HIGH SCHOOL MATHEMATICS

Unit 5.

RELATIONS AND FUNCTIONS

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

MAX BEBERMAN, *Director*

HERBERT E. VAUGHAN, *Editor*

UNIVERSITY OF ILLINOIS PRESS · URBANA, 1960

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Teachers' Edition

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TEACHERS COMMENTARY

The introduction to this unit [pages 5-A through 5-K] is designed to create a nonverbal awareness of what a relation is. The student is expected to learn, at least informally, that a relation is a set of ordered pairs. The definition of a relation as a set of ordered pairs is given explicitly on page 5-1. There follows a section on the algebra of sets [unioning and intersecting sets are somewhat like adding and multiplying numbers], and another which contains illustrations of relations which arise from geometric problems. Section 5.04 introduces the notions of domain, range, and converse of a relation and discusses the properties of reflexivity and symmetry. Functions are relations of a special kind and are discussed in section 5.05. The remainder of the unit deals with related matters: variable quantities, functional dependence [including variation], linear functions, quadratic functions, and systems of equations.

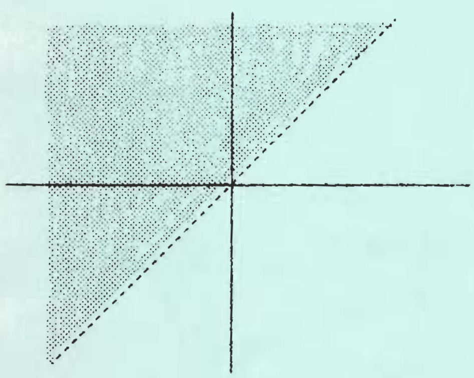
*

Each student should be supplied with [or supply himself with] a ruler, a protractor, a compass, and cross-section paper [4(or 5)-to-the-inch and 10-to-the-inch varieties]. He will also need "lattice paper" which is not obtainable commercially. [4-to-the-inch cross-section paper can be used for this purpose.] Lattice paper is easy to manufacture if you have a mimeograph machine or spirit duplicator available. Make some of the type shown on page 5-B and some like that on page 5-40. Students should keep a supply of cross-section paper and lattice paper in their notebooks. [It may be helpful to have a three-hole punch in the classroom, so that the paper can be readily stored in the notebook.]

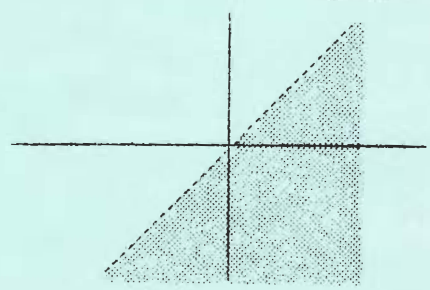
You should also have readily available a cross-section blackboard and a lattice blackboard so that you need not waste time when giving blackboard illustrations. Conditions vary too much from classroom to classroom for us to make specific suggestions which are universally applicable. One general suggestion is that you give prior thought to this problem and consult school supply guides.

Teachers who have used earlier editions of Unit 5 should especially note that the rules for games like TREE which are used in the present edition are different from those used in earlier editions. As stated on TC[5-A, B, C, D, E], when cards with values x and y are played, in that order, the trick belongs to the second player if and only if the graph of (x, y) is one of the heavy dots on the chart. [In earlier editions the trick belonged, in this case, to the first player.] This change has the effect that, for example, the sentence '(x, y) ϵ T' is equivalent to 'y TREES x' [rather than, as previously, to 'x TREES y']. It is believed that this will make easier the passage from the consideration of relations in general to that of functions.

This change also affects some of the work on relations in sections 5.01, 5.03, and 5.04. For example, under the new convention, y has the relation greater-than to x [cf. 'y TREES x'] if and only if (x, y) belongs to the relation greater-than. So, the relation greater-than is $\{(x, y): y > x\}$, and its graph is:



With the convention used in earlier editions the graph is:



For another example [although the definitions of domain and range of a relation are the same as previously], in the discussion of the relation of being-an-uncle-of in section 5.04, the domain of this relation is now the set of all nephews and nieces, and the range is the set of all uncles.

Correction. In the line just below the picture of Morris' cards, the sentence should read: Since the deck is made up of ---

Students may find it helpful to make a copy of the chart on page 5-B. This will save a good deal of page flipping as they work from page 5-C through page 5-K. A copy of the chart on the blackboard for class discussion may also help.

*

On page 5-A, Tony's cards should be marked with these numerals:

3, 9, 7, 10, 6, 5, 4, 2, 8, 5

Morris thinks he has won the 3, 4-trick because he believes that, as in most card games, the higher card of the two played wins the trick. You may wish to have the students construct decks of twenty cards [use 3 × 5 index cards] like that described on page 5-A and actually play one or two games of TREE or UPPER TRIANGLE [page 5-J]. Be careful not to overdo this. The only value in such activity is the help it gives students in understanding the procedure for determining the winner of a trick. When cards with values x and y are played, in that order, the trick belongs to the second player if and only if the graph of (x, y) is one of the heavy dots on the chart. [Of course, another way of looking at it is to notice that the trick belongs to the first player if and only if the graph of (x, y) is one of the light dots on the chart.]

*

Answers for Part A [on page 5-E].

Morris first: 2nd card wins because 5 TREES 1.

Tony first: 2nd card wins because 1 TREES 5.

Morris first: 1st card wins because 10 DOES NOT TREE 2.

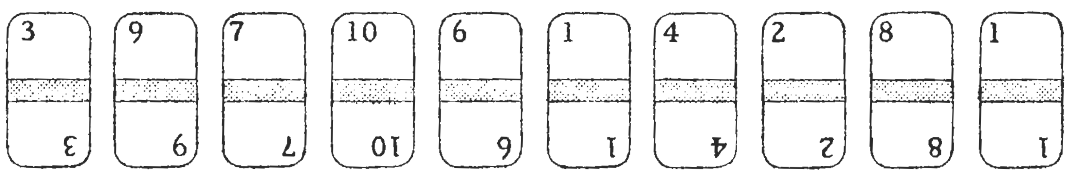
<u>The Games Club</u>.--Morris Thompson, a new student at Zabranchburg High School, is visiting a regular meeting of the Games Club. The president announces:

Today we play TREE.

The members form pairs, and Tony invites Morris to be his opponent. Tony starts to explain the card game TREE to him, but Morris says, "Let's just play a few hands; I'm sure I'll catch on that way."

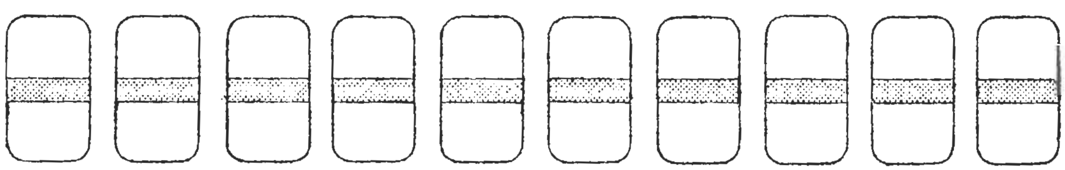
So, Tony hands him a chart which looks like the one on page 5-B. Then, he picks up a deck of twenty cards, shuffles it, and deals ten cards to Morris and ten cards to himself. Morris picks up his cards.

<u>Morris' cards</u>

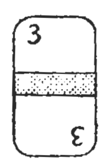


Since the deck is made of of twenty cards, two 1s, two 2s, two 3s, etc., you can tell what Tony's cards are. So, fill in the numerals:

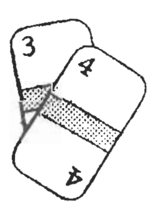
<u>Tony's cards</u>



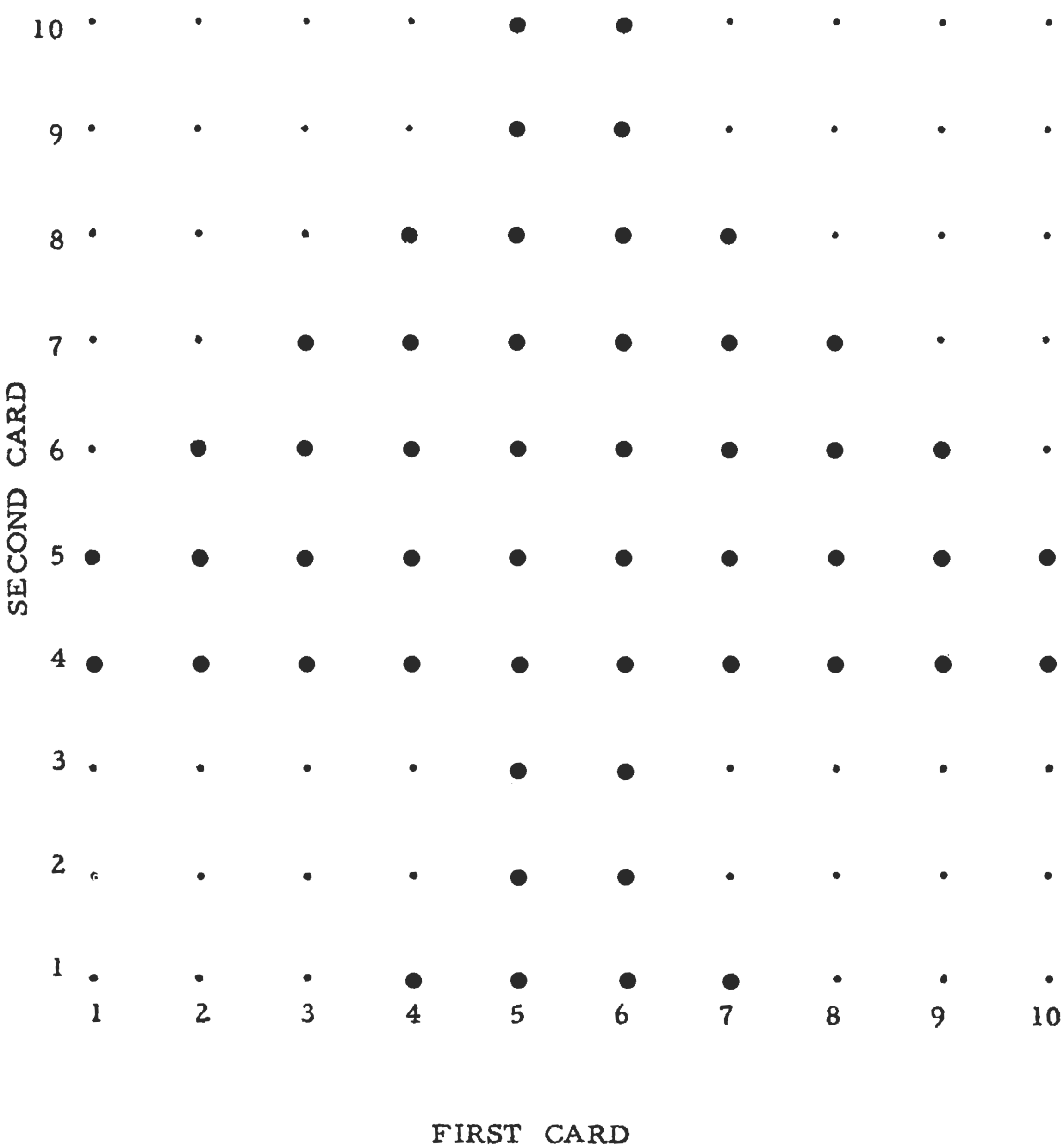
Tony says, "I'll play first," and plays a



Morris, thinking that this game is like most card games, plays a higher card, a 4, and reaches for the cards.



TREE CHART

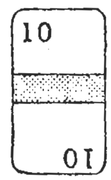


Tony, after glancing at the chart, says, "O.K., you win that trick. 3 loses to 4 because

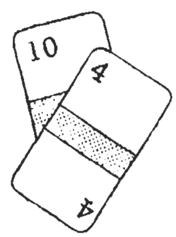
4 TREES 3.

Now, you play first because you won."

So, Morris plays as first card a



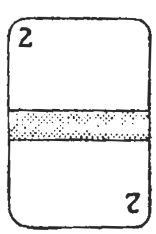
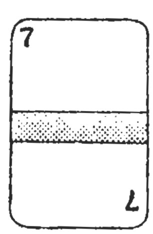
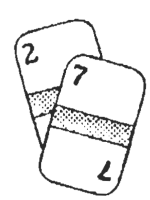
and Tony plays as second card a 4.



Morris, thinking he has won again, starts to reach for the pair of cards but Tony stops him. "No, Morris, look at the chart; I win this trick." 10 loses to 4 because

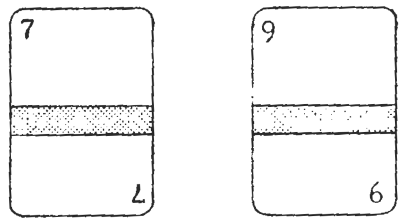
4 TREES 10

Morris is puzzled, but they continue to play with Tony telling the winner of each trick after looking at the chart. Here are their next few plays.

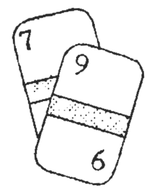
	Tony 1st Card	Morris 2nd Card	
Tony first			1st Card wins because 7 DOES NOT TREE 2. 

	Tony 1st Card	Morris 2nd Card

Tony first

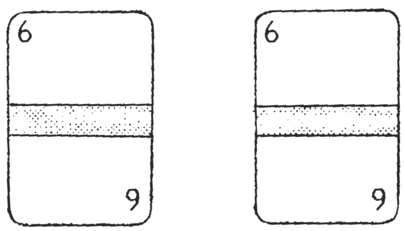


1st Card wins because
9 DOES NOT TREE 7.

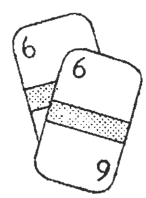


	Tony 1st Card	Morris 2nd Card

Tony first

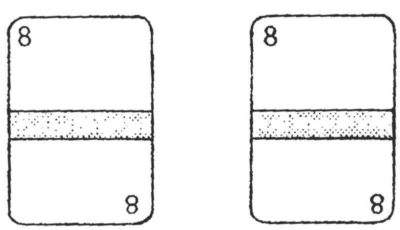


2nd Card wins because
6 TREES 6.

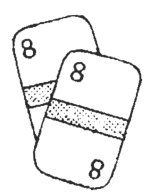


	Morris 1st Card	Tony 2nd Card

Morris first

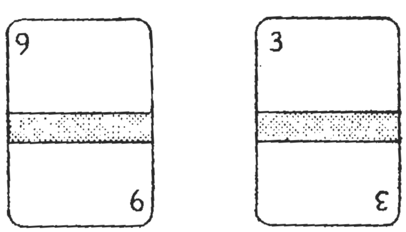


1st Card wins because
8 DOES NOT TREE 8.

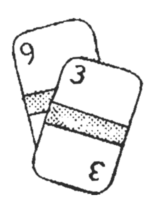


	Morris 1st Card	Tony 2nd Card

Morris first

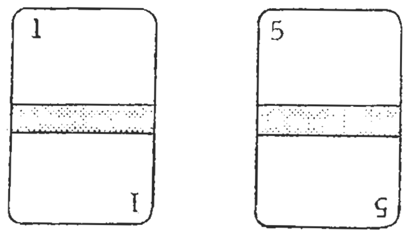


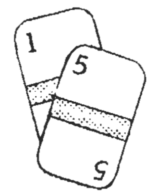
1st Card wins because
3 DOES NOT TREE 9.

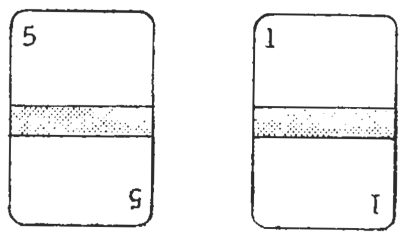


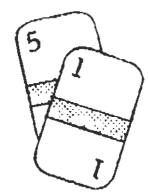
EXERCISES

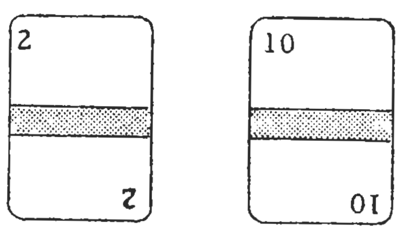
A. Who wins in each of these last three plays? Fill in the blanks at the right.

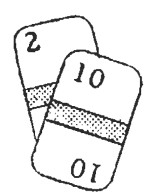
	Morris 1st Card	Tony 2nd Card	
Morris first			___ Card wins because 5 ________________ 1.



	Tony 1st Card	Morris 2nd Card	
Tony first			___ Card wins because 1 ________________ 5.



	Morris 1st Card	Tony 2nd Card	
Morris first			___ Card wins because 10 ________________ 2.



B. By now Morris has caught on to the game and sees how to use the chart in order to tell the winner of a trick, that is, in order to tell when a second card TREES a first card. He sees that

4 TREES 6 and 6 TREES 4,
1 TREES 6 but 6 DOES NOT TREE 1,
3 DOES NOT TREE 8 and 8 DOES NOT TREE 3.

Use the TREE chart on page 5-B to tell which of the following are true statements and which are false statements.

1. 4 TREES 5	2. 7 TREES 3	3. 3 TREES 3
4. 2 TREES 6	5. 10 TREES 1	6. 1 TREES 10
7. 7 TREES 7	8. 5 TREES 10	9. 10 TREES 5
10. 6 DOES NOT TREE 10		11. 4 DOES NOT TREE 4
12. 7 DOES NOT TREE 5		13. 1 DOES NOT TREE 1

* * *

The chart on page 5-B pictures a 10-by-10 lattice. This lattice is the set of ordered pairs which is the cartesian square $D \times D$, where $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. ['D' for 'decade'.] The chart for TREE pictures a subset of $D \times D$. Let's call this subset 'T'. The set of ordered pairs in $D \times D$ which do not belong to T is called <u>the complement of T</u> [with respect to $D \times D$]. 'the complement of T' is commonly abbreviated '$\tilde{T}$'. What is $T \cup \tilde{T}$? What is $T \cap \tilde{T}$?

In order to tell whether the sentence:

4 TREES 5

is true, you can look at the TREE chart and see whether

(5, 4) is an element of T.

In other words [abbreviating 'is an element of' by 'ϵ'], the sentences:

4 TREES 5 and: $(5, 4) \epsilon T$

are equivalent sentences. Do you see that

'4 DOES NOT TREE 9' and '$(9, 4) \epsilon \tilde{T}$'

are equivalent sentences?

* * *

Answers for Part B.

1. T	2. T	3. F	4. T	5. F	6. F	7. T
8. T	9. T	10. T	11. F	12. F	13. T	

As class discussion exercises, try these:

If you want to win the trick, what are the best cards to have in your hand when

(a) it is your turn to play second? [Answer: 4 and 5]

(b) it is your turn to play first? [Answer: 1 and 10]

If you want to win the trick, what are the worst cards to have in your hand when

(a) it is your turn to play second? [Answer: 2, 3, 9, and 10]

(b) it is your turn to play first? [Answer: 5 and 6]

*

Words like 'lattice' and 'cartesian square' were used in Unit 4. [See pages 4-E ff., section 4.01, and the related COMMENTARY.] In making the operation sign for cartesian product, students should make a large boldface times sign to distinguish it from the ordinary times sign. [Read 'D **×** D' as 'D cross D'.] It may be helpful to insert here a few exercises for class discussion such as:

List the ordered pairs in the cartesian square of {1, 2, 3}. [Answer: (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)]

Does (4, 3) belong to the cartesian square of {3, 4, 5, 6}? [Answer: yes]

How many ordered pairs are there in C **×** C where C is the set of integers from 1 through 100? [Answer: 10,000]

List the ordered pairs in the intersection of the cartesian square of {1, 2, 3, 4, 5} and the cartesian square of {3, 4, 5, 6}. [Answer: (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)]

[One occasionally sees an exponent symbol used in denoting a cartesian square. For example, 'D^2' may be used as an abbreviation for 'D **×** D'.]

*

Notice that, in order to avoid ambiguity, we speak of the complement of T with respect to D $\times$ D. When referring to the complement of a set, one does so always with respect to some containing set. In each instance, the containing set consists of those objects with which we are then concerned, and is often called 'the space'. At present the space in which we are interested is D $\times$ D. When it is clear what the space is, one is justified in not referring to it. So, we can here abbreviate 'the complement of T with respect to D $\times$ D' to 'the complement of T' and abbreviate the latter to '$\tilde{T}$'. Note that, with this convention as to the meaning of '$\tilde{T}$', (7, 16) belongs neither to T nor to $\tilde{T}$. However,

$$\forall_{x \in D} \forall_{y \in D} [(x, y) \in T \text{ or } (x, y) \in \tilde{T}].$$

In general, for each space S and each set $E \subseteq S$, the complement of E with respect to S [or: $\tilde{E}$] is $\{x \in S: x \notin E\}$. So, $E \cup \tilde{E} = S$ and $E \cap \tilde{E} = \emptyset$.

[In some texts, the symbol '$\tilde{B}_A$' is used to refer to the complement of B with respect to a containing set A. In others, the symbol 'A − B' is used for the same purpose.].

*

A bit of imagination may help in getting across the notion of complement. Ask students to imagine that D $\times$ D is pictured by a 10-by-10 array of tiny light bulbs. Supply current to some of these bulbs, and you get a light picture of T. Divert the current from these bulbs to other bulbs in D $\times$ D, and you get a light picture of $\tilde{T}$. [See, also, TC[5-K]a.]

In the last four lines on page 5-F, students are asked whether they see that the sentences '4 DOES NOT TREE 9' and '(9, 4) $\in \tilde{T}$' are equivalent. They may not! The sentences <u>are</u> equivalent because when one says that 4 DOES NOT TREE 9 one means that the graph of (9, 4) is not one of the heavy dots on the chart; that is, one means that (9, 4) $\in \tilde{T}$. So, '4 DOES NOT TREE 9' and (9, 4) $\in \tilde{T}$' say the same thing.

Now, it happens that $(9, 4) \in T$. So, the sentences:

$$4 \text{ DOES NOT TREE } 9 \text{ and: } (9, 4) \in T$$

are both false. The sentences '4 TREES 9' and '$(9, 4) \in T$' are equivalent, and true. Also, the sentences '9 DOES NOT TREE 4' and '$(4, 9) \in \widetilde{T}$' are equivalent, and true.

*

To help students understand the idea of the complement of a set with respect to a containing space, try exercises like these in class. [Perhaps you can duplicate such a list in advance and pass it out to the students at this time.]

(1) Suppose the space is A where $A = \{1, 2, 3, 4, 5\}$. If B is $\{1, 2, 5\}$, what is $\widetilde{B}$? [Answer: $\widetilde{B} = \{3, 4\}$]

(2) Suppose the space is A where $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. If $B = \{1, 2, 5\}$, what is $\widetilde{B}$? [Answer: $\widetilde{B} = \{3, 4, 6, 7, 8, 9, 10\}$]

(3) Suppose the space consists of the students in your school. What is the complement of the set of students in your mathematics class? [Answer: the set consisting of those students who are <u>not</u> in the mathematics class]

(4) Suppose the space consists of the students in the classroom. What is the complement of the set consisting of those students who do not sit in the second row? [Answer: the set consisting of those students who <u>do</u> sit in the second row]

(5) Suppose the space is $S \times S$ where $S = \{1, 2, 3, 4, 5\}$. If R is the cartesian square of $\{2, 3, 4, 5\}$, what is $\widetilde{R}$? [Answer: $\widetilde{R} = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (3, 1), (4, 1), (5, 1)\}$]

(6) Suppose the space is G, where $G = \{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$. If $H = \{(2, 2), (3, 2), (2, 4), (4, 2), (4, 4)\}$, what is $\widetilde{H}$? [Answer: $\widetilde{H} = \{(2, 3), (3, 3), (3, 4), (4, 3)\}$]

(7) Suppose the space is A, where A is the cartesian square of $\{a, e, u\}$. If $V = \{(a, a), (a, e), (a, u), (e, a), (u, a), (u, u)\}$ what is $\widetilde{V}$? [Answer: $\widetilde{V} = \{(e, e), (e, u), (u, e)\}$]

(8) Suppose the space is the number plane. What is the complement of the set of all ordered pairs of real numbers such that the second component is not equal to the first component? [Answer: the set of all ordered pairs of real numbers such that the second component <u>is</u> equal to the first component. Using brace-notation, we would write:

$$\widetilde{\{(x, y):\ y \neq x\}} = \{(x, y):\ y = x\}]$$

(9) Suppose the space is the number plane. What is the complement of the first quadrant? [Answer: the set consisting of the ordered pairs of real numbers in any of quadrants II, III, IV, or in either of the axes. In other words:

$$\widetilde{\{(x, y):\ x > 0 \text{ and } y > 0\}} = \{(x, y):\ x \leq 0 \text{ or } y \leq 0\}$$

Recall that the domain of 'x' and 'y' is the set of real numbers.]

(10) Suppose the space is the number plane, and $A = \{(x, y):\ xy = 0\}$. What is the complement of A? [Answer: $\widetilde{A}$ consists of those ordered pairs of real numbers that are not in either of the axes. A shorter way of giving the answer is: $\widetilde{A} = \{(x, y):\ xy \neq 0\}$]

*

Exercises 9 through 14 may be done in at least two ways. The student might just try replacements for 'x' which will convert the sentence into a true sentence. A more efficient way to do Exercise 9, for example, would be to graph S, the set of ordered pairs in $D \times D$ whose second components are twice their first components. The solution set of '$(x, 2x) \in T$' is $S \cap T$. A similar procedure could be used in Exercises 10-14.

*

For Exercises 15 and 16, note that an ordered pair is in $\tilde{\tilde{T}}$ just if it is a point of $D \times D$ which is not in $\tilde{T}$. And, since a point is in $\tilde{T}$ if and only if it is a point of $D \times D$ which is not in T, a point of $D \times D$ is <u>not</u> in $\tilde{T}$ if and only if it is in T. So, an ordered pair is in $\tilde{\tilde{T}}$ if and only if it is in T. In other words, $\tilde{\tilde{T}} = T$.

*

Skill quizzes, and quizzes on content, are given on the pages listed below.

TC[5-I, J]b
TC[5-18]b
TC[5-39]c
TC[5-71]
TC[5-96]d, e
TC[5-122]
TC[5-144, 145, 146]c
TC[5-163, 164, 165]b
TC[5-189, 190]b
TC[5-217, 218]b
TC[5-K]e, f
TC[5-23]ℓ
TC[5-47]d
TC[5-78]
TC[5-106]b, c
TC[5-133]b
TC[5-148, 149]b
TC[5-169]b
TC[5-197]c
TC[5-12]e
TC[5-37]c
TC[5-55, 56]c
TC[5-85]
TC[5-115]b, c
TC[5-138, 139]b
TC[5-156, 157]b
TC[5-183]b
TC[5-207, 208]b

A comprehensive examination over pages 5-A through 5-115 is given on TC[5-117, 118]b, c, d, e, f, g.

A collection of quiz items for all of Unit 5 is given on TC[5-218]a, b, c, d, e, f, g, h, i.

The use of 'ϵ' as an abbreviation for 'is an element of' was introduced on TC[3-112]a. 'ϵ' may also be read as 'is a member of' or as 'belongs to'.

*

Answers for Part C.

1. F

2. T [After the student has answered 'false' to Exercise 1, he should immediately know that the answer for Exercise 2 is 'true'. Similarly for Exercises 3 and 4.]

3. T	4. F	5. F	6. T	7. F
8. F	9. T	10. T	11. T	12. F

*

In Part D the student is asked to find the solution set for each sentence, that is, to find values of 'x' which satisfy the sentence. The domain of 'x' is D.

*

Answers for Part D.

1. $\{5, 6\}$

2. $\{1, 2, 3, 4, 7, 8, 9, 10\}$, or: $\widetilde{\{5, 6\}}$

[When the answer is given as '$\widetilde{\{5, 6\}}$', it must be understood that the set named is the complement of $\{5, 6\}$ with respect to D.]

3. $\widetilde{\{1, 10\}}$, or: $\{2, 3, 4, 5, 6, 7, 8, 9\}$

4. $\{1, 10\}$, or: $\widetilde{\{2, 3, 4, 5, 6, 7, 8, 9\}}$ [Note that once a student has answered Exercise 3, he should be able to answer Exercise 4 immediately.]

5. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, or: D, or: $\widetilde{\emptyset}$

6. $\emptyset$, or: $\widetilde{D}$

7. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, or: D, or: $\widetilde{\emptyset}$

8. $\{1, 4, 5, 6, 7, 8\}$	9. $\{2, 3, 4, 5\}$	10. $\{3, 4, 5\}$
11. $\{5, 6, 7, 8\}$	12. $\{1, 7, 8\}$	13. $\{3, 4, 5\}$
14. $\{2, 3, 5, 6\}$	15. $\{3, 4, 5, 6, 7, 8\}$	16. $\{4, 5\}$

C. True or false?

1. (10, 1) is an element of T
2. (10, 1) is a member of $\tilde{T}$
3. (6, 9) belongs to T
4. (6, 9) belongs to $\tilde{T}$
5. $(7, 3) \in T$
6. $(4, 2) \in \tilde{T}$
7. $(5, 10) \in \tilde{T}$
8. $(9, 6) \in \tilde{T}$
9. $(2, 9) \in \tilde{T}$
10. $(6, 6) \in T$
11. $(8, 8) \in \tilde{T}$
12. $(2, 1) \in T$

D. Find the solution set of each of the following sentences.

Sample. $(x, 8) \in T$

Solution. Since $(4, 8) \in T$, $(5, 8) \in T$, $(6, 8) \in T$, and $(7, 8) \in T$, and no other pair in $D \times D$ with second component 8 belongs to T, the solution set of the sentence '$(x, 8) \in T$' is $\{4, 5, 6, 7\}$.

[Note. In some of the exercises it will be the case that no matter what numeral you substitute for 'x', you will not get a true sentence. This is like having an equation with no roots. In such cases the solution set is the empty set, and when you write your answer, just write: $\emptyset$.]

1. $(x, 3) \in T$ [Answer: $\{5, 6\}$]
2. $(x, 3) \in \tilde{T}$
3. $(x, 6) \in T$
4. $(x, 6) \in \tilde{T}$
5. $(6, x) \in T$
6. $(6, x) \in \tilde{T}$
7. $(x, 4) \in T$
8. $(4, x) \in T$
9. $(x, 2x) \in T$
10. $(2x, x) \in T$
11. $(x, x - 2) \in T$
12. $(x, x + 2) \in \tilde{T}$
13. $(x, 2x - 1) \in T$
14. $(x, 7 - x) \in T$
15. $(x, 7) \in \tilde{\tilde{T}}$
16. $(1, x) \in \tilde{\tilde{T}}$

* * *

You may recall from an earlier unit a quick way of writing a name for the solution set of a sentence. For example, take the sentence:

$$(x, 3) \in T$$

We can name the solution set of this sentence by:

$$\{x \in D: (x, 3) \in T\}$$

This name is read as 'the set of all x in D such that (x, 3) ∈ T'.

* * *

E. True or false?

Sample 1. $\{x \in D: (x, 3) \in T\} = \{x \in D: (x, 8) \in T\}$

Solution. One way to decide whether this sentence is true or false is to list the elements in the sets.

$$\{x \in D: (x, 3) \in T\} = \{5, 6\}$$
$$\{x \in D: (x, 8) \in T\} = \{4, 5, 6, 7\}$$

Since there is at least one element in {4, 5, 6, 7} which is not a member of {5, 6}, we conclude that

$$\{5, 6\} \neq \{4, 5, 6, 7\}.$$

So, the given sentence is false.

Sample 2. $\{x \in D: (x, 3) \in T\} \subseteq \{x \in D: (x, 8) \in T\}$

Solution. We know, from Sample 1, that

$$\{x \in D: (x, 3) \in T\} = \{5, 6\}$$
and
$$\{x \in D: (x, 8) \in T\} = \{4, 5, 6, 7\}.$$

Since there is no member of {5, 6} which is not a member of {4, 5, 6, 7}, the first set is a subset of the second.

$$\{5, 6\} \subseteq \{4, 5, 6, 7\}.$$

So, the given sentence is true.

1. $\{x \in D: (x, 3) \in T\} = \{x \in D: (x, 2) \in T\}$

2. $\{x \in D: (x, 8) \in T\} \subseteq \{x \in D: (x, 3) \in T\}$

3. $\{x \in D: (x, 3) \in T\} \subset \{x \in D: (x, 8) \in T\}$

Correction. In Exercise 3, change '$\subset$' to '$\subseteq$'.

