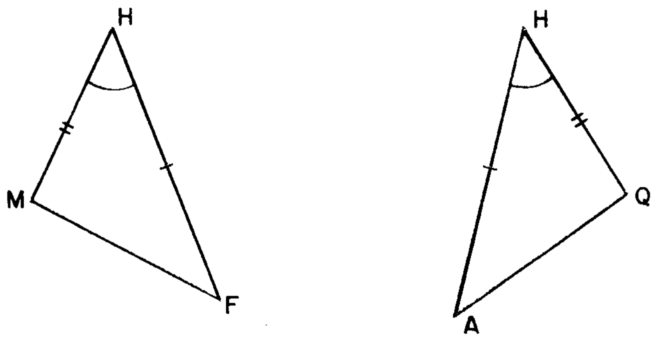
b. Yes, many possible pairs.
F' and Q' will be two points of $\overline{DC}$ such that DF' = CQ'.
There are also possibilities on $\overline{AB}$.

10.

1. AH = AB.	1. Given.
2. $\angle HAF \cong \angle BAF$.	2. Definition of bisect.
3. AF = AF.	3. Identity.
4. $\triangle ABF \cong \triangle AHF$.	4. S.A.S.
5. FH = FB.	5. Definition of congruent triangles.

---

123 When dealing with overlapping triangles a person can, as the text says, avoid getting mixed up by writing congruences down in standard form. Another policy many teachers recommend is that of redrawing figures on scratch paper, separating the triangles. Thus a person can see the crucial triangles more clearly if he draws this figure to assist him in dealing with the figure on page 123.



125 In the last paragraph of Section 5-5 we explicitly state the conventions about the information a student may and may not draw from a figure in solving problems.

<u>A reminder</u>, particularly pertinent in this chapter which contains so many problems: Most students should attempt only a reasonable sampling of the problems provided. The generous array is provided so that you may select according to your class and your own preferences, and so that the very best

student will not want for opportunity to test his ability and to discover interesting mathematical relationships.

Problem Set 5-5

125 1.

1. AC = DB. ∠ACF ≅ ∠DBE. FC = EB.	1. Given.
2. △ACF ≅ △DBE.	2. S.A.S.
3. AF = DE.	3. Corresponding parts of congruent triangles.

2. 4. Given.
   5. Given.
   6. S.A.S.

126 3.

1. HA = FB.	1. The sides of a square are equal in length.
2. AB = BA.	2. Identity.
3. ∠HAB ≅ ∠FBA.	3. Each is a right angle.
4. △HAB ≅ △FBA.	4. S.A.S.
5. AF = BH.	5. Corresponding parts of congruent triangles.

4. No. We do have BF = HF (Definition of midpoint) and since ∠ABW ≅ ∠RHQ we also know that ∠WBF ≅ ∠QHF (Supplements of congruent angles are congruent), but these facts are not enough to prove the triangles congruent.

5.

a. 1. $\overline{AX} \cong \overline{BY}$.	1. Given.
2. $\overline{AB} \cong \overline{AB}$.	2. Identity.
3. ∠XAB ≅ ∠YBA.	3. Each is a right angle.
4. △XAB ≅ △YBA.	4. S.A.S.
5. $\overline{AY} \cong \overline{BX}$.	5. Corresponding parts of congruent triangles.

b. No.

6.

1.	$r = m.$ $x = y.$	1.	Given.
2.	$r + x = m + y.$	2.	Addition from Step 1.
3.	$m\angle HAB = r + x.$ $m\angle FBA = m + y.$	3.	Angle Addition Postulate.
4.	$m\angle HAB = m\angle FBA.$	4.	Steps 2 and 3.
5.	$AB = BA.$	5.	Identity.
6.	$AH = BF.$	6.	Given.
7.	$\triangle HAB \cong \triangle FBA.$	7.	S.A.S.

7.

1.	$\overline{AR} \perp \overline{RX}$, $\overline{BR} \perp \overline{RY}$.	1.	Given.
2.	$m\angle ARX = m\angle YRB = 90.$	2.	Definition of right angle.
3.	$m\angle XRB = m\angle XRB.$	3.	Identity.
4.	$m\angle ARB = m\angle XRY.$	4.	Addition from Steps 2 and 3, and the Angle Addition Postulate.
5.	$AR = RX$, $BR = RY$.	5.	Given.
6.	$\triangle ARB \cong \triangle XRY.$	6.	S.A.S.
7.	$\overline{AB} \cong \overline{XY}.$	7.	Definition of congruent triangles.

---

127 Here is a striking example of the use of a particular correspondence to establish a congruence. We merely show that an isosceles triangle is congruent to itself under a correspondence which interchanges the vertices at the ends of the base. This is considerably simpler than the traditional proof.

128 Proof of Corollary 5-2-1

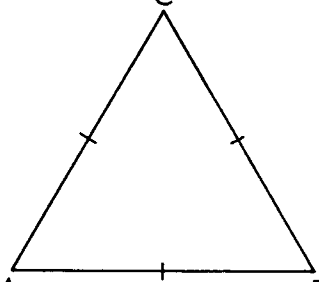
Every equilateral triangle is equiangular.

Given: $\triangle ABC$ such that $AC = BC = AB$.

To prove: $\angle A \cong \angle B \cong \angle C$.

The general procedure is to make successive applications of Theorem 5-1.

Proof:



If $AC = BC$ then, by Theorem 5-1 we have $m\angle A = m\angle B$.

If $AB = AC$ then, by Theorem 5-1 we have $m\angle B = m\angle C$.

Therefore, $m\angle A = m\angle C$ and, $m\angle A = m\angle B = m\angle C$ or $\angle A \cong \angle B \cong \angle C$.

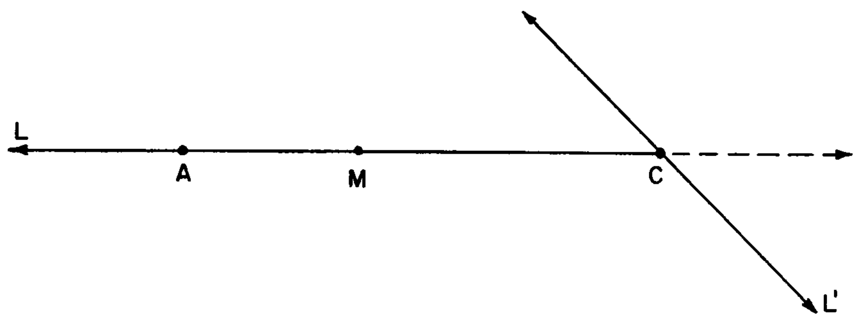
129 In the Angle Bisector Theorem the points B and C, the auxiliary segment $\overline{BC}$ and the point D are introduced into the figure as a part of the proof. We believe their use is natural at this point. Later, in Chapter 6, we elaborate on such auxiliary sets. You may want to mention that dotted segments are often used for auxiliary segments and should not be confused with dotted segments used to indicate segments hidden by a plane in figures involving three dimensions.

130 You may have noticed that the proof of Theorem 5-3 is not complete: we have not shown that D is in the <u>interior</u> of $\angle BAC$, as required by the definition of a bisector. This omission was deliberate, and similar ones will occur in some later proofs. Most such omissions will be concerned with separation properties; that is, with showing that certain points lie on the same or on opposite sides of certain lines or planes, or with showing that a certain point lies between two others on a line. These things are all "obvious" from pictures, and their proofs are often long, difficult and uninteresting. We therefore feel that they should be omitted

from the exposition in the text. You will find problems in Section 6-5 to take care of these betweenness matters which should seem interesting and worthwhile to your strongest students.

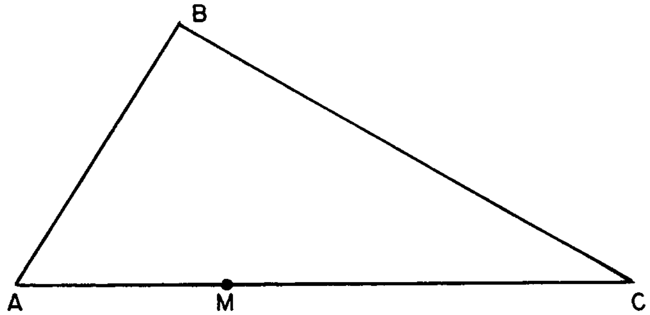
In the case of Theorem 5-3, the omitted proof depends on the following two theorems which are reproduced from Section 6-5, of the text. We suggest that you wait until Chapter 6 to discuss this with your students.

Theorem 6-5. If M is between A and C on a line L then M and A are on the same side of any other line that contains C.



Proof: The proof will be indirect. If M and A are on opposite sides of L' (in the plane that contains L and L') then some point D of L' lies on the segment $\overline{AM}$. Therefore, D is between A and M, by definition of a segment. But D lies on both L and L'. Therefore, D = C. Therefore, C is between A and M. This is impossible, because M is between A and C. (See Theorem 2-3).

Theorem 6-6. If M is between A and C, and B is any point not on the line $\overleftrightarrow{AC}$, then M is in the interior of $\angle ABC$.



Proof: By the preceding theorem, we know that M and A are on the same side of $\overleftrightarrow{BC}$. By another application of the preceding theorem (interchanging A and C) we know that M and C are on the same side of $\overleftrightarrow{AB}$. By definition of the interior of an angle, these two statements tell us that M is in the interior of $\angle ABC$, which was to be proved.

Problem Set 5-6

130 1.

1. Base angles of an isosceles triangle are congruent.
2. The Supplement Postulate.
3. Supplements of congruent angles are congruent.

131 2.

1. FA = FD.	1. Given.
2. $\angle A \cong \angle D$.	2. Base angles of an isosceles triangle are congruent.
3. AB = DC.	3. Given.
4. $\triangle AFB \cong \triangle DFC$.	4. S.A.S.
5. $\angle ABF \cong \angle DCF$.	5. Corresponding parts of congruent triangles.
6. $\angle FBC \cong \angle FCB$.	6. Supplements of congruent angles are congruent.

3.

1. $\angle EBC \cong \angle ECB$.	1. Base angles of an isosceles triangle are congruent.
2. $\angle ABE$ is supplementary to $\angle EBC$. $\angle DCE$ is supplementary to $\angle ECB$.	2. The Supplement Postulate.
3. $\angle EBA \cong \angle ECD$.	3. Supplements of congruent angles are congruent.

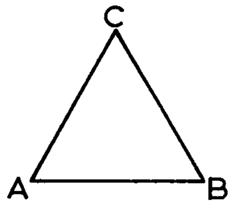
4.

1. $m\angle ABC = m\angle ACB$. $m\angle DBC = m\angle DCB$.	1. Base angles of an isosceles triangle are congruent.
2. $m\angle ABC + m\angle DBC = m\angle ACB + m\angle DCB$.	2. Addition, from Step 1.
3. $m\angle ABD = m\angle ABC + m\angle DBC$. $m\angle ACD = m\angle ACB + m\angle DCB$.	3. Angle Addition Postulate.
4. $\angle ABD \cong \angle ACB$.	4. Steps 2 and 3.

5.

1. $m\angle ACB = m\angle ABC$. $m\angle DCB = m\angle DBC$.	1. Base angles of an isosceles triangle are congruent.
2. $m\angle ACB - m\angle DCB = m\angle ABC - m\angle DBC$.	2. Subtraction, from Step 1.
3. $m\angle ACD = m\angle ACB - m\angle DCB$. $m\angle ABD = m\angle ABC - m\angle DBC$.	3. From the Angle Addition Postulate.
4. $m\angle ACD = m\angle ABD$.	4. Steps 2 and 3.

132 6. Since CA = CB, $\frac{1}{2}$ CA = $\frac{1}{2}$ CB. As X is the midpoint of $\overline{AC}$, CX = $\frac{1}{2}$ AC. Similarly, CY = $\frac{1}{2}$ CB. It follows that CX = CY. Then $\triangle CXY$ is an isosceles triangle with base angles $\angle CXY$ and $\angle CYX$. Theorem 5-2 tells us that these base angles are congruent.



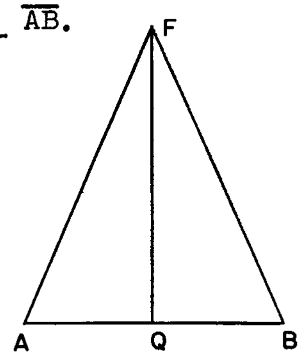
7. Given: $\triangle ABC$ with $AB = BC = CA$.
To prove: $\angle A \cong \angle B \cong \angle C$.

1. $CA = CB$.	1. Given.
2. $\angle A \cong \angle B$.	2. Base angles of an isosceles triangle are congruent.
3. $AB = BC$.	3. Given.
4. $\angle A \cong \angle C$.	4. Base angles of an isosceles triangle.
5. $\angle A \cong \angle B \cong \angle C$.	5. Steps 2 and 4.

8. Given: $\triangle ABC$ with $AB = BC = CA$, and P, Q, R the mid-points of $\overline{AC}$, $\overline{AB}$ and $\overline{BC}$.
To prove: $PR = RQ = QP$.

1. $AC = CB = BA$.	1. Given.
2. $\frac{1}{2} AC = \frac{1}{2} CB = \frac{1}{2} BA$.	2. Multiplication, from Step 1.
3. $CR = RB = \frac{1}{2} CB$, $BQ = QA = \frac{1}{2} AB$, $CP = PA = \frac{1}{2} CA$.	3. Definition of midpoint.
4. $CR = RB = BQ = QA = AP = PC$.	4. Steps 2 and 3.
5. $\angle C \cong \angle B \cong \angle A$.	5. Every equilateral triangle is equiangular.
6. $\triangle CRP \cong \triangle BQR \cong \triangle APQ$.	6. S.A.S.
7. $PR = RQ = QP$.	7. Corresponding parts of congruent triangles.

9. Given: $\overline{FQ}$ is a median of $\triangle FAB$. $\overline{FQ} \perp \overline{AB}$.
Prove: $\triangle FAB$ is isosceles.



1.	AQ = BQ.	1.	Definition of median of a triangle.
2.	∠FQA and ∠FQB are right angles.	2.	Definition of perpendicular.
3.	∠FQA ≅ ∠FQB.	3.	All right angles are congruent.
4.	FQ = FQ.	4.	Identity.
5.	△FQA ≅ △FQB.	5.	S.A.S.
6.	FA = FB.	6.	Corresponding parts.
7.	△FAB is isosceles.	7.	Definition of isosceles triangle.

---

132 In Theorem 5-4, the point F' is shown between D and F, the figure could just as well be drawn so that F is between D and F'.

133 Proof of Theorem 5-5.

If two angles of a triangle are congruent, then the sides opposite these angles are congruent.

Given: △ABC with ∠A ≅ ∠B.

To prove: $\overline{AC} \cong \overline{BC}$.

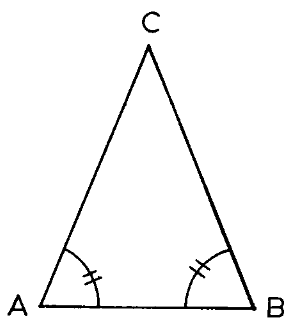
The general procedure is to set up a one-to-one correspondence between the triangle and itself, indicated by ABC ⟷ BAC, and to use the A.S.A. Theorem.

In the correspondence CAB ⟷ CBA we see that

∠A ⟷ ∠B,

$\overline{AB}$ ⟷ $\overline{BA}$,

∠B ⟷ ∠A.



Thus two angles and the included side of △CAB are congruent to the parts that correspond to them. By the A.S.A. Theorem this means that

△CAB ≅ △CBA.

By the definition of a congruence all pairs of corresponding parts are congruent. Therefore,

$$\overline{AC} \cong \overline{BC}.$$

From the definition of isosceles triangle, $\triangle ABC$ is isosceles.

Proof of Corollary 5-5-1

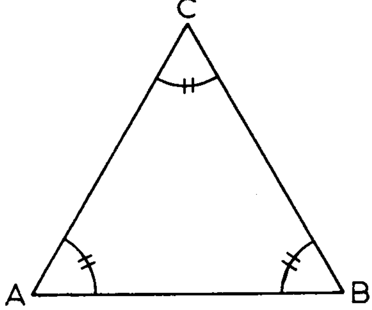
An equiangular triangle is equilateral.

Given: $\triangle ABC$ such that $\angle A \cong \angle B \cong \angle C$.

To prove: $\overline{AB} \cong \overline{BC} \cong \overline{AC}$.

The general procedure is to make successive applications of Theorem 5-5. Of course, you could set up a one-to-one correspondence and use the A.S.A. Theorem if you wished.

Since $\angle A \cong \angle B$, we have from Theorem 5-5 $\overline{AC} \cong \overline{BC}$, and since $\angle C \cong \angle B$, we have from Theorem 5-5 $\overline{AC} \cong \overline{AB}$.
Therefore, $\overline{AB} \cong \overline{BC} \cong \overline{AC}$.



Problem Set 5-7

133 1. a. Need $\angle a \cong \angle b$. (S.A.S.).
   b. Need $\overline{HF} \cong \overline{BF}$. (S.A.S.), or $\angle a \cong \angle b$. (A.S.A.).
   c. A.S.A.
   d. Need $\overline{QR} \cong \overline{WR}$. (S.A.S.), or $\angle A \cong \angle M$. (A.S.A.).
   e. Need $\overline{AR} \cong \overline{MR}$. (A.S.A.).
   f. Need $\overline{XF} \cong \overline{KF}$. (S.A.S.), or $\angle XYF \cong \angle KYF$. (A.S.A.).
   g. Need $\angle XFY \cong \angle KFY$. (A.S.A.), or $\overline{XY} \cong \overline{KY}$. (S.A.S.).

134 2. a. $\angle AHB$.
   b. $\angle AHB$, $\angle ABH$.
   c. $\overline{BF}$.
   d. $\angle F$, $\overline{FH}$ or $\angle HBF$, $\overline{HB}$.

3. a. $\angle AFB$, $\angle B$.
   b. $\overline{AR}$, $\overline{RF}$.
   c. $\overline{AB}$, $\overline{BR}$.
   d. $\angle R$.
   e. $\overline{RF}$.
   f. $\angle AFB$.

135 4. a. $\overline{HB}$, $\overline{BF}$.

b. $\angle AHB$, $\angle HBA$.

c. $\angle HBF$.

d. $\angle HBF$, $\angle F$.

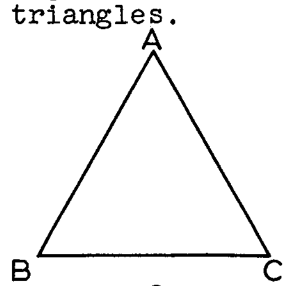
e. $\angle A$.

5.

1. $\overline{GE} \cong \overline{FE}$.	1. Definition of bisect.
2. $\angle a \cong \angle b$.	2. Given.
3. $\angle CEG \cong \angle BEF$.	3. Vertical angles.
4. $\triangle CGE \cong \triangle BFE$.	4. A.S.A.
5. $\overline{CE} \cong \overline{BE}$.	5. Corresponding parts.
6. $\overline{GF}$ bisects $\overline{BC}$.	6. Definition of bisect.

6.

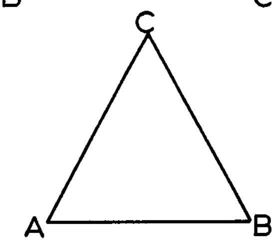
1. $\angle B \cong \angle C$.	1. Given.
2. $BC = CB$.	2. Identity.
3. $\angle C \cong \angle B$.	3. Given.
4. $\triangle ABC \cong \triangle ACB$.	4. S.A.S.
5. $AB = AC$.	5. Corresponding parts of congruent triangles.



7. Given: $\triangle ABC$ with $\angle A \cong \angle B \cong \angle C$.

To prove: $AB = BC = AC$.

Proof: The sides opposite $\angle A$ and $\angle B$ are congruent by Theorem 5-5. Hence, $BC = AC$. Considering $\angle C$ and $\angle A$ in a similar fashion, we find that $AB = BC$. Therefore, $AB = BC = AC$.



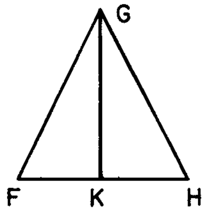
8. Given: $\triangle ABC$ with $\overline{AB} \cong \overline{BC} \cong \overline{CA}$.

To prove: $\triangle ABC \cong \triangle CAB$.

1. $\overline{AB} \cong \overline{CA}$ and $\overline{BC} \cong \overline{AB}$.	1. Given.
2. $\angle B \cong \angle A$.	2. An equilateral triangle is equiangular.
3. $\triangle ABC \cong \triangle CAB$.	3. S.A.S.

(This could also be proved using A.S.A.)

9. Given: $\overline{GK}$ bisects $\angle FGH$.
$\overline{GK} \perp \overline{FH}$ at K.
To prove: $\triangle FGH$ is isosceles.



1. $\angle FGK \cong \angle HGK$.	1. Definition of bisects.
2. $\overline{GK} \cong \overline{GK}$.	2. Identity.
3. $\angle GKF$ and $\angle GKH$ are right angles.	3. Definition of perpendicular.
4. $\angle GKF \cong \angle GKH$.	4. All right angles are congruent.
5. $\triangle GKF \cong \triangle GKH$.	5. A.S.A.
6. $\overline{FG} \cong \overline{HG}$.	6. Corresponding parts.
7. $\triangle FGH$ is isosceles.	7. Definition of isosceles triangle.

10.

1. $\angle FBH \cong \angle RMH$.	1. Supplements of congruent angles are congruent.
2. $\angle FHB \cong \angle RHM$.	2. Vertical angles are congruent.
3. $\overline{HB} \cong \overline{HM}$.	3. Given.
4. $\triangle BFH \cong \triangle MRH$.	4. A.S.A.
5. $\overline{HF} \cong \overline{HR}$.	5. Corresponding parts.

136 11. Yes.

1. $\angle RWM \cong \angle SWM$.	1. Supplements of congruent angles are congruent.
2. $\overline{MW} \cong \overline{MW}$.	2. Identity.
3. $\angle RMW \cong \angle SMW$.	3. Definition of bisect.
4. $\triangle RWM \cong \triangle SWM$.	4. A.S.A.
5. $\angle R \cong \angle S$.	5. Corresponding parts.

12.

1. AF = RB.	1. Given.
2. BF = FB.	2. Identity.
3. AB = RF.	3. Subtraction, from Steps 1 and 2.
4. $\angle A \cong \angle R$.	4. Given.
5. $\angle x \cong \angle y$.	5. Given.
6. $\triangle ABN \cong \triangle RFH$.	6. A.S.A.
7. $\overline{AN} \cong \overline{RH}$.	7. Corresponding parts.

*13. a.

1. $m\angle AXR = m\angle BXF$.	1. Given.
2. $m\angle RXF = m\angle FXR$.	2. Identity.
3. $m\angle AXF = m\angle BXR$.	3. Steps 1 and 2 and the Angle Addition Postulate.
4. $\overline{AX} \cong \overline{BX}$.	4. Definition of midpoint.
5. $\angle A \cong \angle B$.	5. Given.
6. $\triangle AXF \cong \triangle BXR$.	6. A.S.A.
7. $\overline{AF} \cong \overline{BR}$.	7. Corresponding parts.

b. No.

14.

1. $m\angle a = m\angle b$.	1. Given.
2. $m\angle w = m\angle x$.	2. Given.
3. $m\angle a + m\angle w = m\angle b + m\angle s$.	3. Addition.
4. $m\angle MKH = m\angle a + m\angle w$.	4. Angle Addition Postulate.
5. $m\angle MRG = m\angle b + m\angle s$.	5. Angle Addition Postulate.
6. $m\angle MKH = m\angle MRG$.	6. Steps 2, 3, and 4.
7. $\overline{MK} \cong \overline{MR}$.	7. Theorem 5-5.
8. $\angle M \cong \angle M$.	8. Identity.
9. $\triangle MKH \cong \triangle MRG$.	9. A.S.A.
10. $\overline{KH} \cong \overline{GR}$.	10. Definition of a congruence between triangles.

15. No. Neither S.A.S. nor A.S.A. apply.

*16.

1. $m\angle B = m\angle T$.	1. Given.
2. $m\angle Q = m\angle S$.	2. Given.
3. $BQ = TS$.	3. Given.
4. $\triangle BRQ \cong \triangle TRS$.	4. A.S.A.
5. $QR = SR$.	5. Corresponding parts.
6. $\angle XRQ \cong \angle YRS$.	6. Vertical angles.
7. $\triangle XRQ \cong \triangle YRS$.	7. Steps 2, 5, 6, and A.S.A.
8. $RX = RY$.	8. Corresponding parts.

137 In Steps 9 and 10 of the proof of Theorem 5-6 we tacitly assume that H lies in the interior of $\angle ABC$ and the interior of $\angle AE'C$. This is justified by Theorem 6-6, the proof of which appears above.

Problem Set 5-8

139 1.

1. $\overline{AH} \cong \overline{AB}$.	1. Given.
2. $\overline{HF} \cong \overline{BF}$.	2. Given.
3. $\overline{AF} \cong \overline{AF}$.	3. Identity.
4. $\triangle ABF \cong \triangle AHF$.	4. S.S.S.
5. $\angle BAF \cong \angle HAF$.	5. Corresponding parts.

2.

1. $\overline{AB} \cong \overline{FH}$. $\overline{AH} \cong \overline{FB}$.	1. Given.
2. $\overline{AF} \cong \overline{FA}$.	2. Identity.
3. $\triangle ABF \cong \triangle FHA$.	3. S.S.S.
4. $\angle r \cong \angle s$.	4. Corresponding parts.

3.

1. $\overline{AH} \cong \overline{BR}$. $\overline{BH} \cong \overline{AR}$.	1. Given.
2. $\overline{AB} \cong \overline{BA}$.	2. Identity.
3. $\triangle ABR \cong \triangle BAH$.	3. S.S.S.
4. $\angle H \cong \angle R$.	4. Corresponding parts.

140 4. a. S.A.S.
b. Cannot be proved congruent.
c. S.A.S.
d. S.S.S.
e. Cannot be proved congruent.
f. S.A.S.
g. S.A.S.
h. S.S.S.
i. S.A.S.
j. S.A.S.

5. He can specify the lengths of three sides, or the lengths of two sides and the measure of the included angle, or the length of one side and the measure of the two angles including it.

6. It is given that $AC = BC$ and $\angle ACH \cong \angle BCH$, by Theorem 5-2, $\angle A \cong \angle B$, so that $\triangle ACH \cong \triangle BCH$ by A.S.A. Then $\angle AHC$ and $\angle BHC$ are right angles, and, by definition, $\overline{CH} \perp \overline{AB}$.

7. Let $\triangle ABC$ be isosceles with $AC = BC$, and let $\overline{CD}$ be the median to the base. Prove: $\angle ACD \cong \angle BCD$.

1. $AC = BC$.	1. Given.
2. $CD = CD$.	2. Identity.
3. $DA = DB$.	3. Definition of median of a triangle.
4. $\triangle ACD \cong \triangle BCD$.	4. S.S.S.
5. $\angle ACD \cong \angle BCD$.	5. Definition of congruent triangles.

(An alternate proof using S.A.S. is also possible.)

8.

1. $AF = BF$.	1. Given.
2. $\angle AFH \cong \angle BFH$.	2. Definition of bisector.
3. $FH = FH$.	3. Identity.
4. $\triangle AHF \cong \triangle BHF$.	4. S.A.S.
5. $\overline{AH} \cong \overline{HB}$.	5. Corresponding parts.
6. $\angle AHF \cong \angle BHF$.	6. Corresponding parts.
7. $\angle AHF$ and $\angle BHF$ are right angles.	7. Definition of right angle.
8. $\overline{FH} \perp \overline{AB}$.	8. Definition of perpendicular.

(An A.S.A. proof is also possible.)

To the Teacher: It seems improbable that any student will question as to whether the bisector of $\angle AFB$ will in fact intersect the base $\overline{AB}$. If this question does arise, point out that in the preceding exercise it was shown that in an isosceles triangle the median to the base bisects the vertex angle. Hence, we know that the bisector of the vertex angle does intersect the base as the figure indicates. General questions of this sort are discussed in Section 6-4 of Studies II.

[page 141]

142 9.

a. 1.	$\overline{AF} \cong \overline{BR}$.	1.	Given.
2.	$\overline{AR} \cong \overline{BF}$.	2.	Given.
3.	$\overline{RF} \cong \overline{FR}$.	3.	Identity.
4.	$\triangle AFR \cong \triangle BRF$.	4.	S.S.S.
5.	$\angle ARF \cong \angle BFR$.	5.	Corresponding parts.

b. No.

10.

a. 1.	AB = FH.	1.	Given.
2.	AH = FB.	2.	Given.
3.	HB = BH.	3.	Identity.
4.	$\triangle ABH \cong \triangle FHB$.	4.	S.S.S.
5.	$\angle FHB \cong \angle ABH$.	5.	Corresponding parts.
6.	HK = BK.	6.	Definition of bisects.
7.	$\angle HKR \cong \angle BKQ$.	7.	Vertical angles.
8.	$\triangle HKR \cong \triangle BKQ$.	8.	A.S.A.
9.	QK = RK.	9.	Corresponding parts.

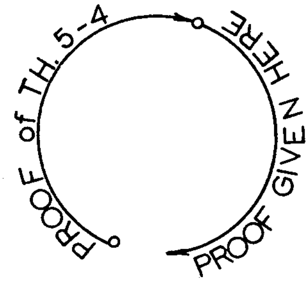
b. Yes. The intersecting lines $\overleftrightarrow{HB}$ and $\overleftrightarrow{RQ}$ determine a plane in which the other segments and points must lie.

11.

1.	$\triangle ASP \cong \triangle BPQ \cong \triangle CQR \cong \triangle DRS$.	1.	S.A.S.
2.	SP = QR. PQ = RS.	2.	Corresponding parts.
3.	QS = SQ.	3.	Identity.
4.	$\triangle PQS \cong \triangle RSQ$.	4.	S.S.S.

143 12. The S.S.S. theorem was used as a reason in the proof of the theorem. However, the very same theorem we are proving (The base angles of an isosceles triangle are congruent.) was used in the proof of the S.S.S. theorem.

143 *13. The A.A.A. theorem was given as a reason in Step 7. But in the proof of A.S.A. (Theorem 5-4), the reason for Step 2 was given as the S.A.S. postulate, which is what we are trying to prove now. Thus, our reasoning looks like this:



145 *14.

1.	$\angle a \cong \angle b$.	1.	Given.
2.	$\angle ARH \cong \angle ARB$.	2.	Supplements of congruent angles.
3.	$\overline{AR} \cong \overline{AR}$.	3.	Identity.
4.	$\angle m \cong \angle w$.	4.	Given.
5.	$\triangle ARH \cong \triangle ARB$.	5.	A.S.A.
6.	$\overline{RH} \cong \overline{RB}$.	6.	Corresponding parts.
7.	$\overline{RF} \cong \overline{RF}$.	7.	Identity.
8.	$\triangle RHF \cong \triangle RBF$.	8.	S.A.S.
9.	$\angle HFR \cong \angle BFR$.	9.	Corresponding parts.
10.	$\angle HFR$ and $\angle BFR$ are right angles.	10.	Definition of right angles.
11.	$\overline{AF} \perp \overline{BH}$.	11.	Definition of perpendicular.

15. Although a lengthy indirect proof is possible, it should not be expected at this point. After we have proved that the sum of the measures of the angles of a triangle is 180, this can be done easily by A.S.A.

16.

1. AW = AB.	1. Given.
2. $\angle A = \angle A$.	2. Identity.
3. $\overline{HB} \perp \overline{AF}$. $\overline{FW} \perp \overline{AH}$.	3. Given.
4. $m\angle AWF = m\angle ABH$.	4. Definition of perpendicular and of right angle.
5. $\triangle AWF \cong \triangle ABH$.	5. A.S.A.
6. FW = HB.	6. Corresponding parts.

17.

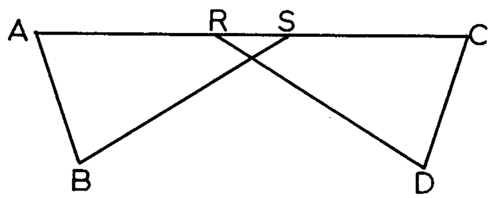
1. $\angle AWF \cong \angle RQF$	1. Given.
2. $m\angle a = \frac{1}{2} m\angle AQF$. $m\angle b = \frac{1}{2} m\angle RQF$.	2. Definition of bisect.
3. $m\angle a = m\angle b$.	3. Steps 1 and 2.
4. $\overline{FQ} \perp \overline{AR}$.	4. Given.
5. $m\angle BFQ = m\angle HFQ$.	5. Definition of perpendicular and of right angle.
6. FQ = FQ.	6. Identity.
7. $\triangle BFQ \cong \triangle HFQ$.	7. A.S.A.
8. $\overline{BQ} \cong \overline{HQ}$.	8. Corresponding parts.

*18. On $\overline{AF}$ take A' such that AF = A'F. Thus $\triangle$ CFA' $\cong$ $\triangle$ BFA by S.A.S. Hence A'C = AB and $m\angle CA'F = m\angle BAF$. Similarly, taking H' on $\overrightarrow{HQ}$ such that H'Q = HQ, $\triangle$ WQH' $\cong$ $\triangle$ RQH, so that WH' = HR and $m\angle WH'Q = m\angle RHQ$. But HR = AB, so WH' = A'C. Since AC = WH and AA' = HH' we get $\triangle$ ACA' $\cong$ $\triangle$ HWH' by S.S.S. This gives $m\angle CAF = m\angle WHQ$ and $m\angle CA'F = m\angle WH'Q$, so that $m\angle FAB = m\angle QHR$. By addition, $m\angle CAB = m\angle WHR$. Thus $\triangle$ ABC $\cong$ $\triangle$ HRW by S.A.S.

146*19.

1. BC = RW.	1. Given.
2. RQ = $\frac{1}{2}$ RW. BF = $\frac{1}{2}$ BC.	2. Definition of median of a triangle.
3. RQ = BF.	3. Steps 1 and 2.
4. AF = HQ.	4. Given.
5. AB = CR.	5. Given.
6. $\triangle ABF \cong \triangle HRQ$.	6. S.S.S.
7. $\angle B \cong \angle R$.	7. Corresponding parts.
8. $\triangle ABC \cong \triangle HRW$.	8. Steps 1, 5, 7 and S.A.S.

*20. a. One figure is:



1. AR = CS.	1. Given.
2. AR + RS = CS + SR.	2. Addition.
3. AR + RS = AS. CS + SR = CR.	3. Definition of betweenness.
4. AS = CR.	4. Steps 2 and 3.
5. AB = CD. BS = DR.	5. Given.
6. $\triangle ABS \cong \triangle CDR$.	6. S.S.S.
7. $\angle BSA \cong \angle DRC$.	7. Definition of congruence between triangles.

b. No.

*21. $\triangle ADB \cong \triangle GDE$, by S.A.S. since AD = GD, BD = ED, and $m\angle ADB = m\angle GDE$.

Hence, AB = GE.

$\triangle CAD \cong \triangle FGD$, since AD = GD, CD = FD, $m\angle CDA = m\angle FDG$.

Hence, AC = GF.

$\triangle BDC \cong \triangle EDF$, since CD = FD, BD = ED, $m\angle BDC = m\angle EDF$.

Hence, BC = ED. Therefore, $\triangle EFG \cong \triangle BCA$ by S.S.S.

*22. Yes.

23.

1.	$m\angle RQA = m\angle SQA$.	1.	Definition of perpendicular and of right angle.
2.	$\overline{RQ} \cong \overline{SQ}$.	2.	Given.
3.	$\overline{AQ} \cong \overline{AQ}$.	3.	Identity.
4.	$\triangle RQA \cong \triangle SQA$.	4.	S.A.S.
5.	$\overline{RA} \cong \overline{SA}$.	5.	Definition of congruent triangles.
6.	$\overline{AC} = \overline{AC}$.	6.	Identity.
7.	$RC = SC$.	7.	Given.
8.	$\triangle RAC \cong \triangle SAC$.	8.	S.S.S.
9.	$\angle RCA \cong \angle SCA$.	9.	Definition of congruent triangles.

