STUDENT'S TEXT

UNIT NO. 43

INTRODUCTION TO ALGEBRA

PART I

SCHOOL MATHEMATICS STUDY GROUP

Introduction to Algebra

Student's Text, Part I

REVISED EDITION

Prepared under the supervision of
a Panel consisting of:

V. H. Haag	Franklin and Marshall College
Mildred Keiffer	Cincinnati Board of Education
Oscar Schaaf	South Eugene High School, Eugene, Oregon
M. A. Sobel	Montclair State College, Upper Montclair, New Jersey
Marie Wilcox	Thomas Carr Howe High School, Indianapolis, Indiana
A. B. Willcox	Amherst College

Stanford, California
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FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

Chapter

Chapter

PREFACE

To The Student:

Have you ever thought of mathematics as something that you could read? Or has mathematics always meant only problems for you to work? Here is a textbook which is written for you to read. It is not just a list of problems.

Reading mathematics is not the same as reading a story. You will find that you may have to read a paragraph several times before the meaning becomes clear. Sometimes you will find it necessary to use paper and pencil to work examples. Occasionally you will need to ask for help.

Careful reading of this textbook will help you understand some of the important ideas of mathematics. You will need to understand these ideas in order to work the problems.

You have a new and enriching experience ahead of you. Make the most of it.

Chapter 1

SETS AND THE NUMBER LINE

1-1. Sets.

This mathematics book is going to begin in a strange way!

We will start by thinking and talking about collections of objects. The idea of a collection or group of things is a familiar one. We are using this idea when we speak of

a herd of cattle,
a flock of geese,
a crowd of people,
a swarm of bees,
a bunch of bananas.

Can you give some more examples?

A carpenter uses a set of tools;
A golfer plays with a set of clubs;
A waiter drops a set of dishes!

There are other ways of talking and thinking about collections. Take the things in your pocket. You could describe this collection by making a list, such as

{pencil, dime, penny, handkerchief, gum}.

As a further example, consider the names of all the cars in a certain parking lot

{Ford, Chevrolet, Plymouth, Buick, Valiant, Lark},

or simply a collection of numbers

$$\{1, 2, 3, 4, 5\}.$$

As we have said, the notion of a collection or set is a familiar one. It may come as a surprise to you to discover that this simple idea is going to be a big help to you in the study and understanding of mathematics. To make things easy, we will use one word in place of all the others like herd, flock, crowd, or bunch. This word is set.

The various sets mentioned above contained such things as insects, animals, cars, people, or numbers. As a general rule we will call the objects in any given set "elements." In the mixed-up set of things in your pocket the pencil is an "element." So is the dime and each of the other articles.

We need some definite way of showing that we are describing sets. If the "elements" can be listed, we will write the list and enclose it within these marks { }, called braces. It will then be clear that when we read such things as

$$\{2, 4, 6, 8, 10\}$$

or $\{$California, Colorado, Connecticut$\}$

or $\{2, 3, 5, 7, 8\}$,

we are talking about sets.

Check Your Reading

1. What are some other words for the word "set"?
2. What do we mean by "element of a set"?

Problem Set 1-1a

1. List the elements of the set of whole numbers greater than 1 and less than 50 that end in
   (a) 9
   (b) 3
   (c) 0
2. In problem 1, which set has the least number of elements?
3. (a) List the elements of the set of letters of the alphabet from d to j, including d and j.
   (b) List the set of vowels in the name "Rhode Island."
   (c) List the set of letters that occur more than once in "Mississippi."
4. List the elements of the set of all common fractions with denominators 2, 3, or 5 that are between (means "not including")
   (a) 0 and 1,
   (b) 1 and 2,
   (c) 2 and 3.

Problem Set 1-1a
(continued)

*5. List the elements of the set of all the states whose names begin with
   (a) the letter C,
   (b) the letter N,
   (c) the letter H,
   (d) the letter B.

This question comes up, "To describe a set is it always necessary to make a list?" Think it over! Suppose the set you were working with was the set of all people living in your town--or the set of all numbers from 1 to 1000. In both cases making lists would be too much work and probably not worth it.

To get around such difficulties we will agree that sets may also be indicated by giving a verbal description of the elements. To show two methods clearly, we will begin with cases where both methods can be used. Check the following examples carefully. See if you think the two methods indicate the same set.

List:	{2, 4, 6, 8, 10}
Verbal description:	all even numbers between 1 and 11

List:	{California, Colorado, Connecticut}
Verbal description:	all states in the U. S. A. whose names begin with C

Why have two ways? As we said before, there are times when one of the ways may be impossible--or nearly so. Here are examples.

Verbal description:	the set of all types of insects
List:	(This is possible, but very difficult.)

Verbal description:	the set of all odd numbers
List:	(You could start this list, but don't try to finish it!)

In some cases there is a way to list a set even though the number of elements is very large. Suppose we want the set of all odd numbers. We begin by listing the first few elements in the set as {1, 3, 5, 7. Then to show that we want to include all the other odd numbers, we write the complete set as

$$\{1, 3, 5, 7, \ldots\}.$$

If we want to show that the set ends at some definite place, we could write

$$\{1, 3, 5, 7, \ldots, 25\}.$$

Here the verbal description is, "The set of all odd numbers up to 25 including 25."
Another example might be

$$\{5, 10, 15, 20, \ldots, 50\}.$$

Can you describe this set in words?

We must be very careful to see that the elements or numbers which we do write show clearly what we mean. It would be very confusing to see something like

$$\{4, 2, 11, 20, \ldots\}$$

since there would be no way of telling what the rest of the numbers are supposed to be.

Two very important sets can be described clearly in this way. One of these is the set

$$\{1, 2, 3, 4, \ldots\}.$$

We call this set the set of all

<u>counting numbers.</u>

If we take the set of all counting numbers and include with this set the number 0, we obtain the set

$$\{0, 1, 2, 3, \ldots\}.$$

This new set we call the set of all

<u>whole numbers.</u>

For convenience we can use a capital letter as a name for any particular set in which we may be interested. For example,

$$N = \{1, 2, 3, 4, \ldots\}$$

is a way of saying that we are using N as a name for the set of all counting numbers. When we write

$$B = \{1, 2, 3\},$$

we are giving the set of numbers 1, 2, and 3 the name B.

Let us now consider the set of all whole numbers. We can call the set W. That is, let

$$W = \{0, 1, 2, 3, \ldots\}.$$

Now suppose we form a new set. Let the elements in this new set be the numbers which we would get if we multiplied each element in the set W by the number 3. Since $3 \times 0 = 0$, $3 \times 1 = 3$, $3 \times 2 = 6$, etc., our new set, which we can call T, would look like this:

$$T = \{0, 3, 6, 9, 12, \ldots\}.$$

This newly formed set we call

the set of all <u>multiples</u> of 3.

Thus $$F = \{0, 5, 10, 15, \ldots\}$$

is the set of all multiples of 5.

The set of all multiples of 2,

$$E = \{0, 2, 4, 6, \ldots\},$$

we call the set of all <u>even</u> numbers.

By adding 1 to each element in the set E we get the set of <u>odd</u> numbers

$$G = \{1, 3, 5, 7, \ldots\}.$$

In each of the above examples three dots have been used to replace the missing elements. This is the usual procedure and does not imply that exactly three elements of the set have been left out.

You may also have noticed that the description of set T above includes 5 actual numerals while set W has only 4. There is no general rule for this. We usually include enough elements to make completely clear what the others are supposed to be.

Check Your Reading

1. Tell two ways of describing a set.
2. What is the meaning of three dots (. . .) in a set description?
3. Describe the set N of counting numbers.
4. Describe the set W of whole numbers. How does it differ from the set N?
5. What is a multiple of a number? Describe the multiples of 5.
6. Describe the set of even numbers.
7. How can we get the set of odd numbers from the set of even numbers?

Problem Set 1-1b

1. Describe each of the following sets by means of a list.
   (a) All counting numbers up to and including 12
   (b) All whole numbers up to and including 10
   (c) All whole numbers greater than 10
   (d) All multiples of 7 which are less than 50
   (e) All multiples of 3 which are less than 30
   (f) All even numbers less than 14
   (g) All even numbers greater than 14
   (h) All odd numbers greater than 10
   (i) All odd numbers less than 40
   (j) All common fractions with denominator 7 with value less than 1 and greater than 0.

2. Give verbal descriptions for the following sets.
   (a) A = {0, 2, 4, 6}
   (b) B = {1, 3, 5, 7, . . .}
   (c) C = {0, 6, 12, 18, . . .}
   (d) D = {0, 1, 2, 3, . . .}
   (e) E = {0, 4, 8, 12, . . .}
   (f) F = {12, 13, 14, . . .}
   (g) G = {23, 25, 27, . . .}
   (h) H = {1, 3, 5, . . ., 17}
   (i) I = {0, 6, 12, . . ., 66}
   (j) J = {28, 30, 32, . . ., 98}
   (k) K = {September, April, June, November}
3. Form a set by dividing each element in the set of even numbers by 2. What is the name of the set which you obtain in this way?

Suppose we are given two sets, A and B. If every element of the set B is also an element of the set A, we say that

the set B is a <u>subset</u> of the set A.

To obtain a clear idea of the meaning of <u>subset</u> let us look at the two sets

$$A = \{2, 4, 6, 8, 10\},$$

and

$$B = \{2, 4, 6, 8\}.$$

It should be clear that every element in the set B is also an element in the set A. In other words we cannot find an element of B which is <u>not</u> an element of A. The set B therefore is a <u>subset</u> of the set A.

Now consider the set consisting of all men who have been president of the United States. Call this the set P. Let

Q = {Washington, Lincoln, Eisenhower}.

Is every element of the set Q an element of the set P also? Do you see that the set Q is a <u>subset</u> of the set P?

Again let us take the following two sets. Let C be the set of all counting numbers from 1 to 20 inclusive. (The word inclusive means that both 1 and 20 are included in the set.) Thus

$$C = \{1, 2, 3, \ldots, 20\}.$$

Then let $$D = \{5, 6, 7, 8, 9, 10\}.$$

Would you say that D is a subset of C? Remember that the three dots in the set C indicate that some of the elements of C are not listed. By this time it should also be clear that the set C is <u>not</u> a subset of the set D, since there are elements in C which are not in D. The element 20 is one of these, so is 15. Can you find others?

Now comes an interesting question. Can we say that any set is a <u>subset</u> of itself? Look carefully at the statement about the word <u>subset.</u> It says that B is a subset of A when every element of B is also an element of A.

Consider any set, for example the set

$$A = \{1, 2, 5, 7, 8\}.$$

Now ask, "Are each of the elements 1, 2, 5, 7, 8 also in the same set A?" Is there any element in A which is <u>not</u> in A? These questions may sound foolish, but the obvious answers give us the following rule:

Every set is a subset of itself.

<u>Check Your Reading</u>

1. What is one subset of the set of all Presidents of the U. S.?
2. How can we tell if a set is a subset of another set?
3. Is a set a subset of itself? Why?

<u>Problem Set 1-1c</u>

1. Given the set $A = \{1, 2, 3, 4, \ldots, 25\}$. Which of the following sets are subsets of A? Give a reason for those which you decide are not subsets of A.

   B: The set of all counting numbers from 1 to 10.

   $C = \{1, 2, 3, 4, 30\}$

Property, in the most familiar sense of the word, is something you have. A property of one is something the number one has; that is, a characteristic of the number one. A similar usage of the word would be "sweetness is a property of sugar."

You have known this property of one for a long time but perhaps you have never thought about how useful it is.

All the numerals below, and many more, are names for the number 1.

$$\frac{2}{2},\ \frac{8}{8},\ \frac{3}{3},\ \frac{5}{5},\ \frac{31}{31}$$

Any number may be given different names by multiplying by one, using different names for one.

$\frac{1}{2}$ may be written as "$\frac{1}{2} \times \frac{3}{3}$", and as "$\frac{3}{6}$".

$\frac{1}{2}$ may also be written as "$\frac{1}{2} \times \frac{5}{5}$", and as "$\frac{5}{10}$".

5 may be written as "$5 \times \frac{8}{8}$", and as "$\frac{40}{8}$".

$\frac{3}{8}$ may be written as "$\frac{3}{8} \times \frac{2}{2}$", and as "$\frac{6}{16}$".

In each of the examples above, the fraction at the right is another numeral for the number at the beginning of the line. You may also have noticed that using the multiplication property of one amounts to "multiplying numerator and denominator by the same number," as you may have said in elementary school. There is nothing wrong with speaking in this way, but you should realize that it is the multiplication property of one that makes it correct; it is not really "another way" at all. In this course, since we are going to emphasize properties of numbers, we shall always speak of the process in terms of using the multiplication property of one.

Problem Set 1-1c
(continued)

D = {1, 2, 3, 4, . . ., 25}

E: The set of all even numbers between 1 and 25.

F: The set of all multiples of 3 which are less than 25.

G = {1, 3, 5, 7, . . .}

H = {1}

2. Let S = {1, 2, 3, 4}.
Form a new set T by multiplying each element in S by itself. We call this set T the set of all <u>squares</u> of elements in the set S.
Now make a list for a set R where R is the set of all elements which are in S <u>and</u> in T. (This means that for an element to be in R, it is necessary for it to be in S and also in T.) Then answer the following questions.
   (a) Is 2 an element of R?
   (b) Is R a subset of T? If your answer is "No," give a reason.
   (c) Is R a subset of S? If your answer is "No," give a reason.
   (d) Is T a subset of S? If your answer is "No," give a reason.

3. Make a list for a new set K where K has all the numbers in it which are either in S <u>or</u> in T from problem 2. In other words, if a number is in S, put it in the new set K. If a number is in T, put that in K also. (We never include the same element more than once in a set. For example, K should have only one 4 in it.) Now answer the following questions about the sets S, T, R, and K:
   (a) Which of the sets S, T, R, and K are subsets of K?
   (b) Which of the sets S, T, R, and K are subsets of R?
   (c) Which of these sets has the most elements in it?
   (d) Which of these sets has the fewest elements in it?

4. (a) If we add any two odd numbers, will their sum be an odd number ? Give the reason for your answer.
   (b) If we multiply any two odd numbers, will their product be an odd number? Give the reason for your answer.

5. Consider the set T = {1, 2, 3, 4}. If we select any element of this set and add to it any element of the set (including the same element), what is the set S of all possible sums? Is S a subset of T? Why?
6. Consider the set Q = {0, 1}. Choose any element of Q and multiply it by any element of Q including the same element. What is the set P of all possible products? Is P a subset of Q?
7. Consider the set R = {0, 1, 2}. Find the set S of all possible sums and the set P of all possible products of pairs of elements of R (including an element and itself) as we did in problems 5 and 6. Is S a subset of R? Is P a subset of R?
8.* If the set resulting from an operation such as the addition or multiplication of pairs as we did in problems 5, 6, and 7 is a subset of the original set, then we say the set is "closed under that operation."
   (a) Is set T of problem 5 "closed under addition"?
   (b) Is set Q of problem 6 "closed under multiplication"? under addition"?
   (c) Is set R of problem 7 "closed under addition"? under multiplication"?
   (d) Is the set N of all counting numbers "closed under addition"? "under multiplication"?

Here is another question. Suppose we were to describe a few sets by using the following words:

Let S be the set of all counting numbers which are less than 5 and at the same time larger than 6.

Let D be the set of all women who have been President of the United States.

Let G be the set of all giraffes who own sports cars.

Let B be the set of all even numbers which are elements of the set {1, 3, 5, 7}.

Notice that we have used the words set in talking about S, D, G, and B. The question, "Are they really sets?"

Two thoughts might occur to you. The first is that they are not sets at all because they do not have any elements. The second idea is that they are sets because they have been described.

Which is right? To settle the argument we will agree that each of these is to be called a set. We shall give this type of set a special name. We call it

the <u>null</u> set

or the <u>empty</u> set

To abbreviate we use the symbol $\emptyset$ to represent this particular set.

> Warning! The set $Z = \{0\}$ is not the same as the null set. It is not empty. It contains one element; the element is the whole number 0.

There is one special thing about the set $\emptyset$ which is important to remember. It is a <u>subset</u> of <u>every</u> set. Why is this true? An example will help us understand this idea. Let A be any set. For example suppose $A = \{1, 2, 3, 4\}$. Now consider the set $\emptyset$. Can we find an element of $\emptyset$ which is <u>not</u> in A? Since $\emptyset$ does not have any elements, the answer is no. Thus $\emptyset$ is a subset of A. Do you see that this would also be true no matter what set is chosen in place of A?

Check Your Reading

1. What do we call a set which contains no elements?
2. What symbol do we use to represent the empty set?

Problem Set 1-1d

1. Which of the following sets would you call by the name $\emptyset$?
   (a) A: The set of all elements which are in both the set $D = \{2, 4, 6, 8\}$ and the set $F = \{1, 2, 3\}$

(b) B: The set of all elements which are in both the set $G = \{2, 5, 7, 9\}$ and the set $H = \{3, 4, 8\}$

(c) The set of all whole numbers which are in the set $\{\frac{2}{3}, \frac{3}{4}, \frac{1}{5}\}$

(d) The set of all whole numbers which are in both the set $\{0, 3, 5\}$ and the set $\{0, 2, 4, 6\}$

(e) The set of all whole numbers which are greater than 6 and less than 7

(f) The set of all whole numbers which are less than one.

2. If you were asked to describe all the sets which are subsets of the set $A = \{1, 2\}$, the correct answer would be:

A has these four subsets:

$\emptyset$ (Remember the null set is a subset of every set)

$\{1\}$

$\{2\}$

$\{1, 2\}$

Now describe all subsets of the set

$B = \{1, 2, 3\}$. How many are there?

*3. If you are really brave, try describing all subsets for the set

$C = \{1, 2, 3, 4\}$. How many are there?

*4. Can you suggest a short-cut for counting the number of subsets of a given set?

1-2. <u>The Number Line</u>.

In your study of arithmetic you began by using the numbers to count. You also worked with rulers and scales, which can be thought of as lines with certain points marked on them. We will now make a connection, or association, between numbers and points on a line.

First we draw a line.

Then we choose two separate points on the line and mark the one on the left with a 0 and the one on the right with a 1.

Using the distance between these two points as a measure, mark off other points to the right of 1 all the same distance apart.

We think of this process as continuing without end even though we cannot show the process beyond the edge of the page. An "etc." at the right of each line indicates this endless process. Now label the points to the right of 1 marking each point in succession with the next whole number.

We call each number the successor of the one on its left. Thus 3 is the successor of 2, 8 is the successor of 7, 51 is the successor of 50, and so forth. What is the successor of 105? of 100,000,005? It is easy to see that the successor of any whole number can be found by adding 1.

You can always add 1 to any number. So every whole number has a successor. Therefore, there cannot be a largest whole number.

What about the set of all whole numbers? If one tried to count the elements in this set, the counting could not possibly come to an end.

If, on the other hand, the elements in any given set can be counted with the counting coming to an end or if the set is the null set, we call it a finite set. If not, as in the case of the set of all whole numbers, we call it an

infinite set.

We say that such a set has infinitely many elements.

It is very important not to confuse the idea of an infinite set with the idea of a very large set. For example, the set of

all counting numbers between 1 and ten billion is not infinite.

What would you say about the set of all grains of salt in a barrel?

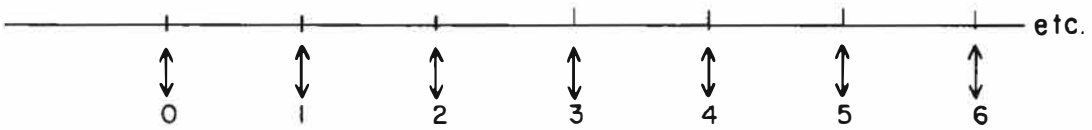
It might be helpful to think of our method of listing sets as a means of distinguishing infinite sets from those which are not. In describing an infinite set one would place the dots on the right.

$$\{1, 2, 3, 4, \ldots\}$$

If a set were not infinite, the final number would be written at the right. Thus, the set of all counting numbers from 1 to 1000 inclusive would be written

$$\{1, 2, 3, 4, \ldots, 1000\}.$$

To return to our line, we must understand that every whole number is now associated with a point on the extended line. Also, every point that we have located so far on this line is matched with a whole number. A matching of points and numbers like this



is an example of what we call a correspondence. The matching of the elements of one set with the elements of another set is a <u>correspondence</u> of the two sets.

We shall call a line on which points are labeled with numbers a <u>number line</u>. On a number line the number associated with the point is called the <u>coordinate</u> of the point.

Check Your Reading

1. Describe a number line. How is it labeled?
2. What is the successor of a whole number?
3. How many whole numbers are there?
4. What is an infinite set?

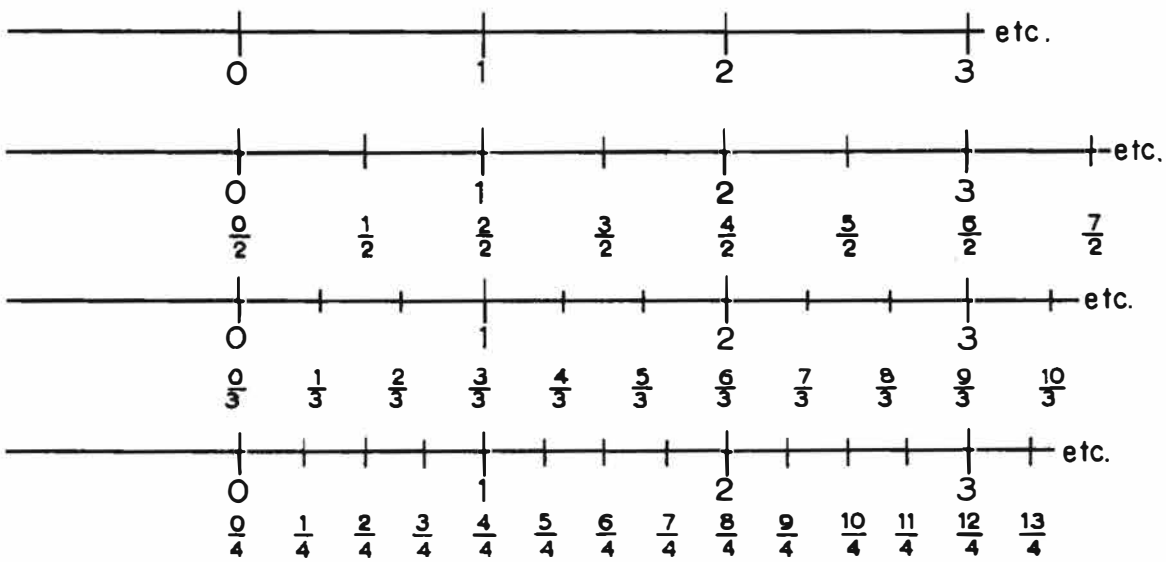
Check Your Reading
(continued)

5. What is a finite set?
6. What do we mean by a correspondence between a set of points and a set of numbers?

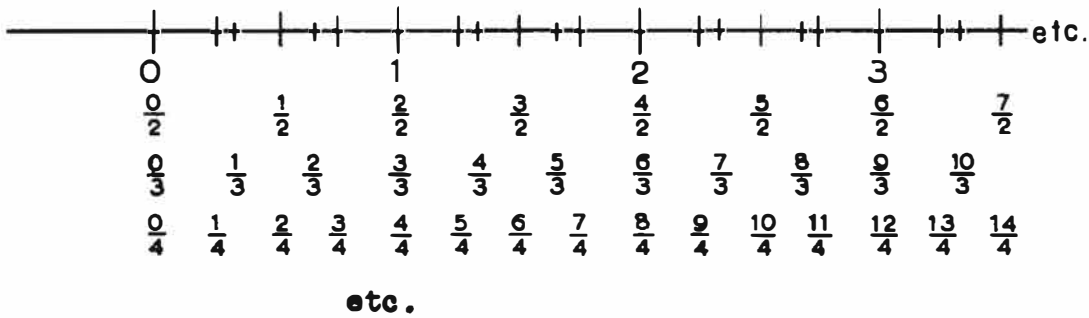
Problem Set 1-2a

1. Describe the set S of successors of the elements of set $F = \{0, 5, 10, \ldots, 45\}$.
2. Classify the following sets (finite or infinite):
   (a) All odd numbers between 0 and 100.
   (b) All whole numbers.
   (c) The squares of all counting numbers.
   (d) All citizens of the United States.
   (e) All counting numbers less than one billion.
   (f) All counting numbers greater than one billion.

Starting with the line on which certain points are labeled with whole numbers, we can label other points by dividing the distance between the points into halves, thirds, fourths, etc.



We should think of this process of dividing our line into smaller and smaller pieces as continuing without end. The "etc." below the diagram indicates an endless process. Now put all the points together on one line and we have a labeling like this



of points corresponding to a set of numbers. This line is another example of a <u>number line.</u>

At this time let us review what is meant by a "fraction." Notice that the coordinate of the point on the number line which corresponds to 2 has many names:

$$\frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \qquad \text{etc.}$$

Each of these is a <u>fraction,</u> and each is a different name for the same number. The number 3 can also be represented by fractions:

$$\frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \qquad \text{etc.}$$

Some fractional forms of $2\frac{1}{2}$ would be

$$\frac{5}{2}, \frac{10}{4}, \frac{25}{10}.$$

In general, we shall mean by a fraction a name or symbol which represents the quotient of two numbers. Remember that "quotient" is the word used to describe the result of a division of one number by another. Thus the fraction $\frac{4}{2}$ stands for the number 4 divided by the number 2. In all cases the number on the

bottom, often called the denominator, is the number we are dividing by.

A number which we can write as a fraction where this fraction shows the quotient of two whole numbers is called a <u>rational number.</u> Division by zero is not included.

$$0,\ \frac{1}{2},\ \frac{2}{3},\ 2,\ 2\frac{1}{2},\ .6,$$

are examples of rational numbers since

$$\begin{aligned} 0 &\text{ may be written as } \frac{0}{1}, \\ 2 &\text{ may be written as } \frac{2}{1}, \\ 2\frac{1}{2} &\text{ may be written as } \frac{5}{2}, \text{ and} \\ .6 &\text{ may be written as } \frac{6}{10}. \end{aligned}$$

How would you write 3.4, 7, $2\frac{2}{3}$, and 4.5 as quotients of whole numbers?

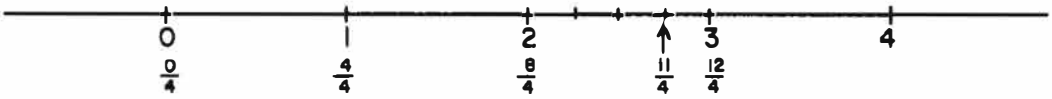
The numbers which can be written in the way described above do not make up the complete set of rational numbers. There are others which we shall meet later. We will find these when we study numbers which correspond to points to the left of 0 on our number line.

From these ideas we can see that the set of whole numbers is a <u>subset</u> of the set of rational numbers. A whole number is a rational number. A rational number is not necessarily a whole number. For example, $\frac{2}{3}$ is a rational number, but not a whole number.

Suppose we are given any rational number of the type we have been talking about. By our description we can represent this as the quotient of two whole numbers. As illustrated above, we can locate the point on the number line which corresponds to this quotient. For an example take the rational number $2\frac{3}{4}$. As a quotient of two whole numbers, we write this

$$\frac{11}{4}.$$

On the number line the point can be located as follows:



A big question is coming up. When we get the answer we shall have discovered a very important mathematical fact. The question can be stated as follows:

> "If we are given two rational numbers, can we always find a third rational number between the two given ones?"

By means of an example we shall learn a general method for finding such a number. Suppose we start with the two numbers

$$\frac{1}{3} \text{ and } \frac{1}{2}.$$

Is there a rational number between these two? Let us write each of these numbers in a different way. We see that $\frac{1}{3}$ has the names

$$\frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \text{ etc.}$$

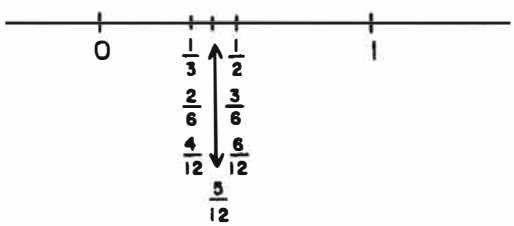
and $\frac{1}{2}$ has the names

$$\frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{6}{12}, \text{ etc.}$$

Using the name $\frac{4}{12}$ for $\frac{1}{3}$ and $\frac{6}{12}$ for $\frac{1}{2}$, it is now easy to see that there is a number between

$$\frac{4}{12} \text{ and } \frac{6}{12}.$$

One such number is $\frac{5}{12}$.
Since we have a number, we also have a point on the number line. The process for finding the number and locating the point is illustrated by the picture.



Now that we have found the new number $\frac{5}{12}$, let us ask the same question about the numbers $\frac{4}{12}$ and $\frac{5}{12}$. Is there a rational number between $\frac{4}{12}$ and $\frac{5}{12}$?

Again,

$\frac{4}{12}$ may be written as $\frac{8}{24}$

and $\frac{5}{12}$ may be written as $\frac{10}{24}$

Certainly $\frac{9}{24}$ lies between $\frac{8}{24}$ and $\frac{10}{24}$.

This process of finding a number between numbers can be carried out for any two numbers no matter how close. Thus we can locate a third point between any two points. Here, then, is a surprising fact.

> There are not only a great many points between any two given points, but infinitely many.

It should now be clear that every rational number corresponds to a point on the number line. What about asking the question the other way around? Does every point on the number line to the right of 0 correspond to a rational number? The answer to this question may come as a shock. It is,

N O !

Later we shall explain this fact to you.

Meanwhile, we assume that every point to the right of 0 has a coordinate. The set of numbers consisting of 0 and all numbers corresponding to points to the right of 0 is called the set of numbers of arithmetic. In chapters 1 through 5 we shall be concerned with this set.

Check Your Reading

1. What is the meaning of the word fraction?
2. Give some examples of rational numbers. Is the set of whole numbers a subset of the set of rational numbers?

Check Your Reading
(continued)

3. How can we find a rational number between the rational numbers $\frac{1}{3}$ and $\frac{1}{2}$? Between $\frac{4}{12}$ and $\frac{5}{12}$?
4. Is it possible to locate a point between any two points on a number line?
5. Does every point on the number line have a coordinate that is a rational number?

Problem Set 1-2b

1. Draw a number line for each of the following. Label the points whose coordinates are 0 and 1 in such a way that there will be space on your paper to label the point whose coordinate is 5. Then label the points whose coordinates are
   (a) 2, 3, 4, 5.
   (b) $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, $\frac{9}{2}$.
   (c) 0, .5, .7, 1.1, 1.5, 1.8, 2.0, 2.7, 3.5, 4, 4.4.
2. Circle all points labeled in 1a, b and c that have whole number coordinates.
3. Draw a number line more than 6 inches long on your paper. Near the left margin mark a point "0", 6 inches from zero mark a point "1". Using your ruler mark all the points between 0 and 1 which correspond to fractions whose denominators are 2, 3, 4, 6, 8, 12.
   (a) What is the largest rational number you have represented on the number line between 0 and 1? Can you suggest one larger? What is the smallest? Can you suggest one smaller?
   (b) What names do you have for the coordinate of the point midway between 0 and 1?
   (c) Name a rational number between $\frac{1}{12}$ and $\frac{2}{12}$. Show how you could find it without reference to the number line.
   (d) If you were to continue the process you started, for how many rational numbers between 0 and 1 would you be able to find points on the number line?

Problem Set 1-2b
(continued)

4. (a) How many rational numbers are there between 2 and 3? between $\frac{2}{500}$ and $\frac{3}{500}$?

   (b) List two rational numbers between 2 and 3; between $\frac{2}{500}$ and $\frac{3}{500}$.

   (c) What is the <u>next</u> rational number after 2?

5. Write three other names for the coordinate of the point associated with $\frac{20}{5}$.

6. Write six numerals which could be used as names for the coordinate of the point associated with $\frac{3}{4}$.

7. On the number line we see that some points lie to the right of others, some to the left of others, some between others. How is the point with coordinate 3.5 located with respect to the point with coordinate 2? Compare 3.5 to 2. How is the point with coordinate 1.5 located with respect to the point with coordinate 2? Compare 1.5 to 2.

8. Classify the following sets (finite or infinite):

   (a) All rational numbers between 1 and 2 whose numerators are whole numbers and whose denominators are whole numbers between 1 and 10.

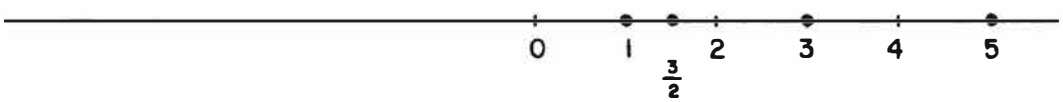
   (b) All rational numbers between 1 and 4.

   (c) All rational numbers between 1 and 2 whose numerators are between 1 and 10.

Let us return to the idea of a set of numbers and imagine a set on the number line. For example, each element of the set

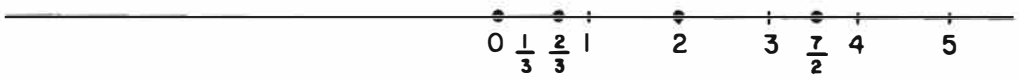
$$A = \{1, \tfrac{3}{2}, 3, 5\}$$

is a number associated with a point on the number line. We call this set of associated points the <u>graph</u> of the set A. Let us indicate the points of the graph by marking them specially with heavy dots:

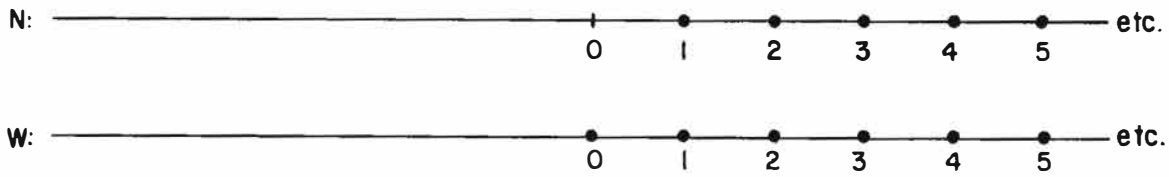


Thus the graph of a set of numbers is the corresponding set of points on the number line whose coordinates are the numbers of the set, and only those points.

For the second example let $B = \{0, \frac{2}{3}, 2, \frac{7}{2}\}$. The graph is the set of heavy dots shown below.



In passing, we note that the graphs of the set N of counting numbers and the set W of whole numbers are:



From these graphs we see immediately that N is a subset of W.

Check Your Reading

1. What do we mean by the graph of a set A?
2. How do we indicate points of a graph on a number line?

Problem Set 1-2c

1. Draw the graphs of the following sets:
   (a) A: The set of all counting numbers less than 5.
   (b) B: The set of all whole numbers less than 10 which are squares of whole numbers.
   (c) $C = \{0, \frac{1}{2}, \frac{4}{3}, \frac{5}{2}, 3\}$.
2. Given the sets $S = \{0, 3, 4, 7\}$ and $T = \{0, 2, 4, 6, 8, 10\}$
   (a) List the set K, the set of all numbers which belong to S <u>and</u> to T. List the set M, the set of all numbers which belong to S <u>or</u> to T.

(b) Draw four number lines, one below the other. On the first line show the graph of the set S; on the second the graph of T; on the third the graph of K; and on the fourth the graph of M.

(c) What way do you see for getting the graphs of K and M from the graphs of S and T?

3. Consider the sets A = {0, 5, 7, 9} and B = {1, $\frac{5}{2}$, 8, 10}.

(a) Draw two number lines, one below the other, and on these lines put the graphs of A and B.

(b) If set C is the set of numbers which are elements of A and of B, what can you say about set C from looking at the graphs of A and B?

(c) What is the name of the set C?

Summary

In this chapter we have studied the following important ideas, terms, and definitions.

1. Braces - Marks used to enclose the list of elements of a set.
2. Coordinate - The number associated with a particular point on the number line.
3. Correspondence - The pairing of the elements of one set with the elements of another set.
4. Counting number - An element of the set: {1, 2, 3, 4, 5, . . .}.
5. Elements - the objects in a set.
6. Even number - An element of the set: {0, 2, 4, 6, . . .}.
7. Finite set - If the elements of a set can be counted with the counting coming to an end or if the set is the null set, we call it a finite set.
8. Fraction - A symbol which represents the quotient of two numbers.
9. Infinite set - When the elements of a set can not be counted, that is, with the counting coming to an end, the set is said to be infinite. The exception to this is the null set which is a finite set.

10. Infinitely many - An infinite set has infinitely many elements.
11. Multiple of a number - A number obtained by multiplying an element of the set of whole numbers by the given number.
12. Null set - a set which has no elements. This is also referred to as the empty set and is denoted by $\emptyset$.
13. Number line - a line whose points have been labeled with numbers.
14. Odd number - An element of the set: {1, 3, 5, 7, 9, . . .}.
15. Rational number - A number which can be represented by a fraction indicating the quotient of two whole numbers, excluding division by zero, is called a rational number. (Such numbers do not make up the complete set of rational numbers.)
16. Set - A collection of objects.
17. Subset - If every element of a set B is an element of a set A, then set B is a subset of set A.
18. Successor - The successor of any whole number is found by adding one to the given whole number.
19. Whole number - An element of the set: {0, 1, 2, 3, 4, 5, . . .}.

Review Problem Set

1. List the set whose elements are the whole numbers which can be divided exactly by 3 and are less than 50.
2. List the set of multiples of 3 which are less than 50.
3. List the set of multiples of 6 which are less than 50.
4. Is the set of multiples of 3 a subset of the set of multiples of 6? Show why your answer is correct.
5. Give a verbal description of the set: {10, 12, 14, 16, . . .}.
6. Give a verbal description of the set: {7, 9, 11, . . ., 59}.
7. Describe the set whose elements are the numbers greater than 113 <u>and</u> less than 112. (Remember the explanation in problem 2 on page 9.)
8. How many elements has each of the following sets?
   (a) The set of all whole numbers from 10 to 27, inclusive.
   (b) The set of all odd numbers between 0 and 50.
   (c) The set of all multiples of 3.

Review Problem Set
(continued)

(d) The set of all multiples of 3 from 0 to 99, inclusive.

(e) The set of all multiples of 10 from 0 to 1,000 inclusive.

(f) The set of all counting numbers.

(g) The set of all counting numbers greater than one billion.

9. Given the sets S = {5, 7, 9} and T = {0, 2, 4, 6, 8, 10}.

(a) List K, the set of all numbers belonging to S <u>and</u> to T. Is K a subset of S? of T? Are S, T, K finite?

(b) List M, the set of all numbers each of which belongs to <u>S or</u> to T. (Remember the explanation in problem 3 on page 9.) Is M a subset of S? Is T a subset of M? Is M finite?

(c) List R, the subset of M which contains all the odd numbers in M. Of which others of our sets is this a subset?

(d) List A, the subset of M which contains all the elements of M which are multiples of 11 greater than zero. Did you find that A has no elements? What do we call this set?

(e) Are sets A and K the same? If not, how do they differ?

(f) From your experience with the last few problems, could you draw the conclusion that subsets of finite sets are also finite?

(g) Let D be the set of all rational numbers from 0 to 10, inclusive. Is D a finite set? Is S a subset of D? Can you conclude that some infinite sets have finite subsets? Do you think all **infinite** sets have some finite subsets? Is D a subset of D? Can an infinite set have an infinite subset?

10. On a number line indicate with heavy dots the points whose coordinates are

(a) $\frac{1}{3}, \frac{3}{3}, \frac{5}{3}, \frac{7}{3}, \frac{9}{3}, \frac{11}{3}, \frac{13}{3}$.

Review Problem Set
(continued)

(b) rational numbers which can be written as common fractions with denominator 6, beginning with $\frac{1}{6}$ and ending with $\frac{20}{6}$. List the set C of numbers which are coordinates of points common to both graphs. Which elements of set C are whole numbers? Which are counting numbers? Which are rational numbers?

11. Between which successive whole numbers do you find $\frac{22}{7}$? Is $\frac{22}{7}$ greater than 3.1? Does the point with coordinate $\frac{22}{7}$ lie to the left of the point with coordinate 3.1? Between what two numbers, expressed in tenths, does $\frac{22}{7}$ lie?

*12. Another name for a counting number is <u>natural number.</u> Every finite set S is related to a finite set T of natural numbers in the following way. There can be found a correspondence between the elements of S and the elements of T in which each element of S corresponds to exactly one element of T and each element of T corresponds to exactly one element of S. Such a correspondence is called <u>one-to-one.</u> For example, the set of all letters of the English alphabet can be put in one-to-one correspondence with the set {1, 2, 3, . . ., 26}. We illustrate one such correspondence by the pairings:

$$\begin{array}{ccccccc} 1, & 2, & 3, & 4, & . . ., & 26 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow \\ a, & b, & c, & d, & . . ., & z \end{array}$$

An infinite set, however, cannot be put into one-to-one correspondence with a finite set. Furthermore, an infinite set Q has the surprising property that a one-to-one correspondence can be found between Q and a <u>proper</u> subset of Q. (A proper subset of Q is not $\emptyset$ and is a subset that does not contain all the elements of Q). How can the infinite set of whole numbers be put into one-to-one correspondence with the set of all multiples of 3 (which is a proper subset of the set of whole numbers)?

Chapter 2

NUMERALS, SENTENCES, AND VARIABLES

2-1. Numbers and Their Names.

We all know people who have several names. For example, John Jones, Jr. may be called:

"Junior" at home,
"Johnny" in the classroom,
"Shorty" by his basketball coach.

We are also aware that the name and the person are not really the same thing at all.

You have been reading, writing, thinking, and working with numbers most of your life. Each number, too, can be given many different names. The indicated sum "6 + 2" and the indicated product "4 × 2" both name the same number: the number 8. Each of the following is a name for the number two:

$$2,\ 3 - 1,\ \text{II},\ 1 \times 2,\ 1 + 1,\ \frac{4}{2},\ \frac{2 + 6}{4}.$$

The names of numbers are called numerals.
In the above list of different names for the number two we call the first one a "common name." Similarly,

"8" is a common name for $6 + 2$
"7" is a common name for $10 - 3$
"$3\frac{1}{2}$" is a common name for $1\frac{1}{4} + 2\frac{1}{4}$.

A problem which occurs very often in arithmetic is the problem of finding a common name for a number which is given in another way. For example, 7×23 is found by arithmetic to be the number 161. Thus, "161" is a common name for 7×23. "152" is a common name for $72 + 80$.

Sometimes we use more complicated kinds of numerals such as:

the indicated sum "3 + 6",
the indicated product "2 × 4",
the indicated difference "6 - 2",
the indicated quotient "$\frac{30}{8}$".

These are called <u>numerical phrases.</u> Each of these involves numerals along with one or more symbols for operations. For example, the phrase "2 × 4" involves the numerals "2" and "4" and the symbol for the operation of multiplication.

Check Your Reading

1. What is a numeral?
2. What do we mean by the term "common name for a number"?
3. How many names can a number have?
4. What is a numerical phrase? Give several examples.
5. Give an example of an indicated sum, an indicated product, an indicated difference, and an indicated quotient.

Oral Exercises 2-1a

Give a common name for each of the following numerical phrases:

1. $10 + 2$
2. $5 - 2$
3. $\frac{4 \times 3}{2}$
4. $\frac{3 - 1 + 5}{7}$
5. $\frac{30}{10}$
6. $\frac{10}{30}$
7. $\frac{8}{10}$
8. $\frac{8}{10} + 1$
9. $\frac{8 + 1}{10}$
10. $\frac{5}{10}$
11. $3\frac{1}{2} + 2\frac{1}{2}$
12. $1\frac{1}{4} + 1\frac{1}{4}$
13. $3 + \frac{1}{2} + \frac{1}{2}$
14. $.25 + \frac{1}{4}$
15. $\frac{1}{3} + \frac{3}{1}$

Give an indicated sum, an indicated product, an indicated difference, and an indicated quotient for which the following are common names:

16. 12
17. 15
18. $\frac{3}{4}$
19. 1

Problem Set 2-1a

1. Each of the following numerical phrases represents a number. What is a common name for each?

(a) $\frac{15}{3}$
(b) $5 + 10$
(c) $7 - 2$
(d) $3\frac{3}{4} + 1\frac{1}{4}$
(e) $\frac{2 \times 6}{3}$
(f) $\frac{2 + 3 - 1}{4}$
(g) $\frac{6}{12}$
(h) $\frac{7}{5} + \frac{3}{2}$

Problem Set 2-1a
(continued)

2. For each of the following write two other numerical phrases that represent the same number that is represented by the given phrase.

(a) $8 - 3$
(b) $\frac{42}{3}$
(c) $9 + 4$
(d) 9×4
(e) 5
(f) $\frac{3}{4} + \frac{1}{4}$
(g) $\frac{5 - 2 + 9}{4}$
(h) $\frac{1.8}{2}$

3. Write an indicated product, an indicated sum, an indicated quotient, and an indicated difference that represents the same number as each of the following:

(a) 10
(b) 35
(c) $\frac{2}{3}$
(d) 0

Let us consider the expression

$$6 + 3 \times 4$$

This involves numerals and symbols for operations. It seems, therefore, to be a <u>numerical phrase.</u> What number does it represent? If you look at it one way, you might say that

$$6 + 3 = 9 \quad \text{and} \quad 9 \times 4 = 36,$$

and that, therefore, this numerical phrase represents 36.

But look at it another way. We know that

$$3 \times 4 = 12 \quad \text{and} \quad 6 + 12 = 18$$

So, on second thought, the number could be 18.

It might be interesting for your class to see how many interpretations you can give for the expression

$$6 + 3 \times 4 - 2. \quad \text{(There are more than two!)}$$

It would certainly be confusing if a numerical phrase had more than one meaning. There are two ways to avoid this confusion. The first is to adopt a rule. In any given expression where there is confusion as to which operation comes

<u>before</u> the other, we agree to multiply and divide before adding and subtracting. Applying this rule we see that our original expression,

$$6 + 3 \times 4,$$

is now a numerical phrase. Since multiplication comes before addition, we see that this phrase represents the number 18.

"$3 \times 5 - 6$" represents the number 9.	In this case, the multiplication is done before the subtraction. 3×5 is 15. Then $15 - 6$ is 9.
"$4 - 6 \div 2$" represents the number 1.	Here, the division is done before the subtraction. $6 \div 2$ is 3. Then, $4 - 3$ is 1.
"$4 \times 3 - 6 \div 2 + 1$" is a numeral for the number 10.	The multiplication and division are done first--- 4×3 is 12, $6 \div 2$ is 3. The expression then reads, "$12 - 3 + 1$." Next, the addition and subtraction are done---$12 - 3$ is 9, and $9 + 1$ is 10.

In the last example above, after the multiplication and division were done, we were left with the expression "$12 - 3 + 1$." This expression involves both addition and subtraction. We then proceeded from left to right, and this is the agreement we shall make concerning all such expressions. Two more examples of this kind are given below.

"$9 - 5 + 2$" represents the number 6.	The expression involves only addition and subtraction. Reading from left to right, $9 - 5$ is 4, and $4 + 2$ is 6.
"$7 + 9 - 3$" is a numeral for the number 13.	Performing the operations in order from left to right, "$7 + 9$" represents the number 16, and "$16 - 3$" is a name for 13.

Check Your Reading

1. Why is the order of operation important in the example $6 + 3 \times 4$?
2. Which do we agree to do first, addition and subtraction or multiplication and division?
3. When we have both addition and subtraction indicated which of these is done first?

Oral Exercises 2-1b

What is a common name for each of the following?

1. $5 + 3 \times 4$	6. $7 - 2 \times 3$	11. $\frac{1}{2} \times 4 - \frac{1}{2}$
2. $5 \times 3 + 4$	7. $7 + 2 \times 3$	12. $4 \times \frac{1}{2} - \frac{1}{2}$
3. $5 - 3 \times 4$	8. $7 \times 2 + 3$	13. $5 \times 2 \div 10$
4. $5 \times 3 - 4$	9. $\frac{1}{2} \times 4 + \frac{1}{2}$	14. $5 + 5 \div 10$
5. $7 \times 2 - 3$	10. $\frac{1}{2} + 4 - \frac{1}{2}$	15. $6 - 2 \div 1$

Problem Set 2-1b

Write a common name for each of the following:

1. $13 + 2 \div 2$	5. $120 - 118 + 2$	9. $40 \div 4 - 1$
2. $21 + 1 \times \frac{1}{2}$	6. $2 \times \frac{9}{2} \div 3$	10. $40 - 4 \times 1$
3. $10 \div 2 + 3$	7. $13 - 3 \times 2$	
4. $\frac{3}{4} \times \frac{1}{2} + \frac{1}{2}$	8. $\frac{3}{7} + \frac{1}{7} \times 4$	

There is another way to avoid confusion in dealing with complicated expressions. This method uses some very helpful symbols called parentheses.

The idea is simple. When we enclose a numerical phrase such as "7 - 2" in parentheses, we mean that the phrase "(7 - 2)" is to be treated as a single numeral.

Let us see how this works with our previous example:

$$6 + 3 \times 4.$$

Suppose that "3×4" is enclosed in parentheses. Then "3×4" is to be treated as a numeral and

$$6 + (3 \times 4)$$

represents the number $6 + 12$, or 18. In this case, the parentheses are not really necessary because we had already agreed that in

$$6 + 3 \times 4$$

multiplication is done first. This means that

$$\text{"}6 + (3 \times 4)\text{"} \quad \text{and} \quad \text{"}6 + 3 \times 4\text{"}$$

are names for the same number.

Now suppose that we want to add 3 to 6 first and then multiply 4 by this sum. What phrase can we write which would indicate this? "$6 + 3 \times 4$" is not correct, nor is "$6 + (3 \times 4)$". We want "$6 + 3$" to be treated as a numeral. The way to do this is to enclose "$6 + 3$" in parentheses. Thus, "$(6 + 3) \times 4$" is the correct phrase.

It is important for you to notice that the two phrases "$6 + (3 \times 4)$" and "$(6 + 3) \times 4$" represent two different numbers.

A convenient way to show that the numbers are not the same is to write

$$(6 + 3) \times 4 \neq 6 + (3 \times 4).$$

Here the symbol "$\neq$" means "is <u>not</u> equal to".

A <u>numerical phrase</u> like $6 \times (2 + 5)$ is often written $6(2 + 5)$ without the symbol "$\times$". When two numerals are placed side-by-side in this way, the operation of multiplication is understood. Thus,

$$5(7 - 2) = 5 \times (7 - 2);$$

likewise $$6(4) = 6 \times 4;$$

and $$(43)(52) = 43 \times 52.$$

Another case in which we need to agree on how an expression should be read is illustrated by the following example:

$$\frac{5(6 - 2)}{13 - 3}$$

It is understood that "13 - 3" is a numeral even though it is not enclosed in parentheses. Therefore, the expression is an indicated quotient of the numbers $5(6 - 2)$ and $13 - 3$. It is a complicated name for the number 2.

Check Your Reading

1. What do we mean when we enclose a numerical phrase such as "5 - 3" in parentheses?
2. Do we really need parentheses to indicate that the multiplication is to be done first in "$6 + 3 \times 4$"?
3. How could we use parentheses to indicate that the addition should be done first in "$6 + 3 \times 4$"?
4. How can we indicate that "$6 + (3 \times 4)$" and "$(6 + 3) \times 4$" represent two different numbers?
5. How do we indicate that two numbers are to be multiplied without using the "times sign"?

Oral Exercises 2-1c

1. Which of the following pairs of numerals name the same number?
   (a) $2 + 4 \times 5$ and 22
   (b) $(2 + 4) \times 5$ and 30
   (c) $3 \times 3 - 1$ and 6
   (d) $2 \times 5 + 1$ and $(2 \times 5) + 1$
   (e) $4 + 3 \times 2$ and $(4 + 3) \times 2$
   (f) $(3 + 2) + 5$ and $3 + (2 + 5)$
   (g) $14 - 4 \times 3$ and 2
   (h) $2(4 + 5)$ and $2 \times 4 + 5$
   (i) $(4 + 1) \times 3$ and $4 + 1 \times 3$
   (j) $(3 \times 2) \times 4$ and $3 \times (2 \times 4)$

You remember that the symbol "$\neq$" represents the verb phrase "is not" or "is not equal to". We can form sentences using this symbol. Here is an example.

$$7 + 2 \neq 5.$$

This sentence states that the numerals "7 + 2" and "5" represent different numbers. We see that it is a true sentence. On the other hand,

$$6 + 3 \neq 9$$

is a false sentence.

The important fact about a sentence involving numerals is that it is either true or false, but not both. Some examples of true sentences are:

$$2 + 5 = 7$$
$$5 \times 3 \neq 10$$
$$2(4 + 6) = 20$$

Some examples of false sentences are:

$$3 + 5 = 7$$
$$5 \times 2 \neq 10$$
$$2(4 + 7) = 15$$

Check these.

There are other symbols which are used in forming mathematical sentences. The symbol ">" is read "is greater than". The sentence,

"8 is greater than 5",

can be written

$$8 > 5.$$

Is this sentence true? The following is also a sentence:

$$4 > 10.$$

Is it a true sentence?

To make a statement that one number is less than another, we use the symbol "<". It is read "is less than." Thus the sentence,

$$4 < 8,$$

is read "4 is less than 8". Is this sentence true? Try to write a false sentence using the symbol "<".

It is important that you remember which of the symbols "<" or ">" says "is less than" and which one says "is greater than".

Can you see that every "is less than" statement can also be written as an "is greater than" statement? For example, the true sentence

$$5 < 7$$

can be written

$$7 > 5.$$

These two sentences express the same idea. Can you write the sentence

$$5 > 3$$

in another way?

Check Your Reading

1. What do we call an expression like "$6 + 2 = 8$"?
2. What verb does the symbol "$=$" represent?
3. What verb phrase does the symbol "$\neq$" represent?
4. What is an important fact about a sentence involving numerals?
5. How do we read the symbol "$>$"?
6. How do we read the symbol "$<$"?
7. Can we use "$>$" and "$<$" to form mathematical sentences?
8. How can you remember which of the symbols "$<$" or "$>$" says "is less than" and which one says "is greater than"?
9. How can the sentence "$5 > 3$" be written in another way?

Oral Exercise 2-2

Which of the following sentences are true sentences? Which are false?

1. $4 + 8 = 10 + 5$
2. $4 + 8 \neq 10 + 5$
3. $8 + 3 = 10 + 1$
4. $8 + 3 \neq 10 + 1$
5. $4 + 8 = 8 + 4$
6. $12 = 4 + (4 + 4)$
7. $\frac{85}{1} \neq 85$
8. $13 + 0 \neq 15 + 0$
9. $13 > 11$
10. $12 < 10$
11. $4 + 2 > 7 - 2$
12. $4 + 2 < 7 - 2$
13. $\frac{1}{2} + \frac{1}{2} \neq 1$
14. $7(6 + 3) \neq 7 \times 6 + 3$
15. $\frac{1}{85} = 85$
16. $65 \times 1 > 65$
17. $(4 + 2)(3 - 1) \neq 4 + 2 \times 3 - 1$
18. $3(2 + 1) < 3 \times 2 + 1$
19. $15 < 15$
20. $13 \times 0 > 13$

Problem Set 2-2

1. Which of the following sentences are true? Which are false?

   (a) $4 + 3 < 3 + 4$

   (b) $5(2 + 3) > (5 + 2) + 3$

   (c) $3(4 - 2) \neq 3 \times 4 - 2$

   (d) $\frac{1}{2}(7 - 3) = (1 - \frac{1}{2}) + (3\frac{3}{4} - 2\frac{1}{4})$

   (e) $2 + 1.3 > 3.3$

   (f) $2 + 1.3 \neq 3.3$

   (g) $2 + 1.3 = 3.3$

   (h) $\frac{12 + 2}{2} < \frac{12}{2} + 2$

   (i) $\frac{12 + 2}{2} > \frac{12}{2} + 2$

   (j) $\frac{12 + 2}{2} \neq \frac{12}{2} + 2$

   (k) $\frac{12 + 2}{2} = \frac{12}{2} + 2$

2. Insert parentheses in each of the following expressions so that the resulting sentence is true:

   (a) $10 - 7 - 3 = 6$

   (b) $3 \times 5 + 7 = 36$

   (c) $3 \times 5 + 7 = 22$

   (d) $3 \times 5 - 4 = 3$

   (e) $3 \times 5 - 4 = 11$

   (f) $3 \times 5 + 2 \times 4 = 23$

   (g) $3 \times 5 + 2 \times 4 = 84$

   (h) $3 \times 5 + 2 \times 4 = 68$

   (i) $3 \times 5 - 2 \times 4 = 36$

   (j) $3 \times 5 - 2 \times 4 = 7$

   (k) $3 \times 5 - 2 \times 4 = 52$

   *(l) $12 \times \frac{1}{2} - \frac{1}{3} \times 9 = 51$

   *(m) $12 \times \frac{1}{2} - \frac{1}{3} \times 9 = 3$

   *(n) $12 \times \frac{1}{2} - \frac{1}{3} \times 9 = 18$

3. Tell which of the following sentences are true and which are false:

   (a) $(3 + 7) = 3 + 7(4)$

   (b) $4(5) + 4(8) < 4(13)$

   (c) $2(5 + \frac{1}{2}) = 2(5) + 2(\frac{1}{2})$

Problem Set 2-2
(continued)

(d) $23 - 5(2) \neq 36$

(e) $\frac{7 + 9}{2} = 7 + \frac{9}{2}$

(f) $\frac{16}{2} + 4 - 3 \neq \frac{16}{2} + (4 - 3)$

(g) $5(7 + 3) > 10(\frac{20}{5} + 1)$

(h) $\frac{12}{3(2)} = \frac{12(2)}{3}$

(i) $3(8 + 2) \neq 6 \times 5$

(j) $12 + (2 \times 3) < 12\ (9)$

4. Write each of the following sentences in words. Underline the verb or verb phrase in each. Is the sentence true?

(a) $4 + 8 = 10 + 5$

(b) $5 + 7 \neq 6 + 5$

(c) $13 < 18 - 7$

(d) $1 + 2 > 0$

2-3. A Property of the Number One.

What do you get when you multiply 5 by 1? You get 5, don't you? We write

$$1 \times 5 = 5.$$

It is true also that $5 \times 1 = 5$. Similarly "$1 \times \frac{3}{4}$" and "$\frac{3}{4} \times 1$" name the number $\frac{3}{4}$. What number is named by "1×4.28" or "4.28×1"?

You surely agree that if you multiply a number by 1, the result is the given number.

Let us ask if any other number behaves in this way. If you multiply 5 by 2, by 8, or by $\frac{3}{4}$, is the result 5 in any of these cases? No. In fact, if you multiply 5 by any number other than one, you do not get 5. It looks as if we have found a property which only the number 1 possesses. We call it the multiplication property of one.

Property, in the most familiar sense of the word, is something you have. A property of one is something the number one has; that is, a characteristic of the number one. A similar usage of the word would be "sweetness is a property of sugar."

You have known this property of one for a long time but perhaps you have never thought about how useful it is.

All the numerals below, and many more, are names for the number 1.

$$\frac{2}{2},\ \frac{8}{8},\ \frac{3}{3},\ \frac{5}{5},\ \frac{31}{31}$$

Any number may be given different names by multiplying by one, using different names for one.

$\frac{1}{2}$ may be written as "$\frac{1}{2} \times \frac{3}{3}$", and as "$\frac{3}{6}$".

$\frac{1}{2}$ may also be written as "$\frac{1}{2} \times \frac{5}{5}$", and as "$\frac{5}{10}$".

5 may be written as "$5 \times \frac{8}{8}$", and as "$\frac{40}{8}$".

$\frac{3}{8}$ may be written as "$\frac{3}{8} \times \frac{2}{2}$", and as "$\frac{6}{16}$".

In each of the examples above, the fraction at the right is another numeral for the number at the beginning of the line. You may also have noticed that using the multiplication property of one amounts to "multiplying numerator and denominator by the same number," as you may have said in elementary school. There is nothing wrong with speaking in this way, but you should realize that it is the multiplication property of one that makes it correct; it is not really "another way" at all. In this course, since we are going to emphasize properties of numbers, we shall always speak of the process in terms of using the multiplication property of one.

The multiplication property of one makes it possible for us to write a common name for $\frac{12}{18}$:

$$\frac{12}{18} = \frac{6 \times 2}{6 \times 3}$$

$$= \frac{6}{6} \times \frac{2}{3}$$

$$= 1 \times \frac{2}{3}$$

$$= \frac{2}{3}.$$

We say that "$\frac{2}{3}$" is a common name for $\frac{12}{18}$.

The two examples below show some other ways in which you can use the multiplication property of one.

<u>Example 1</u>. Write the number $\frac{5}{6}$ as a fraction having "18" as denominator.

The problem can be put in the form

$$\frac{5}{6}(1) = \frac{(\quad)}{18}$$

What fractional numeral for "1" shall we use? Since $18 = 6 \times 3$, "$\frac{3}{3}$" is the correct choice.

$$\frac{5}{6}(\frac{3}{3}) = \frac{15}{18}$$

This shows that "$\frac{15}{18}$" and "$\frac{5}{6}$" are names for the same number. It is important to notice that in writing "$\frac{15}{18}$" instead of "$\frac{5}{6}$" we have changed the name but <u>we have not changed the number.</u>

<u>Example 2</u>. Add the fractions $\frac{2}{3}$ and $\frac{1}{4}$.

Remember that there are at least two other ways to write this problem.

One way looks like this:

$$\begin{array}{r} \frac{2}{3} \\ +\ \frac{1}{4} \\ \hline \end{array}$$

Another way looks like this:

$$\frac{2}{3} + \frac{1}{4}$$

Let's agree to use the second form here.
Then we can write:

$$\begin{aligned} \frac{2}{3} + \frac{1}{4} &= \left(\frac{2}{3} \times \frac{4}{4}\right) + \left(\frac{1}{4} \times \frac{3}{3}\right) \\ &= \frac{8}{12} + \frac{3}{12} \\ &= \frac{11}{12} \end{aligned}$$

So we have shown that the sum of $\frac{2}{3}$ and $\frac{1}{4}$ is $\frac{11}{12}$.
Do you see how we have used the multiplication property of one here? We chose the name "$\frac{4}{4}$" and the name "$\frac{3}{3}$" for one because they gave us the same denominator for both fractions.

Check Your Reading

1. What do we mean by the multiplication property of one?
2. How is this property used to give different names for a number?
3. What numeral for one is used in writing $\frac{5}{6}$ as "$\frac{15}{18}$" ?
4. What names for one are used in writing $\frac{2}{3}$ and $\frac{1}{4}$ as fractions having "12" as denominator?

Oral Exercises 2-3

1. Use the multiplication property of one to change the names of the given numbers on the left in the way indicated.

(a) $\frac{1}{4} \times \_\_\_ = \frac{\quad}{20}$

(b) $\frac{1}{7} \times \_\_\_ = \frac{\quad}{21}$

(c) $\frac{2}{11} \times \_\_\_ = \frac{\quad}{44}$

(d) $\frac{4}{8} \times \_\_\_ = \frac{\quad}{24}$

(e) $\frac{5}{2} \times \_\_\_ = \frac{\quad}{6}$

(f) $\frac{7}{4} \times \_\_\_ = \frac{\quad}{100}$

(g) $3 \times \_\_\_ = \frac{\quad}{5}$

(h) $\frac{3}{7} \times \_\_\_ = \frac{\quad}{35}$

Problem Set 2-3

1. Use the multiplication property of one to change the names of the numbers on the left in the way indicated.

(a) $\frac{3}{5} \times \_\_\_ = \frac{\quad}{40}$

(b) $\frac{5}{6} \times \_\_\_ = \frac{\quad}{24}$

(c) $\frac{9}{10} \times \_\_\_ = \frac{\quad}{80}$

(d) $\frac{25}{4} \times \_\_\_ = \frac{\quad}{12}$

(e) $4 \times \_\_\_ = \frac{\quad}{3}$

(f) $\frac{84}{12} \times \_\_\_ = \frac{\quad}{36}$

(g) $\frac{40}{9} \times \_\_\_ = \frac{\quad}{18}$

(h) $\frac{2}{11} \times \_\_\_ = \frac{\quad}{99}$

2. Use the multiplication property of one to find a simpler numeral for each of the following numbers:

(a) $\frac{1}{4} + \frac{3}{8}$

(b) $\frac{1}{3} + \frac{3}{5}$

(c) $\frac{10}{3} + \frac{7}{6}$

(d) $\frac{3}{4} + \frac{2}{7}$

(e) $2(\frac{3}{5}) + \frac{1}{10}$

(f) $\frac{8}{6} + \frac{12}{9}$

3. Show what numeral for one is used in finding a common name for each number below.

Example: $\frac{18}{30} = \frac{6}{6} \times \frac{3}{5}$

Problem Set 2-3
(continued)

(a) $\frac{9}{24}$

(b) $\frac{15}{21}$

(c) $\frac{4}{32}$

(d) $\frac{14}{35}$

(e) $\frac{5(3 + 4)}{5}$

(f) $\frac{4(7 - 2)}{12}$

(g) $\frac{9(8 + 3)}{15}$

(h) $\frac{4(6 - 5)}{4}$

2-4. Some Properties of Addition and Multiplication.

Try an experiment on yourself. Try to add the numbers 7, 2, and 4 all at the same time. What happens? You will find that you always begin by adding two of the numbers first. Then you will add the third number to this sum. In other words, you may start by saying to yourself

$$7 + 2 = 9 \quad \text{and} \quad 9 + 4 = 13,$$

or, you may say $\quad 2 + 4 = 6 \quad \text{and} \quad 7 + 6 = 13.$

Your experiment should bring out two ideas. One is that addition is really an operation that is applied to only two numbers at a time. For this reason we say

Addition is a Binary Operation.

The other idea is that the two final answers are the same. Both are 13.

Let us use parentheses to illustrate what happened. The first approach would look like this:

$$\begin{aligned} 7 + 2 + 4 &= (7 + 2) + 4 \\ &= 9 + 4 \\ &= 13. \end{aligned}$$

The second approach would be as follows:

$$\begin{aligned} 7 + 2 + 4 &= 7 + (2 + 4) \\ &= 7 + 6 \\ &= 13. \end{aligned}$$

The sentence which states that the end results are the same is:

$$(7 + 2) + 4 = 7 + (2 + 4).$$

We see that this sentence is true.

Examine the following sentences:

$$(6 + 5) + 10 = 6 + (5 + 10),$$

and

$$(\tfrac{1}{2} + \tfrac{1}{2}) + 3 = \tfrac{1}{2} + (\tfrac{1}{2} + 3).$$

These sentences have the same pattern and each of them is true. In fact, all sentences having this pattern are true. We say that this is a property of addition of numbers. We call it the <u>associative property of addition.</u>

How can we state this property <u>in words?</u> One way might be to say that if you add a second number to a first and then add a third number to their sum, the outcome is the same as if you first add the third number to the second, and then add their sum to the first. For example,

$$(3 + 4) + 5 = 3 + (4 + 5).$$

Perhaps the best way to understand the <u>associative property</u> is to study carefully the way in which the parentheses were placed in the above example. Where do you think the parentheses should be placed in the example,

$$2 + 7 + 6 = 2 + 7 + 6,$$

to illustrate the associative property?

<u>Check Your Reading</u>

1. What do we mean when we say addition is a binary operation?
2. What does the associative property of addition tell you about the addition of three numbers?
3. How do parentheses help in expressing the idea of the associative property of addition?
4. What is the pattern in the examples on pages 45 and 46?

Oral Exercises 2-4a

Add each of the following in two different ways, as shown by the example.

Example: $3 + 4 + 5$ (a) $(3 + 4) + 5 = 7 + 5$
$= 12$

(b) $3 + (4 + 5) = 3 + 9$
$= 12$

1. $4 + 2 + 7$
2. $6 + 5 + 3$
3. $2 + 9 + 4$
4. $5 + 6 + 1$
5. $\frac{1}{2} + \frac{3}{4} + \frac{1}{4}$ (which is easier?)
6. $\frac{1}{3} + \frac{2}{3} + \frac{1}{4}$ (which is easier?)
7. $2 + .25 + .75$
8. $4\frac{1}{2} + \frac{1}{2} + \frac{1}{4}$ (which is easier?)

Problem Set 2-4a

1. Place parentheses in the following sentences so that each becomes an example of the associative property.

   (a) $4 + (2 + 7) = 4 + 2 + 7$

   (b) $(6 + 1) + \frac{1}{2} = 6 + 1 + \frac{1}{2}$

   (c) $3 + (4 + 11) = 3 + 4 + 11$

   (d) $5 + 1 + 6 = 5 + 1 + 6$

   (e) $11 + 13 + 121 = 11 + 13 + 121$

   (f) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

2. In each of the following use the associative property of addition to show two different ways of grouping the numbers. Choose the grouping that you think makes an easier computation and complete the addition.

   (a) $\frac{1}{4} + \frac{3}{4} + \frac{2}{3}$

   (b) $\frac{2}{3} + \frac{1}{5} + \frac{4}{5}$

   (c) $\frac{1}{2} + \frac{1}{2} + \frac{5}{6}$

   (d) $\frac{7}{8} + \frac{3}{4} + \frac{1}{4}$

Problem Set 2-4a
(continued)

(e) $2\frac{5}{8} + 1\frac{3}{8} + 2\frac{1}{4}$

(f) $2.7 + 13.2 + .8$

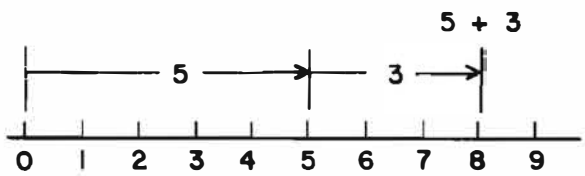
(g) $\frac{7}{8} + \frac{3}{4} + \frac{1}{8}$

3. From the tip of a mouse's nose to the back of his head is 32 millimeters; from the back of his head to the base of his tail is 71 millimeters: from the base of his tail to the tip of his tail is 76 millimeters. What is the length of the mouse from the tip of his nose to the tip of his tail? Is he the same length from the tip of his tail to the tip of his nose? Why do you think this exercise is included in this book?

There is another important property of addition, and it can be seen by using the number line to add numbers. As one example, let us look at

$$5 + 3,$$

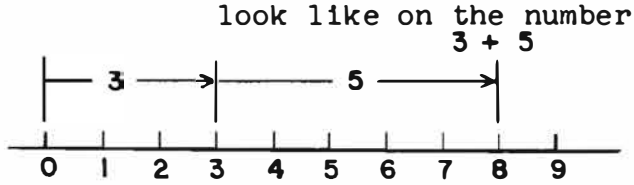
which is a symbol for the number obtained by <u>adding 3 to 5</u>. This can be thought of as moving from 0 to 5 on the number line and then moving from this point three units to the right, thereby locating the point whose coordinate is 5 + 3.



What does the picture of

$$3 + 5$$

look like on the number line?



Describe this addition process. Your description should be similar to the one above.

We see by the pictures that these two addition processes are different. We notice, however, that they end at the same point. That is, "5 + 3" and "3 + 5" are different names for the same number 8. That is,

$$5 + 3 = 3 + 5.$$

Decide whether the following are true sentences by using the number line:

$$0 + 6 = 6 + 0$$

$$2\tfrac{1}{2} + 5 = 5 + 2\tfrac{1}{2}$$

We see that two numbers can be added in either order, the results being the same. This property of addition of numbers is called the <u>commutative property of addition.</u> Below are some more examples that illustrate this property:

$$2\tfrac{1}{3} + 5 = 5 + 2\tfrac{1}{3}$$

$$\tfrac{1}{4} + \tfrac{3}{8} = \tfrac{3}{8} + \tfrac{1}{4}$$

$$104 + 2.4 = 2.4 + 104$$

In problems of addition we often use both the commutative and associative properties. In general the associative property of addition tells us, in effect, that an indicated sum of three numbers does have meaning. It may be written with parentheses around the first two numerals, or the last two numerals, to indicate which of these two numbers are added first; or it may be written without parentheses. All three representations are names for the same number. The commutative property tells us that the addition of two numbers is performed in either order.

Consider the problem:

$$13 + 2 + 87.$$

First, the associative property allows us to insert parentheses, giving us

$$(13 + 2) + 87.$$

Next, the commutative property allows us to interchange the numbers 2 and 13, giving us

$$(2 + 13) + 87.$$

The associative property then permits us to combine 13 and 87, instead of 2 and 13, giving us

$$2 + (13 + 87) = 2 + (100).$$

Check Your Reading

1. What is the commutative property of addition?
2. What is the associative property of addition?
3. How do the commutative and associative properties of addition make some addition problems "easier"?

Problem Set 2-4b

Which of the following are true statements? Which property or properties of addition have you used in arriving at an answer?

1. $8 + 6 = 6 + 8$
2. $7\frac{1}{2} + 2 = 2 + 7\frac{1}{2}$
3. $8.3 + 5.7 = 3.8 + 7.5$
4. $12\frac{2}{3} + 6\frac{1}{2} = 6\frac{1}{2} + 12\frac{2}{3}$
5. $8.3 + 5.7 = 5.3 + 8.7$
6. $(3 + 4) + 5 = 3 + (4 + 5)$
7. $(3 + 4) + 5 = (4 + 3) + 5$
8. $3 + (4 + 5) \neq 3 + (5 + 4)$
9. $(3 + 4) + 5 = 3 + (5 + 4)$
10. $3 + (4 + 5) = 3(4 + 5)$
11. $(7 + 2) + 3 = (7 + 2) + 3$
12. $(7 + 2) + 3 \neq 7 + (2 + 3)$
13. $(7 + 2) + 3 = (2 + 7) + 3$
14. $(7 + 2) + 3 = 3 + (7 + 2)$
15. $(7 + 2) + 3 \neq 3 + (2 + 7)$
16. $8.2 + (3.1 + 2.4) = (3.1 + 8.2) + 2.4$
17. $(9 + 2) + (5 + 4) = (4 + 5) + (9 + 2)$
18. $(9 + 2) + (5 + 4) \neq (4 + 5) + (2 + 9)$
19. $(\frac{1}{2} + \frac{1}{3}) + \frac{1}{4} = (\frac{1}{4} + \frac{1}{2}) + \frac{1}{3}$

Problem Set 2-4b
(continued)

20. Perform each of the following additions in the easiest way, using the properties we have discussed.

(a) $6 + (8 + 4)$

(b) $\frac{2}{5} + \frac{2}{3} + 1 + \frac{1}{3} + \frac{8}{5}$

(c) $5\frac{4}{7} + 6 + 14\frac{3}{7}$

(d) $\frac{3}{4} + \frac{1}{3}$

(e) $2\frac{1}{5} + 3\frac{2}{3} + 6 + 7\frac{4}{5}$

(f) $(2\frac{1}{3} + 1) + \frac{6}{5}$

(g) $(1.8 + 2.1) + (1.6 + .9) + 1.2$

(h) $(8 + 7) + 4 + (3 + 6)$

We have seen that addition is a binary operation and that it has the <u>associative property</u> and the <u>commutative property</u>. Another way of saying this is:

Addition is both <u>ASSOCIATIVE</u> and <u>COMMUTATIVE.</u>

Are there similar properties of multiplication? For example, consider the indicated product

$$2 \times 3 \times 5.$$

Can the multiplication be done all at once? Try it! You will agree that multiplication can be performed with only two numbers at a time. We see that multiplication, like addition, is a binary operation.

Let us look again at our example, $2 \times 3 \times 5$. Is the following a true sentence:

$$(2 \times 3) \times 5 = 2 \times (3 \times 5)?$$

Since the phrases inside the parentheses are to be treated as numerals, the sentence can be written as follows:

$$6 \times 5 = 2 \times 15.$$

"6×5" and "2×15" are both numerals for the same number. Therefore, the sentence is true.

Can you determine in the same way whether or not the following is a true sentence:

$$(7 \times 6) \times \frac{1}{3} = 7 \times (6 \times \frac{1}{3})?$$

By this time you have probably decided that multiplication is associative, just as addition is associative. Perhaps the following table will help to make this clear:

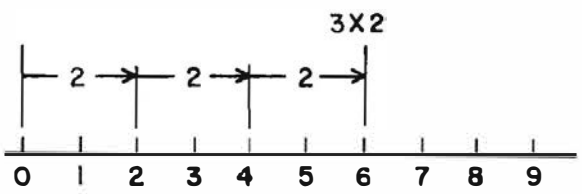
Addition	Multiplication
$(1 + 4) + 2 = 1 + (4 + 2)$	$(1 \times 4) \times 2 = 1 \times (4 \times 2)$
$(4 + 8) + 90 = 4 + (8 + 90)$	$(4 \times 8) \times 90 = 4 \times (8 \times 90)$
$(4 + 1\frac{1}{2}) + 2\frac{1}{2} = 4 + (1\frac{1}{2} + 2\frac{1}{2})$	$(4 \times 1\frac{1}{2}) \times 2\frac{1}{2} = 4 \times (1\frac{1}{2} \times 2\frac{1}{2})$

Check to see that each of the 6 sentences in the table is true.

The number line can help us to decide whether or not multiplication is commutative. We shall use an example to illustrate this.

$$3 \times 2$$

is a symbol for the number obtained by adding three 2's. On the number line this is accomplished by using the length of the line from 0 to 2 as a measure and moving to the right from 0 a distance of three such lengths. In this way we locate the point whose coordinate is 3×2.



Can you guess what the picture of

$$2 \times 3$$

will look like?

2-4

By this time you have probably decided that multiplication is associative, just as addition is associative. Perhaps the following table will help to make this clear:

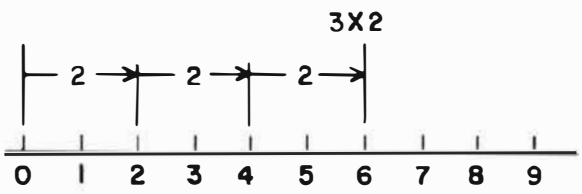
Addition	Multiplication
$(1 + 4) + 2 = 1 + (4 + 2)$	$(1 \times 4) \times 2 = 1 \times (4 \times 2)$
$(4 + 8) + 90 = 4 + (8 + 90)$	$(4 \times 8) \times 90 = 4 \times (8 \times 90)$
$(4 + 1\frac{1}{2}) + 2\frac{1}{2} = 4 + (1\frac{1}{2} + 2\frac{1}{2})$	$(4 \times 1\frac{1}{2}) \times 2\frac{1}{2} = 4 \times (1\frac{1}{2} \times 2\frac{1}{2})$

Check to see that each of the 6 sentences in the table is true.

The number line can help us to decide whether or not multiplication is commutative. We shall use an example to illustrate this.

$$3 \times 2$$

is a symbol for the number obtained by adding three 2's. On the number line this is accomplished by using the length of the line from 0 to 2 as a measure and moving to the right from 0 a distance of three such lengths. In this way we locate the point whose coordinate is 3×2.

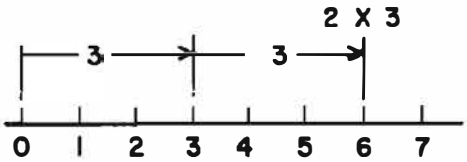


Can you guess what the picture of

$$2 \times 3$$

will look like?

Here it is:



From the picture we see that the two multiplications on the number line are different, but, again, they end at the same point. The coordinate of this point is 6. Thus,

$$3 \times 2 = 2 \times 3.$$

We also see that

$$4 \times \frac{1}{2} = \frac{1}{2} \times 4$$

and

$$1.5 \times 6 = 6 \times 1.5$$

are true sentences.

These examples illustrate the fact that multiplication is commutative.

We see, in summary, that addition is both associative and commutative and that multiplication is both associative and commutative.

Check Your Reading

1. What is meant by "multiplication is a binary operation."?
2. What is the associative property of multiplication?
3. What is the commutative property of multiplication?

Oral Exercises 2-4c

Which of the following sentences are true and what properties were used?

