STUDENT'S TEXT

UNIT NO. 29

CSMP

MATHEMATICS FOR THE ELEMENTARY SCHOOL GRADE 5

PART I

SCHOOL MATHEMATICS STUDY GROUP

YALE UNIVERSITY PRESS

School Mathematics Study Group

Mathematics for the Elementary School, Grade 5

Unit 29

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Student's Text, Part I

REVISED EDITION

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New Haven and London, Yale University Press, 1963

Financial support for the School Mathematics
Study Group has been provided by the
National Science Foundation.

FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

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PREFACE

As one of its contributions to the improvement of mathematics in the schools of this country, the School Mathematics Study Group has prepared a series of sample text materials for grades 4 through 6. These are designed to illustrate a kind of mathematics curriculum that we believe appropriate for elementary schools.

This volume is a portion of these materials which were prepared by a group of 30 individuals, divided almost equally between distinguished college and university mathematicians and master elementary teachers and consultants. A strong effort has been made on the part of all to make the content of this text material mathematically sound, appropriate and teachable. Preliminary versions were used in numerous classrooms both to strengthen and to modify these judgments.

The content is designed to give the pupil a much broader concept, than has been traditionally given at this level, of what mathematics really is. There is less emphasis on rote learning and more emphasis on the construction of models and symbolic representation of ideas and relationships from which pupils can draw important mathematical generalizations.

The basic content is aimed at the development of some of the fundamental concepts of mathematics. These include ideas about:

number; numeration; the operations of arithmetic; and intuitive geometry. The simplest treatment of these ideas is introduced early. They are frequently re-examined at each succeeding level and opportunities are provided throughout the texts to explore them more fully and apply them effectively in solving problems. These basic mathematical understandings and skills are continually developed and extended throughout the entire mathematics curriculum, from grades K through 12 and beyond.

We firmly believe mathematics can and should be studied with success and enjoyment. It it our hope that these texts may greatly assist all pupils and teachers who use them to achieve this goal, and that they may experience something of the joy of discovery and accomplishment that can be realized through the study of mathematics.

Chapter 1

EXTENDING SYSTEMS OF NUMERATION

UNDERSTANDING OUR SYSTEM OF NUMERATION

Place Value Name	Hundred Millions	Ten Millions	One Millions	Hundred Thousands	Ten Thousands	One Thousands	Hundreds	Tens	Ones
Digits			1,	2	3	4,	5	6	7

In our decimal system each place or position in a numeral has a name. This name tells its value - ones, tens, hundreds, etc. For instance, in 24, the 4 means 4 ones. In 421, the 4 means 4 hundreds.

Look at the chart above. Tell what number is represented by each digit in the numeral 1,234,567.

If the 5 in the numeral above is changed to 9, how much was added to the original number?

What happens to the number, if the 3 is replaced with a 0?

READING LARGE NUMBERS

Period	Million			Thousand			Units		
Place Name	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones
Digits			1,	2	7	4,	3	6	5

To make it easier to read numerals for large numbers, the names of the digits, the place-value name and the period name are used. To read the numeral in the table above begin with the period on the left. Read the digit or digits in the first period as one numeral, followed by the name of the period, as "one <u>million</u>".

Then read the second group of digits as one numeral, followed by the name of the period, as "two hundred seventy-four <u>thousand</u>".

Now read the third group of digits as one numeral without the period name, as "three hundred sixty-five".

The complete numeral is read, "one million, two hundred seventy-four thousand, three hundred sixty-five".

In what place is each digit written in the numeral 1,274,365?

How many commas were used in writing this numeral?

Why is each period separated by a comma?

Explain how to place the commas to help you read a numeral.

Read each of the following numerals.

7,862,419	18,771	5,440,103
275,002	9,030,210	4,564,300

Exercise Set 1

1. What number is represented by the symbol 3 in each numeral below?

a)	234,600	d)	413,062
b)	98,532	e)	6,371,524
c)	3,827,129	f)	9,317

2. Write the decimal numeral for each of these.
   a) Six thousand, nine hundred thirty-seven
   b) Nine hundred eight thousand, thirteen
   c) Four hundred thirty thousand, nine hundred ninety-nine
   d) Eight million, three hundred five thousand, two hundred fifty-four
   e) Two million, eight hundred twenty thousand, one

3. Write the name of each numeral in Exercise 1.

Braintwisters

4. Write the decimal numeral for each of these.
   a) Twenty-two million, four hundred seven thousand, three hundred sixty-one
   b) Seven hundred thirty-six million, five hundred twenty-five thousand, two hundred thirteen
   c) Three hundred million, forty thousand, six

5. Write the largest possible nine-place decimal numeral using the digits 3, 4, and 6 just once, and as many zeros as necessary.

EXPANDED NOTATION

To better understand a number, we learned to add the numbers represented by each digit in the numeral for that number. For example, we learned that 352 can be thought of as 300 + 50 + 2.

Since 300 means 3 hundreds, we can write it as (3×100). 50 means 5 tens, which can be written as (5×10). 2 ones can be written as (2×1). Writing 352 as $(3 \times 100) + (5 \times 10) + (2 \times 1)$ is called <u>expanded</u> <u>notation</u>.

Look at the numerals in the chart below. Place values are written at the top of the chart. Use the chart to help you see how these numerals are written in expanded notation.

	1,000,000	100,000	10,000	1,000	100	10	1	
a				4	2	8	3	$= (4 \times 1000) + (2 \times 100) + (8 \times 10) + (3 \times 1)$
b			2	3	5	8	4	$= (2 \times 10{,}000) + (3 \times 1{,}000) + (5 \times 100) + (8 \times 10) + (4 \times 1)$
c		6	2	8	7	3	9	$= (6 \times 100{,}000) + (2 \times 10{,}000) + (8 \times 1{,}000) + (7 \times 100) + (3 \times 10) + (9 \times 1)$
d	7	9	4	3	2	1	5	$= (7 \times 1{,}000{,}000) + (9 \times 100{,}000) + (4 \times 10{,}000) + (3 \times 1{,}000) + (2 \times 100) + (1 \times 10) + (5 \times 1)$

Exercise Set 2

1. Write the decimal numeral for each of these following in expanded notation.

   a) 8,134 d) 2,591,622

   b) 2,236 e) 49,525

   c) 14,892 f) 835,731

2. Write the decimal numeral for each of these.

   a) $(4 \times 1{,}000) + (2 \times 100) + (2 \times 10) + (3 \times 1)$

   b) $(5 \times 1{,}000) + (8 \times 100) + (1 \times 10) + (7 \times 1)$

   c) $(2 \times 10{,}000) + (2 \times 1{,}000) + (9 \times 100) + (6 \times 10) + (5 \times 1)$

   d) $(9 \times 10{,}000) + (3 \times 1{,}000) + (7 \times 10) + (4 \times 1)$

   e) $(8 \times 100{,}000) + (1 \times 10{,}000) + (6 \times 1{,}000) + (5 \times 100) + (9 \times 10) + (2 \times 1)$

3. Write the decimal numeral for each of these. Look carefully at this exercise.

   a) $(6 \times 10) + (3 \times 100) + (5 \times 1)$

   b) $(4 \times 100) + (1 \times 1{,}000) + (7 \times 1) + (3 \times 10)$

   c) $(6 \times 1) + (9 \times 1{,}000) + (2 \times 10)$

   d) $(4 \times 10{,}000) + (8 \times 10) + (2 \times 1) + (2 \times 100) + (7 \times 1{,}000)$

   e) $(8 \times 1{,}000) + (3 \times 10) + (4 \times 100{,}000) + (5 \times 1) + (6 \times 100)$

4. BRAINTWISTER. Fill in the blanks so these mathematical sentences are true.

a) $(4 \times 100) + (5 \times 10{,}000) + (6 \times 1{,}000) + (8 \times 1) + (\quad) = 56{,}478$

b) $(9 \times 1{,}000) + (8 \times 1) + (\quad) + (1 \times 10{,}000) + (8 \times 10) = 19{,}588$

c) $(9 \times 10) + (\quad) + (8 \times 100) + (6 \times 10{,}000) + (\quad) + (2 \times 100{,}000) = 263{,}897$

d) $(5 \times 10) + (\quad) + (2 \times 10{,}000) + (\quad) + (8 \times 1) = 420{,}358$

RENAMING LARGER NUMBERS

Below are examples showing some of the ways a number can be named.

A.

25,000 = 2 ten thousands + 5 thousands
25,000 = 25 thousands
25,000 = 25,000 ones
25,000 = 250 hundreds
25,000 = 2,500 tens

B.

426,315 = 4 hundred thousands + 2 ten thousands + 6 thousands + 3 hundreds + 1 ten + 5 ones
426,315 = 42 ten thousands + 6 thousands + 3 hundreds + 1 ten + 5 ones
426,315 = 426 thousands + 3 hundreds + 1 ten + 5 ones
426,315 = 425 thousands + 13 hundreds + 15 ones
426,315 = 400,000 + 20,000 + 6,000 + 300 + 10 + 5

Exercise Set 3

1. Write four different names for each of these numbers.

   a) 14,651 c) 230,000

   b) 27,748 d) 632,110

2. Write the decimal numeral for each of the following.

   a) Twelve thousands + three hundreds + seventeen ones

   b) Thirty-eight ten thousands + eight thousands + ninety-four tens + two ones

   c) Four ten thousands + twenty-eight hundreds + fifty-three ones

3. Write each of the following as a decimal numeral.

   a) 365 tens + 7 ones

   b) 46 hundreds + 2 tens + 5 ones

   c) 16 thousands + 12 hundreds + 14 tens

   d) 29 ten thousands + 3 thousands + 73 tens + 16 ones

4. Write each of the answers in Exercise 3 in expanded notation.

DECIMAL NAMES FOR RATIONAL NUMBERS

We have learned how to name rational numbers using symbols such as $\frac{2}{7}$ and $\frac{11}{12}$, called fractions. When a fraction has a denominator 10 or 100, as in $\frac{7}{10}$ or $\frac{53}{100}$, there is another way in which we can write its name.

The chart below shows how we can extend the idea of place-value to the right of the ones' place. Using this idea we can name rational numbers like $\frac{7}{10}$ and $\frac{53}{100}$ in a new way.

Hundreds	Tens	Ones	Tenths	Hundredths
			.7	
			.5	3

The name .7 and the name $\frac{7}{10}$ are names for the same rational number. Both names are read in the same way: "seven tenths".

The name .53 and the name $\frac{53}{100}$ are names for the same rational number. Both names are read in the same way: "fifty-three hundredths".

Names like $\frac{7}{10}$ and $\frac{53}{100}$ are called fractions. Names like .7 and .53 are new examples of decimal numerals. We will usually shorten "decimal numeral" to "decimal".

The dot (.) in a decimal is called the decimal point.

In .7, the 7 is written in the tenths' place. In .53, the 5 is written in the tenths' place and the 3 is written in the hundredths' place.

1. Are .7 and $\frac{7}{10}$ names for the same number?
   a. Which name is a decimal?
   b. Which name is a fraction?

2. Are $\frac{53}{100}$ and .53 names for the same number?
   a. Which name is a decimal?
   b. Which name is a fraction?

3. Are .3 and .03 names for the same number? Check your answer by writing each name as a fraction.

4. Are .7 and .70 names for the same number? Check your answer by writing each name as a fraction.

Exercise Set 4

1. Rename each of these as a decimal.

$$\frac{1}{10} \quad \frac{29}{100} \quad \frac{75}{100} \quad \frac{8}{10} \quad \frac{4}{100} \quad \frac{2}{10} \quad \frac{30}{100}$$

2. Rename each of these as a fraction.

.15 .9 .1 .82 .05 .4 .60

3. Copy and finish the following counting chart using decimals.

.01	.02	.03				.07	.08	.09	.10
.11	.12						.18	.19	.20
.21									
		.33							
			.44						
				.55					
					.66				
						.77			
							.88		
								.99	

4. Look at the decimals in the last column of the chart you just completed (.10, .20, .30, etc.) Each of these decimals may be replaced by another decimal. (For example, .1 is another name for .10.) To the right of the chart, write another decimal for each decimal in the last column.

5. Complete each of these.

a) .16, .18, .20, _____, _____, _____.

b) .24, .27, .30, _____, _____, _____.

c) .37, .39, .41, _____, _____, _____.

d) .43, .48, .53, _____, _____, _____.

e) .90, .80, .70, _____, _____, _____.

f) .85, .75, .65, _____, _____, _____.

g) .68, .64, .60, _____, _____, _____.

h) .58, .55, .52, _____, _____, _____.

6. Write T if the mathematical sentence is true. Write F if it is false.

a) $.50 = .5$

b) $.7 < .07$

c) $\frac{23}{100} > .23$

d) $\frac{4}{100} \neq .4$

e) $\frac{45}{100} < .54$

f) $.72 > .8$

g) $\frac{9}{10} < .65$

h) $\frac{50}{100} \neq .05$

BRAINTWISTERS

Can we rename $\frac{2}{5}$ as a decimal? Can we rename $\frac{9}{25}$ as a decimal? We can if first we are able to rename it as a fraction with a denominator of 10 or 100.

We can rename $\frac{2}{5}$ as $\frac{4}{10}$. We can rename $\frac{2}{5}$ as the decimal, _____. Also, we can rename $\frac{9}{25}$ as $\frac{}{100}$. So we can rename $\frac{9}{25}$ as the decimal, _____.

Now rename each of these as a decimal.

$\frac{1}{2}$ $\qquad \frac{9}{20}$ $\qquad \frac{47}{50}$ $\qquad \frac{3}{5}$ $\qquad \frac{18}{25}$ $\qquad \frac{10}{40}$

RENAMING DECIMALS

We have learned to think about a decimal like .73 as 73 hundredths. We also know that in .73, the 7 is in the tenths' place and the 3 is in the hundredths' place. This gives us another way to name .73:

.73 = 7 tenths and 3 hundredths.

In the same way,

.49 = ___ tenths and ___ hundredths.

We also can say

8 tenths and 2 hundredths = .82.

In the same way,

3 tenths and 6 hundredths = _____.

Exercise Set 5

1. Finish each of these

   a) .29 = _____ tenths and _____ hundredths.

   b) .58 = _____ tenths and _____ hundredths.

   c) .41 = _____ tenths and _____ hundredths.

   d) .80 = _____ tenths and _____ hundredths.

   e) .04 = _____ tenths and _____ hundredths.

   f) .36 = _____ hundredths and _____ tenths.

2. Write the decimal for each of these.

   a) 5 tenths and 7 hundredths = _____.

   b) 9 tenths and 3 hundredths = _____.

   c) 1 tenth and 6 hundredths = _____.

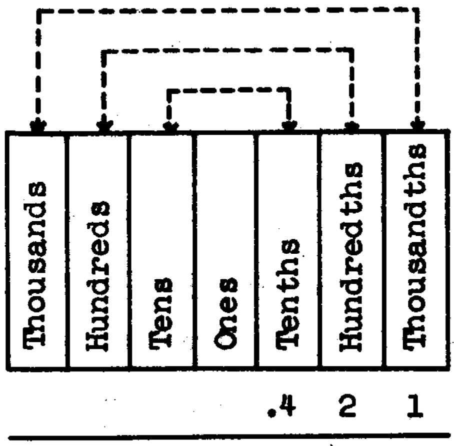
   d) 2 tenths and 0 hundredths = _____.

   e) 0 tenths and 4 hundredths = _____.

   f) 5 hundredths and 3 tenths = _____.

DECIMALS WITH THOUSANDTHS

We have learned how to extend place-value for decimals from tenths to hundredths. Using what we have learned, let us extend the place-value chart another place to the right. This is called the thousandths' place.



The name .421 and the name $\frac{421}{1000}$ are names for the same rational number. Both names are read as "four hundred twenty-one thousandths".

In .421 the 4 is written in the tenths' place, the 2 is written in the hundredths' place and the 1 is written in the thousandths' place.

1. Are $\frac{421}{1000}$ and .421 names for the same number?
   a. Which name is a decimal?
   b. Which name is a fraction?

2. Which is larger, .2, .02, or .002? Check your answer by naming each number as a fraction.

3. Are .2, .20, and .200 all names for the same rational number? Check your answer by writing each as a fraction.

Another way to think about and name .421 is 4 tenths and 2 hundredths and 1 thousandth.

In the same way,

.582 = _____ tenths and _____ hundredths and _____ thousandths.

Finish each of these.

a) .138 = ___ tenth and ___ hundredths and ___ thousandths.

b) .140 = ___ tenth and ___ hundredths and ___ thousandths.

c) .306 = ___ tenths and ___ hundredths and ___ thousandths.

d) .374 = ___ hundredths and ___ thousandths.

e) .009 = ___ tenths and ___ hundredths and ___ thousandths.

Exercise Set 6

1. Rename each of these as a decimal.

$\frac{32}{1000}$ $\frac{5}{1000}$ $\frac{9}{10}$ $\frac{492}{1000}$ $\frac{18}{1000}$ $\frac{174}{1000}$ $\frac{8}{1000}$ $\frac{18}{100}$

2. Rename each of these as a fraction.

.475 .011 .8 .023 .62 .729 .007

3. Write T if the mathematical sentence is true. Write F if it is false.

a) $.6 = .600$

b) $.9 > .009$

c) $\frac{23}{1000} > .23$

d) $\frac{8}{10} < .85$

e) $\frac{52}{100} \neq .052$

f) $.79 = \frac{79}{1000}$

g) $.008 > \frac{8}{1000}$

h) $.072 < .72$

4. Arrange the three numbers in each group in order of size. Name the smallest number first in each case.

a) .003 .3 .03

b) .37 .037 .3

c) .402 .42 .042

d) .560 .506 .056

5. Complete each of these.

a) .058 .060 .062 ____ ____ ____

b) .007 .012 .017 ____ ____ ____

c) .550 .450 .350 ____ ____ ____

d) .755 .760 .765 ____ ____ ____

e) .042 .142 .242 ____ ____ ____

6. Complete

a) .729 = ____ thousandths and ____ hundredths and ____ tenths.

b) .402 = ____ tenths and ____ hundredths and ____ thousandths.

c) .519 = ____ tenths and ____ hundredth and ____ thousandths.

d) .052 = ____ thousandths and ____ hundredths and ____ tenths.

e) .530 = ____ tenths and ___ hundredths and ____ thousandths.

7. Write the decimal for each of these.

a) 5 thousandths and 3 hundredths and 4 tenths = "____."?

b) 0 thousandths and 2 hundredths and 3 tenths = "____."?

c) 6 thousandths and 4 hundredths and 8 tenths = "____."?

d) 5 tenths and 0 hundredths and 5 thousandths = "____."?

e) 4 thousandths and 2 hundredths and 0 tenths = "____."?

OTHER DECIMALS

We have been learning how to read and interpret decimals such as .7 and .39 and .561. We already knew the meaning of decimal numerals such as 82, 7, or 356. Many times we need to use rational numbers which are greater than one but are not whole numbers. We already have fraction names for some of these numbers, names like $\frac{11}{10}$, $\frac{12}{10}$, $\frac{21}{10}$, or $\frac{125}{100}$. Since these all have denominators which are 10 or 100 we should be able to find decimal names for them and for numbers like them.

We might begin by thinking of counting by <u>tenths</u>.

The number line below shows counting by tenths with decimals and with fractions. We need decimal numerals to complete the top line.

decimals	0	.1	.2	.3	.4	.5	.6	.7	.8	.9											
fractions	$\frac{0}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$	$\frac{10}{10}$	$\frac{11}{10}$	$\frac{12}{10}$	$\frac{13}{10}$	$\frac{14}{10}$	$\frac{15}{10}$	$\frac{16}{10}$	$\frac{17}{10}$	$\frac{18}{10}$	$\frac{19}{10}$	$\frac{20}{10}$

$\frac{10}{10} = 1$

$\frac{11}{10}$ = eleven tenths = one and one tenth.

We express this as a decimal numeral by writing 1.1. The numeral 1 on the left stands for 1 <u>one</u>. The numeral 1 on the right stands for 1 <u>tenth</u>.

1. Use this idea to copy and complete the number line shown above. When we are thinking in tenths we usually write 1.0 (one and 0 tenths) instead of 1 and 2.0 instead of 2.

2. Write a decimal for each of the following:

a) 1 ten and 1 one

b) 1 tenth and 1 hundredth

c) 1 one and 1 hundredth

We read 2.3 as "two and three tenths", and 1.25 is read as "one and twenty-five hundredths". The chart below should help us to read and interpret other decimals.

Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
		2	6	. 3	4	5

We read 26.345 as "twenty-six and three hundred forty-five thousandths". In reading a decimal with digits on either side of the decimal point, the decimal point is read as "and".

3. Read each of the following.

a) 263.45

b) 2634.5

c) 2.6345

Sometimes a kind of numeral is used which combines decimals and fractions. The numeral $1\frac{3}{10}$ is an example. It names one and three tenths or 1.3 (decimal) or $\frac{13}{10}$ (fraction). Such a numeral is called a <u>mixed form</u>.

4. a) Read $7\frac{5}{100}$

b) What is a decimal name for this number?

c) Write a mixed form for 7.5.

Exercise Set 7

1. Choose the largest number in each column.

A	B	C	D	E
3.4	8.50	.002	45.405	.209
3.8	8.56	1.92	35.405	.287
2.4	8.65	2.2	45.5	.291
4.4	8.05	2.22	45.05	.289

2. Copy and complete each of these.

a) 7.5 8.0 ____ 9.0 ____ ____ ____ ____

b) 3.40 3.30 ____ ____ ____ ____

c) .20 .40 ____ ____ ____ ____

d) 4.75 4.80 ____ ____ ____ ____

3. Write these as decimals.

$2\frac{3}{10}$, $15\frac{7}{100}$, $32\frac{64}{100}$, $148\frac{37}{1000}$, $52\frac{184}{1000}$

4. Write a mixed form name for each of these.

22.3 72.15 18.047 459.003 78.39

5. Tell the number represented by each numeral 3.

Tell the number represented by each numeral 5.

a) 321.59 b) 71.03 c) 421.36

d) 720.513 e) 49.035 f) 795.309

6. Write a decimal for each of these.

a) 27 and 9 tenths

b) 364 and 57 hundredths

c) 70 and 41 thousandths

d) 38 and 7 hundredths

e) 3 and 0 hundredths

f) 5 and 429 thousandths

g) 83 and 4 tenths

h) 480 and 5 hundredths

i) 20 and 64 hundredths

j) 6 and 7 thousandths

k) 75 and 2 tenths

BASE FIVE NUMERALS

At the beginning of this chapter, you reviewed grouping and regrouping by tens. This is the idea behind our decimal numeral system. However, there are many ways of grouping objects. One of these ways is grouping in sets of five. This gives us the idea of a numeral system based on grouping by fives.

Example A

XXXXX

XXXXX

XXX

Here is a picture of a set of thirteen X's This set can be grouped into 2 sets of five and 3 ones. We shorten this to 23 (read "two three") to name the number of X's in the set. The set can also be grouped into 1 set of ten and 3 ones. We shorten this to 13 to get our ordinary decimal numeral. To show that 23 comes from grouping by fives and not by tens we will write the word "five" to the right and slightly below the numeral.

23_{five} means 2 sets of five and 3 ones.

13 means 1 set of ten and 3 ones.

We call 23_{five} a base five numeral and we read it "two three, base five".

Example B

XXXXX

XXXXX

XXXXX

XXXXX

XX

Look at this picture.

How many sets of five X's are there?

How many X's remain?

How would you write the base five numeral?

How would you read this base five numeral?

Exercise Set 8

1. Draw the following sets of X's. Group in fives and answer these questions for each set.

   How many sets of five are there?

   How many ones remain?

   How would you write the base five numeral?

Use this form.			
Nine X's	XXXXX XXXX	1 five and 4 ones	14_{five}

   a) twelve X's
   b) nineteen X's
   c) four X's
   d) twenty-three X's

2. Draw a picture that will represent X's for

   a) 30_{five}
   b) 42_{five}
   c) 14_{five}
   d) 10_{five}

3. Name the largest number with a base five numeral having two digits.

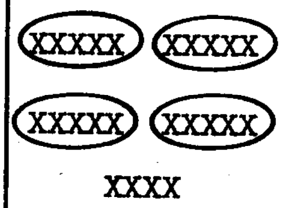
4. Name, in base five, the number which will come just before each of these numbers.

   a) 4_{five} b) 20_{five} c) 32_{five} d) 40_{five}

PLACE VALUE IN BASE FIVE

In the base ten system, the number named 99 is the largest with a two-place numeral. This is because 9 is one less than the base.

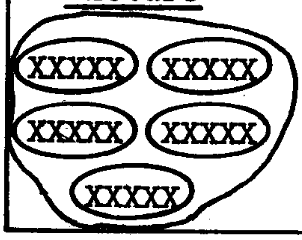
In the base five system, the number named 44_{five} is the largest with a two-place numeral. This is because 4 is one less than the base, as shown in the diagram below.

Picture	Meaning	Notation
	4 fives and 4 ones	44_{five}

There is no two-place symbol in our base ten system to mean <u>ten tens</u>. We give ten tens the name 1 hundred. We write this as the three-place numeral 100.

When we are thinking in base five we think of five groups of five as 1 group of <u>five fives</u>. We can use the name twenty-five for five fives.

How would the base five numeral for five fives or twenty-five be written?

Picture	Meaning	Notation
	1 twenty-five, (five fives) 0 fives, and 0 ones.	100_{five}

Exercise Set 9

1. Copy the X's below and group them in fives and five fives. Write the number of X's in base five notation.

   a) XXX
XXX
XXX
XXX

   b) XXXXXXX
XXXXXXX
XXXXXXX

   c) XXXXXXXXX
XXXXXXXXX
XXXXXXXXX
XXXXXXXXX

2. Copy and complete the following:

   a) 33_{five} means _____ fives and _____ ones.

   b) 142_{five} means _____ twenty-fives and _____ fives and _____ ones.

   c) 104_{five} means _____ twenty-fives and _____ fives and _____ ones.

3. Write the base five numeral for the number that is one larger than each of these.

   a) 4_{five} c) 43_{five} e) 144_{five}

   b) 13_{five} d) 132_{five} f) 204_{five}

4. Write these numbers in base five notation.

   a) The number of this page in this book

   b) The number of cookies in 4 dozen

   c) The total number of pages in this book

5. Make a base five chart of the numerals from 1_{five} to 200_{five}.

BASE FIVE AND BASE TEN NUMERALS

Numeral in Base Five System	Picture in Base Five System	Picture in Base Ten System	Numeral in Base Ten System
a) 22_{five}	XXXXX XX XXXXX	XXXXXXXXXX X X	12
b) 33_{five}	XXXXX XXXXX XXXXX XXX	XXXXXXXXXX XXXXXXXX	18
c) 114_{five}	XXXXX XXXXX XXXXX XXXX XXXXX XXXXX XXXXX	XXXXXXXXXX XXXXXXXXXX XXXXXXXXXX XXXX	34

Study the chart above. What does the numeral 22_{five} tell us?

What does the numeral 12 tell us?

Are 12 and 22_{five} names for the same number?

Why are 33_{five} and 18 names for the same number?

Why are 114_{five} and 34 names for the same number?

The procedure below shows how we may think to change a base five numeral to a base ten numeral.

a) $22_{five} = (2 \text{ fives} + 2 \text{ ones}).$

$= (2 \times 5) + (2 \times 1)$

$= 10 + 2$

$= 12$

b) $33_{five} = (3 \text{ fives} + 3 \text{ ones}).$

$= (3 \times 5) + (3 \times 1)$

$= 15 + 3$

$= 18$

c) $114_{five} = (1 \text{ twenty-five} + 1 \text{ five} + 4 \text{ ones}).$

$= (1 \times 25) + (1 \times 5) + (1 \times 4)$

$= 25 + 5 + 4$

$= 34$

MORE ABOUT BASE FIVE AND BASE TEN NUMERALS

?	Twenty-fives	Fives	Ones

So far, when we have written numerals in base five, we have used the place-values that are shown above. Can you tell what the next place-value will be?

For numerals we will be using right now, the only place values we will work with are twenty-fives, fives, and ones.

Suppose we want to change 111 to a base five numeral. How many groups of twenty-five are there in 111?

What is the remainder?

Write the mathematical sentence for this division process.

Find how many fives there are in 11.

How many ones remain?

Write the mathematical sentence for this division process.

Put both mathematical sentences together in a mathematical sentence which shows how 111 can be grouped by fives and twenty-fives.

What is the base five numeral for 111?

Try changing the following base ten numerals to base five numerals. In each part write the mathematical sentence which shows why your answer is correct.

a) 12 b) 36 c) 44 d) 52

<u>Exercise Set 10</u>

1. Draw a set of 21_{five} X's. Separate these X's into groups of ten. How many X's are there? Write your answer as a base ten numeral.

2. Draw a set of 134_{five} X's. Separate these X's into groups of ten. How many X's are there? Write your answer as a base ten numeral.

3. Change the following base ten numerals to base five numerals.

a) 14	c) 23	e) 42
b) 51	d) 60	f) 33

4. Change the following base five numerals to base ten numerals.

a) 23_{five}	c) 34_{five}	e) 42_{five}
b) 141_{five}	d) 340_{five}	f) 204_{five}

5. Which is greater?

a) 210_{five} or 201	c) 33_{five} or 23
b) 134_{five} or 42	d) 40_{five} or 20

USING GROUPING BY FIVES

We use some groupings of five in our everyday life. Let us look at our system of money. Suppose we have 34 cents. If we use only quarters, nickels, and pennies <u>and the fewest coins</u>, we have one quarter, one nickel, and four pennies. How could we write this using base five notation?

<u>Exercise Set 11</u>

Separate the following amounts of money into quarters, nickels, and cents. Use the smallest number of coins.

How much money?	How many quarters?	How many nickels?	How many pennies?	Base five notation
Examples: 14 cents	0	2	4	24_{five}
43 cents	1	3	3	133_{five}
1) 23 cents				
2) 26 cents				
3) 29 cents				
4) 33 cents				
5) 42 cents				
6) 57 cents				
7) 73 cents				
8) 97 cents				
9) 124 cents				

Exercise Set 12

Copy and complete this chart.

	Arrange in groups of	How many groups?	How many remain?	Notation
Example: X X X X X X X X	three	2	2	22_{three}
1. X X X X X X X X	four			
2. X X X X X X X X X X X	four			
3. X X X X X X X X X	seven			
4. X X X X X X X X X X X X X X X	six			
5. X X X X X X X X X X X X X X	five			
6. X X X X X X X X X XX X X	eight			
7. X X X X X X X	three			
8. X X X X X X XXX X X X X X X X X X X X X X	eight			

9. Draw a set of 20_{six} objects. Separate these objects into groups of ten. How many objects are there? Write your answer in base ten notation.

10. Draw a set of 34_{seven} objects. Separate these objects into groups of ten. How many objects are there? Write your answer in base ten notation.

11. Each mathematical sentence below shows how to change a decimal numeral into a numeral in another base. Write that numeral in the blank as shown in a).

a) $21 = (1 \times 16) + (1 \times 4) + 1.$

$21 = \underline{111_{four}}$

b) $50 = (1 \times 36) + (2 \times 6) + 2.$

$50 = \underline{\qquad}$

c) $26 = (1 \times 16) + (1 \times 8) + (1 \times 2).$

$26 = \underline{\qquad}$

d) $82 = (1 \times 81) + 1.$

$= \underline{\qquad}_{nine}$

$= \underline{\qquad}_{three}$

Exercise Set 13

Copy this chart. Write the numeral for the first twenty-four counting numbers using base eight, base six, base three, and base four.

Base Ten	Base Eight	Base Six	Base Three	Base Four
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				
21				
22				
23				
24				

Exercise Set 14

Complete the table.

Base Ten Numeral	Sixteens	Fours	Ones	Base Four Numeral
31				
17				
59				
Base Ten Numeral	Thirty-sixes	Sixes	Ones	Base Six Numeral
34				
90				
215				
Base Ten Numeral	Nines	Threes	Ones	Base Three Numeral
26				
9				
22				
Base Ten Numeral	Forty-nines	Sevens	Ones	Base Seven Numeral
60				
290				
99				
Base Ten Numeral	Twenty-fives	Fives	Ones	Base Five Numeral
46				
103				
89				
Base Ten Numeral	Sixty-fours	Eights	Ones	Base Eight Numeral
31				
80				
54				

Exercise Set 15

1. Fill in blanks as shown in the example.

	43_{five}	The numeral	4	stands for	4	fives	.
a)	301_{four}	The numeral	3	stands for	3	________	.
b)	423_{five}	The numeral	4	stands for	4	________	.
c)	63_{seven}	The numeral	6	stands for	6	________	.
d)	85_{nine}	The numeral	8	stands for	8	________	.
e)	300_{six}	The numeral	3	stands for	3	________	.

2. Change these numerals into base ten numerals as shown in a).

a) $23_{five} = (2 \times 5) + 3$
$= 10 + 3$
$= 13.$

b) 202_{three}

c) 106_{seven}

d) 210_{four}

e) 18_{nine}

f) 34_{eight}

g) 440_{five}

h) 122_{three}

i) 312_{four}

3. Copy and complete this counting chart.

a) Base five 133_{five} ___ ___ ___ ___ ___ ___
___ ___

b) Base seven 56_{seven} ___ ___ ___ ___ ___ ___
___ ___ ___

c) Base four 31_{four} ___ ___ ___ ___ ___ ___ ___

d) Base six 125_{six} ___ ___ ___ ___ ___ ___
___ ___

4. In what base are we counting?

a) 1, 2, 3, 4, 10, 11, 12, 13, ...

b) 14, 15, 16, 20, 21, 22, 23, 24, 25, 26, 30, ...

c) 1, 2, 3, 10, 11, 12, 13, 20, 21, 22, ...

d) 11, 12, 20, 21, 22, 100, 101, 102, 110, ...

5. Copy the work below. Use the "greater than," "less than," or "equals" sign to complete a true mathematical sentence.

a) 44_{five} —— 102_{three}

b) 100_{seven} —— 54_{nine}

c) 32_{six} —— 25_{eight}

d) 211_{three} —— 21_{four}

e) 77_{eight} —— 223_{five}

6. A place value system of numeration has twenty digits. What is the base?

7. Count by tens in base five from 20_{five} to 400_{five}.

8. Are these odd or even numbers?

a) 12_{three}

b) 21_{three}

c) 101_{three}

d) 111_{three}

e) 121_{three}

f) 102_{three}

BRAINTWISTERS

9. Copy and fill in the blanks.

a) $33_{five} = \_\_\_\__{seven}$

b) $14_{eight} = \_\_\_\__{three}$

c) $25_{six} = \_\_\_\__{four}$

d) $128_{nine} = \_\_\_\__{five}$

10. What is <u>n</u> in each of these mathematical sentences?

a) $n_{five} + 2_{five} = 11_{five}$

b) $23_{four} + 10_{four} = n_{four}$

c) $n_{eight} - 42_{eight} = 25_{eight}$

d) $123_{six} + n_{six} = 130_{six}$

11. Suppose a base three system used the symbol A for the number zero, B for one, and C for two. In this numeral system count from zero through ten.