147 24. a. Nothing about the distances. Since $\triangle VAB$ is isosceles, $\angle VAB \cong \angle VBA$; and similarly for the other two pairs.

b. In this case $\triangle AVB \cong \triangle BVC \cong \triangle AVC$. Therefore, $AB = BC = AC$, so that $\triangle ABC$ is equilateral, and the six indicated angles are congruent.

*25. a. $\triangle AMB \cong \triangle RMQ$ by given data, vertical angles, and the S.A.S. Postulate. Hence, $AB = RQ$. Prove $AQ = RB$ similarly, using $\triangle AMQ \cong \triangle RMB$.

b. Six pairs. ($AB = RQ$, $AQ = RB$, $AC = RX$, $QC = BX$, $BC = QX$, $AX = RC$.)

c. Still true if figure is not planar.

26. a. Four. Twelve.

b. Yes, all four faces are congruent by S.S.S. Equilateral triangles.

Review Problems

148 1. congruent; sides; congruence.

2. (a) S.A.S.

   (b) A.S.A., S.A.S.

3. RTS $\leftrightarrow$ STR, RTS $\leftrightarrow$ RTS.

4. S.A.S., A.S.A.

5.

1. AR = RH.	1. Given.
2. $\angle A \cong \angle H$.	2. Base angles of an isosceles triangle are congruent.
3. AF = BH.	3. Given.
4. $\triangle AFR \cong \triangle HBR$.	4. S.A.S.
5. RB = RF.	5. Definition of congruence of triangles.

6.

1. RB = RF.	1. Given.
2. $\angle RBF \cong \angle RFB$.	2. Base angles of an isosceles triangle are congruent.
3. $\angle ABR \cong \angle HFR$.	3. Supplements of congruent angles are congruent.
4. AB = HF.	4. Given.
5. $\triangle ABR \cong \triangle HFR$.	5. S.A.S.
6. AR = RH.	6. Definition of congruence of triangles.

149 7. $\overline{QX}$. A.S.A.

8. Yes, approximately.

   $\triangle ABC \cong \triangle ABC'$.

   A.S.A.

150 9. $\angle SXQ$ is the angle.

1. $\overline{SX} \cong \overline{SR}$.	1. Given.
2. $\overrightarrow{SQ}$ bisects $\angle RSX$, or $m\angle RSQ = m\angle XSQ$.	2. Definition of angle bisector.
3. $\overline{SQ} \cong \overline{SQ}$.	3. Identity.
4. $\triangle RSQ \cong \triangle XSQ$.	4. S.A.S.
5. $\angle R \cong \angle SXQ$.	5. Corresponding angles of congruent triangles are congruent.

10.

1. $\angle ABF$ and $\angle RHF$ are right angles.	1. If two segments are perpendicular to each other, the angle determined is a right angle.
2. $\angle x \cong \angle y$.	2. Given.
3. $\angle FBQ \cong \angle FHW$.	3. Complements of congruent angles are congruent.
4. QB = WH.	4. Given.
5. FB = FH.	5. Definition of midpoint.
6. $\triangle BFQ \cong \triangle HFW$.	6. S.A.S.

11.

1. $\angle BAH \cong \angle RAH$.	1. Given.
2. AB = AR.	2. Given.
3. AF = AF.	3. Identity.
4. $\triangle ABF \cong \triangle ARF$.	4. S.A.S.
5. FB = FR.	5. Definition of congruence.

12.

1. RB = RF.	1. Given.
2. $m\angle RBF = m\angle RFB$.	2. Base angles of an isosceles triangle are congruent.
3. BF = FB.	3. Identity.
4. AB = HF.	4. Given.
5. AB + BF = HF + FB.	5. Addition, Steps 3 and 4.
6. AB + BF = AF. HF + FB = HB.	6. Definition of between.
7. AF = HB.	7. Steps 5 and 6.
8. AFR $\cong \triangle HBR$.	8. S.A.S.

151 13. In $\triangle$s ABM and FBR,

1. AB = FB.	1. Given.
2. MB = RB.	2. Given.
3. $\angle MBA \cong \angle RBF$.	3. Vertical angles.
4. $\triangle ABM \cong \triangle FBR$.	4. S.A.S.
5. AM = FR.	5. Corresponding parts.

Statements	Reasons
In ⓢ AQR and FQM,	
6. $\angle A \cong \angle F$ and $\angle AMB \cong \angle FRB$.	6. Corresponding parts.
7. $\angle ARQ \cong \angle FMQ$.	7. Supplements of congruent angles are congruent.
8. AR = FM.	8. Addition from Steps 1 and 2.
9. $\triangle AQR \cong \triangle FQM$.	9. A.S.A.

14.

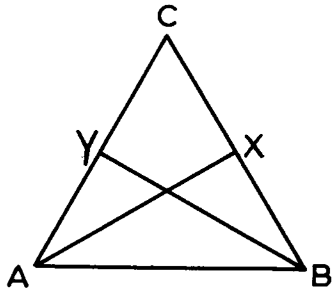
Statements	Reasons
1. AF = HB.	1. $\frac{2}{3}$ AH = $\frac{2}{3}$ AH.
2. $\angle A \cong \angle H$.	2. Given.
3. AR = HQ.	3. Given.
4. $\triangle AFR \cong \triangle HBQ$.	4. S.A.S.
5. $\angle RFA \cong \angle QBH$.	5. Definition of congruence.
6. BW = FW.	6. Theorem 5-5.

15.

Statements	Reasons
1. HA = HB.	1. Given.
2. $m\angle HAB = m\angle HBA$.	2. Theorem 5-2.
3. $\frac{1}{2} m\angle HAB = \frac{1}{2} m\angle HBA$.	3. Multiplication, from Step 2.
4. $m\angle FAB = \frac{1}{2} m\angle HAB$. $m\angle FBA = \frac{1}{2} m\angle HBA$.	4. Definition of bisect.
5. $m\angle FAB = m\angle FBA$.	5. Steps 3, 4.
6. FA = FB.	6. Theorem 5-5.

16.

Statements	Reasons
1. $\overline{AE} \cong \overline{BC}$. $\overline{ED} \cong \overline{CD}$. $\angle E \cong \angle C$.	1. Given.
2. $\triangle AED \cong \triangle BCD$.	2. S.A.S.
3. $\overline{AD} \cong \overline{BD}$.	3. Definition of congruent triangles.
4. $\angle DAB \cong \angle DBA$.	4. Theorem 5-2.



17. Given: $\triangle ABC$ with median $\overline{AX} \perp \overline{BC}$ and median $\overline{BY} \perp \overline{AC}$.

Prove: $\triangle ABC$ is equilateral.

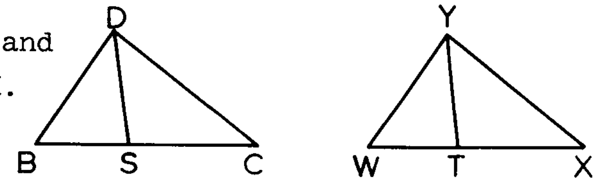
1. $\angle AXB$ and $\angle AXC$ are right angles.	1. Perpendicular lines determine right angles.
2. $\angle AXB \cong \angle AXC$.	2. Right angles are congruent.
3. BX = CX.	3. Definition of median.
4. AX = AX.	4. Identity.
5. $\triangle AXB \cong \triangle AXC$.	5. S.A.S.
6. $\overline{AB} \cong \overline{AC}$.	6. Definition of congruent triangles.
7. $\overline{BA} \cong \overline{BC}$.	7. Proof similar to Steps 1 through 6.
8. $\overline{AC} \cong \overline{BC} \cong \overline{AB}$.	8. Steps 6 and 7.
9. $\triangle ABC$ is equilateral.	9. Definition of equilateral triangle.

152 18.

1. $\overline{AB} \cong \overline{HB}$.	1. Given.
2. $\angle ABR \cong \angle HBF$.	2. Vertical angles are congruent.
3. $\overline{FB} \cong \overline{RB}$.	3. Given.
4. $\triangle ABR \cong \triangle HBF$.	4. S.A.S.
5. $m\angle A = m\angle H$. $m\angle ARB = m\angle HFB$.	5. Corresponding parts.
6. $m\angle MRH = m\angle MFA$.	6. Supplements of congruent angles are congruent.
7. AB + BF = HB + BR or AF = RH.	7. Addition, from Steps 1 and 3.
8. $\triangle MRH \cong \triangle MFA$.	8. A.S.A.
9. $\overline{AM} \cong \overline{HM}$.	9. Corresponding parts

19. Given: $\triangle BCD \cong \triangle WXY$, $\overrightarrow{DS}$ bisects $\angle BDC$ and $\overrightarrow{YT}$ bisects $\angle WYX$.

Prove: $\overline{DS} \cong \overline{YT}$.



1. $\triangle BCD \cong \triangle WXY$.	1. Given.
2. $\angle B \cong \angle W$. $\overline{BD} \cong \overline{WY}$. $\angle BDC \cong \angle WYX$.	2. Definition of congruent triangles.
3. $\angle BDS \cong \angle WYT$.	3. Definition of bisects and Step 2.
4. $\triangle BDS \cong \triangle WYT$.	4. A.S.A.
5. $\overline{DS} \cong \overline{YT}$.	5. Definition of congruent triangles.

20.

1. $\angle X \cong \angle Q$.	1. Given.
2. $XW = QR$.	2. Given.
3. $WR = RW$.	3. Identity.
4. $XR = QW$.	4. Addition, Steps 2 and 3.
5. $\angle a \cong \angle b$.	5. Given.
6. $\triangle XAR \cong \triangle QMW$.	6. A.S.A.
7. $XA = QM$.	7. Corresponding parts.
8. $KX = KQ$.	8. Theorem 5-5.
9. $KA = KM$.	9. Subtraction, Steps 7 and 8.

21.

1. $m\angle 1 + m\angle 3 = m\angle XJT$. $m\angle 2 + m\angle 4 = m\angle XJB$.	1. The Angle Addition Postulate.
2. $m\angle 1 + m\angle 3 = m\angle 2 + m\angle 4$.	2. By algebra from what is given.
3. $m\angle XJT = m\angle XJB$.	3. Steps 1 and 2.
4. $JT = JB$.	4. Given.
5. $JX = JX$.	5. Identity.
6. $\triangle XJT \cong \triangle XJB$.	6. S.A.S.
7. $\angle TXJ \cong \angle BXJ$.	7. Corresponding angles of congruent triangles.
8. $\triangle XJP \cong \triangle XJQ$.	8. A.S.A.
9. $\angle 5 \cong \angle 6$.	9. Corresponding angles of congruent triangles.

22. Yes. The natural proof, showing $\triangle PAQ \cong \triangle PBQ$ holds in either case. The congruence postulates and theorems hold for <u>any</u> two triangles, coplanar or not.

153 23. a. By S.S.S. $\triangle AQP \cong \triangle BQP$. Therefore, $\angle AQP \cong \angle BQP$. Then $\triangle AQR \cong \triangle BQR$ by S.A.S. and $RA = RB$.

b. No. Yes.

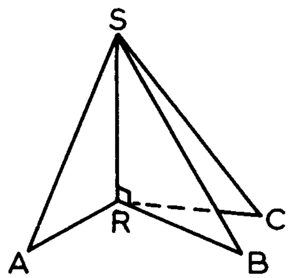
*24. Yes.

1. $AF = FH$ and $BF = FM$.	1. Definition of trisect.
2. $m\angle AFB = m\angle MFH$.	2. Vertical angles.
3. $\triangle AFB \cong \triangle HFM$.	3. S.A.S.
4. $\angle A \cong \angle FHM$.	4. Definition of congruence.
5. $AF = FB$.	5. Given.
6. $FH = FM$.	6. Steps 1 and 5.
7. $\angle M \cong \angle FHM$.	7. Theorem 5-2.
8. $\angle M \cong \angle A$.	8 Steps 4 and 7.
9. $AT = MR$.	9. Multiplication, Step 6.
10. $\triangle ABT \cong \triangle MTH$.	10. S.A.S.

25. Given: $\overline{RA}$, $\overline{RB}$, $\overline{RC}$ each $\perp$ $\overline{RS}$.

$RA = RB = RC$.

Prove: $SA = SB = SC$.



1. $\angle SRA$, $\angle SRB$, $\angle SRC$ are right angles.	1. Perpendicular lines determine right angles.
2. $\angle SRA \cong \angle SRB \cong \angle SRC$.	2. All right angles are congruent.
3. $SR = SR = SR$.	3. Identity.
4. $\triangle SRA \cong \triangle SRB \cong \triangle SRC$.	4. S.A.S.
5. $SA = SB = SC$.	5. Definition of congruent triangles.

*26.

1. △PAB ≅ △QAB.	1. Given.
2. AP = AQ, and ∠BAP ≅ ∠BAQ.	2. Definition of congruent triangles.
3. AX = AX.	3. Identity.
4. △XAP ≅ △XAQ.	4. Steps 2 and 3 and S.A.S.
5. PX = QX.	5. Definition of congruent triangles.

*27.

1. AH = AF.	1. Construction.
2. AB = AC.	2. Given.
3. ∠A ≅ ∠A.	3. Identity.
4. △ABH ≅ △ACF.	4. S.A.S.
5. ∠AHB ≅ ∠AFC.	5. Corresponding parts.
6. BF = CH.	6. Subtraction, Steps 1 and 2.
7. FC = HB.	7. Corresponding parts.
8. △FBC ≅ △HCB.	8. S.A.S.
9. ∠FBC ≅ ∠HCB.	9. Corresponding parts.
10. ∠ABC ≅ ∠ACB.	10. Supplements of congruent angles are congruent.

154 *28.

1. △ADC ≅ △CBA.	1. S.S.S.
2. ∠BAC ≅ ∠DCA.	2. Corresponding parts.
3. △ABD ≅ △CDB.	3. S.S.S.
4. ∠ABD ≅ ∠CDB.	4. Corresponding parts.
5. △ABE ≅ △CDE.	5. A.S.A.
6. AE = CE, BE = DE.	6. Corresponding parts.

*29. Draw $\overline{BC}$. Then:

1. DB = DC, AB = AC.	1. Given.
2. m∠ABC = m∠ACB, m∠DBC = m∠DCB.	2. Base angles of an isosceles triangle are congruent.
3. m∠ABD = m∠ACD.	3. Subtraction, Step 2.
4. ∠BAX ≅ ∠CAY	4. Given.
5. △BAX ≅ △CAY.	5. A.S.A.
6. AX = AY.	6. Corresponding parts.

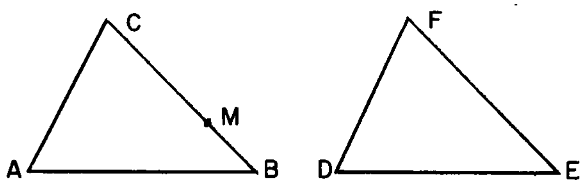
Illustrative Test Items for Chapter 5.

A. 1. Below are listed the 6 pairs of corresponding parts of two congruent triangles. Name the congruent triangles.

$\overline{AB} \cong \overline{MK}$. $\angle A \cong \angle M$.
$\overline{BW} \cong \overline{KF}$. $\angle B \cong \angle K$.
$\overline{AW} \cong \overline{MF}$. $\angle W \cong \angle F$.

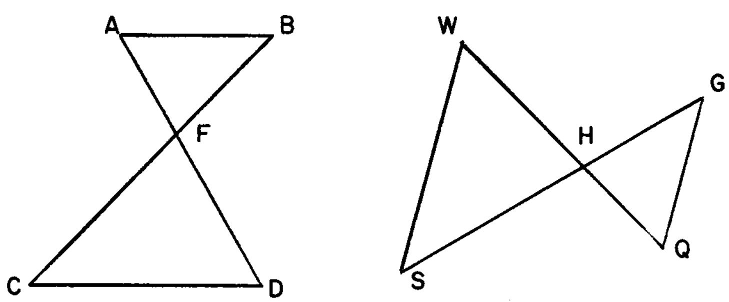
2. Given the figures shown below with $\triangle ABC \cong \triangle DEF$, and M between B and C. Write " + " for each of the following statements which is true. Otherwise, correct the statement to make it true.

a. $\overline{AB} \cong \overline{DE}$.
b. $\angle A = \angle D$.
c. $BC = EF$.
d. $m\angle B = m\angle E$.
e. $\angle ABC \cong \angle ABM$.
f. $\angle ABC = \angle ABM$.
g. $\angle C \cong \angle F$.
h. $\angle ACB \cong \angle DEF$.

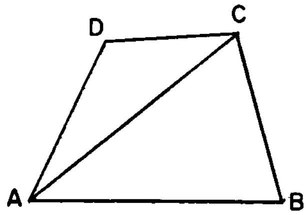


3. Given the two congruent figures shown, complete each correspondence in such a way that a congruence results.

a. ABCD $\leftrightarrow$ ____.
b. BFA $\leftrightarrow$ ____.
c. FCD $\leftrightarrow$ ____.
d. ABFCD $\leftrightarrow$ ____.

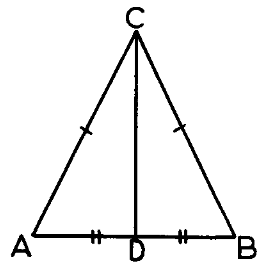


4. Given the figure shown, in accordance with the specifications at the left, list the data that would correctly fill the blanks at the right.
   a. side, angle, side of $\triangle ACD$: $\overline{AC}$,____, $\overline{AD}$.
   b. angle, side, angle of $\triangle ABC$: ____, $\overline{AB}$, ____.



B. 1. Complete the following definitions:
   a. Two angles are congruent angles if ______________.
   b. Two segments are congruent segments if __________.
   c. An _____ triangle is one having two congruent sides.
   d. $\angle XYZ$ is bisected by a ray $\overrightarrow{YS}$ if S is in ________ and if _________________.
   e. A segment whose endpoints are a midpoint of one side of a triangle and the opposite vertex is the ____________ of the triangle.

2. In $\triangle ABC$ as marked in the figure, $\overline{CD}$ is __________ to the base of the triangle and $\angle ACB$ is the _______ of the triangle.



3. Indicate whether each of the following is true or false:
   a. If $\triangle ABC \cong \triangle CAB$, then $\angle A \cong \angle B$.
   b. All equilateral triangles are congruent.
   c. Given a correspondence between two triangles such that two sides and an angle of the first triangle are congruent to the corresponding parts of the

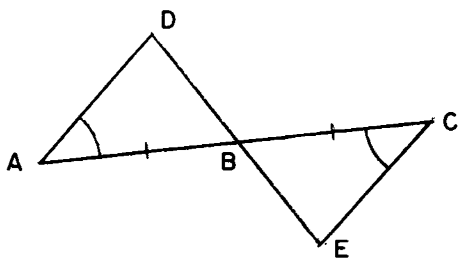
second triangle, then the correspondence is a congruence.

d. If $\angle ABC = \angle XYZ$, then the points A, B, and C coincide respectively with points X, Y, and Z.

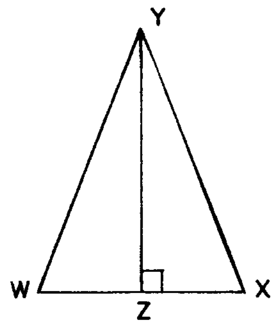
e. An equilateral triangle is isosceles.

C. 1. If like markings indicate congruent parts, in which of the following figures can two triangles be proved congruent? Answer by naming the pair of triangles which can be proved congruent or by writing "none." In the cases where two triangles can be proved congruent give the abbreviation of the congruence theorem or postulate which applies (S.A.S., etc.).

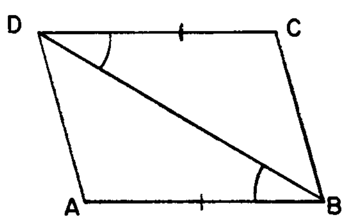
a.



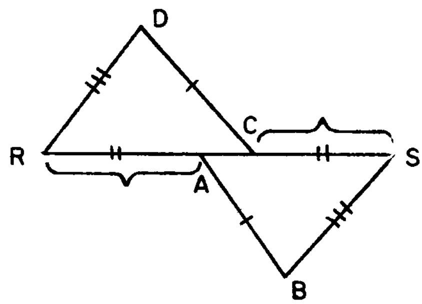
b.



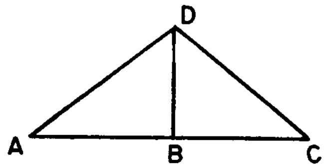
c.



d.



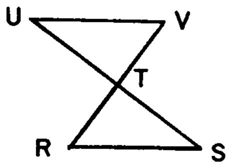
2. In each of the following, if enough is given to establish congruence between the two triangles, state the appropriate reason by writing S.A.S., S.S.S., or A.S.A. If not, name one other pair of parts which, if congruent, would enable you to prove the triangles congruent.



Given:

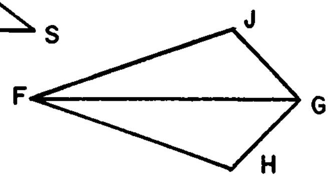
a. $\angle ADB \cong \angle CDB$, $\overline{AD} \cong \overline{CD}$.

b. $\overline{AB} \cong \overline{CB}$.



c. UT = ST, VT = RT.

d. UV = RS, UT = ST.



e. $\angle JFG \cong \angle HFG$, $\angle HGF \cong \angle JGF$.

f. FJ = FH, JG = HG.

3. State whether or not each of the pairs of triangles described below can be proved congruent using postulates and theorems we have had.

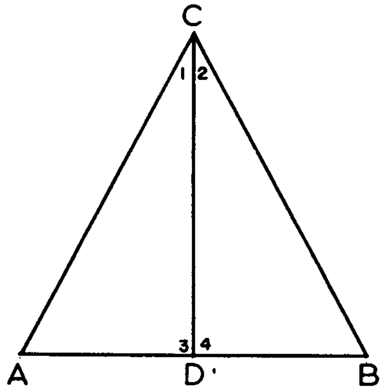
   a. Two isosceles triangles with congruent bases.

   b. Two equilateral triangles with congruent bases.

   c. Two isosceles triangles with congruent bases and a base angle of one congruent to a base angle of the other.

   d. Two isosceles triangles with congruent vertex angles.

4. The information given in the statements refers in each case to the figure. If the given information is sufficient to prove the triangles congruent, write the abbreviation of the congruence statement which would be used as a final reason. Otherwise write "not enough given".



a. AC = BC, AD = DB.

b. $\overline{AC} \cong \overline{BC}$, $\angle 1 \cong \angle 2$.

c. $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$.

d. AC = BC, $\angle A \cong \angle B$.

e. $\overline{AD} \cong \overline{DB}$, $m\angle 3 = m\angle 4$.

f. $\overline{CD}$ bisects $\angle C$.

g. $\overline{CD} \perp \overline{AB}$.

h. $\overline{CD}$ is a median to $\overline{AB}$.

i. AC = BC, $\overline{CD}$ bisects $\angle C$.

j. $\overline{CD} \perp \overline{AB}$, $\overline{CD}$ is the bisector of $\angle C$.

k. $\angle ACD \cong \angle BCD$, $\angle CAD \cong \angle CBD$.

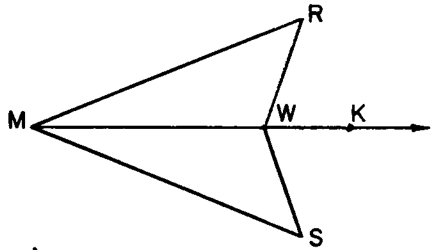
l. $\overleftrightarrow{CD}$ bisects $\overline{AB}$, $\overline{AC} \cong \overline{CB}$.

m. $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$, $\angle A \cong \angle B$.

D. 1. Given: $\angle RMW \cong \angle SMW$.

$\angle RWK \cong \angle SWK$.

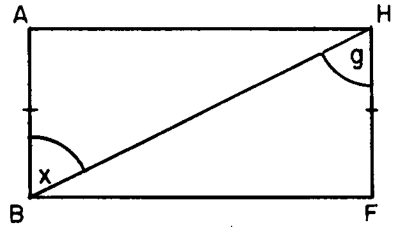
Prove: $\angle R \cong \angle S$.



Proof: (Supply the reasons.)

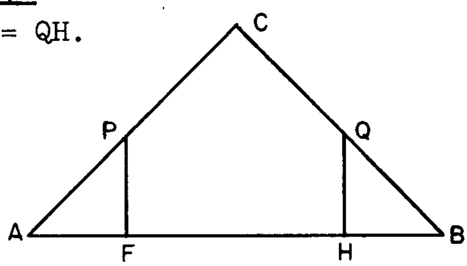
Statements	Reasons
1. $\angle MWR$ is supplementary to $\angle RWK$. $\angle MWS$ is supplementary to $\angle SWK$.	1.
2. $\angle RMW \cong \angle SMW$. $\angle RWK \cong \angle SWK$.	2.
3. $\angle NWR \cong \angle MWS$.	3.
4. $\overline{MW} \cong \overline{MW}$.	4.
5. $\triangle MWR \cong \triangle MWS$.	5.
6. $\angle R \cong \angle S$.	6.

2. In this figure AB = FH and $m\angle x = m\angle g$. Show that $m\angle A = m\angle F$.



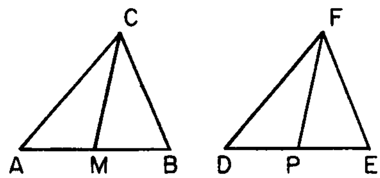
3. Given: $\triangle ABC$, with $\overline{AC} \cong \overline{BC}$, $\overline{AF} \cong \overline{BH}$, $\overline{PF} \perp \overline{AB}$ and $\overline{QH} \perp \overline{AB}$.

   Prove: PF = QH.



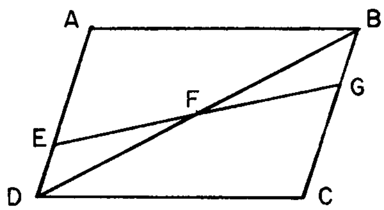
4. Given: The figure with
   AC = DF,
   AB = DE,
   $\overline{CM}$ and $\overline{FP}$ are congruent medians.

   Prove: $\triangle ABC \cong \triangle DEF$.



5. Given: The figure with
   AB = CD,
   AD = CB, and
   F bisects $\overline{BD}$.

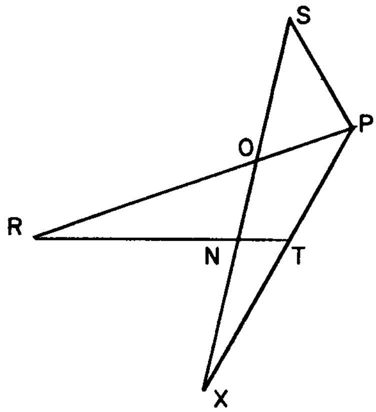
   Prove: EF = GF.



E. 1. Prove the theorem that the median from the vertex of an isosceles triangle is the bisector of the vertex angle of the triangle.

2. Prove: Angle bisectors from corresponding vertices of two congruent triangles are congruent.

3. Prove: A diagonal of a square bisects its angles. (Note: The teacher may prefer to supply the drawing from the answers in order to make lettering uniform.)



4. In the figure,
Given: $\angle RTP \cong \angle XPS$, $PT = SP$
and $\angle PSO \cong \angle TPO$.
Prove: $\overline{RT} \cong \overline{XP}$.

Answers

A. 1. $\triangle ABW$, $\triangle MKF$.

2. a. +.
   b. $m\angle A = m\angle D$.
      or $\angle A \cong \angle D$.
   c. +.
   d. +.
   e. +.
   f. +.
   g. +.
   h. $\angle ACB \cong \angle DFE$,
      or $\angle ABC \cong \angle DEF$.

3. a. QGSW.
   b. GHQ.
   c. HSW.
   d. QGHSW.

4. a. $\angle DAC$.
   b. $\angle CAB$, $\angle B$. (In either order.)

B. 1. a. They have the same measure.
   b. They have the same length.
   c. Isosceles.
   d. The interior of $\angle XYZ$; $\angle XYS \cong \angle ZYS$.
   e. Median.

2. Perpendicular, vertex angle.

3. a. True.
   b. False.
   c. False.
   d. False.
   e. True.

C. 1. a. $\triangle ABD \cong \triangle CBE$: A.S.A.
   b. None.
   c. $\triangle ABD \cong \triangle CDB$: S.A.S.
   d. $\triangle RCD \cong \triangle SAB$: S.S.S.

2. a. S.A.S. (or A.S.A.).
   b. $\overline{AD}$ and $\overline{CD}$, or $\angle ABD$ and $\angle CBD$.
   c. S.A.S.
   d. $\angle V$ and $\angle S$, or $\overline{VT}$ and $\overline{RT}$.
   e. A.S.A.
   f. S.S.S.

3. a. Not necessarily.
   b. Yes.
   c. Yes.
   d. Not necessarily.

4. a. S.S.S.
   b. S.A.S. or A.S.A.
   c. A.S.A.
   d. Not enough given.
   e. S.A.S.
   f. Not enough given.
   g. Not enough given.
   h. Not enough given.
   i. S.A.S. or A.S.A.
   j. A.S.A.
   k. A.S.A. or S.A.S.
   l. S.S.S. or S.A.S.
   m. A.S.A. or S.A.S.

Reasons

D. 1. 1. Supplement Postulate.
   2. Given.
   3. Supplements of congruent angles are congruent.
   4. Identity.
   5. A.S.A.
   6. Definition of a congruence between triangles.

2.

1. $AB = FH$.	1. Given.
2. $m\angle x = m\angle g$.	2. Given.
3. $BH = BH$.	3. Identity.
4. $\triangle ABH \cong \triangle FHB$.	4. S.A.S.
5. $m\angle A = m\angle F$.	5. Definition of a congruence between triangles.

3.

1. $\overline{PF} \perp \overline{AB}$, $\overline{QH} \perp \overline{AB}$.	1. Given.
2. $\angle PFA \cong \angle QHB$.	2. Definition of perpendicular. Any two right angles are congruent.
3. $\overline{AC} \cong \overline{BC}$.	3. Given.
4. $\angle A \cong \angle B$.	4. If two sides of a triangle are congruent, the angles opposite these sides are congruent.
5. $\overline{AF} \cong \overline{BH}$.	5. Given.
6. $\triangle PFA \cong \triangle QHB$.	6. A.S.A.
7. PF = QH.	7. Corresponding parts of congruent triangles.

4.

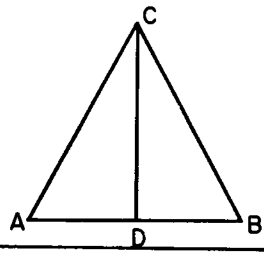
1. AB = DE.	1. Given.
2. $\overline{CM}$ and $\overline{FP}$ are medians.	2. Given.
3. M and P are midpoints of $\overline{AB}$, $\overline{DE}$.	3. Definition of median.
4. AM = DP.	4. Step 1 and definition of midpoint.
5. $\overline{CM} \cong \overline{FP}$.	5. Given.
6. AC = DF.	6. Given.
7. $\triangle AMC \cong \triangle DPF$.	7. S.S.S.
8. $\angle A \cong \angle D$.	8. Corresponding parts.
9. $\triangle ABC \cong \triangle DEF$.	9. S.A.S.

Note: A proof in which the final reason is S.S.S. is also possible if $\triangle CMB$ is proved congruent to $\triangle FPE$.

5.

1. AB = CD, AD = CB.	1. Given.
2. BD = BD.	2. Identical.
3. $\triangle ABD \cong \triangle CDB$.	3. S.S.S.
4. $\angle EDF \cong \angle GBF$.	4. Corresponding parts.
5. DF = BF.	5. Definition of bisects.
6. $\angle EFD \cong \angle GFB$.	6. Vertical angles are congruent.
7. $\triangle EDF \cong \triangle GBF$.	7. A.S.A.
8. EF = GF.	8. Corresponding parts.

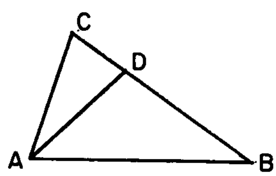
E. 1. Given: $\triangle ABC$ is isosceles with vertex at $\angle C$.
$\overline{CD}$ is a median.
Prove: $\overline{CD}$ bisects $\angle ACB$.

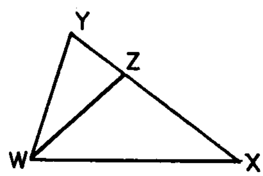


1. $\overline{AC} \cong \overline{BC}$.	1. Definition of isosceles triangle.
2. $\overline{AD} \cong \overline{DB}$.	2. Definition of median.
3. $\overline{CD} \cong \overline{CD}$.	3. Identical.
4. $\triangle ACD \cong \triangle BCD$.	4. S.S.S.
5. $\angle ACD \cong \angle BCD$.	5. Corresponding parts.
6. $\overrightarrow{CD}$ bisects $\angle ACB$.	6. Definition of angle bisector.

(Another way of proving $\triangle ACD \cong \triangle BCD$ is to show $\angle A \cong \angle B$ and use S.A.S.)

2. Given: $\triangle ABC \cong \triangle WXY$.
$\overline{AD}$ and $\overline{WZ}$ are angle bisectors.
Prove: $\overline{AD} \cong \overline{WZ}$.

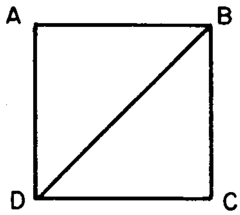




1. $\triangle ABC \cong \triangle WXY$.	1. Given.
2. $\angle CAB \cong \angle YWX$.	2. Corresponding parts.
3. $\angle DAB \cong \angle ZWX$.	3. Step 2, and definition of angle bisector.
4. $\overline{AB} \cong \overline{WX}$.	4. Corresponding parts.
5. $\angle B \cong \angle X$.	5. Corresponding parts.
6. $\triangle ABD \cong \triangle WXZ$.	6. A.S.A.
7. $\overline{AD} \cong \overline{WZ}$.	7. Corresponding parts.

3. Given: ABCD is a square with diagonal $\overline{DB}$.

Prove: $\overline{DB}$ bisects $\angle ADC$ and $\angle ABC$.



1. AB = BC, AD = DC.	1. Definition of square.
2. $\overline{DB} \cong \overline{DB}$.	2. Identity.
3. $\triangle ABD \cong \triangle CBD$.	3. S.S.S.
4. $\angle ABD \cong \angle CBD$, $\angle ADB \cong \angle CDB$.	4. Corresponding parts.
5. $\overline{BD}$ bisects $\angle ADC$ and $\angle ABC$.	5. Definition of bisect.

4.

1. $\angle RTP \cong \angle XPS$.	1. Given.
2. PT = SP.	2. Given.
3. $\angle PSO \cong \angle TPO$.	3. Given.
4. $\triangle RTP \cong \triangle XPS$.	4. A.S.A.
5. $\overline{RT} \cong \overline{XP}$.	5. Corresponding parts.

Answers to Review Exercises

Chapters 1 to 5

1.	-	21.	-	41.	-	61.	-
2.	-	22.	-	42.	+	62.	+
3.	+	23.	-	43.	+	63.	-
4.	+	24.	+	44.	+	64.	-
5.	+	25.	-	45.	-	65.	-
6.	+	26.	+	46.	+	66.	-
7.	-	27.	-	47.	-	67.	+
8.	-	28.	-	48.	+	68.	-
9.	+	29.	+	49.	-	69.	-
10.	-	30.	-	50.	+	70.	+
11.	+	31.	-	51.	-	71.	-
12.	+	32.	-	52.	-	72.	+
13.	-	33.	+	53.	+	73.	+
14.	-	34.	-	54.	+	74.	+
15.	+	35.	+	55.	-	75.	+
16.	+	36.	+	56.	+	76.	-
17.	-	37.	-	57.	-	77.	+
18.	-	38.	-	58.	+	78.	-
19.	+	39.	-	59.	+	79.	+
20.	-	40.	+	60.	+	80.	+

Chapter 6

A CLOSER LOOK AT PROOF

One purpose of this chapter is to allow the student, having had some experience with proof, to observe the material of previous chapters as illustrating the postulational structure of mathematics. Another purpose is to prove Theorems 3-2 to 3-5, as promised in Chapter 3. These proofs are used to introduce indirect proof and existence and uniqueness theorems. This chapter also discusses questions of betweenness that were avoided in Chapter 5.

As we pointed out in the Introduction to the Commentary, this chapter includes material that we believe can be omitted by some classes. If your class is composed chiefly of students for whom the material in Section 6-5 is too abstract, it may be best simply to move on. There is plenty of worthwhile material in later chapters.