1. $7 \times (2 \times 5) = (7 \times 2) \times 5$
2. $7 \times 4 = 4 \times 7$
3. $7 + 4 = 4 + 7$

Oral Exercises 2-4c
(continued)

4. $3(4 + 1) = (4 + 1)3$
5. $\frac{2}{3} + \frac{3}{4} = \frac{3}{4} \times \frac{2}{3}$
6. $3 + (4 + 5) = (3 \times 4) \times 5$
7. $3 + (4 + 1) = (4 + 1) + 3$
8. $3 + (4 + 1) = (1 + 4) + 3$
9. $(6 + 2)(7 - 2) = (7 - 2)(6 + 2)$
10. $(6 + 2)(7 - 2) = (7 - 2)(2 + 6)$
11. $\frac{1}{2}(\frac{1}{4} \times \frac{1}{3}) = (\frac{1}{2} \times \frac{1}{4}) \times \frac{1}{3}$
12. $(\frac{1}{2} + \frac{1}{4}) + \frac{1}{3} = \frac{1}{2} + (\frac{1}{4} + \frac{1}{3})$

Problem Set 2-4c

1. Consider various ways to do the following computations mentally, and find the one that seems easiest (if there is one). Then perform the indicated operations in the easiest way.

Example:

$(4 \times 9) \times 25$	$(4 \times 9) \times 25$
$(9 \times 4) \times 25$	36×25
$9 \times (4 \times 25)$	?
9×100	
900	

Using the method on the left, the arithmetic is easier.

(a) $4 \times 7 \times 25$

(b) $\frac{1}{5} \times (26 \times 5)$

(c) $73 \div 62 + 27$

(d) $2 \times 38 \times 50$

(e) $\frac{1}{2} \times 39 \times 2$

(f) $(\frac{1}{3} \times 43) \times 6$

(g) $\frac{1}{5} \times (18 \times 15)$

(h) $50 \times 97 \times 2$

(i) $\frac{3}{4} \times 19 \times 4$

(j) $(4 \times 8) \times (25 \times 5)$

(k) $(3 \times 4) \times (7 \times 25)$

(l) 12×14

(m) $\frac{1}{2} \times \frac{1}{3} \times \frac{5}{6}$

(n) $6 \times 8 \times 125$

(o) $1.25 \times 5.5 \times 8$

(p) $(2 \times 5) \times 1.97$

(q) $\frac{5}{4} \times 6 \times \frac{4}{3} \times \frac{1}{5}$

Problem Set 2-4c
(continued)

2. Which is easier to compute:

$$\begin{array}{r} 957 \\ \times\ 222 \\ \hline \end{array} \qquad \begin{array}{r} 222 \\ \times\ 957 \\ \hline \end{array} \quad ? \quad \text{Why?}$$

How about

$$\begin{array}{r} 3.89 \\ \times\ 141 \\ \hline \end{array} \qquad \begin{array}{r} 141 \\ \times\ 3.89 \\ \hline \end{array} \quad ? \quad \text{Why?}$$

3. What number can "t" represent so that these sentences will be true?

(a) $t + 3 = 3 + 5$

(b) $t + (3 + 4) = (3 + 4) + 8$

(c) $4 \times 1\frac{1}{2} = t \times 4$

(d) $(\frac{3}{4} + 1)5 = 5 \times t$

(e) $3.7 + 7.3 = 7.3 + t$

(f) $.5 + (2.4 + .6) = (2.4 + .6) + t$

(g) $(7.2 + 5) \times 1\frac{1}{2} = 1\frac{1}{2} \times t$

(h) $5 - 4 = t - 5$

(i) $((3 + 2) + 5) + 4 = 4 + t$

(j) $1\frac{1}{2} + t = \left((7\frac{1}{2} - 2) - 5\right) + 1\frac{1}{2}$

(k) $7 \times (5 \times t) = (4 \times 5) \times 7$

(l) $(8.5 \times 6.2) + 9.5 = 9.5 + (t \times 8.5)$

(m) $8 \div t = 4 \div 8$

4. Are division and subtraction commutative? associative? Give examples to support your decision.

Problem Set 2-4c
(continued)

5. A <u>binary</u> operation on numbers is an instruction which tells how you may obtain from two numbers given in a certain order, a single number. If the operation is <u>commutative,</u> then the same number is obtained regardless of the order in which the numbers are given. We have already seen that the binary operation " + " of arithmetic is commutative.

   Suppose the binary operation "$\oplus$" means "add twice the second number to the first". What single number would be obtained from $2 \oplus 3$? What single number would be obtained from $3 \oplus 2$? Is the operation $\oplus$ commutative?

6. Suppose the operation $*$ means "add 1 to each of the numbers, then multiply the resulting sums". What third number would be obtained from $2 * 5$? What third number would be obtained from $5 * 2$? Does the operation of "$*$" appear to be commutative?

*7. Recall that in testing the associative properties of multiplication and addition we tried to determine if sentences like

$$(3 + 4) + 7 = 3 + (4 + 7)$$

and

$$(3 \times 4) \times 7 = 3 \times (4 \times 7)$$

were true. When we found they were true, we agreed the operations "+" and "x" were associative. Using the definitions given in exercise 5, let us find the number represented by $(2 \oplus 3) \oplus 4$ and compare it with the number represented by $2 \oplus (3 \oplus 4)$.

Suggestion: $2 \oplus 3 = 2 + (2 \times 3) = 8$

$$(2 \oplus 3) \oplus 4 = 8 \oplus 4 = 8 + (2 \times 4) = 16$$

Now determine the number represented by $2 \oplus (3 \oplus 4)$ by first finding the number represented by $3 \oplus 4$ and then completing the operation involving this result and the number 2. Does "$\oplus$" appear to be associative?

Problem Set 2-4c
(continued)

*8. Find the number represented by $(2 * 5) * 3$ and also that represented by $2 * (5 * 3)$, using "$*$" as defined in Problem 6. Compare the results, then repeat the process using three other numbers. From the results do you think "$*$" is associative?

*9. Describe an instruction for operation "$\oplus$" then find the third numbers defined by $3 \oplus 2$ and $2 \oplus 3$. State whether "$\oplus$", as you have defined it, is commutative.

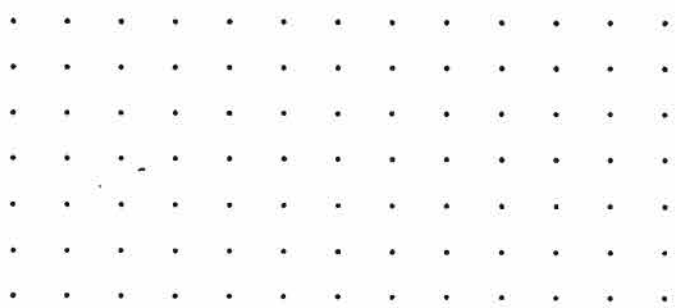
*10. Using the same definition for the operation "$\oplus$" as you used in Problem 9, determine whether "$\oplus$" is associative.

2-5. The Distributive Property

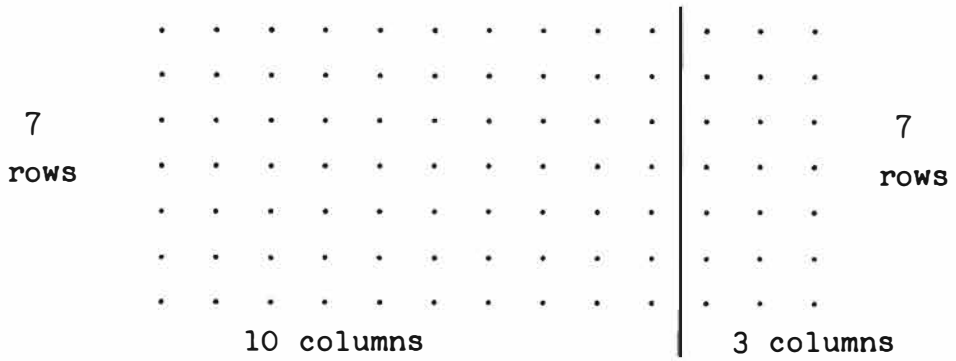
Below are some dots arranged in rows and columns. This is an array.

In this array, there are 3 rows and 4 columns. Thus, there are 12 dots in all. In fact, this array illustrates the basic multiplication fact, $3 \times 4 = 12$.

Below is another array.



Here there are 7 rows and 13 columns. So the total number of dots is 7×13. The array can be "split" into two smaller arrays, like this:



In the array to the left of the dividing mark, the number of dots is 7×10; in the array to the right, the number of dots is 7×3. So, the total number of dots in the original array is the sum of these two numbers, $(7 \times 10) + (7 \times 3)$. We have already stated that the total number of dots is 7×13. Therefore,

$$7 \times 13 = (7 \times 10) + (7 \times 3).$$

The statement in the line above is one illustration of a property called the <u>distributive property of multiplication over addition.</u> The following steps may help to explain the name.

7×13

$7(10 + 3) \longrightarrow$ "10 + 3" is an indicated addition.

$7(10 + 3) \longrightarrow$ Multiplication by 7 is <u>distributed over</u> the addition.

$(7 \times 10) + (7 \times 3)$

The distributive property of multiplication over addition has many uses in mathematics, as we shall see throughout this course. It is helpful in doing certain kinds of computations mentally. For example, it can be used to change an indicated product to an indicated sum.

7×13	indicated product (of 7 and 13)
$7(10 + 3)$	indicated product (of 7 and 10 + 3)
$(7 \times 10) + (7 \times 3)$	indicated sum (of 7×10 and 7×3)
$70 + 21$	indicated sum (of 70 and 21)

Thus, the product 7×13 is equal to the sum $70 + 21$. In this way, it is easy to determine that $7 \times 13 = 91$.

Below is another illustration of the distributive property of multiplication over addition. You will notice the same pattern as before. That is, one of the numbers in an indicated product is expressed as a sum; then the multiplication is distributed over the addition.

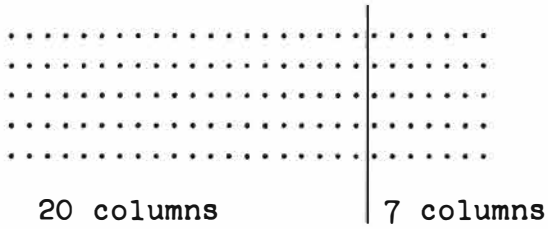
5×27

$5(20 + 7)$

$(5 \times 20) + (5 \times 7)$

$100 + 35$

135



In this case, the indicated product "$5(20 + 7)$" was changed to the indicated sum "$(5 \times 20) + (5 \times 7)$." Do you see again that all of the steps can easily be done in your head?

Check over the following to see how the distributive property of multiplication over addition is used. (An array might be drawn on the chalkboard of your classroom to illustrate the steps.)

$$\begin{aligned} 8 \times 18 &= 8(10 + 8) \\ &= (8 \times 10) + (8 \times 8) \\ &= 80 + 64 \\ &= 144 \end{aligned}$$

The name "distributive property of multiplication over addition" is usually shortened to "distributive property." And we shall use this abbreviation. It is important to remember, however, that multiplication is distributed over addition. Study carefully the following two examples. In one of them, the distributive property applies; in the other, it does not.

$8(3 + 2) = (8 \times 3) + (8 \times 2)$ Here the distributive property <u>does</u> apply. Multiplication is distributed over addition.

$8 + (3 \times 2) \neq (8 + 3) \times (8 + 2)$ Here the distributive property does <u>not</u> apply. Addition is <u>not</u> distributed over multiplication.

In the problems that follow, you will have a chance to test your understanding of the distributive property by changing indicated products to indicated sums. In the table below are three more examples of this kind of change; each is an illustration of the distributive property.

Indicated Product		Indicated Sum
$4(7 + 3)$	=	$4(7) + 4(3)$
$5(100 + 30)$	=	$5(100) + 5(30)$
$5(150)$	=	$5(100) + 5(50)$

Check Your Reading

1. What is meant by the phrase "multiplication is distributive over addition"?
2. How is the distributive property used to find the product 7×13 mentally?
3. Give an example which shows that addition is <u>not</u> distributive over multiplication.

Oral Exercises 2-5a

Test the truth of these sentences and tell which illustrate the distributive property.

1. $3(5 + 2) = 3(5) + 3(2)$
2. $4(2 + 5) = 4(2) + 4(5)$
3. $5(1 + 10) = 5(1) + 5(10)$

Oral Exercises 2-5a
(continued)

4. $3(2) + 6(3) = 9(2 + 3)$
5. $2 + (3 \times 5) = (2 + 3) \times (2 + 5)$
6. $6 + (3 \times 4) = 6 + 12$
7. $6(10 + 7) = 6(10) + 6(7)$

Which of the following are indicated sums and which are indicated products?

8. $3(4 + 5)$
9. $3(4) + 3(5)$
10. $5(6) + 5(3)$
11. $5(6 + 3)$
12. $6 + (3 \times 4)$
13. $(6 + 3)4$

Problem Set 2-5a

Complete the following sentences so that they express the distributive property.

For example: $2(3 + 5) = ?$
$2(3 + 5) = 2(3) + 2(5)$

1. $6(8 + 4) = ?$
2. $9(7 + 6) = ?$
3. $0(8 + 9) = ?$
4. $9(8 + 11) = ?$
5. $5(8 + 4) = ?$
6. $7(2 + 8) = ?$
7. $3(80 + 3) = ?$
8. $4(100 + 7) = ?$
9. $13(10 + 1) = ?$
10. $18(20 + 2) = ?$

Which of the following sentences are true? Which are true by the distributive property?

11. $4 + (8 \times 5) = (4 + 8) \times (4 + 5)$
12. $5 \times (4 + 2) = (5 + 4) \times (5 + 2)$
13. $4(6 + 5) = (6 + 5)4$
14. $5(4 + 2) = (5 \times 4) + (5 \times 2)$
15. $7 \times (6 \times 4) = (7 \times 6) \times (7 \times 4)$

Problem Set 2-5a
(continued)

16. $5(11 + 3) = 55 + 15$

17. $4(3 + 11) = (4)(3) + (4)(11)$

18. $6(4 + 5) = 6(4) + 5$

Show how you could use the distributive property to perform these multiplications mentally.

Example: $8(42) = 8(40 + 2)$

$= 8(40) + 8(2)$

19. $7(33)$

20. $6(109)$

21. $13(21)$

22. $8(13)$

23. $14(16)$

24. $15(23)$

In the previous section, our work with arrays led us to the distributive property. In working with arrays, we worked only with counting numbers. However, the distributive property applies to all of the numbers with which we have worked. For example,

$$\begin{aligned} 4 \times 8\tfrac{1}{2} &= 4(8 + \tfrac{1}{2}) \\ &= (4 \times 8) + (4 \times \tfrac{1}{2}) \\ &= 32 + 2 \\ &= 34. \end{aligned}$$

Again in this example, the following may be noted:

(1) The indicated product "$4(8 + \frac{1}{2})$" was changed to the indicated sum "$(4 \times 8) + (4 \times \frac{1}{2})$".

(2) The distributive property makes it easy to compute $4 \times 8\frac{1}{2}$ mentally.

Below is another example.

$$\begin{aligned} 6 \times 5\tfrac{1}{3} &= 6(5 + \tfrac{1}{3}) \\ &= 6(5) + 6(\tfrac{1}{3}) \\ &= 30 + 2 \\ &= 32. \end{aligned}$$

Finally, here is a table containing two more examples of changing an indicated product to an indicated sum.

Indicated Product		Indicated Sum
$6(\frac{1}{2} + \frac{3}{2})$	=	$6(\frac{1}{2}) + 6(\frac{3}{2})$
$12(1\frac{3}{4} + 2)$	=	$12(1\frac{3}{4}) + 12(2)$

Check Your Reading

1. How is the distributive property used to multiply $4 \times 8\frac{1}{2}$?
2. Express the indicated product, $4 \times 8\frac{1}{2}$, as an indicated sum.

Oral Exercises 2-5b

Use the distributive property to express the following indicated products as indicated sums.

1. $5(7 + \frac{1}{5})$
2. $4(8 + \frac{1}{2})$
3. $12(2 + \frac{1}{2})$
4. $6(5 + \frac{1}{3})$
5. $4(6 + \frac{3}{4})$
6. $6(\frac{1}{2} + \frac{1}{3})$

We have been using the distributive property to change an indicated product to an indicated sum. For example,

$$8(10 + 2) = 8(10) + 8(2).$$

Of course, it would also be correct to write

$$8(10) + 8(2) = 8(10 + 2).$$

This suggests that the distributive property can be used to "go the other way," to change an indicated sum to an indicated product. In the case above, the indicated sum "8(10) + 8(2)" was changed to the indicated product "8(10 + 2)."

Below are some examples of the use of the distributive property in this way.

<u>Example</u> 1. $13(8) + 13(2)$ indicated sum

$13(8 + 2)$ indicated product

This example suggests again a way in which the distributive property may simplify computation. Instead of multiplying 8 by 13, then multiplying 2 by 13, and then adding, one can simply multiply 10 by 13.

<u>Example</u> 2. $19(7) + 19(3)$

<u>One way</u>	<u>Another way</u> (using distributive property)
$19(7) + 19(3)$	$19(7) + 19(3)$
$133 + 57$	$19(7 + 3)$
190	$19(10)$
	190

Which way is easier?

<u>Example</u> 3. $(.3)(1.73) + (.3)(.27)$

$(.3)(1.73 + .27)$

$(.3)(2)$

$.6$

<u>Example</u> 4.
$$\begin{aligned} 138 + 92 &= 6(23) + 4(23) \\ &= 23(6 + 4) \\ &= 23(10) \\ &= 230 \end{aligned}$$

<u>Check Your Reading</u>

1. Use the distributive property to express the indicated sum, "$8(10) + 8(2)$", as an indicated product.
2. Why is it easier to compute $19(7) + 19(3)$ after an application of the distributive property?

Oral Exercises 2-5c

Express the following indicated sums as indicated products.

1. $2(3) + 2(5)$
2. $18(3.2) + 18(.8)$
3. $(3.1)(7) + (3.1)(3)$
4. $6(19.2) + 6(.8)$
5. $3(37) + 3(3)$
6. $\frac{1}{2}(5) + \frac{1}{2}(3)$
7. $\frac{2}{3}(8) + \frac{2}{3}(4)$
8. $14(.6) + 14(.4)$
9. $9(76) + 9(4)$

Problem Set 2-5c

Perform the indicated operations in the easier way. Show your method.

1. $110(8) + 110(92)$
2. $12(\frac{1}{3} + \frac{1}{4})$
3. $27(\frac{7}{8}) + 27(\frac{1}{8})$
4. $\frac{1}{5}(\frac{7}{8}) + \frac{1}{5}(\frac{1}{8})$
5. $3(\frac{1}{12}) + 3(\frac{11}{12})$
6. $6(\frac{2}{3} + \frac{3}{2})$
7. $9(11 + 9)$
8. $100(98) + 100(2)$
9. $5(\frac{1}{5} + \frac{4}{3})$
10. $7(8 + \frac{4}{7})$
11. $16 + (8 \times 5)$
12. $0(17 + 83)$
13. $88(200 + 1)$
14. $\frac{8}{9}(0 + 9)$
15. $9(\frac{3}{5}) + 9(\frac{2}{5})$
16. $7(3.2) + 7(.8)$
17. $8(43) + 8(57)$
18. $7(\frac{1}{3}) + 7(\frac{2}{3}) + 7(5)$

Show how you would use the distributive property to perform these multiplications mentally. Can you finish the work mentally?

19. $8(13)$
20. $7(108)$
21. $12(13)$
22. $12(24)$
23. $25(14)$
24. $80(12)$
25. $(75)(1001)$
26. $4(6\frac{1}{2})$
27. $(9)(8\frac{1}{3})$
28. $(18)(1\frac{2}{9})$
29. $(\frac{9}{17})(1\frac{1}{9})$
30. $13(2002)$
31. $30(52)$
32. $101(101)$

Problem Set 2-5c
(continued)

Write a common name for each of the following.

33. $3 \times (17 + 4) \times \frac{1}{3}$

34. $\frac{3(17 + 4)}{3}$

35. $\frac{5(8 + 4)}{5}$

Use the distributive property to write the following indicated sums as indicated products. See the example below:

Example: $4(7) + 12$
$4(7) + 4(3)$
$4(7 + 3)$

36. $2(5) + 2(6)$

37. $3(5) + 6$

38. $14 + 21$

39. $5(7) + 15$

40. $4(9) + 12$

41. $20 + 4(8)$

42. $25 + 5(6)$

43. $25 + 30$

44. $18 + 24$

*45. Write a common name for

$$(\frac{1}{2} + \frac{2}{3})11 + (\frac{1}{2} + \frac{2}{3})7.$$

(Hint: Think of $\frac{1}{2} + \frac{2}{3}$ as one numeral, and don't start working until you have thought of a way of doing this exercise which isn't much work.)

*46. Write a common name for

(a) $(\frac{3}{5} + \frac{2}{3})8 + (\frac{2}{3} + \frac{3}{5})7$

(b) $(\frac{2}{7} + \frac{1}{2})10 + (\frac{2}{7} + \frac{1}{2})4$

2-6. Variables

"Take a number, multiply it by 3, and then add 4." This is a simple exercise. If you choose the number 7, to start with, you get 25 as the answer to the exercise. The steps could be shown as follows:

$$7$$
$$3(7)$$
$$3(7) + 4$$

Notice that "3(7) + 4" is a numeral for the number 25. It is not the simplest numeral for 25, but it shows clearly the form of the exercise. Sometimes the form of an exercise is more important than the answer.

Let us consider a second example. "Choose a counting number between 1 and 100, add 5, and divide by 2." This time the starting number is to be taken from the set {2, 3, 4, ..., 99}. If you choose the number 89, the following steps show the process.

$$89$$
$$89 + 5$$
$$\frac{89 + 5}{2}$$

Suppose that your friend John follows the instructions of the exercise. He does not tell you his starting number. You, of course, do not know his final answer. But you do know the pattern of his work. You might think of it as follows:

John's number

John's number + 5

$$\frac{\text{John's number} + 5}{2}$$

Suppose we let the letter "n" represent John's starting number. Then we think as follows:

$$n$$
$$n + 5$$
$$\frac{n + 5}{2}$$

Here "n" represents one number; it happens that you do not know which number it is except that it comes from the set {2, 3, 4, ..., 99}.

In our example we chose the letter "n". We could just as well have chosen a different letter such as "c" or "x".

When a letter is used in the way "n" was used in the above example, it is called a variable.

A <u>variable</u> is a numeral which represents a definite, but unspecified, number chosen from a given set of numbers.

A number that a variable can represent is called a <u>value</u> of the variable. The set of numbers from which the value of the variable may be chosen is called the <u>domain</u> of the variable. Thus, in the example at the beginning of this section, we used n to represent a number. But we were told that n was a counting number and that it had to be taken from the set {2, 3, 4, ..., 99}. That is, the value of the variable had to be chosen from the set {2, 3, 4, ..., 99}. Therefore, in this example, the set {2, 3, 4, ..., 99} is the domain of the variable n.

You remember that we called an expression like "3 + 5" or "8 - 6" a <u>numerical phrase.</u> When a phrase contains a variable, we call it an <u>open phrase.</u> For example, "n + 7" is an open phrase. In the table below are some more examples of open phrases and their word meanings.

Open phrase	Word phrase
$n + 5$	a number increased by 5
$n - 8$	8 less than a number
$6n$	6 times a number
$\frac{n}{12}$	a number divided by 12

Each of the four open phrases in the above table involve the variable "n." In each case, the value of the open phrase depends on the value of n. Suppose that the value of n must be chosen from the set {8, 12, 18}. What we are saying of course is that the set {8, 12, 18} is the domain of n for these examples.

Then if n is 12 (n certainly may be 12, since 12 is in the domain), the value of "n + 5" is 17. What are the corresponding values of "n - 8," "6n," and "$\frac{n}{12}$"? The table below summarizes the answers to these questions together with the same questions in the cases where n is 18 and where n is 8.

n	n + 5	n - 8	6n	$\frac{n}{12}$
12	17	4	72	1
18	23	10	108	$1\frac{1}{2}$
8	13	0	48	$\frac{2}{3}$

Check Your Reading

1. Is a variable a numeral or is it a number?
2. Does a variable represent a definite number?
3. In what way is a variable different from numerals such as "5", "$1\frac{1}{2}$", and ".78"?
4. What is a variable?
5. What is a numerical phrase? Give an example.
6. What is an open phrase? Give an example.
7. What do we mean by a <u>value</u> of a variable?
8. What do we mean by the <u>domain</u> of a variable?
9. What letters can we use for variables? Give some examples.
10. What do we mean by the value of an open phrase?

Oral Exercises 2-6a

Find the possible values of these open phrases if the domain of n is the set: $\{2, 5, 10\}$.

1. $n + 7$
2. $5(n)$ (this can also be written $5n$)
3. $n(11)$
4. $3(n - 1)$
5. $5(n) - 7$
6. $9 - 3(n) + 6(n + 1)$
7. $8(n) - 3(n)$ (this can also be written $8n - 3n$)
8. $7(n + 2) + 4(n)$
9. $\frac{n}{8}$
10. $\frac{n + 6}{4}$
11. $\frac{2(n) - 3}{5}$
12. $\frac{6(n + 3)}{5}$
13. $\frac{(n - 1)(n + 2)}{n + 3}$
14. Would your thinking have been changed in any of the above exercises if we had used a, b, x, or y, instead of n as the variable?

State a mathematical phrase, using a variable, to represent each word phrase.

15. Two more than some number.
16. Seven less than some number.
17. Six times some number.
18. Some number divided by 4.
19. Four more than twice some number.

State a word phrase describing each open phrase.

20. $n + 5$
21. $c - 2$
22. $4x$
23. $\frac{y}{5}$
24. $2n + 3$
25. $3a - 2$
26. $7(n - 2)$
27. $\frac{s}{4} + 5$
28. $(n + 5)(n - 2)$

Problem Set 2-6a

Find the possible values of these open phrases if the domain of n is the set: $\{2, 5, 10\}$.

1. $\frac{2(n + 1)}{3}$

2. $\frac{3(n) - 2}{4}$

3. $\frac{(n - 1)(n + 3)}{n + 1}$

4. $\frac{3(n)(n + 2)}{2(n) + 4}$ (This can also be written $\frac{3n(n + 2)}{2n + 4}$)

5. $\frac{(n - 1)(n + 5)(n + 10)}{2(n + 8)}$

If a has the value 2, b has the value 5, and c has the value 3, find the values of these open phrases.

6. $\frac{3c + 4b}{a}$

7. $7a - 2b + \frac{2c}{a}$

8. $\frac{6b - c}{3} + \frac{3b}{a}$

9. $(3a + b)(b - c)$

10. $7c(a + b)$

11. $(3a - 2c)(2b - 10)$

12. $\frac{1}{2}a + \frac{1}{5}b + \frac{2}{3}c$

13. $\frac{(\frac{7a}{2} + \frac{3b}{2}) - \frac{5c}{2}}{2}$

Using a variable, write a mathematical phrase for each word phrase.

14. Seven less than three times some number.

15. Six more than one half of a number.

16. One half of the sum of a number and six.

17. Seven more than the quotient of some number divided by eleven.

18. Six less than the quotient of a number divided by three.

19. The sum of twice a number and three.

20. The sum of a number and seven all divided by twelve.

Problem Set 2-6a
(continued)

Write a word phrase for each mathematical phrase.

21. $8n + 4$

22. $\frac{2s}{3} - 4$

23. $8(2n - 5)$

24. $\frac{12(n - 5) + 0}{3}$ (Hint: One way would be to state it as an instruction--
take a number, subtract.....)

You have been introduced to the meaning of "variable". How is a variable used? You will be interested to discover that variables are used not so much to find <u>answers</u> as to discover <u>forms</u> of problems or number relationships.

Let's consider a number game:

What number do you get if you take the number of years in your age, multiply by 3, add 12, divide by 3, subtract 2? (Don't tell us your age!)

Perhaps you thought of the following numbers, as the instructions were given: 14, 42, 54, 18, 16. We ask you your final number and when you say 16, we tell you that your age is 14. How did we know?

Try another number, say 34, the age of your father. Let's write your reasoning process this time as well as the result of each calculation. It would look like this:

Instruction	Reasoning Process	Answer
Multiply by 3	$3(34)$	102
Add 12	$3(34) + 12$	114
Divide by 3	$\frac{3(34) + 12}{3}$	38
Subtract 2	$\frac{3(34) + 12}{3} - 2$	36

There are two things we wish to notice. First, how do the two answers you have found so far relate to the numbers you started with? Try some other numbers and repeat the process. Do each of these answers have the same relationship to the original numbers?

We can answer this last question better if we now give our attention to the second point of importance. Look at the last expression in the second column of the table,

$$\frac{3(34) + 12}{3} - 2.$$

This expression shows the <u>form</u> of the problem. No matter what number we start with, the form is the same. If we start with 14, we get

$$\frac{3(14) + 12}{3} - 2.$$

If we start with 40, we get

$$\frac{3(40) + 12}{3} - 2.$$

Suppose we don't tell the number with which we started. What would the form of the problem be? Do you remember that a variable represents an unspecified number from a specified set? Our problem develops as follows, if we denote the beginning number by the variable "n".

$$n$$

$$3n$$

$$3n + 12$$

$$\frac{3n + 12}{3}$$

$$\frac{3n + 12}{3} - 2$$

Does the final open phrase agree with your idea of what the form should be? This open phrase,

$$\frac{3n + 12}{3} - 2,$$

provides us with any possible answer to the problem, depending on your starting number. Let us examine the open phrase "$\frac{3n + 12}{3} - 2$" more closely.

$$3n + 12 = 3n + 3(4),$$

and by the distributive property

$$3n + 3(4) = 3(n + 4).$$

Therefore,

$$\begin{aligned} \frac{3n + 12}{3} &= \frac{3(n + 4)}{3} \\ &= \frac{3}{3} \times (n + 4) \\ &= n + 4 \end{aligned}$$

and our <u>open phrase</u> can be written more simply

$$\begin{aligned} \frac{3n + 12}{3} - 2 &= (n + 4) - 2 \\ &= n + 2. \end{aligned}$$

This tells us that the answer given will always be two more than the starting number. Does this agree with your earlier observation? This was our "secret". This was the way we could tell your age when you gave us your final answer.

There will be many places in this course and in future mathematics where the <u>pattern</u> or <u>form</u> of a problem is much more important than any single answer.

If the domain of the variable in our problem above had been the set S = {1, 2, 3, . . ., 1,000,000}, then our <u>form</u> "$\frac{3n + 12}{3} - 2 = n + 2$" could immediately provide us with any one of a million different answers. If the domain had been the set T of all whole numbers, the <u>form</u> would have provided us with <u>infinitely</u> many answers. Do you agree that this is so?

The discussion above illustrates the great power of methods based upon pattern or form rather than on simple arithmetic. This approach is made possible by the introduction of the idea of <u>variables.</u>

Check Your Reading

1. Are variables used more to find <u>answers</u> or to discover <u>forms</u> of problems or number relationships?
2. If your "starting number" in the number game of the preceding section is 5, what is the final answer?
3. If the domain of the variable in the number game had been the set {1, 2, 3, . . ., 10}, how many different answers would the form "$\frac{3n + 12}{3} - 2 = n + 2$" provide?

Problem Set 2-6b

1. If the sum of a certain number t and 3 is doubled, which of the following would be a correct form:

$$2t + 3 \quad \text{or} \quad 2(t + 3)?$$

2. If 5 is added to twice a certain number n and the sum is divided by 3, which is the correct form:

$$\frac{2n + 5}{3} \quad \text{or} \quad \frac{2n}{3} + 5\ ?$$

Problem Set 2-6b
(continued)

3. If one-fourth of a certain number x is added to one-third of four times the same number, which is the correct form:

$$\frac{1}{4}(x) + \frac{1}{3}(4x) \quad \text{or} \quad \frac{x}{4} + \frac{4}{3}(x)?$$

4. If the number of gallons of milk purchased is y, which is the correct form for the number of quart bottles that will contain it:

$$4y \quad \text{or} \quad \frac{y}{4}\ ?$$

5. If a is the number of feet in the length of a certain rectangle and b is the number of feet in the width of the same rectangle, which is the correct form for the perimeter:

$$a + b \quad \text{or} \quad ab?$$

6. Find a common name for

   (a) $\frac{9}{5}C + 32$, when $C = 100$

   (b) $\frac{h(a + b)}{2}$, when $h = 4$, $a = 6$, $b = 8$

   (c) $P(1 + rt)$, when $P = 500$, $r = 0.04$, $t = 3$

   (d) $\frac{r\ell - a}{r - 1}$, when $a = 4$, $r = 2$, $\ell = 48$

   (e) ℓwh, when $\ell = 24$, $w = 12$, $h = 5$

7. Write the form of the answer, using a variable, for this series of instructions. "Think of a number, add 2, multiply by 3, subtract 6, divide by 3. What is the final number?" Simplify the form if you can. Do you end up with the original number?

8. Do as in Problem 7 for this series of instructions: Think of a number, multiply by 4, add 8, divide by 2, subtract 4, subtract twice your original number. What does the pattern reveal?

Problem Set 2-6b
(continued)

Tell which of the following sentences are true if x has the value 7:

9. $x + 2 = 2 + x$
10. $(x + 3) + 4 = x + (3 + 4)$
11. $x - 5 = 2x - 10$
12. $3(x + 4) = 3x + 12$
13. $x - 10 = 3$
14. $\frac{3}{7}x + 6 = x$
15. $(x - 5)(x - 4) = x - 1$
16. $8x + 9x = 17x$
17. $3x + 2 > 5x$
18. $(x - 4)(x + 7) < \frac{4x}{x}$
19. $11x - 2x > 63$
20. $x(3x + 7) = 3x^2 + 7x$ (Remember: $x^2 = (x)(x)$)

Summary

Among the important ideas of this chapter are the following:

1. Numerals are names for numbers, as distinguished from numbers themselves.
2. The presence of this symbol "=" between two numerals indicates that the numerals represent the same number.
3. The symbol "$\neq$" represents the verb "is not equal to" and is used to indicate that two numerals do not name the same number.
4. The symbol "$>$" represents the verb phrase "is greater than".
5. The symbol "$<$" represents the verb phrase "is less than".
6. Indicated multiplication, for example, of 7 by 3, may be written 7×3, $7(3)$ or $(7)(3)$.

7. In order that the meaning of the numeral shall not be in doubt, we agree to give multiplication precedence over addition and subtraction, unless otherwise indicated, in expressions involving these operations.

8. Parentheses may be used to enclose those parts of an expression to be taken as a numeral.

9. A numerical phrase is any numeral given by an expression which involves other numerals along with the signs for operations.

10. Numerical phrases may be combined to form numerical sentences, which may be either true or false, but not both.

11. A property of an operation is "something which it has" or one of its characteristics.

12. A binary operation is one which is always performed upon two numbers. Addition and multiplication are binary operations.

13. The associative property of addition.
Example: $(2 + 3) + 4 = 2 + (3 + 4)$

14. The commutative property of addition.
Example: $2 + 3 = 3 + 2$

15. The multiplication property of one.
Example: $8 \times 1 = 8$

16. The associative property of multiplication.
Example: $(2 \times 3) \times 4 = 2 \times (3 \times 4)$

17. The commutative property of multiplication.
Example: $2 \times 3 = 3 \times 2$

18. The distributive property (of multiplication over addition).
Example: $2(3 + 4) = 2(3) + 2(4)$

19. A variable is a numeral which represents a definite, though unspecified, number from a given set of numbers.

20. A number which a given variable can represent is called a value of the variable.

21. The set of values of a variable is called its domain.

Review Problem Set

1. Write five different numerals for the number three.

2. What do we mean when we refer to a "common name" of a number?

3. What is the order that we agreed upon for carrying out operations in problems involving addition, subtraction, multiplication, and division?

4. What is a common name for the numerical phrase "$7 + 5 \times 4$"?

5. Which of the following sentences are true?

   (a) $(5 - 3) \times 2 = (5 - 3)2$

   (b) $6 + 4 \times 3 \neq 6 + (4 \times 3)$

   (c) $(5 + 1)(6 + 2) = 5 + (1)(6) + 2$

   (d) $42 = 4(2)$

   (e) $6 \times 2 + 4 \times 3 \neq 6(2 + 4)3$

   (f) $\frac{13 + 1}{7} < 1$

6. Which symbol, "$>$" or "$<$", means "is less than"?

7. Write the following sentences using the symbol "$>$".

   (a) $5 < 7$ (b) $x < 3$ (c) $M < N$

8. What do we mean by a binary operation?

9. Which of the following phrases are indicated products of two numbers?

   (a) 3×6 (d) $(3 + 6)(5 + 2)$

   (b) $3(6 + 5)$ (e) $3 \times 6 \times 5 \times 2$

   (c) $3 \times 6 + 5$ (f) $3 \times 6 + 5 \times 2$

10. Which of the following illustrate the associative property of addition? of multiplication? of neither?

    (a) $4 + (3 + 1) = (4 + 3) + 1$

    (b) $(4 + \frac{1}{2})(3 + 2)(\frac{1}{4}) = (\frac{1}{2} + 4)(2 + 3)(\frac{1}{4})$

(c) $(\frac{3}{4} \times \frac{1}{2}) \times \frac{1}{3} = \frac{3}{4}(\frac{1}{2} \times \frac{1}{3})$

(d) $(\frac{1}{2} + \frac{1}{3})\frac{1}{4} \neq \frac{1}{4}(\frac{1}{2} + \frac{1}{3})$

11. For each of the following statements tell what numeral for one has been used in changing the name of the number.

(a) $\frac{5}{6} = \frac{35}{42}$

(b) $\frac{1}{3} + \frac{5}{12} = \frac{4}{12} + \frac{5}{12}$

(c) $\frac{10}{12} = \frac{15}{18}$

(d) $\frac{24}{32} = \frac{3}{4}$

(e) $\frac{5(x + 2)}{20} = \frac{x + 2}{4}$

12. Which of the following are true sentences? Tell what **property or properties of addition and multiplication are involved in those sentences which you consider to be correct.**

(a) $(3 + 2)7 = 7(2 + 3)$

(b) $(3\frac{1}{2} + 5\frac{1}{2}) + 6\frac{1}{4} = 5\frac{1}{2} + (3\frac{1}{2} + 6\frac{1}{4})$

(c) $5(9 - 8) = 5$

(d) $6(7 + 2) + 4 \neq 4 + (7 + 2)6$

(e) $3 \times 4 + 2 \times 3 = 3(2 + 4)$

(f) $9.2 - 8 = 8 - 9.2$

(g) $3 \times \frac{8}{8} > \frac{5}{5}$

(h) $3 + 4 \times 6 + 3 = 3 + (4)(6)$

(i) $13(2 + 3) \neq (13 + 2)(13 + 3)$

13. Perform the indicated operation in the "easiest" way.

(a) $19(\frac{7}{8} + \frac{1}{8})$

(b) $15(12)$

(c) $9(\frac{2}{5}) + 9(\frac{3}{5}) + 9(29)$

(d) $(203)(101)$

14. What is a variable?

Review Problem Set
(continued)

15. Given that the domain of x is the set $\{4, 5, 6\}$ find the value of each of the following open phrases, for each value of x.

(a) $x + 3$

(b) $7(x + 1) - 5$

(c) $\frac{(2x + 1)(x + 2)}{4}$

(d) $\frac{x - 4}{x + 4}$

(e) $\frac{x^2 + x}{x}$ Can you find an "easy" way?

16. Write an open phrase that gives the form of the following number game.

Think of a number, double it, add two, multiply by four, divide by eight and add seven.

Can you simplify the form? What "trick" can you use to find the answer quickly?

Chapter 3

OPEN SENTENCES AND TRUTH SETS

3-1. Open Sentences.

In chapter 2 we called expressions such as "n + 3", "c - 5", etc., open phrases. They are phrases which contain variables. In the same way we call sentences which contain one or more variables

open sentences.

Thus, the statement "n + 3 = 5" is an example of an open sentence.

Suppose we were asked whether the sentence "n + 3 = 5" is true. Of course, it is impossible to decide until we know what number the variable "n" represents. The sentence

"He is President of the United States."

presents the same type of difficulty. Again we cannot decide whether it is true until someone indicates the person referred to by the pronoun "he".

The open sentence "n + 3 = 5" becomes a true sentence when n is 2. Therefore, we call the number 2 a truth number of this particular open sentence. Can you find other truth numbers for this same sentence?

If we are given an open sentence, how can we determine a value of the variable which will make it a true sentence? Suppose, for example, we consider the open sentence

$$2x + 3 = 18$$

We could begin by guessing various numbers and then trying each one of these in our open sentence in place of the variable "x". In this way we might hope to hit on a "truth" number. This could be a long process. It might also happen that we just never would guess a right number.

It would be much better if we thought carefully about the results of our first guesses and used these to guide us. Let us

use our example to see how the process might work. Our open sentence is

$$2x + 3 = 18$$

As a first guess, we might try 6 for x. Letting x be 6, the sentence

$$2x + 3 = 18$$

becomes

$$2(6) + 3 = 18$$

This is not a true sentence because it says that 15 equals 18. Since the number 15 is less than 18, this means our guess, 6, is not a truth number and is probably too small.

Suppose, then, that we try the number 8. By the same process we see that the sentence

$$2x + 3 = 18$$

becomes

$$2(8) + 3 = 18$$

which is again a false sentence. It now says 19 equals 18, which seems to indicate that 8 is too large.

It seems natural that we would now try a number between 6 and 8, possibly 7. Our sentence now becomes

$$2(7) + 3 = 18$$

which is still a false sentence. What number do you think we should try now? Do you think it should be a number between 7 and 8? If so, why?

If we replace x by $7\frac{1}{2}$, we find that the sentence

$$2x + 3 = 18$$

becomes

$$2(7\tfrac{1}{2}) + 3 = 18$$

which <u>is</u> a true sentence and we have discovered a number, $7\frac{1}{2}$, that makes our open sentence true.

Check Your Reading

1. What is an "open sentence"?
2. Are the following expressions open sentences? Why? Are

they true sentences? Why?

$n + 3 = 5$,

He is President of the United States.

3. Does 6 make the sentence "$2x + 3 = 18$" true? Does $7\frac{1}{2}$? What kind of number do we say $7\frac{1}{2}$ is in relation to the sentence?

Oral Exercises 3-1a

1. Which of these sentences are true if x is $\frac{1}{2}$?

(a) $x + \frac{3}{2} = 2$

(b) $2x = 1$

(c) $2(x + 1) = 3$

(d) $2(x + 1) = x + \frac{5}{2}$

(e) $x + 2 = \frac{5}{2}$

(f) $4x + \frac{3}{2} = 7x$

(g) $4(x + 1) = x$

(h) $2x + 1 = 2(x + 1)$

(i) $x + 2 = 3x + 2 - x$

2. Find a truth number of each of these sentences.

(a) $x + 4 = 6$

(b) $x + 4 = \frac{11}{2}$

(c) $2(x + 2) = 6$

(d) $2(x + 2) = 7$

(e) $3x - 2 = 7$

(f) $3x - 2 = 6$

(g) $3x - 2 = 0$

3. Find the numbers, if any, that make each of these open sentences true.

(a) $x + 4 = 7$

(b) $x + 5 = 7$

(c) $x + 6 = 7$

(d) $6 - x = 2$

(e) $6 - y = 3$

(f) $4m = 14$

(g) $4m = 18$

(h) $5m = 0$

(i) $5m = 5$

(j) $x + 3\frac{1}{3} = 7$

(k) $2x + 5 = 12$

(l) $3m - 2 = 8$

Problem Set 3-1a

1. Tell if the given value of the variable is a truth number of the open sentence.

(a) $x + 5 = 8$; x is 5

(b) $x - 4 = 12$; x is 16

Problem Set 3-1a
(continued)

(c) $2x = 6$; x is 3
(d) $(x + 5)(x + 3) = 24$; x is 1
(e) $(x - 5)(x - 4) = 0$; x is 5
(f) $3x - 2 = 12$; x is 5
(g) $2x + 5 = 3x - 7$; x is 12

2. Find a truth number for each of these open sentences.

(a) $x + 7 = 15$	(d) $2(x + 4) = 17$	(g) $5x + 4 = 12$
(b) $x + 7 = \frac{15}{2}$	(e) $4x - 7 = 18$	(h) $5x + 4 = 19$
(c) $2(x + 4) = 12$	(f) $(x - 5)(x + 8) = 0$	(i) $12 = 2n + 7$

3. Find a truth number of each of the following open sentences.

(a) $2(n + 2) = 14$	(j) $y + 25 = 2y + 7$
(b) $2(n + 4) = 18$	(k) $y + 7 = 2y + 7$
(c) $14 - 3y = 8$	(l) $3a + 5 = 2a + 10$
(d) $17 - 3y = 8$	(m) $3a + 10 = 2a + 15$
(e) $3 + 2y = 10$	(n) $6a + 5 = 5(a + 2)$
(f) $19 + 4y = 61$	(o) $3t - 2 = 5$
(g) $20 - 3y = 8$	(p) $4t - 3 = 8$
(h) $y + 13 = 2y + 7$	(q) $6t + 5 = 14$
(i) $y + 19 = 2y + 7$	(r) $11t - 14 = 44$

In each of the above exercises we found only one number that made the given sentence true. There are open sentences, however, which have many truth numbers. There are also open sentences for which there are no truth numbers. Consider, for example, the expression

$$x > 3$$

As discussed in chapter 2, the symbol ">" means "is greater than". Since our sentence contains the variable x, it is an open sentence. We now wish to determine those values of x which make the sentence true. Can you think of some of them?

We see that the sentence is true when $x = 4$, $x = 5$, $x = 4\frac{1}{2}$, $x = 100$, and so on. In fact the sentence is true when x is any number which is greater than 3.

As a further example, consider the open sentence

$$2n + 1 > 7.$$

Is this a true sentence when $n = 5$? In this case we are really asking the question, is the sentence

$$11 > 7$$

a true sentence? What can you say about this same open sentence when $n = 3$? For $n = 3$ our open sentence becomes

$$2(3) + 1 > 7.$$

Can you see that this is not a true sentence? For $n = 1$ our open sentence becomes

$$2(1) + 1 > 7,$$

which is also a false sentence. From this we should see that the open sentence $2n + 1 > 7$ is true whenever n is a number greater than three. Another way of describing this is to let T be the set of all numbers greater than 3. Then we could say that our sentence is true whenever n is an element of the set T.

Suppose we consider the open sentence

$$x + 2 = x + 3.$$

Let us look for a truth number. The question we are really asking is, "What number plus two equals the same number plus 3?" Notice that a given variable appearing several times in any one sentence must <u>always represent the same number.</u> Thus if x were to have the value 5, our sentence would be

$$5 + 2 = 5 + 3.$$

Is this a true sentence? Do you see that it is impossible to find a value of x which would make our open sentence true? We see, then, that there are open sentences which have no truth numbers. Can you suggest some others?

Let us now examine the open sentence

$$3 + c + 4 = c + 7.$$

What can we say about the truth numbers for this sentence? Can you think of any values of c which will make this sentence false?

From the above examples we can see that there are various types of open sentences. Some of them have one truth number. Others have many truth numbers, and some have no truth numbers at all.

Check Your Reading

1. Does an open sentence always have one truth number? How many truth numbers can an open sentence have?
2. How many numbers of arithmetic make the sentence $2n + 1 > 7$ a true sentence?
3. How many numbers of arithmetic make the sentence $x + 2 = x + 3$ a true sentence?
4. How many numbers of arithmetic make the sentence $x + 1 = 6$ a true sentence?

Oral Exercises 3-1b

1. Which of the following open sentences are true if the value of x is 9?

(a)	$x = 9$	(i)	$x \neq x$
(b)	$x = 5$	(j)	$x = x$
(c)	$4x \neq 36$	(k)	$3x = 2x$
(d)	$x < 4$	(l)	$x + 5 > x + 4$
(e)	$x + 5 = 14$	(m)	$x + 3 > 7$
(f)	$x - 5 > 0$	(n)	$x = x + 1$
(g)	$x + 2 = x + 8$	(o)	$3x < 7$
(h)	$3x - 2x = x$	(p)	$2x = x + 1$

2. Which of the open sentences in Exercise 1 have no truth number?
3. Which of the open sentences in Exercise 1 have exactly one truth number?
4. Which of the open sentences in Exercise 1 have infinitely many truth numbers?

Problem Set 3-1b

For each of the following open sentences describe the set of numbers such that each element of the set makes the open sentence true. (Remember a set may be described by listing the elements in the set.)

1. $x = 5$
2. $x + 3 > 7$
3. $m = m + 1$
4. $3x > 15$
5. $x + 2 < 10$
6. $8 = x + 5$
7. $2y = y + y$
8. $3k = 0$
9. $x + 1 > x$
10. $m - m = 0$
11. $4 < 2x$
12. $4x + 3 > 4x$
13. $2 = 3a$
14. $5(y - 6) = 0$
15. $0 \times b = 5$
16. $(a + 3) + 2 = a + (3 + 2)$
17. $\frac{x}{3} < 9$

*18. $3y + 2 < 17$

*19. $2y + 2 \neq y + 2$

*20. $2y + 2 = y + 2$

Sometimes a variable occurs in an expression which has the form s^2. This means "s multiplied by s", and is read "s squared". Consider the open sentence

$$s^2 = 9.$$

This becomes a false sentence if we let s be 5, and it is true if s is 3. Can you see why?

An interesting thing happens when we try to find the truth numbers for the following open sentence

$$x^2 + 2 - 3x = 0$$

in which the domain of the variable is the set

$$S = \{0, 1, 2, 3, 4\}$$

What elements of the set S are truth numbers for our open sentence? If you keep working at this problem, you may find that more than one of these elements is a truth number.

Check Your Reading

1. What is a common name for $(3)^2$? For $(4)^2$?
2. What does "s^2" mean? How do we read it?
3. What number is a truth number of the sentence "$s^2 = 9$"?
4. What number is a truth number of the sentence "$x^2 + 2 - 3x = 0$"? Are there more than one?

Oral Exercises 3-1c

1. For the following sentences, let the domain of the variable be the set $S = \{0, 1, 2, 3, 4, 5\}$. Find the truth numbers.

(a) $s^2 = 0$	(f) $y^2 = 2$
(b) $y^2 = 4$	(g) $m^2 = \frac{1}{2}$
(c) $x^2 = 25$	(h) $x^2 + 1 = 10$
(d) $9 = x^2$	(i) $x^2 + 2 = 6$
(e) $y^2 = 30$	(j) $2x^2 + 4 = 36$

2. What is a common name for these expressions?

(a) $(5)^2$	(f) a^2, if a is 12
(b) $(10)^2$	(g) m^2, if m is 8
(c) 6^2	(h) n^2, if n is 14
(d) 11^2	(i) n^2, if n is 1
(e) x^2, if x is 9	(j) n^2, if n is 0

3. Find the values of the variables that make these open sentences true. Let the counting numbers be the domain of the variable.

(a) $x^2 = 4$	(g) $x^2 = 14^2$
(b) $x^2 = 16$	(h) $x^2 = 0$
(c) $64 = x^2$	(i) $x^2 = 1$
(d) $x^2 = 225$	(j) $x^2 + 1 = 26$
(e) $x^2 = 10000$	(k) $x^2 - 1 = 8$
(f) $x^2 = 3^2$	

Problem Set 3-1c

Find the values of the variables which make the following open sentences true. Let the domain of the variables be the set of numbers of arithmetic.

1. $x^2 = 9$
2. $25 = x^2$
3. $25 = x^2 + 16$
4. $x^2 = 3 + 1$
5. $(x + 1)^2 = 4$
6. $x^2 = 0$
7. $x^2 = 1$
8. $x^2 = x$
9. $x^2 = (7)^2$
10. $x^2 = (5.1)^2$
11. $x^2 + 6 - 5x = 0$
12. $0 = 2x^2 + 4 - 6x$

*13. $x^2 = \frac{9}{16}$

*14. $x^2 = 1 + \frac{9}{16}$

*15. $(x + 1)^2 = \frac{16}{9}$

3-2. Truth Sets of Open Sentences.

In the preceding section we have been working with the problem of finding truth numbers for certain open sentences. Let us review quickly some of the examples we used by means of the following table:

Open Sentence	Truth Numbers
$n + 3 = 5$	2
$x > 3$	all numbers greater than three
$x + 2 = x + 3$	no numbers at all
$3 + c + 4 = c + 7$	all numbers
$x^2 + 2 - 3x = 0$	1, 2

From these examples and others which we have studied we can see that the truth numbers for any given open sentence can be thought of as forming a set. We call this set the truth set of the open sentence.

As you remember from chapter one, sets may be indicated either by a list or by a verbal description. You should also

remember that a set with no elements in it is called the null set, or the empty set. The symbol for this set is "$\emptyset$".

In the language of sets our table would look as follows:

Open Sentence	Truth Set
$n + 3 = 5$	$T = \{2\}$
$x > 3$	T: the set of all numbers greater than three
$x + 2 = x + 3$	$T = \emptyset$
$3 + c + 4 = c + 7$	T: the set of all numbers
$x^2 + 2 - 3x = 0$	$T = \{1, 2\}$

It is important to remember that in forming the truth set for a given open sentence we must include all of the truth numbers. Furthermore there must be no elements in our set which are not truth numbers.

In dealing with truth sets of open sentences we must also pay particular attention to what we called the domain of the variable. You recall that the domain of a variable is the set of all numbers from which the value of the variable may be chosen.

In certain examples and problems the domain of the variable will be stated specifically. On the other hand, when no mention is made of the domain we shall assume in this chapter that the domain in question will be the set of all numbers of arithmetic for which the given sentence has meaning.

Suppose, for example, we consider the following problem: Let the open sentence be

$$x > 3$$

and let the domain of x be the set

$$D = \{0, 1, 2, 3, 4, 5\}.$$

Find the truth set.

This means that we are to pick out all the elements from the set D which make our open sentence true. Can you do this?

We see that our truth set is the set $T = \{4, 5\}$. We also see that the set of numbers $F = \{0, 1, 2, 3\}$ is the set of numbers which make the open sentence false. One way of looking at this particular example is to say that the open sentence sorts the

set D into two subsets, the truth set T, and the set F.

Let's take another example. Consider the open sentence

$$x + 4 = 7.$$

Nothing is said here about the domain of x. In this case we assume therefore that the domain consists of all the numbers of arithmetic. What is the truth set? What is the set of numbers which make the sentence false?

Once again we see that this open sentence sorts the set of all numbers into two subsets:

$T = \{3\}$ and F the set of all numbers except 3.

From these examples we see that an open sentence actually does act as a sorter. Just as you might sort a deck of cards into two sets, black and red, an open sentence sorts a set of numbers into two sets: one, the truth set, and another, the set of all numbers in the given domain which make the sentence false.

Check Your Reading

1. If we are going to describe the truth set of a sentence, what must we be careful to consider?
2. What do we call the set from which values of the variable may be selected?
3. What is the truth set of a sentence?
4. What is the truth set of the open sentence $x > 3$ if the domain of x is the set $\{0, 1, 2, 3, 4, 5\}$?
5. If the domain of the variable of a sentence is not stated, what will the domain be?

Oral Exercises 3-2a

If the domain of x is the set $\{0, 1, 2\}$ find the truth set of each of the following sentences.

1. $x + 2 = 2$
2. $x > 0$
3. $x + x = 2$
4. $x < 2$
5. $3x = 6$
6. $3x + 1 = 4$
7. $3x + 2 = 8$
8. $3 = 3 + 3x$
9. $x + 4 = 2$
10. $3x = 2x$

Oral Exercises 3-2a
(continued)

11. $x + 5 > 12$
12. $x + 3 > 3$
13. $x + 5 > 5$
14. $2x + 5 = 8$
15. $8 + x = 10$
16. $2(x + 4) = x + 10$
17. $5x + 8 = 2(2x + 5)$
18. $3x > 0$
19. $3 < 3x$
20. $3x > 6$

Problem Set 3-2a

1. For each of the open sentences determine the truth set T, and the set F of numbers which make the sentence false, if the domain of the variable is the set $W = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

   (a) $x + 2 = 5$
   (b) $3n + 2 = 14$
   (c) $x > 5$
   (d) $x \neq 5$
   (e) $y < 6$
   (f) $m + \frac{1}{2} = 4$

2. Do as in Problem 1, but list only the truth set T. The domain of the variable is the set $W = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

   (a) $y < 5$
   (b) $2x + 3 = 7$
   (c) $x^2 + 12 - 7x = 0$
   (d) $t = t + 1$
   (e) $x^2 = x$
   (f) $\frac{5m + 10}{3} = 10$
   (g) $x > 8$
   (h) $x = 8$
   (i) $x < 8$
   (j) $x \neq 8$

3. Below several domains are described. For each, consider the open sentence $x + 2 < 9$ and then determine the two sets T and F as in exercise 1.

   (a) $\{2, 3, 4, 5, 6, 7, 8, 9\}$
   (b) $\{0, 10, 20, 30, 40, 50\}$
   (c) $\{3, 5, 7, 9, 11\}$
   *(d) the set of numbers greater than 4
   *(e) the set of numbers less than 10
   *(f) the set of numbers greater than 8

Problem Set 3-2a
(continued)

4. Follow the same instructions as in problem 3 but list only the set T.

(a) $\{0, 1, 2, 3, 4\}$

(b) $\{6, 7, 8\}$

(c) $\{7, 8, 9\}$

*(d) the set of numbers greater than 7

*(e) the set of counting numbers

*(f) the set of whole numbers

5. In each of the following test whether or not the given number belongs to the truth set of the open sentence.

(a) $7 + x = 12$; 5

(b) $3 + t \neq 8$; 4

(c) $m + 2 < 7$; 6

(d) $2s + 3 > 10$; 3

(e) $x^2 + 2x + 1 = 0$; 1

(f) $(x - 2)^2 = 4$; 4

(g) $\frac{3x + 2}{5} = 4$; 1

(h) $3(x - 1) + 4 = 12$; $\frac{11}{3}$

(i) $2(x + 3) = 2x + 5$; 3

6. Write an open sentence whose truth set contains

(a) No elements

(b) Exactly one element

(c) all numbers of arithmetic

(d) an infinite number of elements but not all numbers of arithmetic

7. In the following exercise consider the domain $R = \{0, 1, 2, 3, 4, 5\}$. Find, and list when possible, the truth sets of the following sentences.

(a) $4 + x = 6$

(b) $4x + 3 = 6$

(c) $2x > 5$

(d) $x + 4 < 6$

8. Do problem 7 again, but this time consider the domain to be <u>all</u> numbers between 0 and 5. Which of these truth sets are finite sets?

9. Find the truth sets of the following sentences:

(a) $\frac{1}{3} + x = 5$

(b) $4 + \frac{6}{x} = 7$

(c) $x + \frac{1}{x} = 2$

(d) $2x + 1 = 2(x + 1)$

(e) $\frac{1}{3}x = 4$

(f) $\frac{2}{3}x = 4$

Problem Set 3-2a
(continued)

(g) $\frac{3}{3}x = 4$

(h) $\frac{4}{3}x = 4$

(i) $\frac{5}{3}x = 4$

(j) $\frac{x}{x} = 1$

(k) $\frac{x}{x} = 2$

In our discussion of sentences in chapter 2 we used the symbols

$$=,\ \neq,\ <,\ >.$$

We must be sure that we completely understand the exact meanings of these symbols. Now suppose we wish to write an open sentence which says that a number n is greater than <u>or</u> equal to 7. Can you think of a short way of doing this? A convenient way, and one which is most commonly used, is the following:

$$n \geq 7,$$

where, as we see, the symbol is a combination of "=" and ">". Likewise, the open sentence

$$n \leq 10$$

means "n" is less than or equal to "10".
The truth set of the first sentence consists of the number 7 and all numbers greater than 7. The truth set of the second sentence consists of the number 10 and all numbers less than 10. As before, we assume that the domain of the variable is the set of all numbers of arithmetic for which the sentence has meaning, since we have not indicated otherwise. However, if we chose as the domain of x, the set of all even numbers; that is, if

$$D = \{0, 2, 4, 6, \ldots\},$$

then the open sentence

$$x \leq 10$$

would have as its truth set

$$T = \{0, 2, 4, 6, 8, 10\}.$$

What is the truth set of the open sentence

$$n \geq 7,$$

if the domain of x is the set of all odd numbers?

Check Your Reading

1. What does the symbol "$\geq$" mean? What does "$\leq$" mean?
2. What does the sentence "$x \leq 10$" say?
3. If the domain of x is the set of all even numbers, what is the truth set of the open sentence "$x \leq 10$"?

Oral Exercises 3-2b

If the domain of x is the set [0, 1, 2, 3, 4, 5], what is the truth set for each of these sentences?

1. $x \geq 2$	6. $x \leq 0$	11. $x + 5 \geq 14$
2. $x > 3$	7. $x \leq 7$	12. $x + 2 \leq 4$
3. $x \geq 5$	8. $x \leq 4$	13. $2(x + 1) \geq 3$
4. $x \geq 7$	9. $x + 1 \leq 6$	14. $3x + 1 \geq 2$
5. $x \geq 0$	10. $x + 5 \geq 0$	15. $x + 4x \leq 12$

Problem Set 3-2b

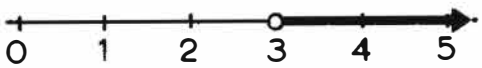
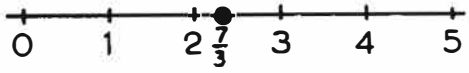
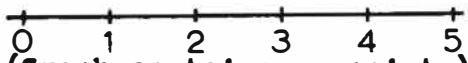
Find the truth sets of the following sentences.

1. $2(x + 2) = 6$	6. $x + 4 > 4 + x$	11. $3x \leq 6$
2. $2x + 4 > 6$	7. $x + 4 = 4 + x$	*12. $7x + 2 \geq 12$
3. $2x + 4 \leq 6$	8. $x + 4 \geq 4 + x$	*13. $4x - 2 \leq 2x - 1$
4. $2x + 4 \geq 6$	9. $3x \geq 0$	*14. $4x - 2 = 2x - 1$
5. $2(x + 2) < 6$	10. $3x \geq 3$	15. $x \geq x + 2$

3-3. Graphs of Truth Sets.

As we have seen in chapter 1, the graph of a set, S, of numbers is the set of all points on the number line that correspond to the numbers in the set.

Thus, the graph of the truth set of an open sentence is the set of all of the points on the number line whose coordinates make the open sentence true. Let us see what the graphs of a few open sentences look like.

Sentences	Truth Set	Graph
$x = 2$	$\{2\}$	
$x > 3$	All numbers greater than 3	
$1 + x < 4$	All numbers from 0 to 3, including 0, not including 3.	
$x \geq 2$	All numbers greater than or equal to 2.	
$3 + x = 7$	$\{4\}$	
$y(y + 1) = 3y$	$\{0, 2\}$	
$3y = 7$	$\{\frac{7}{3}\}$	
$x + 1 = x$	$\emptyset$	 (Graph contains no points)

You will notice that we indicate that a point is in the graph if it is marked with a heavy dot, but is not included if it is circled. The heavy lines indicate that all the points of that portion of the number line belong to the graph. The arrows indicate that the heavy line is assumed to continue without end.

Problem Set 3-3

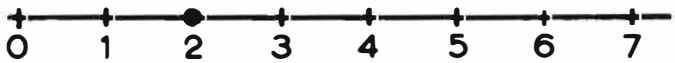
1. Draw the graph of the truth set of each of the following open sentences:

(a) $y = 3$	(f) $2x = 5$	(k) $m \geq 3$
(b) $x + 1 = 2$	(g) $2x > 5$	(l) $\frac{y}{3} = 4$
(c) $x > 2$	(h) $3 + y > 4$	(m) $5 < 5n + 5$
(d) $3 + y = 4$	(i) $3 + y < 4$	(n) $x \neq 0$
(e) $3 + y \neq 4$	(j) $m \leq 3$	(o) $x \neq 2$

Problem Set 3-3
(continued)

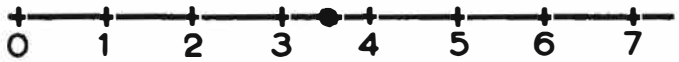
2. Determine whether the indicated set of points is the graph of the truth set of the given open sentence.

(a) $2 + x = 4$



(b) $3x = 5$

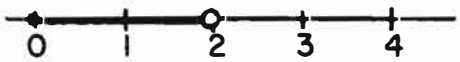
(c) $2y = 7$



(d) $x > 1$

(e) $x \leq 3$

3. For each of the following find an open sentence whose truth set is represented by the given graph.

Example: 

The sentence whose truth set is represented by this graph is $x < 2$.

(a)

(b)

(c)

(d)

3-4. Compound Open Sentences and their Graphs.

In mathematics we are often called upon to deal with expressions of the following form:

$$n + 1 = 5 \quad \text{and} \quad n < 7$$

Your first impression may be that we have written two sentences. But if you read the sentence from left to right, it is one compound sentence with the connective and between two clauses. We shall use the word clause to mean a sentence which is part of a compound sentence. In the compound sentence above we call $n + 1 = 5$ a clause and $n < 7$ a clause. Each clause has its own truth set.

In section 2 we learned that every open sentence has associated with it a truth set. How, then, shall we determine the truth set of the compound sentence itself? We do this by first finding the truth sets for the individual clauses on each side of the connecting word "and". In our example the clause on the left is

$$n + 1 = 5.$$

Do you see that its truth set is

$$L = \{4\}?$$

The truth set for the clause on the right is

R: the set of all numbers less than 7.

For convenience we have used the letters L and R to indicate the truth sets of the left and right clauses. To determine the truth set of the compound open sentence we use two ideas. First we say that a compound sentence with the connecting word "and" is true when the left clause is true and the right clause is true. Otherwise it is false. For example the compound sentence

$$4 + 3 = 7 \quad \text{and} \quad 3 + 10 = 13$$

is a true sentence but

$$4 + 3 = 7 \quad \text{and} \quad 5 + 10 = 11$$

is a false sentence. Second, we remember that the truth set for any open sentence is the set of all numbers which make that particular sentence true. From this we can then say that

> the truth set of a compound sentence with connecting word "and" consists of all those numbers and only those numbers which are in both of the individual truth sets.

In other words, if an element is in the set L, and if it is also in the set R, it then belongs to the set C, where we use the letter C to represent the truth set of the compound open sentence. If an element is not in both sets L and R, then it is not in C. How, then, would you describe the set C for the example which we have been studying? Do you see that the only

element in R which is also in L is the number 4? It should then be clear that

$$C = \{4\}.$$

As a further example, consider the compound open sentence

$$x \geq 5 \quad \text{and} \quad x < 9.$$

For this example let us say that the domain of x is to be the set of whole numbers. To find C, that is, to find the truth set for the compound sentence we follow the process used above. Remembering the meaning of the symbol "$\geq$" we see that the truth set for the clause on the left is

$$L = \{5, 6, 7, \ldots\}.$$

See if you also agree that the truth set for the clause on the right is

$$R = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}.$$

What is the truth set of the compound sentence? Do you agree that it should be $C = \{5, 6, 7, 8\}$? This is the truth set of the compound sentence because these are the elements that are in L and also in R.

As we have pointed out before, there are some open sentences which have no truth numbers. Their truth set, in other words, is the empty set. This can also happen in the case of a compound sentence. Here is an example.

$$n + 2 = 7 \quad \text{and} \quad n < 4.$$

The truth set of the left clause is $L = \{5\}$. Is the number 5 an element of the truth set of the clause on the right? Since it is not, we see that the truth set of this compound sentence is $\emptyset$.

Check Your Reading

1. When is a compound sentence with connective "and" true?
2. When is it false?
3. The truth set of the sentence "$n + 1 = 5$" is $\{4\}$, and the truth set of "$n < 7$" is the set of all numbers less than 7. What is the truth set of "$n + 1 = 5$ and $n < 7$"?

Oral Exercises 3-4a

1. Is the sentence "$8 - 1 = 7$" true? Is the sentence "$5 + 4 = 9$" true? Is the sentence "$8 - 1 = 7$ and $5 + 4 = 9$" true? Why?

2. Is the sentence "$13 - 7 = 6$" true? Is the sentence "$11 + 12 = 25$" true? Is the sentence "$13 - 7 = 6$ and $11 + 12 = 25$" true? Why?

3. Is the sentence "$5\frac{1}{2} + 3\frac{1}{3} = 9$" true? Is the sentence "$9 + 18 = 37$" true? Is the sentence "$5\frac{1}{2} + 3\frac{1}{3} = 9$ and $9 + 18 = 37$" true? Why?

4. Tell which of the following compound sentences are true and which are false. If false, why?
   (a) $8 + 19 = 17$ and $17 - 9 = 8$
   (b) $16 \neq 8 + 8$ and $45 = 1149 - 1104$
   (c) $37 + 8 = 45$ and $9 - 6 = 2$
   (d) $5 + 8 = 12$ and $19 \times 725 = 15225$
   (e) $9 \times (13 \times 7) - (48 + 96) = 324$ and $3 + 4 = 12$

5. Let the domain of x be the set of whole numbers. What is the truth set of the following compound sentences?
   (a) $x > 3$ and $x < 5$
   (b) $x \geq 3$ and $x \leq 5$
   (c) $x = 10$ and $x < 14$
   (d) $x > 10$ and $x < 5$

Problem Set 3-4a

1. Which of these compound sentences are true? Which are false? Why?
   (a) $7 + 2 = 9$ and $8 < 10$
   (b) $7 + 2 \neq 9$ and $8 = 10$
   (c) $7 + 2 = 9$ and $8 \neq 10$
   (d) $7 + 2 = 9$ and $8 = 10$
   (e) $6 > 4 + 1$ and $3 + 1 = 4$
   (f) $6 = 4 + 1$ and $3 + 1 = 4$
   (g) $6 > 4 + 1$ and $3 + 1 \neq 4$
   (h) $6 \neq 4 + 1$ and $3 + 1 \neq 4$
   (i) $4 < 5$ and $5 = 4 + 2$
   (j) $.6 = .5 + .1$ and $.9 \neq .7 + .2$

Problem Set 3-4a
(continued)

(k) $\frac{3}{2} + \frac{1}{2} \neq 4$ and $\frac{3}{2} + \frac{1}{2} > 1$

(l) $3 + 2 > 9 \times \frac{1}{3}$ and $4 \times \frac{3}{2} \neq 5$

2. Let the domain of the variable be the set of whole numbers. Find the truth sets of the following compound sentences.

(a) $x \geq 5$ and $x = 12$
(b) $x < 7$ and $x \geq 4$
(c) $n^2 = 9$ and $n \geq 0$
(d) $n + 4 = 6$ and $n > 5$
(e) $n + 4 = 6$ and $n < 5$
(f) $n + 4 = 6$ and $n < 2$
(g) $n + 4 > 6$ and $n < 5$
(h) $t \geq 3$ and $t \leq 3$

We have just studied compound sentences connected by the word "and". There is another type of compound open sentence which we shall now study. This is the type which has the connecting word "or". For example,

$$x + 5 = 7 \quad \text{or} \quad x > 10$$

is a sentence of this type. To determine the truth set of this type of sentence we begin the process as before. We must first determine the truth sets of the individual clauses to the right and left of the connecting word "or". But here is the difference! If our compound sentence has the connecting word "or", we say that this type of open compound sentence is true if either the left clause is true, or the right clause is true, or if both clauses are true. Otherwise, it is false. For example the compound sentence

$$4 + 3 = 7 \quad \text{or} \quad 5 + 10 = 11$$

is a true sentence. Also the compound sentence

$$4 + 3 = 7 \quad \text{or} \quad 8 + 7 = 15$$

is a true sentence. But the compound sentence

$$4 + 3 = 9 \quad \text{or} \quad 8 + 10 = 9$$

is a false sentence. This tells us that

> the truth set of a compound sentence with the connecting word "or" consists of all those numbers which are in at least one of the truth sets of the individual clauses.

Consider the example

$$x + 5 = 7 \quad \text{or} \quad x > 10.$$

If we examine the individual clauses to determine their truth sets, we see that

$$x + 5 = 7$$

has the truth set

$$L = \{2\}.$$

The right clause has the truth set

R: the set of all numbers greater than 10.

What is the truth set of the compound sentence? Do you see that it can be described as

C: the number 2 and all numbers greater than 10?

In other words, we include the elements in the set L and the elements in the set R.

As a second example, consider the open compound sentence

$$x > 5 \quad \text{or} \quad x + 2 = 10.$$

Here we see that L is the set of all numbers greater than 5. It should also be clear that

$$R = \{8\}.$$

In this case the number 8 is already included in the truth set of the left clause. In describing sets we do not count an element twice. Therefore we can say that the truth set of the compound sentence is

C: the set of all numbers greater than 5.

These ideas will become easier to understand as we take up the study of graphs of the truth sets of compound open sentences.

Check Your Reading

1. When is a compound sentence with the connective "or" true?

2. When is it false?

Oral Exercises 3-4b

1. Is the sentence "$8 - 1 = 7$" true? Is the sentence "$5 + 4 = 9$" true? Is the sentence "$8 - 1 = 7$ or $5 + 4 = 9$" true? Why?

2. Is the sentence "$9 + 8 = 17$" true? Is the sentence "$13 - 7 = 5$" true? Is the sentence "$9 + 8 = 17$ or $13 - 7 = 5$" true? Why?

3. Is the sentence "$22 - 11 = 33$" true? Is the sentence "$9 \times 12 = 96$" true? Is the sentence "$22 - 11 = 33$ or $9 \times 12 = 96$" true? Why?

4. In each of the following state whether the sentence is true or whether it is false and tell why.

   (a) $7 + 3 = 10$ or $99 \times 108 = 17$

   (b) $\frac{15}{3} = \frac{3}{15}$ or $22 - 10 = 12$

   (c) $\frac{3 + 4}{3 \times 4} = 1$ or $\frac{16}{2} = 8$

   (d) $\frac{999}{111} = 888$ or $3 \times 2 = 16$

   (e) $9 \times 9 = 81$ or $7(325) = 2275$

5. If the domain of the variable is the set of whole numbers, find the truth set of each of the following sentences.

   (a) $x < 4$ or $x < 2$

   (b) $x < 4$ or $x = 2$

   (c) $x = 13$ or $x < 5$

   (d) $x + 2 = 4$ or $x + 1 < 2$

   (e) $3x = 6$ or $x = 1$

   (f) $4x < 8$ or $x = 2$

   (g) $6x < 2$ or $x = 5$

   (h) $6x = 2$ or $x < 1$

   (i) $6x > 2$ or $x \leq 1$

   (j) $3x \leq 3$ or $x \geq 1$

Problem Set 3-4b

1. Which of the following sentences are true? Which are false? Why?

   (a) $5 + 4 = 9$ or $5 + 2 \neq 7$

   (b) $3 + 2 \neq 6$ or $3 + 2 < 4$

   (c) $\frac{2}{4} + \frac{3}{4} = 1$ or $\frac{1}{4} + \frac{5}{4} = 1$

Problem Set 3-4b
(continued)

(d) $.2 + 1.5 = 3.5$ or $.2 + 1.5 = 1.7$

(e) $3 + 4 = 7$ or $3 + 4 > 7$

(f) $3 + 4 > 7$ or $3 + 4 < 7$

(g) $9(8) = 81$ or $8(72) + 5 \geq 581$

(h) $\frac{3(5 + 4)}{27} = 1$ or $7(6 - 2) \neq (6 - 2)7$

(i) $(8 - 5)(5 + 8) = (8 + 5)(8 - 5)$ or $5(5) = 10$

(j) $(3)(3) = 6$ or $7\big((6)(2)\big) \neq \big((7)(6)\big)(2)$

2. Describe the truth sets of the following sentences:

(a) $x = 5$ or $x > 6$
(b) $3x < 9$ or $x = 3$
(c) $2x > 4$ or $5x \leq 10$
(d) $x < 5$ and $4x > 20$
(e) $3x < 1$ or $2(3x + 1) = 8$

3. Which of the following sentences are true when x is 8?

(a) $3x < 2$ or $x + 9 = 17$

(b) $5x < 5$ and $4x > 2$

(c) $3x - 1 = 23$ or $2x \geq 16$

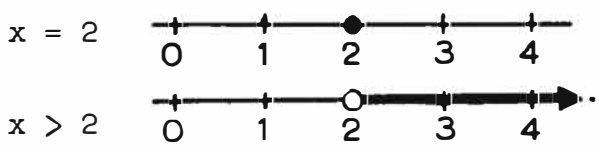
(d) $2x - 4 > 12$ or $2(x + 3) < 22$

(e) $\frac{3}{2}x < 10$ or $9x = 72$

Graphs of truth sets of compound open sentences present special features. For example, consider the open sentence

$$x > 2 \text{ or } x = 2.$$

The right and left clauses of the sentence and the corresponding graphs of their truth sets are:



Remember that a compound sentence with connective <u>or</u> is true if at least one of its parts is true. If a number belongs to the truth set of the sentence "$x = 2$" or to the truth set of the sentence "$x > 2$" or to both truth sets, it is a number belonging to the truth set of the compound sentence "$x = 2$ or $x > 2$". So the truth set of the compound sentence is

C: the set of all numbers greater than or equal to 2.

Any number less than 2 makes both parts of the compound sentence false and so does not belong to the truth set.

The graph of the truth set of

$$x = 2 \quad \text{or} \quad x > 2$$

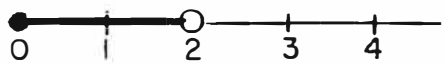
formed by combining the two graphs above, is

Do you see that this is the same as the graph of the truth set of $x \geq 2$?

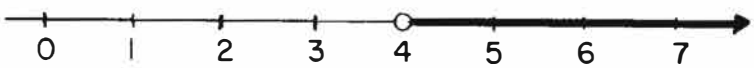
What is the graph of the truth set of the sentence

$$x < 2 \quad \text{or} \quad x > 4?$$

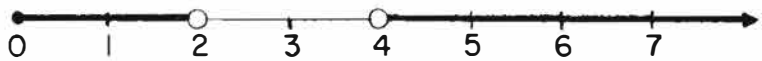
Let us draw the graph of the truth set of the sentence "$x < 2$".



Now draw the graph of the truth set of the sentence "$x > 4$".



Then, we place both graphs together on the one line; we now have the graph of the compound sentence $x < 2$ or $x > 4$.



The graph of the truth set of a compound sentence with the connective <u>or</u> consists of all points which belong to at least one of the graphs of the two clauses.

Check Your Reading

1. How do we draw the graph of the truth set of a compound sentence with connective "or" if we have the graphs of the two simple sentences?
2. What is the graph of the truth set of the sentence "$x = 2$"? Of the sentence "$x > 2$"? Of the sentence "$x = 2$ or $x > 2$"?
3. What is the graph of the truth set of the sentence "$x < 2$"? Of the sentence "$x > 4$"? Of the sentence "$x < 2$ or $x > 4$"?

Problem Set 3-4c

Draw the graphs of the truth sets of the following compound sentences:

1. $x < 1$ or $x > 3$
2. $x = 3$ or $x < 3$
3. $x \leq 6$
4. $x > 6$ or $x \leq 3$
5. $x \neq 3$ (Hint: First write this sentence with two clauses.)
6. $x > 3$ or $x = 2$
7. $x < 7$ or $x = 10$
8. $x < 7$ or $x = 5$
9. $x < 7$ or $x > 6$
10. $x = 5$ or $x = 3$
11. $x \neq 5$ or $x \neq 3$

What is the truth set of an open sentence such as "$x > 2$ <u>and</u> $x < 4$"?

The graphs of the truth sets of the individual clauses are:

$x > 2$

and

$x < 4$

Since a compound sentence with connective <u>and</u> is true only when both clauses are true, a number must belong to the truth sets of both

$$x > 2$$

and

$$x < 4$$

to belong to the truth set of the compound sentence

$$x > 2 \quad \text{and} \quad x < 4.$$

Can you see that the <u>truth set</u> is the set of all of the numbers that are greater than 2 and are less than 4? The graph of the truth set of

$x > 2$ and $x < 4$ is 

Notice that this includes only the numbers between 2 and 4 (it does not include the 2 and the 4). All other numbers make the sentence false. For example, 5 is in the truth set of $x > 2$, but not in the truth set of $x < 4$, and so it is not in the truth set of the compound sentence.

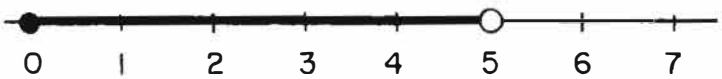
Now draw the graph of the truth set of the sentence

$$x \geq 2 \quad \text{and} \quad x < 5.$$

Remember that this is a compound sentence.

The graphs of the individual clauses are:

$x \geq 2$ 

$x < 5$ 

Which points of each graph are common to both graphs? These common points are the graph of the compound sentence

$x \geq 2$ and $x < 5$ 

The graph of the truth set of a compound sentence with connective <u>and</u> consists of all of the points which are common to the graphs of the two individual clauses.