You [or your students] may wonder why we have written:

(1) $\{x \in D: (x, 3) \in T\}$

rather than merely:

(2) $\{x: (x, 3) \in T\}$

[In Unit 4, pages 4-9 ff., we introduced restrictions similar to '$\in D$'. There we wrote, for example, '$\{(x, y), x$ and y integers: $x = y - 13\}$'. In this unit we shall write, instead of this, '$\{(x, y) \in I \times I: x = y - 13\}$', using 'I' as a name for the set of integers.] Such names as (1) have the advantage that, for example,

$$\overline{\{x \in D: (x, 3) \in T\}} = \{x \in D: (x, 3) \notin T\}.$$

Thus, complementation with respect to the space D is associated with denial, as expressed by the slash ['$\notin$'], just as union is associated with alternation, as expressed by 'or':

$$\{x \in D: (x, 3) \in T\} \cup \{x \in D: (x, 5) \in T\} = \{x \in D: (x, 3) \in T \text{ or } (x, 5) \in T\}$$

and intersection is associated with conjunction, as expressed by 'and':

$$\{x \in D: (x, 3) \in T\} \cap \{x \in D: (x, 5) \in T\} = \{x \in D: (x, 3) \in T \text{ and } (x, 5) \in T\}$$

In contrast, $\{x: (x, 3) \notin T\}$ consists of all real numbers which do not belong to D as well as those which belong to $\{x \in D: (x, 3) \notin T\}$.

The same result might have been obtained, in this case, by writing

(3) $\{x: x \in D \text{ and } (x, 3) \in T\}$

instead of (1). However, (1) has the advantage that it first points out to the reader what elements come into the total picture [that is, the members of the space under consideration], and it then tells him how to select from these the members of the subset which it names.

Also, as has been pointed out on TC[3-27]c, restrictions like '$\in D$' are sometimes necessary if one is to avoid nonsense. For example, '$\{x: x/x = 1\}$' is unsatisfactory because the result of substituting '0' for 'x' in '$x/x = 1$' is neither true nor false. Instead, it is meaningless. So, we write '$\{x \neq 0: x/x = 1\}$' to indicate that our usual convention that the domain of 'x' is the set of real numbers is here replaced by the convention that the domain of 'x' is the set of nonzero real numbers. Similarly, '$\{(x, y): x = \sqrt{y}\}$' is unsatisfactory because, although -3, for example, belongs to the domain of 'y', '$\sqrt{-3}$' is nonsense [when, as now, we are dealing with real numbers]. So, we write '$\{(x, y), y \geq 0: x = \sqrt{y}\}$', instead. [Read this as 'the set of (x, y), y nonnegative, such that $x = \sqrt{y}$'.]

*

Samples 1 and 2 of Part E provide the students with a review of the notions of subset and equality of sets. For each set A, for each set B, A = B if and only if each member of A is a member of B, and each member of B is also a member of A. Hence [Sample 1], in order to show that $\{x \in D: (x, 3) \in T\} \neq \{x \in D: (x, 8) \in T\}$, it is sufficient either to find at least one member of $\{x \in D: (x, 3) \in T\}$ which is not a member of $\{x \in D: (x, 8) \in T\}$, or to find one member of $\{x \in D: (x, 8) \in T\}$ which is not a member of $\{x \in D: (x, 3) \in T\}$.

Sample 2 deals with the notion of subset. For each set A, for each set B, $A \subseteq B$ if and only if each member of A is a member of B. To show that $A \not\subseteq B$, we must find at least one member of A which is not a member of B. Since there is no member of $\{5, 6\}$ which is not a member of $\{4, 5, 6, 7\}$ [alternatively: since each member of A is a member of B], $\{5, 6\} \subseteq \{4, 5, 6, 7\}$. This method of testing to decide if a first set is a subset of a second set is helpful in showing that the empty set is a subset of every set. Since, by definition, the empty set contains no members, there is no member of the empty set which is not a member of any given set. Hence, the empty set is a subset of every set. [See Exercise 10 of Part E on page 5-I.]

*

Here is a summary of some of the properties of the empty set. For each set A, $\emptyset \subseteq A$, $A \times \emptyset = \emptyset = \emptyset \times A$, and [for each set B] if $A \times B = \emptyset$ then $A = \emptyset$ or $B = \emptyset$. Also, $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$, and, with respect to any space S, $\tilde{\emptyset} = S$ and $\tilde{S} = \emptyset$.

*

Note that by mentioning subsets we can shorten the description of equality of sets. Thus, for each set A, for each set B, A = B if and only if $A \subseteq B$ and $B \subseteq A$.

*

Some authors use '$\subset$' for 'is a subset of', rather than '$\subseteq$'. We prefer to reserve '$\subset$' for 'is a proper subset of'. So, for each set A, for each set B, $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$. Note the analogy between $\subset$, $\subseteq$ and $<$, $\leq$.

*

Answers for Part E [on pages 5-H and 5-I].

1. T

2. F

3. T

Quiz.

Consider the game, CROSS, played with the same cards as TREE. The chart used for CROSS is pictured below on a 10-by-10 lattice, where $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Second										
10	×	×	•	•	•	•	•	•	×	×
9	×	×	×	•	•	•	•	×	×	×
8	•	×	×	×	•	•	×	×	×	•
7	•	•	×	×	×	×	×	×	•	•
6	•	•	•	×	×	×	×	•	•	•
5	•	•	•	×	×	×	×	•	•	•
4	•	•	×	×	×	×	×	×	•	•
3	•	×	×	×	•	•	×	×	×	•
2	×	×	×	•	•	•	•	×	×	×
1	×	×	•	•	•	•	•	•	×	×
	1	2	3	4	5	6	7	8	9	10

First

A. True or False?

1. $\{x \in D: (x, 5) \in C\} \subseteq \{x \in D: (x, 7) \in C\}$
2. $\{x \in D: (4, x) \in C\} = \{x \in D: (x, 4) \in C\}$
3. $\{x \in D: (1, x) \notin C\} \subseteq \{x \in D: (9, x) \notin C\}$
4. $\{x \in D: (x, 2) \in \tilde{C}\} \subseteq \{x \in D: (x, 10) \in \tilde{C}\}$
5. $\{x \in D: (2, x) \in \tilde{C}\} = \{x \in D: (x, 6) \notin C\}$

B. Simplify.

1. $\{x \in D: (x, 10) \in C\} \cap \{x \in D: (x, 3) \in C\}$
2. $\{x \in D: (x, 8) \notin C\} \cup \{x \in D: (9, x) \in C\}$
3. $\{x \in D: (5, x) \in \tilde{C}\} \cap \{x \in D: (x, 2) \in C\}$
4. $\{x \in D: (4, x) \in C\} \cap \{x \in D: (x, 10) \in C\}$
5. $\{x \in D: (x, 7) \notin \tilde{C}\} \cup \{x \in D: (1, x) \in C\}$

*

Answers for Quiz.

A. 1. T 2. T 3. F 4. T 5. F

B. 1. $\{2, 9\}$ 2. $\{4, 7\}$ 3. $\{4, 5, 6, 7\}$ 4. $\emptyset$ 5. $\tilde{\emptyset}$

TC[5-I, J]b

> <u>Correction.</u> In the Solution for Sample 1, the second line should read:
>
> that is, we want to know what elements they have in

4. T 5. T 6. T 7. T 8. T 9. T 10. T

*

Answers for Part F.

1. $\widetilde{\{1, 2, 9, 10\}}$ or $\{3, 4, 5, 6, 7, 8\}$ 2. D 3. $\{4, 5, 6\}$

4. $\widetilde{\{4, 5, 6\}}$ 5. $\{5\}$ 6. $\widetilde{\{1, 8\}}$ 7. $\emptyset$ 8. $\emptyset$

9. $\widetilde{\{1, 10\}}$ 10. $\{4, 5, 6\}$

*

In solving Exercises 3 and 4 of Part F students may discover that for each set A, for each set B, $\tilde{A} \cup \tilde{B} = \widetilde{A \cap B}$. This is a theorem from the algebra of sets, and is one of two such theorems which are called 'DeMorgan's Laws'. Other theorems can be obtained from this by interchanging '$\cup$' and '$\cap$'. Other theorems of the algebra of sets are discussed in section 5.02. [Augustus DeMorgan was a nineteenth century mathematician who made important contributions to logic.]

*

UPPER TRIANGLE [page 5-J] is played with the same cards as TREE, and also with the same type of rule. That is, the second card wins the trick just if the ordered pair of cards corresponds to a heavy dot in the chart for UPPER TRIANGLE. [If each player plays a 3, the first player wins.]

*

Answers for Part G [on page 5-J].

1. $\{1, 2\}$ 2. $\widetilde{\{1, 2\}}$ 3. $\emptyset$ 4. D 5. $\{1, 2, 3, 4, 5\}$

6. $\{\tilde{1}\}$ 7. $\{7, 8, 9, 10\}$ 8. $\{7, 8, 9, 10\}$ 9. $\{2, 3, 4\}$

10. $\{3, 4\}$ 11. $\{2, 4, 6, 8, 10\}$ 12. $\{4, 6, 8, 10\}$

4. $\{x \in D: (x, 3) \in T\} \subseteq \{x \in D: (x, 2) \in T\}$

5. $\{x \in D: (x, 2) \in T\} \subseteq \{x \in D: (x, 3) \in T\}$

6. $\{x \in D: (x, 2) \in \tilde{T}\} = \{x \in D: (x, 3) \in \tilde{T}\}$

7. $\{x \in D: (x, 8) \in T\} = \{x \in D: (3, x) \in T\}$

8. $\{x \in D: (4, x) \notin T\} = \{x \in D: (7, x) \notin T\}$

9. $\{x \in D: (4, x) \in \tilde{T}\} \subseteq \{x \in D: (7, x) \in \tilde{T}\}$

10. $\{x \in D: (6, x) \in \tilde{T}\} \subseteq \{x \in D: (x, 4) \in \tilde{T}\}$

F. Simplify.

Sample 1. Simplify: $\{x \in D: (x, 2) \in T\} \cap \{x \in D: (x, 8) \in T\}$

Solution. We are interested in the intersection of the sets, that is, we want to know that elements they have in common. A way to find this out is to list the elements in the sets.

$$\{x \in D: (x, 2) \in T\} = \{5, 6\},$$
$$\{x \in D: (x, 8) \in T\} = \{4, 5, 6, 7\}$$

Since 5 belongs to each set and 6 belongs to each set, and since these are the only elements which belong to both, it follows that $\{5, 6\} \cap \{4, 5, 6, 7\} = \{5, 6\}$. So, '$\{x \in D: (x, 2) \in T\} \cap \{x \in D: (x, 8) \in T\}$' simplifies to '$\{5, 6\}$'.

1. $\{x \in D: (x, 5) \in T\} \cap \{x \in D: (x, 7) \in T\}$

2. $\{x \in D: (x, 5) \in T\} \cup \{x \in D: (x, 7) \in T\}$

3. $\{x \in D: (4, x) \in T\} \cap \{x \in D: (9, x) \in T\}$

4. $\{m \in D: (4, m) \notin T\} \cup \{m \in D: (9, m) \notin T\}$

5. $\{x \in D: (x, 10) \in T\} \cap \{x \in D: (1, x) \in T\}$

6. $\{a \in D: (7, a) \in \tilde{T}\} \cup \{b \in D: (3, b) \in T\}$

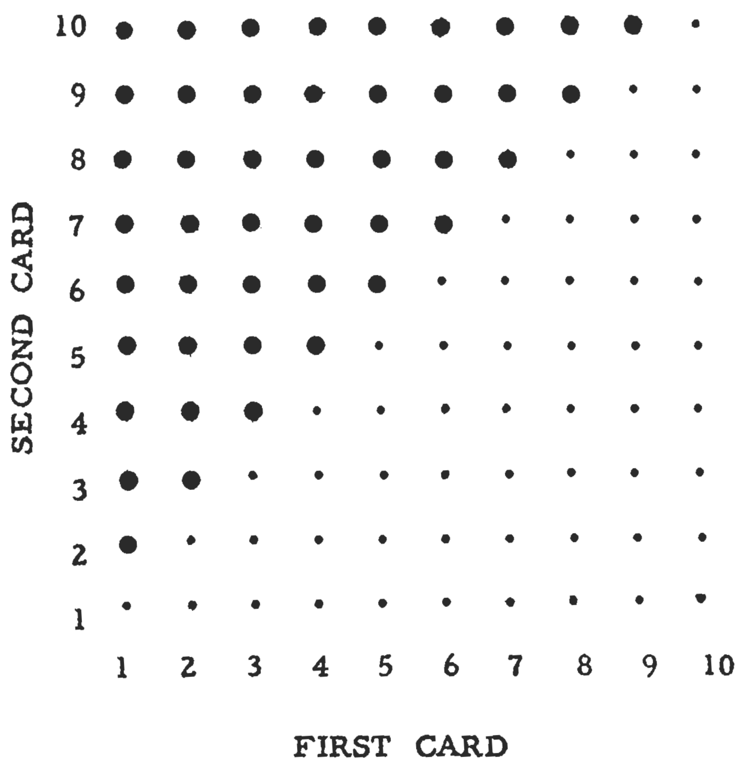
7. $\{x \in D: (8, x) \in \tilde{T}\} \cap \{x \in D: (x, 3) \in T\}$

8. $\{k \in D: (5, k) \in \tilde{T}\} \cap \{k \in D: (k, 4) \in T\}$

9. $\{n \in D: (n, 6) \in T\} \cup \{n \in D: (6, n) \in \tilde{T}\}$

10. $\{x \in D: (2, x) \in T\} \cap [\{x \in D: (x, 9) \in T\} \cup \{x \in D: (10, x) \in T\}]$

G. Another game which is played with the same cards as TREE is UPPER TRIANGLE. Here is the chart for UPPER TRIANGLE.



If the first player plays a 5 and the second player a 6, the second player wins the trick because (5, 6) corresponds with one of the heavy dots in the "upper triangle". For short, we say that

$$(5, 6) \in U.$$

If an 8 is played first and a 6 is played second, the 6 loses because $(8, 6) \in \tilde{U}$. What happens if each player plays a 3? Give the solution set of each of the following sentences.

1. $(x, 3) \in U$
2. $(x, 3) \in \tilde{U}$
3. $(10, x) \in U$
4. $(x, x) \in \tilde{U}$
5. $(x, 2x) \in U$
6. $(1, x) \in U$
7. $(6, x) \in U$
8. $(x, 7) \in \tilde{U}$
9. $(x, 3x - 2) \in U$
10. $(x, 4x - 7) \in U$
11. $(x, \frac{x}{2} + 1) \in \tilde{U}$
12. $(x, \frac{x}{2} - 1) \in \tilde{U}$

Answers for Quiz.

A.

1. $3a - 8$
2. $9x - 6$
3. $7a - 2$
4. 118
5. 1230
6. 4
7. n^2
8. $16 + 8x$
9. 1900
10. 1
11. x
12. 8/21
13. $21c/8$
14. $4b$
15. $2(d + 2)$

B.

1. $5(n - 2)$
2. $x(2 + 3y)$
3. $13r(2s - 1)$
4. $(b - 4)(b + 4)$
5. $4(c^2 - 15)$
6. $(d - 9)(d - 2)$
7. $(e - 9)(e + 2)$
8. $(f + 6)(f - 3)$
9. $(g + 6)(g + 3)$
10. $(h - 18)(h - 1)$
11. $(j + 18)(j - 1)$
12. $(k - 4)(k - 4)$
13. $(n - 8)(n + 2)$
14. $(4 + p)(4 + p)$
15. $(8 - q)(2 + q)$
16. $r(s - 3)(s - 3)$
17. $3(t - 4)(t - 4)$
18. $(\frac{1}{2}u - \frac{1}{3}x)(\frac{1}{2}u + \frac{1}{3}x)$

C.

1. 3
2. 3
3. $\{y: y > 3\}$
4. 3.5
5. -15
6. 1
7. $\{d: 1 < d\}$
8. 6, -6
9. 0, 1

Skill Quiz.

A. Simplify.

1. $3(a - 4) + 4$
2. $6(2x - 1) - 3x$
3. $5a - 2(1 - a)$
4. $(65 + 18) + 35$
5. $8 \times 123 + 2 \times 123$
6. $(4 + x^2) - x^2$
7. $(n - 5)(n + 5) + 25$
8. $(4 + x)^2 - x^2$
9. $75 \times 19 + 19 \times 25$
10. $\frac{2}{3} + \frac{3}{4} - \frac{5}{12}$
11. $\frac{2x}{3} - \frac{5x}{12} + \frac{3x}{4}$
12. $\frac{10}{27} \times \frac{36}{35}$
13. $\frac{35c^2}{36} \div \frac{10c}{27}$
14. $12ab \div (3a)$
15. $\frac{4(d + 2)^2}{2(d + 2)}$

B. Factor.

1. $5n - 10$
2. $2x + 3xy$
3. $26rs - 13r$
4. $b^2 - 16$
5. $4c^2 - 60$
6. $d^2 - 11d + 18$
7. $e^2 - 7e - 18$
8. $f^2 + 3f - 18$
9. $g^2 + 9g + 18$
10. $h^2 - 19h + 18$
11. $j^2 + 17j - 18$
12. $k^2 - 8k + 16$
13. $n^2 - 6n - 16$
14. $p^2 + 8p + 16$
15. $16 + 6q - q^2$
16. $rs^2 - 6rs + 9r$
17. $3t^2 - 24t + 48$
18. $\frac{1}{4}u^2 - \frac{1}{9}x^2$

C. Solve. [For the inequations, give the solution set, using the simplest sentence possible as set selector.]

1. $3x - 8 = 13 - 4x$
2. $10 - 2v = v + 1$
3. $2y + 7 > y + 10$
4. $\frac{r}{3} = \frac{7}{6}$
5. $\frac{s + 3}{4} = \frac{s}{5}$
6. $\frac{15}{4c} = \frac{3}{4c} + 3$
7. $14 - 3d < 2 + 9d$
8. $e^2 = 36$
9. $a(a - 1) = 0$

*

the relation of being-$^+2$-greater-than [the second component of each of its ordered pairs is $^+2$ greater than its first component].

[On terminology.--Many people, including some mathematicians, feel that a relation is something other than a set of ordered pairs [although all would probably agree that this something "gives" one such a set]. However, it seems to be very difficult for one who feels this way about relations to convey his meaning of 'relation' to someone who doesn't already feel the same way. Since the concept of a relation as being merely a set of ordered pairs is adequate for mathematics [and for much of logic], and since this concept involves nothing new to students who have already worked with sets and ordered pairs, we have chosen this simpler one of the two alternatives. Similar remarks apply to the notion of operation and to the yet-to-be-introduced notion of function. The definition of 'function' which we adopt in Section 5.05 is that a function is a set of ordered pairs no two of which have the same first component. So, an operation is a function, and a function is a special kind of relation. [Functions are sometimes called many-one relations.] There seems to be no explicit convention as to when one uses the word 'operation' in preference to 'function'.]

*

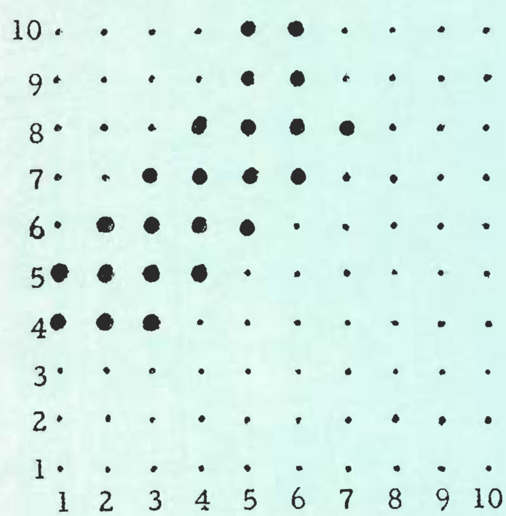
[Please see TC[5-10]d for an important note on Supplementary Exercises.]

*

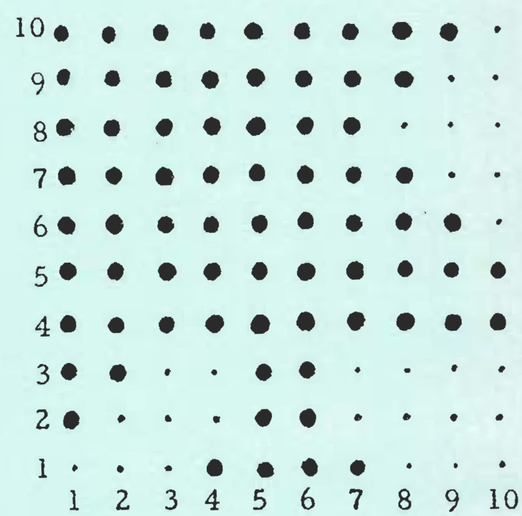
Note that the Miscellaneous Exercises are really review exercises for Units 1, 2, 3, and 4. With a very few exceptions, they can be assigned independently of the work you are doing in Unit 5. We have included work on manipulation, solving of equations and inequations, word problems, business arithmetic, percent, conversion of units, and radicals. [The exercises on radicals are arranged in developmental fashion, and should be assigned just before the work on quadratic equations.] On TC[5-K]e we give a suggested Skill Quiz. Results on this quiz may provide you with an opportunity for making differentiated assignments from the Miscellaneous Exercises.

Answers for Part H [on page 5-K].

1.



2.



3. (a) 50 (b) 50 (c) 45 (d) 55 (e) 100 (f) 0 (g) 100
(h) 0 (i) 23 (j) 77 (k) 77 (ℓ) 72 (m) 28 (n) 28

*

In doing parts (b), (d), (e), (g), and (j) of Exercise 3 students will probably become aware of and use the generalization: For each subset A of a space S, $n(A) + n(\widetilde{A}) = n(S)$. [As on page 4-12, '$n(A)$' means the number of elements in A.] They may also use this to get a short cut for solving part (ℓ) [$n(T \cup U) = 100 - n(\widetilde{T \cup U})$]. So, they may solve (m) first, and use the result to solve part (ℓ). On the other hand, they may answer part (ℓ) by recalling the generalization given on page TC[4-12]a and using the instance:

$$n(T \cup U) = n(T) + n(U) - n(T \cap U),$$

together with the results of parts (a), (c), and (i). Point out, if they do not use this to answer part (ℓ), that they may use it to check the consistency of their answers for parts (ℓ), (a), (c), and (i). The consistency of their answers for parts (k), (b), (d), and (n) can be checked in a similar manner.

Notice that Part H begins with the sentence:

> For each two games that the members of the Games Club invent, they can make up more games just by combining the charts for the two games.

Ask your students whether there could be pairs of games whose intersection will not give a new game. The intersection of TREE and UPPER TRIANGLE INTERSECTION TREE will not give a new game since $U \cap T \subseteq T$. Also, ask the students: If the intersection of two games does not give a new game, can the union of these two games give a new game? [If the set of points for one game is a subset of the set of points for another game, neither the intersection of these two sets nor their union will give a new game.]

*

Here is a description of a device you may find useful in illustrating the notions of union, intersection, and complement, and demonstrating principles such as DeMorgan's Laws. Draw a large-scale graph of T, say, on a sheet of typing paper. Then cut out 1/4-inch squares, each with its center at a graph of a member of T, and with its sides parallel to the edges of the paper. [This gives you a sheet with fifty square holes cut in it.] Indicate the graphs of the members of $\tilde{T}$ by moderately heavy dots. Label the sheet 'T'. Now, do the same thing for U, for $\tilde{T}$, and for $\tilde{U}$, making sure that, when you place one sheet on another, graphs of the same ordered pair coincide.

Now, if you place, say, the T-sheet on the U-sheet, members of $T \cap U$ are identified by holes, and members of $T \cup U$ by holes or thin spots. It is clear, for example, that $T \cap U$ is a subset of both T and U, and that both T and U are subsets of $T \cup U$. Using the T-sheet and the $\tilde{T}$-sheet shows that $T \cup \tilde{T} = D \times D$ and that $T \cap \tilde{T} = \emptyset$. That $\widetilde{T \cap U} = \tilde{T} \cup \tilde{U}$ can be seen by comparing what one sees in placing the T-sheet over the U-sheet with what one sees on placing the $\tilde{T}$-sheet over the $\tilde{U}$-sheet. In the same way, one can see that $\widetilde{T \cup U} = \tilde{T} \cap \tilde{U}$. Other uses of these sheets, perhaps supplemented by sheets for additional relations, may occur to you as you use them.

*

H. For each two games that the members of the Games Club invent, they can make up more games just by combining the charts for the two games. An example of such a new game is UPPER TRIANGLE INTERSECTION TREE. A chart for this game is a picture of $D \times D$ with the graph of $U \cap T$ drawn on it.

1. Make such a chart.

2. On another chart make a graph of $U \cup T$.

3. How many points are there in each of these sets?

(a) T (b) $\tilde{T}$ (c) U

(d) $\tilde{U}$ (e) $U \cup \tilde{U}$ (f) $U \cap \tilde{U}$

(g) $T \cup \tilde{T}$ (h) $T \cap \tilde{T}$ (i) $T \cap U$

(j) $\widetilde{T \cap U}$ (k) $\tilde{T} \cup \tilde{U}$ (ℓ) $T \cup U$

(m) $\widetilde{T \cup U}$ (n) $\tilde{T} \cap \tilde{U}$

I. 1. The experts in the Games Club play a game which differs from UPPER TRIANGLE only in that it is played with a deck of 200 cards, two 1s, two 2s, two 3s, etc. The set of ordered pairs in their "upper triangle" is a subset of a 100-by-100 lattice. [How many ordered pairs does their "upper triangle" contain?] Pick out the ordered pairs listed below which belong to the experts' UPPER TRIANGLE.

(a) (17, 4) (b) (31, 74) (c) (16, 17)

(d) (17, 16) (e) (16, 16) (f) (98, 1)

(g) (99, 100) (h) (61, 59) (i) (21, 3)

(j) (75, 76) (k) (42, 39) (ℓ) (81, 97)

2. The experts wanted a short name for the set of ordered pairs pictured on a chart for their game. They chose 'G'. Do you see why?

In the case of genetic relationships, our convention leads, for example, to identifying fatherhood with $\{(x, y) \in P \times P: \text{ y is the father of x}\}$. [Here, 'P' denotes the set of people.] So, 'John is the father of Henry' is equivalent to '(Henry, John) $\in$ Fatherhood. Unfortunately this conflicts with the common usage of logicians who commonly make 'John is the father of Henry' equivalent to '(John, Henry) $\in$ Fatherhood. [Their convention is that a relation R is $\{(x, y): \text{ x bears the relation R to y}\}$.] Although this divergence in usage is unfortunate, we believe it best to remain consistent with mathematical usage as it has been established in the important case of functional relations.

5.01 Relations.--One of the important aims of scientific work is to find answers to questions about how one thing is related to another. For example, an economist might want to know how lifetime income is related to years of formal education, or how wheat production is related to annual rainfall. A psychologist wants to know how the number of trials it takes a rat to learn a maze is related to the type of food the rat eats. A geneticist tries to describe the relation of eye color of children to eye color of parents. A physicist wants to express the relation of the range of a projectile to its initial velocity.

In asking such questions, the investigator is really trying to find out how one thing is affected by another. In order to discover this, the economist will, for example, collect data which he might tabulate like this:

Subject	Years of Schooling	Lifetime Income ($)
Mr. A	16	320,000
Miss B	14	200,000
Mr. C	8	150,000
Mrs. D	16	250,000
.	.	.
.	.	.
.	.	.

In other words, he collects ordered pairs [(16, 320 000), (14, 200 000), etc.], and then in order to study the relationship of income to education [or, how education affects income], he attempts to describe the set of ordered pairs. People investigating other such problems involving relationships also list ordered pairs, and their study of relationships leads them to describe sets of ordered pairs.

This tie-up between studying a relationship and describing a set of ordered pairs suggests that a set of ordered pairs be called a relation.

Let's take a very simple example. Suppose someone claims that he has investigated a certain relationship among the numbers in $\{-3, -2, -1, 0, 1, 2\}$ and has found that

−3 has this relationship to −2

−1 bears this relationship to 0, and so do −2 and −3,

−1 is related in this way to 1,

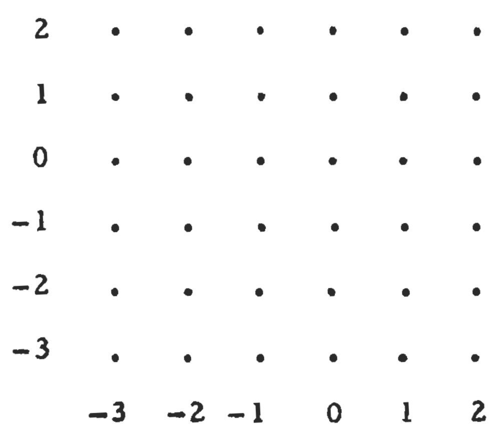
−3 bears this relationship to −1, and so does −2,

−3, −2, and 0 all are related this way to 1,

1, 0, −1, −2, and −3 all bear this relationship to 2.

You try to get more information; so, you ask, "Does 2 have this relationship to 0?" He answers, "No, it doesn't. I've told you about <u>all</u> the numbers in the set which have this relationship."

"Well, your description is a bit complicated. Perhaps I can follow it better if I record the facts in some kind of systematic way. This relationship you investigated is something which holds for ordered pairs of numbers in {−3, −2, −1, 0, 1, 2}. For the six numbers in the set, there are exactly ... um ..., yes, 6 times 6, or 36 ordered pairs.



Now, I'll darken some of these dots. I go to the −3-column, to darken the dots corresponding to those numbers which have this relationship to −3. According to what you have told me, there are no such numbers. Next, I go to the −2-column. Since −3 is the only number which has this relationship to −2, I will darken the graph of (−2, −3), and do nothing else to this column. Next, the −1-column. From your description, I darken just the graphs of (−1, −3) and (−1, −2). Continuing in this fashion, I get the following picture.

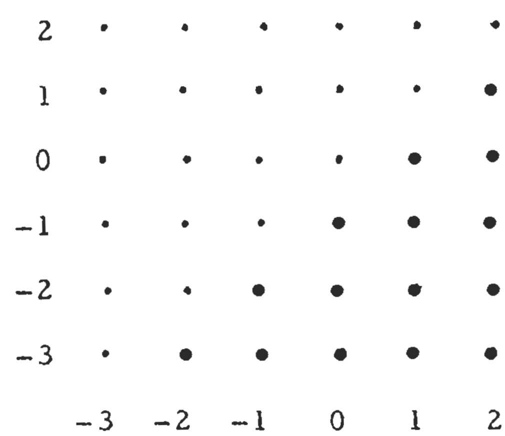
To avoid possible later confusion, note that when we speak of a relation among the members of a set, we do not mean to imply that all members of the set "get into the act". For example, cousinhood is a relation among people even though not everyone has a cousin. In this connection, you may want to glance over the introductory paragraph in section 5.04.

*

The objective to be reached by studying the material on pages 5-1 through 5-4 is student acceptance of the facts that one can come to know a relation by studying a certain set of ordered pairs and that it is natural to say that the relation is this set of ordered pairs. See TC[5-K]c, d.

*

One who identifies relations with sets of ordered pairs must decide whether a given relation R is $\{(x, y): y$ bears the relation R to $x\}$ or $\{(x, y): x$ bears the relation R to $y\}$. If he asks the question 'What relation does a nonnegative number bear to its square?', he gets as answer 'The square rooting relation.'. [A nonnegative number is the square root of its square.] Now, the square rooting relation is the function $\sqrt{}$ and, in graphing this function, the convention is to draw a picture of $\{(x, y), x \geq 0: y = \sqrt{x}\}$. So, by this convention, the square rooting relation is $\{(x, y), x \geq 0:$ y bears the square rooting relation to $x\}$. Since it would be quite unrealistic to consider changing this basic mathematical convention, and since we feel it to be very important to maintain consistency, we choose to identify each relation R with $\{(x, y): y$ bears the relation R to $x\}$. Consequently, in the Introduction, '5 TREES 3', for example, is equivalent to '$(3, 5) \in T$'. One result of this is that the relation of being greater than which each number bears to each smaller number is $\{(x, y): y > x\}$. So, for each x, for each y, $(x, y) \in\, >$ if and only if $y > x$. Similarly, the relation of being a factor of, with respect to the set I^+ of positive integers is $\{(x, y) \in I^+ \times I^+: y$ is a factor of x with respect to $I^+\}$. This relation is usually denoted by '$|$', so, $y \mid x$--that is, y is a factor of x with respect to I^+, if and only if $(x, y) \in\, |$.