159 Section 6-1 should be quite understandable to students now, particularly if they reread Section 1-2. In general, we encourage students to direct their attention to the geometric rather than the algebraic issues involved in proofs since the student is supposed to be familiar with the fundamentals of algebra, but is just learning geometry. For this reason we are more explicit in stating geometric principles rather than algebraic principles as reasons in proofs. The teacher can use any formulation of algebraic principles that he considers suitable for his class.

160 Our viewpoint is that in a first approach to deductive reasoning, it is desirable to treat logic informally and to encourage the student to appreciate the nature of logical reasoning by engaging in it. Consequently, we avoid putting into this text any apparatus of logic that we can readily get along without. However, you may wish to mention some relevant principles of logic yourself. Thus when treating indirect proof, you may wish to refer at the appropriate time to the

Law of the Excluded Middle, which asserts that either a statement is true or its negation is true. This also can be expressed: a statement must be either true or false.

The essential logical principle which is implicit in the indirect method may be expressed formally as follows: If statement A implies a false statement, then A itself is false. For example, let A be the statement "It is not raining". Then A implies the statement, "The people coming in the door are dry". The latter statement is false, since the people actually are wet. Thus we conclude that statement A, "It is not raining", is false. You can test other examples of the indirect method to see that they are applications of the principle above.

A common type of argument which involves the indirect method may be put in the following form:

(1) One of the statements A or B is known to be true.

(2) A implies X.

(3) X is known to be false.

(4) Therefore, A is false.

(5) Therefore by (1), B must be true.

Usually (1) will be an application of the Law of the Excluded Middle, as in "$AB = CD$ or $AB \neq CD$", or "today is Tuesday or today is not Tuesday".

Often (3) will be justified by pointing out that statement X contradicts an accepted principle or a known truth. For example, if X is the statement "Two lines have two points in common", X is false since it contradicts Postulate 1. This is an illustration of the Law of Contradiction, which asserts that a statement and its negative (or contradictory) cannot both be true. Thus if X contradicts Y, and Y is true, X must be false.

Sometimes we encounter an argument similar to the type described above, in which we have several alternatives, rather than just two. Thus (1) might have the form: One of the statements A, B or C is known to be true. Then we would proceed to "demolish" the alternatives as above. We show that A implies a false statement and must be false. Similarly we show B false. Then we conclude that C must be true. A common example of such an argument might begin with the statement: $AB < CD$ or $AB = CD$ or $AB > CD$.

160 Some students may be confused by such a statement as: We suppose something is false in order to prove it true. It may help to soft-pedal the word "false" and say that if we don't know whether a statement is true, it is reasonable to take its opposite (or negative) and see what follows from it. Our approach is to explore possibilities, not to say categorically that the given statement is false or equivalently that its opposite is true.

The very phrase "suppose so and so" may be confusing to some students. The word "suppose" may suggest to them that we are supposing it as a *fact* rather than *considering* it as a hypothesis. Remind them that in everyday life we often reason from premises without knowing that they are true. For example, when not sure of today's date we might argue so: I know today is Saturday and I think the date is June 15th, but I'm not sure. If today is June 15th, then June 1st also was a Saturday. But I remember that June 1st was a school day. Therefore, today can't be June 15th. Sometimes we actually reason from false premises, as when we argue that if Lincoln had not been shot, the course of American history would have been such and such; or that if the Lusitania had not been torpedoed, the United States would never have entered World War I.

You may be able to help your students by using, in informal classroom speech, such phrases as: Assume for the sake of argument; Pretend, and see where you end up; Work on the theory that . . . , and see the kind of jam you get into.

[page 160]

Problem Set 6-2a

161 1. a. My Mother is not color blind.

b. My brother is left-handed.

c. Jane drank some hot chocolate.

2. All.

3. (1) This set is not a stainless steel product.

(2) This set is a stainless steel product.

(3) This set will not rust.

(4) This set did rust.

162 4. y is true, z is true.

5. w, u and x are not true. Yes, indirect reasoning is used in reaching each conclusion.

6. Let A be "someone is a member of the swimming club".
Let B be "someone can play the piccolo".
Let C be "someone is a turtle".
Let D be "someone wears striped trunks in the club pool".

Then the problem may be diagrammed this way:

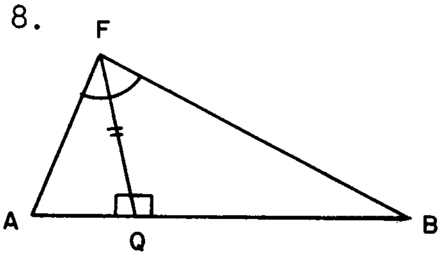
(1) If A is true, then B is true.
(2) If B is true, then C is not true.
(3) If D is true, then A is true.
(4) D is true.

The conclusion is that "C is not true" is true. Hence, in terms of the problem, the conclusion is "I am not a turtle".

7. a. Red, white.

b. Yes. A is not green.

8.



Given scalene $\triangle ABF$. To prove that the bisector of any angle, F, is not perpendicular to $\overline{AB}$. If we assume that the bisector of $\angle F$ is perpendicular to $\overline{AB}$, then $\triangle AFQ \cong \triangle BFQ$ (A.S.A.) and $AF = BF$. The assumption that $\overline{FQ}$ is perpendicular to $\overline{AB}$ led to the contradiction that the scalene $\triangle ABF$ is isosceles.

163 Notice from the proofs in Section 6-2 that uniqueness is usually established by indirect proof. Showing that there is only one of something can be accomplished by showing that there cannot be two.

Note that it is possible to establish uniqueness without, or before, establishing existence. For example, the proof of uniqueness in Theorem 3-3 can be made logically independent of the question of existence, as follows: Suppose that there are two planes containing L and P. Let Q and R be two points of L. Then both planes contain P, Q, R which are non-collinear points. This contradicts Postulate 7. Hence our supposition is false and there is at most one plane containing L and P.

In ordinary life, too, knowledge of uniqueness can be independent of knowledge of existence. A person with just one day of his vacation left knows very well that he will not spend more than one day sailing. But he does not know that he <u>will</u> spend that one day sailing.

Existence means that there is at least one. Uniqueness means that there is at most one. Existence and uniqueness means that there is one and only one, or exactly one.

Problem Set 6-2b

166 1. Yes. Postulates 6 and 7.

2. 3. $\overleftrightarrow{WB}$ and $\overleftrightarrow{HK}$. $\overleftrightarrow{WB}$ and $\overleftrightarrow{AF}$. $\overleftrightarrow{HK}$ and $\overleftrightarrow{AF}$.

167 3. 6. $\overleftrightarrow{AQ}$ and $\overleftrightarrow{BQ}$, $\overleftrightarrow{AQ}$ and $\overleftrightarrow{CQ}$, $\overleftrightarrow{AQ}$ and $\overleftrightarrow{DQ}$, $\overleftrightarrow{BQ}$ and $\overleftrightarrow{CQ}$, $\overleftrightarrow{BQ}$ and $\overleftrightarrow{DQ}$, $\overleftrightarrow{CQ}$ and $\overleftrightarrow{DQ}$.

4. $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PT}$ are the same line.

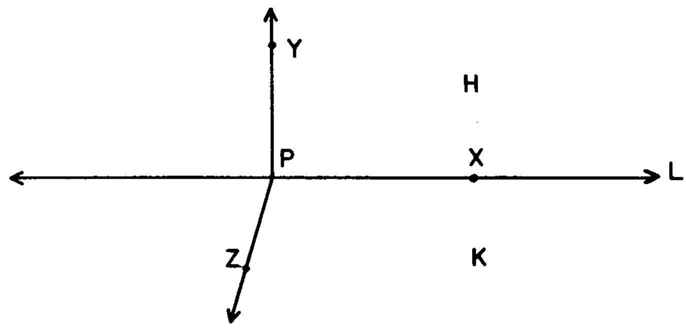
5. Yes. By Postulate 7. ABQ. $\overleftrightarrow{AB}$. B.

6. If A, B, C, D are not coplanar, we list the planes ABC, ABD, ACD, BCD. However, if A, B, C, D are coplanar, there is only one plane determined.

---

168 Many students may feel that the formal proof of uniqueness in Theorem 6-1 is mere hair splitting. For them it probably is best not to belabor the point. After they have had more contact with uniqueness principles they may better appreciate the point.

Some students may object that the uniqueness proof is unnecessarily complicated, that the Angle Construction Postulate "guarantees" that there is a unique line M in plane B perpendicular to L at P. This is not quite correct. The Angle Construction Postulate asserts that there is a unique <u>ray</u> $\overrightarrow{PY}$ with Y in half-plane H such that $m\angle XPY$ is 90. Then <u>line</u> $\overleftrightarrow{PY} \perp L$. Suppose then we apply the same process to the half-plane K opposite to H.



The Angle Construction Postulate now asserts that there is a unique ray $\overleftrightarrow{PZ}$ with Z in half-plane K such that $m\angle XPZ$ is 90. Then line $\overleftrightarrow{PZ} \perp L$. No one of our postulates or theorems tells us that the lines $\overleftrightarrow{PY}$ and $\overleftrightarrow{PZ}$ are identical. The uniqueness part of Theorem 6-1 takes care of this. Actually it does more - it proves that no conceivable process of "construction" or definition can yield a second line perpendicular to L at P in plane E.

169 The question at the end of the paragraph following Theorem 6-1: Can you identify a uniqueness theorem which has no corresponding existence theorem? Yes, Theorem 3-1: Two different lines intersect in at most one point. Theorem 3-2 could be reworded to yield another example: If a plane does

not contain a line, then the plane and the line intersect in at most one point.

169 In Theorem 6-2 we have put together in compact form, an important theorem and converse, by using the language of sets. The theorem and its converse establish a characteristic or distinguishing property of any point of the perpendicular bisecting line of a given segment - that is, a property which holds for, and only for, points of this line. This property then is a characterization of the perpendicular bisector as a set of points. Other such characterization theorems will appear later.

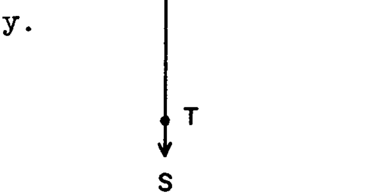
In Theorem 6-2 note the importance of the restriction that all points considered lie in a plane. If this restriction is removed, we get a corresponding result in space: The perpendicular bisecting plane of a segment is the set of all points that are equidistant from the endpoints of the segment. This is Theorem 8-7 of Chapter 8. Note that Theorems 8-1 and 8-2 give further "equidistance" properties of lines and planes.

173 Case 2 of Theorem 6-4: U = Q.

Use the first 5 statements of Case 1.

6. $\angle RUP \cong \angle RUT$. 6. Statement 2, and U = Q.

7. $\overleftrightarrow{PU} \perp L$. 7. Definition of perpendicularity.

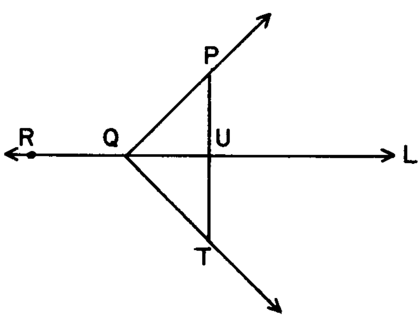


Case 3 of Theorem 6-4: Q is between R and U.

Insert a step between steps 2 and 3:

$\angle PQU \cong \angle TQU$. Supplements of congruent angles.

Refer in Reason 6 to the new statement rather than to Statement 2.



Problem Set 6-3

174 1. $\overleftrightarrow{EC}$ is the perpendicular bisector of $\overline{BD}$ and so EB = ED by Theorem 6-2.

2. x = 7, y = 5, z = 10.

3. Since P and M are points which are each equidistant from A and B, $\overleftrightarrow{PM}$ is the perpendicular bisector of $\overline{AB}$ by Theorem 6-2 and Postulate 1. Then QA = QB by Theorem 6-2.

4.

1. PT = PR + RT.	1. Definition of betweenness.
2. RT = RQ.	2. Theorem 6-2.
3. PT = PR + RQ.	3. Substituting RQ for RT in Statement 1.

175 7. No. Yes.

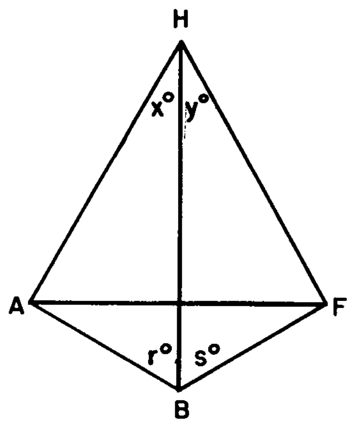
*8.

1. AC = BC.	1. Given.
2. $m\angle A = m\angle B$.	2. Base angles.
3. $\frac{1}{2}m\angle A = \frac{1}{2}m\angle B$.	3. Division, from Statement 2
4. $\angle DAB \cong \angle EBA$.	4. From step 3.
5. AF = BF.	5. If two angles of a triangle are congruent, the sides opposite them are congruent.
6. $\overleftrightarrow{CF}$ is perpendicular bisector of $\overline{AB}$.	6. Theorem 6-2 and Postulate 1.

*9.

Given: $\overline{HB}$ bisects $\angle$s AHF and ABF.

Prove: $\overline{HB}$ bisects $\overline{AF}$.



1. x = y, r = s.	1. Definition of bisect.
2. HB = HB.	2. Identity.
3. Δ ABH $\cong$ Δ FBH.	3. A.S.A.
4. HA = HF, BA = BF.	4. Corresponding parts.
5. $\overline{HB} \perp \overline{AF}$. $\overline{HB}$ bisects $\overline{AF}$.	5. Theorem 6-2.

*10.

1. RC = SC. $\angle$ RCA $\cong$ $\angle$ SCA.	1. Given.
2. AC = AC.	2. Identity.
3. Δ RCA $\cong$ Δ SCA.	3. S.A.S.
4. RA = SA.	4. Corresponding parts.
5. Q is mid-point of $\overline{RS}$.	5. Given.
6. $\overline{AQ} \perp \overline{RS}$.	6. Theorem 6-2.

176 This discussion of the introduction of auxiliary sets is a departure from the conventional treatment. It is important and deserves attention. Consider how often students assume they can, by "construction", justify referring to a line whose existence has not been proved and which, in fact, may not exist (see Example 2).

Notice that we say "introduce" line $\overleftrightarrow{AB}$ or segment $\overline{PQ}$ and avoid using such words as "draw" or "construct". As soon as we have shown the existence of line $\overleftrightarrow{AB}$ (or segment $\overline{PQ}$) we have the logical right to reason about it and to derive properties of it in our geometry. This is independent of whether we choose to draw or represent it in a diagram.

Having proved the existence and uniqueness of a certain geometric object in our theory, we sometimes ask how it could be constructed physically from given data using prescribed operations or procedures. Thus the discussion of Theorem 6-4 gives a precise description of the construction of the perpendicular to a given line from a given external point using ruler and protractor. In this instance, the construction is given before the proof to help the student grasp it.

(Once this important distinction between the common meaning of "draw" and the meaning of "introduce" described above is established with your students, it seems agreeable to use the term "draw" for convenience. An occasional reminder of the distinction should be made, however, so that the correct concept becomes the one suggested by whatever word is used.)

Notice in Section 6-4 that we do not say that auxiliary segments always are shown as dotted segments. The dotted segment seems necessary only when the figure becomes so complicated that the method of proof becomes obscure.

178 If A, C, D and E are non-coplanar in Example 1, the
179 proof based on introducing $\overline{DE}$ does not hold. The proof in which $\overline{AC}$ is introduced does hold, however.

Problem Set 6-4

181 1.

Statements	Reasons
1. Consider $\overline{AC}$.	1. Two points determine a line.
2. AD = CD.	2. Given.
3. $m\angle DAC = m\angle DCA$.	3. Base angles of an isosceles triangle are congruent.
4. $m\angle DAB = m\angle DCB$.	4. Given.
5. $m\angle BAC = m\angle BCA$.	5. Subtraction using statements 3 and 4.
6. AB = CB.	6. If two angles of a triangle are congruent, the sides opposite are congruent.

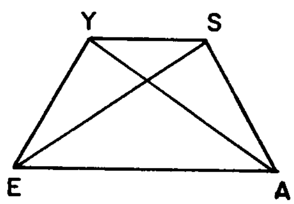
This proof does not work if points A, B, C and D are not coplanar. Step 5 would not be valid.

2.

1. Draw $\overline{XA}$. *	1. Two points determine a line.
2. XA = XA.	2. Identical.
3. XY = AB and AY = XB.	3. Given.
4. Δ YXA $\cong \Delta$ BAX.	4. S.S.S.
5. m$\angle$ YXA = m$\angle$ BAX. m$\angle$ BXA = m$\angle$ YAX.	5. Corresponding parts.
6. m$\angle$ YXO = m$\angle$ BAO.	6. Subtraction in Statement 5.
7. $\angle$Y $\cong \angle$B.	7. Corresponding parts.
8. ΔXOY $\cong \Delta$ AOB.	8. A.S.A.

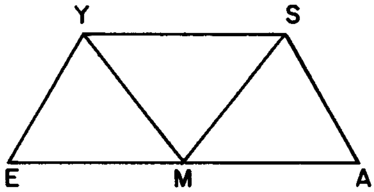
*A similar proof is possible if $\overline{YB}$ is drawn.

3.



1. Draw $\overline{ES}$ and $\overline{AY}$.	1. Two points determine a line.
2. YE = SA. $\angle$ E $\cong \angle$ A.	2. Given.
3. YS = SY.	3. Identity.
4. Δ YSA $\cong \Delta$ SYE.	4. S.A.S.
5. $\angle$YSA $\cong \angle$ SYE.	5. Corresponding parts.

4.



1. Let M be the midpoint of EA.	1. A segment has exactly one midpoint.
2. Consider $\overline{MY}$ and $\overline{MS}$.	2. Two points determine a line.
3. $\overline{EM} \cong \overline{AM}$.	3. Definition of midpoint.
4. $\angle E \cong \angle A$ and $\overline{YE} \cong \overline{SA}$.	4. Given.
5. Δ YEM $\cong \Delta$ SAM.	5. A.S.A.
6. $\overline{YM} \cong \overline{SM}$.	6. Corresponding parts.
7. $m\angle$ MYS = $m\angle$ MSY.	7. Base angles of an isosceles triangle are congruent.
8. $m\angle$ EYM = $m\angle$ ASW.	8. Corresponding parts.
9. $m\angle$ EYS = $m\angle$ ASY.	9. Addition of Statements 7 and 8.

5.

1. Consider $\overline{AD}$.	1. Two points determine a line.
2. AC = AB. CD = DB.	2. Given.
3. AD = AD.	3. Identity.
4. Δ ACD $\cong \Delta$ ABD.	4. S.S.S.
5. $\angle$ACD $\cong \angle$ ABD.	5. Corresponding parts.

---

182 This is very unusual material for a tenth grade geometry text. We introduce it to indicate that the assertions we make can be justified from our postulates (without recourse to diagrams), and to give some typical examples of how we can logically justify betweenness and separation properties which usually are read from figures. There are two pitfalls here. First, it is best not to try to teach this material to students who are perfectly satisfied with the proofs as originally

given. They probably are not yet ready for this kind of critical thinking and their progress in geometry will not be impeded by passing on to the next chapter. There is an opposite danger for the very critical student. He may become distrustful of diagrams and fail to develop a sound geometrical intuition. He should be reminded that our theory of geometry is suggested by physical space, is applicable to it, and that many theorems can be discovered and most can be appreciated by the study of diagrams and models. (See Chapter 7, Section 7-1, on making conjectures in geometry.) Point out that a geometric proof in which one step depends on the diagram, although not mathematically perfect, is still incomparably superior to what is considered logical in most areas of human discourse.

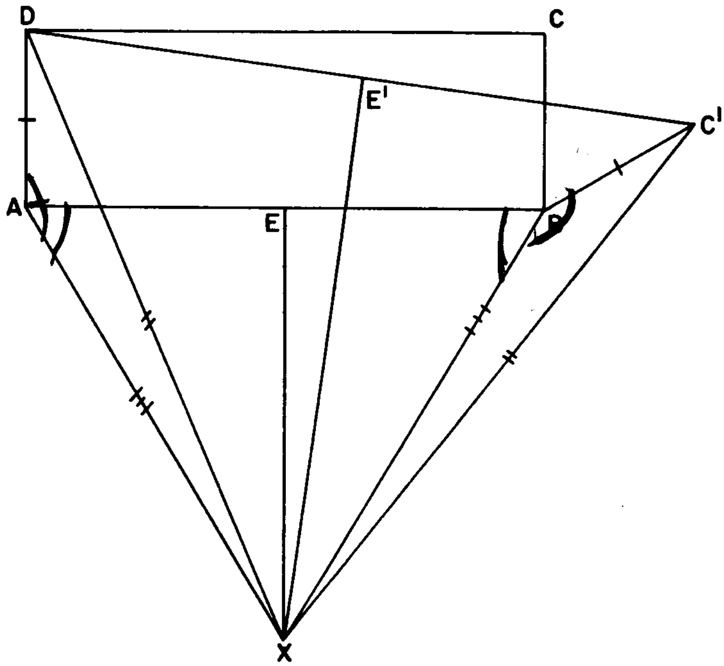
Having clarified the basic point in this section we don't hesitate in later chapters to use the diagram to justify properties of betweenness and separation. The complete justification of all such properties used is still quite difficult and requires a deeper study of the foundations of geometry. (See Studies II.)

As we mentioned earlier, you will have to decide how much time your class should devote to Section 6-5. If you do not choose to have your class as a whole study the section, your better students may find that the exposition and the problems provide excellent supplementary work.

Problem Set 6-5

1. a. $\angle$ B. By Theorem 6-6.
   b. $\angle$ C. By Theorem 6-6.
   c. $\angle$ A, $\angle$ B, and $\angle$ C. By the definition of the interior of a triangle.

185 *2. The argument (using the first drawing) depends on the assumption from the drawing that E is inside $\angle XBC'$. In a careful drawing (see below) X will appear on the opposite side of $\overleftrightarrow{BC'}$ from E.



3. The three possibilities are:
   a. A is on L. In this case L intersects both $\overline{AC}$ and $\overline{AB}$.
186 b. A is in H_1. In this case A is on the same side of L as B, and C is on the other side of L. In this case L intersects $\overline{AC}$. This follows from the Plane Separation Postulate.
   c. A is in H_2. In this case A is on the other side of L from B so L intersects $\overline{AB}$.
4. a. Since D is between A and C, D is in the interior of $\angle ABC$, by Theorem 6-6, and the definition of the interior of an angle implies that A and D are on the same side of $\overleftrightarrow{BC}$.
   Theorem 6-5 implies that D and F are on the same side of $\overleftrightarrow{BC}$.

b. Since $\overline{BF}$ intersects $\overleftrightarrow{AC}$ at D, it follows from the Plane Separation Postulate that F belongs to H_2. Since $\overline{BE}$ intersects $\overleftrightarrow{AC}$ at C, it follows from the Plane Separation Postulate that E belongs to H_2.

187 5. a. A and D are on the same side of $\overleftrightarrow{BC}$ because it is given that D is in the interior of $\angle ABC$. Theorem 6-5.

E is in H_2 by the Plane Separation Postulate. Theorem 6-5.

Each point of $\overrightarrow{BD}$ with the exception of B lies in H_1 but no point of $\overline{EC}$ does. Also, B does not lie on $\overline{EC}$.

b. Each point of $\overline{EC}$ other than E lies on the same side of $\overleftrightarrow{AB}$ as C and D, but each point on the ray opposite $\overrightarrow{BD}$ with the exception of B lies on the other side of $\overleftrightarrow{AB}$. Note that C and D are on the same side of $\overleftrightarrow{AB}$ since D is in the interior of $\angle ABC$.

c. It follows from Problem 3 that $\overleftrightarrow{BD}$ intersects either $\overline{AC}$ or $\overline{EC}$. It follows from parts _a_ and _b_ above that $\overleftrightarrow{BD}$ does not intersect $\overline{EC}$.

d. Each point of $\overline{AC}$ other than A lies on the same side of $\overleftrightarrow{AB}$, with C and D by Theorem 6-5 and the Plane Separation Postulate, but each point of the ray opposite $\overrightarrow{BD}$, with the exception of B, lies on the other side of $\overleftrightarrow{AB}$.

*6. Since D is in the interior of $\angle ABC$, it follows from the Angle Addition Postulate that $m\angle ABD + m\angle DBC = m\angle ABC$. Since all of these measures are positive it is impossible that either

(1) $m\angle ABD + m\angle ABC = m\angle DBC$ or

(2) $m\angle ABC + m\angle DBC = m\angle ABD$.

Since (1) is impossible, A is not in the interior of $\angle DBC$. Since (2) is impossible, C is not in the interior of $\angle ABD$.

188 *7. a. D lies in the plane determined by A, B, C since it is on the line $\overleftrightarrow{BC}$. E lies in this plane since it is on the line $\overleftrightarrow{AB}$.

b. A and B are on the same side of $\overleftrightarrow{ED}$ and C is on the opposite side from A and B. Hence, $\overleftrightarrow{ED}$ intersects $\overline{AC}$ at a point X between A and C.

c. $\overleftrightarrow{BC}$.

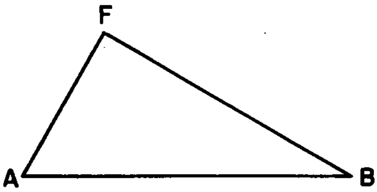
8. a. True.

b. True.

c. False.

d. True.

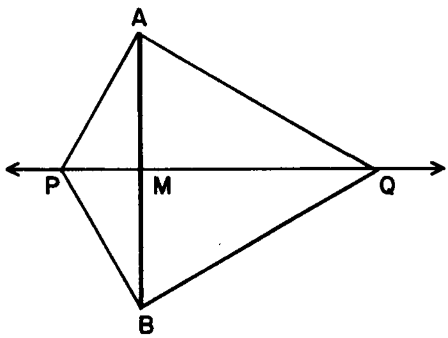
Illustrative Test Items for Chapter 6

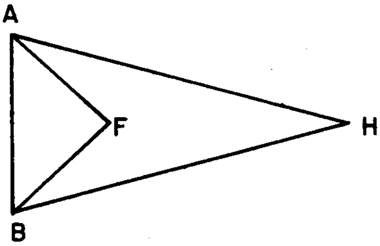
1. 

In Δ ABF, every point of $\overline{AF}$ except ___ and ___ is in the interior of $\angle$ ___. ___ of the points of $\overline{AB}$ is in the interior of $\angle$ ABF.

2. Snow melts at temperatures above 32°. There is snow on the ground and the temperature outside is 40°. Write a logical conclusion.

3. Given that PA = PB, QA = QB and $\overleftrightarrow{PQ}$ meets $\overline{AB}$ at M as shown in the figure. Without using congruent triangles, prove M is the midpoint of $\overline{AB}$.



4. 

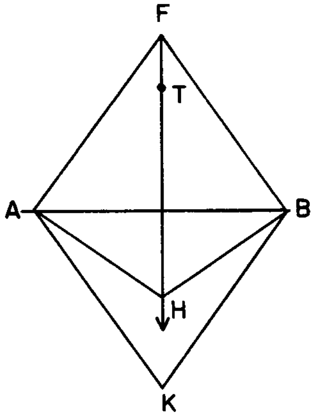
In this plane figure there are two isosceles triangles with the same base, $\overline{AB}$. $\overleftrightarrow{HF}$ _____ $\overline{AB}$ and is _____ to $\overline{AB}$. Every point of $\overleftrightarrow{HF}$ is _____ from A and B.

5. If, for the sake of argument, you accept the following hypothesis, which of the following are logical conclusions?

Hypothesis: Every piece of Alpha candy is delicious.

Conclusions:
a. Since this piece of candy is delicious, it must have been made by Alpha Company.
b. This Alpha caramel is delicious.
c. Since this piece of candy is not delicious, it could not have been made by Alpha Company.

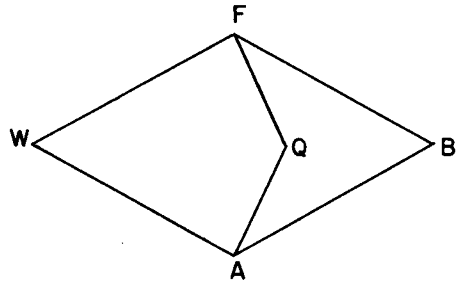
6.



AF = BF. The points given in the picture are coplanar.
AH = BH.
AK = BK.

Does line $\overleftrightarrow{FH}$ pass through K?
If AT = 3, then BT = ______.
State a theorem which supports your conclusions.

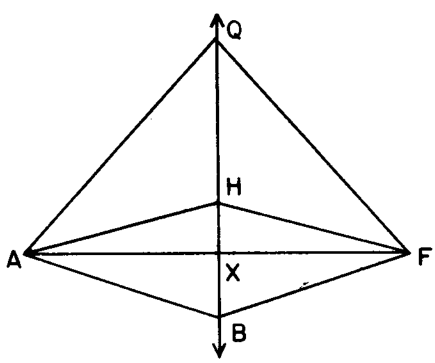
7. FB = AB.
FQ = AQ.
WF = WA.
Are W, Q and B necessarily collinear if these three points are coplanar?



8. Given that A, B, C, F are four non-coplanar points, list all the planes determined by subsets of A, B, C, F.

9. Prove that the perpendicular bisector of one side of a scalene triangle cannot include the opposite vertex of the triangle.

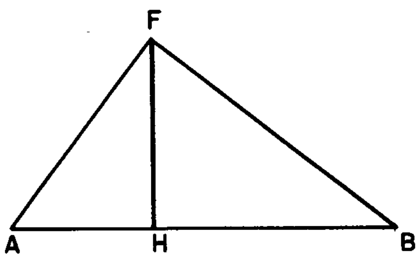
10.



In this figure,
$\overleftrightarrow{QB} \perp \overleftrightarrow{AF}$,
AX = FX,
XH = XB.
Prove: HF = BF and
QA = QF.

Answers

1. A and F; ABF; None.
2. The snow is melting.
3. Since PA = PB and QA = QB, P and Q lie in the perpendicular bisector of $\overline{AB}$. Therefore, $\overleftrightarrow{PQ}$ is the perpendicular bisector of $\overline{AB}$.
4. Bisects. Perpendicular. Equidistant.
5. b, c.
6. Yes. 3. Theorem 6-2.
7. No. Not unless the entire figure is a plane figure.
8. ABC, ABF, ACF, BCF.
9.



Given that Δ ABF is scalene. Assume that $\overline{FH}$ could bisect $\overline{AB}$ and be perpendicular to $\overline{AB}$. Then Δ AFH $\cong$ Δ BFH by S.A.S. and FA $\cong$ FB, so Δ AFB is isosceles. The assumptions lead to the contradiction that a scalene triangle is isosceles. Hence the assumptions were false.

10. $\overleftrightarrow{QB}$ is the perpendicular bisector of $\overline{AF}$. Therefore, QA = QF. Since XH = XB is given, X is the midpoint of $\overline{HB}$ and $\overleftrightarrow{FX}$ is its perpendicular bisector. Hence, HF = BF.

Chapter 7

GEOMETRIC INEQUALITIES

The material covered in this chapter is quite similar to that found in corresponding chapters of other geometry texts. The main difference is that we compare two segments or two angles merely by comparing their lengths or measures. Thus, although our inequalities describe geometric relations, they involve only real numbers. This is another advantage of our early introduction of real numbers. Because students do not always know principles of inequalities well, we restate the order postulates first given in Section 2-2, giving examples to show how they are applied.

189 The idea that a conjecture must not be considered true until (unless) it has been proved, bears emphasis. To put it bluntly, a conjecture is a guess. The kind of conjectures we pay attention to are the shrewd, reasonable ones that are based upon inductive thinking or insight. But conjectures, no matter how reasonable they seem, remain guesses until they are proved.

It may be good for your students to be reminded that conjecturing is an important part, even if only the first stage, of mathematical work. After all, a man who develops new mathematics often must try to decide what the truth is before he can present a logical proof of it. There is no reason to look down on the art of making conjectures. There is, however, no excuse for confusing guessing with proving.

Goldbach's conjecture that every even number is the sum of two primes is a simple non-geometric example that you can mention. After many generations the conjecture is still not a theorem.

The example of Section 7-1 should suggest two things to the student. First, he should try to make reasonable conjectures. Second, he should express his conjectures in good mathematical language. The second goal may be the more difficult to achieve.

Problem Set 7-1

190 1. The opposite sides are unequal in length with the side opposite the largest angle having the greater length.

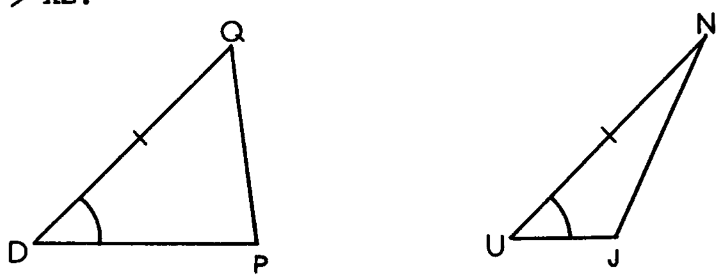
2. AB + BC > AC. BC + AC > AB. AB + AC > BC. The sum of the lengths of two sides of a triangle is greater than the length of the third side.

3. RS + ST + TQ > RQ. The sum of the lengths of three sides of a quadrilateral is greater than the length of the fourth side.

4. It increases.

5. DF > XZ.

6.



7. From B drop a perpendicular to E at a point D of E. Then D will lie on some $\overrightarrow{AC}$ and for this $\overrightarrow{AC}$, m ∠ BAC = m ∠ BAD is a minimum. If $\overrightarrow{AF}$ is the opposite ray to $\overrightarrow{AD}$, m ∠ BAF is a maximum.

191 8. The procedure does not work since m ∠ DAE is larger than either m ∠ BAD or m ∠ EAC. This shows up clearly if m ∠ BAC is close to 180.

It may be helpful to state the order principles in English as well as in algebraic symbolism. For example, 0-2 may be stated: If the first of three numbers is less than the second and the second is less than the third, then the first is less than the third. Similarly, 0-3 asserts: If the same number is added to each of two unequal numbers, the sums are unequal in the same order. Or 0-3 may be stated: If the same number is added to each side of an inequality, the inequality remains true. You recognize that these order principles are essentially the same as the "Axioms of Inequality" which appear in most geometry texts. The order principles refer to real numbers rather than geometric quantities.

192 Example 6, simple as it seems, is quite important and often used. In many geometric problems it is necessary to prove a relation such as $a < c$ or $c > a$. In the conventional treatment we refer to a diagram and conclude $c > a$ by the principle, "The whole is greater than any of its parts". Ordinarily, we prove a relation like $c > a$ by applying Example 6, that is, we show $c = a + b$, where b is positive. (Actually in our applications a, b and c will all be positive.) We might reword this as, $a + b > a$ when $b > 0$, since $c = a + b$. Even more simply we can say, "The sum of two positive numbers is greater than either number." Thus, the final justification is a property of real numbers. An important application of Example 6 occurs later in Step 8 of the proof of Theorem 7-1.

Example 6. If $a + b = c$ and b is positive then $a < c$.

Reasons only:

1. Given.
2. Definition of positive.
3. Relation between < and >.
4. Postulate 0-3.
5. Substituting c for $a + b$.

Example 7. If $a + b < c$, then $a < c - b$.

Proof:

1. $a + b < c$.	1. Given.
2. $a + b + (-b) < c + (-b)$.	2. Postulate 0-3.
3. $a < c - b$.	3. Algebra.

"Algebra" means here that the principle involved is well known to the student in the sense that it involves the "field" properties; that is, the basic properties of addition, multiplication, subtraction and division but not order or inequality properties. He knows that $a + b + (-b) = a$, and that $c + (-b) = c - b$. Step 3 also involves substitution.

{page 192}

Example 8. If $a < b$, then $c - a > c - b$ for every c. This may be stated: If unequal numbers are subtracted from the same number, the differences are unequal in reverse order.

Proof:

1. $a < b$.	1. Given.
2. $a + (c - a - b) < b + (c - a - b)$.	2. Postulate 0-3.
3. $c - b < c - a$.	3. Algebra.
4. $c - a > c - b$.	4. Relation between < and >.