<u>Check Your Reading</u>

1. What must be true about a number that makes a compound sentence with connective "and" true?

2. What is the graph of the truth set of "$x > 2$"? What is the graph of the truth set of "$x < 4$"? What is the graph of the truth set of "$x > 2$ and $x < 4$"?

Check Your Reading
(continued)

3. What is the graph of the truth set of "$x \geq 2$"? What is the graph of the truth set of "$x < 5$"? What is the graph of the truth set of "$x \geq 2$ and $x < 5$"?

4. What points does the graph of a compound sentence with connective "and" consist of?

Oral Exercises 3-4d

1. Will the number 6 be in the truth set of "$x < 4$"? Will the point labeled 6 be on the graph of the truth set of "$x < 4$ and $x > 2$"?

2. Will the number 3 be in the truth set of "$x < 4$"? Will it be in the truth set of "$x > 2$"? Will the point 3 be on the graph of the truth set of "$x < 4$ and $x \geq 2$"?

3. Will the number 7 be in the truth set of "$x > 2$"? Will it be in the truth set of "$x < 4$"? Will the point labeled 7 be on the graph of the truth set of "$x < 4$ and $x > 2$"?

4. Is the number 1 in the truth set of "$x < 2$"? Is it in the truth set of "$x > 4$"? Will the point labeled 1 be on the graph of the truth set of "$x < 2$ and $x > 4$"?

Problem Set 3-4d

Draw the graphs of the truth sets of the following compound sentences:

1. $x > 4$ and $x < 6$
2. $x = 4$ and $x > 2$
3. $x < 4$ and $x < 2$
4. $x > 4$ and $x < 2$
5. $x = 4$ and $x = 2$
6. $x \geq 2$ and $x < 3$
7. $x \geq 1$ and $x \leq 5$
8. $x < 2$ and $x \neq 0$
9. $x \neq 3$ and $x < 4$
10. $x < 2$ and $x \geq 5$

To save words, we shall in the future often refer to the graph of the truth set of an open sentence as the <u>graph of a sentence</u>. It is simpler and no confusion should result if we

recall what is really meant by the description. In the same way we shall find it convenient to speak of the point 3, or the point $\frac{1}{2}$, when we mean the point with coordinate 3, or the point with coordinate $\frac{1}{2}$. Whenever there is the possibility of confusion, we shall give the complete description.

Problem Set 3-4e

In each of the following sentences (a) write its truth set (b) draw the graph of its truth set.

1. $x = 2$ or $x = 3$
2. $x = \frac{1}{2}$ or $x + 2 = 5$
3. $x = 2$ and $x + 1 = 4$
4. $x + 1 = 4$ and $x + 2 = 5$
5. $x = 2$ or $x + 1 = 4$
6. $x + 1 = 4$ or $x + 2 = 5$
7. $x > 3$ or $x = 3$
8. $x > 3$ or $x = 1$
9. $x < 5$ or $x > 7$
10. $x < 3$ or $x = 3$
11. $x \neq 3$ and $x \neq 4$
12. $x \geq 2$ and $x \neq 2$
13. $x \geq 2$ and $x \leq 2$
14. $x \geq 2$ or $x \leq 2$
15. $x = 3$ or $x < 2$ or $x > 4$
16. $x = 3$ or ($x < 2$ and $x > 4$)
17. $x < 3$ and $x > 2$ and $x < 4$

Summary

We have examined some sentences and have seen that each one can be identified as either true or false, but not both. We have introduced a set of symbols to indicate relations between numbers.

"=" says "is" or "is equal to"
"$\neq$" says "is not" or "is not equal to"
"<" says "is less than"
">" says "is greater than"
"$\leq$" says "is less than or is equal to"
"$\geq$" says "is greater than or is equal to"

An <u>open sentence</u> is a sentence containing one or more variables.

The <u>domain</u> of the variable is the set of all numbers from which the value of the variable may be chosen.

Summary
(continued)

The truth set of an open sentence containing one variable is the set of all those numbers in the domain which makes the sentence true. The open sentence acts as a sorter. That is, it sorts the numbers of the domain into two sets: a set of numbers which make the sentence true, and a set which make the sentence false.

We find that in mathematics we study compound sentences of two types according to the connective.

Connective	The sentence is true if:
or	at least one clause is true.
and	both clauses are true.

The graph of a sentence is the graph of the truth set of the sentence.

The graph of a compound sentence with connective or is the set of all points belonging to at least one of the graphs of the two individual clauses.

The graph of a compound sentence with connective and is the set of all points which are common to both of the graphs of the individual clauses.

Review Problem Set

1. Determine whether the given value of the variable is an element of the truth set of the sentence.

   (a) $3x + 5 = 4x$, if x is 5

   (b) $2(2x + 1) > 7$, if x is $\frac{3}{2}$

   (c) $3x - 2 < 7$ and $x = 4$, if x is 4

   (d) $5m \leq 17$, if m is $\frac{13}{4}$

   (e) $185n + 1047 = 1047 + 185n$, if n is $\frac{4}{3}$

   (f) $3a + 14 < 14$ or $a > 10$, if a is 0

   (g) $2m - 4 = 6$ and $3m > 14$, if m is 5

Review Problem Set
(continued)

(h) $4w + \frac{3}{8} \geq 10$ and $w + w < 1$, if w is $\frac{10}{3}$

(i) $3y - 2 \leq 3$, if y is $\frac{4}{3}$

(j) $5a(a + 5) = 70$, if a is 2

2. Describe the truth set of each of these sentences:

(a) $x + 5 = \frac{17}{2}$

(b) $x + 7 > \frac{1}{2}$

(c) $3y + 10 \geq 10 + 3y$

(d) $3x^2 = 3x$

(e) $x^2 = 16$

(f) $2y^2 = 18$

(g) $\frac{2}{3} + 2m = 5$

(h) $3x = 9$ or $2x - 6 > 0$

(i) $4w + \frac{3}{8} \geq 10$ and $w + w < 1$

(j) $x + 5 > 7$ and $x + 3 < 10$

3. Draw the graph of the truth sets of these sentences:

(a) $3x = 10$

(b) $5 + x < 8$

(c) $x > 5$ or $x < 2$

(d) $x = 5$ and $x < 3$

(e) $x = 5$ or $x < 3$

(f) $3x + 2 \leq 8$

(g) $2x < 1$ or $5x > \frac{15}{2}$

(h) $2x > 1$ or $5x < \frac{15}{2}$

(i) $2x > 1$ and $5x < \frac{15}{2}$

(j) $2x < 1$ and $5x > \frac{15}{2}$

4. State the truth set of each of the following open sentences and draw the graph of each truth set.

(a) $z + 8 = 14$

(b) $2 + v < 15$

(c) $2x = 3$

(d) $6 > t + 3$ and $5 + t = 4$

(e) $6 > t + 3$ or $2 + t = 1$

(f) $x^2 = x$

(g) $x + 2 = 3$ or $x + 4 = 6$

(h) $\frac{x}{2} > 3$

(i) $t + 4 = 5$ or $t + 5 \neq 5$

(j) $3x^2 = 12$

(k) $9 + t < 12$ or $5 + 1 \neq 6$

(l) $t + 6 \leq 7$ and $t + 6 \geq 7$

(m) $3(x + 2) = 3x + 6$

(n) $t + 2 \neq 3$ and $8 + 2 < 5$

5. If A and B are true sentences and C and D are false sentences, determine which of the following compound sentences are true:

(a) A or B

(b) A or D

(c) B and C

(d) C and D

Review Problem Set
(continued)

(e) C or D
(f) A and B
(g) A and D
(h) A or C
(i) B or C
(j) B and D

6. Write a common name for the following expressions if a is 2, b is $\frac{3}{2}$, c is $\frac{1}{3}$.

(a) $3a + 2b + 6c$
(b) $ab + c$
(c) $12c + \frac{a}{2} + 6b$
(d) $\frac{1}{2}a + b + c$
(e) $4bc - a$
(f) $(a + 2)(b + \frac{1}{2})$
(g) $\frac{3c + 2b}{a}$
(h) abc
(i) $9c^2 + 4b^2 - 5a$

7. Determine which of the following are true for every value of the variable if the domain of each variable is $\{0, \frac{1}{2}, 3, 4, 10\}$:

(a) $a + b = b + a$
(b) $a(b + c) = ab + c$
(c) $(a + b) + c = a + (b + c)$
(d) $a(bc) = (ab)c$
(e) $ab = ba$
(f) $ab = 0$

8. If C is the set of counting numbers, W is the set of whole numbers, and A is the set of numbers of arithmetic, which of the following statements are true?

(a) A is a subset of C
(b) W is a subset of A
(c) C is a subset of A
(d) C is a subset of W
(e) $\emptyset$ is a subset of A
(f) C is a subset of C

9. Use the multiplication property of one to change the names of the numbers on the left in the way indicated.

example: $\frac{1}{4} \times \_\_\_ = \frac{\_\_}{12}$

$\frac{1}{4} \times \frac{3}{3} = \frac{3}{12}$

(a) $\frac{2}{3} \times \_\_\_ = \frac{\_\_}{9}$
(b) $\frac{3}{5} \times \_\_\_ = \frac{\_\_}{25}$
(c) $\frac{5}{7} \times \_\_\_ = \frac{\_\_}{14}$
(d) $\frac{1}{8} \times \_\_\_ = \frac{\_\_}{56}$

Review Problem Set
(continued)

(e) $\frac{5}{4} \times \frac{\quad}{\quad} = \frac{\quad}{16}$ (f) $\frac{8}{3} \times \frac{\quad}{\quad} = \frac{\quad}{12}$

10. We have agreed on a certain order of operations in problems which involve both addition (or subtraction) and multiplication (or division). Write a common name for each of the following:

(a) $14 + 2 \times 3$ (d) $16 \div 4 - 2$

(b) $16 - 4 \times 3$ (e) $4 + 8 \times \frac{1}{2}$

(c) $8 \times 3 + 2$ (f) $\frac{7}{6} - \frac{1}{6} \times 7$

11. Use the distributive property to write the following indicated sums as indicated products and indicated products as indicated sums.

(a) $3(10 + 3)$ (d) $3a + 5a$

(b) $5(8) + 3(8)$ (e) $5a + 5b$

(c) $(4 + \frac{1}{2})6$ (f) $a(6 + b)$

Chapter 4

PROPERTIES OF OPERATIONS

4-1. Identity Elements.

What are the truth sets of these open sentences:

$$5 + x = 5$$

$$3 + y = 3$$

$$2\frac{1}{2} + a = 2\frac{1}{2}$$

$$0 + b = 0 \quad ?$$

Did you find that the truth set of each sentence is {0}?

What values of n make this sentence true:

$$n + 0 = n \quad ?$$

Can you find any value of n which makes this sentence false? This sentence is true for every value of n.

Here we have an interesting property, called the addition property of zero. We can state this property in words like this:

The sum of any number and zero
is equal to the given number.

We can state it even more briefly as follows:

For every number a, $a + 0 = a$.

When we add zero to a number, the result is always identical with the number to which we added zero. Therefore, we call zero the

identity element for addition.

Check Your Reading

1. For what values of n is the sentence "$n + 0 = n$" true?
2. State the addition property of zero.
3. What is the identity element for addition? What does this mean?

Chapter 4

PROPERTIES OF OPERATIONS

4-1. Identity Elements.

What are the truth sets of these open sentences:

$$5 + x = 5$$

$$3 + y = 3$$

$$2\tfrac{1}{2} + a = 2\tfrac{1}{2}$$

$$0 + b = 0 \quad ?$$

Did you find that the truth set of each sentence is {0}?

What values of n make this sentence true:

$$n + 0 = n \quad ?$$

Can you find any value of n which makes this sentence false? This sentence is true for every value of n.

Here we have an interesting property, called the addition property of zero. We can state this property in words like this:

The sum of any number and zero
is equal to the given number.

We can state it even more briefly as follows:

For every number a, $a + 0 = a$.

When we add zero to a number, the result is always identical with the number to which we added zero. Therefore, we call zero the

identity element for addition.

Check Your Reading

1. For what values of n is the sentence "$n + 0 = n$" true?
2. State the addition property of zero.
3. What is the identity element for addition? What does this mean?

Oral Exercises 4-1a

Find the truth set of each of the following open sentences.

1. $4 + y = 4$
2. $m + 6 = 6$
3. $a + 0 = 0$
4. $1 + x = 1$
5. $0 + b = 7$
6. $4 + x = 5$
7. $0 + b = b$
8. $b + 0 = b$
9. $a(0) = 0$
10. $a + 1 = a$

In Chapter 2 we saw that if a number is multiplied by 1, the result is equal to the given number.

How can we state the multiplication property of one more briefly than we did before? Perhaps we can do this by using a variable. For what values of a is

$$a(1) = a$$

a true sentence? It is true for every value of the variable a. Therefore we can state the <u>multiplication property of one</u> as follows:

<u>For every number a,</u> $a(1) = a$.

Because the result of our multiplication is always identical with the number we multiply by one, we call the number one the

<u>identity element for multiplication.</u>

You remember that the number one can be given many names. For example, all of the following are names for one:

$$\frac{4}{4},\ \frac{3}{3},\ \frac{6}{6},\ \frac{11}{11},\ \frac{105}{105}.$$

We will often use names like these for one. But no matter what name we use, the number one is the identity element for multiplication. For instance,

$$"n(\tfrac{4}{4}) = n"$$

is just another way of saying

$$"n(1) = n".$$

Check Your Reading

1. For what values of a is the sentence "$a(1) = a$" true?
2. State the multiplication property of one.
3. For what values of n is the sentence "$n(1) = 1$" true?
4. What is the identity element for multiplication? What does this mean?
5. What are some other names for 1?

Oral Exercises 4-1b

Find the truth sets of the following open sentences.

1. $4x = 4$
2. $\frac{3y}{5} = \frac{3}{5}$
3. $\frac{6m}{m} = 1$
4. $.8b = .8$
5. $2m + 2 = 4$
6. $m(1) = m$
7. $(1)m = m$
8. $5m = 15$
9. $m + 1 = m$
10. $m(5) = 5$
11. $5m = 0$

Look at the following sentences:

$$(5)0 = 0$$
$$(3)0 = 0$$
$$(0)0 = 0$$
$$(m)0 = 0$$

Do you see that each time we multiplied a number by zero, the result was zero? Notice especially the last sentence,

$$(m)0 = 0.$$

Can you find a number m that will make this sentence false? This sentence is true for <u>every number</u> <u>m</u>.

This property of numbers is called the <u>multiplication property of zero</u>. Here is how it might be stated in words:

Any number times zero equals zero.

We may also state it like this:

<u>For every number</u> <u>a</u>, $a(0) = 0$.

Check Your Reading

1. For what values of m is the sentence "$m(0) = 0$" true?
2. State the multiplication property of zero.
3. If $a = 0$ or $b = 0$, what do you know about ab?

Oral Exercises 4-1c

We have now studied the following properties. Can you give the name of each one?

1. For every number x, $x + 0 = x$.
2. For every number x, $x(1) = x$.
3. For every number x, $x(0) = 0$.

The properties that we have studied are very useful. Let's look at some examples that will show some of the ways they are used.

Example 1. Add the fractions $\frac{5}{6}$ and $\frac{3}{8}$.

$$\begin{aligned}\frac{5}{6} + \frac{3}{8} &= \frac{5}{6}(1) + \frac{3}{8}(1)\\ &= \frac{5}{6}\left(\frac{4}{4}\right) + \frac{3}{8}\left(\frac{3}{3}\right)\\ &= \frac{20}{24} + \frac{9}{24}\\ &= \frac{29}{24}\end{aligned}$$

Do you see that we could have used "48" as common denominator? In that case we would have used $\frac{"8"}{8}$ and $\frac{"6"}{6}$ as names for 1.

Example 2. What is a common name for the number

$$\frac{4}{\frac{1}{2}} \quad ?$$

We use the <u>multiplication property of one</u> and multiply $\frac{4}{\frac{1}{2}}$ by $\frac{2}{2}$.

$$\frac{4}{\frac{1}{2}} = \frac{4}{\frac{1}{2}} \times 1$$

$$= \frac{4}{\frac{1}{2}} \times \frac{2}{2}$$

$$= \frac{4 \times 2}{\frac{1}{2} \times 2}$$

$$= \frac{8}{1}$$

$$= 8$$

"8" is a common name for the number $\frac{4}{\frac{1}{2}}$.

Why did we multiply by $\frac{2}{2}$? What if we had multiplied by $\frac{3}{3}$, which is another name for one? Do you see that this would not have given us the common name "8"?

<u>Example 3</u>. What is a common name for the number

$$\frac{\frac{2}{3}}{\frac{4}{5}} \ ?$$

$$\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{\frac{2}{3}}{\frac{4}{5}} \times \frac{15}{15}$$

Why is "$\frac{15}{15}$" used for <u>one</u>?

$$= \frac{\frac{2}{3} \times 15}{\frac{4}{5} \times 15}$$

$$= \frac{10}{12}$$

$$= \frac{5}{6}$$

Thus, "$\frac{5}{6}$" is a common name for the number $\frac{\frac{2}{3}}{\frac{4}{5}}$.

Check Your Reading

1. Show how the multiplication property of one is used to add $\frac{5}{6}$ and $\frac{3}{8}$.
2. Show how the number $\frac{4}{\frac{1}{2}}$ can be written in a simpler way by using the multiplication property of one.
3. Show how the number $\frac{\frac{2}{3}}{\frac{4}{5}}$ can be written in a simpler way by using the multiplication property of one. What is the common denominator? How do we choose the form of the numeral for 1?

Oral Exercises 4-1d

Use at least one of the following properties, the <u>addition property of zero</u>, the <u>multiplication property of one</u>, or the <u>multiplication property of zero</u> to do these exercises.

1. Give a common name for each of the following and tell which properties are used.

   (a) $6 + 0$
   (b) $(125)(1)$
   (c) $(5)(0)$
   (d) $\frac{5}{3} + 0$
   (e) $(2.81)(1)$
   (f) $0(5 + 6)$
   (g) $1(5.2 + 0)$

2. Which of the following sentences are true?

   (a) $(8)(0) = 0$
   (b) $(9)(0) = 9$
   (c) $(15)(0) \neq 0$
   (d) $(6)(0) = 0$
   (e) $(y)(0) = 0$
   (f) $(1)(0) \neq 1$
   (g) $(1)(9) < 9$
   (h) $(1)(1) = 2$

3. Use the <u>multiplication property of one</u> to change the names of the given numbers on the left in the way indicated.

   (a) $\frac{1}{3} \times \_\_\_ = \frac{\quad}{12}$
   (b) $\frac{2}{9} \times \_\_\_ = \frac{\quad}{36}$
   (c) $\frac{10}{3} \times \_\_\_ = \frac{\quad}{9}$
   (d) $\frac{m}{n} \times \_\_\_ = \frac{\quad}{3n}$

Oral Exercises 4-1d
(continued)

(e) $\frac{x}{4} \times \_\_\_ = \frac{\quad}{12}$

(f) $\frac{2}{3} \times \_\_\_ = \frac{\quad}{6a}$

(g) $\frac{2}{a+1} \times \_\_\_ = \frac{\quad}{3(a+1)}$

(h) $\frac{4}{9} \times \_\_\_ = \frac{\quad}{9(b+3)}$

Problem Set 4-1d

1. Write a simpler numeral for each of the following and indicate what property was used.

   (a) $(m)(0)$

   (b) $(b)(1)$

   (c) $n + 0$

   (d) $n + 1$

2. Which of the following sentences are true for all values of the variable?

   (a) $(y)(\frac{5}{5}) > y$

   (b) $(1)(m) = m$

   (c) $\frac{12}{12}(m) \neq m$

   (d) $m + 0 = 0$

   (e) $m + 1 = m$

   (f) $\frac{6m}{6} + 1 = m$

3. Use the <u>multiplication property of one</u> to change the names of the numbers on the left in the way indicated.

   (a) $\frac{18}{5} \times \_\_\_ = \frac{\quad}{65}$

   (b) $\frac{5}{n} \times \_\_\_ = \frac{\quad}{4n}$

   (c) $\frac{m}{x} \times \_\_\_ = \frac{\quad}{5x}$

   (d) $\frac{x}{10} \times \_\_\_ = \frac{\quad}{30}$

   (e) $\frac{2y}{5} \times \_\_\_ = \frac{\quad}{15}$

   (f) $\frac{a}{2} \times \_\_\_ = \frac{\quad}{4b}$

   (g) $2 \times \_\_\_ = \frac{\quad}{a}$

4. Use the <u>multiplication property of one</u> to complete the following so that they are true sentences.

   (a) $\frac{3}{4}(\_\_\_) + \frac{3}{8} = \frac{\quad}{8} + \frac{3}{8}$

   (b) $\frac{5}{9} + \frac{1}{3}(\_\_\_) = \frac{5}{9} + \frac{\quad}{9}$

   (c) $\frac{1}{3}(\_\_\_) + \frac{1}{4}(\_\_\_) = \frac{\quad}{12} + \frac{\quad}{12}$

   (d) $\frac{2}{\frac{1}{3}}(\_\_\_) = \frac{\quad}{1}$

Problem Set 4-1d
(continued)

(e) $\dfrac{\frac{1}{5}}{3}(\_\_\_) = \dfrac{1}{}$

(f) $\dfrac{7}{\frac{2}{3}}(\_\_\_) = \dfrac{}{2}$

(g) $\dfrac{\frac{1}{3}}{6}(\_\_\_) = \dfrac{1}{}$

5. Use the <u>multiplication property of one</u> to complete the following so that they are true for all values of the variables. The domain of each variable is the set of all numbers except 0.

(a) $\frac{1}{3}(\_\_\_) + \frac{a}{6b} = \frac{}{6b} + \frac{a}{6b}$

(b) $\frac{1}{2a}(\_\_\_) + \frac{1}{3b}(\_\_\_) = \frac{}{6ab} + \frac{}{6ab}$

(c) $\dfrac{\frac{1}{2}a}{3}(\_\_\_) = \dfrac{a}{}$

(d) $\dfrac{b}{\frac{1}{4}a}(\_\_\_) = \dfrac{}{a}$

6. Use the <u>multiplication property of one</u> to find a simpler numeral for each of the following numbers:

(a) $\frac{3}{7} + \frac{5}{14}$

(b) $\frac{5}{6} + \frac{7}{9}$

(c) $\frac{x}{3} + \frac{3}{2}$

(d) $\frac{2a}{5} + \frac{b}{3}$

(e) $\dfrac{3}{\frac{1}{2}}$

(f) $\dfrac{\frac{3}{4}}{\frac{1}{3}}$

(g) $\dfrac{(a)(\frac{1}{3})}{\frac{4}{1}}$

(h) $\dfrac{(b + 1)(\frac{1}{5})}{\frac{3}{2}}$

7. Find the truth sets of the following open sentences:

(a) $(0)m = 0$

(b) $9a = 0$

(c) $2a = 2$

(d) $(1)m = 1$

(e) $a + 0 = a$

(f) $3 + a = 3$

(g) $(1)y = 5$

(h) $\frac{3a}{3} = 3$

(i) $\frac{4b}{4} + 2 = 7$

(j) $\frac{4}{4}(b + 3) = 6$

(k) $\frac{12}{12}(a + 5) = 16$

4-2. Closure.

You have been working with the numbers of arithmetic for many years. You have also worked with certain operations on these numbers. For example, addition and multiplication are operations, and each of these operations is performed on a pair of numbers.

If a pair of numbers is selected, the operation of addition will produce a third number. For example, if 7 and 3 are chosen, the operation of addition produces the number 10.

7 and 3 are numbers.
$7 + 3$ is a number. ("10" is another name for this number.)

If $3\frac{1}{2}$ and $2\frac{1}{4}$ are selected, the operation of addition produces the number $5\frac{3}{4}$.

$3\frac{1}{2}$ and $2\frac{1}{4}$ are numbers.
$3\frac{1}{2} + 2\frac{1}{4}$ is a number. ("$5\frac{3}{4}$" is another name for this number.)

In fact, no matter what pair is chosen from the numbers of arithmetic, you know from your earlier work in mathematics that the operation of addition will produce another number of arithmetic. We can use variables to say this, as follows:

If a and b represent numbers of arithmetic,
then $a + b$ is a number of arithmetic.

In a similar way, the operation of multiplication produces a number of arithmetic for any pair of numbers. For example, if 7 and 3 are chosen, the operation of multiplication produces the number 21.

7 and 3 are numbers.
7×3 is a number. ("21" is another numeral for it.)

$3\frac{1}{2}$ and $2\frac{1}{4}$ are numbers.
$(3\frac{1}{2})(2\frac{1}{4})$ is a number. ("$7\frac{7}{8}$" is another numeral for it.)

We could continue picking pairs of numbers, but you already know that for every pair of numbers, multiplication can be

performed, producing another number. Using the variables a and b, we can say:

If a and b represent numbers of arithmetic
then ab is a number of arithmetic.

Thus, whenever a and b represent two numbers, we have a perfect right to say that

<u>a + b is a number</u> and <u>ab is a number.</u>

We shall find this way of talking very useful. As another example, suppose x represents a number. Then do you see that we can say

3x is a number, and
3x + 6 is a number?

With subtraction the situation changes. The subtraction "9 - 5" can be done but "3 - 7" can't. (Do you see why?) Can 12 be subtracted from 10? So you see that in arithmetic subtraction sometimes can't be done. Division is another operation that sometimes can't be done. There is one number that can't be divided into 1. Can you remember which one? Thus with the numbers of arithmetic the operations of addition and multiplication can always be done but subtraction and division sometimes can not. Later on in this book, when we go into a larger system of numbers called "the real numbers", we <u>will</u> be able to do any subtraction in this larger system.

There are times when we work with a subset of the numbers of arithmetic rather than with the entire set of numbers of arithmetic. For example, one such subset is the set of counting numbers

{1, 2, 3, 4, . . .}.

The variables a and b might be used to represent two members of this set; that is, a and b represent counting numbers. Then a + b represents the sum of a and b, and we know of course that a + b is a number of arithmetic. Is a + b also a counting number? Below are three examples:

If a is 9, b is 4, a + b is 13.
If a is 5, b is 5, a + b is 10.
If a is 1, b is 186, a + b is 187.

Notice each time that the sum is not only a number of arithmetic but is also a counting number--that is, a number from the set with which we started. In fact, though we cannot show every case, if the variables a and b represent any two counting numbers, then $a + b$ is also a counting number. This fact is expressed by saying that

the counting numbers are <u>closed under addition.</u>

To get a better idea of what "closed under addition" means, let us look at another example. Let us consider the operation of addition together with the set

$$A = \{0, 1\},$$

which is a subset of the numbers of arithmetic and has only two elements. Is set A closed under addition? In this case, we can actually list all possible sums:

$$0 + 0 \qquad 0 + 1 \qquad 1 + 0 \qquad 1 + 1.$$

Each of these sums is a number of arithmetic of course. <u>But is each sum also a member of set A</u> (the set with which we started)? The answer is "no", because the number $1 + 1$ (or 2) is not in set A. So, set A is <u>not</u> closed under addition.

Let us consider the operation of multiplication together with the set

$$A = \{0, 1\}.$$

Is set A closed under multiplication? This time we list all possible products instead of all possible sums:

$$0 \times 0 \qquad 0 \times 1 \qquad 1 \times 0 \qquad 1 \times 1.$$

All of these products are numbers of arithmetic. But are they all members of set A? Simpler names for these products are, in order, 0, 0, 0, 1. Each of these is a number in set A. Therefore, set A is closed under multiplication.

Notice that when we speak of "closed", we are speaking of a set of numbers together with an operation. The three examples above may be summarized in a table.

SET	OPERATION	
Counting numbers	addition	The set is closed under addition.
{0, 1}	addition	The set is not closed under addition.
{0, 1}	multiplication	The set is closed under multiplication.

For a final example in this section, let us return to the set of counting numbers. Is the set of counting numbers closed under division? Since there is an infinite number of elements, we cannot list all possible quotients. But some examples may give us an idea as to what the answer to the question is. Let us choose a pair of counting numbers, say 8 and 4.

$$\frac{8}{4} = 2.$$

The number 2 is a counting number. That is, the result of dividing the counting number 8 by the counting number 4 is the counting number 2. This might lead somebody to say that the counting numbers are closed under division. However, suppose we select the numbers 5 and 10.

$$\frac{5}{10}$$

is a number that is <u>not</u> in the set of counting numbers. This means that the set of counting numbers is not closed under division.

Check Your Reading

1. When is a set closed under addition?
2. Is the set of counting numbers closed under division? Why?
3. What is meant by "the set of numbers represented on the number line is closed under multiplication"?
4. State the property of closure for addition. Do the same for multiplication.

Oral Exercises 4-2

1. Is the set of all even numbers closed under addition? Under multiplication?

Oral Exercises 4-2
(continued)

2. Is the set of all odd numbers closed under addition? Under multiplication?

3. Is the set of whole numbers closed under addition? Under multiplication?

4. Is the set of rational numbers closed under addition? Under multiplication?

5. Which of these sets are closed under division?
   (a) counting numbers
   (b) whole numbers greater than 4
   (c) even numbers excluding zero
   (d) rational numbers excluding zero

Problem Set 4-2

1. For each of the following determine whether the set is closed under addition; subtraction; multiplication; division. If your answer in any case is "no", give an example to support your answer.
   (a) counting numbers
   (b) whole numbers greater than 1
   (c) rational numbers excluding zero
   (d) even numbers excluding zero
   (e) odd numbers
   (f) multiples of 6 excluding zero
   (g) rational numbers that can be represented by fractions with denominator 3 and numerator a counting number.
   (h) rational numbers that can be represented by fractions with numerator 3 and denominator a counting number.

2. The set A is given as $A = \{0, 1\}$. Use the addition and multiplication tables given for A to determine whether A is closed under these operations

+	0	1
0	0	1
1	1	2

X	0	1
0	0	0
1	0	1

Problem Set 4-2
(continued)

3. Consider the set B = {0, 1, 2}. Is this set closed under multiplication? Under addition? (Remember, an element may be multiplied by, or added to, itself.)

4. Let us think of the symbol "$*$" as indicating the operation "take the average of the two numbers". Thus $8 * 10 = 9$. Which of the following sets of numbers are closed under the operation "$*$"?
   (a) The set of counting numbers
   (b) The set of odd numbers
   (c) The set of even numbers
   (d) The set of rational numbers
   (e) The set of numbers of arithmetic greater than 2

5. Let the symbol "$\triangle$" indicate the operation "take the first of a pair of numbers". Thus $3 \triangle 9 = 3$ and $7 \triangle 4\frac{1}{2} = 7$. Which of the following sets of numbers are closed under the operation "$\triangle$"?
   (a) The counting numbers
   (b) The whole numbers
   (c) The rational numbers
   (d) The numbers of arithmetic greater than 3

*6. Below are multiplication and addition tables of a set C whose elements are shown in the top horizontal row and in the left column of each table. Is the set C closed under multiplication? Under addition?

+	a	b	c	d
a	a	c	d	b
b	b	d	c	a
c	c	a	b	d
d	d	b	a	c

X	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	b	e
d	a	d	e	m

Problem Set 4-2
(continued)

*7. Is either the operation of addition or the operation of multiplication shown by the table in Problem 6 commutative? Give reasons for your answer.

4-3. Commutative and Associative Properties of Addition and Multiplication.

In Chapter 2, we discovered that addition is commutative. The following true sentences are examples of this property:

$$5 + 7 = 7 + 5$$

$$108 + 92 = 92 + 108$$

$$12 + 0 = 0 + 12$$

$$\frac{1}{2} + \frac{1}{4} = \frac{1}{4} + \frac{1}{2}$$

In fact, in finding the sum of <u>any</u> two numbers, the order in which the numbers are added does not affect the sum. So we can state the commutative property of addition as follows:

<u>For every number a and every number b,</u>
$$a + b = b + a.$$

What we are saying, of course, is that "$a + b = b + a$" is a true sentence no matter what numbers a and b represent. For example,

if $a = 3.5$ and $b = 12$, we get
$$3.5 + 12 = 12 + 3.5,$$
which is a true sentence.

If $a = 103$ and $b = 27$, we get
$$103 + 27 = 27 + 103,$$
which is a true sentence.

If m and n are numbers, then $2m$ and $3n$ are numbers. Therefore, since $a + b = b + a$ for every a and b, we know that $2m + 3n = 3n + 2m$ for every m and n. Perhaps the

scheme below will help to make this clearer.

$$\begin{array}{ccccccc} a & + & b & = & b & + & a \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ 3m & + & 2n & = & 2n & + & 3m \end{array}$$

Here are a few more examples that illustrate the use of the commutative property of addition:

$$4 + 6x = 6x + 4$$

$$(2 + 3a) + 2b = 2b + (2 + 3a)$$

$$xz + 3y = 3y + xz$$

$$bc + df = df + bc$$

Verify that the last of these, for example, is true if b is 3, c is 5, d is 2, and f is 4.

Check Your Reading

1. State the commutative property of addition.
2. If m and n are numbers, what can you say about $2m$ and $3n$?

Oral Exercises 4-3a

Which of the following are true for all values of the variables?

1. $2c + 6 = 6 + 2c$
2. $3y + 2 = 2y + 3$
3. $2a + 7 = 7 + 2a$
4. $682 + 9y = 9 + 682$
5. $b + d = d + b$
6. $xy + mn = mn + xy$
7. $3x + y = 3y + x$
8. $3x = 9$

Problem Set 4-3a

Which of the following are true for all values of the variables? Give a reason for each answer.

1. $(m + 2) + 5 = 5 + (m + 2)$
2. $(x + y) + z = z + (y + x)$

Problem Set 4-3a
(continued)

3. $3(x + y) + 2z = 2z + 3(y + x)$

4. $(a + b + c) + d = d + (a + b + c)$

5. $(3a + 2b) + (2m + 2n) = (2n + 2m) + (3b + 2a)$

Below are some examples illustrating other properties of addition and multiplication of numbers that we discovered in Chapter 2:

$4(5) = 5(4)$ — an example that illustrates the commutative property of multiplication

$3 + (2 + 7) = (3 + 2) + 7$ — an example that illustrates the associative property of addition

$3(2 \times 4) = (3 \times 2)4$ — an example that illustrates the associative property of multiplication

We have given just one example illustrating each of these properties. We could give more. In fact, we could use any numbers to illustrate these properties.

We usually state these properties as follows:

<u>Commutative Property of Multiplication</u>

For every number a and every number b,

$$a(b) = b(a)$$

<u>Associative Property of Addition</u>

For every number a, every number b, and every number c,

$$a + (b + c) = (a + b) + c.$$

<u>Associative Property of Multiplication</u>

For every number a, every number b, and every number c,

$$a(bc) = (ab)c.$$

Remember that bc means "b times c" or "$b \times c$" or "(b)(c)". We also use $b \cdot c$ to mean "b times c". We call b a <u>factor</u> of bc because there is a number c such that the product of b and c

gives bc. For the same reason c is a factor of bc. For example, 4 is a factor of 12. Why?

Check Your Reading

1. State the commutative property of multiplication.
2. State the associative properties of addition and multiplication.
3. Show the application of the commutative property of multiplication to the numeral "$(x + y)(z + w)$".
4. Why is 4 a factor of 12?

Oral Exercises 4-3b

1. Which of the following are true for every value of the variable (or variables)? Which of the 4 properties helped you reach a decision?

   (a) $(3) + (d) = (d) + (3)$

   (b) $c + (2 + 4) = (c + 2) + 4$

   (c) $m(2 \times 3) = (m \times 2) \times 3$

   (d) $ms = sm$

   (e) $ms = ms$

   (f) $x + (y + z) = (xy)z$

   (g) $5 + 8x = 8 + 5x$

   (h) $x(y + 2) = (y + 2)x$

   (i) $m(n + 3) = (3 + n)m$

   (j) $(2 + x)y = (2 + y)x$

   (k) $m(2 + 1) = 2(m + 1)$

   (l) $(uv^2)z = u(vz^2)$

   (m) $(2a + c) + d = 2a(c + d)$

   (n) $2c + b = b + c(2)$

Oral Exercises 4-3b
(continued)

2. Use the commutative and associative properties to find a simpler name for each of the following:

(a)	$7(3a)$	(g)	$(\frac{2}{3})(15m)$
(b)	$(5m)4$	(h)	$(.1q)8$
(c)	$5(3c)$	(i)	$(16y)(\frac{3}{2})$
(d)	$9(3x)$	(j)	$(2m)m$
(e)	$(3x)x$	(k)	$(\frac{1}{2}a)(\frac{1}{4})$
(f)	$(8y)2$	*(l)	$(8a)(\frac{3}{4}a)$

You may wonder why we have been so interested in the properties of addition and multiplication. Are not these properties true of <u>all</u> operations on numbers? Let us try the operation of division. Is division commutative? Is the sentence "$\frac{a}{b} = \frac{b}{a}$" true for every number a and every number b? Let us try $a = 6$ and $b = 2$. This leads to

$$\frac{6}{2} = \frac{2}{6}$$

and

$$3 = \frac{1}{3}$$

which is a false sentence. This shows that division is not commutative since we have found a pair of numbers which make the open sentence "$\frac{a}{b} = \frac{b}{a}$" false.

Check Your Reading

1. How do we show that division is not commutative?

Problem Set 4-3c

1. Is division associative? Test

$$(a \div b) \div c = a \div (b \div c).$$

Let $a = 18$, $b = 6$, $c = 2$.

Problem Set 4-3c
(continued)

2. Is subtraction commutative? Try to write an open sentence expressing this idea. Can you find numbers for the variables that show it is false?

3. Which of the following are true for every value of the variable (or variables)? Which properties of addition or multiplication are helpful in arriving at an answer?

   (a) $(\frac{a}{2} + \frac{b}{2}) + \frac{c}{2} = \frac{a}{2} + (\frac{b}{2} + \frac{c}{2})$

   (b) $.5b(200) = 200(.5b)$

   (c) $(2mn)r = 2m(rn)$

   (d) $(a + b)(c + d) = (c + d)(a + b)$

   (e) $(2a + b)(3c + d) = (d + 3c)(b + 2a)$

   (f) $a + (2b + c) = \big(a + b(2)\big) + c$

   (g) $a + (2bc + d) = \big(a + c(2b)\big) + d$

   (h) $2a + (2b + c) = \big(b(2) + c\big)2a$

   (i) $(a + b)(3a + 4c) = (3a + 4c)(b + a)$

   (j) $\big(a + (b + c)\big)\big((m + n) + n\big) = \big(m + (n + n)\big)\big(b + (a + c)\big)$

   (k) $a(bc) + 2mnr(xy) = (ba)c + (2mnrx)y$

4. Suppose the symbol "$*$" denotes the operation indicated by $a * b = \frac{a + b}{2}$. You know this operation as that of finding the average of two numbers.

   (a) Is "$*$" a commutative operation?

   (b) Is "$*$" an associative operation?

5. Suppose the symbol "$\triangle$" indicates the operation "take the first of a pair of numbers".

   (a) Is "$\triangle$" a commutative operation?

   (b) Is "$\triangle$" an associative operation?

4-4. Distributive Property.

In Chapter 2 many examples such as

$$5(27 + 3) = 5(27) + 5(3)$$

$$13(c + 4) = 13(c) + 13(4)$$

$$10(6 + \tfrac{1}{2}) = 10(6) + 10(\tfrac{1}{2})$$

were given to illustrate the distributive property. We could give many more examples like the above, using any three numbers. In general, we state the distributive property as follows:

<u>For every number</u> <u>a</u>, <u>every number</u> <u>b</u>,
<u>and every number</u> <u>c</u>,

$$a(b + c) = ab + ac.$$

It is very important to understand what this means. We are saying that "$a(b + c) = ab + ac$" is a true sentence no matter what numbers a, b, and c are.

Let us verify that the statement "$a(b + c) = ab + ac$" is true if $a = 2$, $b = 3$, and $c = 5$. In this instance

$$\begin{aligned} a(b + c) &= 2(3 + 5) \\ &= 2(8) \\ &= 16, \end{aligned}$$

and

$$\begin{aligned} ab + ac &= 2(3) + 2(5) \\ &= 6 + 10 \\ &= 16. \end{aligned}$$

Thus we have verified that "$a(b + c)$" and "$ab + ac$" name the same number if $a = 2$, $b = 3$, and $c = 5$.

The distributive property can also be written with the phrase "$ab + ac$" on the left side and the phrase "$a(b + c)$" on the right side. That is, we can also state the property as follows:

<u>For every number</u> <u>a</u>, <u>every number</u> <u>b</u>,
<u>and every number</u> <u>c</u>,

$$ab + ac = a(b + c).$$

Check Your Reading

1. In what four ways may the distributive property be stated?
2. For what values of a is this sentence true:
$$5a + 3a = 8a \quad ?$$
3. For what values of a and x is this sentence true:
$$(a + 3)x = ax + 3x \quad ?$$

Oral Exercises 4-4b

In the following phrases, use the distributive property to state each sum as a product, and each product as a sum.

1. $(4 + \frac{2}{3})3$
2. $6(c) + 4(c)$
3. $2a + .3a$
4. $(7 + 3)m$
5. $7(4) + 8(4)$
6. $(9 + 3)2$
7. $2(\frac{5}{2}) + 2(4)$
8. $(m + 6)a$
9. $6a + 6b$
10. $(a + b)8$
11. $a(3x) + b(3x)$
12. $6(a) + 6(1)$
13. $6(a) + 6$
14. $3x + x$
15. $.7m + .4m$
16. $(a + c)b$
17. $5(a + b)$
18. $\frac{4}{3}r + \frac{4}{3}s$

Problem Set 4-4b

1. Which of the following sentences are true? Which are false? (You may be able to decide without finding a common name for each numerical phrase.)
   (a) $8(20 + 4) = 8(20) + 8(4)$
   (b) $12(5) + 7(5) = (12 + 7)5$
   (c) $3(22) + 3(6) = (22 + 6)3$
2. Which of these sentences are true for every value of the variables?
   (a) $4(25 + m) = 4(25) + m$
   (b) $2(23) + 2(x) = 2(23 + x)$
   (c) $2(x + 2) = 2x + 2$

Problem Set 4-4b
(continued)

(d) $(3 + y)5 = 15 + y$

(e) $a(b + 3) = ab + 3a$

(f) $(4 + c)a = 4a + c$

(g) $(3 + a)b = 3b + ab$

(h) $3a + 3b = 3(a + b)$

(i) $ab + cb = (a + c)b$

3. Change the following indicated products to indicated sums.

(a) $3(10 + 5)$

(b) $3(x + 2)$

(c) $(m + 3)2$

(d) $5(4 + c)$

(e) $(11 + 1)k$

(f) $(a + b)4$

(g) $a(b + 2)$

(h) $a(b + c)$

(i) $(x + y)m$

(j) $(a + 2)a$

4. Write the following indicated sums as indicated products.

(a) $3(5) + 3(7)$

(b) $3(5) + 7(5)$

(c) $15(4) + a(4)$

(d) $2(b) + 2(c)$

(e) $a(2) + a(5)$

(f) $c(d) + a(d)$

(g) $a(b) + a(4)$

(h) $f(m) + a(n)$

(i) $2a + a^2$ (Recall: a^2 may be written $(a)(a)$.)

(j) $x^2 + xy$

(k) $4c + 3c$

(l) $a(x) + x$ (Hint: $x(1) = x$.)

(m) Write the indicated product for exercise (k) above with a simpler number name.

Problem Set 4-4b
(continued)

5. Use the associative, commutative, and distributive properties, and the multiplication property of 1, to write the following open phrases in simpler form.

(a)	$5b + 2b$	(f)	$8.9b + 3.2b$
(b)	$4a + a(7)$	(g)	$3y + y$
(c)	$c(2) + c(3)$	(h)	$m + 2m$
(d)	$\frac{1}{7}m + \frac{3}{7}m$	(i)	$2a + 3b$
(e)	$.4n + .6n$	(j)	$3.7n + n(.4)$

Some further uses of the distributive property are shown by the examples that follow. They should be studied carefully.

<u>Example 1</u>. Write $2r(s + t)$ as an indicated sum.
Here we may think of $2r$ as one number. Then the phrase "$2r(s + t)$" involves the three numbers $2r$, s, and t. By the distributive property we can write:

$$\begin{aligned} 2r(s + t) &= 2r(s) + 2r(t) \\ &= 2rs + 2rt \end{aligned}$$

For every number r, every number s, and every number t,

$$2r(s + t) = 2rs + 2rt$$

<u>Example 2</u>. Write $3u(v + 3y)$ as an indicated sum.

$$\begin{aligned} 3u(v + 3y) &= 3u(v) + 3u(3y) && \text{distributive property} \\ &= 3u(v) + (3)(3)(u)(y) && \text{commutative property of multiplication} \\ &= 3uv + 9uy. \end{aligned}$$

So $\quad 3u(v + 3y) = 3uv + 9uy.$

Example 3. Write $2rs + 2rt$ as an indicated product.
Here are three ways to do this:

(1) $2rs + 2rt = 2(rs) + 2(rt)$
$= 2(rs + rt)$

(2) $2rs + 2rt = r(2s) + r(2t)$
$= r(2s + 2t)$

(3) $2rs + 2rt = 2r(s) + 2r(t)$
$= 2r(s + t)$

All of these ways are correct. The last one usually is the most useful.

Example 4. Write $6xy + 6x$ as an indicated product.

$6xy + 6x = 6x(y) + 6x(1)$ Both "6xy" and "6x" can be written as "6 times something".

$6xy + 6x = 6x(y + 1)$.

Check Your Reading

1. Can $3uv$ be considered as one number? Why?
2. What are three ways that $2rs + 2rt$ may be written as a product?

Oral Exercises 4-4c

Which of the following sentences are true for every value of every variable?

1. $5(c + 4) = 5c + 20$
2. $2y + 4y = (2 + x)y$
3. $3ab + 3c = 3a(b + c)$
4. $3x(a + b) = 3ax + 3b$
5. $2ab + 2ac = 2a(b + c)$
6. $3xy + 6xz = x(3y + 6z)$
7. $m(m + n) = m^2 + n$

Oral Exercises 4-4c
(continued)

8. $3xy + 6xz = 3x(y + 2z)$

9. $2ax + 2bx = (2a + b)x$

10. $5a + 30b = (5a)(6b)$

Problem Set 4-4c

1. Write each of the following indicated product- indicated sum.

(a) $(6 + 3p)m$	(f) $5a(5 + a)$
(b) $2h(k + 1)$	(g) $2m(x + 3y)$
(c) $6(2s + 3r)$	(h) $(a + m)3m$
(d) $(x + y)2a$	(i) $(x + y)4x$
(e) $7a(a + 1)$	(j) $k(2x + 5)$

2. Write each of the indicated sums as an indicated product.
Example: $2ax + 2ay = 2a(x) + 2a(y)$
$= 2a(x + y)$

(a) $2a + 2b$	(f) $cx + 4cy$
(b) $2mn + 5n$	(g) $2ax + 5x$
(c) $2mn + 2n$	(h) $3ab + 9a^2$ (Hint: Write $9a^2$ as $(3a)(3a)$.)
(d) $6bc + 6c$	(i) $3x + 3x^2$
(e) $4mn + 4mp$	(j) $xz^2 + x^2z$

3. Write each of the indicated sums as an indicated product.

(a) $6b + 3b$	(d) $5ab + 6ac$
(b) $2a + 3a$	(e) $mx^2 + x^2$
(c) $5ax + 5a$	(f) $18x^2 + 6x$

4. Write each of the indicated sums as an indicated product.

(a) $3x^2 + 2x^2$	(d) $9bc + 6c$
(b) $ax^2 + bx^2$	(e) $12ab + 8bc$
(c) $5x^2 + 5c$	(f) $6a + 6a^2$

Problem Set 4-4c
(continued)

5. Indicate how the distributive property can be used to
   (a) find the perimeter of a rectangle with a length of 7 inches and a width of 3 inches.
   (b) find the amount of money collected at a game if 125 tickets were sold at one window and 375 tickets at the other window. The price per ticket was $1.50.

Another important use of the distributive property is shown by the following example:

Write $(m + 2)(m + x)$ as an indicated sum, using the distributive property.

To begin with, we shall think of $(m + 2)$ as <u>a single number.</u> Then, first writing the distributive property as a pattern to follow, we have:

$$\begin{array}{ccccccccc} a & \cdot & (b & + & c) & = & a & \cdot b & + & a \cdot c \\ \downarrow & & \downarrow & & & & \downarrow & \downarrow & & \downarrow \quad \downarrow \\ (m + 2) & & (m & + & x) & = & (m + 2) & m & + & (m + 2)x \end{array}$$

Using the distributive property again,

$$\begin{aligned}(m + 2)m + (m + 2)x &= (m)(m) + (2)(m) + (m)(x) + (2)(x) \\ &= m^2 + 2m + mx + 2x\end{aligned}$$

So $(m + 2)(m + x) = m^2 + 2m + mx + 2x$ for <u>every number</u> <u>m</u> and <u>every number</u> <u>x</u>.

Below is another example of this kind.

$$\begin{aligned}(x + a)(x + 2) &= (x + a)x + (x + a)2 \\ &= (x)x + (a)x + (x)2 + (a)2 \\ &= x^2 + ax + 2x + 2a.\end{aligned}$$

So $(x + a)(x + 2) = x^2 + ax + 2x + 2a$ for every number x and every number a.

Check Your Reading

1. How many times do we apply the distributive property in the example above? Is $(x + a)$ used as one number or two numbers in the first application?

2. Is $x + a$ used as one number or two numbers in the second application of the distributive property? What is the final indicated sum?

Oral Exercises 4-4d

Apply the distributive property once, using the first indicated sum as a single number. Example:

$$(m + n)(a + b) = (m + n)a + (m + n)b$$

1. $(a + c)(a + 4)$
2. $(x + a)(x + 3)$
3. $(x + 1)(a + b)$
4. $(3a + 4)(a + 5)$
5. $(7 + x)(x + 7)$
6. $(3a + b)(c + d)$
7. $(mn + x)(a + b)$
8. $(ab + c)(b + c)$
9. $(8 - x)(8 + x)$

Problem Set 4-4d

Apply the distributive property as many times as necessary to write these as indicated sums in the simplest form. Use the other properties discussed in this chapter as well when they are needed to write the simplest form of the phrase.

1. $(a + 1)(a + b)$
2. $(a + 5)(a + b)$
3. $(2x + c)(x + 4)$
4. $(3 + m)(5 + a)$
5. $(a + b)(c + d)$
6. $(2c + d)(a + d)$
7. $(20 + 5)(40 + 3)$
8. $(x + 2) + (x + 5)$
9. $(20 + 5)(x + 3)$
10. $(a + 2b)(2a + c)$
11. $m(m + n)$
12. $(x + y)(m + n)$
13. $(3r + 1)(r + a)$
14. $(3r + 1)(r + 3a)$
15. $(mn + b)(mn + a)$
16. $(xy + a)(y + b)$
17. Show that $(a + 2)(a + x) = a^2 + 2a + ax + 2x$ when a is 5 and x is 2.

Problem Set 4-4d
(continued)

18. Show that $(2x + 3)(x + a) = 2x^2 + 3x + 2ax + 3a$ when x is 3 and a is 0.

19. Do you see that in Problem 7 above you really multiplied $(25)(43)$? How could you similarly multiply the following numbers?

(a)	$(12)(24)$	(g)	$(33)(23)$	(m)	$(45)(202)$
(b)	$(22)(21)$	(h)	$(11)(4.2)$	(n)	$(25)(1003)$
(c)	$(16)(12)$	(i)	$(3\frac{1}{2})(2\frac{1}{2})$	(o)	$(4\frac{1}{3})(3\frac{3}{4})$
(d)	$(18)(61)$	(j)	$(35)(35)$	(p)	$(6.4)(408)$
(e)	$(25)(32)$	(k)	$(1.5)(36)$	(q)	$(6\frac{1}{4})(8\frac{2}{3})$
(f)	$(42)(36)$	(l)	$(4.5)(4.5)$	(r)	$(3\frac{3}{4})(4.8)$

Summary

The numbers of arithmetic can be added and multiplied. We have learned that these numbers and their operations have basic properties which we shall list below and always refer to as the <u>properties of the numbers of arithmetic.</u>

1. For every number a and every number b, $a + b$ is a number.

2. For every number a and every number b, ab is a number.

3. <u>Commutative Property of Addition:</u> For every number a and every number b, $a + b = b + a$.

4. <u>Commutative Property of Multiplication:</u> For every number a and every number b, $ab = ba$.

5. <u>Associative Property of Addition:</u> For every number a, every number b, and every number c, $(a + b) + c = a + (b + c)$.

6. <u>Associative Property of Multiplication:</u> For every number a, every number b, and every number c, $(ab)c = a(bc)$.

7. <u>Distributive Property:</u> For every number a, every number b, and every number c, $a(b + c) = ab + ac$.

8. <u>Addition Property of Zero:</u> For every number a,

$$a + 0 = a.$$

9. <u>Multiplication Property of One:</u> For every number a,

$$a(1) = a.$$

10. <u>Multiplication Property of Zero:</u> For every number a,

$$a(0) = 0.$$

Review Problem Set

1. Name the identity element for addition and state its addition and multiplication properties.

2. Name the identity element for multiplication, state its multiplication property, and give three different numerals for this number.

3. Write simpler numerals for the following expressions, using the properties of one and zero.

   (a) $x + 0 + b(1)$

   (b) $a(1) + 0(a + b)$

   (c) $\frac{3}{8} + \frac{4}{3}$

   (d) $\dfrac{\frac{3}{8}}{\frac{4}{3}}$

   (e) $\left(\dfrac{\frac{5}{8}}{\frac{3}{4}}\right)\left(14(0)\right)$

4. Complete these sentences so that they will be true for all m and n other than 0.

   (a) $\frac{1}{2} \times \_\_\_ = \frac{\quad}{96}$

   (b) $\frac{4}{3} \times \_\_\_ = \frac{\quad}{45}$

   (c) $\frac{2}{5} \times \_\_\_ = \frac{\quad}{5m}$

   (d) $\frac{m}{n} \times \_\_\_ = \frac{\quad}{5n}$

   (e) $\frac{5}{n} \times \_\_\_ = \frac{\quad}{2mn}$

Review Problem Set
(continued)

5. Find the truth sets of the following sentences:

(a) $6a = 0$

(b) $6a = 6$

(c) $6a + 7(0) = 6a$

(d) $\frac{6}{6}a + \frac{5}{5} = \frac{5}{5}$

(e) $\frac{6}{6}a + \frac{5}{5}(0) = \frac{5}{5}$

(f) $6(\frac{6}{6}a) + 6 = 6$

(g) $6(a + 6) = 42$

(h) $6(a + 6) = 46$

(i) $a + 6 = a + \frac{6}{6}$

(j) $66a = \frac{0}{6}(6a)$

6. Is the set of whole numbers, whose numerals have a units digit of 0, closed under addition? Under multiplication? Under subtraction?

7. State the associative, commutative, and distributive properties.

8. Which of the following sentences are true for all values of the variables? For each such sentence, decide which property or properties is illustrated by the sentence.

(a) $(x + y)(m + n) = (x + m)(y + n)$

(b) $x + y + m = y + x + m$

(c) $(x + y)(m + n) = (x + y)(n + m)$

(d) $(x + y)(m + n) = (m + n)(x + y)$

(e) $x + ym = xy + m$

(f) $x(y + m) = y(x + m)$

(g) $x^2m = mx^2$

(h) $x(m + n) = mx + nx$

(i) $x + ym = (x + y)(x + m)$

(j) $x(ym) = (xy)m$

Review Problem Set
(continued)

9. Write open phrases, using n as the variable, for the following word phrases.
   (a) 5 more than twice a number
   (b) 6 less than $\frac{1}{3}$ of a number
   (c) 3 times a number, less 6
   (d) a number multiplied by the sum of 5 and the number itself
   (e) the square of a number, increased by twice the number
   (f) the product of the sum of a number and 5 by the sum of the number and 2
   (g) the square of the sum of a number and 7
   (h) a number multiplied by 2 less than the number
   (i) the quotient of a number divided by 8
   (j) the quotient of the sum of a number and 6, divided by the number

10. Find the values of these open phrases if m is 2, n is 5, and p is 0.
   (a) mnp
   (b) $\frac{m + p}{n}$
   (c) $\frac{np}{m(n^2 + p)}$
   (d) $p\frac{(mn + n^2 + m^2)}{m + 7n}$
   (e) $\frac{m + n}{n + m}\left(p + \frac{m}{m}(n + m)\right)$

11. Change the indicated products to indicated sums and the indicated sums to indicated products, using the distributive property.
   (a) $3(m + n)$
   (b) $6m + 6n$
   (c) $a(b + c)$
   (d) $(x + y)(b + c)$
   (e) $7x(y + 1)$
   (f) $a(ax + m)$
   (g) $a^2b + ac$
   (h) $(a + 8)xy$
   (i) $(a + r)(4a + 1)$
   (j) $(a + 3)(b + x)$
   (k) $(3 + a)x + (3 + a)y$
   (l) $6ax + 6ay$
   (m) $8y^2 + 4y^2$
   (n) $12x + 18$
   (o) $ab(a + b)$
   (p) $xy + xz$

Review Problem Set
(continued)

(q) $2a(3a + 5)$

(r) $(m + n)3m$

(s) $(a + 3b)(a + c)$

(t) $abcx + abcy$

(u) $18(a + b) + 6(a + b)$

(v) $(x + y)m + (x + y)n$

12. Indicate how the distributive property can be used to find a simpler name for

(a) $3 \times 2\frac{3}{7}$

(b) $5 \times 4\frac{2}{13}$

(c) $4 \times 1\frac{5}{8}$

13. Find the truth sets of the following sentences. Draw their graphs. Determine which is a subset of the other.

$x + 5 = 6$ $\qquad$ $(x - 1)^2 = x - 1$

14. Find the truth sets of the following sentences. Draw their graphs and determine which is a subset of the other.

$3x < 6$ $\qquad$ $5x \geq 0$

15. Find the truth sets of the following sentences. Draw their graphs and determine which is a subset of the other.

$2x + 5 = 12$ $\qquad$ $8x + 1 < 17$

Chapter 5

OPEN SENTENCES AND WORD SENTENCES

5-1. Open Phrases to Word Phrases.

Have you ever heard anybody say that he was going to "talk algebra"? This is not so silly as it seems. In a way, algebra is a language.

To see what this means, think first of a number. We know, for instance, that 3 is a number. We can use this number to talk about things, like 3 books, 3 people, 3 inches, and so on. Of course, we would not want to say that 3 books is a number. It is a word phrase that just relates the number 3 to a collection of books.

In the same way, n is a number. We can use this number to talk about things, like n books, n people, n inches, and so on. Again, we don't want to say that n books is a number. It is just a word phrase that relates the number n to a collection of books.

When we say "n books", we mean that "n" is the "number of books". Because algebra is a kind of language, we often say that a phrase like "number of books" is a translation of the variable n. This shows that we have translated from algebra to words. We can also say the phrase "number of books" tells us what the variable represents.

Here are some numbers "n" might represent:

number of students
in your class

number of dollars
in your allowance

number of years
in your age

We have given only three different translations of n. It is possible to give many, many more. If it seems hard to think of a translation for n, just ask yourself the question, "Number of what?". When you answer the question, you are translating the variable.

Look at the open phrase "n + 5". We could say that it is a phrase in the language of algebra. What does the phrase represent? How can we explain in words what the phrase represents?

Here are some of the things the phrase might represent:

n	n + 5
a number	the sum of a number and 5
number of students in your class	number of students if 5 more came in
number of dollars in your allowance	number of dollars if your allowance is increased 5 dollars
number of years in your age	number of years in the age of somebody who is 5 years older than you

Do you see that the translation of "n + 5" depends on what translation we make for "n"? But no matter what translation into words we make, n is a number and n + 5 is another number.

"2x" is another open phrase. So is "2x + 3". As soon as we translate "x", we can translate both of these phrases. Here are two ways:

x	2x	2x + 3
number of points Jim made	number of points Bill made if he made twice as many as Jim	number of points Bill made if he made twice as many as Jim, and 3 more or number of points Bill made if he made 3 more than twice as many as Jim
number of pounds one person can take on an airplane	number of pounds two people can take on an airplane	number of pounds two people take on an airplane if they have 3 pounds too much

Can you think of a different way to translate the open phrase "2x" and the open phrase "2x + 3"? It should be possible for each person in your class to give a translation different from that of anybody else.

Check Your Reading

1. Is the phrase 3 books a number?
2. Is the phrase n books a number?
3. How many translations of "n" can be made?
4. How many translations of "2n + 5" can be made?

Oral Exercises 5-1

1. Given the open phrases, "t + 1", "t - 2", "2t", "2t + 3", and "$\frac{t}{2}$". Translations of the phrases can be made as soon as the variable "t" has been translated. Three different possible translations for "t" are given below. For each one give a translation of the given open phrases.

	t+1	t-2	2t	2t+ 3	$\frac{t}{2}$
(a) number of quarts of berries that can be picked in one hour					
(b) number of records you can buy for $3					
(c) number of feet in the diameter of a given circle					

Problem Set 5-1

In the following problems write a translation of the open phrase. Give a different translation of the variable in each problem.

1. $n + 7$
2. $n - 7$
3. $x + 2$
4. $x - 2$
5. $2n$
6. $2n + 1$
7. $2n - 1$
8. $\frac{n}{3}$
9. $\frac{n + 1}{3}$
10. $2r + 5$
11. $2r - 5$
12. $x + 7x$
13. $\frac{t}{2} + 3$
14. $3r + 1$
15. $\frac{2t - 1}{3}$

5-2. Word Phrases to Open Phrases.

In the last lesson, we translated open phrases into word phrases. For instance, we translated the open phrase "n + 5". Before we did this, we had to translate the variable "n".

We can also go the other way. We can take a word phrase and translate it into an open phrase in algebra.

Suppose you want to talk about your age 7 years from now. This is easy since you know your age now. You might think like this:

Number of years in your age now	14
Number of years in your age 7 years from now	14 + 7.

So you can say that in 7 years your age will be 21.

Suppose you want to talk about Bill's age 7 years from now. You don't know his age now. But you can use a variable. The thinking might go like this:

Number of years in Bill's age now	k
Number of years in Bill's age 7 years from now	k + 7.

Notice that the phrase "k + 7" means "number of years in Bill's age 7 years from now". Do you see that we are translating from words to algebra?

Suppose we are asked to write an open phrase which is a translation of the word phrase, "Five more than a number n". In thinking about this expression we could say that we begin with a number n and then add 5 to it.
This suggests that we write

$$n + 5$$

Because of the fact that addition is commutative, we know that the phrase "5 + n" is also a name for the same number. We shall, however, always use the form "n + 5" for the word phrase above. Thus we shall translate "Eight more than the number x" as "x + 8", and "Twelve more than the number y as "y + 12".

Let us now consider an expression such as "Six less than the number n". In the same manner as above we may assume that we have started with a number n and then subtract 6 from this number. Thus our open phrase is

$$n - 6$$

In this case we cannot write the phrase "6 - n" as a translation for the same word phrase. Do you see why?

Thus we must translate "Eight less than the number x" as "x - 8" and "Twelve less than the number y" as "y - 12".

Here are some other examples:

A LINE SEGMENT 3 FEET LONGER THAN ANOTHER LINE SEGMENT

We know we have a line segment 3 feet longer than another line segment. But we don't know the number of feet in that other segment. So let t be the number of feet in that other segment. Then the translation is

$$t + 3.$$

THE NUMBER OF CENTS IN y QUARTERS

To translate this, we can think: How many cents (or pennies) in one quarter? in two quarters? in three quarters? We have y quarters. So the translation is

$$25y.$$

5 POUNDS LESS THAN TWICE THAT NUMBER OF POUNDS

What number of pounds? We don't know. So let n be that number of pounds. What is twice that number? That's easy. It is 2n. But we don't want this much. We want 5 pounds less. So the translation is

$$2n - 5.$$

Oral Exercises 5-2

Below are some word phrases that can be translated into open phrases. If there are any you are not sure of, it would be a good idea to write them down.

1. If the number of years in Bill's age now is k, what is the number of years in Bill's age 7 years from now?
2. What is the number of cents in t quarters? in n dollars?
3. A number 5 more than n
4. A number 5 less than n
5. A number 5 times as large as n
6. The sum of n and 5

Oral Exercises 5-2
(continued)

7. The product of 14 and x
8. 7 dollars more than the number of dollars in the bank
9. 7 dollars less than the number of dollars in the bank
10. Number of years in Sam's age 4 years from now
11. Number of years in Sam's age 3 years ago
12. Number of years in Sam's age when he is twice as old as he is now
13. Number of years in Sam's age when he was half as old as he is now
14. Number of inches in x feet (HINT: How many inches in one foot? In two feet? In three feet?)
15. Number of feet in y yards
16. Number of inches in t yards
17. Number of cents in k nickels
18. Number of cents in d dimes
19. Number of cents in 2y quarters
20. Number of nickels in 3n dollars

Problem Set 5-2

Below are some word phrases. For each, choose a variable, tell what the variable represents, and then write a translation of the word phrase into an open phrase.

1. 7 more than 3 times the number of dollars Fred has
2. 7 less than 3 times the number of dollars Ann has
3. The number of inches in the length of a rectangle that is twice as long as it is wide
4. The sum of a number and twice that number
5. The sum of a counting number and the next two counting numbers

Problem Set 5-2
(continued)

6. The sum of an even number and the next even number

7. The number you get when you add 3 to some number and then multiply by 2 (HINT: Think of it like this: Choose some number. Add 3 to the number. Then multiply by 2.)

8. The number you get when you multiply some number by 2 and then add 3

9. The number of square inches in the area of a rectangle that is 10 inches longer than it is wide

10. The number of inches in the perimeter of a rectangle that is 10 inches longer than it is wide (HINT: You can find the perimeter of this rectangle by adding the number of inches in all four sides. You may have heard people say that "perimeter" of a rectangle is the "distance around" the rectangle.)

11. The perimeter of a square

12. The number of cents in t dollars and $(t + 2)$ quarters

13. The number of cents in d dimes and $(d + 5)$ quarters

14. The cost of n pounds of coffee at 60¢ per pound and $(n + \frac{1}{2})$ pounds of coffee at 70¢ per pound

15. The amount of salt in n ounces of solution whose concentration is 10%

16. The number of cents in k dollars, $(k - 2)$ quarters, and $2k$ nickels

17. The cost of g gallons of ethyl gasoline at 36.9¢ per gallon and $(g + 2)$ gallons of regular gasoline at 30.9¢ per gallon

18. The cost of q quarts of oil at 50¢ per quart and $(7q)$ gallons of gasoline at $32\frac{1}{2}$¢ per gallon

19. The total value of c cents, $2c$ dimes, $\frac{1}{2}(c + 5)$ nickels, and $(c - 7)$ dollars

Problem Set 5-2
(continued)

20. The cost of y candy bars at 10¢ each and $(y + 2)$ packages of chewing gum at 5¢ each

21. The cost of a certain number of candy bars at 3 for 10¢ and a certain number of packages of potato chips at 4 for 50¢, if there are 4 more packages of potato chips than there are candy bars

22. The total value of a certain number of dimes, two more quarters than dimes, and half as many nickels as dimes

23. The number of pounds of salt in a certain number of pounds of 25% solution

24. The perimeter of a rectangle whose width is $\frac{3}{4}$ of the length

25. The volume of a box whose length is twice the width and whose depth is $\frac{3}{8}$ of the width

---

5-3. Open Sentences to Word Sentences.

You know that "$2x = 500{,}000$" is an open sentence. Could this sentence be translated into words? We could say, "Two times x is five hundred thousand". But this isn't very interesting. Instead we might translate the variable as we did when we were translating phrases.

For example, we might translate "x" as "the number of soldiers in the army of a country". Then what is the translation of the sentence? We could show our work like this:

x	$2x = 500{,}000$
number of soldiers in the army of a country	Two times the number of soldiers in the army of a country is 500,000.

Do you see what the translation of "$2x = 500{,}000$" is? Can you give a different translation?

Here are some other examples:

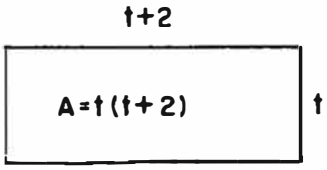
(1) Give a translation of the open sentence "$3x - 2 = 784$".

x	$3x - 2 = 784$
number of books in the library	The number of books in the library is tripled. Two books are lost. There are 784 books left.

(2) Translate $t(t + 2) = 425$.

t	$t + 2$	$t(t + 2) = 425$
number of feet in the width of a rectangle	number of feet in the length of the rectangle if the length is 2 feet more than the width	The area of the rectangle is 425 square feet, since we know that the formula for the area of a rectangle is area = width × length

The "picture" at the right may help in seeing what the translation means.



If we wanted to give the translation of "$t(t + 2) = 425$" in one complete sentence, we could write:

> A rectangle whose length is 2 feet more than its width has an area of 425 square feet.

(3) Translate $n + (2n - 5) = 54$

The translation might look like this:

> A board 54 feet long is cut into two pieces so that one piece is 5 feet shorter than twice the other piece. Do you see that the translation says these things:

n	$2n - 5$	$n + (2n - 5) = 54$
number of feet in one piece of the board	number of feet in the other piece	The length of the whole board is 54 feet.

Oral Exercises 5-3

Translate these open sentences.

1. $t + 8 = 90$

2. $5y = 105$

Oral Exercises 5-3
(continued)

3. $\frac{x}{10} = 47$
4. $2n + 8 = 62$
5. $x - 5 = 12$
6. $n + (2n + 1) = 30$
7. $r(r - 3) = 18$
8. $r(r - 3) \neq 18$
9. $t + (3t - 1) = 20$
10. $t + (3t - 1) \neq 20$

Problem Set 5-3

Below are some open sentences. For each, decide what the variable represents, and then write a word translation for the open sentence. Make it as sensible a translation as possible.

1. $5n = 25$
2. $y + 5 = 20$
3. $t - 5 = 20$
4. $\frac{t}{5} = 20$
5. $2n + 3 = 47$
6. $2n - 3 = 47$
7. $x + x + x + x = 90$
8. $4n + 7n = 44$
9. $5k + 12k = 51$
10. $(n)(2n) = 300$
    (HINT: This sentence might suggest the area of a rectangle.)
11. $n(n + 2) = 300$
12. $w(w - 4) = 16$

*13. $(3x + 1) + x = 46$

*14. $(2y + 3) + (y + 3) = 30$

5-4. Word Sentences to Open Sentences.

Open sentences are very useful in solving many types of problems. In order to use open sentences to solve problems that are stated in words we must first translate from word sentences to open sentences. We shall study several examples so as to become familiar with the way the process works. To begin with, suppose we consider the following:

A board 44 inches long is to be cut into two pieces so that one piece is 3 inches longer than the other piece. How long should the short piece be?

You can see that this is a problem. If we wanted to solve the problem, we could try guessing. For instance, we might

guess that 18 should be the number of inches in the short piece. It isn't hard to decide whether our guess is right or wrong. The thinking might go like this:

First Guess

LENGTH OF SHORT PIECE 18 (Remember, this is just a guess.)

LENGTH OF LONG PIECE 18 + 3 (How do we know this?)

or 21

Add the numbers 18 and 21. Do you get 44?

Second Guess

LENGTH OF SHORT PIECE 20

LENGTH OF LONG PIECE 20 + 3 (Why?)

or 23

If you add the numbers 20 and 23, you still don't get 44. So we still have not solved the problem.

We could spend a long time guessing without solving the problem. But we really don't have to guess. We can just admit that we don't know the number of inches in the short piece and let k be that number. Then our work looks like this:

Using Variable

LENGTH OF SHORT PIECE k

LENGTH OF LONG PIECE $k + 3$ (Why?)

We don't know what number k is or what number $(k + 3)$ is. But we do know that when we add these numbers the sum must be 44. So we can write:

$$k + (k + 3) = 44.$$

If we found the truth set of this sentence, the answer to our problem would be in this set. We're not going to worry about trying to find the truth set here; we'll discover some good ways to do this in a later chapter. (Of course, if you would like to try to find the truth set of this sentence, go right ahead.)

Here is what we really need to notice from this example. We have translated from a word problem about a board to an algebra problem about numbers, like this:

Words		Algebra
A board 44 inches long is cut into two pieces so that one piece is 3 inches longer than the other piece.	$\longrightarrow$	$k + (k + 3) = 44$
How long should the short piece be?	$\longrightarrow$	What is the truth set of "$k + (k + 3) = 44$"?

You see, we might say that we are making two translations. We are translating a word sentence about a board into an open sentence in algebra. This is what we are going to try to do in this lesson.

We are also translating a question about the board into a question about the truth set of an open sentence. We are not going to try to answer these questions in this lesson. Later, we will.

Here is another problem.

The width of a rectangle is 10 inches less than the length. The area of the rectangle is 400 square inches. What is the length of the rectangle?

First Guess

NUMBER OF INCHES IN LENGTH 18 (Just a guess)

NUMBER OF INCHES IN WIDTH 18 - 10 (How do we know this?)
or 8

NUMBER OF SQUARE INCHES IN AREA 18 × 8 (How is the area of a rectangle found?)
or 144

Since we got an area of 144 square inches instead of 400 square inches, we know that our guess was not an answer to the problem.

Second Guess

NUMBER OF INCHES IN LENGTH _____ (You make a guess)

NUMBER OF INCHES IN WIDTH _____

_____

NUMBER OF SQUARE INCHES IN AREA _____

_____

Was your guess right? How do you know?

Using a Variable

NUMBER OF INCHES IN LENGTH t

NUMBER OF INCHES IN WIDTH t - 10

We don't know what number t is or what number (t - 10) is, but we do know that their product must be 400. So we can write the open sentence:

$$t(t - 10) = 400.$$

If we could find the truth set of this open sentence, the answer to our problem would be in this set. As we said before, we are not going to try to find this truth set now. We just stop with the open sentence.

WARNING!

Before you go on to the problems, be sure that you understand the following:

Suppose somebody says, "5 is less than 9". How would we write it in mathematics? We could write "5 < 9". That is all. It doesn't say how much less. Do you remember that "<" is a translation of "is less than"?

But suppose somebody says "5 is 4 less than 9". How do we write this in mathematics?

$$5 = 9 - 4$$

Here we know how much less 5 is than 9. It is 4 less. So if we subtract 4 from 9, the result is the same as, or is equal to, 5. There is no need for the "<" here.

Check Your Reading

1. How do we write "5 is less than 9" using symbols?

2. How do we write "5 is 4 less than 9"?

Check Your Reading
(continued)

3. How do we write "15 is greater than 9"?

4. How do we write "15 is 6 greater than 9"?

Oral Exercises 5-4

1. The width of a rectangle is 8 inches less than the length. The area of the rectangle is 180 square inches. What is the length of the rectangle?

   (a) What variable do you choose to represent the number of inches in the length?
   (b) What is the open phrase representing the number of inches in the width?
   (c) What is the area of the rectangle in terms of the variable?
   (d) Write an open sentence that could be used in solving the problem.

2. A board 39 inches long is to be cut into two pieces so that one piece is 3 inches longer than the other piece. How long should the short piece be?

   (a) What variable do you choose to represent the number of inches in the length of the short piece?
   (b) What is the open phrase representing the number of inches in the longer piece?
   (c) What is the sum of the lengths in terms of the variable?
   (d) Write an open sentence that could be used to solve this problem.

Translate the following into "sentences in algebra".

3. 30 is 13 greater than 17.

4. 14 is 3 less than 17.

5. 14 is less than 10.

6. 14 is 7 less than 10.

7. 42 is 10 greater than 32.

8. 42 is 10 greater than x.

Oral Exercises 5-4
(continued)

9. 42 is less than the sum of x and 7.
10. 12 is 9 less than 21.
11. 12 is 9 less than x.
12. x is 4 more than 12.
13. y is 12 less than 32.
14. y is greater than the number which is 12 less than 32.
15. 2m is 3 more than m.
16. s is 5 less than 2s.
17. s is less than the sum of 21 and 5.
18. 15 is 2x less than 3x.
19. 5 is y more than 4y.
20. A number is three less than twice the number.
21. A number is less than the sum of 3 and 5 times the number.
22. 3 times a number is more than the sum of 5 and 2 times the number.

Problem Set 5-4

Below are some problems written in words. For each one write an open sentence which we could use to solve the problem. Remember that you can stop with the open sentence; you do not need to find the truth set. Be careful to translate the variable you use in each problem.

1. The sum of a number and twice the number is 80. What is the number?
2. The difference between a number and a number 3 times as large is 15. What is the number? (HINT: Be sure that you subtract the smaller number from the larger one.)
3. 106 is the product of a number and another number 5 greater than the first. What is the first number?
4. 82 is the product of a number and another number 6 less than the first. What is the first number?

Problem Set 5-4
(continued)

5. A board 58" long is cut into two pieces so that one piece is twice as long as the other piece. How long is the shorter piece?

6. A board 60" long is cut into three pieces. If the first piece is 10" long and the second piece is twice as long as the <u>first</u> piece, how long is the third piece?

7. A board 60" long is cut into three pieces. If the first piece is 10" long and the second piece is twice as long as the <u>third</u> piece, how long is the third piece?

8. A board 40" long is cut into 3 pieces so that the second piece is twice as long as the first piece and the third piece is three times as long as the first piece. How long is the first piece?

9. The perimeter of a rectangle is 66 feet and the width of the rectangle is 13 feet. What is the length of the rectangle?

10. A rectangle has a width of w feet and a length which is 5 feet longer than the width. If the perimeter is 60 feet, find the width.

11. The length of a rectangle is 7 feet longer than the width, and the perimeter of the rectangle is 50 feet. What is the width of the rectangle?

12. The width of a rectangle is 6 feet less than the length, and the perimeter of the rectangle is $72\frac{1}{2}$ feet. What is the length of the rectangle?

13. The length of a rectangle is twice the width, and the area of the rectangle is 98 square feet. What is the width of the rectangle?

14. Mary and David ran for class president. Mary got 30 votes more than David, and there were 516 votes in all. How many votes did David get?

15. A man left a total of $10,500 for his wife, his son, and his daughter. The wife got $5,000. The daughter got twice as much as the son. How much did the son get?

Problem Set 5-4
(continued)

16. Roger changes 175 pennies into nickels at the store. How many nickels does he get?

17. Roger changes 175 pennies into nickels and dimes at the store. He gets x nickels and 3 dimes. How many nickels does he get?

18. If Roger wishes to change 175 pennies for x nickels and 2x dimes, how many nickels does he get?

19. John has 60 cents, all in nickels and dimes. If he has 6 nickels, how many dimes does he have?

20. John has 60 cents, all in nickels and dimes. He has twice as many nickels as dimes. How many dimes does he have?

21. Ann has $1.55, all in quarters and dimes. The number of dimes is one less than three times the number of quarters. How many quarters does she have?

22. Dick has $2.00, all in nickels, dimes, and quarters. If he has 2 quarters and twice as many dimes as nickels, how many nickels does he have?

23. Dick has $1.65, all in nickels, dimes, and quarters. He has one more quarter than he has dimes. The number of nickels he has is one more than twice the number of dimes. How many dimes does he have?

24. Roger uses a baseball bat which is 2 inches longer than the bat Mickey uses. The sum of the two lengths is 64 inches. How long is Roger's bat?

25. Willie has 21 home runs, 3 triples, and three times as many singles as doubles. If he has 153 <u>total</u> bases, how many doubles does he have?

26. John is now three times as old as Dick. Three years ago, the sum of their ages was 22 years. How old is Dick now?

*27. Jim has $3.25 in nickels, dimes, and quarters. He has two more quarters than dimes. How many nickels does he have?

*28. Bob is twice as old as Bill. Three years from now, the sum of their ages will be 30 years. How old is Bill now?

Problem Set 5-4
(continued)

*29. A rectangular table is three times as long as it is wide. If the length were 3 feet less and the width were 3 feet more, the table would be square in shape. How wide is the table?

*30. A passenger train goes 20 miles per hour faster than a freight train. After 5 hours, the passenger train has gone 100 miles farther than the freight train. How fast is the freight train going? (HINT: The number of miles equals the number of miles per hour multiplied by the number of hours.)

*31. John has a pocket full of change. He has three times as many dimes as quarters and twice as many nickels as quarters. How much money does he have?