12. Change each of the following to decimal numerals.

a) BBB

b) CAB

c) CBA

d) ABC

Chapter 2

FACTORS AND PRIMES

FACTORS AND PRODUCTS

Let's think of two numbers, for example 4 and 5. Use multiplication to get a third number, 20. We write this

$$4 \times 5 = 20.$$

4 is called a <u>factor</u> of 20.

5 is called a <u>factor</u> of 20.

20 is called the <u>product</u> of 4 and 5.

If we use the name, 4×5, for 20, we are <u>writing 20 as a product of two factors</u>. Sometimes we call 4×5 a <u>product expression</u> for 20.

The multiplication sentence

$$30 = 2 \times 3 \times 5$$

says that

30 is the product of 2 and 3 and 5.

It also says that

2 is a factor of 30, and 3 is a factor of 30 and 5 is a factor of 30.

A product expression for 30 is $2 \times 3 \times 5$.

Exercise Set 1

1. List three different names for each of the following whole numbers: (Use product expressions.)

   a. ten
   b. twelve
   c. sixteen
   d. twenty-one
   e. nine

2. Copy the following statements and fill in the blanks.

   a. 5 is a factor of 15 because 15 = __________.
   b. 15 = 5 × 3 shows that __________ is another factor of 15.
   c. 24 is the product of 6 and __________.
   d. __________ is a factor of every number.
   e. Every number greater than 1 has at least __________ different factors.

3. How many different arrays can be formed with

   a. 10 objects?
   b. 20 objects?

   List the number of rows and columns in each array. (Remember that the number of rows is always named first.)

Exercise Set 2

1. Express the following numbers as a product of two factors. Find three different ways for each.

   a. 24

   b. 30

   c. 28

2. Write the decimal numeral for each product.

a.	$6 \times 9 =$	f.	$5 \times 9 =$
b.	$7 \times 6 =$	g.	$8 \times 6 =$
c.	$9 \times 7 =$	h.	$9 \times 8 =$
d.	$8 \times 8 =$	i.	$7 \times 8 =$
e.	$7 \times 7 =$	j.	$6 \times 6 =$

3. Complete each mathematical sentence below to make a true statement.

a.	$3 \times \_\_\_\_ = 21$	f.	$\_\_\_\_ \times 4 = 28$
b.	$\_\_\_\_ \times 8 = 56$	g.	$8 \times \_\_\_\_ = 32$
c.	$4 \times \_\_\_\_ = 4$	h.	$4 \times \_\_\_\_ = 36$
d.	$9 \times \_\_\_\_ = 81$	i.	$\_\_\_\_ \times 6 = 24$
e.	$\_\_\_\_ \times 9 = 72$	j.	$7 \times \_\_\_\_ = 63$

4. Express each of the following numbers as a product of two factors in every possible way.

   a. 12 (There are 6 ways.)

   b. 35 (There are 4 ways.)

   c. 42 (There are 8 ways.)

   d. 18 (There are 6 ways.)

   e. 45 (There are 6 ways.)

   f. 24 (There are 8 ways.)

TESTING NUMBERS AS FACTORS

Is 3 a factor of 57? Is 3 a factor of 37? We may see by using division (Method A).

$$
\begin{array}{r|r}
3\overline{)\,57} & \\
\underline{30} & 10 \\
27 & \\
\underline{27} & \underline{9} \\
 & 19
\end{array}
\qquad\qquad
\begin{array}{r|r}
3\overline{)\,37} & \\
\underline{30} & 10 \\
7 & \\
\underline{6} & \underline{2} \\
1 & 12
\end{array}
$$

$57 = (19 \times 3)$ | $37 = (12 \times 3) + 1$

3 <u>is</u> a factor of 57 | 3 <u>is</u> <u>not</u> a factor of 37.

Here is another method we may use to see if one number is a factor of another (Method B). Is 7 a factor of 67?

$$\text{I know } 9 \times 7 = 63$$
$$67 = 63 + 4.$$

Therefore $(9 \times 7) + 4 = 63 + 4 = 67$.

Since 4 is less than 7, 4 is the remainder when 67 is divided by 7. This shows that 7 is not a factor of 67.

Exercise Set 3

1. Use Method A to answer these. Write your answer in a complete sentence.

   a. Is 8 a factor of 81?

   b. Is 4 a factor of 52?

   c. Is 7 a factor of 59?

2. Use Method B to answer each of these. Write your answer in a complete sentence.

   a. Is 7 a factor of 58?

   b. Is 9 a factor of 75?

   c. Is 8 a factor of 56?

3. Use either Method A or Method B to answer these. Write your answer in a complete sentence.

   a. Is 3 a factor of 51?

   b. Is 9 a factor of 138?

   c. Is 6 a factor of 73?

   d. Is 7 a factor of 217?

   e. Is 8 a factor of 94?

A. Starting from $6 \times 5 = 30$, we can get

$$(2 \times 3) \times 5 = 30.$$

B. Starting from $2 \times 15 = 30$, we can get

$$2 \times (3 \times 5) = 30.$$

The associative property also shows us how to get B from A.

$$6 \times 5 = 30$$
$$(2 \times 3) \times 5 = 30$$
$$2 \times (3 \times 5) = 30 \quad \text{(Associative Property)}$$
$$2 \times 15 = 30.$$

If we show <u>no</u> grouping and just write

$$2 \times 3 \times 5 = 30,$$

we see clearly that 2, 3, and 5 are factors of 30.

By thinking of both groupings, we see that 6 and 15 are also factors of 30, because we get

$$2 \times 15 = 30 \quad \text{and}$$
$$6 \times 5 = 30.$$

Writing the product expression of 3 or more factors without parentheses can give us as much information as writing all possible groupings. We will use parentheses only when we want to show particular groupings.

THE COMMUTATIVE PROPERTY OF MULTIPLICATION

When we know $6 = 2 \times 3$, we also know

$6 = 3 \times 2$.

If we know that $24 \times 32 = 768$, then we know that

$32 \times 24 = 768$.

If we know $30 = 2 \times 3 \times 5$, then we also know

$30 = 2 \times 5 \times 3$,

$30 = 5 \times 2 \times 3$,

$30 = 3 \times 2 \times 5$,

$30 = 3 \times 5 \times 2$, and

$30 = 5 \times 3 \times 2$.

Any one of these ways of expressing 30 as a product of three factors tells us that 2, 3, and 5 are factors of 30. When we know one way, we can list all six; but we will find nothing new from the other five ways.

From now on in this unit we will not say two ways of writing a product expression are <u>different ways</u> unless they show a <u>different set of factors</u>.

WAYS TO WRITE DIFFERENT PRODUCT EXPRESSIONS FOR THE SAME NUMBER

There are two <u>different</u> ways to express 6 as a product of two factors. We can use the factors 1 and 6, or 2 and 3.

$$6 = 1 \times 6$$

$$6 = 2 \times 3$$

There are <u>five</u> different ways to write 30 as a product of <u>three</u> factors. The factors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30. Using these factors, name the 5 different ways.

The factors we get depend upon the way we write the product expression. If we write $60 = 2 \times 3 \times 10$, we will find one set of factors. If we write $60 = 2 \times 6 \times 5$, we will get a different set of factors:

$60 = 2 \times 3 \times 10$		$60 = 2 \times 6 \times 5$	
The factors are:		The factors are:	
2	(given)	2	(given)
3	(given)	6	(given)
10	(given)	5	(given)
6	(2×3)	12	(2×6)
20	(2×10)	10	(2×5)
30	(3×10)	30	(6×5)

1 is a factor because 1 is a factor of every number. 60 is a factor because every number has itself for a factor.

If $60 = 2 \times 3 \times 10$, the factors are:	If $60 = 2 \times 6 \times 5$, the factors are:
1, 2, 3, 6, 10, 20, 30, 60	1, 2, 5, 6, 10, 12, 30, 60

Exercise Set 4

1. Each number below is written as a product of two factors. Use this to write the number as a product of three factors.

   a. $12 = 4 \times 3$

   Answer: $12 = 1 \times 4 \times 3$ or $12 = 2 \times 2 \times 3$

   b. $8 = 4 \times 2$

   c. $18 = 9 \times 2$

   d. $16 = 4 \times 4$

   e. $18 = 6 \times 3$

   f. $36 = 6 \times 6$

   g. $36 = 4 \times 9$

2. Write two different product expressions for each of these numbers. Use three factors in each product expression. Then use each product expression to find as many different factors of the number as you can. Part a. is done for you.

   a. 12

   Answers: $12 = 2 \times 2 \times 3$ Factors we can find: 2, 3, 4, 6, 12

   $12 = 1 \times 2 \times 6$ Factors we can find: 1, 2, 6, 12

   b. 18

   c. 36

   d. 16

3. In exercise 2, when we used $12 = 2 \times 2 \times 3$, we find that if we put 1 in our list we have all of the factors of 12. Find whether this is true for each of the product expressions in exercise 2.

4. How can we express a number as a product of <u>three</u> factors in <u>all</u> different ways? We might first express the number as a product of <u>two</u> factors in different ways.

a. 10

$10 = 2 \times 5$, so $10 = 1 \times 2 \times 5$

$10 = 1 \times 10$, so $10 = 1 \times 1 \times 10$

I can find two different ways.

b. 12

$12 = 3 \times 4$, so $12 = 1 \times 3 \times 4$, and

$12 = 3 \times 2 \times 2$.

$12 = 2 \times 6$, so $12 = 1 \times 2 \times 6$, and

$12 = 2 \times 2 \times 3$ (already found).

$12 = 1 \times 12$, so $12 = 1 \times 1 \times 12$, and

$12 = 1 \times 2 \times 6$, (already found)

and $12 = 1 \times 3 \times 4$ (already found).

I can find <u>four</u> different ways.

$1 \times 3 \times 4$

$2 \times 2 \times 3$

$1 \times 2 \times 6$

$1 \times 1 \times 12$

Use the method shown in a and b to find as many ways as you can to express these numbers as products of three factors.

c. 16 e. 20 g. 44

d. 18 f. 11 h. 42

Using 1 as a factor in a product expression tells us nothing we don't know about the factors of the number. For example:

a. We know that 1 and 15 are factors of 15, since every number has as factors, itself and 1. Writing $15 = 1 \times 15$ tells us nothing more about the factors of 15.

b. If we write $12 = 4 \times 3 \times 1$, we know no more about the factors of 12 than if we write $12 = 4 \times 3$.

c. If we write $36 = 9 \times 4 \times 1$ or $36 = 1 \times 4 \times 1 \times 9$, we know no more about the factors of 36 than if we write $36 = 4 \times 9$.

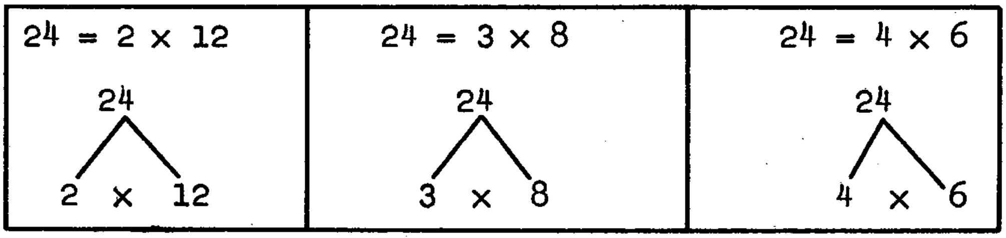
Because of this, when we want to know more about the factors of a number, we look for factors greater than 1 but less than the number itself.

FACTOR TREES

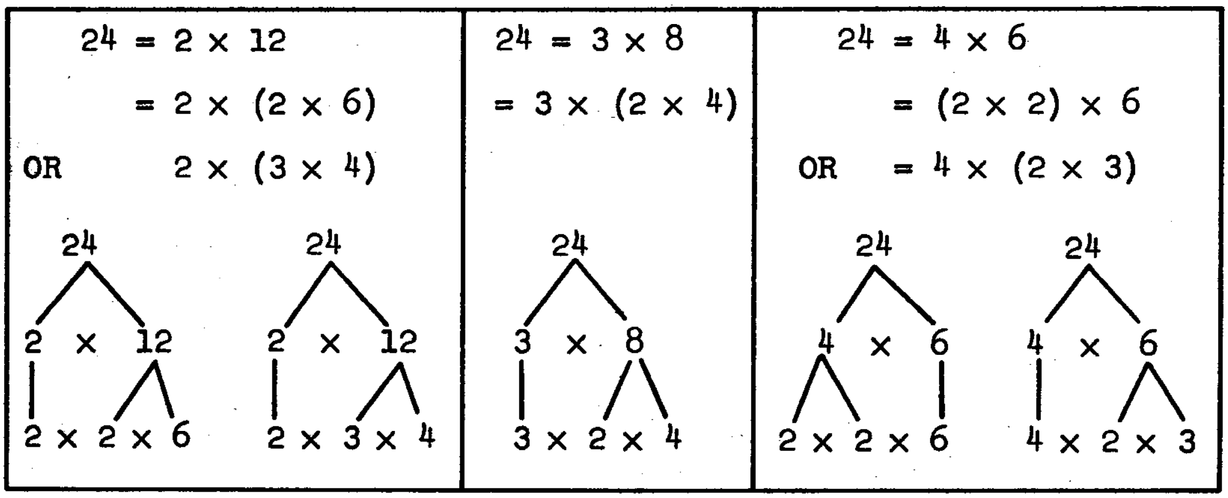
A "factor tree" is a diagram which shows factors of a given number. Let's look at the number 24. We can give product expressions with two factors (each one greater than 1) as follows:

$$24 = 2 \times 12$$
$$24 = 3 \times 8$$
$$24 = 4 \times 6$$

These product expressions can be pictured by "factor trees" which look like this.

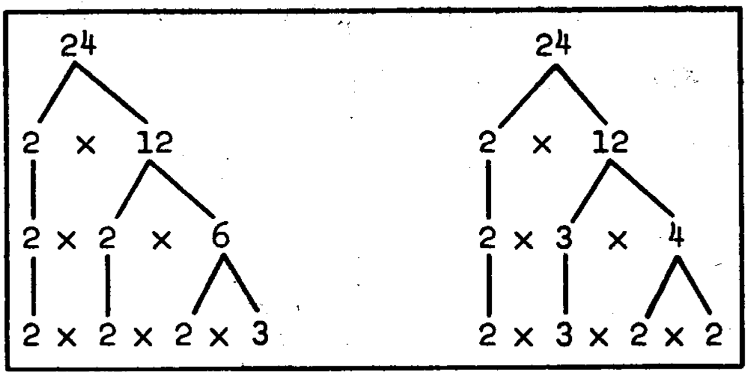


We can picture each product expression using 3 factors (each > 1) by using the "factor tree."

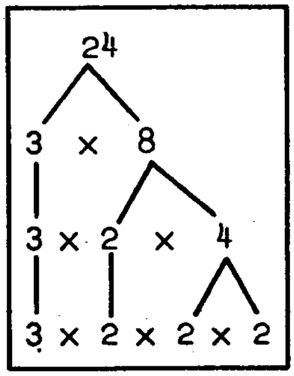


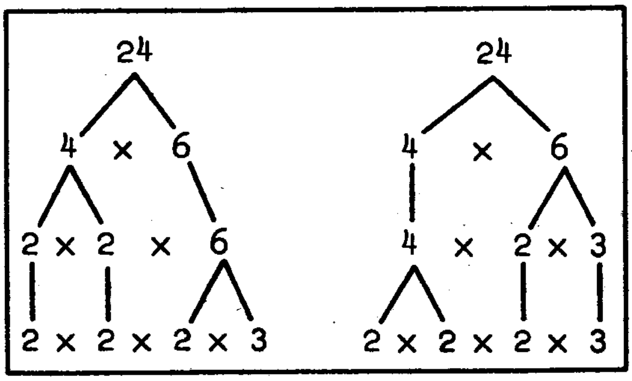
A B C D E

We can extend the factor trees at the bottom of page to picture how 24 can be expressed as a product of 4 factors.



A B





C D E

Is it possible to extend the factor tree to another row that would show 24 as a product of 5 factors (not using 1 as a factor)?

What do you notice about the last row in the factor trees in A, B, C, D, and E above?

Exercise Set 5

1. Draw two factor trees (if there are two) for each of the following numbers. Extend each tree as far as possible. Do not use the factor 1.

   a. 24   e. 60

   b. 30   f. 23

   c. 28   g. 48

   d. 35   h. 72

2. List the smallest number which has all of these numbers as factors.

   a. 2, 3, 5

   b. 2, 5, 7

   c. 2, 4, 8

   d. 2, 6, 12

   e. 2, 3, 4

   f. 4, 6, 8

   g. 5, 7

   h. 2, 5, 7, 10

BRAINTWISTERS

3. 6 is a factor of 678. This means that 678 must have other factors. What are they?

4. 12 is a factor of 2,844. What other factors must 2,844 have?

5. I am thinking of a number. It has 4 and 10 as factors. List all factors which you can be sure it has.

A prime number is a whole number which is greater than 1 but cannot be expressed as the product of two smaller factors.

2, 3, 5, 7, 11 are examples of primes.

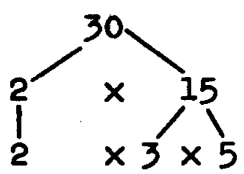
The name "prime number" is usually shortened to "prime".

A whole number which is not prime, and is greater than 1, is called a composite number.

A composite number is one which can be expressed as a product of two smaller factors.

4, 6, 8, 9, 10 are examples of composite numbers.

A "factor tree" can picture prime numbers. This factor tree tells us that 2, 3, and 5 are prime numbers.



Questions for Class Discussion

1. In each classroom in a school, the seats form an array. There are never more than 7 rows of 5 seats each. What is the largest number of seats there can be in a classroom?

2. I am thinking of two numbers. One is no greater than 8, and the other is no greater than 7. What do you know about their product?

3. A number is no greater than 4. If it is multiplied by itself, how great can the product be?

4. The product of two numbers is 64. One of them is greater than 8. What do you know about the other?

5. The product of two numbers is 100. One is less than 10. What do you know about the other?

6. A certain factor of 144 is greater than 12 . What do you know about the unknown factor?

BRAINTWISTER

7. The number 6 is equal to the sum of its factors, not including 6 itself. $6 = 1 + 2 + 3$. There is another whole number less than 30 which is equal to the sum of its factors, not including itself. Find it.

TESTING FOR PRIMES

The factors of a number can be arranged in pairs. This diagram shows these pairs of factors of 24.

1, 2, 3, 4, 6, 8, 12, 24

If one of a pair of factors of 24 is less than 5, the other is greater than 5. Why?

If one of a pair of factors of 36 is greater than 6, the other is less than 6. Why?

At least one factor in every pair of factors of 48 is less than 7. Why?

We can use this idea to make the work easier in finding factors. It also helps in locating primes.

Suppose we want to find factors of 23. We can test 2, 3, 4, by dividing or by knowing multiplication facts. None of these is a factor of 23. We know, then, that 23 is prime because: if 23 had a factor greater than 4, the other factor would have to be 4 or smaller. Otherwise, their product would be at least $5 \times 5 = 25$.

To know that 23 is prime, we do not need to test any other numbers as factors. We do not even need to test 4. Do you see why?

Exercise Set 6

1. To find whether 41 is prime or composite, what numbers must we test as possible factors?

2. Use division to find whether or not 41 is prime.

Test the following numbers as you did 41. If the number is composite, express it as a product of prime factors. If it is prime, write "prime".

Example: 19 prime

21 composite, $21 = 3 \times 7$.

3. 22
4. 27
5. 31
6. 33
7. 39
8. 53
9. 55
10. 67
11. 69
12. 83
13. 87
14. 143

THE PRIME FACTOR CHART

	Prime Factors				
No.	2	3	5	7	
2	2				prime
3		3			prime
4	2				2 × 2
5			5		prime
6	2	3			2 × 3
7				7	prime
8	2				2×2×2
9		3			3 × 3
10	2		5		2 × 5
11					prime
12	2	3			2×2×3
13					prime
14	2			7	2 × 7
15		3	5		3 × 5
16	2				2×2×2×2
17					prime
18	2	3			2×3×3
19					prime
20	2		5		2×2×5
21		3		7	3 × 7
22	2				2 × 11
23					prime
24	2	3			2×2×2×3
25			5		5 × 5

	Prime Factors				
No.	2	3	5	7	
26	2				2 × 13
27		3			3×3×3
28	2			7	2×2×7
29					prime
30	2	3	5		2×3×5
31					prime
32	2				2×2×2×2×2
33		3			3 × 11
34	2				2 × 17
35			5	7	5 × 7
36	2	3			2×2×3×3
37					prime
38	2				2 × 19
39		3			3 × 13
40	2		5		2×2×2×5
41					prime
42	2	3		7	2×3×7
43					prime
44	2				2×2×11
45		3	5		3×3×5
46	2				2 × 23
47					prime
48	2	3			2×2×2×2×3
49				7	7 × 7
50	2		5		2×5×5

Exercise Set 7 (Oral)

Using your prime factor chart, answer the questions.

1. Look at all the primes in the chart that are greater than 2. There is always at least one number between any two of them. Why?

2. Look at the numbers between 7 and 49 with 7 as a prime factor. Each number also has 2, 3, or 5 as a factor. Why must this happen?

3. Can the numbers from 2 to 50 have prime factors which are not shown on the chart? Give an example if there is one.

4. What numbers in the chart are prime numbers in addition to the numbers 3, 5, and 7?

TESTING 2, 3, AND 5 AS FACTORS OF A NUMBER

From our study of the Prime Factor Chart we observed:

1. In the set of counting numbers, {1, 2, 3, 4 ...}, a number will have 2 as a factor if the units' digit in its numeral is 0, 2, 4, 6 or 8.

   Examples of counting numbers which have a factor of 2 are: 40, 182, 364, 56, 218.

2. In the set of counting numbers, a number will have 3 as a factor if the sum of the digits in its numeral can be divided by 3.

   Examples of counting numbers which have a factor of 3 are:

   951 (Because 9 + 5 + 1 = 15 and 15 can be divided by 3.)

   543 (Because 5 + 4 + 3 = 12.)

   864 (Because 8 + 6 + 4 = 18. 864 also has 2 for a factor because the units' digit is 4.)

3. In the set of counting numbers, a number will have 5 as a factor if the units' digit of its numeral is 0 or 5.

   Examples of counting numbers which have a factor of 5 are: 4,835, 495, and 860.

   495 would also have 3 as a factor because the sum of the digits of its numeral can be divided by 3.

   860 would have a factor of 2 because the units' digit in its numeral is 0.

Exercise Set 8

Find one prime factor of each of the following numbers.

1. 785
2. 7,012
3. 8,001
4. 7,136
5. 4,895
6. 4,083
7. 67,210
8. 60,105

Find two different prime factors of each of the following numbers.

9. 405
10. 6,780
11. 3,042
12. 5,055
13. 4,314
14. 6,060

Write 2, 3, and 5 in the correct places in this chart. Exercise 15 is done for you.

	Number	These numbers are factors	These numbers are not factors
15.	365	5	2, 3
16.	492		
17.	835		
18.	3,681		
19.	370		
20.	86,910		

BRAINTWISTERS

For each exercise below, what are all the numbers less than 100 which have these numbers and no others as prime factors?

21. 3 and 5

22. 3 and 7

23. 5 and 7

24. 2 and 11

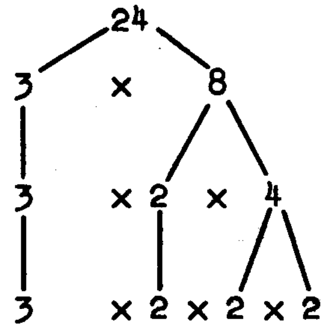
COMPLETE FACTORIZATION

Every composite number is the product of smaller numbers. If one of these numbers is composite, then it also is the product of smaller numbers. If we continue this, we must come to a product expression in which no number is composite and every factor is a prime. Doing this is called <u>complete factorization</u> of a composite number.

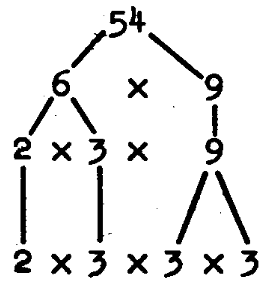
An example of complete factorization:

A picture, using the factor tree is:

$24 = 3 \times 8$	(3 is prime.) (8 is composite.)
$= 3 \times 2 \times 4$	(3 and 2 are prime.) (4 is composite.)
$= 3 \times 2 \times 2 \times 2$	(All are prime.)
$= 2 \times 2 \times 2 \times 3$	(Rearranged for convenience)



$54 = 6 \times 9$	(6 and 9 are composite.)
$= 2 \times 3 \times 9$	(2 and 3 are prime.) (9 is composite.)
$= 2 \times 3 \times 3 \times 3$	(All are prime.)



This suggests that every number greater than 1 is either prime or is a product of primes.

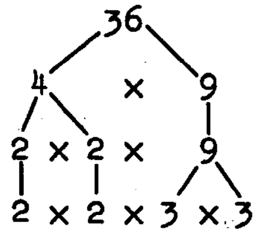
How can we find a way to express any number as a product of primes, for example 36?

We may know some way to express the number as a product.

$$36 = 4 \times 9$$

Then we can write each composite factor as a product expression. Continue until we have only prime factors.

$$36 = 2 \times 2 \times 9$$
$$= 2 \times 2 \times 3 \times 3$$



This product expression $2 \times 2 \times 3 \times 3$ is the complete factorization of 36.

Another way to express a number as a product of primes is by testing small prime numbers such as 2, 3, 5, 7, etc. to see if they are factors of the numbers.

Example:

$$36 = 2 \times 18 \text{ (starting with } 2\text{).}$$

Then we look for prime factors of 18 starting with 2.

$$36 = 2 \times (2 \times 9)$$

Then we look for prime factors of 9, starting with 2. Since 2 is not a factor, we next test 3.

$$36 = (2 \times 2) \times (3 \times 3)$$
$$= 2 \times 2 \times 3 \times 3.$$

Either of these ways may be called <u>factoring</u>. Sometimes it is easier to use one process. Sometimes it is easier to use the other process. With practice, you can find shortcuts by combining them.

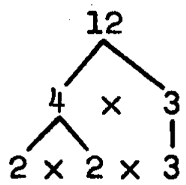
Exercise Set 9

Express each number below as a product of two smaller factors. If possible, then express one of these factors as a product of smaller factors. Continue until you have expressed the number as a product of primes. This is one factoring process. Show your work by drawing a "factor tree".

Example: $12 = 4 \times 3$

$= (2 \times 2) \times 3$

or



$12 = 2 \times 2 \times 3$

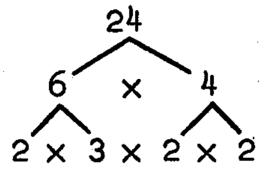
1. 16
2. 18
3. 20
4. 25
5. 27
6. 28
7. 30
8. 35
9. 40
10. Do Exercises 1 through 9 again, but this time start with a different pair of factors if there is another pair.

11. Following the example shown, express each number as a product of primes. Draw a factor tree for parts b, d, f.

Example: $24 = 6 \times 4$

$= 2 \times 3 \times 4$

$= 2 \times 3 \times 2 \times 2$



$24 = 2 \times 2 \times 2 \times 3$

a. 30 c. 84 e. $128 = 8 \times 16$

b. 72 d. 96 f. $288 = 12 \times 24$

g. $225 = 15 \times 15$

12. Use any factoring process to write each number as a product expression of primes.

a. 144 Answer: $144 = 2 \times 72$

$= 2 \times 2 \times 36$

$= 2 \times 2 \times 2 \times 18$

$= 2 \times 2 \times 2 \times 2 \times 9$

$= 2 \times 2 \times 2 \times 2 \times 3 \times 3$

b. 225 e. 385 h. 189

c. 588 f. 127 i. 143

d. 363 g. 585

13. Without multiplying, write each number as a product expressio of primes.

a. 18×60 d. 50×50

b. 42×84 e. 125×64

c. 21×78 f. 25×320

A PROPERTY OF PRODUCTS OF PRIMES

The results of the last exercises suggest that we have found a general property. We might state it as:

> Except for the order in which factors are written, a composite number can be <u>expressed</u> as a product of primes <u>in only one way</u>.

You will not find any exceptions to this property because there is a way to show that it is always true. We do not attempt to show in this book why this is true. However, as you use it you should become more sure that it is true. The statement in the "box" is called <u>The Fundamental Theorem of Arithmetic</u>.

FINDING ALL FACTORS

If we know how to express a number as a product of primes, then we can find the <u>set</u> <u>of</u> <u>all</u> <u>factors</u> of the number.

Suppose we write

$$60 = 2 \times 2 \times 3 \times 5.$$

Here are some of the things we can find:

1. The prime factors of 60 are 2, 3, and 5.

2. By multiplying in pairs the factors shown in the product expression for 60, we see that 4, (2×2) 6, (2×3) 10, (2×5) and 15, (3×5) are also factors of 60.

3. By multiplying in threes the factors shown in the product expression for 60, we see that 12, $(2 \times 2 \times 3)$ 20, $(2 \times 2 \times 5)$ and 30, $(2 \times 3 \times 5)$ are also factors of 60.

   The factors shown in $2 \times 2 \times 3 \times 5$ are primes. For this reason, we must have found by our method, <u>every</u> factor of 60.

4. We know then that

   $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$

   is the set of all factors of 60.

5. **From the set of all factors of 60, we can get <u>every</u> way of naming 60 as a product of two factors.**

1 2 3 4 5 6 10 12 15 20 30 60

$$1 \times 60 = 60$$

$$2 \times 30 = 60$$

$$3 \times 20 = 60$$

$$4 \times 15 = 60$$

$$5 \times 12 = 60$$

$$6 \times 10 = 60$$

Exercise Set 10

1. Find the set of all factors of each number.

   a. 24

   Answer: $24 = 2 \times 2 \times 2 \times 3$

   Set of factors of 24 $= \{1, 24, 2, 3, 4, 6, 8, 12\}$

   $= \{1, 2, 3, 4, 6, 8, 12, 24\}$

   b. 30
   c. 72
   d. 84
   e. 96
   f. 128
   g. 225
   h. 144
   i. 363
   j. 385
   k. 89
   l. 189
   m. 143

2. Use what you found in exercise 1 to get all of the different ways to write each number in that exercise as a product expression of two factors.

   a. 24

   Answer:

   Set of factors of 24 $= \{1, 2, 3, 4, 6, 8, 12, 24\}$

   $24 = 1 \times 24 = 2 \times 12 = 3 \times 8 = 4 \times 6.$

3. Find whether each number listed below is a factor of $2 \times 2 \times 3 \times 7 \times 11 \times 11$:

a. 6

<u>Answer</u>:

Yes, because $2 \times 2 \times 3 \times 7 \times 11 \times 11$

$$= (2 \times 3) \times (2 \times 7 \times 11 \times 11)$$

$$= 6 \times (2 \times 7 \times 11 \times 11)$$

The factor belonging with 6 is

$$2 \times 7 \times 11 \times 11.$$

b. 14

c. 28

d. 210

e. 242

COMMON FACTORS

Suppose Set S is the set of all factors of 12 and Set R is the set of all factors of 18.

$S = \{1, 2, 3, 4, 6, 12\}$

$R = \{1, 2, 3, 6, 9, 18\}$

Then the set of all factors of <u>both</u> 12 and 18 is

$S \cap R = \{1, 2, 3, 6\}$

The members of this set are called the <u>common factors</u> of 12 and 18.

What are the common factors of 16 and 36?

$K = \{1, 2, 4, 8, 16\}$ is the set of all factors of 16

and

$L = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ is the set of all factors of 36,

$K \cap L = \{1, 2, 4\}$ is the set of all common factors of 16 and 36.

The common factors of 16 and 36 are 1, 2, and 4.

Exercise Set 11

1. Two numbers are given in each exercise below. Find all factors of each number; then find the common factors of the two numbers. The first exercise is an example of what you are to do.

   a. 12 and 30.

   Let A = the set of all factors of 12.

   $A = \{1, 2, 3, 4, 6, 12\}$

   Let B = the set of all factors of 30.

   $B = \{1, 2, 3, 5, 6, 10, 15, 30\}$

   $A \cap B = \{1, 2, 3, 6\}$

   1, 2, 3, and 6 are the common factors of 12 and 30.

   b. 40 and 30

   c. 36 and 27

   d. 60 and 40

   e. 52 and 72

   f. 75 and 120

   g. 72 and 108

2. For each <u>intersection</u> in exercise 1:

   a. What is the largest or greatest factor in each set of common factors?

   b. Is each other member of the set of common factors a factor of the largest member?

   c. Are there any members of the intersection set which are <u>not</u> factors of the largest member?

FINDING THE GREATEST COMMON FACTOR

If we know the set of common factors of two numbers, we can easily find the greatest common factor of the two numbers. The greatest number in the set of common factors is called the greatest common factor.

The set of common factors of 12 and 18 is

$$\{1, 2, 3, 6\}.$$

The largest among these numbers is 6. It is called the greatest common factor of 12 and 18.

The set of common factors of 16 and 36 is

$$\{1, 2, 4\}.$$

The greatest common factor of 16 and 36 is 4.

There is a way to find the greatest common factor of two numbers without first finding the intersection of the sets of factors of each number.

First we express the numbers, say 30 and 42, as products of primes.

$$30 = 2 \times 3 \times 5$$

$$42 = 2 \times 3 \times 7.$$

The factors of 30 can all be found by forming "pieces" of this expression. Pieces of $2 \times 3 \times 5$ are 2, 3, 5, 2×3, 2×5, 3×5, and $2 \times 3 \times 5$. The factors of 42 can all be found in the same way. The pieces of $2 \times 3 \times 7$ are 2, 3, 7, 2×3, 2×7, 3×7, and $2 \times 3 \times 7$. The common factors of 30 and 42 must be expressed by those pieces which are found in both expressions. The greatest common factor must be the largest piece found in both expressions.

The largest piece in the prime product expressions for both 32 and 40 is 2×3 or 6. Then 6 must be the greatest common factor of 32 and 40.

Here is another example. To find the greatest common factor of 90 and 50 we write:

$$90 = 2 \times 3 \times 3 \times 5$$
$$50 = 2 \times 5 \times 5.$$

By rewriting $90 = (2 \times 5) \times (3 \times 3)$ we see that 2×5 is the largest piece that can be found in both expressions. The expression $2 \times 5 \times 3$ can be found in one and $2 \times 5 \times 5$ in the other. But neither can be found in <u>both</u>. We know then that 10 is the greatest common factor of 90 and 50.

If we have found the greatest common factor in this way we can quickly find all common factors. Do you see how? The common factors must be those which can be expressed as pieces of both prime product expressions. They must then be the <u>pieces of the largest piece</u>. This means that the common factors are simply the <u>factors of the greatest common factor</u>.

Since 6 is the greatest common factor of 30 and 42, the set of common factors is $\{1, 2, 3, 6\}$.

Since 10 is the greatest common factor of 90 and 50, the set of common factors is $\{1, 2, 5, 10\}$.