I have darkened a dot for each ordered pair of numbers whose second component has your relationship to its first component. So, the picture makes it easier for me to understand the relationship. I can tell at a glance, for example, that the only numbers to which −2 has this relationship are −1, 0, 1, and 2. And, now, I see an easy way to describe the <u>relation</u> whose graph I have drawn. It's the set of ordered pairs whose components belong to $\{-3, -2, -1, 0, 1, 2\}$ and such that the second component of each ordered pair is less than its first component. For short, it's

$$\{(x, y),\ x \text{ and } y \text{ integers from } -3 \text{ through } 2 : y < x\}.$$

If I use 'S', say, as a name for the set of integers from −3 through 2, and remember that the ordered pairs whose components both belong to S are just the members of S × S, then I can write an even shorter name for this relation:

$$\{(x, y) \in S \times S:\ y < x\}$$

So, it's not as complicated as your description made it look."

There are many relations among the members of the set S. Some others, besides the one just discussed, are

$$\{(x, y) \in S \times S:\ y \neq x\},$$

$$\{(x, y) \in S \times S:\ y = x + 2\},$$

$$\{(x, y) \in S \times S:\ y = -1 \text{ or } x = 1\},$$

and

$$\{(-2, -1), (0, -2), (2, 1)\}.$$

Each relation among the members of S is a subset of S × S; and, each subset of S × S is such a relation.

As you have seen, one way of describing a relation is, first, to tell about things you are interested in, and, then to list the ordered pairs of these things which belong to the relation. Instead of listing, you can use a graph. Suppose, for example, you were watching two members of the Games Club playing TREE, and that these players knew the game so well that they were playing it without a chart. In order to find out how the game is played, you might begin by noticing that the cards have values which belong to the set D of integers from 1 through 10. Then, you could keep track of the tricks played and who won each of them by starting a list like this:

(3, 4), second player

(10, 4), second player

(2, 7), first player

(7, 9), first player

(6, 6), second player

(8, 8), first player

•
•
•

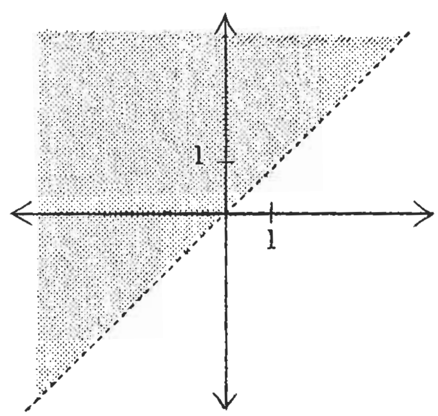
and continuing it for several games. You would probably find it difficult to discover any pattern from this list. So, you might graph the ordered pairs corresponding to tricks won by the second player, using a heavy dot for this purpose, and graph the tricks won by the first player, using light dots for these. If you kept this up long enough, you would get the picture shown on page 5-B. The heavy dots make up a graph of the relation T, a subset of $D \times D$. [The light dots make up a graph of the relation $\breve{T}$.] You have discovered that when cards with values x and y are played, in that order, the trick belongs to the second player if and only if $(x, y) \in T$. So, by discovering the relation T, you've learned how TREE is played.

NAMING A RELATION

As we have noted, a relation is a set--a set of ordered pairs. In order to talk about a relation we need a name for it, or a picture we can point to. It is usually convenient to have a name and, in the Introduction, we used the arbitrary names 'T' and 'U' [or 'G'] for the relations used in playing the games TREE and UPPER TRIANGLE. We could as well have called them 'Tom' and 'Ursula'. In either case, the name doesn't tell us anything about the relation. The relation has to be described in some way before we can know what the name means.

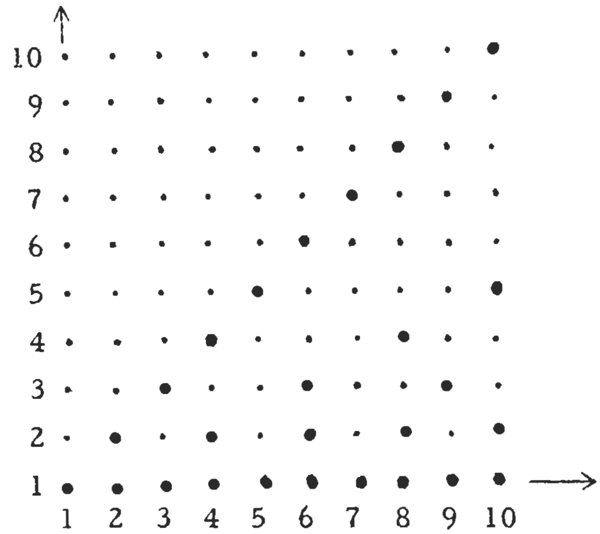
For example, we often have occasion to talk about the relation [among the real numbers] of being greater than. Since it is customary to abbreviate 'is greater than' by '>', mathematicians sometimes use the symbol '>' as a name for this relation. [Notice that in doing this one is using as a noun a symbol which is primarily used as a predicate. This could be confusing, but it is usually very easy to tell from context how the symbol is being used. For example, in the sentence '9 > 7', the '>' is a predicate; in the sentence '(7, 9) ϵ >', the '>' is a noun.] If you wanted to tell someone what '>' means, you would need a way of describing the relation. [Like 'T' and 'U', '>' is an arbitrary name and conveys no information until some kind of description is given.]

One way of describing the relation > is to draw a graph.



A person looking at the picture may feel fairly sure that he knows what '>' means. Of course, you would have to tell him that the picture is meant "to go on forever", and his main problem is what you mean by this. You may feel that this is not much of a problem, but that is because you are already well acquainted with the relation >.

However, consider the relation which mathematicians name by '|'. Here is a picture of part of the relation |.



This picture, too, is meant to go on forever. Can you add parts of the next two rows and columns? Which of these ordered pairs do you think belong to | -- (11, 11), (12, 12), (18, 12), (22, 11), (12, 3), (1540, 308), (7, 35)? Perhaps you have already discovered what relation the picture is meant to describe. [It is a relation you studied in Unit 4.] If so, you know what '|' means.

So, although a picture is sometimes helpful in describing a relation, it does have limitations. A much less ambiguous method is to use brace-notation. Thus, instead of giving a picture of >, we can say that it is

$$\{(x, y) \in R \times R:\ y - x \text{ is positive}\},$$

where R is the set of real numbers. Also, instead of the picture of |, we can say that it is

$$\{(x, y) \in I^+ \times I^+:\ y \text{ is a factor of } x \text{ with respect to } I^+\},$$

where I^+ is the set of positive integers.

A brace-notation name for a relation enables you to tell whether or not an ordered pair belongs to the relation. The first part of the name--the part on the left of the colon--tells you which ordered pairs are under consideration. The second part of the name is a sentence which can be thought of as a device for selecting from these ordered pairs just those which belong to the relation. [For this reason, such a sentence is often called 'a set selector'.]

Correction. In part (a) of Exercise 2 on page 5-7, change 'M' to 'B'.

In preparation for Exercise 1(f) of Part A, you may need to remind your students that, for each $x > 0$, $\sqrt{x}$ is the positive number whose square is x, and $-\sqrt{x}$ is the negative number whose square is x. Also, for Exercise 5, students may need to be reminded of the distinction between rational and irrational numbers [page 4-43 et. seq.] and of the fact [page 4-48] that, for each positive integer n, if $\sqrt{n}$ is not an integer, then $\sqrt{n}$ is irrational.

*

Answers for Part A [on pages 5-7, 5-8, and 5-9].

1. (a) T (b) T (c) T (d) F (e) F (f) T

   [Note the interesting connections in Exercise 1 between (a) and (c), and among (a), (d), and (e).]

2. (a) T (b) T (c) F [$7.8 \notin I$] (d) F (e) T (f) F

3. (3, 0), (8, −3), (−2, 3) [For each integer k, $(3 + 5k, -3k) \in A$.]

4. [Since 3 is a factor of 3 and of 6 with respect to I but is not a factor of 7, $D = \emptyset$. So, the instructions cannot be carried out.]

*

The set selectors in Exercises 3 and 4 [for which the domain of 'x' and 'y' is the set of integers] are examples of Diophantine equations. [See TC[4-41, 42]b and, for problems leading to such equations, TC[3-81] and TC[4-106, 107, 108]b and c.] The facts concerning the existence of solutions of such equations can be stated as follows. For integers a, b, and c, there are integers x and y such that $ax + by = c$ if and only if each common factor of a and b is a factor of c. It is easy to see that there are no solutions if this condition is not satisfied. It is more difficult to see that there are solutions whenever the condition is satisfied.

The bracketed remark made above in connection with the answer for Exercise 3 suggests how you can obtain all solutions of such an equation when one has been found. [In the case of Exercise 3, the components of a solution "in integers" other than (3, 0) must differ from 3 and 0 by integers, and if the sum of 3 times the first component and 5 times the second component is still to be 9, then the sum of 3 times the first difference and 5 times the second difference must be 0.] An interesting question to ask now in connection with Exercise 3 is 'Does A contain any point in the first quadrant?'. The answer is 'no', because there is no integer k such that $3 + 5k > 0$ and $-3k > 0$.

In Unit 2 we adopted the convention that unless otherwise specified the domain of a pronumeral is the set of real numbers. So, we can omit '$\in R \times R$' from the first part of a brace-notation name for a relation among the real numbers. Thus, for example, when you see:

$$\{(x, y): y \geq 3x + 7\},$$

you are looking at an abbreviation of:

$$\{(x, y) \in R \times R: y \geq 3x + 7\}$$

If we wanted to make a great deal of use of this particular relation, we might assign a short, arbitrary name to it, say, 'K'. Then, we could say that $(2, 15) \in K$ and $(5, 20) \notin K$. Indeed, we might even say that $15\ K\ 2$ or $20 \not{K}\ 5$. In most cases, one's interest in a given relation is likely to be too short-lived to make it profitable to give it a short, arbitrary name.

EXERCISES

A. Answer each of the following questions.

1. Suppose M is the relation $\{(x, y): x^2 + y^2 = 25\}$. Which of the following sentences are true?

   (a) $(3, 4) \in M$ (b) $(5, 10) \in \widetilde{M}$

   (c) $-3\ M\ -4$ (d) $(3, 4.08) \in M$

   (e) $(2.9, 3.9) \in M$ (f) $-\sqrt{21}\ \widetilde{M}\ -\sqrt{2}$

2. Suppose B is the relation $\{(p, q) \in I \times I: |p| + 1 = |q|\}$, where I is the set of integers. Which of the following sentences are true?

   (a) $(4, 5) \in M$ (b) $-5\ B\ 4$

   (c) $(7.8, -8.8) \in B$ (d) $(8, -7) \in B$

   (e) $(-7, 8) \in B$ (f) $-1\ \widetilde{B}\ 0$

3. Give three ordered pairs which belong to A where

$$A = \{(x, y) \in I \times I: 3x + 5y = 9\}.$$

4. Give three ordered pairs which belong to D where

$$D = \{(x, y) \in I \times I: 3x + 6y = 7\}.$$

5. Give three ordered pairs of irrational numbers which belong to F, where

$$F = \{(x, y): x^2 + y^2 - 7 = 0\}.$$

6. True or false?

(a) $(5, 30) \in \{(x, y): y > 5x + 4\}$

(b) $(2, -2) \in \{(x, y): x^2 = y^2\}$

(c) $(6, 3) \in \{(p, q), p \geq -3: q = \sqrt{p + 3}\}$

(d) $(5, 4) \in \{(a, b): a + 2b + 7 = 0\}$

(e) $(6, 3) \in \{(p, q), q \geq -3: p = \sqrt{q + 3}\}$

(f) $(6, 3) \in \{(q, p), q \geq -3: p = \sqrt{q + 3}\}$

7. For each relation, find an ordered pair which belongs to it.

(a) $\{(x, y): y - x = 7\}$

(b) $\{(x, y): x - y = 7\}$

(c) $\{(x, y): x^2 + y^2 = 100\}$

(d) $\{(x, y): y = |x| + 2\}$

(e) $\{(x, y): y = 3x^2\}$

(f) $\{(x, y): x = 3y^2\}$

(g) $\{(a, b): b > 3a - 1\}$

(h) $\{(c, d): |c| + |d| > 5\}$

(i) $\{(p, q): p + 3q = 9 \text{ and } p \geq 7\}$

(j) $\{(r, s): r = 2s \text{ or } r^3 + 3s^2 = 11\}$

(k) $\{(r, s): r = s \text{ and } 2r + 5s = 7\}$

(ℓ) $\{(x, y), x \in I^+ \text{ and } y \in I^-: 3x + 4y = 4\}$, where I^+ is the set of positive integers and I^- is the set of negative integers.

(m) $\{(x, y), x \in A \text{ and } y \in B: x + 2y = 7\}$, where A is the set of multiples of 3 and B is the set of rational numbers.

(n) $\{(x, y) \in A \times B: x + y = 45\}$, where A is the set of multiples of 5 and B is the set of multiples of 6. [Note that '$(x, y) \in A \times B$' is just an abbreviation for '(x, y), $x \in A$ and $y \in B$'. The relation in question is a subset of the cartesian product of $A \times B$.]

5. $(\sqrt{2}, \sqrt{5})$, $(-\sqrt{2}, \sqrt{5})$, $(\sqrt{2}, -\sqrt{5})$

6. (a) T (b) T (c) T (d) F (e) F (f) T

7. [Since there are many correct answers for each of most of the parts of Exercise 7, there is little point in giving answers here. The usual procedure of substituting for one variable in the set selector and solving the resulting sentence will work for most parts.

   (j) Each solution in (r, s) of '$r = 2s$' belongs to the relation and, also, each solution in (r, s) of '$r^3 + 3s^2 = 11$' belongs to the relation.

   (k) The only correct answer is: (1, 1)

   (ℓ) (4, −2), (8, −5), (12, −8), ... all belong to the relation. In fact, the relation is $\{(x, y):$ for some $z \in I^+$, $x = 4z$ and $y = 1 - 3z\}$.

   (m) The relation is $\{(x, y):$ for some $k \in I$, $x = 3k$ and $y = (7 - 3k)/2\}$. So, some of the ordered pairs in the relation are (−3, 5), (0, 7/2), and (3, 2).

   (n) The relation is $\{(x, y):$ for some $k \in I$, $x = 15(2k + 1)$ and $y = 30(1 - k)\}$. So, some of the ordered pairs in the relation are (−15, 60), (15, 30), (45, 0), and (75, −30).

   (o) Here is an easy kind of answer: (George Washington, Mrs. George Washington).

   (p) Perhaps the baseball coach will help you grade this question.]

(o) $\{(x, y) \in M \times W$: x was a married president of the United States and y was his wife$\}$, where M is the set of all men who ever lived and W is the set of all women who ever lived.

(p) $\{(x, y) \in T \times P$: y has played on x$\}$, where T is the set of major league baseball teams and P is the set of major league baseball players, past and present.

* * *

Parts (o) and (p) of Exercise 7 illustrate the fact that the concept of a relation is a very general one. Any set of ordered pairs is a relation, not just sets of ordered pairs of numbers. Note that in the sentence:

(1) y has played on x,

the letters 'y' and 'x' are pronouns just as they are in a sentence like:

(2) $y = 7 + x$

The pronouns in (2) hold places for numerals [that is, for names of numbers], and so we have called such pronouns pronumerals. What do the pronouns in (1) hold places for? If we look back at part (p) of Exercise 7, we see that 'y' holds a place for names of members of the set P, that is, names of major league baseball players. What does 'x' hold a place for? It would be inappropriate to say that the pronouns in (1) are pronumerals, and it is hardly worthwhile to invent special names for such pronouns. It is customary to call all such pronouns, including pronumerals, variables.

A variable is a pronoun. A pronumeral is a kind of variable; actually, it is a numerical variable. The important thing to remember is that a variable is a mark which holds a place in a sentence or in an expression for names of things. The things [numbers, sets, points, people, teams, etc.] whose names are substituted for the variable are called values of the variable; and the set of all values of a variable is called the domain of the variable [or, in some books, the range of the variable].

* * *

B. Draw graphs of these relations. [Use the convention, adopted in Unit 4, of drawing a horizontal line to picture the first component axis, and a vertical line for the second.]

1. $\{(x, y): y = 2x + 4\}$
2. $\{(x, y): x = 2y + 4\}$
3. $\{(x, y): y = 3\}$
4. $\{(x, y): x = 3\}$
5. $\{(a, b): 3a + 2b - 6 = 0\}$
6. $\{(r, s): s \geq |r|\}$
7. $\{(x, y): x = 3 \text{ or } y = 2\}$
8. $\{(x, y): x = 3 \text{ and } y = 2\}$
9. $\{(x, y): y = x^2\}$
10. $\{(x, y): y \geq x^2\}$
11. $\{(x, y): x > 2 \text{ and } y < 3\}$
12. $\{(x, y): y < x < 3\}$
13. $\{(x, y): 2y = x + 5 \text{ and } x \geq 3\}$
14. $\{(x, y): y = 2x - 2 \text{ and } y > 4\}$
15. $\{(x, y): (2y = x + 5 \text{ and } x \geq 3) \text{ or } (y = 2x - 2 \text{ and } y \geq 4)\}$
16. $\{(x, y): 2x + 3y = 26 \text{ or } 5x - 2y = 8\}$
17. $\{(x, y): 2x + 3y = 26 \text{ and } 5x - 2y = 8\}$

[Supplementary exercises on drawing graphs of relations are in Part A on page 5-238. Supplementary exercises which give practice in recognizing graphs of relations are in Part B on pages 5-238 through 5-239.]

INTERSECTIONS AND UNIONS OF RELATIONS

In Exercise 16 of Part B you were required to find the ordered pairs which satisfied at least one of two sentences. So, Exercise 16 required you to graph the union of two relations, that is, to graph

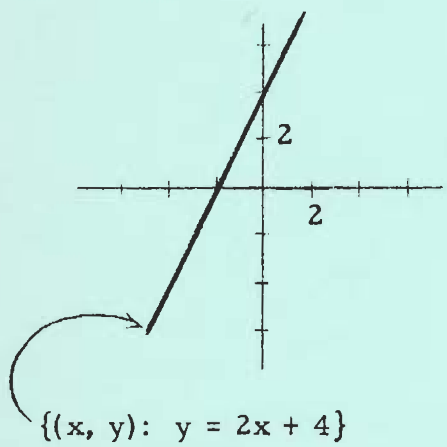
$$\{(x, y): 2x + 3y = 26\} \cup \{(x, y): 5x - 2y = 8\}.$$

On the other hand, in Exercise 17 you were required to find the ordered pairs which satisfied both of two sentences. This is a problem in graphing the intersection of two relations, that is, in graphing

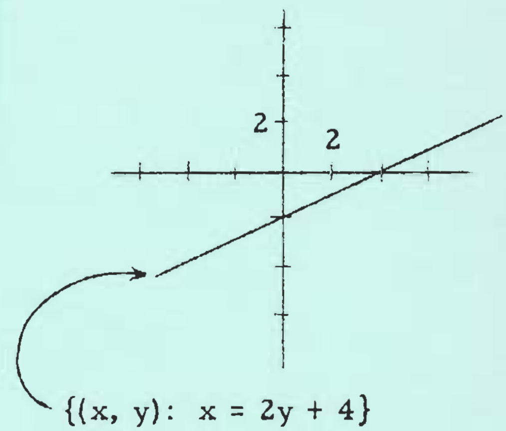
$$\{(x, y): 2x + 3y = 26\} \cap \{(x, y): 5x - 2y = 8\}.$$

Answers for Part B.

1.



2.



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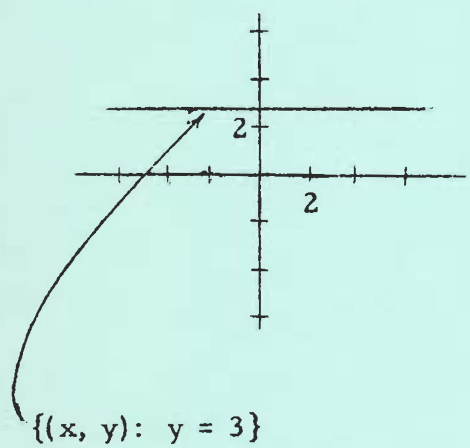

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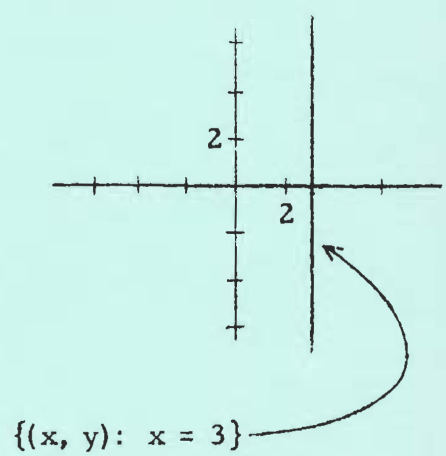
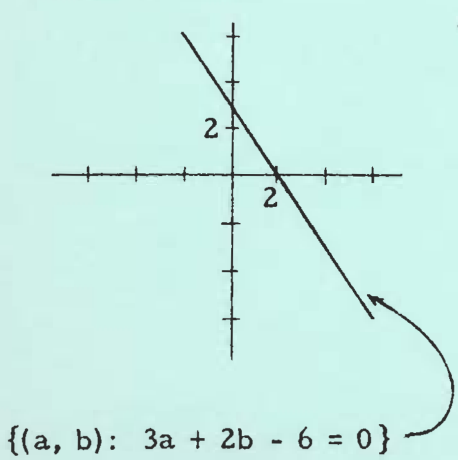


5.



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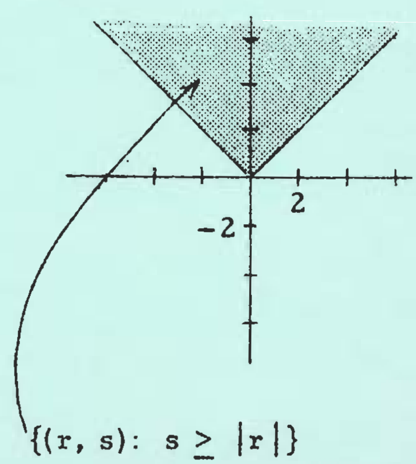

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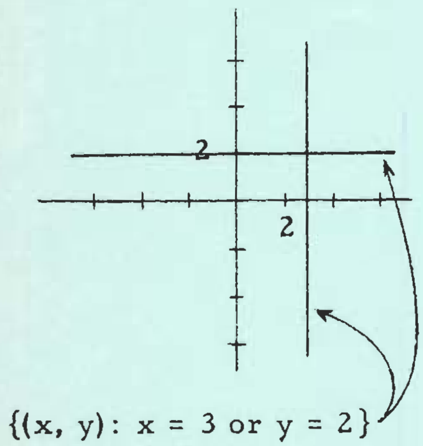
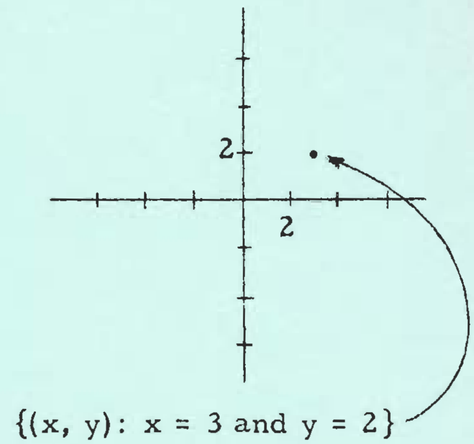


8.



*

Your students will probably notice

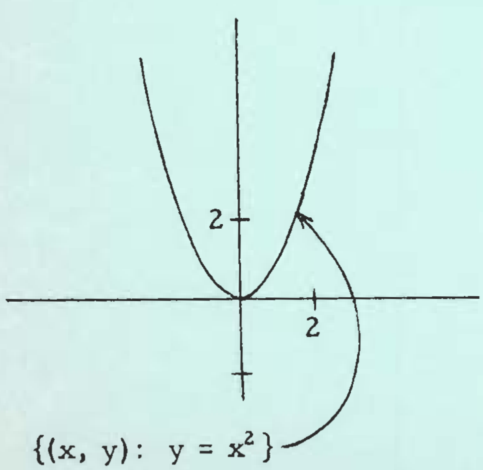
that $\{(x, y): x = 3 \text{ or } y = 2\} = \{(x, y): x = 3\} \cup \{(x, y): y = 2\}$

and $\{(x, y): x = 3 \text{ and } y = 2\} = \{(x, y): x = 3\} \cap \{(x, y): y = 2\}$

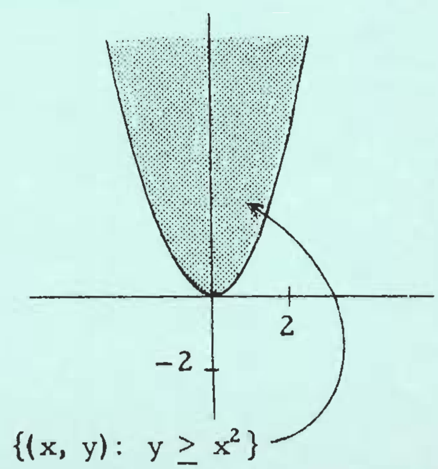
Finding other names using '$\cup$' or '$\cap$' for the relations given in Exercises 11-17 will provide a good review of the connections between 'and' and 'intersection', and between 'or' and 'union'. It may also make the graphing a little easier. See TC[5-H]a.

*

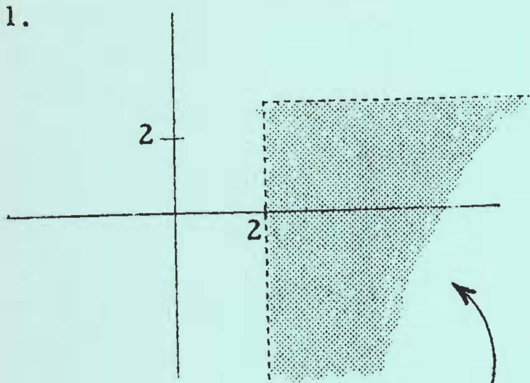
9.



10.

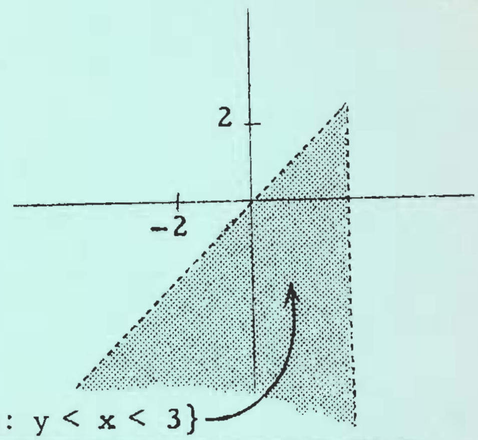


11.



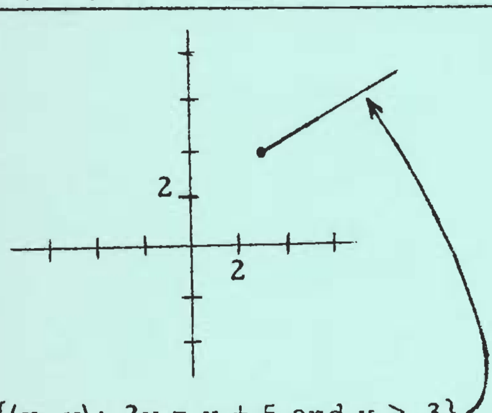
$\{(x, y): x > 2 \text{ and } y < 3\}$

12.



$\{(x, y): y < x < 3\}$

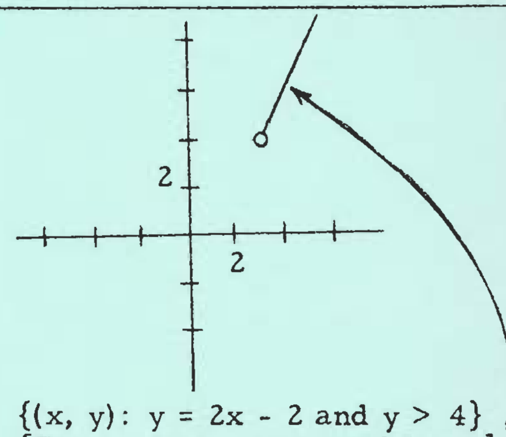
13.



$\{(x, y): 2y = x + 5 \text{ and } x \geq 3\}$

[This relation is a ray.]

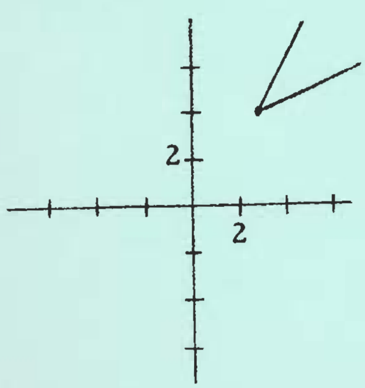
14.



$\{(x, y): y = 2x - 2 \text{ and } y > 4\}$

[This relation is a half-line.]

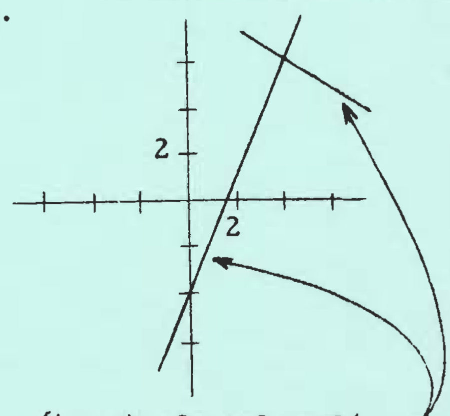
15.



$\{(x, y): (2y = x + 5 \text{ and } x \geq 3)$ $\text{or } (y = 2x - 2 \text{ and } y \geq 4)\}$

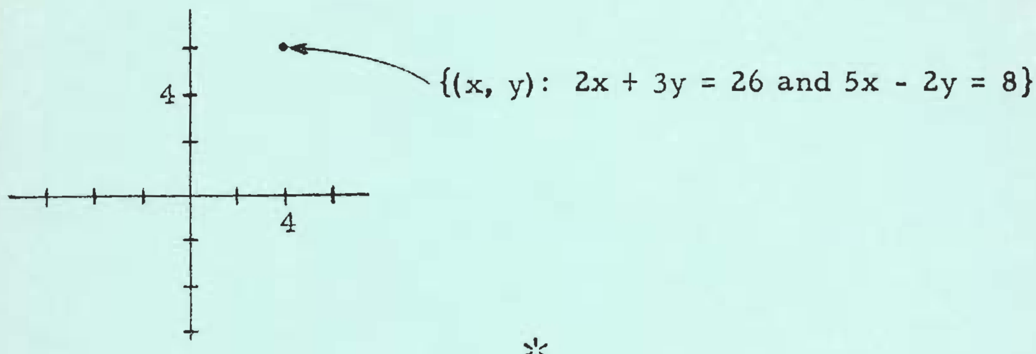
[This relation is an angle.]

16.



$\{(x, y): 2x + 3y = 26$ $\text{or } 5x - 2y = 8\}$

17.



*

Note the bracketed sentences following the last exercise in Part B. Our practice with regard to Supplementary Exercises in Units 1-4 was to provide you with exercises for students who needed more drill. In Unit 5 we are modifying this practice to include in the Supplementary Exercises not only more drill work but also material for the student who wants harder problems or who has the time and interest to pursue additional topics. We strongly urge that you acquaint yourself with Supplementary Exercises well in advance of the time you may want to assign them. Then you will be prepared to make supplementary [and differentiated] assignments on a moment's notice and you will avoid the unfortunate situation of assigning impossibly difficult problems to slower students.