5-5. Other Translations.

Earlier we worked with open sentences such as

$$x + 30 < 40.$$

Of course, we could read this as "x plus thirty is less than forty". But we can also translate this sentence just as we did sentences using "=".

To do this we first tell what the variable represents. For instance, we might say that "x" represents "the number of dollars". Then the translation of the sentence can be shown like this:

x	$x + 30 < 40$
number of dollars in my wallet	If 30 more dollars are collected, there will still be less than 40 dollars in my wallet.

Just as before, there are many ways to translate the variable. Here is another translation.

x	$x + 30 < 40$
number of points made by a football team during the first half	If the team makes 30 more points in the last half, they will still have less than 40 points.

It is worth noticing again that no matter what word translation we make, <u>x is a number and $x + 30$ is another number.</u>

Oral Exercises 5-5a

Try to translate these open sentences into word sentences. Several people might give a different translation for each sentence.

1. $a < 3$
2. $a > 3$
3. $n + 1 > 17$
4. $n + 1 < 17$
5. $3t + 4 < 12$
6. $x > 10$ and $x < 15$
7. $m \leq 12$ and $m \geq 3$
8. $n + (n + 5) > 35$
9. $a > b$ and $a + b > c$
10. $5n + 10(n + 2) + 25(n - 1) > 400$
11. $\frac{1}{2}a(a + 2) > 20$ (Think of a triangle or trapezoid.)
12. $(\ell + 1)\ell \leq 37$
13. $\pi(a + 1)^2 \geq 40$
14. $\pi a^2(a + 2) < 17$
15. $\pi(x - 2)^2 4 \geq 6x$
16. $.10x = 9$
17. $.30x + .40(x + 2) = 3.40$
18. $\frac{1}{2}(b + 1)(b) \leq 15$
19. $2(\ell - 2) + 2\ell < 19$
20. $a + 2a + (3a - 1) > 12\frac{1}{2}$

Problem Set 5-5a

Write a translation for each one of the following open sentences. Be sure to start by carefully telling what the variable represents.

1. $t < 6$
2. $t > 6$
3. $y + 15 < 60$
4. $y + 15 > 60$
5. $10y > 80$
6. $25r < 200$
7. $2x + 5 > 50$
8. $2x + 5 < 50$
9. $a + 2a + 3a > 48$

*10. $a + 2a + 3a \geq 48$

In the problems above, you were translating from an open sentence to a word sentence. We already know that we can also translate from a word sentence to an open sentence.

Look at the following:

Fred has more than 5 dollars.
How many dollars does he have?

It seems strange to ask how many dollars he has. We can't tell exactly how many he has. But at least we can translate this word problem about dollars to an algebra problem about numbers.

We know he might have 7 dollars, since $7 > 5$.
We know he might have 100 dollars, since $100 > 5$.
We know he doesn't have only 4 dollars, since it is not true that $4 > 5$.

It would be impossible to list all of the numbers that might be the number of dollars he has. But we can use a variable, say x, and make a translation like this:

English	algebra
number of dollars Fred has	x
He has more than 5 dollars.	$x > 5$

The answer to the question, "How many dollars does he have?", will be one of the numbers in the truth set of "$x > 5$".

As another example, think of an object falling through the air toward the earth. Suppose we know that during the second second it falls 32 feet more than it did during the first second, and that it falls 48 feet or less during the first two seconds together. How far did it fall during the first second?

A Guess

NUMBER OF FEET FALLEN DURING 1st SECOND 30

NUMBER OF FEET FALLEN DURING 2nd SECOND 30 + 32 (Why?)

or 62

How do we know our guess couldn't possibly be right? To find how far the object fell during the first two seconds, find the sum of 30 and 62, which is 92. But 92 is more than 48. We know that it fell 48 feet or less during the first two seconds. "$30 + 62 \leq 48$" is not true.

You may want to make other guesses, but we know that we can use a variable, say y, to represent "number of feet fallen during the first second".

Using a Variable

NUMBER OF FEET FALLEN DURING 1st SECOND y

NUMBER OF FEET FALLEN DURING 2nd SECOND $y + 32$ (Why?)

When we add the numbers y and $y + 32$, the sum must be 48 or less. In other words, the sum must be less than or equal to 48. So we can write the following open sentence:

$$y + (y + 32) \leq 48.$$

y is one of the numbers in the truth set of this sentence. We will not try to find the truth set at this time.

In the last example, do you see that once again we have taken a problem about a thing (a falling object) and translated it to an algebra problem about numbers?

Finally, here is a third example.

Ann was counting votes in a school election where Joe and Bob were running for president. She had to leave before the

counting was over. When she left, Joe had 52 votes and Bob had 40 votes. Later she learned that Bob won the election and that there were 220 votes in all. How many votes did Joe get?

Joe got 52 votes or more.	He had 52 votes when Ann left. He might have gotten more after she left.
Joe got 109 votes or less.	There were only 220 votes. If Joe got 110, it would have been a tie. If Joe got more than 110, he would have won. But we know he lost.

In translating this word problem about votes to an algebra problem about numbers, we can let "n" represent "number of votes Joe got". Then,

$$n \geq 52 \quad \text{AND} \quad n \leq 109.$$

n is one of the numbers in the truth set of "$n \geq 52$ and $n \leq 109$". It's true that there are many numbers in this truth set, and we don't know which one is the number of votes that Joe got. But at least we have told, in the language of algebra, what we know about the number n. From what set of numbers must n be chosen? Would 40 be a possible answer to the question? Why? Would 100.5 or $77\frac{1}{4}$ be suitable answers? What is the domain of the variable?

Check Your Reading

1. In the first example above, how do we know that Fred does not have 5 dollars?
2. In the third example above, could Joe have had 30 votes? 120 votes? Could Bob have had 30 votes? 120 votes? Why? Could he have $71\frac{1}{2}$ votes? Why?

Oral Exercises 5-5b

In each of the following choose a variable, tell what it represents, and then state an open sentence that would help solve the problem.

1. John has more money than Tom. Tom has $50. How much money

Oral Exercises 5-5b
(continued)

does John have?

2. 300 students attend Washington School. Less than one half live in the city. How many live in the city?

3. An airplane can fly no higher than 30,000 feet above sea level. How high is it, if it is flying now?

4. If the altitude of Denver is 5,280 feet above sea level and the plane in exercise 3 is circling the Denver airport, what is the airplane's altitude above sea level?

5. Bill is ten pounds heavier than John. Their total weight is greater than 220 pounds. How heavy is John?

6. Mary has more brothers than Jane. How many brothers does Mary have?

Problem Set 5-5b

Below are a number of problems. For each one, write an open sentence that would help to solve the problem. You do not need to find the truth set. In each problem, be sure to tell what the variable represents.

1. Tom has saved more than $200. How much money does he have?

2. Fewer than 100 people went to the park. How many people went?

3. The sum of four times a number and nine times the number is greater than 100. What is the number?

4. The product of a number and 7 is greater than 45 or equal to 45. What is the number?

5. The difference between eight times a number and three times the number is less than or equal to ten. What is the number?

6. The altitude of Denver, Colorado, is more than 5,000 feet. What is the altitude of Denver?

7. The population of the United States is about 180,000,000, and this is greater than twice the population of Mexico. How many people live in Mexico?

Problem Set 5-5b
(continued)

9. Bill is 5 years older than Norma, and the sum of their ages is less than 23. How old is Norma?

10. John said, "It will take me more than 2 hours to mow the lawn and I can't spend more than 4 hours on the job if I want to go swimming." How many hours can he expect to spend on the job?

11. The National Safety Council says that between 250 and 300 people will be killed on the highways over the weekend. How many people do they expect to be killed? (HINT: We can suppose they mean "no fewer than 250 and no more than 300", although this is not what "between" means in mathematics.)

12. A boat, going downstream, goes 12 miles per hour faster than the current. The boat's speed downstream is less than 30 miles per hour. What is the speed of the current?

*13. On a half-hour TV show, the advertiser says there must be at least (meaning not less than) 3 minutes for commercials, and the network says there must be less than 7 minutes for commercials. How many minutes can be used for advertising?

*14. A square and an equilateral triangle have equal perimeters. A side of the triangle is 5 inches longer than a side of the square. What is the length of a side of the square?

15. Less than one-half of the students at school attended the game. 152 left in the middle of the game. How many remained?

Summary

In this chapter, we have seen that we can "switch back and forth" between the language of words and the language of algebra. This is one of the reasons that mathematics is the important subject that it is. People can solve many problems about many different kinds of things by translating them to algebra problems

and then solving the algebra problems.

For example, here is a kind of problem that might come up in the study of medicine or biology.

> When a man is 15,000 feet in the air, the number of breaths he takes each minute is $1\frac{1}{2}$ times the number of breaths he takes each minute at sea level. A certain man is found to breathe 30 times each minute at 15,000 feet. How many times a minute does he breathe when he is at sea level?

To solve this problem, the translation might go like this:

NUMBER OF BREATHS HE TAKES EACH MINUTE AT SEA LEVEL $\underline{\quad n \quad}$
NUMBER OF BREATHS HE TAKES EACH MINUTE AT 15,000 FT. $\underline{1\frac{1}{2}n}$(Why?)

But the problem says that we know the number of breaths he takes at 15,000 feet. It is 30. So we can write the sentence:

$$(1\tfrac{1}{2})n = 30.$$

If we were to find the truth set of this sentence, we would have an answer to our problem in biology.

We saw in this chapter that we can translate both phrases and sentences. For instance, suppose that we use "t" to represent "the number of quarts of water a certain jar can hold". Here are some translations.

PHRASES

(1) Number of quarts held by a jar holding twice as much $\underline{2t}$
(2) Number of quarts held by a jar holding 3 quarts more $\underline{t + 3}$
(3) Number of quarts held by a jar holding 2 quarts less than three times as much $\underline{3t - 2}$
(4) Number of quarts in the jar if it is filled and one quart is taken out $\underline{t - 1}$

SENTENCES

(1) If the jar is filled and one quart is taken out, there will be 5 quarts left in the jar. $\underline{t - 1 = 5}$

(2) If the jar is half-full, there will be less than 4 quarts in the jar. $\frac{1}{2}t < 4$

(3) If another jar of the same size is used, the two jars together will hold more than 6 quarts. $2t > 6$

(4) The number of quarts the jar will hold is greater than or equal to 6. $t \geq 6$

After working in this chapter, it is not hard to see why we say that mathematics is a kind of language. We have now learned to say many things in this language.

Maybe you were bothered sometimes in this chapter because we had problems that we translated but did not go on to solve. We will be doing this very soon. But in order to do a better job of it, we need to learn about some new numbers. This is what we will be doing in the next chapter.

Review Problem Set

1. Write a translation of "t" and then translate each of the following phrases and sentences using the same translation of "t".

(a) $2t$	(e) $t - 1 = 5$
(b) $t + 3$	(f) $\frac{1}{2}t < 4$
(c) $3t - 2$	(g) $2t > 6$
(d) $t - 1$	(h) $t \geq 6$

2. Translate the following word phrases into open phrases using one variable in each case. What does the variable represent in each problem?

   (a) the number of days in w weeks

   (b) the product of a number and twice the number

   (c) five more than three times the number of students

   (d) the number you get when you subtract 5 from a number and then multiply by 7

   (e) one half the area of a rectangle that has one side twice as long as the other

Review Problem Set
(continued)

(f) the total value of a number of nickels and twice that number of dimes

(g) the total cost of a certain number of pounds of chocolates costing \$1.40 per lb. and of a certain number of pounds of jelly beans costing 30¢ per lb., if there is one pound more of jelly beans than of chocolates

(h) the area of a triangle whose altitude is $3\frac{1}{2}$ inches longer than the base

(i) the volume of a cylinder whose height is $5\frac{3}{4}$ inches greater than the radius

(j) the result of increasing a number by $2\frac{1}{3}$, multiplying the sum by 8.9, and then dividing the resulting product by 3.4

(k) 20% of a certain number of gallons of salt solution

(l) the age of Mary's brother four years ago if he is now twice as old as Mary is

(m) the total cost of a certain number of loaves of bread at 25¢ each and a certain number of boxes of cookies at 32¢ each, if there are two more boxes of cookies than there are loaves of bread

(n) the total value of a certain number of quarters, a certain number of dimes, and a certain number of nickels, if there are 3 more dimes than there are quarters and 2 less nickels than there are quarters

(o) the area of a triangle whose height is 2 inches less than half the base

(p) the result of multiplying a certain number by 7, dividing the product by $\frac{3}{2}$, and increasing the resulting quotient by twice the original number

Review Problem Set
(continued)

(q) the area of a trapezoid whose altitude is one inch less than the shorter of the two bases, and whose larger base is $3\frac{1}{2}$ inches greater than the shorter base

(r) the perimeter of a triangle whose longest side is twice the length of the second longest side, while the second longest is twice the length of the shortest

3. Write an open sentence, using one variable in each case, which could be used in solving the following problems. What does the variable represent?

(a) The sum of a number and three times the number is 45. Find the number.

(b) The sum of a whole number and its successor is 45. What are the numbers?

(c) Jim is two inches taller than John. Five years ago they were the same height. How tall is John?

(d) Mary, who is 16, is four years older than her sister. How old is her sister?

(e) Pike's Peak is more than 14,000 feet above sea level. How high is Pike's Peak?

(f) The sum of two consecutive odd numbers (meaning "an odd number and the next larger odd number") is 75. What are the numbers?

(g) A teacher says, "If I had 3 times as many students in my class as I have, I would have at least 26 more than I now have." How many students does he have in his class?

(h) The largest angle of a triangle is 20° more than twice the smallest, and the third angle is 70°. The sum of the angles of a triangle is 180°. How large is the largest angle?

Review Problem Set
(continued)

(i) One even number is 48 greater than another and the sum of the two numbers is 1124. What are the numbers?

(j) John is 5 years older than his brother. Five years ago the sum of their ages was 18. What will the sum of their ages be 6 years from now?

(k) Two squares differ in area by 27 square units. A side of the larger square is one unit greater than a side of the smaller. What is the side of each square?

(l) Sam has 5 hours at his disposal. How far can he ride his bicycle into the country at 8 miles per hour if he is to return by the same route at 12 miles per hour?

(m) A rectangle is 7 times as long as it is wide. Its perimeter is 150 inches. How wide is the rectangle?

(n) If $\frac{2}{3}$ of a number is added to 32 the result is $38\frac{1}{2}$. What is the given number?

(o) 20 lbs. of water is added to a certain number of lbs. of 10% salt solution. The resulting solution is $2\frac{1}{2}$% salt. Find the number of lbs. of the original 10% solution.

(p) On a 20% discount sale a chair was sold for $29.95. What was the price before the discount?

(q) Two trains leave Chicago at the same time. One travels north at $62\frac{1}{2}$ miles per hour while the other travels north at 39.7 miles per hour. How many hours will it take them to become $125\frac{1}{2}$ miles apart?

(r) John has 50 coins which are nickels, pennies, and dimes. He has four more dimes than pennies and six more nickels than dimes. How many of each kind of coin does he have?

Review Problem Set
(continued)

4. Find the truth sets of each of the following sentences and draw their graphs.

(a) $3x > 0$

(b) $x + 2 = 5$

(c) $2a - 1 = 2$

(d) $\frac{3}{4} = \frac{3}{4}x$

(e) $5m < 1$

(f) $\frac{3}{8}s = \frac{7}{8}$

(g) $x = 2$ and $2x < 7$

(h) $m + 2 = m + 4$

(i) $x = 4$ or $2x > 8$

(j) $x > 2$ and $3x < 6$

(k) $y + 2\frac{1}{2} = 3\frac{1}{4}$ or $\frac{5}{4}y > \frac{15}{16}$

(l) $3x + \frac{21}{2} = 3(\frac{7}{2} + x)$

5. Write each of the following expressions as an indicated product, using the distributive property.

(a) $a(2b) + a(3a)$

(b) $a(3) + a$

(c) $x(2b) + x$

(d) $ax(\frac{4}{3}) + ax(\frac{1}{3})$

(e) $\frac{4}{3}ax + \frac{1}{3}ax$

(f) $2a(\frac{7}{8}) + 2x(\frac{3}{4})$

(g) $\frac{5}{2}m + \frac{5}{2}m(\frac{9}{10})$

(h) $2ax + x$

(i) $7(2) + 6(2)$

(j) $(a + b)x + (a + b)y$

6. Write each of the following expressions as an indicated sum, using the distributive property.

(a) $x(3x + 1)$

(b) $2a(3a + 2)$

(c) $a^2(b^2 + 7b)$

(d) $10a(a + b)$

(e) $12cd(c + 1)$

(f) $\frac{4}{3}m(\frac{2}{3} + m)$

(g) $\frac{4}{5}x(\frac{x}{2} + \frac{3}{8})$

(h) $3.3(2.5x + 1.2)$

(i) $(a + b)(c + d)$

(j) $(5 + 4)2$

7. State whether each of the following sets is closed under the operation of addition. Give explanations or examples to support your answers.

(a) $\{1, 2, 3\}$

(b) $\{0, 2, 4, \ldots\}$

(c) $\{1, 3, 5, 7, \ldots\}$

(d) $\{2, 4, 6, \ldots, 30\}$

Review Problem Set
(continued)

(e) $\{5, 10, 15, \ldots, 30\}$

(f) $\{0\}$

(g) $\{0, 1\}$

(h) $\{1, \frac{3}{2}, 3, \frac{5}{2}, 5, \ldots\}$

(i) $\{0, 5, 10\}$

(j) $\{0, 10\}$

8. Find the value of each of the following open phrases if m is 2, n is $\frac{1}{2}$, and p is 3.

(a) $m(np)$

(b) $m(n + p)$

(c) $\frac{1}{2}m + \frac{1}{3}n + \frac{1}{4}p$

(d) $m^2 - 4n + \frac{2}{3}p$

(e) $\dfrac{5m + 2p^2}{n}$

(f) $np + \frac{2}{3}m$

(g) $\dfrac{5np + \frac{1}{2}(m + p)}{4}$

9. Simplify each of the following expressions. Use the properties that make the work easiest.

(a) $(\frac{4}{3} \times \frac{29}{12})(\frac{3}{4})$

(b) $(\frac{5}{8})(\frac{7}{12}) + (\frac{5}{8})(\frac{13}{12})$

(c) $\dfrac{\frac{7}{5}}{\frac{24}{5}}$

(d) $3(1 + \frac{1}{4})$

(e) $(\frac{5}{7} + \frac{100}{20}) + \frac{2}{7}$

(f) $\frac{3}{5}(\frac{65}{12}) + \frac{3}{5}(\frac{115}{12})$

(g) $(\frac{5}{19})(\frac{14}{37})(\frac{1}{2} - \frac{1}{3} - \frac{1}{6})$

(h) $\frac{9}{16}(14 + \frac{2}{3}) - \left(\frac{9}{16}(14) + \frac{9}{16}(\frac{2}{3})\right)$

(i) $(489 \times \frac{8}{47})(5 - \frac{20}{4})$

(j) $\dfrac{\frac{1}{7}(42 + 49)}{\frac{11}{12}(\frac{49}{11}) + \frac{11}{12}(\frac{23}{11})}$

CHALLENGE PROBLEMS

1. By putting one of the signs, +, ×, -, in each of the blanks, and inserting parentheses to indicate grouping, work out <u>all</u> the numbers which can be obtained from:

   8_____3_____2.

   As examples, $8 - (3 \times 2) = 2$, and $(8 + 3) \times 2 = 22$.

2. Look for the pattern in the following calculation:

$$
\begin{aligned}
19 \times 13 &= 19(10 + 3) \\
&= 19(10) + 19(3) && \text{(what property?)} \\
&= 19(10) + (10 + 9)3 \\
&= 19(10) + \big(10(3) + 9(3)\big) && \text{(what property?)} \\
&= \big(19(10) + 10(3)\big) + 9(3) && \text{(what property?)} \\
&= (19 + 3)10 + 9(3) && \text{(what property?)}
\end{aligned}
$$

   The final result may be expressed as a "rule for multiplying teens" (whole numbers from 11 through 19): Add to the first number the units digit of the second, and multiply by 10; then add to this the product of the units digits of the two numbers. Use the rule to find (a) 15×14, (b) 13×17, (c) 11×12.

3. In problem 2 you discovered a "rule for multiplying teens". Using a and b, respectively, to stand for the units digits of the two numbers, you should now be able to write an open sentence which expresses the product p in terms of a and b. When you have written your sentence, use the distributive property to verify the correctness of your choice.

4. Here you are going to see how to test whether a whole number is exactly divisible by 9. Keep a record, as you go, of the properties of addition and multiplication which are used. Try the following:

$$
\begin{aligned}
2357 &= 2(1000) + 3(100) + 5(10) + 7(1) \\
&= 2(999 + 1) + 3(99 + 1) + 5(9 + 1) + 7(1) \\
&= 2(999) + 2(1) + 3(99) + 3(1) + 5(9) + 5(1) + 7(1) \\
&= \big(2(999) + 3(99) + 5(9)\big) + \big(2(1) + 3(1) + 5(1) + 7(1)\big) \\
&= \big(2(111) + 3(11) + 5(1)\big)\ 9 + (2 + 3 + 5 + 7) \\
&= (222 + 33 + 5)9 + (2 + 3 + 5 + 7)
\end{aligned}
$$

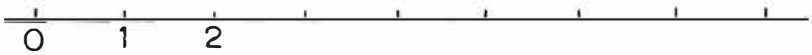
Is 2357 divisible by 9? Try the same procedure with 35874. Can you formulate a general rule to tell when a whole number is divisible by 9?

5. When x takes as values the elements of the set $\{1, 2, 4\}$, $2x$ takes as values the elements of the set $\{2, 4, 8\}$ and $x + 1$ the elements of the set $\{2, 3, 5\}$. Now try to answer the following:

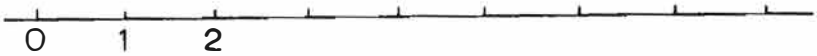
   (a) If x belongs to the set of numbers with graph

   then $2x$ belongs to the set of numbers with graph

   and $x + 1$ to the set with the graph

   (b) If x belongs to the set of numbers between $\frac{1}{2}$ and 5, then to what set does $2x$ belong? $x + 1$?

   (c) If $2x$ belongs to the set with graph

   to what set does x belong?

   (d) If x belongs to the set with graph

   to what set does $\frac{1}{x}$ belong?

6. Let us imagine the coordinates of points on the number line to be written in black. Then write another set of coordinates in red for the same points, assigning red 0 to the point with black 0, red 1 to the point with black $\frac{3}{4}$, red 2 to the point with black $\frac{3}{2}$, etc. In this way each point now has a black coordinate and a red one.

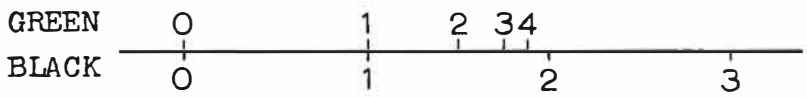
   (a) What will be the red coordinates of the points whose black coordinates are 1, 2, $\frac{5}{2}$, $\frac{7}{4}$, 10, 20?

   (b) What will be the black coordinates of the points whose red coordinates are 1, 2, $\frac{5}{2}$, $\frac{7}{4}$, $\frac{3}{4}$, $\frac{1}{2}$?

   (c) Can you find a point to the right of 0 for which the red coordinate is 3 times the black? What are its coordinates?

   (d) Try to write an open sentence, using b and r for the black and red coordinates, respectively, which would enable you to find one if you knew the other.

7. We start with the number line and coordinates marked in black. We begin by making black 0 also green 0, and black 1 also green 1. Then we make green 2 half as far to the right from green 1 as the distance between green 0 and 1, then green 3 half as far to the right from green 2 as the distance between green 1 and 2. We keep this up, and get something that looks like this:



(a) What is the black coordinate of the point with green coordinate 3? Does every whole number become the green coordinate of a point if this process is continued?

(b) What green coordinate, if any, does the point with black coordinate 3 get?

(c) Can you describe the location of the point which you would want to have green coordinate 7?

8. On the number line we can see that the average of two numbers is associated with the midpoint between the two points associated with the numbers. In this problem we work instead with thirds.

(a) Find the points on the number line $\frac{1}{3}$ of the way from 1 to 2, 2 to 3, 3 to 5, 4 to 6, 5 to 8, 1 to 8, 1 to $\frac{5}{2}$, $\frac{1}{2}$ to $\frac{4}{3}$.

(b) Try to write an open sentence which would enable you to find the number c which is $\frac{1}{3}$ of the way from a to b, where b is greater than a.

(c) Can you do the same sort of thing to find the number d which is $\frac{2}{3}$ of the way from a to b?

9. A man, with five dollars in his pocket, stops at a candy store on his way home with the intention of taking his wife two pounds of candy. He finds candy by the pound box selling for \$1.69, \$1.95, \$2.65, and \$3.15. If he leaves the store with two one-pound boxes of candy,

(a) What is the smallest amount of change he could have?

(b) What is the greatest amount of change he could have?

(c) What sets of two boxes can he not afford?

10. Our numerals for numbers are always in <u>decimal</u> (ten) form. This was illustrated in problem 4 when we wrote

$$2357 = 2(10^3) + 3(10^2) + 5(10) + 7(1).$$

Is there any reason why we cannot write a numeral in powers of a number other than 10, say 8? What set of digits would be required to form numerals in this "8 scale"? (In the decimal scale we need {0, 1, 2, . . ., 9}.) We would then write 2357_{eight} to mean

$$2(8^3) + 3(8^2) + 5(8) + 7(1).$$

What number has this as a numeral? Can you write 207 in the form of a numeral in the 8 scale?

$$207 = 3(64) + 15 = 3(8^2) + 15,$$

and

$$15 = 1(8) + 7(1), \ . \ . \ .;$$

can you finish the work?

11. If you understand the difference between the sets {0} and $\emptyset$, then you will be able to explain what is wrong with this argument: "I can prove that every girl has three heads. You will agree that no girl has two heads; and certainly every girl has one more head than no girl. So by simple addition, every girl has three heads."

12. A number is usually represented by certain special symbols, but there is no reason why we could not use letters of the alphabet to represent numbers. In fact, before special symbols were invented, some people, such as the early Greeks and Romans, used letters for numbers.

Consider the following code:

0	1	2	3	4	5	6	7	8	9
a	b	c	d	e	f	g	h	i	j
k	l	m	n	o	p	q	r	s	t
u	v	w	x	y	z.				

You see that 6 has the names g and q. What are the possible numerals for 2?

Now let us try to read the message

$$9034 = 7424.$$

Trying every possible numeral for the numbers, we have

$$9034 = 7424$$

$$\left.\begin{matrix}\text{jade}\\ \text{tkno}\\ \text{uxy}\end{matrix}\right\} \quad \text{is} \quad \left\{\begin{matrix}\text{hece}\\ \text{romo}\\ \text{ywy}\end{matrix}\right.$$

By carefully choosing numerals, we have the statement "jane is home." Another message could be "june is home", or "judo is homy." Are there other possible messages?

(a) Use the above code to decipher the message $74 = 703674$.
(b) Devise some messages of your own.

Using the same code, decide which of the following are true sentences:

(c) $3\ v(10) + m = y(10) + g$.
(d) $h + m = m + h$.
(e) $p(h) > r(f)$.
(f) $g + 5 \neq 5(g)$.
(g) $8(m) = m(8)$. (Read this as "eight times m equals m times eight.")

13. Certain mathematical puzzles are in the form of addition problems in which each letter of the puzzle represents the same numeral each time it occurs, and a different numeral from every other letter. For example,

$$\begin{array}{r} \text{I} \\ \text{AM} \\ + \ \text{THE} \\ \hline \text{BOSS} \end{array}$$

has several correct "solutions," that is, ways of substituting numerals for letters which satisfy the rules just laid down and make the sum correct. One such solution is:

$$\begin{array}{r} 2 \\ 45 \\ + \ 986 \\ \hline 1033. \end{array}$$

Can you find the solution for the famous addition puzzle

$$\begin{array}{r} \text{SEND} \\ + \quad \text{MORE} \\ \hline \text{MONEY?} \end{array}$$

(It has only one solution.)

14. A teacher proposed the following to his class: "Write a column of four four-digit numbers. If you let me write three more to extend the column to seven, I can give you the sum of all seven before writing down the extra three." The class gave him:

8432
2765
3961
4028

The teacher gave the sum 34025 and then put below the column:

1567
7234
6038

Was the teacher correct? The class then wrote:

8025
4567
3902
5678

The teacher gave a sum 35675. With what numbers should he extend the column? Can you discover the principle used?

15. Complete the following multiplication table and decide how many new multiplications you actually have to perform. (Make liberal use of the commutative and distributive properties.) Hint: There are less than 3.

X	12	10	2	$\frac{5}{4}$	$\frac{3}{4}$
12					9
10				$\frac{25}{2}$	
2					$\frac{3}{2}$
$\frac{5}{4}$	15		$\frac{5}{2}$	$\frac{25}{16}$	
$\frac{3}{4}$		$\frac{15}{2}$		$\frac{15}{16}$	$\frac{9}{16}$

16. What can you say about a set S of whole numbers if it has the two characteristics

(1) 2 is an element of S

(2) whenever a number is an element of S its successor is also an element of S.

17. For each of the following defined operations on numbers decide whether the operation is commutative. That is, is $a \circ b = b \circ a$ true for every a and every b? Give illustrations to support your decisions.

(a) "$a \circ b$" means "$2a + b$"

(b) "$a \circ b$" means "$\frac{a + b}{2}$"

(c) "$a \circ b$" means "$(a - a)b$

(d) "$a \circ b$" means "$a + \frac{1}{3}b$"

(e) "$a \circ b$" means "$(a + 1)(b + 1)$"

18. Decide which of the operations listed in problem 17 are associative. In other words is the sentence $(a \circ b) \circ c = a \circ (b \circ c)$ a true sentence for every a, b, and c. Give examples to illustrate your decision in each case.

Write open sentences that would make possible the solution of the following problems.

19. If one man can paint a house in 5 days and another man can paint the same house in 3 days, then if they work at the same rate how long would it take the two together to paint the house?

If the first man is always $\frac{3}{5}$ as fast as the second, how much time can be saved on <u>any</u> job by the two men working together instead of the first man working alone?

20. There are three pipes to a storage tank. When operating separately one fills the tank in 5 hours, the second one fills the tank in 3 hours and the third drains the tank in 4 hours. If all three pipes are left open when the tank is empty, after how many hours will it start to overflow?

GLOSSARY CHAPTERS 1-5

ADDITION - A binary operation which can be applied to any two numbers.

ADDITION PROPERTY OF ZERO - For every number a, $a + 0 = a$.

ASSOCIATIVE PROPERTY OF ADDITION - For every number a, every number b, and every number c, $(a + b) + c = a + (b + c)$.

ASSOCIATIVE PROPERTY OF MULTIPLICATION - For every number a, every number b, and every number c, $a(bc) = (ab)c$.

BINARY OPERATION - An operation that is applied to two numbers.

CLOSURE - A subset of the numbers of arithmetic has closure with respect to a binary operation if the number produced by applying the operation to any two numbers of the subset is also an element of that set.

COMMUTATIVE PROPERTY OF ADDITION - For every number a and every number b, $a + b = b + a$.

COMMUTATIVE PROPERTY OF MULTIPLICATION - For every number a and every number b, $ab = ba$.

COMPOUND SENTENCE - A sentence consisting of two clauses with a connective. We are particularly interested in the types using connectives "or" and "and".

COORDINATE - The number that is associated with a particular point on the number line.

COUNTING NUMBER - An element of the set $\{1, 2, 3, 4, 5, \ldots\}$.

DISTRIBUTIVE PROPERTY - For every number a, every number b, and every number c, $a(b + c) = ab + ac$.

DOMAIN OF A VARIABLE - The set of numbers from which the value of the variable may be chosen.

ELEMENTS OF A SET - the objects in the set.

EMPTY SET - A set which has no elements, sometimes called the null set. The symbol "$\emptyset$" is used to indicate the empty set.

EVEN NUMBER - An element of the set $\{0, 2, 4, 6, \ldots\}$.

FACTOR - We call b a factor of bc because there is a number c such that the product of b and c gives bc.

FINITE SET - If the elements of a set can be counted with the counting coming to an end or if the set is the null set, we call it a finite set.

FRACTION - A symbol which represents the quotient of two numbers.

GRAPH OF THE TRUTH SET OF AN OPEN SENTENCE - The set of points whose coordinates make the open sentence true.

INFINITE SET - A set whose elements cannot be counted, that is, with the counting coming to an end. The exception to this is the null set which is a finite set.

MULTIPLES OF A NUMBER - A set of numbers which includes numbers obtained by multiplying the given number by a whole number.

MULTIPLICATION - A binary operation which can be applied to any two numbers.

MULTIPLICATION PROPERTY OF ONE - For every number a, $a(1) = a$.

MULTIPLICATION PROPERTY OF ZERO - For every number a, $a(0) = 0$.

NULL SET - The empty set.

NUMERAL - A name for a number.

NUMERICAL PHRASE - A numerical phrase is any numeral given by an expression which involves other numerals along with signs for operations.

ODD NUMBER - An element of the set $\{1, 3, 5, 7, 9, \ldots\}$, obtained by adding 1 to each element of the set of even numbers.

OPEN PHRASE - A mathematical phrase which contains one or more variables.

OPEN SENTENCE - A mathematical sentence which contains one or more variables.

PARENTHESES - ( ) - Symbols to show that the numeral inside is the name for one number.

PROPERTY - A property of an operation is "something which it has" or one of its characteristics.

RATIONAL NUMBERS - A set of numbers including those which can be represented by a fraction indicating a quotient of two whole numbers, excluding division by zero.

SENTENCE - In mathematics we use sentences to make statements about numbers.

SET - A collection of objects.

SUBSET - If every element of set B is an element of set A, then set B is a subset of set A.

SUCCESSOR - The successor of any whole number is the number that is found by adding 1 to the given whole number.

SYMBOLS - The symbol "=" between two numerals indicates that the numerals represent the same number.

The symbol "$\neq$" is used to indicate that two numerals do not name the same number.

The symbol ">" represents the verb phrase "is greater than".

The symbol "<" represents the verb phrase "is less than".

TRUTH NUMBER - A value of the variable which will make a sentence true.

TRUTH SET OF AN OPEN SENTENCE - The set of numbers which make the open sentence true.

VARIABLE - A numeral which represents a definite, but unspecified, number chosen from a given set of numbers.

WHOLE NUMBER - An element of the set {0, 1, 2, 3, 4, . . .}.

Chapter 6

THE REAL NUMBERS

6-1. The Real Numbers.

Integers

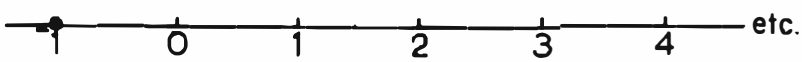
You remember that earlier we "labeled" points on a line with names of numbers. A better way to say this is to say that we associated numbers with the points of a line called the number line. In this chapter, we are going to use the number line to introduce some new numbers.

To begin with, remember that we graphed the whole numbers on the number line, like this:

The number 0 is associated with a point. All of the other numbers are associated with points to the right of 0. (We mean, of course, points to the right of the point associated with 0.) Maybe you have wondered about numbers associated with points to the left of 0.

Very soon we'll see that a new kind of number is needed to solve certain problems. These new numbers will be called negative numbers, and we can associate them with points of the number line to the left of 0.

Let's start by noticing the interval (or the "piece" of the line) between 0 and 1. We will again use this interval as a unit of measure, but this time points will be marked to the left of 0. Using this interval as a unit of measure, the first point located to the left of 0 is shown below:



We label this point " $^{-}1$ " and read it as "NEGATIVE ONE". Notice how high the "dash" in " $^{-}1$ " is written. This dash is a signal to us that we are talking about a point to the left of zero.

The next point located is labeled " $^{-}2$ " and is read "NEGATIVE TWO".

The next point located is labeled " $^{-}3$ " and is read "NEGATIVE THREE".

We could go on and on locating points like this. That's what the "etc." means. How would you locate the point to be labeled " $^{-}7$ "? How would you locate the point to be labeled " $^{-}15$ "?

Using this way of labeling points on the number line, would there be a point labeled " $^{-}1{,}000{,}000$ "? How would this "label" be read?

All of these "labels" we've been giving to points to the left of 0 will be used as names of numbers (that is, as <u>numerals).</u> We'll soon see how these numbers behave and how useful they can be.

We can now take the whole numbers, {0, 1, 2, 3, ...}, together with the new numbers we have named and form a set of numbers that can be shown like this:

$$\{\ \ldots,\ {}^{-}3,\ {}^{-}2,\ {}^{-}1,\ 0,\ 1,\ 2,\ 3,\ \ldots\}.$$

This set is called the <u>set of integers.</u>

Any one of the numbers in the set is called an <u>integer.</u> For example, 8 is an integer, $^{-}43$ is an integer, and so on. (On the other hand, $\frac{1}{2}$ is <u>not</u> an integer.)

In the set of integers shown above, do you remember the meaning of the "three dots" after 3? They mean, of course, that we could go on forever locating integers to the <u>right</u> of 3 on the number line. The three dots before $^{-}3$ mean that we could go on and on locating integers to the <u>left</u> of $^{-}3$ on the number line.

There are some special <u>subsets</u> of the set of integers that can be shown like this:

$$\underbrace{\ldots,\ {}^{-}3,\ {}^{-}2,\ {}^{-}1,}_{\text{negative integers}}\ \underset{\text{zero}}{0},\ \underbrace{1,\ 2,\ 3,\ \ldots}_{\text{positive integers}}$$

The positive integers are associated with points to the right of zero. We can show the set of positive integers like this:

$$\{1,\ 2,\ 3,\ \ldots\}.$$

The negative integers are associated with points to the left of zero. We can show the set of negative integers like this:

$$\{\ \ldots,\ {}^{-}3,\ {}^{-}2,\ {}^{-}1\}.$$

However, many times the set of negative integers is shown like this:

$$\{{}^{-}1,\ {}^{-}2,\ {}^{-}3,\ \ldots\}.$$

Zero itself is an integer, but it is neither positive nor negative.

We can make the following statement:

> The set of positive integers, the set of negative integers, and zero make up the <u>set of integers.</u>

Check Your Reading

1. What kind of symbol do we use for a number which is the coordinate of a point to the left of zero on the number line?
2. What elements make up the set of integers? How many are there?
3. What elements make up the set of positive integers? How many are there?
4. What elements make up the set of negative integers? How many are there?
5. Are there any other integers besides the positive and negative integers?
6. Is zero positive or is it negative?

Oral Exercises 6-1a

1. Name five elements of the set of positive integers.
2. Name five elements of the set of negative integers.
3. Name five elements in the set of whole numbers.
4. Name five elements in the set of counting numbers.
5. Describe the set for which $\emptyset$ is a symbol.

These are all subsets of the set of integers.

Problem Set 6-1a

1. (a) List each of the following sets.

   W, the set of whole numbers

   P, the set of positive integers

   L, the set of non-negative integers

   (Hint: "non" means "not" so "non-negative numbers" means "numbers that are not negative.")

   I, the set of integers

   N, the set of counting numbers

   Q, the set of non-positive integers

   S, the set of negative integers

   (b) Which of the above sets are the same?

   (c) Which of the above are subsets of I? of Q? of L? of P?

2. Draw the graphs of the following sets:

   (a) $\{0, 3, 5, {}^{-}2, {}^{-}4\}$.

   (b) The set of positive integers less than 7.

   (c) The set of negative integers ≥ 5.

   (d) All integers greater than ${}^{-}5$ but less than 4.

   (e) The set of counting numbers less than 1.

3. Of the two points whose coordinates are given, which is to the right of the other on the number line?

   (a) $3, {}^{-}4$

   (b) $5, {}^{-}4$

   (c) ${}^{-}2, {}^{-}4$

   (d) ${}^{-}2, 0$

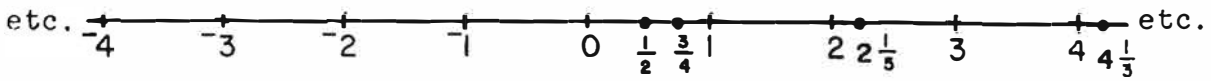
   (e) $0, {}^{-}4$

   (f) $5, 0$

Problem Set 6-1a
(continued)

4. Translate this problem into a sentence in algebra after selecting a variable and telling what it represents. You need not find the truth set of the sentence.
   (a) Bill has 77 pigeons. This is 25 more than twice the number he had 3 years ago. How many did he have 3 years ago?
   (b) The first of two trains travels at a certain rate. The second travels 10 miles per hour more than twice as fast. Starting at the same station and traveling in opposite directions for 4 hours, what are their rates, if they are then 340 miles apart?
   (c) Find the width of a rectangle if its perimeter is 196 inches and its length is 62 inches.

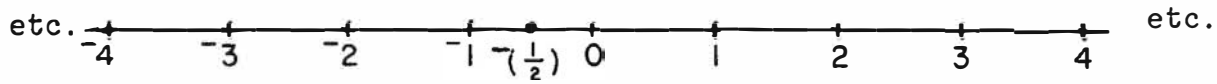
In the last section, it was pointed out that $\frac{1}{2}$ is <u>not</u> an integer. Some examples of numbers that are <u>not</u> integers are $\frac{3}{4}$, $2\frac{1}{5}$, and $4\frac{1}{3}$. You may remember, though, that we have called numbers such as these <u>rational numbers.</u> They also are associated with points of the number line. We can show the graph of the four numbers mentioned in this paragraph like this:



Of course, there are many, many other rational numbers. In fact, as we saw in Chapter 1, there are infinitely many of such rational numbers.

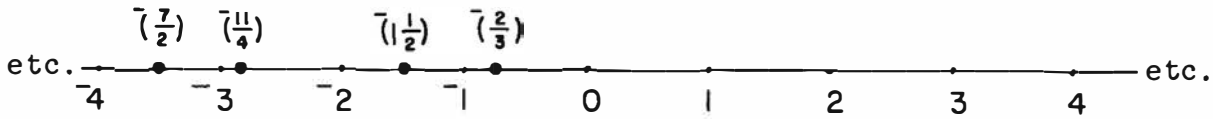
All of the rational numbers we have worked with so far have been associated with points to the right of 0 (and then, of course, 0 itself). But you were warned earlier that there are other rational numbers.

To begin with, after graphing the negative integers, it seems natural to put the label " $^{-}(\frac{1}{2})$ " with a point of the number line as shown below:



"$^{-}(\frac{1}{2})$" is read "NEGATIVE ONE-HALF" and $^{-}(\frac{1}{2})$ is a rational number.

Below you see some other rational numbers graphed on the number line:



Read the name of each one of these numbers.

The rational numbers to the right of 0 are called <u>positive rational numbers.</u> The rational numbers to the left of 0 are called <u>negative rational numbers.</u> Again, we shall see that these new numbers are very useful.

You may remember that we agreed in Chapter 1 that every whole number is also a rational number. For example, we said that 3 is not only a whole number but is also a rational number. (We can give it the name " $\frac{3}{1}$ ".) For much the same reason, we say that <u>every integer is also a rational number.</u> For example, $^{-}2$ is not only an integer it is also a rational number. Of course, it is <u>not</u> true that every rational number is an integer.

We can now say:

> The set of positive rational numbers,
> the set of negative rational numbers,
> and zero form the <u>set of all rational numbers.</u>

Check Your Reading

1 What point is associated with the number $^{-}(\frac{1}{2})$? How do we say the name of this number?

2. What sets form the set of all rational numbers? Give some examples from each set.

Oral Exercises 6-1b

If set R = [$^{-}5$, $^{-}4$, $^{-}(\frac{10}{2})$, $^{-}(\frac{3}{2})$, $^{-}1$, 0, $\frac{1}{4}$, 1, $\frac{3}{2}$, 2, $\frac{7}{3}$, 6],

1. List I, the set of integers in R.

Oral Exercises 6-1b
(continued)

2. List W, the set of whole numbers in R.
3. List A, the set of positive integers in R.
4. List P, the set of negative integers in R.
5. List N, the set of counting numbers in R.
6. List G, the set of positive rational numbers in R.
7. List L, the set of negative rational numbers in R.
8. List Y, the set of non-negative integers in R.

Problem Set 6-1b

1. Draw the graphs of the following sets:
   (a) $\{0,\ 2,\ {}^{-}3,\ {}^{-}(\frac{1}{2}),\ \frac{1}{2}\}$
   (b) $\{{}^{-}(\frac{2}{3}),\ \frac{2}{3},\ \frac{5}{2},\ {}^{-}(\frac{5}{2})\}$
   (c) $\{{}^{-}(\frac{3}{2}),\ 5,\ {}^{-}7,\ {}^{-}(\frac{11}{3})\}$
   (d) $\{{}^{-}1,\ {}^{-}(1 + \frac{1}{2}),\ (1 + \frac{1}{2})\}$

2. Of the two points whose coordinates are given, which is to the right of the other on the number line?
   (a) $5,\ 4$
   (b) $0,\ {}^{-}5$
   (c) ${}^{-}4,\ {}^{-}7$
   (d) ${}^{-}(\frac{1}{2}),\ 1$
   (e) ${}^{-}(\frac{5}{2}),\ {}^{-}(\frac{10}{4})$
   (f) ${}^{-}4,\ {}^{-}(\frac{15}{4})$
   (g) ${}^{-}(\frac{16}{3}),\ {}^{-}(\frac{21}{4})$

3. Translate this sentence and write it as an algebraic sentence. You need not find the answer.
   What is the length of a rectangle if the length is 4 times the width and the perimeter is 24 inches?
   Be sure you have selected and described a variable.

So far in this chapter we have looked at the set of rational numbers. Every rational number is <u>one</u> of the following: a positive number, a negative number, or zero. Some rational numbers are integers, and some are not. Give some examples of

rational numbers that are integers. Give some examples of rational numbers that are <u>not</u> integers.

You may have the feeling that every point on the number line can be associated with a rational number. Strange as it seems, this is not true. <u>There are points on the number line that cannot be associated with rational numbers.</u>

A number that is not rational but is associated with a point on the number line is called an <u>irrational number</u>. $\sqrt{2}$ is an example of an irrational number. $\sqrt{3}$ and π are also irrational numbers. There are many, many other numbers in the set of irrational numbers. Is $\sqrt{25}$ one of them?

We can now form a new set of numbers that will include the rational numbers and the irrational numbers. This set is called the

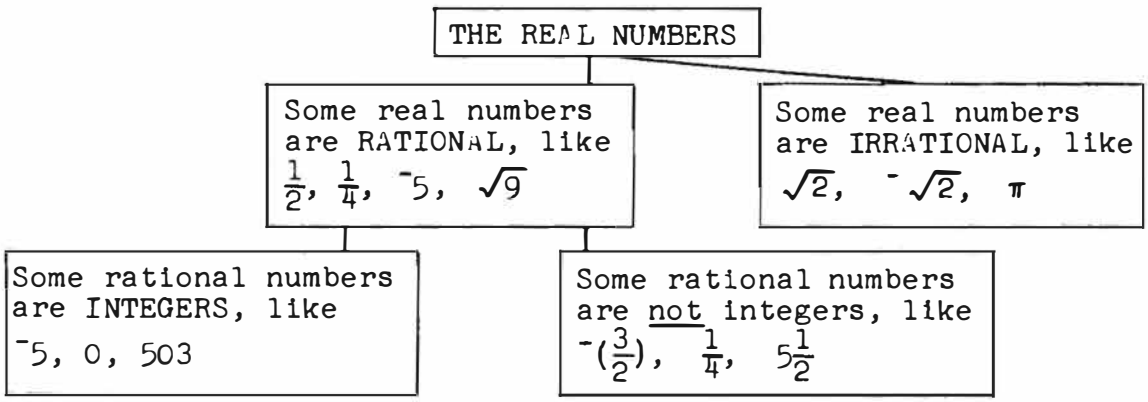
<u>Set of Real Numbers.</u>

All the rational numbers are in the set of real numbers. All the irrational numbers are in the set of real numbers. So we can say that the rational numbers form a subset of the real numbers, and the irrational numbers form another subset of the real numbers.

The points associated with the real numbers make up the whole number line, which is called the <u>real number line.</u>

We can think of the real number line as the graph of the set of real numbers.

The following diagram may help to review the kinds of numbers we have discussed:



Check Your Reading

1. What do we call the set of numbers that corresponds to the set of all points on the number line?
2. What are two principal subsets of the set of real numbers?
3. Is $^{-}(\frac{3}{2})$ a rational number? an integer? a real number?
4. Are all real numbers rational numbers? Are all rational numbers real numbers?

Problem Set 6-1c

1. (a) Is $^{-}2$ a whole number? an integer? a rational number? a real number?

   (b) Is $^{-}(\frac{10}{3})$ a whole number? an integer? a rational number? a real number?

   (c) Is $\sqrt{2}$ a whole number? an integer? a rational number? a real number?

   (d) Is 0 a whole number? an integer? a rational number? a real number? a positive number? a negative number?

2. Which of the following are true and which are false?
   (a) The set of real numbers is a subset of the set of integers.
   (b) The set of rational numbers is a subset of the set of real numbers.
   (c) The set of integers is a subset of the set of rational numbers.
   (d) The set of non-negative rational numbers is a subset of the set of counting numbers.
   (e) The set of whole numbers is a subset of the set of numbers of arithmetic.
   (f) The set of positive integers is a subset of the set of counting numbers.
   (g) The set of rational numbers is a subset of the set of real numbers.
   (h) The set of rational numbers is a subset of the set of positive real numbers.

Problem Set 6-1c
(continued)

3. The number π is the ratio of the circumference of a circle to its diameter. ("Ratio" means "the first number divided by the second number".) Thus, a circle whose diameter is of length 1, has a circumference of length π. The number π is an irrational number. Imagine such a circle resting on the number line at the point 0. If the circle is rolled on the line, without slipping, one complete revolution to the right, it will stop on a point. What is the coordinate of this point? If rolled to the left, it will stop on what point? Locate these points, approximately, on the real number line.

4. The first several digits of π are given by 3.1415926... Two rational numbers are given below which approximate π. Divide the numerator by the denominator and determine the number of digits of agreement.
   (a) $\frac{22}{7}$
   (b) $\frac{355}{113}$

5. In each of the following problems write an open sentence which represents the problem. You need not find the answer. Be sure to select a variable and describe what the variable represents.
   (a) Mary is twice as old as her brother, and her brother is twice as old as their baby sister; the sum of their ages is 15 years. How old is each?
   (b) Two boys riding bicycles started from the same point and rode in opposite directions for two hours. They were then 90 miles apart. If one boy traveled twice as fast as the other, what was the speed of each?
   (c) The sum of two consecutive even integers is 86. Find the integers.

6-2. Order on the Real Number Line.

$$6 > 5.$$

This true sentence reads "6 is greater than 5". It means "6 is to the right of 5" on the number line.

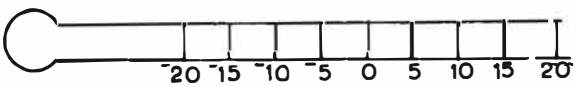
$$4 > {}^{-}8.$$

This true sentence reads "4 is greater than $^{-}8$". It means "4 is to the right of $^{-}8$" on the number line.

"Is to the right of" on the number line and "is greater than" describe the same order. What shall we mean by "is greater than" for any two real numbers, whether they are positive, negative, or 0?

Our answer is: "is to the right of" on the number line.

Here is a common example. Scales on thermometers use numbers above 0 and numbers below 0, as well as 0 itself. We know that the warmer the weather the higher up the scale we read the temperature. If we place the thermometer as is shown below, we see that it looks like a model of a part of the real number line.



When we say "is greater than" ("is a higher temperature than"), we mean "is to the right of" on the thermometer scale.

On this scale, which number is greater, $^{-}5$ or $^{-}10$? 5 or $^{-}10$? $^{-}15$ or 0?

For any two real numbers a and b,

a <u>is greater than</u> b

means the same as

a <u>is to the right of</u> b on the number line.

No matter which way we want to say it, we can write: $a > b$.

Oral Exercises 6-2a

1. Describe the meaning, on the number line, of "is less than" for real numbers, as we did above for ">".
2. What is the meaning, on the number line, of "$\geq$" for real numbers?
3. What is the meaning, on the number line, of "$\leq$" for real numbers?

Problem Set 6-2a

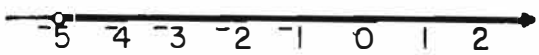
1. Determine which of the following sentences are true and which are false.

(a) $7 < 4$	(f) $3.5 < {}^{-}4$
(b) ${}^{-}(\frac{1}{2}) < 0$	(g) ${}^{-}4 \neq 3.5$
(c) ${}^{-}3 < 4$	(h) $3 \leq {}^{-}1$
(d) ${}^{-}6 > {}^{-}3$	(i) $2 \geq {}^{-}(\frac{7}{2})$
(e) ${}^{-}3 < {}^{-}2.8$	(j) ${}^{-}5 \geq {}^{-}(\frac{10}{2})$

2. Graph the truth set of each of the following open sentences:

   For example:

   $x > {}^{-}5$

   $x \leq {}^{-}2$

(a) $y > 2$	(d) $x \geq {}^{-}5$
(b) $y \geq {}^{-}2$	(e) $x = 3$ or $x < {}^{-}1$
(c) $y \neq 2$	(f) $c < 2$ and $c > {}^{-}2$

3. In the blanks below use "=", "<", or ">" to make a true sentence.

   (a) $\frac{3}{5}$ ____ $\frac{6}{10}$

   (b) $\frac{3}{5}$ ____ $\frac{3}{6}$

   (c) $\frac{9}{12}$ ____ $\frac{8}{12}$

Problem Set 6-2a
(continued)

(d) $\frac{9}{12}$ ___ $\frac{4}{6}$

(e) $^{-}(\frac{3}{5})$ ___ $\frac{3}{6}$

(f) $^{-}(\frac{3}{5})$ ___ $^{-}(\frac{4}{5})$

(g) $^{-}(\frac{3}{5})$ ___ $^{-}(\frac{3}{6})$

4. (a) During a cold day the temperature rises 10 degrees from $^{-}4$. What is the final temperature?
   (b) On another day the temperature rises 5 degrees from $^{-}10$. How high does it go?
   (c) During one day the temperature rises from $^{-}15$ to 35. How much does it rise?
5. Translate the following sentences into algebraic sentences.
   (a) A number is greater than or equal to $^{-}18$.
   (b) One number is greater than another.
   (c) One number is 5 greater than another.
   (d) One number less 7 is less than the same number increased by 4.
   (e) The sum of a number and 5 is less than the sum of 8 and the number.
   (f) 5 less than a number is four less than twice the number.
   (g) 3 times a number is 8, or 4 less than the same number is greater than 2.
   (h) A number is 6 less than $^{-}4$.

Suppose you try to guess the number of marbles in a bowl. Then the marbles are counted so that the number of marbles is known. It is easy to see that exactly <u>one</u> of the following three statements would be true:

(1) Your number <u>is greater than</u> the number of marbles.
(2) Your number <u>is less than</u> the number of marbles.
(3) Your number <u>is equal to</u> the number of marbles.

We could say that after the marbles were counted, your number was <u>compared</u> with the number of marbles.

If a is a real number and b is a real number then exactly one of the following is true:

$$a > b$$
$$a < b$$
$$a = b.$$

The way in which the real numbers are ordered on the number line makes it easy to see why exactly one of the statements above must be true. In fact, this is a property of order for the real numbers. It is sometimes called the

<u>comparison property.</u>

Check Your Reading

1. What is the comparison property of real numbers?
2. If two numbers are not the same, what else can be said about them?

Problem Set 6-2b

1. In the blanks below, use "=", "< "or "> "to make a true sentence in each case.

(a) 2 ____ $^-3$

(b) 2 ____ 1.6

(c) $\frac{3}{5}$ ____ $\frac{3}{6}$

(d) $\frac{3}{5}$ ____ $\frac{6}{10}$

(e) $^-(\frac{3}{5})$ ____ $^-(\frac{3}{6})$

(f) $\frac{4}{2}$ ____ $^-(\frac{5}{2})$

(g) $\frac{4}{5}$ ____ $\frac{11}{10}$

(h) $\frac{13}{15}$ ____ $\frac{2}{3}$

(i) 2 ____ 1.5

(j) $^-2$ ____ $^-1.5$

2. Write true sentences using "<" for the following pairs of real numbers.

(a) 6 and 5

(b) $^-3$ and 0

(c) $^-(\frac{2}{3})$ and $^-(\frac{5}{9})$

(d) 2 and $(1.4)^2$

(e) π and 3

3. Write true sentences using "> "for the following pairs of real numbers.

(a) $^-7$ and $^-5$

(b) $^-8$ and 0

(c) 8 and 0

(d) $\frac{1}{3}$ and .3

(e) 2 and $\frac{19}{8}$

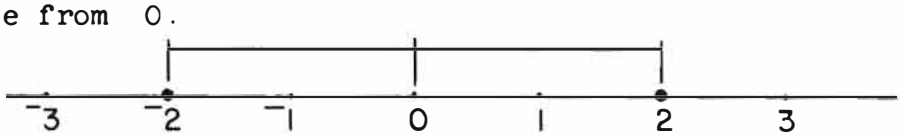
Problem Set 6-2b
(continued)

4. Write an algebraic sentence for each problem after choosing and describing a variable.
   (a) A coat was sold for \$33. This was at a discount of $\frac{1}{3}$ of the original price. What was the original price? (Hint: It wasn't \$44.)
   (b) John's age is 4 years less than twice his brother's age. If John is 12 years old, find his brother's age.
   (c) The bottom of a box is 8 inches by 12 inches. The volume of the box is 864 cubic inches. Find the height of the box.

6-3. Opposites.

We have now labeled points on the number line to the left of 0 as well as to the right of 0. This means that we can "pair off" points which are at the same distance from 0.

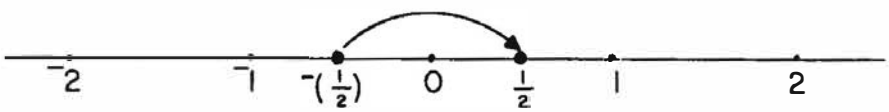
For example, think of the point associated with the number 2. It is easy to see that there is another point at the same distance from 0.



This other point lies on the opposite side of 0, and the number associated with it is ⁻2.

Since the two points we have just located lie on opposite sides of 0, and at the same distance from 0, it seems natural to say that the numbers 2 and ⁻2 are opposites. Each number is the opposite of the other. That is, ⁻2 is the opposite of 2 also, 2 is the opposite of ⁻2.

In the example above, we started with the number 2. We could just as easily start with any number and find its opposite. For instance, suppose we start with the number $^{-}(\frac{1}{2})$. Then find another point at the same distance from 0 but on the opposite side. This is also easy to do; and, as in the diagram below, you see that the coordinate of this point is $\frac{1}{2}$.



Therefore, $\frac{1}{2}$ is the opposite of $^{-}(\frac{1}{2})$. This means also that $^{-}(\frac{1}{2})$ is the opposite of $\frac{1}{2}$. Or we could just say that $\frac{1}{2}$ and $^{-}(\frac{1}{2})$ are opposites.

We agree that the opposite of zero is zero.

Here are some statements about pairs of numbers that are opposites:

(1) $^{-}7$ is the opposite of 7.

(2) $12\frac{1}{4}$ is the opposite of $^{-}12\frac{1}{4}$.

(3) 73.2 is the opposite of $^{-}73.2$.

(4) $^{-}1{,}000{,}002$ is the opposite of 1,000,002.

You can see that writing "the opposite of" every time means a lot of writing. So we will agree that when we want to write "opposite of 2" we can just write "-2". If we want to write "opposite of $^{-}3$", we can write "$-^{-}3$" or "$-(^{-}3)$".

Notice that the dash we use to mean "opposite of" is written lower than the dash we use in writing the name of a negative number. The dash we use to mean "opposite of" looks like the "minus" sign used in subtraction. But it is important to understand that we are not using it here to mean subtraction.

Now we can write the four statements above like this:

(1) $^{-}7 = -7$.

(2) $12\frac{1}{4} = -^{-}12\frac{1}{4}$. Sometimes we write this $12\frac{1}{4} = -(^{-}12\frac{1}{4})$.

(3) $73.2 = -(^{-}73.2)$.

(4) $^{-}1{,}000{,}002 = -1{,}000{,}002$.

Check Your Reading

1. What is the opposite of 2? of $^{-}2$?
2. What is the opposite of 0?
3. What is the symbol we use for "the opposite of"?
4. What does -2 mean? What does $-^{-}3$ mean?
5. Recall the difference between the meaning of the upper dash " $^{-}$ " and the centered dash " - ".

Problem Set 6-3a

1. Each of the numerals below names a number. For each one, give a common name for the number. For example, if you were given the numeral "$-(^{-}9)$", a common name would be "9". If you were given "-9" a common name would be "$^{-}9$" or "negative 9".

(a)	-55	(h)	$-(^{-}2\frac{1}{2})$
(b)	$-(^{-}55)$	(i)	$-2\frac{1}{2}$
(c)	-33.5	(j)	$-1{,}000{,}000{,}000$
(d)	-0	(k)	$-(^{-}1{,}000{,}000{,}000)$
(e)	$-(^{-}100)$	(l)	$-(3 + 5)$
(f)	$-\left(^{-}(\frac{3}{4})\right)$	(m)	$-(7 + 2)$
(g)	$-\frac{2}{3}$	(n)	$-(8.4 + 7.6)$

2. In looking over your answers from Problem 1, which of the following statements do you think is true?
   The opposite of a positive number is a positive number.
   The opposite of a positive number is a negative number.
   The opposite of a positive number is zero.
3. Which of these statements do you think is true?
   The opposite of a negative number is a positive number.
   The opposite of a negative number is a negative number.
   The opposite of a negative number is zero.
4. Is the opposite of zero a positive number, a negative number, or zero?
5. Are "-9" and "$^{-}9$" names for the same number? Write each in words.

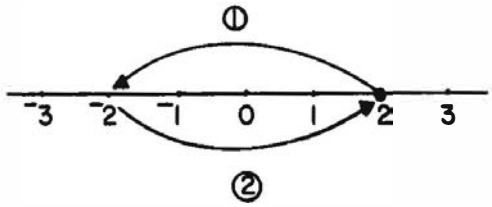
When you see the numeral "-(-2)", it may look confusing. In words, we could read it as "the opposite of the opposite of 2". Here is one way we might think about it:

$-(-2)$

start with 2

① find the opposite, getting $^-2$

② then find the opposite of $^-2$, getting 2



So we can say that "the opposite of the opposite of 2 is 2". Or, more briefly, we can write:

$$-(-2) = 2.$$

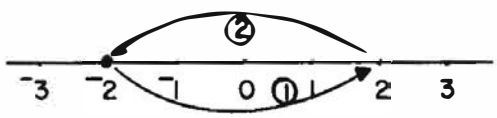
In the example above, we started with a positive number. We could just as well start with a negative number. For instance, how could we decide upon a common name for "$-(-^-2)$"?

$-(-^-2)$

start with $^-2$

① find the opposite, getting 2

② then find the opposite of 2, getting $^-2$



This shows that we can write:

$$-(-^-2) = {}^-2.$$

These two examples suggest that

$-(-y) = y$, for any real number y.

We have already seen that this statement is true when y represents 2 and when y represents $^-2$. Is it true if y represents 10? if y represents $^-10$? Is it true if y represents 0?

We cannot use every number, but with a little work on the number line, you should soon see that $-(-y) = y$ for any real number y.

Check Your Reading

1. State this sentence in words: $-(-y) = y$ for any real number y.
2. Describe how we would find the common name for $-(-^-2)$.

Oral Exercises 6-3b

1. Each of the numerals below is the name of a number. For each one, state a common name of the number. For example, if you were given the numeral "-(-3)", a common name would be "3".

(a) $-(-20)$ (d) $-(^{-}5)$

(b) $-(-^{-}20)$ (e) $-(-210)$

(c) $-\left(-^{-}(\frac{1}{2})\right)$ (f) $-(-^{-}37.5)$

Problem Set 6-3b

1. Each of the numerals below is the name of a number. For each one, write a common name of the number.

(a) $-(-40)$ (d) $-\left(-(5 + 3)\right)$

(b) $-\left(-^{-}(\frac{7}{2})\right)$ (e) $-(-0)$

(c) $-\left(-^{-}(5 + 3)\right)$ (f) $-\left(-(-9)\right)$

2. If y represents a positive number, does -y represent a positive number, a negative number, or zero?
3. If y represents a negative number, does -y represent a positive number, a negative number, or zero?
4. If y represents zero, does -y represent a positive number, a negative number, or zero?
5. If -y is positive, is y positive, negative, or zero?
6. If -y is negative, is y positive, negative, or zero?
7. If -y is zero, is y positive, negative, or zero?
8. Write an algebraic sentence whose truth set includes the answer to this problem. You need not find the answer.

   A rectangular lawn measures 20 feet by 30 feet. A walk of uniform width is put along both ends and one side. The perimeter of the entire area of lawn and walk is then 150 feet. How wide is the walk?

   Hint: Draw a diagram.

We know that the real numbers are ordered on the number line. That is, if we have any two different numbers, one will be less

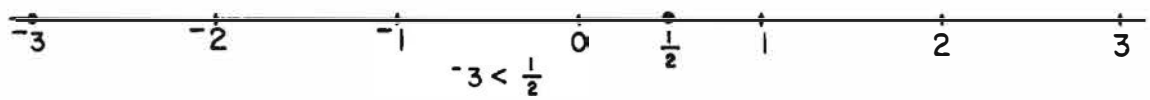
than the other. For example, using the pair of numbers $\frac{1}{2}$ and $^{-}3$, we can say:

$$^{-}3 < \frac{1}{2}.$$

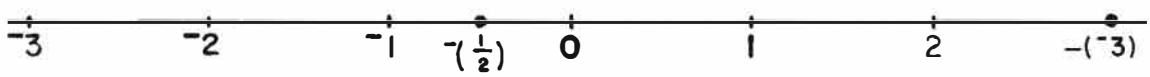
Let's take the opposite of each one of these two numbers. The opposite of $^{-}3$ is 3. The opposite of $\frac{1}{2}$ is $-(\frac{1}{2})$. Since $-(\frac{1}{2}) < 3$, we can say:

$$-(\tfrac{1}{2}) < -(^{-}3).$$

The number lines below may help in seeing what happened in this "experiment".



$^{-}3 < \frac{1}{2}$



$-(\frac{1}{2}) < -(^{-}3)$

Let's try the experiment again, this time using the numbers 2 and 100.

$$2 < 100,$$

but $\quad -100 < -2.$

Let's try one more pair of numbers, $^{-}8$ and $^{-}2$.

$$^{-}8 < {}^{-}2,$$

but $\quad 2 < 8.$

The examples above are specific illustrations of the following true statement:

<u>For any number a and for any number b,</u> if $a < b$

then $-b < -a$

If you feel that you are not sure of the meaning of this statement, try letting a and b be some numbers different from the ones we used above. Then see if the statement says what we want it to say.

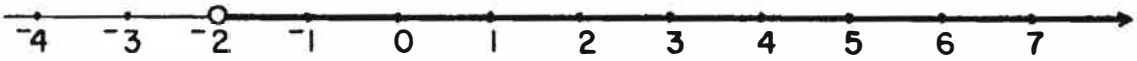
Sometimes the fact that if $a < b$, then $-b < -a$, is a big help in finding the truth sets of open sentences. Suppose you were trying to graph the truth set of:

$$-x < 2$$

We know that "$-x < 2$" and "$-2 < x$" mean the same. So, we can draw the graph of:

$$-2 < x$$

Remember, the sentence "$-2 < x$" can be written "$x > -2$" . Of course, the opposite of 2 is $^-2$, and it is easy to draw the graph of $^-2 < x$. The graph looks like this:



Could you show somebody why, for example, $^-3$ is <u>not</u> in the truth set?

Here is another example. Suppose you were trying to graph the truth set of:

$$-x > {}^-5$$

"$-x > {}^-5$" may also be written as "$^-5 < -x$." If $^-5 < -x$,
then $-(-x) < -({}^-5)$,
$x < 5$.

Actually, "$-x > {}^-5$" and "$x < 5$" have the same truth set. So for each sentence the graph of the truth set is as follows:

Here is a different problem. Below is a graph of a set of numbers.

Notice that this set of numbers does <u>not</u> include the number 3 or any number greater than 3. Also, the set does not include the number $^-3$ or any number less than $^-3$. So, if we let x represent a number in this set, x must be greater than $^-3$ and less than 3. We can say that this graph is the graph of the truth set of:

$$x > {}^-3 \quad \text{and} \quad x < 3.$$

Check Your Reading

1. The relationship between $^-3$ and $\frac{1}{2}$ is $^-3 < \frac{1}{2}$. What is the relationship between the opposites of $^-3$ and $\frac{1}{2}$?
2. The relationship between a and b is $a < b$. What is the relationship between $-a$ and $-b$?
3. How do we draw the graph of the truth set of $-x < 2$? What can we do to make the work easier? Is $^-3$ in the truth set?

Oral Exercises 6-3c

1. In each of the following pairs, decide which is the greater number; then take the opposites of the two numbers and again decide which is greater.

(a) 2.97, -2.97	(d) -1, 1	(g) 0, -0
(b) -12, 2	(e) -370, -121	(h) -.1, -.01
(c) -358, -762	(f) .12, .24	(i) .1, .01

2. What is the meaning of $x \neq 3$? Express this sentence using "<" and ">".

Problem Set 6-3c

1. Write two true sentences for the following **pairs of numbers and their opposites, using the relation " < ".**

   Example: 2, 7

   $2 < 7$ and $-7 < -2$.

   (a) 3, -1

   (b) $\frac{3}{4}$, $^{-}(\frac{1}{2})$

   (c) $\frac{2}{7}$, $^{-}(\frac{1}{6})$

   (d) $\sqrt{2}$, $^{-}\pi$

   (e) π, $\frac{22}{7}$ (π is approximately 3.1416)

   (f) $3(\frac{4}{3} + 2)$, $\frac{5}{4}(20 + 8)$

   (g) $2(8 + 5)$, $^{-}(5 + 4)$

   (h) $-\frac{(8 + 6)}{7}$, $^{-}2$

2. Choose the greater of each of the following numbers and its opposite.

   (a) $^{-}7.2$

   (b) 3

   (c) $-(^{-}5)$

   (d) $^{-}\sqrt{2}$

   (e) 17

   (f) -.01

   (g) -(-2)

   (h) $(1 - \frac{1}{4})^2$

   (i) $1 - (\frac{1}{4})^2$

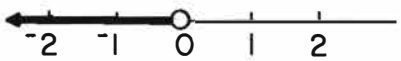
   (j) $-(\frac{1}{2} - \frac{1}{3})$

3. Write two open sentences for each of the following graphs, one involving x, and the other involving -x.

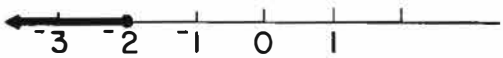
   (a) 

   (b) 

   (c) 

   (d) 

   (e) 

   (f) 

   (g) 

4. Graph the truth sets of the following open sentences:

   (a) $x > 3$

   (b) $x > {}^{-}3$

   (c) $-x > 3$ (Hint: first express the relation involving the opposites of these numbers.)

   (d) $-x > {}^{-}3$

Problem Set 6-3c
(continued)

5. Describe the truth set of each open sentence.

   (a) $-x \neq 3$ (Remember "$\neq$" means "$<$" or "$>$".)
   (b) $-x \neq {}^{-}3$
   (c) $x < 0$
   (d) $-x < 0$
   *(e) $-x \geq 0$
   *(f) $-x \leq 0$

6. Write open sentences that would help solve the following problems. Be sure to tell what the variable represents.

   (a) John's score is greater than negative 100. What is his score?
   (b) He doesn't have any money, but he is no more than $200 in debt. How much money does he have?
   (c) Paul has paid $10 on his bill, but still owes more than $25. What was the original amount of the bill?

What is the "opposite of 5"? By this time, this is an easy question. You know that the opposite of 5 is ${}^{-}5$.

Since we have agreed to use "-5" to mean "opposite of 5", we can say:

$$-5 = {}^{-}5.$$

In other words, -5 and ${}^{-}5$ are names for the same number. Maybe you have noticed this already. Of course, it is also true that -2 and ${}^{-}2$ are equal, -7 and ${}^{-}7$ are equal, $-\frac{1}{2}$ and ${}^{-}(\frac{1}{2})$ are equal, and so on.

This means that we don't really have to use two different dashes anymore. From now on, for example, we can use "-5" to mean either "opposite of 5" or "negative 5".

We must be careful with the expression "-x". If x is a negative number, then its opposite, $(-x)$, is a positive number. If x is a positive number, its opposite, $(-x)$, is a negative number. If x is zero, its opposite, $(-x)$, is zero.

6-4. Absolute Value.

Think of the number 5. Its opposite is -5.
This gives us the pair of numbers 5 and -5.
Which number of the pair is greater?

Think of the number -8. Its opposite is 8.
This gives us the pair of numbers -8 and 8.
Which number of the pair is greater?

Think of the number $-\frac{1}{2}$. Its opposite is $\frac{1}{2}$.
This gives us the pair of numbers $-\frac{1}{2}$ and $\frac{1}{2}$.
Which number of the pair is greater?

When you answered the questions above, you were working with a new and useful operation in mathematics. It is called "taking the absolute value" of a number.

The absolute value of any non-zero real number is the greater of that number and its opposite. The absolute value of zero is zero.

Thus, we could reword the questions above and ask:

What is the absolute value of 5?
What is the absolute value of -8?
What is the absolute value of $-\frac{1}{2}$?

Instead of writing "absolute value" each time, we use a new symbol. For example:

"$|5| = 5$" means "absolute value of 5 is 5".
"$|-8| = 8$" means "absolute value of -8 is 8".
"$|0| = 0$" means "absolute value of 0 is 0".
"$|n|$" means "absolute value of the number n".

Check Your Reading

1. What is the absolute value of a non-zero number?
2. What is the absolute value of zero?
3. What does the symbol $|n|$ mean?

Oral Exercises 6-4a

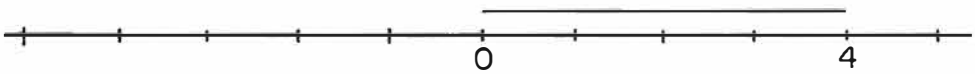
1. What is the absolute value of each of the following numbers?

(a) -7	(c) $(6 - 4)$	(e) $-(14 + 0)$
(b) $-(-3)$	(d) 14×0	(f) $-(-(-3))$

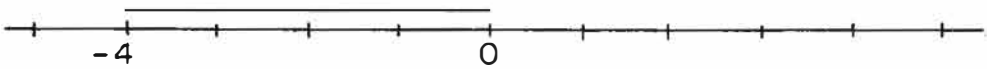
2. What is the absolute value of x if x is 3? If x is -2?
3. If x is a non-negative real number, what kind of number is $|x|$?
4. If x is a negative real number, what kind of number is $|x|$?
5. Is $|x|$ a non-negative number for every x?
6. For a negative number x, which is greater, x or $|x|$?
7. Is the absolute value of a number ever a negative number?
8. When is the absolute value of a number not a positive number?

Using the number line sometimes helps in working with absolute values. Look at the following examples.

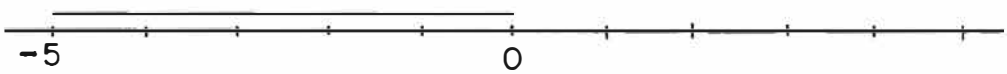
(1) $|4| = ?$ What is the distance between 0 and 4? How do the answers to these questions compare?



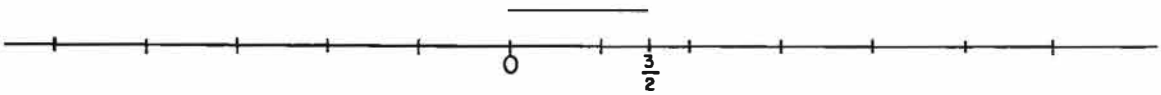
(2) $|-4| = ?$ What is the distance between 0 and -4? How do the answers to these questions compare?



(3) $|-5| = ?$ What is the distance between 0 and -5? How do the answers to these questions compare?



(4) $|\frac{3}{2}| = ?$ What is the distance between 0 and $\frac{3}{2}$? How do the answers to these questions compare?



Do you see that the absolute value of a number is the distance between the number and 0 on the number line? By "distance", we mean just the "number of units". Here, the word "distance" has nothing to do with direction.

If x is 7, what is $|x|$?

If x is 12, what is $|x|$?

If x is 8,750, what is $|x|$?

If x is 0, what is $|x|$?

In these four examples, x has been either a positive number or zero. A shorter way of saying this is to say that x has been a <u>non-negative</u> number. Note that a non-negative number is a number that is not negative, that is, a number that is either positive or zero. In each of the examples above, it turned out to be true that $|x| = x$. After thinking about other non-negative numbers, you should see that we can say:

$$|x| = x, \text{ if } x \geq 0.$$

Is it always true that $|x| = x$? Let's try some negative numbers.

If x is -5, what is $|x|$? $|x| = 5$. Notice that $5 = -(-5)$.

If x is -3, what is $|x|$? $|x| = 3$. Notice that $3 = -(-3)$.

If x is -45, what is $|x|$? $|x| = 45$. Notice that $45 = -(-45)$.

In these three examples, x has been a negative number. Each time, $|x|$ has been, <u>not</u> x, but $-x$. We express this by saying:

$$|x| = -x, \text{ if } x < 0.$$

You may have noticed that an absolute value is always a non-negative number. So it may seem strange ever to say "$|x| = -x$". But remember, if x is a negative number, $-x$ is a positive number. Therefore,

$$|x| = x, \text{ if } x \geq 0$$

and $$|x| = -x, \text{ if } x < 0$$

is just another way of saying that $|x|$ is always a non-negative number.

As one more example, let's look at $|-20|$:

$$|-20| = 20.$$

This agrees with what was said above, since $-20 < 0$, and $|-20| = -(-20)$.

Check Your Reading

1. Which of the following describes all non-negative numbers?

   $x \leq 0$ $\quad$ $x \geq 0$ $\quad$ $x < 0$ $\quad$ $x > 0$

2. Which of the following open sentences are true for all real numbers x?

   $|x| \geq 0$ $\quad$ $-x \leq |x|$

   $x \leq |x|$ $\quad$ $-|x| \leq x$

3. State in words those sentences in Question 2 which you decided are true.

Oral Exercises 6-4b

1. For a negative number x, which is greater, x or -x?

2. Which of the following sentences are true?

   (a) $|-7| < 3$ $\quad$ (e) $|-5| \neq |2|$

   (b) $|-2| \leq |-3|$ $\quad$ (f) $-3 < 17$

   (c) $|4| < |1|$ $\quad$ (g) $-2 < |-3|$

   (d) $2 \leq |-3|$ $\quad$ (h) $|-2|^2 = 4$

3. State each as a simple numeral.

   (a) $|2| + |3|$ $\quad$ (f) $7 - |-3|$

   (b) $|-2| + |3|$ $\quad$ (g) $|-5| \times 2$

   (c) $-(|2| + |3|)$ $\quad$ (h) $-(|-5| - |-2|)$

   (d) $-(|-2| + |3|)$ $\quad$ (i) $|-3| - |2|$

   (e) $|-7| - (7 - 5)$ $\quad$ (j) $|-2| + |-3|$

Oral Exercises 6-4b
(continued)

(k) $-(|-3| - 2)$ (n) $-(|-7| - 6)$

(l) $-(|-2| + |-3|)$ (o) $|-5| \times |-2|$

(m) $3 - |3 - 2|$ (p) $-(|-2| \times 5)$

(q) $-(|-5| \times |-2|)$

Problem Set 6-4

1. What is the truth set of each open sentence?
   (a) $|x| = 1$ (c) $|x| + 1 = 4$
   (b) $|x| = 3$ (d) $5 - |x| = 2$
2. Graph the truth sets of the following sentences.
   (a) $|x| < 2$ (c) $|x| > 2$
   (b) $x > -2$ and $x < 2$ (d) $x < -2$ or $x > 2$
3. Graph the integers less than 5 whose absolute values are greater than 2. Is -5 an element of this set? Is 0 an element of this set? Is -10 an element of this set? Is 4 an element of this set?
4. If R is the set of all real numbers, P the set of all positive real numbers, and I the set of all integers, write three numbers which are
   (a) in P but not in I,
   (b) in R but not in P,
   (c) in R but not in P or in I,
   (d) in P but not in R.
5. Compare the truth sets of the two sentences.

$$|x| = 0, \qquad |x| = -1.$$

*6. Three boys, Sam, Bob, and Pete, were talking. Sam said, "Bob is older than I am." Pete said, "Bob is twice as old as I am and Sam is 3 years older than I am." Bob said, "My father is more than twice as old as all of our ages put together, and he is 45." How old was each boy? Write the sentence whose truth set will lead to the answer to this problem. It is not necessary to find the answer.