192 Example 10. If $x < y$ and $z < 0$ then $xz > yz$.

Proof:

1. $z < 0$.	1. Given.
2. $z + (-z) < 0 + (-z)$.	2. Postulate 0-3.
3. $0 < -z$.	3. Algebra.
4. $-z > 0$.	4. Relation between < and >.
5. $x < y$.	5. Given.
6. $x(-z) < y(-z)$.	6. Postulate 0-4.
7. $-xz < -yz$.	7. Algebra.
8. $-xz + (xz + yz) < -yz + (xz + yz)$	8. Postulate 0-3.
9. $yz < xz$.	9. Algebra.
10. $xz > yz$.	10. Relation between < and >.

We have just proved: If unequal numbers are multiplied by the same negative number, then the products are unequal in the opposite order. Actually all the familiar "Axioms of Inequality" can be derived from the four order postulates.

194 Step 6 of the proof of Theorem 7-1 tacitly assumes that F is in the interior of $\angle$ BCD. This is justified in Problem 4 of Problem Set 6-5. It is probably true that no kind of mathematics can be effectively presented in a completely rigorous form to a tenth-grade class. We should not feel guilty about teaching tenth-grade students merely as much as they can learn. The betweenness problem here will probably go unnoticed by most students. It should be called to the attention only of very capable and critical students. (Such students will probably be rare.)

The formal justification of Step 8 involves an application of Example 6 of Section 7-2: If $a + b = c$ and b is positive, then $a < c$. (See the Commentary above.) We have Step 7,

$$m\angle BCD = m\angle B + m\angle FCD,$$

and $m\angle FCD$ is positive (all angle measures are positive by the Angle Measurement Postulate). Thus, by Example 6

$$m\angle B < m\angle BCD \text{ or } m\angle BCD > m\angle B.$$

Hereafter we usually apply Example 6 in such situations without explicit reference.

The following lemma usually is applicable in proving an angle larger than another.

Lemma. If D is in the interior of $\angle ABC$, then $m\angle ABC > m\angle ABD$.

Proof: The argument above applies. By the Angle Measurement Postulate

$$m\angle ABC = m\angle ABD + m\angle CBD$$

and $m\angle ABC > m\angle ABD$ by Example 6.

Similarly we can prove an analog for lengths of segments.

Lemma. If C is between A and B, then $AB > AC$.

Problem Set 7-3a

195 1. a. $\angle ACB$ and $\angle CAB$.
   b. $\angle FCB$.
2. a. $\angle DBC$ and $\angle EBA$.
   b. $m\angle DBC > m\angle A$, by Theorem 7-1.
   c. $m\angle DBC > m\angle C$, by Theorem 7-1.
   d. $m\angle DBC + m\angle CBA = 180$, by Postulate 14.
3. a. 40.
   b. is greater than 73.
   c. is equal to 112.
   d. is less than 112.
   e. is equal to 30.
   f. is equal to 90.
   g. This is impossible, since, by Theorem 6-3, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ are not both perpendicular to $\overleftrightarrow{AB}$.

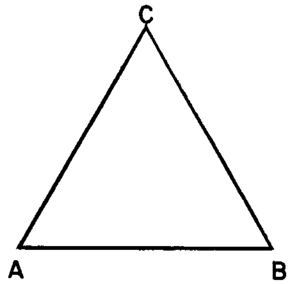
196 4. No. It is not true for the exterior angle at each of the other vertices. Another exception is a rectangle.

*5. By the Supplement Postulate, $a + w = 180$. But $b < w$, by the Exterior Angle Theorem. Adding a to each side of this inequality, we get $a + b < a + w$ which becomes $a + b < 180$, which was to be proved. Similarly, $b + c < 180$ and $a + c < 180$.

*6. Given: Δ ABC with $\overline{AC} \cong \overline{BC}$.

To prove: $m\angle A < 90$.

$m\angle B < 90$.



Proof: By the previous problem we have $m\angle A + m\angle B < 180$. But the base angles of an isosceles triangle are congruent, so $2(m\angle A) < 180$, and $m\angle A < 90$. Also, $m\angle B < 90$, since the measures of the base angles are equal.

---

197 The S.A.A. Theorem usually is proved after the Parallel Postulate is introduced, since it follows readily from the theorem that the sum of the angle measures of a triangle is 180. Since the S.A.A. Theorem does not depend on the Parallel Postulate (Chapter 9) we introduce it here and can apply it whenever needed.

An S.S.A. theorem also holds when the angle is an obtuse angle, but there is little value is bringing this fact to the attention of a class. Outstanding students might enjoy proving the fact, however.

Problem Set 7-3b

199 1. Since AQ = BQ, $\angle$ QBA $\cong \angle$ QAB.

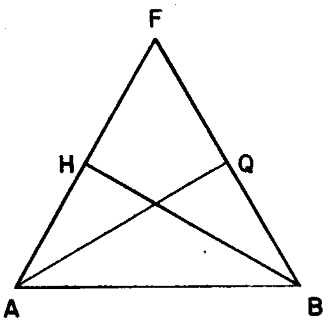
Also AB = AB and $\angle$ H $\cong \angle$ F. Therefore, Δ ABH $\cong \Delta$ BAF by S.A.A.

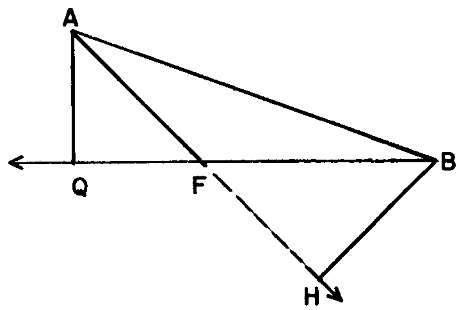
2.

1. AB = HF.	1. Given.
2. BF = BF.	2. Identity.
3. AF = HB.	3. Addition, Steps 1 and 2.
4. $\angle$s K and Q are right angles.	4. Definition of perpendicular lines.
5. AK = HQ.	5. Given.
6. $\triangle$ AKF $\cong \triangle$ HQB.	6. Hypotenuse - Leg Theorem.
7. KF = QB.	7. Corresponding parts.

3. $\triangle$ FAH $\cong \triangle$ AFX by Hypotenuse-Leg Theorem, hence $\angle$ BFA $\cong \angle$ FAB. Therefore, FB = AB.

200 4.





Given: $\overline{HB} \perp \overleftrightarrow{AF}$, $\overline{QA} \perp \overleftrightarrow{BF}$, HB = QA.

Prove: $\triangle$ FAB is isosceles.

Since AB = AB, $\triangle$ ABH $\cong \triangle$ BAQ by Hypotenuse-Leg Theorem and so $\angle$ HAB $\cong \angle$ QBA. It follows that FA = FB and $\triangle$ FAB is isosceles.

5. $\angle$ AKF $\cong \angle$ ABQ (Supplements of congruent angles), $\angle$ A = $\angle$ A, AQ = AF. Hence, $\triangle$ AQB $\cong \triangle$ AFK by S.A.A. Then QB = FK.

6. Since $\angle$ a $\cong \angle$ c, AB = FB. Also in $\triangle$s ABH and FBH BH = BH, and $\angle$s BAH and BFH are right angles. Therefore, these triangles are congruent by the Hypotenuse-Leg Theorem Hence, AH = FH.

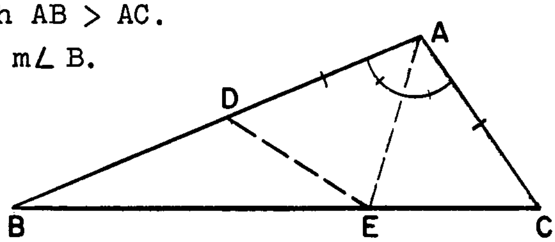
---

200 In the proof of Theorem 7-4, Statement (3) involves Example 6 of Section 7-2. (See comment above on Theorem 7-1, Step 8.)

One frequently sees Theorem 7-4 proved by the following method:

Given: $\triangle ABC$ with $AB > AC$.

To prove: $m\angle C > m\angle B$.



Take D, between A and B, such that AD = AC. Bisect $\angle A$, and let E be the intersection of the bisector with the line $\overleftrightarrow{BC}$. Show that $\triangle ADE \cong \triangle ACE$, by the S.A.S. Postulate. It follows that $m\angle ADE = m\angle ACE$. By the Exterior Angle Theorem,

$$m\angle ADE > m\angle DBE.$$

Therefore, $m\angle C > m\angle B$, which was to be proved.

This proof tacitly assumes that the bisector of $\angle A$ really does intersect $\overleftrightarrow{BC}$ in a point between B and C. See Problem 5 of Problem Set 6-5 for consideration of this matter.

Problem Set 7-3c

203 1. $\angle G$. $\angle K$.

2. $\overline{AC}$. $\overline{BC}$.

204 3. a. 40.

b. 80.

c. $\overline{AB}$.

4. a. $ML > KL$.

b. $ML < MK$.

c. $KL > ML > MK$.

d. None.

e. $ML > KL$ and $ML > KM$.

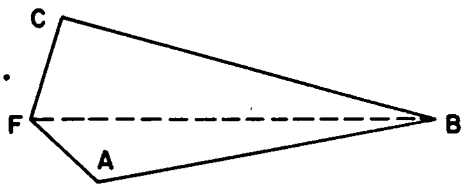
f. $ML \geq KM$ and $ML \geq KL$.

5. In $\triangle ABC$, $\overline{AC}$ is the longest side, since it is opposite the angle with the greatest measure. In $\triangle ADC$, $\overline{AD}$ is the longest side, for the same reason. Therefore, $AD > AC$ and $\overline{AD}$ is the longest of the five segments.

6. $\overline{BC}$, $\overline{AB}$, $\overline{AC}$.

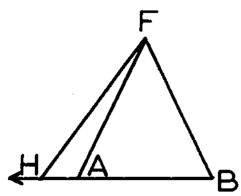
(Note to the teacher: You may expect to get a reaction from the student, to the effect that the figure is incorrect, since m∠ A + m∠ B + m∠ C < 180. This is a fine opportunity to point out that we cannot prove, on the basis of the postulates given so far, that the sum of the measures of the angles in a triangle is 180. When we get to the Parallel Postulate in Chapter 9, we will be in a position to prove the angle sum theorem. In any case, given the hypothesis that such a triangle exists, we can assert the conclusion that its sides are ordered in the given manner.)

7. Given: $\overline{AF}$ is the shortest side.
$\overline{CB}$ is the longest side.
To prove: m∠ CFA > m∠ CBA.



1. In Δ ABF, AB > AF.	1. Given.
2. m∠ BFA > m∠ ABF.	2. Theorem 7-4.
3. In Δ BCF, CB > CF.	3. Given.
4. m∠ CFB > m∠ CBF.	4. Theorem 7-4.
5. m∠ BFA + m∠ CFB > m∠ ABF + m∠ CBF.	5. Adding Steps 2, 4.
6. m∠ CFA > m∠ CBA.	6. Step 5 and the Angle-Addition Postulate.

205 *8. Given: FA = FB.
A is between H and B.
To prove: FH > FB.



1. FA = FB.	1. Given.
2. m∠ FAB = m∠ B.	2. Base angles of an isosceles triangle are congruent.
3. m∠ FAB > m∠ H.	3. Theorem 7-1.
4. m∠ B > m∠ H.	4. Steps 2 and 3.
5. FH > FB.	5. Theorem 7-5.

9. a. If a team can win some games, it has some spirit.
   b. If two angles are congruent, they are right angles.
   c. Any two supplementary angles are congruent.
   d. The intersection of two half-planes is the interior of an angle.
   e. If Joe is seriously ill, he has scarlet fever.
   f. If a man lives in Ohio, he lives in Cleveland, Ohio.
   g. If two triangles are congruent, then the three angles of one are congruent to the corresponding angles of the other.
   h. If the sum of the measures of two angles is 90, the angles are complementary.

Statement		Converse	Statement		Converse
a.	F	T	e.	T	F
b.	T	F	f.	T	F
c.	F	F	g.	F	T
d.	T	F	h.	T	T

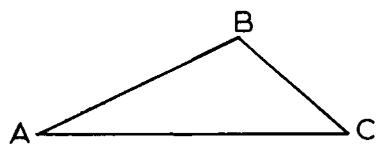
10. No. The converse should be, "If I will be burned, I hold a lighted match too long." The hypothesis does not contain "if", and the conclusion does not contain "then".

11. a. No. 9b, 9d, 9e, 9f.
    b. Yes. 9a, 9g.

---

206 Note that the distance between a line and a point is a number. Theorem 7-7 really involves three inequalities:

(1) $AB + BC > AC$,
(2) $BC + AC > AB$,
(3) $AC + AB > BC$.



The text proves (1), and this is sufficient since a relabeling of the figure will give (2) and (3).

Problem Set 7-3d

207 1. AT and AF. AT and TF.

The statement of Theorem 7-6.

208 2. HB < HA < HF. The statement of Theorems 7-6 and 7-5.

3. 3, 13.

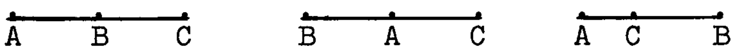
4. $k - j < x < k + j$.

5.

1. DB < CD + CB. DB < AD + AB. CA < CD + AD. CA < CB + AB.	1. The sum of the lengths of two sides of a triangle is greater than the length of the third side of the triangle.
2. 2DB + 2CA < 2CD + 2AD + 2CB + 2AB.	2. Addition.
3. DB + CA < CD + AD + CB + AB.	3. Division.

6. 1. If the points are noncollinear, the inequality follows from Theorem 7-7.

2. If the points are collinear, then either (1) B is on the segment $\overline{AC}$, in which case AB + BC = AC, or (2) A is between B and C, in which case BC > AC, so AB + BC > AC, or (3) C is between A and B, in which case AB > AC, so AB + BC > AC.



209 *7. Case 1. (n = 3). We know from the preceding problem (Problem 6) that the result is true in this case; that is, $A_1A_2 + A_2A_3 \geq A_1A_3$.

Case 2. (n = 4).

1. $A_1A_2 + A_2A_3 + A_3A_4 \geq A_1A_2 + A_2A_4$ because it follows from Case 1 that $A_2A_3 + A_3A_4 \geq A_2A_4$.

2. $A_1A_2 + A_2A_4 \geq A_1A_4$ by Case 1.

3. $A_1A_2 + A_2A_3 + A_3A_4 \geq A_1A_4$ follows from Steps 1 and 2.

General Case (n is arbitrarily large).

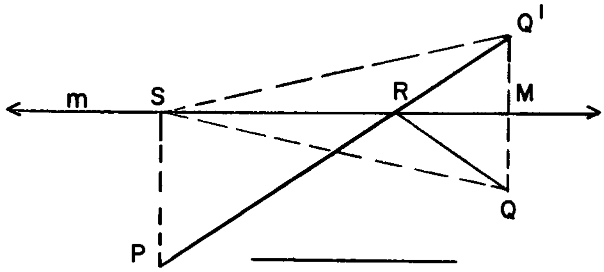
1. We continue as in Cases 1 and 2 to show that $A_1A_2 + A_2A_3 + \ldots\ldots + A_{n-2}A_{n-1} \geq A_1A_{n-1}$.
2. $A_1A_{n-1} + A_{n-1}A_n \geq A_1A_n$ by Case 1.
3. $A_{n-1}A_n \geq A_1A_n - A_1A_{n-1}$ by Step 2.
4. $A_1A_2 + A_2A_3 + \ldots\ldots\ldots + A_{n-2}A_{n-1} + A_{n-1}A_n > A_1A_n$ from Steps 1 and 3.

*8. XA + XC > PA + PC except when X is on the segment AC, in which case the equality sign holds. Similarly, XB + XD > PB + PD except when X is on $\overline{BD}$, in which case equality holds. Therefore, XA + XB + XC + XD > PA + PB + PC + PD except when X is on $\overline{AC}$ and also on $\overline{BD}$, and this can happen only if X = P, which is excluded by hypothesis.

The result also holds if X is not in the plane of A, B, C and D.

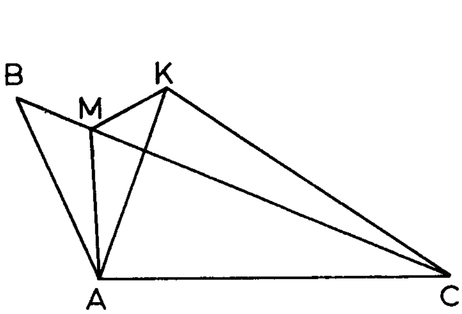
*9. Consider the reflection Q' of Q with respect to m. Then m is the ⊥-bisector of $\overline{QQ'}$ and intersects $\overline{QQ'}$ at a point which we call M. The point R on m to make PR + RQ a minimum is the point where $\overline{PQ'}$ intersects m.

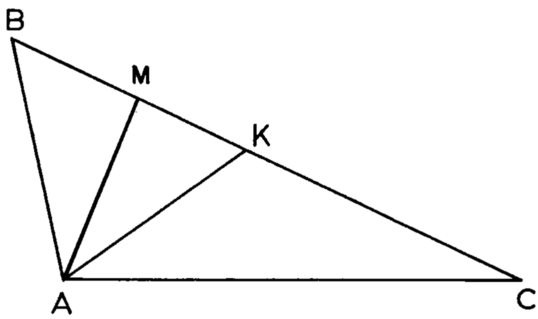
Let S be any point of m other than R. If S ≠ M, then $\triangle$ SMQ' ≅ $\triangle$ SMQ by S.A.S. SQ' = SQ, so PS + SQ = PS + SQ'.
If S = M, then again PS + SQ = PS + SQ'.
In $\triangle$ PSQ', PS + SQ > PQ' = PR + RQ' = PR + RQ.
∴ PS + SQ > PR + RQ.



210 The proof of theorem 7-8 is among the harder ones; you may want to skip it and merely authorize the use of the theorem in solving problems.

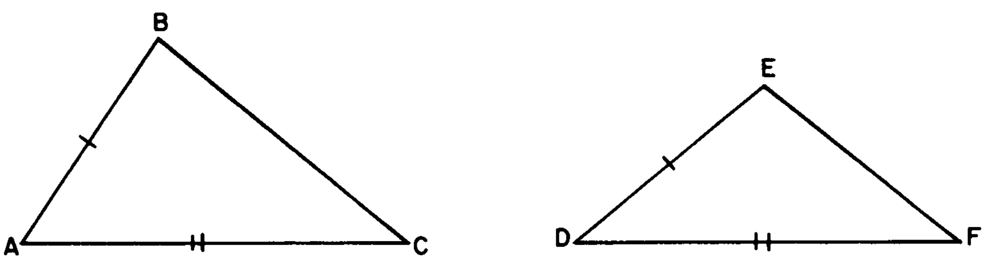
We have assumed properties from the diagram without formal justification. This will be done often hereafter as we indicated in the Commentary at the end of Chapter 6. The proof in the text tacitly assumes that K, M, C are noncollinear. The proof applies to the case indicated by the left-hand figure below as well as to that shown in the text. If K, M, C are collinear (see right-hand figure below), then B, K, C are collinear and K lies between B and C. Thus BC > CK and since CK = EF we have BC > EF.





211 Proof of Theorem 7-9.

Restatement: Given Δ ABC and Δ DEF. If AB = DE, AC = DF and BC > EF, then m∠ A > m∠ D.



Proof: Since m∠ A and m∠ D are numbers, there are only three possibilities: (1) m∠ A = m∠ D (2) m∠ A < m∠ D, and (3) m∠ A > m∠ D.

(1) If m∠ A = m∠ D, then Δ BAC ≅ Δ EDF and BC = EF. This contradicts the hypothesis, therefore it is impossible that m∠ A = m∠ D.

(2) If m∠ A < m∠ D, then BC < EF by Theorem 7-8. The last is false. Therefore, it is impossible that

m∠ A < m∠ D.

Only possibility (3) remains, and the theorem is proved.

Problem Set 7-3e

212 1. If two triangles have two sides of one congruent to two sides of the other, the third side of the first is longer than the third side of the second if and only if the included angle in the first is larger than the included angle in the second.

2. In ⚠ ACD and BCD, AC = BC, DC = DC and BD < AD, and so m ∠ x > m ∠ y by Theorem 7-9.

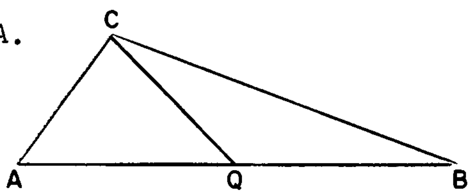
3.

1. RA = RF.	2. Given.
2. RB = RB.	2. Identity.
3. m∠ ARB < m∠ BRF.	3. Given.
4. AB < BF.	4. Theorem 7-8.

4.

1. RA = RF.	1. Definition of median.
2. RB = RB.	2. Identity.
3. m∠ FRB > m∠ ARB.	3. Supplement Postulate.
4. FB > AB.	4. Theorem 7-8.
5. m∠ A > m∠ F.	5. Theorem 7-4.

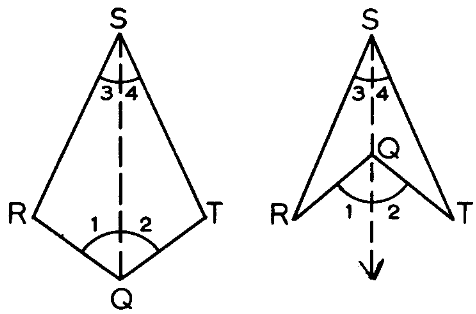
5. In ⚠ ACQ and BCQ, AQ = BQ, CQ = CQ and BC > AC. Then by Theorem 7-9 ∠ CQB > ∠ CQA. Since the two angles are supplementary, ∠ CQB is obtuse.



213 6. In ⚠ AHF and FQA, FH = AQ, AF = AF, and AH > FQ. Therefore, by Theorem 7-9, m ∠ AFH > m∠ FAQ. Then in Δ ABF, AB > FB by Theorem 7-5.

7. Given: QR = QT, SR = ST.
Prove: m ∠ RQT > m ∠ RST.

RS > RQ, ST > TQ.
m∠ 1 > m∠ 3,
m∠ 2 > m∠ 4.
m∠ RQT > m∠ RST.



8.

1. AB = FB.	1. Definition of median.
2. BH = BH.	2. Identity.
3. m∠ ABH > m∠ HBF. (or, m∠ HBF > m∠ ABH. See below.)	3. Given.
4. AH > FH.	4. Theorem 7-8.

Also, if the median were drawn so that ∠ ABH < ∠ FBH, then AH < FH.

<u>Alternate</u> <u>Proof:</u> Assume that HA = HF. Then △ AHB ≅ △ FHB by S.S.S., so ∠ ABH ≅ ∠ FBH and $\overline{HB} \perp \overline{AF}$. This contradicts the given information, so that HA ≠ HF.

9.

1. AB > AC.	1. Given.
2. ∠ ACB > ∠ ABC.	2. Theorem 7-4.
3. In △ BCD and △ FBC, FC = DB.	3. Given
4. CB = CB.	4. Identity.
5. FB > CD.	5. Theorem 7-8.

---

214 In reading Section 7-4, consider the following. A blasting worker may ask for more soup at 11 a.m., and mean nitroglycerine. He may ask for more soup at noon, and mean food. If confusion could arise in any given case, he would be explicit. His listener will normally interpret his language in light of the circumstances. Likewise, the fact that the context usually points to the proper meaning of altitude makes the use of the word for three different ideas

permissible, and perhaps even desirable.

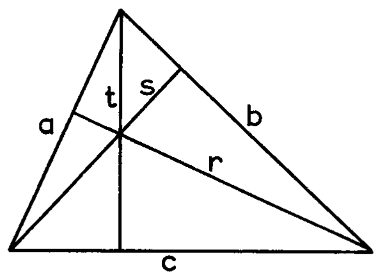
Problem Set 7-4

215 1. a. An altitude of a triangle is the perpendicular segment joining a vertex of the triangle to the line that contains the opposite side.

b. A median of a triangle is a segment whose end-points are one vertex and the mid-point of the opposite side.

216 3. They are the same segments and hence have the same length.

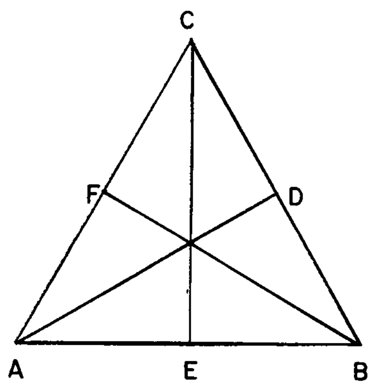
4.



$a > t$, $b > r$, $c > s$ by Theorem 7-6, and $a+b+c > r+s+t$. If the triangle is oblique the proof still holds. If the triangle is a right triangle, simply replace two of the $>$ symbols by the $\geq$ symbol.

5. Given: $\triangle ABC$ with $AC=AB=CB$.
$\overline{CE} \perp \overline{AB}$, $\overline{AD} \perp \overline{CB}$,
$\overline{BF} \perp \overline{AC}$.

Prove: $CE = BF = AD$.
$\triangle ABD \cong \triangle BCF \cong \triangle CAE$
by S.A.A. and so
$AD = BF = CE$.

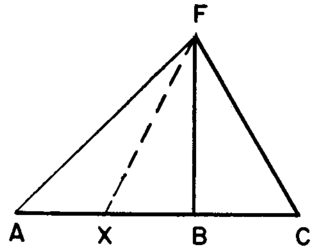


Review Problems

216 1. Yes, if the trunk is perpendicular to the ground. There are really three congruent triangles by Hypotenuse-Leg Theorem.

2. $\overline{CE}$. In $\triangle$ ADC, $\overline{AC}$ is the shortest side since it is opposite the smallest angle. In $\triangle$ ACE, $\overline{CE} < \overline{AC}$ for the same reason. Therefore, $\overline{CE}$ is the shortest segment in the figure.

3. Given: $\overline{FB} \perp \overline{AC}$.
$\overline{AB} > \overline{BC}$.
Prove: $\overline{AF} > \overline{FC}$.
Locate X on $\overleftrightarrow{AC}$ so that BX = BC.



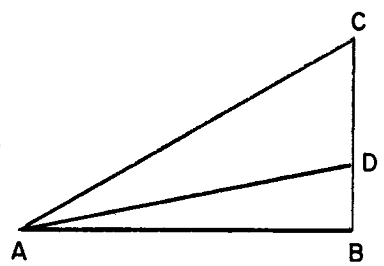
$\angle$ FXB $> \angle$ A by Theorem 7-1. $\angle$ C $= \angle$ FXB $> \angle$ A. Therefore, AF > CF.

4.

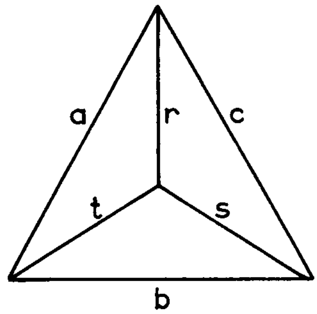
1. AF = HB.	1. Given.
2. BF = BF.	2. Identity.
3. AB = HF.	3. Subtraction in Statements 1 and 2.
4. $\triangle$ ABK $\cong \triangle$ HFQ.	4. Hypotenuse-Leg Theorem.
5. $\angle$ Q $\cong \angle$ K.	5. Corresponding parts.

Yes. There will be two triangles which are congruent by S.A.A.

217 5. Since AC > AB, m$\angle$ B > m$\angle$ C. $\angle$ ADC is an exterior angle of $\triangle$ ABD and so m$\angle$ ADC > m$\angle$B. Therefore, m$\angle$ ADC > m$\angle$ C. Hence, AC > AD.

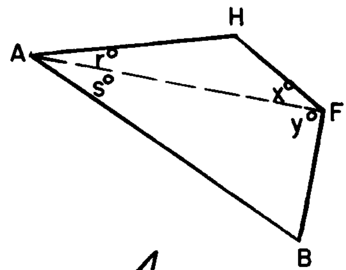


6. Let a, b and c be the lengths of the sides as shown.

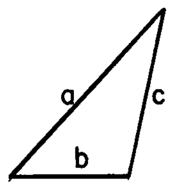


$t + r > a$.
$t + s > b$.
$r + s > c$.
$2(t + r + s) > a + b + c$.
$t + r + s > \frac{1}{2}(a + b + c)$.

7. $x > r$ since $\overline{HF}$ is the shortest segment. $y > s$ since $\overline{AB}$ is the longest segment.
$x + y > r + s$, by addition.
Therefore, $m\angle F > m\angle A$.



8. Let a be the length of the longest side of the triangle and b and c the lengths of the other sides.

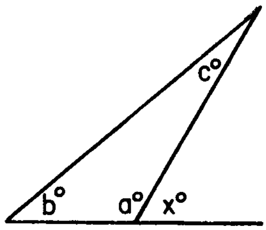


1. $a < b + c$.	1. Theorem 7-7.
2. $a = a$.	2. Identity.
3. $2a < a + b + c$.	3. Addition.
4. $a < \frac{a + b + c}{2}$	4. Division.

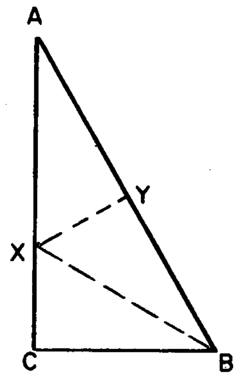
*9. In $\triangle ABF$ $w < a$ (Given that $AF > AB$). $c < w$ ($\angle AFB$ is an exterior angle of $\triangle FBH$.) And so, $c < a$. Also, $a < a + x$, which gives us that $m\angle A < m\angle ABH$. We now have $m\angle H < m\angle A < m\angle ABH$ and as a result we know that the three sides of $\triangle ABH$ are unequal.

*10. Since $m\angle CAB < m\angle ABG$ by Theorem 7-1, $m\angle C + m\angle CBA + m\angle CAB < 1 + m\angle CBA + m\angle ABG = 1 + 180 = 181$.

218 *11. The conclusion is obvious if each angle is acute so we suppose we have a figure as shown so that $a > 90$. Then $x < 90$ and $a + b + c < (a + x) + x < 180 + 90 = 270$.

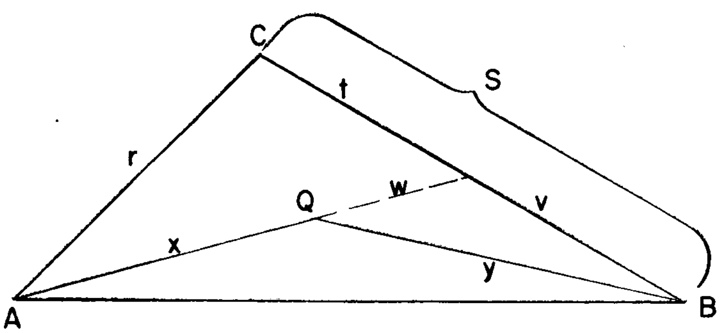


*12. $\overline{XB}$ bisects $\angle CBA$. $\overline{XY} \perp \overline{AB}$. $m\angle XBY = m\angle A$, $\angle XYA \cong \angle XYB$, $XY = XY$, therefore $\Delta AXY \cong \Delta BXY$ (S.A.A.) and $AY = BY$. $\Delta XBC \cong \Delta XBY$. (S.A.A.) and so $BC = BY$. Therefore, $AB = 2BC$.



13. We prove that $r + s > x + y$.

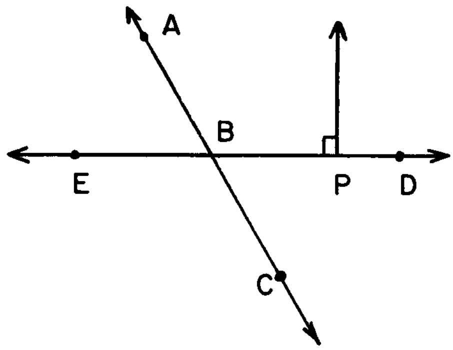
1. $r + t > x + w$. $w + v > y$.	1. Theorem 7-7.
2. $r + t + w + v > x + w + y$.	2. Addition.
3. $r + t + v > x + y$.	3. Subtraction.
4. $r + s > x + y$.	4. Statement 3 and the fact that $t + v = s$.



*14. If $\angle$ ABE is a right angle, P = Q = B. Hence, we suppose, with no loss of generality, that $\angle$ ABE is acute. Its vertical angle is also acute, so

m $\angle$ ABE < 90,

m $\angle$ CBD < 90.



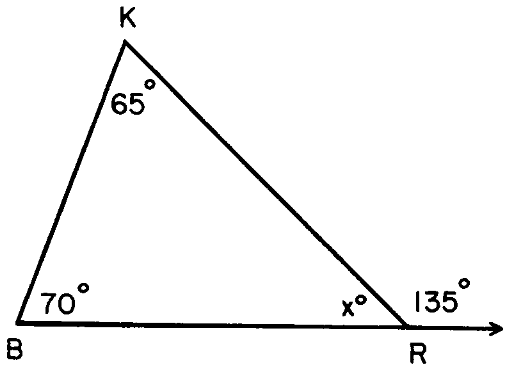
We show that P is on the same side of B as E by showing that it cannot be on the side with D. If P were on the side with D, $\angle$ ABE would be an exterior angle of Δ ABP. This leads to the contradiction that

m $\angle$ APB < m $\angle$ ABE.

However, this is impossible since m $\angle$ APB = 90 and m $\angle$ ABE < 90. Hence, P is on the same side of B as E. Similarly, it may be shown that Q is on the same side of B as D by considering Δ BCQ and showing that the assumption that it lies on the side with E leads to the contradiction that the acute exterior $\angle$ CBD has measure less than the right $\angle$ CBQ.

Illustrative Test Items for Chapter 7

1. Consider this figure and list correct responses to fill blanks below.

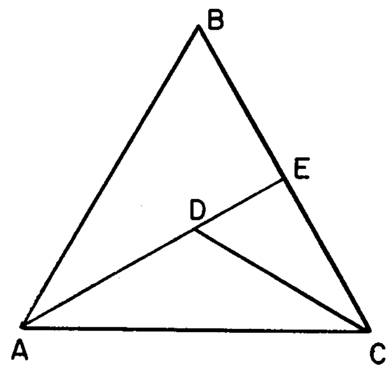


   a. x = _____.
   b. ______ is the longest side of Δ KBR.
   c. ______ is the shortest side of Δ KBR.

2. In Δ XYZ, if XY = 18, YZ = 10 and XZ = 15, which angle of the triangle has the largest measure?

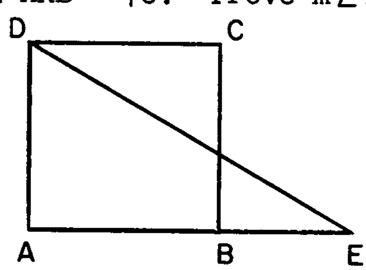
3. A triangle has sides of lengths x and x + y. Can the third side of the triangle be of length y? State a theorem to support your conclusions.

4. Given: Δ ABC.
   E is a point between B and C.
   D is a point between A and E.
   Prove: ∠ ADC > ∠ B.

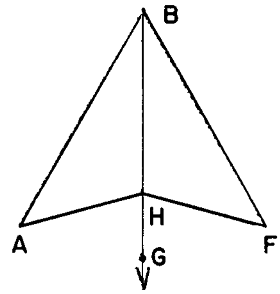


5. Given Δ ABC with median $\overline{RB}$ and m ∠ ARB = 73. Prove m∠A > m∠C.

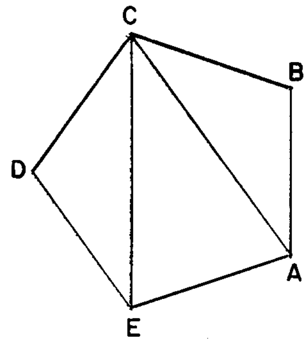
6. As shown in this figure, ABCD is a square and E is a point on $\overleftrightarrow{AB}$ such that B is between A and E.
   Prove: ED > AC.



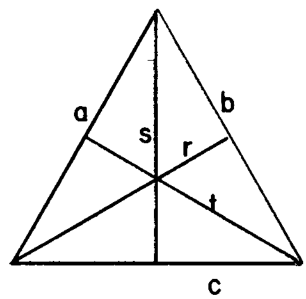
7. If, in this figure, $\overrightarrow{BH}$ bisects $\angle$ ABF and $\angle$ A $\cong$ $\angle$ F, prove the ray opposite $\overrightarrow{HB}$ bisects $\angle$ AHF.