Now try 24 and 60.

$$24 = 2 \times 2 \times 2 \times 3$$
$$60 = 2 \times 2 \times 3 \times 5.$$

The pieces which these expressions have in common are 2, 3, 2×2, 2×3, and $2 \times 2 \times 3$. This last is the largest, so 12 is the greatest common factor of 24 and 60. The set of all common factors is $\{1, 2, 3, 4, 6, 12\}$.

Exercise Set 12

1. Find the greatest common factor by first finding the intersection of the sets of factors. Exercise a. is answered for you as an example.

   a. 12 and 40

   $12 = 2 \times 2 \times 3$

   All factors of 12 $A = \{1, 2, 3, 4, 6, 12\}$

   $40 = 2 \times 2 \times 2 \times 5$

   All factors of 40 $B = \{1, 2, 4, 5, 8, 10, 20, 40\}$

   $A \cap B = \{1, 2, 4\}$

   The greatest common factor of 12 and 40 is 4.

   b. 16 and 6

   c. 90 and 12

2. Find the greatest common factor by first writing each number as a product of primes.

   a. 2 and 6
   b. 7 and 35
   c. 16 and 8
   d. 20 and 36
   e. 48 and 30
   f. 60 and 45
   g. 72 and 60

   h. $2 \times 2 \times 2 \times 3 \times 3 \times 5$ and $2 \times 3 \times 5 \times 7$

   i. $3 \times 3 \times 3 \times 3 \times 7 \times 7 \times 11$ and $2 \times 3 \times 3 \times 13$

   j. $m = 2 \times 2 \times 3 \times 3 \times 3 \times 3 \times 7 \times 7 \times 7 \times 7 \times 7$ and $n = 2 \times 2 \times 3 \times 3 \times 7$

BRAINTWISTER

3. a. Can a pair of numbers with 2, 3, and 5 among their common factors have 20 as a greatest common factor? Why?

   b. If 2 and 3 are among the common factors of a pair of numbers, name one other common factor which the pair must have.

   Answer the same question if the common factors are:

   c. 3 and 5

   d. 9 and 5

   e. 9 and 4

   f. 4 and 6

   g. 6 and 14

   h. 12 and 9

4. a. The greatest common factor of 728 and 968 is 8. Write the set of common factors of 728 and 968.

   b. The greatest common factor of 330 and 294 is 6. Write the set of common factors of 330 and 968.

FACTORING AND FRACTIONS

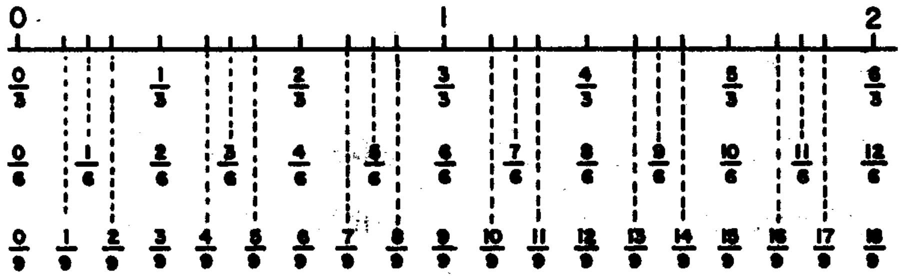
When we studied fractions we learned that there are many fractions which name the same rational number. For example

$$\frac{2}{3},\ \frac{4}{6},\ \text{and}\ \frac{6}{9}$$

are all names for the same number.

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9}\ .$$

This number line may help to remind you why this is so.



The diagram shows scales in units, thirds, sixths, and ninths. It shows that if a segment has a measure $\frac{2}{3}$ then it also has measure $\frac{4}{6}$ and $\frac{6}{9}$. By studying the diagram you should be able to answer the following questions:

1. John has a pencil $\frac{1}{3}$ of a foot long. Mary has a piece of chalk $\frac{1}{6}$ of a foot long. John measures the side of a large book with his pencil. Mary measures the same side with her chalk. John finds that the edge measures 4 in pencil lengths. What does it measure in feet? What number should Mary find as the measure of the edge in chalk lengths? How would she probably express this length in feet?

2. List the two other names for $\frac{5}{3}$ shown on the diagram. List two more names not shown on the diagram. Is there a name for $\frac{1}{2}$ shown on the diagram? If there is, what is it? What scales would you add to the diagram to show two other names for $\frac{1}{6}$?

In using fractions it is often very important to be able to answer questions like these:

a. Is $\frac{30}{48} = \frac{25}{30}$?

b. Is $\frac{15}{25} < \frac{24}{30}$?

We can answer questions like these if we can tell when two fractions are names for the same number. We know that

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12} = \frac{1 \times n}{2 \times n}$$

and that

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15} = \frac{12}{18} = \frac{2 \times n}{3 \times n}.$$

We can also use this idea to find smaller numerators and denominators.

$$\frac{18}{24} = \frac{2 \times 9}{2 \times 12} = \frac{9}{12} = \frac{3 \times 3}{3 \times 4} = \frac{3}{4}$$

$$\frac{18}{24} = \frac{3 \times 6}{3 \times 8} = \frac{6}{8} = \frac{2 \times 3}{2 \times 4} = \frac{3}{4}$$

Thus $$\frac{18}{24} = \frac{9}{12} = \frac{6}{8} = \frac{3}{4}.$$

This suggests that we can answer our question about $\frac{30}{48}$ and $\frac{25}{40}$ by factoring. We can start by writing both 30 and 48 as products of primes.

$$30 = 2 \times 3 \times 5$$

$$48 = 2 \times 2 \times 2 \times 2 \times 3$$

Now $\frac{30}{48} = \frac{2 \times 3 \times 5}{2 \times 2 \times 2 \times 2 \times 3} = \frac{(2 \times 3) \times 5}{(2 \times 3) \times (2 \times 2 \times 2)}$

$$= \frac{6 \times 5}{6 \times 8} = \frac{5}{8}.$$

Also $\frac{25}{40} = \frac{5 \times 5}{2 \times 2 \times 2 \times 5} = \frac{5 \times 5}{5 \times (2 \times 2 \times 2)}$

$$= \frac{5 \times 5}{5 \times 8} = \frac{5}{8}$$

We find then that $\frac{30}{48} = \frac{25}{40} = \frac{5}{8}$.

Now for our second question, b).

$$\frac{15}{25} = \frac{3 \times 5}{5 \times 5} = \frac{3}{5}$$

$$\frac{24}{30} = \frac{2 \times 2 \times 2 \times 3}{2 \times 3 \times 5} = \frac{(2 \times 3) \times (2 \times 2)}{(2 \times 3) \times 5}$$

$$= \frac{2 \times 2}{5} = \frac{4}{5}.$$

Since we know that $\frac{3}{5} < \frac{4}{5}$, we also know that $\frac{15}{25} < \frac{24}{30}$.

Exercise Set 13

1. Find the fraction with the smallest possible denominator for each of the following.

   Example: $\frac{60}{350} = \frac{2 \times 2 \times 5 \times 3}{2 \times 5 \times 5 \times 7} = \frac{(2 \times 5) \times (2 \times 3)}{(2 \times 5) \times (5 \times 7)} = \frac{2 \times 3}{5 \times 7}$.

   Since 2×3 and 5×7 have no common factors except 1, $\frac{6}{35}$ must be the fraction we wanted to find.

   a. $\frac{6}{16}$
   b. $\frac{7}{19}$
   c. $\frac{12}{20}$
   d. $\frac{21}{35}$
   e. $\frac{26}{14}$
   f. $\frac{16}{27}$
   g. $\frac{2 \times 3 \times 5 \times 5 \times 7}{2 \times 5 \times 7 \times 11}$
   h. $\frac{3 \times 5 \times 7}{2 \times 11}$
   i. $\frac{9 \times 4 \times 5}{16 \times 3 \times 7}$

2. Find each of the measures given below. Express each using the smallest possible denominator.

   Example: The measure of 5 days in weeks is $\frac{5}{7}$. This is the expression with the smallest denominator.

   a. The measure of 36 seconds in minutes.
   b. The measure of 14 hours in days.
   c. The measure of 30 days in years.
   d. The measure of 6 ounces in pounds.
   e. The measure of 42 inches in yards.

BRAINTWISTERS

3. Suppose that m and n are counting numbers. Mark T for true or F for false for each of the following sentences about $\frac{m}{n}$.

   a. If m and n are both <u>even</u> then $\frac{m}{n}$ can always be expressed using a denominator smaller than n.

   b. If m and n are both <u>odd</u> then $\frac{m}{n}$ cannot be expressed using a smaller denominator.

   c. If <u>no</u> prime is a factor of <u>both</u> m and n, then the greatest common factor of m and n is 1.

   d. If <u>no</u> prime is a factor of <u>both</u> m and n, then $\frac{m}{n}$ cannot be expressed using a smaller denominator.

   e. If $\frac{m}{n} = \frac{4}{6}$ then 4 is a factor of m and 6 is a factor of n.

   f. If $\frac{m}{n} = \frac{2}{3}$ then 2 is a factor of m and 3 is a factor of n.

<u>Supplementary Exercise Set A</u>

1. Write as a product of primes:

   a. 63×120

   b. 65×92

   c. 210×180

2. a. How many times does 2 appear if 24×7075 is written as a product of primes?

   b. How many times does 3 appear?

3. Find three pairs of numbers with the number given as greatest common factor.

   a. 9

   b. 10

   c. 12

4. There is a composite number less than 125. It does not have 2, 3, 5, or 7 as a factor. What is the number?

5. Find the greatest common factor of these <u>triples</u> of numbers.

   a. 6, 9, 30

   b. 8, 12, 25

   c. 25, 30, 50

6. I am thinking of an operation on counting numbers. I will call the result of operating on m and n, $m \cdot n$ ("m dot n"). Here are some facts about the operation "dot."

$6 \cdot 4 = 2$	$4 \cdot 3 = 1$	$5 \cdot 15 = 5$	$8 \cdot 12 = 4$
$n \cdot 1 = 1$	$10 \cdot 15 = 5$	$18 \cdot 26 = 2$	$42 \cdot 25 = 1$

a. What is a rule for finding $m \cdot n$?

b. Is the operation "dot" commutative?

c. Is it associative?

Supplementary Exercise Set B

1. Suppose you know a large prime number, n. Then you can be sure that n + 1 is not a prime. Why?

2. In this exercise write only base five numerals. Write as a product of primes, if possible.

   a. $(30)_{five}$

   b. $(131)_{five}$

   c. $(100)_{five}$

3. a. Using base five numerals, is there a simple test to find whether 2 is a factor of a number?

   b. Is there a simple test for 3 as a factor?

   c. Is there a simple test for $(10)_{five}$?

4. Which are prime and which are composite?

   a. $(10)_{four}$

   b. $(10)_{seven}$

   c. $(13)_{seven}$

   d. $(10)_{eight}$

   e. $(15)_{eight}$

   f. $(100)_{seventeen}$

5. Find a rule for testing 3 as a factor using base six numerals.

Supplementary Exercise Set C

1. Primes with only one number between them are called twin primes. 11 and 13 are twins, so are 17 and 19.

   a. What are the next two pairs of twin primes?

   The primes 3, 5, and 7 might be called triplet primes. If 15 were prime then 11, 13, 15 would be triplets.

   b. Do you know any other triplets besides 3, 5, and 7?

   c. In your chart of prime factors, find one other triplet other than 3, 5, and 7, if you can.

2. The number 6 has an interesting property noticed by Greek mathematicians over 2,000 years ago. It is this: the number 6 is the sum of all of its factors except 6.

$$1 + 2 + 3 = 6.$$

The Greeks admired this rare property and called such numbers perfect numbers. No one has ever been able to find a way to get all perfect numbers. No one knows whether there are any odd perfect numbers.

Find the next perfect number greater than 6.

3. All primes except 2 are odd. The sum of any two odd primes is <u>even</u>. Suppose we ask what even numbers are sums of two (perhaps equal) odd primes? The smallest number which could be is 6. It is, because $3 + 3 = 6$. Also $8 = 3 + 5$, $10 = 3 + 7$, $12 = 5 + 7$.

Show that every number from 6 through 30 is a sum of two odd primes.

No one has ever found an even number greater than 4 which is not the sum of two odd primes. Most mathematicians believe that <u>every</u> such even number is the sum of two odd primes. No one has been able to show that there cannot be any exceptions.

EXTENDING MULTIPLICATION AND DIVISION I

Chapter 3

REVIEWING IDEAS OF MULTIPLICATION

To express the product of two numbers using a mathematical sentence, we can write:

$$5 \times 4 = 20.$$

We read this either as:

5 times 4 is equal to 20

or

5 times 4 equals 20.

20 is the product of the numbers 5 and 4. 5 and 4 are factors of 20.

$$\underset{\text{factor}}{\underset{\uparrow}{5}} \times \underset{\text{factor}}{\underset{\uparrow}{4}} = \underset{\text{product}}{\underset{\uparrow}{20}}$$

We have found that any number has many names. The expression, 5×4, is another name for 20. When we use a name showing multiplication like 5×4 for 20, we call it a <u>product expression</u>. Both 20 and 5×4 name the product of 5 and 4. In this chapter we will learn ways of finding the decimal name for the products of large numbers.

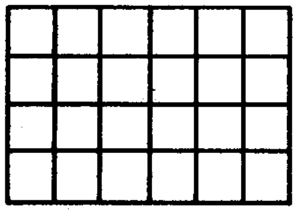
Exercise Set 1

Copy the following table and fill in the blanks with the products. (Use decimal numerals.)

x	6	8	5	10	4	0	9	2	7	3	1
4											
7											
1											
9											
3											
6											
10											
5											
0											
8											
2											

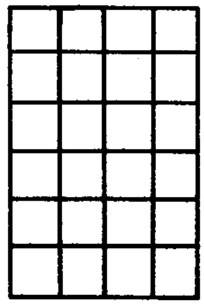
COMMUTATIVE PROPERTY OF MULTIPLICATION

A 4 by 6 array can be turned to form a 6 by 4 array.



4 by 6 array

$4 \times 6 = 24$



6 by 4 array

$6 \times 4 = 24$

This shows that $4 \times 6 = 6 \times 4$.

A 24 by 35 array can be turned to form a 35 by 24 array. This shows $24 \times 35 = 35 \times 24$. When we write 24×35 in place of 35×24, we are using the <u>commutative property of multiplication</u>.

By using the commutative property, we have fewer multiplication facts to learn.

If we know $5 \times 9 = 45$, then we know $9 \times 5 = 45$.

If we know $7 \times 8 = 56$, then we know $8 \times 7 = 56$.

If this property is used, how many multiplication facts are to be learned? How do you know?

What are the properties of 0 and 1 for multiplication?

How can we use these properties so we have even fewer multiplication facts to remember?

ASSOCIATIVE PROPERTY OF MULTIPLICATION

We know that we can multiply three numbers, such as 4 and 2 and 3, in that order, in either of two ways:

$$(4 \times 2) \times 3 = 8 \times 3 = 24$$

$$4 \times (2 \times 3) = 4 \times 6 = 24$$

Each way of grouping the numbers gives the same product. So, we may write:

$$(4 \times 2) \times 3 = 4 \times (2 \times 3).$$

When we replace one way of grouping the numbers by the other way, we are using the <u>associative property of multiplication</u>.

Because of the associative property of multiplication, we can write

$$4 \times 2 \times 3 = 24$$

without using any parentheses. We know that either grouping of the factors will give the same product.

We have learned how to multiply using 10, or 100, or 1000 as a factor in examples like these:

$3 \times 10 = 30$ $\quad$ $7 \times 100 = 700$ $\quad$ $9 \times 1000 = 9000$

$23 \times 10 = 230$ $\quad$ $57 \times 100 = 5700$ $\quad$ $39 \times 1000 = 39{,}000$

We also know our "multiplication facts," such as: $4 \times 3 = 12$, $7 \times 5 = 35$, $6 \times 8 = 48$.

Now let us review how we can use these two things along with the associative property of multiplication to find products of numbers such as 4 and 20, or 6 and 700, or 5 and 3000.

Example 1

4×20	$= 4 \times (2 \times 10)$	(Think of 20 as 2×10.)
	$= (4 \times 2) \times 10$	(Use associative property.)
	$= 8 \times 10$	(Product of 4 and 2 is 8.)
	$= 80$	(Product of 8 and 10 is 80.)

Example 2

6×700	$= 6 \times (7 \times 100)$	(Think of 700 as 7×100.)
	$= (6 \times 7) \times 100$	(Use associative property.)
	$= 42 \times 100$	(Product of 6 and 7 is 42.)
	$= 4200$	(Product of 42 and 100 is 4200.)

Example 3

5×3000	$= 5 \times (3 \times 1000)$	(Think of 3000 as 3×1000.)
	$= (5 \times 3) \times 1000$	(Use associative property.)
	$= 15 \times 1000$	(Product of 5 and 3 is 15.)
	$= 15{,}000$	(Product of 15 and 1000 is 15,000.)

Products of numbers such as 60 and 70, or 700 and 30 can be found using the associative property of multiplication along with the commutative property of multiplication.

Example 4

60×70	$= (6 \times 10) \times (7 \times 10)$	(Rename 60 and 70.)
	$= (6 \times 7) \times (10 \times 10)$	(Use the associative and commutative properties.)
	$= 42 \times 100$	(The product of 6 and 7 is 42; the product of 10 and 10 is 100.)
	$= 4200$	(The product of 42 and 100 is 4200.)

Example 5

700×30	$= (7 \times 100) \times (3 \times 10)$	(Rename 700 and 30.)
	$= (7 \times 3) \times (100 \times 10)$	(Use the associative and commutative properties.)
	$= 21 \times 1000$	(The product of 7 and 3 is 21; the product of 100×10 is 1000.)
	$= 21{,}000$	(The product of 21 and 1000 is 21,000.)

Do you know a way in which you can find the product of numbers like 60 and 70, or 700 and 30 more quickly? If not see if you can find one.

Exercise Set 2

1. Write each of the following products as decimal numerals.

a. 3×10
b. 4×100
c. $1,000 \times 7$
d. 100×12
e. $32 \times 1,000$
f. 10×56
g. 200×4
h. 33×100
i. 4×600
j. 800×3
k. $8 \times 2,000$
l. 500×6
m. 300×2
n. 7×80

2. Find the product of each of the pairs of numbers by using the commutative and associative properties of multiplication.

Example: 50 and 40

$$\begin{aligned} 50 \times 40 &= (5 \times 10) \times (4 \times 10) \\ &= (5 \times 4) \times (10 \times 10) \\ &= 20 \times 100 \\ &= 2,000 \end{aligned}$$

a. 30 and 70
b. 80 and 60
c. 200 and 300
d. 90 and 700
e. 300 and 40
f. 50 and 700
g. 600 and 80
h. 300 and 9,000

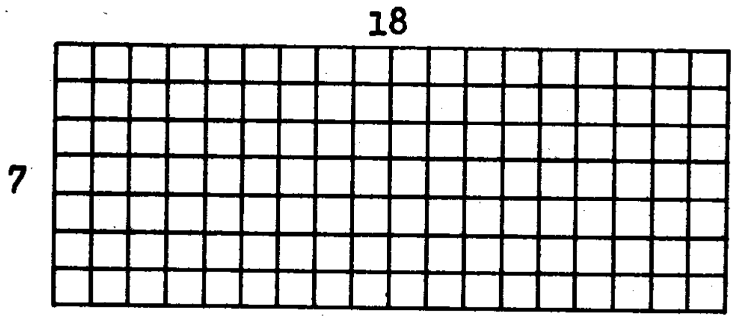
Exercise Set 3

Find n in each sentence. (Use a decimal numeral.)

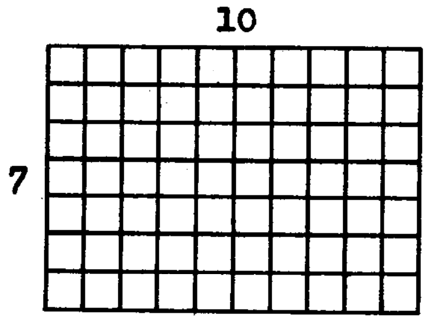
1. $40 \times 30 = n$
2. $50 \times 70 = n$
3. $60 \times 80 = n$
4. $30 \times 50 = n$
5. $60 \times 40 = n$
6. $20 \times 600 = n$
7. $500 \times 30 = n$
8. $400 \times 7 = n$
9. $70 \times 800 = n$
10. $80 \times 900 = n$
11. $200 \times 300 = n$
12. $500 \times 700 = n$
13. $300 \times 800 = n$
14. $700 \times 40 = n$
15. $30 \times 600 = n$
16. $70 \times 90 = n$
17. $80 \times 700 = n$
18. $90 \times 30 = n$
19. $80 \times 50 = n$
20. $20 \times 12,000 = n$

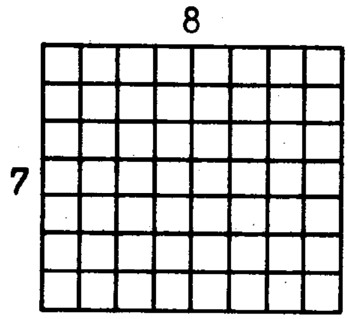
DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION

To find the product of 7 and 18, think of a 7 by 18 array.



Separate it into two arrays showing products you already know. For example:





7 by 10 array

$7 \times 10 = 70$

7 by 8 array

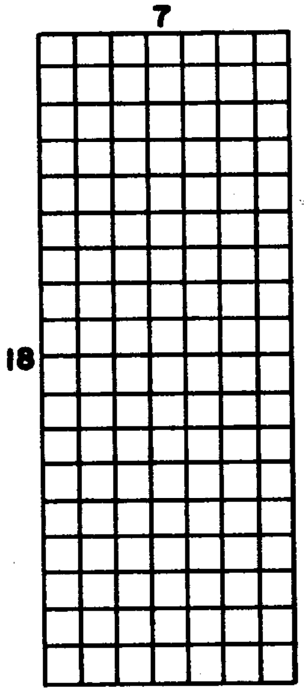
$7 \times 8 = 56$

These arrays help us see that

$$\begin{aligned} 7 \times 18 &= 7 \times (10 + 8) \\ &= (7 \times 10) + (7 \times 8) \\ &= 70 + 56 \\ &= 126. \end{aligned}$$

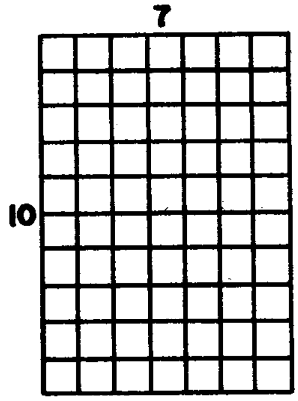
When we write $(7 \times 10) + (7 \times 8)$ in place of $7 \times (10 + 8)$, we are using the <u>distributive property of multiplication over addition</u>.

Now, suppose we find the product of 18 and 7.

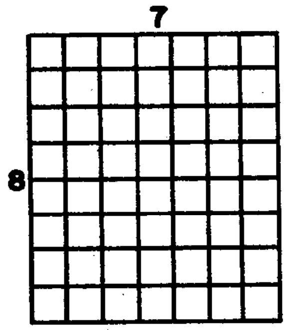


(18 by 7 array)

Can we separate the array like this?



(10 by 7 array)



(8 by 7 array)

Find the products separately and add them to get the total number of elements in the 18 by 7 array.

$$\begin{aligned} 18 \times 7 &= (10 + 8) \times 7 \\ &= (10 \times 7) + (8 \times 7) \\ &= 70 + 56 \\ &= 126 \end{aligned}$$

The commutative property of multiplication tells us that a 7 by 18 array has the same number of elements as an 18 by 7 array, thus: $7 \times 18 = 18 \times 7$.

Since $7 \times 18 = 7 \times (10 + 8)$

$= (7 \times 10) + (7 \times 8)$,

and $18 \times 7 = (10 + 8) \times 7$

$= (10 \times 7) + (8 \times 7)$,

then $(7 \times 10) + (7 \times 8) = (10 \times 7) + (8 \times 7) = 126$ elements.

Here are other illustrations of how we may use the distributive property of multiplication over additon.

1.	$20 \times 37 = 20 \times (30 + 7)$	(Rename 37 as 30 + 7.)
	$= (20 \times 30) + (20 \times 7)$	(Distribute 20 over 30 and 7.)
	$= 600 + 140$	(Use multiplication facts and place value.)
	$= 740$	(Use addition facts and place value.)
2.	$42 \times 30 = (40 + 2) \times 30$	(Rename 42 as 40 + 2.)
	$= (40 \times 30) + (2 \times 30)$	(Distribute 30 over 40 and 2.)
	$= 1200 + 60$	(Use multiplication facts and place value.)
	$= 1260$	(Use addition facts and place value.)
3.	$4 \times 285 = 4 \times (200 + 80 + 5)$	(Rename 285 as 200 + 80 + 5.)
	$= (4 \times 200) + (4 \times 80) + (4 \times 5)$	(Distribute 4 over 200, 80, and 5.)
	$= 800 + 320 + 20$	(Use multiplication facts and place value.)
	$= 1140$	(Use addition facts, associative property, and place value.)

Exercise Set 4

1. Using the properties of multiplication, express the following products as decimal numerals.

   Example: $6 \times 21 = 6 \times (20 + 1)$
   $= (6 \times 20) + (6 \times 1)$
   $= 120 + 6$
   $= 126$

   a. 3×27
   b. 42×6
   c. 2×128
   d. 7×341
   e. 217×8
   f. 4×285
   g. 22×10
   h. 47×30
   i. 20×62
   j. 71×30
   k. 40×57
   l. 60×23
   m. 78×10
   n. 20×91
   o. 86×30
   p. 39×50

2. Name the property of multiplication illustrated by each mathematical sentence.

   a. $8 \times 18 = 18 \times 8$
   b. $2 \times (9 \times 6) = (2 \times 9) \times 6$
   c. $10 \times 32 = (10 \times 30) + (10 \times 2)$

3. Find n in each mathematical sentence. Use what you know about the properties of multiplication to help you.

   a. $15 \times 30 = (10 \times 30) + (n \times 30)$
   b. $18 \times 5 = 5 \times n$
   c. $36 \times (10 \times 2) = 10 \times (2 \times n)$

4. On your paper, write <u>true</u> if the mathematical sentence is true. Write <u>false</u> if the mathematical sentence is false.

   a. $8 \times (7 + 5) = (8 \times 7) + (8 + 5)$

   b. $12 \times 10 = 10 \times 12$

   c. $33 \times 42 = (30 + 3) \times (40 + 2)$

   d. $(10 \times 3) \times 4 = 10 \times (4 \times 3)$

   e. $(10 \times 5) \times 7 = 10 \times (5 + 7)$

5. Each of the expressions below is equal to (40×60). Which <u>does</u> <u>not</u> illustrate the distributive property? Write its letter.

   a. $(20 \times 60) + (20 \times 60)$

   b. $(40 \times 30) + (40 \times 30)$

   c. $(4 \times 10) \times (6 \times 10)$

   d. $(25 \times 60) + (15 \times 60)$

BECOMING SKILLFUL IN MULTIPLYING

We have learned that we can use mathematical sentences to show our thinking when we multiply. For example,

$$4 \times 285 = n.$$

We can find the number which n represents in this way.

$$\begin{aligned} 4 \times 285 &= 4 \times (200 + 80 + 5) \\ &= (4 \times 200) + (4 \times 80) + (4 \times 5) \\ &= 800 + 320 + 20 \\ &= 1140 \end{aligned}$$

Then, $4 \times 285 = 1140$.

The numbers 800, 320, and 20, are called <u>partial</u> <u>products</u>.

Here is a shorter way to find the product of 285 and 4. We can write the partial products under each other as we multiply. Then, we can add them. For example, if $4 \times 285 = n$, we find the number which n represents in this way.

$$\begin{array}{rl} 285 & \\ \underline{\times \quad 4} & \\ 20 & \longleftarrow (4 \times 5) \\ 320 & \longleftarrow (4 \times 80) \\ \underline{800} & \longleftarrow (4 \times 200) \\ 1140 & \end{array}$$

Many of us should be able to write the product in an even shorter way.

$$\begin{array}{r} 285 \\ \underline{\times \quad 4} \\ 1140 \end{array}$$

Then, $4 \times 285 = 1140$.

What must we remember in order to do this?

Now let us consider this mathematical sentence.

$$3 \times 408 = n$$

We may write:

$$\begin{aligned} 3 \times 408 &= 3 \times (400 + 8) \\ &= (3 \times 400) + (3 \times 8) \\ &= 1200 + 24 \\ &= 1224 \end{aligned}$$

So, $n = 1224$, and $3 \times 408 = 1224$.

If we used shorter ways to find the product, we could write:

$$\begin{array}{rl} 408 & \\ \times\ 3 & \\ \hline 24 & \longleftarrow (3 \times 8) \\ 1200 & \longleftarrow (3 \times 400) \\ \hline 1224 & \end{array} \qquad \text{or} \qquad \begin{array}{r} 408 \\ \times\ 3 \\ \hline 1224 \end{array}$$

In the shorter way at the left, above, why are there just two partial products?

In each of the shorter ways shown above, is there any time when you did or could use the zero property for multiplication?

Exercise Set 5

A. Find n. If you need to, show the partial products.

1. $5 \times 63 = n$
2. $4 \times 56 = n$
3. $6 \times 93 = n$
4. $3 \times 256 = n$
5. $6 \times 307 = n$
6. $8 \times 209 = n$
7. $9 \times 347 = n$
8. $6 \times 986 = n$
9. $7 \times 837 = n$
10. $8 \times 2{,}609 = n$

B. Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

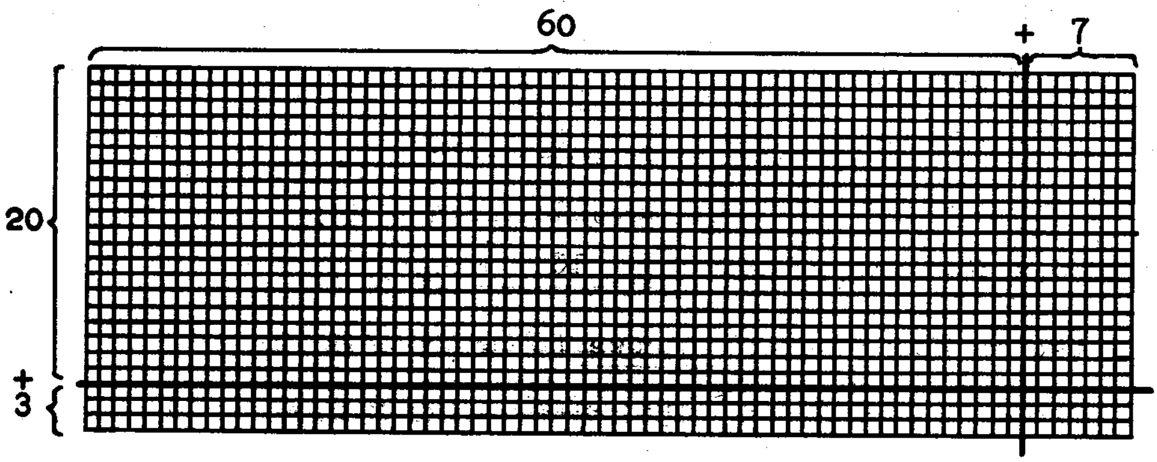
11. A building has 72 windows. If it takes 3 minutes to wash one window, how many minutes will it take to wash all of them?

12. A traffic light changes its color every 18 seconds. How many seconds will it take for the light to make 7 changes?

13. A phonograph record revolves 33 times a minute. How many revolutions will the record make if it plays for 3 minutes?

14. John and his father went on a fishing trip. It took them 6 hours to get to the lake. John's father was driving 55 miles per hour. How far did they have to drive before they could fish?

Working Together



23 by 67 array

We can show, by using the distributive property, how to multiply two numbers greater than 10 but less than 100.

$n = 23 \times 67$

$= 23 \times (60 + 7)$ (Think of 67 as 60 + 7.)

$= (23 \times 60) + (23 \times 7)$ (Distribute 23 over 67. The heavy vertical line shows how the array is separated into smaller arrays.)

$= (20 + 3) \times 60 + (20 + 3) \times 7$ (Think of 23 as 20 + 3.)

$= (20 \times 60) + (3 \times 60) + (20 \times 7) + (3 \times 7)$ (The heavy horizontal line then shows how the array is separated into 4 smaller arrays. The heavy lines drawn on the array above illustrate these four arrays.)

$= 1200 + 180 + 140 + 21$ (These show the number of elements in each of the four arrays.)

$= 1541$ (The total number of elements in a 23 by 67 array is 1541.)

The vertical form also can be used with larger numbers. Look at this example.

$23 \times 67 = n$

$$\begin{array}{rl} 67 & \\ \times 23 & \\ \hline 21 & \longleftarrow (3 \times 7) \\ 180 & \longleftarrow (3 \times 60) \\ 140 & \longleftarrow (20 \times 7) \\ 1200 & \longleftarrow (20 \times 60) \\ \hline 1541 & \longleftarrow (23 \times 67) \end{array}$$

$23 \times 67 = 1541$

See if you can identify each of the partial products shown above with parts of the array.

Using the vertical form, compute the following.

$$\begin{array}{r} 54 \\ \times\ 32 \\ \hline \end{array} \qquad \begin{array}{r} 25 \\ \times\ 18 \\ \hline \end{array} \qquad \begin{array}{r} 37 \\ \times 42 \\ \hline \end{array}$$

Exercise Set 6

A. Compute using the vertical form. Show the partial products.

Example: 32×54

$$\begin{array}{r} 54 \\ \times\ 32 \\ \hline 8 \\ 100 \\ 120 \\ 1500 \\ \hline 1728 \end{array}$$

1. 45×23
2. 64×25
3. 37×26
4. 61×59
5. 28×92
6. 37×12
7. 24×37
8. 26×97
9. 37×86
10. 49×81
11. 57×77
12. 66×88
13. 44×95
14. 82×28
15. 37×75
16. 91×67

B. Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

17. A set of books weighs 12 pounds. If a school ordered 38 sets, what would be the total weight of the books ordered?

18. Mr. Jones, a farmer, sent 27 crates of eggs to the market. There were 24 dozen eggs in each crate. How many dozen eggs did he send to market?

19. During our vacation last summer, we traveled for 28 hours. We drove at 59 miles per hour. How far did we travel during the 28 hours?

20. The candy store packed 86 boxes of candy. Each box contained 64 pieces of candy. How many pieces of candy were needed to pack all the boxes?

A SHORTER FORM FOR MULTIPLYING

Look at this example.

$$25 \times 72 = n$$

Here are two forms for finding the decimal numeral for n:

Longer Form

$$\begin{array}{rl} 72 & \\ \times\ 25 & \\ \hline 10 & (5 \times 2) \\ 350 & (5 \times 70) \\ 40 & (20 \times 2) \\ 1400 & (20 \times 70) \\ \hline 1800 & \end{array}$$

Shorter Form

$$\begin{array}{rl} 72 & \\ \times\ 25 & \\ \hline 360 & (5 \times 72) \\ 1440 & (20 \times 72) \\ \hline 1800 & \end{array}$$

$$n = 1800$$

$$25 \times 72 = 1800$$

Explain how the partial products in the longer and shorter forms are related to each other.