Summary

(1) Points to the left of 0 on the number line are associated with negative numbers.

(2) The real numbers are those numbers that can be associated with points of the real number line. They include rational numbers and irrational numbers.

(3) The integers form a special subset of the rational numbers.

(4) Positive integers are integers associated with points to the right of zero.

(5) Negative integers are integers associated with points to the left of zero.

(6) The rational numbers to the right of zero are called positive rational numbers.

(7) The rational numbers to the left of zero are called negative rational numbers.

(8) "a is greater than b" and "a is to the right of b on the number line" have the same meaning for any two real numbers a and b.

(9) For any two real numbers a and b, exactly one of the following is true: $a > b$, $a < b$, $a = b$.

(10) The opposite of 0 is 0. The opposite of any other real number is the number which is at an equal distance from 0 on the number line and on the opposite side of 0.

(11) The opposite of the opposite of a number is just the number itself. That is, $-(-x) = x$.

(12) The absolute value of 0 is 0. The absolute value of any other real number n is the greater of n and -n. "Absolute value of n" is written "$|n|$".

(13) $|n|$ is the distance between 0 and n on the real number line.

(14) $|n| = n$, if $n \geq 0$.
$|n| = -n$, if $n < 0$.

Review Problem Set

1. Consider the following sets.

   R*: the set of all real numbers
   P: the set of positive real numbers
   Q: the set of negative real numbers
   R: the set of all rational numbers
   N: the set of counting numbers
   W: the set of whole numbers
   I: the set of integers
   J: the set of irrational numbers

   In each of the following pairs of sets tell which set is a subset of the other. For some pairs neither set may be a subset of the other.

   (a) I, W
   (b) N, W
   (c) N, R
   (d) R*, R
   (e) R, I
   (f) P, Q
   (g) R*, J
   (h) I, J
   (i) R*, W
   (j) P, W

2. Give two meanings of the symbol ">", one related to numbers and the other related to the position of points on the number line.

3. Place the correct symbol (>, < , or =) between these numbers so that a true sentence results.

   (a) 3 ____ 5
   (b) -3 ____ 5
   (c) -3 ____ -5
   (d) 3 ____ -5
   (e) $\frac{3}{4}$ ____ $\frac{26}{36}$
   (f) $-(\frac{3}{4})$ ____ $\frac{26}{36}$
   (g) $-(\frac{3}{4})$ ____ $-(\frac{26}{36})$
   (h) $\frac{3}{4}$ ____ $\frac{27}{37}$
   (i) $-(\frac{3}{4})$ ____ $-(\frac{45}{56})$
   (j) $\frac{3}{4}$ ____ $\frac{96}{128}$

4. Name three numbers that are not rational numbers. Is $\sqrt{16}$ irrational? $\sqrt{49}$? $\sqrt{5}$?

Review Problem Set
(continued)

5. Graph the truth sets of these sentences.

(a) $x < 5$	(d) $\lvert x \rvert > 0$
(b) $-x < 5$	(e) $\lvert x \rvert < 0$
(c) $\lvert x \rvert = 4$	(f) $\lvert x \rvert = 0$

6. Tell what you know about the order of any two real numbers a and b. What property is involved?
7. If $x < 3$, what can we state about the order of $-x$ and -3?
8. Graph these sets.

   (a) $\{-(\frac{4}{3}),\ 0,\ -(-\frac{4}{3}),\ 2,\ -2\}$

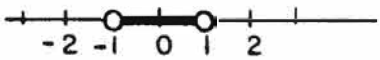
   (b) The set of all positive integers less than 2

   (c) The set of all counting numbers less than $^-2$

   (d) The set of all integers between $^-4$ and 2

   (e) The set of all numbers between $^-4$ and 2
9. Write an open sentence for each graph.

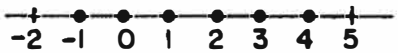
(a) 

(c) 

(b) 

(d) 

10. Describe in words the sets for which the following are the graphs.

(a) 

(b) 

11. What number, added to the same number increased by $4\frac{1}{2}$, will result in a sum of 7.3? Write the sentence whose truth set includes the answer to this problem. You need not find the answer. Tell what the variable represents.
12. Do as in Problem 11 for this problem.

    If a number is increased by 7.6 times the number, and the resulting sum is 58, what is the number?
13. Do as in Problem 11.

    The product of a number and the number increased by $3\frac{1}{2}$ is 84. What is the number?

Review Problem Set
(continued)

14. Do as in Problem 11.

    One book has 310 pages more than another. The number of pages in the combined volumes is more than 1000 pages. How many pages are in each volume?

15. An airplane flies due east at an average speed of 200 miles per hour. Another plane leaves from the same starting point one hour later. It flies in the same direction and overtakes the first 800 miles away. What was the average speed of the second plane?

Chapter 7

ADDITION OF REAL NUMBERS

7-1 Using the Real Numbers in Addition.

Ever since the first grade, you have been adding numbers--the numbers of arithmetic. Now we are ready to work with a larger set of numbers--the real numbers. Your work in adding the numbers of arithmetic should give a clue as to how we add any two real numbers.

To begin with, think of an ice cream salesman in business for twelve days. On some days, he makes money; then we say that he shows a profit. On other days, he loses money: then we say that he shows a loss. On still other days, he may show neither a profit nor a loss.

At the bottom of the page are two columns. The one on the left gives, in words, the profit or loss for each one of the twelve days. The column on the right shows the arithmetic used in figuring the profit or loss for two-day periods.

Notice that to find the profit or loss for a two-day period, we "put together", or add, the profit or loss for one day and the profit or loss for the other day. The numbers that we add are positive in some cases, negative in others, and zero in still others.

Monday:	Profit of \$7	$7 + 5 = 12$	12 is the number showing his net income for 2 days.
Tuesday:	Profit of \$5		
Wednesday:	Profit of \$6	$6 + (-4) = 2$	
Thursday:	Loss of \$4		
Friday:	Loss of \$7	$(-7) + 4 = -3$	
Saturday:	Profit of \$4		
Sunday:	Day of rest	$0 + (-3) = -3$	
Monday:	Loss of \$3		
Tuesday:	Loss of \$4	$(-4) + (-6) = -10$	
Wednesday:	Loss of \$6		
Thursday:	Profit of \$5	$5 + (-5) = 0$	
Friday:	Loss of \$5		

In these examples, we have found the sum of two positive numbers, the sum of two negative numbers, and the sum of a positive number and a negative number. We have also found a sum involving zero.

How would you complete each of the following statements?

$\frac{1}{2} + 2 =$ $\qquad$ $(-4) + 2 =$

$0 + \frac{4}{3} =$ $\qquad$ $(-\frac{7}{5}) + 0 =$

$3 + (-3) =$ $\qquad$ $(-4) + (-\frac{1}{2}) =$

$3 + (-1) =$

Maybe you thought of the numbers above as representing profits and losses, as in the example about the ice cream salesman. You may want to keep thinking about positive and negative numbers in this way for a while. However, you will probably find, as you study this chapter, that you will be able to add real numbers without thinking about profits and losses at all.

Check Your Reading

1. When you added two negative numbers, was the sum a positive or negative number?
2. When you added two positive numbers, was the sum a positive or a negative number?
3. When you added a positive number and a negative number, how did you decide whether the sum was a negative number, a positive number, or zero?
4. When you add zero to a real number, how do you decide whether the answer is a negative or a positive number?

Oral Exercises 7-1

1. Think of gains as positive numbers and losses as negative numbers to answer the following questions.

   (a) Harry earned \$5 yesterday and spent \$3 today. What is his present financial situation?

Oral Exercises 7-1
(continued)

(b) Bill had 50¢ when he went to school today. He spent 40¢ for his lunch and he was charged 25¢ for supplies. What is his financial situation?

(c) A certain stock market price gained two points one day then lost 5 points the next day. What was the net change?

(d) Miss Jones lost 6 pounds during the first week of her dieting, lost 3 pounds the second week, gained 4 pounds the third week, and gained 5 pounds the last week. What was her net gain or loss?

(e) A football team lost 6 yards on the first play and gained 8 yards on the second play. What was the net yardage on the two plays?

2. Find the following sums by thinking of the positive numbers as profits and the negative numbers as losses.

(a) $7 + 2$	(g) $(-5) + (-8)$
(b) $(-4) + (-3)$	(h) $8 + 0$
(c) $5 + (-8)$	(i) $(-6) + 0$
(d) $8 + (-5)$	(j) $(-2) + ((-3) + 6)$
(e) $(-5) + 8$	(k) $(7 + (-4)) + (-2)$
(f) $(-8) + 5$	(l) $(-2) + ((-4) + 5)$

Problem Set 7-1

1. Write the common name for each of the following sums. Think of the positive numbers as profits and the negative numbers as losses.

(a) $4 + 7$	(e) $7 + (-2)$
(b) $(-4) + 8$	(f) $(-7) + 2$
(c) $(-2) + (-10)$	(g) $2 + (-7)$
(d) $(-2) + 7$	(h) $(-2) + (-7)$

Problem Set 7-1
(continued)

(i) $7 + 0$

(j) $4\frac{1}{2} + (-2\frac{1}{2})$

(k) $(-6.3) + (-4.5)$

(l) $(-5) + 0$

(m) $((-4) + 0) + 3$

(n) $(8 + (-3)) + (-4)$

(o) $((-3) + (-4)) + 5$

(p) $((-2) + 4) + (-1)$

(q) $8 + ((-3) + (-4))$

(r) $(-3) + ((-4) + 5)$

(s) $(-2) + (4 + (-1))$

2. In each of the following what "profit" or "loss" will make the open sentence true?

(a) $a + 2 = 5$

(b) $a + (-2) = 5$

(c) $a + 2 = -5$

(d) $b + (-2) = -5$

(e) $(-4) + c = 2$

(f) $(-3) + m = (-6)$

(g) $4 + n = 4$

(h) $(-4) + n = (-4)$

(i) $m + (-1) = (-2)$

(j) $c + (-3) = 3$

(k) $b + 4 = (-7)$

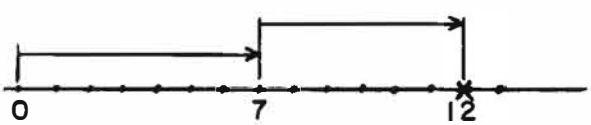
(l) $(-2\frac{1}{2}) + a = (-5)$

(m) $b + (1.5) = 1.1$

7-2. Addition and the Number Line.

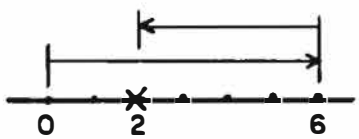
In earlier chapters, we used the number line to show some facts about numbers. Now let's use the number line to show the arithmetic of the ice cream salesman's record.

Mon. and Tues.



$7 + 5 = 12$

Wed. and Thurs.



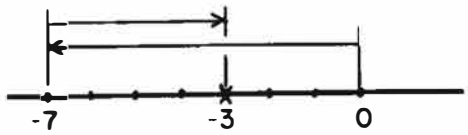
$6 + (-4) = 2$

Notice that in the first example (Mon. and Tues.) we start at zero. We first move 7 units to the right. Then we move 5 more units to the right. The final position on the number line is shown by a "x". This position shows us that the result of

adding 5 to 7 is 12.

In the second example (Wed. and Thurs.) we again start at zero. First we move 6 units to the right. Then we move 4 units to the left. (Why do we move 4 units to the left instead of 4 units to the right?) What is the result? Would we get the same result if we moved $|-4|$ units to the left? Remember that $|-4| = 4$. Do you see, then, that to add (-4) we move $|-4|$ units to the left? To add 6 we move $|6|$ units to the right.

Fri. and Sat.

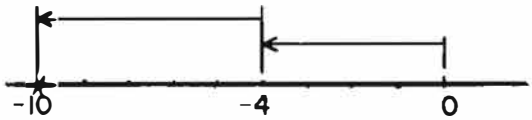


$(-7)+4=-3$

Sun. and Mon.

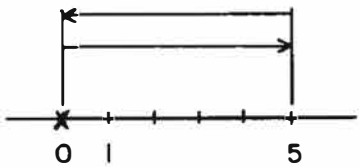
$0+(-3)=-3$

Tues. and Wed.



$(-4)+(-6)=-10$

Thurs. and Fri.



$5+(-5)=0$

In working with the number line, remember that we show addition of a positive number by moving to the right. From the above examples, you can see that addition of a negative number is shown by moving to the left. In which direction do we move to show addition of zero? In each case the number of units moved is the absolute value of the number.

Below is a summary of things to remember when showing addition of two real numbers a and b on the number line.

(1) Start at zero.

(2) From zero, move $|a|$ units to the right if a is positive, $|a|$ units to the left if a is negative, no units if a is zero.

(3) From this point, move $|b|$ units to the right if b is positive, $|b|$ units to the left if b is negative, no units if b is zero. This determines another point, or the "final position", on the number line.

(4) The coordinate of the "final position" point is the sum of the two numbers.

In the third example (Fri. and Sat.), the sum of (-7) and 4 is obtained by moving $|-7|$ units to the <u>left</u> of zero and then moving $|4|$ units to the <u>right</u>. The final position is at (-3).

Check Your Reading

1. What is the starting point when we add on the number line?
2. In which direction do we move on the number line to indicate the addition of a positive number?
3. In which direction do we move on the number line to indicate the addition of a negative number?
4. What is the number of units moved when we add a number x ?
5. What does the final position on the number line indicate?

Oral Exercises 7-2a

1. If a thermometer registered -15° F and the temperature rises 10 degrees, what does the thermometer then register?
2. A thermometer registered 10° F at noon and dropped 6 degrees in three hours. What was the temperature at 3 p.m.?
3. How would you represent a drop of 3° by a number?
4. If a thermometer registers -5° C and then rises 6°, what is the new temperature?
5. If a thermometer registers 3° C and then drops 4 degrees, what is the new temperature?
6. Describe how you would find these sums on the number line.

(a) $(-5) + 2$	(f) $(-11) + 15$
(b) $(-5) + (-2)$	(g) $4 + 12$
(c) $5 + 2$	(h) $6 + (-7)$
(d) $5 + (-2)$	(i) $6 + (-6)$
(e) $(-6) + (-7)$	(j) $(-7) + 0$

Oral Exercises 7-2a
(continued)

(k) $(4 + (-6)) + (-8)$

(l) $((-5) + (-2)) + (-7)$

(m) $((-4) + 8) + (-4)$

(n) $(0 + (-2)) + 2$

(o) $(7 + (-2)) + 3$

Problem Set 7-2a

1. Find the following sums. Use the number line to aid you if necessary.

   (a) $5 + 2$

   (b) $5 + (-2)$

   (c) $(-2) + 5$

   (d) $(-2) + (-5)$

   (e) $(-5) + 2$

   (f) $(-7) + (-3)$

   (g) $(-7) + 3$

   (h) $6 + 0$

   (i) $8 + (-10)$

2. Think of the following numbers as "gains" and "losses" and then find the sums.

   (a) $(-5) + 0$

   (b) $(-6) + 9$

   (c) $6 + (-12)$

   (d) $(-6) + 12$

   (e) $0 + 0$

   (f) $((-7) + 4) + 1$

   (g) $(8 + (-2)) + (-3)$

   (h) $((-5) + 9) + (-4)$

3. Which of the following sentences are true? Use "gains" or "losses", or the number line, to help you decide.

   (a) $(-3) + 4 = 7$

   (b) $(-3) + 4 > 7$

   (c) $(-3) + 4 < 7$

   (d) $(-2) + 5 \neq 6 + (-4)$

   (e) $(-1) + 6 < 1 + (-6)$

   (f) $(-1) + 6 > 1 + (-6)$

   (g) $(-1) + 6 = 1 + (-6)$

   (h) $6 + (-7) = 7 + (-8)$

   (i) $3(2) + (-4) \neq (-5) + 6$

   (j) $3(2) + (-4) > (-5) + 6$

   (k) $3(2) + (-4) < (-5) + 6$

   (l) $3(2) + (-4) = (-5) + 6$

Problem Set 7-2a
(continued)

4. Perform the following additions of real numbers. (use the number line to help you if you need to.)

(a) $(-5) + 2 + 7$

(b) $(-5) + (-2) + 6$

(c) $6 + (-2) + 4$

(d) $(-7) + (4 + (-3))$

(e) $((-7) + 3) + (-4)$

(f) $7 + ((-4) + 3)$

(g) $(7 + (-3)) + 4$

(h) $((-7) + (-3)) + 4$

(i) $(-7) + ((-4) + 3)$

(j) $((-7) + (-4)) + (-3)$

5. Which of the following sentences are true? Which are false?

(a) $(-5) + (-7) = -(|-5| + |-7|)$

(b) $(-\frac{1}{2}) + (-\frac{3}{2}) = (|-\frac{1}{2}| + |-\frac{3}{2}|)$

(c) $(-6) + 2 = -(|-6| - |2|)$

(d) $(-4) + 7 = (|7| - |-4|)$

(e) $2 + (-8) = (|-8| - |2|)$

(f) $(-5) + 5 = (|-5| - |5|)$

(g) $(-7) + 0 = (|-7| + 0)$

(h) $0 + (-\frac{3}{2}) = -(0 + |-\frac{3}{2}|)$

6. If the domain of the variable is the set of real numbers, find the truth sets of the following open sentences.

(a) $m + 5 = 1$

(b) $a + (-2) = 4$

(c) $(-3) + a = -5$

(d) $4 + b = 2$

(e) $(-2) + m = 8$

(f) $n + (-1) = 7$

(g) $b + 4 = -3$

(h) $b + (-4\frac{1}{2}) = 1\frac{1}{2}$

(i) $a + 2.5 = (-2.5)$

(j) $a + (-3) > 1$

(k) $m + (-2) > (-2)$

(l) $(-3) + m < (-5)$

(m) $(-5) + m = (-3) + m + (-2)$

(n) $(-4) + b \neq (-5) + b + 1$

Let us go back to the fifth example in the sales of ice cream (Tues. and Wed.) in which we add

$$(-4) + (-6) = -10.$$

By this time you certainly know <u>how</u> to find the sum, which is -10.

In this as in the other examples you have been able to find the sum either by thinking in terms of profit and loss or by using points on the number line.

We would now like to see how addition of real numbers can be described, or defined, without reference to a number line or some other special device such as profit and loss. Can we do this using only addition and subtraction of <u>numbers of arithmetic</u> and the taking of opposites? Can we, in other words, define addition of real numbers in terms of operations with which you are already familiar?

In forming our definition we must keep two things in mind. First, we must include all possible situations. Second, our definition must not change or contradict any ideas which we have already developed about the addition of numbers of arithmetic.

With this last notion in mind, we should begin as follows:

The sum of any two non-negative real numbers is equal to the sum as defined for numbers of arithmetic. This means, of course, that additions such as $7 + 5 = 12$ and $3 + 0 = 3$, etc., may be treated as before.

We must then consider all cases involving one or more negative numbers. Let's first look at the example $(-4) + (-6)$. Working with the number line we saw that

$$(-4) + (-6) = (-10).$$

If we consider absolute values, we would get

$$|-4| + |-6| = 10.$$

The resulting sum is the <u>opposite</u> of the number line answer. This suggests a way of defining the sum of two negative numbers. We could say that the sum of two negative real numbers is equal to the <u>opposite</u> of the sum of absolute values. That is,

$$(-4) + (-6) = -(|-4| + |-6|).$$

Now let's look at another type. On the number line we found that

$$6 + (-4) = 2.$$

Here the sum was obtained by moving 4 units to the <u>left</u> of 6. Do you see that we would also get a result of 2 if we <u>subtracted</u> the number 4 from 6? Once again we can make use of absolute values. Since $|-4| = 4$ and $|6| = 6$, we could say that

$$6 + (-4) = |6| - |-4|.$$

A similar example, which we also worked on the number line gave us

$$(-7) + (4) = (-3).$$

However, in this case if we subtract absolute values we obtain

$$|-7| - |4| = 3.$$

Our answer is the <u>opposite</u> of the number line result. What, then, is the basic difference between this and the previous example? In both instances we were finding the sum of a positive and a negative number. Where is the difficulty? Look at the examples! In the first case the positive number 6 has the larger absolute value. In the second case the negative number (-7) has the larger absolute value. That is,

$$|6| > |-4| \qquad \text{but} \qquad |-7| > |4|.$$

We can put the two ideas together and come up with the following: To add two real numbers one of which is positive and the other negative we subtract the smaller absolute value from the larger. However, if the negative number has the larger absolute value, we must take the <u>opposite</u> of the result. For the two cases we have

$$6 + (-4) = |6| - |-4|$$

$$(-7) + 4 = -(|-7| - |4|).$$

It also follows that

$$(-4) + 6 = |6| - |-4|$$

and

$$(4) + (-7) = -(|-7| - |4|).$$

The two remaining types to consider are

$$5 + (-5) \quad \text{and} \quad 0 + (-3).$$

From the profit and loss example we recall that

$$5 + (-5) = 0.$$

We also note that $|5| = |-5|$. This suggests that the sum of a positive and a negative number with equal absolute values is equal to zero. It can also be shown that $(-5) + 5 = 0$.

For the last type, the profit and loss example gave us

$$0 + (-3) = -3.$$

On the other hand, the sum of absolute values gives us

$$|0| + |-3| = 3.$$

Once again we take the opposite. That is,

$$0 + (-3) = -(|0| + |-3|).$$

Here we see that the sum becomes the original negative number itself. That is, $0 + (-3) = -3$. Similarly $(-3) + 0 = -3$.

We have accomplished two important steps in the study of real numbers. We have learned <u>how</u> to add them. The examples have suggested to us a way in which addition may be defined in terms of the familiar operations on numbers of arithmetic.

This definition will be long, since it is necessary to include all of the different cases. We shall probably not use it very often in finding actual sums. Your own ideas on how to do this as suggested by the number line will often be more efficient.

The definition, however, is important. It must be used in proving certain properties of real numbers, and will help us develop a greater understanding of these numbers. We shall therefore state the definition in the form of a summary of the ideas we have obtained above. As you read the following, try to associate each part with one of the given examples.

ADDITION OF REAL NUMBERS

(a) The sum of two non-negative real numbers is equal to the sum as defined for numbers of arithmetic.

(b) The sum of two negative real numbers is equal to the <u>opposite</u> of the sum of the absolute values. That is,

if a and b are negative, then

$$a + b = -(|a| + |b|).$$

(c) The sum of two real numbers, one of which is positive and the other negative is equal to

(1) The difference of absolute values if the positive number has the larger absolute value.

(2) The <u>opposite</u> of the difference of absolute values if the positive number has the smaller absolute value.

(3) Zero, if the absolute values are equal.

That is, for any positive number a and negative number b

if $|a| > |b|$, then $a + b = |a| - |b|$

if $|a| < |b|$, then $a + b = -(|b| - |a|)$

if $|a| = |b|$, then $a + b = 0$

For any negative number a and positive number b

if $|a| > |b|$, then $a + b = -(|a| - |b|)$

if $|a| < |b|$, then $a + b = |b| - |a|$

if $|a| = |b|$, then $a + b = 0$

(d) The sum of two real numbers, one of which is negative and the other zero is equal to the opposite of the sum of absolute values. That is, for any negative number a

$$a + 0 = -(|a| + |0|)$$

and

$$0 + a = -(|0| + |a|).$$

(In both cases the result is equal to the original negative number.)

All cases have been covered. We can now state two significant results.

(1) It is possible to add any two real numbers.

(2) The resulting sum is always a real number.

Check Your Reading

1. How is the sum of two non-negative real numbers defined?
2. How is the sum of two negative numbers defined?
3. What determines whether the sum of a positive and negative number is positive or negative?
4. What operations and numbers are used to define the addition of real numbers?

Problem Set 7-2b

Use the definition of addition to add the following as in the example.

Example:
$$\begin{aligned}(-5) + 3 &= -(|-5| - |3|)\\ &= -(5 - 3)\\ &= -2\end{aligned}$$

1. $(7) + (-3)$
2. $(-7) + (-3)$
3. $(-7) + 0$
4. $3 + (-7)$
5. $7 + (-7)$
6. $(-3) + 7$
7. $(-3) + (-7)$
8. $0 + (-7)$
9. $(-3) + 3$

Use the definition of addition to decide which of the following sentences are true and which are false.

10. $(-8) + 6 = -(|-8| - |6|)$
11. $6 + (-9) = (|-9| - |6|)$
12. $(-2) + (-3) = -(|-2| + |-3|)$
13. $(-10) + 0 = -(|-10| + 0)$
14. $13 + (-13) = (|13| + |-13|)$
15. $(-\frac{2}{3}) + \frac{1}{3} = -(|-\frac{2}{3}| - |\frac{1}{3}|)$
16. $\frac{3}{5} + (-\frac{1}{5}) = -(|\frac{3}{5}| - |-\frac{1}{5}|)$
17. $(-6) + 6 = 0$
18. $0 + (-3) = |-3| - |0|$
19. $(-5) + (-4) = (|-5| + |-4|)$

7-3. Addition Property of Zero; Addition Property of Opposites.

In part (d) of the definition, the statement in parentheses tells us that the sum of any negative real number and zero is equal to the negative number we started with. The same thing is true for all numbers of arithmetic. Combining these ideas

we can state the following property

For any real number a, $a + 0 = a$.

A convenient name for this property is The Addition Property of Zero.

The definition of addition also tells us that the sum of a positive and negative number is zero if the absolute values are equal. We already know that every real number and its opposite have equal absolute values and that of these two numbers one is positive and the other negative. Thus another property follows.

For any real number a, $a + (-a) = 0$.

This property is often called The Addition Property of Opposites.

Check Your Reading

1. What is the sum of any real number and zero?
2. What is the sum of any real number and its opposite?
3. State the addition property of zero for real numbers.
4. State the addition property of opposites for real numbers.

Oral Exercises 7-3

Which of the following are true sentences? Which are false?

1. $7 + (-7) = 0$
2. $8 + 8 = 0$
3. $(-4\frac{2}{3}) + 4\frac{2}{3} = 0$
4. $(-9) + 0 = -9$
5. $(-7) + 0 = 7$
6. $(8) + (-8) = 0$
7. $(-9) + \big(-(-9)\big) = 0$
8. $(-15) + (-15) = 0$
9. $\big(-(-7)\big) + 0 = 7$
10. $7(0) = 7$
11. $\big(-(-15)\big) + 15 = 0$
12. $7(0) + (-7) = 0$
13. $7\big(7 + (-7)\big) = 0$
14. $49 + (-49) = 0$

Problem Set 7-3

For what value (or values) of the variable is each of the following open sentences true?

1. $(-14) + t = 0$
2. $8 + x = 8$

Problem Set 7-3
(continued)

3. $r + (-4\frac{2}{3}) = (-4\frac{2}{3})$
4. $s + 0 = (-8)$
5. $9 + \ell = 0$
6. $(-45) + 0 = c$
7. $a + 0 = 0$
8. $a + (-3) > 0$
9. $3 + b < 0$
10. $a + (-3) > 3$
11. $2 + b < (-2)$

7-4. Properties of Addition.

Our definition of addition was an outgrowth of the results which were obtained on the number line and the profit-loss example. At the same time we wanted the definition to be such that the properties which applied to numbers of arithmetic would also hold for all real numbers including the negatives. We have just seen that the addition property of zero does hold. We have also come up with a new property, the addition property of opposites.

It now remains to determine whether or not the commutative and associative properties are also retained. Let's first look at the former. We already know that for any two numbers of arithmetic the sum

$$a + b = b + a.$$

Is this true for all real numbers? Since parts of our definition of addition involve subtraction, we might not be sure. Suppose we examine one case. Let a be positive and b be negative and assume that $|a| > |b|$. By definition we know that

$$a + b = |a| - |b|.$$

What about $b + a$? The definition tells us to take the difference of absolute values. The difference of two numbers of arithmetic is obtained by subtracting the smaller number from the larger. Thus we see that

$$b + a = |a| - |b| \quad \text{also.}$$

If we considered all the other cases which involve negative numbers, we could show in a similar way that the commutative

property of addition holds for each of these. Hence we can say that

For any real numbers, a and b, $a + b = b + a$.

Through the use of the definition and arguments like those above, we can also show that addition of real numbers is associative. This means that

For any real numbers, a, b, and c
$$(a + b) + c = a + (b + c).$$

We shall not attempt to demonstrate this property for any of the cases involving negative numbers. However, the following example should help bring out the point.

If $a = -8$, $b = 6$, and $c = -7$, $(a + b) + c$ is $(-2) + (-7)$. This is equal to -9. But $a + (b + c)$ is $(-8) + (-1)$. This is also equal to -9.

Oral Exercises 7-4

1. Add a negative number to a positive number; now add the same positive number to the negative number. Did you get the same sum?
2. What property of addition was shown in Question 1?
3. (a) Add (-3) and (-4); then add 5 to the result.
   (b) Add (-4) and 5; then add that number to (-3).
   (c) Did you get the same sum in (a) and (b)?
   (d) What property of addition is illustrated here?
4. State the commutative property of addition for real numbers.
5. State the associative property of addition for real numbers.
6. Tell what property or properties of addition are illustrated by each of the following:
   (a) $(-5) + (-7) = (-7) + (-5)$
   (b) $\big(4 + (-3)\big) + 2 = 4 + \big((-3) + 2\big)$
   (c) $(-3) + \big((-2) + (-5)\big) = \big((-3) + (-2)\big) + (-5)$
   (d) $\big((-4) + (-5)\big) + 2 = 2 + \big((-4) + (-5)\big)$

Oral Exercises 7-4
(continued)

(e) $(-4) + 1 + (-7) = (-4) + (-7) + 1$

(f) $(m + n) + q = m + (n + q)$

(g) $a + (b + c) = (b + c) + a$

(h) $(x + y) + z = (y + z) + x$

(i) $x + (a + b) = (a + x) + b$

(j) $(a + b) + (x + y) = (a + y) + (b + x)$

Problem Set 7-4

1. Find each of the following sums. Watch for easy groupings of numbers that make use of the properties of addition.

   (a) $(-3) + 7 + 5 + 3 + (-5)$

   (b) $14 + 6 + (-7) + 4 + 3$

   (c) $5 + (-8) + 6 + (-3) + 2$

   (d) $(-9) + 5 + 6 + (-3)$

   (e) $11 + (-17) + 9 + (-3) + 4$

   (f) $c + 2 + (-c) + 5$

   (g) $r + 4 + (-r) + (-4)$

   (h) $r + 6 + (-r) + (-3)$

2. Which of the following sentences are true? Which are false?

   (a) $(-5) + (-2) = (-2) + (-5)$

   (b) $(7 + (-2)) + (-6) = 7 + ((-2) + (-6))$

   (c) $(-6) + 3 = 3 + (-6)$

   (d) $(-8) + 2 = (-2) + 8$

   (e) $(-8) + (6 + (-4)) = 8 + 6 + (-4)$

   (f) $(-9) + ((-3) + 6) = ((-9) + (-3)) + (-6)$

   (g) $(7 + 5) + 4 = 4 + (7 + 5)$

   (h) $(14 + 5) + (-2) = 14 + (2 + (-5))$

Problem Set 7-4
(continued)

3. In each of the following find the real number (or numbers) which makes the open sentence true.

(a) $(-6) + (-3) = (-3) + r$

(b) $(-5) + y = 7 + (-5)$

(c) $15 + \big((-9) + 2\big) = (15 + x) + 2$

(d) $\big((-2) + 5\big) + c = (-2) + \big(5 + (-3)\big)$

(e) $14\frac{1}{3} + x = 2\frac{1}{2} + 14\frac{1}{3}$

(f) $(14 + 8) + (-4) = 14 + \big((-4) + b\big)$

(g) $a + (-16) = (-12) + 10 + (-4)$

(h) $\big(5 + (-5)\big) + m = 5 + \big((-5) + m\big)$

(i) $\big(3.2 + (-n)\big) + 2.8 = 3.2 + \big(2.8 + (-n)\big)$

(j) $|-5| + a = |-4| + |-5|$

(k) $b + (-9) = |-6| + |-3|$

7-5. <u>Addition Property of Equality.</u>

"3 + 4" is the name of a number. If 8 is added to the number, we get a number which can be named "(3 + 4) + 8".

"9 - 2" is the name of a number. If 8 is added to the number, we get a number which can be named "(9 - 2) + 8".

Did you notice something strange about the pair of statements above? They were really saying the same thing. In both cases, we had the number 7, and we added 8. Of course, the result in both cases was 15. It is true that we used different names for the number 7 and the number 15. However, we were talking about the same numbers.

We can write:

$$(3 + 4) = (9 - 2)$$

so $$(3 + 4) + 8 = (9 - 2) + 8.$$

This is an example of a property of equality.

Here are some other examples.

Example 1.

$$(-7) + 2 = -5$$
$$\big((-7) + 2\big) + (-2) = (-5) + (-2)$$

Example 2.

$$(-3) + (-2) = (-8) + 3$$
$$\big((-3) + (-2)\big) + 2 = \big((-8) + 3\big) + 2$$

Sometimes people talk about "adding the same number to both sides". In the two examples above, we could say we "added -2 to both sides" in the first one, and "added 2 to both sides" in the second one.

"Sides" is a kind of slang word in mathematics. In the sentence "(-7) + 2 = -5", we might speak of "(-7) + 2" as being on the left side of the verb "=", and "-5" as being on the right side of the verb "=".

Here is another example:

$$(3 \times 2) + (-4) = 2.$$

This true sentence says that we have two different names for the same number. If 4 is added to the number, the sentence then reads:

$$\big((3 \times 2) + (-4)\big) + 4 = 2 + 4.$$

This is also a true sentence. You can see that the "left side" and the "right side" of the sentence both show that 4 has been added to the number we started with.

All of these examples show a property of equality called the

<u>addition property of equality,</u>

which can be written:

<u>For any real numbers a, b, and c,</u> if $a = b$, then $a + c = b + c$.

Of course, it could also be stated:

<u>For any real numbers a, b, and c,</u> if $a = b$, then $c + a = c + b$.

Oral Exercises 7-5

1. What happens when we add the same number to each side of the true sentence "$9 + (-4) = 3 + 2$"?

2. What happens when we add a positive number to the right side only of the sentence in question 1?

3. What happens when we add a negative number to the left side only of the sentence in question 1?

4. Which of the following is a statement of the addition property of equality?

   (a) For all real numbers a, b, and c,
   if $a = b$,
   then $c + a = c + b$.

   (b) For all real numbers a, b, and c,
   if $a = b$,
   then $(a + b) + c = a + (b + c)$.

   (c) For all real numbers a, b, and c,
   if $a = b$,
   then $a + c = b + c$.

5. For what values of the variables are the following sentences true?

   (a) $3 + x = x + 3$

   (b) $(-3) + 4 + (-5) + x = (-4) + x$

   (c) $x + (-8) + (-7) + (-6) = (-14) + (-7) + x$

6. In each of the following what real number must be added to the given number so that the result is x?

(a) $x + 5$	(f) $(-7) + x$
(b) $x + (-6)$	(g) $(-3) + 4 + x$
(c) $x + (-11)$	(h) $x + 3 + (-8)$
(d) $x + 8$	(i) $(-5) + x + (-7)$
(e) $9 + x$	(j) $12 + x + (-24)$

Oral Exercises 7-5
(continued)

7. In each of the following sentences assume that both "sides" are names for the same number. Now tell what number could be added to both sides so that the new sentence will have the variable alone on one side.

Example: $x + 4 = 5 + (-2)$

We would say "Add (-4) to both sides", because this would give us

$$x + 4 + (-4) = 5 + (-2) + (-4)$$
$$x + 0 = 5 + (-2) + (-4)$$
$$\text{or} \quad x = 5 + (-2) + (-4).$$

Here x is alone on the left side.

(a) $x + 3 = 10$	(g) $16 = 16 + a$
(b) $(-4) + x = 11$	(h) $28 + a + (-12) = -7$
(c) $x + (-16) = 16$	(i) $(-6) + (-5) = x + 5$
(d) $x + (-8) = -8$	(j) $(-14) + x = 14 + (-6)$
(e) $(-5) + m + (-4) = 12$	(k) $(-4) + y + (-3) = -15$
(f) $-18 = b + (-3)$	(l) $y + (-5) + 5 = -8$

Problem Set 7-5

1. Which of these sentences are true? Which are false?

(a) $\big((-7) + 2\big) + (-2) = (-5) + (-2)$

(b) $\big((-14) + (-3)\big) + 3 = \big((-20) + 3\big) + 3$

(c) $\big((-6) + 9\big) + (-9) = (-6) + (-9)$

(d) $\big(2 + (-10)\big) + 6 = (-8) + 6$

(e) $\big((-8) + 5\big) + \big(5 + (-10)\big) = (-3) + (-5)$

(f) $\big((-15) + 23\big) + (-23) = \big(8 + (-16)\big) + (-23)$

(g) $\big((-2\frac{1}{3}) + 7\frac{2}{3})\big) + 2\frac{1}{3} = 5\frac{1}{3} + 2\frac{1}{3}$

(h) $(3.6 + 4.2) + (-3.6) = 7.8 + (-3.6)$

Problem Set 7-5
(continued)

2. In each of the following sentences assume that both sides are names for the same number. Determine what number we could add to both sides so that when the sums are simplified one side consists of only the variable.

   (a) $x + 4 = 10$

   (b) $(-6) + x + 4 = 12$

   (c) $5 + y = 12 + (-2)$

   (d) $4 + 8 + (-7) = a + (-2)$

   (e) $a + (-2) + (-7) = (-8) + (-4)$

   (f) $6 + b = (-11) + (-5)$

   (g) $4 + x + (-6) = (-8) + 8$

   (h) $(-30) + (-10) + y = (-40)$

   (i) $-(3(4 + 2)) + m = 0$

   (j) $\frac{3}{4} + (-8) = y$

3. What real number will make each open sentence true?

   (a) $((-5) + (-2)) + 4 = (-7) + x$

   (b) $(14 + (-3)) + 3 = y + 3$

   (c) $((-7) + 4) + (-4) = m + (-4)$

   (d) $(4 + (-11)) + 11 = n + 11$

   (e) $(3\frac{3}{4} + 2\frac{1}{4}) + (-2\frac{1}{4}) = a + 2\frac{1}{4}$

   (f) $(1.5 + (-.5)) + 1 = b + 1$

   (g) $(m + (-8)) + 8 = 3 + 8$

   (h) $7.2 + (-5.2) + (-5.2) = m + 5.2$

   (i) $(-5) + y + 5 = 9$

   (j) $7 + (-14) + x = (-4) + 12$

7-6. Truth Sets of Open Sentences.

Earlier we worked with open sentences and found truth numbers. In nearly all cases we found these numbers by "guessing" certain values and then testing to see if these were correct.

You may have wondered how you could find the truth set of an open sentence without guessing. A method of doing this would be quite helpful in the case of more complicated sentences for which the guessing process might be very difficult. Such a method will also have other advantages, as we shall soon see.

Suppose, for example, you were asked to find the truth set for the sentence

$$x + \frac{3}{5} = -2$$

From now on, unless otherwise specified, we will assume in solving sentences that the domain of the variable is the set of all real numbers for which the sentence has meaning. It may be that you can guess a number which will make this sentence true. But this may not be easy. What if there is no truth number? What if the sentence has more than one truth number?

We can use some of the properties we have studied to take care of all of these questions. We shall also obtain a useful method for finding truth sets in general.

Let's begin by supposing that there <u>is</u> a truth number x which makes the sentence true. Then for this number x we would know that the left side "$x + \frac{3}{5}$" and the right side "-2" are equal. This allows us to use the addition property of equality and <u>add</u> a real number to both sides. In other words, if the sentence

$$x + \frac{3}{5} = -2$$

is true for a certain number x, then

$$x + \frac{3}{5} + (-\frac{3}{5}) = -2 + (-\frac{3}{5})$$

is also true for the same x. The number we chose to add to both sides was $(-\frac{3}{5})$, which is the <u>opposite</u> of $\frac{3}{5}$. Do you see the reason for this choice?

Because of the addition property of opposites and the addition property of zero our sentence may now be written

$$x + 0 = -2 + (-\tfrac{3}{5}).$$

Adding we get

$$x = -2 + (-\tfrac{3}{5})$$

and

$$x = -\tfrac{13}{5}$$

where the last form was obtained by adding the real numbers on the right. From this we see that if a certain number x is a truth number for the original sentence, then it is also a truth number for the sentence

$$x = -\tfrac{13}{5}$$

This tells us that we don't have to guess. If the original sentence has a truth number, then this truth number must be $-\frac{13}{5}$. We still have to find out whether or not $-\frac{13}{5}$ <u>is</u> a truth number. We can do this by substituting $(-\frac{13}{5})$ in the original sentence. When we do this,

$$x + \tfrac{3}{5} = -2 \qquad \text{becomes} \qquad (-\tfrac{13}{5}) + \tfrac{3}{5} = -2.$$

Adding real numbers on the left we get

$$-(\tfrac{10}{5}) = -2,$$

which is a true sentence. Thus we know that $-\frac{13}{5}$ is a truth number. From what has gone before we also know that it is the only possible truth number. This means that we have found the <u>complete</u> truth set. It is

$$\{-\tfrac{13}{5}\}$$

As a second example, find the truth set of

$$x + 4 = -2.$$

Though it is fairly easy to guess a truth number for this one, we shall use the addition property once more to illustrate the method. You can then see whether or not your guess was correct. See if you can understand all of the following steps!

Assume there is a value of x which makes this sentence true. Then for this particular x the sentences

$$x + 4 = -2$$

$$x + 4 + (-4) = -2 + (-4) \qquad \text{(opposites again)}$$

$$x + 0 = -6$$

and

$$x = -6$$

are all true. Thus -6 is the only possible truth number, if there is one. Again we substitue, and

$$x + 4 = -2 \qquad \text{becomes} \qquad -6 + 4 = -2$$

which is a true sentence. Do you see, then, that $\{-6\}$ is the <u>complete</u> truth set? Is this the number you guessed to begin with?

<u>Example 3</u>. Find the truth set of $4 + (-2) = x + (-5)$.

If

$$4 + (-2) = x + (-5)$$

is true for some x,

then

$$\big(4 + (-2)\big) + 5 = \big(x + (-5)\big) + 5$$

is true for the same x.

$$\big(4 + (-2)\big) + 5 = x + \big((-5) + 5\big)$$

$$\big(4 + (-2)\big) + 5 = x + 0$$

$$\big(4 + (-2)\big) + 5 = x$$

$$(2) + 5 = x$$

Then $7 = x$ is true for the same x.

Is 7 a truth number of "$4 + (-2) = x + (-5)$"? If x is 7, the right side of the sentence is the numeral "$7 + (-5)$", whose common name is "2".

The left side of the sentence is the numeral "$4 + (-2)$", whose common name is "2".

Therefore, the complete truth set of "$4 + (-2) = x + (-5)$" is

$$\{7\}.$$

In the above example the original sentence

$$4 + (-2) = x + (-5)$$

could have been written as

$$2 = x + (-5)$$

to begin with. If more than one number appears on any one side of the sentence, such numbers may be combined before the addition property is used. This may simplify the steps involved.

Check Your Reading

1. What property guarantees that if "$x + \frac{3}{5} = -2$" is true for a certain number, then "$x + \frac{3}{5} + (-\frac{3}{5}) = (-2) + (-\frac{3}{5})$" is true for the same number?
2. Why was $(-\frac{3}{5})$ chosen to add to both sides of the sentence "$x + \frac{3}{5} = -2$"?
3. How can we be sure than $-\frac{13}{5}$ is a truth number for "$x + \frac{3}{5} = -2$"?

Oral Exercises 7-6

What number should be added to both sides of the sentences below in order to obtain a simpler sentence?

1. $x + \frac{2}{3} = 3$
2. $m + \frac{1}{6} = \frac{2}{3}$
3. $2y + (-9) = (-5)$
4. $(-3) = y + 3$
5. $3z + 6 = 9$
6. $2x + (-1.5) = .5$
7. $y + \big((-3) + (-2)\big) = 8$
8. $4m + 3 = 15$
9. $17z + (-2) = 32$
10. $35 = 11k + 2$

Problem Set 7-6

1. Find the truth number of each of the following open sentences. If you cannot guess the truth number, use the addition property of opposites to write a simpler sentence.

   (a) $x + 5 = -3$

   (b) $n + 3 = 5$

Problem Set 7-6
(continued)

(c) $4 + t = -7$

(d) $(-5) + (-7) = m + 3$

(e) $y + 8 = 5 + 3$

(f) $n + (-\frac{1}{12}) = 1 + \frac{1}{4}$

(g) $(.32) + (-.77) = 1.05 + w$

(h) $(-5) + 3x + (-8) = 15 + (-20) + (1)$

2. Find the truth number of each of the following open sentences.

(a) $(-5) + a = (-5) + 4$

(b) $a + (-3) = 6 + ((-2) + (-1))$

(c) $b + (-6) = ((-3) + 2) + (-6)$

(d) $5 + (-4) = m + (-4)$

(e) $(-7) + (-5) = ((-4) + (-3)) + n$

(f) $5\frac{1}{2} + (-1\frac{1}{2}) = 5\frac{1}{2} + b$

(g) $(-2.6) + c = (-1.6) + (-1.0) + 3.1$

(h) $3.5 + c = 5.0 + (-1.5) + c$

3. Find the truth number for each of the following open sentences.

(a) $2x + (-5) = -3$

(b) $11 + 3y = 26$

(c) $13m + 5 = 31$

(d) $z + \frac{1}{5} = \frac{6}{5}$

(e) $5x + (-11) = 14$

(f) $24 = 3 + 7k$

(g) $2z + 3 = 8$

(h) $x + (-\frac{1}{3}) = \frac{1}{3}$

7-7. Additive Inverse.

Can you complete each of the following so as to have a true sentence?

$$4 + (-4) =$$

$$(-8) + 8 =$$

$$x + (-x) =$$

Can you answer the following questions?

What number added to -7 gives a sum of zero?

What number makes the open sentence "$x + 9 = 0$" true?

These questions were probably not very hard. We have already learned that the sum of a number and its opposite is zero.

In mathematics, there is another important name we can start using when talking about statements like "$4 + (-4) = 0$".

-4 is called an additive inverse of 4, since $4 + (-4) = 0$.
4 is called an additive inverse of -4, since $(-4) + 4 = 0$.
-7 is called an additive inverse of 7, since $7 + (-7) = 0$.
7 is an additive inverse of -7. Why?
Give an additive inverse of 25.

If we have two real numbers x and y whose sum is zero, like this:

$$x + y = 0$$

then y is the additive inverse of x
and x is the additive inverse of y.

Oral Exercises 7-7a

1. What number added to (-5) gives the sum zero?
2. What number added to 6 gives the sum zero?
3. What is the additive inverse of (-7)?
4. What is the additive inverse of 10?
5. In the sentence $x + 5 = 0$
   (a) Choose an additive inverse that will help you find the truth set.

Oral Exercises 7-7a
(continued)

(b) Employ the addition property of equality and the additive inverse to find the truth set of this sentence.

(c) What is the truth set of the sentence?

6. If the sum of two numbers is zero, what can we say about the numbers?

7. Give the additive inverse of each of the following.

(a) 4	(f) $-x$	(k) $(-6) + 3m$
(b) -9	(g) $3m$	(l) $(-4) + 2y + 5$
(c) 25	(h) $-(5k)$	(m) $a + b$
(d) -12	(i) $3 + 5$	(n) $3m + n$
(e) x	(j) $x + 5$	(o) $4y - x + 2$

Problem Set 7-7a

1. Find the truth set of each of the following open sentences by addition of an additive inverse.

(a) $x + (-6) = (-8)$

(b) $(-14) = x + 3$

(c) $(-4) + x = (-9) + (-2)$

(d) $7 + (-3) = 11 + x$

(e) $(-7) + (-5) + x = (-6) + 4 + (-6)$

(f) $7 + x + (-5) = (-4) + 1 + (-9)$

(g) $9 + 2 + (-4) = (-7) + (-3) + x$

2. Find the truth set of each of the following open sentences.

(a) $3m + 5 = 14$

(b) $(-4) + 4x + (-2) = 10$

(c) $(-8) + 2 = (-7) + 3k + (-8)$

(d) $(12) + (3) = 3m + (-15)$

(e) $(-5) + 4x = 7$

Problem Set 7-7a
(continued)

(f) $6 + 3y + (-7) = 2$

(g) $9 + (-7) = 7m + (-12)$

3. Find the truth set of each of the following open sentences.

(a) $4k + 3 = 11$

(b) $6 + 7m = 41$

(c) $2z + (-11) = -10$

(d) $\frac{19}{2} = 5n + (-\frac{1}{2})$

(e) $\frac{1}{2}x + (-3) = -2$

(f) $26 = 8y + 2$

(g) $13z + (-2) = 11$

(h) $x + (-6) = -12$

4. Find the truth set of each of the following open sentences.

(a) $3k + 6 = 6$

(b) $2m + (-4) = 0$

(c) $8 + 5k = 10 + (-2)$

(d) $7x + (-3) = -2$

(e) $-4 = 4z + (-5)$

(f) $10 + (-8) = 3 + x$

(g) $3x + (-1) = 24 + (-4)$

(h) $13y + 6 = 19$

5. Translate these sentences into algebra.

(a) A number added to its additive inverse has the sum zero.

(b) The sum of a number and the additive inverse of 5 is 15.

(c) Three times a number, increased by the additive inverse of $\frac{1}{2}$ is the additive inverse of the number.

Problem Set 7-7a
(continued)

(d) The sum of a number and negative 3 multiplied by the sum of the same number and 3 is equal to 9 less than the square of the number.

(e) Five less than a certain number is the same as the sum of the number and the additive inverse of 5.

(f) Is the sentence in (a) true for every number?
Is the sentence in (e) true for every number?

Write an open sentence for each of the following problems.

6. John had four times as many pennies as nickels and twice as many dimes as nickels. Then when his uncle gave him 7 cents he had a total of 94 cents. How many of each coin did he finally have?

7. The largest angle of a triangle is 20° more than twice the smallest. The third angle is 70°. The sum of the angles of a triangle is 180°. How large is each angle?

8. A rectangle is 6 times as long as it is wide. Its perimeter is 112 inches. What are its dimensions?

When you were answering questions about additive inverses, you might possibly have wondered if a number could have more than one additive inverse. For example, it is easy to see that -4 is an additive inverse of 4. But suppose somebody claimed there was <u>another</u> number, different from -4, that is also an additive inverse of 4.

Could you <u>prove</u> to this person that he is wrong? Could you prove to him that -4 is the <u>only</u> additive inverse of 4?

You could, of course, ask this other person to name this <u>other</u> additive inverse of 4. However, you don't really need to. We can let "z" be that "other additive inverse of 4" that he claims to have.

<u>If</u> it is true that z is an additive inverse of 4, then

$4 + z = 0$. This sentence must be true if z is an additive inverse of 4.

$(-4) + (4 + z) = (-4) + 0$	Here we are using the Addition Property of Equality. If you are wondering why we added -4 to each side, see if you can discover why from the next step.
$((-4) + 4) + z = (-4) + 0$	Here we have used the Associative Property of Addition.
$0 + z = (-4) + 0$	Here we have used the Addition Property of Opposites. Do you see now why we added -4 back in the second step? It was the only way way we could get that "0" on the left side.
$z = -4$	Addition Property of Zero.

You may have that "So what?" feeling; but look at the last line, "$z = -4$". It shows that the "other" additive inverse of 4 turned out to be not a different one at all. It is once again -4. So, no matter what anybody <u>claims,</u> we have proved that 4 has one and <u>only one</u> additive inverse, namely, -4. We know, in other words, that $\{-4\}$ is the complete truth set of the sentence $4 + z = 0$.

Could we go through the same kind of argument to show that 5 has only one additive inverse, that 20 has only one additive inverse, and so on? We could, but we don't have to do it for one number at a time. Instead, we can talk about <u>any</u> real number x.

We know that if x is <u>any</u> real number, then $-x$ is an additive inverse of x, because

$$x + (-x) = 0.$$

If there is some other additive inverse of x, we can call it "z" and write:

$$x + z = 0.$$
$$(-x) + (x + z) = (-x) + 0$$
$$((-x) + x) + z = (-x) + 0$$
$$0 + z = (-x) + 0$$
$$z = -x$$

We see that the "other" additive inverse is still the same $-x$.

If you are puzzled about some of the steps above, compare this with the proof that 4 has only one additive inverse. The

reasons are written out there, and the reasons here are the same.

In the final line, you see

$$z = -x.$$

This shows that if z is an additive inverse of x, then z is the same as, or is equal to, $-x$. In other words, there is no <u>other</u> additive inverse of x, except $-x$.

We can state it this way:

> Any real number x has one and <u>only</u> one additive inverse. This additive inverse is $-x$.

We have proved this statement to be true. We proved it to be true by using properties that are true for all real numbers. When a statement is proved in this way, it is often called a <u>theorem.</u>

By means of this theorem we can discover another important property. Suppose we consider a sum of any two real numbers. Call this

$$a + b$$

Since this sum is a real number, it has an additive inverse. We can write it as $-(a + b)$, and we know that

$$(a + b) + \big(-(a + b)\big) = 0.$$

Now consider the expression

$$a + b + (-a) + (-b)$$

By rearranging the numbers, which we can do because of associative and commutative properties, we see that this expression is also equal to zero. This tells us that since

$$(a + b) + \big((-a) + (-b)\big) = 0$$

then the expression $(-a) + (-b)$ must also be an additive inverse of

$$a + b.$$

But our theorem tells us there is only one of these. Thus we conclude that

$$-(a + b) = (-a) + (-b).$$

There is an interesting way to state this property. We can say that

> the opposite of a sum of two real numbers
> is equal to the sum of the opposites.

Check Your Reading

1. What property justifies the conclusion that $-x$ is an additive inverse of x, where x is any real number?
2. What is proved when we show that some other additive inverse z is also $-x$?
3. What theorem is established by the two previous statements?
4. Showing that $(a + b) + ((-a) + (-b))$ equals zero establishes what fact about $((-a) + (-b))$?

Problem Set 7-7b

1. What property establishes "$(3 + (-4)) + (-(3 + (-4))) = 0$" as a true sentence?
2. Use the properties of real numbers to show that $(3 + (-4)) + ((-3) + 4)$ equals zero.
3. Since exercises 1 and 2 establish both $(-(3 + (-4)))$ and $((-3) + 4)$ as additive inverses of $(3 + (-4))$, what further conclusion can be drawn?
4. What property establishes that "$(-5) + 5 = 0$" is a true sentence?
5. What property establishes that "$(-5) + (-(-5)) = 0$" is a true sentence?
6. The sentences of exercises 4, 5, and what theorem establish that $5 = -(-5)$?
7. Use $-(a + b) = (-a) + (-b)$ to write another expression for the following:

(a) $-(a + 3)$	(d) $-(3x + 2y)$	(g) $-((-a) + (-b))$
(b) $-(x + y)$	(e) $-((-2) + a)$	(h) $-(5x + 3)$
(c) $-(2m + 3)$	(f) $-(x + (-3y))$	

Summary

In this chapter, we have discussed addition of real numbers. If a and b are any two real numbers, their sum may be written

$$a + b.$$

However, it is important to remember that $(a + b)$ is itself a real number.

We also discussed the following properties:

Commutative Property of Addition

For any two real numbers a and b, $a + b = b + a$.

Associative Property of Addition

For any real numbers a, b, and c, $(a + b) + c = a + (b + c)$.

Addition Property of Opposites

For every real number a, $a + (-a) = 0$.

Addition Property of Zero

For every real number a, $a + 0 = a$.

Addition Property of Equality

For any real numbers a, b, and c, if $a = b$
then $a + c = b + c$.

It is important to remember that the addition property of equality can be used in finding the truth sets of certain open sentences.

Review Problem Set

1. Perform the following additions on the number line.

   (a) $4 + (-3)$
   (b) $2 + (-6)$
   (c) $7 + 0$
   (d) $0 + (-4)$
   (e) $1.5 + 2.5$
   (f) $6 + 0$

2. In each of the following describe the sum by using the definition of addition. Check your answers by using any convenient method.

   (a) $3 + (-5)$
   (b) $(-5) + (-11)$
   (c) $0 + (-15)$
   (d) $\sqrt{2} + (-\sqrt{2})$
   (e) $18 + (-14)$
   (f) $(-\pi) + \pi$
   (g) $(-\frac{2}{3}) + \frac{1}{2}$
   (h) $(-35) + (-65)$
   (i) $12 + 7$
   (j) $(-6) + 10$
   (k) $1 + (-\frac{3}{2})$
   (l) $200 + (-201)$

3. Find a common name for each of the following.

   (a) $3 + (6 - 2)$
   (b) $8 - (4 + 1)$
   (c) $(8 - 4) + 1$
   (d) $(4 \times 2) + 5$
   (e) $4 \times (2 + 5)$
   (f) $24 - (3 \times 5)$
   (g) $(24 - 3) \times 5$
   (h) $(5 \times 0) + 6$
   (i) $5 \times (0 + 6)$
   (j) $5 + (-5) + (-10)$
   (k) $6 + (-4) + (-1)$
   (l) $(-7) + (-9) + 10$
   (m) $8 + (-6) + (-8)$
   (n) $|-9| + |4| + (-|-6|)$
   (o) $|-4| + |8| + 4 + (-4)$

4. Which of the following are true sentences? Which are false?

   (a) $(4 + 6) + 5 = 4 + (6 + 5)$
   (b) $6(\frac{1}{4} + \frac{3}{4}) = 6$
   (c) $6(1.5 + 3.5) = 6(1.5) + (3.5)$
   (d) $-(|-1.5| - |0|) = -1.5$

Review Problem Set
(continued)

(e) $\left(4 + (-6)\right) + 6 = 4 + \left((-6) + 6\right)$

(f) $(-5) + \left(-(-5)\right) = -10$

(g) $(-7) + \left((-5) + (-3)\right) = \left((-7) + (-5)\right) + 3$

(h) $-\left(6 + (-2)\right) = (-6) + (-2)$

(i) $(-3) + 7 = -\left(3 + (-7)\right)$

5. Which of the following are true for all values of the variables? Which are false?

(a) $(3 + 2) + m = 3 + (m + 2)$

(b) $2(r + 3) = 2r + 3$ where r is a positive number

(c) $4r(2y + z) = 8ry + 4rz$ where r, y, and z are positive

(d) $-x + 0 = x$

(e) $-a + 5 = 5 + (-a)$

(f) $(-x) + (-y) = -(x + y)$

6. For each of the following find the real number value (or values) of the variable which makes the given sentence true.

(a) $x + 2 = 7$

(b) $3 + x = 0$

(c) $3 + y = -7$

(d) $(-2) + a = 0$

(e) $a + 5 = 0$

(f) $3 + 5 + y = 0$

(g) $b + (-7) = 3$

(h) $x + (-\frac{1}{2}) = 0$

(i) $(-\frac{5}{6}) + x = -\frac{5}{6}$

(j) $c + (-3) = -7$

(k) $y + \frac{2}{3} = -\frac{5}{6}$

(l) $\left(y + (-2)\right) + 2 = 3$

Review Problem Set
(continued)

(m) $(3 + x) + (-3) = -1$

(n) $\frac{1}{2}b + (-4) = 6$

(o) $6 + x + (-4) = 5 + (-2)$

(p) $5 + x + (-7) = 4$

(q) $7 + m + (-3) = m + 9 + (-5)$

(r) $\big(10 + (-6)\big) + b = (-10) + (6 + b)$

(s) $|-5| + |6| + a = 0$

(t) $b + |-4| + |-3| = |-7|$

7. Find the truth set of each of the following open sentences.

(a) $m + 7 = 12$

(b) $a + (-5) = 8$

(c) $a + (-4) + (-5) = (-9) + 3$

(d) $(-6) + 7 = (-8) + x$

(e) $(-1) + 2 + (-3) = 4 + x + (-5)$

(f) $(-2) + x + (-3) = x + (-\frac{5}{2})$

(g) $x + (-3) = |-4| + (-3)$

(h) $(-\frac{4}{3}) + (x + \frac{1}{2}) = x + (x + \frac{1}{2})$

(i) $x + (-3) = |-5| + |-3|$

(j) $|-4| + b = |-6| + |4|$

(k) $|-5| + a = a + |9| + |-2|$

(l) $|-8| + m = (-9) + |-1| + m$

(m) $x + 2 + x = (-3) + x$

8. Tell why each of the following sentences is true. Name the property, or properties, that are illustrated by each sentence, whether associative, commutative, addition property of 0, or addition property of opposites.

(a) $3 + \big((-3) + 4\big) = 0 + 4$

(b) $\big(5 + (-3)\big) + 7 = \big((-3) + 5\big) + 7$

Review Problem Set
(continued)

(c) $\left(7 + (-7)\right) + 6 = 6$

(d) $|-1| + |-3| + (-3) = 1$

(e) $(-2) + \left(3 + (-4)\right) = \left((-2) + 3\right) + (-4)$

(f) $(-|-5|) + 6 = 6 + (-5)$

(g) $\left((-2) + 6\right) + (-8) = (-2) + \left(6 + (-8)\right)$

(h) $8 + |-5| + a = |-5| + 8 + a$

(i) $|-6| + (-6) + 0 = 0$

(j) $a + 4 + (-a) = 4$

9. Use the commutative and associative properties to obtain the following sums in an easy way.

(a) $(-\frac{1}{2}) + 7 + (-2) + (-\frac{3}{2}) + 2$

(b) $\frac{5}{3} + (-3) + 6 + \frac{1}{3} + (-2)$

(c) $125 + (-17) + (-13) + (-25)$

(d) $(-3) + 8 + 11 + (-5) + (-3) + 12 + (-4)$

(e) $\frac{2}{3} + \frac{3}{2} + (-\frac{5}{3}) + (-\frac{1}{2}) + |-2|$

(f) $|-5| + 21 + (-5) + (-8) + (-7)$

(g) $(-9) + |-7| + 12 + |-2| + 7$

(h) $-|-10| + (-15) + 15 + (-3) + (-|-6|)$

10. Write open sentences for the following: (Be sure to identify the variable.)

(a) Jim learned that on a certain day the low tide registered 0.6 feet below sea level and that it rose 5.1 feet during a six hour period. How far above sea level did the tide register after it rose to the high tide?

(b) Dave shot at a target and hit 10 inches above the center on the first shot. The second shot hit 3 inches below the first shot. How far above the center was the second shot?

Review Problem Set
(continued)

(c) A submarine that was cruising at 254 feet below sea level rose 78 feet. How far below sea level was it after it rose to the new position?

(d) A man left a $50,000 estate. His will stated that his son was to receive twice as much as his daughter and his widow was to receive as much as both together received. How much did each receive?

(e) Mr. Johnson owed the bank $200, then had to borrow a small amount again. How much did he owe the bank?

Chapter 8

MULTIPLICATION OF REAL NUMBERS

8-1. Products.

In the previous chapter, addition was defined so that addition of real numbers "behaves" in the same way that addition of the numbers of arithmetic behaves. That is, it has the same properties.

In this chapter, we shall try to decide how any two real numbers should be multiplied so that multiplication of real numbers also behaves like multiplication of the numbers of arithmetic. That is, we would like the following properties to be true for any real numbers a, b, and c:

$ab = ba$	commutative property of multiplication
$(ab)c = a(bc)$	associative property of multiplication
$(a)(1) = a$	multiplication property of one
$(a)(0) = 0$	multiplication property of zero
$a(b + c) = ab + ac$	distributive property

We would like these properties to be true for real numbers; and this will help us decide how multiplication of real numbers shall be defined.

Here are some possible products of real numbers:

$$(2)(3), \quad (3)(0), \quad (0)(0), \quad (-3)(0), \quad (3)(-2), \quad (-2)(-3).$$

The first three examples use only numbers that are not negative.

As for the fourth one, $(-3)(0)$, if we want the multiplication property of zero to be true, we must be able to say "$(-3)(0) = 0$". So we can make the following definition:

$$(-3)(0) = 0.$$

Of course, if we want the commutative property of multiplication to hold, we must also say:

$$(0)(-3) = 0.$$

In fact, we make the following definition for <u>any real number</u> a:

$$(a)(0) = 0.$$

$$(0)(a) = 0.$$

Check Your Reading

1. What is a simpler name for the product $(3)(0)$?
2. Give a simpler name for the product $(0)(3)$.
3. Give a simpler name for the product $(-3)(0)$.
4. Give a simpler name for the product $(0)(-3)$.
5. What is the truth set of the sentence "$(a)(0) = 0$"?
6. What is the truth set of the sentence "$(0)(a) = 0$"?
7. State (in words) the definition made in this section for the product of zero and any real number.

Oral Exercises 8-1a

1. Give the common name for each of the following:

(a) $(7)(0)$	(f) $(m)(0)$
(b) $(0)(0)$	(g) $(-x)(0)$
(c) $(0)(-3)$	(h) $(-5.2)(1.0)$
(d) $(0)(-\frac{19}{24})$	(i) $0(8 + (-10))$
(e) $(-92.75)(0)$	(j) $((-6) + 6)3$

2. Which of the following sentences are true? Which are false?

(a) $(0)(5) = 0$	(d) $(2.14)(0) > 0$
(b) $(5)(0) \neq 0$	(e) $(0)(-7.9) < 0$
(c) $(-7)(0) < 0$	

Oral Exercises 8-1a

(continued)

3. Which of the following sentences are true for every value of the variable?

(a) $(0)(a) = 0$

(b) $(-a)(0) = 0$

(c) $(0)(-n) < 0$

(d) $(0)\big(m + (-m)\big) = 0$

(e) $(0)(m + m) > 0$

(f) $\big((-a) + (-a)\big)(0) < 0$

(g) $\big((-x) + y\big)(0) \neq 0$

In the previous section, we considered products in which one number was zero. In this section, let us look at some products in which one number is positive and one number is negative. Let us begin by looking at the following list of products:

$(3)(2) = 6$
$(3)(1) = 3$
$(3)(0) = 0$
$(3)(-1) =$
$(3)(-2) =$

In each case, 3 is multiplied by another number. Reading down the list, the "other number" decreases by 1 each time. Do you see that the product decreases by 3 each time (6, 3, 0)?

No simpler name has been given for $(3)(-1)$ because we have never before considered such a product. But if the product is to continue its pattern of decreasing by 3 each time in the list, what would $(3)(-1)$ be? In a similar way, we can decide what $(3)(-2)$ might be.

Continuing the pattern in the table above, it turns out that $(3)(-1)$ is -3, and $(3)(-2)$ is -6. And we can give an even more convincing argument for defining these products in this way. Remember that there are certain properties of multiplication which we want to hold for all real numbers. One of these properties is the distributive property of multiplication

over addition. See if you can follow the steps in the following argument:

$0 = (3)(0)$	We have already agreed that the product of zero and any real number is zero.
$0 = (3)(2 + (-2))$	If the first sentence is true, so is this one. "$2 + (-2)$" is just another name for the number 0.
$0 = (3)(2) + (3)(-2)$	This is what we <u>want</u> to be able to say, since we want the distributive property to hold for all real numbers. The question is: How must we define $(3)(-2)$ so that this will be true?
$0 = (6) + (3)(-2)$	Here we have simply used the name "6" for the product $(3)(2)$.

In order for the last two numerical sentences to be true, $(3)(-2)$ must be a number that can be added to 6 to give zero. In other words, $(3)(-2)$ must be the <u>additive inverse</u> of 6. As proved in Chapter 7, a number has only one additive inverse; the additive inverse of 6 is -6. Therefore, in order for the distributive property to hold in the above example, $(3)(-2)$ must be defined to be -6. Notice that this agrees with the result obtained from the "pattern" in the list at the beginning of this section.

For any particular product involving a positive number and a negative number, we can take steps similar to those above. For example, consider the product $(2)(-7)$. The four sentences below correspond to the four sentences above.

$$0 = (2)(0)$$

$$0 = (2)(7 + (-7))$$

$$0 = (2)(7) + (2)(-7)$$

$$0 = 14 + (2)(-7)$$

Thus, if "$0 = 14 + (2)(-7)$" is to be true, the product $(2)(-7)$ must be defined to be -14.

We have arrived at definitions for two particular products:

$$(3)(-2) = -6 \qquad\qquad (2)(-7) = -14.$$

From these, we can easily decide upon two other particular definitions. Remember that we want the <u>commutative property</u> of multiplication to hold for all real numbers. Therefore, we would want the following definitions:

$$(-2)(3) = -6 \qquad\qquad (-7)(2) = -14.$$

Following the same line of thinking that we followed for $(3)(-2)$ and $(2)(-7)$, we could arrive at definitions for many other particular products. But probably these two examples have already suggested to you common names for such products as

$$(7)(-10), \quad (-\tfrac{1}{2})(42), \quad (1)(-1), \quad \text{and} \quad (5)(-5).$$

In fact, based on our experience in this section, you can probably supply a word for the blank in the following sentence:

> The product of a positive number and a negative number is a _____ number.

And with this experience, you probably feel that you know <u>how</u> to multiply a positive number by a negative number and a negative number by a positive number. This is what you are asked to do in the problems that follow. In a later section, we shall give a formal definition for such products.

<u>Check Your Reading</u>

1. Give, in order, simpler names for the following products: $(3)(2)$, $(3)(1)$, $(3)(0)$, $(3)(-1)$, $(3)(-2)$.
2. "$2 + (-2)$" is a simpler name for what number?
3. What number is the additive inverse of 6?
4. What two properties were used in arriving at the definitions in this section?
5. Is the product of a positive number and a negative number negative or positive?

Oral Exercises 8-1b

1. Give a common name for each of the following:

(a) $6(3)$

(b) $9(0)$

(c) $3(6)$

(d) $0(8)$

(e) $4(-5)$

(f) $8(-3)$

(g) $6(-\frac{1}{2})$

(h) $\frac{5}{4}(-\frac{1}{3})$

(i) $(-1.5)(8)$

(j) $9(-0.4)$

(k) $(-1.6)(.2)$

(l) $0(7.83)$

(m) $0(a)$

(n) $2(-3)$

(o) $1(-5)$

(p) $a(0)$

(q) $(-5)(1)$

2. Which of the following indicated operations can be performed using only numbers of arithmetic?

(a) $6\cdot(-7)$

(b) $|-3|\cdot(4)$

(c) $|-5|\cdot|6|$

(d) $(-2)\cdot|-2|$

3. (a) What is a common name for $(3)(-2)$?

(b) Are $|3|$ and $|-2|$ numbers of arithmetic?

(c) What is a name for $|3|\cdot|-2|$?

(d) Explain why "$-(|3|\cdot|-2|)$" is a name for $(3)(-2)$.

4. Which of the following sentences are true?

(a) $(-4)(3) = -(-|4|\cdot|3|)$

(b) $(5)(-7) = (|5|\cdot|-7|)$

(c) $(3)\cdot|-7| = |-3|\cdot 7$

(d) $(4 + (-5))(2) = -(|-1|\cdot|2|)$

Problem Set 8-1b

1. Which of the following sentences are true?

(a)	$(5)(4) = (20)$	(h)	$2(-2.4) = (2.8)$
(b)	$(5)(0) = 5$	(i)	$8(-1.1) = (-8.8)$
(c)	$(0)(5) = 0$	(j)	$1(\frac{3}{8}) = \frac{4}{8}$
(d)	$4(-4) = (16)$	(k)	$3(-4) = (-4) + 3$
(e)	$7(-\frac{2}{3}) = (-\frac{14}{3})$	(l)	$2(-8) = (-9) + (-7)$
(f)	$5(-6) = (-30)$	(m)	$2(-7) = (-7) + (-7)$
(g)	$3(-\frac{7}{8}) = (\frac{7}{24})$	(n)	$6(-6) = 0$

2. Write a common name for each of the following.

(a)	$(-5)\big((-6) + 7\big)$	(d)	$\frac{1}{5}(-75) + \frac{1}{5}(-25)$
(b)	$\frac{2}{3}\big((-\frac{1}{2})6\big)$	(e)	$7(-2) + 7(2)$
(c)	$\frac{2}{3}\big(-\frac{3}{8} + (-\frac{9}{16})\big)$	(f)	$28\big(5(-\frac{1}{7})\big)$

We have not yet discussed a product in which both numbers are negative. Below is a list of products; in each of these products, the number -3 is multiplied by another number.

$(-3)(2) = -6$
$(-3)(1) = -3$
$(-3)(0) = 0$
$(-3)(-1) =$
$(-3)(-2) =$

Reading down this list, the product increases by 3 each time (-6, -3, 0). No simpler name has been given for the product $(-3)(-1)$, since we have not yet discussed such products.

However, if the product is to continue its pattern of increasing by 3 each time, what would $(-3)(-1)$ be? In a similar way, we can decide what $(-3)(-2)$ might be.

Continuing the pattern in the list above, it turns out that $(-3)(-1)$ is 3, since 3 represents an increase of 3 over the previous product of 0; also, it turns out that $(-3)(-2)$ is 6, since 6 represents an increase of 3 over the previous product of 3. We can also use properties of multiplication to show that the definition "$(-3)(-2) = 6$" is a desirable one to make.

$0 = (-3)(0)$	The product of zero and any real number is zero.
$0 = (-3)(2 + (-2))$	The truth of this sentence follows from the truth of the first one, since "$2 + (-2)$" is simply another name for the number 0.
$0 = (-3)(2) + (-3)(-2)$	<u>If</u> the distributive property is to hold (and we want it to), then the truth of this statement must follow from the truth of the second statement.
$0 = (-6) + (-3)(-2)$	Here we have just used the name "-6" for the product $(-3)(2)$, which we agreed to in the previous section.

"$0 = (-6) + (-3)(-2)$" will be true only if $(-3)(-2)$ is a number that can be added to -6 to give zero. In other words, $(-3)(-2)$ must be the <u>additive inverse</u> of -6. The additive inverse of -6 is 6. Therefore, in order for the distributive property to hold in the above example, $(-3)(-2)$ must be defined to be 6. This definition agrees with the result we obtained from the "pattern" in the list at the beginning of this section.

The same kind of argument can be given for any other particular product involving two negative numbers. Consider, for example, $(-7)(-5)$.

$0 = (-7)(0)$	Why?
$0 = (-7)(5 + (-5))$	Why?
$0 = (-7)(5) + (-7)(-5)$	We want to be able to make this statement so that a certain property will hold. What property is it?
$0 = -35 + (-7)(-5)$	

Thus, if "$0 = -35 + (-7)(-5)$" is to be true, the product $(-7)(-5)$ must be defined to be the additive inverse of -35. That is, we must make the definition: "$(-7)(-5) = 35$", since 35 is the only additive inverse of -35.

We have now arrived at the following two particular definitions:

$$(-3)(-2) = 6 \qquad\qquad (-7)(-5) = 35$$

Following the same line of reasoning that we followed for these two definitions, we could define the product of any two negative numbers. But these two examples may already have suggested to you common names for such products as the following:

$$(-2)(-8), \qquad (-5)(-5), \qquad (-8)(-\tfrac{1}{2}), \qquad (-1)(-1).$$

In fact, based on our experience in this section, you can probably supply a word for the blank in the following sentence:

> The product of a negative number and another negative number is a _____ number.

Probably, then, you can find a simpler name for the product of any two particular negative numbers. In the next section, we shall make a formal definition for such products.

Check Your Reading

1. Give, in order, simpler names for the following products: $(-3)(2)$, $(-3)(1)$, $(-3)(0)$, $(-3)(-1)$, $(-3)(-2)$.
2. Give a simpler name for the product $(-7)(-5)$.
3. What two properties were used in arriving at the definitions in this section?
4. Is the product of two negative numbers a positive number or a negative number?

Oral Exercises 8-1c

1. Give a common name for each of the following.

(a)	$(6)(5)$	(h)	$(-1)(7)$
(b)	$(-4)(3)$	(i)	$(-1)(-1)$
(c)	$(-3)(4)$	(j)	$(-1)(0)$
(d)	$(-3)(-4)$	(k)	$(-3)(b)$
(e)	$(3)(4)$	(l)	$(-7)(-b)$
(f)	$(0)(6)$	(m)	$(-4)(b)(0)$
(g)	$(8)(0)$	(n)	$(0)(-1)(-5)$

2. (a) What is a common name for $(-2)(-3)$?
   (b) Are $|-2|$ and $|-3|$ numbers of arithmetic?
   (c) What is a name for $|-2|\cdot|-3|$?
   (d) Explain why $|-2|\cdot|-3|$ is a name for $(-2)(-3)$.
3. Which of the following are true? Which are false?
   (a) $(-4)(5) = |-4|\cdot|5|$
   (b) $(-1.2)(-.5) = |-1.2|\cdot|-.5|$
   (c) $\big(6 + (-8)\big)(-4) = |-2|\cdot|-4|$
   (d) $\big((5) + (-5)\big)\cdot(-4) = -(0\cdot|-4|)$
   (e) $(-3)\big(4 + (-\frac{2}{3})\big) = (-3)(4) + (-3)(-\frac{2}{3})$
   (f) $(-5)\big((-4)(2)\big) = \big((-5)(-4)\big)(2)$

Problem Set 8-1c

1. Give the common name for each of the following.

(a) $5(8)$

(b) $6(-4)$

(c) $7(0)$

(d) $8(-5)$

(e) $(-6)(4)$

(f) $(3)(-3)$

(g) $(-3)(-3)$

(h) $(-5)(0)$

(i) $(-2)(-7)$

(j) $(-8)(-9)$

(k) $(-3)(-4)$

(l) $(-6)(-\frac{1}{2})$

(m) $(-12)(-\frac{1}{3})$

(n) $(-9)(-0)$

(o) $(-\frac{1}{4})(-\frac{1}{3})$

(p) $(-.5)(-8)$

(q) $(-.7)(-5)$

(r) $(-.6)(-.3)$

(s) $(-4)(-.2)$

2. Which of the following sentences are true? Which are false?

(a) $5(10) = 50$

(b) $(6)(0) = 6$

(c) $1(-5) = 5$

(d) $3(-4) = -12$

(e) $(8)(0) = 0$

(f) $(-9)(0) = 0$

(g) $(-2)(-3) = 6$

(h) $(-5)(-4) = -20$

(i) $(-1)(-10) = 10$

(j) $(-1)(1) = 0$

(k) $(-0)(-5) = 5$

(l) $(-8)(-7) = 56$

(m) $(-\frac{1}{3})(-\frac{1}{5}) = (-\frac{1}{15})$

(n) $(-2\frac{1}{2})(-2) = 5$

(o) $(-\frac{3}{4})(-\frac{2}{5}) = \frac{5}{20}$

(p) $(-1.5)(-3) = 4.5$

(q) $(-.6)(-.4) = (-.24)$

(r) $(-4)(-3) = (-4) + (-3)$

3. Find the truth set of each of the following.

(a) $4m = -12$

(b) $-8x = 24$

(c) $x(x + 3) = -2$, if the domain of x is the set of integers. (There are two truth numbers.)