*

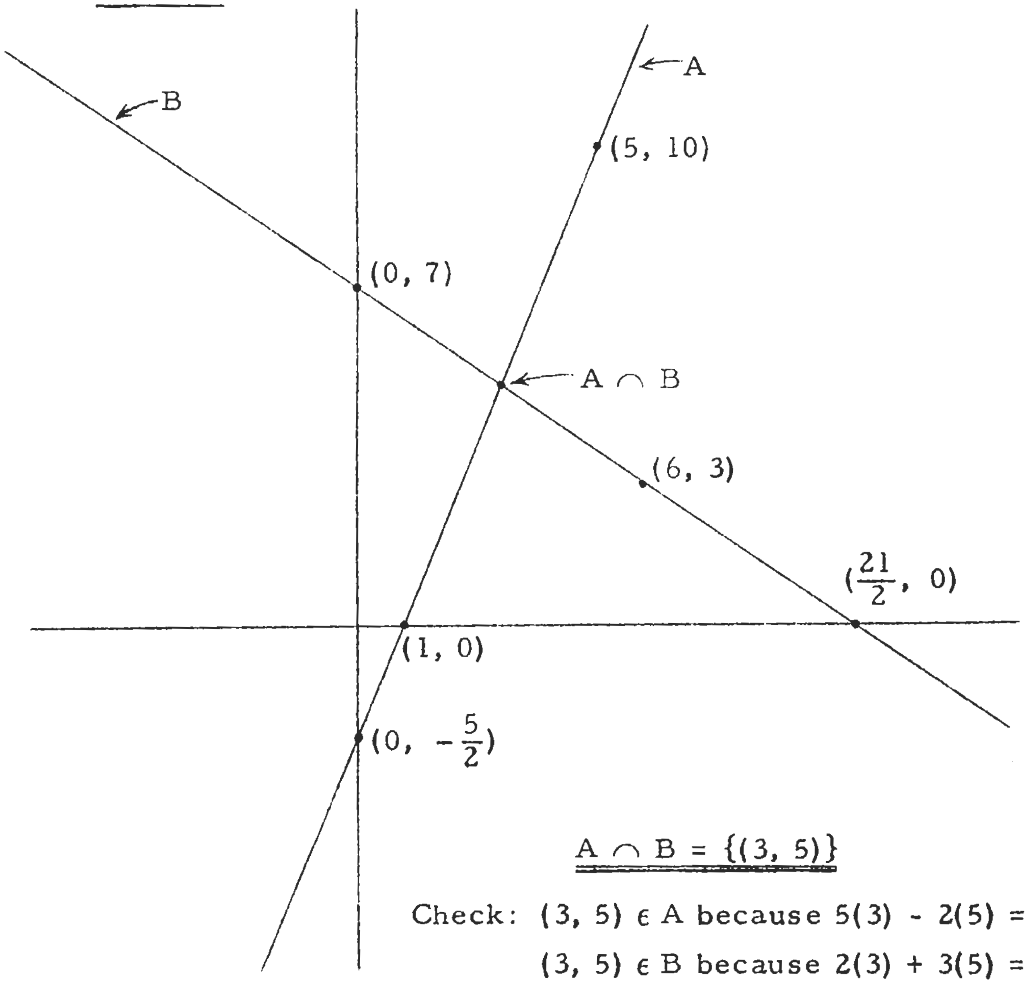
The comparison made [in the lower third of page 5-10] between Exercises 16 and 17 of Part B can also be made between Exercises 7 and 8 of Part B. Students should be asked to point out this second pair of exercises.

EXERCISES

A. Draw graphs and find the ordered pairs in $A \cap B$. Check by substitution.

Sample. $A = \{(x, y): 5x - 2y = 5\}$
$B = \{(x, y): 2x + 3y = 21\}$

Solution.



$A \cap B = \{(3, 5)\}$

Check: $(3, 5) \in A$ because $5(3) - 2(5) = 5$.
$(3, 5) \in B$ because $2(3) + 3(5) = 21$.

The picture above suggests that there is only one ordered pair, (3, 5), which belongs to both A and B. This is the case. So, $A \cap B$ is the set consisting of just the ordered pair (3, 5).

1. $A = \{(x, y): 7x + 2y - 35 = 0\}$
   $B = \{(x, y): 4x - y - 5 = 0\}$

2. $A = \{(x, y): 3x + y = 3\}$
   $B = \{(x, y): 8x + 5y = 1\}$

3. $A = \{(p, q): 3p - 2q + 1 = 0\}$
   $B = \{(r, s): 2r - 3s - 11 = 0\}$

4. $A = \{(x, y): |y| = 3\}$
   $B = \{(x, y): x + 5y = 17\}$

5. $A = \{(x, y): |x| = 4\}$
   $B = \{(x, y): |x| = |y|\}$

6. $A = \{(x, y): y = x^2\}$
   $B = \{(x, y): 2x = y\}$

*

Find the ordered pairs in $A \cup B$.

7. $A = \{(x, y) \in I \times I: x + y < 5, \ x > 0, \text{ and } y > 0\}$
   $B = \{(x, y) \in I \times I: x + y \geq 5, \ x < 5, \text{ and } y < 5\}$

8. $A = \{(x, y) \in I \times I: |x + 2y| = 10 \text{ and } xy \geq 0\}$
   $B = \{(x, y) \in I \times I: |x - 2y| = 10 \text{ and } xy \leq 0\}$

[Supplementary exercises on the graphs of intersections, unions, and complements are in Part C, page 5-240.]

B. Draw graphs of A, B, and $A \cap B$, where

$$A = \{(x, y): y < x + 2\},$$

and

$$B = \{(x, y): y > |x - 2|\}.$$

C. 1. Graph the relations A, B, and C, where

$$A = \{(x, y): |x| < 8 \text{ and } |y| = 8\},$$

$$B = \{(x, y): |y| < 8 \text{ and } |x| = 8\},$$

and

$$C = \{(x, y): |x| + |y| = 10\}.$$

2. List the members of $(A \cup B) \cap C$.

3. List the members of $(A \cap C) \cup (B \cap C)$.

D. 1. Graph the relations A, B, and C, where

$$A = \{(x, y): y \leq \tfrac{1}{2}x\},$$

$$B = \{(x, y): y \geq -\tfrac{1}{2}x\},$$

and

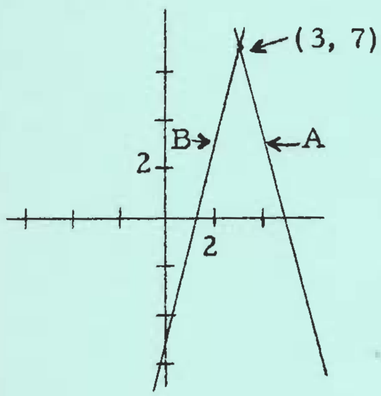
$$C = \{(x, y): x \leq 0 \text{ and } y = 0\}.$$

2. Graph $(A \cap B) \cup C$.

3. Graph $(A \cup C) \cap (B \cup C)$.

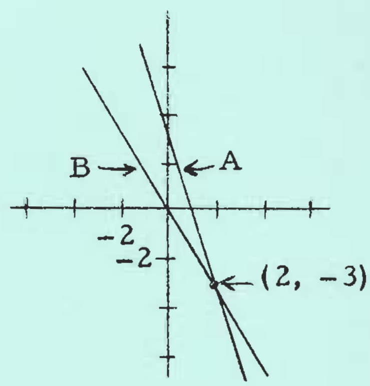
Answers for Part A [which begins on page 5-11].

1.



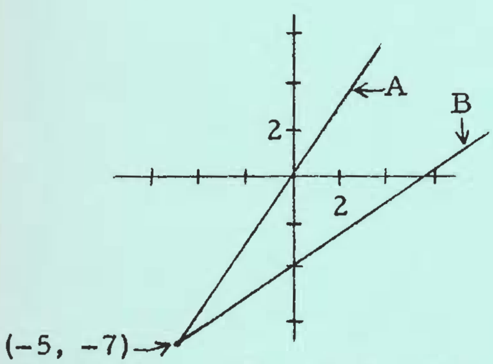
$A \cap B = \{(3, 7)\}$

2.



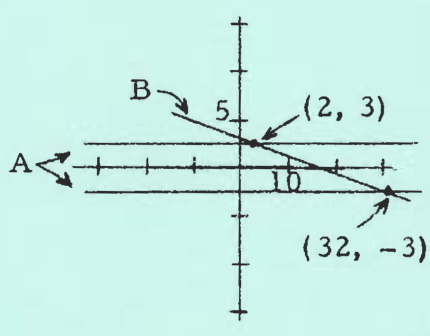
$A \cap B = \{(2, -3)\}$

3.



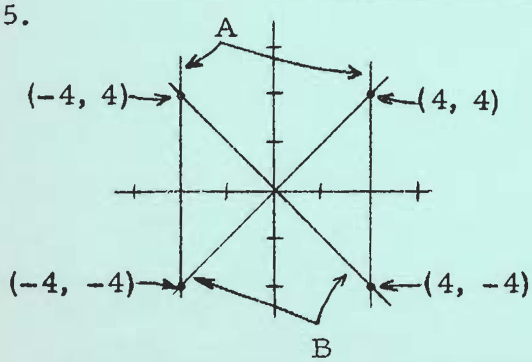
$A \cap B = \{(-5, -7)\}$

4.



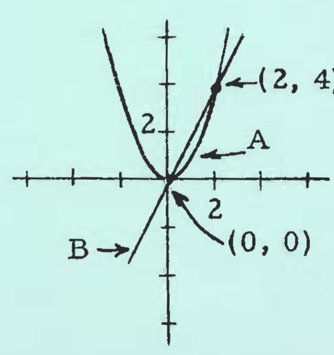
$A \cap B = \{(2, 3), (32, -3)\}$

5.



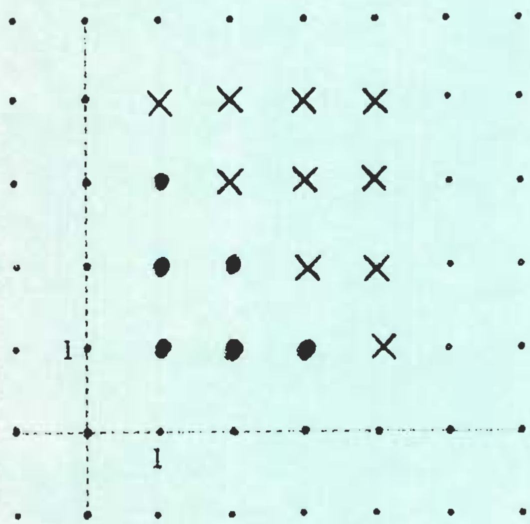
$A \cap B = \{(4, 4), (4, -4),$
$(-4, 4), (-4, -4)\}$

6.



$A \cap B = \{(0, 0), (2, 4)\}$

7. There are 16 ordered pairs in $A \cup B$. Students can answer this question by drawing the graphs or by listing the pairs. You may also want to accept '$\{(x, y) \in I \times I: 0 < x < 5 \text{ and } 0 < y < 5\}$' as an answer.

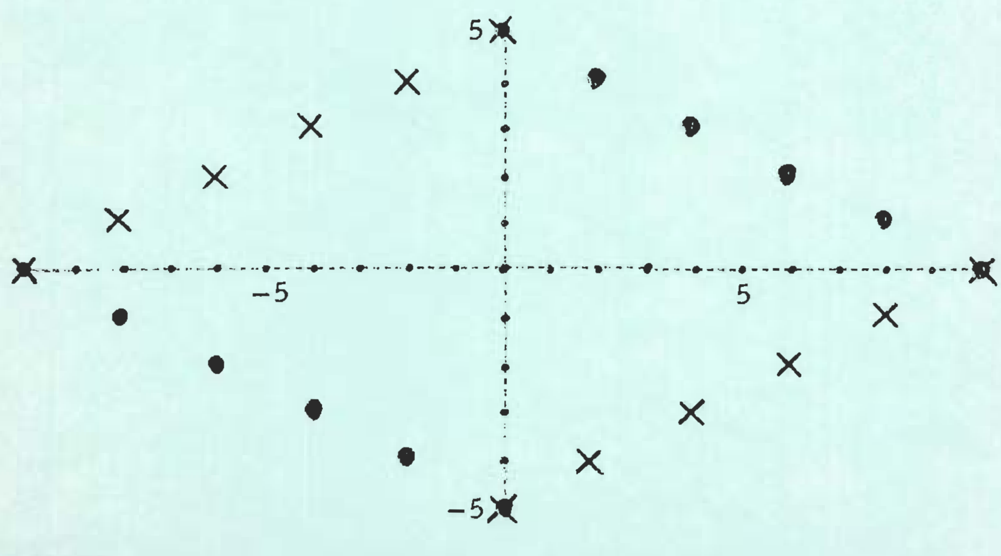


The graph of A consists of dots.

The graph of B consists of crosses.

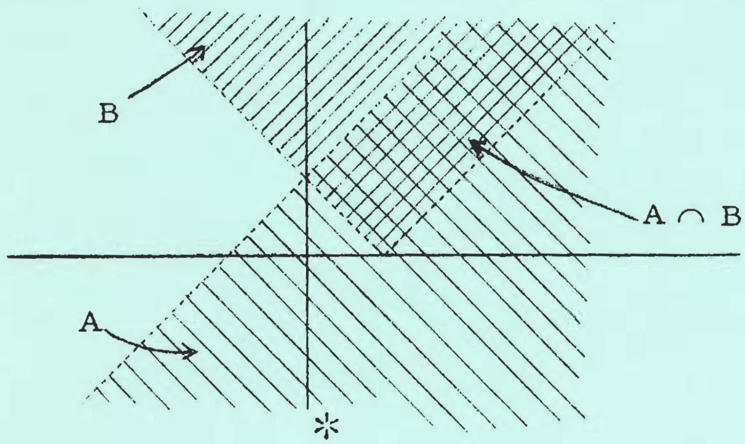
The ordered pairs in $A \cup B$ are those whose graphs are marked by either a dot or a cross [or both].

8. The graph of A consists of dots, the graph of B consists of crosses, and the ordered pairs in $A \cup B$ are those whose graphs are marked by either a dot or a cross [or both].



Answer for Part B [on page 5-12].

[This exercise is preparation for the geometric problem in section 5.03.]



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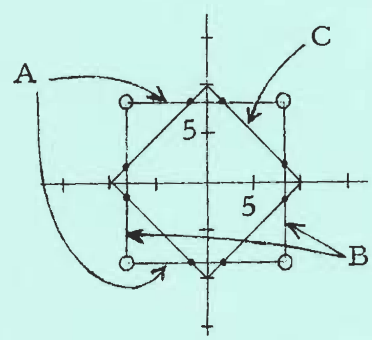
Answers for Part C [on page 5-12].



2. $A \cup B$ is a square with the corners missing.

   $(A \cup B) \cap C$

   $= \{(8, 2), (2, 8), (-2, 8), (-8, 2), (-8, -2), (-2, -8), (2, -8), (8, -2)\}$

3. $A \cap C = \{(2, 8), (-2, 8), (-2, -8), (2, -8)\}$

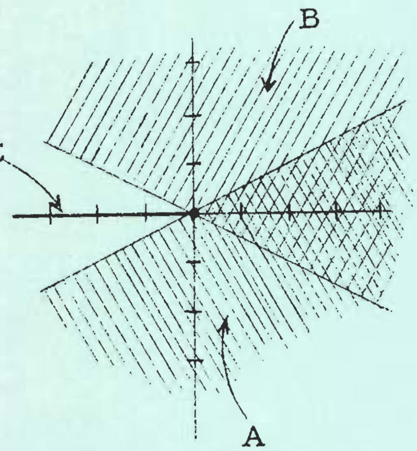
   $B \cap C = \{(8, -2), (8, 2), (-8, 2), (-8, -2)\}$

   So, $(A \cap C) \cup (B \cap C) = (A \cup B) \cap C$.

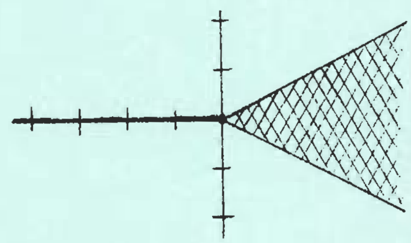
*

Answers for Part D [on page 5-12].

1.



2.



3. $(A \cup C) \cap (B \cup C) = (A \cap B) \cup C$

*

Part C suggests that the operation intersecting is distributive with respect to the operation unioning.

Part D suggests that the operation unioning is distributive with respect to the operation intersecting.

These suggestions are developed in section 5.02 which begins on page 5-13.

Quiz

A. True or false?

1. $(3, -8) \in \{(x, y): y < 3x - 15\}$ 2. $(2, -1) \in \{(x, y): x = y^2 + 1\}$

3. $(5, 1) \in \{(r, s): s + 7r - 41 = 1 + 4s\}$

4. $(8, 0) \in \{(u, v), u \geq 0: v < \sqrt{u} - 3\}$

5. $(17, 3) \notin \{(a, b): a + 3 < 5b + 4\}$

B. Draw graphs and find the ordered pairs in $A \cap B$.

1. $A = \{(x, y): 4x - 3 = 2y\}$, $B = \{(x, y): 6x + 3y = -\frac{9}{2}\}$

2. $A = \{(r, s): |s| = 3\}$, $B = \{(r, s): 3s = r\}$

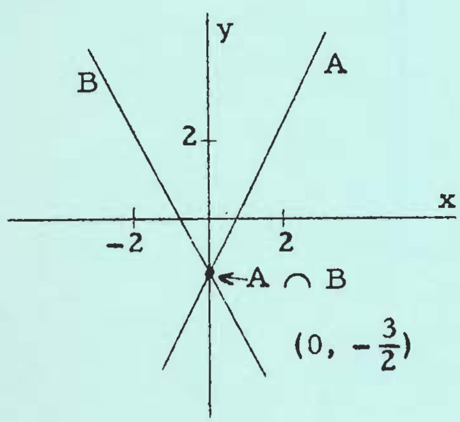
3. $A = \{(x, y) \in I \times I: x + y = 7\}$, $B = \{(x, y) \in I \times I: -3 \leq y \leq 3\}$

*

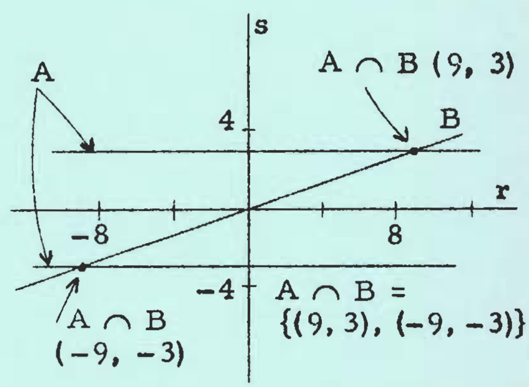
Answers for Quiz.

A. 1. T 2. T 3. F 4. F 5. T

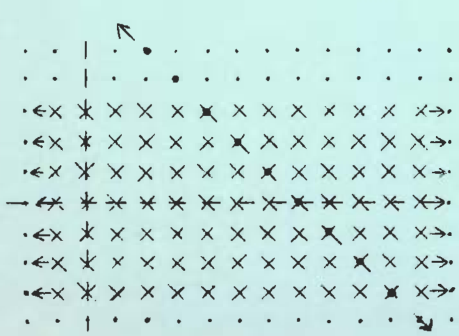
B. 1.



2.



3.



The graph of A consists of heavy dots.

The graph of B consists of crosses.

The ordered pairs in $A \cap B$ are those whose graphs are marked by a dot and a cross. These pairs are $(10, -3)$, $(9, -2)$, $(8, -1)$, $(7, 0)$, $(6, 1)$, $(5, 2)$, and $(4, 3)$.

Section 5.02 has two purposes:

(1) to remind students of some principles which they have already discovered for operating on sets and to acquaint them with other such principles, and

(2) to point out analogies between operations on sets and operations on numbers, and show students that theorems about sets can be derived from basic principles in much the same way that, in Unit 2, they derived theorems about numbers from basic principles.

[A list of basic principles and theorems about sets is given in the SUMMARY on pages 5-22 and 5-23.]

To save class time, exercises on proving theorems have, for the most part, been made optional [see Note at foot of page 5-18].

*

There are logical difficulties involved in the notion of "the set of all sets", and besides, the discussion of complementing requires specification of some containing set or space [see TC[5-F]b]. Hence, in discussing (2) on page 5-13, we specify that the domain of 'x', 'y', and 'z' should consist of all subsets of some set S, and refer to (2) as the dpiu <u>for subsets of S</u>.

*

Part D on page 5-12 [referred to on page 5-14] suggests the distributive principle for unioning over intersecting, for subsets of the number plane:

$$\forall_x \forall_y \forall_z \; (x \cap y) \cup z = (x \cup z) \cap (y \cup z)$$

This principle continues to hold when the domain of 'x', 'y', and 'z' is taken to be the set of subsets of any set S. In that case, it is <u>the distributive principle for unioning over intersecting, for subsets of S</u>.

*

5.02 Principles for sets.--Your answers to Exercises 2 and 3 of Part C on page 5-12 may have suggested to you an important generalization about sets. Did you find that, for the sets A, B, and C,

$$(1) \quad (A \cup B) \cap C = (A \cap C) \cup (B \cap C)?$$

You should have, for this is an instance of the distributive principle for intersecting over unioning, for subsets of the number plane. Does the pattern of sentence (1) remind you of the pattern of instances of one of the principles for numbers you studied in an earlier unit?

If we use 'x', 'y' and 'z' as variables whose domain consists of all subsets [lines, rays, segments, curves, regions, etc.] of the number plane, then (1) is an instance of:

$$(2) \quad \forall_x \forall_y \forall_z \ (x \cup y) \cap z = (x \cap z) \cup (y \cap z)$$

In general, if the domain of 'x', 'y', and 'z' consists of all subsets of some set S then (2) is the distributive principle for intersecting over unioning, for subsets of S. Let's check this in another case, where S is the set of all whole numbers,

$$A = \{5, 8, 13, 15\},$$

$$B = \{7, 8, 12, 14\},$$

and

$$C = \{3, 8, 12, 15\}.$$

[Then A, B, and C are subsets of S.] Is it the case that

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)?$$

To answer this question, we first find the members of $(A \cup B) \cap C$. Next, we find those of $(A \cap C) \cup (B \cap C)$. If we find that $(A \cup B) \cap C$ and $(A \cap C) \cup (B \cap C)$ have the same members then we will know that the answer to the question is 'yes'. Now,

$$A \cup B = \{5, 8, 13, 15\} \cup \{7, 8, 12, 14\}$$
$$= \{5, 7, 8, 12, 13, 14, 15\}.$$

So, since

$$C = \{3, 8, 12, 15\},$$
$$(A \cup B) \cap C = \{5, 7, 8, 12, 13, 14, 15\} \cap \{3, 8, 12, 15\}$$
$$= \{8, 12, 15\}.$$

On the other hand,

$$A \cap C = \{5, 8, 13, 15\} \cap \{3, 8, 12, 15\}$$
$$= \{8, 15\},$$

and

$$B \cap C = \{7, 8, 12, 14\} \cap \{3, 8, 12, 15\}$$
$$= \{8, 12\}.$$

So,

$$(A \cap C) \cup (B \cap C) = \{8, 15\} \cup \{8, 12\}$$
$$= \{8, 12, 15\}.$$

Hence, the answer is 'yes'.

Look at Exercises 2 and 3 of Part D on page 5-12. Do they suggest another principle for subsets of the number of plane? What might you call this principle? State it. Do you think that there is a similar principle for sets of whole numbers? Check by using easy sets like A, B, and C of page 5-13.

EXERCISES

A. You learned in your work with numbers that the operation addition, for numbers, is commutative and associative, and that the operation multiplication also has these properties. Do you think that the set operations, intersecting and unioning, for subsets of a given set S, have these properties? For each property and each operation, write a statement which says that the operation has the property. Then, for each of these four statements, pick a few sets [sets of numbers, sets of ordered pairs, sets of students in your class, etc.], form an instance of the statement, and check the instance.

B. The number 0 has two interesting properties which we expressed by principles--the principle for adding 0:

$$\forall_x \ x + 0 = x,$$

and the principle for multiplying by 0:

$$\forall_x \ x \times 0 = 0.$$

Are there analogous principles for sets? State them and check instances.

Answers for Part A.

[See page 5-16 for statements of the commutative and associative principles for unioning and intersecting.]

*

Answers for Part B.

$\forall_x\ x \cup \emptyset = x$; $\forall_x\ x \cap \emptyset = \emptyset$

*

Answers for Part C [on page 5-15].

$\forall_x\ x \cup x = x$; $\forall_x\ x \cap x = x$

C. The operations with sets have a property which the operations with numbers do not have. For example, what do you get if you union a set with itself? If you intersect a set with itself? State principles which express the facts that unioning and intersecting have this property. [This property is sometimes called the idempotent property.]

BASIC PRINCIPLES AND THEOREMS

In the preceding exercises you have discovered that some of the principles for operating with subsets of a given set S are analogous to principles for operating with real numbers. Here is a list of ten basic principles for real numbers which you studied in Unit 2.

Commutative principles

$\forall_x \forall_y\ x + y = y + x$

$\forall_x \forall_y\ x \cdot y = y \cdot x$

Associative principles

$\forall_x \forall_y \forall_z\ (x + y) + z = x + (y + z)$

$\forall_x \forall_y \forall_z\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$

Distributive principle

$\forall_x \forall_y \forall_z\ (x + y) \cdot z = x \cdot z + y \cdot z$

Principles for 0 and 1

$\forall_x\ x + 0 = x$

$\forall_x\ x \cdot 1 = x$

Principle of Opposites

$\forall_x\ x + -x = 0$

Principle for Subtraction

$\forall_x \forall_y\ x - y = x + -y$

Principle of Quotients

$\forall_x \forall_{y \neq 0}\ (x \div y) \cdot y = x$

If in seven of these principles for real numbers, you replace '+' by '$\cup$', '$\cdot$' by '$\cap$', '0' by '$\emptyset$', and '1' by 'S', and you adopt the convention that the domain of the variables 'x', 'y', and 'z' is the set of all subsets of S, you get the following principles for operating with subsets of S:

Commutative principles

$\forall_x \forall_y\ x \cup y = y \cup x$

$\forall_x \forall_y\ x \cap y = y \cap x$

Associative principles

$\forall_x \forall_y \forall_z\ (x \cup y) \cup z = x \cup (y \cup z)$

$\forall_x \forall_y \forall_z\ (x \cap y) \cap z = x \cap (y \cap z)$

Distributive principle

$\forall_x \forall_y \forall_z\ (x \cup y) \cap z = (x \cap z) \cup (y \cap z)$

Principles for $\emptyset$ and S

$\forall_x\ x \cup \emptyset = x$

$\forall_x\ x \cap S = x$

The remaining three principles for real numbers [the po, the ps, and the pq] do not "translate" successfully. However, there are three other basic principles for subsets of S--the other distributive principle:

$$\forall_x \forall_y \forall_z\ (x \cap y) \cup z = (x \cup z) \cap (y \cup z)$$

and:

Principles for Complements

$\forall_x\ \tilde{x} \cup x = S$

$\forall_x\ \tilde{x} \cap x = \emptyset$

Just as in Unit 2 you derived theorems from the basic principles for real numbers, so from these ten principles one can derive all theorems about unioning, intersecting, and complementing subsets of S. To see how this could be done, let's find test-patterns for the uniqueness principle for intersecting:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x \cap z = y \cap z,$$

and the principle for intersecting with $\emptyset$:

$$\forall_x\ x \cap \emptyset = \emptyset$$

For a discussion of the kind of testing-pattern shown at the top of page 5-17, see TC[2-64]a, b.

*

Answers for Part D [on pages 5-17 and 5-18].

1. [We shall not rewrite the test-pattern here. In doing so, be sure to make the indicated replacements in <u>both</u> <u>columns</u> of the pattern.] Yes

2. After rewriting one has the same statements he started with. [The fact that they are listed in a different order is unimportant.]

3 By making the replacements specified in Exercise 1.

*

A sentence like '$\forall_x \forall_y\ (x \cup y) \cap (\tilde{x} \cap \tilde{y}) = \emptyset$' contains only two kinds of symbols--logical symbols [that is, symbols such as '$\forall$', 'x', '$=$', 'if... then---', etc.] and symbols relating to sets [that is, symbols like '$\cup$', '$\cap$', '$\sim$', '$\emptyset$', and 'S']. Such a sentence is transformed into its dual by the replacements specified in Exercise 2 of Part D. For example, the dual of the sentence mentioned above is '$\forall_x \forall_y\ (x \cap y) \cup (\tilde{x} \cup \tilde{y}) = S$'. In doing the exercises of Part D the student should discover that [since dualizing does not modify the grammatical structure of a sentence] the result of dualizing the sentences in a test-pattern is again a test-pattern; and that, since the dual of each basic principle is again a basic principle, it follows that the dual of each theorem is a theorem. This point should be discovered by the time the students have completed Exercise 3. Exercise 4 on page 5-18 gives an opportunity for checking the validity of this discovery. [As will be seen in the COMMENTARY for page 5-21, in dualizing a sentence which contains '$\subseteq$' one must, in addition to making the replacements specified in Exercise 2 of Part D, replace '$\subseteq$' by '$\supseteq$'.]

*

Replacement Table for Dualizing

$\cap$	$\cup$	$\emptyset$	S	$\subseteq$	$\supseteq$
↓	↓	↓	↓	↓	↓
$\cup$	$\cap$	S	$\emptyset$	$\supseteq$	$\subseteq$

Here is a test-pattern for the uniqueness principle. [You may want to compare it with the test-pattern near the bottom of page 2-64.]

Suppose that	$x = y$.
Since	$x \cap z = x \cap z$, $[\forall_a\ a = a]$
it follows that	$x \cap z = y \cap z$.
Hence,	if $x = y$ then $x \cap z = y \cap z$.

Here is a test-pattern for the principle for intersecting with $\emptyset$. [Since there is no operation for subsets which is analogous to the oppositing operation for real numbers, this test-pattern is quite different from the one on page 2-66 for the principle for multiplying by 0.]

$\tilde{x} \cup \emptyset = \tilde{x}$	$[\forall_x\ x \cup \emptyset = x]$
$(\tilde{x} \cup \emptyset) \cap x = \tilde{x} \cap x$	$[\forall_x \forall_y \forall_z$ if $x = y$ then $x \cap z = y \cap z]$
$(\tilde{x} \cap x) \cup (\emptyset \cap x) = \tilde{x} \cap x$	$[\forall_x \forall_y \forall_z\ (x \cup y) \cap z = (x \cap z) \cup (y \cap z)]$
$\emptyset \cup (\emptyset \cap x) = \emptyset$	$[\forall_x\ \tilde{x} \cap x = \emptyset]$
$(\emptyset \cap x) \cup \emptyset = \emptyset$	$[\forall_x \forall_y\ x \cup y = y \cup x]$
$\emptyset \cap x = \emptyset$	$[\forall_x\ x \cup \emptyset = x]$
$x \cap \emptyset = \emptyset$	$[\forall_x \forall_y\ x \cap y = y \cap x]$

* * *

<u>D</u>. 1. Rewrite the test-pattern for the principle for intersecting with $\emptyset$, but write '$\cap$' instead of '$\cup$', '$\cup$' instead of '$\cap$', and 'S' instead of '$\emptyset$'. [Your last line should be '$x \cup S = S$'.] Are the bracketed expressions on the right of your new pattern principles for subsets of S?

2. Rewrite the ten basic principles for subsets of S, but write '$\cap$' for '$\cup$', '$\cup$' for '$\cap$', 'S' for '$\emptyset$' and '$\emptyset$' for 'S'. What do you notice?

3. Suppose you have derived a theorem about subsets of S from the ten basic principles. How can you find another theorem?

4. Here is a test-pattern for the theorem '$\forall_x\ x = x \cup x$', <u>the idempotence theorem for unioning</u>.

$x = x \cup \emptyset$	$[\forall_x\ x \cup \emptyset = x]$
$= \emptyset \cup x$	$[\forall_x \forall_y\ x \cup y = y \cup x]$
$= (\tilde{x} \cap x) \cup x$	$[\forall_x\ \tilde{x} \cap x = \emptyset]$
$= (\tilde{x} \cup x) \cap (x \cup x)$	$[\forall_x \forall_y \forall_z\ (x \cap y) \cup z = (x \cup z) \cap (y \cup z)]$
$= S \cap (x \cup x)$	$[\forall_x\ \tilde{x} \cup x = S]$
$= (x \cup x) \cap S$	$[\forall_x \forall_y\ x \cap y = y \cap x]$
$= x \cup x$	$[\forall_x\ x \cap S = x]$

Write a test-pattern for '$\forall_x\ x = x \cap x$', <u>the idempotence theorem for intersecting</u>.

<u>E</u>. One of the important theorems you proved about the oppositing operation for real numbers was the 0-sum theorem:

$$\forall_x \forall_y \text{ if } x + y = 0 \text{ then } -x = y$$

For subsets of S, we have the theorem:

$$\forall_x \forall_y \text{ if } x \cup y = \emptyset \text{ then } y = \emptyset$$

Prove this theorem by completing the test-pattern which is started below.