Exercise Set 7

Compute using a vertical form. Use the shorter form if you can.

Example: 37×54

$$
\begin{array}{r}
54 \\
\times \underline{\ \ 37} \\
378 \\
\underline{1620} \\
1998
\end{array}
$$

1. 12×34
2. 21×43
3. 41×25
4. 15×37
5. 37×18
6. 24×37
7. 32×48
8. 12×98
9. 35×56
10. 86×72
11. 34×62
12. 84×53
13. 76×38
14. 83×95
15. 46×73
16. 66×37
17. 53×46
18. 72×33
19. 38×25
20. 36×49

USING A SHORTER FORM TO MULTIPLY LARGER NUMBERS

These examples will help you to learn how to find products of larger numbers.

<u>Example 1:</u> n = 43 × 237

237			237	
× 43		OR	× 43	
21	(3 × 7)		711	(3 × 237)
90	(3 × 30)		9480	(40 × 237)
600	(3 × 200)		10191	(43 × 237)
280	(40 × 7)			
1200	(40 × 30)		n = 10,191	
8000	(40 × 200)			
10191	(43 × 237)			

<u>Example 2:</u> n = 34 × 5032

5032			5032	
× 34		OR	× 34	
8	(4 × 2)		20128	(4 × 5032)
120	(4 × 30)		150960	(30 × 5032)
20000	(4 × 5000)		171088	(34 × 5032)
60	(30 × 2)			
900	(30 × 30)			
150000	(30 × 5000)		n = 171,088	
171088	(34 × 5032)			

Exercise Set 8

A. Use a vertical form to compute the following.

1. 26×201
2. 41×607
3. 42×121
4. 64×328
5. 270×37
   Hint: By using the commutative property of multiplication we know that $270 \times 37 = 37 \times 270$.
6. 863×27
7. 96×8021
8. 45×378
9. 37×856
10. 54×2805
11. 317×47
12. 598×36
13. 58×4566
14. 638×21
15. 956×57

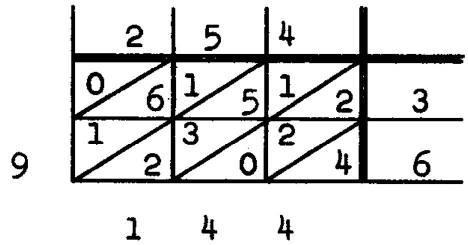
B. Use mathematical sentences to solve the following problems. Express each answer in a complete sentence.

16. If your father earns $840 a month, how much does he earn in a year?

17. An automobile averages 16 miles per gallon of gasoline. The gasoline tank holds 17 gallons. How many miles will the automobile go on 17 gallons?

18. **BRAINTWISTER:** During the time of Columbus, a different multiplication form was used in Europe. This was called the Gelosia or Lattice method.

The solution of $n = 254 \times 36$ is shown by the diagram.



Can you find the value of n from the diagram? Test your knowledge of the Gelosia method by showing that

$$56 \times 672 = 37,632.$$

PROBLEM SOLVING

A coin book has 35 slots for coins on each page. If the book has 12 pages and 287 coins have been placed in the slots, how many more are needed to complete the book?

Here is a way to solve this problem using two mathematical sentences.

$12 \times 35 = p$ $\qquad$ $420 - 287 = n$

$$\begin{array}{r} 35 \\ \underline{\times 12} \\ 70 \\ \underline{350} \\ 420 \end{array} \qquad \begin{array}{r} 420 \\ \underline{-287} \\ 133 \end{array}$$

There are 133 coins needed to complete this book.

Here is a way to solve this problem using one mathematical sentence.

$$(12 \times 35) - 287 = n$$

$$\begin{array}{r} 35 \\ \underline{\times 12} \\ 70 \\ \underline{350} \\ 420 \end{array} \qquad \begin{array}{r} 420 \\ \underline{-287} \\ 133 \end{array}$$

There are 133 coins needed to complete the book.

Exercise Set 9

Use mathematical sentences to help you solve the following problems. Express each answer in a complete sentence.

1. A typewriter prints 12 symbols to an inch across a page. How many symbols can be printed on a sheet of paper 8 inches wide without using spaces between the symbols if there are 65 rows of symbols possible?

2. John bought a notebook for 25¢, a pencil for 7¢, and an arithmetic book for $2.50. He gave the clerk $5.00. How much change did he receive?

3. Jane takes the bus to and from school 5 days per week. The fare each way is 25¢. How much is her fare for the week?

4. The Brown family of six planned to fly to Washington on their vacation. Each person was allowed 40 pounds of free baggage. The Browns had 263 pounds of baggage. What was the number of pounds of extra baggage?

5. There are 24 pages in Mary's stamp album. On each page there is room for 18 stamps. Mary has 279 stamps. How many stamps does she need to fill her album?

6. A parking lot had 25 rows with 16 spaces in each row. The size of the lot was increased with spaces for 225 cars. Since the addition, how many cars can be parked on this lot?

REVIEWING IDEAS OF DIVISION

Division is the operation we use to find an unknown factor when the product and one factor are known.

The following sentences suggest division.	This is how we can read them.
$n \times 4 = 20$	What number times 4 is equal to 20?
$4 \times n = 20$	4 times what number is equal to 20?
$20 \div 4 = n$	20 divided by 4 is equal to what number?
$20 \div n = 4$	20 divided by what number is equal to 4?

In each case we are to find the unknown factor. We may use the same process.

A form for computing:

$$\underset{\text{Product}}{\underset{\uparrow}{20}} \quad \div \quad \underset{\text{Known Factor}}{\underset{\uparrow}{4}} \quad = \quad \underset{\text{Unknown Factor}}{\underset{\uparrow}{n}}$$

$$4\overline{)\,20\,}^{\;5}$$
$$\underline{20}$$

$n = 5$

We have learned to become skillful with multiplication. Now we want to learn ways of making the process of division easier.

WORKING WITH MULTIPLES OF 10 AND 100

Copy the table and complete it.

x	10	20	30	40	50	60	70	80	90	100
1										
2		40					140			
3				120						
4										
5									450	
6			180				420			
7	70									700
8		160			400					
9								720		
10						600				

Study the table you have just completed. How did you know to write 1000 in the lower right hand box?

How can this table be used to find the unknown factor in a division example?

Look at this example.

$$150 \div 3 = n$$

We think: $3 \times n = 150$. In the table, find the "3-row" and follow it until you see 150. Then look up the column and find the other factor, 50. Thus, $3 \times \underline{50} = 150$. So, $150 \div 3 = \underline{50}$.

Exercise Set 10

Find n in each of these.

1. $540 \div 9 = n$
2. $270 \div 3 = n$
3. $600 \div 10 = n$
4. $720 \div 8 = n$
5. $490 \div 7 = n$
6. $350 \div 5 = n$
7. $180 \div 6 = n$
8. $210 \div 3 = n$
9. $640 \div 8 = n$
10. $400 \div 5 = n$
11. $120 \div 2 = n$
12. $810 \div 9 = n$
13. $360 \div 9 = n$
14. $540 \div 6 = n$
15. $240 \div 4 = n$
16. $400 \div 5 = n$

Exercise Set 11

x	100	200	300	400	500	600	700	800	900	1000
1						600			900	
2						1200	1400			
3	300	600	900	1200	1500	1800	2100	2400	2700	3000
4					2000	2400				
5			1500			3000				
6						3600			5400	6000
7		1400		2800		4200		5600		
8				3200		4800				
9				3600		5400		7200		9000
10						6000				

After you complete this table, your teacher will discuss it with you.

Find n in the following examples. Use the table you have just completed.

1. $1500 \div 5 = n$
2. $4900 \div 7 = n$
3. $6000 \div 6 = n$
4. $3200 \div 4 = n$
5. $7200 \div 8 = n$
6. $900 \div 3 = n$
7. $2700 \div 9 = n$
8. $10{,}000 \div 10 = n$
9. $5600 \div 7 = n$
10. $2400 \div 8 = n$

Exercise Set 12

Using the tables you just completed, find the unknown factor in each of these mathematical sentences.

1. $80 \div 2 = n$
2. $280 \div 7 = n$
3. $5400 \div 9 = p$
4. $6400 \div 8 = s$
5. $3500 \div 5 = m$
6. $490 \div 7 = r$
7. $810 \div 9 = n$
8. $320 \div 4 = p$
9. $270 \div 3 = s$
10. $1400 \div 2 = r$
11. $6300 \div 7 = n$
12. $4200 \div 6 = s$
13. $640 \div 8 = n$
14. $270 \div 9 = m$
15. $6300 \div 9 = r$
16. $4000 \div 8 = m$
17. $450 \div 5 = n$
18. $420 \div 7 = s$
19. $1200 \div 4 = t$
20. $5000 \div 10 = p$

Exercise Set 13

Copy each row of exercises below. Complete the blanks so that each mathematical sentence is true.

	Use the largest whole number.	Use the largest multiple of 10.	Use the largest multiple of 100.
1.	(a) $4 \times$ ___ $= 12$	(b) $4 \times$ ___ $= 120$	(c) $4 \times$ ___ $= 1200$
2.	(a) $6 \times$ ___ $= 36$	(b) $6 \times$ ___ $= 360$	(c) $6 \times$ ___ $= 3600$
3.	(a) $8 \times$ ___ $= 24$	(b) $8 \times$ ___ $= 240$	(c) $8 \times$ ___ $= 2400$
4.	(a) $9 \times$ ___ $= 45$	(b) $9 \times$ ___ $= 450$	(c) $9 \times$ ___ $= 4500$
5.	(a) $5 \times$ ___ $= 30$	(b) $5 \times$ ___ $= 300$	(c) $5 \times$ ___ $= 3000$
6.	(a) $3 \times$ ___ $= 27$	(b) $3 \times$ ___ $= 270$	(c) $3 \times$ ___ $= 2700$
7.	(a) $7 \times$ ___ $= 56$	(b) $7 \times$ ___ $= 560$	(c) $7 \times$ ___ $= 5600$
8.	(a) $4 \times$ ___ $= 32$	(b) $4 \times$ ___ $= 320$	(c) $4 \times$ ___ $= 3200$

Exercise Set 14

1. Copy and complete with the correct multiple of 10.

   Example: $\underline{70} \times 5 = 350$

a.	___ × 6 = 420	f.	___ × 9 = 810
b.	8 × ___ = 480	g.	___ × 8 = 400
c.	___ × 9 = 270	h.	___ × 6 = 180
d.	___ × 3 = 240	i.	7 × ___ = 210
e.	2 × ___ = 180	j.	___ × 6 = 240

2. Copy and complete with the correct multiple of 100.

   Example: $\underline{400} \times 4 = 1600$

a.	___ × 3 = 1500	f.	___ × 5 = 4500
b.	___ × 6 = 2400	g.	9 × ___ = 7200
c.	4 × ___ = 3200	h.	___ × 6 = 4800
d.	___ × 7 = 4900	i.	___ × 7 = 6300
e.	___ × 8 = 1600	j.	6 × ___ = 3600

3. Copy and complete with the correct multiple of 10 or 100.

   Example: $\underline{80} \times 6 = 480$

a.	7 × ___ = 6300	f.	___ × 2 = 1600
b.	___ × 4 = 2800	g.	___ × 9 = 6300
c.	___ × 5 = 4500	h.	___ × 8 = 6400
d.	___ × 3 = 270	i.	7 × ___ = 5600
e.	10 × ___ = 6000	j.	___ × 5 = 2500

Exercise Set 15

Copy each row of exercises below. Complete the blanks so that each mathematical sentence is true.

	Use the largest whole number.	Use the largest multiple of 10.	Use the largest multiple of 100.
1.	(a) $\underline{\quad} \times 6 < 25$	(b) $\underline{\quad} \times 6 < 252$	(c) $\underline{\quad} \times 6 < 2526$
2.	(a) $\underline{\quad} \times 4 < 31$	(b) $\underline{\quad} \times 4 < 315$	(c) $\underline{\quad} \times 4 < 3158$
3.	(a) $\underline{\quad} \times 9 < 28$	(b) $\underline{\quad} \times 9 < 283$	(c) $\underline{\quad} \times 9 < 2834$
4.	(a) $\underline{\quad} \times 8 < 44$	(b) $\underline{\quad} \times 8 < 446$	(c) $\underline{\quad} \times 8 < 4465$
5.	(a) $\underline{\quad} \times 3 < 26$	(b) $\underline{\quad} \times 3 < 263$	(c) $\underline{\quad} \times 3 < 2639$
6.	(a) $\underline{\quad} \times 8 < 76$	(b) $\underline{\quad} \times 8 < 765$	(c) $\underline{\quad} \times 8 < 7657$
7.	(a) $\underline{\quad} \times 8 < 60$	(b) $\underline{\quad} \times 8 < 600$	(c) $\underline{\quad} \times 8 < 6000$
8.	(a) $\underline{\quad} \times 7 < 45$	(b) $\underline{\quad} \times 7 < 456$	(c) $\underline{\quad} \times 7 < 4568$

Exercise Set 16

Copy each row of exercises below. Complete the blanks so that each mathematical sentence is true.

	Use the largest whole number.	Use the largest multiple of 10.	Use the largest multiple of 100.
1.	(a) $\_\_ \times 7 < 23$	(b) $\_\_ \times 7 < 238$	(c) $\_\_ \times 7 < 2385$
2.	(a) $6 \times \_\_ = 54$	(b) $6 \times \_\_ = 540$	(c) $6 \times \_\_ = 5400$
3.	(a) $\_\_ \times 5 < 21$	(b) $\_\_ \times 5 < 219$	(c) $\_\_ \times 5 < 2197$
4.	(a) $5 \times \_\_ < 37$	(b) $5 \times \_\_ < 375$	(c) $5 \times \_\_ < 3759$
5.	(a) $\_\_ \times 7 = 49$	(b) $\_\_ \times 7 = 490$	(c) $\_\_ \times 7 = 4900$
6.	(a) $8 \times \_\_ < 78$	(b) $8 \times \_\_ < 782$	(c) $8 \times \_\_ < 7828$
7.	(a) $\_\_ \times 7 < 65$	(b) $\_\_ \times 7 < 654$	(c) $\_\_ \times 7 < 6547$
8.	(a) $8 \times \_\_ < 50$	(b) $8 \times \_\_ < 500$	(c) $8 \times \_\_ < 5000$

Exercise Set 17

1. Complete with the largest multiple of 10 that may be used to make the sentence true.

a. ___ × 5 < 103	f. 8 × ___ < 500
b. ___ × 6 < 191	g. ___ × 9 < 650
c. ___ × 7 < 220	h. ___ × 7 < 583
d. 4 × ___ < 175	i. 9 × ___ < 750
e. 5 × ___ < 311	j. ___ × 6 < 549

2. Complete with the largest multiple of 100 that may be used to make the sentence true.

a. ___ × 6 < 2500	f. 4 × ___ < 3000
b. ___ × 5 < 600	g. ___ × 9 < 4852
c. ___ × 4 < 1000	h. ___ × 3 < 1000
d. 6 × ___ < 2000	i. 4 × ___ < 1846
e. 7 × ___ < 4000	j. 2 × ___ < 1946

3. Complete with the largest multiple of 100 that may be used to make the sentence true. If this is not possible then use the largest multiple of 10.

a. 8 × ___ < 5000	f. 4 × ___ < 304
b. ___ × 4 < 2196	g. 6 × ___ < 4507
c. 7 × ___ < 568	h. ___ × 8 < 412
d. 6 × ___ < 596	i. ___ × 4 < 3597
e. ___ × 8 < 2502	j. 9 × ___ < 8200

BECOMING SKILLFUL IN DIVIDING

We shall use what we know about multiples of numbers to learn more about dividing one number by another.

Suppose we are to find n in either of these sentences.

$$
\begin{array}{ccccccc}
n & \times & 4 & = & 332 & \text{or} & 332 \div 4 = n \\
\uparrow & & \uparrow & & \uparrow & & \\
\text{Unknown} & & \text{Known} & & \text{Product} & & \\
\text{Factor} & & \text{Factor} & & & &
\end{array}
$$

To find n in either sentence we divide 332 by 4. We can use one of the forms below. You may select the one you would like to use. Use either Form I or Form II.

<u>Form I:</u> <u>Form II:</u>

$$
\begin{array}{rcrl}
\underline{83} & & & \\
3 & & & \\
\underline{80} & & & \\
4\overline{)332} & & 4\,\big)\,332 & \\
\underline{320} & \longleftarrow (80 \times 4) \longrightarrow & \underline{320} & \;80 \\
12 & & 12 & \\
\underline{12} & \longleftarrow (3 \times 4) \longrightarrow & \underline{12} & \;\underline{3} \\
0 & & 0 & \;83
\end{array}
$$

Mathematical Sentence: $83 \times 4 = 332$ or $332 \div 4 = 83$.

We can check our answer:

$$
\begin{array}{r}
83 \\
\underline{\times \;\; 4} \\
332
\end{array}
$$

Exercise Set 18

Find n. Use either Form I or Form II. Check your answers.

1. $n \times 4 = 52$
2. $n \times 6 = 84$
3. $n \times 9 = 117$
4. $5 \times n = 75$
5. $7 \times n = 98$
6. $n \times 4 = 84$
7. $n \times 8 = 560$
8. $5 \times n = 390$
9. $n \times 9 = 837$
10. $9 \times n = 135$
11. $n \times 4 = 208$
12. $7 \times n = 217$
13. $3 \times n = 153$
14. $n \times 9 = 828$
15. $n \times 7 = 574$
16. $7 \times n = 231$
17. $8 \times n = 448$
18. $4 \times n = 192$
19. $n \times 7 = 595$
20. $n \times 3 = 279$

FINDING QUOTIENTS AND REMAINDERS

We have used sentences like this

$$47 = (5 \times n) + r$$

in working with story problems.

We have seen how we can find the largest possible n and the smallest r in ways like these.

$$\begin{array}{rrl} & 9 & \leftarrow \text{quotient} \\ \text{divisor} \rightarrow 5 & \overline{)47} & \leftarrow \text{dividend} \\ & \underline{45} & \\ & 2 & \leftarrow \text{remainder} \end{array} \qquad \begin{array}{rr|l} \text{divisor} \rightarrow 5 &)\,47 & \leftarrow \text{dividend} \\ & \underline{45} & \underline{9} \\ \text{remainder} \longrightarrow & 2 & 9 \leftarrow \text{quotient} \end{array}$$

We have found that $47 = (5 \times 9) + 2$.

We can see that this sentence is true by thinking

$$47 = 45 + 2.$$

We can use these same ways to find quotients and remainders when we work with larger dividends.

Now look at this mathematical sentence.

$$437 = (n \times 9) + r$$

$$\begin{array}{rr} & \underline{48} \\ & 8 \\ & \underline{40} \\ 9 &)437 \\ & \underline{360} \\ & 77 \\ & \underline{72} \\ & 5 \end{array} \quad \begin{array}{c} \\ \\ \\ \\ \longleftarrow (40 \times 9) \longrightarrow \\ \\ \longleftarrow (8 \times 9) \longrightarrow \\ \\ \end{array} \quad \begin{array}{r|r} 9)437 & \\ \underline{360} & 40 \\ 77 & \\ \underline{72} & \underline{8} \\ 5 & 48 \end{array}$$

Which number is the quotient?

Which number is the dividend?

Which number is the divisor?

Which number is the remainder?

Is the remainder less than the divisor?

We have found that

$$437 = (48 \times 9) + 5.$$

We can check to see if the sentence is true by multiplying 48 and 9, and adding 5. Our answer should be 437.

$$\begin{array}{r} 48 \\ \times\ 9 \\ \hline 432 \\ +\ 5 \\ \hline 437 \end{array}$$

Exercise Set 19

A. Use either Form I or Form II to find n and r.

Then rewrite the sentence using the numbers you found.

1. $600 = (n \times 7) + r$
2. $138 = (n \times 9) + r$
3. $213 = (7 \times n) + r$
4. $450 = (n \times 8) + r$
5. $271 = (n \times 3) + r$
6. $107 = (3 \times n) + r$
7. $230 = (n \times 7) + r$
8. $162 = (n \times 6) + r$
9. $738 = (9 \times n) + r$
10. $200 = (n \times 6) + r$
11. $372 = (n \times 9) + r$
12. $725 = (8 \times n) + r$
13. $373 = (n \times 9) + r$
14. $288 = (4 \times n) + r$
15. $451 = (n \times 8) + r$

B. Use mathematical sentences to solve these problems. Express each answer in a complete sentence.

16. At camp, John made a collection of 176 small stones. He put the same number of stones in each of 4 small boxes. How many did he put in each box? How many were left over?

17. There were 256 children visiting the Natural History Museum. Nine guides showed children around the museum. How many groups containing the same number of children could be formed? Are there any children left over?

Exercise Set 20

1. Name the divisor, dividend, quotient, and remainder for each of the following.

a.
```
     32
      2
     30
8 ) 258
    240
     18
     16
      2
```

b.
```
6 ) 732 |
    600 | 100
    132 |
    120 |  20
     12 |
     12 |   2
      0 | 122
```

2. **Use a number to complete the following so they are true statements.**

   a. If the remainder is ___, then the divisor is a factor of the dividend.

   b. If the remainder is not ___, then the divisor is not a factor of the dividend.

   c. If $1026 = (7 \times 146) + 4$, then the remainder is ____.

   d. If $842 = (6 \times n) + r$ with $r < 6$, then $n =$ ____, and $r =$ ____.

3. Divide the first number by the second. Then write the mathematical sentence. **For example,** 258 divided by 8 gives a quotient 32 and a remainder 2. The mathematical sentence is $258 = (32 \times 8) + 2$. Check the last 5 sentences.

a. 512 by 8	f. 756 by 7	k. 859 by 3
b. 382 by 7	g. 527 by 3	l. 604 by 6
c. 251 by 4	h. 805 by 4	m. 2597 by 7
d. 456 by 6	i. 927 by 9	n. 2001 by 5
e. 812 by 9	j. 625 by 5	o. 7024 by 8

FINDING MULTIPLES OF LARGER NUMBERS

Copy and complete the following table.

x	10	20	30	40	50	60	70	80	90	100
10	100				500					
20				800					1800	
30			900				2100			
40				1600					3600	
50					2500					5000
60				2400			4200			
70			2100			4200				
80				3200			5600			
90			2700					7200		
100	1000								9000	

Exercise Set 21

Use your table to find n.

1. 800 ÷ 20 = n
2. 2800 ÷ 40 = n
3. 2800 ÷ 70 = n
4. 20 × n = 1800
5. n × 70 = 5600
6. 70 × 90 = n
7. 4500 ÷ 50 = n
8. n × 100 = 8000
9. 60 × n = 5400
10. 2700 ÷ 90 = n
11. n × 50 = 1500
12. 80 × 80 = n
13. 4900 ÷ 70 = n
14. 50 × n = 2000
15. 80 × n = 7200
16. 6000 ÷ 60 = n
17. 3600 ÷ 40 = n
18. 30 × n = 1800
19. n × 90 = 6300
20. n × 100 = 10,000

Exercise Set 22

1. Complete with the largest multiple of 10 which makes the sentence true.

a. ___ × 20 < 720
b. ___ × 10 < 836
c. ___ × 30 < 506
d. ___ × 50 < 918
e. 20 × ___ < 432
f. ___ × 60 < 3290
g. ___ × 70 < 3040
h. ___ × 60 < 5500
i. ___ × 80 < 5000
j. 90 × ___ < 6500
k. 80 × ___ < 4700
l. 50 × ___ < 3500

2. Complete with the largest multiple of 100 which makes the sentence true.

a. 40 × ___ < 8442
b. 20 × ___ < 5591
c. 10 × ___ < 2146
d. ___ × 30 < 6723
e. ___ × 6 < 3290
f. ___ × 3 < 2872
g. 50 × ___ < 36,012
h. ___ × 70 < 45,000
i. 20 × ___ < 5640
j. 70 × ___ < 26,500
k. 80 × ___ < 60,000
l. 90 × ___ < 75,000

3. Find the largest multiple of 100 which makes the sentence true. If there is no multiple of 100, then find the largest multiple of 10.

a. 20 × ___ < 731
b. ___ × 46 < 4830
c. ___ × 30 < 742
d. 30 × ___ < 12,200
e. 50 × ___ < 26,200
f. 40 × ___ < 2449
g. 60 × ___ < 45,000
h. 70 × ___ < 30,000
i. ___ × 90 < 7500
j. 90 × ___ < 75,460

USING DIVISORS THAT ARE MULTIPLES OF 10

Exploration

We are going to learn to divide when the divisors are multiples of 10. Look at each of the examples below. Can you tell what was done in each example?

Example 1:

Divide 480 by 20.

$\underline{24}$			
4			
20			
$20\overline{)480}$		$20\overline{)480}$	
$\underline{400}$ ←	(20 × 20)	→ $\underline{400}$	20
80		80	
$\underline{80}$ ←	(4 × 20)	→ $\underline{80}$	$\underline{4}$
0		0	24

$$480 = 20 \times 24$$

We think of n as the largest multiple of 10, so that (n × 20) is not greater than 480.

We then think of n as the largest number so that (n × 20) is not greater than 80.

We describe the results of the process by the mathematical sentence:

$480 = (24 \times 20) + 0$ or $480 = 24 \times 20.$

We can check the work by multiplication.

$$\begin{array}{r} 24 \\ \underline{\times 20} \\ 480 \end{array}$$

Example 2:

Divide 9,285 by 40.

$$
\begin{array}{r}
\underline{232} \\
2 \\
30 \\
200 \\
40\,\overline{)\,9285} \\
\underline{8000} \\
1285 \\
\underline{1200} \\
85 \\
\underline{80} \\
5
\end{array}
\qquad\qquad
\begin{array}{r|l}
40\,\overline{)\,9285} & \\
\underline{8000} & 200 \\
1285 & \\
\underline{1200} & 30 \\
85 & \\
80 & \underline{\;\;2} \\
\hline
5 & 232
\end{array}
$$

We think of n as the largest multiple of 100 so that $(n \times 40)$ is not greater than 9,285.

Next, we think of n as the largest multiple of 10 so that $(n \times 40)$ is not greater than 1,285.

Finally, we think of n as the largest number so that $(n \times 40)$ is not greater than 85.

We describe the results of the process by the mathematical sentence

$$9{,}285 = (40 \times 232) + 5.$$

We can check our work by multiplication and addition.

$$
\begin{array}{r}
232 \\
\underline{\times\;\;40} \\
9280 \\
\underline{+\;\;\;\;\;5} \\
9285
\end{array}
$$

Exercise Set 23

A. For each of the following exercises, divide the first number by the second. Then write a mathematical sentence which describes how we can express the results.

1. 720 by 30
2. 840 by 20
3. 680 by 40
4. 570 by 10
5. 1160 by 40
6. 990 by 90
7. 780 by 60
8. 3850 by 50
9. 5810 by 70
10. 5360 by 80
11. 783 by 10
12. 1600 by 30
13. 1956 by 20
14. 1897 by 40
15. 3162 by 50
16. 5599 by 70
17. 2600 by 60
18. 8746 by 90
19. 7543 by 80
20. 5757 by 70

B. Solve the following problems.

21. A shipping carton holds 20 books. How many cartons will be needed to ship an order of 900 books?
22. An auditorium can seat 1680 persons. If each row seats 40 persons, how many rows are in this auditorium?
23. How many trips must an elevator (capacity 20 persons) make to carry 254 people? (Hint: One trip may not carry a full load.)
24. The room mothers are boxing candy to sell at the annual carnival. They bought 2,880 pieces of candy and each box will hold 30 pieces. How many boxes of candy do the room mothers have to sell?

A SHORTER FORM FOR DIVIDING

There is a shorter way to write your quotient in division. It will allow you to do your work more quickly.

Study the examples below.

a. Longer Form

$$
\begin{array}{r}
\underline{139} \\
9 \\
30 \\
100 \\
6\,\overline{)\,836} \\
\underline{600} \\
236 \\
\underline{180} \\
56 \\
\underline{54} \\
2
\end{array}
$$

b. Shorter Form

$$
\begin{array}{r}
139 \\
6\,\overline{)\,836} \\
\underline{600} \\
236 \\
\underline{180} \\
56 \\
\underline{54} \\
2
\end{array}
$$

In b, to show the partial quotient 100, we can write 1 in the hundred's place. Instead of writing 30, we can write 3 in the ten's place. Then we can write 9 in the one's place.

We describe the results of either process by the mathematical sentence

$$836 = (139 \times 6) + 2.$$

c. Longer Form

$$\begin{array}{r|l} 6\overline{)836} & \\ \underline{600} & 100 \\ 236 & \\ \underline{180} & 30 \\ 56 & \\ \underline{54} & \underline{9} \\ \hline 2 & 139 \end{array}$$

d. Shorter Form

$$\begin{array}{r} \underline{139} \\ 6\overline{)836} \\ \underline{600} \\ 236 \\ \underline{180} \\ 56 \\ \underline{54} \\ 2 \end{array}$$

In d, to show the partial quotient 100, we can write 1 in the hundred's place. Instead of writing 30, we can write 3 in the ten's place. Then we can write 9 in the one's place.

We describe the results of either process by the mathematical sentence

$$836 = (139 \times 6) + 2.$$

What do you notice about b and d?

Exercise Set 24

For each of the following, divide the first number by the second. Write a mathematical sentence to describe the result.

1. 963 by 3
2. 848 by 4
3. 499 by 3
4. 648 by 4
5. 4882 by 6
6. 6896 by 8
7. 4928 by 6
8. 6524 by 9
9. 7932 by 8
10. 3654 by 4

A SHORTER FORM FOR DIVIDING BY LARGER DIVISORS

Study the examples below.

a. Longer Form

$$\begin{array}{r} \underline{261} \\ 1 \\ 60 \\ 200 \\ 30\,\overline{)7833} \\ \underline{6000} \\ 1833 \\ \underline{1800} \\ 33 \\ \underline{30} \\ 3 \end{array}$$

b. Shorter Form

$$\begin{array}{r} 261 \\ 30\,\overline{)7833} \\ \underline{6000} \\ 1833 \\ \underline{1800} \\ 33 \\ \underline{30} \\ 3 \end{array}$$

In b, to show the partial quotient 200, we can write 2 in the hundred's place. Instead of writing 60, we can write 6 in the ten's place. Then we can write 1 in the one's place.

We can describe the results of either process by the mathematical sentence

$$7833 = (261 \times 30) + 3.$$

c. Longer Form d. Shorter Form

$$\begin{array}{r|l}
30\,\overline{)\,7833} & \\
\underline{6000} & 200 \\
1833 & \\
\underline{1800} & 60 \\
33 & \\
\underline{30} & \underline{1} \\
\hline
3 & 261
\end{array}
\qquad\qquad
\begin{array}{r}
261 \\
30\,\overline{)\,7833} \\
\underline{6000} \\
1833 \\
\underline{1800} \\
33 \\
\underline{30} \\
3
\end{array}$$

In d, to show the partial quotient 200, we can write 2 in the hundred's place. Instead of writing 60, we can write 6 in the ten's place. Then we can write 1 in the one's place.

We can describe the results of either process by the mathematical sentence

$$7833 = (261 \times 30) + 3.$$

What do you notice about examples b and d?

Find the quotient and remainder in each of these, using both a longer form and the shorter form.

$$40\,\overline{)\,8153} \qquad\qquad 30\,\overline{)\,10517}$$

For each example, did you get the same quotient and remainder using both forms? You should have!

Exercise Set 25

For each of the following, divide the first number by the second. Write a mathematical sentence to describe the result of the process.

1. 5820 by 10
2. 9240 by 40
3. 13,440 by 20
4. 17,550 by 30
5. 23,350 by 50
6. 58,980 by 60
7. 57,840 by 80
8. 40,680 by 90
9. 27,760 by 80
10. 21,000 by 50
11. 3,462 by 10
12. 18,464 by 20
13. 19,056 by 40
14. 27,291 by 70
15. 29,083 by 30
16. 32,240 by 60
17. 15,989 by 90
18. 42,750 by 80
19. 40,876 by 50
20. 31,452 by 70

Practice Exercises

1. Write each of the following as the product of two factors. Write 3 different product expressions for each number.

   Example: $30 = 1 \times 30$, 2×15 , 5×6

   a) 52

   b) 116

   c) 128

   d) 88

   e) 176

   f) 90

   g) 81

   h) 126

   i) 225

   j) 100

2. Solve the following:

   a) $8 \times (9000 + 6)$

   b) $(32 + 78) - 41$

   c) 9×847

   d) $.6 + .45 + 1.7 + 8$

   e) $(74 \times 600) + (74 \times 95)$

   f) $835 - 585$

   g) $301 \div 7$

   h) $7 \times 7 \times 912$

   i) $.61 + .09 + 8.5 + .48$

   j) $976 \div 8$

3. Write the number that n represents.

a) $90 \times 370 = n$

b) $49{,}003 - n = 39{,}936$

c) $n \times 9 = 936$

d) $887 + 875 + 699 - n = 0$

e) $n \div 9 = 98$

f) $7 \times n = 637$

g) $835 - 257 = n$

h) $(104 \times 9) + n = 950$

i) $97 \times 8697 = n$

j) $2275 = (n \times 35) + 0$

4. Solve the following:

a) $n \div 8 = 5632$

b) $52 \times (6000 + 40) = n$

c) $6408 = (8 \times n) + 0$

d) $70 \times 490 = n$

e) $7 \times n = 673$

f) $32 + n + 41 = 162$

g) $n + 184 = 986$

h) $503 = (6 \times n) + 5$

i) $764 = (34 \times 22) + n$

j) $3 \times 3 \times 465 = n$

5. Solve:

a) $997 = (33 \times n) + 7$

b) $9076 \times 6 \times 6 = n$

c) $5472 = (8 \times n) + 0$

d) $164 = (41 \times 4) + n$

e) $5838 = (6 \times n) + 0$

f) $n = (7 \times 906) + 3$

g) $6 \times 465 \times 3 = n$

h) $48 \times 7080 = n$

i) $360 = (72 \times n) + 0$

j) $5 \times 4 \times 68 = n$

6.