(d) $|x| < |-3| \cdot |-2|$

(e) $3|x| < \big((-3)(-2)\big)|x|$

Problem Set 8-1c
(continued)

4. Find two integers whose product is 5, if one integer is 4 more than the other.

5. Find a common name for each of the following.

(a) $2\cdot|-5|$

(b) $3\cdot|4|$

(c) $|5|\cdot|-4|$

(d) $|-3|\cdot|-2|$

(e) $|15|\cdot|8|$

(f) $0\cdot|-9|$

(g) $|-1|\cdot|-7|$

(h) $0\cdot|4|$

6. Which of the following sentences are true?

(a) $2\cdot|4| = 2\cdot 4$

(b) $|3|\cdot|-4| = -(3 \times 4)$

(c) $(-5)(6) = -|5 \times 6|$

(d) $|2|\cdot|6| = -(2 \times 6)$

(e) $|-5|\cdot 4 = -20$

(f) $|-2|\cdot|-6| = -(2\cdot 6)$

(g) $(8)(3) = |8 \times 3|$

(h) $(7)(-4) = -\left(|7|\cdot|-4|\right)$

(i) $0(-5) = 0\cdot|5|$

(j) $(-5)(-5) = |-5|\cdot|-5|$

(k) $(-4)(-6) = -\left(|-4|\cdot|-6|\right)$

(l) $(-3.5)(2) = -\left(|-3.5|\cdot|2|\right)$

(m) $(-\frac{9}{92})(-\frac{1}{34}) = -\left(|-\frac{9}{92}|\cdot|-\frac{1}{34}|\right)$

(n) $|4|\cdot|-3| = 1$

(o) $|-6|\cdot|-5| = 20 + 10$

(p) $|-5|\cdot|4| = (-25) + 5$

(q) $|3|\cdot|-8| = 30 + (-6)$

Problem Set 8-1c
(continued)

7. Given the set $S = \{-3, -2, -1, 0, 1, 2, 3\}$.

   (a) Find the set of all possible products of pairs of elements of the set S.

   (b) Is the set S closed under the operation of multiplication?

   (c) Is the set of all of the integers closed under the operation of multiplication? Can you think of two integers whose product is not an integer?

8. Given the set $R = \{-2, -1, -\frac{1}{2}, 0\}$.

   (a) Find the set of all possible products of pairs of elements of the set R.

   (b) Is the set R closed under the operation of multiplication?

   (c) Is the set of all of the negative real numbers closed under the operation of multiplication? Can you think of two negative numbers whose product is a negative number?

9. Given the set $A = \{. . ., -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, . . .\}$.

   (a) Find the set of all possible products of pairs of elements of the set A.

   (b) Is the set A closed under the operation of multiplication?

From the work in previous sections, you know <u>how</u> to multiply two real numbers. That is, given any two real numbers, you can find their product. There are several possible cases:

One number may be positive and one number negative.
Both numbers may be positive.
Both numbers may be negative.

One of the numbers may be zero.

Both of the numbers may be zero.

Just as we did with addition, we shall <u>define</u> multiplication of real numbers. This will not make us any better at finding particular products, but it will enable us to prove certain properties of multiplication---something that just "knowing how" would never do. We must be sure, of course, that our definition includes all possible cases.

All of this may sound puzzling, but a few examples will lead directly to the definition.

$(-3)(-5) = 15.$	We have already agreed to this; we know "how" to multiply two negative numbers.
Is it true that $(-3)(-5) = \lvert -3\rvert \cdot \lvert -5\rvert$?	This is true, because both expressions name the number 15.
Is it true that $(-4)(-10) = \lvert -4\rvert \cdot \lvert -10\rvert$?	

Suppose that the variables a and b represent <u>any two negative numbers.</u> Then do you see that

$$ab = |a| \cdot |b| \ ?$$

Now suppose that the variables a and b represent any two <u>positive</u> numbers. In this case, it is also true that

$$ab = |a| \cdot |b| \ .$$

For example,

$$(10)(2) = |10| \cdot |2| .$$

Also, if a = 0 or if b = 0 (or if a and b are both zero), it is true that

$$ab = |a| \cdot |b| \ .$$

For example,

$(8)(0) = \lvert 8\rvert \cdot \lvert 0\rvert .$	Remember that $\lvert 0\rvert$ is 0.
$(0)(-2) = \lvert 0\rvert \cdot \lvert -2\rvert .$	
$(0)(0) = \lvert 0\rvert \cdot \lvert 0\rvert .$	

Thus, of all possible products of real numbers, we have found some in which the product is the same as the product of the absolute values. This is true if both numbers are positive; it is true if both numbers are negative; and it is true if one or both of the numbers are zero. Is it always true?

Let us look at some different examples.

$(3)(-5) = -15.$	In this case, one number is positive and one number is negative.
Is it true that $(3)(-5) = \lvert 3\rvert \cdot \lvert -5\rvert$?	This is <u>not</u> true, since one expression names the number -15 and the other names the number 15.
Is it true that $(3)(-5) = -(\lvert 3\rvert \cdot \lvert -5\rvert)$?	This <u>is</u> true, because each of the expressions is a numeral for -15.

In this case, then, the product is not the same as the product of the absolute values. However, it is the same as the <u>opposite</u> of the product of absolute values. A little thinking should convince you that, if the variables a and b represent real numbers, one number positive and one number negative, then

$$ab = -(\lvert a\rvert \cdot \lvert b\rvert).$$

For example,

$$(4)(-7) = -(\lvert 4\rvert \cdot \lvert -7\rvert)$$

$$(-3)(9) = -(\lvert -3\rvert \cdot \lvert 9\rvert)$$

And now a definition of multiplication of real numbers may be stated as follows:

If a and b represent two real numbers, one positive and one negative	$ab = -(\lvert a\rvert \cdot \lvert b\rvert)$.
and in all other cases,	$ab = \lvert a\rvert \cdot \lvert b\rvert$.

Notice that the definition enables us to restate the product of any two real numbers in terms of numbers of arithmetic and their opposites, since $|a|$ and $|b|$ and $|a|\cdot|b|$ are numbers of arithmetic, for any real numbers a and b. This will enable us to establish properties of multiplication of real numbers.

Also notice that the definition assures us that <u>any</u> two real numbers may be multiplied. Every possible case is included in the definition.

Check Your Reading

1. In discussing the product of two real numbers, one possibility is that both numbers are positive. What other possibilities are there?
2. Is $|n|$ a number of arithmetic, for any real number n?
3. When is it true that $ab = |a|\cdot|b|$?
4. When is it true that $ab = -(|a|\cdot|b|)$

Oral Exercises 8-1d

Use the definition of multiplication of real numbers to express the following products in terms of numbers of arithmetic.

Example: $(-2)(3) = -(|2|\cdot|3|)$

1. $(6)(-5)$
2. $(-8)(-3)$
3. $(2)(4)$
4. $(5)(0)$
5. $(-7)(1)$
6. $(-\frac{2}{3})(2)$
7. $(-5)(-\frac{1}{2})$
8. $(0)(-2)$
9. $(8)(-1)$
10. $(-2)(-3)$
11. $(\frac{1}{3})(-\frac{1}{2})$
12. $(-1)(-1)$
13. $(0)(-\frac{7}{3})$
14. $(8)(6)$
15. $(-\frac{1}{4})(-\frac{1}{2})$
16. $(3)(-7)$
17. $(-11)(2)$
18. $(-5)(-\frac{2}{3})$
19. $(-6)(-6)$
20. $(-7)(0)$

Problem Set 8-1d

1. Copy each of the following and complete with either "$|a|\cdot|b|$" or "$-(|a|\cdot|b|)$" whichever will result in a true sentence.

   (a) If $a < 0$ and $b < 0$, then $ab =$
   (b) If $a < 0$ and $b > C$, then $ab =$
   (c) If $a > 0$ and $b < 0$, then $ab =$
   (d) If $a > 0$ and $b > 0$, then $ab =$
   (e) If $a = 0$ and $b = 0$, then $ab =$
   (f) If $a = 0$ and $b \neq 0$, then $ab =$

2. Determine a common name for each of the following.
   Example. $(-2)(8)(-\frac{1}{2}) = 8$.

   (a) $(5)(-3)(-3)$
   (b) $(5)(-3)(-3)(-1)$
   (c) $(5)(-3)(-3)(-1)(-1)$
   (d) $(-10)(-10)$
   (e) $(-10)(-10)(-10)$
   (f) $(-10)(-10)(-10)(-10)$
   (g) $(-10)(-10)(-10)(-10)(-10)$
   (h) $(24)\frac{1}{3}$
   (i) $(24)(-\frac{1}{3})$
   (j) $(-24)(-\frac{1}{3})$
   (k) $(1.8)(-2.3)$
   (l) $(-1.8)(2.3)$
   (m) $(-1)(-1.8)(2.3)(-1)$
   *(n) $(\sqrt{2})(\sqrt{2})$
   *(o) $(-\sqrt{2})(-\sqrt{2})$

8-2. Properties of Multiplication.

We have discussed properties many times. And we have seen that properties of numbers are important in learning how numbers behave. Now that we have defined multiplication of real numbers, we must be sure that the properties of multiplication that we listed for the numbers of arithmetic are true for the entire set of real numbers.

Let us first consider the multiplication property of one. Is it true that $a \cdot 1 = a$, for any real number a?

We already know that $a \cdot 1 = a$ if a is a positive number or zero, since such numbers are numbers of arithmetic. You may be willing to believe that it is true if a is a negative number. But it can actually be proved to be true, just from our definition of multiplication.

For example, suppose we want to prove that $(-4)(1) = -4$.

In the product $(-4)(1)$, one of the numbers is positive and one of the numbers is negative. If you will refer to the definition in the previous section, you will see that we made the following agreement: if a and b are two numbers, one positive and one negative, then

$$ab = -(|a| \cdot |b|)$$

So, for the product $(-4)(1)$,

$$\begin{aligned}(-4)(1) &= -(|-4| \cdot |1|) \\ &= -((4)(1)) \\ &= -(4) \\ &= -4\end{aligned}$$

Thus, we have proved that $(-4)(1) = -4$.

Of course, we have worked with just one particular product: $(-4)(1)$. But we can prove that for any negative number a, $(a)(1) = a$.

Since a is negative and 1 is positive, by the definition of multiplication,

$$\begin{aligned}(a)(1) &= -(|a| \cdot |1|) \\ &= -(|a| \cdot 1) \\ &= -(|a|)\end{aligned}$$

However, since a is negative, $|a| = -a$.

So $-|a| = -(-a)$

$= a$.

Therefore, $(a)(1) = a$.

Thus, from the definition of multiplication accepted in the previous section, we have proved that, if a is a negative number, it must be true that $(a)(1) = a$. Do you see now what was meant when it was said that the definition makes it possible to prove properties.

Since we already knew that $(a)(1) = a$ if a is a positive number, or if a is zero, we can now say:

For any real number a,

$$(a)(1) = a.$$

Following are some other properties which are true for the entire set of real numbers. Each one of these properties could be proved from our definition of multiplication, just as we proved the multiplication property of one. However, the proofs are long in most cases; so we just list the properties with some examples.

Commutative Property of Multiplication

For any real numbers a and b, $ab = ba$.

Examples:

$$(5)(-4) = (-4)(5)$$
$$(0)(86) = (86)(0)$$
$$(-7)(-5) = (-5)(-7)$$

Associative Property of Multiplication

For any real numbers a, b, and c, $(ab)c = a(bc)$.

Examples:

$$\big((3)(2)\big)(-4) = (3)\big((2)(-4)\big)$$
$$\big((7)(-3)\big)(-2) = (7)\big((-3)(-2)\big)$$
$$\big((-2)(-3)\big)(-5) = (-2)\big((-3)(-5)\big)$$

Distributive Property

For any real numbers a, b, and c, $a(b + c) = ab + ac$.

Examples:

$$5\big(2 + (-3)\big) = 5(2) + 5(-3)$$
$$5\big((-2) + (-3)\big) = 5(-2) + 5(-3)$$
$$(-5)\big((-2) + (-3)\big) = (-5)(-2) + (-5)(-3)$$

Recall that there is another form of the distributive property:

$$(b + c)a = ba + ca.$$

Examples:

$$\big(3 + (-4)\big)(-4) = (3)(-4) + (-4)(-4)$$
$$(-5)(-7) + (-2)(-7) = \big((-5) + (-2)\big)(-7)$$
$$(3)(-8) + (3)(5) = 3\big((-8) + 5\big)$$

Check Your Reading

1. What is the truth set of "$(a)(1) = a$"?
2. What name is given to the property expressed by "$(a)(1) = a$"?
3. "$(5)(-4) = (-4)(5)$" is a specific illustration of what property of the real numbers?
4. "$\big((3)(2)\big)(-4) = (3)\big((2)(-4)\big)$" is a specific illustration of what property of the real numbers?
5. For any real numbers a, b, and c, "$a(b + c) = ab + ac$" is a true sentence. What is the name of the property expressed by this sentence?
6. Name the four properties of real numbers established in this section.

Oral Exercises 8-2a

1. Give a common name for each of the following.

(a)	$(4)(1)$	(j)	$(-3)(2)(1)(0)$
(b)	$(1)(-5)$	(k)	$(1.5)(-2)(1)$
(c)	$(1)(12)$	(l)	$(-2)(1)(-\frac{1}{2})$
(d)	$(1)(-10)$	(m)	$(7.8)(-4.5)(1)(0)$
(e)	$(-9)(1)$	(n)	$(-b)(1)(3)$
(f)	$(-7)(2)(1)$	(o)	$\frac{1}{2}\big(6 + (-4)\big)$
(g)	$(3)(1)(6)$	(p)	$(-\frac{2}{3})(5 + 4)$
(h)	$(5)(-3)(1)$	(q)	$\big((-8) + (-10)\big)\frac{1}{3}$
(i)	$(-6)(1)(-4)$	(r)	$\big((-9) + (-3)\big)(-\frac{2}{3})$

Problem Set 8-2a

1. Tell what property or properties (associative or commutative property of multiplication, distributive property, multiplication property of one, or multiplication property of zero) are illustrated in each of the following true sentences.

(a) $(2)(3) = (3)(2)$

(b) $(-2)(5) = (5)(-2)$

(c) $\big((-4)(5)\big)(-3) = (-4)\big((5)(-3)\big)$

(d) $\big((-3)(a)\big)(-2) = (-3)\big((-2)(a)\big)$, for all numbers a

(e) $(a)(3) = (1)(a)(3)$, for all numbers a

(f) $(-2)(a)(1) = (-2)(a)$, for all numbers a

(g) $(-4)(b)(1) = (-4)(b)$, for all numbers b

(h) $(2)(-b)(0) = (0)(-b)(2)$, for all numbers b

(i) $(-3)(0)(1) = (-3)(1)(0)$

(j) $2\big(3 + (-4)\big) = (2)(3) + (2)(-4)$

Problem Set 8-2a
(continued)

(k) $(-3)\big((-1) + 2\big) = (-1)(-3) + 2(-3)$

(l) $(-5)\big(a + (-b)\big) = \big(a + (-b)\big)(-5)$, for all numbers a and b

2. Find a common name for each of the following by using the properties of multiplication.

(a) $(-\frac{1}{2})(-4)$

(b) $(-\frac{1}{2})(2)(-5)$

(c) $(-\frac{3}{2})(-4)(\frac{2}{3})(1)$

(d) $(-\frac{1}{3})(3)(7)$

(e) $(-\frac{1}{3})(-3)(-7)$

(f) $(4)(-6)(1)(\frac{7}{4})$

(g) $|-3|(-2)(4)$

(h) $|-3||-2|(-6)$

(i) $(-3)|-2|(4)$

(j) $(-2)^2(3)$

(k) $(-2)(3)^2$

(l) $(|-3|)^2$

(m) $(-3) + (2)^2$

(n) $(-3)^2 + (2)^2$

(o) $(3)(-2.1)(0)(7.3)(1)$

(p) $(-4)(b)(1)(2c)(0)$

(q) $9\big(4 + (-2)\big)$

(r) $(-3)\big((-7) + 4\big)$

(s) $\frac{2}{3}\big((-6) + (-3)\big)$

(t) $\big((-\frac{5}{3}) + (-\frac{1}{3})\big)(-\frac{1}{2})$

(u) $\big((-9.25) + 6.5\big)(0)$

(v) $(-5)x + 3x$

(w) $7a + (-2)a$

3. Write each of the following indicated products as an indicated sum.

(a) $2\big(3 + (-2)\big)$

(b) $(-3)\big((-4) + (-6)\big)$

(c) $4\big((a) + (-5)\big)$

(d) $(1)\big((-5) + (-10)\big)$

(e) $(1)\big(0 + (-1)\big)$

(f) $(1)\big((-1) + 1\big)$

(g) $5\big((-6)a + (-2)\big)$

(h) $(-6)\big(10x + (-3)\big)$

(i) $\big(6a + (-2)\big)(4)$

(j) $\big((-8) + (-2)\big)(-7)$

(k) $\big((1.2) + (-1.1)\big)6m$

(l) $\big((-\frac{2}{3}) + (\frac{3}{4})\big)\big((-\frac{2}{5})(2x)\big)$

Problem Set 8-2a
(continued)

4. Write the following indicated sums as indicated products.

(a)	$2(5) + 2(4)$	(f)	$(x)(-9) + (-6)(x)$
(b)	$(-3)(7) + (-3)(3)$	(g)	$(1)(4) + (1)(6)$
(c)	$(4)(-6) + 4(-9)$	(h)	$(1)(-3) + (1)(9)$
(d)	$(-7)(-5) + (-7)(-4)$	(i)	$(-2)(a) + (-2)(b)$
(e)	$(-8)y + (5)y$	(j)	$(3)(a) + (4)(y)$

What is the product of 5 and -1? It is easy to see that the product is -5. We could say that when 5 is multiplied by -1, the product is the <u>opposite of 5</u>.

Check the following true sentences:

$$(-1)(7) = -7$$
$$(-1)(3) = -3$$
$$(-1)(-8) = 8$$
$$(-1)(-2) = 2$$

Notice that in each of these cases, a number was multiplied by -1; the product turned out to be the <u>opposite</u> of that number. In the last sentence, for example, -2 was multiplied by -1; the product, 2, is the opposite of -2.

From these examples, you might guess that if any real number is multiplied by -1, the product is the opposite of that number. We can use the distributive property to <u>prove</u> that this is really the case.

We are trying to prove that $(-1)a = -a$.

This means that we must show that $(-1)a$ is the opposite of a, that is, that $(-1)a$ is the number which, when added to a, gives zero.

For any real number a,

To prove: $a + (-1)a = 0$.

$a + (-1)a = 1(a) + (-1)a$	Here we have used "1(a)" for "a". We have the right to do this, because of the multiplication property of 1.
$= 1 + (-1)\,a$	Here the distributive property has been used.
$= (0)a$	$1 + (-1)$ is the same as 0, since the sum of a number and its opposite is zero.
$= 0$	We have agreed that the product of any number and zero is zero.

We have now shown that, for any real number a, if $(-1)a$ is added to a, the sum is zero. In other words, $(-1)a$ is the <u>additive inverse</u> of a, or $-a$. More briefly, we can say:

For any real number a,

$$(-1)a = -a.$$

For any numbers a and b, $a + b$ is a number. From the proof above, we can say

$$(-1)(a + b) = -(a + b).$$

But since the distributive property holds for all real numbers, we also know that

$$\begin{aligned}(-1)(a + b) &= (-1)a + (-1)b \\ &= (-a) + (-b).\end{aligned}$$

This proves that, for any numbers a and b,

$$-(a + b) = (-a) + (-b).$$

Do you see that this is the same result we obtained in section 7-7? In words, we can say that <u>the opposite of a sum is the sum of the opposites.</u> Below is an example in which this fact is applied.

Write an expression for the opposite of $-x + 2$.

$$\begin{aligned}-(-x + 2) &= -(-x) + -(2) \\ &= x + (-2).\end{aligned}$$

The fact that $(-1)a = -a$, for any real number a, can be used to prove other useful properties. Consider the product $(-a)(b)$.

$$\begin{aligned}(-a)(b) &= \big((-1)a\big)(b)\\ &= (-1)(ab)\\ &= -ab.\end{aligned}$$

Therefore, <u>for any real number a and any real number b,</u> $(-a)(b) = -ab$.

Next, consider the product $(-a)(-b)$.

$$\begin{aligned}(-a)(-b) &= \big((-1)a\big)\big((-1)b\big)\\ &= (-1)(-1)(a)(b)\\ &= (1)(ab)\\ &= ab.\end{aligned}$$

Therefore, <u>for any real number</u> a <u>and any real number</u> b, $(-a)(-b) = ab$.

Below are some examples in which the above properties are applied.

<u>Example</u> 1. $(y)(-x) = -xy$

<u>Example</u> 2. $(-4y)(-y) = 4y^2$

<u>Example</u> 3. $\big(-(x + y)\big)\big(-(x + y)\big) = (x + y)^2$

<u>Check Your Reading</u>

1. What is the product of 5 and -1?
2. What is the product of -2 and -1?
3. For any real number a, what is the product of a and -1?
4. For any real numbers a and b, $-(a + b) = (-a) + (-b)$. How may this property be stated in words?
5. What is the opposite of the number $-x + 2$?

Complete each of the following sentences.

6. For any real numbers a and b, $(-a)(b) =$
7. For any real numbers x and y, $(y)(-x) =$
8. For any real numbers a and b, $(-a)(-b) =$
9. For any real number y, $(-4y)(-y) =$

Oral Exercises 8-2b

Give a common name for each of the following.

1. $(-1)(2)$
2. $(-1)(-3)$
3. $(-1)a$
4. $(-1)(-a)$
5. $-(x + 5)$
6. $-(-x + 5)$
7. $-(x - 5)$
8. $-(-x - 5)$
9. $-(a + 2)$
10. $(-1)(a + 2)$
11. $-(x + y)$
12. $-(x - y)$
13. $(x)(-y)$
14. $(-x)(y)$
15. $(-x)(-y)$
16. $(-1)(x + y)$
17. $(-1)(x - y)$
18. $a(-1)$
19. $(-1)x + x$
20. $\big(m + (-2)\big)(-2)$

Problem Set 8-2b

1. State the opposite of each of the following numbers without just placing an "opposite" sign in front of the number. Check your results by adding each of the given numbers and its opposite.

   (a) $-m$
   (b) $-m + 4$
   (c) $-2x + (-3)$
   (d) $5y + 7x$
   (e) $-y + (-x)$
   (f) $-\big(3m + (-4)\big)$
   (g) $-(-1)(5x + \frac{3}{2})$
   (h) $-8x^2 + (-16x)$
   (i) $3y^2 + (-7y)$
   (j) $-\big(-(2y + 11)\big)$

2. Give a common name for each of the following.

   (a) $-(-8y)$
   (b) $-(m + 2)$
   (c) $-(-1)(5x)$
   (d) $(-1)\big(-(7y)\big)$
   (e) $(-1)\big(m + (-3)\big)$
   (f) $(-1)(x + 2) + (-1)\big(-(x + 2)\big)$
   (g) $-\big(5 + (-4)\big)x$
   (h) $-\big(4y + (-7y)\big)$
   (i) $(-ab)(k)$
   (j) $(-m)(-3a)$
   (k) $(y)(-4y)$
   (l) $(-2m)(-nx)$

Problem Set 8-2b
(continued)

3. Find the truth set of each of the following sentences.

(a) $-(x + \frac{1}{2}) = -2$ (c) $y + (-1)y = 5$

(b) $5 + \left(-(3y + 4)\right) = -2$ (d) $3m + 2 + \left(-(3m + 2)\right) = 0$

4. The opposite of the sum of two numbers is 17. One of the numbers is 5 more than the other. What are the numbers?

5. A mother and her daughter went shopping. The mother spent \$5 more than three times as much as the daughter. Together they spent \$49. How much did each spend?

6. One number is three times the opposite of the other. Their sum is -86. What are the numbers?

8-3. Using the Multiplication Properties.

Now that we know how to multiply any two real numbers and we have some important properties of multiplication, we can work with many kinds of open phrases in algebra.

Here are some examples:

Example 1. Use the distributive property to multiply $(-6)\left(2 + (-x)\right)$.

$$\begin{aligned}(-6)\left(2 + (-x)\right) &= (-6)(2) + (-6)(-x)\\ &= -12 + 6x\end{aligned}$$

Do you see why $(-6)(-x)$ is the same as $6x$? It could be written out this way:

$$\begin{aligned}(-6)(-x) &= (-6)\left((-1)(x)\right)\\ &= \left((-6)(-1)\right)(x)\\ &= (6)(x)\end{aligned}$$

Example 2. Use the distributive property to write $xy + x$ as a product.

$$xy + x = x(y + 1)$$

Do you see that the distributive property tells us this sentence is true for any real numbers x and y?

We have written $xy + x$ as the product $x(y + 1)$, which may be thought of as the product of the number x and the number $(y + 1)$.

Example 3. Use the distributive property to "simplify" the open phrase "$5x + (-2)x$".

$$5x + (-2)x = \big(5 + (-2)\big)x$$
$$= (3)x$$

or $3x$. You will probably agree that the open phrase "$3x$" seems "simpler" than "$5x + (-2)x$".

In Example 3, the distributive property was used to write "$5x + (-2)x$" as "$3x$".

In an expression like "$5x + (-2)x$", "$5x$" and "$(-2)x$" are called terms of the phrase. When we simplify "$5x + (-2)x$" to "$3x$", we often say that we are collecting terms.

Below you will find some other examples of simplifying by collecting terms.

Example 4. Simplify "$(-4)n + 20n$".

$$(-4)n + 20n = \big((-4) + 20\big)n$$
$$= 16n$$

Example 5. Simplify "$8x + 3y$".

Do you see why the terms in this phrase cannot be collected? We can say that the phrase "$8x + 3y$" is already in simplest form.

Example 6. Simplify "$10x + (-2)y + 7x + 14y$".

Using the associative and commutative properties, we can rearrange the terms, like this:

$$10x + 7x + (-2)y + 14y.$$

Then, using the distributive property, we have

$$(10 + 7)x + \big((-2) + 14\big)y, \quad \text{or}$$
$$17x + 12y.$$

Example 7. Simplify "$4r + (-2)z + (-7)r + 8z$".

$$\begin{aligned} & 4r + (-2)z + (-7)r + 8z \\ = \; & 4r + (-7)r + (-2)z + 8z \\ = \; & \big(4 + (-7)\big)r + \big((-2) + 8\big)z \\ = \; & (-3)r + 6z \end{aligned}$$

Check Your Reading

1. Give a simpler expression for "$(-6)(-x)$."
2. Supply an appropriate word for each of the blanks in the following sentence:
   "$xy + x$" is an indicated _____; the distributive property may be used to write it as an indicated _____.
3. How many "terms" are there in the expression "$5x + (-2)x$"?
4. If the "terms are collected" in "$5x + (-2)x$" by applying the distributive property, the result is _____.
5. Which of the following two expressions can be simplified by use of the distributive property: $(-4)n + 20n$, $8x + 3y$. ?

Oral Exercises 8-3a

1 Use the distributive property to perform the indicated multiplications.

(a) $4\big((-3) + b\big)$

(b) $(-2)\big((-3) + (-b)\big)$

(c) $(-m)\big(4m + (-5)\big)$

(d) $m\big((-a) + (-c)\big)$

(e) $\big((-7) + (-2)a\big)(-4)$

(f) $\big((-x) + x\big)(-x)$

Oral Exercises 8-3a
(continued)

2. Use the distributive property to write the following indicated sums as indicated products.

(a) $2a + 2b$ (c) $9x + (-12)y$ (e) $m + m^2$

(b) $(-5)a + (-5)b$ (d) $(-4)m + (-5)m$ (f) $(-2)x + 4x$

Problem Set 8-3a

1. Use the distributive property, as in Example 1, to multiply the following.

(a) $5(3 + a)$

(b) $4\big((-3) + b\big)$

(c) $(-2)\big((-3) + (-b)\big)$

(d) $(4)\big((-2) + (-c)\big)$

(e) $(a + b)(-3)$

(f) $(-2)\big(a + (-b)\big)$

(g) $(-5)\big((-m) + (-n)\big)$

(h) $\big((-2.1a) + 1.2b\big)(-4)$

(i) $m\big((-a) + (-c)\big)$

2. Use the distributive property to write the following indicated sums as indicated products as in Example 2.

(a) $2a + 2b$

(b) $(-5)(a) + (-5)(m)$

(c) $(-2)(b) + (-2)(c)$

(d) $(-3)(c) + (-4)(c)$

(e) $(-1)(m) + (-1)(-n)$

(f) $(-2)(m) + (-1)(m)$

(g) $ab + am$

(h) $xy + ry$

(i) $mr + m$

(j) $(-\frac{2}{3})(m) + (-\frac{2}{3})(n)$

(k) $1.5a + 1.5b$

(l) $2m + (-4)n$ Hint: $(-4)n$ may be written as $(2)(-2)n$.

(m) $6a + 9b$

(n) $9x + (-12y)$

(o) $(-10m) + (-15n)$

(p) $6a + (-15b)$

(q) $9ax + 12ay$

(r) $(-4ac) + (-6ab)$

(s) $m + m^2$ Hint: m^2 can be written $(m)(m)$.

(t) $a^2 + (-2a)$

(u) $(-5m) + 6n$

Problem Set 8-3a
(continued)

3. Simplify the following expressions by collecting terms as in the examples.

   (a) $6m + 4m$

   (b) $7a + (-3)a$

   (c) $9a + (-15)a$

   (d) $(-5)y + 14y$

   (e) $(-3)m + (-6)m$

   (f) $4a + 3b$

   (g) $(4a + a)$ (Hint: a may be written as $(1)a$.)

   (h) $(-5)x + 2y$

   (i) $(-6)a + a$

   (j) $(-a) + (-4)a$ (Hint: $(-a)$ may be written as $(-1)a$.)

   (k) $2t + (-4)w + 3t + (-2)w$

   (l) $(-5)a + (-2)b + 6a + 5b$

   (m) $(-2)m + 6b + 2m + (-2)b$

   (n) $y + 6x + (-7)x + (-y)$

   (o) $8a + (-5)a + 2a + (-3)b$

   (p) $(-2)m + (-3)n + 6m + m$

   (q) $(-4)a + 5a + (-3)a + 7a$

   (r) $(-3)y + 2x + 4z + (-3)w$

4. Show that for all real numbers a, b, c, and d,

$$a(b + c + d) = ab + ac + ad.$$

Hint: write $b + c + d = b + (c + d)$ and apply the distributive property. Then apply it again to get the result.

Problem Set 8-3a
(continued)

5. Use the result of Problem 4 to change each indicated sum to an indicated product and each indicated product to an indicated sum.

(a) $(2k)(a + m + 5)$ (d) $(-6)c + (-6)b + (-6)a$

(b) $(-6)\big(a + (-b) + (-7)\big)$ (e) $(5a)\big(a + (-2) + (-a)\big)$

(c) $(-3)x + (-7)x + 10x$ (f) $5a^2 + 10a + (-5)a$

6. Find the truth set of each of the following sentences.

(a) $3x + (-7)x = (-4)(x + 2)$ (d) $(-7)k + 4k + 3k = 0$

(b) $15m + (-14)m = 5$ (e) $y^2 + (-y) = y\big(y + (-1)\big)$

(c) $6x + (-2)x + (-3)x = x$ (f) $(2y)\big(|-5| + (-8)\big) = 18$

7. The sum of three consecutive odd integers is 153. What are the integers?

8. Esau wants to cut a rectangular piece from a sheet of plywood such that the perimeter of his piece of plywood is 24 inches and the number of inches in the width and the number of inches in the length of the piece are consecutive integers. Is this possible? Set up and solve an open sentence to answer this question.

We have already seen how the properties of multiplication allow us to simplify phrases by collecting terms. There are other ways to simplify phrases, however.

For example, consider the phrase "$(3x)(2x)$".

$(3x)(2x) = 3\cdot x\cdot 2\cdot x$	The associative property tells us that we may associate in any way; so the symbols of grouping may be omitted.
$= 3\cdot 2\cdot x\cdot x$	The commutative property gives us the right to change the order of the numbers.

$= (3\cdot 2)\cdot(x\cdot x)$	The associative property allows us to group the numbers in this way.
$= 6x^2$	Here we have just used the common name "6" for "$3\cdot 2$" and the simpler name "x^2" for "$x\cdot x$".

All of the steps above are important, because they show how the properties of multiplication allow us to make the simplification. It is not necessary to write all the steps each time, however. It is possible to do most of the work in your head and just write:

$$(3x)(2x) = 6x^2.$$

Here are two other examples:

<u>Example 1</u>. Simplify "$(3xy)(-7ay)$".

$$\begin{aligned}(3xy)(-7ay) &= 3\cdot x\cdot y\cdot(-7)\cdot a\cdot y\\ &= 3\cdot(-7)\cdot a\cdot x\cdot y\cdot y\\ &= \big(3\cdot(-7)\big)\cdot a\cdot x\cdot(y\cdot y)\\ &= (-21)axy^2\end{aligned}$$

<u>Example 2</u>. Simplify "$(-2ax)(-7ax)$".

$$(-2ax)(-7ax) = 14a^2x^2$$

<u>Check Your Reading</u>

1. What property or properties are used in multiplying $(2x)(3x)$?
2. Can you write "$3\cdot 2\cdot a\cdot a$" in a shorter form?

Oral Exercises 8-3b

1. Simplify each of the following phrases:

(a) $(2)(2m)$

(b) $(-3)(4xy)$

(c) $3x + (-4x)$

(d) $(6)(-2a)$

(e) $(-2)(-4st)$

(f) $(4x)(-7y)$

(g) $(-5a)(-3b)$

(h) $(-1)(m)$

(i) $(-b)(4c)$

(j) $7x + 2y + (-4x)$

(k) $(6a)(-c)$

(l) $(-5m)(-n)$

(m) $(0)(-5xy^2)$

(n) $(6am^2)(2m)(0)$

(o) $(2ax)(-by)$

(p) $(-4am)(-3am)$

Problem Set 8-3b

1. Simplify each of the following phrases as in Examples 1 and 2.

(a) $(2a)(4am)$

(b) $(-4y)(3y)$

(c) $(3b)(-2ab)$

(d) $(-4cd)(-6d)$

(e) $(-3c^2)(d)$

(f) $(-x)(6ay)$

(g) $(-c)(-4abd)$

(h) $(2ax)(-6by)$

(i) $(-3mn)(-7amn)$

(j) $(abc)(abxy)$

(k) $(am)(-9mx^2)$

(l) $(-9a^2m)(-8mn^2)$

(m) $(\frac{4}{3}m^2)(-\frac{3}{4}n^2)$

(n) $(-\frac{2}{3}ab^2)(-\frac{4}{5}ac^2)$

(o) $(-1.5c)(-3b)$

(p) $(-4.75d)(3.12cdm^2)(0)$

(q) $(-5a)\ 7a + (-2b)$

2. Find the truth set of each of the following.

(a) $2\big(3x + (-7x)\big) = 8$

(b) $(-2m)\big((-2m) + 7\big) + (-4m^2) = -14$

(c) $|x + 2| = -7$

(d) $4 = (-2)\big(3x + (-2)\big)$

Now we can use the distributive property to work with more complicated phrases. Study the following example.

By the distributive property,

$$(-3a)\big(2a + (-5c)\big) = (-3a)(2a) + (-3a)(-5c)$$
$$= -6a^2 + 15ac.$$

Problem Set 8-3c

Use the distributive property to simplify the following:

1. $3x(2x + 4z)$
2. $6x\big((-3a) + 2b\big)$
3. $(-2m)\big(m + (-3n)\big)$
4. $(-3m)\big((-x) + (-y)\big)$
5. $(-5m)\big((-4a) + (-1)c\big)$
6. $a\big(2 + (-3)\big)$
7. $a\big(b + (-2c)\big)$
8. $(-m)\big((-c) + (-d)\big)$
9. $-\big((-b) + c\big)$
10. $-(4x + 3y)$
11. $-\big((-2x) + (3z)\big)$
12. $-\big((-4c) + (-6d)\big)$
13. $(-\frac{2}{3}c)(6a + 9b)$
14. $\frac{5}{2}a\big((-4m) + 6n\big)$
15. $(-1.5m)\big((-2)mx + (-3)my\big)$
16. $(-1.4c)\big((-3)ab + 3ab\big)$
17. $-(a + b + c)$
18. $-\big(b + (-c) + m\big)$
19. $-\big(b + (-d) + (-t)\big)$
20. $(-2m)\big((-2)x + 3y + (-4)z\big)$

Sometimes the distributive property is used more than once in working with a phrase. Below are two examples.

Example 1. Multiply "$(x + 3)(x + 2)$".

$(x + 3)(x + 2) = (x + 3)x + (x + 3)2$	by the distributive property
$= x^2 + 3x + 2x + 6$	by the distributive property
$= x^2 + (3 + 2)x + 6$	by the distributive property
$= x^2 + 5x + 6$	

<u>Example 2.</u> $\big(a + (-7)\big)(a + 3) = \big(a + (-7)\big)a + \big(a + (-7)\big)3$

$$= a^2 + (-7)a + 3a + (-21)$$

$$= a^2 + (-4)a + (-21)$$

Problem Set 8-3d

Use the distributive property as in Examples 1 and 2 above to simplify the following phrases.

1. $(a + 3)(a + 2)$
2. $\big(a + (-2)\big)\big(a + (-3)\big)$
3. $\big(a + (-5)\big)\big(a + (-3)\big)$
4. $(b + 4)(b + 6)$
5. $\big(c + (-5)\big)(c + 7)$
6. $(m + 8)\big(m + (-1)\big)$
7. $\big(m + (-5)\big)\big(m + (-4)\big)$
8. $(m + 1)(m + 1)$
9. $\big(t + (-1)\big)(t + 1)$
10. $(x + 3)(x + 3)$
11. $\big(a + (-5)\big)(a + 5)$
12. $(x + 3)(x + 5)$
13. $(x + 3)\big(x + (-5)\big)$
14. $(k + 7)\big(k + (-6)\big)$
15. $\big(k + (-7)\big)(k + 9)$
16. $(b + 1)\big(b + (-2)\big)$
17. $\big(b + (-1)\big)(b + 8)$
18. $\big(z + (-2)\big)\big(z + (-5)\big)$
19. $\big(z + (-3)\big)\big(z + (-7)\big)$
20. $\big(z + (-1)\big)\big(z + (-1)\big)$
21. $\big(m + (-3)\big)\big(m + (-3)\big)$
22. $(a + 5)(a + 5)$
23. $\big(a + (-5)\big)\big(a + (-5)\big)$
24. $(b + 2)(b + 2)$
25. $\big(4 + (-b)\big)(2 + b)$
26. $\big(6 + (-a)\big)\big(3 + (-a)\big)$
27. $\big(6 + (-a)\big)(6 + a)$
28. $(2a + 3)(4a + 5)$
29. $(4m + 2n)(3m + n)$
30. $(a + b)(c + d)$
31. $\big(x + (-a)\big)\big(x + (-b)\big)$
32. $(x + 2a)\big(x + (-3a)\big)$
33. $(3a + 7)\big(4a + (-3)\big)$
34. $\big(2m + (-3)\big)\big(m + (-4)\big)$
35. $(2m + n)(2m + n)$
36. $(2k + b)(2k - b)$
37. $\big(z + (-3a)\big)\big(z + (-2a)\big)$
38. $(3 + 2z)\big(3 + (-2z)\big)$
39. $(2x + 3a)(3x + 2a)$
40. $\big(2x + (-3a)\big)\big(2x + (-2a)\big)$

8-4. Multiplicative Inverse.

Determine a common name for each of the following products:

$$4\cdot\frac{1}{4}, \qquad 9\cdot\frac{1}{9}, \qquad \frac{1}{7}\cdot 7, \qquad \frac{2}{3}\cdot\frac{3}{2}.$$

In each case, a common name for the product is "1." In other words, in each case we have two numbers whose product is one.

Find a truth number for each of the following open sentences:

$$14a = 1, \quad 2n = 1, \quad \frac{1}{5}(t) = 1, \quad \frac{5}{2}a = 1.$$

In determining a truth number of each of the above sentences, you are again determining a pair of numbers whose product is one.

In all of the examples above, only positive numbers were used. Now consider the following product:

$$(-4)(-\frac{1}{4}).$$

Do you see that here again we have two numbers whose product is one? Determine a truth number of each of the following sentences:

$$(-2)n = 1, \qquad (-\frac{1}{5})t = 1.$$

In this section we have used a knowledge of arithmetic, together with the ability to multiply any two real numbers, to determine pairs of numbers such that the product of each pair is one. This is what you are asked to do in the exercises that follow.

Oral Exercises 8-4a

1. Find a common name for each of the following:

   (a) $5\cdot\frac{1}{5}$

   (b) $3\cdot\frac{1}{3}$

Oral Exercises 8-4a
(continued)

(c) $\frac{1}{4} \cdot 4$

(d) $\frac{1}{8} \cdot 8$

(e) $\frac{3}{4} \cdot \frac{4}{3}$

(f) $(-\frac{1}{6})(-6)$

(g) $(-7)(-\frac{1}{7})$

(h) $(-\frac{5}{6})(-\frac{6}{5})$

(i) $(-2\frac{1}{2})(-\frac{2}{5})$

2. Give the truth set of each of the following sentences.

(a) $4n = 1$

(b) $(-3)n = 1$

(c) $(-\frac{1}{3})a = 1$

(d) $\frac{1}{5}b = 1$

(e) $\frac{4}{5}m = 1$

(f) $(-\frac{4}{7})m = 1$

(g) $\frac{5}{2}x = 1$

(h) $(-\frac{4}{3})y = 1$

(i) $(2\frac{1}{4})b = 1$

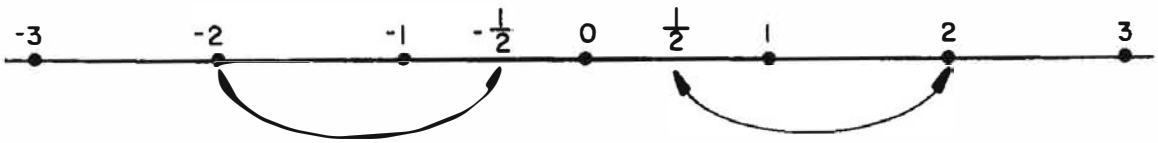
(j) $(0)b = 1$

We have been "pairing off" numbers whose product is one. For example,

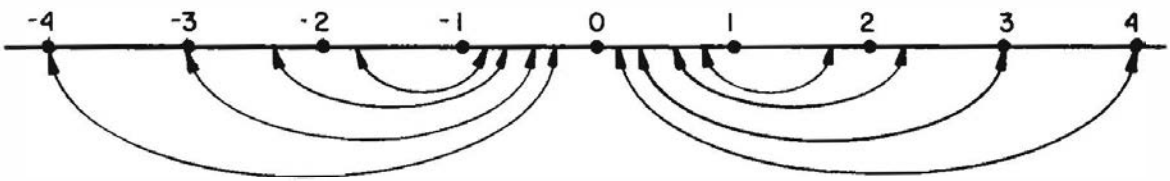
$\frac{1}{2}$ was paired with 2, because $(\frac{1}{2})(2) = 1$.

-2 was paired with $-\frac{1}{2}$, because $(-2)(-\frac{1}{2}) = 1$.

On the number line, these pairings can be pictured like this:



Below are pictured some other pairings of numbers whose product is one.



Notice that a positive number is paired with a positive number, and a negative number is paired with a negative number. Why is it impossible to find a pair of numbers, one positive and the other negative, whose product is one?

Although it is not indicated in the diagram above, there are two numbers each of which is paired with itself. Do you know which numbers these are?

Also notice that 0 is not paired with any number at all. Pairing the number 0 with another number so that the product is one, is the same as finding a truth number of this open sentence: $(0)x = 1$. What property tells us that this sentence has no truth number, and that 0 cannot be paired with any number so that the product is one?

The name <u>multiplicative inverse</u> is used to describe a pair of numbers whose product is one.

> If c and d are real numbers such that $cd = 1$,
> then d is called a multiplicative inverse of c,
> and c is called a multiplicative inverse of d.
>
> For example,
> 2 and $\frac{1}{2}$ are real numbers such that $(2)(\frac{1}{2}) = 1$.
> So 2 is called a multiplicative inverse of $\frac{1}{2}$,
> and $\frac{1}{2}$ is called a multiplicative inverse of 2.

It was pointed out earlier that 0 has no multiplicative inverse. From your knowledge of arithmetic, it probably seems reasonable to you that every other number besides 0 does have a multiplicative inverse. We now state this as a formal property:

> For every real number c <u>different from 0,</u>
> there is a multiplicative inverse.

Furthermore, no number has more than one multiplicative inverse. So we can say that the multiplicative inverse is unique, that is, every number except 0 has <u>one</u> and <u>only one</u> multiplicative inverse. This gives us the right to speak of

> <u>the</u> multiplicative inverse of a number.

The multiplicative inverse of -2 is $-\frac{1}{2}$.
The multiplicative inverse of $-\frac{1}{2}$ is -2.
What is the multiplicative inverse of -5?
The multiplicative inverse of $\frac{3}{4}$ is $\frac{4}{3}$.
What is the multiplicative inverse of $\frac{7}{2}$?
What number has no multiplicative inverse?

We have seen the word "inverse" earlier, when we discussed additive inverses. There is a close relation between additive inverse and multiplicative inverse.

0 is the <u>identity element</u> of addition. For any number c, $c + 0 = c$.
d is the <u>additive inverse</u> of c, if $c + d = 0$.

1 is the <u>identity element</u> of multiplication. For any number c, $c(1) = c$.
d is the <u>multiplicative inverse</u> of c, if $cd = 1$.

As we proved earlier, the additive inverse of a number is unique. That is, every number has one and only one additive inverse, and that inverse is the opposite of the number.

In this section, we have agreed that the multiplicative inverse of a number is also unique. That is, every number except zero has one and only one multiplicative inverse. For the time being, you must rely on your knowledge of arithmetic to determine this multiplicative inverse in each case.

Check Your Reading

1. What is the product of 2 and $\frac{1}{2}$?
2. What is the multiplicative inverse of 2?
3. What is the multiplicative inverse of $\frac{1}{2}$?
4. In this section, the phrase "multiplicative inverse" was defined. When is a number d said to be the multiplicative inverse of a number c?
5. Does any number have more than one multiplicative inverse?

6. Does every number have a multiplicative inverse?
7. Does any number have more than one additive inverse?
8. Does every number have an additive inverse?
9. What is the identity element of multiplication?
10. What is the identity element of addition?

Oral Exercises 8-4b

1. What is the multiplicative inverse of 1?
2. What is the multiplicative inverse of (-1)?
3. What is the multiplicative inverse of zero?
4. Why does zero have no multiplicative inverse?
5. Is the multiplicative inverse of a positive real number always a positive real number?
6. Is the additive inverse of a positive real number always a positive real number?
7. There are two elements each of which is its own multiplicative inverse. Which elements are these?
8. What is the additive inverse of a?
9. What is the additive inverse of (-a)?
10. Does every real number have a multiplicative inverse? an additive inverse?
11. How can you tell that one number is the multiplicative inverse of another?

Problem Set 8-4b

1. Which of the following sentences are true?

(a) $4(\frac{1}{4}) = 1$	(e) $(\frac{2}{3})(\frac{3}{2}) = 1$	(i) $(-\frac{5}{1})(5) = 1$
(b) $(-3)(\frac{1}{3}) = 1$	(f) $(-\frac{1}{3})(3) = 1$	(j) $(-\frac{5}{1})(-\frac{1}{5}) = 1$
(c) $(-5)(-\frac{1}{5}) = 1$	(g) $(3\frac{1}{2})(-\frac{2}{7}) = 1$	(k) $(-\frac{4}{1})(0) = 1$
(d) $(6)(-\frac{1}{6}) = 1$	(h) $(-2\frac{1}{3})(-\frac{3}{7}) = 1$	(l) $(\frac{17}{1})(\frac{1}{17}) = 1$

Problem Set 8-4b
(continued)

2. Find the truth set of each of the following open sentences.

(a) $3x = 1$

(b) $-2x = 1$

(c) $\frac{1}{5}x = 1$

(d) $(3)(a)(2) = 1$

(e) $(-4)(t)(3) = 1$

(f) $(-2)(-2)(n)(-1) = 1$

(g) $3a = \frac{1}{5}(5)$

(h) $-2t = 4(\frac{1}{4})$

(i) $\frac{1}{8}(y) = (2)(\frac{1}{2})(-\frac{1}{3})(-3)$

(j) $3 + x = 0$

(k) $(3)(x) = 1$

8-5. Multiplication Property of Equality.

"(2)(6)" is the name of a number. If the number is multiplied by 2, we get a new number whose name may be written "(2)((2)(6))".

"(4)(3)" is the name of a number. If the number is multiplied by 2, we get a new number whose name may be written "(2)((4)(3))".

In both cases, we were really doing the same thing. We started with two names for the number 12. We multiplied 12 by 2. So, of course, both times we obtained names for the number 24. In other words,

since $(2)(6) = (4)(3)$ is true,

then $(2)\big((2)(6)\big) = (2)\big((4)(3)\big)$ is true.

This is an example of the multiplication property of equality.

Here are some other examples.

Example 1. Since $(2)(-3) = -6$ is true,

then $(\frac{1}{2})\big((2)(-3)\big) = (\frac{1}{2})(-6)$ is true.

<u>Example 2</u>. Since $(-3)(8) = (6)(-4)$ is true,

then $(-\frac{1}{3})\big((-3)(8)\big) = (-\frac{1}{3})\big((6)(-4)\big)$ is true.

The multiplication property of equality may be written like this:

<u>For real numbers a, b, and c,</u> if $a = b$, then $ac = bc$.

Of course, it may also be written:

<u>For real numbers a, b, and c,</u> if $a = b$, then $ca = cb$.

Do you remember that we also have an addition property of equality? How is it stated? We found it useful in finding truth sets of open sentences. The multiplication property of equality is useful in this way, too.

<u>Example:</u> Find the truth set of

$$-2x = 10.$$

If there is an x such that

$$-2x = 10 \quad \text{is true,}$$

then the same x makes

$$(-\tfrac{1}{2})(-2x) = (-\tfrac{1}{2})(10)$$

and

$$\big((-\tfrac{1}{2})(-2)\big)x = (-\tfrac{1}{2})(10)$$

and

$$x = -5 \quad \text{true also.}$$

Here we have used the multiplication property of equality.

<u>Check Your Reading</u>

1. Is (2)(6) the same number as (4)(3)?
2. If each of these products, (2)(6) and (4)(3), is multiplied by 2, what is true about the results?
3. What property is illustrated in Question 2?

Oral Exercises 8-5

Tell how you would use the multiplicative inverse and the multiplication property of equality to write a simpler sentence to help find the truth set of each of the following. You need not find the truth set.

1. $2m = 4$
2. $3n = -6$
3. $4n = 20$
4. $(3)b = 17$
5. $(-5)a = 15$
6. $(-3)a = 19$
7. $(-2)a = -10$
8. $(-4)a = -9$
9. $\frac{1}{2}x = 4$
10. $(-\frac{1}{3})x = 2$
11. $(-\frac{1}{2})x = 1$
12. $(-\frac{1}{3})y = -3$
13. $(-\frac{2}{3})c = 0$
14. $(-\frac{5}{6})c = 1$
15. $(-\frac{3}{4})b = 12$

Problem Set 8-5

Use the multiplicative inverse and the multiplication property of equality to find the truth set of each of the following open sentences.

1. $2a = 12$
2. $(-3)a = 15$
3. $5a = -25$
4. $(-8)a = -16$
5. $(-6)x = 0$
6. $-m = 7$
7. $\frac{1}{2}b = 4$
8. $(-\frac{1}{4})b = 8$
9. $\frac{1}{3}b = -3$
10. $(-\frac{1}{2})m = -5$
11. $4 = \frac{2}{3}m$
12. $(-\frac{5}{6})m = 10$
13. $4n = 7$
14. $(-3)n = 17$
15. $(-12)b = -30$
16. $(-3)c = \frac{3}{4}$
17. $\frac{7}{2} = 9c$
18. $\frac{2}{3} = \frac{4}{5}b$
19. $(-\frac{5}{6})c = \frac{10}{11}$
20. $(-\frac{4}{3})b = (-\frac{9}{10})$

8-6. Solutions of Open Sentences.

Find the truth set of

$$2x + 5 = 27.$$

Perhaps you could do this by "guessing". However, we shall soon find cases where guessing the truth set would be very, very difficult. So let's try finding the truth set in another way.

The phrase "solve the open sentence" is often used instead of "find the truth set of the open sentence". Here, then, is how we would solve the open sentence "$2x + 5 = 27$".

If there is an x that makes

$$2x + 5 = 27 \quad \text{true,}$$

then the same x makes

$$(2x + 5) + (-5) = 27 + (-5)$$

and

$$2x + \big((5) + (-5)\big) = 27 + (-5)$$

and

$$2x = 22 \quad \text{true, also.}$$

Here we have used the addition property of equality. We added (-5). (Of course, we could say that (-5) was added "to both sides".)

Now if there is an x that makes

$$2x = 22 \quad \text{true,}$$

then the same x makes

$$\tfrac{1}{2}(2x) = \tfrac{1}{2}(22)$$

and

$$(\tfrac{1}{2}\cdot 2)(x) = \tfrac{1}{2}(22)$$

and

$$x = 11 \quad \text{true, also.}$$

Here we have used the multiplication property of equality. We multiplied by $\frac{1}{2}$. We could say that we multiplied "both sides" by $\frac{1}{2}$.

It is very easy to see that the number 11 makes "$x = 11$" true. We could now check to see if the number 11 makes the original sentence, "$2x + 5 = 27$" true also. Actually, however, provided we have made no mistakes in arithmetic, it is not necessary to do this. Let's see why.

We have just shown that if there is a number x that makes "$2x + 5 = 27$" true, then it also makes each of the following sentences true:

$(2x + 5) + (-5) = 27 + (-5)$	addition property of equality
$2x + \big((5) + (-5)\big) = 27 + (-5)$	associative property of addition
$2x = 22$	
$\frac{1}{2}(2x) = \frac{1}{2}(22)$	multiplication property of equality
$(\frac{1}{2}\cdot 2)(x) = \frac{1}{2}(22)$	associative property of multiplication
$x = 11$	

However, we could just as easily "go the other way". That is, we could start with the sentence "$x = 11$". If there is an x that makes "$x = 11$" true, then it also makes each of the following sentences true.

$2(x) = 2(11)$	multiplication property of equality
$2x = 22$	
$2x + 5 = 22 + 5$	addition property of equality
$2x + 5 = 27$	

In other words, any x that makes "$2x + 5 = 27$" true also makes "$x = 11$" true; <u>and</u> any x that makes "$x = 11$" true also makes "$2x + 5 = 27$" true. The two sentences have the <u>same truth set.</u> Since the truth set of "$x = 11$" is easy to see, we can find it instead of the truth set of "$2x + 5 = 27$".

Two open sentences with the same truth set are called

<u>equivalent sentences.</u>

In the example, "$2x + 5 = 27$" and "$x = 11$" are equivalent sentences. They have the same truth set. Since $\{11\}$ is the truth set of "$x = 11$", it is also the truth set of "$2x + 5 = 27$".

It is easy to form equivalent sentences. If the same real number is <u>added</u> to both sides of an open sentence, or if both sides are <u>multiplied</u> by the same real number <u>except zero,</u> the result is an open sentence equivalent to the original one.

Here is another example. Solve

$$3x + 7 = x + 15.$$

$$3x + 7 = x + 15$$

is equivalent to

$$(3x + 7) + (-7) = (x + 15) + (-7)$$

addition property of equality (We added (-7).)

and

$$3x + \big((7) + (-7)\big) = x + \big((15) + (-7)\big)$$

and

$$3x = x + 8.$$

This is equivalent to

$$3x + (-x) = x + 8 + (-x)$$

addition property of equality (We added $(-x)$.)

and

$$2x = 8.$$

This is equivalent to

$$\tfrac{1}{2}(2x) = \tfrac{1}{2}(8)$$

multiplication property of equality (We multiplied by $\frac{1}{2}$.)

and

$$x = 4.$$

Thus, we have a list of equivalent sentences. They all have the same truth set. It is easy to see that the truth set of "$x = 4$" is $\{4\}$. If we have not made a mistake in arithmetic somewhere down the line, $\{4\}$ is also the truth set of "$3x + 7 = x + 15$". Being human, we <u>do</u> sometimes make mistakes in arithmetic. So it is still a good idea to check to see if $\{4\}$ is the truth set of the original sentence. We'll leave this for you to do.

Here is a final example. Solve

$$7 + 3x + (-5) + 9x = 37 + 5x.$$

See if you can give the reason for each of the following steps.

$$\begin{aligned}
7 + 3x + (-5) + 9x &= 37 + 5x \\
12x + 2 &= 37 + 5x \\
12x + \big((2) + (-2)\big) &= 5x + \big((37) + (-2)\big) \\
12x &= 5x + 35 \\
12x + (-5x) &= 35 + \big((5x) + (-5x)\big) \\
7x &= 35 \\
\tfrac{1}{7}(7x) &= \tfrac{1}{7}(35) \\
x &= 5
\end{aligned}$$

Therefore, "$x = 5$" and "$7 + 3x + (-5) + 9x = 37 + 5x$" are equivalent sentences. They have the same truth set. $\{5\}$ is the truth set of "$x = 5$". We sometimes say that 5 is a <u>solution</u> of the sentence "$x = 5$".

As a guard against mistakes in arithmetic, let us check to see if 5 is a solution of "$7 + 3x + (-5) + 9x = 37 + 5x$".

If x is 5, then the left phrase is $7 + 3(5) + (-5) + 9(5)$, which is a name for 62. If x is 5, the right phrase is $37 + 5(5)$, also a name for 62. We have thus shown that if x is 5, then $7 + 3x + (-5) + 9x = 37 + 5x$ is a true sentence.

From these examples, you see that solving a sentence is something like a game. The rules of the game are just the properties of real numbers. We usually try to use the properties to get an equivalent sentence in which the variable stands "alone" as in "$x = 5$" and "$x = 11$" and "$x = 4$".

Problem Set 8-6

Find the truth set of each of the following open sentences by writing in each of the Problems 1 - 4 a series of sentences equivalent to the original sentence. Show that your solution is correct by reconstructing the original sentence by reversing the steps you took in solving the sentence.

1. $5x + (-4x) = 7$
2. $6c + 3c = 16 + 2$
3. $8r + (-2r) + 4 = 16$
4. $17 + (-5) = 4m$

Find the truth set of each of the following open sentences. Check to see if each element in the truth set you obtain does make the original sentence a true sentence.

5. $19 + (-4) = (-3m) + 5m$
6. $3m + (-6) + 2m = 9$
7. $2m + 5m + 16 = (-5)$
8. $(-4m) + (-6m) + (-10) = 15$
9. $a + (-6a) = 7 + (-2a)$
10. $12n + (-27) = 5n + (-4)$
11. $(-5b) + (-14) = 6 + (-10b)$
12. $(-8a) + (-12) = (-5a) + 5$
13. $6m + 9 = 5m + (-8)$
14. $10m + 8 = 9m + 9$
15. $17x = (-11) + 18x + (-12)$
16. $(-5) + 3x + \frac{1}{3}x = 5$
17. $(-6\frac{2}{3}) + 4x + 9\frac{4}{5} = 4x + 7\frac{1}{2}$
18. $(-a) + 2a + (\frac{27}{4}) = (-a) + \frac{13}{4}$
19. $(-5a) + (-\frac{21}{4}) = (-3a) + 5\frac{1}{4}$
20. $(-3y) + \frac{17}{3} + 2y = 5 + (-y) + \frac{2}{3}$
21. $\frac{5}{8} + \frac{1}{4}|y| = \frac{7}{8} - \frac{4}{16}$
22. $\frac{9}{16} + |y| = -\frac{3}{16}$

8-7. Products and the Number Zero.

What is a common name for each of the following products:

$$(7)(0), \qquad (0)(-3), \qquad (0)(0), \qquad (\tfrac{1}{2})(0)\ ?$$

Each of these products is a particular illustration of the multiplication property of zero---For any number x, $x(0) = (0)x = 0$.

Because of the multiplication property of zero, the following statements can be made about a product xy of real numbers x and y:

If $x = 0$, then $xy = 0$.	This is true, since $(0)y = 0$ for any number y.
If $y = 0$, then $xy = 0$.	This is true, since $x(0) = 0$ for any number x.

In the statements above, we have shown that if x is zero, or if y is zero, then we can draw a conclusion concerning the product xy---namely, the product xy is zero.

However, if it is known that the product xy is zero, then can any conclusion be drawn concerning the numbers x and y?

Suppose $xy = 0$.	
Also suppose $x \neq 0$.	
Let c be the multiplicative inverse of x.	Since $x \neq 0$, x has a multiplicative inverse.
If $xy = 0$, then $c(xy) = c(0)$	Multiplication property of equality
$(cx)y = 0$	
$(1)y = 0$	"1" may be used for "cx," because the product of any number and its multiplicative inverse is one.
$y = 0$.	

Thus, if $xy = 0$, and $x \neq 0$, then $y = 0$. In the same way, it can be shown that if $xy = 0$ and $y \neq 0$, then $x = 0$. Therefore, we can make the following statement:

If $xy = 0$, then $x = 0$ or $y = 0$.

It is, of course, clear that both x and y may be 0.

The following list of examples illustrate an important use of the statement above.

Example 1. Find the truth set of "$(7)(x) = 0$".

If $x = 0$, the sentence reads "$(7)(0) = 0$", which is true. Is there any other number that will make the sentence true?
The truth set is $\{0\}$.

Example 2. Find the truth set of "$(y)(8) = 0$".

If $y = 0$, the sentence reads "$(0)(8) = 0$", which is true. Is there any other number that will make the sentence true?
The truth set is $\{0\}$.

Example 3. Find the truth set of "$8\big(x + (-3)\big) = 0$".

Notice that, on the left side, we have the product of the number 8 and the number $x + (-3)$. The product will be zero if either of these numbers is zero. 8, of course, cannot be zero. However, if $x = 3$, $\big(x + (-3)\big)$ is the number zero; and the sentence reads "$8\big(3 + (-3)\big) = 0$", which is true. Is there any other number that will make the sentence true?
The truth set is $\{3\}$.

Example 4. Solve "$(x + 5)(x + 2) = 0$".

On the left side, we have the product of the numbers $(x + 5)$ and $(x + 2)$. If x is (-5), the sentence reads "$(0)(-3) = 0$", which is true. If x is (-2), the sentence reads "$(3)(0) = 0$", which is true.
The truth set is $\{-2, -5\}$.

Check Your Reading

1. State the multiplication property of zero.
2. If $x = 0$, what conclusion can be drawn concerning the product xy?
3. If $y = 0$, what conclusion can be drawn concerning the product xy?
4. If $xy = 0$ and $x \neq 0$, what conclusion can be drawn?
5. If $xy = 0$ and $y \neq 0$, what conclusion can be drawn?
6. If $xy = 0$, what conclusion can be drawn?
7. What is the truth set of "$(x + 5)(x + 2) = 0$"?

Oral Exercises 8-7

1. Which of the following sentences are true?

(a) $7(0) = 0$

(b) $(-5)(0) = -5$

(c) $(0)(-7) = 0$

(d) $(0)(0) = 0$

(e) $(\frac{2}{3})(0) = \frac{2}{3}$

(f) $(-\frac{5}{4})(0) = 0$

(g) $(2.5)(0) = 2.5$

(h) $(6.325)(0) = 0$

2. Find the truth sets of the sentences:

(a) $7y = 0$

(b) $-4m = 0$

(c) $7(x + 1) = 0$

(d) $-6(y + 3) = 0$

(e) $(m + 1)(m + 2) = 0$

(f) $(n + 4)(n + \frac{1}{2}) = 0$

Problem Set 8-7

1. Find the truth set of each of the following open sentences.

(a) $(5)(b) = 0$

(b) $(-7)(a) = 0$

(c) $(\frac{2}{3})(m) = 0$

(d) $(b)(-\frac{5}{6}) = 0$

(e) $(3.2)(c) = 0$

(f) $(-4.34)(b) = 0$

(g) $0(x) = 0$

(h) $(-5)y = 0$

Problem Set 8-7
(continued)

2. Find the truth set of each of the following open sentences.

(a) $\left(x + (-4)\right) = 0$

(b) $5\left(x + (-3)\right) = 0$

(c) $-2\left(a + (-7)\right) = 0$

(d) $5(m + 2) = 0$

(e) $8(b + 6) = 0$

(f) $-6(a + 1) = 0$

(g) $(n + \frac{2}{3}) = 0$

(h) $-2(n + .8) = 0$

(i) $-1\left(m + (-1)\right) = 0$

(j) $-9\left(b + (-\frac{5}{4})\right) = 0$

(k) $7(\frac{7}{8} + a) = 0$

(l) $3\left(\frac{5}{6} + (-a)\right) = 0$

(m) $-\frac{4}{5}(\frac{3}{4} + m) = 0$

(n) $\left(.91 + (-r)\right) = 0$

3. Find the truth set of each of the following.

(a) $(x + 4)(x + 3) = 0$

(b) $(m + 2)(m + 6) = 0$

(c) $\left(a + (-5)\right)(a + 6) = 0$

(d) $(b + 10)\left(b + (-5)\right) = 0$

(e) $\left(n + (-3)\right)\left(n + (-5)\right) = 0$

(f) $(n + 9)\left(n + (-9)\right) = 0$

(g) $(a + \frac{2}{3})\left(a + (-\frac{1}{8})\right) = 0$

(h) $\left(m + (-\frac{1}{2})\right)\left(m + (-\frac{5}{6})\right) = 0$

(i) $\left(b + (-\frac{3}{4})\right)(b + \frac{9}{12}) = 0$

(j) $(a + 3.4)(a + 2.18) = 0$

(k) $(c + 5.15)\left(c + (-3.12)\right) = 0$

(l) $\left(y + (-1.75)\right)\left(y + (-2.25)\right) = 0$

(m) $n(n + 2) = 0$

(n) $a\left(a + (-3)\right) = 0$

(o) $b(b + \frac{4}{5}) = 0$

(p) $c\left(c + (-\frac{7}{3})\right) = 0$

(q) $\left(m + (-1)\right)m = 0$

(r) $n(n + 1) = 0$

Problem Set 8-7
(continued)

*(s) $(w + 6)(2w + 5) = 0$

*(t) $(m + 5)(\frac{1}{2}m + 4) = 0$

*(u) $\left(1.5n + (-6)\right)\left(.5n + (-2)\right) = 0$

*(v) $\left(\frac{2}{3}a + (-1)\right)(\frac{3}{4}a + \frac{3}{8}) = 0$

*(w) $(\frac{5}{8}m + 1)(\frac{1}{12}m + 2) = 0$

4. Translate the following into open sentences and find their truth sets.

(a) Mr. Johnson bought 30 feet of wire and later bought 55 more feet of the same kind of wire at the same price per foot. He found that he paid $4.20 more than his neighbor paid for 25 feet of the same kind of wire at the same price per foot. What was the cost per foot of the wire?

(b) Four times an integer is ten more than twice the successor of that integer. What is the integer?

(c) In a stock car race, one driver, starting with the first group of cars drove for 5 hours at a certain speed and was then 120 miles from the finish line. Another driver, who set out with a later heat, had traveled at the same rate as the first driver for 3 hours and was 250 miles from the finish. How fast were these men driving? Draw a diagram to help you write the open sentence.

(d) The perimeter of a triangle is 44 inches. The second side is three inches more than twice the length of the third side, and the first side is five inches longer than the third side. Find the lengths of the three sides of this triangle.

(e) If an integer and its successor are added, the result is one more than twice that integer. What is the integer?

Problem Set 8-7
(continued)

(f) In a farmer's yard were some pigs and chickens, and no other creatures except the farmer himself. There were, in fact, sixteen more chickens than pigs. Observing this fact, and further observing that there were 74 feet in the yard, not counting his own, the farmer exclaimed happily to himself--for he was a mathematician as well as a farmer, and was given to talking to himself--"Now I can tell how many of each kind of creature there are in my yard." How many were there? (Hint: Pigs have 4 feet, chickens 2 feet.)

(g) At the target shooting booth at a fair, Montmorency was paid 10¢ for each time he hit the target, and was charged 5¢ each time he missed. If he lost 25¢ at the booth and made ten more misses than hits, how many hits did he make?

Summary.

In this chapter, we have discussed multiplication of real numbers. The product of two real numbers a and b may be written in any of the following ways:

$$ab, \qquad (a)(b), \qquad a \cdot b .$$

We stated a definition for the product of any two real numbers. This definition assures us that the product of any two real numbers is also a real number

Here are some properties of multiplication:

<u>Multiplication Property of One</u>
For any real number a, $(a)(1) = a$.

<u>Multiplication Property of Zero</u>
For any real number a, $(a)(0) = 0$.

<u>Commutative Property of Multiplication</u>
For any real numbers a and b, $ab = ba$.

Associative Property of Multiplication
For any real numbers a, b, and c, $(ab)c = a(bc)$.

Distributive Property
For any real numbers a, b, and c, $a(b + c) = ab + ac$.

Multiplication Property of Equality
For real numbers a, b, and c, if $a = b$, then $ac = bc$.

For any real number a, $(-1)a = -a$.

For any real numbers a and b, $(-a)b = -ab$
and $(-a)(-b) = ab$.

For any real numbers a and b, $-(a + b) = (-a) + (-b)$.

If $xy = 1$, x is called the multiplicative inverse of y, and y is called the multiplicative inverse of x. The number 0 has no multiplicative inverse. Every real number except 0 has one and only one multiplicative inverse.

Two open sentences that have the same truth set are called equivalent sentences.

If $x = 0$, then $xy = 0$.
If $y = 0$, then $xy = 0$.

If $xy = 0$, then $x = 0$ or $y = 0$.

Review Problem Set

1. Find a common name for each of the following sums.

   (a) $7 + (-10)$
   (b) $12 + 12$
   (c) $(-10) + (-15)$
   (d) $6 + (-16)$
   (e) $a + (-a)$
   (f) $(-2a) + 2a$

2. Find a common name for each of the following products.

   (a) $(2)(4)$
   (b) $(2)(-8)$
   (c) $(-6)(-5)$
   (d) $(-4)(6m)$
   (e) $(-8m)(-3a)$
   (f) $(-2m)(2.5m)$

Review Problem Set
(continued)

(g) $(-\frac{2}{3}m)(-\frac{3}{4}y)$

(h) $(-1)(\frac{4}{5}x)(-\frac{2}{3}y)$

(i) $(-3)(-4a)(2b)$

(j) $(-2m)(12b)(\frac{1}{2})$

(k) $(8a)(7\frac{1}{2})(-\frac{3}{4}a)$

(l) $(19m)(-1.75n)(-2.25a)(0)$

(m) $(2x)(-4a)(-6am)(-\frac{5}{4})$

3. Use the distributive property to change the form of each of the following.

(a) $2(a + 2b)$

(b) $4\big((-c) + 4d\big)$

(c) $-6\big((-7c) + 6d\big)$

(d) $8a\big((-4m) + (-3n)\big)$

(e) $(-7a)\big((-3b) + (-4c)\big)$

(f) $(-4a)\big((-4a) + (\frac{1}{2}b)\big)$

(g) $(-1)\big((-3a) + 6b\big)$

(h) $(a)\big(4m + (-2n)\big)$

(i) $(-b)\big((-2c) + (-3d)\big)$

(j) $(-m)\big((-5m) + 10n)\big)$

(k) $(-2c)\big((-5c) + (-10d)\big)$

(l) $\big((-5m) + 8b\big)(-3b)$

(m) $\big(4a + (-4c)\big)(-d)$

(n) $(a + 3)(a + 4)$

(o) $\big(a + (-3)\big)(a + 4)$

(p) $(m + 6)\big(m + (-6)\big)$

(q) $\big(y + (-2)\big)\big(y + (-9)\big)$

(r) $(y + \frac{1}{2})(y + \frac{3}{4})$

(s) $\big(z + (-5)\big)(z + 5)$

(t) $\big(w + (-\frac{3}{4})\big)\big(w + (-\frac{3}{4})\big)$

Review Problem Set
(continued)

(u) $(a + 1.5)(a + 1.5)$
(v) $\big(b + (-2.1)\big)(b + 2.1)$
(w) $(-2y)\big((-3)y + 2z + 4\big)$
*(x) $\big((-3y) + 4\big)\big(2y + (-3)z + 4\big)$
*(y) $\big((-2m) + (-1)\big)\big((-4)m + (-n) + 1\big)$

4. Use the distributive property to collect terms in each of the following.

(a) $3x + 10x$
(b) $(-9)a + (-4)a$
(c) $11k + (-2)k$
(d) $(-27)b + 30b$
(e) $17n + (-16)n$
(f) $x + 8x$
(g) $(-15)a + a$
(h) $\frac{7}{8}a + \frac{9}{8}a$
(i) $5p + 4p + 8p$
(j) $7x + (-10)x + 3x$
(k) $12a + 5c + (-2)c$
(l) $6a + 4b + c$
(m) $9p + 4q + (-3)p + 7q$
(n) $6p + (-4r) + (-8p) + (-2r)$
(o) $3a + (-9b) + 5a + (-8a)$
(p) $(-5m) + (-6c) + (3m) + (6c) + (2m)$
(q) $r + 2t + (-r) + 5s$
(r) $\frac{5}{2}a + \frac{5}{6}b + (-\frac{3}{2}a) + \frac{1}{3}b$
(s) $(-\frac{7}{3})m + \frac{2}{3}n + (-\frac{5}{3})m + \frac{1}{3}n + a$
(t) $\frac{14}{3}a + 4b + \frac{7}{3}a + (-4)b$

Review Problem Set
(continued)

5. Use the multiplication property of <u>one</u> to simplify each of the following.

(a) $\frac{3}{4} + \frac{4}{5}$

(b) $\frac{5}{6} + \frac{7}{8}$

(c) $\frac{3}{2} + \frac{3}{5}$

(d) $\dfrac{\frac{5}{8}}{\frac{2}{3}}$

(e) $\dfrac{\frac{9}{5}}{\frac{2}{3}}$

(f) $\dfrac{3\frac{1}{4}}{2\frac{2}{5}}$

6. Use the distributive property to write the following indicated sums as indicated products.

(a) $3a + 3b$

(b) $(-5)c + (-5)d$

(c) $10m + 5n$

(d) $(-10a) + (-15b)$

(e) $mn + nay$

(f) $-mx + 2my$

(g) $2bx + 4by$

(h) $4am + 6an$

(i) $(-6)bx + (-9)bw$

(j) $\frac{2}{3}at + \frac{4}{3}bt$

(k) $(-\frac{5}{8})b + (-\frac{5}{8})c$

(l) $2.5m + 5.0n$

7. Which of the following are true? Which are false?

(a) $(2)(5 + 3) = (5 + 3)(2)$

(b) $(-1)(2) = 2$

(c) $(-1)(-3) = 3$

(d) $(-1)(0)(-4) = 4$

(e) $(-2.7)(-1)(3.95)(0) = 6.65$

(f) $(-5)(1) > 5$

(g) $(6)(-3) \neq 18$

(h) $6(-3) > 18$

(i) $6(-3) < 18$

Review Problem Set

(continued)

8. Which of the following sentences are true for all values of the variables?

(a) $a\big(b + (-c)\big) = \big((-c) + b\big)a$

(b) $2a + (-b) + 3c = (-b) + 3c + 2a$

(c) $5m + (-5)n = 5(m + n)$

(d) $(-1)(c) = c$

(e) $(b)(-1)(0) = 0$

(f) $(-m)(-1) = m$

(g) $(0)(m) = m$

(h) $(3.25)(-1)(-t) = 3.25t$

(i) $(0)m < 0$

(j) $a(-1) \neq 1$

(k) $2a + (-5a) \neq 3a$

(l) $10m + (-12n) = 2\big(5m + (-6n)\big)$

9. Which sentences are true for the given value of the variable?

(a) $2m = -10; \quad m = 5$

(b) $3m = 1; \quad m = 0$

(c) $\frac{1}{3}a = 3; \quad a = 9$

(d) $\frac{3}{4}a = 0; \quad a = 0$

(e) $-\frac{2}{3}a = 1; \quad a = \frac{3}{2}$

(f) $\frac{5}{6}a = 1; \quad a = \frac{6}{5}$

Review Problem Set

(continued)

(g) $2a + 4 = (-2)$; $a = (-1)$

(h) $\frac{3}{2}m + 6 = (-6)$; $m = \frac{2}{3}$

(i) $-\frac{7}{8}m + (-3) = (-4)$; $m = (-\frac{7}{8})$

(j) $(-2a)(-3) \neq 6$; $a = 1$

(k) $2(-2.4m) > (-9.6)$; $m = (-4.8)$

(l) $4|x| < 0$; $x = -2$

(m) $|m| + (-2) = 0$; $m = -2$

10. Find the truth set of each of the following sentences.

(a) $6x + 9x = 30$

(b) $12y + (-5y) = 35$

(c) $(-3)a + (-7)a = 40$

(d) $x + 5x = 3 + 6x$

(e) $3y + 8y = -99$

(f) $-15z + 12z = 24$

(g) $14x + (-14)x = 15$

(h) $(-3)a + 3a = 0$

(i) $13k + (-14)k + 9k = 0$

(j) $x + 2x + 3x = 42$

(k) $a + 2a = 3$

(l) $(-8)y + 9y \geq 5$

(m) $7a + (3)a < 10$

(n) $|x| + 3|x| < 0$

(o) $-|m| + 2|m| > 5$

(p) $4y + 3 = 3y + 5 + y + (-2)$

(q) $12x + (-6) = 7x + 24$

(r) $8x + (-3)x + 2 = 7x + 8$ (Collect terms first.)

(s) $6z + 9 + (-4)z = 18 + 2z$

(t) $12n + 5n + (-4) = 3n + (-4) + 2n$

(u) $15 = 6x + (-8) + 17x$

(v) $5y + 8 = 7y + 3 + (-2y) + 5$

Review Problem Set
(continued)

(w) $7(x + 4) = 0$

(x) $8x\big(x + (-1)\big) = 0$

(y) $(m + 7)\big(2m + (-4)\big) = 0$

*(z) $(2y)(y + 3)(2y + 3) = 0$

11. Write open sentences for each of the following, then find the truth set.

(a) The sum of twice a number and 5 is 47. What is the number?

(b) A farmer had his wheat hauled to an elevator in two trucks, with the same capacity, which carried a full load each trip. He stored 490 bushels in the elevator. One truck made 3 trips, the other made 4.

(1) How much grain did each truck hold?

(2) How many bushels did each truck haul altogether?

(c) Mrs. Abbott tried to remember how much she paid per can for two cans of peaches she bought when she shopped for groceries. She could only remember that she had bought a can of coffee which cost 83¢ and that she received 4¢ change from $1.50 which she gave the clerk. How much did Mrs. Abbott pay for each can of peaches? (Disregard the sales tax.)

(d) One angle of a triangle is twice as large as the second. The third angle contains 12° more than the smaller of the first two angles. How many degrees are in each angle?

(e) A freight train and a passenger train, running on a double track, left Washington, D. C. for New York. The freight train left at 6:00 A.M. and traveled at an average speed of 40 miles per hour. The passenger train left at 7:00 A.M. and averaged 60 miles per hour.

Review Problem Set
(continued)

(1) How long will it take the passenger train to overtake the freight?

(Hint: If t is the number of hours that the passenger train ran before overtaking the freight train, then $(t + 1)$ is the time that the freight train ran before it was overtaken by the passenger train.)

(2) At what time did the passenger train overtake the freight?

(3) How far were they from Washington when the passenger train overtook the freight?

(f) The length of a rectangular flower bed is 8 feet more than twice the width. Its perimeter is 196 feet. What are its dimensions?

12. Draw the graphs of the truth sets of the following sentences.

(a) $|x| > 5$

(b) $|y| < 0$

(c) $x < 2$ and $x > -1$

(d) $x + 1 = 5$ or $x + 1 = 4$

(e) $x \leq 3$ and $x > 0$

(f) $x \neq 5$

Chapter 9

PROPERTIES OF ORDER

9-1. The Order Relation for Real Numbers.

Suppose you were asked to name the letters of the alphabet. Would you be apt to say, "c, f, y, s, q, t, k, etc."? This is very unlikely. In all probability you would begin, "a, b, c, d, e, f, g, etc." So would everybody else.

There must be a reason for this. It is because you are in the habit of using what we call an order relation. Whenever you look up a word in the dictionary, or a topic in the index of a book such as this one, you will be taking advantage of an order relation; that is, alphabetical order. The concept of order occurs in a variety of situations. For example a baseball team has a certain batting order. Can you think of other instances?

As you have already seen, the idea of order plays an important part in the study of numbers. You will remember the type of sentence which we wrote, for example, as

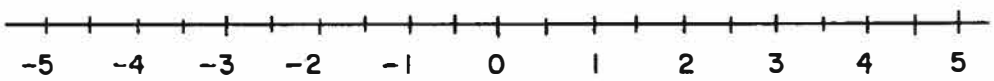
$$4 < 7.$$

Here the symbol "<", meaning "is less than", shows a relation between 4 and 7. We call a relation of this type an order relation.