8. Prove that the perimeter of the pentagon (shown in this figure) is greater than the perimeter of Δ ACE.



9. For the given figure prove that the sum of the altitudes is less than the perimeter of the triangle. (Use a, b, c, as lengths of the sides of the triangle and r, s, t, as lengths of the altitudes, as indicated.)



10. Indicate whether true or false.
    a. The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.
    b. The base angles of an isosceles triangle are acute.
    c. Any exterior angle of a triangle is larger than any interior angle of the triangle.
    d. If an angle of one triangle is larger than an angle of a second triangle, then the side opposite the angle in the first triangle is longer than the side opposite the angle in the second.
    e. A triangle can be formed with sides of lengths 351, 513, 162.

f. An altitude of a triangle lies in the interior of the triangle.

g. If AB > AC in Δ ABC, then m∠ C > m∠ B.

h. Two triangles are congruent if they have two angles and a side of one congruent to the corresponding parts of the other.

i. If the three angles of a triangle have unequal measures, then no two sides of the triangle are congruent.

j. A median of a triangle is perpendicular to the side to which it is drawn.

k. In Δ ABC both $\overline{AB}$ and $\overline{AC}$ can be perpendicular to $\overline{BC}$.

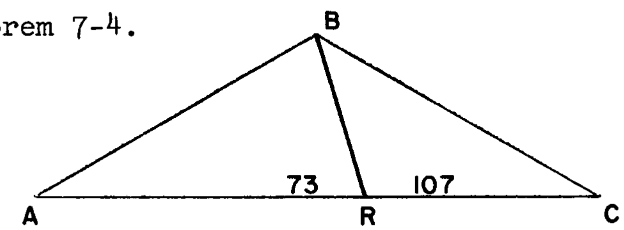
l. The shortest segment from P to $\overleftrightarrow{AB}$ is the perpendicular from P to $\overleftrightarrow{AB}$.

11. Prove: If D is a point between B and C, then $\overline{AD}$ is shorter than one of $\overline{AC}$, $\overline{AB}$.

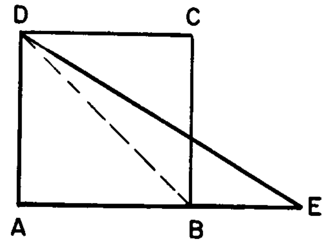
12. Prove that one of the congruent sides of an isosceles triangle is longer than the segment which connects the vertex with any point in the base.

Answers

1. x = 45. $\overline{KR}$. $\overline{KB}$.
2. ∠ Z.
3. No. The sum of the lengths of two sides of a triangle is greater than the length of the third side.
4. ∠ ADC is an exterior angle of Δ DEC and so ∠ ADC > ∠ DEC. ∠ DEC is an exterior angle of Δ ABE and so ∠ DEC > ∠ B. Therefore, ∠ ADC > ∠ B.
5. BC > AB by Theorem 7-8.
   m ∠ A > m ∠ C by Theorem 7-4.



6. Since $m\angle A = 90$, $\angle$ DBE is obtuse. By Corollary 7-1-1 $\angle$ E is acute. Then in $\triangle$ DBE, DE > DB by Theorem 7-5. $\triangle ABD \cong \triangle BCA$ by S.A.S. so AC = DB. Hence, DE > AC.



7. Let G be a point on $\overrightarrow{BH}$ beyond H so that $\overrightarrow{HG}$ is the ray opposite $\overrightarrow{HB}$. $\triangle ABH \cong \triangle FBH$ by S.A.A. Theorem. Then $\angle AHB \cong \angle FHB$ and hence $\angle AHG \cong \angle GHF$ since supplements of congruent angles are congruent.

8. $\left.\begin{array}{l} ED + DC > EC \\ AB + BC > AC \end{array}\right\}$ Theorem 7-7.

   EA = EA.

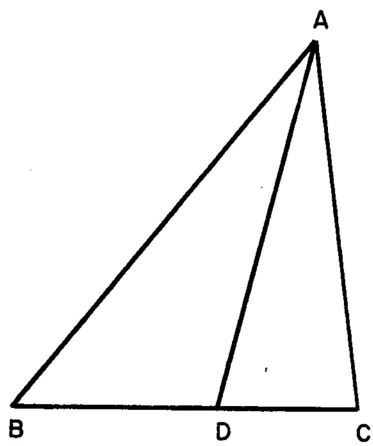
   ED + DC + AB + BC + EA > EC + AC + EA, by addition.

9. $r < c$, $t < b$, $s < a$ by Theorem 7-6, then $r + t + s < a + b + c$ by addition.

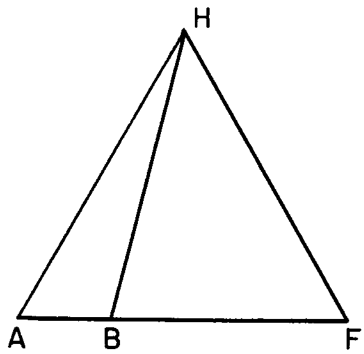
10.

a.	T	e.	F	i.	T
b.	T	f.	F	j.	F
c.	F	g.	T	k.	F
d.	F	h.	T	l.	T

11. If $\overline{AD} \perp \overline{BC}$ then AD < AB and AD < AC by Theorem 7-6. If $\overline{AD}$ is not perpendicular to $\overline{BC}$ then either $\angle$ ADB or $\angle$ ADC must be obtuse. Say $\angle$ ADB is obtuse, then $\angle$ ADC is acute. But $m\angle ADC > m\angle B$. Hence, $\angle$ B is acute. Thus, AD < AB by Theorem 7-5.



12. Given: Δ AHF with AH = FH and B a point between A and F.
To prove: AH > HB.



Proof:

1. m ∠ HBA > m ∠ F.	1. Theorem 7-1.
2. m ∠ A = m ∠ F.	2. Base angles of an isosceles triangle are congruent.
3. m ∠ HBA > m ∠ A.	3. Substitution.
4. AH > BH.	4. Theorem 7-5.

Chapter 8

PERPENDICULAR LINES AND PLANES IN SPACE

This is a good time to ask yourself whether it is likely that your class will cover all the topics in the text. You will want to plan ahead to give your class a suitable program. You could make, rather quickly if necessary, an intuitive presentation of the propositions of Chapter 8 by using familiar physical objects. Having students draw some figures after looking at simple models will improve their ability to handle three-dimensional problems.

On the other hand, deductive work in three-space may seem more important to you than many alternatives. Part of the time you plan to allot to deductive work can be spent on proofs in three-space, even if this entails omitting some deductive work in two-space.

219 It is worth spending time to make the basic definition of the chapter meaningful. A sizeable model will make your demonstration more effective. Use the floor as a plane, several pointers for concurrent lines in the plane, and a window pole for the perpendicular. Have students concentrate on one particular pointer. Move the pole to show that the pole can be in many positions, even in the plane, and be perpendicular to the particular pointer. But the pole - in all but one position - is not perpendicular to the other pointers. When the pole is perpendicular to all of the pointers, it is perpendicular to the plane. If some students discover the idea of Theorem 8-3 at this time, that's fine!

While such demonstrations can do much to assist students in understanding spatial relationships, a most effective means is the assigning of smaller models to be constructed by each student. Coat hangers, thin wire, straws, string and cardboard can be used to make models of the next theorems to be studied. (See Problem Set 8-1a, Problem 10.)

One particularly meaningful device which students can make at an early stage is the following. Each student has a piece of cardboard on which he draws a segment and marks a point on that segment. Next he inserts several common pins such that each pin is perpendicular to the segment at the point. The teacher can check each model at a glance. The model helps to illustrate the basic definition of Section 8-1 and Theorem 8-5.

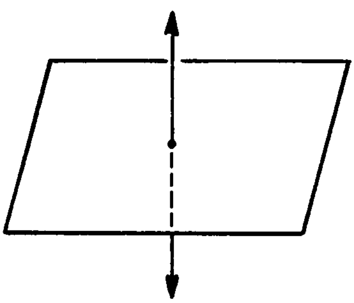
Some excellent materials, mainly sticks and connectors, for constructing models in three-space are available from suppliers of scientific and mathematics equipment. Many teachers find these to be advantageous over ready-made models.

Problem Set 8-1

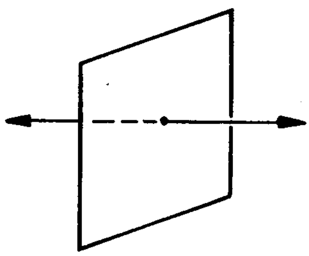
220 1. a. Yes.

b. No, there would be points in space which are not in plane B.

2. a.

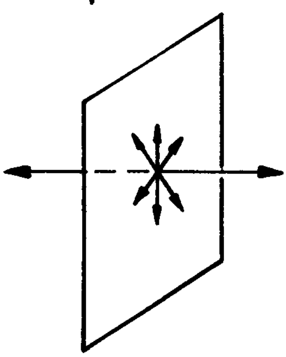


b.



c. Yes.

3.



Each of the three lines is perpendicular to the original line.

{page 220}

4. The statement is true. (Refer to the discussion of using the word <u>if</u> in definitions, Chapter 2, page 41 of the text.)

5. $\angle$ ABR, $\angle$ ABS, $\angle$ TBA.

6. No. The definition requires that the line be perpendicular to <u>every</u> line containing Q and lying in E.

7. a. Yes. b. $\angle$ TSP and $\angle$ TSR.

8. a. Yes, three points are always coplanar.

   b. Not necessarily.

*9. a.

1. PB = QB.	1. Given.
2. PA = QA.	2. Given.
3. BA = BA.	3. Identity.
4. Δ PAB $\cong$ Δ QBA.	4. S.S.S.
5. $\angle$ PAB $\cong$ $\angle$ QAB.	5. Corresponding parts.
6. AX = AX.	6. Identity.
7. Δ PAX $\cong$ Δ QAX.	7. Statements 2, 5, 6 and S.A.S.
8. PX = QX.	8. Corresponding parts.

   b. No.

---

222 As a model for Theorem 8-1 you can use the tip of a light fixture and a spot on the floor as points, and a window pole as a line. You can even tag the pole with A, B, and a movable X.

Problem Set 8-2a

224 1. Yes. Statement of Theorem 8-1. 6.

2. Yes. Yes. Yes. Statement of Theorem 8-1.

---

225 Some students should enjoy making a model for Theorem 8-2. We suggest a thin stick punched through a sheet of cardboard, with different colored strings leading from the ends of the stick to A, B, and C. Then use thumb-tacks for points X, Y and Z.

226 You can devise a model for Theorem 8-3 by punching a pointer through a sheet of cardboard to represent L and E. Then lay pencils on the cardboard to represent L_1, L_2, and L_3.

Problem Set 8-2b

227 1. This follows directly from Theorem 8-2.

2. The line of intersection is perpendicular to the floor. Many, in fact, every line in the floor going through the point at which L intersects the floor will be perpendicular to L. No. It is perpendicular only to lines of the floor that contain the point of intersection of L and the floor.

228 3. a. Three. The sides of the square all lie in a plane. $\overline{AB}$ and $\overline{FB}$ determine another plane, and $\overline{AB}$ and $\overline{BH}$ determine a third.

b. We know $\overline{BH} \perp \overline{HR}$, $\overline{HR} \perp \overline{RF}$, $\overline{RF} \perp \overline{FB}$, $\overline{BF} \perp \overline{BH}$ (from the square) and $\overline{FB} \perp \overline{AB}$ (Given.). From the last two of these we note that one line, $\overleftrightarrow{FB}$, is perpendicular to two other lines at their point of intersection so we know that $\overleftrightarrow{FB} \perp$ plane ABH. It is also true that $\overline{RH} \perp$ plane ABH, but the student probably cannot prove this now.

4. a. Three. Planes ABF, RHB, and AHRF.

b. $\overline{HB} \perp \overline{RH}$. (Given.) $\overline{HB} \perp \overline{AF}$. (Theorem 6-2 and Postulate 1.) Therefore, $\overline{HB} \perp$ plane AHRF. This follows from Theorem 8-3.

229 5.

1.	$\overline{FB} \perp$ plane P.	1.	Given.
2.	$\overline{FB} \perp \overline{AB}$.	2.	Definition of a line perpendicular to a plane.
3.	$m\angle FBA = m\angle FBR = 90$.	3.	Definition of perpendicular lines.
4.	BR = BA.	4.	Given.
5.	FB = FB.	5.	Identity.
6.	$\triangle ABF \cong \triangle RBF$.	6.	S.A.S.
7.	FA = FB.	7.	Corresponding parts.
8.	$\angle FAR \cong \angle FRA$.	8.	Base angles of an isosceles triangle.

*6. Yes.

1.	AT = TF.	1.	Property of the edge of a cube.
2.	AB = BF.	2.	Same as Reason 1.
3.	BR = BL.	3.	Given.
4.	AR = FL.	4.	Subtraction, Steps 2 and 3.
5.	$\triangle ATR \cong \triangle FTL$.	5.	S.A.S.
6.	TR = TL.	6.	Definition of congruence.
7.	$\overline{KT} \perp \overline{AT}$. $\overline{KT} \perp \overline{FT}$.	7.	Property of a cube.
8.	$\overline{KT} \perp$ plane ABFT.	8.	Theorem 8-3.
9.	$\overline{KT} \perp \overline{RT}$ and $\overline{KT} \perp \overline{TL}$.	9.	Definition of a line perpendicular to a plane.
10.	$\triangle KTR \cong \triangle KTL$.	10.	S.A.S.
11.	KR = KL.	11.	Corresponding parts.

230 7.

1.	$\overleftrightarrow{WX} \perp \overleftrightarrow{RQ}$.	1.	Definition of a line perpendicular to a plane.
2.	$\overleftrightarrow{RQ} \perp \overleftrightarrow{AB}$.	2.	Given.
3.	$\overleftrightarrow{RQ} \perp E$.	3.	Theorem 8-3.

231 By the time you reach Theorem 8-4 it might be best to proceed without a complete or elaborate model. Students should be encouraged to perceive spatial relationships in a diagram rather than to become completely dependent on spatial models.

231 You may use a spoked wheel and axle to make Theorem 8-5 intuitively familiar: any line perpendicular to the axle at the hub must be in the plane of the wheel.

232 Proof of Theorem 8-7

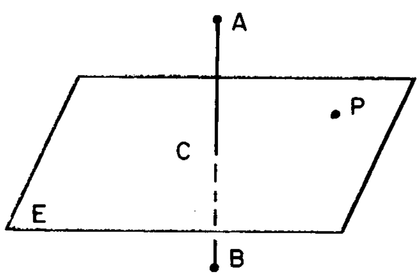
The perpendicular bisecting plane of a segment is the set of all points equidistant from the end-points of the segment.

Restatement: Let E be the perpendicular bisecting plane of $\overline{AB}$. Let C be the mid-point of $\overline{AB}$. Then

(1) If P is in E, then $PA = PB$, and

(2) If $PA = PB$, then P is in E.

Proof of (1): If $P = C$, then we already know that $PA = PB$. If $P \neq C$, then $\overleftrightarrow{CP}$ lies in E by Postulate 6, and $\overleftrightarrow{AB} \perp \overleftrightarrow{CP}$ by the definition of a line perpendicular to a plane. It follows that $\angle ACP \cong \angle BCP$, and, since $CA = CB$ and $CP = CP$, we have $\triangle ACP \cong \triangle BCP$ by S.A.S. Therefore, $PA = PB$.

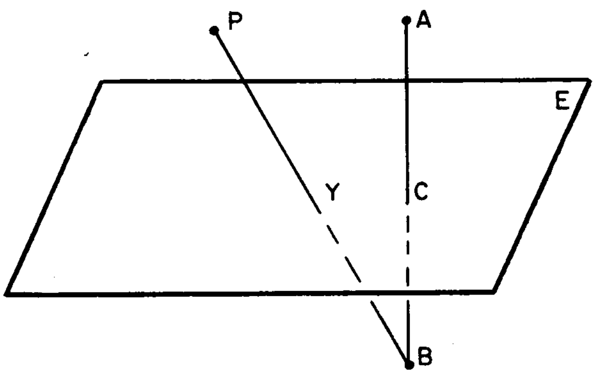


Proof of (2): If $P = C$, then certainly P is in E. If $P \neq C$, then $\Delta ACP \cong \Delta BCP$ by S.S.S. Theorem. Thus $\angle ACP \cong \angle BCP$ and $\overleftrightarrow{CP} \perp \overleftrightarrow{AB}$. E contains $\overleftrightarrow{CP}$ by Theorem 8-5, and P lies in E.

Alternate proof of (1): Let C be the mid-point of $\overline{AB}$. If $P = C$, then certainly $PA = PB$. If $P \neq C$, then $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$ (in plane ABP) and therefore $PA = PB$ by Theorem 6-2.

Alternate proof of (2): If $P = C$, then certainly P is in E. If $P \neq C$, then $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$ (in plane ABP). Since E contains $\overleftrightarrow{CP}$ by Theorem 8-5, P lies in E.

The proof of part (2) of Theorem 8-7 as given above requires that Theorem 8-5 be proved previously. A simple indirect proof uses Theorem 8-1 and the Space Separation Postulate in the following way:



Given P such that $PA = PB$. Suppose P does not lie in E. Then it lies in one of the two half-spaces into which E separates space. A and B lie in opposite half-spaces, since $\overline{AB}$ intersects E at C, by hypothesis. Then P is in the half-space opposite to either A or B, say B. Then $\overline{PB}$ meets E in a point Y. By (1), Y is equidistant from A and B, and by hypothesis, P is equidistant from A and B. Then by Theorem 8-1, B is equidistant from A and B! This absurdity implies that our supposition is false, and so P is in E.

Problem Set 8-2c

233 1. a. Infinitely many.

b. One.

2. Yes. Yes. No.

3. The conclusion follows directly from Theorem 8-5.

4. Points W, X, Y and Z are given equidistant from the ends of $\overline{AB}$. By Theorem 8-7, they all belong to the perpendicular bisecting plane of $\overline{AB}$ and are therefore coplanar.

234 5. a. $\overline{BW}$. $\overline{BK}$. $\overline{BR}$. 90. $\angle$ BKF.

b. Not necessarily. W, K and R could be any points in E.

*6.

1. There exists a plane E' perpendicular to L at M.	1. Theorem 8-4.
2. If E = E', each line in E' through M is perpendicular to L.	2. Definition of a line perpendicular to a plane.
3. If E $\neq$ E', the intersection of E and E' is a line L'.	3. Postulate 8.
4. L $\perp$ L'.	4. Definition of a line perpendicular to a plane.

234 The proof of Theorem 8-8 uses the word "let" in two somewhat different senses. "Let Lines L_1 and L_2 be perpendicular" means "Call the two given perpendiculars L_1 and L_2". "Let M be the mid-point of $\overline{AB}$" means "Consider the mid-point of $\overline{AB}$, and call it M". (The mid-point exists by Theorem 2-5).

Review Problems

236 1. a. F. e. T.
b. F. f. F.
c. F. g. T.
d. T. h. T.

2. AR > RB. $m\angle B > m\angle A$ ($m\angle B = 90$).

3. Theorem 8-8. Yes. Yes.

237 4. Yes. No. No. Yes. No.

5. Theorem 6-3.

6. Only one. $\overleftrightarrow{MQ}$ and $\overleftrightarrow{WF}$ are coplanar by Theorem 8-8, so that M, Q, W and F are coplanar. If two points are in a plane the line containing them is in the same plane. Hence $\overleftrightarrow{MW}$ and $\overleftrightarrow{QF}$ are coplanar with $\overleftrightarrow{MQ}$ and $\overleftrightarrow{WF}$.

7. a. Three. Plane ABF, plane RHB and plane RHF. Two intersecting lines determine a plane.

b. $\overline{AF} \perp \overline{RH}$ and $\overline{AF} \perp \overline{BH}$ so, $\overline{AF} \perp$ plane RHB by Theorem 8-3.

238 8. $\Delta XAP \cong \Delta XBP$ by S.A.S.
Hence XA = XB. Similarly we know XB = XC.
Hence X is equidistant from A, B, C.

9.

1. $L \perp$ plane ABC.	1. Given.
2. $L \perp \overline{QA}, \overline{QB}, \overline{QC}$.	2. Definition of a line perpendicular to a plane.
3. $\overline{PQ} \cong \overline{PQ}$.	3. Identity.
4. PA = PB = PC.	4. Given.
5. $\Delta PAQ \cong \Delta PBQ \cong \Delta PCQ$.	5. Hypotenuse-Leg Theorem.
6. QA = QB = QC.	6. Corresponding parts.
7. For any point $X \neq Q$ on L, $\Delta XAQ \cong \Delta XBQ \cong \Delta XCQ$.	7. S.A.S.
8. XA = XB = XC.	8. Corresponding parts.

10. On the ray opposite to $\overrightarrow{QB}$ let R be the point such that QR = QB. Then $\triangle PQR \cong \triangle PQB$ by S.A.S. $\therefore$ PR = PB. $\overline{AP} \perp \overline{PR}$, and $\overline{AP} \perp \overline{PB}$ since $\overline{AP} \perp$ plane PBC. Therefore, $\triangle APR \cong \triangle APB$ (S.A.S.) and AR = AB. $\therefore$ $\overline{AQ} \perp \overline{RB}$, $(\overline{AQ} \perp \overrightarrow{BC})$ by Theorem 6-2 and Postulate 1.

239 11. Connect A with X, the point of $\overrightarrow{BF}$ such that BX = BH. Then $\triangle ABH \cong \triangle ABX$ (S.A.S.) and AX = AH. Since $\overline{AB} \perp \overline{BF}$, $m\angle ABF > m\angle F$, and since $\angle AXF$ is an exterior angle of $\triangle ABX$, $m\angle AXF > m\angle ABX > m\angle F$. Then AF > AX and, substituting, we have AF > AH.

12. Suppose $\overrightarrow{AB}$ were perpendicular to each of the three rays $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$. Then by Theorem 8-3 and 8-5, the three rays would be coplanar. If $\overrightarrow{AD}$ and $\overrightarrow{AE}$ were each perpendicular to $\overrightarrow{AC}$ and all were in a plane, then $\overrightarrow{AD}$, $\overrightarrow{AE}$ would be opposite rays and not perpendicular. Hence each ray cannot be perpendicular to the other three.

13.

1. $\overleftrightarrow{YB} \perp n$.	1. Given.
2. $\overleftrightarrow{YP} \perp \overleftrightarrow{AB}$, or $\overleftrightarrow{AB} \perp \overleftrightarrow{YB}$.	2. Definition of a line perpendicular to a plane.
3. $\overleftrightarrow{XB} \perp m$.	3. Given.
4. $\overleftrightarrow{XB} \perp \overleftrightarrow{AB}$, or $\overleftrightarrow{AB} \perp \overleftrightarrow{XB}$.	4. Reason 2.
5. $\overleftrightarrow{AB} \perp E$.	5. Statements 2, 4 and Theorem 8-3.

Illustrative Test Items for Chapter 8

A. 1. Can the distance from a given point to a given plane vary?

2. Identify the set of points which are equidistant from two points A and B?

3. Through a given point not in a plane, how many lines can be perpendicular to the plane?

4. At a point on a line how many lines can be perpendicular to the line?

5. At a point on a line how many planes can be perpendicular to the line?

6. Is it possible for a line which intersects a plane in only one point not to be perpendicular to any line in the plane?

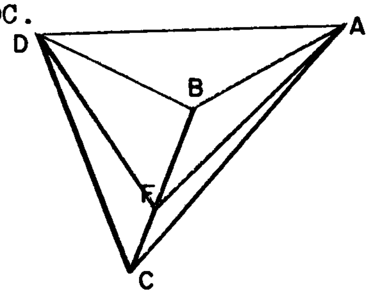
7. Can a line be perpendicular to a line in a plane and yet not be perpendicular to the plane?

8. Three points A, B, C are each equidistant from two points P and Q. Fill in the blanks to make true statements.

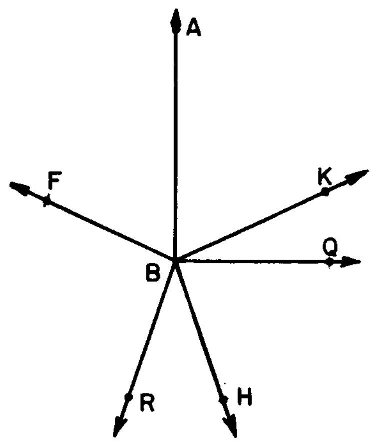
   a. If A, B, C are collinear then ____________ is equidistant from P and Q.

   b. If A, B, C are not collinear then ____________ is equidistant from P and Q.

B. 1. Points A, B, C, and D are not coplanar. ΔABC is isosceles with $AB = AC$. ΔDBC is isosceles with $DB = DC$. F is the mid-point of $\overline{BC}$. In the figure at least one segment is perpendicular to a plane. What segment? What plane?



2. Given in this figure that $\overleftrightarrow{BK} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{QB} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{HB} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{RB} \perp \overleftrightarrow{AB}$ and $\overleftrightarrow{BF} \perp \overleftrightarrow{AB}$.

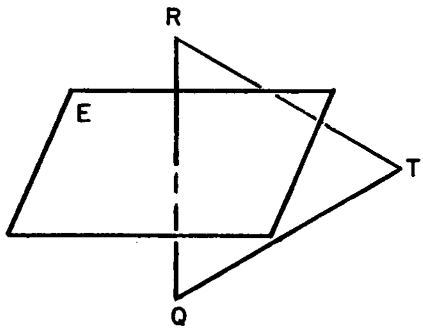


a. $\overleftrightarrow{BK}$ and $\overleftrightarrow{AB}$ determine a plane ABK. IS $\overleftrightarrow{BQ}$ perpendicular to plane ABK? If your answer was "yes", state a theorem that supports your conclusion.

b. Do $\overleftrightarrow{FB}$, $\overleftrightarrow{RB}$, $\overleftrightarrow{HB}$ all lie in plane KBQ? Explain.

c. There will be ________ different planes determined by the given lines.

3. In this figure, plane E bisects $\overline{RQ}$ and $E \perp \overrightarrow{RQ}$. Also RT = QT. Explain why T lies in plane E.



C. Indicate whether true or false:

1. A line perpendicular to a plane is perpendicular to every line in the plane.

2. If a line is perpendicular to two lines of a plane it must be perpendicular to the plane.

3. Through a point on a plane only one plane can be passed.

4. There are infinitely many lines perpendicular to a given line at a given point on the line.

5. Two lines perpendicular to the same plane are coplanar.

6. Through a point on a line two planes can be passed perpendicular to the line.

7. Thirteen points each equidistant from the end-points of a segment are coplanar.

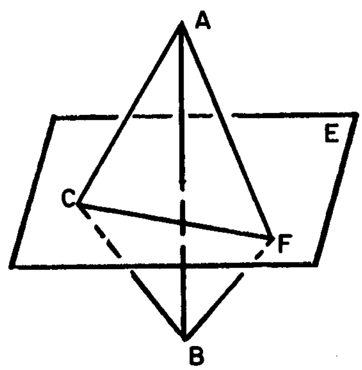
8. If two lines L_1 and L_2 are each perpendicular to line L, at a given point of L, there is a plane containing L_1 and L_2 that is perpendicular to L.

9. All lines perpendicular to a line at a given point of the line are coplanar.

10. A line perpendicular to a line in a plane is perpendicular to the plane.

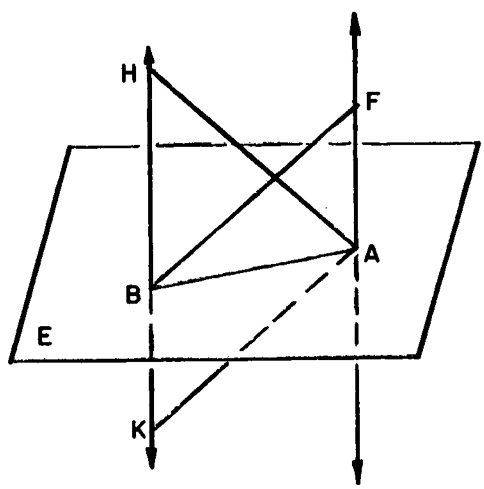
11. If $\overleftrightarrow{AB}$ and plane E are each perpendicular to $\overleftrightarrow{FH}$ at point P, then $\overleftrightarrow{AB}$ lies in plane E.

D. 1. In this figure E is the perpendicular bisecting plane of $\overline{AB}$. If $\overline{CF}$ lies in E and CF = CB = FB, prove Δ ACF is equilateral.



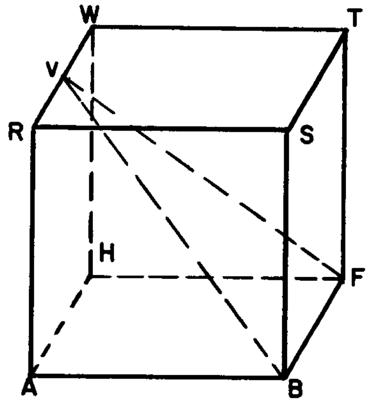
2. Given in this figure:
$\overleftrightarrow{HK} \perp$ E at B.
$\overleftrightarrow{FA} \perp$ E at A.
HA = FB = AK.

Prove: Δ HBA, Δ FAB, and Δ KBA are in one plane and are congruent to each other.



3. V is the mid-point of edge $\overline{RW}$ of the cube shown in this figure.

Prove VB = VF.



Answers

A. 1. No, it is the length of the unique perpendicular segment from the point to the plane.

2. The perpendicular bisecting plane of $\overline{AB}$.

3. One.

4. Infinitely many.

5. One.

6. No.

7. Yes.

8. a. If A, B, C are collinear then each point of the line containing A, B, C is equidistant from P and Q.

   b. If A, B, C are not collinear, then each point of the plane containing A, B, and C is equidistant from P and Q.

B. 1. $\overline{BC} \perp$ plane DFA.

2. a. No. $\overline{BQ}$ cannot be proved perpendicular to the plane ABK on the basis of the information given.

b. Yes, Theorem 8-5.

c. Six; ABK, ABQ, ABH, ABR, ABF, and the plane perpendicular to $\overleftrightarrow{AB}$ at B.

3. This follows from Theorem 8-7.

C. 1. F. 7. T.

2. F. 8. T.

3. F. 9. T.

4. T. 10. F.

5. T. 11. T.

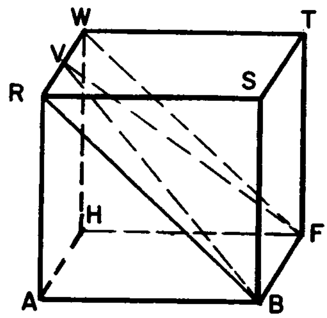
6. F.

D. 1.

1. AC = CB. AF = FB.	1. Theorem 8-7.
2. AC = CF = AF.	2. Hypothesis and Step 1.
3. $\triangle$ ACF is equilateral.	3. Definition of equilateral triangle.

2. $\overleftrightarrow{HK}$ and $\overleftrightarrow{FA}$ are coplanar (Theorem 8-8). Since all vertices of $\triangle$ HBA, $\triangle$ FAB and $\triangle$ KBA are points of these lines, the triangles are in one plane. $\angle$ HBA, $\angle$ KBA and $\angle$ FAB are right angles (Definition of a line perpendicular to a plane). BA = BA (Identity). $\triangle$ HBA $\cong$ $\triangle$ FAB $\cong$ $\triangle$ KBA (Hypotenuse-Leg Theorem).

*3.



Use auxiliary segments $\overline{RB}$ and $\overline{WF}$.

1. $\triangle RAB \cong \triangle WHF$.	1. S.A.S.
2. RB = WF.	2. Corresponding sides.
3. RV = VW.	3. Definition of mid-point.
4. $\angle VRB$ and $\angle VWF$ are right angles.	4. $\overleftrightarrow{RW} \perp$ planes of faces RABS and WHFT.
5. $\triangle RVB \cong \triangle WVF$.	5. S.A.S.
6. VB = VF.	6. Corresponding sides.

Chapter 9

PARALLEL LINES IN A PLANE

In this chapter we introduce the Parallel Postulate and the familiar theorems on parallels and quadrilaterals. The treatment is not significantly different from that of most traditional texts, except in this respect: The explicit use of the postulates and theorems of our early chapters and the careful formulation of definitions.

By this time the student should be quite adept at making proofs. Consequently, this chapter simply states the easier theorems and leaves their proofs for the student to accomplish. Proofs not supplied in the text are provided in this commentary. Please note, however, that students may often discover proofs different from the one given here, or in the text, and, of course, such proofs should receive appropriate recognition and acceptance.

As we proceed to study more complicated material we shall relax the degree of precision with which we treat it. We shall sometimes state definitions which are not wholly precise and give proofs that are not logically complete, with the expectation that they will be understood with the aid of diagrams. In succeeding chapters this is done more extensively. In the present chapter we point out several instances of unprecise treatment and indicate appropriate clarification.

The discussion of parallel lines in a plane, though by no means difficult, encompasses probably the most significant property of Euclidean geometry, namely, the "Parallel Postulate", stated on page 262. By way of introduction ask the students to tell what they mean by parallel lines. The answers will no doubt vary, and some will probably be incorrect. Most answers will probably be descriptions, rather than definitions. It is hoped that from a discussion of this sort the class will get the feeling that they are working with something that is intuitively very simple, but that at the same time the concept of parallelism is not one that can easily be "pinned down" by the student.

241 Point out to the students the definition of parallel lines gives two conditions that must be met by the lines, (1) they must lie in the same plane and (2) they must not intersect. Ask the student for an example of two lines that satisfy condition (2), but fail to satisfy condition (1) and hence are not parallel. Skew lines is the example.

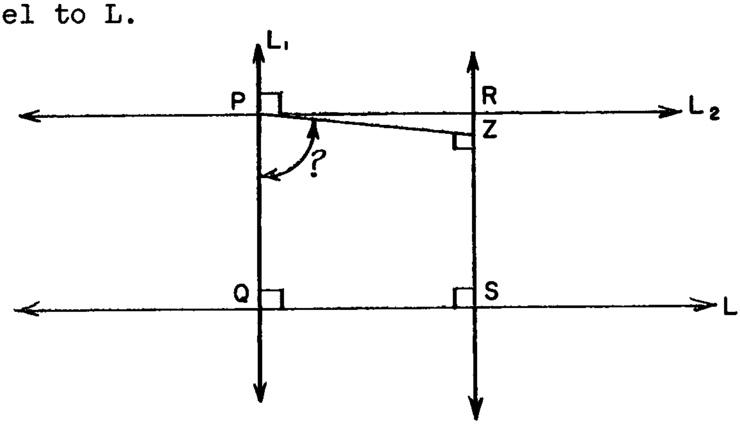
Remind the students that parallel lines do not meet. You will sometimes hear the expression: "Parallel lines meet at infinity". This does not mean that the lines do meet. Mathematicians abhor exceptions, for example, two lines do not always meet in the Euclidean plane, and just as it is convenient to introduce complex numbers into algebra so that every quadratic equation has a root, so it is convenient to adjoin to the points of the plane, certain "ideal" points so that we can say two lines always meet.

Notice, however, that such lines are no longer Euclidean lines. To each Euclidean line we adjoin an ideal point to form a new kind of line, called a projective line, that is no longer a Euclidean line. This is done in such a way that the same ideal point is adjoined to each line of a family of parallel lines. If two Euclidean lines are parallel then their associated projective lines meet in an ideal point. If two Euclidean lines are not parallel they meet in a point P and their associated projective lines meet in the same point P. This avoids an exception, but all the properties of real points do not carry over automatically to ideal points. When we say two projective lines meet at an ideal point, it follows that their associated Euclidean lines do not meet at all. If we adjoin these ideal points to the set of real points in the Euclidean plane, we get a new "plane", which has different properties from the Euclidean plane, and which we may call a "projective plane" in the sense that "point", "line", and "plane" would satisfy the set of incidence postulates usually made for projective geometry. But this is not the geometry we are studying; in Euclidean Geometry we do have parallel lines, in Projective Geometry there are no parallel lines.

242 Theorem 9-2 gave us <u>one</u> method for constructing a line parallel to another line through an external point. The method was used in Theorem 9-3 to prove the existence of at least one line parallel to a given line from a point not on the line.