Add:

1) $\begin{array}{r} 578 \\ 4,549 \\ 496 \\ \underline{27,083} \end{array}$

2) $\begin{array}{r} 6,324 \\ 796 \\ 39,137 \\ \underline{4,034} \end{array}$

3) $\begin{array}{r} 304 \\ 76,451 \\ 3,517 \\ \underline{25,064} \end{array}$

4) $80\overline{)2320}$

5) $50\overline{)650}$

Subtract:

6) $\begin{array}{r} 58,931 \\ \underline{6,336} \end{array}$

7) $\begin{array}{r} 6,719 \\ \underline{2,480} \end{array}$

8) $\begin{array}{r} 5,833 \\ \underline{3,097} \end{array}$

9) $60\overline{)7260}$

Multiply:

10) $\begin{array}{r} 354 \\ \underline{26} \end{array}$

11) $\begin{array}{r} 836 \\ \underline{54} \end{array}$

12) $\begin{array}{r} 8235 \\ \underline{35} \end{array}$

13) $\begin{array}{r} 709 \\ \underline{61} \end{array}$

14) $\begin{array}{r} 126 \\ \underline{16} \end{array}$

15) $\begin{array}{r} 789 \\ \underline{56} \end{array}$

Subtract:

16) $\begin{array}{r} 5837 \\ \underline{2528} \end{array}$

17) $\begin{array}{r} 25,813 \\ \underline{5,804} \end{array}$

18) $8\overline{)2472}$

19) $20\overline{)8160}$

20) $60\overline{)4200}$

Review

SET I

Part A

1. Write each of these as a decimal. Example: a is done for you.

a) $\frac{7}{10} = .7$	d) $25\frac{13}{100} =$	g) $\frac{102}{100} =$
b) $\frac{34}{100} =$	e) $4\frac{1}{10} =$	h) $5\frac{16}{1000} =$
c) $16\frac{9}{10} =$	f) $\frac{45}{10} =$	i) $2\frac{10}{100} =$

2. Write the decimal numeral for each of these:

   a) $(9 \times 100) + (8 \times 10) + (6 \times 1)$

   b) $(3 \times 1{,}000) + (4 \times 100) + (2 \times 10) + (5 \times 1)$

   c) $(4 \times 1{,}000) + (2 \times 100) + (2 \times 10) + (3 \times 1)$

   d) $(9 \times 10{,}000) + (3 \times 1{,}000) + (1 \times 100) + (7 \times 10) + (4 \times 1)$

   e) $(6 \times 100{,}000) + (3 \times 10{,}000) + (4 \times 1{,}000) + (7 \times 10) + (4 \times 1)$

   f) $(5 \times 100{,}000) + (8 \times 10{,}000) + (9 \times 1{,}000) + (6 \times 10)$

   g) $(1 \times 10{,}000) + (5 \times 1{,}000) + (8 \times 10) + (7 \times 1)$

   h) $(8 \times 10{,}000) + (9 \times 10) + (4 \times 1)$

3. Which of these numbers are divisible by 10?

a) 353	d) 4,000	g) 960	j) 5,800
b) 637	e) 30	h) 16	k) 190
c) 21	f) 42	i) 462	l) 382

Which of these numbers are divisible by 5?

a) 38	d) 3055	g) 1114	j) 215
b) 700	e) 105	h) 680	k) 23
c) 90	f) 77	i) 53	l) 190

Which of these numbers are divisible by 2 ?

a)	94	d)	894	g)	201	j)	27
b)	1112	e)	7,000	h)	50	k)	1,128
c)	423	f)	633	i)	192	l)	729

4. Complete the following to make them true sentences.

   a) $68 \times 11 = 680 +$ ______

   b) $28 \times 64 = 512 +$ ______

   c) $74 \times 14 = (74 \times 7) +$ ______

   d) $571 \times 318 = (500 \times 318) + (70 \times 318) +$ ______

   e) $74 \times 386 = 21{,}000 + 5{,}600 + 420 +$ ______ $+ 320 +$ ______

5. Use <u>2</u> as many times as you can as a repeated factor of each of these numbers. Example a is done for you.

   a) $28 = 2 \times 2 \times 7$

   b) 16 =

   c) 24 =

   d) 14 =

   e) 20 =

   f) 42 =

   g) 22 =

   h) 6 =

   i) 12 =

   j) 32 =

   What do you notice about all of the factors above?

6. In each of the following explain what the 4 represents. A sample problem is done for you.

   a) In 242_{five} <u>4</u> <u>represents</u> <u>4</u> <u>sets of five</u>

   b) In 40_{eight}

   c) In 104_{five}

   d) In 47

   e) In 1024_{seven}

   f) In 542_{six}

   g) In 432_{eight}

7. Write each of the following as decimal numerals.

a) Twenty-six thousand, eight hundred twelve

b) Forty thousand, three hundred sixty

c) Eight hundred fifty-seven thousand, ninety-one

d) Four million, seven hundred sixty-three thousand

e) One million, one thousand, one

Part B

Write a mathematical sentence (or two sentences if necessary) and solve. Write an answer sentence.

1. The Jackson School bought 7 new wall maps. Each map cost \$9.95. What was the total cost of the maps?

2. Jim had \$3.25. Tom had 75 cents more than Jim. How much money did the two boys have together?

3. Joanne went to a party dressed as a witch. She paid 85 cents for black cloth for a dress, 72 cents for a broom, and 29 cents for a mask. How much did she pay for the entire costume?
She gave the clerk five dollars. How much change did she get?

4. The pupils in Peggy's class are making bookcovers. There were 26 books to cover. They had a dozen and a half sheets of colored paper. How many more sheets of paper will they need in order to have a sheet for each book?

5. The Hoover School was built in 1934. The Lincoln School was built in 1960. The Hoover School is how many years older than the Lincoln School?

6. There are 32 children in Mr. Lang's class. For a party each child received 4 cookies. How many cookies did the class have?

Suggested Activities

Group Activity

Relays - Working with Multiples

The object of the game is to locate points named by multiples of the number on the number line. The first member of each team draws the line and locates the first point, for example using multiples of 7 he would locate and name 7. The next player in each team would go up to locate 14, the third player names 21, and so on. The team that can correctly name the most points in a determined time period wins. This may also be used for counting in other bases.

Individual Projects

Prepare and show your class a magic trick with numbers. Tricks with numbers fall into three main groups--lightning calculations, predicitions, or mind reading effects. You will find information about number tricks in many books about mathematics. One clue--try looking up some of the "mysteries of nine."

Review

SET II

Part A

1. Using the symbols >, <, or = make the following true sentences.

   a) .40 ______ .4
   b) .6 ______ .06
   c) $\frac{34}{100}$ ______ .34
   d) $\frac{5}{100}$ ______ .5
   e) $\frac{54}{100}$ ______ .45
   f) .64 ______ .7
   g) $\frac{8}{10}$ ______ .65
   h) $\frac{5}{100}$ ______ .05
   i) .4 ______ .36
   j) .3 ______ .40

2. Write these numerals in expanded notation.

   a) 114 =
   b) 2,236 =
   c) 7,330 =
   d) 5,050 =
   e) 6,803 =
   f) 49,527 =
   g) 827,666 =
   h) 412,305 =

3. On the number line below, the points for 0 and 1 are labeled. Label the other points with base five numerals.

Fill in the blanks with the numerals 20_{five} and 24_{five} to make each of the following true sentences.

____________ is less than ____________; ____________ is greater than ____________. ____________ is to the left of ____________; ____________ is to the right of ____________

4. A = {1, 3, 5, 7, 9, 11, 13}.

Sets T, S, E and P are subsets of A.

a) The members of Set T are divisible by 3.

b) The members of Set S are divisible by 1.

c) The members of Set E are divisible by 2.

d) The members of Set P are prime numbers.

e) Rewrite Set A and rename its members as product expressions. Call it Set M.

B = {0, 2, 4, 6, 8, 10, 12, 14, 16}.

Sets F, R, Q and H are subsets of B.

a) The members of Set F are divisible by 2.

b) The members of Set R are divisible by 3.

c) The members of Set Q are divisible by 1.

d) The members of Set H are prime numbers.

e) Write the members of the Set $A \cup B$.

f) Write the members of the Set $A \cap B$.

5. Rename each of these decimals. The first one is done for you

a) 6.84 = _6_ ones + _8_ tenths + _4_ hundredths.

b) 12.62 = ____ ones + ____ tenths + ____ hundredths.

c) .07 = ____ ones + ____ tenths + ____ hundredths.

d) 1.01 = ____ ones + ____ tenths + ____ hundredths.

6. This is one way of changing a base five numeral to a base ten numeral.

$114_{five} = (1 \text{ twenty five}) + (1 \text{ five}) + (4 \text{ ones})$

$114_{five} = (1 \times 25) + (1 \times 5) + (4 \times 1)$

$114_{five} = 25 + 5 + 4$

$114_{five} = 34$

Using the same procedure change the following base five numerals to base ten numerals.

a) 23_{five}

b) 44_{five}

c) 12_{five}

d) 123_{five}

Part B

Write a mathematical sentence (or two sentences if necessary) and solve. Write an answer sentence.

1. Roy bought four fish for his aquarium. He paid 60 cents for one, 28 cents for another, 35 cents for another, and 45 cents for the fourth one. How much money did he spend for all the fish?

2. The Smith family went on a vacation. The first day they drove an average of 41 miles an hour. They traveled 9 hours. How many miles did they drive the first day?

3. Janis and her sister made 75 pieces of fudge for a party. After the party only 19 pieces of fudge were left. How many pieces of fudge were eaten at the party?

4. Mrs. Gray has the milkman deliver 3 quarts of milk each day. The milk costs 26 cents a quart. What is the total milk bill for a week?

5. Shirley had been saving quarters. She now has 10 quarters. If she changes them to nickles, how many will she get?

6. Mr. Norman pays 16 dollars a month for garage rent. How much rent does he pay in one year?

Braintwisters

1. A frog is climbing out of a well twenty feet deep. He climbs four feet every day and slips down three feet every night. How long does it take the frog to get to the top?

2. You have 8 sections of silver chain, each of four links. The cost of cutting open a link is 10¢ and of welding it together again is 25¢. What is the least you can pay to have the eight pieces joined together in a single chain?

3. Sally had a piece of ribbon $4_{\underline{?}}$ inches long. She found another piece $4_{\underline{?}}$ inches long. Now she has $13_{\underline{?}}$ inches of ribbon. What number base was Sally using?

4. Two boys were comparing sticks. One boy had a stick $6_{\underline{?}}$ inches long. The other boy's stick was $3_{\underline{?}}$ inches longer or $12_{\underline{?}}$ inches long. What number base were they using?

Review

SET III

Part A

1. Write each of the following expressions using symbols.

   Example: The number n increased by 6 $\quad n + 6$

   a) The number n increased by 8
   b) The number 7 multiplied by n
   c) The sum of n and 9
   d) The number n decreased by 4
   e) The product of 6 and n
   f) The number n divided by 3
   g) The number which is the result of 10 subtracted from n.

2. What number is represented by each of the expressions in Problem 1 if $n = 12$.

3. Answer each of the following with a complete sentence. The first one is worked for you.

   a) How many 4's are there in six 8's?

      <u>There are twelve 4's in six 8's</u>.
   b) How many 7's are there in three 14's?
   c) How many 6's are there in fifteen 4's?
   d) How many 3's are there in four 12's?
   e) How many 8's are there in fourteen 4's?

4. Find what number y represents in each of these. Tell what operation is needed to find y. Example a is done for you.

a) $108 + y = 144$	$y = 36$	subtraction
b) $87 + 116 = y$	______	__________
c) $30 \times 74 = y$	______	__________
d) $y = 54 \times 18$	______	__________
e) $2,563 + y = 8,010$	______	__________
f) $58 \times 867 = y$	______	__________
g) $y - 2649 = 6763$	______	__________
h) $30,600 - y = 408$	______	__________

5. Name the first ten members of each of the following sets:

S = {The set of multiples of 100}

T = {The set of multiples of 1,000}

6. Complete these sentences with a multiple of 100 or 1,000 needed to make them true sentences. Here is one possibility.

Example: $\underline{2,000} \times 5 < 12,110$

a) $\_\_\_\_ \times 6 > 932$

b) $9 \times \_\_\_\_ < 40,121$

c) $\_\_\_\_ \times 4 < 5,210$

d) $70 \times \_\_\_\_ < 15,316$

e) $6 \times \_\_\_\_ > 27,880$

f) $\_\_\_\_ \times 33 = 3,300$

g) $25 \times \_\_\_\_ > 2,312$

h) $\_\_\_\_ \times 140 < 293,000$

i) $30 \times \_\_\_\_ = 6,000$

j) $\_\_\_\_ \times 25 = 5,000$

*Answers will vary

7. Complete each of these: Example a is done for you.

a) .58 = 58 hundredths or 5 tenths plus 8 hundredths

b) .33 = ____ hundredths or ___ tenths plus ___ hundredths

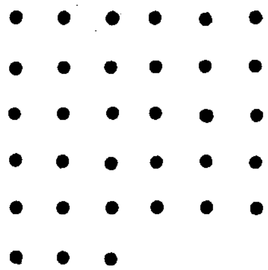
c) .07 = ____ hundredths or ___ tenths plus ___ hundredths

d) .70 = ____ hundredths or ___ tenths plus ___ hundredths

e) .09 = ____ hundredths or ___ tenths plus ___ hundredths

f) .99 = ____ hundredths or ___ tenths plus ___ hundredths

8. How many dots are there in this diagram? Write the answer in each of the following number bases.



a) Base ten ____

b) Base five ____

c) Base six ____

d) Base four ____

e) Base nine ____

f) Base seven ____

g) Base eight ____

Part B

Write a mathematical sentence (or two sentences if necessary) and solve. Write an answer sentence.

1. Mark said, "Tonight I am going to sleep 9 hours and 30 minutes. How many minutes will Mark sleep?

2. An army division has 345 platoons. There are 38 soldiers in each platoon. How many soldiers are there in the division?

3. Mr. Jones bought 12 gallons of gasoline. He paid 33 cents a gallon. How much money did he spend for gasoline?

4. Mary and Martha were selling greeting cards at 50 cents a box. The first day Mary sold 16 boxes and Martha sold 10 boxes. How much money did they make altogether that day?

5. There were two fifth grade classes in the Marshall School. There were 57 fifth grade pupils in the two classes. 23 of these were girls. How many boys were there?

6. Dick rides his bicycle to and from school in 10 minutes. He walks to and from school in 26 minutes. How much time will he save riding his bicycle to school all week?

Suggested Activities

Group Project

Column Relays - Have the class choose teams and form team columns facing the board. A dittoed sheet of problems is handed to the first person in line. He moves to the board, reads, and works the first problem then returns the problem sheet to the second person in line as he moves to the rear of the line. Each person moves up, works his problem, and returns to line until all members have had a turn. One point is scored for each correct answer.

Example: $16 \times \underline{\ n\ } = 212$ or $325 \quad 30 = \underline{\quad\quad} + 25$

Other questions may be given on:

a) writing expanded notations

b) changing to other bases

c) writing decimals as fractions and vice versa.

Chapter 4

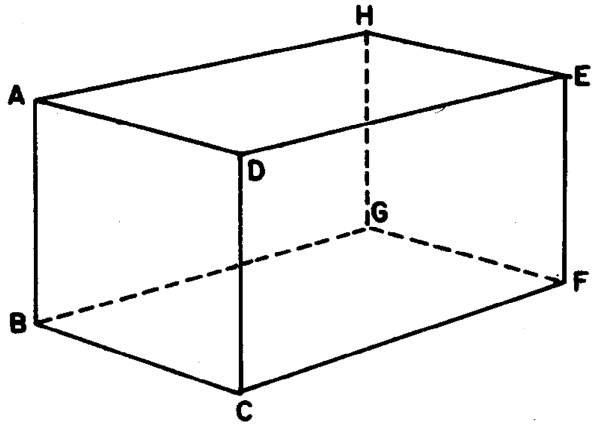
CONGRUENCE OF COMMON GEOMETRIC FIGURES

REVIEW OF GEOMETRIC FIGURES

Rectangular Prism

Exploration

Look at a chalkbox.



1. a) Place your finger on the top face.
   Place your finger on the bottom face.
   How many faces has a chalkbox?

   b) Trace any edge of the box with your finger tip.
   How many edges has the box?

   c) Point to a vertex of the box.
   How many vertices has the box?

2. Suppose we name each corner (vertex) of the box with the letter given in the above sketch.

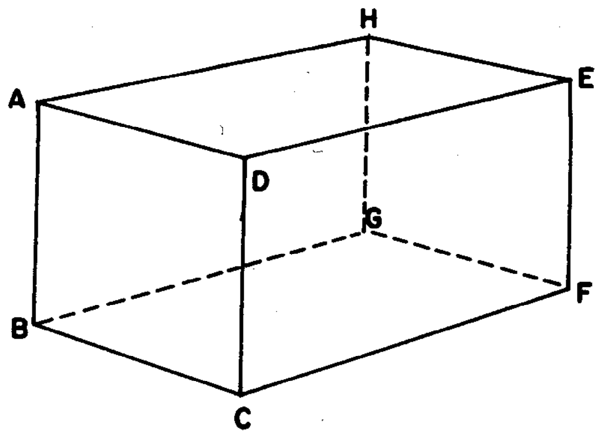
   a) Name 3 edges of this rectangular prism.

   b) Name 4 faces of this rectangular prism.

c) You can see that a vertex represents a point; an edge represents a line segment; and a face represents a part of a plane.

Every line segment has two endpoints. We label the endpoints with capital letters.

Then we may name a line segment by using the letters at its endpoints with a bar over them. Thus: $\overline{AD}$ or $\overline{GF}$.



3. What geometric figures can you find that are formed by the edges of the box?
How many rectangles did you find? How many squares did you find?

4. Name the intersection of the top face and the front face.
What is the intersection of the set of points on the bottom face and the set of points on the front face?

5. What is the intersection of $\overline{CF}$ and $\overline{GF}$?

What is the intersection of $\overline{AB}$ and the top face?

Name three sets whose intersection is the point H.

What is the intersection of $\overline{AD}$ and $\overline{BC}$?

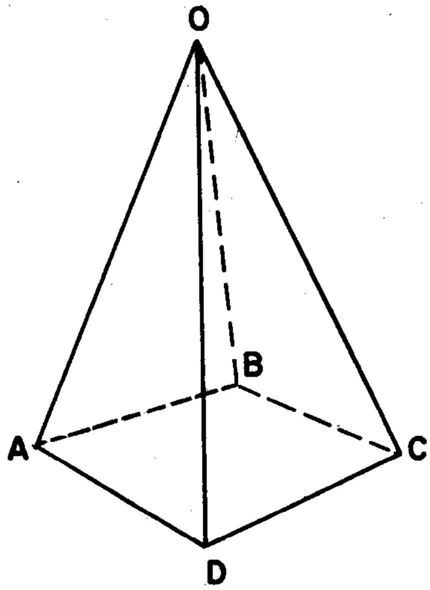
Name some other pairs of sets whose intersection is the empty set.

6. Name the geometric figure which is the union of the sets $\overline{DC}$, $\overline{DE}$, $\overline{EF}$, and $\overline{CF}$.

Name the geometric figure which is the union of the sets $\overline{HG}$, $\overline{GF}$, $\overline{FE}$, $\overline{HE}$.

Pyramid

Exploration



1. a) How many faces has this pyramid?

   b) How many edges does the pyramid have?

   c) How many vertices has the figure?

   d) Which edges outline the bottom face.

   e) Name the figure formed by the edges of the bottom face.

2. a) Which faces intersect on $\overline{OD}$?

   b) Which faces intersect on $\overline{OC}$? On $\overline{OB}$? On $\overline{AB}$?

   c) Do faces OAD, OBC, OAB, ODC, and ABCD represent planes?

   d) Which of these planes intersect at O?

3. a) Name the geometric figure outlined by the edges $\overline{OD}$, $\overline{OC}$, $\overline{DC}$.

   b) Trace these edges with your finger tip. Name them.

   c) Place your finger tip in the interior of $\triangle$OAD.

4. Name the intersection of the edges of the four triangular faces.

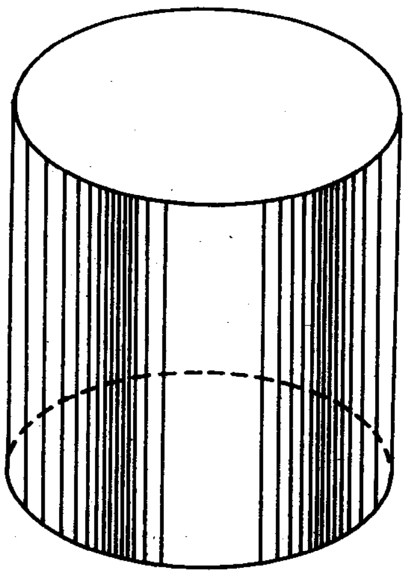
5. a) Could a pyramid have just 3 faces? Remember that the base is called a face, too.

   b) Could a pyramid have just 4 faces?

   c) Could a pyramid have just 999 faces?

Cylinder

Exploration



1. Nearly every time you select a can of food at the store, you are handling an object like a geometric figure called a cylinder.

   a) What are the "top" and "bottom" of a cylinder called?

   b) What is the name of the geometric figure which outlines a base of this kind of cylinder?

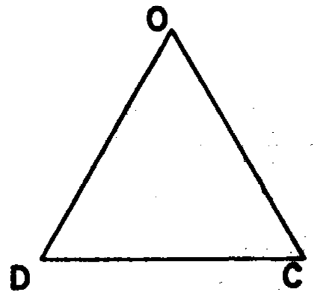
2. How many such figures are outlined on this cylinder? Trace them with your finger tip.

3. Do the bases of a cylinder have to be circular regions?

4. Could the bases of a cylinder be square regions?

5. Could each base of a cylinder have 1001 sides?

Exploration



1. a) Copy figure ODC on a sheet of paper.
   What set of points form ΔODC?

   b) Trace ΔODC with your finger tip.
   Place your finger in the interior of the triangle.

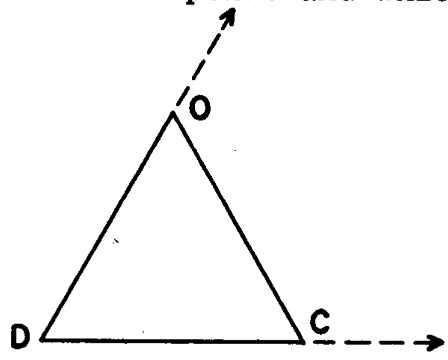
   c) Name the angle whose vertex is at D.

   d) Name the angle whose vertex is at O.

   e) How many names were given for the angle whose vertex is at D?

   f) How many names were given for the angle whose vertex is at O?

2. a) Recall that an angle is the set of points on two rays which have a common endpoint and which are not on the same line.



Trace the rays (that is, part of them) with your finger tip.

b) Name the rays that form $\angle ODC$.

c) Name the common endpoint.

d) Does $\overrightarrow{DC}$ end at C?

e) How many endpoints does $\overrightarrow{DC}$ have?

f) Why was the letter D placed in the middle (between O and C) in the name, $\angle ODC$?

3. a) Make another drawing to show the rays which form $\angle OCD$. Why is the letter C placed between the letters O and D in the name $\angle OCD$?

b) Make another drawing to show the rays which form $\angle DOC$. Why is the letter O placed between the letters D and C in the name, $\angle DOC$?

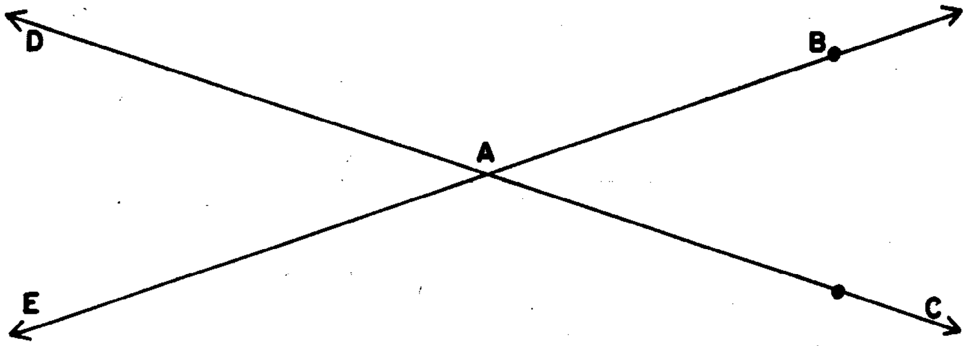
4. In the drawing for Exercise 2 which line segment (except for its end points) is in the interior of $\angle ODC$?

5. Draw an angle on your paper. Color the interior of the angle red. If only the interior of the angle is to be red, should the rays of the angle be made red?

Half Plane

Exploration

1. a) Copy the figure below.



b) Color the line EB red.

c) Color the portion of the plane below $\overleftrightarrow{EB}$ (the part which contains C) blue. Do not get any blue on the line EB.

d) What would be a good name for the part of your figure which is colored blue?

e) What is the name for the part of your figure which is colored red?

f) What would be a good name for the part of your figure which is not colored?

2. a) Color the half plane above $\overleftrightarrow{DC}$ (the part which contains E) yellow. Do not get any yellow on line CD.

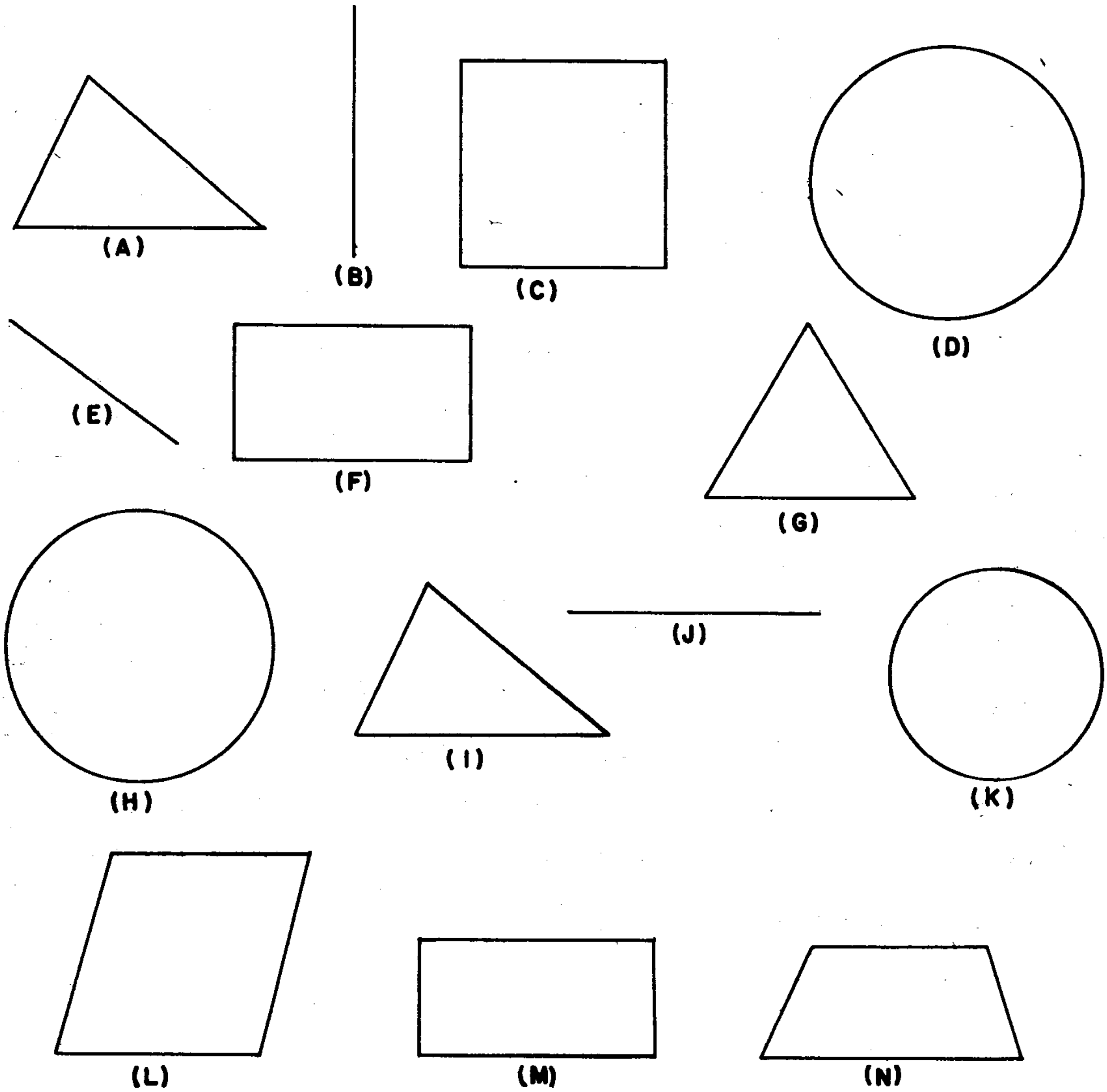
b) What color is the interior of $\angle BAC$?

CONGRUENT FIGURES

Congruence

Exploration

1. Can you find pairs of figures which look as if one of them could fit exactly on the other?



2. Which figure will fit exactly on

Triangle A | Rectangle F

Segment B | Triangle G

Square C | Figure L

Circle D | Figure N

Figure M

3. How can you use tracing paper to see whether your answers are correct?

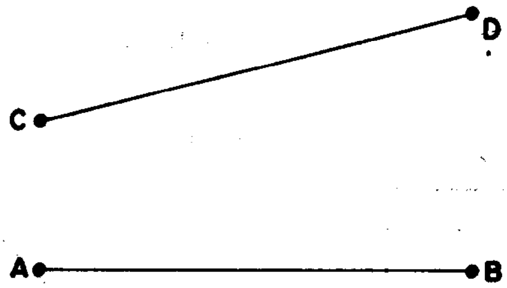
Summary

A geometric figure is a set of points. We know that we cannot make a point on a piece of paper but only a model or a picture of a point. When we draw a line or a triangle we are drawing a model. In this text when we say, "Look at the triangle," we really mean, "Look at this model of the triangle."

Two geometric figures are <u>congruent</u> to each other if they have exactly the same size and shape. This means that if we make a tracing of one figure and place it on top of the other figure and if it fits exactly, then we say that the two figures are congruent.

Congruent Line Segments

Exploration



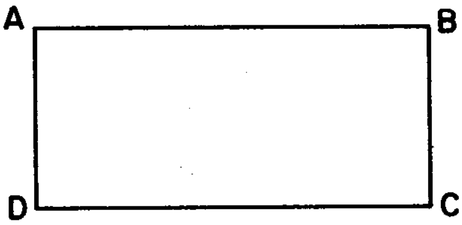
Trace $\overline{AB}$ on a thin sheet of paper. Can you place this tracing of $\overline{AB}$ so that it fits exactly on $\overline{CD}$? Did you place the tracing of the point A on the point C or the point D? Does it matter?

Recall that $A = B$ means A and B are names for the same thing. We cannot write $\overline{AB} = \overline{CD}$ because the points of $\overline{AB}$ are not points of $\overline{CD}$. For example, there is no point on $\overline{CD}$ that is the same point as the point A on $\overline{AB}$. But we would like to write briefly that a tracing of one segment fits exactly on the other. We will write $\overline{AB} \cong \overline{CD}$ to say that the two segments are congruent.

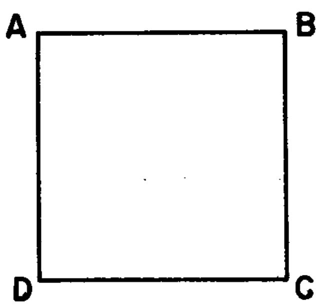
Exercise Set 1

Can you find two congruent segments in each figure? Can you find more than two? Trace segments on a thin sheet of paper to help you decide. Write your answers like this: $\overline{MN} \cong \overline{PQ}$

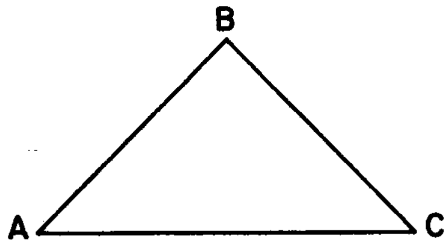
1.



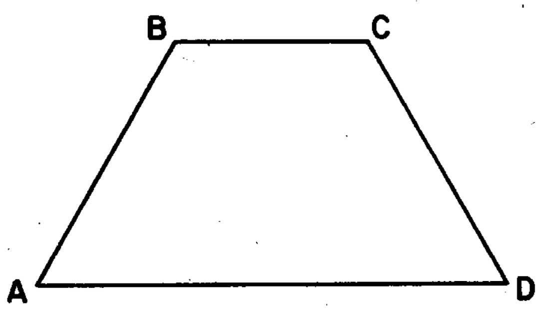
2.



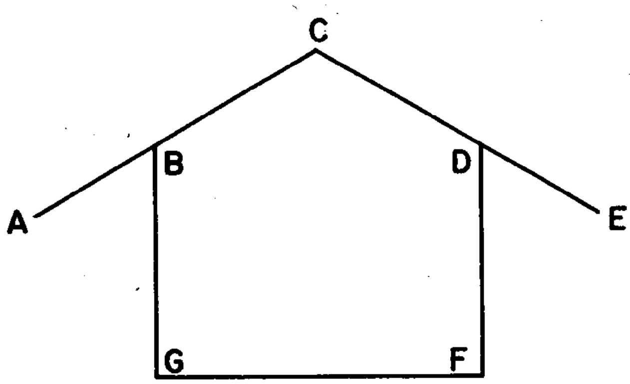
3.



4.



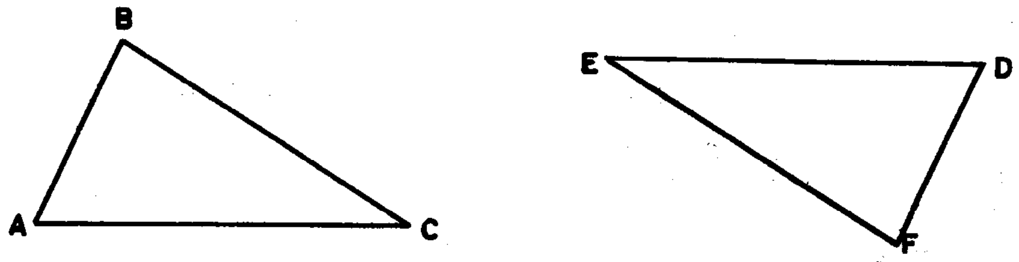
5.



Congruent Triangles

Exploration

You have learned that we call two figures congruent if a tracing of one figure can be placed to fit exactly on the other. (The tracing may be "turned over.") Let us see whether the following two triangles are congruent?



Trace $\triangle ABC$ on a sheet of thin paper and see whether it will fit exactly on $\triangle DFE$.

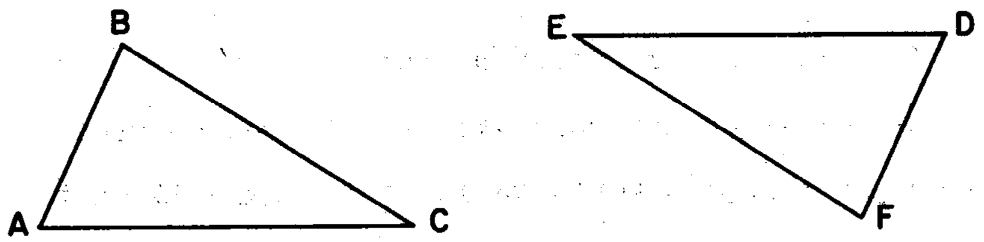
Notice that the triangles will fit exactly if

1. Vertex A is placed on vertex ____ of $\triangle DFE$.
2. Vertex B is placed on vertex ____ of $\triangle DFE$.
3. Vertex C is placed on vertex ____ of $\triangle DFE$.