In Chapter 6 the relation of order was extended from the numbers of arithmetic to the real numbers, which include the negative numbers as well as the positive numbers and zero. This was done by using the number line. We agreed that:

"is less than" for real numbers means

"is to the left of" on the real number line.



Thus, by referring to the above line we see that -3 is less than -1; -4 is less than 2, etc. It should be clear that the following sentences are also true.

$$-2 < 2. \qquad -1 < 0. \qquad -4 < -3.$$

Write down any real number at all! Feel free to choose a negative number, or a number in the form of a fraction, whatever you wish. Without saying what your number is, ask a friend to write any real number. Call **your number a, and your friend's** number b. Now look at both numbers. Then state which of the following is a true sentence:

$$a < b$$
$$a = b$$
$$b < a$$

You have undoubtedly found that one, and only one, of the sentences is true, and that the other two sentences are false. Now no matter how many times you repeat the experiment with other choices of numbers, you will find that the same situation occurs. It will always happen that <u>one</u> of the sentences (not the same one every time, of course) is true, and that the other two are false. Try the experiment several times. Make a record each time as to which sentence is true and which two sentences are false.

The fact that it always happens that for any real number a and any real number b one of the sentences

$$a < b$$
$$a = b$$
$$b < a$$

is true and the other two are false is an important property which is called the

<u>comparison property of order.</u>

We shall now discover another property. Suppose we are given any three different real numbers a, b, and c. Assume it is known that

$$a < b.$$

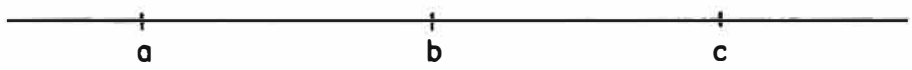
Suppose we also know that

$$b < c.$$

What is the relation between a and c?

Let us examine the number line. The following illustration represents the two conditions

$$a < b \quad \text{and} \quad b < c.$$



In this drawing, a is clearly to the left of c. By definition this represents the relation

$$a < c.$$

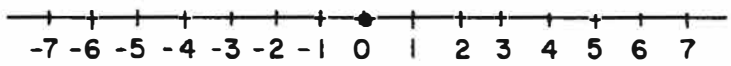
This is an example of the application of a property of the order relation "is less than". It is called the

<u>transitive property of order.</u>

We can restate the property as follows:

> If a, b, and c are any three real numbers and if $a < b$ and $b < c$, then $a < c$.

In Chapter 6 we learned of another property which connects the order relation with the operation of taking opposites. A portion of the number line will help us review this property.



It is certainly clear that

$$2 < 5.$$

But what about the relation between -2 and -5? The number line tells us that

$$-5 < -2.$$

Let's look at -1 and 4. Certainly

$$-1 < 4.$$

Now let's take opposites. The opposite of -1 is 1. The opposite of 4 is -4. The order relation between these two is

$$-4 < 1.$$

Finally, consider -7 and -3. We see that

$$-7 < -3$$

and

$$3 < 7$$

where we have again taken opposites.
The property which these examples have illustrated should now be clear. It can be stated as follows:

> If a and b are any real numbers such that $a < b$, then $-b < -a$.

Check Your Reading

1. What meaning on the number line is associated with "is less than"?

2. What is the order of a and b if it is false that "$a \geq b$"?

3. If the order of a and b is given by "$a < b$", what is the order of $-a$ and $-b$?

Oral Exercises 9-1

1. If $a < -2$ and $-2 < 4$, what is the order of a and 4?
2. If $-3 < c$ and $c < d$, what is the order of -3 and d?
3. If $x + (-2) < 5$ and $5 < x + 2$, then what is the order of $x + (-2)$ and $x + 2$?
4. If $y + (-3) < 0$ and $0 < y + 1$, then what is the order of $y + (-3)$ and $y + 1$?
5. If $m + (-1) < 5$, what is the order of $-\big(m + (-1)\big)$ and -5?
6. If $t + (-3) < 6$, what is the order of $3 + (-t)$ and -6?
7. If "$5 < m$" is false and "$m < 5$" is false, what is the relationship between m and 5?
8. If $b < -1$ what is the order of b and 5?
9. If $4 > a$, what is the order of $-a$ and -4?
10. If $c > 0$, what is the order of 0 and $-c$?

Problem Set 9-1

1. In each of the following determine the order relation between the two numbers.

   (a) -5, -2

   (b) $-\frac{3}{2}$, $-\frac{4}{3}$

   (c) -5, $.01$

   (d) $\frac{5}{16}$, $.3124$

   (e) a, b if a is positive and b is negative

   (f) x, $x + 1$ for all real numbers x

Problem Set 9-1
(continued)

2. In each of the following determine the order of the two numbers.

   (a) $-a, -2$ if $2 < a$

   (b) $b, 2$ if $2 < a$ and $a < b$

   (c) $-x, 3$ if $x > -3$

   (d) $y, 2$ if it is false that "$y \geq 2$"

   (e) $-|a|, 0$ if $0 < |a|$

   (f) $1, x^2$ if $1 < x$ and $x < x^2$

3. In each of the following determine the order relation between the two numbers.

   (a) $x + (-1), 3$ if $-3 < 1 + (-x)$

   (b) $0, z$ if $0 < y$ and $y < z$

   (c) $2, m$ if it is false that "$m \leq 2$"

   (d) $-(a + b), b + (-a)$ if $a + b > a + (-b)$

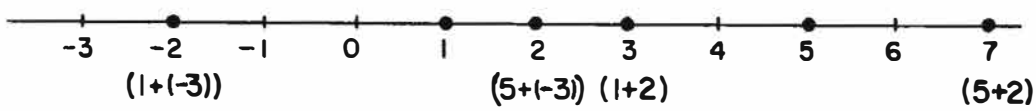
   (e) $-|-3|, -2$

   (f) $-a, -b$ if $a = b$

9-2. Addition Property of Order

We have just seen that there is a definite connection between the order relation and the operation of taking opposites. We now wish to study the connection between order and the operation of addition.

As before it will be helpful to use the number line.



To begin, consider the numbers 1 and 5. On the number line it is evident that

$$1 < 5$$

It should also be clear that

$$(1 + 2) < (5 + 2).$$

What about the relation between $1 + (-3)$ and $5 + (-3)$?

The number line tells us that

$$\big(1 + (-3)\big) < \big(5 + (-3)\big).$$

In both cases the numbers remain the same distance apart. Suppose we had chosen two other numbers to begin with rather than 1 and 5, say -2 and 4, such that the first number is still less than the second, i.e.

$$-2 < 4.$$

What if we now add -4 to each number? We see that

$$\big(-2 + (-4)\big) < \big(4 + (-4)\big)$$

since the number line tells us that

$$-6 < 0.$$

These examples illustrate what we call the

<u>addition property of order</u>

This property states that

if a, b, and c are real numbers and
if $a < b$, then $(a + c) < (b + c)$.

It is interesting to see that we can combine the addition property and the transitive property to prove a third property. Suppose we have four real numbers a, b, c, and d, which are related as follows

$$a < b \quad \text{and} \quad c < d.$$

We would like to find out the relation between $(a + c)$ and $(b + d)$. This can be done in the following way. By the addition property we know that since

$$a < b, \quad \text{then} \quad (a + c) < (b + c) .$$

In the same way we know that since

$$c < d, \quad \text{then} \quad (c + b) < (d + b) .$$

By the commutative property this last inequality can be written as

$$(b + c) < (b + d) ,$$

Therefore since we know that

$$(a + c) < (b + c) \quad \text{and} \quad (b + c) < (b + d)$$

the transitive property tells us that

$$(a + c) < (b + d).$$

Check Your Reading

1. State the addition property of order.
2. Is the statement "If $a < b$, then $a + c < b + c$" true when $c = 0$? Why? If c is negative is the statement still true?
3. If a is -3 and b is 1, state an order relationship involving a and b. If c is 2, state an order relationship involving $(a + c)$ and $(b + c)$.
4. If $a < b$ and $c < d$, what is the order of $a + c$ and $b + d$?

Oral Exercises 9-2a

1. Which of the following statements are true?

   (a) If $m < n$, then $m + 2 < n + 2$.

   (b) If $-2 < 5$, then $-2 + n < 5 + n$.

Oral Exercises 9-2a
(continued)

(c) If $-5 < a$, then $a > -5$.

(d) If $a < b$ and $c < d$, then $a < d$.

(e) If a and b are different numbers, then exactly one of the following is true: $a < b$ or $b < a$.

(f) If $m < n$ and $p < m$, then $p < n$.

(g) If $-a < b$, then $-b < a$.

(h) If $-a + 5 > b + 5$, then $-b > a$.

(i) If $m < -n$, then $-m < n$.

2. (a) Does the relation indicated by the symbol "=" have the transitive property? Give an example.

(b) Does the relation indicated by the symbol ">" have the transitive property? Give an example.

(c) Does the relation indicated by the symbol "≠" have the transitive property? Give an example. to support your answer.

(d) Restate the comparison property so that it describes the relationship of two different numbers.

3. In each of the following indicate which symbol, "<" or ">", should be put in the place occupied by the question mark, so that the resulting statement is true.

(a) -3 ? -5

(b) If $m < n$ and n ? x, then $m < x$.

(c) If $-3 < a$, then 3 ? $-a$.

(d) If $m < n$, then $-n$? $-m$.

(e) If $a + 5 < b + 5$, then a ? b.

(f) If $a + (-5) < b + (-5)$, then $-a$? $-b$.

(g) If $a < b$, then $a + 5 < b + 5$ and $a + 5 + (-5)$? $b + 5 + (-5)$.

Oral Exercises 9-2a
(continued)

(h) If $x + 5 < 2$, then $x + 5 + (-5)$? $2 + (-5)$.

(i) If $5 + m$? $3m$, then $5 + m + (-m) < 3m + (-m)$.

*(j) If $(-a) < b + 5$ and $c < (-a)$, then $c + (-5)$? b.

Problem Set 9-2a

1. Write each of the following statements with either ">" or "<" replacing the question mark so that the resulting statement is true.

   (a) If (-5) ? a, then $5 > (-a)$.

   (b) If $b < 4$, then $b + 5$? $4 + 5$.

   (c) If $b + 5 < (-7)$, then $b + 5 + (-5)$? $(-7) + (-5)$.

   (d) If $b < 3$, then $-(b + 2)$? $-(3 + 2)$.

   (e) If $a < b$ and $b < c$, then $a + 2$? c.

   (f) If $b + 5 < (-7)$, then b ? (-12). (see part (c))

   (g) If $a < -3$ and $0 < b$, then $a + 2$? b.

   (h) If $a = b$, then $a + c$? b.

   (i) If $3 < x$, and if $3 + m = x$, then 0 ? m.

   (j) x ? a if $x < b$ and $b < a$.

2. Which of the following sentences are true? Which are false? (Hint: the addition property of order, if wisely used may help.)

   (a) $3 + 4 < \frac{12}{4} + 4$

   (b) $(-6) + 7 < (-3) + 7$

   (c) $5 + (-\frac{24}{25}) < 3\frac{1}{8} + (-\frac{24}{25})$

   (d) $\frac{15}{16} + (-\frac{3}{8}) < (-\frac{3}{8}) + \frac{30}{31}$

Problem Set 9-2a
(continued)

(e) $3 + (-12) + 47 < 47 + (-18) + 9$

(f) $18 + (-432) + (-79) < 24 + (-432) + (-79)$

(g) $(-273) + 1\frac{3}{8} + (-382) < 1\frac{3}{8} + (-114) + (-382)$

(h) $(-\frac{5}{3})(\frac{6}{5}) + (-5) < (-\frac{5}{2}) + (-5)$

(i) $(-5.3) + (-2)(-\frac{4}{3}) < (0.4) + \frac{8}{3}$

(j) $(\frac{5}{2})(-\frac{3}{4}) + 2 \leq (-\frac{15}{8}) + 2$

We can use the addition property of order in problems involving truth sets for open sentences. For example suppose we were asked to find the truth set of the sentence

$$x + 5 < 8.$$

If there is a real number x for which our sentence is true, then for this x the following sentence is also true:

$$x + 5 + (-5) < 8 + (-5).$$

This follows from the addition property of order. This sentence can be written

$$x < 3.$$

The truth set for the sentence "$x < 3$" can be described quite easily. It is the set of <u>all real numbers less than 3</u>.

We have shown that any number x which makes the sentence "$x + 5 < 8$" true also makes the sentence "$x < 3$" true. Now we must check to see that any real number x which makes "$x < 3$" true also makes the sentence "$x + 5 < 8$" true. Once again we can use the addition property. If there is a real number x for which

$$x < 3,$$

then for this x $\quad x + 5 < 3 + 5$

or $\quad x + 5 < 8.$

The "check" which we used for the problem is not strictly necessary. In Chapter 8 it was stated that if a real number is added to both sides of an open sentence, where the open sentence expresses equality, then the resulting open sentence will be equivalent to the first one. In the same way it can be shown that this is true for open sentences which involve the relation of inequality. Thus, the two sentences

$$x + 5 < 8$$

and

$$x < 3$$

are equivalent. As you recall, this means that they have the same truth set. Therefore, if we can determine the truth set of "$x < 3$", we can be sure that this is also the truth set of "$x + 5 < 8$".

Can you see that the truth set of

$$x < 10$$

is the same as the truth set of

$$x + 7 < 17?$$

What is this truth set?

As another example, let us find the truth set of the sentence

$$5x + 9 < 4x + 3.$$

If there is a real number x for which this is true, then we can add (-9) to both sides and obtain

$$5x + 9 + (-9) < 4x + 3 + (-9).$$

By addition we get $5x + 0 < 4x + (-6)$.

If we now add $(-4x)$ to both sides the sentence becomes

$$5x + (-4x) < 4x + (-4x) + (-6),$$

which is the same as

$$x < (-6).$$

The last sentence is equivalent to our first one since we added the real numbers $(-4x)$ and (-9) to both sides.
Thus the truth set of

$$x < (-6)$$

is the same as the truth set of

$$5x + 9 < 4x + 3.$$

This is the set of <u>all real numbers less than -6</u>. Draw its graph.

Check Your Reading

1. What number could we "add to both sides of the sentence" $x + 5 < 8$ to help us find the truth set of the sentence? What is the truth set of the sentence?
2. Are the sentences "$x < 3$" and "$x + 5 < 8$" the same sentence? Are they equivalent sentences? What do we mean by equivalent sentences? How can we tell if two sentences are equivalent?
3. When finding the truth set of the sentence $5x + 9 < 4x + 3$, why do we add $(-4x)$ to both sides of the sentence? Why do we add (-9) to both sides of the sentence? What properties are we using when we do this? What is a simpler name for $\left(5x + (-4x)\right)$? What property do we use to obtain it?
4. How would you draw the graph of the truth set of the sentence "$5x + 9 < 4x + 3$"?

Oral Exercises 9-2b

1. Do you agree with the following statements?
   (a) If we are finding the truth set of $3x + 11 < 12$, a simpler sentence is formed if we add $(-3x)$ to both sides.

Oral Exercises 9-2b
(continued)

(b) A good way to begin to find the truth set of the sentence $(-5) + 3x + 4 < 2x + 8$ would be to write the equivalent sentence
$5 + (-5) + 3x + 4 + (-4) < 2x + 8 + 5 + (-4)$.

(c) A helpful step in finding the truth set of the sentence $3x < 2x + 9$ would be to add $(-2x)$ to both sides.

2. For each of the following sentences use the addition property of order to obtain an equivalent sentence having the variable or a product containing the variable, alone on one side:

(a) $3 + x < (-4) + 2$

(b) $2n + (-8) < (-27)$

(c) $(-8) + 12 < (-3n) + 4$

(d) $7 + (-\frac{3}{2}) + 2 < \frac{8}{3} + 2x$

(e) $.8 + 14 + (-\frac{2}{3}) < 3y + (-\frac{1}{2}) + y$

3. For each of the following sentences the simplest equivalent sentence would be one which would have the variable alone on one side and a specified number on the other side. Tell how you would use the addition property of order to achieve this in each case.

(a) $3 + 2x < x + (-2)$

(b) $4y + (-4) < 2 + 3y + (-3)$

(c) $(-4n) + 14 < 7 + (-3n) + (-7)$

(d) $\frac{y}{2} + \frac{4}{3} < 1\frac{3}{4} + (-\frac{y}{2})$

(e) $.7 + 3.2y + (-.4) < (-.4) + .7 + 2.2y$

Problem Set 9-2b

1. Find the truth set of each of the following sentences.

   (a) $(-5) + x + 4 < 7$

   (b) $3x < 2x + 2$

   (c) $\frac{1}{2} + n + (-\frac{3}{2}) < \frac{4}{3} + (-\frac{7}{3})$

   (d) $2y + 2 < 3y + 2$

   (e) $\frac{3}{5} + (-\frac{3}{10}) < x + (-\frac{4}{5})$

   (f) $\frac{4}{3} + 2x < 3x$

   (g) $3x + (-4) + \frac{1}{2} \leq (-8) + 2x + \frac{1}{2}$

   (h) $\frac{5}{3} + x \leq (-\frac{2}{3}) + 2x$

   (i) $3y + |-3| < 3y + (-2)$

   (j) $(-x) + 7 < \frac{1}{2} + (-3)$

   (k) $(-x) + 4 < (-3) + |-3|$

   (l) $(-5) + (-x) < \frac{2}{3} + |-\frac{4}{3}|$

   (m) $(-2) + 2x < (-3) + 3x + 5$

   (n) $x + |-\frac{3}{2}| \geq (-\frac{3}{4}) + \frac{5}{4}$

   (o) $5x + 2 = 2x + (-7)$

   (p) $(-7x) + |-5| = 4x + (-6)$

   (q) $(-4) + (-2x) + (-7) < (-3x) + 4$

   (r) $\frac{1}{3} + y + 2 \leq \frac{5}{2} + 2 + y$

   (s) $(-5x) + (-4) = (-8x) + (-16)$

   (t) $|-5|x + 2.5 < |-7.5| + 4x$

2. Draw the graphs of the truth sets of parts (a), (d), (g), (i), (p), (r) in Problem 1.

Problem Set 9-2b
(continued)

3. Which of the following statements are true? Which are false?

   (a) If $b < 0$, then $3 + b < b$.

   (b) If $b < 0$, then $3 + b < 3$.

   (c) If $b < 0$, then $3 + b + (-4) < 3 + (-4)$.

4. Three more than five times a number is greater than seven increased by four times the number. What set of numbers will make this sentence true?

5. A student has test grades of 82 and 91. What must he score on a third test to have an average higher than 90?

There is a strong connection between the relation "is less than" and the idea of equality. This connection can be seen if we consider a few examples. Consider the number 2 and the number 7. It is clear that

$$2 < 7,$$

since the number 2 is located to the left of the number 7 on the number line. But suppose you were asked to give another reason why it is clear that 2 is less than 7 without using the number line. We might say the following: The number 2 is less than 7 <u>because there is a number which we have to add to 2 in order to get 7</u>. Evidently the number is 5, since

$$2 + 5 = 7.$$

Notice that the number 5, which we added, is a <u>positive</u> number Thus, we see that there is a close connection between the sentence

$$2 < 7$$

and the sentence

$$2 + 5 = 7.$$

Both of these sentences bring out the order relation between 2 and 7. Let us consider some examples given by the following table:

Sentence involving "<"	Sentence involving "="
$2 < 6$	$2 + 4 = 6$
$(-2) < 6$	$(-2) + 8 = 6$
$(-10) < (-3)$	$(-10) + 7 = (-3)$
$\frac{1}{2} < 4$	$\frac{1}{2} + 3\frac{1}{2} = 4$

The sentences on the left have been changed to the sentences on the right by adding a positive number to the left side in every case. In other words if we have a sentence involving the order "is less than", then there seems to be a positive number which if added to the left side makes this into a sentence involving "equals".

The two kinds of sentences which we have been considering are frequently given special names. The type involving the "=" symbol is called an <u>equation.</u> Sentences with the symbols "<" or ">" are called <u>inequalities.</u> Thus "$2 < 6$" is an example of an inequality. The sentence "$2 + 4 = 6$" is an equation.

The examples also show that a certain type of equation involving addition implies an order relation. For instance, when we examine the true sentence $2 + 4 = 6$, we can see that a positive number, 4, has been added to 2 to make the sum, 6, so we know that 2 is less than 6. This connection between the two types of sentences can be stated as a property. To show the connection completely the statement goes two ways.

(1) If a real number a is less than a real number b, then there is a <u>positive</u> real number c such that

$$a + c = b.$$

(2) If $a + c = b$, where a, b, and c are real numbers and c is positive then

$$a < b.$$

Look back at the examples in our table! Statement (1) begins with the inequality on the left and "connects" it to the equation on the right. Statement (2) begins with the equation on the right and "connects" it to the inequality on the left.

Check Your Reading

1. The sentence "$2 < 7$" expresses the order between 2 and 7. What is a sentence involving "=" which expresses the same order?

2. If "$a + c = b$" expresses the order $a < b$, which of the following is correct: "$c < 0$", "$c = 0$", or "$c > 0$".

Oral Exercises 9-2c

1. Give a statement of equality to replace each of the following order relations by "adding" a positive number to one side of each given sentence.

   (a) $3 < 7$
   (b) $-2 < 4$
   (c) $-5 < -4$
   (d) $-\frac{12}{5} < -\frac{9}{5}$
   (e) $.99 < .999$
   (f) $-.3999 > -.4000$
   (g) $x < x + 2$
   (h) $k + 1 > k$

2. State the order relation corresponding to each of the following statements of equality.

   (a) $x + 4 = 6$
   (b) $7 + w = 9.5$
   (c) $x + 3 = y$
   (d) $2x + 11 = y$
   (e) $m = 3 + n$
   (f) $\left(x + (-1)\right) + 3 = y + 2$
   (g) $x + 2 = y + 5$
   (h) $k + m = 1$ and $m > 0$

Problem Set 9-2c

1. For each pair of numbers, determine their order and find a positive number which when added to the smaller gives the larger.

   (a) -15 and -24
   (b) $\frac{63}{4}$ and $-\frac{5}{4}$

Problem Set 9-2c
(continued)

(c) $\frac{6}{5}$ and $\frac{7}{10}$

(d) $-\frac{1}{2}$ and $\frac{1}{3}$

(e) -254 and -345

(f) $-\frac{33}{13}$ and $-\frac{98}{39}$

(g) 1.47 and -0.21

(h) $(-\frac{2}{3})(\frac{4}{5})$ and $(\frac{3}{2})(-\frac{5}{4})$

2. Which of the following statements are true? Which are false?

(a) If $a + 1 = b$, then $a < b$

(b) If $a + (-1) = b$, then $a < b$

(c) If $(a + c) + 2 = (b + c)$, then $a + c < b + c$

(d) If $(a + c) + (-2) = (b + c)$, then $b + c < a + c$

(e) If $a < -2$, then there is a positive number d such that $-2 = a + d$

(f) If $-2 < a$, then there is a positive number d such that $a = (-2) + d$.

(g) If $b < 0$, then $3 + b < b$

(h) If $b < 0$, then $3 + b < 3$

3. A proof of the statement "If $a + c = b$ and $0 < c$, then $a < b$" is given below.
Give the missing reasons.

Assume	$0 < c$	
then	$a + 0 < a + c$	why?
then	$a < a + c$	why?
and	$a < b.$	why?

9-3. Multiplication Property of Order.

We have seen that whenever a number a is less than a number b, then

$$a + c < b + c,$$

no matter what the number c is. That is, it makes no difference whether c is positive, negative, or zero.

Now we ask the following question. Suppose once again that

$$a < b.$$

When we multiply both of these numbers by a real number is the order of the resulting numbers the same as for the original numbers. That is, will the product ca be always less than the product cb regardless of what the number c is?

To begin, let's consider an example, letting $a = 5$ and $b = 8$. It is certainly true that

$$5 < 8.$$

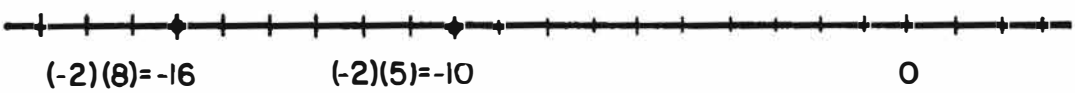
Suppose we multiply both sides by 2. Is the following sentence true?

$$2(5) < 2(8)$$

The answer is easy, since we know that 10 is less than 16. However suppose we multiply by (-2). What about the following sentence?

$$(-2)(5) < -2(8)$$

A glance at the number line



shows us that $(-2)(8)$ is less than $(-2)(5)$, and our sentence above is false. Suppose we try another experiment, this time beginning with two negative numbers, say -7 and -2. We see that $-7 < -2$. If we multiply both sides by a positive number, say 3, what happens? Is the following sentence true?

$$3(-7) < 3\ (-2)$$

Check the number line to see if $(-21) < (-6)$. Finally, let's multiply by (-3). What is the relation between

$$(-3)(-7) \quad \text{and} \quad (-3)(-2)?$$

In Chapter 8 we learned that the product on the left is positive and the product on the right is positive. It should be clear that the sentence

$$21 < 6$$

is a <u>false</u> sentence, and that the sentence

$$6 < 21 \quad \text{is true.}$$

The previous examples seem to point to the fact that there is a multiplication property of order, but that it differs in one major respect from the addition property. Let us look at another example that may help us to state this property. Try this! Take (-5) and (6). It's clear that $(-5) < 6$. Now multiply both numbers by 4. Then check the relation between

$$4(-5) \quad \text{and} \quad 4(6)$$

Next multiply (-5) and 6 by (-4) and check the relation between

$$(-4)(-5) \quad \text{and} \quad (-4)(6).$$

By now you will be getting an idea as to how the <u>multiplication property of order</u> seems to work. Actually the property can be stated as follows:

For any real numbers a and b,
such that $a < b$,
$ca < cb$ if c is a <u>positive</u> number,
but $cb < ca$ if c is a <u>negative</u> number.

We observe that multiplication by a negative number <u>reverses</u> the order whereas multiplication by a positive number keeps the order the same.

Your ideas about this property have been gained by working with examples and using the number line. It is interesting to see that we can show this property in a more general way. We use, instead, two other properties which we have already studied in this chapter. First we begin with two real numbers a and b such that

$$a < b.$$

Now remember the property which "connects" the order "is less than" to "equals". We say that if

$$a < b,$$

then there is a positive number, call it p, such that

$$a + p = b.$$

Now suppose we multiply both sides of this equation by a positive number, for example, 3.
We get

$$3(a + p) = 3b,$$

which becomes $3a + 3p = 3b$. What property is used here
The second part of the same "connecting property" then tells us that since

$$3a + 3p = 3b \quad \text{and} \quad 3p \text{ is positive}$$

then $3a < 3b$.

We know that $3p$ is positive, because p and 3 are both positive. Therefore we know that $3a < 3b$. The number 3 was chosen for convenience. It could have been any positive number. Now suppose we multiply by a negative number, for example -5. The multiplication gives us

$$(-5)(a) \quad \text{and} \quad (-5)(b).$$

What is the relation between these two products? Now comes an important idea! We learned in a previous chapter that

$(-5)(a)$ is the opposite of $(5)(a)$ and $(-5)(b)$ is the opposite of $(5)(b)$.

Do you remember what happens to the order relation when we take opposites? We learned that order is reversed. Therefore we can say the following:

Since we know that $5a < 5b$, because 5 is positive,

then $(-5)(b) < (-5)(a)$.

We see that multiplication by a negative number brings about a reversal of order.

The multiplication property is helpful in finding the answer to another question. Suppose

$$a < b.$$

Then, do we know about the relation between

$$a^2 \quad \text{and} \quad b^2 \text{ ?}$$

We can answer the question for two cases. Let's first suppose that both a and b are positive. We know that

if $a < b$ then $(a)(a) < (a)(b)$.

We also know that $(a)(b) < (b)(b)$. (Why?)

Now we use the transitive property to say that

since $(a)(a) < (a)(b)$ and $(a)(b) < (b)(b)$ then

$(a)(a) < (b)(b)$.

So $$a^2 < b^2.$$

Now suppose that both a and b are negative. We can use the second part of the multiplication property.

This property tells us that if $a < b$, and if c is negative, then $cb < ca$.

Thus if $a < b$, multiplication of both sides by the negative number a gives us

$$ab < a^2.$$

Likewise multiplication of both sides of "$a < b$" by the negative number b gives us

$$b^2 < ab.$$

It follows by the transitive property that since

$$b^2 < ab \quad \text{and} \quad ab < a^2, \quad \text{then} \quad b^2 < a^2.$$

Putting these results together we see that

If $a < b$, then

$a^2 < b^2$ if both a and b are positive,

and $b^2 < a^2$ if both a and b are negative.

These results will be very useful later on.

Check Your Reading

1. Since 5 is less than 8, what is the order between 2(5) and 2(8)? What is the order of -2(5) and -2(8)?
2. Is the sentence "$5a < 8a$" true for every value of a?
3. State the multiplication property of order.

Oral Exercises 9-3a

1. If $a < 5$, what is the order of $2a$ and 10?
2. If $-3 < b$, what is the order of $-2b$ and 6?
3. If $0 < p$, what is the order of 0 and $-p$?
4. If $m < n$, what is the order of $3m$ and $3n$?
5. If $-5 < q$, what is the order of 10 and $-2q$?
6. If $1 + (-x) < 3$, what is the order of $x + (-1)$ and -3?
7. If $a + (-b) < -5$, what is the order of $-3(a + (-b))$ and 15?
8. If $a + b > 5$, what is the order of $5a + 5b$ and 25?
9. If $-\frac{1}{2}a > 2$. what is the order of a and -4?
10. If $-4x < 8$, what is the order of x and -2?

Problem Set 9-3a

1. Make the following statements true by inserting the correct symbol ">" or "<".

   (a) If $c < 5$, then $2c$ ____ 10.

   (b) If $r < -3$, then $(-3)r$ ____ 9.

   (c) If $2 < s$, then 14 ____ $5s$.

   (d) If $-5 < q$, then 15 ____ $(-3)q$.

   (e) If $d < m$ and $q > 0$, then dq ____ mq.

   (f) If $s < t$ and $y < 0$, then yt ____ ys.

2. (a) If $x < -5$, what is the order of 15 and $-3x$?

   (b) If $a < -2$, what is the order of a and -1?

   (c) If $0 < z$, what is the order of -2 and $z + (-2)$?

   (d) If $-x < \frac{1}{5}$, what is the order of x and $-\frac{1}{5}$?

   (e) If $x + 2 < 2x + (-3)$, what is the order of 5 and x?

   (f) If $y + 3 = z$, what is the order of y and z?

Problem Set 9-3a
(continued)

3. (a) If $\frac{1}{2}x = 3$, what is the relationship between x and 6?

(b) If $-2z = 6$, what is the relationship between z and -3?

(c) If $3x < x + 2$, what is the order of $2x$ and 2?

(d) If $2y < 2$, what is the order of y and 1?

(e) If $m^2 > 0$, what is the order of $m^2 + 1$ and 1?

(f) If $s < t$ and $t < 0$, what is the order of st and t^2?

4. (a) If $x > 5$, what is the order of x^2 and 25?

(b) If $x < 5$ and x is positive, what is the order of x^2 and 25?

(c) If $z < -5$, what is the order of z^2 and 25?

(d) If $z > -5$ and z is negative, what is the order of z^2 and 25?

(e) If $a < 5$ and $b < 7$, what is the order of $a + b$ and 12?

(f) If $|a| > |b|$, what is the order of $|a|^2$ and $|b|^2$?

The multiplication property of order is useful in finding truth sets for certain types of open sentences, such as

$$2x < 8.$$

If there is a real number x for which the above sentence is true, then for this x we have:

$$\frac{1}{2}(2x) < \frac{1}{2}(8)$$

$$\left(\frac{1}{2}(2)\right)x < \frac{1}{2}(8)$$

$$x < 4$$

What is the truth set of "$x < 4$"? It is the set of <u>all numbers less than</u> <u>4</u>. Is this the same as the truth set of

$$2x < 8?$$

Before answering the second question we will state a property which is based on a similar property for equations. If the two sides of an open sentence involving inequalities are each multiplied by the same <u>non-zero</u> real number using the multiplication property of order then the resulting open sentence has the same truth set as the first sentence.

Let us consider a case where the number we are multiplying by is negative. The sentence "$-2x < 10$" has the same truth set as the sentence

$$(-\tfrac{1}{2})10 < (-\tfrac{1}{2})(-2x),$$

which is the same as

$$-5 < x.$$

Does "$-5 < x$" have the same truth set as "$-2x < 10$"? We say then that "$-2x < 10$" and "$-5 < x$" are equivalent sentences.

Let us try an example in which we use both the addition and multiplication properties of order. The sentence

$$(-11) + (-4x) < (-7x) + 3$$

is equivalent to

$$(-11) + (-4x) + 7x + 11 < (-7x) + 3 + 7x + 11, \quad \text{(Why?)}$$

and this sentence can be simplified to

$$3x < 14. \qquad \text{(give reasons).}$$

This is equivalent to

$$\tfrac{1}{3}(3x) < \tfrac{1}{3}(14), \qquad \text{(Why?)}$$

and this can be simplified to

$$x < \frac{14}{3}.$$

Do you agree that this last sentence is equivalent to the first sentence? Our truth set can be described as <u>all numbers less</u> than $\frac{14}{3}$.

Check Your Reading

1. By what number would we multiply both sides of "$2x < 8$" to obtain a simpler equivalent open sentence?
2. Are the sentences "$-2x < 10$" and "$x > -5$" equivalent open sentences? Why?

Oral Exercises 9-3b

In order to find the truth set of each of the following sentences we need a simpler equivalent sentence. By what number would you multiply in each case to obtain this simpler sentence?

1. $3x < 12$
2. $\frac{x}{3} < 7$
3. $-\frac{1}{2}x < 14$
4. $15 < -3x$
5. $-12 < 4x$
6. $7 < \frac{2}{3}x$
7. $-\frac{5}{8}x < \frac{2}{3}$
8. $x + 5 < 12$
9. $2x + 4 < 8$
10. $(-6) + \frac{1}{4} < 3 + 3x$

Problem Set 9-3b

1. Find the truth set of each of the following sentences. Try some of the numbers in the set to see if they make the sentence true.

   (a) $4x < 12$
   (b) $\frac{1}{3}x < \frac{1}{2}$
   (c) $-2x < -\frac{3}{8}$
   (d) $x + 2 < 3$
   (e) $(-4) + 7 < -4x + 3x$
   (f) $\frac{4}{7} \leq 4x$
   (g) $-12 < \frac{2}{3}x$
   (h) $-\frac{x}{16} < \frac{1}{2}$
   (i) $3x = 0$
   (j) $x + 11 = x$

2. Draw the graphs of the truth sets in (a), (f) and (i) of Problem 1.

Problem Set 9-3b
(continued)

3. Find the truth sets of the following sentences.

   (a) $x + 5 < 2$

   (b) $(-3x) + (-3) < 8$

   (c) $(-17) + 12 \leq 2x + 4$

   (d) $3x + 2 + (-2x) = x + 5$

   (e) $\frac{1}{2}x + (-7) = -\frac{3}{2}x + 4$

   (f) $x + (-5) < 2x + 7$

   (g) $4.2 + 7x < (-3x) + 7.3$

   (h) $17x + (-7) < 3 + 12x + (-2x) + 7x$

   (i) $(-12x) + \frac{3}{4} = 5x + \frac{1}{2}$

   (j) $(-11) + (-7x) < 4x + 12 + (-4)$

In each of Problems 4, 5, 6, and 7, write and solve an open sentence to find out what you can about the answer to the question in the problem.

4. When Joe and Moe were planning to buy a sailboat, they asked a salesman about the cost of a new type of boat that was being designed. The salesman replied, "It will cost less than $380." If Joe and Moe had agreed that Joe was to contribute $130 more than Moe when the boat was purchased, how much would Moe have to pay?

5. Three more than six times a number is greater than seven increased by four times the number. What is the number?

6. A teacher says, "If I had three times as many students in my class as I do have, I would have less than 46 more than I now have". How many students does he have in his class?

7. Bill is 5 years older than Norma and the sum of their ages is less than 23. How old is Norma?

Summary

In this chapter we have discussed the following properties involving the order relation "is less than".

1. The Comparison Property

   For any real number a and any real number b one and only one of the following sentences is true.

$$a < b$$

$$b < a$$

$$a = b.$$

2. The Transitive Property of Order

   If a, b, and c are any three real numbers and if $a < b$ and $b < c$, then $a < c$.

3. The Property Related to the Taking of Opposites

   If a and b are any real numbers such that $a < b$, then $(-b) < (-a)$.

4. The Addition Property of Order

   If a, b, and c are any real numbers and if $a < b$, then $(a + c) < (b + c)$.

5. The Property Connecting Order with Equality

   Part I. If a real number a is less than a real number b, then there is a positive real number c such that

$$a + c = b.$$

   Part II. If $a + c = b$, where a, b, and c are real numbers and c is positive, then

$$a < b.$$

6. The Multiplication Property of Order

   For any real numbers a and b if

$$a < b,$$

   then $ca < cb$ if c is a positive number,

   but $cb < ca$ if c is a negative number.

Review Problem Set.

1. Which of the following statements are true? Which are false?

   (a) If $a + 1 = b$, then $b < a$.

   (b) If $a + (-1) = b$, then $b + 1 = a$.

   (c) If $a + (-1) = b$, then $a < b$.

   (d) If $(a + c) + 2 = b + c$, then $a + c < b + c$.

   (e) If $(a + c) + (-2) = b + c$, then $a + c = (b + c) + 2$.

   (f) If $(a + c) + (-2) = b + c$, then $b + c < a + c$.

   (g) If $a < (-2)$ then there is a positive number d such that $(-2) + d = a$.

   (h) If $(-2) < a$, then there is a number d such that $ad < -2d$.

   (i) If $3 < 5$, then $3a < 5a$ for every number a.

   (j) If b is a positive number, then $-bc$ is a negative number.

2. Find the truth set of each of the following sentences.

   (a) $(-4) + 7 < (-2)x + (-5)$

   (b) $4x + (-3) > 5 + (-2x)$

   (c) $\frac{2}{3} + (-\frac{5}{6}) < (-\frac{1}{6}) + (-3)x$

   (d) $\frac{1}{2}x + (-2) < (-5) + \frac{5}{2}x$

   (e) $\frac{7}{3}x + 5 + (-2x) = (-x) + (-7) + (-\frac{2}{3}x)$

   (f) $m + (-\frac{1}{2}) \leq \frac{2}{3}m + 7$

   (g) $-5n = 4n + 7 + (-6n)$

   (h) $2x < 3 + (-2)(-\frac{4}{3})$

   (i) $4x + 7 + (-2x) > (-2) + 5 + (-3x)$

   (j) $-(2 + x) < 3 + (-7)$

3. Draw the graphs of the truth sets of parts (a) and (b) of Problem 2.

Review Problem Set
(continued)

*4. If a rectangle has area 12 square inches and one side has length less than 6 inches, find out what you can about the length of the adjacent side.

*5. If a rectangle has area 12 square inches and one side has length between 4 and 6 inches, find out what you can about the length of the adjacent side.

6. If $x \neq 0$, then x is either negative or positive. If x is positive, then what kind of number is x^2? If x is negative, what about x^2? State a general result about x^2 if $x \neq 0$. What is a general result about x^2 for any real number x?

7. Each of the following expressions is either an indicated sum or an indicated product. Write each indicated sum as an indicated product and write each indicated product as an indicated sum, by using the distributive property.

(a) $3a\big(a + (-2ab) + b\big)$

(b) $4x^2 + 2xy + (-2x)$

(c) $(-4a)\big((-3) + (-2a) + x\big)$

(d) $a(x + 2) + (-4)(x + 2)$

(e) $(x + 2)\big(x + (-1)\big)$

(f) $(3x + 1)(2x + 2)$

(g) $\big(7y + (-2)\big)\big((-2y) + 4\big)$

(h) $4mx + 2m + 6am$

(i) $\big(2r + (-4)\big)(3t + 1)$

(j) $(x + 2m)^2$

8. Simplify these expressions using the distributive property.

(a) $3x + (-4)x + (-10x)$

(b) $7a + 2b$

(c) $(-4y) + (-8y) + 12y$

(d) $2rst + (-6stm)$

(e) $3x + 4y + (-2x) + (-7y)$

Review Problem Set
(continued)

9. Two cars start from the same point at the same time and travel in the same direction at average speeds of 34 and 45 miles per hour respectively. In how many hours will they be 35 miles apart?

10. Henry and Charles were opposing candidates in a class election. Henry received 30 votes more than Charles, and 516 members of the class voted. How many votes did Charles get?

11. A man left \$10,500 for his widow, a son and daughter. The widow received \$5,000 and the daughter received twice as much as the son. How much did the son get?

12. Each of the following expressions is written as an opposite. Write each expression as an indicated sum, as shown in the example.

Example: $-\left(3a + (-4) + 2b\right)$

This can be written as

$$(-3a) + 4 + (-2b)$$

(a) $-\left(3a + (-2b)\right)$

(b) $-\left(-2x + 3a + (-7)\right)$

(c) $-\left(2a + (-3) + 2a + 3\right)$

(d) $-\left(-(3a + (-2b))\right)$

(e) $-(2x + 1)\left(x + (-1)\right)$ (Hint: Find the product first.)

13. Prove the following properties of real numbers.

(a) $-(a + b) = (-a) + (-b)$ (Hint: Add $(a + b)$ to $(-a) + (-b)$. What does your result show?)

(b) For real numbers a, b, and c, if $a + c = b + c$ then $a = b$ (Hint: Use the addition property of equality.)

Chapter 10

SUBTRACTION AND DIVISION OF REAL NUMBERS

10-1. The Meaning of Subtraction.

In arithmetic we did a great deal of subtracting. In this process, however we always subtracted a positive number from an equal or larger positive number. Now that we are working with real numbers, which include negative numbers, we will want a method of subtraction for all of these numbers as well. That is, we shall be interested in finding out how to subtract any real number from any real number. This process will have to apply to the subtraction of a larger number from a smaller number as well as to subtraction involving negative numbers.

How can we find a rule which will work in all cases? We will begin by studying the process with which you are already familiar.

Suppose you buy something at the store which costs 83 cents and you give the clerk one dollar. How does she count out your change? She first says, "83." Then she puts down two cents and says, "85". She then puts down one nickel and says, "90", and finally she puts down a dime and says, "One dollar, thank you!"

The clerk wants to find the difference between 83 and 100. That is, her problem is really one of subtraction. But what does she actually do? She finds what amount needs to be added to 83 to get 100. In other words, she finds her subtraction answer by adding! She has mentally changed the wording of the problem from the question, "One hundred minus eighty-three equals what?" to "Eighty-three plus what equals one hundred?" The sentence

$$100 - 83 = x$$

has become

$$83 + x = 100.$$

Now, how do we find the truth number of the sentence "$83 + x = 100$"? We can use the addition property as follows:

$$83 + x = 100$$
$$83 + x + (-83) = 100 + (-83)$$
$$x = 100 + (-83)$$
$$x = 17$$

The truth set of our sentence is {17}. The number 17 is also the answer to our subtraction problem.

Notice that the next to the last step in the above discussion is the following:

$$x = 100 + (-83)$$

This tells us that in this case we can get our answer by <u>adding</u> the <u>opposite</u> of 83 to 100. Thus "100 - 83" and "100 + (-83)" are names for the same number.

In the problem just finished we were subtracting the positive number 83 from the larger positive number 100. Now try a few more examples of the same kind. In the left column is the answer you know from arithmetic; in the right column, see if you get the same answer by <u>adding</u> to the first number the <u>opposite</u> of the second:

$20 - 9 = 11$	$20 + (-9) = 11$
$8 - 6 = 2$	$8 + (-6) = (\ \ )$
$8.5 - 3.2 = (\ \ \ )$	$8.5 + (\ \ \ ) = 5.3$

So you see that subtraction problems of this kind, which you already know how to do in arithmetic, can also be done by finding the <u>opposite</u> of the second number and <u>adding</u> this to the first number.

Now let's try one we don't know how to do in arithmetic--- "From 4 subtract 6". This doesn't make sense in arithmetic because we can't subtract 6 from the smaller number 4. But remember we now have all the real numbers to work with. Can we use these to make some sense of "4 - 6"? There is an answer and it comes from the <u>second</u> way we did the subtraction problems above. We can get the opposite of the second number 6 and add it to the first number 4. Do you see that "4 + (-6)" does make sense? Since it does, we can now say this: "4 - 6" makes sense if we agree that it means "4 + (-6)"

Can you get "(-4) - 6" to make sense? Should it mean "(-4) + (-6)"? Can you do this addition and get another name for "(-4) + (-6)"? What should "5 - (-3)" mean?

From these examples we can now see what "a - b" ought to mean for any two real numbers a and b. It ought to mean "a + (-b)". There are two reasons for this. First, any number b has an opposite -b, and we can always add -b to any number a (Why?). So <u>this way</u> we can always subtract <u>any</u> number b from <u>any</u> number a. Second, when we do subtraction this way for numbers like a = 100 and b = 83 we get the same answer as we did when we subtracted in arithmetic.

So we make the following definition of subtraction for real numbers.

> To subtract a real number b from a real number a, add the opposite of b to a. Thus, for real numbers a and b
>
> $$a - b = a + (-b).$$

Examples:

subtraction problem		addition of opposites
20 - 9	=	20 + (-9)
10 - 15	=	10 + (-15)
(-8) - 6	=	(-8) + (-6)
(-10) - (-7)	=	(-10) + 7
7 - (-4)	=	7 + 4
(-5) - 2	=	-5 + ( )
5 - (-2)	=	5 + ( )

Perform the operations in the right sides of the above equations.

To help keep the two uses of the symbol "-" clear, let us comment on each:

In the phrase

$a - b$,

"-" stands <u>between</u> two numerals and indicates the operation of subtraction. We read the phrase as

"a minus b".

In the phrase

$a + (-b)$,

"-" is part of one numeral and indicates the <u>opposite of</u>. We read the phrase as

"a plus the opposite of b".

As an example, consider the phrase "(-2) - 5". Here the symbol "-" occurs twice. The first one, in front of "2", does <u>not</u> stand between two numerals and therefore means "the opposite of" The second does stand between two numerals, namely "(-2)" and "5", and indicates subtraction. Thus $(-2) - 5 = (-2) + (-5)$.

Frequently after this we shall abbreviate "(-2) - 5" to "-2 - 5" and "(-a) - b" to "-a - b". Thus "-a - b" is to be understood as meaning "(-a) + (-b)".

Here is a subtraction example. See if you can understand the reasons given for each step.

<u>Example</u>. Find a simpler expression for $(7x + 1) - 5x$.

$$\begin{aligned}
(7x + 1) - 5x &= (7x + 1) + \big(-(5x)\big) && \text{Definition of subtraction} \\
&= (7x + 1) + (-5)x && -(ab) = (-a)b \\
&= 7x + (-5)x + 1 \\
&= \big(7 + (-5)\big)x + 1 \\
&= 2x + 1
\end{aligned}$$

Here is another example. See if you can state the reasons for each step.

<u>Example.</u> Find a simpler expression for $3a - (a + 4b)$.

$$\begin{aligned}
3a - (a + 4b) &= 3a + \big(-(a + 4b)\big) \\
&= 3a + \big((-a) + (-4b)\big) \\
&= \big(3a + (-a)\big) + (-4b) \\
&= \big(3a + (-1)a\big) + (-4b) \\
&= \big(3 + (-1)\big)a + (-4b) \\
&= 2a - 4b.
\end{aligned}$$

Check Your Reading

1. Give an open sentence involving the operation of addition which is equivalent to the open sentence "100 - 83 = x."
2. How can the problem "From 4 subtract 6" be stated as an addition problem?
3. For any real numbers a and b, the expression "a - b" names the same number as what other expression?
4. Which "-" in the following expression indicates the operation of subtraction?

   "-2 - 5".
5. How can the expression "-a - b" be restated so as to indicate the sum of two real numbers?

Oral Exercises 10-1

State the following subtractions in terms of adding an opposite.
Example: $8 - 2 = 8 + (-2)$

(a) $5 - 4$	(f) $4a - 3a$	(k) $\pi - (-\pi)$
(b) $11 - 12$	(g) $-2x - (-2)$	(l) $8k - (-11k)$
(c) $-4 - 8$	(h) $7y - (-2y)$	(m) $6x - 2x$
(d) $-11 - (-5)$	(i) $(8 - 12) - 2$	(n) $0 - (-3m)$
(e) $24 - (-8)$	(j) $2 - (8 - 12)$	(o) $6\sqrt{2} - 9\sqrt{2}$

Problem Set 10-1

1. Give a simpler expression for each of the following. Remember that the opposite of a sum is the sum of the opposites.

   Example: $-(x^2 - 3) = -\left(x^2 + (-3)\right)$
   $= (-x^2) + \left(-(-3)\right)$
   $= -x^2 + 3.$

Problem Set 10-1
(continued)

(a) $-(a + 7)$

(b) $-(a - 7)$

(c) $-(x^2 + x + 2)$

(d) $-(x^2 - x - 2)$

(e) $-(-x + y)$

(f) $-(-(x + y))$

2. Write each of the following differences in terms of the addition of opposites, and write a simplified answer.

(a) $15 - 25$

(b) $132 - (-18)$

(c) $-12 - (-24)$

(d) $-7b - 12b$

(e) $-3x - (-4x)$

(f) $\frac{3}{4} - (-\frac{1}{2})$

(g) $7m - (m + 12)$

(h) $-4x - (2x - b)$

(i) $\frac{3}{4}x - \frac{1}{5}x$

(j) $7.4m - (12 - 3.5m)$

3. Find a simpler expression for each of the following:

(a) $(5x + 2) - 3x$

(b) $(5 + 4w) - 5w$

(c) $(2y + 5) - (2y + 5)$

(d) $x(2x + 1) - 2x^2$

(e) $4 - (2x + 3)$

(f) $3a - (4 + a)$

(g) $2y - (3 + 2y)$

(h) $4 - (x^2 - 2x + 4)$

(i) $(2x + 3x) - 5x$

(j) $4m + (-3m) - 7m$

(k) $(5 - 2m) + (6m - 8)$

(l) $(2y + 1) - (2y - 1)$

(m) $a + b - (-(a + b))$

(n) $6 + |-6| - (-6)$

(o) $-\frac{3}{4} + \frac{1}{20} - (-\frac{4}{5})$

(p) $5 - (3a + 2b - 5)$

(q) $(x + 1)^2 - (x^2 + 2x + 1)$

(r) $\frac{3 + 5}{2 \times 4} - (-(-1))$

4. Find the truth sets of these sentences.
(Hint: First change subtraction to addition of the opposite.)

(a) $x - 5 = -4$

(b) $2x - 3 = 4 - 9$

(c) $11 - 18 = 4 - 2m$

(d) $4 - \frac{2}{3} + x = 5 - \frac{1}{2}x$

(e) $-18 = 5 - (2y - 4)$

(f) $x - 5 < 4$

(g) $3x - 7 < 4 - 11$

(h) $3x - 1 < 3x - 4$

Problem Set 10-1
(continued)

5. Write a phrase for each of the following involving only addition.

   (a) Subtract -8 from 15.

   (b) From -25 subtract -4.

   (c) What number is 6 less than -9?

   (d) From 22 deduct -30.

   (e) -12 is how much greater than -17?

   (f) How much greater is 8 than -5?

   (g) What is 5 less 10?

   (h) What number added to -8 gives 7?

6. A marksman hears the bullet hit the target 2 seconds after he fires. He knows the speed of the bullet is 3300 feet per second and the speed of sound is 1100 feet per second. How far away was the target?

   (Hint: Let t be the number of seconds it takes for the bullet to reach the target.)

7. A service station manager wishes to mix "regular" gas at 30¢ per gallon with "ethyl" gas at 35¢ per gallon to fill a 500 gallon tank with gas selling for 32¢ per gallon. How much of each type of gas need he put in the tank?

8. Two men start walking from the same place in the same direction. The second man starts one hour later than the first and walks 3 miles per hour. The first man walks 2 miles per hour. How long will the second man have walked when he catches up to the first?

10-2. Properties of Subtraction.

In previous chapters we have discussed the commutative and associative properties of addition and multiplication. Do you think that subtraction is commutative? To answer this we must determine whether or not the following sentences are true:

$$5 - 2 = 2 - 5$$

$$6 - 9 = 9 - 6$$

$$(-4) - 2 = 2 - (-4)$$

From our definition of the subtraction process we see that the left side of our first sentence represents the number 3, but the right side of the same sentence represents the number (-3). Therefore, the sentence is false. What can you say about the other two sentences? Do you see then that subtraction is not a commutative operation?

Is subtraction associative? Consider the following sentence.

$$10 - (7 - 2) = (10 - 7) - 2.$$

On the left side the phrase inside the parentheses represents the number 5. Thus, the left side has the same value as

$$10 - 5, \text{ which is } 5.$$

The phrase in parentheses on the right represents the number 3. The right side has the same value as

$$3 - 2, \text{ which is } 1.$$

Is the original sentence true?

Since we have shown that

$$10 - (7 - 2) = (10 - 7) - 2$$

is a false sentence, it is clear that the placement of parentheses, that is, the grouping of terms does make a difference. Thus, subtraction is not an associative operation for real numbers.

We found that with addition the **associative** property allows us to group terms in any way we wish. Thus, an expression like

$$10 + 7 + 2$$

has a perfectly clear meaning. Even though addition is a <u>binary operation,</u> we know that

$$10 + 7 + 2 = 19$$

with or without parentheses.

However, if we had an expression such as

$$10 - 7 - 2,$$

how would we interpret it? From the discussion above we see that it might be equal to 5 or to 1, depending on how the terms are grouped. To avoid this confusion we shall agree to a rule, or convention. We shall say that an expression like

$$10 - 7 - 2$$

shall always mean

$$(10 - 7) - 2.$$

From our definition of subtraction this enables us to state that

$$\begin{aligned} 10 - 7 - 2 &= (10 - 7) - 2 \\ &= 10 + (-7) + (-2) \\ &= 1. \end{aligned}$$

<u>Check Your Reading</u>

1. Is the following sentence true or false: "5 - 2 = 2 - 5"?
2. Which of the following operations are commutative: addition, subtraction, multiplication ?
3. Is the following sentence true or false: "10 - (7 - 2) = (10 - 7) - 2" ?
4. Which of the following operations are **associative**: addition, subtraction, multiplication ?

Problem Set 10-2a

1. Write a simpler expression for each of the following:

   (a) $24 - 35 - 47$

   (b) $3a - 4a - 8a$

   (c) $14x - 11x + \frac{3}{2}x$

   (d) $2x - (3x - x)$

   (e) $.22x - (5 - .78x)$

   (f) $\frac{2}{3}y - 2x - 7y + \frac{1}{2}x$

   (g) $3x\big((-2x) + (-3) + (-2y)\big)$

   (h) $3x(-2x - 3 - 2y)$

   (i) $(-1)\big((-7m) + 2x - 4\big)$

   (j) $-\big((-7m) + 2x - 4\big)$

   (k) $(a + 2b) - (3a - b) - (a - 2b)$

2. Determine which of the following sentences are true for all values of the variables.

   (a) $a + b = b + a$

   (b) $a - b = b - a$

   (c) $a + (-b) = -b + a$

   (d) $-a + b = b - (-a)$

   (e) $(a + b) + c = a + (b + c)$

   (f) $(a - b) + c = a - (b + c)$

   (g) $a - (b - c) = (a - b) - c$

   (h) $a - b < a + b$

   (i) $3x + (-7y) + 4 = 3x - 7y + 4$

It should be clear that once we have changed a statement about subtraction into one involving addition, all the properties of addition will hold.

The sentence

$$7 - 4 = 4 - 7$$

is certainly false. On the other hand, the sentence

$$7 + (-4) = (-4) + 7$$

is definitely a true sentence. Both sides represent the number 3. The following example illustrates how the commutative and associative properties may be used in a problem involving subtraction. Suppose we are given the expression

$$(\frac{6}{5} + 2) - \frac{1}{5}$$

and are asked to represent the same number in a simpler form. That is, suppose we are asked to write the <u>common name</u> for this number. In the future we shall frequently use the word "simplify" to indicate the same process. Using our definition of subtraction we see that

$$(\frac{6}{5} + 2) - \frac{1}{5} = (\frac{6}{5} + 2) + (-\frac{1}{5}) .$$

By the commutative law for addition we can reverse the numerals in the first parentheses. This gives us

$$(2 + \frac{6}{5}) + (-\frac{1}{5}) .$$

By the associative property this can be changed to

$$2 + \left(\frac{6}{5} + (-\frac{1}{5})\right) .$$

Our expression now becomes

$$2 + 1$$

which we see is equal to 3.

In some of the problems of the previous section, we used the fact that

$$-(b + c) = (-b) + (-c).$$

That is, the opposite of a sum is equal to the sum of the opposites. You may remember that we developed this in both chapters 7 and 8. We can now use this fact to establish a fact about subtraction.

If a, b, and c are any real numbers, then "a - (b + c)" names a number. Can we find another numeral for this number? The following steps show how this might be done.

$a - (b + c) = a + \big(-(b + c)\big)$	Remember that subtracting a number is the same as adding the opposite.
$= a + \big((-b) + (-c)\big)$	The opposite of a sum is equal to the sum of the opposites.
$= \big(a + (-b)\big) + (-c)$	Addition is associative.
$= (a - b) - c$	Definition of subtraction
$= a - b - c.$	This means the same as "(a - b) - c".

What we have shown then is that for <u>any real numbers a, b, and c</u>,

$$a - (b + c) = a - b - c.$$

Here is an example in which this fact is applied:

$$\begin{aligned}(5y - 3) - (6y - 8) &= (5y - 3) - \big(6y + (-8)\big)\\ &= (5y - 3) - 6y - (-8)\\ &= (5y - 3) - 6y + (8)\\ &= 5y + (-6y) + (-3) + (8)\\ &= 5 - y.\end{aligned}$$

In our examples and exercises we have also used the distributive property---multiplication "distributes over addition." We know that subtraction is not commutative and is not associative, but addition is. Perhaps subtraction fails again; perhaps multiplication does not distribute over subtraction. We can try an example in order to see what happens. Is "3(5 - 7) = (3)(5) - (3)(7)" a true sentence? Since each side names the number -6, the sentence is true. We try

another example. Is "$(-2)(4 - 9) = (-2)(4) - (-2)(9)$" true? Again, it is.

Perhaps in every example the sentence would turn out to be true. So we look at the general case. If a, b, and c are real numbers, is it true that

$$a(b - c) = ab - ac?$$

$a(b - c)$	$= a\big(b + (-c)\big)$	
	$= ab + a(-c)$	Multiplication is distributed over addition.
	$= ab + (-ac)$	For any numbers a and c, $a(-c) = -ac$.
	$= ab - ac$	Subtracting ac is the same as adding the opposite of ac.

Therefore, we have proved that

<u>for any real numbers a, b, and c,</u>

$$a(b - c) = ab - ac.$$

Multiplication does distribute over subtraction as well as over addition.

Here is an example in which multiplication is distributed over subtraction.

$$\begin{aligned}(-3)(2x - 5) &= (-3)(2x) - (-3)(5)\\ &= -6x - (-15)\\ &= -6x + 15.\end{aligned}$$

Check Your Reading

1. Which of the following sentences is true?

   $7 - 4 = 4 - 7$; $\qquad 7 + (-4) = (-4) + 7$.

2. Give a simpler name for the number $(\frac{6}{5} + 2) - \frac{1}{5}$.

3. If a, b, and c are any real numbers, "$a - (b + c)$" names a number. Give another numeral for this number.

4. Is multiplication distributed over addition?
   Is multiplication distributed over subtraction?

5. If a, b, and c are any real numbers, "a(b - c)" names a number. Give another numeral for this number.

Oral Exercises 10-2b

1. State the opposite of each of the following numbers:

(a) b
(b) $-c$
(c) $-3c$
(d) $11x$
(e) $3x + 2$
(f) $2x - 5$
(g) $a + 2b - c$
(h) $(-1)(x - 2y)$
(i) $-x + w$

2. Simplify:

(a) $-(-4)$
(b) $x - (y + z)$
(c) $a - (x + 7)$
(d) $-3x + 6x$
(e) $-2(x - 2)$
(f) $-\frac{1}{2}(2a - 4)$
(g) $(-1)(3y - 5)$

Problem Set 10-2b

1. Write the opposite of each of the following:

(a) $7 - 2x$
(b) $a + b$
(c) $-(4 - 2c)$
(d) $(-2)(5a)$
(e) $1 - .01x$
(f) $\frac{3}{4}x - 2y - \frac{5}{4}x + 2y$
(g) $3x(2x + 3)$
(h) $7m(3m - 2)$

2. Simplify each of the following:

(a) $3 - (x + 2)$
(b) $(2y + 5) - (5y - 3)$
(c) $5(a - 2)$
(d) $(-1)(5m - n)$
(e) $-(5m - n)$

Problem Set 10-2b
(continued)

(f) $(-1)(-7x - 3y + 4)$

(g) $(-2)(3x - 2b)$

(h) $-100(x - .01x)$

(i) $7y - (3y + 2)$

(j) $(8t + 10) + (-3t - 2)$

(k) $-x(x - y)$

(l) $(9a + 2b - 7) - (3a - 7b + 5)$

(m) $3x - x^2 - x(1 - x)$

(n) $(x - 1)(x + 1) - (x^2 - 1)$

(o) $2m^2 - 6m(m - 1) - m$

(p) $(x + 2)(x + 1) - (x + 2)x$

3. Which of the following are true for all values of the variables?

(a) $-(w + z) = -w - z$

(b) $a(b - c) = ab - ac$

(c) $-a(b + c) = -ab + ac$

(d) $(a - b) - c = a + ((-b) + (-c))$

(e) $-(a - b) = b + a$

(f) $(a + b)(c - d) = (a + b)c - (a + b)d$

(g) $(a - b)(c) = c(b - a)$

(h) $-(-c) = |c|$

4. Find the truth set of each of the following sentences:

(a) $-(x + 2) = 4$

(b) $-(2x + 3) = 7$

(c) $2x \geq 8$

(d) $-3x > -6$

(e) $-\frac{1}{2}x \leq \frac{3}{2}$

(f) $-|x| = 3$

(g) $(-2)(x - 5) = 0$

(h) $-3(x - 4) = 6$

(i) $2y < 2(y - 4)$

(j) $2 - (3x + 4) = 5x - (4 -$

Problem Set 10-2b
(continued)

5. Write simpler expressions for each of the following:

   (a) $(a^2 - 2ab + b^2) - (3a^2 - 2ab - b^2)$

   (b) $(3x - 2y + m) - (6x - 7y + m)$

   (c) $(7a + \frac{3}{4}b - 5) - (6 - 2a + \frac{1}{2}b)$

   (d) $-(2k + 6k^2 - 4) - 3(11 - k + k^2)$

   (e) $-5n(n - 4) + 3(2n - 1)(n + 1)$

6. Translate each of the following into open sentences or phrases.

   (a) John's age 8 years ago.

   (b) A man is 6 times as old as his son.

   (c) Five times a certain distance is 36 miles.

   (d) The length of a rectangle is 2 feet more than twice the width.

   (e) The number of feet in 3y yards.

   (f) The value of a certain number of pounds of candy at $1.10 per pound.

   (g) The total value of some gasoline which is a mixture of two different kinds of gasoline, one kind worth 30 cents per gallon and the other kind worth 35 cents per gallon, if the number of gallons of 35-cent gasoline is 40 more than the number of gallons of 30-cent gasoline.

   (h) The number of cents in 2d dollars.

   (i) 15 dollars more than twice the number of dollars I have.

7. Write the expression that shows the form for the following exercise. Then use the properties of operations and numbers to write this form in the simplest way.

Problem Set 10-2b
(continued)

Take a number, multiply by 7, add 12, subtract 4, add your original number, multiply by $\frac{1}{8}$, multiply by 2, subtract 4, multiply by $\frac{1}{2}$.

What answer would you have if you started with 2? with 11? with -3?

8. John has \$1.65 in his pocket, all in nickels, dimes, and quarters. He has one more quarter than he has dimes, and the number of nickels he has is one more than twice the number of dimes. How many quarters does he have?

9. A milkman has a tray of pint and half-pint bottles. There are 6 times as many pint bottles as half-pint bottles. The total amount of milk contained in the bottles is 39 quarts. How many half-pint bottles are there?

10. From $11a + 13b - 7c$ subtract $8a - 5b - 4c$.

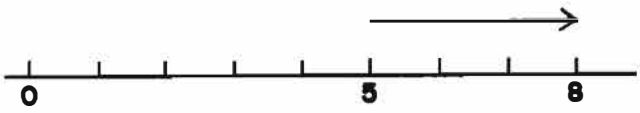
11. What must be added to $3s - 4t + 7u$ to obtain $-9s - 3u$?

12. Prove that for any real numbers a, b, c,
(i) if $a = b + c$, then $a - b = c$:
(ii) if $a - b = c$, then $a = b + c$.

(Hint: we may write $a - b = a + (-b)$, by our definition of subtraction, and then use the addition property of equality.)

10-3. Finding Distances by Subtraction.

In the first part of this chapter, it was found that any real number may be subtracted from any other real number. It is possible to use subtraction to determine the distance from one point on the number line to another point. Suppose, for example, we refer to the number line in asking the question, "What is the distance from 5 to 8?"



Starting at 5, in order to get to 8, a move of 3 units <u>to the right</u> must be made. Moving to the right is moving "in the positive direction" and is represented by a positive number---in this case, the number 3. In answer to the original question, then, the distance from 5 to 8 is 3.

The question "What is the distance from 8 to 5?" is a different one. Reference is made to the number line below.

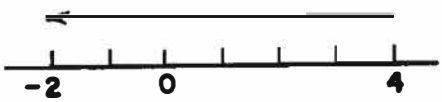
In this case, starting at 8, a move of 3 units <u>to the left</u> must be made. Moving to the left is moving "in the negative direction" and is represented by a negative number---in this case, the number -3. We can say then that the distance from 8 to 5 is -3.

Such distances can be determined without drawing a number line at all. The statements below, arranged in two columns, indicate how this might be done.

DISTANCE	SUBTRACTION
The distance from 5 to 8 is 3.	8 - 5 = 3
The distance from 8 to 5 is -3.	5 - 8 = -3

Do you see that these statements indicate that distances can be determined by the operation of subtraction?

Let us take another example, again using the number line to show exactly what is meant. What is the distance from 4 to -2?



As a glance at the number line makes clear, the distance from 4 to -2 is -6. This represents a move of 6 units to the left in going from 4 to -2. On the other hand, the distance from -2 to 4 is 6.

DISTANCE	SUBTRACTION
The distance from 4 to -2 is -6.	$-2 - (4) = -6$
The distance from -2 to 4 is 6.	$4 - (-2) = 6$

What is the distance from -1 to 6? From the experiences above, this can be found by subtraction, as follows: $6 - (-1) = 7$. Therefore, the distance from -1 to 6 is 7. By way of contrast, the distance from 6 to -1 is given by the following subtraction: $-1 - 6 = -7$.

As a matter of fact, for any two real numbers a and b, the following statement can be made:

The distance <u>from</u> a <u>to</u> b is the number b - a.

The number b - a also tells the direction of movement in going from a to b on the number line. If b - a is positive, the move from a to b is to the right; if b - a is negative, the move from a to b is to the left.

What is the distance from x to 4? The distance is 4 - x.

What is the distance from 4 to x? The distance is x - 4.

Often, in speaking of distance, the direction is of little importance; the only concern is with the number of units. In such a case, the expression "distance between" is used.

The distance between 5 and 8 is 3. The distance between 8 and 5 is 3.	The distance from 5 to 8 is 3, and the distance from 8 to 5 is -3. The "distance between" refers only to the number of units between the points; so the positive number 3 is used.

The distance between 4 and -2 is 6.

The distance between -2 and 4 is 6.

The distance from 4 to -2 is -6, and the distance from -2 to 4 is 6. The distance between the two numbers is the positive number 6.

In other words, in speaking of "distance between," a positive number is used.

What is the distance between a and b?

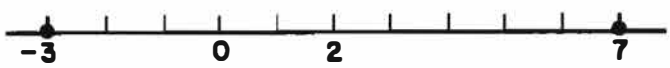
The distance from a to b is $b - a$.
The distance from b to a is $a - b$.

The distance between a and b is the positive of the two numbers, $b - a$ and $a - b$. But the positive of these two numbers is $|a - b|$.

Therefore, the distance between a and b is $|a - b|$.

Below is a problem in which knowledge of distance between two points on the number line is used in solving an open sentence.

Find the truth set of "$|x - 2| = 5$."



$|x - 2|$ represents the distance between x and 2. The open sentence states that this distance is 5. There are exactly two numbers such that the distance between each of them and 2 is 5. These numbers are -3 and 7. Therefore, the truth set is $\{-3, 7\}$.

Check Your Reading

1. What is the distance from 5 to 8?
2. What is the distance from 8 to 5?
3. What is the distance between 8 and 5?
4. What is the distance from a to b?
5. What is the distance from b to a?
6. What is the distance between a and b?
7. What arithmetic operation is associated with determining distance on the number line?
8. The expression "$|x - 2|$" represents the distance between what two numbers?
9. What is the truth set of "$|x - 2| = 5$"?

Oral Exercises 10-3

1. What is the distance
   (a) from -3 to 5?
   (b) between -3 and 5?
   (c) from 6 to -2?
   (d) between 6 and -2?
   (e) from 5 to 1?
   (f) between 5 and 1?
   (g) from -8 to -2?
   (h) between -8 and -2?
   (i) from 7 to 0?
   (j) between 7 and 0?

2. What is the distance
   (a) from x to 5?
   (b) between x and 5?
   (c) from -2 to x?
   (d) between -2 and x?
   (e) from -1 to -x?
   (f) between -x and -1?
   (g) from 0 to x?
   (h) between 0 and x?

Problem Set 10-3

1. What is the distance

   (a) from (-5) to 1?

   (b) from 1 to (-5)?

   (c) between (-5) and 1?

   (d) between 1 and (-5)?

   (e) from 8 to 3?

   (f) from 3 to 8?

   (g) between 8 and 3?

   (h) between 3 and 8?

2. What is the distance

   (a) from (-8) to (-3)?

   (b) between 4 and 7?

   (c) from 0 to 5?

   (d) between (-5) and 6?

   (e) from (-1) to (-9)?

   (f) from $(\frac{5}{2})$ to $(\frac{7}{2})$?

   (g) between (-12) and (-8)?

   (h) from (-8) to 0?

3. Think of 5 and x on a number line;

   (a) What is the distance from 5 to x?

   (b) If the distance from 5 to x is positive, which is true, $x < 5$ or $5 < x$?

   (c) If the distance from 5 to x is negative, which is true $x < 5$ or $5 < x$?

   (d) If the distance from 5 to x is 4, what is x?

   (e) If the distance from 5 to x is -4, what is x?

   (f) If the distance <u>between</u> 5 and x is 4, what is x?

   (g) What is the truth set of $|x - 5| = 4$?

4. Think of 5 and x on a number line;

   (a) If the distance between 5 and x is less than 4 and x is to the right of 5, what is x?

   (b) If the distance between 5 and x is less than 4 and x is to the left of 5, what is x?

   (c) If the distance between 5 and x is less than 4, what is x?

   (d) What is the truth set of $|x - 5| < 4$?

Problem Set 10-3
(continued)

5. What are the two numbers x on the number line such that

$$|x - 4| = 1?$$

6. Find the truth set of each of the following equations; graph each of these sets:

(a) $|x - 6| = 8$
(b) $y + |-6| = 10$
(c) $|10 - a| = 2$
(d) $|x| < 3$
(e) $|v| > -3$
(f) $|y| + 12 = 13$
(g) $|z| + 12 = 6$
(h) $|x - (-19)| = 3$
(i) $|y + 5| = 9$

7. What is the truth set of the sentence

$$|x - 4| < 1?$$

Draw the graph of this set on the number line.

8. What is the truth set of the sentence

$$|x - 4| > 1?$$

9. Graph the truth set of the compound sentence

$$x > 3 \text{ and } x < 5$$

on the number line. Is this set the same as the truth set of $|x - 4| < 1$? (We usually write "$3 < x < 5$" for the sentence "$x > 3$ and $x < 5$".)

10. For each sentence in the left column pick the sentence in the right column which has the same truth set:

$\|x\| = 3$	$x \leq -3$ or $x \geq 3$
$\|x\| < 3$	$x = -3$ or $x = 3$
$\|x\| \leq 3$	$x > -3$ and $x < 3$
$\|x\| > 3$	$x \geq -3$ and $x \leq 3$
$\|x\| \geq 3$	$x < -3$ or $x > 3$

10-4. The Meaning of Division.

Division is a familiar process in arithmetic. As we did with subtraction, we must now describe division for all pairs of real numbers, the negative as well as the positive numbers. Let's begin with a simple problem. You will already know the answer.

"Divide 15 by 3."

We can write this operation in any of the following ways:

$$15 \div 3 \quad \text{or} \quad 3\overline{)15} \quad \text{or} \quad \frac{15}{3}$$

To arrive at an answer we could ask ourselves the question

"What number do we multiply by 3 to get 15?"

The answer is clearly 5. But suppose we had the following problem:

"Divide (-20) by 4."

The question this time is, "What number do we multiply by 4 to get (-20)?" This may take a little more thought, but it should occur to us from our recent study of the multiplication of real numbers that

$$(4)(-5) = (-20).$$

What is our answer this time? Now suppose we were asked to

divide (-21) by (-7).

Since

$$(-7)(3) = (-21),$$

do you see that the answer, or "quotient", in this case is 3?

Oral Exercises 10-4a

1. For each of the following, state a question that has to do with multiplication, then answer it.

   (a) $\frac{12}{4}$ (c) $\frac{12}{-4}$

   (b) $\frac{-12}{4}$ (d) $\frac{-12}{-4}$

Oral Exercises 10-4a
(continued)

(e) $\frac{4a}{4}$

(f) $\frac{12m}{3m}$

(g) $\frac{27x^2}{9}$

(h) $\frac{26a}{-13}$

(i) $\frac{4}{\frac{1}{2}}$

(j) $\frac{-6a}{-2}$

2. Each of the following phrases or questions can be represented by $\frac{a}{b}$ or $\frac{b}{a}$. Decide which it will be; then state each as a question of the type "What number multiplied by a (or b) gives the product b (or a)?"

(a) b divided by a.
(b) The quotient of a by b.
(c) The ratio of b to a.
(d) The number expressed by a fraction whose numerator is a and whose denominator is b.
(e) The result of division where the divisor is b and the dividend is a.

Problem Set 10-4a

Find a simpler expression for each of the following: (Mentally use the method of changing each of the following to a question about multiplication. The domains of the variables do not include values for which the denominator is zero.)

1. $\frac{4}{-1}$
2. $\frac{6a}{3}$
3. $\frac{-30}{6}$
4. $\frac{49}{-7}$
5. $\frac{20m}{-4}$
6. $\frac{-21x}{x}$
7. $\frac{-35a^2}{5a}$
8. $\frac{x^2}{x}$
9. $\frac{-45ax^2}{-9x}$
10. $\frac{3(a + b)}{3}$

Problem Set 10-4a
(continued)

11. $\frac{a}{a}$

12. $\frac{6a}{-2}$

13. $\frac{ax}{a}$

14. $\frac{16a}{-4}$

15. $\frac{4}{\frac{1}{3}}$

16. $\frac{-a}{a}$

17. $\frac{3x}{x}$

18. $\frac{6a^2}{-3a}$

19. $\frac{0}{5}$

20. $\frac{-8}{\frac{1}{2}}$

You will recall that we defined subtraction of a real number as addition of the opposite of the number. Subtraction, that is, was defined in terms of addition.

Since division is related to multiplication in much the same way as subtraction is related to addition, we might expect to define division in terms of multiplication.

In order to do this, however, we must develop further the idea of multiplicative inverse. As we saw in Chapter 8,

> every real number a except zero has one and only one multiplicative inverse b such that $ab = 1$.

What is the multiplicative inverse of -3?

$(-3)(-\frac{1}{3}) = 1$. Therefore, the multiplication inverse of -3 is $-\frac{1}{3}$.

What is the multiplicative inverse of $\frac{7}{8}$?

$(\frac{7}{8})(\frac{8}{7}) = 1$. Therefore, the multiplicative inverse of $\frac{7}{8}$ is $\frac{8}{7}$.

The multiplicative inverse of a real number is also called the <u>reciprocal</u> of the number. In other words, the statements above can be reworded as follows:

The reciprocal of -3 is $-\frac{1}{3}$. The reciprocal of $\frac{7}{8}$ is $\frac{8}{7}$.

The symbol "$\frac{1}{x}$" is used to denote the reciprocal of the number x. This, together with the fact that the reciprocal of a real number is another name for the multiplicative inverse of the number, enables us to make the following statements:

(1) The number 0 has no reciprocal.

(2) Every real number except 0 has one and only one reciprocal.

(3) For any number x except 0, $x(\frac{1}{x}) = (\frac{1}{x})x = 1$.

The following questions and answers bring out some other important points concerning reciprocals, and they will help you to work the problems that follow.

What is the reciprocal of -5?

For any number x except 0, the reciprocal is $\frac{1}{x}$.
Therefore, a symbol for the reciprocal of -5 is "$\frac{1}{-5}$."
However, the multiplicative inverse of -5 is $-\frac{1}{5}$
since $(-5)(-\frac{1}{5}) = 1$.
Therefore, the reciprocal of -5 is $-\frac{1}{5}$.
This shows that "$\frac{1}{-5}$" and "$-\frac{1}{5}$" are names for the same number.

What is the reciprocal of -7?

The reciprocal of -7 is $-\frac{1}{7}$.

What is the reciprocal of $\frac{2}{3}$?

A symbol for the reciprocal of $\frac{2}{3}$ is "$\frac{1}{\frac{2}{3}}$"

However, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, since $(\frac{2}{3})(\frac{3}{2}) = 1$.

Therefore, "$\frac{1}{\frac{2}{3}}$" and "$\frac{3}{2}$" are names for the same number.

What is the reciprocal of $\frac{a}{b}$, where a is not zero and b is not zero?

The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

Therefore, $\frac{1}{\frac{a}{b}} = \frac{b}{a}$.

Check Your Reading

1. What is the multiplicative inverse of -3?
2. What is the reciprocal of -3?
3. What symbol is used for the reciprocal of x?
4. What number has no reciprocal?
5. Provided that x is not zero, the symbol "$x(\frac{1}{x})$" names what number?
6. What is the reciprocal of $\frac{7}{8}$?
7. If a is not zero and b is not zero, what is the reciprocal of $\frac{a}{b}$?
8. State another name for the number $\frac{1}{-5}$.
9. State another name for the number $\frac{1}{\frac{2}{3}}$
10. Provided that a is not zero and b is not zero, state another name for the number $\frac{1}{\frac{a}{b}}$.

Oral Exercises 10-4b

1. State the reciprocals of the following:

(a) 5 (c) $\frac{1}{2}$

(b) -7 (d) $-\frac{1}{6}$

Oral Exercises 10-4b
(continued)

(e) $\frac{3}{5}$

(f) $\frac{1}{-3}$

(g) $\frac{7}{3}$

(h) -12

(i) c

(j) $\frac{1}{d}$

2. Which of the following sentences are true? Which are false?

(a) $\frac{1}{-6} = -\frac{1}{6}$

(b) $\frac{1}{\frac{2}{3}} = \frac{1}{6}$

(c) $\frac{1}{\frac{1}{2}} = \frac{1}{2}$

(d) $\frac{1}{\frac{5}{7}} = \frac{7}{5}$

(e) $\frac{1}{\frac{1}{x}} = x$ for all x except $x = 0$

(f) $\frac{3}{-4} = -\frac{3}{4}$

3. Why is $\frac{13}{3}$ a reciprocal of $\frac{3}{13}$?

4. Zero has no reciprocal. Suppose that zero does have a reciprocal, call it n. Then

$$0 \cdot n = 1. \quad \text{Why?}$$

What is the truth set of this open sentence?

We are now ready to give a definition of division, and the idea of reciprocal will be used in this definition.