Suppose that $x \cup y = \emptyset$.

Then $y = y \cup (x \cup y)$ $[\forall_x\ x \cup \emptyset = x]$

$= \_\_\_\_\_\_\_\_$ $[\forall_x \forall_y\ x \cup y = y \cup x]$

.
.
.

<u>Note</u>: The rest of this section contains optional material on subsets. Mastery of this material is not necessary for understanding of later work in Unit 5. However, some of the theorems proved here will be helpful, so, we list them in a summary on page 5-22. Be sure to familiarize yourself with this summary even if you don't study the optional material.

4. [The test-pattern asked for is obtained by dualizing the one given in the exercise. It begins with '$x = x \cap S\ [\forall_x\ x \cap S = x]$', and ends with '$= x \cap x [\forall_x x \cup \emptyset = x]$'.]

*

Answer for Part E.

Suppose that $x \cup y = \emptyset$.

$$
\begin{aligned}
\text{Then} \qquad y &= y \cup (x \cup y) && [\forall_x\ x \cup \emptyset = x] \\
&= (x \cup y) \cup y && [\forall_x \forall_y\ x \cup y = y \cup x] \\
&= x \cup (y \cup y) && [\forall_x \forall_y \forall_z\ (x \cup y) \cup z = x \cup (y \cup z)] \\
&= x \cup y && [\forall_x\ x = x \cup x] \\
&= \emptyset. && [\text{Assumption: } x \cup y = \emptyset]
\end{aligned}
$$

So, if $x \cup y = \emptyset$ then $y = \emptyset$.

*

Notice that the theorem proved in Part E shows that the analogy between adding and unioning, multiplying and intersecting, 0 and $\emptyset$, and 1 and S cannot be extended to include an operation on subsets of S analogous to the operation oppositing on real numbers. For if there were such an operation *, then the analogue, '$\forall_x\ x \cup * x = \emptyset$' of the principle of opposites would have to be a theorem about subsets of S. But, if so, by the theorem proved in Part E, '$\forall_x * x = \emptyset$' would be a theorem. And this last is analogous to the statement '$\forall_x\ -x = 0$' about real numbers. Since <u>this</u> statement is not a theorem about real numbers, the only candidate for an analogue of oppositing [the operation on sets which maps each set on $\emptyset$] is <u>not</u> such an analogue. Hence, oppositing has no analogue in the algebra of sets.

Skill Quiz.

A. Simplify.

1. $(3a + 2)(3a - 2) + (2a - 3)(3a + 1)$
2. $57a^2bc^3 \div (3abc^2)$
3. $7x - 3y - (-4y + 8x)$
4. $\frac{7n}{5} + \frac{n}{3}$
5. $\frac{(c + 2)^2}{33} \cdot \frac{44}{4c + 8}$
6. $51e^5(f + g)^2 \div [17e^6(f + g)]$
7. $3(r - s)^2 - 3(s^2 + 2rs + r^2)$
8. $\frac{9 - d}{54} + \frac{d - 6}{36}$

B. Factor completely.

1. $49 - y^2$
2. $a^2 - a - 12$
3. $5de^2 - 5d$
4. $16f^2 - 40f + 25$
5. $12n^2 - nr - 6r^2$
6. $12t^2 + 89jt - 56j^2$

C. Solve. [For the inequations, give the solution set, using the simplest sentence possible as set selector.]

1. $3q^2 = 108$
2. $5s - 12 = 9s + 6$
3. $3z + 2 > 11$
4. $\frac{7}{12} = \frac{k}{4}$
5. $(2 - a)(3 + a) = \frac{5a - 15}{-2.5}$
6. $5p + 7 < 7p - 5$

*

Answers for Quiz.

A.
1. $15a^2 - 7a - 7$
2. $19ac$
3. $-x + y$
4. $\frac{26n}{15}$
5. $\frac{c + 2}{3}$
6. $\frac{3(f + g)}{e}$
7. $-12rs$
8. $\frac{d}{108}$

B.
1. $(7 - y)(7 + y)$
2. $(a - 4)(a + 3)$
3. $5d(e - 1)(e + 1)$
4. $(4f - 5)^2$
5. $(4n - 3r)(3n + 2r)$
6. $(12t - 7j)(t + 8j)$

C.
1. 6, −6
2. −4.5
3. $\{z: z > 3\}$
4. 7/3
5. 0, 1
6. $\{p: 6 < p\}$

and either of the principles for complements from the other. For example:

$$\tilde{x} \cup \emptyset = \tilde{x} \qquad [\forall_x\, x \cup \emptyset = x]$$

$$\tilde{x} \cup \tilde{S} = \tilde{x} \qquad [\tilde{S} = \emptyset]$$

$$\widetilde{x \cap S} = \tilde{x} \qquad [\text{DeMorgan's Laws}]$$

$$\widetilde{\widetilde{x \cap S}} = \tilde{\tilde{x}}$$

$$x \cap S = x \qquad [\forall_x\, \tilde{\tilde{x}} = x]$$

and:

$$\tilde{\tilde{x}} \cup \tilde{x} = S \qquad [\forall_x\, \tilde{x} \cup x = S]$$

$$\tilde{\tilde{x}} \cup \tilde{x} = \tilde{\emptyset} \qquad [\tilde{\emptyset} = S]$$

$$\widetilde{\tilde{x} \cap x} = \tilde{\emptyset} \qquad [\text{DeMorgan's Laws}]$$

$$\widetilde{\widetilde{\tilde{x} \cap x}} = \tilde{\tilde{\emptyset}}$$

$$\tilde{x} \cap x = \emptyset \qquad [\forall_x\, \tilde{\tilde{x}} = x]$$

Finally, either of DeMorgan's Laws, and either of '$\tilde{S} = \emptyset$' and '$\tilde{\emptyset} = S$' can be derived from the other by using '$\forall_x\, \tilde{\tilde{x}} = x$'. For example:

$$\tilde{\tilde{x}} \cup \tilde{\tilde{y}} = \widetilde{\tilde{x} \cap \tilde{y}} \qquad [\forall_x \forall_y\, \widetilde{x \cap y} = \tilde{x} \cup \tilde{y}]$$

$$x \cup y = \widetilde{\tilde{x} \cap \tilde{y}} \qquad [\forall_x\, \tilde{\tilde{x}} = x]$$

$$\widetilde{x \cup y} = \widetilde{\widetilde{\tilde{x} \cap \tilde{y}}}$$

$$\widetilde{x \cup y} = \tilde{x} \cap \tilde{y} \qquad [\forall_x\, \tilde{\tilde{x}} = x]$$

and:

$$S = \tilde{\emptyset}$$

$$\tilde{S} = \tilde{\tilde{\emptyset}}$$

$$\tilde{S} = \emptyset \qquad [\forall_x\, \tilde{\tilde{x}} = x]$$

So, we could take as basic principles, for example, the first of each of the pairs given on page 5-17 together with '$\forall_x \forall_y\, \widetilde{x \cap y} = \tilde{x} \cup \tilde{y}$', '$\forall_x\, \tilde{\tilde{x}} = x$', and '$S = \tilde{\emptyset}$'.

$\emptyset \cup S = S$ $\quad$ $[\forall_x\, \emptyset \cup x = x]$

$\emptyset \cap S = \emptyset$ $\quad$ $[\forall_x\, x \cap S = x]$

So, $\emptyset \cup S = S$ and $\emptyset \cap S = \emptyset$.

But, if $\emptyset \cup S = S$ and $\emptyset \cap S = \emptyset$ then $\tilde{\emptyset} = S$. [the complement theorem]

Hence, $\tilde{\emptyset} = S$.

$\tilde{x} \cup x = S$ $\quad$ $[\forall_x\, \tilde{x} \cup x = S]$

$\tilde{x} \cap x = \emptyset$ $\quad$ $[\forall_x\, \tilde{x} \cap x = \emptyset]$

So, $\tilde{x} \cup x = S$ and $\tilde{x} \cap x = \emptyset$

But, if $\tilde{x} \cup x = S$ and $\tilde{x} \cap x = \emptyset$ then $\tilde{\tilde{x}} = x$. [the complement theorem]

Hence, $\tilde{\tilde{x}} = x$.

*

DeMorgan's Laws cast more light on the duality discovered in solving Part D on pages 5-17 and 5-18. For example, using these laws and the principle '$\forall_x\, \tilde{\tilde{x}} = x$', one can derive the second distributive principle from the first. Here is a test-pattern.

$(\tilde{x} \cup \tilde{y}) \cap \tilde{z} = (\tilde{x} \cap \tilde{z}) \cup (\tilde{y} \cap \tilde{z})$ $\quad$ $[\forall_x \forall_y \forall_z\, (x \cup y) \cap z = (x \cap z) \cup (y \cap z)]$

$\widetilde{(x \cap y)} \cap \tilde{z} = \widetilde{(x \cup z)} \cup \widetilde{(y \cup z)}$ $\quad$ [DeMorgan's Laws]

$\widetilde{(x \cap y) \cup z} = \widetilde{(x \cup z) \cap (y \cup z)}$ $\quad$ [DeMorgan's Laws]

$\widetilde{\widetilde{(x \cap y) \cup z}} = \widetilde{\widetilde{(x \cup z) \cap (y \cup z)}}$

$(x \cap y) \cup z = (x \cup z) \cap (y \cup z)$ $\quad$ $[\forall_x\, \tilde{\tilde{x}} = x]$

[The fourth line derives from the third by way of the logical principle '$\forall_x \forall_y$ if $x = y$ then $\tilde{x} = \tilde{y}$.' <u>Cf.</u> the corresponding uniqueness principles for intersecting, for which a test-pattern is given on page 5-17.]

In entirely similar manners one can derive either of the commutative and associative principles from the other. Moreover, if we use, in addition to DeMorgan's Laws and '$\forall_x\, \tilde{\tilde{x}} = x$', the principles '$\tilde{S} = \emptyset$' and '$\tilde{\emptyset} = S$', we can derive either of the principles of $\emptyset$ and S from the other,

Below is a test-pattern for the complement theorem.

Suppose that $x \cup y = S$ and $x \cap y = \emptyset$.

It follows that $\tilde{x} \cap (x \cup y) = \tilde{x} \cap S$,

so, $(\tilde{x} \cap x) \cup (\tilde{x} \cap y) = \tilde{x} \cap S$, [$\ell$dpiu]

and $\emptyset \cup (\tilde{x} \cap y) = \tilde{x} \cap S$. [$\forall_x \tilde{x} \cap x = \emptyset$]

Hence, $\tilde{x} \cap y = \tilde{x}$. [$\forall_x \emptyset \cup x = x$; $\forall_x x \cap S = x$]

But, since $x \cap y = \emptyset$,

$(\tilde{x} \cap y) \cup (x \cap y) = \tilde{x} \cup \emptyset$,

so, $(\tilde{x} \cup x) \cap y = \tilde{x} \cup \emptyset$, [dpiu]

and $S \cap y = \tilde{x} \cup \emptyset$. [$\forall_x \tilde{x} \cup x = S$]

Hence, $y = \tilde{x}$, [$\forall_x S \cap x = x$; $\forall_x x \cup \emptyset = x$]

and $\tilde{x} = y$.

Consequently, if $x \cup y = S$ and $x \cap y = \emptyset$ then $\tilde{x} = y$.

[As above, we shall continue to use (without bothering to derive them) theorems, such as '$\forall_x \emptyset \cup x = x$', which can be derived from basic principles or previously proved theorems by using the commutative principles.]

*

Here are test-patterns for '$\tilde{S} = \emptyset$', '$\tilde{\emptyset} = S$', and '$\forall_x \tilde{\tilde{x}} = x$'.

$S \cup \emptyset = S$ [$\forall_x x \cup \emptyset = x$]

$S \cap \emptyset = \emptyset$ [$\forall_x S \cap x = x$]

So, $S \cup \emptyset = S$ and $S \cap \emptyset = \emptyset$.

But, if $S \cup \emptyset = S$ and $S \cap \emptyset = \emptyset$ then $\tilde{S} = \emptyset$. [the complement theorem]

Hence, $\tilde{S} = \emptyset$.

☆ MORE THEOREMS ABOUT SUBSETS

The theorem of Part E, on page 5-18, about subsets of S shows that there is no satisfactory analogue, for subsets, of the oppositing operation for real numbers [Why?]. However, there is a theorem about complementing which is somewhat analogous to the 0-sum theorem. We shall call it <u>the complement theorem</u>:

$$\forall_x \forall_y \text{ if } x \cup y = S \text{ and } x \cap y = \emptyset \text{ then } \tilde{x} = y$$

The complement theorem can be derived from the ten basic principles for subsets, but we shall not do so here. You can easily convince yourself by an example that the complement theorem is true. Take $S = \{1, 2, 3, 4, 5, 6\}$ and $A = \{1, 2, 3\}$, and try to find a set B such that $A \cup B = S$ and $A \cap B = \emptyset$. What must B be? Is $B = \tilde{A}$?

You can use the complement theorem to prove several useful theorems about complements. For example, from the principles for $\emptyset$ and S on page 5-16, it follows that $S \cup \emptyset = S$ and $\emptyset \cap S = \emptyset$. Hence [since $\emptyset \cap S = S \cap \emptyset$], it follows from the complement theorem that $\tilde{S} = \emptyset$. [Since $S \cup \emptyset = \emptyset \cup S$, it follows in the same manner that $\tilde{\emptyset} = S$.]

As another simple example, notice that the principles for complements on page 5-16 and the complement theorem tell you at once that

$$\forall_x \tilde{\tilde{x}} = x.$$

A more interesting application of the complement theorem is to prove:

<u>DeMorgan's Laws</u>

$$\forall_x \forall_y \widetilde{x \cup y} = \tilde{x} \cap \tilde{y}$$

$$\forall_x \forall_y \widetilde{x \cap y} = \tilde{x} \cup \tilde{y}$$

You may have discovered these when solving exercises in Part F on page 5-I. The first can be interpreted as saying that the members of S which don't belong to either x or y are just those members of S which belong neither to x nor to y. What does the second tell you about the members of S which don't belong both to x and to y?

Here is a test-pattern for the first of DeMorgan's Laws:

$$(x \cup y) \cap (\tilde{x} \cap \tilde{y})$$

$$\quad \Big\} \forall_x \forall_y \forall_z \ (x \cup y) \cap z = (x \cap z) \cup (y \cap z)$$

$$= [x \cap (\tilde{x} \cap \tilde{y})] \cup [y \cap (\tilde{x} \cap \tilde{y})]$$

$$\quad \Big\} \forall_x \forall_y \ x \cap y = y \cap x$$

$$= [x \cap (\tilde{x} \cap \tilde{y})] \cup [y \cap (\tilde{y} \cap \tilde{x})]$$

$$\quad \Big\} \forall_x \forall_y \forall_z \ (x \cap y) \cap z = x \cap (y \cap z)$$

$$= [(x \cap \tilde{x}) \cap \tilde{y}] \cup [(y \cap \tilde{y}) \cap \tilde{x}]$$

$$\quad \Big\} \forall_x \forall_y \ x \cap y = y \cap x$$

$$= [\tilde{y} \cap (\tilde{x} \cap x)] \cup [\tilde{x} \cap (\tilde{y} \cap y)]$$

$$\quad \Big\} \forall_x \ \tilde{x} \cap x = \emptyset$$

$$= [\tilde{y} \cap \emptyset] \cup [\tilde{x} \cap \emptyset]$$

$$\quad \Big\} \forall_x \ x \cap \emptyset = \emptyset$$

$$= \emptyset \cup \emptyset$$

$$\quad \Big\} \forall_x \ x \cup x = x$$

$$= \emptyset$$

[Having gotten this far, we know that '$\forall_x \forall_y (x \cup y) \cap (\tilde{x} \cap \tilde{y}) = \emptyset$' is a theorem. So, we know that '$\forall_x \forall_y (x \cap y) \cup (\tilde{x} \cup \tilde{y}) = S$' is also a theorem [Why?]. We can use this in the remainder of this proof.]

$$(x \cup y) \cup (\tilde{x} \cap \tilde{y}) = (\tilde{x} \cap \tilde{y}) \cup (x \cup y) \qquad [\forall_x \forall_y \ x \cup y = y \cup x]$$

$$= (\tilde{x} \cap \tilde{y}) \cup (\tilde{\tilde{x}} \cup \tilde{\tilde{y}}) \qquad [\forall_x \ \tilde{\tilde{x}} = x]$$

$$= S \qquad [\forall_x \forall_y \ (x \cap y) \cup (\tilde{x} \cup \tilde{y}) = S]$$

We have shown that

$$(x \cup y) \cup (\tilde{x} \cap \tilde{y}) = S \text{ and } (x \cup y) \cap (\tilde{x} \cap \tilde{y}) = \emptyset.$$

So, by the complement theorem, $\widetilde{x \cup y} = \tilde{x} \cap \tilde{y}$.

EXERCISES

A. Write a test-pattern for each of the given generalizations.

1. $\forall_x \ \tilde{\tilde{x}} = x$
2. $\forall_x \forall_y \ \widetilde{x \cap y} = \tilde{x} \cup \tilde{y}$

* * *

So far we have discussed only the operations $\cup$, $\cap$ and $\tilde{}$. In addition to these, there is the relation $\subseteq$ to be considered. For this

In order to save space horizontally, we have, in giving the first part of the test pattern for the first of DeMorgan's Laws, reverted to the form first introduced on page 2-35.

*

Answers for Part A.

1. [Such a test-pattern is given on TC[5-19]b.]

2. [The test-pattern asked for can be obtained merely by dualizing the test-pattern given on page 5-20 for the first of DeMorgan's Laws. The first lines obtained will be:

$$\left.\begin{array}{l} (x \cap y) \cup (\tilde{x} \cup \tilde{y}) \\ = [x \cup (\tilde{x} \cup \tilde{y})] \cap [y \cup (\tilde{x} \cup \tilde{y})] \end{array}\right\} \forall_x \forall_y \forall_z\ (x \cap y) \cup z = (x \cup z) \cap (y \cup z)$$

and it will end with:

So, by the complement theorem, $\widetilde{x \cap y} = \tilde{x} \cup \tilde{y}$.]

*

The complement theorem is a special case of <u>the</u> <u>cancellation</u> <u>theorem</u>:

$$\forall_x \forall_y \forall_z \text{ if } y \cup z = x \cup z \text{ and } y \cap z = x \cap z \text{ then } x = y$$

[More precisely, the complement theorem is an easy consequence of the cancellation theorem, the principles for complements, and the commutative principles. For, an immediate consequence of the cancellation theorem is:

$$\forall_y \forall_z \text{ if } y \cup z = \tilde{z} \cup z \text{ and } y \cap z = \tilde{z} \cap z \text{ then } \tilde{z} = y]$$

Below is a test-pattern for the cancellation theorem.

Suppose that $y \cup z = x \cup z$ and $y \cap z = x \cap z$.

It follows that $(y \cup z) \cap \tilde{z} = (x \cup z) \cap \tilde{z}$,

So, $(y \cap \tilde{z}) \cup (z \cap \tilde{z}) = (x \cap \tilde{z}) \cup (z \cap \tilde{z})$, [dpiu]

and $(y \cap \tilde{z}) \cup \emptyset = (x \cap \tilde{z}) \cup \emptyset.$ $[\forall_x\ x \cap \tilde{x} = \emptyset]$

Hence, $y \cap \tilde{z} = x \cap \tilde{z}.$ $[\forall_x\ x \cup \emptyset = x]$

But, since $y \cap z = x \cap z,$

$$(y \cap \tilde{z}) \cup (y \cap z) = (x \cap \tilde{z}) \cup (x \cap z),$$

so, $y \cup (\tilde{z} \cap z) = x \cup (\tilde{z} \cap z),$ [ℓdpui]

$$y \cup \emptyset = x \cup \emptyset, \quad [\forall_x\ \tilde{x} \cap x = \emptyset]$$

and $y = x.$ $[\forall_x\ x \cup \emptyset = x]$

Hence, $x = y.$

Consequently, if $y \cup z = x \cup z$ and $y \cap z = x \cap z$ then $x = y$.

15. $\forall_x \forall_y \forall_z\ x \sim (y \cup z) = (x \sim y) \sim z$

[$x \cap \widetilde{y \cup z} = x \cap (\tilde{y} \cap \tilde{z}) = (x \cap \tilde{y}) \cap \tilde{z}$]

16. $\forall_x \forall_y \forall_z\ (x \sim y) \cap z = (x \cap z) \sim y$

[Use the principle of relative complements, the api and the cpi.]

17. $\forall_x \forall_y \forall_z\ (x \sim y) \cup (x \sim z) = x \sim (y \cap z)$

[$(x \cap \tilde{y}) \cup (x \cap \tilde{z}) = x \cap (\tilde{y} \cup \tilde{z}) = x \cap \widetilde{y \cap z}$]

18. $\forall_x \forall_y \forall_z\ (x \cup y) \sim z = (x \sim z) \cup (y \sim z)$

[$(x \cup y) \cap \tilde{z} = (x \cap \tilde{z}) \cup (y \cap \tilde{z})$]

19. $\forall_x \forall_y \forall_z\ (x \cap y) \sim z = (x \sim z) \cap (y \sim z)$

[$(x \cap y) \cap \tilde{z} = (x \cap \tilde{z}) \cap (y \cap \tilde{z})$]

20. $\forall_x \forall_y \forall_z\ (x \sim y) \sim z = (x \sim z) \sim (y \sim z)$

[$(x \cap \tilde{z}) \cap \widetilde{y \cap \tilde{z}} = (x \cap \tilde{z}) \cap (\tilde{y} \cup z) =$
$[(x \cap \tilde{z}) \cap \tilde{y}] \cup [(x \cap \tilde{z}) \cap z] = (x \cap \tilde{z}) \cap \tilde{y} = (x \cap \tilde{y}) \cap \tilde{z}$]

21. $\forall_x \forall_y \forall_z\ (x \sim y) \cap z = (x \cap z) \sim (y \cap z)$

[Similar to 18.]

22. $\forall_x \forall_y \forall_z$ [$z \subseteq x$ if and only if $x \sim (y \sim z) = (x \sim y) \cup z$]

[$x \sim (y \sim z) = x \cap (\tilde{y} \cup z) = (x \cap \tilde{y}) \cup (x \cap z)$. So, $x \sim (y \sim z) = (x \sim y) \cup z$ if and only if $(x \cap \tilde{y}) \cup (x \cap z) = (x \cap \tilde{y}) \cup z$. But, in any case, $(x \cap \tilde{y}) \cap (x \cap z) = (x \cap \tilde{y}) \cap z$. So, by the cancellation theorem, $x \sim (y \sim z) = (x \sim y) \cup z$ if and only if $x \cap z = z$.]

6. $\forall_x \forall_y$ $y \subseteq x$ if and only if $y \cap \tilde{x} = \emptyset$

[$y = (y \cap x) \cup (y \cap \tilde{x})$, so, if $y \cap \tilde{x} = \emptyset$ then $y \cap x = y$, and $y \subseteq x$. And, if $y \subseteq x$, so that $y = y \cap x$, then $y \cap \tilde{x} = (y \cap x) \cap \tilde{x} = \ldots$.]

7. $\forall_x \forall_y$ $x \cup y = x \cup (y \sim x)$

[$x \cup (y \sim x) = (x \cup y) \cap (x \cup \tilde{x})$, by the principle for relative complements and the ℓdpui.]

8. $\forall_x \forall_y$ $x \sim y = (x \cup y) \sim y$

[$(x \cup y) \cap \tilde{y} = x \cap \tilde{y}$]

9. $\forall_x \forall_y$ $x \sim y = x \sim (x \cap y)$

[$x \cap \widetilde{x \cap y} = x \cap (\tilde{x} \cup \tilde{y}) = x \cap \tilde{y}$]

10. $\forall_x \forall_y$ $\widetilde{x \sim y} = \tilde{x} \cup y$

[$\widetilde{x \cap \tilde{y}} = \tilde{x} \cup y$]

11. $\forall_x \forall_y$ $x \sim (x \sim y) = x \cap y$

[$x \cap \widetilde{x \cap \tilde{y}} = x \cap (\tilde{x} \cup y) = x \cap y$]

12. $\forall_x \forall_y$ $x \sim (y \sim x) = x$

[$x \cap (\tilde{y} \cup x) = x$ ($x \subseteq z \cup x$, for any z)]

13. $\forall_x \forall_y \forall_z$ $x \sim (y \sim z) = (x \sim y) \cup (x \cap z)$

[$x \cap (\tilde{y} \cup z) = (x \cap \tilde{y}) \cup (x \cap z)$]

14. $\forall_x \forall_y$ if $x \sim y = y \sim x$ then $x = y$

[$x = (x \cap \tilde{y}) \cup (x \cap y)$, so, if $x \cap \tilde{y} = y \cap \tilde{x}$ then $x = (y \cap \tilde{x}) \cup (x \cap y) = y \cap (\tilde{x} \cup x) = y \cap S = y$]

The last two suggest that the dual of relative complementation is a sort of analogue of division. So [although this notation is not standard], one might introduce the principle:

$$\forall_x \forall_y \; x \dot{\div} y = x \cup \tilde{y}$$

This would have the advantage that we could, after introducing this principle and the principle for relative complements, obtain duals by interchanging the binary operators '$\sim$' and '$\dot{\div}$' [in addition, of course, to interchanging '$\cup$' and '$\cap$', '$\emptyset$' and 'S', and '$\subseteq$' and '$\supseteq$'].

*

Here, for the use of <u>those of your students who wish to explore further</u> into the algebra of sets, are additional theorems which can be derived from the basic principles. To save space, we give only one of each pair of dual theorems. [We also give hints for proofs.]

1. $\forall_x \forall_y \forall_z \; (x \cup y) \cup z = (x \cup z) \cup (y \cup z)$

   [Replace 'z' on the left by 'z $\cup$ z', and use the cpu and the apu.]

2. $\forall_x \forall_y \forall_z$ if $x \subseteq z$ and $y \subseteq z$ then $x \cup y \subseteq z$

   [If $z \cap x = x$ and $z \cap y = y$ then $(z \cap x) \cup (z \cap y) = x \cup y$. Use the ℓdpiu.]

3. $\forall_x \forall_y$ if $x \cup y = x \cap y$ then $x = y$

   [Since $x \cap y \subseteq x$, if $x \cup y = x \cap y$ then $x \cup y \subseteq x$. So, since $x \subseteq x \cup y$, $x \cup y = x$. Hence $y \subseteq x$. Etc..]

4. $\forall_x \forall_y \forall_z$ if $x \cup y = y \cap z$ then ($x \subseteq y$ and $y \subseteq z$)

   [Similar to preceding.]

5. $\forall_x \forall_y \; x = (x \cap y) \cup (x \sim y)$

   [Use the principle for relative complements, the ℓpdiu, etc..]

*

Note that, by using the inclusion principle and its dual, the theorems of Exercises 4 and 3 of Part B on page 5-21 may be transformed into:

$$\forall_x \forall_y\ (x \cup y) \cap x = x \text{ and: } \forall_x \forall_y\ (x \cap y) \cup x = x,$$

respectively. These theorems are called <u>absorption</u> <u>laws</u>.

*

Although, as noted on TC[5-18], the operation oppositing, for real numbers, has no analogue, there is, in a limited way, an analogue of subtracting. To see this, we introduce:

<u>Principle</u> <u>for</u> <u>relative</u> <u>complements</u>

$$\forall_x \forall_y\ x \sim y = x \cap \tilde{y}$$

[Read '$x \sim y$' as 'the complement of y relative to x'.]

Among the interesting theorems are:

(1) $\forall_x\ x \sim \emptyset = x$ (2) $\forall_x\ S \sim x = \tilde{x}$

(3) $\forall_x \forall_y\ x \subseteq y$ if and only if $x \sim y = \emptyset$

(4) $\forall_x \forall_y\ (x \cup y) \sim y = x$ if and only if $y \subseteq \tilde{x}$

(5) $\forall_x \forall_y\ (x \sim y) \cup y = x$ if and only if $y \subseteq x$

[Proofs are most easily obtained by first using the principle for relative complements to replace relative complements by intersections.]

The duals of these are, of course, theorems. To obtain the dual of (1), '$\forall_x\ x \sim \emptyset = x$', we first use the principle for relative complements and get '$\forall_x\ x \cap \tilde{\emptyset} = x$'. Then, we make the replacements necessary for dualizing, and get '$\forall_x\ x \cup \tilde{S} = x$'. Here are the duals for the 5 theorems:

(1′) $\forall_x\ x \cup \tilde{S} = x$ (2′) $\forall_x\ \emptyset \cup \tilde{x} = \tilde{x}$

(3′) $\forall_x \forall_y\ y \subseteq x$ if and only if $x \cup \tilde{y} = S$

(4′) $\forall_x \forall_y\ (x \cap y) \cup \tilde{y} = x$ if and only if $\tilde{x} \subseteq y$

(5′) $\forall_x \forall_y\ (x \cup \tilde{y}) \cap y = x$ if and only if $x \subseteq y$

Since $y \cap x = x$,

it follows that $z \cap x = x$.

But, if $z \cap x = x$ then $x \subseteq z$. $[\forall_x \forall_y \text{ if } x \cap y = y \text{ then } y \subseteq x]$

Hence, $x \subseteq z$.

Consequently, if $x \subseteq y$ and $y \subseteq z$ then $x \subseteq z$.

*

As noted on TC[5-17, 18], in dualizing a sentence which contains '$\subseteq$', one replaces this symbol by '$\supseteq$' [or, replaces, for example, '$y \subseteq x$' by '$x \subseteq y$']. With this extension of the notion of dual, to show that the dual of a theorem is a theorem it is sufficient to show that the dual of the inclusion principle is a theorem. That is, it is sufficient to show that the following generalization is a theorem:

$$\forall_x \forall_y \; x \subseteq y \text{ if and only if } x \cup y = y$$

Here is a test-pattern.

Suppose that $x \cup y = y$.

Since $x \subseteq x \cup y$, $[\forall_x \forall_y \; x \subseteq x \cup y]$

it follows that $x \subseteq y$.

Consequently, if $x \cup y = y$ then $x \subseteq y$.

Suppose that $x \subseteq y$.

If $x \subseteq y$ then $y \cap x = x$. $[\forall_x \forall_y \text{ if } y \subseteq x \text{ then } x \cap y = y]$

Consequently, $y \cap x = x$.

Hence, $x \cup y = (y \cap x) \cup y$

$= (y \cap x) \cup (y \cap S)$ $[\forall_x \; x \cap S = x]$

$= y \cap (x \cup S)$ $[\forall_x \forall_y \forall_z \; x \cap (y \cup z) = (x \cap y) \cup (x \cap z)]$

$= y \cap S$ $[\forall_x \; x \cup S = S]$

$= y.$ $[\forall_x \; x \cap S = x]$

Consequently, if $x \subseteq y$ then $x \cup y = y$.

Hence, if $x \cup y = y$ then $x \subseteq y$ and if $x \subseteq y$ then $x \cup y = y$ --that is, $x \subseteq y$ if and only if $x \cup y = y$.

> Correction. On page 5-22, the last part of Theorem 5 should be:
>
> then $\tilde{x} = y$

Answers for Part B.

1. $x \cap \emptyset = \emptyset$ $\quad [\forall_x\ x \cap \emptyset = \emptyset]$

   if $x \cap \emptyset = \emptyset$ then $\emptyset \subseteq x$ $\quad [\forall_x \forall_y$ if $x \cap y = y$ then $y \subseteq x]$

   $\emptyset \subseteq x$

2. $S \cap x = x$ $\quad [\forall_x\ S \cap x = x]$

   if $S \cap x = x$ then $x \subseteq S$ $\quad [\forall_x \forall_y$ if $x \cap y = y$ then $y \subseteq x]$

   $x \subseteq S$

3. $x \cap (x \cap y) = (x \cap x) \cap y$ $\quad [\forall_x \forall_y \forall_z\ (x \cap y) \cap z = x \cap (y \cap z)]$

   $= x \cap y$ $\quad [\forall_x\ x \cap x = x]$

   if $x \cap (x \cap y) = x \cap y$ then $x \cap y \subseteq x$ $\quad [\forall_x \forall_y$ if $x \cap y = y$ then $y \subseteq x]$

   $x \cap y \subseteq x$

4. $(x \cup y) \cap x = (x \cup y) \cap (x \cup \emptyset)$ $\quad [\forall_x\ x \cup \emptyset = x]$

   $= x \cup (y \cap \emptyset)$ $\quad [\forall_x \forall_y \forall_z\ x \cup (y \cap z) = (x \cup y) \cap (x \cup z)]$

   $= x \cup \emptyset$ $\quad [\forall_x\ x \cap \emptyset = \emptyset]$

   $= x$ $\quad [\forall_x\ x \cup \emptyset = x]$

   if $(x \cup y) \cap x = x$ then $x \subseteq x \cup y$ $\quad [\forall_x \forall_y$ if $x \cap y = y$ then $y \subseteq x]$

   $x \subseteq x \cup y$

5. Suppose that $x \subseteq y$ and $y \subseteq z$.

   Then, $y \cap x = x$ and $z \cap y = y$.