We notice then that when the vertices are matched the sides also match. Complete the following:

4. $\overline{AB}$ is congruent to side ____ of $\triangle DFE$.
5. $\overline{AC}$ is congruent to side ____ of $\triangle DFE$.
6. $\overline{BC}$ is congruent to side ____ of $\triangle DFE$.

We call the vertices A and D, B and F, C and E <u>corresponding vertices</u> since when A is placed on D, B on F, and C on E, one triangle fits exactly on the other. We call sides AB and DF corresponding sides since they join corresponding (matching) vertices.



7. Name the other pairs of corresponding sides.

We can use the same symbol "$\cong$" that we used for congruent line segments to show that one triangle is congruent to another.

If the triangles fit when

point A is placed on point D,

point B is placed on point F,

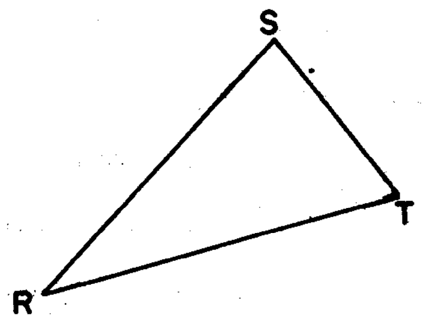
point C is placed on point E,

we shall show this by writing

$\triangle ABC \cong \triangle DFE$.

8. Is $\triangle ABC \cong \triangle DEF$? (This means: Can you place the triangles so that A is on D, B is on E, and C is on F?)

9. Use your tracing of $\triangle ABC$ to see whether the following triangle is congruent to $\triangle ABC$. Are the triangles congruent?



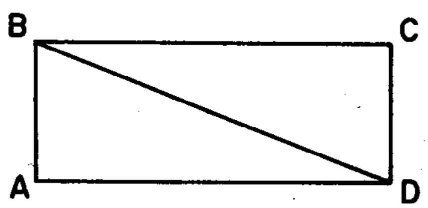
10. List the corresponding vertices.

A and ____, B and ____, C and ____.

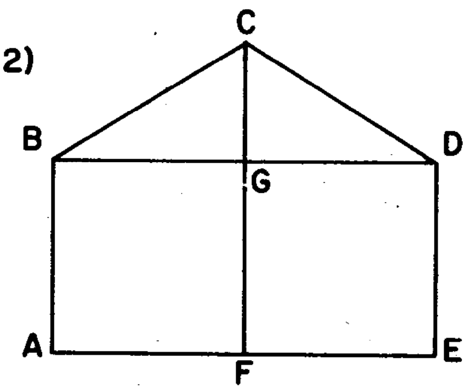
Exercise Set 2

By tracing one triangle on a sheet of thin paper find the triangles which are congruent to each other. Be sure to name corresponding vertices in order. In Exercise 1, state your answer like this: $\Delta BAD \cong \Delta DCB$. In Exercises 3, 5, and 6 you may have to trace more than one triangle.

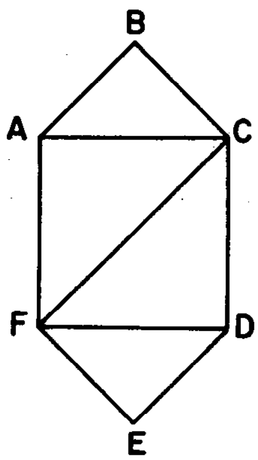
1)



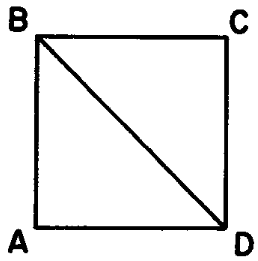
2)



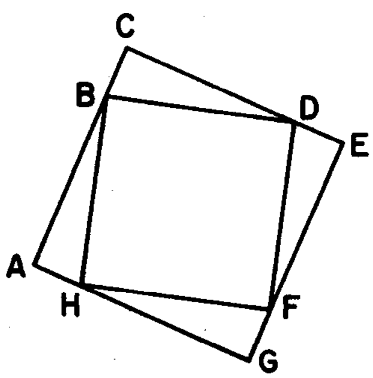
3)



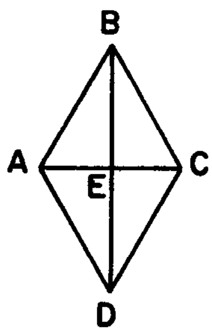
4)



5)



6)



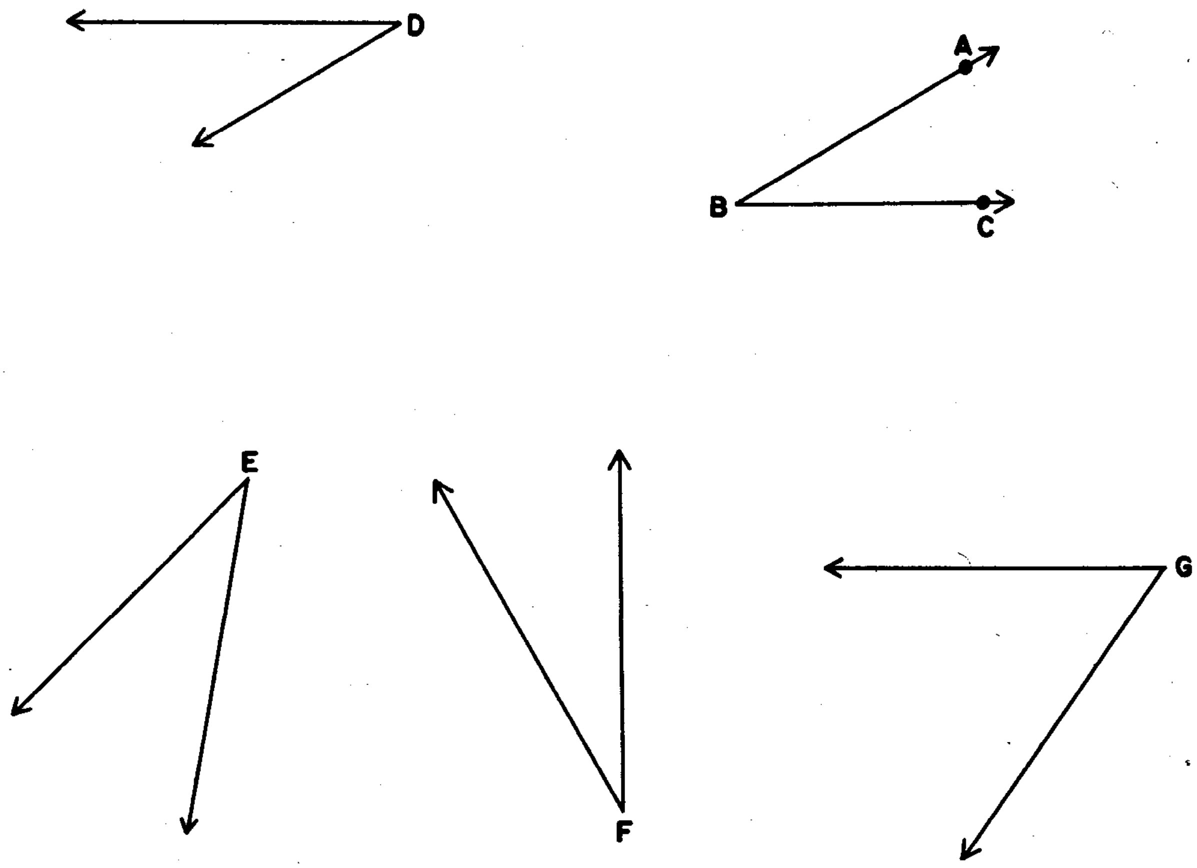
Congruent Angles

Exploration

We say two angles are congruent to each other if we can place the vertex of a tracing of one angle on the vertex of the other angle and the rays of the tracing can be placed to lie exactly along the rays of the second angle.

<u>Exercise Set 3</u>

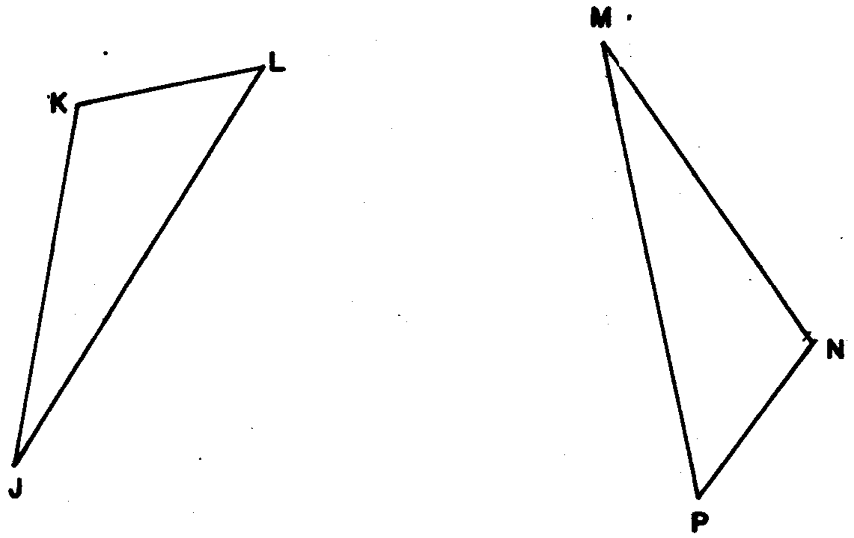
By tracing $\angle ABC$ on a sheet of thin paper, determine which of the following angles are congruent to $\angle ABC$.



Corresponding Angles

Exploration

Triangles JKL and MNP are congruent.



Trace $\triangle$MNP and place this tracing so it fits exactly on $\triangle$JKL.

Where does $\angle$N fall?

$\angle$N and $\angle$K are corresponding angles.

Where does $\angle$L fall?

$\angle$L and $\angle$P are corresponding angles.

Where does $\angle$J fall?

$\angle$J and $\angle$M are corresponding angles.

Corresponding angles of congruent triangles are those which fit together when a tracing of one triangle is placed so it fits exactly on the other.

Summary

In this section we learned some facts about congruent line segments, congruent angles, and congruent triangles. We learned that:

1. Line segments are congruent if a tracing of one can be placed to fit exactly along the other.

2. Triangles are congruent if a tracing of one can be placed to fit exactly along the other. The tracing may be "turned over."

3. In naming congruent triangles, vertices must be named in the proper order.

4. Two angles are congruent if we can place the vertex of a tracing of one angle on the vertex of the other angle, and the rays of the tracing can be made to lie exactly along the rays of the second angle.

5. When two triangles are congruent the corresponding angles are congruent and the corresponding sides are congruent.

Do you agree that this summary tells what we found? Can you think of anything that should be added?

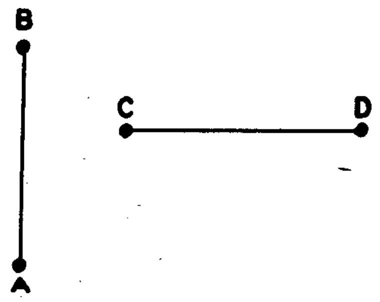
COPYING A LINE SEGMENT

Comparing Lengths of Line Segments

Exploration

1. Do you remember how to use your compass to compare the lengths of two line segments? Look at $\overline{AB}$ and $\overline{CD}$.

   Which appears to be longer, $\overline{AB}$ or $\overline{CD}$?



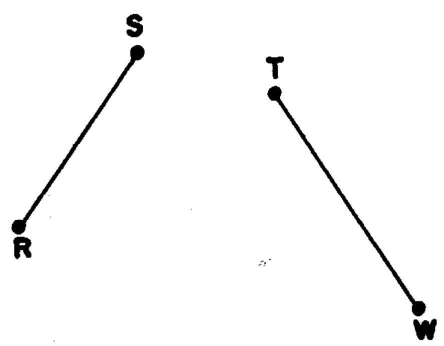
2. Use your compass to compare the length of $\overline{AB}$ with that of $\overline{CD}$. What do you observe now?

3. Does your observation agree with the guess you made by just looking at the line segment?

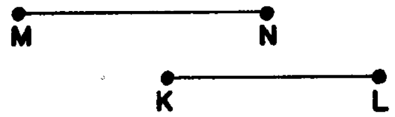
<u>Exercise Set 4</u>

Use your compass to find answers to the following questions.

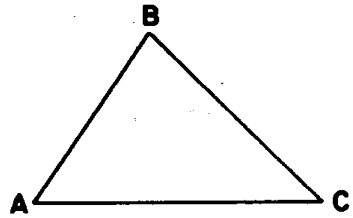
1. How does the length of $\overline{TW}$ compare with that of $\overline{RS}$? Which is longer? How do you know?



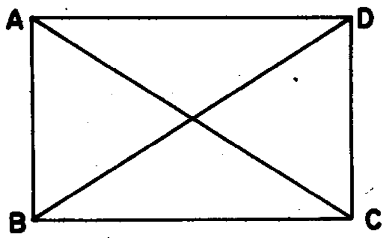
2. Is the length of $\overline{MN}$ greater than, equal to, or less than the length of $\overline{KL}$?



3. Which side of $\triangle ABC$ is the longest?



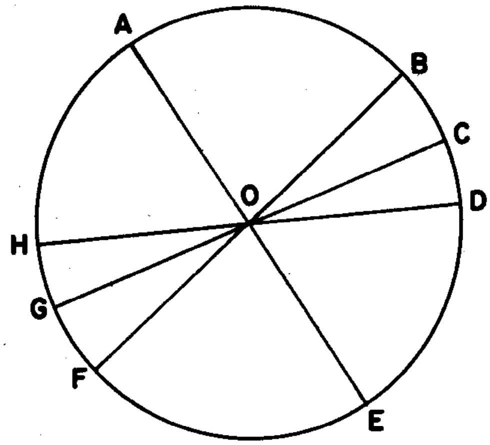
4. Compare the length of $\overline{AC}$ with that of $\overline{BD}$.



5. a) Compare the lengths of $\overline{AE}$, $\overline{FB}$, $\overline{GC}$, $\overline{HD}$.

   Compare the lengths of $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OF}$, $\overline{OH}$.

   b) Since O names the center of the circle, do your results agree with what you already knew about circles?

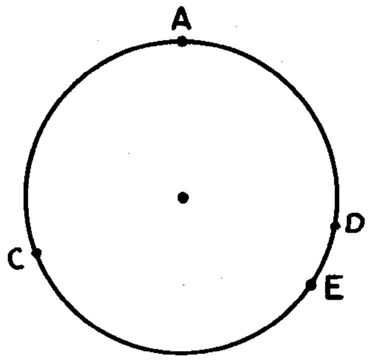


Copying a Line Segment Using the Compass

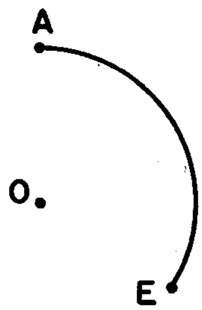
Exploration

Recall that every point on a circle is the same distance from the center of the circle. We call a connected part of a circle an <u>arc</u> of a circle, and we call the center of the circle the <u>center of the arc</u>.

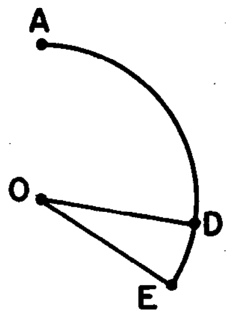
In this picture the part of the circle from A to E which does not include C represents arc AE. The points A and E are the endpoints of the arc. The arc may be named arc AE or arc EA. (If there is a possibility of confusion we name this arc, arc ADE.)



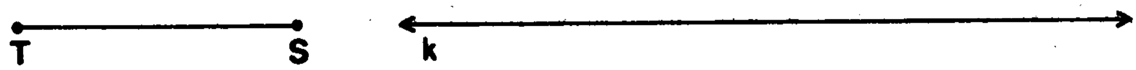
You do not have to draw a complete circle to make an arc of a circle. You could draw arc AE with your compass like this:



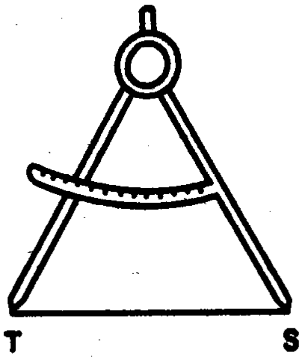
Every point on an arc of a circle is the same distance from its center. The lengths of $\overline{OA}$, $\overline{OD}$, and $\overline{OE}$ are the same, since O names the center.

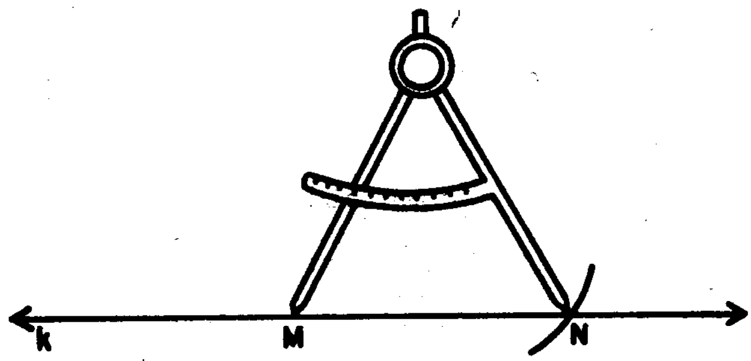


You may use an arc to help make a copy of a line segment. Suppose you are given a line segment TS which you wish to copy on line k. (Sometimes we name a line with a small letter.)



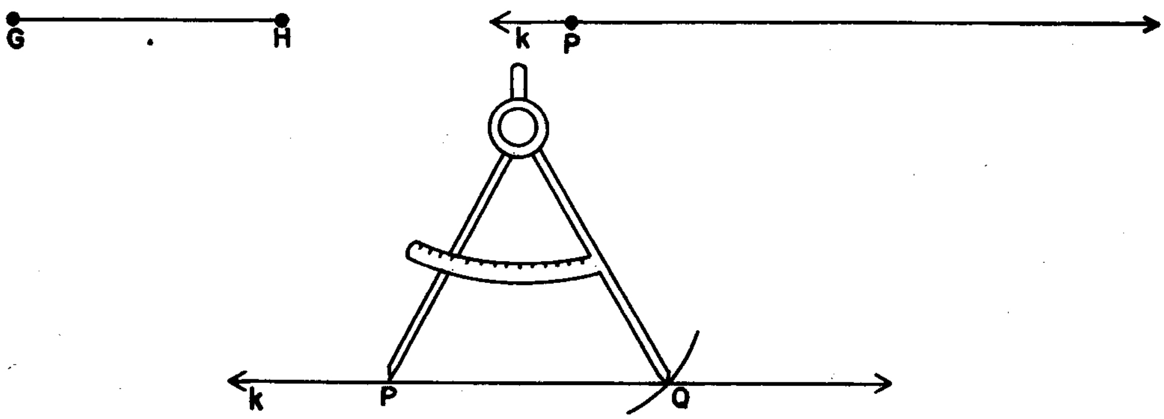
How is the compass placed on $\overline{TS}$? Since you haven't been told where on line k to copy $\overline{TS}$ you may place it anywhere on the line.





The sharp metal point of the compass was placed at M. The pencil point of the compass made an arc intersecting the line K at a point we name N. Is $\overline{MN} \cong \overline{TS}$? Why?

Sometimes you are asked to copy a line segment at a special place. If you are given $\overline{GH}$, and told to copy it on line k so that one endpoint of the new segment is at point P, then the picture would look like this:

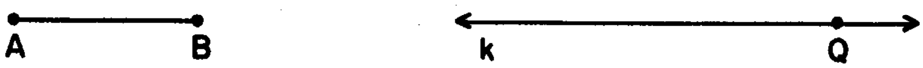


If $\overline{PQ}$ is a copy of $\overline{GH}$, then $\overline{PQ} \cong \overline{GH}$.

Exercise Set 5

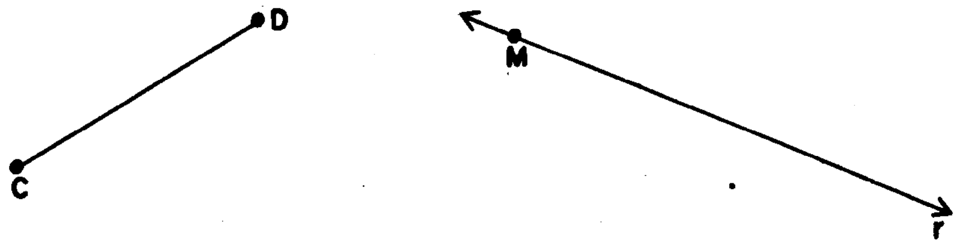
Trace $\overline{AB}$ and $\overleftrightarrow{k}$ on a sheet of paper.

1. Copy $\overline{AB}$ on line k so that one endpoint of the line segment is at Q.

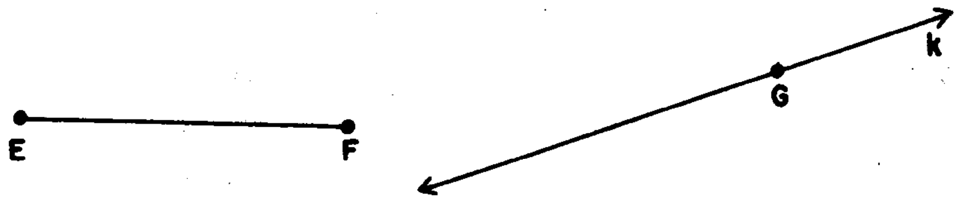


2. Copy each segment so that one endpoint is at the point named on the line.

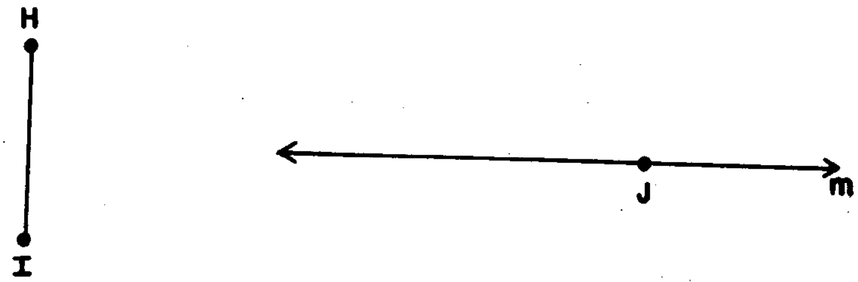
a.



b.

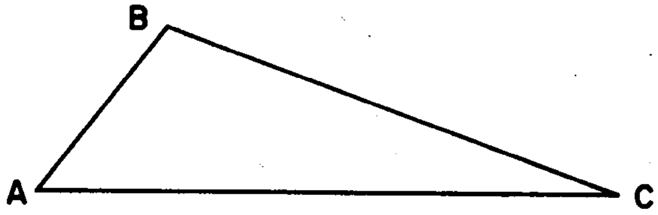


c.



How many segments can you make on line m with one endpoint at J and with the length the same as the length of $\overline{HI}$?

3. a) Copy this figure on a piece of paper.



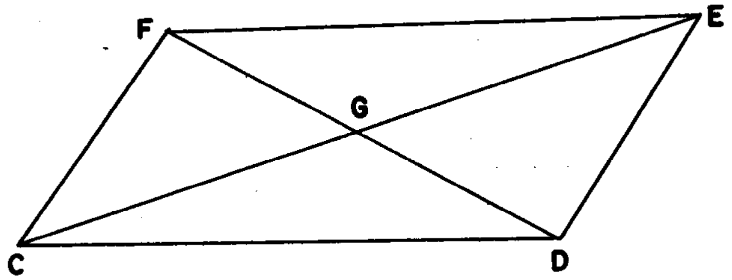
b) Copy $\overline{AB}$ on $\overline{AC}$ of your drawing so that one endpoint of the new segment is at A. Name the other endpoint D.

c) Copy $\overline{AB}$ on $\overline{AC}$ of your drawing so that one endpoint of the new segment is at C. Name the other endpoint E.

d) Copy $\overline{BC}$ on $\overline{AC}$ of your drawing so that one endpoint of the new segment is at A. Name the other endpoint F.

e) Copy $\overline{BC}$ on $\overline{AC}$ of your drawing so that one endpoint of the new segment is at C. Name the other endpoint G.

4. a) Copy this figure on a piece of paper.



b) Copy $\overline{CF}$ on $\overline{CD}$ of your figure using C as an endpoint. Label the other endpoint H.

c) Copy $\overline{FD}$ on $\overline{EC}$ of your figure using C as an endpoint. Label the other endpoint I.

d) Copy $\overline{FG}$ on $\overline{CG}$ of your figure using G as an endpoint. Label the endpoint J.

e) Can you copy $\overline{CE}$ on $\overline{FD}$ of your figure using F as an endpoint?

Why?

Can you do it using D as an endpoint?

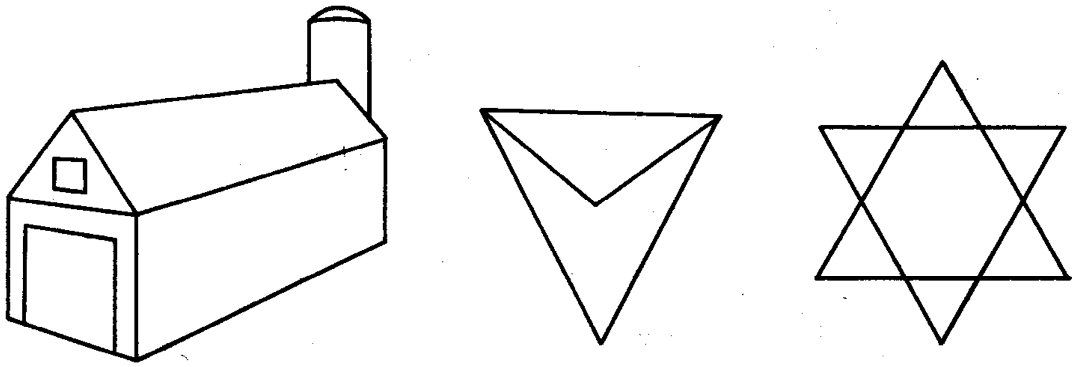
Can you do it using any point on $\overline{FD}$ as the endpoint?

TRIANGLES

Seeing Triangles

Exploration

Here are sketches of a barn, a folded paper napkin, and a six pointed star.

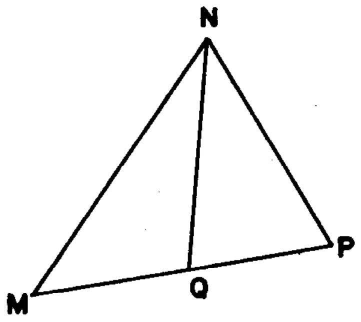


Trace the triangles in each picture with the tip of your finger. How many triangles did you find in the picture of the six pointed star? Did you find as many as eight?

Exercise Set 6

Trace with your finger the triangles in the following figures. Tell how many you found in each case.

1.



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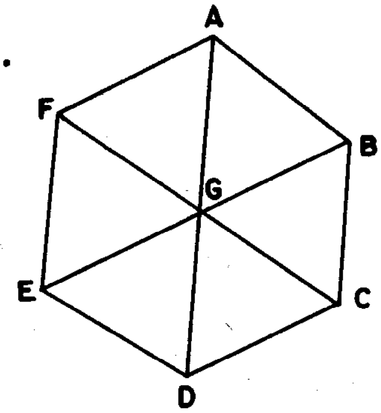

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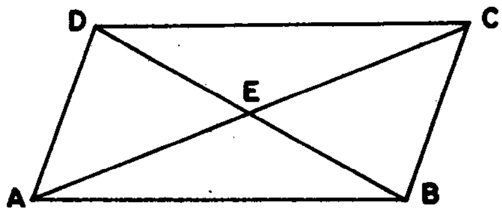

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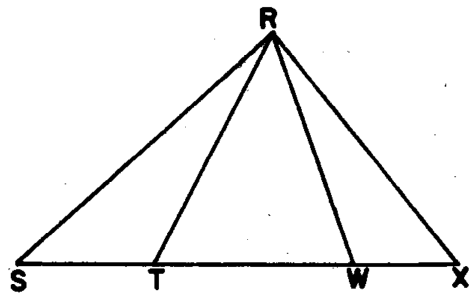

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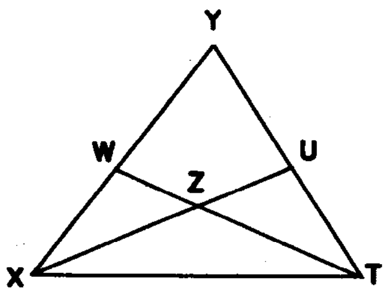
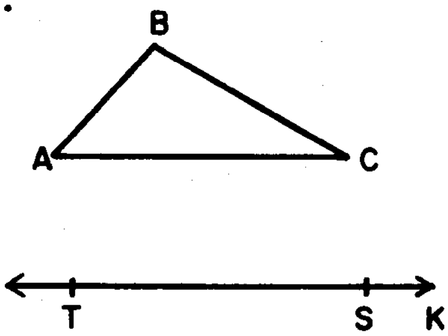


Copying a Triangle

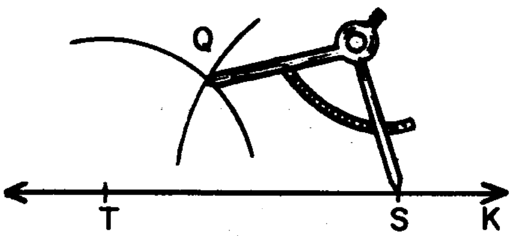
Exploration

1. Trace ΔABC on another sheet of paper. Trace $\overleftrightarrow{K}$ on this same sheet of paper. We may start by copying $\overline{AC}$ on line K. Call the ends of the segment T and S. Your copy should look like this.

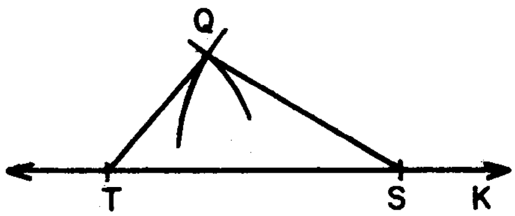


2. Then place the points of your compass at A and B. Move your compass so that the sharp point is on point T. Swing the pencil point to make an arc.

3. Copy $\overline{BC}$. This time put the sharp point of your compass at S and swing the pencil point to make an arc. Label the intersection of the two arcs Q.



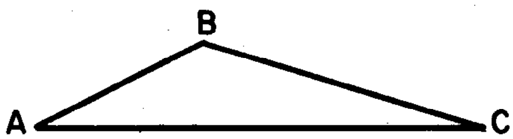
Draw $\overline{TQ}$ and $\overline{QS}$. Your copy of ΔABC will be named ΔTQS. Is $\Delta TQS \cong \Delta ABC$? How can you be sure?

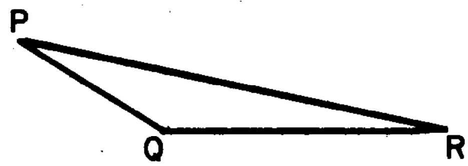


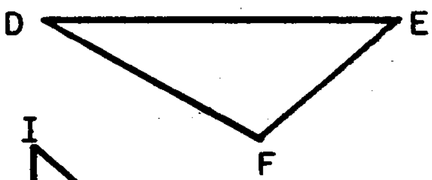
Exercise Set 7

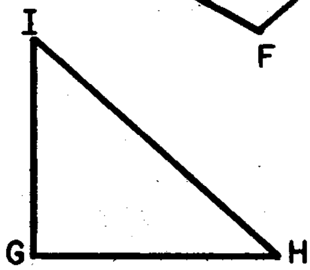
In each of the following exercises draw your own line k and choose some point on it to be an endpoint of the line segment you copy on $\overleftrightarrow{k}$.

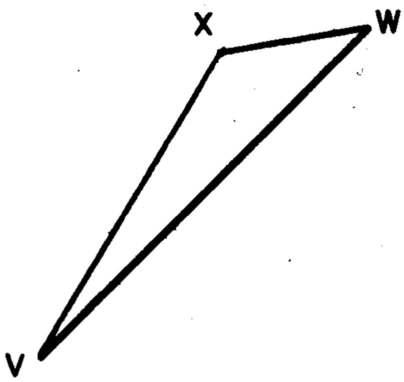
1. Copy each of the following triangles using a compass and straightedge.

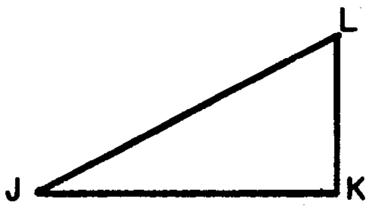


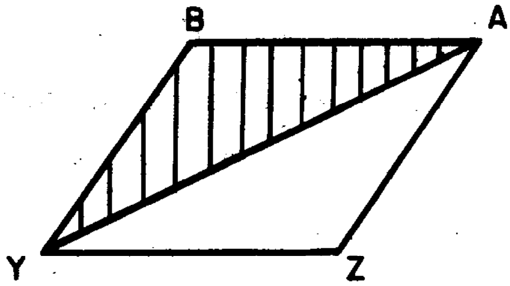












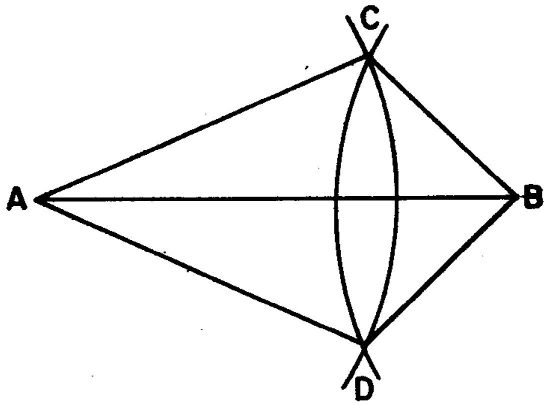
Copy the triangle whose interior is shaded.

BRAINTWISTER

2. a) How does the length of $\overline{AC}$ compare with that of $\overline{AD}$ in the figure below?

   b) How does the length of $\overline{CB}$ compare with that of $\overline{DB}$?

   c) What can you predict about ΔABC and ΔABD?



Constructing a Triangle, Given Three Segments

Exploration

You have been copying triangles. However, you might be given these line segments and be asked to construct a

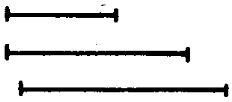
triangle whose sides have the lengths of these segments. Of course, you would need to choose your own line k and point P on it. Does it matter which of the three given segments you copy on line k? If you copy $\overline{RS}$ on line k, which two segments will you use for finding the intersection of the arcs? Could you copy $\overline{TM}$ on line k? Could you copy $\overline{NQ}$ on line k?

If each child in the class constructs a triangle using $\overline{RS}$ $\overline{TM}$ $\overline{NQ}$ as lengths of sides, what can you predict about all the resulting triangles?

Exercise Set 8

If possible, in each exercise construct a triangle using the lengths of the given line segments for the lengths of the sides of the triangle. If it is not possible, tell why.

1.