Some enterprising students will feel that Theorem 9-3 establishes uniqueness as well as existence of L_2, especially in light of the paragraph following the proof. After all, Theorem 6-1 assures that L_2 as a perpendicular to L_1 at P is unique. Should this arise you may counter with a statement of this sort: "If this seems astonishing to you, perhaps you are reading more meaning in Theorem 9-2 than is actually there. Notice that Theorem 9-2 does <u>not</u> say two lines in a plane are parallel <u>only</u> if they are <u>both</u> perpendicular to the same line. Is it possible then that two lines could also be parallel under some other conditions?"

If more discussion seems necessary you may decide to present the following: Let the figure be that of Theorem 9-3. From point R on L_2 drop a perpendicular to L, meeting L at S. Note that we do <u>not</u> know that $\overleftrightarrow{RS} \perp L_2$. From P make $\overleftrightarrow{PZ} \perp \overleftrightarrow{RS}$. Now we have $\overleftrightarrow{PZ} \parallel L$ and $L_2 \parallel L$ by Theorem 9-2. We seem to have two lines through P parallel to L.



{page 242}

The student will probably claim that $\overleftrightarrow{PR} \perp \overleftrightarrow{RS}$ and therefore $\overleftrightarrow{PZ}$ and L_2 coincide (Theorem 6-3). While you may agree with him that this sounds promising, ask him to prove that $\overline{PR} \perp \overline{RS}$, the fact his argument is based on. Whatever he may refer to as convincing evidence from his general store of knowledge you easily can maintain the essential point of the whole discussion: that nothing in our previous postulates or theorems will disprove our argument. The sort of reasons which refute it - the sum of the measures of the angles of a quadrilateral is 360, or of a triangle is 180, alternate interior angles (Theorem 9-8), corresponding angles (Theorem 9-9), and so on - have not been proved yet (and in fact, can not be until the Parallel Postulate is assumed).

You would probably not want to go further into this with your class, especially at this time - and probably not even this far. But we should state the point to this discussion, for the reader, at least. The point is that the statements which would refute the above argument are all logically equivalent to Postulate 16. Neither Postulate 16 nor any of these equivalent statements is deducible as a theorem from Postulates 1-15. It was the discovery of this fact that finally led geometers to the realization that some postulate of parallelism is necessary. (See Talks on Introduction to Non-Euclidean Geometry and on Miniature Geometries.)

245 Notice that we give a precise definition of alternate interior angles rather than a "definition" in terms of a picture. Observe that our definition depends on the separation concept as developed in Chapter 3.

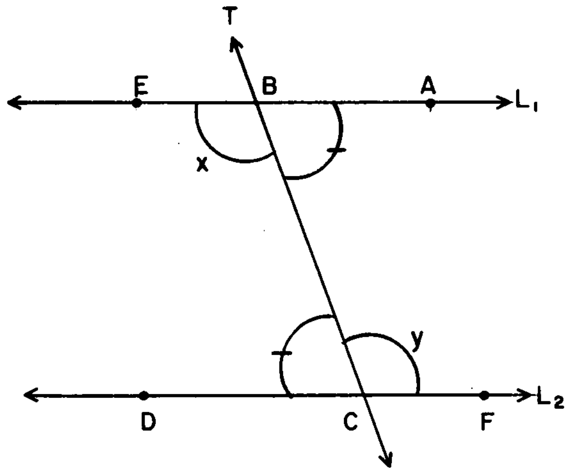
246 Proof of Theorem 9-4

Given a transversal to two lines, if one pair of alternate interior angles are congruent, then the other pair of alternate interior angles are also congruent.

Given: Lines L_1 and L_2 cut by transveral T such that $\angle ABC \cong \angle BCD$.

To Prove: $\angle x \cong \angle y$.

By the Supplement Postulate $\angle ABC$ and $\angle x$ are supplementary, as are $\angle BCD$ and $\angle y$. Since $\angle ABC \cong \angle BCD$, then $\angle x \cong \angle y$, because supplements of congruent angles are congruent.



Problem Set 9-1

248 1. a. No. b. No.

2. They do not intersect, they are both perpendicular to a third line, they form alternate interior angles with a transversal.

   (Note: The third condition includes the second as a special case.)

3. No.

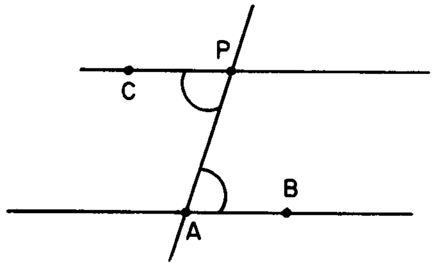
4. Not necessarily.

5. a. No, the 80° angles are not alternate interior angles, and the alternate interior angles are not equal.

   b. Two sizes: 80° and 100°.

6. $L_1 \parallel L_2$, $M_1 \parallel M_2$.

249 7. Select any two points A, B on L. Draw $\overleftrightarrow{PA}$. Draw $\angle CPA \cong \angle BAP$ so that C and B are on opposite sides of $\overleftrightarrow{PA}$. Then $\overleftrightarrow{CP} \parallel L$ by Theorem 9-5.



8. a. Yes. b. No. c. Yes. d. Yes. e. Yes, since a line containing the center of the earth is perpendicular to certain other lines containing the center.
f. No. g. Yes. h. Yes.

9. Yes. (Such lines are called <u>skew</u> lines.)

10. $\triangle ABD \cong \triangle BAC$ by S.A.S. Then DB = CA. Then $\triangle DCB \cong \triangle CDA$ by S.S.S. and $m\angle BCD \cong m\angle ADC$. (It is not possible to prove that $\angle BCD$ and $\angle ADC$ must be right angles. Attempts to do so suggest the need for some further postulate.)

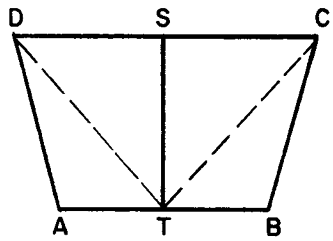
250 11. Proof: $\triangle APR \cong \triangle PBQ \cong \triangle RQC \cong \triangle QRP$ by S.S.S. By corresponding parts $m\angle a = m\angle A$, $m\angle b = m\angle B$ and $m\angle c = m\angle C$. Since the sum of the measures of $\angle a$, $\angle b$ and $\angle c$ is 180 by Postulates 13 and 14, the sum of the measures of $\angle A$, $\angle B$ and $\angle C$ is 180.

It may seem surprising that we can prove that the sum of the measures of the angles of $\triangle ABC$ is 180 before we have introduced the Parallel Postulate. In this Problem the hypothesis assumes the existence of a triangle in which the length of each segment joining the mid-points of two sides is one-half the length of the third side. This cannot be proved before assuming the Parallel Postulate. We should note, however, that if we do assume that such a triangle exists, and from this show that the sum of the measures of the angles is 180, we can prove the Parallel Postulate. (See the commentary above on equivalence of statements to the Parallel Postulate. See, also, Talks on Introduction to Non-Euclidean Geometry, Corollary 7.)

250 12. Proof: $\triangle PAR \cong \triangle QAR$ by S.A.S. Then $\angle ARP \cong \angle ARQ$ and $\overleftrightarrow{AR} \perp \overleftrightarrow{PQ}$. By a similar proof using $\triangle ABD$ and $\triangle ACD$, $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$. Then $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$ by Theorem 9-2.

(Note: A proof based on isosceles triangles without drawing $\overline{AD}$ is also possible.)

13.



1. $\triangle DAT \cong \triangle CBT$.	1. S.A.S.
2. $DT = CT$.	2. Corresponding parts.
3. $m\angle DTA = m\angle CTB$.	3. Corresponding parts.
4. $\triangle DST \cong \triangle CST$.	4. S.S.S.
5. $m\angle DTS = m\angle CTS$.	5. Corresponding parts.
6. $m\angle STA = m\angle STB$.	6. Addition.
7. $\overleftrightarrow{ST} \perp \overleftrightarrow{AB}$.	7. Definition of perpendicular lines.
8. $m\angle TSD = m\angle TSC$.	8. Corresponding parts.
9. $\overleftrightarrow{ST} \perp \overleftrightarrow{CD}$.	9. Definition of perpendicular lines.
10. $\overline{DC} \parallel \overline{AB}$.	10. Theorem 9-2.

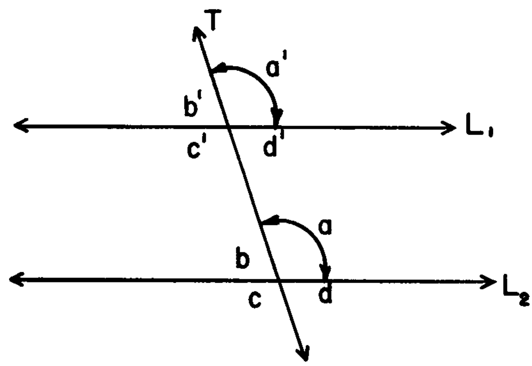
252 Proof of Theorem 9-6

Given two lines and a transversal, if one pair of corresponding angles are congruent, then the other three pairs of corresponding angles have the same property.

Given: Lines L_1 and L_2 cut by transversal T such that a pair of corresponding angles, $\angle a$ and $\angle a'$, are congruent.

To Prove: $\angle b \cong \angle b'$, $\angle c \cong \angle c'$, $\angle d \cong \angle d'$.

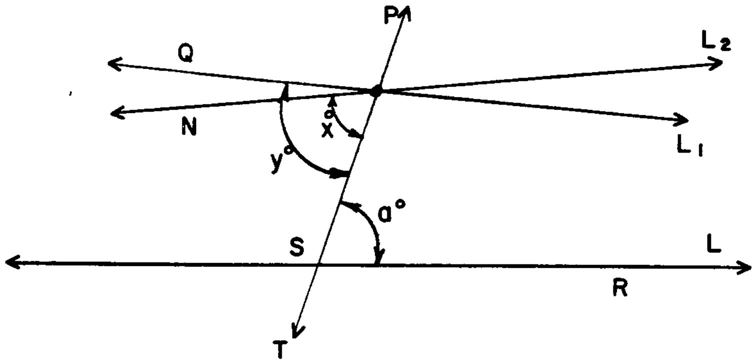
Given that $\angle a \cong \angle a'$. By the Supplement Postulate $\angle a$ is supplementary to $\angle b$, and $\angle a'$ is supplementary to $\angle b'$. Since supplements of congruent angles are congruent, $\angle b \cong \angle b'$. Similarly we show $\angle c \cong \angle c'$ and $\angle d \cong \angle d'$.



252 The method of proof of Theorem 9-7 is merely to use the property of vertical angles to establish a pair of alternate interior angles congruent, and by Theorem 9-5, the lines are parallel.

Because the converses of Theorems 9-5 and 9-7 are reasonable and are readily accepted by students as intuitively true, you may find that the dependence on the Parallel Postulate remains unrecognized, even after the converses have been proved. As preparation for the proof of Theorem 9-8 and preliminary to the Parallel Postulate a consideration similar to the following could be discussed.

It seems reasonable that the converse of Theorem 9-5 is true. Let's examine its reasonableness if we assume that the parallel to a line through a point not on the line is not unique. Then we could suppose two such parallels exist, as in the figure.



Now how reasonable is the converse of Theorem 9-5? According to it, $a = x$ and $a = y$, so that $x = y$. But by the Angle Construction Postulate $x \neq y$. This contradiction means that if we want the converse of Theorem 9-5, and many more such "reasonable" theorems, to hold, then we must accept the uniqueness of the parallel.

Problems 7 and 8 of Problem Set 9-3 present a more complete picture of the situation by showing that the Parallel Postulate can be proved if Theorem 9-8 or Theorem 9-12 is assumed. From all of this the student should become convinced some postulate of parallelism must be stated. The importance of the Parallel Postulate is best seen, perhaps after the sequence of theorems through Theorem 9-13 is finished and the student can look at the sequence, including the Postulate, in its entire development.

252 The Parallel Postulate seems reasonable on the basis of our experience in the world about us. There is no <u>theoretical</u> reason why we could not assume the existence of two parallels to a given line through an external point. From this point on, Parallel Postulates different from ours result in the development of different geometries, called Non-Euclidean Geometries. (See Chapter 1 of Studies II and the Talks on Miniature Geometries and Introduction to Non-Euclidean Geometry.)

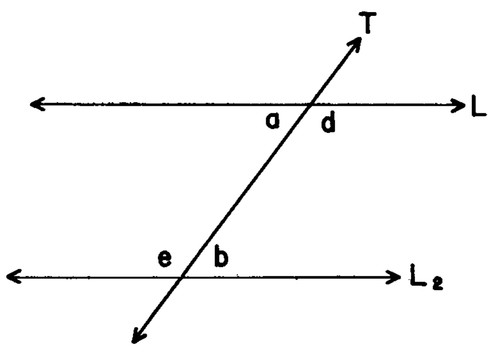
253 Now that we have the uniqueness of a parallel through an external point it is possible to prove the converse of Theorem 9-5. Note carefully in the proof in the text how the fact that this parallel is unique is used to establish the validity of Theorem 9-8.

254 Proofs of Theorems 9-9, 9-10, 9-11 and 9-12

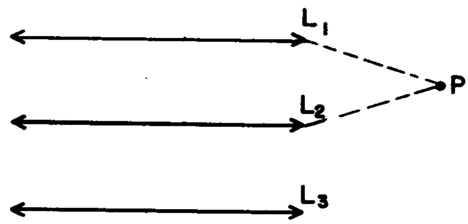
<u>Theorem 9-9</u>. If $L_1 \parallel L_2$, we then know by Theorem 9-8 that the alternate interior angles are congruent. By application of the property that vertical angles are congruent, we can establish the pairs of corresponding angles to be congruent.

The term "interior angles on the same side of the transversal" can be defined formally as follows: Let L be a transversal of L_1 and L_2, intersecting them in P and Q. Let A be a point of L_1 and B a point of L_2 such that A and B are on the same side of L. Then $\angle PQB$ and $\angle QPA$ are called <u>interior angles on the same side of the transversal</u> L. Compare this with the definition of alternate interior angles.

254 <u>Theorem 9-10</u>. Given $L_1 \parallel L_2$. Then it follows from Theorem 9-8 that $\angle a \cong \angle b$. Also, $\angle a$ and $\angle d$ are supplementary. Hence, $m\angle a + m\angle d = 180 = m\angle b + m\angle d$. Therefore $\angle b$ and $\angle d$ are supplementary. In a like manner $\angle e$ can be proved supplementary to $\angle a$.



255 Theorem 9-11. Given: In a plane, $L_1 \parallel L_3$ and $L_2 \parallel L_3$. To Prove: $L_1 \parallel L_2$. We use the indirect method of proof and assume that L_1 is not parallel to L_2. If this is true, then these two lines will meet at some point P. This means that there are now two lines through P (L_1 and L_2) parallel to L_3. This contradicts the Parallel Postulate, hence, L_1 must be parallel to L_2.

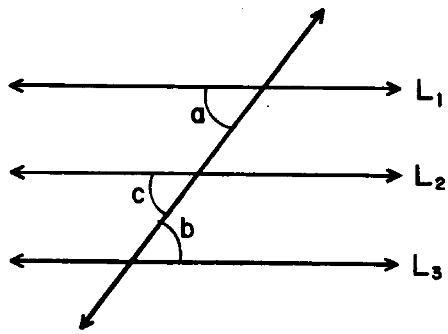


Remark on proof of Theorem 9-11. This theorem can be proved directly as follows:

Given: In a plane, $L_1 \parallel L_3$, $L_2 \parallel L_3$.

To Prove: $L_1 \parallel L_2$.

Let T be a transversal intersecting L_1, L_2 and L_3. Such a transversal exists, since any line in the plane of L_1, L_2, L_3 which meets L_1 in only one point must meet L_2 and L_3 by the Parallel Postulate. Consider the alternate interior angles formed as indicated in the figure.

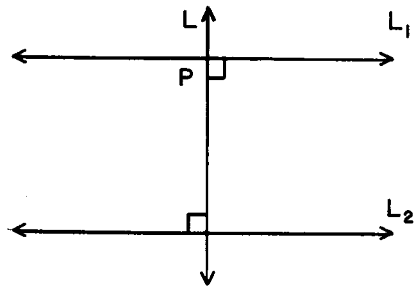


$L_1 \parallel L_3$, hence	(1) $\angle a \cong \angle b$	by Theorem 9-8.
$L_2 \parallel L_3$, hence	(2) $\angle c \cong \angle b$	by Theorem 9-8.
Therefore,	(3) $\angle a \cong \angle c$,	
and	(4) $L_1 \parallel L_2$	by Theorem 9-7.

255 Theorem 9-12. Lines L, L_1 and L_2 are coplanar.

Given: $L_1 \parallel L_2$ and $L \perp L_1$ at P.

To Prove: $L \perp L_2$.



L intersects L_2, otherwise L and L_1 would be parallel to L_2 and contain P. This contradicts the Parallel Postulate. Therefore L is a transversal of L_1 and L_2. By Theorem 9-8 it follows that L and L_2 form a right angle. Thus $L \perp L_2$.

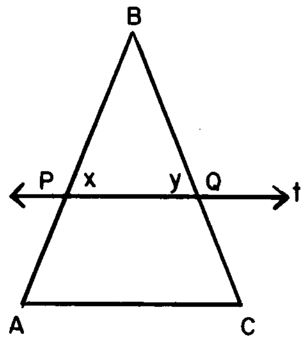
Problem Set 9-3

255 1.

1. $m\angle A = m\angle B = m\angle C = 90$.	1. Given.
2. $\overline{AD} \parallel \overline{CB}$.	2. Theorem 9-2.
3. $m\angle C = m\angle D = 90$.	3. Theorem 9-10.

2. Given: Isosceles Δ ABC with AB = BC and $t \parallel \overline{AC}$ and intersecting $\overline{AB}$ and $\overline{BC}$ at P and Q.

Prove: Δ PBQ is isosceles.



1. $\angle A \cong \angle C$.	1. Theorem 5-2.
2. $t \parallel \overline{AC}$.	2. Given.
3. $\angle x \cong \angle A$ and $\angle y \cong \angle C$ so that $\angle x \cong \angle y$.	3. Theorem 9-9.
4. $\overline{PB} \cong \overline{BQ}$, or ΔPBQ is isosceles.	4. Theorem 5-5.

255 3. $\angle PQT \cong \angle S$ by Theorem 9-9.
$\angle RTS \cong \angle S$ by Theorem 5-2.

From these two statements $\angle PQT \cong \angle RTS$. Then $\overline{PQ} \cong \overline{PT}$ by Theorem 5-5.

256 4. a. Suppose M does not intersect L_2. Then, by definition, $M \parallel L_2$. But L_1 is given $\parallel L_2$. Hence there are through P two parallels to L_2 -- an impossibility by the Parallel Postulate. The assumption that M does not intersect L_2 is therefore false, so that M does intersect L_2.

b. Suppose $L_1 \parallel L_2$. $R \parallel L_2$ by the given information. Also by the given information both L_1 and R contain P. Since there cannot be two parallels to a line through a point, the assumption $L_1 \parallel L_2$ is false, and L_1 intersects L_2.

5. a. $\angle Y \cong \angle BQY$ and $\angle B \cong \angle BQY$ by Theorem 9-8. Therefore, $\angle B \cong \angle Y$.

257 b. Consider $\overleftrightarrow{YX}$ forming $\angle PYZ$ with sides extending in the same direction as those of $\angle ABC$. Then, from part (a), $m\angle PYZ = m\angle ABC$. But $m\angle PYZ + m\angle XYZ = 180$, and therefore $m\angle ABC + m\angle XYZ = 180$.

It should be intuitively clear what is meant when we say two parallel rays extend in the same or opposite directions. A formal definition is easily given. If $\overrightarrow{AB} \parallel \overrightarrow{CD}$ and B and D are on the same side (opposite sides) of $\overleftrightarrow{AC}$ we say $\overrightarrow{AB}$ and $\overrightarrow{CD}$ extend in the same (opposite) directions.

6. If the sides of one angle are perpendicular respectively to the sides of another angle, then the angles are either congruent or supplementary.

257 *7. Draw a transversal $\overleftrightarrow{PQ}$ of L_1 and M and also of L_2 and M forming angles a, b and c as shown. If $L_2 \parallel M$, then $\angle b \cong \angle c$; and since $L_1 \parallel M$, $\angle a \cong \angle c$ by Theorem 9-8. Therefore, $\angle a \cong \angle b$. But then $L_1 = L_2$ by the Angle Construction Postulate, so there cannot be a second parallel to M through Q.

258 *8. Consider a line t perpendicular to M from P. By Theorem 9-12, $t \perp L_1$. Assume L_2 parallel to M. Then $t \perp L_2$. Since L_1 and L_2 cannot both be perpendicular to t at P, L_2 cannot be parallel to M as was assumed.

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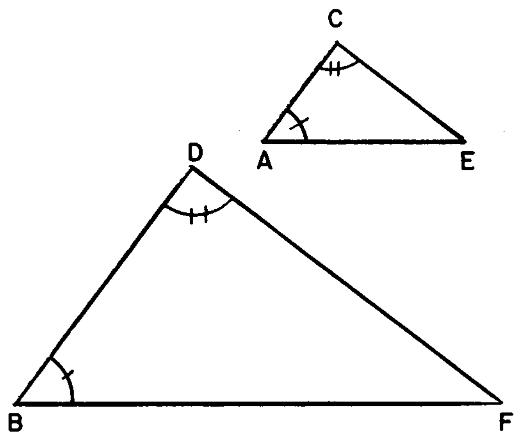
258 Observe that although the proof of Theorem 9-13 is more precise than that given in most texts, it still depends on the figure to show that $\angle x$ and $\angle x'$ are alternate interior angles.

Theorem 9-13 is the first major consequence of our Parallel Postulate. The proof is directly related to the fact that there is but one line parallel to the base of the triangle through the opposite vertex. If there were more than one, or no parallels, the sum of the measures of the angles of a triangle would be less than 180 or greater than 180 as is the case in the Non-Euclidean Geometries. (See Talk, Introduction to Non-Euclidean Geometry.) It is interesting that in Euclidean spherical geometry the sum of the measures of the angles of a spherical triangle is greater than 180.

259 Proofs of the Corollaries

<u>Corollary 9-13-1</u>. Given a correspondence between two triangles. If two angles of the first triangle are congruent to the corresponding parts of the second, then the third angles are congruent.

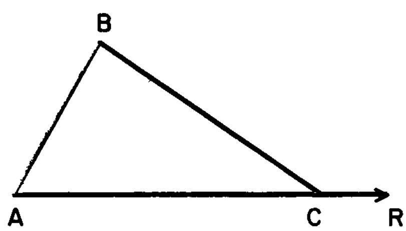
Given: ΔACE and ΔBDF, such that $\angle A \cong \angle B$ and $\angle C \cong \angle D$.
To Prove: $\angle E \cong \angle F$.



We now know, from Theorem 9-13 that the sum of the measures of the angles of a triangle is 180. Given that the sums of the measures of two angles in each triangle are equal, then the differences between this sum and 180 in each case are equal. Thus $m\angle E = m\angle F$ and $\angle E \cong \angle F$.

260 Corollary 9-13-2. This proof follows directly from Theorem 9-13. If the sum of the measures of the angles of a triangle is 180, and one angle has a measure of 90, then the sum of the measures of the remaining two angles must be 90. By definition, then, these angles are complementary.

260 Corollary 9-13-3.
Given: ΔABC with exterior angle $\angle BCR$.
To Prove: $m\angle BCR = m\angle A + m\angle B$.



By the Supplement Postulate $m\angle BCR = 180 - m\angle BCA$.

From Theorem 9-13 it follows that $m\angle A + m\angle B = 180 - m\angle BCA$.
Therefore $m\angle BCR = m\angle A + m\angle B$.

Problem Set 9-4

260 1. a. 85. d. 180 - (r + a).

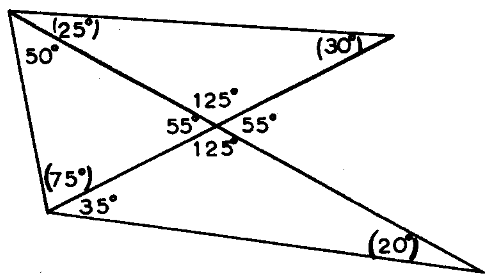
b. 1. e. 90.

c. 180 - 2n. f. $90 - \frac{1}{2}k$.

2. $m\angle P = 4.2$.

3. The Parallel Postulate assures us that L is the only parallel to $\overline{AC}$ through B. It is also used to prove that alternate interior angles are congruent when parallels are cut by a transversal, and this theorem in turn is used in the proof of the angle-sum theorem.

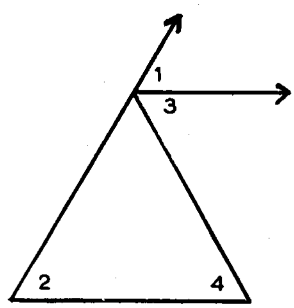
261 4. (Numbers in parentheses were given in the original problem.)



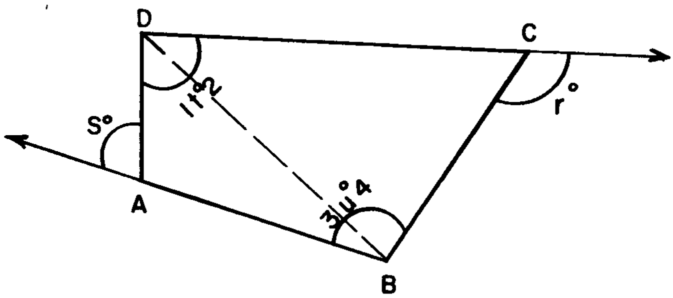
5. a. Yes. b. No.

6. By theorems on transversals of parallels $\angle EBD \cong \angle A$ and $\angle DBC \cong \angle C$. But $\angle EBD \cong \angle DBC$. Therefore $\angle A \cong \angle C$. Hence $AB = BC$.

7. We have $m\angle 1 = m\angle 3$ by hypothesis and $m\angle 2 = m\angle 4$ by Theorem 5-2. But $m\angle 1 + m\angle 3 = m\angle 2 + m\angle 4$ by Corollary 9-13-3. Taking half of each sum we have $m\angle 1 = m\angle 2$, and the bisector is parallel to the base by Theorem 9-7.



262 8. For convenience we indicate angles as shown in the figure.



1.	$r = m\angle 2 + m\angle 4$. $s = m\angle 1 + m\angle 3$.	1.	Corollary 9-13-3.
2.	$r + s = (m\angle 1 + m\angle 2) + (m\angle 3 + m\angle 4)$.	2.	Addition.
3.	$m\angle 1 + m\angle 2 = t$ and $m\angle 3 + m\angle 4 = u$.	3.	Angle Addition Postulate.
4.	$r + s = t + u$.	4.	Statements 2 and 3.

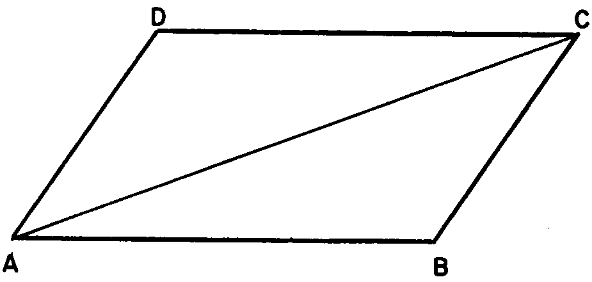
*9. Since $QB = QA$, $\angle B \cong \angle 1$. Since $\angle 2$ and $\angle 1$ are complements, $\angle 2$ and $\angle B$ are also. But $\angle B$ and $\angle C$ are complements, hence, $\angle 2 \cong \angle C$ because complements of the same angle are congruent. Now $QA = QC$, and, hence, $QB = QC$.

*10. In ΔABC, $m\angle B = 90 - a$.
In ΔATS, $m\angle ATS = \frac{180 - a}{2}$.
In ΔBTR, $m\angle BTR = \frac{180 - (90 - a)}{2} = \frac{90 + a}{2}$,

$$m\angle STR = 180 - (m\angle ATS + m\angle BTR)$$
$$= 180 - (\frac{180 - a}{2} + \frac{90 + a}{2}$$
$$= 180 - 135 = 45.$$

265 Proofs of Theorems 9-14 through 9-18

Theorem 9-14.



Given: Parallelogram ABCD with diagonal $\overline{AC}$.

To Prove: $\triangle ABC \cong \triangle CDA$.

1. $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{CD}$.	1. Definition of a parallelogram.
2. $\angle DCA \cong \angle CAB$. $\angle DAC \cong \angle ACB$.	2. Alternate interior angles.
3. $\overline{AC} = \overline{CA}$.	3. Identity.
4. $\triangle ABC \cong \triangle CDA$.	4. A.S.A. Theorem.

The proof using diagonal $\overline{BD}$ is of course, similar to this.

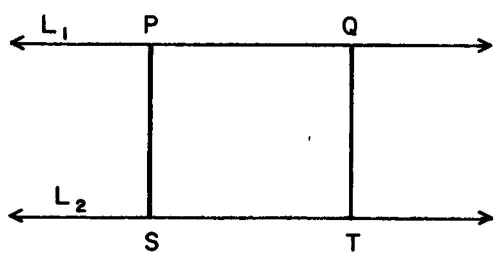
Observe we are reading from the figure that D and B are on opposite sides of $\overleftrightarrow{AC}$.

265 Theorem 9-15 is an immediate consequence of Theorem 9-14: Since the triangles are congruent it follows that the corresponding sides are congruent.

Corollary 9-15-1.

Given: $L_1 \parallel L_2$ and P and Q on L_1.

To Prove: P and Q are equidistant from L_2.

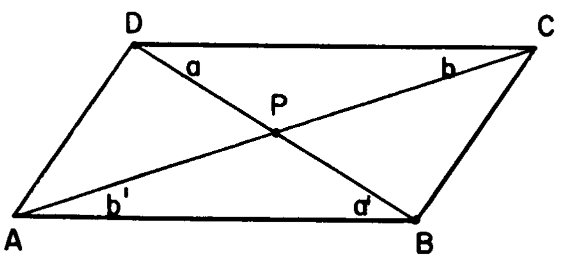


1.	From P and Q drop perpendiculars $\overline{PS}$ and $\overline{QT}$ to L_2.	1.	Theorem 6-4 and definition of distance from a point to a line.
2.	$\overline{PS} \parallel \overline{QT}$.	2.	Theorem 9-2.
3.	PQTS is a parallelogram.	3.	Definition of parallelogram.
4.	PS = QT.	4.	Theorem 9-15.

266 <u>Theorem</u> <u>9-16</u>. Since the triangles into which a diagonal divides a parallelogram are congruent, then the corresponding angles are congruent. In the figure of Theorem 9-14, $\angle D \cong \angle B$. Considering diagonal $\overline{DB}$, we can show, in the same manner, $\angle A \cong \angle C$.

266 <u>Theorem</u> <u>9-17</u>. Consider any two consecutive angles of a parallelogram as the interior angles on the same side of a transversal cutting two parallel lines. Then Theorem 9-17 is immediate by Theorem 9-10 (given two parallel lines and a transversal, interior angles on the same side of the transversal are supplementary).

266 <u>Theorem</u> <u>9-18</u>.



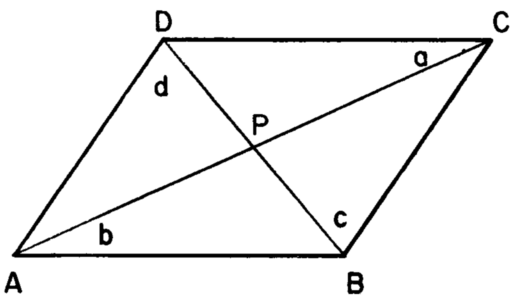
Given: Parallelogram ABCD with diagonals $\overline{AC}$ and $\overline{BD}$. (We assume from the figure that the diagonals intersect at P. For a proof see answers to Problems 19 and 20 of Problem Set 9-6.)

To Prove: $\overline{AC}$ and $\overline{BD}$ bisect each other.

1.	$\angle a \cong \angle a'$. $\angle b \cong \angle b'$.	1.	Alternate interior angles.
2.	$\overline{AB} \cong \overline{CD}$.	2.	Theorem 9-15.
3.	$\Delta ABP \cong \Delta CDP$.	3.	A.S.A. Theorem.
4.	$\overline{AP} \cong \overline{CP}$. $\overline{DP} \cong \overline{BP}$.	4.	Corresponding parts.
5.	$\overline{AC}$ and $\overline{BD}$ bisect each other.	5.	Definition of bisect.

As is pointed out in the text, there is a natural break, or summary point, after Theorem 9-18. Teachers should keep in mind that a careful selection of problems can emphasize the common characteristic of Theorems 9-14 through 9-18, and similiarly for Theorems 9-19, 9-20, and 9-21. At the same time, the fact that Theorems 9-14 through 9-25 all involve quadrilaterals is strengthened by the arrangement of the text. Thus Problem Set 9-6 supplies problems for both Section 9-5 and Section 9-6.

266 Proofs of Theorems 9-19, 9-20, and 9-22



<u>Theorem</u> <u>9-19</u>.

Given: Quadrilateral ABCD with $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$.

To Prove: ABCD is a parallelogram.

1. Draw diagonals $\overline{AC}$ and $\overline{DB}$.
2. By the S.S.S. Theorem $\Delta ABC \cong \Delta CDA$ and $\Delta DAB \cong \Delta BCD$.
3. Therefore $\angle a \cong \angle b$ and $\angle c \cong \angle d$.
4. Then by Theorem 9-5, $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$.
5. ABCD is a parallelogram by definition.

266 Theorem 9-20.

Given: Quadrilateral ABCD with $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$.

To Prove: ABCD is a parallelogram.

Since $\overline{AB} \parallel \overline{CD}$, $\angle a \cong \angle b$ by alternate interior angles $\overline{AC} = \overline{CA}$, and $\Delta ABC \cong \Delta CDA$ by the S.A.S. Postulate. Therefore $\overline{DA} \cong \overline{BC}$ and by Theorem 9-19 ABCD is a parallelogram.

266 Theorem 9-21.

Given: Quadrilateral ABCD with diagonals $\overline{DB}$ and $\overline{AC}$ bisecting each other at P.

To Prove: ABCD is a parallelogram.

1. $\overline{DP} \cong \overline{PB}$. $\overline{AP} \cong \overline{PC}$.	1. Given.
2. $\angle CPB \cong \angle DPA$. $\angle DPC \cong \angle BPA$.	2. Vertical angles are congruent.
3. $\Delta DPC \cong \Delta BPA$. $\Delta CPB \cong \Delta APD$.	3. S.A.S. Postulate.
4. $\overline{AB} \cong \overline{CD}$. $\overline{AD} \cong \overline{CB}$.	4. Corresponding parts.
5. ABCD is a parallelogram.	5. Theorem 9-19.

267 Theorem 9-22 states a fact that surprises many students. Perhaps some students will enjoy making a model to demonstrate visually, rather than just logically, that the length of the segment joining the mid-points of two sides is one-half the length of the third side.

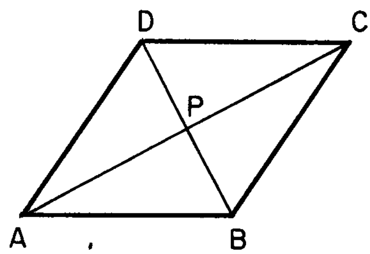
268 In some texts a rectangle is defined in the following way: If one angle of a parallelogram is a right angle then the figure is a rectangle. If this definition is used, you would want the Theorem. If one angle of a parallelogram is a right angle then all four angles are right angles, which in effect is Theorem 9-23. Using this theorem you see that the suggested definition is equivalent to our definition of rectangle.

268 Proofs of Theorems 9-23, 9-24, and 9-25

<u>Theorem 9-23</u>. By Theorem 9-17 the consecutive angles of a parallelogram are supplementary, and since one angle is a right angle its supplement must be a right angle. Two successive applications of Theorem 9-17 will establish that the other two angles are right angles. Or we could apply the theorem that opposite angles of a parallelogram are congruent.

Theorem 9-23 gives us an efficient way to prove that a quadrilateral is a rectangle. First prove that it is a parallelogram and then prove that one angle is a right angle.