   Since, $z \cap y = y$,

   $(z \cap y) \cap x = y \cap x$,

   and $z \cap (y \cap x) = y \cap x$. $\quad [\forall_x \forall_y \forall_z\ (x \cap y) \cap z = x \cap (y \cap z)]$

we need one more basic principle:

Inclusion principle

$\forall_x \forall_y$ [$y \subseteq x$ if and only if $x \cap y = y$]

One consequence of the inclusion principle is, of course:

$\forall_x \forall_y$ if $x \cap y = y$ then $y \subseteq x$

Using this it is easy to show that, for example, '$\forall_x\, x \subseteq x$' is a theorem. Here is a test-pattern:

	$x \cap x = x$	[$\forall_x\, x \cap x = x$]
But,	if $x \cap x = x$ then $x \subseteq x$.	[$\forall_x \forall_y$ if $x \cap y = y$ then $y \subseteq x$]
Hence,	$x \subseteq x$.	

Another consequence of the inclusion principle is:

$\forall_x \forall_y$ if $y \subseteq x$ then $x \cap y = y$

Using this we can show, for example, that '$\forall_x \forall_y$ if $x \subseteq y$ and $y \subseteq x$ then $x = y$' is a theorem. Here is a test-pattern:

Suppose that $x \subseteq y$ and $y \subseteq x$.

Then,	$y \cap x = x$ and $x \cap y = y$.	[$\forall_x \forall_y$ if $y \subseteq x$ then $x \cap y = y$]
So,	$x \cap y = x$ and $x \cap y = y$.	[$\forall_x \forall_y\, x \cap y = y \cap x$]
Hence,	$x = y$.	

Consequently, if $x \subseteq y$ and $y \subseteq x$ then $x = y$.

* * *

B. Write test-patterns for each of the given generalizations.

1. $\forall_x\, \emptyset \subseteq x$ 2. $\forall_x\, x \subseteq S$ 3. $\forall_x \forall_y\, x \cap y \subseteq x$

4. $\forall_x \forall_y\, x \subseteq x \cup y$ [Hint. Here is a first step in a test-pattern for Exercise 4:

$(x \cup y) \cap x = (x \cup y) \cap (x \cup \emptyset)$ [$\forall_x\, x \cup \emptyset = x$]]

5. $\forall_x \forall_y$ if $x \subseteq y$ and $y \subseteq z$ then $x \subseteq z$

SUMMARY

BASIC PRINCIPLES

Commutative principles

$\forall_x \forall_y\ x \cup y = y \cup x$

$\forall_x \forall_y\ x \cap y = y \cap x$

Associative principles

$\forall_x \forall_y \forall_z\ (x \cup y) \cup z = x \cup (y \cup z)$

$\forall_x \forall_y \forall_z\ (x \cap y) \cap z = x \cap (y \cap z)$

Distributive principles

$\forall_x \forall_y \forall_z\ (x \cup y) \cap z = (x \cap z) \cup (y \cap z)$

$\forall_x \forall_y \forall_z\ (x \cap y) \cup z = (x \cup z) \cap (y \cup z)$

Principles for $\emptyset$ and S

$\forall_x\ x \cup \emptyset = x$

$\forall_x\ x \cap S = x$

Principles for complements

$\forall_x\ \tilde{x} \cup x = S$

$\forall_x\ \tilde{x} \cap x = \emptyset$

Inclusion principle

$\forall_x \forall_y\ [y \subseteq x \text{ if and only if } x \cap y = y]$

THEOREMS

1. (a) $\forall_x \forall_y \forall_z$ if $x = y$ then $x \cup z = y \cup z$

   (b) $\forall_x \forall_y \forall_z$ if $x = y$ then $x \cap z = y \cap z$

2. (a) $\forall_x\ x \cup S = S$ (b) $\forall_x\ x \cap \emptyset = \emptyset$

3. (a) $\forall_x\ x = x \cup x$ (b) $\forall_x\ x = x \cap x$

4. (a) $\forall_x \forall_y$ if $x \cup y = \emptyset$ then $y = \emptyset$ (b) $\forall_x \forall_y$ if $x \cap y = S$ then $y = S$

5. $\forall_x \forall_y$ if $x \cup y = S$ and $x \cap y = \emptyset$ then $x = y$

6. (a) $\tilde{S} = \emptyset$ (b) $\tilde{\emptyset} = S$

7. $\forall_x\ \tilde{\tilde{x}} = x$

*

Open-Book Quiz.

Use the basic principles and the theorems on pages 5-22 and 5-23 to do the following exercises.

1. Simplify.

   (a) $\widetilde{A} \cap \widetilde{(B \cup C)}$ (b) $(A \cup \widetilde{A}) \cap (A \cap \widetilde{A})$

2. (a) Of what theorem [Theorem _?_] is the sentence:

   $$A \subseteq A \cup B$$

   an instance?

   (b) Is it also the case that $B \subseteq A \cup B$?

   (c) From what theorem [Theorem _?_] does the sentence:

   $$\text{if } C \subseteq A \text{ then } C \cup A = A$$

   follow immediately?

   (d) Simplify: $(A \cap B) \cup A$

*

Answers for Quiz.

1. (a) $A \cup (B \cup C)$ (b) $\emptyset$

2. (a) Theorem 10(a) (b) Yes [Commutativity and Theorem 10(a)]

   (c) Theorem 11 [only-if part]

   (d) A[Theorems 10(b) and 3(a)]

In order to derive the inclusion principle from P1 through P7 it is convenient to use a stronger result than P7*:

$$\text{P7**} \quad \forall_e[e \in x \Longleftrightarrow e \in y] \Longleftrightarrow x = y$$

Since we have already derived P7*, all that remains to do to derive P7** is to derive:

$$\text{if } x = y \text{ then } \forall_e[e \in x \Longleftrightarrow e \in y]$$

To do so, suppose that $x = y$.

Since $\quad e \in x \Longleftrightarrow e \in x,$

it follows that $\quad e \in x \Longleftrightarrow e \in y.$

So, $\quad \forall_e[e \in x \Longleftrightarrow e \in y].$

Hence, $\quad$ if $x = y$ then $\forall_e[e \in x \Longleftrightarrow e \in y].$

In deriving the inclusion principle we shall make use of a principle of logic which assures us that the sentence:

$$\forall_e(\text{if } e \in y \text{ then } e \in x) \Longleftrightarrow \forall_e[(e \in x \text{ and } e \in y) \Longleftrightarrow e \in y]$$

is valid. [Although the validity of this sentence can be justified on the basis of simple principles of logic with which we are already acquainted, to do so would take us too far afield.]

Here is a derivation of the inclusion principle:

$$\left.\begin{array}{l} y \subseteq x \\ \Longleftrightarrow \forall_e \text{ if } e \in y \text{ then } e \in x \end{array}\right\} \text{[P6]}$$

$$\Longleftrightarrow \forall_e[(e \in x \text{ and } e \in y) \Longleftrightarrow e \in y]$$

So, $\quad y \subseteq x \Longleftrightarrow \forall_e[(e \in x \text{ and } e \in y) \Longleftrightarrow e \in y].$

But, $\quad e \in x \cap y \Longleftrightarrow (e \in x \text{ and } e \in y).$ $\quad$ [P2]

So, $\quad y \subseteq x \Longleftrightarrow \forall_e[e \in x \cap y \Longleftrightarrow e \in y].$

But, $\forall_e[e \in x \cap y \Longleftrightarrow e \in y] \Longleftrightarrow x \cap y = y.$ $\quad$ [P7**]

So, $\quad y \subseteq x \Longleftrightarrow x \cap y = y.$

Therefore, $\quad \forall_x \forall_y [y \subseteq x \Longleftrightarrow x \cap y = y].$

Here is a derivation of the intersecting principle for complements:

$$e \notin \emptyset \qquad [P4]$$

Hence, if $e \notin \tilde{x} \cap x$ then $e \notin \emptyset$.

Consequently, if $e \in \emptyset$ then $e \in \tilde{x} \cap x$.

$$e \in \tilde{x} \cap x$$

$$\Longleftrightarrow (e \in \tilde{x} \text{ and } e \in x) \qquad [P2]$$

$$\Longleftrightarrow (e \notin x \text{ and } e \in x) \qquad [P3]$$

So, $e \in \tilde{x} \cap x \Longleftrightarrow (e \notin x \text{ and } e \in x)$.

But, it is not the case that $(e \notin x \text{ and } e \in x)$.

So, $e \notin \tilde{x} \cap x$.

Hence, if $e \notin \emptyset$ then $e \notin \tilde{x} \cap x$.

Consequently, if $e \in \tilde{x} \cap x$ then $e \in \emptyset$.

So, $e \in \tilde{x} \cap x \Longleftrightarrow e \in \emptyset$.

Therefore, $\forall_e \, [e \in \tilde{x} \cap x \Longleftrightarrow e \in \emptyset]$.

But, if $\forall_e \, [e \in \tilde{x} \cap x \Longleftrightarrow e \in \emptyset]$ then $\tilde{x} \cap x = \emptyset$. [P7*]

So, $\tilde{x} \cap x = \emptyset$.

Therefore, $\forall_x \, \tilde{x} \cap x = \emptyset$.

[Each of the two conditionals, above, which are preceded by 'Consequently,' is a consequence of the conditional directly above it by virtue of the principle of logic by which a conditional sentence is a consequence of its contrapositive.]

Here is a derivation of the unioning principle for complements:

$$
\begin{aligned}
& e \in \tilde{x} \cup x \\
\Longleftrightarrow\ & (e \in \tilde{x} \text{ or } e \in x) \quad \} \text{P1} \\
\Longleftrightarrow\ & (e \notin x \text{ or } e \in x) \quad \} \text{P3}
\end{aligned}
$$

So, $e \in \tilde{x} \cup x \Longleftrightarrow (e \notin x \text{ or } e \in x)$.

But, $e \notin x$ or $e \in x$.

So, $e \in \tilde{x} \cup x$.

Hence, if $e \in S$ then $e \in \tilde{x} \cup x$.

Also, $e \in S$. [P5]

Hence, if $e \in \tilde{x} \cup x$ then $e \in S$.

Consequently, $e \in \tilde{x} \cup x \Longleftrightarrow e \in S$.

Therefore, $\forall_e [e \in \tilde{x} \cup x \Longleftrightarrow e \in S]$.

But, if $\forall_e [e \in \tilde{x} \cup x \Longleftrightarrow e \in S]$ then $\tilde{x} \cup x = S$. [P7*]

So, $\tilde{x} \cup x = S$.

Therefore, $\forall_x\ \tilde{x} \cup x = S$.

[Each of the two conditionals above, which are preceded by 'Hence', is a consequence of the sentence directly above it. The principle of logic appealed to here is the principle according to which a conditional sentence is implied by its consequent.]

Therefore, $\forall_e [e \in x \cup \emptyset \iff e \in x]$.

But, if $\forall_e [e \in x \cup \emptyset \iff e \in x]$ then $x \cup \emptyset = x$. [P7*]

So, $x \cup \emptyset = x$.

Hence, $\forall_x\, x \cup \emptyset = x$.

Here is a derivation of the principle for intersecting with S:

Suppose that $e \in x$.

Since $e \in S$,

$e \in x$ and $e \in S$.

Since $e \in x \cap S \iff (e \in x \text{ and } e \in S)$, [P2]

$e \in x \cap S$.

Hence, if $e \in x$ then $e \in x \cap S$.

Suppose that $e \in x \cap S$.

Since $e \in x \cap S \iff (e \in x \text{ and } e \in S)$, [P2]

$e \in x$ and $e \in S$.

So, $e \in x$.

Hence, if $e \in x \cap S$ then $e \in x$.

Consequently, $e \in x \cap S \iff e \in x$.

Therefore, $\forall_e [e \in x \cap S \iff e \in x]$

But, if $\forall_e [e \in x \cap S \iff e \in x]$ then $x \cap S = x$. [P7*]

So, $x \cap S = x$.

Hence, $\forall_x\, x \cap S = x$.

$\Longleftrightarrow ((e \in x \text{ and } e \in z) \text{ or } (e \in y \text{ and } e \in z))$ } P2

$\Longleftrightarrow (e \in x \cap z \text{ or } e \in y \cap z)$ } P1

$\Longleftrightarrow e \in (x \cap z) \cup (y \cap z)$

So, $e \in (x \cup y) \cap z \Longleftrightarrow e \in (x \cap z) \cup (y \cap z)$.

Hence, $\forall_e[e \in (x \cup y) \cap z \Longleftrightarrow e \in (x \cap z) \cup (y \cap z)]$.

But, if $\forall_e[e \in (x \cup y) \cap z \Longleftrightarrow e \in (x \cap z) \cup (y \cap z)]$ then

$(x \cup y) \cap z = (x \cap z) \cup (y \cap z)$. [P7*]

So, $(x \cup y) \cap z = (x \cap z) \cup (y \cap z)$.

Hence, $\forall_x \forall_y \forall_z (x \cup y) \cap z = (x \cap z) \cup (y \cap z)$.

Here is a derivation of the principle for unioning with $\emptyset$:

Suppose that $e \in x$.

Then $e \in x \text{ or } e \in \emptyset$.

Since $e \in x \cup \emptyset \Longleftrightarrow (e \in x \text{ or } e \in \emptyset)$, [P1]

$e \in x \cup \emptyset$.

Hence, if $e \in x$ then $e \in x \cup \emptyset$.

Suppose that $e \in x \cup \emptyset$.

Since $e \in x \cup \emptyset \Longleftrightarrow (e \in x \text{ or } e \in \emptyset)$, [P1]

$e \in x \text{ or } e \in \emptyset$.

But $e \notin \emptyset$.

So, $e \in x$.

Hence, if $e \in x \cup \emptyset$ then $e \in x$.

Consequently, $e \in x \cup \emptyset \Longleftrightarrow e \in x$.

A derivation of the commutative principle for intersecting can be obtained from the derivation above merely by replacing '$\cup$' by '$\cap$', 'or' by 'and', and 'P1' by 'P2'. Line (2) in the derivation so obtained is the logically valid sentence '(e ϵ x and e ϵ y) $\Longleftrightarrow$ (e ϵ y and e ϵ x)'.

The associative principle for unioning is derived as follows:

$$
\begin{aligned}
& e \in (x \cup y) \cup z && \} \text{ P1} \\
\Longleftrightarrow\ & (e \in x \cup y \text{ or } e \in z) && \} \text{ P1} \\
\Longleftrightarrow\ & ((e \in x \text{ or } e \in y) \text{ or } e \in z) && \} \text{ [Why?]} \\
\Longleftrightarrow\ & (e \in x \text{ or } (e \in y \text{ or } e \in z)) && \} \text{ P1} \\
\Longleftrightarrow\ & (e \in x \text{ or } e \in y \cup x) && \} \text{ P1} \\
\Longleftrightarrow\ & e \in x \cup (y \cup z)
\end{aligned}
$$

So, $e \in (x \cup y) \cup z \Longleftrightarrow e \in x \cup (y \cup z)$.

Hence, $\forall_e\ [e \in (x \cup y) \cup z \Longleftrightarrow e \in x \cup (y \cup z)]$.

But, if $\forall_e\ [e \in (x \cup y) \cup z \Longleftrightarrow e \in x \cup (y \cup z)]$ then

$(x \cup y) \cup z \ = \ x \cup (y \cup z)$. [P7*]

So, $(x \cup y) \cup z \ = \ x \cup (y \cup z)$.

Hence, $\forall_x \forall_y (x \cup y) \cup z \ = \ x \cup (y \cup z)$.

A derivation of the associative principle for intersecting can be obtained from the above derivation by replacing '$\cup$' by '$\cap$', 'or' by 'and', and 'P1' by 'P2'.

Here is a derivation of the distributive principle for intersecting over unioning:

$$
\begin{aligned}
& e \in (x \cup y) \cap z && \} \text{ P2} \\
\Longleftrightarrow\ & (e \in (x \cup y) \text{ and } e \in z) && \} \text{ P1} \\
\Longleftrightarrow\ & ((e \in x \text{ or } e \in y) \text{ and } e \in z)
\end{aligned}
$$

Here is a derivation of the commutative principle for unioning:

$$e \in x \cup y$$

(1) $\Longleftrightarrow (e \in x \text{ or } e \in y)$ } P1

(2) $\Longleftrightarrow (e \in y \text{ or } e \in x)$

(3) $\Longleftrightarrow e \in y \cup x$ } P1

(4) So, $e \in x \cup y \Longleftrightarrow e \in y \cup x.$

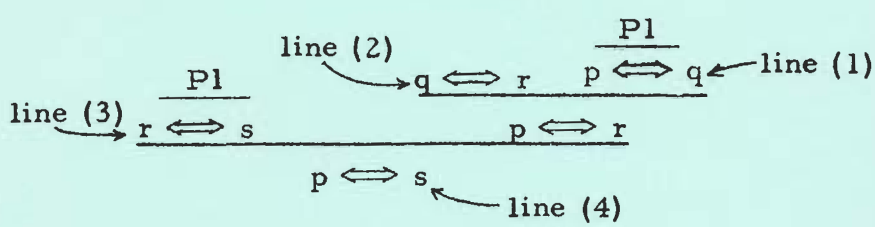
(5) Hence, $\forall_e\, [e \in x \cup y \Longleftrightarrow e \in y \cup x].$

(6) But, if $\forall_e\, [e \in x \cup y \Longleftrightarrow e \in y \cup x]$ then $x \cup y = y \cup x.$ [P7*]

(7) Hence, $x \cup y = y \cup x.$

(8) Consequently, $\forall_x \forall_y\, x \cup y = y \cup x.$

Line (1) is a consequence of P1. Line (2) is the logically valid sentence '$(e \in x \text{ or } e \in y) \Longleftrightarrow (e \in y \text{ or } e \in x)$'. Line (3) is a consequence of P1, and line (4) follows from the first three lines by virtue of the substitution principle for biconditionals. Schematically:



Line (5) is justified by the fact that the preceding lines constitute a pattern which shows that '$\forall_e\, [e \in x \cup y \Longleftrightarrow e \in y \cup x]$' is a consequence of P1. Line (6) is a consequence of P7*. Line (7) follows lines (5) and (6) by modus ponens, another principle of logic, this time one dealing with conditional sentences. Finally, line (8) is justified by the fact that lines (5), (6), and (7) constitute a pattern which shows that the generalization '$\forall_x \forall_y\, x \cup y = y \cup x$' is a consequence of line (5) and P7*.

principle for biconditional sentences. Using the tree-form of presenting derivations we can schematize these steps in the derivation as follows:

$$\dfrac{\dfrac{P6}{q \Longleftrightarrow s} \qquad \dfrac{\dfrac{P6}{p \Longleftrightarrow r} \qquad (p \text{ and } q) \Longleftrightarrow (p \text{ and } q)}{(p \text{ and } q) \Longleftrightarrow (r \text{ and } q)}}{(p \text{ and } q) \Longleftrightarrow (r \text{ and } s)}$$

[line (1): $p \Longleftrightarrow r$; line (2): $q \Longleftrightarrow s$; line (3): $(p \text{ and } q) \Longleftrightarrow (p \text{ and } q)$; line (4): $(p \text{ and } q) \Longleftrightarrow (r \text{ and } s)$]

In deriving line (5) from line (4) we use a logical principle concerning conjunctions and universal generalizations according to which a biconditional sentence of the form '$[\forall_e\ p(e)]$ and $[\forall_e\ q(e)] \Longleftrightarrow \forall_e[p(e)$ and $q(e)]$' is valid merely on logical grounds. Line (6) follows from line (5) because '$e \in x \Longleftrightarrow e \in y$' is just an abbreviation for '(if $e \in y$ then $e \in x$) and (if $e \in x$ then $e \in y$)'. Line (7) sums up lines (4), (5), and (6), according to the scheme below.

$$\dfrac{c \Longleftrightarrow d \qquad \dfrac{b \Longleftrightarrow c \qquad a \Longleftrightarrow b}{a \Longleftrightarrow c}}{a \Longleftrightarrow d}$$

[line (5): $b \Longleftrightarrow c$; line (4): $a \Longleftrightarrow b$; line (6): $c \Longleftrightarrow d$; line (7): $a \Longleftrightarrow d$]

Line (8) is a consequence of P7, and line (9) is obtained from this and line (7) by the substitution principle for biconditionals. Line (10) is justified by the fact that the preceding lines constitute a pattern [compare page 2-33 of Unit 2] which shows that the generalization '$\forall_x \forall_y$ if $\forall_e [e \in x \Longleftrightarrow e \in y]$ then $x = y$' is a consequence of P6 and P7. [When we use this generalization in later proofs we shall refer to it by 'P7*'.]

*

We can now exhibit derivations of the basic principles, on page 5-22, from P1 through P7. [Bearing in mind the analogy between '$\Longleftrightarrow$' and '=', it is clear that we can adopt a form analogous to that introduced on page 2-35 of Unit 2.]

So, one can use a biconditional sentence to justify replacing one sentence by another just as he uses an equation as a premiss to justify his replacing one expression by another.

As an illustration of how these logical principles are used, we shall derive:

$$\forall_x \forall_y \text{ if } \forall_e \text{ [e} \in \text{x if and only if e} \in \text{y] then x = y}$$

from P6 and P7. [To save space, we shall abbreviate 'if and only if' by '$\Longleftrightarrow$'.]

(1) $y \subseteq x \Longleftrightarrow \forall_e$ if $e \in y$ then $e \in x$ [P6]

(2) $x \subseteq y \Longleftrightarrow \forall_e$ if $e \in x$ then $e \in y$ [P6]

(3) So, since $y \subseteq x$ and $x \subseteq y \Longleftrightarrow y \subseteq x$ and $x \subseteq y$,

(4) $y \subseteq x$ and $x \subseteq y \Longleftrightarrow [\forall_e$ if $e \in y$ then $e \in x]$ and $[\forall_e$ if $e \in x$ then $e \in y]$

(5) $\Longleftrightarrow \forall_e$ [(if $e \in y$ then $e \in x$) and (if $e \in x$ then $e \in y$)]

(6) $\Longleftrightarrow \forall_e [e \in x \Longleftrightarrow e \in y]$.

(7) So, $y \subseteq x$ and $x \subseteq y \Longleftrightarrow \forall_e [e \in x \Longleftrightarrow e \in y]$.

(8) But, if $y \subseteq x$ and $x \subseteq y$ then $x = y$. [P7]

(9) So, if $\forall_e [e \in x \Longleftrightarrow e \in y]$ then $x = y$.

(10) Consequently, $\forall_x \forall_y$ if $\forall_e [e \in x \Longleftrightarrow e \in y]$ then $x = y$.

In the derivation above, lines (1) and (2) are consequences of P6, and line (3), since it is of the form '$p \Longleftrightarrow p$', is valid on merely logical grounds. Line (4) then follows from the first three lines by virtue of the substitution

So, one principle of logic concerning biconditionals says that from a biconditional sentence one can infer either of two conditional sentences:

$$\frac{\text{p if and only if q}}{\text{if q then p}}, \quad \text{or:} \quad \frac{\text{p if and only if q}}{\text{if p then q}}$$

[This principle of logic has already been used on page 5-21.]

And, a second principle of logic says that if a conditional sentence and its converse have both been derived from given premisses, then the related biconditional sentence is also a consequence of these premisses. For short:

$$\frac{\text{if q then p} \qquad \text{if p then q}}{\text{p if and only if q}}$$

The principal value of biconditional sentences stems from an analogy between such sentences and equations. As explained on TC[1-56]a and b, and illustrated on TC[2-31, 32]b, TC[2-64]a and b, and TC[2-65]b and c, the logical principles which govern the use of '=' all follow from the fact that '$\forall_x\ x = x$' is valid merely on logical grounds, and from the following substitution principle for '=':

> From an equation and a second sentence one can infer any sentence which can be obtained by replacing an occurrence in the second sentence of either side of the equation by the other side of the equation. [Example: From the equation '$3 = 2 + 1$' and the sentence '$a < 3$', one can infer the sentence '$a < 2 + 1$'.]

Now, any sentence of the form 'p if and only if p' is valid merely on logical grounds, and there is a substitution principle for 'if and only if':

> From a biconditional sentence and a second sentence one can infer any sentence which can be obtained by replacing an occurrence in the second sentence of either component of the biconditional sentence by the other component of the biconditional sentence. [Example: From the biconditional sentence '$e \in y \cup z$ if and only if ($e \in y$ or $e \in z$)' and the sentence '$e \in x$ and $e \in y \cup z$', one can infer the sentence '$e \in x$ and ($e \in y$ or $e \in z$)'.]

For completeness, and to illustrate some techniques of proof, we show here how to derive the basic principles on page 5-22 from the alternative principles, P1 through P7, given on page 5-23. Since the new basic principles relate unions, intersections, and complements of <u>sets</u> to alternations ['or'], conjunctions ['and'], and denials ['not'] of <u>sentences</u>, it turns out that each of the basic principles of 5-22 is <u>derived from</u> the new ones by using an appropriate principle of logic which concerns sentences and looks much like the basic principle which is being derived. For example, to derive:

$$\forall_x \forall_y\ x \cup y = y \cup x,$$

one uses the principle of logic according to which each sentence of the form:

[p or q] if and only if [q or p]

is valid merely on logical grounds, and, so, is acceptable without need of proof. To a beginner, whose understanding of logic is, as yet, shallow, such logical principles are likely to appear trivial, and their use in deriving the corresponding principles about sets may seem to beg the question. Unfortunately, the charge of triviality can be countered only by a rather deep and lengthy analysis [which would be out of place here] of the meanings of such words as 'or', 'and', 'not', and 'if - then'. Once this has been accomplished, it is readily seen that the derivations given are not circular; naturally, if one uses the logical word 'or' in defining '$\cup$', he must expect to use principles of logic concerning alternation sentences in proving theorems about the operation of unioning.

*

In order to make use of P1 through P7 in deriving the basic principles on page 5-22, we shall need to use some logical principles concerning biconditional sentences--sentences of the form 'p if and only if q'. Such sentences can be considered to be abbreviations of sentences of the form:

(if q then p) and (if p then q)

A biconditional sentence is just an abbreviation for the conjunction of a conditional sentence and its converse.

8. (a) $\forall_x \forall_y \widetilde{x \cup y} = \tilde{x} \cap \tilde{y}$ (b) $\forall_x \forall_y \widetilde{x \cap y} = \tilde{x} \cup \tilde{y}$

9. (a) $\forall_x \emptyset \subseteq x$ (b) $\forall_x x \subseteq S$

10. (a) $\forall_x \forall_y x \subseteq x \cup y$ (b) $\forall_x \forall_y x \cap y \subseteq x$

11. $\forall_x \forall_y$ $x \subseteq y$ if and only if $x \cup y = y$

12. $\forall_x x \subseteq x$

13. $\forall_x \forall_y$ if $x \subseteq y$ and $y \subseteq x$ then $x = y$

14. $\forall_x \forall_y \forall_z$ if $x \subseteq y$ and $y \subseteq z$ then $x \subseteq z$

* * *

The relation of membership in a set [denoted by 'ϵ'] plays no role in the preceding discussion of subsets. For a different approach to the subject, one which makes much use of ϵ, we can start from the following basic principles. [In stating these principles, we use 'x' and 'y', as above, for variables whose domain consists of the subsets of S, and 'e' as a variable whose domain consists of the members of S. So, the domain of 'e' is S.]

P1. $\forall_x \forall_y \forall_e$ [$e \in x \cup y$ if and only if ($e \in x$ or $e \in y$)]

P2. $\forall_x \forall_y \forall_e$ [$e \in x \cap y$ if and only if ($e \in x$ and $e \in y$)]

P3. $\forall_x \forall_e$ [$e \in \tilde{x}$ if and only if $e \notin x$]

P4. $\forall_e e \notin \emptyset$

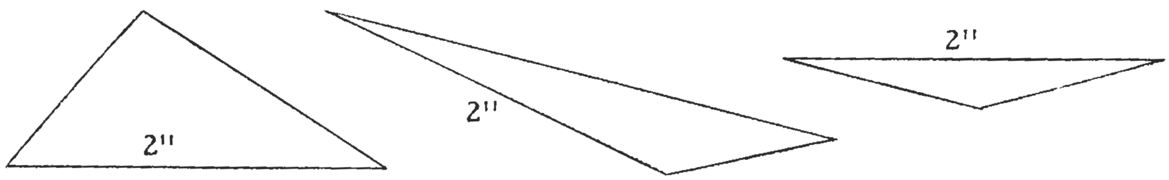
P5. $\forall_e e \in S$

P6. $\forall_x \forall_y$ [$y \subseteq x$ if and only if $\forall_e$ if $e \in y$ then $e \in x$]

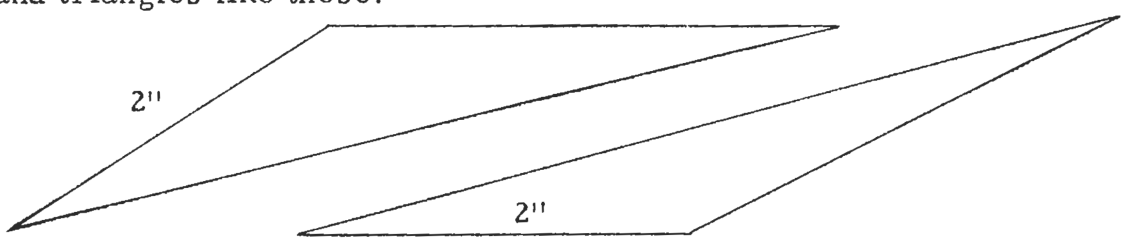
P7. $\forall_x \forall_y$ if $y \subseteq x$ and $x \subseteq y$ then $x = y$

One can use these seven new basic principles to prove the eleven basic principles stated at the beginning of this SUMMARY. So, all the theorems which can be derived from the original basic principles can also be derived from these new basic principles.

5.03 Relations and geometric figures.--Suppose that each person in your class draws a triangle, and that each of these triangles has one side which is 2 inches long. What are the inch-measures of the other two sides? This sounds like a silly question because, as you can readily imagine, there are lots of possibilities for the inch-measures of two sides of a triangle whose third side has inch-measure 2. You could have triangles like these:



and triangles like these:



and many more. We can't predict the inch-measures of the other two sides of such triangles. But, is there anything we can say about these inch-measures? For example, if you knew that the inch-measure of one of the two sides was 3, what could you say about the inch-measure of the other side? Could it be 4? 1? 0.5? 5? 7?

As you are probably beginning to suspect, there is some relation of the inch-measure of one side to the inch-measure of a second side of a triangle whose third side is 2 inches long. We shall now find out just what this relation is. As mentioned earlier, a relation is a set of ordered pairs. The relation in question is a set of ordered pairs of inch-measures of sides of triangles each having a 2-inch third side. Since the inch-measure of a side of a triangle can be any number of arithmetic greater than 0, this is a relation among the nonzero numbers of arithmetic. Our problem, now, is to find out which ordered pairs of nonzero numbers of arithmetic have components which can be the inch-measures of two sides of a triangle whose third side has inch-measure 2.

One way to approach this problem is to try to test every ordered

Section 5.03 provides the student with a few opportunities to apply what he has learned about relations. Since the subject matter is geometric, we cannot presuppose more geometric knowledge than what was learned or used in Units 2 and 3. But, we do expect that students will pick up additional geometric information in this section. Such information will be called upon in exercises throughout the remainder of Unit 5.

*

As mentioned in earlier units, measures of geometric entities are numbers [in particular, numbers of arithmetic]. A side of a triangle is a segment, and one of the properties of a segment is its length. The length of a given segment might be called '3 feet', or '36 inches', or '1 yard', or ... depending upon the unit. We say that the foot-measure of the length of this segment is 3, that its inch-measure is 36, that its yard-measure is 1, ... Note the difference in terminology:

the length of the segment is 3 feet

and:

the foot-measure [of the length] of the segment is 3

Clearly, it makes no sense to talk about the measure of a given segment because a segment has many measures each depending upon the unit selected as the unit of measure. However, when the unit [say, the inch] has been specified somewhere in context, it is customary to use 'the measure' as an elision [say, for 'the inch-measure']. [See the third sentence of the first paragraph on page 5-26 for an application of this convention.]