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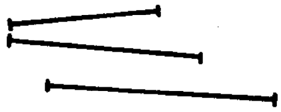

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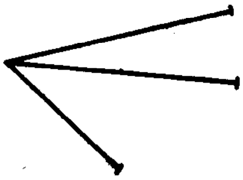

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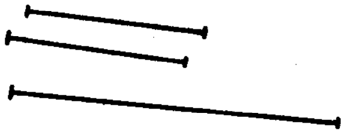

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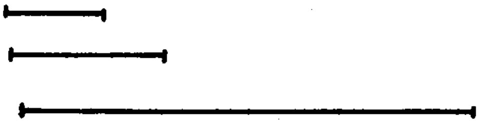

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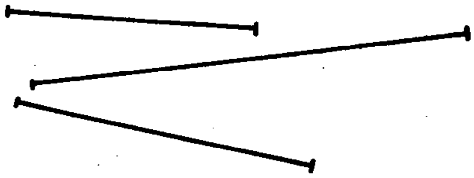

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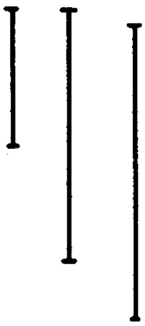

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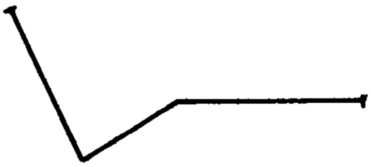

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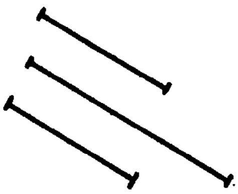

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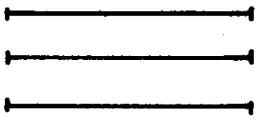


How Many Sides Determine Exactly One Triangle

Exploration

Be sure to read all the instructions for each problem before you start. This will help you in arranging your drawings on your paper.

1. a) Draw five congruent line segments, each about four inches long. Call them $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, $\overline{GH}$, and $\overline{KL}$.

   b) Draw a triangle using $\overline{AB}$ for one side.

   c) Draw a differently shaped triangle on each of the other segments.

   d) If you had fifty congruent segments, could you draw a triangle on each of them, each one different in shape and size from the other 49 triangles?

2. a) Draw five new congruent segments.

   b) Draw a special sixth segment different in length.

   c) On each of the first five segments draw a triangle. This time, make the second side of each triangle congruent to your sixth segment.

   Try to make each triangle different in size and shape from all others. Can you do this?

3. a) Draw three new congruent segments.

   b) Draw a fourth segment not congruent to any one of the first three.

   c) Draw a fifth segment not congruent to any one of these four segments. Choose the length of this fifth segment carefully. We want to construct a triangle on each of your first three segments with sides congruent to the fourth and fifth segments.

   d) Draw three triangles on the first three segments. In each triangle, make the second side congruent to the fourth segment, and the third side congruent to the fifth segment.

   e) Can you make each triangle different in size and shape from any of the others?

   f) What is true about all your triangles?

   Because all of the triangles are congruent, we say that three sides <u>determine</u> exactly one triangle.

4. Did two sides <u>determine</u> exactly one triangle?

5. Did one side <u>determine</u> exactly one triangle?

COPYING AN ANGLE USING STRAIGHTEDGE AND COMPASS

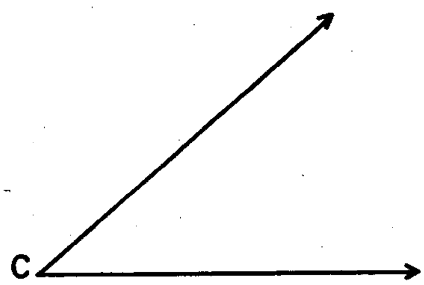
Exploration

You have learned how to copy line segments and triangles using the straightedge and compass. Now you will learn how to copy an angle using the straightedge and compass.

1. Do you remember how to copy a triangle using the straightedge and compass? Draw a triangle and copy it.

2. When you copied the triangle, did you also copy its angles?

3. Suppose you wish to copy $\angle C$. (When we name an angle by a single letter we mean the angle whose vertex is the point named by that letter.) How could you make part of $\angle C$ two sides of a triangle? Draw a dashed line to complete a triangle. The dashed line will help to keep in mind the angle you are copying.



4. Make a copy of the triangle you made in Exercise 3.

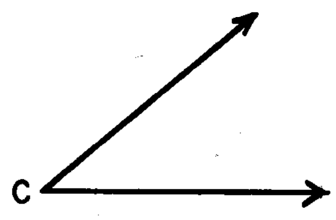
5. Which angle of the triangle that you made in Exercise 4 do you think is congruent to $\angle C$? Trace this angle and place it on $\angle C$ to see whether it is a copy.

6. In Exercise 3 you made $\angle C$ an angle of a triangle. Would some <u>special</u> triangle have made the construction easier? Can you think of a special triangle which would have required fewer changes in the distance between the points of your compass?

7. List the things you do in copying an angle, and then see how your list compares with the list in the following summary.

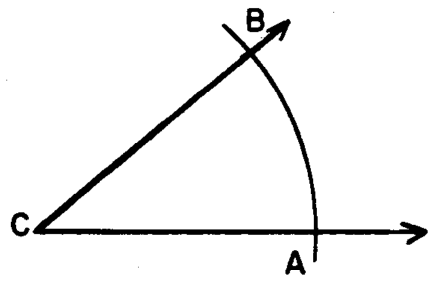
Summary

To copy an angle such as $\angle C$ make it an angle of a triangle. Next, copy the triangle by making the three sides the same lengths as the three sides of the first triangle.

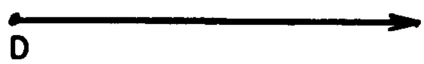


The following procedure can be used:

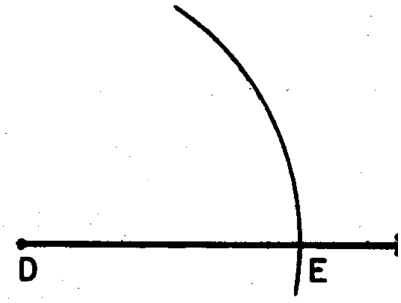
1. The vertex of the angle we wish to copy is point C. With C as a center, construct an arc cutting the sides at points we will call A and B.



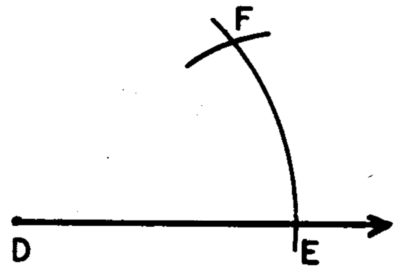
2. Draw the dashed line segment AB. $\triangle ABC$ is the triangle you are to copy.



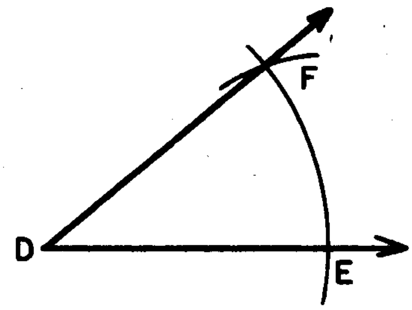
3. Draw a ray (leave enough room so you can construct the triangle using part of this ray) and call the endpoint, D.



4. With point D as the center and with the same setting of your compass as in Step 1, construct an arc. Call the point where this arc intersects the ray, point E.



5. Change the setting of your compass so that its point are at points A and B of $\angle BCA$. Keep this setting and place the point of the compass at E and draw an arc which intersects the first arc. Call the point of intersection of the two arcs F.

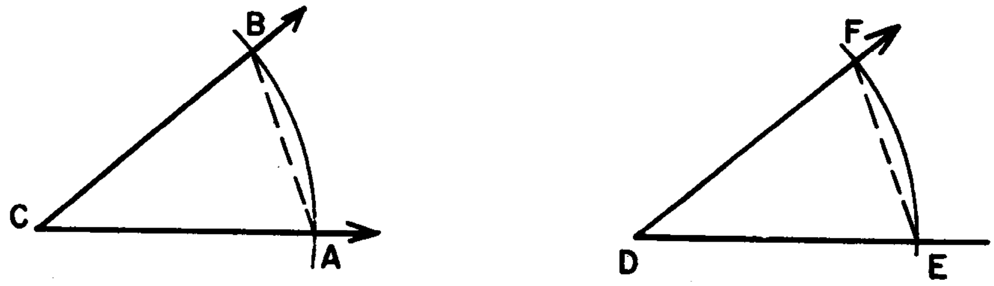


6. Draw $\overrightarrow{DF}$.

Have you made $\angle FDE \cong \angle BCA$? Let us see.

Draw $\overline{BA}$ and $\overline{FE}$.

Is $\Delta FDE \cong \Delta BCA$? Why?

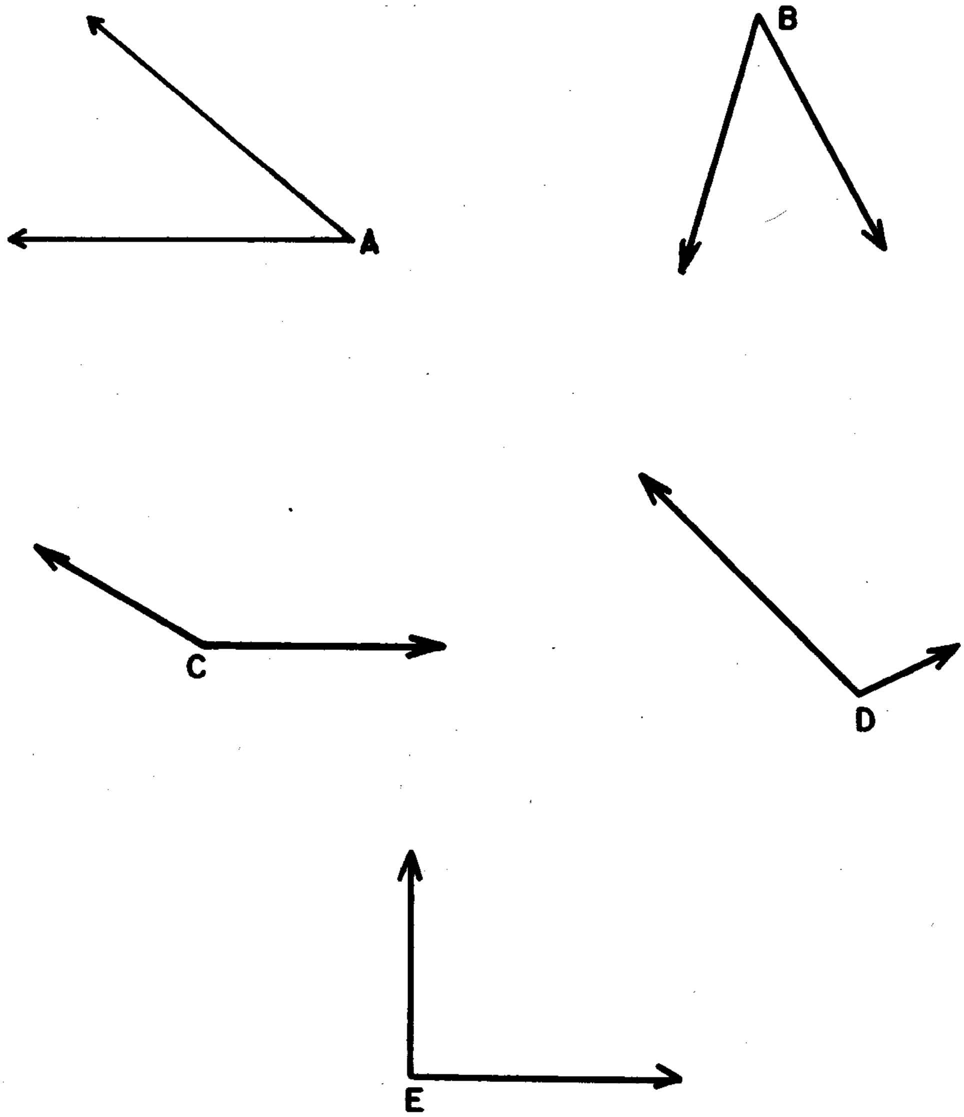


Is $\angle FDE \cong \angle BCA$? Why?

We know $\triangle FDE \cong \triangle BCA$ because we have made three sides of one triangle congruent to three sides of the other triangle. We have chosen two sides the same length for convenience. Now, since we know that corresponding angles of congruent triangles are congruent, we know that $\angle FDE \cong \angle BCA$.

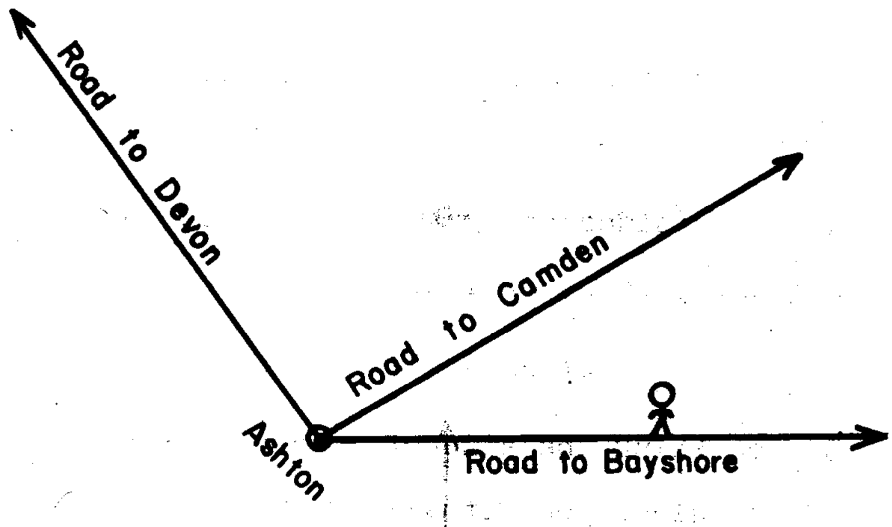
Exercise Set 9

1. Make an angle about like $\angle A$ on your paper. Copy it by using the steps we have outlined. Then do the same for the other angles.



COMPARING SIZES OF ANGLES

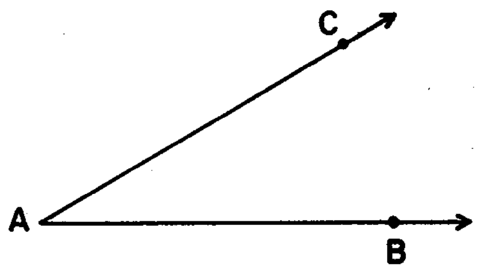
Three roads run from a point in the town of Ashton--one to Bayshore, one to Camden and one to Devon. The man in the sketch is walking toward Ashton. When he comes to the intersection in Ashton, he will choose whether he will follow the road to Camden or the road to Devon. We sometimes say, "The Camden road angles off from the Bayshore road." If he goes to Camden he turns off "at an angle" of one size. If he goes to Devon, he turns off "at an angle" of a different size. Let us see what we mean by the "size" of an angle.

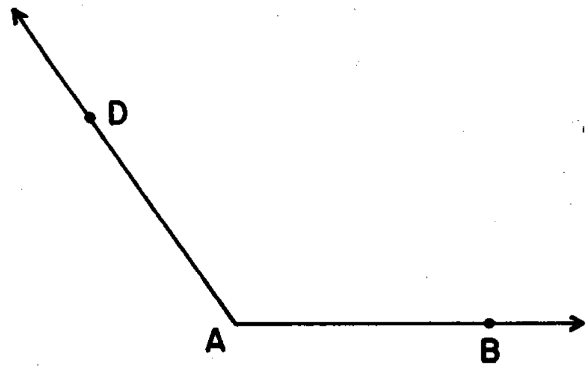


Angles With a Common Ray

Exploration

The first sketch below shows the Bayshore and Camden roads. The second shows the Bayshore and Devon roads. Think of the roads as representing rays with endpoint A. Which angle do you think has the larger size?





1. Recall what we mean by the word "angle." How have we defined it?

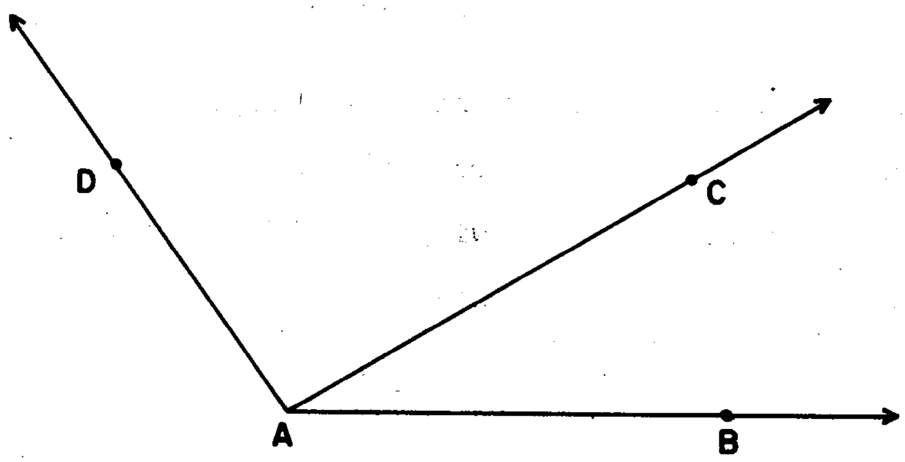
2. Name the sides of $\angle BAC$ and $\angle BAD$. Are the sides segments, rays, or lines?

3. Do the sides of an angle have a definite length?

4. Do you think the size of an angle depends on the lengths of the sides you actually draw?

   It is clear that the size of an angle cannot depend on the length of its sides, since rays have no definite length.

To see what is meant by "One angle is larger in size than another angle," look at the sketch of the roads to Bayshore, Camden, and Devon.



5. Name the sides of $\angle BAC$.

   Name the sides of $\angle BAD$.

   What ray is a side of both angles?

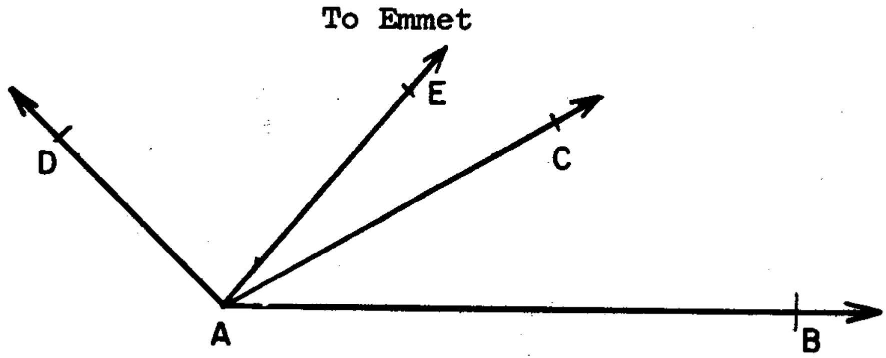
6. Is point C in the interior, or in the exterior of $\angle BAD$?

7. Is $\overline{AC}$ (except for point A) in the interior, or in the exterior of $\angle BAD$?

Because a) $\angle BAD$ and $\angle BAC$ both have side $\overrightarrow{AB}$, and

b) point C is in the interior of $\angle BAD$,

we say that the size of $\angle BAD$ is larger than the size of $\angle BAC$. (Or we can say that the size of $\angle BAC$ is smaller than the size of $\angle BAD$.)



8. Name all the angles in the sketch. (There are six.)

9. Look at ∠CAE. What rays are its sides?

10. Are E and C in the interior of ∠BAD? Because E and C are in the interior of ∠BAD we say, "The size of ∠BAD is larger than the size of ∠CAE." (Or, "The size of ∠CAE is smaller than the size of ∠BAD.")

11. Name an angle whose size is smaller than the size of ∠DAC. Name another one that appears to be smaller. How can you be sure your answer is right?

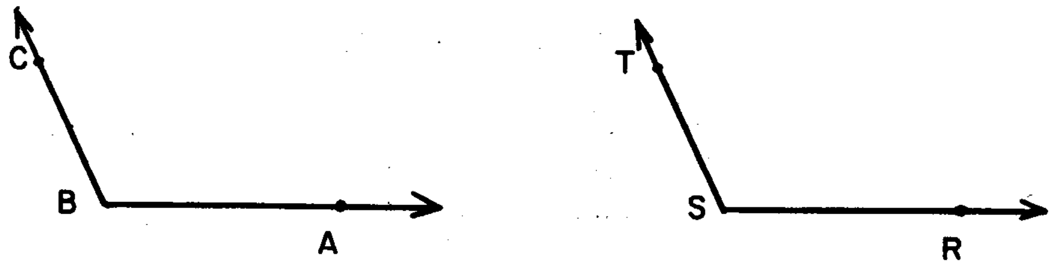
12. Name an angle of larger size than ∠EAD. Name another one. How can you be sure?

13. Name three angles, each of larger size than ∠EAC.

14. Suppose another town, Farley, is on the Ashton-Camden Road. Copy the sketch and represent Farley by point F.

15. What can you say about the sizes of ∠CAE and ∠FAE? About ∠DAF and ∠DAC? ∠BAC and ∠FAB?

16. In this sketch, $\angle ABC$ is congruent to $\angle RST$.



a) Trace $\angle ABC$ on tracing paper. Place B on S and $\overrightarrow{BC}$ on $\overrightarrow{ST}$. Put $\overrightarrow{BA}$ on the R-side of $\overleftrightarrow{TS}$. Must $\overrightarrow{BA}$ lie on $\overrightarrow{SR}$?

b) Is either of these angles larger than the other?

c) If two angles are congruent, can the size of one be larger than the size of the other?

Summary

The examples above show:

1. The size of one angle is smaller than the size of a second angle:

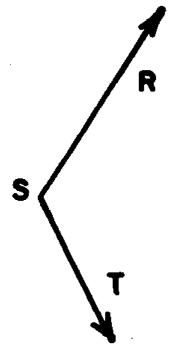
   a) If the angles have one ray in common, and a point on the other ray of the first angle lies in the interior of the second angle.

   b) If a point on each ray of the first angle lies in the interior of the second angle.

2. Congruent angles have the same size.

Exercise Set 10

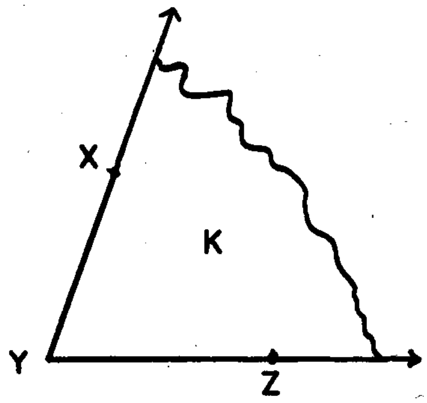
1. a) Trace $\angle RST$. Choose a point in the interior of $\angle RST$. Call this point, W. Draw $\overrightarrow{SW}$.



   b) Compare the size of $\angle RST$ with the size of $\angle RSW$.

   c) Compare the size of $\angle RST$ with the size of $\angle WST$.

2. a) Trace $\angle XYZ$ and point K. Point K is in the ________ of $\angle XYZ$. Draw $\overrightarrow{YK}$.



   b) Compare the sizes of $\angle XYZ$ and $\angle XYK$.

   c) Compare the sizes of $\angle KYZ$ and $\angle XYZ$.

3. a) Cut along $\overrightarrow{YX}$ and $\overrightarrow{YZ}$ and tear along the jagged curve. Fold along $\overrightarrow{YK}$. Does $\overrightarrow{YZ}$ fall along $\overrightarrow{YX}$?

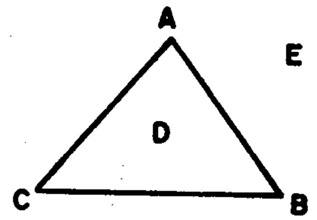
   b) Is $\angle XYK \cong \angle KYZ$?

4. In the interior of $\angle ZYX$, place a point N near Z and draw $\overrightarrow{YN}$. Fold along $\overrightarrow{YN}$. Which has the larger size, $\angle XYN$ or $\angle NYZ$?

5. Draw an angle. Name it $\angle MPR$. Choose a point (call it S) so that you can be sure the size of $\angle SPM$ is smaller than the size of $\angle MPR$. Where did you place S?

6. Using the angle of exercise 5, choose a point (call it T) so you can be sure that the size of $\angle TPM$ is larger than the size of $\angle MPR$. Where did you place T?

7. a) Is point D in the interior of $\angle BAC$ shown in this figure?



   b) Is it in the interior of $\angle ABC$? of $\angle ACB$?

8. a) Is E in the interior of $\angle ACB$ shown in the figure?

   b) Is it in the interior of $\angle BAC$? Of $\angle CBA$?

9. a) Draw ΔABC and label a point D as in the previous sketch. Then draw $\overrightarrow{AD}$.

   b) What two angles are smaller in size than $\angle CAB$?

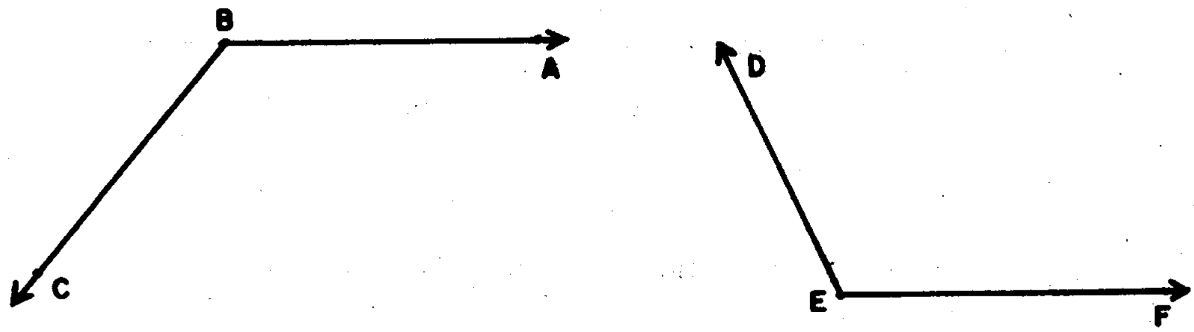
10. a) Draw a ΔABC and label a point E as in the sketch above. Draw $\overrightarrow{BE}$.

   b) What angle of ΔABC is smaller in size than $\angle EBC$?

Angles Without a Common Ray

Exploration

You know how the sizes of two angles are compared when the two angles have one ray in common, or when the rays (except for the vertex) of one are in the interior of the other. How shall we compare the sizes of two angles which are not placed in either of these ways?



1. Copy $\angle$DEF by tracing it on thin paper. Copy the letters, too.

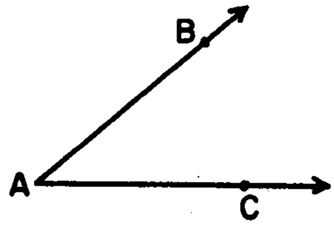
2. a) How should the rays of $\angle$DEF be placed on $\angle$ABC to compare the sizes of the angles? You may want to turn your tracing over.

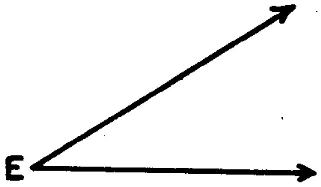
   b) Is there more than one way to place $\angle$DEF in order to compare its size with that of $\angle$ABC?

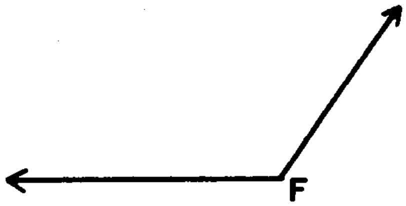
3. How do the sizes of $\angle$ABC and $\angle$DEF compare?

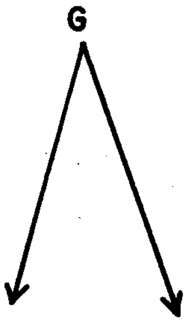
Exercise Set 11

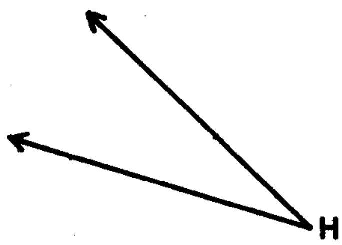
1. Trace $\angle CAB$ on thin paper. Then compare the size of $\angle CAB$ with the size of each angle below.

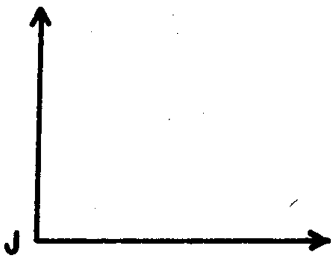


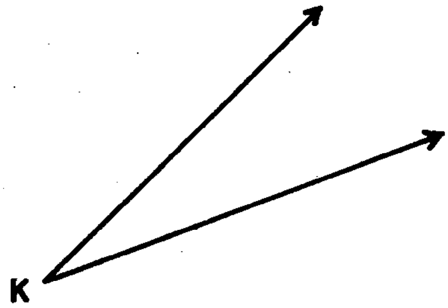








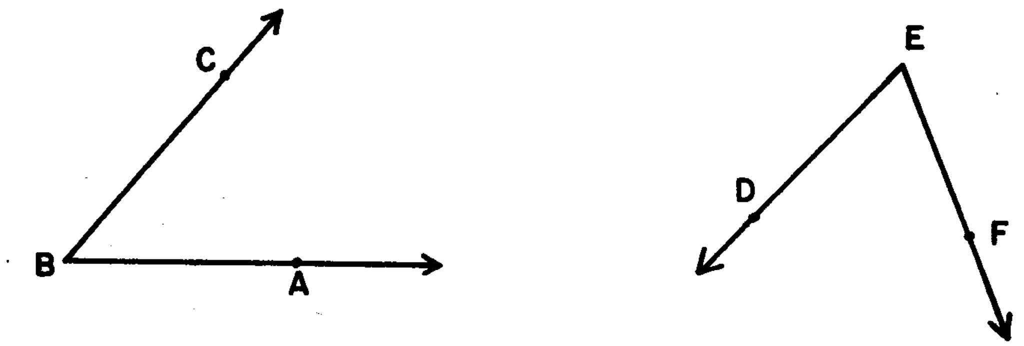




Using the Congruent Angle Construction

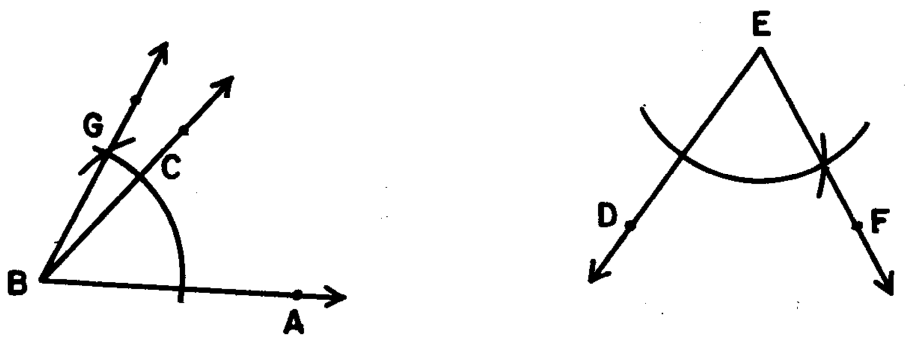
Exploration

You know how to construct an angle congruent to a given angle, and you know that congruent angles have the same size. Can you use what you know to compare the sizes of two angles, no matter what their positions?



1. a) Look at $\angle ABC$ and $\angle DEF$. Where should $\angle DEF$ be copied so as to compare the sizes? What point should you use as vertex?

   b) What ray should you use as one side of the copy?



2. a) In the figures, $\angle ABG$ was constructed congruent to $\angle DEF$, so they have the same size. What angles can we compare now?

   b) What does this tell us about the sizes of $\angle ABG$ and $\angle ABC$?

3. a) In what other position could be copy $\angle DEF$ to compare its size with the size of $\angle ABC$? Could we use some point other than B as vertex?

   b) Could we use a ray different from $\overrightarrow{BA}$ as one side?

   c) Could the comparison be the same?

4. a) Could we copy $\angle ABC$ instead of $\angle DEF$?

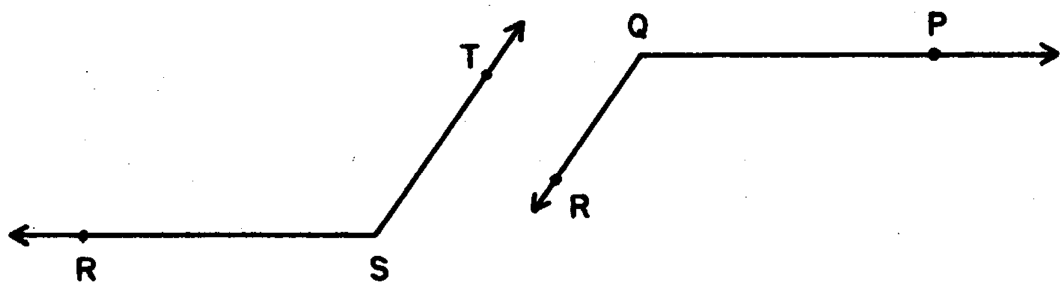
   b) If so, what point should be the vertex?

   c) What ray should be a side?

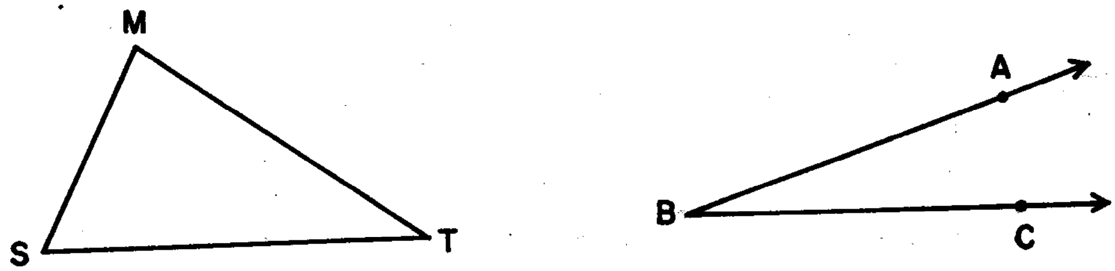
Exercise Set 12

1. Copy $\angle ABC$ and $\angle DEF$ by tracing them on thin paper. Use your compass and straightedge to construct an angle congruent to $\angle DEF$ so you can compare the sizes of the angles.

2. Compare the sizes of $\angle RST$ and $\angle PQR$.



3. Compare the sizes of $\angle ABC$ and $\angle MTS$.

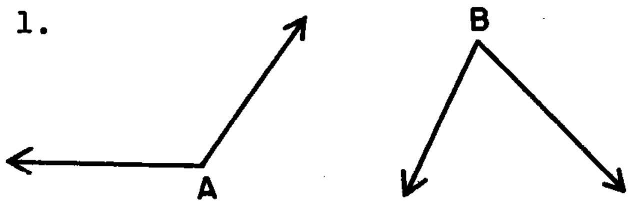


When you understand what is meant by "The size of $\angle A$ is larger than the size of $\angle B$," and what is meant by "$\angle A \cong \angle B$," you can often tell by looking at two angles which has the larger size. You can also tell whether they may be congruent.