In dividing 15 by 3, reference was made to the question, "What number multiplied by 3 gives 15?" In other words, the following open sentences have the same truth number:

$$x = 15 \div 3, \qquad 3x = 15.$$

The multiplication property of equality may be applied to the sentence "$3x = 15$." Since the product of 3 and its reciprocal is 1, we multiply both sides by the number $\frac{1}{3}$, as follows:

Oral Exercises 10-4b
(continued)

(e) $\frac{3}{5}$

(f) $\frac{1}{-3}$

(g) $\frac{7}{3}$

(h) -12

(i) c

(j) $\frac{1}{d}$

2. Which of the following sentences are true? Which are false?

(a) $\frac{1}{-6} = -\frac{1}{6}$

(b) $\frac{1}{\frac{2}{3}} = \frac{1}{6}$

(c) $\frac{1}{\frac{1}{2}} = \frac{1}{2}$

(d) $\frac{1}{\frac{5}{7}} = \frac{7}{5}$

(e) $\frac{1}{\frac{1}{x}} = x$ for all x except $x = 0$

(f) $\frac{3}{-4} = -\frac{3}{4}$

3. Why is $\frac{13}{3}$ a reciprocal of $\frac{3}{13}$?

4. Zero has no reciprocal. Suppose that zero does have a reciprocal, call it n. Then

$$0 \cdot n = 1. \quad \text{Why?}$$

What is the truth set of this open sentence?

We are now ready to give a definition of division, and the idea of reciprocal will be used in this definition.

In dividing 15 by 3, reference was made to the question, "What number multiplied by 3 gives 15?" In other words, the following open sentences have the same truth number:

$$x = 15 \div 3, \qquad 3x = 15.$$

The multiplication property of equality may be applied to the sentence "$3x = 15$." Since the product of 3 and its reciprocal is 1, we multiply both sides by the number $\frac{1}{3}$, as follows:

Also, the definition leads to two simple but important properties of division.

<u>For any real number</u> a, $\frac{a}{1} = a \cdot 1$ The reciprocal of 1 is 1.

$= a.$

<u>For any real number</u> a, except 0,

$$\frac{a}{a} = a \cdot \frac{1}{a}$$

$$= 1.$$

The definition of division is really quite simple. Let us see how it might be applied in a few specific cases.

<u>Example</u> 1. Divide 10 by 2.

$\frac{10}{2} = (10)(\frac{1}{2}) = 5.$ $\frac{1}{2}$ is the reciprocal of 2.

<u>Example</u> 2. Divide -3 by $\frac{1}{5}$.

$\frac{-3}{\frac{1}{5}} = (-3)(5) = -15.$ 5 is the reciprocal of $\frac{1}{5}$.

<u>Example</u> 3. Simplify $\frac{x - 2}{x - 2}$.

For any number a except 0, $\frac{a}{a} = 1$.

Therefore, provided $x \neq 2$, $\frac{x - 2}{x - 2} = 1$.

Examples 1 and 2 above show once again that division by any real number except zero can be performed by multiplying by the reciprocal of the number. This is closely related to a fact developed earlier---subtracting any real number can be performed by adding the opposite of the number. This relation is pointed up by the following two columns.

For any number a, the additive inverse is -a, the opposite of a.	For any number a except zero, the multiplicative inverse is $\frac{1}{a}$, also called the reciprocal of a.

For any numbers a and b,
$a - b = a + (-b)$

Thus, subtraction is defined in terms of addition.

For any numbers a and b,
$b \neq 0, \quad a \div b = a(\frac{1}{b})$.

Thus, division is defined in terms of multiplication.

Check Your Reading

1. Dividing 15 by 3 is the same as multiplying 15 by what number?
2. The quotient $a \div b$ is the same as the product of a and what number?
3. For what cases is the quotient $a \div b$ not defined?
4. In what other way may the quotient $2 \div 3$ be written?
5. In what other way may the quotient $a \div b$, $b \neq 0$, be written?
6. Give a simpler name for $\frac{a}{1}$.
7. Give a simpler name for $\frac{a}{a}$, where a is not zero.
8. Give a simpler name for $\frac{x - 2}{x - 2}$, where x is not 2.
9. How is the difference $a - b$ defined in terms of addition?
10. How is the quotient $a \div b$ defined in terms of multiplication?

Oral Exercises 10-4c

1. Which of the following sentences are true for all values of the variables? If there is only one exception, state it. Give an example for those which you think are true for all values of the variables.

   (a) $\frac{1}{b} \cdot b = 1$

   (b) $\frac{a}{a} = 0$

   (c) $\frac{1}{a}(b + c) = \frac{b}{a} + \frac{c}{a}$

Oral Exercises 10-4c
(continued)

(d) If $-3x = a$, then $(-\frac{1}{3})(-3x) = (-\frac{1}{3})(a)$.

(e) If $\frac{a}{b} = c$, then $a = cb$.

(f) $\frac{x + 3}{x + 3} = 1$

(g) $\frac{2x + 3}{1} = 2x + 3$

2. By what number would you multiply each side of the equation to obtain an equivalent sentence whose truth set is easily found.

(a) $15x = 5$

(b) $-3y = -12$

(c) $\frac{1}{3}w = 28$

(d) $-\frac{2}{3}m = 14$

(e) $4 = \frac{3}{8}x$

(f) $-.9x = 12$

(g) $y = -8 + 4 - 7$

3. Use the definition of division to express each of the following in a form that involves multiplication.
For example:
"$\frac{3}{\frac{1}{2}}$" is equivalent to "$3 \cdot 2$".

(a) $\frac{1}{\frac{1}{5}}$

(b) $\frac{6}{\frac{3}{4}}$

(c) $\frac{-7}{\frac{2}{3}}$

(d) $\frac{x}{\frac{1}{x}}$

(e) $\frac{-3y}{-\frac{2}{3y}}$

(f) $-\frac{2}{3} \div \frac{4}{5}$

4. Give the reciprocal of

(a) -5,

(b) $-\frac{1}{5}$,

(c) $|-2|$,

(d) $\frac{-1}{|2|}$,

(e) $\frac{1}{a}$, if $a \neq 0$.

Problem Set 10-4c

1. Use the definition of division to write each of the following as an expression involving multiplication.

Example: $\dfrac{\frac{1}{2}}{\frac{2}{3}} = \frac{1}{2} \cdot \frac{3}{2}$

(a) $\frac{5}{6} \div \frac{2}{5}$

(b) $\dfrac{\frac{3}{4}}{\frac{1}{5}}$

(c) $b\overline{a}$ $\quad b \neq 0$

(d) $\dfrac{\frac{a}{b}}{\frac{x}{y}}$ $\quad b \neq 0, \quad y \neq 0, \quad x \neq 0$

(e) $\frac{a}{x} \div \frac{x}{a}$ $\quad a \neq 0, \quad x \neq 0$

(f) $\dfrac{\frac{3}{n-1}}{\frac{2}{a}}$ $\quad a \neq 0, \quad n \neq 1$

2. By what number would you multiply each side of the equation to obtain an equivalent equation whose truth set is easily found.

(a) $13x = 7$

(b) $-5x = 17$

(c) $\frac{3}{4}m = 9$

(d) $\frac{1}{4} = \frac{1}{3}y$

(e) $-\frac{5}{7}z = -\frac{3}{5}$

(f) $-7 = \frac{15}{6}a$

3. Use the definition of division to change the following indicated products into indicated quotients.

(a) $42 \cdot \frac{1}{6}$

(b) $5 \cdot \frac{1}{6}$

(c) $\frac{1}{2}c$

(d) $x \cdot \frac{1}{y}$

(e) $\frac{2}{3} \cdot bc$

(f) $5 \cdot 6$

4. Find the truth set of each of the following sentences:

(a) $6y = 41$

(b) $2x = 71$

(c) $-5m = -20 + 4$

(d) $3x = 7 + 7x$

(e) $\frac{1}{5}y = 5 - 10$

(f) $\frac{a}{3} - 2 = -21 + 3$

Problem Set 10-4c
(continued)

5. In the first two exercises of problem 4 show how the definition of divison is of assistance in obtaining the truth set.

6. Find two consecutive odd positive integers whose sum is less than or equal to 83.

7. A rectangle is 7 times as long as it is wide. Its perimeter is 144 inches. How wide is the rectangle?

8. John is three times as old as Dick. Three years ago the sum of their ages was 22 years. How old is each now?

9. Find two consecutive even integers whose sum is 46.

10. One-half of a number is 3 more than one-sixth of the sam number. What is the number?

11. A plane which flies at an average speed of 200 m.p.h. (when no wind is blowing) is held back by a head wind and takes $3\frac{1}{2}$ hours to complete a flight of 630 miles. What is the average speed of the wind?

12. Prove that for any real numbers a, b, c, where $b \neq 0$,
(i) if $a = bc$, then $\frac{a}{b} = c$,
(ii) if $\frac{a}{b} = c$, then $a = bc$.
(Hint: we may write $\frac{a}{b} = a \cdot \frac{1}{b}$, by our definition of division, and then use the multiplication property of equality.)

In this section we shall work with the expressions $\frac{-a}{b}$ and $\frac{a}{-b}$, both of which represent the quotient of one number by another.

Let us begin with the expression $\frac{-a}{b}$.

$\frac{-a}{b} = (-a)(\frac{1}{b})$ Definition of division.

$= (-1)(a)(\frac{1}{b})$ For any number a, $(-1)a = -a$.

$= (-1)(\frac{a}{b})$ By definition of division, $(a)(\frac{1}{b}) = \frac{a}{b}$.

$= -\frac{a}{b}$. For any number x, $(-1)x = -x$.

Now let us consider the expression $\frac{a}{-b}$.

$$\frac{a}{-b} = (a)(\frac{1}{-b}).$$

For any number x, except 0, the reciprocal is $\frac{1}{x}$. Therefore, the reciprocal of $-b$ is represented by $\frac{1}{-b}$. However, the reciprocal of $-b$ is $-\frac{1}{b}$, since $(-b)(-\frac{1}{b}) = (-1)(-1)(b)(\frac{1}{b}) = 1$.
So "$-\frac{1}{b}$" may be used for "$\frac{1}{-b}$".

$$\begin{aligned}\frac{a}{-b} &= (a)(-\frac{1}{b})\\ &= (a)(-1)(\frac{1}{b})\\ &= (-1)(a)(\frac{1}{b})\\ &= (-1)(\frac{a}{b})\\ &= -\frac{a}{b}.\end{aligned}$$

Therefore, "$\frac{-a}{b} = -\frac{a}{b}$" and "$\frac{a}{-b} = -\frac{a}{b}$". This shows that, for any real numbers a, b, and c, $b \neq 0$, the following are names for the same number:

$$\frac{-a}{b}, \quad \frac{a}{-b}, \quad -\frac{a}{b}. \quad (b \neq 0)$$

In most instances, the name "$-\frac{a}{b}$" is considered the simplest of the three as in the examples.

<u>Example</u> 1. $\frac{3}{-5} = -\frac{3}{5}$.

<u>Example</u> 2. $\frac{-3}{5} = -\frac{3}{5}$.

<u>Example</u> 3. $\frac{2}{-x} = -\frac{2}{x}$. Here we must specify that x is not zero.

As we have seen, however, simplification does not always mean the same thing. And in some instances, a change of names such as that illustrated in Example 4 is desirable.

<u>Example</u> 4. $-\frac{3}{2-x} = \frac{3}{-(2-x)}$

$= \frac{3}{-2+x}$

$= \frac{3}{x-2}$.

In this case, if the expression is to name a number, x cannot be 2.

Check Your Reading

1. Give two other names for the number $-\frac{a}{b}$.
2. Give two other names for the number $\frac{-3}{5}$.
3. Give another name for $-\frac{3}{2-x}$ in which the denominator is "$x - 2$."

Oral Exercises 10-4d

1. Give two other names for each of the following:

(a) $\frac{-2}{13}$

(b) $\frac{3}{-11}$

(c) $-\frac{4}{3}$

(d) $\frac{-5}{7a}$ if $a \neq 0$

(e) $\frac{6}{-7a}$ if $a \neq 0$

(f) $\frac{3m}{2(-n)}$ if $n \neq 0$

2. Give the opposite of each of the following in simplest form:

(a) $\frac{-8}{9}$

(b) $\frac{6}{-11a}$ if $a \neq 0$

(c) $\frac{-2a}{3b}$ if $b \neq 0$

(d) $\frac{3x}{7(-y)}$ if $y \neq 0$

(e) $\frac{5a}{-4}$

(f) $-\frac{8m}{7n}$ if $n \neq 0$

Oral Exercises 10-4d
(continued)

3. Perform the indicated operations and give the answers in simplest form.

(a) $\frac{5}{3} + \frac{-2}{3}$

(b) $\frac{3}{7} - \frac{2}{-7}$

(c) $\frac{3}{7} + \frac{2}{-7}$

(d) $\frac{3a}{b} - \frac{a}{-b}$ if $b \neq 0$

(e) $\frac{4}{-7} + \frac{-4}{7}$

(f) $\frac{-3}{2x} - \frac{5}{-2x}$ if $x \neq 0$

Problem Set 10-4d

In each of the problems below the domains of the variables are restricted to exclude division by zero.

1. Give two other names for each of the following:

(a) $\frac{-3}{5}$

(b) $\frac{1}{-7}$

(c) $-\frac{2}{3}$

(d) $-\frac{5}{6}$

(e) $\frac{7}{-3}$

(f) $\frac{-7}{4}$

(g) $\frac{-3a}{4b}$

(h) $-\frac{2m}{3n}$

(i) $\frac{5}{-6a}$

2. Give the simplest name for each of the following:

(a) $\frac{-3a}{4b}$

(b) $-\frac{2m}{3n}$

(c) $\frac{5}{-6a}$

(d) $\frac{16}{3(-x)}$

(e) $\frac{13(-a)}{2k}$

(f) $\frac{(-a)(-b)}{-c}$

3. Perform the indicated operations.

(a) $\frac{3}{4} + \frac{-1}{4}$

(b) $\frac{6}{5} + \frac{3}{-5}$

(c) $-\frac{8}{3} + \frac{-2}{3}$

(d) $\frac{5}{-6} + \frac{5}{6}$

(e) $\frac{6}{-5} + \frac{-2}{5}$

(f) $\frac{8}{3} + \frac{5}{-3}$

Problem Set 10-4d
(continued)

4. Perform the indicated operations.

(a) $\frac{5}{8} + \frac{1}{-4}$

(b) $\frac{1}{12} + \frac{-2}{3}$

(c) $\frac{3}{a} + \frac{5}{-a}$

(d) $\frac{2}{3b} + \frac{a}{-3b}$

(e) $\frac{-a}{b} + \frac{a}{-b}$

(f) $\frac{5}{-3m} + \frac{-2a}{3m}$

5. Write the opposite of each of the following:

(a) $\frac{-3}{4}$

(b) $\frac{8}{-13}$

(c) $-\frac{2}{3}$

(d) $\frac{5}{-a}$

(e) $\frac{-2a}{3b}$

(f) $-\frac{3}{5x}$

6. Perform the indicated operations.

(a) $\frac{5}{8} - \frac{-3}{8}$

(b) $\frac{6}{7} - \frac{2}{-7}$

(c) $\frac{7}{3} - (-\frac{5}{3})$

(d) $\frac{5}{2a} - \frac{-3}{2a}$

(e) $\frac{6b}{5c} - \frac{2b}{-5c}$

(f) $\frac{-a}{b} - \frac{a}{-b}$

7. Perform the indicated operations.

(a) $\frac{1}{a - b} + \frac{2}{b - a}$

(b) $\frac{2}{x - y} - \frac{3}{y - x}$

(c) $\frac{5a}{a - b} + \frac{2a}{b - a}$

(d) $\frac{6a}{m + n} + \frac{2a}{-m - n}$

(e) $\frac{6a}{m + n} - \frac{2a}{-m - n}$

(f) $\frac{a}{a - b} - \frac{b}{b - a}$

10-5. Common Names.

In Chapter 2 we referred to some special names for rational numbers, which we called "common names". The common name is, in a sense, the simplest name for a number. For example,

a common name for $\frac{20}{5}$ is 4

a common name for $\frac{14}{21}$ is $\frac{2}{3}$

How do we obtain these common names? We use the multiplication property of 1. Remember, the multiplication property of 1 tells us that if a is any real number, then

$$(a)(1) = a.$$

Using this property and also the property that for any non-zero real number a

$$\frac{a}{a} = 1,$$

we can see that

$$\frac{20}{5} = \frac{4\cdot 5}{5} = 4 \cdot \frac{5}{5} = 4(1) = 4.$$

We can also show that

$$\frac{14}{21} = \frac{2\cdot 7}{3\cdot 7} = \frac{2}{3} \cdot \frac{7}{7} = \frac{2}{3}(1) = \frac{2}{3}.$$

The step in the above process in which we wrote

$$\frac{2 \cdot 7}{3 \cdot 7} = \frac{2}{3} \cdot \frac{7}{7}$$

represents an operation which is familiar from our study of fractions in arithmetic. By means of our definition of division it is now possible for us to prove that this operation will work in all cases provided that the denominators are not zero.

The theorem, which we will now prove, states that

for any real numbers a, b, c, and d,
if $b \neq 0$, and if $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

To prove this, we first see from our definition of division that

$$\frac{a}{b} = a \cdot \frac{1}{b} \quad \text{and} \quad \frac{c}{d} = c \cdot \frac{1}{d},$$

and therefore

$$\frac{a}{b} \cdot \frac{c}{c} = (a \cdot \tfrac{1}{b})(c \cdot \tfrac{1}{d})$$

$$= (ac)\left((\tfrac{1}{b})(\tfrac{1}{d})\right).$$ (associative and commutative properties of multiplication)

Now we must show that $(\frac{1}{b})(\frac{1}{d}) = \frac{1}{bd}$, that is, that the product of the reciprocals of two numbers is the reciprocal of the product of these numbers. If we multiply $(\frac{1}{b})(\frac{1}{d})$ by bd we obtain the product 1. (Why?) Hence $(\frac{1}{b})(\frac{1}{d})$ is the reciprocal of bd, as is $\frac{1}{bd}$.

Now we see that

$$\frac{a}{b} \cdot \frac{c}{d} = (ac)(\tfrac{1}{b} \cdot \tfrac{1}{d})$$

$$= (ac)(\tfrac{1}{bd})$$

$$= \frac{ac}{bd}.$$ Using, once more, the definition of division

Thus, we have proved our theorem.

In arithmetic you were told, "To multiply two fractions, we multiply the numerators to get the new numerator and we multiply the denominators to get the new denominator."

The theorem, which states that

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd},$$

shows exactly why your teacher was right in saying this.

The following examples will further illustrate the process of finding a common name.

<u>Example</u> 1. Simplify $\frac{3y - 3}{2y - 2}$. Note: When we write this phrase, we must assume that the <u>domain</u> of the variable y cannot include the number 1. Can you see why?

$$\frac{3y - 3}{2y - 2} = \frac{3(y - 1)}{2(y - 1)} \quad \text{by the distributive property}$$

$$= \frac{3}{2}(1) \quad \text{since } \frac{y - 1}{y - 1} = 1, \text{ if } y \neq 1$$

$$= \frac{3}{2}.$$

<u>Example</u> 2. Simplify $\frac{(2x + 5) - (5 - 2x)}{8}$.

$$\frac{(2x + 5) - (5 - 2x)}{8} = \frac{2x + 5 - 5 + 2x}{8}$$

$$= \frac{4x}{8}$$

$$= \frac{4 \cdot x}{4 \cdot 2}$$

$$= \frac{4}{4} \cdot \frac{x}{2}$$

$$= \frac{x}{2}.$$

<u>Example</u> 3. Simplify $\frac{2x + 4}{6x - 12}$. In order for this expression to name a number, $6x - 12$ cannot be 0. Therefore, x cannot be 2. We can say, then, that the domain of x includes all numbers except 2.

Provided that x is not 2, $\frac{2x + 4}{6x - 12} = \frac{2(x + 2)}{6(x - 2)}$

$$= \frac{2}{6} \cdot \frac{(x + 2)}{(x - 2)}$$

$$= \frac{1}{3} \cdot \frac{(x + 2)}{(x - 2)}$$

$$= \frac{(x + 2)}{3(x - 2)}.$$

Check Your Reading

1. In the text two basic properties were used to change the name of the number $\frac{20}{5}$ to 4. What were these two properties?

2. If $b \neq 0$ and $d \neq 0$, give another name for the product $\frac{a}{b} \cdot \frac{c}{d}$.

3. If the phrase "$\frac{3y - 3}{2y - 2}$" is to name a number, there is one value of y that cannot be permitted. What is this value?

4. Give a simpler name for $\frac{3(y - 1)}{2(y - 1)}$.

5. What number is excluded from the domain of x in the phrase "$\frac{2x + 4}{6x + 12}$"?

Oral Exercises 10-5

1. Tell how each of the following fractions can be written as the product of a simpler fraction and a numeral for 1. Then simplify.

Example: If $y \neq -2$, $\frac{4(y + 2)}{3(y + 2)}$ can be written $\frac{4}{3}\left(\frac{y + 2}{y + 2}\right)$, where $\frac{y + 2}{y + 2}$ is a numeral for 1. (Why cannot y be -2?) Thus, a common name is $\frac{4}{3}$.

(a) $\frac{2}{4}$

(b) $\frac{14}{16}$

(c) $\frac{3x}{3y}$

(d) $\frac{-x}{x^2}$

(e) $\frac{3(x + 2)}{(x - 1)3}$

(f) $\frac{2x + 4}{6x + 12}$

(g) $\frac{2x - 4}{3x - 6}$

(h) $\frac{x + 5}{y + 5}$

(i) $\frac{3x + 4}{3x + 5}$

(j) $\frac{(x + 2)(x + 3)}{x + 3}$

Problem Set 10-5

1. Simplify each of the following. Indicate the restrictions on the domains.

(a) $\frac{10}{35a}$

(b) $\frac{6a}{6b}$

(c) $\frac{-9xy}{3xm}$

(d) $\frac{3x}{x^2}$

(e) $\frac{7(x + 2)}{x + 2}$

(f) $\frac{2x + 4}{2x + 1}$

(g) $\frac{(x + 2)(x - 3)^2}{(x + 2)(x - 3)}$

(h) $(x - 2)(x + 4)\frac{9x^2}{(3x)^2}$

(i) $\frac{x}{b} \cdot \frac{b + 1}{x}$

(j) $\frac{3x + 3}{5x + 5}$

(k) $\frac{2a - a^2}{a}$

(l) $\frac{xy + y}{y}$

(m) $\frac{2x - 3}{2y - 3}$

(n) $\frac{x - 1}{3x - 3}$

(o) $\frac{a - b}{b - a}$

(p) $\frac{2x - 4}{4 - 2x}$

2. Find the truth set of each of the following sentences. Indicate the restrictions on the domains.

(a) $\frac{9x^2}{3x} = 6$

(b) $\frac{3x(x - 1)}{x - 1} = 3$

(c) $\frac{3x(x + 1)}{x + 1} = 3$

(d) $\frac{3x(x + 1)}{x + 1} = 0$

*(e) $\frac{(2x - 5) + (4x - 1)}{4x - 4} < 5$

*(f) $\frac{2x + 2 - (3x - 5)}{21x - 3x^2} = 3x$

*(g) $\frac{7(x - \frac{2}{7}) + 3x - 4}{8x - 2(4 + 4x)} = -\frac{1}{2}$

Problem Set 10-5
(continued)

3. Mr. Brown is employed at an initial salary of \$3600, with annual increments of \$300, while Mr. Jones starts at the same time at an initial salary of \$4500, with annual increments of \$200. After how many years will both men be earning the same salary?

4. Bob is twice as old as Bill. Three years from now the sum of their ages will be 30 years. How old is each boy now?

10-6. Fractions.

When we are asked to simplify a given expression it is important that we understand exactly what is meant. "Simplify", we recall, means "find a common name for". To do this properly we shall have to agree to certain basic ideas as to just what a "common name" should really be. There are three important ideas, or conventions, which we will now discuss.

1) A common name contains no indicated division if it can be avoided. For example:

   $\frac{15}{3}$ should be "simplified" to 5.

2) If a common name must contain an indicated division, then the resulting expression should be written in "lowest terms". By this we mean that a fraction such as

   $\frac{6}{9}$ should be changed to $\frac{2}{3}$ if we want the common name.

Note: In Chapter 1 we described a "fraction" as a numeral which indicates the quotient of two numbers. Thus, a fraction involves two numerals, a numerator and a denominator. When there is no chance for confusion, we shall often use the word "fraction" to mean the number itself.

3) We have learned that the fraction

$$\frac{-a}{b}$$

may be written as

$$\frac{a}{-b} \text{ or } -\frac{a}{b}.$$

We will call the third form, $-\frac{a}{b}$, the common name.

For example, we write $\frac{-2}{3}$ and $\frac{2}{-3}$ as $-\frac{2}{3}$.

The theorem which ends with the sentence

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$$

tells us how to write the indicated product of <u>two</u> fractions as <u>one</u> fraction. We have used this theorem in applying the <u>multiplication property of 1</u> to the fraction

$$\frac{14}{21}.$$

In this case we used the theorem in an "opposite" sense by splitting one fraction into two, that is

$$\frac{14}{21} = \frac{2\cdot 7}{3\cdot 7} = \frac{2}{3}\cdot\frac{7}{7}$$

A direct application can be found in the following example.

Simplify $\frac{x}{3}\cdot\frac{5}{6}$.

$$\frac{x}{3}\cdot\frac{5}{6} = \frac{x\cdot 5}{3\cdot 6}$$

$$= \frac{5x}{18}$$

<u>Example:</u> Simplify $(\frac{3}{4})\frac{(x+2)}{5}$

$$(\frac{3}{4})\frac{(x+2)}{5} = \frac{3\cdot(x+2)}{4\cdot 5}$$

$$= \frac{3x+6}{20}$$

Sometimes we use this theorem "both ways" in the same problem.

<u>Example:</u> Simplify $\frac{3}{2}\cdot\frac{14}{9}$

$$\begin{aligned}\frac{3}{2}\cdot\frac{14}{9} &= \frac{3.14}{2\cdot 9}\\ &= \frac{3.(2\cdot 7)}{2\cdot(3\cdot 3)}\\ &= \frac{7\cdot(2\cdot 3)}{3\cdot(2\cdot 3)} \qquad \text{Why?}\\ &= \frac{7}{3}\cdot\left(\frac{6}{6}\right)\\ &= \frac{7}{3}\end{aligned}$$

Check Your Reading

1. What is a shorter way of saying "find a common name for . . ."?
2. State three things to consider to be sure an expression has been "simplified".
3. What theorem shows us how to "simplify" $\left(\frac{3}{4}\right)\frac{x+2}{5}$?

Oral Exercises 10-6a

Simplify the following expressions. What restrictions on the domain of the variables need to be made?

1. $\frac{-1}{7}$
2. $\frac{3}{9}$
3. $\frac{-3}{5}$
4. $\frac{4x}{-3}$
5. $\frac{1}{3}\cdot\frac{7x}{2}$
6. $\frac{2x}{3}\cdot\frac{x+1}{2}$
7. $\frac{3}{x}\cdot\frac{x(x-1)}{3}$
8. $\frac{-2(x-3)(x-1)}{-2(x-1)}$
9. $\frac{24x}{-2(x-1)}$
10. $\frac{-(x+1)}{-3(x+1)}$

Oral Exercises 10-6a
(continued)

11. $\frac{x - 1}{1 - x}$

12. $\frac{3(a + b)}{3(-a - b)}$

13. $\frac{30a^2(a - 2)}{5a(a - 2)}$

14. $\frac{x + 1}{y + 1}$

Problem Set 10-6a

1. Simplify. When variables are present indicate the restrictions which must be made on their domains.

(a) $\frac{3}{8}\cdot\frac{7}{2}$

(b) $\frac{4}{7}\cdot\frac{21}{10}$

(c) $\frac{1}{9}\left((-5)(-2)\right)$

(d) $\frac{1}{n}\cdot\frac{1}{n}$

(e) $n \cdot \frac{1}{n}$

(f) $\frac{1}{n}\cdot\frac{1}{x}$

(g) $\frac{1}{n + n}$

(h) $\frac{-x}{4}\cdot\frac{-x}{3}$

(i) $(-\frac{10}{3})(-\frac{3}{2})$

(j) $(-\frac{10}{3}) + (-\frac{3}{2})$

(k) $(4a^2)(\frac{a}{3})$

(l) $\frac{3}{x}$

2. Simplify each of the following expressions. Indicate what restrictions have to be made on the domains of the variables.

(a) $\frac{2}{9}\cdot\frac{18}{35}\cdot\frac{7}{4}$

(b) $\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a^2}$

(c) $\frac{6x}{11y}\cdot\frac{7y}{5x}$

Problem Set 10-6a
(continued)

(d) $\frac{9b}{c} \cdot \frac{c}{81b}$

(e) $\frac{7ab}{5a^2} \cdot \frac{15ab}{b^2}$

(f) $\frac{2r^2}{9rs} \cdot \frac{45s^2}{14rs}$

(g) $\frac{7(x + y)(x - y)}{14(x - y)}$

3. Use the distributive property to write the numerators and denominators of the following as indicated products, then simplify the fractions. Write the restrictions on the domains of the variables.

(a) $\frac{2u + 2v}{5u + 5v}$

(b) $\frac{cm + cn}{3m + 3n}$

(c) $\frac{ax - ay}{bx - by}$

(d) $\frac{2x - 4}{6 - 3x}$

(e) $\frac{9x - 6}{15x - 10}$

(f) $\frac{7m^2 + 2m}{21m + 6}$

4. Find the truth sets of the following open sentences:

(a) $3x - (\frac{-1}{7}) = \frac{8}{7}$

(b) $\frac{7}{3}x = \frac{7}{12}$

(c) $3x - \frac{2}{3} = x$

(d) $\frac{5}{6}x > \frac{5}{-12}$

(e) $5|x| = \frac{5}{3}$

(f) $3x = \frac{9(x - 1)}{x - 1}$

5. A passenger train averages 20 miles per hour more than a freight train. At the end of 5 hours the passenger train has traveled 100 miles farther than the freight train. How fast did the freight train travel?

Problem Set 10-6a
(continued)

6. On a 20 percent discount sale, a chair is marked \$30.00. What was the price of the chair before the sale?

7. One-half of a number is 3 more than one-sixth of the same number. What is the number?

The property which we most frequently use in simplifying the sum or difference of two fractions is the multiplication property of 1. You will also recognize many of the other properties which are being used in the following simplication.

$\frac{x}{3} + \frac{y}{5} = \frac{x}{3}(1) + \frac{y}{5}(1)$	(multiplication property of one)
$= (\frac{x}{3})(\frac{5}{5}) + (\frac{y}{5})(\frac{3}{3})$	$(\frac{a}{a} = 1$ if $a \neq 0)$
$= \frac{5x}{15} + \frac{3y}{15}$	$(\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd})$
$= 5x(\frac{1}{15}) + 3y(\frac{1}{15})$	(definition of division)
$= (5x + 3y)(\frac{1}{15})$	(distributive property)
$= \frac{5x + 3y}{15}$	(definition of division)

You should understand each of the steps above but in practice a more condensed form is permissible, such as

$$\frac{x}{3} + \frac{y}{5} = \frac{x}{3}(\frac{5}{5}) + \frac{y}{5}(\frac{3}{3})$$

$$= \frac{5x}{15} + \frac{3y}{15}$$

$$= \frac{5x + 3y}{15}$$

Can you supply the reasons for the steps of the following simplification?

$$\frac{5x}{9} - \frac{2x}{3} = \frac{5x}{9} + \left(-\frac{2x}{3}\right)$$

$$= \frac{5x}{9} + \left(\frac{-2x}{3}\right)$$

$$= \frac{5x}{9} + \left(\frac{-2x}{3}\right)(1)$$

$$= \frac{5x}{9} + \frac{-2x}{3} \cdot \frac{3}{3}$$

$$= \frac{5x}{9} + \frac{-6x}{9}$$

$$= 5x\left(\frac{1}{9}\right) + (-6x)\left(\frac{1}{9}\right)$$

$$= \left(5x + (-6x)\right)\frac{1}{9}$$

$$= -x \cdot \frac{1}{9} = \frac{-x}{9} = -\frac{x}{9}$$

If we were asked to find the truth set of an open sentence containing fractions, such as

$$\frac{2x}{3} = \frac{x}{3} + 2$$

we could use the multiplication property of equality as follows:

$$\frac{2x}{3} = \frac{x}{3} + 2$$

may be written as

$$3\left(\frac{2x}{3}\right) = 3\left(\frac{x}{3} + 2\right),$$

where both sides have been multiplied by 3. This becomes

$$\frac{3(2x)}{3} = \frac{3(x)}{3} + 3(2)$$

and then we have

$$2x = x + 6 \quad \text{since} \quad \frac{3}{3} = 1.$$

The truth value for this sentence can easily be found by adding $(-x)$ to both sides. Do you see that the truth value is 6?

As another example, consider the sentence

$$\frac{2x}{3} + \frac{1}{3} = \frac{x}{4} + 2 .$$

To make our sentence easier to work with, we can multiply both sides by 12. We then have

$$12(\frac{2x}{3} + \frac{1}{3}) = 12(\frac{x}{4} + 2) ,$$

which becomes

$$12(\frac{2x}{3}) + 12(\frac{1}{3}) = 12(\frac{x}{4}) + 12(2) .$$

This can be written

$$\frac{3 \cdot 4 \cdot 2x}{3} + \frac{3 \cdot 4}{3} = \frac{3 \cdot 4 \cdot x}{4} + 24 ,$$

which becomes

$$(\frac{3}{3})8x + (\frac{3}{3})4 = (\frac{4}{4})3x + 24 .$$

Finally we can write

$$8x + 4 = 3x + 24 .$$

The truth value of this sentence may now be found quite easily. Do you see that it is 4?

In using the multiplication property of equality we multiplied both sides by the number 12. Can you see why this is a good number to use? We know that 12 can be written as $3 \cdot 4$.

Check Your Reading

1. Before the expression $\frac{x}{3} + \frac{y}{5}$ can be simplified what are the forms of 1 that would be used to help in the simplification?
2. Why is $\frac{5x}{15}$ the same as $5x(\frac{1}{15})$?
3. What property can be applied to "$5x(\frac{1}{15}) + 3y(\frac{1}{15})$" to simplify the expression?

Oral Exercises 10-6b

1. State what form of 1 would be used to make the indicated change in each of the following fractions. Then name the new numerator. Assume that the domains of the variables do not include any values which make the denominator zero.

   (a) $\frac{3x}{4}(\text{——}) = \frac{\quad}{8}$

   (b) $\frac{5x}{3}(\text{——}) = \frac{\quad}{24}$

   (c) $\frac{3}{x}(\text{——}) = \frac{\quad}{3x}$

   (d) $\frac{5a}{-y}(\text{——}) = \frac{\quad}{ay}$

   (e) $\frac{7a}{x + y}(\text{——}) = \frac{\quad}{3(x + y)}$

   (f) $\frac{5}{a - b}(\text{——}) = \frac{\quad}{(a - b)(a + b)}$

   (g) $\frac{6a}{3a(m + n)}(\text{——}) = \frac{\quad}{3a(m + n)^2}$

   (h) $\frac{b(x - y)}{a(x + y)}(\text{——}) = \frac{\quad}{ab(x + y)(x - y)}$

   (i) $\frac{4}{2 - x}(\text{——}) = \frac{\quad}{x - 2}$

   (j) $\frac{2 - x}{3 - x}(\text{——}) = \frac{\quad}{x - 3}$

2. Simplify each of the following sums. Assume that the domains of the variables are restricted to exclude division by zero.

   (a) $\frac{x}{4} + \frac{3x}{4}$

   (b) $\frac{3a}{6} - \frac{a}{6}$

   (c) $\frac{4m}{5} + \frac{n}{5}$

   (d) $\frac{5x + 2}{4} + \frac{7x - 1}{4}$

Oral Exercises 10-6b
(continued)

(e) $\frac{4y - 8}{11} - \frac{8y + 2}{11}$

(f) $\frac{\frac{3}{4}t + 2}{6} - \frac{\frac{3}{4}t - 11}{6}$

(g) $\frac{7}{8x} + \frac{3}{8x} + \frac{y}{8x}$

(h) $\frac{-11}{x + y} + \frac{12}{x + y} - \frac{3y}{x + y}$

(i) $x + \frac{x}{4} - \frac{x}{2}$

(j) $\frac{3}{x} - \frac{2}{3x} + \frac{1}{x}$

Problem Set 10-6b

1. Simplify. Assume that $a \neq 0$.

(a) $\frac{5}{9} - \frac{2}{3}$

(b) $\frac{4}{a} + \frac{5}{a}$

(c) $\frac{4}{a} + \frac{5}{2a}$

(d) $\frac{4}{a} + \frac{5}{a^2}$

(e) $\frac{4a}{7} - \frac{a}{35}$

(f) $\frac{4}{7} - \frac{a}{35}$

(g) $\frac{x + 8}{10} + \frac{x - 4}{2}$

2. Simplify the following expressions. Assume that the domains of the variables do not include values for which denominators are zero.

(a) $\frac{a}{a + b} + \frac{b}{a + b}$

(b) $\frac{3x}{4} + \frac{2y}{4}$

(c) $a^2 - b^2 - (a^2 - b^2)$

Problem Set 10-6b
(continued)

(d) $\frac{3x}{7} + \frac{y}{14}$

(e) $\frac{4a}{36} - \frac{a}{4}$

(f) $x + \frac{1}{x}$

(g) $b + \frac{1}{a + b}$

(h) $\frac{7}{a + b} + 5$

(i) $\frac{\frac{3}{4}}{\frac{2x}{5}}$

(j) $\frac{3}{x + 1} - \frac{4}{x + 1}$

3. Find the truth set of each of the following sentences:

(a) $\frac{x}{5} + 3 = \frac{2}{3}$

(b) $\frac{7}{8} - \frac{1}{4}z = \frac{1}{2}$

(c) $\frac{3}{5}x - \frac{1}{2} = \frac{8}{15}x$

(d) $\frac{y}{2} + \frac{1}{3}y = \frac{2y}{5}$

(e) $\frac{x}{7} + \frac{1}{2} > \frac{9}{14}$

(f) $\frac{3}{4}x - \frac{2}{3} = \frac{1}{4} + \frac{5}{6}x$

(g) $\frac{3}{7} + \frac{5}{4}y = \frac{3}{14} + \frac{3}{2}y + \frac{1}{7}$

(h) $\frac{1}{3}z + \frac{1}{3} = 2 + \frac{1}{2}z$

(i) $4y + 7 = 4y - 3$

(j) $-\frac{3}{4} + \frac{4}{5}u = \frac{11}{8} + u - \frac{5}{8}$

4. Mary bought 15 three-cent stamps and some four-cent stamps. If she paid \$1.80 for all the stamps, was she charged the correct amount?

5. John has 50 coins which are nickels, pennies, and dimes. He has four more dimes than pennies, and six more nickels than dimes. How many of each kind of coin does he have? How much money does he have?

Problem Set 10-6b
(continued)

6. John, who is saving his money for a bicycle, said, "When I have one dollar more than three times the amount I now have, I will have enough money for my bicycle." If the bicycle costs \$76, how much money does John have now?

7. The sum of two numbers is 240, and one number is $\frac{3}{5}$ times the other. Find the two numbers.

8. The numerator of the fraction $\frac{4}{7}$ is increased by an amount x. The value of the resulting fraction is $\frac{27}{21}$. By what amount was the numerator increased?

9. $\frac{13}{24}$ of a number is 13 more than $\frac{1}{2}$ of the number. What is the number?

10. Joe is $\frac{1}{3}$ as old as his father. In 12 years he will be $\frac{1}{2}$ as old as his father then is. How old is Joe? His father?

11. The sum of two positive integers is 7 and their difference is 3. What are the numbers? What is the sum of the reciprocals of these numbers? What is the difference of the reciprocals?

12. A fraction, such as $\frac{2}{3}$, is often called the <u>ratio</u> of 2 to 3, or the ratio $\frac{2}{3}$. We also call a sentence in the form

$$\frac{a}{b} = \frac{c}{d}$$

a <u>proportion.</u> It is read "a, b, c, d are in proportion." These words are convenient when we are using division to show the relative size of two numbers. Since a ratio is a fraction, and a proportion is a simple sentence involving two fractions, these two words are just names for things with which we are already familiar.

<u>Example:</u> Two partners in a firm are to divide the profits in the ratio $\frac{3}{5}$. If the man receiving the larger share receives \$8550, how much does the other partner receive?

Problem Set 10-6b
(continued)

If the smaller share is p dollars, then $\frac{p}{8550} = \frac{3}{5}$.
If there is a number p such that the sentence is true, then

$$p = \frac{3}{5} \cdot 8550$$

$$= 5130.$$

If $p = 5130$, then $\frac{p}{8550} = \frac{5130}{8550} = \frac{3}{5} \times \frac{1710}{1710} = \frac{3}{5}$.
Hence, the smaller share is \$5130.

Notice how saying that the shares are in the ratio $\frac{3}{5}$ leads naturally to writing the proportion $\frac{p}{8550} = \frac{3}{5}$.

(a) In a certain school the ratio of boys to girls was $\frac{7}{6}$. If there were 2600 students in the school, how many girls were there?

(b) In a shipment of 800 radios, $\frac{1}{20}$ of the radios were defective. What is the ratio of defective radios to non-defective radios in the shipment?

(c) The ratio of faculty to students in a college is $\frac{2}{19}$. If there are 1197 students, how many faculty members are there?

(d) If two numbers are in the ratio $\frac{5}{9}$, explain why we may represent those numbers as $5x$ and $9x$. What are the numbers if $x = 7$? if $x = 100$?

(e) Prove that if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

(f) Prove that if $ad = bc$ and $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$.

(g) Show, using the properties of one, that the proportion $\frac{6}{15} = \frac{2}{5}$ is true.

(h) Assuming that the proportion $\frac{x}{y} = \frac{p}{q}$ is true, use parts (e) and (f) to find seven other true proportions.
For example, if $\frac{x}{y} = \frac{p}{q}$, then $xq = yp$ or $qx = yp$.
Hence, by part (f), $\frac{q}{y} = \frac{p}{x}$.

We will now see how to simplify the quotient of two fractions. Several methods are possible for doing this. Let us look at the following examples.

<u>Example 1</u>. Simplify

$$\frac{\frac{5}{3}}{\frac{7}{2}}.$$

We shall use the multiplication property of 1, in which we use $\frac{6}{6}$ for 1. The reason for using $\frac{6}{6}$ will be made clearer as the work goes on.

$$\frac{\frac{5}{3}}{\frac{7}{2}} = \frac{\frac{5}{3}}{\frac{7}{2}} \cdot \frac{6}{6} \qquad \text{(Multiplication property of 1)}$$

$$= \frac{\frac{5}{3} \cdot 6}{\frac{7}{2} \cdot 6} \qquad \left(\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}\right)$$

$$= \frac{\frac{5}{3} \cdot \frac{6}{1}}{\frac{7}{2} \cdot \frac{6}{1}} \qquad \left(a = \frac{a}{1}\right)$$

$$= \frac{\frac{5 \cdot 2 \cdot 3}{3}}{\frac{7 \cdot 3 \cdot 2}{2}}$$

$$= \frac{10 \cdot \frac{3}{3}}{21 \cdot \frac{2}{2}}$$

$$= \frac{(10)(1)}{(21)(1)}$$

$$= \frac{10}{21}$$

Example 2. Simplify

$$\frac{\frac{3}{4}x + \frac{1}{2}}{\frac{2}{3}} .$$

We shall again use the property of 1, but this time we let

$$1 = \frac{\frac{3}{2}}{\frac{3}{2}} ,$$

where we choose $\frac{3}{2}$ because it is the reciprocal of $\frac{2}{3}$.

$$\frac{\frac{3}{4}x + \frac{1}{2}}{\frac{2}{3}} = \frac{\left(\frac{3}{4}x + \frac{1}{2}\right)}{\frac{2}{3}} \cdot \frac{\frac{3}{2}}{\frac{3}{2}}$$

$$= \frac{\left(\frac{3}{4}x + \frac{1}{2}\right)\left(\frac{3}{2}\right)}{\frac{2}{3} \cdot \frac{3}{2}}$$

$$= \frac{\frac{9}{8}x + \frac{3}{4}}{1}$$

$$= \frac{9}{8}x + \frac{3}{4} .$$

Check Your Reading

1. In Example 1 why was $\frac{6}{6}$ used as the form of 1?
2. In Example 2 what happened as a result of using $\frac{\frac{3}{2}}{\frac{3}{2}}$ for 1?
3. What is the reciprocal of $\frac{2}{3}$?

Oral Exercises 10-6c

1. State two forms of 1 that could be used to simplify each of the following expressions. (The values of the variables for which denominators are zero are excluded.)

(a) $\dfrac{\frac{1}{2}}{\frac{3}{4}}$

(b) $\dfrac{\frac{a}{b}}{\frac{c}{d}}$

(c) $\dfrac{\frac{3x}{2}}{\frac{y}{3}}$

(d) $\dfrac{\frac{1}{3x}}{\frac{2}{5x}}$

(e) $\dfrac{\frac{4x}{3y}}{\frac{3y}{4x}}$

(f) $\dfrac{\frac{-7a^2}{4b}}{\frac{-2a}{7b}}$

(g) $\dfrac{\frac{2r + t}{3m}}{\frac{-7}{5m}}$

(h) $\dfrac{\frac{x + 2}{3}}{\frac{x - 1}{6}}$

(i) $\dfrac{\frac{4(x - 2)}{3x}}{\frac{x - 2}{x}}$

(j) $\dfrac{\frac{(x - 1)(x - 2)}{(x - 3)}}{\frac{3x}{(x - 1)}}$

Problem Set 10-6c

In this problem set the values of the variables for which denominators are zero are excluded.

1. Simplify.

(a) $\dfrac{\frac{2}{3}}{\frac{5}{6}}$

(b) $\dfrac{\frac{4}{3}}{\frac{1}{12}}$

(c) $\dfrac{\frac{3}{5}}{\frac{7}{15}}$

(d) $\dfrac{\frac{1}{2}}{\frac{3}{8}}$

(e) $\dfrac{\frac{1}{3}}{\frac{1}{10}}$

(f) $\dfrac{\frac{3}{4}}{\frac{2}{3}}$

Problem Set 10-6c
(continued)

2. Simplify.

(a) $\dfrac{\frac{2x}{3}}{\frac{5x}{2}}$

(b) $\dfrac{\frac{3m}{5}}{\frac{m}{2}}$

(c) $\dfrac{\frac{5a}{3}}{\frac{5a}{6}}$

(d) $\dfrac{\frac{1}{a}}{\frac{1}{b}}$

(e) $\dfrac{\frac{2}{a}}{\frac{1}{2a}}$

(f) $\dfrac{\frac{1}{x-1}}{\frac{1}{x-2}}$

3. Simplify.

(a) $\dfrac{\frac{1}{2}x + \frac{1}{4}}{\frac{2}{3}}$

(b) $\dfrac{\frac{2x-1}{2}}{\frac{x+1}{3}}$

(c) $\dfrac{\frac{a}{3} + \frac{a}{5}}{\frac{a}{15}}$

(d) $\dfrac{a}{\frac{3}{b}}$

(e) $\dfrac{\frac{m}{6}}{n}$

(f) $\dfrac{\frac{x-1}{2}}{\frac{x+1}{3}}$

4. Simplify.

(a) $\dfrac{\frac{3a}{4}}{\frac{a}{3}}$

(b) $\frac{3a}{4} + \frac{a}{3} - 2a$

(c) $(-3a^2b)(2bx)$

(d) $-(2a - 3b + 2c - a + b)$

(e) $\dfrac{\frac{x+1}{6}}{\frac{(x+1)^2}{3}}$

(f) $\dfrac{-28a^2bx}{-12ab^2x}$

Problem Set 10-6c
(continued)

5. Simplify.

(a) $(x + 2)(x - 2)$

(b) $\dfrac{\dfrac{(y - 2)(y - 3)}{9y}}{\dfrac{3xy(y - 3)}{81xy^2}}$

(c) $(\frac{3}{8}x - 2y + 4) - (\frac{1}{2}x - \frac{7}{5}y - 8)$

(d) $\dfrac{\frac{4x}{3} - \frac{1}{5}}{\frac{7}{15}}$

(e) $\dfrac{\frac{4a}{3} + 2a}{\frac{1}{2} + \frac{a}{3}}$

(f) $\dfrac{\frac{5y}{4} - \frac{2y}{3}}{\frac{4}{15} + \frac{y}{5}}$

6. Find the truth set of each of the following sentences:

(a) $(x - 4)(x + 8) = 0$

(b) $\dfrac{\dfrac{(x - 5)^2}{2}}{\dfrac{x - 5}{4}} = 0$

(c) $-7x + 4 - 3x = -8 - 2x + 2$

(d) $\frac{x}{2} + \frac{1}{3} - \frac{x}{3} = \frac{3}{5}$

(e) $\dfrac{\frac{7x}{-3}}{\frac{4x}{2}} + \frac{x}{3} - 5 = 7$

Problem Set 10-6c
(continued)

(f) $|-3|x = \big||4| - |-8|\big|$

(g) $|-4| + x - (-7) < 8 - (-x) - (-|8|)$

(h) $\dfrac{\frac{3x(x-2)}{4}}{12x} + 3 - x = 9$

(i) $\dfrac{3(x+4)}{4(x+4)} = \dfrac{3}{4}$

(j) $x(x-3)(x+2) = 0$

7. Draw the graphs of the truth sets of parts (f), (g), (i), (j) of Problem 6.

You can probably see how to obtain the answers to the following problems without using a variable. Use this as a check of your work after you solve them by writing an open sentence and finding the truth set of the sentence.

8. The sum of three successive positive integers is 1080. Find the integers.

9. The sum of two successive positive integers is <u>less</u> than 25. Find out what you can about pairs of integers which satisfy this condition.

10. Find two consecutive even integers whose sum is 46.

11. The sum of a whole number and its successor is 45. What are the numbers?

12. The sum of two consecutive odd numbers is 75. What are the numbers?

Problem Set 10-6c
(continued)

13. One number is 5 times another. The sum of the two numbers is 15 more than 4 times the first. What are the numbers?

14. Two trains leave New York at the same time; one travels north at 60 m.p.h. and the other south at 40 m.p.h. After how many hours will they be 125 miles apart?

Summary

1. Subtraction of real numbers is defined as follows:
   To subtract a real number b from a real number a, add the opposite of b to a.
   Thus, for any real numbers a and b

$$a - b = a + (-b).$$

2. Subtraction is neither associative nor commutative. However, a rule has been established for dealing with expressions of the form

$$a - b - c.$$

We say that

$$a - b - c = (a - b) - c$$
$$= a + (-b) + (-c).$$

3. For any real numbers a, b, and c,

$$a - (b + c) = a - b - c.$$

4. Multiplication is distributive over subtraction. Thus, if a, b, and c are any real numbers, then

$$a(b - c) = ab - ac.$$

5. For any two real numbers a and b, the distance <u>from</u> a to b on the number line is $b - a$. The distance <u>between</u> a and b on the number line is $|b - a|$.

6. The multiplicative inverse of a real number is also called the <u>reciprocal</u> of the number. The symbol "$\frac{1}{x}$" is used to denote the reciprocal of x.

7. Division of real numbers is defined as follows:

For any real numbers a and b, $b \neq 0$

$$a \div b = a \cdot \frac{1}{b}$$

8. $a \div b$ can also be written $\frac{a}{b}$.

9. We adopted the following conventions with regard to finding a common name of a phrase.

(i) A common name contains no indicated division if it can be avoided.

(ii) If a common name must contain an indicated division then the expression shall be in "lowest terms".

10. For any real numbers a and b, $b \neq 0$,

$$\frac{-a}{b}, \qquad \frac{a}{-b}, \qquad \text{and} \qquad -\frac{a}{b}$$

are names for the same number;

$-\frac{a}{b}$ is the common name.

Review Problem Set

1. Find the reciprocal of each number.

(a) $\frac{3}{4}$	(d) .33	(g) $\sqrt{2}$
(b) 0.3	(e) 1	(h) $a^2 + 1$
(c) -0.3	(f) -1	(i) $\frac{1}{x^2 + 4}$

Review Problem Set
(continued)

(j) $y^2 + 1$

(k) $-\frac{1}{2}$

(l) $\frac{5}{6}$

(m) $\frac{1}{100}$

(n) -6.8

(o) $.45$

2. For what values of a do the following expressions have no reciprocals? (Hint: What number has no reciprocal?)

(a) $a - 1$

(b) $a + 1$

(c) $a^2 - 1$

(d) $a(a + 1)$

(e) $\frac{a}{a + 1}$

(f) $a^2 + 1$

(g) $\frac{1}{a^2 + 1}$

(h) $\frac{(a + 1)^2}{3(a + 1)}$

(i) $\frac{(a - 3)(a + 3)}{(a - 3)(a + 3)}$

*3. Consider the sentence

$$(a - 3)(a + 1) = a - 3,$$

which has the truth set $\{0,3\}$. Check this. If both sides of the sentence are multiplied by the reciprocal of $(a - 3)$, that is, by $\frac{1}{a - 3}$, and some properties of real numbers are used (which properties?), we obtain

$$a + 1 = 1.$$

If we let a be 3 in this last sentence we get $3 + 1 = 1$, which is clearly a false sentence. Why doesn't this new sentence have the same truth set as the original sentence?

Review Problem Set
(continued)

4. Obtain the simplest expression for each of the following:

   (a) $(19x^2 + 12x - 15) - (20x^2 - 3x - 1)$

   (b) $(8a - 13) - (7a + 12)$

   (c) $(14a^2 - 5a + 1) - (6a^2 - 9)$

   (d) $(3n + 12p - 8a) - (5a - 7n - p)$

   (e) $(7x^2 - 7) - (3x + 9)$

   (f) $(a^2 - b^2) - (a^2 - 2ab + b^2)$

   (g) From $11a + 13b - 7c$ subtract $8a - 5b - 4c$.

   (h) What is the result of subtracting $-3x^2 + 5x - 7$ from $-3x + 12$?

   (i) What must be added to $3s - 4t + 7u$ to obtain $-9s - 3u$?

5. Consider three pairs of numbers: (a) $a = 2$, $b = 3$; (b) $a = 4$, $b = -5$; (c) $a = -4$, $b = -7$. Does the sentence $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$ hold true in all three cases?

6. Is the sentence $\frac{1}{b} < \frac{1}{a}$ true in all three cases of Problem 5? Plot the pairs of reciprocals on the number line.

7. Is it true that if $b < a$ and if a and b are positive, then $\frac{1}{a} < \frac{1}{b}$? Try this for some particular values of a and b.

8. Is it true that if $b < a$ and a, b are negative, then $\frac{1}{a} < \frac{1}{b}$? Substitute some particular values of a and b.

Review Problem Set
(continued)

9. Could you tell immediately which reciprocal is greater than another if one of the numbers is positive and the other negative?

10. If $b < a$, what can you say of $a - b$? Complete this statement: If a is to the right of b on the number line, then the difference $a - b$ is _____.

11. If $(a - b)$ is a positive number, which of the statements, $a < b$, $a = b$, $a > b$, is true? What if $(a - b)$ is a negative number? What if $(a - b)$ is zero?

12. If a, b, and c are real numbers and $b < a$, what can we say about the order of $a - c$ and $b - c$?

13. Simplify these expressions using the distributive property where necessary.

(a) $a^2 + 3a^2$	(g) $0 - 5a$
(b) $\pi - (-\pi)$	(h) $(-25pq) - pq$
(c) $8k - (-11k)$	(i) $(-12y) - 4y$
(d) $6\sqrt{2} - 9\sqrt{2}$	(j) $(-3b) - (-3b)$
(e) $6x - 2x$	(k) $(-4y) - 0$
(f) $9x^2 + (-4x^2)$	(l) $0 - (-3m)$

14. The temperature drops 15° from an initial temperature of 4° above zero. Express this statement as a subtraction of real numbers and find the resulting temperature.

15. A submarine has been cruising at 50 feet below the surface, It then goes 30 feet deeper. Express this change as a subtraction of real numbers and find the resulting depth.

Review Problem Set
(continued)

16. -16 is 25 less than some number. Find the number.

17. If the time at 12 o'clock midnight is considered as the starting time, that is, at 12 o'clock midnight $t = 0$, what is the length of the time interval from 11 o'clock P.M. to 2 o'clock A.M.? From 6 o'clock A.M. to 4 o'clock A.M. the next day?

18. John and Rudy ride bicycles on a straight road on which there is a point marked 0. John rides 10 miles per hour and Rudy rides 12 miles per hour. Find the distance between them after 3 hours if

    (a) They start from the 0 mark at the same time and John goes east and Rudy goes west.

    (b) John is 5 miles east and Rudy is 6 miles west of the 0 mark when they start and they both go east.

    (c) John starts from the 0 mark and goes east. Rudy starts from the 0 mark 15 minutes later and goes west.

    (d) Both start at the same time. John starts from the 0 mark and goes west and Rudy starts 6 miles west of the 0 mark and also goes west.

19. If b is the multiplicative inverse (reciprocal) of a,

    (a) What values of b do we obtain if a is larger than 1?

    (b) What values of b do we obtain if a is between 0 and 1?

    (c) What is b if a is 1?

    (d) What is b if a is -1?

    (e) What values of b do we obtain if a is less than -1?

    (f) What values of b do we obtain if $a < 0$ and $a > -1$?

Review Problem Set
(continued)

(g) What kind of number is b if a is positive?

(h) What kind of number is b if a is negative?

(i) What is b if a is 0?

(j) If b is the reciprocal of a, what can you say about a?

20. (a) What is the value of $87 \times (-9) \times 0 \times \frac{2}{3} \times 642$?

(b) Is $8 \cdot 17 = 0$ a true sentence?

(c) If $n \cdot 50 = 0$, what can you say about n?

(d) If $p \cdot 0 = 0$, what can you say about p?

(e) If $p \cdot q = 0$, what can you say about p or q?

(f) If $p \cdot q = 0$, and we know that $p > 10$, what can we say about q?

(g) If $(x - 5) \cdot 7 = 0$, what must be true about $(x - 5)$?

(h) Explain how we know that the only value of y which will make $9 \times y \times 17 \times 3 = 0$ a true sentence is 0.

(i) How can we, without just guessing, determine the truth set of the equation $(x - 8)(x - 3) = 0$?

21. Find the truth set of each of the following equations:

(a) $(x - 20)(x - 100) = 0$

(b) $(x + 6)(x + 9) = 0$

(c) $x(x - 4) = 0$

(d) $4(x + 34) = 0$

(e) $(x - 1)(x - 2)(x - 3) = 0$

(f) $2(x - \frac{1}{2})(x + \frac{3}{4}) = 0$

(g) $(3x - 5)(2x + 1) = 0$

(h) $9|x - 6| = 0$

(i) $-3x + 4.5 = -7x - 3.5$

Review Problem Set
(continued)

(j) $-4.3x - 2.7 < -2.3x + 4.3$

(k) $|-4| - |-3x| > -4 + 3x$

(l) $\frac{3}{5}x + \frac{3}{4} = \frac{8}{10}x - \frac{\frac{1}{4}}{\frac{3}{3}}$

22. Simplify each of the following expressions. Assume that the domains of the variables exclude values for which denominators are zero.

(a) $\dfrac{\frac{2}{3} + \frac{1}{6}}{\frac{5}{6}}$

(b) $\dfrac{2}{\frac{1}{8} + \frac{5}{12}}$

(c) $\dfrac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{3} - \frac{1}{4}}$

(d) $\dfrac{\frac{-3ax}{17}}{\frac{5x}{8}}$

*(e) $\dfrac{\frac{3}{a} - \frac{5}{2a}}{\frac{3}{b} + \frac{5}{2b}}$

(f) $\dfrac{\frac{a - b}{2}}{\frac{a - b}{-4}}$

*(g) $\dfrac{\frac{6x}{5} - \frac{x}{4}}{\frac{x}{2} + \frac{2x}{5}}$

*(h) $\dfrac{\frac{x + 8}{9}}{\frac{3}{x + 2}}$

(i) $\dfrac{\frac{y}{y + 1}}{\frac{y - 1}{3y}}$

(j) $(\frac{3}{8} - \frac{1}{2}) + (\frac{5}{4} + \frac{1}{16})$

(k) $(\frac{1}{9} + \frac{2}{3})(\frac{1}{3} - \frac{5}{6})$

*(l) $\dfrac{a + 6}{a - 6} \cdot \dfrac{a + 2}{a - 2}$

(m) $\dfrac{a^2 - 3}{9} \div \dfrac{a^2 - 3}{9}$

(n) $\dfrac{1 - \frac{2}{x} + \frac{1}{x^2}}{1 - \frac{1}{x^2}}$

(o) $\dfrac{\frac{a}{b} + \frac{b}{a}}{\frac{1}{a} + \frac{1}{b}}$

(p) $\dfrac{x - 2}{3 - x} \cdot \dfrac{x - 3}{2 - x}$

Review Problem Set
(continued)

(q) $\frac{x - 2}{3 - x} + \frac{2 - x}{x - 3}$

(r) $\dfrac{\frac{3}{x - 1} + 1}{\frac{-5}{x + 1} - 1}$

(s) $\dfrac{\frac{2}{a - 1} + \frac{1}{a + 1}}{\frac{1}{a - 1} + \frac{2}{a + 1}}$

*23. Find the truth set of each of the following sentences:

(a) $\dfrac{\frac{a - 1}{2}}{\frac{3}{4}} = 1$

(b) $\dfrac{2x}{\frac{1}{8} + \frac{5}{12}} = 1$

(c) $\dfrac{2x}{\frac{1}{8} + \frac{5}{12}} = 0$

(d) $\dfrac{\frac{6x}{5} - \frac{x}{4}}{\frac{1}{2} + \frac{4}{5}} = x$

24. Write the first step in using the distributive property to expand $(3x + 5)(2x - 3)$.

25. Use properties of addition to show that the following sentence is true.

$$\big(25 + (-10)\big) + (-25) = \big(25 + (-25)\big) + (-10).$$

26. Show that $\frac{3}{8} < \frac{9}{20}$ and $\frac{9}{20} < \frac{7}{15}$ are true sentences. Then tell why you know immediately that $\frac{3}{8} < \frac{7}{15}$ is true.

27. If the length of each edge of a square is multiplied by 2, by what number is the perimeter multiplied? By what number is the area multiplied?

Review Problem Set
(continued)

28. $A = \{0, |-1|\}$ and $B = \{-1, 0, 1\}$.

(a) Under which of the operations (addition, subtraction, multiplication, division) is set A closed? set B?

(b) If C is the set of numbers obtained by squaring elements belonging either to set A or set B, enumerate set C. Is it a subset of A? of B?

29. Given the fraction $\frac{3x + 5}{2x - 7}$; what is the only value of x for which this is not a real number?

30. Let a and b be positive numbers such that $\frac{a}{b} = \frac{2}{3}$.

(a) If $a < 24$, what inequality does b satisfy?

(b) If $b < 24$, what inequality does a satisfy?

31. The product of two numbers is 2. If one of the numbers is less than 3, what is the other? If one is less than -3, what is the other?

32. Does division have the associative property? That is, is $(a \div b) \div c = a \div (b \div c)$? Give reasons for your answer.

33. Is division commutative? Give reasons for your answer.

34. If $x = a + \frac{1}{a}$, and $a = \frac{1}{2}$, what is the value of $ax + a^2$?

35. If a is between p and q, is $\frac{1}{a}$ between $\frac{1}{p}$ and $\frac{1}{q}$? Explain.

36. Translate each of the following phrases into open phrases. Describe the variable carefully where necessary.

(a) The number of feet in $6y$ yards

(b) The number of inches in $2f$ feet

(c) The number of pints in $4k$ quarts

(d) A girl's age 10 years ago

(e) The number of ounces in k pounds and t ounces

Review Problem Set
(continued)

(f) The number of square inches in f square feet

(g) The number of cents in d dollars and k quarters

(h) The number of cents in d dollars, k quarters, t dimes and n nickels

(i) The successor of a whole number

(j) The reciprocal of a number

(k) The number of feet traveled in k miles

(l) Twice the number of feet traveled in k miles

37. Write meaningful word sentences which are translations of the following open sentences.

(a) $x < 80$

(b) $y = 3600$

(c) $z > 100{,}000{,}000$

(d) $u + v + w = 180$

(e) $z(z + 18) = 360$

(f) $x(3x) \leq 300$

(g) $(x + 1)^2 > x(x + 2)$

(h) $30(20.00) \leq (30 - x)(24.00)$

(i) $3a = 4b$

(j) $n + (n + 1) + (n + 2) + (n + 3) + (n + 4) < 90$ and $n > 13$

38. Write open phrases corresponding to the following word phrases, being careful to describe what number the variable represents.

(a) A number diminished by 3

(b) A rise of 20 degrees in temperature

(c) Cost of n pencils at 5 cents each

Review Problem Set
(continued)

(d) The amount of money in my pocket if I have x dimes, y nickels, and 6 pennies

(e) A number increased by twice the number

(f) A number increased by twice another number

(g) The number of days in w weeks

(h) Cost of purchasing x melons at 29 cents each and y pounds of hamburger at 59 cents a pound

(i) Area of a rectangle having one side 3 inches longer than another

(j) One million more than twice the population of a certain city in Kansas

(k) Annual salary equivalent to x dollars per month

(l) Arthur's allowance, which is one dollar more than twice Betty's

(m) The distance traveled in h hours at an average speed of 40 m.p.h.

(n) The real estate tax on property having a valuation of y dollars, the tax rate being \$25.00 per \$1000 valuation

(o) Donald's weight, which is 40 pounds more than Earl's

(p) Speed of a car which is one mile per hour less than that of a following car

(q) Cost of x pounds of steak at \$1.59 per pound

(r) Catherine's earnings for z hours at 75 cents an hour

(s) Cost of g gallons of gasoline at 33.2 cents a gallon

Review Problem Set
(continued)

39. Write open sentences corresponding to the following word sentences, and carefully describe the variable used.

(a) Mary, who is 16, is 4 years older than her sister.

(b) Bill bought b bananas at 9 cents each and paid 54 cents.

(c) If a number is added to twice the number, the sum is less than 39.

(d) Arthur's allowance is one dollar more than twice Betty's but is two dollars less than 3 times Betty's.

(e) The distance from Dodge City to Oklahoma City, 260 miles, was traveled in t hours at an average speed of 40 miles an hour.

(f) The auto trip from St. Louis to Memphis, 300 miles, was made in t hours, the maximum speed being 50 miles an hour.

(g) Pike's Peak is more than 14,000 feet above mean sea level.

(h) A book, 1.4 inches thick, has n pages; each page is 0.003 inches thick, and each cover is $\frac{1}{10}$ inches thick.

(i) Three million is over one million more than twice the population of any city in Colorado.

(j) A square having a side x inches long has a smaller area than a rectangle which is $x + 1$ inches long and $x - 1$ inches wide.

(k) The tax on real estate is calculated at \$24.00 per \$1000 valuation. The tax assessment on property valued at y dollars is \$348.00.

(l) Donald's weight, 152 pounds, is at least 40 pounds more than Earl's.

Review Problem Set
(continued)

(m) The sum of a counting number and its successor is 575.

(n) The sum of a counting number and its successor is 576.

(o) The sum of two numbers, the second greater than the first by 1, is 576.

(p) A board 16 feet long is cut in two pieces such that one piece is one foot longer than twice the other.

(q) Catherine earns $2.25 baby-sitting for 3 hours at x cents an hour.

(r) A familiar formula for making coffee is: "Use one tablespoon of coffee for each cup of water, and add one tablespoon of coffee for the pot." Use C for the number of cups of water, and T for the number of tablespoons of coffee.

(s) In 4 years Mary will be twice as old as she was 6 years ago.

(t) A two-digit number is 7 more than 3 times the sum of the digits.

(u) A number is increased by 17 and the sum is multiplied by 3. The resulting product is 192.

(v) If 17 is added to a number and the sum is multiplied by 3, the resulting product is less than 192.

40. Mr. York is reducing. During each month for the past 8 months he has lost 5 pounds. His weight is now 175 pounds. What was his weight m months ago if $m < 8$? Write an open sentence stating that m months ago his weight was 200 pounds.

Review Problem Set
(continued)

41. In a "guess the number" game Betty is asked to pick a counting number less than or equal to 7.

(a) With x for the number, write the inequalities which indicate the restrictions on the number.

(b) If Betty picks a counting number less than or equal to 7 and Paul picks a counting number less than or equal to 5, what can we say about the sum of Betty's number and 3 times Paul's number?

(c) If Betty picks a counting number less than or equal to 7 and Paul picks a whole number less than or equal to 5, what can we say about the sum of Betty's number and three times Paul's number?

42. (a) At an auto parking lot, the charge is 35 cents for the first hour, or fraction of an hour, and 20 cents for each succeeding (whole or partial) one-hour period. What is the parking fee for 4 hours of parking?

(b) If t is the number of one-hour periods parked <u>after</u> the initial hour, write an open phrase for the parking fee.

(c) With the same charge for parking as in the preceding problem, if h is the <u>total</u> number of one-hour periods parked, write an open phrase for the parking fee.

43. (a) Two water-pipes are bringing water into a reservoir. One pipe has a capacity of 100 gallons per minute, and the second 40 gallons per minute. If water flows from the first pipe for x minutes and from the second for y minutes, write an open phrase for the total flow in gallons.

Review Problem Set
(continued)

(b) In part (a), if the flow from the first pipe is stopped after two hours, write the expression for the total flow in gallons in y minutes, where y is greater than 120.

*(c) With the same data, write an open sentence stating that the total flow is 20,000 gallons. Find five or more pairs of numbers for the variables which yield true numerical sentences.

44. The reading on a Fahrenheit thermometer is 32° more than 1.8 times the reading on a Centigrade thermometer; if the temperature is less than 50° Fahrenheit, what is the temperature Centigrade?

45. The amount of \$205 is to be divided among Tom, Dick and Harry. Dick is to have \$15 more than Harry and Tom is to have twice as much as Dick. How must the money be divided?

46. Last year's tennis balls cost d dollars a dozen. This year the price is c cents per dozen higher than last year. What will half a dozen balls cost at the present price?

47. A man distributes \$ 24 between his two children in amounts proportioned to their ages. The older is 7, and the younger 3. How much should each receive?

48. In a class of 10 pupils the average grade was 72. The students with the two highest grades, 94 and 98, were transferred to another class, and the teacher decided to find the average of the grades of the 8 remaining students. What was the new average?

Review Problem Set
(continued)

49. Consider the set S of all the even integers (positive, negative, and zero). Which of the five operations, (1) addition, (2) subtraction, (3) multiplication, (4) division, (5) finding the average---applied to pairs of elements of S, will give only elements of S? Describe your conclusion in terms of "closure"

50. A haberdasher sold two shirts for \$3.75 each. On the first he lost 25 percent of the cost and on the second he gained 25 percent of the cost. How much did he pay for each shirt? Did he gain or did he lose from the sale?

51. A boy has 95 cents in nickels and dimes. If he has 12 coins, how many of each coin does he have?

52. William has 5 hours at his disposal. How far can he ride his bike into the spooky woods if he goes in at the rate of 4 miles per hour and returns at the rate of 15 miles per hour?

53. A plane which flies at an average speed of 200 m.p.h. (when no wind is blowing) is held back by a head wind and takes $3\frac{1}{2}$ hours to complete a flight of 630 miles. What is the average speed of the wind?

CHALLENGE PROBLEMS

1. Let us write "$\succ$" for the phrase "is further from 0 than" on the real number line. Does "$\succ$" have the comparison property, that is, if a and b are different real numbers, is it true that $a \succ b$ or $b \succ a$ but not both? Does "$\succ$" have a transitive property? For which subset of the set of real numbers do "$\succ$" and ">" have the same meaning?

Challenge Problems
(continued)

2. Prove that the absolute value of the product ab is the product $|a| \cdot |b|$ of the absolute values for any real numbers a and b. Hint: You must consider all possible combinations of positive and negative numbers and zero.

3. Prove: The number 0 has no reciprocal.

   Hint: Assume 0 does have a reciprocal and see what happens when you apply the definition of reciprocal.

4. Prove: The reciprocal of a positive number is positive, and the reciprocal of a negative number is negative.

   Hint: The product of a number and its reciprocal is 1.

5. Prove: The reciprocal of the reciprocal of a non-zero real number a is a.

   Hint: Consider the product $(\frac{1}{a})\left(\frac{1}{\frac{1}{a}}\right)$ and the product $a(\frac{1}{a})$. Compare the results.

6. For each of the following pairs of expressions, fill in the symbol "$>$", "$=$", or "$<$", which will make a true sentence.

   (a) $|9 - 2| \ ? \ |9| - |2|$
   (b) $|2 - 9| \ ? \ |2| - |9|$
   (c) $|9 - (-2)| \ ? \ |9| - |-2|$
   (d) $|(-2) - 9| \ ? \ |-2| - |9|$
   (e) $|(-9) - 2| \ ? \ |-9| - |2|$
   (f) $|2 - (-9)| \ ? \ |2| - |-9|$
   (g) $|(-9) - (-2)| \ ? \ |-9| - |-2|$
   (h) $|(-2) - (-9)| \ ? \ |-2| - |-9|$

7. Write a symbol between $|a - b|$ and $|a| - |b|$ which will make a true sentence for all real numbers a and b. Do the same for $|a - b|$ and $|b| - |a|$. For $|a - b|$ and $\big||a| - |b|\big|$.

Challenge Problems
(continued)

8. Describe the resulting sentences in Problem 7 in terms of distances on the number line.

9. What are the two numbers x on the number line such that

$$|x - 4| = 1,$$

that is, the two numbers x such that the distance between x and 4 is 1?

10. What is the truth set of the sentence

$$|x - 4| < 1,$$

that is, the set of numbers x such that the distance between x and 4 is less than 1? Draw the graph of this set on the number line.

11. Draw the graph of the truth set of the compound open sentence

$$x > 3 \text{ and } x < 5$$

on the number line. Is this set the same as the truth set of $|x - 4| < 1$? (We usually write "$3 < x < 5$" for the sentence "$x > 3$ and $x < 5$".)

12. Find the truth set of each of the following sentences; draw the graph of each of these sets:

(a) $|x - 6| = 8$

(b) $y + |-6| = 10$

(c) $|10 - a| = 2$

(d) $|x| > 3$

(e) $|v| > -3$

(f) $|y| + 12 = 13$

(g) $|y - 8| < 4$ (Read this: the distance between y and 8 is less than 4.)

(h) $|z| + 12 = 6$

(i) $|x - (-19)| = 3$

(j) $|y + 5| = 9$

Challenge Problems
(continued)

13. Prove $\frac{1}{(-a)} = -\left(\frac{1}{a}\right)$, if $a \neq 0$.

Hint: We have proved $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$. Now consider the reciprocal of -1. How could it be used to help in this proof?

14. If $a < b$ and a and b are both positive real numbers prove that $\frac{1}{a} > \frac{1}{b}$.

Hint: Multiply both sides of inequality "$a < b$" by $\left(\frac{1}{a} \cdot \frac{1}{b}\right)$.

15. Prove that if $a < b$, where a and b are both negative, then $\frac{1}{a} > \frac{1}{b}$.

16. Prove that if $a < 0$ and $b > 0$ then $\frac{1}{a} < \frac{1}{b}$.

17. Prove that $\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$ for real numbers a, b, and c $(c \neq 0)$.

Hint: Consider the definition of division, then the distributive property.

18. Prove that $\frac{a}{c} + \frac{b}{d} = \frac{ad + bc}{cd}$ for real numbers a, b, c, and d, $(c \neq 0,\ d \neq 0)$.

Hint: Use the multiplication property of 1, choosing the form of 1 carefully.

19. Given the set $\{1, -1, j, -j\}$ and the following multiplication table:

$\times$	1	-1	j	-j
1	1	-1	j	-j
-1	-1	1	-j	j
j	j	-j	-1	1
-j	-j	j	1	-1

Challenge Problems
(continued)

(a) Is the set closed under multiplication?

(b) Verify that this multiplication is commutative for the cases $(-1, j)$, $(j, -j)$, and $(-1, -j)$.

(c) Verify that this multiplication is associative for the cases $(-1, j, -j)$ and $(1, -1, j)$.

(d) Is it true that $a \times 1 = a$, where a is any element of $\{1, -1, j, -j\}$?

(e) Find the reciprocal of each element in this set.

<u>If x is an unspecified member of the set, find the truth sets of the following, making use of your results in part (e).</u>

(f) $j \times x = 1$

(g) $-j \times x = j$

(h) $j^2 \times x = -1$

(i) $j^3 \times x = -j$

20. A man travels 360 miles due west at a rate of 3 minutes per mile and returns by plane at a rate of 3 miles per minute. What was his total traveling time? What was his average rate of speed for the entire trip?

21. A set of ten numbers has a sum t. If each number is increased by 4, then multiplied by 3, and then decreased by 4, that is the new sum? If you had twenty numbers instead of ten and the same conditions, what would be the new total?

GLOSSARY CHAPTERS 6-10

ABSOLUTE VALUE - The absolute value of any non-zero real number is the greater of that number and its opposite. The absolute value of zero is zero.

ADDITION PROPERTY OF EQUALITY - For any real numbers a, b, and c, if $a = b$, then $a + c = b + c$.

ADDITION PROPERTY OF OPPOSITES - For any real number a, $a + (-a) = 0$.

ADDITION PROPERTY OF ORDER - If a, b, and c are real numbers and if $a < b$, then $a + c < b + c$.

ADDITION PROPERTY OF ZERO - For any real number a, $a + 0 = a$.

ADDITIVE INVERSE - If there are real numbers x and y such that $x + y = 0$, then y is the additive inverse of x, and x is the additive inverse of y.

ASSOCIATIVE PROPERTY OF ADDITION - For any real numbers a, b, and c, $(a + b) + c = a + (b + c)$.

ASSOCIATIVE PROPERTY OF MULTIPLICATION - For any real numbers a, b, and c, $a(bc) = (ab)c$.

COMPARISON PROPERTY - For any real number a and any real number b one and only one of the following sentences is true:

$$a < b$$
$$a = b$$
$$a > b.$$

COMMUTATIVE PROPERTY OF ADDITION - For any two real numbers a and b, $a + b = b + a$.

COMMUTATIVE PROPERTY OF MULTIPLICATION - For any real numbers a and b, $a \cdot b = b \cdot a$.

DISTRIBUTIVE PROPERTY - For any real numbers a, b, and c, $a(b + c) = ab + ac$.

EQUATION - A sentence with the symbol "=" is called an equation.

EQUIVALENT SENTENCES - Two open sentences with the same truth set are called equivalent sentences.

INEQUALITY - A sentence with the symbol ">" or "<" is called an inequality.

INTEGERS - The set of counting numbers, zero, and the opposites of the counting numbers make up the set of integers.

IRRATIONAL NUMBERS - A number that is not rational but is associated with a point on the number line is called an irrational number.

MULTIPLICATION PROPERTY OF EQUALITY - For real numbers a, b, and c, if $a = b$, then $ac = bc$.

MULTIPLICATION PROPERTY OF ONE - For any real number a, $a \cdot 1 = a$.

MULTIPLICATION PROPERTY OF ORDER - If a and b are real numbers such that $a < b$, then $ca < cb$ if c is a positive number, but $cb < ca$ if c is a negative number.

MULTIPLICATION PROPERTY OF ZERO - For any real number a, $a \times 0 = 0$.

MULTIPLICATIVE INVERSE - If a and b are two real numbers such that $ab = 1$, a is the multiplicative inverse of b and b is the multiplicative inverse of a.

NEGATIVE REAL NUMBERS - The set of real numbers associated with points to the left of zero on the number line is the set of negative real numbers.

OPPOSITE - The opposite of any non-zero real number is the number which is at an equal distance from 0 on the number line and on the opposite side of 0. The opposite of zero is zero.

POSITIVE REAL NUMBERS - The set of real numbers associated with points to the right of zero on the number line is the set of positive real numbers.

RATIONAL NUMBER - A number that can be represented by a fraction indicating a quotient of two integers, excluding division by zero, is called a rational number.

REAL NUMBERS - The set of numbers which includes the rational numbers and the irrational numbers is the set of real numbers.

RECIPROCAL - The multiplicative inverse of a real number is called the reciprocal of the number.

TRANSITIVE PROPERTY OF ORDER - If a, b, and c are three real numbers such that $a < b$ and $b < c$, then $a < c$.

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