*

If students do not already know how to use a compass to construct a triangle whose side-lengths are given, this is the appropriate time to teach them to do so.

*

Note that on page 5-25 we call attention to the isomorphism between the system of numbers of arithmetic and the system of nonnegative real numbers. It will be very convenient to use the nonnegative real numbers as measures of geometric entities, and to depend upon the isomorphism to translate the results thus obtained in terms of nonzero numbers of arithmetic.

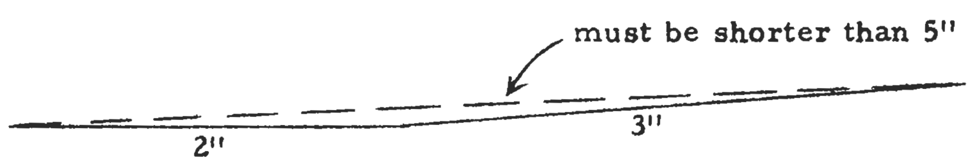
*

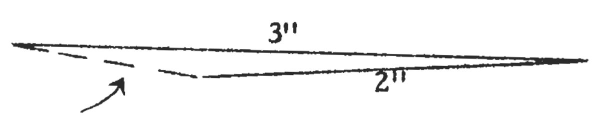
When giving the side-measures [top of page 5-25] of a triangle, we use a counterclockwise orientation, as mentioned. A clockwise orientation would serve just as well except that the counterclockwise one is customary. [Just as going to the right is the customary orientation for the positive direction.] It is essential to pick one orientation, and to stick with it so that each triangle can be associated with only one ordered pair. This need is made manifest in the discussion on page 5-26. Without a preferred orientation, the triangle pictured could be said to correspond with both (3, 4.7) and (4.7, 3). With the counterclockwise orientation it corresponds just with (3, 4.7).

Instead of drawing a second triangle corresponding with (4.7, 3), a student might say: Just turn the drawing over. Certainly, if the given triangle was pictured on a page whose reverse side was blank, one could see a picture of the second triangle by following this suggestion. Also, one could see it by carrying out our suggestion in the bracketed remark about reflection.

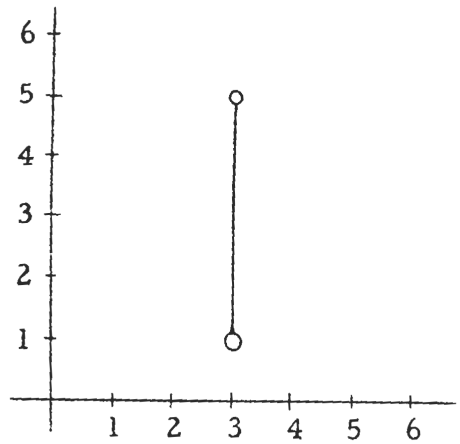
pair. That is, pick an ordered pair, say, (4, 7), and see if you can draw a triangle whose sides have inch-measures 4, 7, and 2, going counterclockwise. If you can, (4, 7) belongs to the relation. If you can't, you know that (4, 7) doesn't belong. Now, obviously, you can't test all of the ordered pairs since there are infinitely many of them. Also, some of them have such large components that it wouldn't be practical to test them by making a drawing. So, we shall have to be clever about this, use our imagination a great deal, search for patterns, and perhaps make just a few drawings to test some crucial hypotheses.

Let's start by deciding to describe the relation by making a graph of it. Since the relation is a subset of the cartesian square of the set of nonzero numbers of arithmetic, we shall graph the relation on a picture of the first quadrant of the number plane. [Recall that the positive real numbers behave just like the nonzero numbers of arithmetic. So, instead of an ordered pair of nonzero numbers of arithmetic we can think of the corresponding ordered pair of positive real numbers.] Now, what points shall we plot? Consider ordered pairs with first component 3. That is, consider triangles whose first and third sides have inch-measures 3 and 2. What are the possible inch-measures for the other side? Here are pictures showing two rather extreme positions for the 3-inch side.



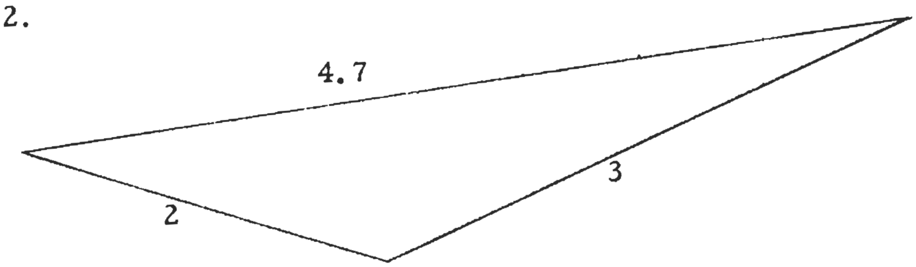


It seems clear that the inch-measure of the second side cannot exceed 5 or be less than 1. In fact, it can't be as large as 5 or as small as 1. [Why?] The second side must have an inch-measure between 1 and 5, and it seems reasonable that there are triangles with sides whose inch-measures are 2, 3, and any number between 1 and 5. So, a subset of the relation we are looking for is the interval whose end points are (3, 1) and (3, 5). Let's graph this interval.

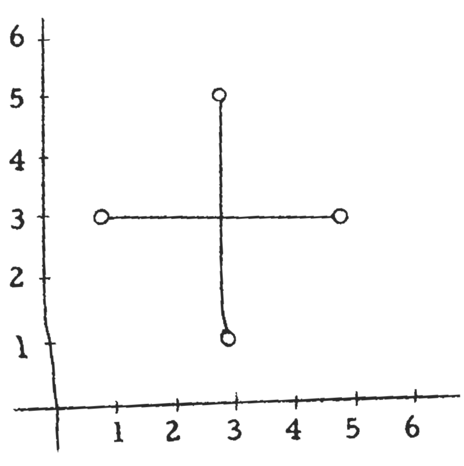


[The hollow dots mean that the corresponding points are not in the relation.]

Now, let's use a bit more imagination to get additional points of the relation. Notice that (3, 4.7) belongs to the relation. Does (4.7, 3) also belong? Here is a picture of a triangle whose sides measure 3, 4.7, and 2.



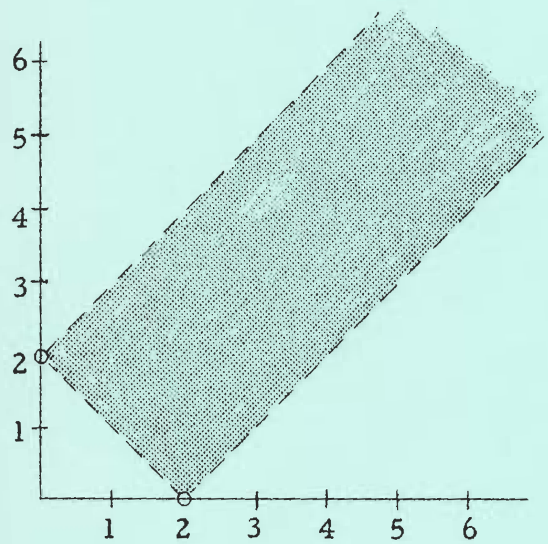
Can you draw a triangle whose sides measure 4.7, 3, and 2? Do so. [You can hold an edge of a mirror next to the picture of the triangle shown above and look at its reflection. The reflection is a triangle whose sides measure 4.7, 3, and 2.] So, for each point of the interval graphed above, the point with components in the opposite order also belong to the relation. This gives us more points to graph.



After establishing that the relation in question is symmetric, we next try to find out if the relation is reflexive. This one is not because it does not contain pairs such as (1, 1), (0.5, 0.5), and (0.28, 0.28). The properties of symmetry and reflexivity are discussed in greater detail in section 5.04. The reference to them on page 5-27 is just an appetite whetter.

*

The completed graph should look like this:



[Compare this with the answer for the exercise in Part B on page 5-12.]

*

Answers for Part A [on pages 5-28 and 5-29].

1. [The graph is like the one given above, but with the scale changed so that the corners are at (0, 5) and (5, 0).]

2. $\{(x, y): |x - 5| < y < x + 5\}$

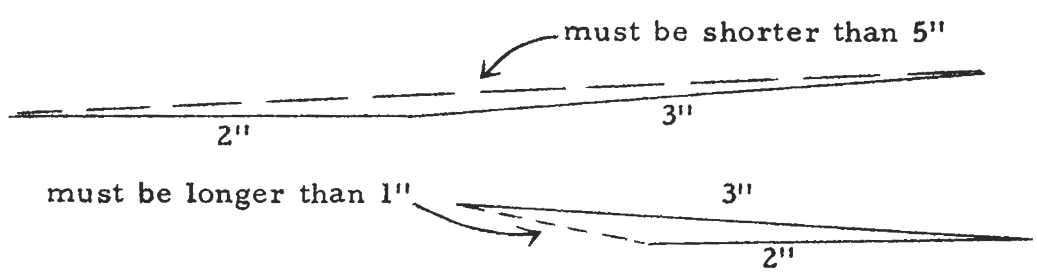
We have discovered an important property of this relation, the property of being symmetric. This is a useful discovery because it helps us find more points in the relation. For example, suppose we know that (101, 100) belongs to the relation. From this, and the fact that the relation is symmetric, we can deduce that what other point belongs to the relation?

There are some points in the relation for which the property of symmetry gives us no additional information. What points are these? What special name do you have for triangles whose side-measures are the components of such points? Is (75, 75) such a point? How about (5, 5)? (2, 2)? Does (0.5, 0.5) belong to the relation? What about (1, 1)? Add more points to the picture on the preceding page by graphing those points in the relation which have equal components. [What special name do we give to triangles with a side of measure 2 whose other side-measures are the components of (2, 2)?]

[In the next section you will learn that a relation is said to be reflexive if and only if it contains all those ordered pairs with equal components which belong to the smallest cartesian square of which the relation is a subset. Although the relation we are now investigating is symmetric, it is not reflexive. Tell why.]

You should now complete the graph of the relation.

Although you have a graph of this relation, it may also be helpful to have a brace-notation name for it. To get such a name, we need a sentence containing two variables, say 'x' and 'y', whose solution set is the relation. Recall the situation discussed on page 5-25. There you discovered that if y is the inch-measure of one side of a triangle

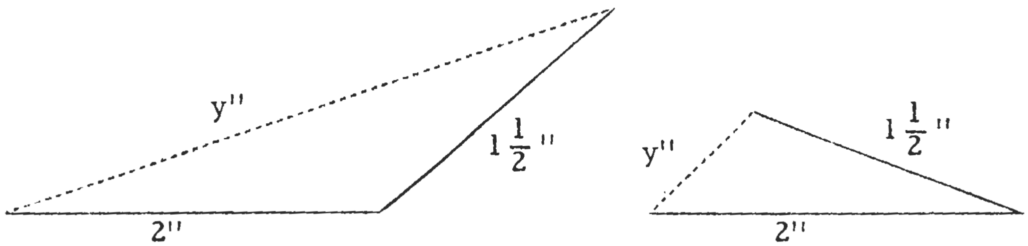


whose other sides are 2 inches and 3 inches long, then

$$y < 3 + 2 \quad \text{and} \quad y > 3 - 2.$$

It also seemed reasonable that for any number which satisfied these inequations, there is such a triangle.

Now, consider this situation:



Here we see that a number can be the inch-measure of a side of a triangle whose other sides are 2 inches and $1\frac{1}{2}$ inches long if and only if it satisfies the sentence:

$$y < 1\frac{1}{2} + 2 \quad \text{and} \quad y > 2 - 1\frac{1}{2}$$

So, in general, y and x can be inch-measures of two sides of a triangle whose third side is 2 inches long if and only if

$$(*) \quad y < x + 2 \quad \text{and} \quad \begin{cases} \text{either } (x \geq 2 \text{ and } y > x - 2) \\ \text{or } (x < 2 \text{ and } y > 2 - x). \end{cases}$$

We can simplify this considerably by noting that

$$\text{if } x \geq 2 \text{ then } x - 2 = {}^{+}|x - 2|$$

and

$$\text{if } x < 2 \text{ then } 2 - x = {}^{+}|x - 2|.$$

Since either $x \geq 2$ or $x < 2$, (*) boils down to:

$$y < x + 2 \text{ and } y > {}^{+}|x - 2|,$$

or, for short:

$${}^{+}|x - 2| < y < x + 2$$

Thus, the relation we have been discussing is

$$\{(x, y): \ |x - 2| < y < x + 2\}.$$

EXERCISES

A. 1. Make a quick sketch of a graph of the relation of the inch-measure of one side of a triangle to the inch-measure of another if the third side is 5 inches long.

2. Write a brace-notation name for this relation.

Answers for Part D [on page 5-29].

The maximum distance obtainable between P and Q is 10, and this occurs when just P, R, and Q are collinear. [In that case, the figure is a triangle, not a quadrilateral.] The minimum distance obtainable between P and Q is 2. [In that case, Q, P, R, and S are collinear.] Thus, the measure of the segment $\overline{PQ}$ can be d if and only if $2 \leq d \leq 10$. [Challenge students to answer these questions when the measure of $\overline{PR}$ is, say, 3.]

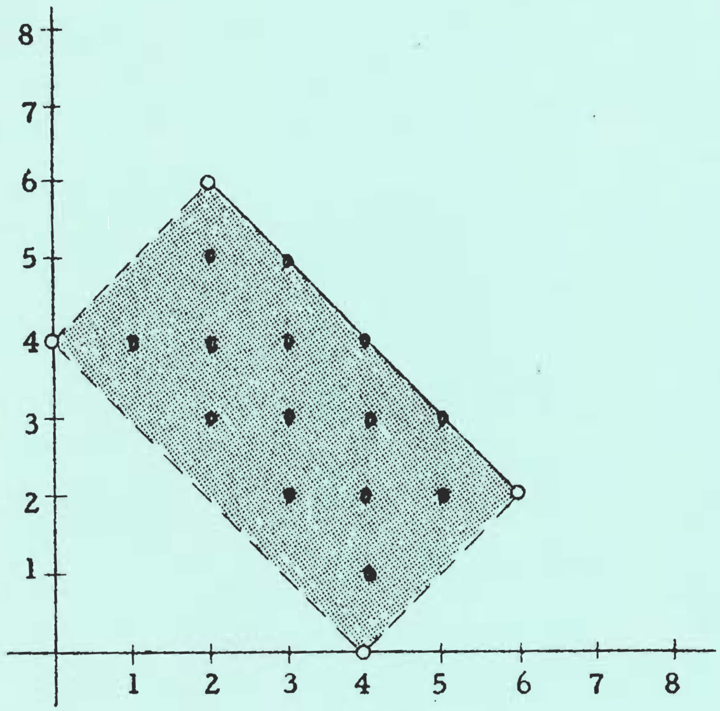
*

Answers for Part E [on page 5-29].

1. No; $2 + 5 \not> 8$	2. No; $9 + 11 \not> 20$	3. No; $8 + 8 \not> 16$
4. Yes	5. Yes	6. Yes
7. No; $1 + 2 \not> 3$	8. Yes	9. No; $6 + 4 \not> 12$

Answers for Part ☆C [on page 5-29].

1.



2. The graphs of points corresponding with the triangles which have inch-perimeter not exceeding 12, one side of inch-measure 4, and the other sides having whole numbers as inch-measures are shown on the figure for Exercise 1. However, not all such triangles are differently-shaped. Those which have the same set of side-measures are considered to have the same shape. Hence, there are just eight triangles which meet the conditions of Exercise 2. They are the triangles whose side-measures are the components of the ordered triples (3, 2, 4), (3, 3, 4), (4, 1, 4), (4, 2, 4), (4, 3, 4), (4, 4, 4), (5, 2, 4), and (5, 3, 4). Notice that symmetry helps in eliminating duplications.

A brace-notation name for the relation referred to in Exercise 1 is:

$$\{(x, y): |x - y| < 4 < x + y \leq 12 - 4\}$$

*

3. (b) $2 < n < 12$

(c) $14 < q < 18$ [or: $|16 - q| < 2 < 16 + q$]

(d) $40 < r < 80$ [or: $|20 - r| < 60 < 20 + r$]

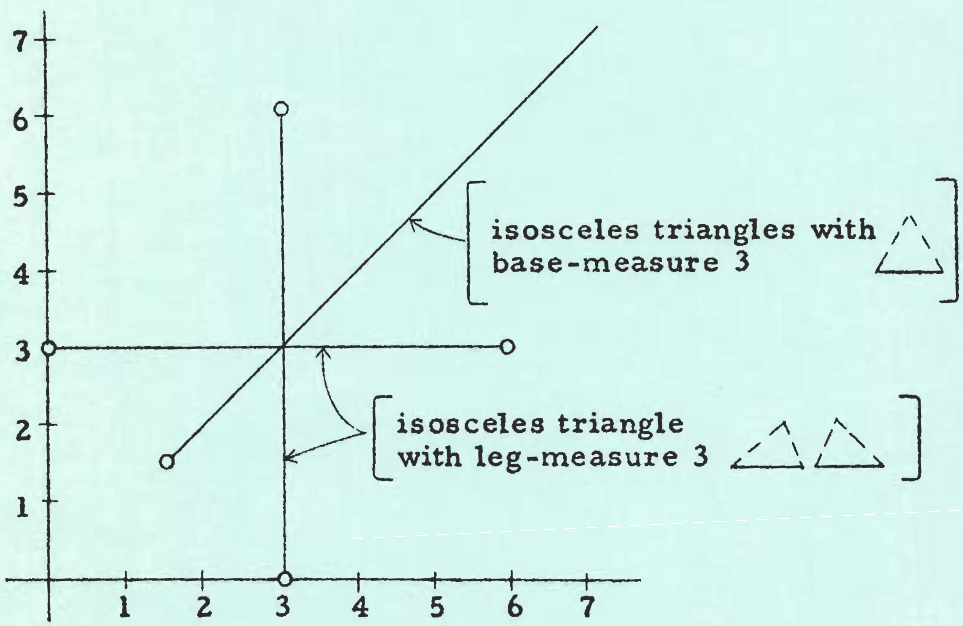
(e) $|a - b| < 11 < a + b$ [or: $|11 - b| < a < 11 + b$]

(f) $|x - z| < y < x + z$

[There are, of course, at least three correct answers for each part of Exercise 3.]

*

Answer for Part ☆B.



Note that this relation is a subset of the relation of the measure of one side to the measure of a second side of a triangle whose third side measures 3. Clearly, the set of all isosceles triangles each of which has a side of measure 3 is a subset of the set of all triangles each of which has a side of measure 3. Note also that the intersection of the two intervals and the half-line contains the point which corresponds with the equilateral triangle of side-measure 3.

*

3. The inch-measures of the sides of some triangle are

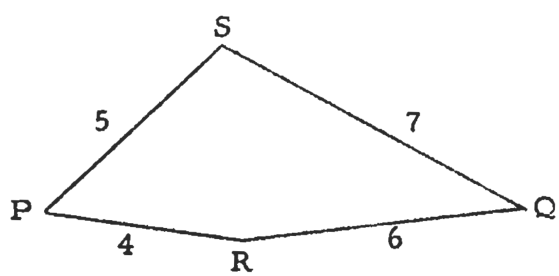
(a) 6, m, and 2 if and only if $4 < m < 8$,
(b) 7, n, and 5 if and only if ________,
(c) 16, 2, and q if and only if ________,
(d) 20, 60, and r if and only if ________,
(e) 11, a, and b if and only if ________,
(f) x, y, and z if and only if ________.

☆ B. Draw a graph of the relation of the inch-measure of one side to the inch-measure of another side of an isosceles triangle whose third side is 3 inches long.

☆ C. 1. Draw a graph of the relation of the inch-measure of one side to the inch-measure of another side of a triangle whose third side is 4 inches long and whose inch-perimeter does not exceed 12.

2. How many such differently-shaped triangles are there whose sides have whole numbers for inch-measures?

D. Four sticks are fastened at their ends to form a quadrilateral as shown.



The quadrilateral is not fixed. That is, it is possible to change its shape by moving, for example, P towards Q, without bending the sticks. How far apart can you move P from Q? How close together can you bring P and Q?

E. For each ordered triple listed below, tell whether its components can be inch-measures of the sides of a triangle.

1. (2, 5, 8) 2. (9, 11, 20) 3. (8, 8, 16)
4. (16, 16, 8) 5. (3, 3, 3) 6. (3, 4, 5)
7. (1, 2, 3) 8. (10, 20, 29) 9. (6, 12, 4)

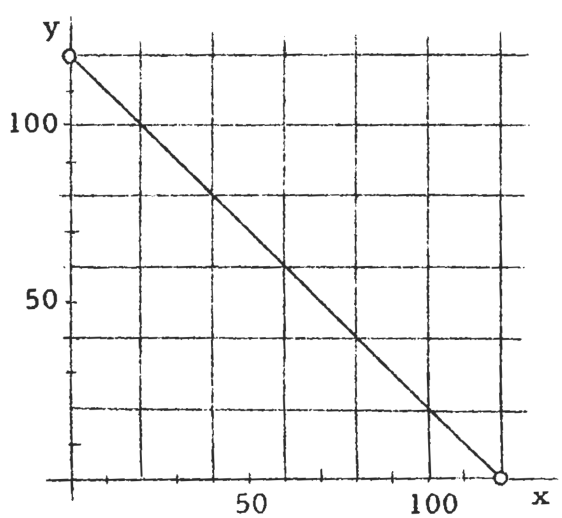
F. Here is another question dealing with a relation between measures of parts of a geometric figure.

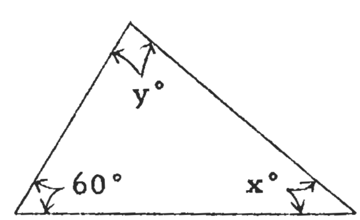
> What is the relation of the degree-measure of one angle to the degree-measure of another angle of a triangle whose third angle is an angle of 50°?

1. Draw at least six triangles each with an angle of 50°, find the degree-measure of each of the other two angles, and plot the corresponding ordered pairs.
2. Are the six ordered pairs you plotted in Exercise 1 enough to show you the pattern for the rest of the relation? If not, draw more triangles with a 50° angle, measure each of the other two angles, and plot ordered pairs until you do see the pattern of the relation. Is the relation symmetric?
3. Complete the following to get a brace-notation name for the relation of the degree-measure of one angle to the degree-measure of another angle of a triangle whose third angle is an angle of 50°.

$$\{(x, y):\ x > 0,\ \ y > 0,\ \text{ and } x + y = \quad \}.$$

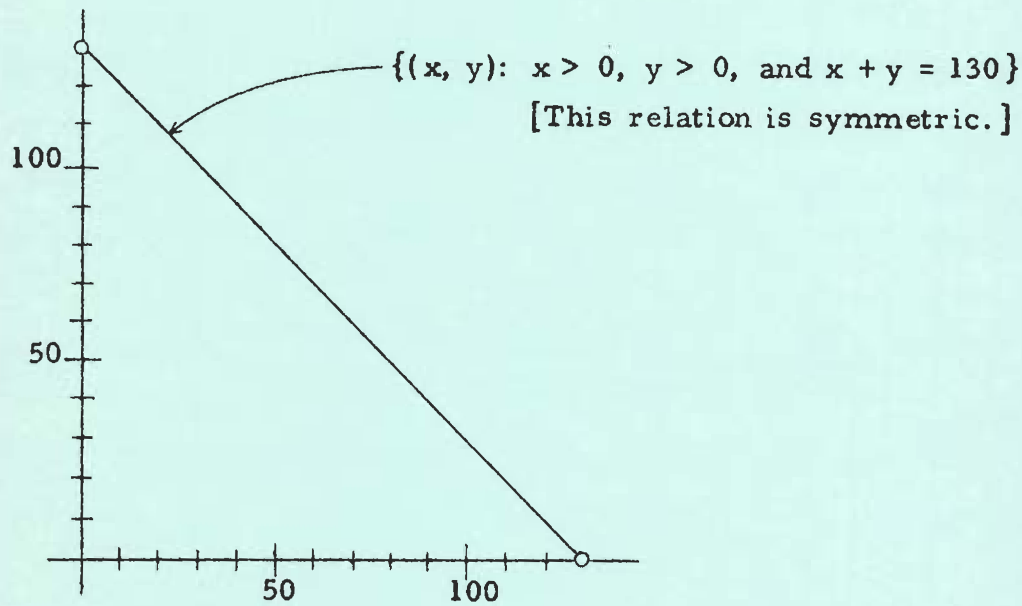
G. Here is a graph of the relation between the degree-measure of one angle to the degree-measure of another angle of a triangle whose third angle is an angle of 60°.





Answers for Part F.

[Students are supposed to discover that the sum of the degree-measures of the other two angles is 130. The fact that students will get the same relation regardless of the side-lengths shows that the sum of the angle-measures is independent of the lengths of the sides.]



Answers for Part G [which begins on page 5-30].

1. $\{(x, y): x > 0, y > 0, \text{ and } x + y = 120\}$

2. (a) 100° (b) 20° (c) 90° (d) 10°
   (e) 60° (f) 81° (g) 80° (h) not a triangle

3. Students should sketch the interval whose end points are (0, 100) and (100, 0). A brace-notation name for this relation is:
   $\{(x, y): x > 0, y > 0, \text{ and } x + y = 100\}$

4. (a) (50, 50), (80, 20), (20, 80); (60, 60) [Students are supposed to be aware of the fact that two angles of a triangle have the same measure if and only if the triangle is isosceles. [Avoid saying 'equal angles'.] The fact that there is only one point of the 60°-relation which corresponds to an isosceles triangle shows that if one angle of an isosceles triangle is an angle of 60° then so are all of them.]

   (b) (60, 60) [This illustrates the fact that a triangle is equilateral if and only if each of its angles is an angle of 60°.]

5. (a) $(33\frac{1}{3}, 66\frac{2}{3})$, $(66\frac{2}{3}, 33\frac{1}{3})$, (40, 60), (60, 40);

   $(33\frac{1}{3}, 66\frac{2}{3}, 80)$, (40, 60, 80)

   (b) (40, 80), (80, 40), (30, 90), (90, 30);
   (40, 60, 80), (30, 60, 90)

6. (a) 30° (b) 60° (c) 28° (d) 29°

1. Write a brace-notation name for this relation.

2. Use the graph or the name of the relation to complete the following table. [∠A, ∠B, and ∠C are the three angles of a triangle.]

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
∠A	60°	60°	60°			60°	60°	60°
∠B	20°	100°		60°	60°	39°	40°	120°
∠C			30°	110°	60°			

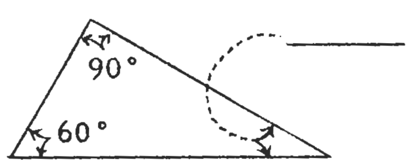
3. Sketch on the chart on page 5-30 a graph of the relation of the degree-measure of one angle to the degree-measure of another angle of a triangle whose third angle is one of 80°. Write its name.

4. (a) Which points of the 80°-relation and of the 60°-relation correspond with isosceles triangles?
   (b) Which points of the 60°-relation and of the 80°-relation correspond with equilateral triangles?

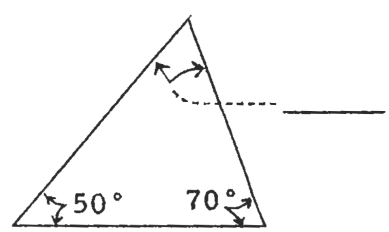
5. (a) Which points of the 80°-relation correspond with triangles in which the degree-measure of one angle is twice the degree-measure of another? What are the degree-measures of the three angles of such triangles?
   (b) Repeat for the 60°-relation.

6. Predict the missing angle-measures, and use a protractor to check your predictions.

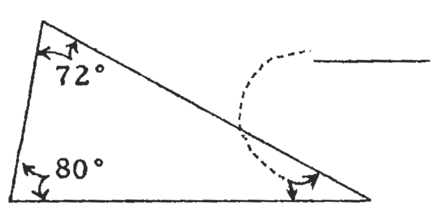
(a)



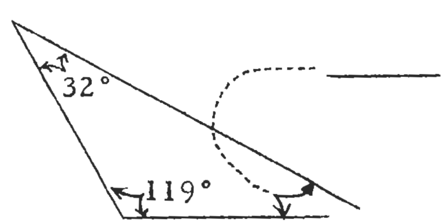
(b)



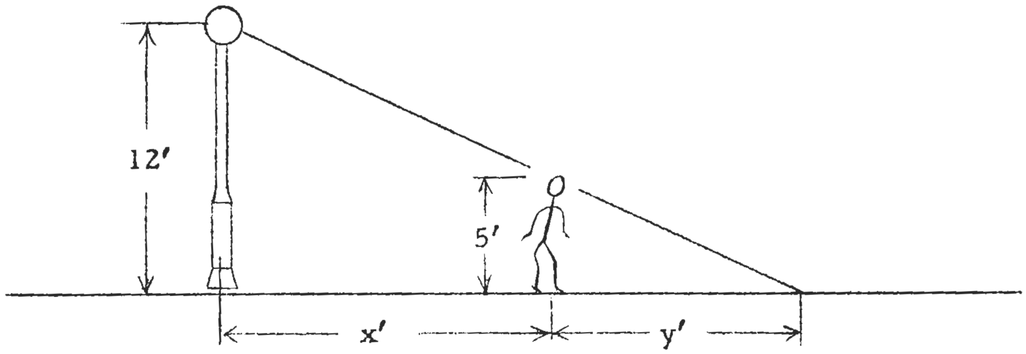
(c)



(d)



☆H. You may have had the experience of walking away from a lamppost and watching your shadow lengthen as you get farther from the post.

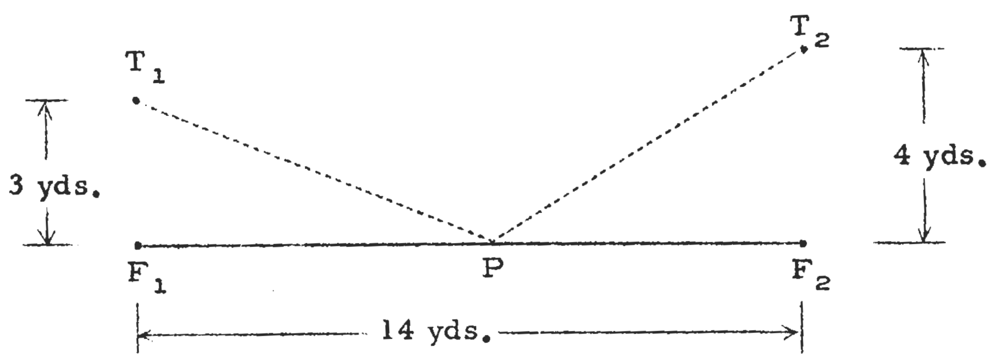


1. Make a graph of the relation of the foot-measure of the shadow of a 5-foot walker to his foot-distance from the 12-foot lamppost.

2. How far is the walker from the post when his shadow is as long as he is tall?

3. How far is the walker from the post when his shadow is twice as long as he is tall?

☆<u>I.</u> Here is a map showing a fence and two trees. If you walk from tree



T_1 to a point P on the fence and from there to tree T_2, the distance walked is the sum of the distances from T_1 to P and from P to T_2.