268 <u>Theorem 9-24</u>.



Given: Rhombus ABCD with diagonals $\overline{AC}$ and $\overline{BD}$.

To Prove: $\overline{AC} \perp \overline{BD}$.

By the definition of rhombus AB = AD and CB = CD; that is, A and C are equidistant from B and D. Since A and C are coplanar with B and D, by Theorem 6-2 $\overleftrightarrow{AC}$ is the perpendicular bisector of $\overline{BD}$. Hence, $\overline{AC} \perp \overline{BD}$.

An alternate proof uses the S.S.S. Theorem to get congruent any two of the triangles having a common side. Then the angles of a linear pair are congruent, and the diagonals are perpendicular.

268 <u>Theorem 9-25</u>. Using the figure of Theorem 9-24 we have:

Given: ABCD with $\overline{AC} \perp \overline{BD}$ and $\overline{AC}$ and $\overline{BD}$ bisecting each other.

To Prove: ABCD is a rhombus.

By hypothesis, $\overleftrightarrow{AC}$ is the perpendicular bisector of $\overline{BD}$, so that AB = AD and CB = CD by Theorem 6-2. Similarly, AD = CD so that AB = AD = CD = CB. By definition, ABCD is a rhombus.

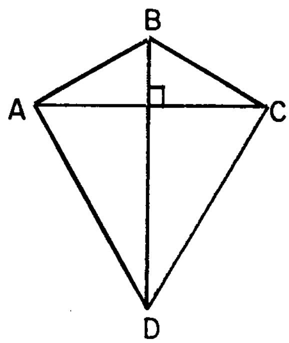
An alternate proof uses the fact that $\triangle APB \cong \triangle APD \cong \triangle CPB \cong \triangle CPD$ by S.A.S.

After the class has become familiar with the properties of quadrilaterals stated on the previous pages you might propose the following two problems for them to work. Neither of these can be solved since there is a counter-example (an example satisfying all of the given conditions that does not satisfy the desired result) for each one.

(1) Given quadrilateral ABCD such that $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \cong \overline{BC}$, prove this quadrilateral is a parallelogram. Do not inform the students that this cannot be proved. Let them search for themselves for a while and perhaps realize that the counter-example is an isosceles trapezoid. This figure satisfies all of the given conditions, but certainly is not a parallelogram.

(2) Given a quadrilateral ABCD such that the diagonals are perpendicular to each other. Prove that the quadrilateral is a rhombus (or a square). This problem, also, cannot be solved. A counter-example is a kite, like this:

It can be formed from two non-congruent isosceles triangles having the same base fitted together as in the figure. A more general figure is also possible.



Problem Set 9-6

269 1. a. All four quadrilaterals.
b. All four.
c. Square, rhombus.
d. All four.
e. Square, rhombus.
f. All four.
g. Square, rhombus.
h. All four.

269 i. Rectangle, square.

j. Rectangle, square.

2. $x + 30 + 2x - 60 = 180$ and $x = 70$. Therefore, $m\angle A = m\angle F = 80$; $m\angle B = m\angle H = 100$.

3. Since the opposite angles of a parallelogram are congruent, $\angle H \cong \angle A$ and also $\angle R \cong \angle A$, so that $\angle R \cong \angle H$. Since interior angles on the same side of a transversal which cuts parallel lines are supplementary, $\angle M$ is supplementary to $\angle A$. By substitution we see that $\angle M$ is supplementary to $\angle H$.

270 4.

a.	Yes.	No.	No.	No.
b.	Yes.	No.	No.	No.
c.	Yes.	Yes.	No.	No.
d.	Yes.	No.	No.	No.
e.	No.	No.	No.	No.
f.	No.	No.	No.	No.
g.	Yes.	No.	Yes.	No.
h.	Yes.	No.	Yes.	No.
i.	Yes.	Yes.	No.	No.
j.	Yes.	Yes.	Yes.	Yes.
k.	Yes.	No.	No.	No.
l.	Yes.	No.	No.	No.

5. $AD = BC$ and $AB = DC$ since opposite sides of a parallelogram are congruent. Then $\triangle APD \cong \triangle CRB$ and $\triangle APB \cong \triangle CRD$ by S.A.S. Then by corresponding parts $\overline{RD} \cong \overline{PB}$ and $\overline{PD} \cong \overline{RB}$. Having opposite sides congruent, DPBR is a parallelogram.

270 6.

1.	$\overline{FE} \parallel \overline{AD}$. $\overline{FE} \parallel \overline{BC}$.	1.	Definition of a parallelogram.
2.	$\overline{AD} \parallel \overline{BC}$.	2.	Theorem 9-11.
3.	$\overline{FE} \cong \overline{AD}$. $\overline{FE} \cong \overline{BC}$.	3.	Theorem 9-15.
4.	$\overline{AD} \cong \overline{BC}$.	4.	Statement 3.
5.	ABCD is a parallelogram.	5.	Statements 2 and 4 and Theorem 9-20.

271 7.

1.	PXRY is a parallelogram.	1.	Definition of a parallelogram.
2.	PX = RY, RX = PY.	2.	Theorem 9-15.
3.	$\angle$ XPS $\cong$ $\angle$ T.	3.	Theorem 9-9.
4.	$\angle$ S $\cong$ $\angle$ T.	4.	Theorem 5-2.
5.	$\angle$ XPS $\cong$ $\angle$ S.	5.	Angles congruent to the same angle. Statements 3 and 4.
6.	PX = SX.	6.	Theorem 5-5.
7.	PY = TY.	7.	By steps similar to steps 2-6.
8.	PX + XR + RY + YP = SX + XR + RY + YT, or PX + XR + RY + YP = RS + RT.	8.	Statements 6 and 7 by addition.

8.

1.	DQ = BQ.	1.	Theorem 9-18.
2.	$\overline{DC} \parallel \overline{BA}$.	2.	Definition of a parallelogram.
3.	$\angle$ EDQ $\cong$ $\angle$ FBQ.	3.	Theorem 9-8.
4.	$\angle$ DQE $\cong$ $\angle$ BQF.	4.	Vertical angles are congruent.
5.	Δ DQE $\cong$ Δ BQF.	5.	A.S.A.
6.	EQ = FQ.	6.	Corresponding parts.
7.	$\overline{EF}$ is bisected by Q.	7.	Definition of bisect.

271 9. Through D draw a parallel to $\overline{CB}$ meeting $\overline{AB}$ at X. Then DCBX is a parallelogram in which case CB = DX. Since it was given that AD = CB, therefore DX = DA and $\angle$ DXA $\cong$ $\angle$ A. But, by corresponding angles $\angle$ DXA $\cong$ $\angle$ B. Therefore $\angle$ A $\cong$ $\angle$ B.

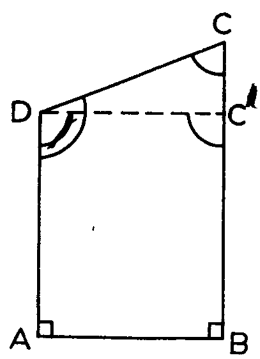
272 10. a. Δ DCQ $\cong$ Δ KBQ by A.S.A. or S.A.A. so that Q is mid-point of $\overline{DK}$. In Δ ADK, $\overline{PQ} \parallel \overline{AK}$ and $PQ = \frac{1}{2}AK = \frac{1}{2}(AB + BK)$. BK = CD since they are corresponding parts of congruent triangles. Hence, $PQ = \frac{1}{2}(AB + CD)$.

b. 8 inches. c. $5\frac{1}{4}$.

12.

1. Draw $\overline{DB}$.	1. Two points determine a segment.
2. $\overline{RQ} \parallel \overline{DB}$; $RQ = \frac{1}{2}$ DB.	2. Theorem 9-22.
3. $\overline{SP} \parallel \overline{DB}$; $SP = \frac{1}{2}$ DB.	3. Theorem 9-22.
4. RQ = SP.	4. Statements 2 and 3.
5. $\overline{RQ} \parallel \overline{SP}$.	5. Theorem 9-11.
6. SPQR is a parallelogram.	6. Theorem 9-20.
7. $\overline{SQ}$ and $\overline{PR}$ bisect each other.	7. Theorem 9-18.

273 13. Let C' be between B and C such that AD = BC'. Then $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC'}$ so that ABC'D is a parallelogram and m$\angle$ ADC' = m$\angle$ BC'D. Making this replacement in m$\angle$ ADC' < m$\angle$ ADC, we have m$\angle$ BC'D < m$\angle$ ADC. By the Exterior Angle Theorem m$\angle$ C < m$\angle$ BC'D. Therefore m$\angle$ C < m$\angle$ ADC.



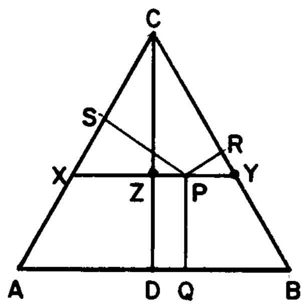
273 *14. Given: $\triangle ACB$ with $AC = BC$, $\overline{PX} \perp \overline{AC}$, $\overline{PY} \perp \overline{BC}$, $\overline{BT} \perp \overline{AC}$.

To Prove: $PX + PY = BT$.

Statements	Reasons
1. Draw a perpendicular, $\overline{PQ}$, from P to $\overline{BT}$.	1. Theorem 6-4.
2. $\overleftrightarrow{PX} \parallel \overleftrightarrow{QT}$, and $\overleftrightarrow{XT} \parallel \overleftrightarrow{PQ}$.	2. Theorem 9-2.
3. PQTX is a parallelogram.	3. Definition of a parallelogram.
4. $PX = QT$.	4. Theorem 9-15.
5. $\angle QPB \cong \angle A$.	5. Theorem 9-9.
6. $\angle YBP \cong \angle A$.	6. Theorem 5-2.
7. $\angle QPB \cong \angle YBP$.	7. Statements 5 and 6.
8. $\triangle QPB \cong \triangle YBP$.	8. S.A.A. Theorem.
9. $PY = BQ$.	9. Corresponding sides.
10. $PX + PY = QT + BQ$, or $PX + PY = BT$.	10. Steps 4 and 9.

*15. Given: P interior to equilateral $\triangle ABC$. $\overline{PQ}$, $\overline{PR}$, $\overline{PS}$ and $\overline{CD}$ are perpendiculars as shown.

To Prove: $PQ + PR + PS = CD$.



Statements	Reasons
1. Draw $\overline{XY}$, through P, $\perp \overline{CD}$ intersecting $\overline{AC}$, $\overline{CD}$, $\overline{BC}$ as shown.	1. Theorems 6-3 and 6-4.
2. $\overline{PZ} \parallel \overline{QD}$, $\overline{PQ} \parallel \overline{ZD}$.	2. Theorem 9-2.
3. PQDZ is a parallelogram.	3. Definition of a parallelogram.
4. $PQ = ZD$.	4. Theorem 9-15.
5. $PR + PS = CZ$.	5. Problem 14.
6. $PQ + PR + PS = CZ + ZD$, $PQ + PR + PS = CD$.	6. Steps 4 and 5.

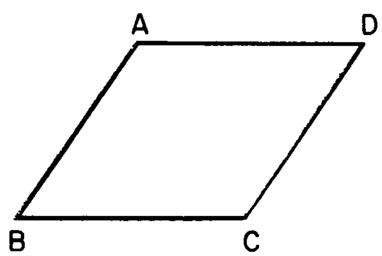
273 16. EFOD is a parallelogram, by definition. Hence EF = DO and $\overline{EF} \parallel \overline{DO}$. Similarly DO = CB and CB = OA. Therefore EF = OA and $\overline{EF} \parallel \overline{OA}$. So EFAO is a parallelogram and $\overline{FA} \parallel \overline{EO}$. Since $\overline{CD} \parallel \overline{EO}$, we have $\overline{FA} \parallel \overline{CD}$.

274 17. a. ABB'A' is a parallelogram so that AA' = BB'. Similarly BCC'B' is a parallelogram and BB' = CC'. Thus AA' = CC' and AA'C'C is a parallelogram. Hence $\overleftrightarrow{AC} \parallel \overleftrightarrow{A'C'}$.

b. The proof does not apply if the figure is not in a plane because it has not been proved that if two lines <u>in space</u> are parallel to a third line they are parallel to each other.

18. By S.A.S. the four triangles are congruent. Hence the four sides $\overline{KL}$, $\overline{LM}$, etc. are congruent. But of the three angles at N, for example, two are complementary. Therefore the third is a right angle. Likewise the other angles of KLMN are right angles and the figure is a square.

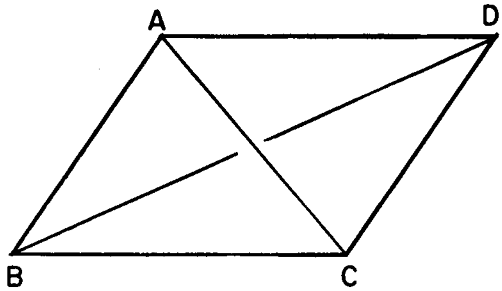
*19. 1. A and D are on the same side of $\overleftrightarrow{BC}$ because $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.



2. Similarly C and D are on the same side of $\overleftrightarrow{AB}$.

3. D is in the interior of $\angle A$ by the definition of the interior of an angle.

274 *20. 1. In the parallelogram ABCD shown, D is in the interior of $\angle ABC$ by the preceding problem.



2. $\overrightarrow{BD}$ intersects $\overline{AC}$ by Problem 5 of Problem Set 6-5.

3. Similarly, $\overrightarrow{DB}$ intersects $\overline{AC}$.

4. Hence $\overline{BD}$ intersects $\overline{AC}$.

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275 It is intuitively evident that B is the mid-point of $\overline{AC}$. This can be proved formally as follows. One of A, B, C must be between the other two (Theorem 2-2). If A is between B and C we have $BC > AB$, contradicting $BC = AB$. Similarly if C is between A and B we get $AB > BC$ which is impossible. Thus B is between A and C and B is the mid-point of $\overline{AC}$ by definition.

275 Caution the students that the statement of Theorem 9-26 does not say that the segments intercepted on one transversal are <u>congruent</u> <u>respectively</u> to segments intercepted on another transversal. The segments of any one transversal are congruent to each other.

In the proof of Theorem 9-26, we have tacitly assumed that T_2 does not contain B; otherwise, T_4 could not be parallel to T_2. The case in which T_2 contains B is easily disposed of using congruent triangles, ΔDBA and ΔFBC, since $\angle DBA$ and $\angle FBC$ are vertical angles.

In Problem Set 9-7, Problem *7 is intended to provide the capable student with some insight into the problem of incommensurability.

280 You may wish to point out to your class that the centroid of a triangle has a significant physical interpretation. It is a "central point" of the triangle. If the triangle and its interior are given a physical existence, in the form of a piece of cardboard or wood, for example, the center of gravity of each is at the centroid of the triangle, and the triangular piece will balance on a pin at this point. Also, if the triangular piece is freely suspended from a vertex and a plumb line attached to that vertex, the plumb line will always come to rest over the centroid of the triangle.

Problem Set 9-7

280 1. a. By Theorem 9-26, $RS = ST$; and then by the same theorem $ZY = YX$.

b. No.

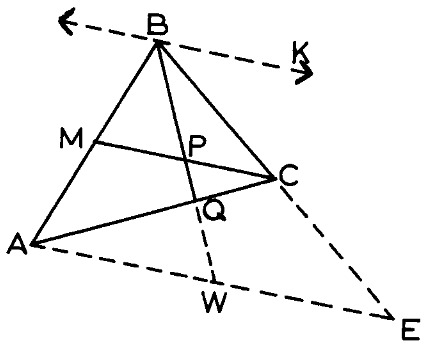
281 2. The right edge of sheet A is a transversal divided into congruent segments by ruled parallels. By Corollary 9-26-1, any other transversal, in particular $\overline{OQ}$, will be divided into congruent segments by the same parallels.

3. Congruent corresponding angles assure parallel lines through N_1, N_2, ..., N_5. Considering a sixth parallel through A, Corollary 9-26-1 explains why $\overline{AB}$ will be divided into congruent segments.

4. 12, 5, 6.

282 5. 10, 5, 5.

282 *6. Extend $\overline{BC}$ making $CE = BC$ and draw $\overline{AE}$. Extend $\overline{BP}$ to meet $\overline{AE}$ at W. Draw $\overleftrightarrow{BK} \parallel \overline{AE}$. Now $\overline{MC} \parallel \overline{AE}$ by Theorem 9-22 and $BP = PW$ by Theorem 9-26. By Theorem 9-22 again, $AW = 2MP = 2PC = WE$. Hence $\overline{BW}$ is a median of ΔABE and meets the median $\overline{AC}$ at a point Q where $AQ = 2QC$.



*7. a. 3.

b. 7.

c. 9.

d. 1207.

e. No set of parallels can include $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$ and $\overleftrightarrow{CT}$.

283 *8.

1. Through C draw a line $\overleftrightarrow{CL} \parallel \overleftrightarrow{DY}$.	1. Theorem 9-3.
2. $BC = AD$.	2. Theorem 9-15.
3. $BY = DX$.	3. Halves of equal numbers are equal.
4. BYDX is a parallelogram.	4. Theorem 9-20.
5. $\overleftrightarrow{DY} \parallel \overleftrightarrow{XB}$.	5. Definition of a parallelogram.
6. $\overleftrightarrow{CL} \parallel \overleftrightarrow{XB}$.	6. Theorem 9-11.
7. $CQ = QT$.	7. Theorem 9-26.
8. $AT = TQ$.	8. By steps corresponding to Steps 1-7.
9. $AT = TQ = QC$.	9. Steps 7 and 8.

Review Problems

283 1.

a.	S.	i.	A.	q.	A.
b.	S.	j.	N.	r.	A.
c.	S.	k.	S.	s.	S.
d.	A.	l.	S.	t.	S.
e.	S.	m.	S.	u.	S.
f.	A.	n.	S.	v.	S.
g.	A.	o.	A.	w.	A.
h.	S.	p.	S.	x.	A.

285 2.

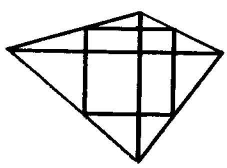
a.	Yes,	No,	No,	No.
b.	No,	No,	No,	No.
c.	Yes,	No,	Yes,	No.
d.	Yes,	Yes,	Yes,	Yes.
e.	No,	No,	No,	No.
f.	Yes,	No,	No,	No.
g.	No,	No,	No,	No.
h.	No,	No,	No,	No.
i.	Yes,	No,	No,	No.
j.	Yes,	No,	Yes,	No.
k.	Yes,	No,	No,	No.
l.	Yes,	No,	Yes,	No.

3. a. supplementary. b. congruent.

286 4. (d) are parallel.

5. (b) a rectangle.

*6. (a) If and only if the diagonals of ABCD are congruent and perpendicular. Answer (c) is incorrect. Although the inscribed quadrilateral is a square if ABCD is a square, it is untrue that the inscribed quadrilateral is a square only if ABCD is a square. See the figure.



286 7. a. 40, 70. c. 90, 6.
b. 60, 120.

287 8. DG = 4. (GF = 8 = $\frac{1}{3}$AF. AF = 24. DF = 12.
DG = DF - GF = 4.)

9. 9 inches.

10. a. 55.
b. $\frac{3a}{2}$, $180 - \frac{3a}{2}$ or $\frac{360 - 3a}{2}$.

11. $m\angle A = m\angle ACD - m\angle ABC = 2b - 2a$.
$m\angle E = m\angle ECD - m\angle EBC = b - a$.
Therefore $m\angle E = \frac{1}{2} m\angle A$.

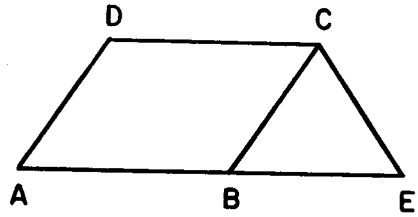
288 12. 65.

13. $\triangle AOC \cong \triangle BOD$ by S.A.S.
$\angle C \cong \angle D$ since they are corresponding parts.
$\overline{AC} \parallel \overline{BC}$ since $\angle C$ and $\angle D$ are congruent alternate interior angles.

14.

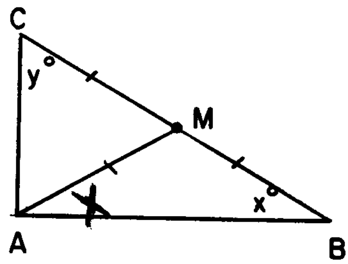
1. AP = RC.	1. Given.
2. AD = CB, $\overline{AD} \parallel \overline{CB}$.	2. Opposite sides of a parallelogram are congruent and parallel.
3. $\angle DAP \cong \angle BCR$.	3. Alternate interior angles.
4. $\triangle DAP \cong \triangle BCR$.	4. S.A.S.
5. $\overline{DP} \cong \overline{BR}$.	5. Corresponding parts.
6. $\overline{PB} \cong \overline{RD}$.	6. Proof similar to Steps 1-6.
7. DPBR is a parallelogram.	7. Theorem 9-19.

288 15. The statement can be disproved by a counter-example. If parallelogram ABCD has side $\overline{CB}$ in common with isosceles triangle CBE in which $\overline{CB} \cong \overline{CE}$ and B is between A and E, then quadrilateral AECD meets the requirements of the hypothesis of the problem but is not a parallelogram.



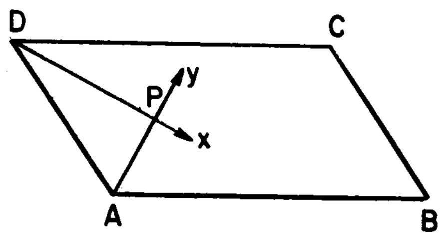
*16. Given: CM = MB, $\overline{AM} \cong \overline{CM}$.
Prove: $\triangle$ ABC is a right triangle.

Let $m\angle B = x$ and $m\angle C = y$ as shown in the figure.



1.	$m\angle MAB = x$, $m\angle CAM = y$.	1.	Base angles of an isosceles triangle.
2.	$m\angle CAB = x + y$.	2.	Angle Addition Postulate.
3.	$2x + 2y = 180$.	3.	Theorem 9-13.
4.	$x + y = 90$.	4.	Division.
5.	$m\angle CAB = 90$.	5.	Steps 2 and 4.

17. Given: ABCD is a parallelogram. $\overrightarrow{DX}$ bisects $\angle ADC$. $\overrightarrow{AY}$ bisects $\angle DAB$. $\overrightarrow{DX}$ and $\overrightarrow{AY}$ intersect at P.
Prove: $\overrightarrow{DX} \perp \overrightarrow{AY}$.



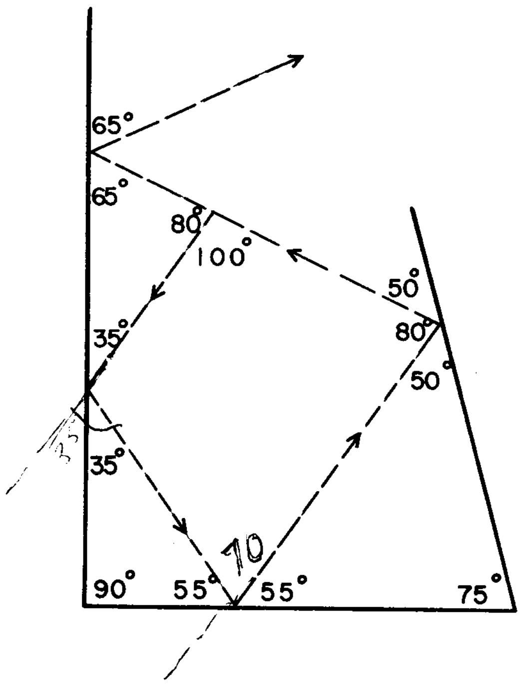
288 17.

1.	m∠ ADC + m∠ BAD = 180.	1.	Theorem 9-17.
2.	$\frac{1}{2}$m∠ ADC + $\frac{1}{2}$m∠ BAD = 90.	2.	Division, from Statement 1.
3.	m∠ ADP + m∠ DAP = 90.	3.	Step 2 and definition of bisect.
4.	m∠ DPA = 90.	4.	Theorem 9-13 and Statement 3.
5.	$\overrightarrow{DX} \perp \overrightarrow{AY}$.	5.	Definition of perpendicular.

289 *18.

1.	Consider $\overline{AC}$, $\overline{PK}$, $\overline{PE}$ and $\overline{KM}$.	1.	A segment is determined by two points.
2.	$\overline{PK} \parallel \overline{AC}$ and PK = $\frac{1}{2}$AC.	2.	Theorem 9-22.
3.	ACDE is a parallelogram.	3.	Theorem 9-20.
4.	ED = AC.	4.	Theorem 9-15.
5.	EM = $\frac{1}{2}$AC.	5.	Given, and Statement 4.
6.	EM = PK.	6.	Statements 2 and 5.
7.	$\overline{EM} \parallel \overline{AC}$.	7.	Definition of parallelogram.
8.	$\overline{PK} \parallel \overline{EM}$.	8.	Theorem 9-11.
9.	PEMK is a parallelogram.	9.	Theorem 9-20.
10.	$\overline{KE}$ bisects $\overline{PM}$.	10.	Theorem 9-18.

289 19.



290 20. The diagonals of quadrilateral ABDC bisect each other so ABDC is a parallelogram. For the same reason, AFBC is a parallelogram. F, B, D are collinear because only one parallel to $\overline{AC}$ can contain point B.

Illustrative Test Items for Chapter 9

A. If a statement is always true write the word TRUE. If it is not always true write the word FALSE.

1. The diagonals of a square are perpendicular to each other.
2. A square is a parallelogram.
3. If the diagonal of a quadrilateral divides it into two congruent triangles, then the quadrilateral is a parallelogram.
4. Lines which never meet are parallel.
5. If two consecutive angles of a quadrilateral are right angles, then the quadrilateral is either a trapezoid or a rectangle.
6. Two lines which are each perpendicular to a third line are parallel.
7. Given a correspondence between two triangles. If the triangles have two sides and an angle of one congruent to the corresponding parts of the other, then the correspondence is a congruence.
8. Every right triangle has two acute angles.
9. If a diagonal of a parallelogram divides it into two isosceles triangles, the parallelogram is a rhombus.
10. If each two opposite sides of a quadrilateral are congruent segments, the quadrilateral is a parallelogram.
11. Opposite angles of a parallelogram are congruent.
12. The measure of an exterior angle of a triangle equals the sum of the measures of the two remote interior angles.

13. The perimeter of the triangle formed by joining the mid-points of the sides of a given triangle is half the perimeter of the given triangle.

14. If the diagonals of a quadrilateral are perpendicular and congruent, the quadrilateral is a rhombus.

15. A line that bisects one side of a triangle bisects another side also.

16. The diagonals of a parallelogram are congruent.

17. A diagonal of a parallelogram bisects two of its angles.

18. A quadrilateral with three right angles is a rectangle.

19. A set of parallel lines intercepts congruent segments on any transversal.

20. Given two parallel lines and a transversal, two interior angles on the same side of the transversal are supplementary.

21. If two angles of a triangle are congruent to two angles of another triangle, then the third angles are congruent.

22. If a line bisects one side of a triangle and is parallel to a second side, then it bisects the third side.

23. If a quadrilateral has a pair of sides parallel and the other pair of sides congruent, then the quadrilateral is a parallelogram.

24. If a parallelogram has one right angle, it has four right angles.

B. 1. Would the following information about a quadrilateral be <u>sufficient</u> <u>to</u> <u>prove</u> it a parallelogram? a rectangle? a rhombus? a square?

a. Each two opposite sides are parallel.

b. Each two opposite sides are congruent.

c. Three of its angles are right angles.

d. Its diagonals bisect each other.

e. Its diagonals are congruent.

f. Its diagonals are perpendicular and congruent.

g. Its diagonals are perpendicular bisectors of each other.

h. It is equilateral.

i. It is equiangular.

j. It is equilateral and equiangular.

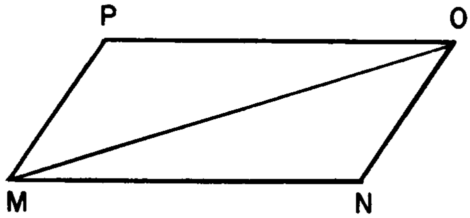
k. Each two opposite angles are congruent.

l. Each two consecutive angles are supplementary.

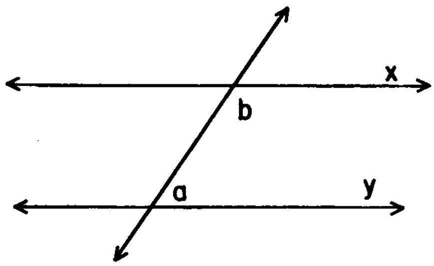
2. Write on your paper these names of quadrilaterals: parallelogram, rhombus, rectangle, square. After each name write the number of every statement below which <u>always</u> applies to it.

   1. Each two opposite sides are parallel.
   2. Each two opposite angles are congruent.
   3. Each two opposite sides are congruent.
   4. Diagonals have equal lengths.
   5. Diagonals bisect each other.
   6. Diagonals are perpendicular.
   7. All sides are congruent.
   8. All angles are congruent.
   9. All angles are bisected by the diagonals.

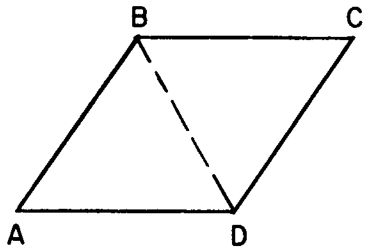
C. 1. a. In quadrilateral MNOP having diagonal $\overline{MO}$, if $\angle$ OMP $\cong$ $\angle$ MON, what two segments are parallel?



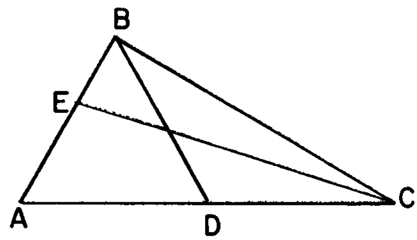
b. If the parallel lines x and y are cut by a transversal, and if m$\angle$ b is 10 greater than m$\angle$ a, find m$\angle$ b.



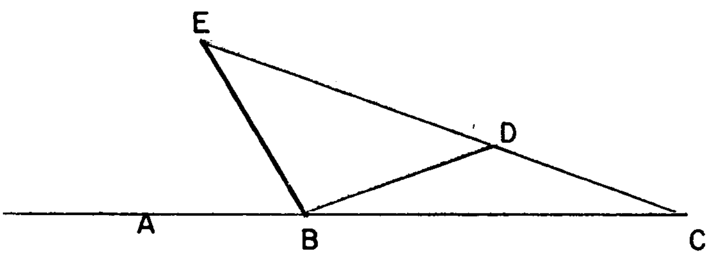
2. Given: ABCD is a rhombus. m$\angle$ BAD = 60, AD = 5.
Find: BD.



3. Given: Δ ABC with medians $\overline{BD}$ and $\overline{EC}$. BD = 8, EC = 9.
Find: The lengths of the shorter segments of each median.

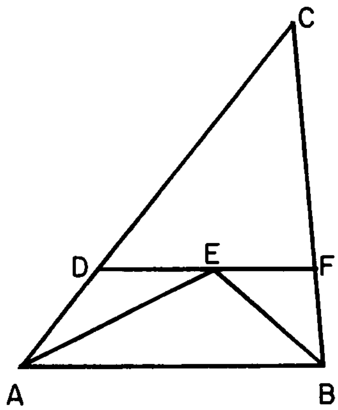


4. If in the figure, DB = DC = BE and m$\angle$ ECB = 30, find m$\angle$ ABE.



5. Two sides of a parallelogram are 6 and 10. Find the length of the segment connecting the mid-point of the shorter side with the intersection of the diagonals.

6. In ΔABC, $\overrightarrow{AE}$ bisects $\angle A$, $\overrightarrow{BE}$ bisects $\angle B$, and $\overleftrightarrow{DF} \parallel \overleftrightarrow{AB}$. $m\angle CAB = 40$ and $m\angle CBA = 60$. What is $m\angle BEF$?



7. In ΔABC, $\overrightarrow{AE}$ bisects $\angle A$. $\overrightarrow{BE}$ bisects $\angle B$, and $\overleftrightarrow{DF} \parallel \overleftrightarrow{AB}$. $m\angle C = 110$ and $m\angle CDF = 50$. What is $m\angle BEF$?

8. Two angles of a triangle have a total measure of 100. What is the measure of either of the obtuse angles formed at the intersection of the bisectors of these two angles?

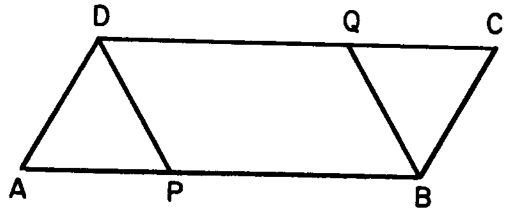
9. If the measure of one of the congruent angles of an isosceles triangle is 70, what is the measure of the smallest angle of the triangle?

10. Find the measure of each acute angle of a right triangle if the measure of one of them is three times that of the other.

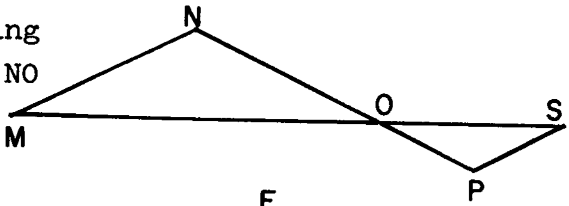
D. 1. Consider the following theorem: Given two lines and a transversal. If one pair of alternate interior angles are congruent, then the lines are parallel.

Given: Lines L_1 and L_2 cut by a transversal L to form congruent alternate interior angles.
To Prove: $L_1 \parallel L_2$.
Proof: Suppose L_1 intersects L_2 in a point P. This situation leads to a contradiction of what theorem?

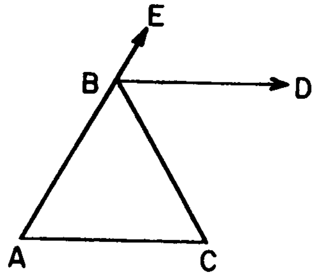
2. Given: In the figure ABCD is a parallelogram with AP = QC.
Prove: $\overline{DP} \parallel \overline{QB}$.



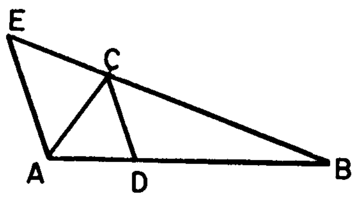
3. Given: $\overline{NP}$ intersecting $\overline{MS}$ at O with MN = NO and OP = PS.
Prove: $\overleftrightarrow{MN} \parallel \overleftrightarrow{PS}$.



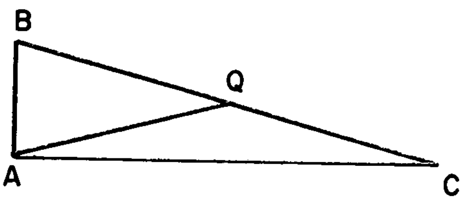
4. Given: $\overrightarrow{BD}$ bisects $\angle$ EBC, and $\overleftrightarrow{BD} \parallel \overleftrightarrow{AC}$.
Prove: AB = BC.



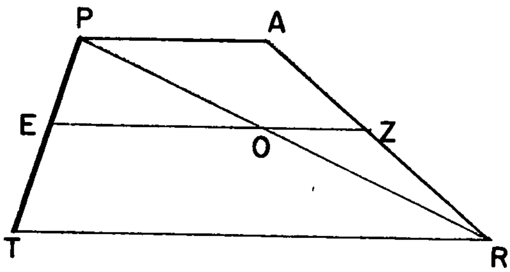
5. Given: In Δ ABC, $\overrightarrow{CD}$ bisects $\angle$ ACB. $\overleftrightarrow{AE} \parallel \overleftrightarrow{CD}$. $\overleftrightarrow{AE}$ intersects $\overleftrightarrow{CB}$ at E.
Prove: AC = EC.



6. Given: $\angle$ BAC is a right angle. QB = QA.
Prove: QB = QC.



7. Prove: If a line is parallel to the bases of a trapezoid and bisects one of the non-parallel sides, then it bisects either diagonal of the trapezoid.



Answers

A.