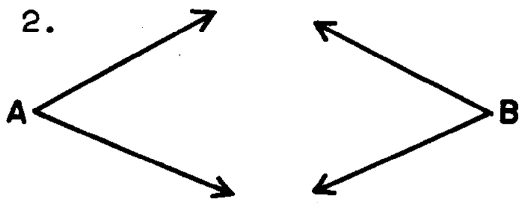
Exercise Set 13

Compare the sizes of $\angle A$ and $\angle B$ in each pair below. If you can't decide which is larger, trace one angle on thin paper and place the tracing on the other angle, or use your compass and straightedge to construct congruent angles.

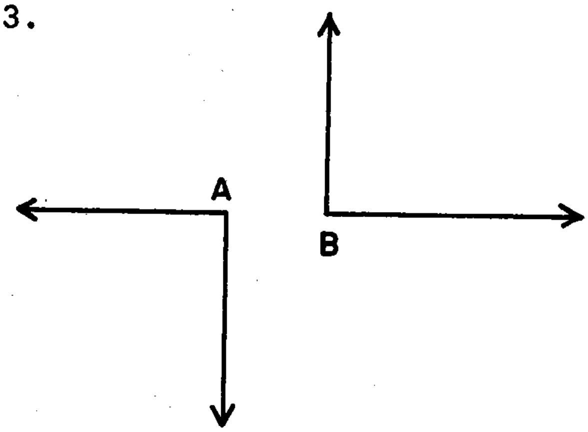
1.



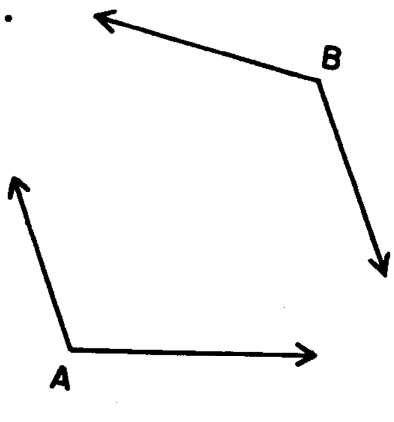
2.



3.



4.



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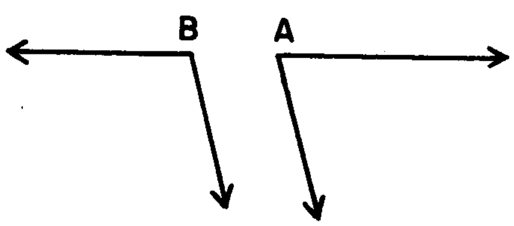
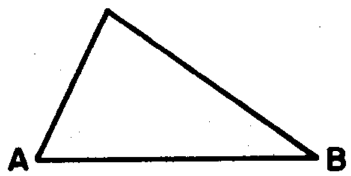
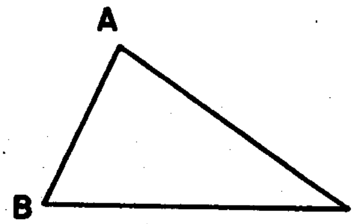


In the figures below, $\angle A$ and $\angle B$ are angles of triangles or angles of other polygons. In each figure, compare the sizes of $\angle A$ and $\angle B$ as you did in Exercises 1 to 5.

6.



7.



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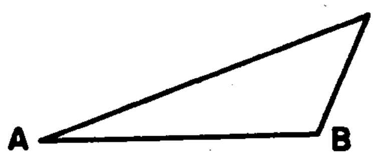
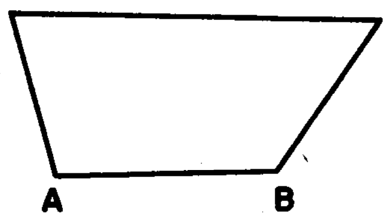
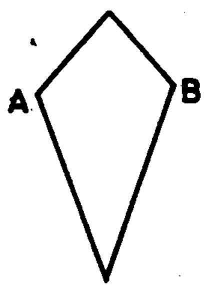


9.

10.



11.



Chapter 5

EXTENDING MULTIPLICATION AND DIVISION II

MULTIPLYING LARGE NUMBERS

In Chapter 3 you learned how to find the product of two numbers. Now we want to find shorter ways to find these products. Let's look at these multiplication examples.

<u>Example</u> <u>1</u>: Multiply 437 and 39

$$\begin{array}{r}437\\ \underline{\times\ 39}\end{array} \qquad\qquad \begin{array}{r}437\\ \underline{\times\ 39}\end{array}$$

$$\left.\begin{array}{r}63\\270\\3600\end{array}\right\} \xrightarrow{(9 \times 437)} 3933$$

$$\left.\begin{array}{r}210\\900\\\underline{12000}\end{array}\right\} \xrightarrow{(30 \times 437)} \underline{13110}$$

$$17043 \qquad\qquad 17043$$

<u>Example</u> <u>2</u>: Multiply 456 and 805

$$\begin{array}{r}805\\ \underline{\times\ 456}\end{array} \qquad\qquad \begin{array}{r}805\\ \underline{\times\ 456}\end{array}$$

$$\left.\begin{array}{r}30\\4800\end{array}\right\} \xrightarrow{(6 \times 805)} 4830$$

$$\left.\begin{array}{r}250\\40000\end{array}\right\} \xrightarrow{(50 \times 805)} 40250$$

$$\left.\begin{array}{r}2000\\\underline{320000}\end{array}\right\} \xrightarrow{(400 \times 805)} \underline{322000}$$

$$367080 \qquad\qquad 367080$$

Explain how to get each of the partial products in the shorter form of these examples.

Exercise Set 1

Use a vertical form to compute the following.

1. 86 x 923
2. 48 x 654
3. 57 x 874
4. 473 x 52
5. 36 x 504
6. 56 x 780
7. 68 x 5346
8. 76 x 3498
9. 4038 x 79
10. 57 x 7239
11. 625 x 834
12. 658 x 762
13. 846 x 648
14. 607 x 546
15. 971 x 356
16. 656 x 750
17. 720 x 856
18. 384 x 507
19. 834 x 720
20. 345 x 637

Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

21. There are 64 rows of seats in the auditorium. There are 45 seats in each row. How many people can be seated in the auditorium?

22. John kept a record of how much gasoline his family car used on their vacation last summer. They used 167 gallons. If they can travel 18 miles on each gallon of gas, how many miles did they travel during their vacation?

23. A brick wall is 126 bricks long and 42 bricks high. How many bricks are there in the wall?

24. If 76 nails are used in making a shoe, how many nails are needed to make 23 pairs of these shoes?

25. A helicopter makes a round trip of 102 miles three times daily to collect and deliver mail in the San Francisco Bay area. How many miles does it travel in a year? (Note: Use 365 days.)

MULTIPLYING LARGER NUMBERS

Example 1: Multiply 4365 and 7439.

```
    7439
  x 4365
   37195
  446340
 2231700
29756000
32471235
```

How many partial products are there in this example?

Example 2: Multiply 5063 and 8309.

```
    8309
  x 5063
   24927
  498540
41545000
42068467
```

Notice that there are only 3 partial products in this example. Explain how each of these partial products was obtained.

Multiply the numbers in the following example and compare the product with the product in example 2.

```
  5063
x 8309
```

Are the products the same? Why?

Are the partial products the same? Why?

Exercise Set 2

Use a vertical form to find the product of each of these pairs of numbers.

1. 537 and 4372
2. 200 and 317
3. 96 and 897
4. 4569 and 5007
5. 957 and 8060
6. 357 and 892
7. 5430 and 739
8. 709 and 5080
9. 101 and 523
10. 3586 and 367
11. 3542 and 4673
12. 234 and 3112
13. 909 and 673
14. 231 and 706
15. 3570 and 4987
16. 8971 and 6173
17. 2003 and 2131
18. 3672 and 4819
19. 8080 and 5599
20. 2712 and 3486

Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

21. A cab driver makes many trips to and from a large city airport. He drives about 315 miles a day. About how many miles does he drive in 28 days?

22. A grapefruit orchard has 32 rows of grapefruit trees with 45 trees in each row. How many trees are there in the orchard?

23. A jet plane travels 485 miles per hour on the average. One month it is flown 114 hours. If that is an average month, how many miles is it flown in a year?

24. The Lincoln family spent \$224 for an 8-day trip. If they spent the same amount each day, how much should they plan to save for next year's 21-day trip?

25. There were 103 passengers on a jet plane going from New York to Toronto. Each passenger was allowed to take 66 pounds of luggage without charge. If each passenger took the full amount, how many pounds of free luggage were carried?

A SHORTER FORM FOR MULTIPLYING

Study the following examples. See what has been done to shorten the way we record the partial products. Why can we do this?

<u>Example</u> <u>1</u>:

$$\begin{array}{r}
5476 \\
\underline{\times\ 3528} \\
43808 \\
109520 \\
2738000 \\
\underline{16428000} \\
19319328
\end{array}
\qquad\qquad
\begin{array}{r}
5476 \\
\underline{\times\ 3528} \\
43808 \\
10952 \\
27380 \\
\underline{16428} \\
19319328
\end{array}$$

<u>Example</u> <u>2</u>:

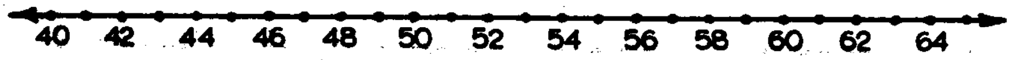
$$\begin{array}{r}
439 \\
\underline{\times\ 605} \\
2195 \\
\underline{263400} \\
265595
\end{array}
\qquad\qquad
\begin{array}{r}
439 \\
\underline{\times\ 605} \\
2195 \\
\underline{2634} \\
265595
\end{array}$$

Exercise Set 3

Use a vertical form to find the product of each of these pairs of numbers.

1. 47 and 63
2. 92 and 78
3. 478 and 356
4. 4234 and 6209
5. 465 and 688
6. 407 and 629
7. 634 and 6070
8. 97 and 401
9. 392 and 847
10. 54 and 286
11. 25 and 2359
12. 465 and 750
13. 3049 and 4340
14. 89 and 76
15. 7294 and 325
16. 58 and 1289
17. 73 and 496
18. 207 and 639
19. 36 and 74
20. 66 and 247

EXPRESSING NUMBERS TO THE NEARER MULTIPLE OF TEN



We have used a number line to help us see that:

53 is nearer to 50 than 60.

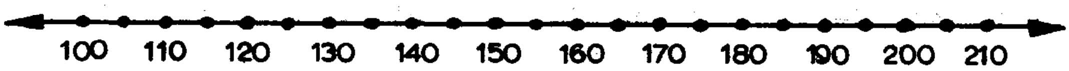
58 is nearer to 60 than 50.

We have discovered a way to find the nearest multiple of 10 to a number without using a number line.

What is the nearest multiple of 10 to each of these numbers?

92	61	383	134
49	34	285	288
75	46	567	476
83	58	684	341
17	25	139	675

EXPRESSING NUMBERS TO THE NEARER MULTIPLE OF ONE HUNDRED



We have used a number line to help us see that:

142 is nearer to 100 than 200.

167 is nearer to 200 than 100.

We have discovered a way to find the nearest multiple of 100 to a number without using a number line.

What is the nearest multiple of 100 to each of these numbers?

145	253	450	666
155	203	230	623
186	850	346	650
174	290	304	857
156	224	572	749

REVIEW OF DIVISION

Exploration

In Chapter 3 we learned about a shorter form for dividing. The boxes below show several forms for dividing 836 by 6.

Longer Forms

$$\begin{array}{rr}
 & \underline{139} \\
 & 9 \\
 & 30 \\
 & 100 \\
6 & \overline{)836} \\
 & \underline{600} \\
 & 236 \\
 & \underline{180} \\
 & 56 \\
 & \underline{54} \\
 & 2
\end{array}
\qquad\qquad
\begin{array}{rr|r}
6 & \overline{)836} & \\
 & \underline{600} & 100 \\
 & 236 & \\
 & \underline{180} & 30 \\
 & 56 & \\
 & \underline{54} & \underline{9} \\
 & 2 & 139
\end{array}$$

A Shorter Form

$$\begin{array}{rr}
 & 139 \\
6 & \overline{)836} \\
 & \underline{600} \\
 & 236 \\
 & \underline{180} \\
 & 56 \\
 & \underline{54} \\
 & 2
\end{array}$$

When 836 is divided by 6, what is the quotient? What is the remainder?

Find a mathematical sentence that tells us that when we divide 836 by 6, the quotient is 139 and the remainder is 2.

We may say that 100 and 30 and 9 are parts of the quotient. Using place value, explain how the shorter form tells us this.

In this chapter we are going to learn about dividing by larger numbers. We also will learn things that can help us become more skillful when we divide.

Can you find a short way to divide 928 by 6 so that you need to write only the quotient and remainder?

If you cannot discover this short way of dividing, this chapter will help you with it later.

Exercise Set 4

For each of the following, divide the first number by the second. Write a mathematical sentence to describe the result.

1. 579 by 8
2. 6847 by 9
3. 4496 by 8
4. 4701 by 8
5. 1728 by 9
6. 2505 by 5
7. 4758 by 9
8. 1690 by 5
9. 5670 by 6
10. 3549 by 5
11. 5535 by 7
12. 6572 by 8

DIVIDING BY NUMBERS GREATER THAN 10 AND LESS THAN 100

Exploration

Let us divide 859 by 23. First, we will use one of the long forms. After we do this, maybe you can see how we can use a shorter form.

$$\begin{array}{rr} & 30 \\ 23 & \overline{)859} \\ & \underline{690} \\ & 169 \end{array}$$

$$\begin{array}{rr|l} 23 & \overline{)859} & \\ & \underline{690} & 30 \\ & 169 & \end{array}$$

A. Will the quotient be at least 10?

Will the quotient be as great as 100?

What does this information tell us?

B. We can use multiples of 10 to help us find part of the quotient.

What are the multiples of 10 that are less than 100?

We try to find the largest multiple of 10 that will be part of the quotient.

What is 10×23?	What is 30×23?
What is 20×23?	What is 40×23?

Have we found the largest multiple of 10 that will be part of the quotient? What is it?

How do we know that 30 is the largest multiple of 10 that will be part of the quotient?

Now explain the work shown in the boxes near the top of the page.

C. Now we will find the remaining part of the quotient.

How do we know that the remaining part of the quotient will be less than 10?

We try to find the largest number so that that number times 23 will no greater than 169. What is it?

How did you find that 7 is the largest number to use?

Now explain how the work in the boxes below was completed.

$$
\begin{array}{r}
\underline{37} \\
7 \\
30 \\
23\,\overline{)\,859} \\
\underline{690} \\
169 \\
\underline{161} \\
8
\end{array}
\qquad\qquad
\begin{array}{r|r}
23\,\overline{)\,859} & \\
\underline{690} & 30 \\
\cline{1-1}
169 & \\
161 & \underline{7} \\
\hline
8 & 37
\end{array}
$$

We divided 859 by 23.

What is the quotient?

What is the remainder?

Write a mathematical sentence that tells us these things.

Show how to check your work.

Now let us divide 1724 by 67. Two forms for doing this are shown in the boxes below.

$$\begin{array}{rr}
 & \underline{25} \\
 & 5 \\
 & 20 \\
67 & \overline{)1724} \\
 & \underline{1340} \\
 & 384 \\
 & \underline{335} \\
 & 49
\end{array}
\qquad\qquad
\begin{array}{rr|r}
67 & \overline{)1724} & \\
 & \underline{1340} & 20 \\
 & 384 & \\
 & \underline{335} & \underline{5} \\
 & 49 & 25
\end{array}$$

Answer these questions about the division.

How do we know that the quotient must be greater than 10 but less than 100?

Multiples of 10 help us find the first part of the quotient. How can we find the largest multiple of 10 to use as the first part of the quotient? What is it?

How do we know that the remaining part of the quotient will be less than 10?

How can we find the remaining part of the quotient? What is it?

We divided 1724 by 67.

What is the quotient?

What is the remainder?

Write a mathematical sentence that tells us these things.

Exercise Set 5

Divide the first number by the second number. Write a mathematical sentence to describe the result.

1. 604 by 82
2. 340 by 41
3. 2681 by 39
4. 2464 by 57
5. 695 by 94
6. 4090 by 73
7. 5136 by 66
8. 184 by 27
9. 6434 by 75
10. 5103 by 88

Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

11. It cost \$128 for a bus to take 32 fifth-graders to the state capitol. How much does each pupil have to pay?

12. A box holds 24 books. How many boxes will be needed to hold 984 books?

13. A store had a sale on one model of a bicycle. 68 bicycles of this model were sold for a total amount of \$2,856. What was the sale price of a bicycle?

14. Jane has 630 stamps that she wants to put into envelopes. If she puts 45 stamps in each envelope, how many envelopes will she need?

15. An automobile is moving at a speed of 28 feet per second. How many seconds will it take it to move 980 feet?

FINDING SHORTER WAYS OF DIVIDING

Exploration

Let us think about dividing 836 by 6.

We have learned how to shorten our work from either one of the two forms at the left to the one at the right.

$$
\begin{array}{r}
\underline{139} \\
9 \\
30 \\
100 \\
6\overline{)836} \\
\underline{600} \\
236 \\
\underline{180} \\
56 \\
\underline{54} \\
2
\end{array}
\qquad\qquad
\begin{array}{r|r}
6\overline{)836} & \\
\underline{600} & 100 \\
236 & \\
\underline{180} & 30 \\
56 & \\
\underline{54} & \underline{9} \\
2 & 139
\end{array}
\qquad\qquad
\begin{array}{r}
139 \\
6\overline{)836} \\
\underline{600} \\
236 \\
\underline{180} \\
56 \\
\underline{54} \\
2
\end{array}
$$

We divided 836 by 6.

What is the quotient?

What is the remainder?

What mathematical sentence tells us these things?

Explain how we used place value to shorten the writing of the quotient numeral in the form at the right.

Now let us see how we can shorten our work even more.

$$\begin{array}{rcrl}
139 & & 139 & \\
6\,\overline{)\,836} & & 6\,\overline{)\,836} & \\
\underline{600} & \longrightarrow & \underline{6} & (\ 6 \quad \text{hundreds}) \\
236 & & 236 & \\
\underline{180} & \longrightarrow & \underline{18} & (18 \quad \text{tens}) \\
56 & & 56 & \\
\underline{54} & \longrightarrow & \underline{54} & (54 \quad \text{ones}) \\
2 & & 2 &
\end{array}$$

We have used place value to help us shorten the writing of the quotient numeral. In the form at the right we also use place value to help us shorten other parts of our work.

How did we use place value to shorten the writing of 600?

How did we use place value to shorten the writing of 180?

Why is 54 written the same way in both forms?

Can we shorten our work even more than we have already? Look at the forms below.

A.

```
   139
6 )836
   6
   236
   18
    56
    54
     2
```

B.

```
   139
6 )836
   6
   23
   18
    56
    54
     2
```

C.

```
    1 3 9    r  2
6 ) 8 (2)3 (5)6
```

In Form B, explain how you could use each of these "helpers", along with place value, to work the example.

When dividing the hundreds, think:

$8 \div 6$. The quotient is 1; the remainder is 2.

When dividing the tens, think:

$23 \div 6$. The quotient is 3; the remainder is 5.

When dividing the ones, think:

$56 \div 6$. The quotient is 9; the remainder is 2.

Could you use these same "helpers" with Form C? Explain. What does " r 2 " mean in Form C?

If you have a good memory, you don't even have to write the (2) and the (5) in Form C. If you can remember them, all you need to write is the quotient and the remainder:

139 r 2.

Let us study together three forms of dividing for the example, $1670 \div 7$.

A.

$$
\begin{array}{r}
238 \\
7\overline{)1670} \\
\underline{14} \\
270 \\
\underline{21} \\
60 \\
\underline{56} \\
4
\end{array}
$$

B.

$$
\begin{array}{r}
238 \\
7\overline{)1670} \\
\underline{14} \\
27 \\
\underline{21} \\
60 \\
\underline{56} \\
4
\end{array}
$$

C.

$$
\begin{array}{l}
2\ \ 3\ \ 8 \quad r\ 4 \\
7\overline{)\ 1\ 6\ {}^{(2)}7\ {}^{(6)}0}
\end{array}
$$

Explain how you could use each of these "helpers", along with place value, in forms B and C.

When dividing the hundreds, think:

$16 \div 7$. The quotient is 2; the remainder is 2.

When dividing the tens, think:

$27 \div 7$. The quotient is 3; the remainder is 6.

When dividing the ones, think:

$60 \div 7$. The quotient is 8; the remainder is 4.

Exercise Set 6

Find each quotient and remainder using the shortest form you can.

1. $3\overline{)79}$
2. $4\overline{)95}$
3. $5\overline{)92}$
4. $2\overline{)95}$
5. $7\overline{)920}$
6. $8\overline{)123}$
7. $6\overline{)1334}$
8. $9\overline{)1417}$
9. $7\overline{)9250}$
10. $4\overline{)9455}$
11. $3\overline{)8624}$
12. $5\overline{)9620}$
13. $6\overline{)8427}$
14. $8\overline{)96834}$
15. $4\overline{)26547}$

USING SHORTER FORMS WHEN DIVISORS ARE MULTIPLES OF TEN

Exploration

Here are some of the ways we can shorten our work when we divide 8469 by 30.

A.	B.	C.
282	282	282
30) 8469	30) 8469	30) 8469
6000	60	60
2469	2469	246
2400	240	240
69	69	69
60	60	60
9	9	9

Here are some of the ways we can shorten our work when we divide 9382 by 70.

A.	B.	C.
134	134	134
70) 9382	70) 9382	70) 9382
7000	70	70
2382	2382	238
2100	210	210
282	282	282
280	280	280
2	2	2

Study carefully each set of examples on the preceding page.

What is the quotient and remainder when 8469 is divided by 30? Write a mathematical sentence that tells this.

What is the quotient and remainder when 9382 is divided by 70? Write a mathematical sentence that tells this.

When dividing 8469 by 30, how could you use each of these as "helpers"?

$8 \div 3$ $\quad$ $24 \div 3$ $\quad$ $6 \div 3$

When dividing 9382 by 70, how could you use each of these as "helpers"?

$9 \div 7$ $\quad$ $23 \div 7$ $\quad$ $28 \div 7$

Which form do you understand best for working each example?

If you can use a shorter form than the ones given on the preceding page, use the chalkboard to show and explain it to other pupils in the class.

Exercise Set 7

Divide. Use the shortest form that you can.

1. $30\overline{)1628}$

2. $70\overline{)6586}$

3. $40\overline{)9274}$

4. $80\overline{)9000}$

5. $60\overline{)8563}$

6. $20\overline{)7459}$

7. $50\overline{)7496}$

8. $90\overline{)38642}$

9. $20\overline{)6538}$

10. $80\overline{)7163}$

11. $70\overline{)5872}$

12. $90\overline{)88429}$

WORKING WITH DIVISORS BETWEEN 10 AND 100

Exploration

We have been working with divisors that are multiples of 10. We have used "helpers" to find parts of the quotient. We can use the same kind of "helper" when working with divisors between 10 and 100.

Here is an example for us to try: $975 \div 23$.

Our quotient must be between 10 and 100. Why?

Is 23 nearer to 20 or to 30?

Since 23 is nearer to 20, let us use $9 \div 2$ as a "helper" to try to find the first part of the quotient. For $9 \div 2$, we think "4".

$$23\,\overline{)\,975}$$

Does the 4 written above the 7 tell us that the first part of the quotient is 40? Why? Can the remaining part of the quotient be as great as 10? Explain.

$$\begin{array}{r} 4 \\ 23\,\overline{)\,975} \\ \underline{920} \\ 55 \end{array}$$

Now let us use $5 \div 2$ as a "helper" to find the remaining part of the quotient. For $5 \div 2$, we think "2". Why is the 2 written above the 5? What is the quotient when we divide 975 by 23? What is the remainder? Is the remainder less than the divisor?

$$\begin{array}{r} 42 \\ 23\,\overline{)\,975} \\ \underline{920} \\ 55 \\ \underline{46} \\ 9 \end{array}$$

Does $975 = (42 \times 23) + 9$?

The check at the right will tell us.

Check

$$\begin{array}{r} 23 \\ \underline{\times\ 42} \\ 46 \\ \underline{92} \\ 966 \\ \underline{+\ \ 9} \\ 975 \end{array}$$

Now let us try this example: $1939 \div 68$

Our quotient must be between 10 and 100. Why?

Is 68 nearer to 60 or to 70?

Since 68 is nearer to 70, let us use $19 \div 7$ as a "helper" to try to find the first part of the quotient. For $19 \div 7$, think "2".

$$68\overline{)1939}$$

Does the 2 written above the 3 tell us that the first part of the quotient is 20? Why?

Can the remaining part of the quotient be as great as 10? Explain.

$$\begin{array}{r} 2 \\ 68\overline{)1939} \\ \underline{1360} \\ 579 \end{array}$$

Now let us use $57 \div 7$ as a "helper" to find the remaining part of the quotient.

For $57 \div 7$, think "8".

Why is the 8 written where it is?

What is the quotient when we divide 1939 by 68. What is the remainder?

Is the remainder less than the divisor?

$$\begin{array}{r} 28 \\ 68\overline{)1939} \\ \underline{1360} \\ 579 \\ \underline{544} \\ 35 \end{array}$$

Write the mathematical sentence that goes with this example.

Show the check for the work.

Exercise Set 8

Divide. Check your answers

1. $63 \overline{)2042}$
2. $36 \overline{)2014}$
3. $29 \overline{)1962}$
4. $88 \overline{)5748}$
5. $67 \overline{)5729}$
6. $73 \overline{)3198}$
7. $92 \overline{)3423}$
8. $44 \overline{)914}$
9. $21 \overline{)1498}$
10. $78 \overline{)1828}$
11. $55 \overline{)823}$
12. $84 \overline{)6766}$
13. $49 \overline{)3419}$
14. $97 \overline{)4388}$

QUOTIENTS GREATER THAN 100

We will study these examples together.

$$8754 \div 32$$

$$\begin{array}{r} 273 \\ 32\,\overline{)\,8754} \\ \underline{6400} \\ 2354 \\ \underline{2240} \\ 114 \\ \underline{96} \\ 18 \end{array}$$

How do we know the quotient will be between 100 and 1000?

Is 32 nearer to 30 or to 40?

Explain how we could use each of these helpers to find parts of the quotient.

$8 \div 3$. $\qquad$ $23 \div 3$. $\qquad$ $11 \div 3$.

The first part of the quotient is 200. How do we know that it could not be as much as 300?

The second part of the quotient is 70. How do we know that it could not be as much as 80?

Explain why each digit of the quotient numeral is placed where it is.

What is the quotient?

What is the remainder?

Is the remainder less than the divisor?

Write the mathematical sentence for this example.

Show the check for your work.

$15014 \div 57$

$$\begin{array}{r}
263 \\
57 \overline{)\,15014} \\
\underline{11400} \\
3614 \\
\underline{3420} \\
194 \\
\underline{171} \\
23
\end{array}$$

How do we know the quotient will be between 100 and 1000?

Is 57 nearer to 50 or to 60?

How can we use each of these "helpers" to find parts of the quotient?

$15 \div 6$. $36 \div 6$. $19 \div 6$.

How can we know that the first part of the quotient is not as great as 300?

How can we know that the second part of the quotient is not as great as 70?

Explain why each digit of the quotient numeral is placed where it is.

What is the quotient?

What is the remainder?

Is the remainder less than the divisor?

Write the mathematical sentence for this example.

Show the check for your work.

Explain the work for these examples.

Be sure to tell why a zero had to be written in each quotient numeral.

$17286 \div 54$

```
      320
54 ) 17286
     16200
     -----
      1086
      1080
     -----
         6
```

$18376 \div 89$

```
      206
89 ) 18376
     17800
     -----
       576
       534
     -----
        42
```

For each example:

Write the mathematical sentence

Show a check for the work.

Exercise Set 9

Divide. Use the shortest form that you can.

1. $38\overline{)7094}$
2. $82\overline{)11732}$
3. $65\overline{)8446}$
4. $93\overline{)91405}$
5. $47\overline{)13954}$
6. $56\overline{)22342}$
7. $74\overline{)60026}$
8. $18\overline{)16001}$
9. $75\overline{)34249}$
10. $21\overline{)9687}$
11. $89\overline{)82810}$
12. $53\overline{)23055}$
13. $27\overline{)12060}$
14. $32\overline{)7840}$
15. $67\overline{)44046}$

Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

16. A cattle rancher has 9,792 acres of land. He estimates that it takes 38 acres of land to provide grass for one cow. What is the largest number of cows he can have on his ranch?

17. There are 31 rows of seats on one side of a football field. There are seats for 6,572 people. If each row has the same number of seats, how many seats are in each row?

18. A machine made 9,503 pencils in 43 minutes. How many pencils did it make in 1 minute?

19. A book company can pack 58 books in each box. How many boxes will be needed to pack 39,018 books?

20. There were 50,902 visitors to a park in 62 days. If the same number of people visited the park each day, how many people visited the park each day?

MORE ABOUT USING HELPERS WHEN DIVIDING

Exploration

The "helpers" we use when dividing will not always lead us to a correct part of the quotient.

We will see this in an example, such as:

$$905 \div 24.$$

To try to find the first part of the quotient we can use $9 \div 2$ as a "helper," and think "4."

Is 40 the first part of the quotient?

How can you tell that 40 is too great?

$$\begin{array}{r} 4 \\ 24\,\overline{)\,905} \\ \underline{960} \end{array}$$

Let us now use 30 as the first part of the quotient.

Explain the work in the box.

$$\begin{array}{r} 3 \\ 24\,\overline{)\,905} \\ \underline{720} \\ 185 \end{array}$$

To try to find the remaining part of the quotient we can use $18 \div 2$ as a "helper," and think "9."

Is 9 the remaining part of the quotient?

How can you tell that 9 is too great?

$$\begin{array}{r} 39 \\ 24\,\overline{)\,905} \\ \underline{720} \\ 185 \\ \underline{216} \end{array}$$

Let us now use 8 as the remaining part of the quotient.

How do we know that 8 is too great?

$$\begin{array}{r} 38 \\ 24\,\overline{)\,905} \\ \underline{720} \\ 185 \\ \underline{192} \end{array}$$

Is 7 the remaining part of the quotient?

How does the work in the box show this?

We divided 905 by 24.

What is the quotient?

What is the remainder?

Is the remainder less than the divisor?

$$\begin{array}{r} 37 \\ 24\,\overline{)\,905} \\ \underline{720} \\ 185 \\ \underline{168} \\ 17 \end{array}$$

Now let us work with the example: $1915 \div 36$.

To try to find the first part of the quotient, we can use $19 \div 4$ as a "helper," and think "4." Look carefully at the work in the box. How can we know that 40 is not the greatest multiple of 10 we can use as the first part of the quotient?

$$\begin{array}{rr} & 4 \\ 36 & \overline{)\,1915} \\ & \underline{1440} \\ & 475 \end{array}$$

Let us now use 50 as the first part of the quotient. Is this the greatest multiple of 10 we can use? Explain.

$$\begin{array}{rr} & 5 \\ 36 & \overline{)\,1915} \\ & \underline{1800} \\ & 115 \end{array}$$

To try to find the remaining part of the quotient, we can use $11 \div 4$ as a "helper" and think "2."
How can we tell that 2 is not the greatest number to use for the remaining part of the quotient?

$$\begin{array}{rr} & 52 \\ 36 & \overline{)\,1915} \\ & \underline{1800} \\ & 115 \\ & \underline{72} \\ & 43 \end{array}$$

Let us use 3 as the remaining part of the quotient. Is this the greatest number we can use? Explain.
We divided 1915 by 36.
What is the quotient?
What is the remainder?

$$\begin{array}{rr} & 53 \\ 36 & \overline{)\,1915} \\ & \underline{1800} \\ & 115 \\ & \underline{108} \\ & 7 \end{array}$$

"Helpers do now always lead us to correct parts of the quotient.

Exercise Set 10

Divide

1. $75 \overline{)3156}$
2. $18 \overline{)1656}$
3. $54 \overline{)9160}$
4. $38 \overline{)4645}$
5. $37 \overline{)2539}$
6. $28 \overline{)2688}$
7. $21 \overline{)1428}$
8. $81 \overline{)3491}$
9. $93 \overline{)4876}$
10. $37 \overline{)1554}$
11. $14 \overline{)537}$
12. $58 \overline{)38918}$
13. $75 \overline{)32631}$
14. $92 \overline{)19780}$
15. $94 \overline{)58270}$
16. $75 \overline{)34149}$

Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

17. A machine produces 348 spoons an hour. How many dozen will it produce in 8 hours of continuous operation?

18. An auditorium is to be used for a meeting of 958 persons. If each row seats 21 persons, how many rows will be needed?

19. Robert reads approximately 96 words a minute. How many minutes will it take him to read a story of 1056 words?

20. A grapefruit orchard has 864 trees in 32 rows. How many trees are there in each row?

SHORTENING OUR WORK

Exploration

We can use place value to shorten our work with division examples when divisors are between 10 and 100.

Think of dividing 17836 by 45.

A.

$$\begin{array}{r} 396 \\ 45\,\overline{)\,17836} \\ \underline{13500} \\ 4336 \\ \underline{4050} \\ 286 \\ \underline{270} \\ 16 \end{array}$$

B.

$$\begin{array}{r} 396 \\ 45\,\overline{)\,17836} \\ \underline{135} \\ 4336 \\ \underline{405} \\ 286 \\ \underline{270} \\ 16 \end{array}$$

C.

$$\begin{array}{r} 396 \\ 45\,\overline{)\,17836} \\ \underline{135} \\ 4336 \\ \underline{405} \\ 286 \\ \underline{270} \\ 16 \end{array}$$

Does $17836 = (396 \times 45) + 16$?

Explain how Form B is shorter than Form A.

Explain how Form C is shorter than Form B.

Exercise Set 11

Divide. Use the shortest form you can.

1. $77\overline{)565}$
2. $32\overline{)2176}$
3. $19\overline{)7300}$
4. $58\overline{)7441}$
5. $29\overline{)9365}$
6. $86\overline{)43688}$
7. $18\overline{)6804}$
8. $86\overline{)27413}$
9. $58\overline{)39092}$
10. $28\overline{)15288}$
11. $92\overline{)45310}$
12. $14\overline{)7116}$
13. $25\overline{)14345}$
14. $73\overline{)61366}$
15. $19\overline{)7330}$

Use mathematical sentences to help solve the following problems. Express each answer in a complete sentence.

16. The committee has 685 tickets for the school play. They put 15 tickets in each package. How many packages of tickets did they have? Were there any left over? If so, how many?

17. Mr. Jones sold 32 television sets for $11,040. If these were all of the same model, what was the price of one set?

18. Ann wants to make 12 curtains. She needs 42 inches of material for each curtain. How many yards of material does she need?

19. The Boy Scouts were having a party. Their mothers baked 134 cupcakes for the party. If each of the 67 boys had the same number of cupcakes, how many would each boy eat?

20. Jean packed 288 oranges into boxes. If each box holds 36 oranges, how many boxes did she fill?