1.	True.	9.	True.	17.	False.
2.	True.	10.	True.	18.	True.
3.	False.	11.	True.	19.	False.
4.	False.	12.	True.	20.	True.
5.	True.	13.	True.	21.	True.
6.	False.	14.	False.	22.	True.
7.	False.	15.	False.	23.	False.
8.	True.	16.	False.	24.	True.

B. 1.

a.	Yes.	No.	No.	No.
b.	Yes.	No.	No.	No.
c.	Yes.	Yes.	No.	No.
d.	Yes.	No.	No.	No.
e.	No.	No.	No.	No.
f.	No.	No.	No.	No.
g.	Yes.	No.	Yes.	No.
h.	Yes.	No.	Yes.	No.
i.	Yes.	Yes.	No.	No.
j.	Yes.	Yes.	Yes.	Yes.
k.	Yes.	No.	No.	No.
l.	Yes.	No.	No.	No.

2. Parallelogram. 1, 2, 3, 5.
   Rhombus. 1, 2, 3, 5, 6, 7, 9.
   Rectangle. 1, 2, 3, 4, 5, 8.
   Square. 1, 2, 3, 4, 5, 6, 7, 8, 9.

C. 1. a. $\overline{MP} \parallel \overline{NO}$. b. 95.

2. 5.

3. The length of the shorter segment of $\overline{BD} = 2\frac{2}{3}$.
   The length of the shorter segment of $\overline{EC} = 3$.

4. $m\angle ABE = 90$.

5. 5.

6. 30.

7. 10.

8. 130.

9. 40.

10. $22\frac{1}{2}$, $67\frac{1}{2}$.

D. 1. The Exterior Angle Theorem.

2.

1. AD = CB, AB = CD.	1. Theorem 9-15.
2. $\angle A \cong \angle C$.	2. Theorem 9-16.
3. AP = CQ.	3. Given.
4. $\Delta APD \cong \Delta CQB$.	4. S.A.S.
5. $\overline{PD} \cong \overline{QB}$.	5. Corresponding parts.
6. PB = DQ.	6. Subtraction, Statements 1 and 3.
7. DQBP is a parallelogram.	7. Theorem 9-19.
8. $\overline{DP} \parallel \overline{QB}$.	8. Definition of a parallelogram.

3.

1. $\angle NOM \cong \angle POS$.	1. Theorem 4-7.
2. $\angle S \cong \angle POS$. $\angle M \cong \angle NOM$.	2. Theorem 5-2.
3. $\angle M = \angle S$.	3. From Statements 1 and 2.
4. $\overleftrightarrow{MN} \parallel \overleftrightarrow{PS}$.	4. Theorem 9-5.

4. $\angle EBD$ and $\angle A$ are congruent because they are corresponding angles formed by parallel lines and the transversal $\overleftrightarrow{AE}$. $\angle CBD$ and $\angle C$ are congruent since they are alternate interior angles of parallel lines. Since the given bisector makes $\angle EBD \cong \angle CBD$, then $\angle A \cong \angle C$, and the opposite sides $\overline{AB}$ and $\overline{BC}$ are congruent.

5.

1. $\overleftrightarrow{AE} \parallel \overleftrightarrow{CD}$.	1. Given.
2. $\angle ACD \cong \angle BCD$.	2. Definition of bisect.
3. $\angle EAC \cong \angle ACD$.	3. Alternate interior angles.
4. $\angle E \cong \angle BCD$.	4. Corresponding angles.
5. $\angle E \cong \angle EAC$.	5. Statements 2, 3, and 4.
6. $AC = EC$.	6. Theorem 5-5.

6.

1. $\angle CAQ$ is a complement of $\angle BAQ$.	1. Definition of complement.
2. $\angle C$ is a complement of $\angle B$.	2. Corollary 9-13-2.
3. $\angle B \cong \angle BAQ$.	3. Base angle of an isosceles triangle.
4. $\angle CAQ \cong \angle C$.	4. Complements of congruent angles are congruent.
5. $QC = QA$.	5. Theorem 5-5.
6. $QB = QA$.	6. Given.
7. $QB = QC$.	7. Steps 4 and 5.

7. Given: The figure with trapezoid TRAP having $\overline{PA} \parallel \overline{TR}$, $PE = ET$ and $\overline{EZ} \parallel \overline{TR}$.
Prove: $PO = OR$.

1. $\overleftrightarrow{PA}$, $\overleftrightarrow{EZ}$ and $\overleftrightarrow{TR}$ are parallel.	1. Theorem 9-11.
2. $PE = ET$.	2. Given.
3. $PO = OR$.	3. Theorem 9-27.

Chapter 10

PARALLELS IN SPACE

This Chapter develops the properties of parallelism and perpendicularity of lines and planes in space and applies these properties to the study of projection of figures on a plane. Essentially the treatment is conventional. A minimum program would cover Section 10-1, studying the essential properties of parallelism of lines and planes and the related properties of perpendicularity. Section 10-2, which probably is more difficult, is devoted to dihedral angles and in particular to their application to the concept of perpendicular planes. Sections 10-1 and 10-2 give good coverage of the basic subject matter. Section 10-3, which could be taken if time and class ability permit, does not add to the student's basic knowledge of parallelism and perpendicularity but applies it to the interesting geometric problem of projecting figures into a plane.

In this Chapter you will see a very strong analogy
between the material concerning parallel lines in a plane as
described in Chapter 9, and the discussion of parallel planes
in space. For example Theorem 10-2, on a line perpendicular
292 to one of two parallel planes, is analogous to Theorem 9-12;
and Theorem 10-3, two planes perpendicular to the same line
293 are parallel, is analogous to Theorem 9-2, expressed in the
form: In a plane, two lines perpendicular to the same line
are parallel. In some cases the proofs are a bit more
involved, since we are working in space and not just in a
plane.

[pages 292-293]

Problem Set 10-1

296 1.
- a. True.
- b. True.
- c. False.
- d. True.
- e. True.
- f. True.
- g. False.
- h. True.
- i. True.
- j. False.
- k. True.
- l. False.

297 2. Let $\overline{AZ}$ intersect plane n at T. Draw $\overline{AX}$, $\overline{BT}$, $\overline{TY}$, and $\overline{CZ}$. Then $\overline{BT} \parallel \overline{CZ}$ and $\overline{TY} \parallel \overline{AX}$ by Theorem 10-1. From Theorem 9-26, in plane ACZ, AT = TZ; and in plane AZX, XY = YZ.

3.

1. $s \parallel r$.	1. Given.
2. $\overline{AB} \perp r$.	2. Given.
3. $\overline{AB} \perp s$.	3. Theorem 10-2.
4. $\overline{AB} \perp \overline{CX}$ and $\overline{AB} \perp \overline{CY}$.	4. Theorem 8-3.
5. $\Delta ACX \cong \Delta ACY$.	5. S.A.S.
6. AX = AY.	6. Corresponding parts.

4.

1. $m \perp \overline{AB}$, $n \perp \overline{AB}$.	1. Given.
2. $m \parallel n$.	2. Theorem 10-3.
3. $m \perp \overline{CD}$.	3. Given.
4. $n \perp \overline{CD}$.	4. Theorem 10-2.

298 5. By Theorem 10-5, AB = CD. Consider $\overline{BD}$. $\overline{AB} \perp \overline{BD}$ and $\overline{CD} \perp \overline{BD}$ by definition of a line perpendicular to a plane. Then $\Delta ABD \cong \Delta CDB$ by S.A.S. and AD = CB by corresponding parts.

298 *6. By Theorem 10-3 we know E || F. By Theorem 10-1 we know $\overleftrightarrow{AD} \parallel \overleftrightarrow{BK}$ and $\overleftrightarrow{CA} \parallel \overleftrightarrow{HB}$. Since BK = AD and BH = AC, we know we have two parallelograms. These are rectangles since $\overline{AB}$ is perpendicular to both planes and therefore to lines in the planes through A and B. $\angle$s CAD and HBK are plane angles of the dihedral angle D-AB-H and are congruent. Then $\triangle$ CAD $\cong$ $\triangle$ HBK by S.A.S. However, we do not know the measure of any of the angles of the two triangles and so cannot find the length of $\overline{CD}$.

*7. Let points D and G be such that AD = BG and E and F be such that AE = BF. Draw $\overline{DE}$ and $\overline{GF}$. Then:

(1) $\overline{AE} \parallel \overline{BF}$ and $\overline{AD} \parallel \overline{BG}$ by Theorem 10-1.

(2) AEFB and ADGB are parallelograms since they have two sides parallel and equal in length.

(3) EF = AB and DG = AB because opposite sides of a parallelogram have equal lengths. Also $\overline{EF} \parallel \overline{AB}$ and $\overline{DG} \parallel \overline{AB}$.

(4) Therefore EF = DG and $\overline{EF} \parallel \overline{DG}$ making EDGF a parallelogram by Theorem 9-20.

(5) ED = FG.

(6) $\triangle$ ADE $\cong$ $\triangle$ GFD by S.S.S.

(7) $\angle$ DAE $\cong$ $\angle$ GBF.

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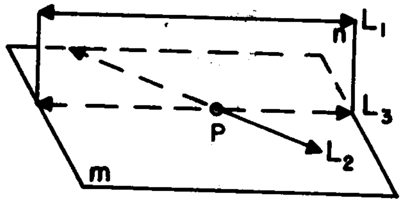
*8. Given two skew lines L_1 and L_2, at any point P on L_2 draw the one line L_3 parallel to L_1. Then L_2 and L_3 intersect and determine a plane parallel to L_1.



Proof: L_1 and L_3 are coplanar and determine a plane n because they are parallel. L_1 and L_2 cannot be coplanar because they are skew. Hence, L_2 and L_3 are distinct intersecting lines determining a plane m. Planes m and n have the line L_3 in common, hence it _is_ their intersection. L_1, which is in n, could intersect m only at some point of L_3; and this is impossible since $L_1 \parallel L_3$. Hence $L_1 \parallel m$.

*9. $\overline{QP} \parallel \overline{SM}$ by Corollary 10-4-2. In the plane of $\overline{RL}$ and $\overline{QP}$, $\overline{QP} \perp \overline{PL}$; and in the plane of $\overline{SM}$ and $\overline{QP}$, $\overline{QP} \perp \overline{PM}$. Since $\overline{QP}$ is perpendicular to both $\overline{PL}$ and $\overline{PM}$, $\overline{QP} \perp E$. Then both $\overline{RL}$ and $\overline{SM}$ are perpendicular to E by Corollary 10-4-1.

---

299 The notion of dihedral angle may seem strange to a student on first acquaintance. You might point out that just as angles arise in the practical problem of measuring the difference in direction of two lines, so dihedral angles are suggested when we have to specify the "difference in direction" of two planes. If you are designing a gable roof for your house, somehow you must specify the size of the angular opening between the sides of the roof.

In designing a building, an architect must specify the relative direction of plane surfaces. Ordinarily walls are perpendicular to floors, but in many modern buildings, planes appear which are not perpendicular to each other. There is implicit in this situation the notion of dihedral angle and the problem of measuring dihedral angles. Consider the every-day situation of specifying how steep a hill is. When we say a hill has an inclination of 25°, this can be interpreted as a statement about the angle formed by the plane of the hill and a horizontal plane.

You can illustrate dihedral angles very easily by using the covers or leaves of a book to represent the faces and the binding to represent the edge. You can use this to give the students some feeling for relative size of dihedral angles, bisection, perpendicularity, and so on.

299 Dihedral angles are important for theoretical reasons as well as for practical ones. Observe that planes are as important in space as lines are in a plane. If angles formed by lines are worth studying in a plane, it is natural to try to make a similar study of angles formed by planes in space. In studying the properties of points, lines and planes in space we naturally try to generalize planar concepts about lines to spatial concepts about lines and planes. Thus we study "angles formed by planes", perpendicularity of lines and planes and of planes and planes, and parallelism of lines and planes and of planes and planes.

299 Notice in the definition of dihedral angles, that we cannot just speak of the union of two half-planes, but that we must include their common edge in the union. This is because a half-plane does not contain its edge. Similarly the side or face of a dihedral angle is defined, not as a half-plane, but as the union of a half-plane and its edge. (This is sometimes called a "closed" half-plane to emphasize that the half-plane has been "closed up" by adjoining its bounding line - in contrast a half-plane in our sense is called an "open" half-plane.) Observe that the intersection of the two faces is their common edge, just as the intersection of the two sides of an (ordinary) angle is their common end-point.

[page 299]

300 Suggested definitions: Dihedral angles $\angle$ A-PQ-B and $\angle$ A'-PQ-B' are <u>vertical</u> if A and A' are on opposite sides of $\overleftrightarrow{PQ}$, and B and B' are on opposite sides of $\overleftrightarrow{PQ}$.

The <u>interior</u> of dihedral angle $\angle$A-PQ-B consists of all points which are on the same side of plane APQ as B and are on the same side of plane BPQ as A. The <u>exterior</u> of a dihedral angle consists of all points which are not in the interior of the angle and are not in the angle itself.

Notice that the rafters of a gable roof form plane angles of the dihedral angle formed by the sides of the roof.

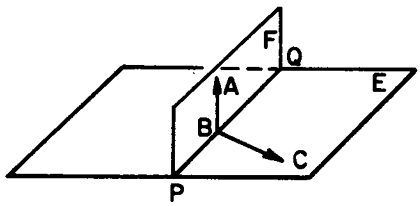
300 Some of your students may have difficulty in grasping the idea that a spatial object like a dihedral angle can be measured by its plane angle which is only a "planar" figure. You might point out that two dihedral angles will be "congruent", that is can be made to "fit", if and only if their plane angles are congruent, that is have equal measure. This can be illustrated with models of sheets of cardboard, folded lengthwise to form dihedral angles. Observe that they can be made to coincide if, and only if, corresponding plane angles can be made to coincide, that is if and only if the plane angles have equal measure. Similarly if you form a dihedral angle which is "twice as large" as a second (say by putting two "congruent" dihedral angles together), you can convince the student that the plane angle of the first has a measure which is twice as large as that of the second.

301 The formal significance of the above discussion is this. Although the text proper does not define congruence of dihedral angles, a general definition of congruence for any two figures is given in Appendix VIII, Rigid Motion. (See also the Talk on the Concept of Congruence.) Using this definition we can prove the theorem that two dihedral angles are congruent if and only if their plane angles are congruent.

301 We could have given a definition of right dihedral angle very similar to that for right angle. First by analogy with the idea of linear pair of angles (see page 82, Student Text), we can define "planar pair" of dihedral angles as follows: Dihedral angles $\angle$ A-PQ-B and $\angle$ A'-PQ-B form a planar pair if A and A' are on opposite sides of $\overleftrightarrow{PQ}$. Then if the dihedral angles of a planar pair have the same measure each is defined to be a right dihedral angle.

Proofs of the Corollaries

302 <u>Corollary</u> <u>10-6-1</u>. If a line is perpendicular to a plane, then any plane containing this line is perpendicular to the given plane.



Given: $\overleftrightarrow{AB} \perp E$, F contains $\overleftrightarrow{AB}$.
To Prove: $F \perp E$.

In E draw $\overleftrightarrow{BC} \perp \overleftrightarrow{PQ}$. Since $\overleftrightarrow{AB} \perp E$, then by the definition of a line perpendicular to a plane, $\overleftrightarrow{AB} \perp \overleftrightarrow{PQ}$, likewise $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$. Hence $\angle$ ABC is a plane angle of $\angle$ A-PQ-C, since $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$ are perpendicular to $\overleftrightarrow{PQ}$ at B. Since $\angle$ ABC is a right angle we see that $F \perp E$ by the definition of perpendicular planes.

302 <u>Corollary</u> <u>10-6-2</u>. If two planes are perpendicular, then any line in one of them perpendicular to their line of intersection is perpendicular to the other plane.

Given: $F \perp E$, $\overleftrightarrow{AB} \perp \overleftrightarrow{PQ}$.
To Prove: $\overleftrightarrow{AB} \perp E$.

Using the figure above, in E draw $\overleftrightarrow{BC} \perp \overleftrightarrow{PQ}$. Then by the definition of a plane angle, $\angle$ ABC is a plane angle of $\angle$ A-PQ-C. $F \perp E$ by hypothesis. Hence $\angle$ A-PQ-C is a right dihedral angle, and its plane angle, $\angle$ ABC, is a right angle, and $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$. Since it was given that $\overleftrightarrow{AB} \perp \overleftrightarrow{PQ}$, we now have $\overleftrightarrow{AB}$ perpendicular to two lines in E passing through its foot, hence by Theorem 8-3, $\overleftrightarrow{AB} \perp E$.

Problem Set 10-2

302 1. $\angle$ C-AB-D, $\angle$ A-BC-D, $\angle$ A-CD-B, $\angle$ B-AD-C, $\angle$ B-AC-D, $\angle$ A-BD-C.

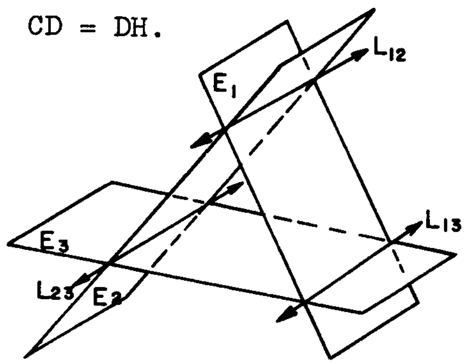
303 2. $\angle$ CPB is a plane angle of $\angle$ C-PA-B. Since $m\angle$ CPB $= 90$, $m\angle$ C-PA-B $= 90$. $m\angle$ CAB $= 60$ since Δ CAB can be proved to be equilateral.

3. a. 1
   b. 1
   c. 0
   d. 0
   e. 0
   f. 1
   g. 0
   h. 0
   i. 0
   j. 0
   k. 1.

304 4. $\overline{XP} \perp r$ and $\overline{YP} \perp s$ by Corollary 10-6-2. Then $\overline{XP} \perp \overline{QP}$ and $\overline{YP} \perp \overline{QP}$ by the definition of a line perpendicular to a plane. By Theorem 8-3, $\overline{QP} \perp E$. Since $\overline{XP} \perp m$, $\overline{XP} \perp \overline{PQ}$ and $\angle$ XPQ is a right angle. Therefore $\angle$ X-AB-Q is a right dihedral angle, and by definition of perpendicular planes $s \perp m$.

5. $x = 45$; $m = 45$; $y = 90$. CD = DH.

305 *6. If $E_3 \parallel L_{12}$, then E_3 and L_{12} do not meet. Then L_{12} and L_{13} do not meet; and since they both lie in E_1, they are parallel. Similarly, L_{12} and L_{13} are parallel. Also L_{13} and L_{23} do not intersect, for if they did intersect at a point P this point would lie in each of E_1, E_2, E_3, and E_3 would meet L_{12} at P which in this case is not possible.



If E_3 intersects L_{12} at point P, P lies on each of E_1, E_2 and E_3, and hence in L_{12}, L_{13}, and L_{23}. Hence all three lines are concurrent at P.

305 *7. Point X lies in plane ABC and also in plane A'B'C', and hence on their intersection. Similarly, Y and Z lie on the intersection on these two planes, or the points X, Y, Z lie on a line u, which was to be proved.

Remark 1. The two non-parallel planes always intersect, but it might happen that $\overleftrightarrow{B'C'}$ and $\overleftrightarrow{BC}$ are parallel lines, so there would be no point X. This would happen if and only if $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are both parallel to the line u. This could not happen for two pairs of side-lines for we could not have two lines through a vertex parallel to u.

Remark 2. The Theorem is also valid if plane ABC = plane A'B'C', but we have not proved it.

Desargues' Theorem is an interesting and important incidence theorem relating concurrence of lines with collinearity of points. The theorem is also valid when the two triangles are coplanar, but is much harder to prove. In this case the student can get an intuitive appreciation of its correctness by imagining the figure to collapse into a plane.

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306 The theory of projections is important in engineering, particularly in drafting. Speaking broadly it may be considered part of the subject of "map" making or the representation of geometrical objects on a given surface, usually taken as a plane. (See Problem 4 of Problem Set 10-3. for an indication of the use of projections in giving planar representations of a solid object.) The study of projection throws light on familiar visual experiences. For example, if we look at a circle, inclined so that its plane is oblique to the line of sight, it appears as an ellipse - that is, we see it as if it were projected on a plane which is perpendicular to the line of vision.

307 Observe that the definition of S' the projection of a set of points S as the set of projections of all points of S means two things. Namely, that the projection of every point of S must be <u>in</u> S', and, in addition, that such projections form the <u>whole</u> of S'. That is each point of S' must be the projection of some point of S. Otherwise S' would contain the projection of S and additional points besides. As a homely illustration of a similar situation consider the statement that the Yale Mathematics Department is the same as the Olympic Hockey Team. Disregarding its improbability, this statement asserts two things. First that every member of the Yale Mathematics Department is a member of the Olympic Hockey Team. But further, that every member of the Olympic Hockey Team is a member of the Yale Mathematics Department - otherwise the Olympic Hockey Team would be a larger set than the Yale Mathematics Department. To summarize: in identifying a set S' as the projection of S we will have to prove a characterization theorem for S' involving a theorem and its converse.

308 The conventional phrase is to project a point or figure "onto" a plane rather than "into" a plane. We have changed this in order to be consistent with mathematical usage in the theory of mappings or transformations. A mapping is a correspondence which associates with each point of a given set S a unique point of a set S'. We describe this by saying that each point of S is "mapped into" its associated point of S' and that S is "mapped into" S'. We say S is "mapped onto" S' only when the whole of S' is involved, that is when each point of S' is the associated point of some point of S. Since this distinction between "into" and "onto" is quite firmly established in higher mathematics we thought it wise to use the appropriate technical term "into" even at this elementary level.

308 The answer to why M intersects L: M and L both lie in F. Suppose $M \parallel L$. Then by Corollary 10-4-1 $M \perp E$ implies $L \perp E$. This contradicts the hypothesis. Therefore M must intersect L.

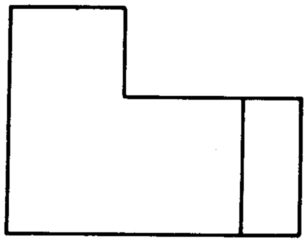
Problem Set 10-3

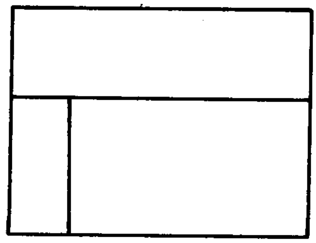
309 1. a. Yes. d. Yes.
b. No. e. No.
c. Yes; yes; yes. f. No.

2. a. Not necessarily. c. Yes.
b. No. d. Yes.

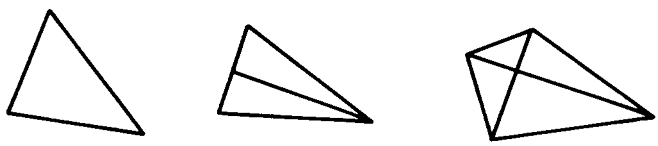
310 3. $\overline{AX}$ and $\overline{BY}$ are perpendicular to plane m. Hence $\overline{AX} \parallel \overline{BY}$ and ABYX is a plane figure. Since the projection of a segment is a segment (or a point) N is in this plane. Since $\overline{MN} \perp m$, $\overline{MN} \parallel \overline{AX}$ and $\overline{MN} \parallel \overline{BY}$. Then $XN = NY$ so that N is the mid-point of $\overline{XY}$ because parallels which intercept congruent segments on one transversal intercept congruent segments on any transversal.

310 4. a. b.





311 5. Since the intersection point shown within the large triangle in the problem may be on a vertex, on an edge or on the extension of an edge, or elsewhere in the exterior of the large triangle the projection may appear as follows:



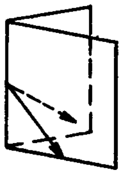
6. Let $\overline{BE}$ be the perpendicular to plane m at B. Then $\overline{AB} \perp \overline{BE}$, and it is given that $\overline{AB} \perp \overline{BC}$. Hence $\overline{AB} \perp$ plane EBC. By definition of projection $\overline{CD} \perp$ m. Then $\overline{CD} \parallel \overline{BE}$ so that D is in the plane EBC. Then $\overline{DB}$ is in this plane and $\overline{AB} \perp \overline{BD}$ or $\angle$ ABD is a right angle.

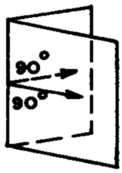
312 *7. By definition of projection $\overline{QQ'} \perp$ m and therefore also $\overline{QQ'} \perp \overline{Q'X}$ so that Δ QQ'X is a right triangle. Then QQ' < QX. But $\overline{AQ} \cong \overline{AQ}$, and $\overline{AX} \cong \overline{AQ'}$. In triangles QAX and QAQ', m$\angle$ QAQ' < m$\angle$ QAX by Theorem 7-9, which was to be proved.

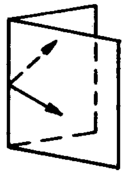
*8. The projection is a regular hexagon with segments from its vertices to its center.

Review Problems

312 1. a. Yes
b. Yes.
c. Yes.







313 2. No. No.

3. Yes. A plane angle of a dihedral angle is the angle formed by two rays, one in each side of the dihedral angle and perpendicular to its edge at the same point. No. 90.

4.

a.	S.	h.	S.
b.	S.	i.	S.
c.	S.	j.	S.
d.	A.	k.	S.
e.	S.	l.	A.
f.	S.	m.	S.
g.	A.	n.	S.

314 5.

1.	$\overline{AF} \perp E$.	1.	Definition of projection.
2.	Plane ABF $\perp$ E.	2.	Corollary 10-6-1.
3.	$\overline{HB} \perp \overline{FB}$.	3.	Definition of perpendicular.
4.	$\overline{HB} \perp$ plane ABF.	4.	Corollary 10-6-2.
5.	$\overline{HB} \perp \overline{AB}$.	5.	Definition of line perpendicular to a plane.
6.	$\angle$ ABH is a right angle.	6.	Definition of perpendicular.

315 6.

1.	$\overline{BD} \parallel \overline{CE}$.	1.	Theorem 10-1.
2.	$\angle ADB \cong \angle E$.	2.	Corresponding angles of parallel lines.
3.	$\angle A \cong \angle E$.	3.	Hypothesis and base angles of an isosceles triangle.
4.	$\angle ADB \cong \angle A$.	4.	Steps 2 and 3.
5.	$BD = AB$.	5.	Theorem 5-5.

7.

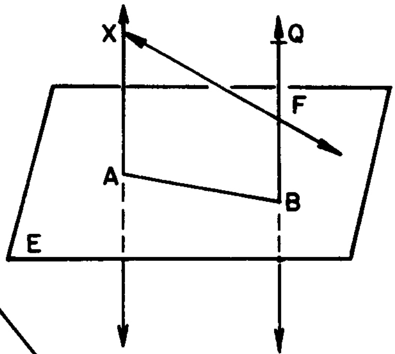
1.	$\overline{RX} \parallel \overline{BD}$ and $RX = \frac{1}{2}BD$. $\overline{YZ} \parallel \overline{BD}$ and $YZ = \frac{1}{2}BD$.	1.	Theorem 9-22.
2.	$\overline{RX} \parallel \overline{YZ}$.	2.	Corollary 10-4-2.
3.	$RX = YZ$.	3.	Step 1.
4.	R, X, Y, Z are coplanar.	4.	Theorem 9-1.
5.	RXYZ is a parallelogram.	5.	Two sides both congruent and parallel.

*8. plane, $\parallel$, plane, plane, $\perp$, $\perp$.
plane, $\perp$, plane, plane, $\parallel$, $\perp$.
plane, $\parallel$, plane, line, $\perp$, $\perp$.
plane, $\parallel$, line, plane, $\perp$, $\perp$.
plane, $\perp$, line, line, $\parallel$, $\perp$.
line, $\parallel$, line, plane, $\perp$, $\perp$.
line, $\perp$, plane, plane, $\parallel$, $\perp$.
plane, $\perp$, line, plane, $\parallel$, $\perp$.

*9. X is the mid-point of $\overline{BD}$ and of $\overline{AC}$. $\overline{AB}$, $\overline{BF}$, $\overline{XY}$, $\overline{DH}$, $\overline{CG}$ are parallel segments. (Theorem 9-2). Y is the mid-point of $\overline{FH}$ and $\overline{EG}$. (Theorem 9-26). In trapezoid AEGC, $XY = \frac{1}{2}(AE + CG)$ (See Problem 10 of Problem Set 9-6). In trapezoid BFHD, $XY = \frac{1}{2}(BF + DH)$. $\therefore AE + CG = BF + DH$.

Illustrative Test Items for Chapter 10

A. 1. Given: $\overleftrightarrow{XA} \perp E$ at A. $\overleftrightarrow{QB} \perp E$ at B. F is a point on $\overleftrightarrow{QB}$. Are X, A, B, F coplanar? State a theorem to support your conclusion. What is $m\angle XAB$? If $m\angle BFX = 135$, what is $m\angle AXF$?

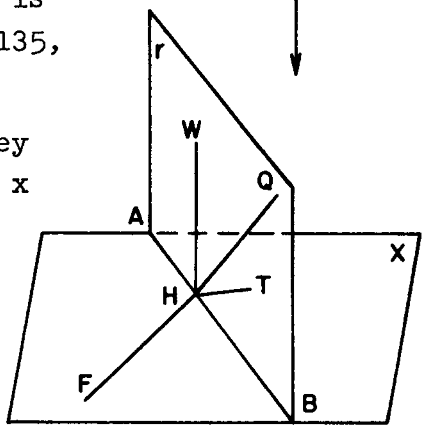


2. Plane x $\perp$ plane r. They intersect in $\overleftrightarrow{AB}$. In x $\overline{FH} \perp \overleftrightarrow{AB}$. $\overline{WH}$, $\overline{QH}$, $\overline{TH}$ lie in plane r.

$m\angle FHW$ = ________.

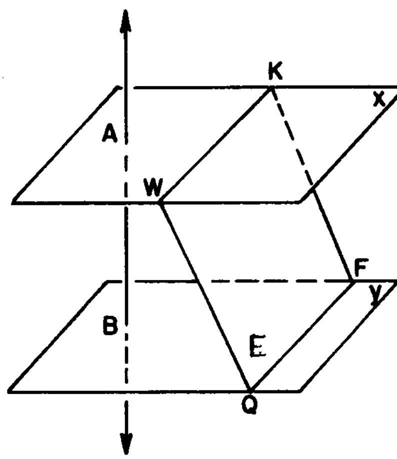
$m\angle FHQ$ = ________.

$m\angle FHT$ = ________.



Support your conclusions with suitable principles. On the basis of the given information we cannot say that any of these three angles is a plane angle of dihedral $\angle$ W-AB-F. $\angle$ WHF would be a plane angle of $\angle$ W-AB-F if $\overline{WH} \perp$ ______.

3. In the figure, plane x $\perp \overleftrightarrow{AB}$ and plane y $\perp \overleftrightarrow{AB}$. Is x $\parallel$ y? State a theorem to support your conclusion. Plane E intersects x in $\overleftrightarrow{WK}$ and y in $\overleftrightarrow{QF}$. $\overleftrightarrow{WK}$ ____ $\overleftrightarrow{QF}$. If a line L is perpendicular to $\overleftrightarrow{WK}$ and intersects $\overleftrightarrow{QF}$, what kind of angles does L make with $\overleftrightarrow{QF}$?



B. Indicate whether true or false.

a. If a plane is perpendicular to each of two lines, the two lines are coplanar.

b. If a plane intersects two other planes in parallel lines, then the two planes are parallel.

c. Two planes perpendicular to the same line are parallel.

d. If each of two planes is parallel to a line, the planes are parallel to each other.

e. The projection of a line into a plane is always a line.

f. Two lines are parallel if they have no point in common.

g. For each acute angle there is a plane such that the projection of the acute angle into the plane is an obtuse angle.

h. The length of the projection of a segment into a plane is always less than the length of the segment.

i. Two lines parallel to the same plane are parallel to each other.

j. If each of two intersecting planes is perpendicular to a third plane, their line of intersection is perpendicular to the third plane.

k. If a line not contained in a plane is perpendicular to a line in the plane, then it is perpendicular to the plane.

l. If a plane bisects a segment, every point of the plane is equidistant from the ends of the segment.

m. At a point on a line there are infinitely many lines perpendicular to the line.

n. Through a point outside a plane there is exactly one line perpendicular to the plane.

o. If plane E is perpendicular to $\overleftrightarrow{AB}$ and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, then $E \perp \overleftrightarrow{CD}$.

p. A plane perpendicular to one of two perpendicular planes is never perpendicular to the other plane.

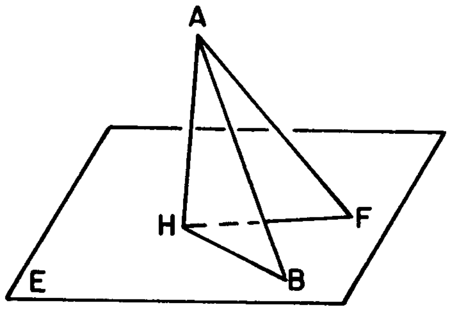
q. If plane M is perpendicular to plane N and Δ ABC lies in plane M, then the projection of Δ ABC into plane N is a line segment.

r. It is possible for the measure of a plane angle of an acute dihedral angle to be 90.

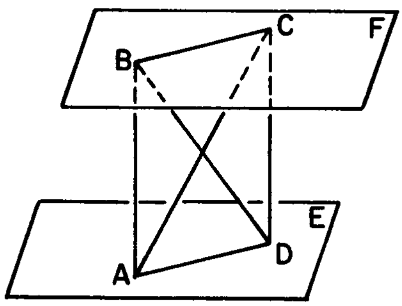
s. Any two plane angles of a given dihedral angle are congruent.

t. If a line is not perpendicular to a plane, then each plane containing this line is not perpendicular to the plane.

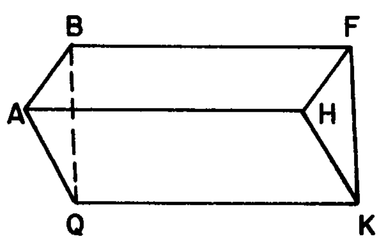
C. 1. Given: H is the projection of A into plane E. $\overline{HB}$ is the projection of $\overline{AB}$ into E. $\overline{HF}$ is the projection of $\overline{AF}$ into E. AF = AB.
Prove: HF = HB.



2. Given: E || F.
$\overleftrightarrow{AB} \perp$ E at A.
$\overleftrightarrow{CD} \perp$ E at D.
Prove: AC = BD.



3. Given: $\overline{AH} \parallel \overline{BF} \parallel \overline{QK}$.
AH = BF = QK.
Prove: Δ ABQ $\cong$ Δ HFK.



Answers

A. 1. Yes. Statement of Theorem 8-8 or 10-4. 90. 45.

2. 90. 90. 90. Statements of Corollary 10-6-2 and the definition of a line perpendicular to a plane. $\overleftrightarrow{AB}$.

3. Yes. Statement of Theorem 10-3. $\overleftrightarrow{WK} \parallel \overleftrightarrow{QF}$. Right angles.

B.

a.	T.	h.	F.	o.	T.
b.	F.	i.	F.	p.	F.
c.	T.	j.	T.	q.	T.
d.	F.	k.	F.	r.	F.
e.	F.	l.	F.	s.	T.
f.	F.	m.	T.	t.	F.
g.	T.	n.	T.		

C. 1.

1. $\overline{AH} \perp E$.	1. Definition of projection.
2. $\angle AHF$ and $\angle AHB$ are right angles.	2. Definition of a line perpendicular to a plane.
3. $AH = AH$.	3. Identity.
4. $\triangle AHF \cong \triangle AHB$.	4. Leg-Hypotenuse Theorem.
5. $HF = HB$.	5. Corresponding parts.

2.

1. $\overline{AB} \parallel \overline{CD}$.	1. Theorem 10-4.
2. A, B, C, D are coplanar and so determine a quadrilateral.	2. Theorem 8-8 or Theorem 9-1.
3. $\overline{AB} \perp F$. $\overline{CD} \perp F$.	3. Theorem 10-2.
4. $AB = CD$.	4. Theorem 10-5.
5. ABCD is a parallelogram.	5. Two sides congruent and parallel.
6. $\angle BAD$ is a right angle.	6. Definition of a line perpendicular to a plane.