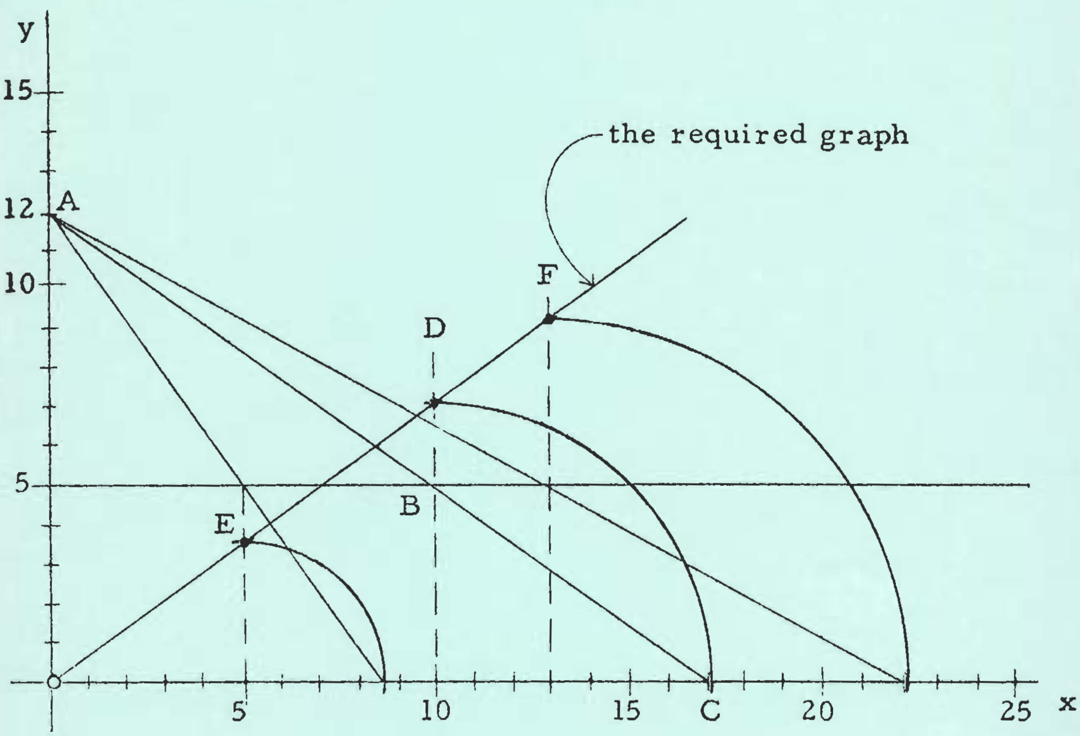
1. Make a graph of the relation of the distance walked to the distance between F_1 and P.

2. Use the graph to tell that location of P for which the distance walked is the smallest.

[Supplementary exercises are in Part D, pages 5-241 through 5-242.]

Answers for Part ☆H.

1. Students may find it helpful to develop a graphical technique for finding points which belong to this relation, a technique which does not require them to make measurements in order to find the components of the points. One such technique is shown in the diagram. [Of course, students should use cross-section paper.]



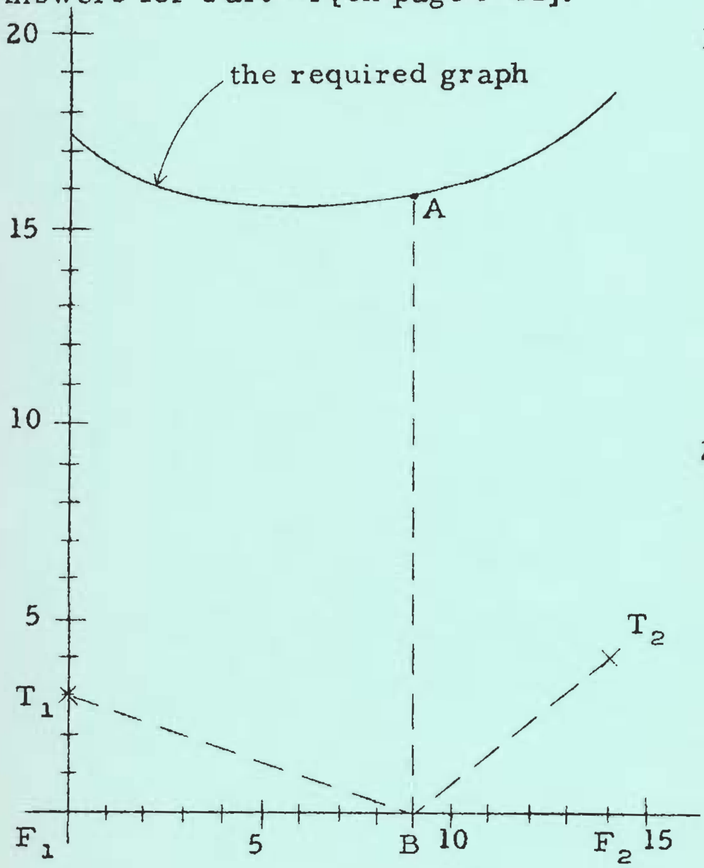
Here is how point D of the graph was located. First, the ray $\overrightarrow{AB}$ was drawn intersecting the x-axis in the point C. [Note that A is 12 units above the horizontal and B is 5 units above the horizontal.] The abscissa of B is the distance between the post and the walker. So, D is a point on the vertical line through B. The point C is the far end of the shadow. So, the distance between the projection of B on the x-axis and point C is the measure the shadow, that is, it is the ordinate of D. Use a compass to fi... point D. [The diagram also shows how points E and F were located.]

[Students who have done some work with similar triangles may recognise that, for each (x, y) which satisfies the conditions of the problem, $y/(x + y) = 5/12$, that is, $y = 5x/7$. So, the relation in question is $\{(x, y): x > 0 \text{ and } y = 5x/7\}$.]

2. This question, in effect, asks for the first components of that point in the relation whose second component is 5. That is the point (7, 5). [Actually, geometric intuition alone will tell most students that the far end of the 5-foot shadow ought to be 12 feet from the foot of the post. So, the walker is 7 feet from the post. It is rewarding to find that the graph of the relation is consistent with intuition.

3. 14 feet.

Answers for Part ☆I [on page 5-32].



1. The diagram illustrates a helpful graphical technique for locating points in this relation. [The distance between A and B is the sum of the distance between T_1 and B and between B and T_2.] The relation is $\{(x, y): 0 \leq x \leq 14 \text{ and } y = \sqrt{9 + x^2} + \sqrt{16 + (14 - x)^2}\}$.

2. We seek the first component of the ordered pair whose graph is the lowest point. This ordered pair is $(6, \sqrt{45} + \sqrt{80})$. So, the required location of P is 6 yards from F_1. Another way of solving Exercise 2 [without using differential calculus] is to reflect T_2 in the line through F_1 and F_2. Let T_3 be the image of T_2. The sum of the distances from T_1 to P to T_3 is the same as the sum of the distances from T_1 to P to T_2. But, the smallest of these sums is obtained when T_1, P, and T_3 are collinear. [This is a consequence of the fact that the measure of one side of a triangle is less than the sum of the measures of the other two sides. Avoid saying, in seriousness, that the shortest distance between two points is a straight line. A straight line is not a distance!] By similar triangles we find that this minimizing location of P is such that the distance between F_1 and P is 6.

The purpose of these Exploration Exercises is to prepare students for the notions of the domain, range, and field of a relation. In answering (a) for each exercise of Part A, they name the domain of the given relation; in answering (b) they name its range. The union of the domain and range of a relation is its field, and in Part B on page 5-34, students are asked to name the fields of the relations given in the exercises of Part A. The exercises of Part C prepare students for a new kind of generalization sentence--existential generalization sentences. Such sentences appear as set-selectors in the definitions of domain and range on page 5-35. They have also been discussed, along with universal generalization sentences, in the COMMENTARY for page 2-27 in Unit 2.

*

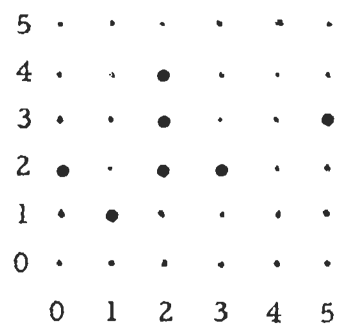
Answers for Part A [on pages 5-33 and 5-34].

1. (a) $\{6, 7, 8\}$ (b) $\{6, 7, 8\}$
2. (a) $\{2, 3, 4, 5\}$ (b) $\{6, 7, 8, 9\}$
3. (a) $\{-2, -1, 0, 1, 2\}$ (b) $\{5\}$
4. (a) the set of real numbers (b) $\{x: 1 \le x \le 2\}$
5. (a) $\{4\}$ (b) the set of real numbers
6. (a) the set of real numbers (b) $\{x: x \ge 0\}$
7. (a) $\{x: |x| < 2\}$ (b) $\{x: |x| < 4\}$
8. (a) $\{x: x \ge 2\}$ (b) $\{x: x \le -1\}$
9. (a) $\{x: |x| \le 5\}$ (b) $\{x: |x| \le 5\}$
10. (a) $\{x: |x| \le 6\}$ (b) $\{x: |x| \le 6\}$
11. (a) $\{x: |x| \le 6\}$ (b) $\{x: 0 \le x \le 6\}$
12. (a) $\{x: 0 \le x \le 6\}$ (b) $\{x: |x| \le 6\}$
13. (a) $\{x: 0 < |x| < 6\}$ (b) $\{x: 0 < |x| < 6\}$
14. (a) $\{x: |x| \le 6\}$ (b) $\{x: |x| \le 6\}$

EXPLORATION EXERCISES

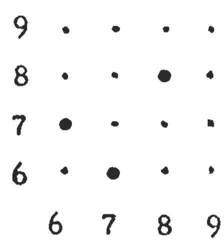
A. Here are graphs or brace-notation names of several relations. For each relation, (a) describe the set of things which are first components, and (b) the set of things which are second components of the members of the relation.

Sample.

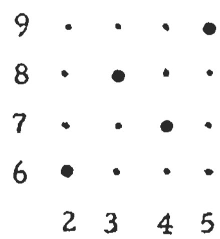


Solution. (a) first components: $\{0, 1, 2, 3, 5\}$
(b) second components: $\{1, 2, 3, 4\}$

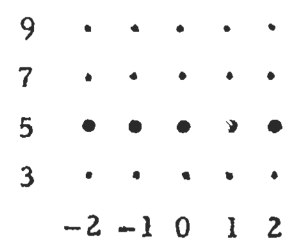
1.



2.



3.



4. $\{(x, y): 1 \leq y \leq 2\}$ 5. $\{(x, y): x = 4\}$ 6. $\{(x, y): y = x^2\}$

7. $\{(x, y): |x| < 2 \text{ and } |y| < 4\}$

8. $\{(x, y): y = 1 - x \text{ and } x \geq 2\}$

9. $\{(a, b): a^2 + b^2 = 25\}$

10. $\{(a, b): a^2 + b^2 = 36\}$

11. $\{(a, b): a^2 + b^2 = 36 \text{ and } b \geq 0\}$

12. $\{(a, b): a^2 + b^2 = 36 \text{ and } a \geq 0\}$

13. $\{(a, b): a^2 + b^2 = 36 \text{ and } ab > 0\}$

14. $\{(a, b): a^2 + b^2 = 36 \text{ and } ab \geq 0\}$

15. $\{(x, y) \in U \times C\colon$ y is a city in $x\}$, where U is the set of all states in the United States and C is the set of all cities in the United States.

16. $\{(x, y) \in U \times C\colon$ y is the capital of $x\}$

17. $\{(x, y) \in P \times P\colon$ y is an uncle of $x\}$, where P is the set of all people.

B. For each of the relations given in Part A, describe the set of things which are either first or second components of the members of the relation.

C. True or false?

1. There is a real number x such that $3 + 2x = 15$.
2. There is a real number y such that $y^2 + 4^2 = 5^2$.
3. There is a real number y such that $y^2 + 6^2 = 5^2$.
4. There is a real number y such that $y + 1 = y + 2$.
5. There is a real number y such that $y + 1 = 1 + y$.
6. There is an integer q such that $22 - 7 = 5q$.
7. There is an integer q such that $22 - 8 = 5q$.
8. There is a real number y such that, for each real number x, $xy = 0$.
9. There is a real number y such that, for each real number $x \neq 0$, $xy = 1$.
10. For each real number $x \neq 0$, there is a real number y such that $xy = 1$.
11. For each real number x, there is a real number y such that $xy = 0$.
12. There is a real number y such that, for each real number x, $x + y = 0$.
13. For each real number x, there is a real number y such that $x + y = 0$.

15. (a) the set of states in the United States

    (b) the set of cities in the United States other than Washington, D. C.

16. (a) the set of states in the United States

    (b) the set of state capitals in the United States

17. (a) the set of people who have uncles [Note that this is not the set of people who are either nephews or nieces: some of these have only aunts !]

    (b) the set of people who are uncles [or: the set of male people who have a nephew or niece]

*

Answers for Part B.

1. $\{6, 7, 8\}$
2. $\{2, 3, 4, 5, 6, 7, 8, 9\}$
3. $\{-2, -1, 0, 1, 2, 5\}$
4. the set of real numbers
5. the set of real numbers
6. the set of real numbers
7. $\{x: |x| < 4\}$
8. $\{x: x \leq -1 \text{ or } x \geq 2\}$
9. $\{x: |x| \leq 5\}$
10. $\{x: |x| \leq 6\}$
11. $\{x: |x| \leq 6\}$
12. $\{x: |x| \leq 6\}$
13. $\{x: 0 < |x| < 6\}$
14. $\{x: |x| \leq 6\}$
15. the set of all political organizations in the United States which are either cities or states, other than Washington, D. C.
16. the set of all political organizations in the United States which are either states or state capitals
17. the set of all people who have or are uncles

*

Answers for Part C.

1. T 2. T 3. F 4. F 5. T 6. T 7. F

8. T 9. F 10. T 11. T 12. F 13. T

Students may, at first, experience some difficulty in distinguishing between the meanings of the sentences in Exercises 8 and 11 [or those in Exercises 9 and 10, or those in Exercises 12 and 13]. Using the existential quantifier '∃' introduced in the COMMENTARY for Unit 2 on TC[2-27]p and, now, in the text, on page 5-35, these exercises can be written:

$$\left.\begin{array}{l} 8.\ \exists_y(\forall_x\ xy = 0) \\ 11.\ \forall_x(\exists_y\ xy = 0) \end{array}\right\} \quad \left.\begin{array}{l} 9.\ \exists_y(\forall_{x \neq 0}\ xy = 1) \\ 10.\ \forall_{x \neq 0}(\exists_y\ xy = 1) \end{array}\right\} \quad \left.\begin{array}{l} 12.\ \exists_y(\forall_x\ x + y = 0) \\ 13.\ \forall_x(\exists_y\ x + y = 0) \end{array}\right\}$$

[The parentheses are unnecessary, but they may make it easier to grasp the sense of the sentences.]

A similar pair of sentences is considered on TC[2-27]i. When rewritten they are:

$$\left.\begin{array}{l} (3')\ \exists_y(\forall_x\ x \leq y) \\ (4')\ \forall_x(\exists_y\ x \leq y) \end{array}\right\}$$

The first says that there is a greatest real number [and, so, is false]; the second says that for each real number there is a real number greater than or equal to it [and, so, is true--each number is less than or equal to itself.].

Of each such pair of sentences, the first makes a stronger claim than the second--more precisely, the first implies the second. The first sentence of such a pair asserts that there is a number which bears a certain relation to each number [or, in the case of Exercise 9, to each nonzero number.] The second sentence asserts only that, for each number, there is a number which bears the relation in question to it. So, for example, since the product of each number by 0 is 0, the sentence of Exercise 8 is true--there is a number [0] such that the product of each number by this single number is 0. Consequently, the sentence of Exercise 11 is also true--for each number there is a number [0] such that the product of the first by the second is 0. The sentence of Exercise 9 is false--there is no single number such that the result of multiplying each nonzero number by it is 1. However, the sentence of Exercise 10 is true--for each nonzero number, there is a number [the reciprocal of the first] such that the product of the first by the second is 1. The sentence of Exercise 12 is false--there

is no single number such that the result of adding it to each real number is 0. On the other hand, the sentence of Exercise 13 is true--for each real number, there is a number [the opposite of the first] such that the sum of the first and the second is 0.

Since, for each pair of sentences of the kind being considered here, the second sentence is a consequence of the first, if the first sentence of such a pair is true then the second must be true. And, if the first is false then, still, the second <u>may</u> be true. That, in this case, the second may also be false is shown by the pair:

$$\left.\begin{array}{l} \exists_y (\forall_x \; xy = 1) \\ \forall_x (\exists_y \; xy = 1) \end{array}\right\}$$

*

If you discussed the imagined graph of U as suggested on TC[5-35, 36]a, you might ask students how they could use the graph to tell if Mr. Adams belonged to the range of U. Find Mr. Adams' name in the vertical list and draw a horizontal line through it. If this line passes through a graph of a point in U, he is in the range. Otherwise, he is not. Next, ask students to consider the set of all points in U whose graphs are on this horizontal line. [This could be the empty set.] What can be said about the first components of these points? They are the nieces and nephews of Mr. Adams. To find out if Bill Smith is in the domain of U, draw a vertical line through his name in the horizontal list. If it hits the graph of the relation, Bill Smith has an uncle. The subset of U whose members have graphs on this vertical line are ordered pairs whose second components are the uncles of Bill Smith. Extend the work a bit by asking students to imagine a vertical line being drawn through the name of Alice Smith, Bill's sister. Will her line hit the same points as Bill's? No, but the second components of the points whose graphs her line contains will be the same as those for Bill's line, if we may assume that brothers and sisters have the same uncles. Ask students to consider the lines they would draw to determine if Mr. Adams belonged to the range and to the domain. Of course, these lines cross each other, but do they cross in the graph of a point which belongs to U? Ask if U contains any point with equal components, but avoid hassles over marriage customs!

*

Some excellent references dealing with relations and their properties are

Tarski, Introduction to Logic (New York: Oxford University Press, 1956),

Suppes, Introduction to Logic (New York: Van Nostrand, 1957),

Huntington, The Continuum (New York: Dover, 1957),

Cogan et al., Modern Mathematical Methods and Models, Volume II (Buffalo, New York: Mathematical Association of America, University of Buffalo, 1958), and

Luce, Some Basic Mathematical Concepts (New Haven: School Mathematics Study Group, 1959).

Some pedagogical suggestions concerning the development of the descriptions (1) and (2) of $\mathfrak{D}_R$ and $\mathfrak{R}_R$ may be in order since they involve a new type of sentence, existential generalizations. We have a relation R among the members of a set S. We wish to know which members of S are involved in the relation, and, in particular which members of S are first components and which are second components of ordered pairs of R. Let us pick a member of S, say x, and ask if x belongs to the domain of R. To answer this question, we search among the members of S. If we find a member of S, say y, such that (x, y) belongs to R, we say:

Yes, $x \in \mathfrak{D}_R$ because there is a $y \in S$ such that $(x, y) \in R$.

Similarly, to find the range of R, we start by picking a member of S, say x, and ask if x belongs to $\mathfrak{R}_R$. To answer this question, we search among the members of S for a member, say y, such that (y, x) belongs to R. If we are successful in the search, we say:

Yes, $x \in \mathfrak{R}_R$ because there is a $y \in S$ such that $(y, x) \in R$.

*

The symbol '$\exists_{y \in S}$' is pronounced as 'there is a y in S such that'. [For a detailed discussion of universal and existential generalizations, see the essay which begins on TC[2-27]e. The "backwards 'E' " reminds one of the first letter of 'exists' in the phrase 'there exists ... '.]

*

In studying the diagram on page 5-36, students should understand that the graphs of $\mathfrak{D}_R$ and $\mathfrak{R}_R$ are graphs of subsets of S, and that the graph of R is a graph of a set of ordered pairs whose first components belong to $\mathfrak{D}_R$ and whose second components belong to $\mathfrak{R}_R$. The graphs of $\mathfrak{D}_R$ and $\mathfrak{R}_R$ are not graphs of axes. [See page 5-62ff.]

*

Answers for questions in the text [on page 5-36].

Bill Smith is in $\mathfrak{D}_U$ if and only if the sentence: (Bill Smith, x) $\in$ U has at least one solution. Mr. Adams is in $\mathfrak{D}_U$ if and only if there is at least one solution of: (Mr. Adams, x) $\in$ U

whose domain is P, the set of all people. Similarly, if M is the set of all male human beings, the sentence:

$$\forall_{x \in M} (\exists_{y \in P} \text{ x is an uncle of y})$$

asserts that each male human being is an uncle of some person. Here, 'ϵ M' indicates that 'x' is a variable whose domain is M, and 'ϵ P' indicates that 'y' is a variable whose domain is P. When the domain of a variable is not so indicated it is to be understood that the domain is the set of real numbers [or, if a restriction is attached, a subset of this set]. [Sometimes, as in section 5.02, we may adopt, for a time, some other convention.]

Now, although, as has just been pointed out, questions concerning the domain of a variable are really questions about the use of language, questions concerning the domain [and range] of a relation are questions about the subject matter. Thus, the domain of the <u>relation</u>

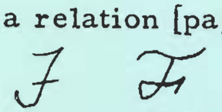
$$\{(x, y) \in P \times P: \text{ y is an uncle of x}\}$$

is the set of all people who have uncles. This is a proper subset of the domain, P, of the <u>variable</u> 'x'.

*

Note the script letters for domain and range. We want abbreviations for 'the domain of R' and 'the range of R'. ['$\mathcal{D}_R$' is read as 'the domain of R' and '$\mathcal{R}_R$' is read as 'the range of R'.] The script letters and the subscripts serve this purpose. Students should be cautioned about the importance of using script letters instead of Roman letters in these cases. Naturally, they need not make copies of the particular script letters we use in the text. All they need do is to make letters which are clearly distinguishable from the block upper case letters. For example:

In the case of the field of a relation [page 5-37], letters like:



will do. Students will get practice in making these letters when they do the exercises in Part A on page 5-37.

*

It is important to distinguish between the two uses of the word 'domain'. Students have learned that the domain of a variable is the set of entities which can serve as values of the variable, that is, the set of entities whose names can be used to replace the variable in an expression or in a sentence in which the variable occurs. When one builds a language for the purpose of talking about a particular subject matter, he usually specifies at the outset which symbols he will use as variables and what the domains of these variables are. To ask about the domain of a variable is to ask about one of the ground rules used in setting up the language. In Units 1 through 4, one of our ground rules is that the domain of each variable is a set of real numbers and, unless otherwise specified, is the set of [all] real numbers. Ways of indicating a restriction of the domain are illustrated by:

$\forall_{x \neq 0}\ \frac{0}{x} = 0$	TC[2-84]a
$\{x \neq 0: \frac{xx}{x} = \frac{x}{x}\}$	TC[3-27]c
$3x \cdot x = 21 \cdot x,\ [x \neq 0]$	TC[3-45, 46, 47]a
$\{(x, y),$ x and y integers$: x = y - 1\}$	4-9
$\{x \in D: (x, 3) \in T\}$	TC[5-H]a
$\{(x, y) \in S \times S: y < x\}$	5-3

In Unit 5 we have a variety of subject matters--real numbers, people, sets, geometric figures, measures, relations .... One way of making clear what is the subject matter of any particular discussion is to choose, in each case, special symbols as variables. For example, we might decide, once and for all, to use upper case Roman letters as variables whose domain is the set of all people, lower case Greek letters as variables whose domain is the set of all geometric figures, etc. However, this is typographically impractical, and even if it were not, would hardly be worth the trouble. Instead, when we need variables whose domain is not the set of real numbers we shall, as in Exercises 7(ℓ) through 7(p) of Part A on pages 5-8 and 5-9, assign an arbitrary name, say 'D', to the domain and use phrases like '$\in$ D' to indicate that a letter is being used as a variable with this domain. Thus in:

$$\{(x, y) \in P \times P: \text{y is an uncle of x}\},$$

'$\in P \times P$' indicates that both 'x' and 'y' are here being used as variables

The relation U may seem a bit strange since the relations worked with in detail thus far have been numerical ones. It may help to ask students what the ordered pairs are which belong to U. For one thing, they are ordered pairs of people. The first component is a person, and so is the second component. Since these are ordered pairs which belong to the relation of being an uncle of [or: unclehood], the second component of each such ordered pair is an uncle, and the first component is one of his nieces or nephews.

Students may wonder if you can graph the relation U. The answer is 'yes', although, practically speaking, it is impossible. However, it is instructive to discuss the steps you would follow in making such a graph. First, you might make a picture of the cartesian square of the set of all people. One way to do this is to make two lists--one vertical and the other horizontal--of all the people in the world. Then, just as one graphs an ordered pair of numbers, you could graph an ordered pair of people. If you wanted to graph the ordered pair

(Al Brown, Stan Moore),

you would look for Al Brown's name in the horizontal list and draw a vertical line through it; then, look for Stan Moore's name in the vertical list and draw a horizontal line through it. The dot in which the two lines cross is the graph of (Al Brown, Stan Moore). There is also a graph of (Stan Moore, Al Brown) which is different from the graph of (Al Brown, Stan Moore) [assuming Al Brown $\neq$ Stan Moore].

Then, to graph the relation U you would have to select from the cartesian square of P and mark on the picture just those ordered pairs for which the second component is an uncle of the first component.

A brace-notation name for U is: $\{(x, y) \in P \times P\colon$ y is an uncle of $x\}$

*

Answers for questions in the text.

Each person who is not an uncle [in particular, each female person] is an example of a member of P who is not the second component of any member of U.

A nephew who is so only by virtue of having an aunt is not the first component of any member of U.

The domain of U is the set of people who have uncles.

The range of U is the set of people who are uncles [that is, the set of male persons who have nephews or nieces].

5.04 Properties of relations. --In the preceding section we mentioned two of the properties [symmetry and reflexiveness] which relations can have. We want to give precise descriptions of these properties, and in order to do so, we need more terminology.

DOMAIN AND RANGE OF A RELATION

We have said that a relation R among the elements of a set S is a subset of the cartesian square $S \times S$. It may not be the case that each member of S is the first component of a member of R, nor that each member of S is the second component of a member of R. For example, consider the relation U of being-an-uncle-of among the members of the set P of all people. There are people who are not uncles, that is, there are members of P who are not second components of members of U. [Give an example.] Also, there are people who are neither nephews nor nieces. [Is it possible that a nephew not be the first component of a member of U?]

Given a relation R among the members of a set S, it is often convenient to talk about the set of those members of S which are first components of members of R. This set is called the domain of R. [What is the domain of U?] The set of members of S which are second components of members of R is called the range of R. [What is the range of U?] Concisely,

(1) $\mathcal{D}_R = \{x \in S\colon \text{ there is a } y \in S \text{ such that } (x, y) \in R\}$,

(2) $\mathcal{R}_R = \{x \in S\colon \text{ there is a } y \in S \text{ such that } (y, x) \in R\}$.

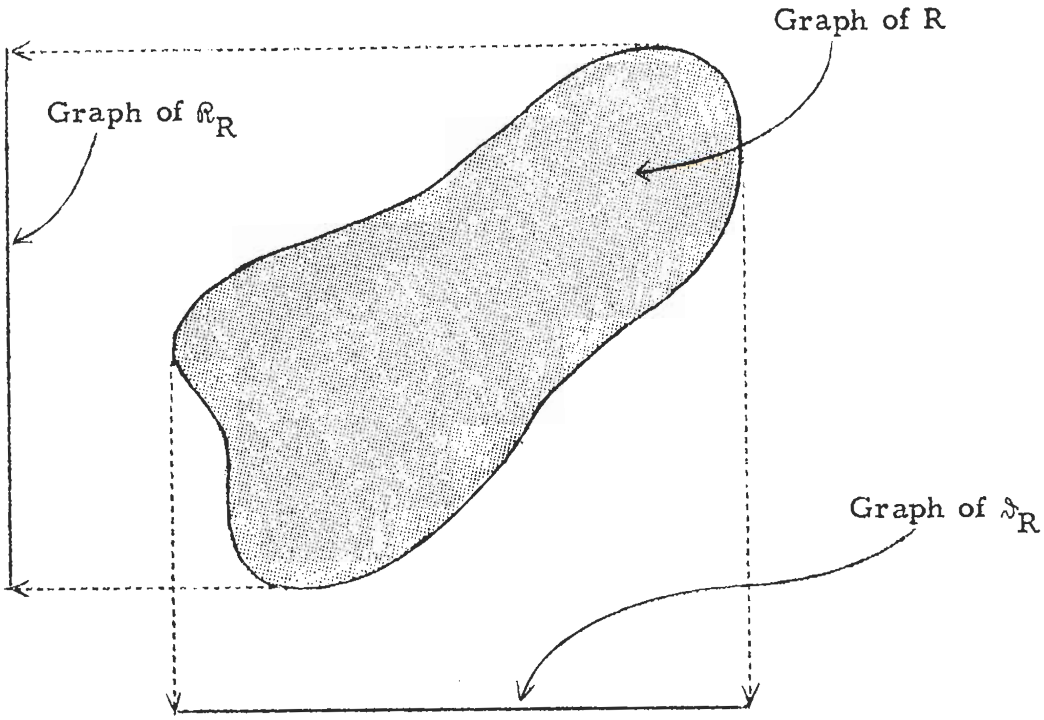
Notice that 'there is a $y \in S$ such that' is a short way of saying 'there is at least one $y \in S$ such that', and it is customary to abbreviate both of these expressions by:

$$\exists_{y \in S}$$

[The symbol '$\exists$' is called an existential quantifier. You are already acquainted with the universal quantifier '$\forall$'.]

So, $\mathcal{D}_R = \{x \in S\colon \exists_{y \in S} (x, y) \in R\}$,

and, $\mathcal{R}_R = \{x \in S\colon \exists_{y \in S} (y, x) \in R\}$.



In the case of the relation U,

$$\mathfrak{D}_U = \{x \in P: \exists_{y \in P} (x, y) \in U\},$$

and $$\mathfrak{R}_U = \{x \in P: \exists_{y \in P} (y, x) \in U\}.$$

The domain of U is that subset of P which consists of all people who have uncles. The range of U is that subset of P which consists of all people who are uncles. How do we tell if a given element of P, say, Mr. Adams, belongs to $\mathfrak{R}_U$? By definition, Mr. Adams $\in \mathfrak{R}_U$ if and only if the sentence:

$$\exists_{y \in P} (y, \text{Mr. Adams}) \in U$$

is true. And, this sentence is true if and only if the sentence:

$$(y, \text{Mr. Adams}) \in U$$

has at least one solution. [It may be the case that this last sentence has more than one solution. This additional information may be of interest, but it is irrelevant to the question of whether Mr. Adams belongs to $\mathfrak{R}_U$.] What sentence must have at least one solution if Bill Smith is in $\mathfrak{D}_U$? If Mr. Adams is in $\mathfrak{D}_U$?

Skill Quiz.

A. Simplify.

1. $\frac{\frac{3}{4}}{2}$ 2. $\frac{\frac{3}{4}+\frac{1}{2}}{\frac{3}{7}}$ 3. $\frac{\frac{1}{5}}{\frac{1}{8}-\frac{2}{9}}$ 4. $\frac{\frac{5}{3}}{\frac{8}{9}+3}$ 5. $\frac{\frac{4}{7}-1}{\frac{4}{7}+1}$

B. Simplify.

1. $3(2a - b) + (b - a) - 2(6 + c)$ 2. $x(x - y) + y(x - y) - 7(xy) + y^2$
3. $4(x - y)(x + y) - 7(x - y)(x - y) + 6(x - y)^2$
4. $2ab(a + c) - 2bc(a + b)$ 5. $10\left(\frac{x - y}{5}\right) + (x + y)(2 + 3y)$

C. Factor.

1. $x^2 + 5x - 14$ 2. $x^2 - 1$ 3. $3x^2 - 24x + 45$
4. $x^2 + 4xy + 4y^2$ 5. $5 - x^2 + 4x$

D. Solve. [In the case of inequations, give the solution set, using the simplest sentence possible as set selector.]

1. $8x - 14 + 3x = 7$ 2. $5x + 16 < 2x + 9$
3. $x^2 + x = 12$ 4. $8y + 5 < 6y - 7$
5. $x^2 + 4 = 4x$

*

Answers for Quiz.

A. 1. $\frac{3}{8}$ 2. $\frac{35}{12}$ 3. $-\frac{72}{35}$ 4. $\frac{3}{7}$ 5. $-\frac{3}{11}$

B. 1. $5a - 2b - 2c - 12$ 2. $x^2 - 7xy$ 3. $3x^2 + 2xy - 5y^2$
4. $2a^2b - 2b^2c$ 5. $4x + 3xy + 3y^2$

C. 1. $(x + 7)(x - 2)$ 2. $(x - 1)(x + 1)$ 3. $3(x - 5)(x - 3)$
4. $(x + 2y)(x + 2y)$ 5. $(5 - x)(1 + x)$

D. 1. $\frac{21}{11}$ 2. $\{x: x < -\frac{7}{3}\}$ 3. $-4, 3$
4. $\{y: y < -6\}$ 5. 2

[illegible] of the range, thereby making f have an inverse.

[illegible]

[illegible]

and thus

[illegible]

[illegible]

[illegible]

Answers for Class A [on pages 4-37 and 4-38].

1. [illegible]

2. [illegible]

3. [illegible]

member of the range because adding 1 has an inverse.

[One <u>can</u> solve Sample 2 by saying that

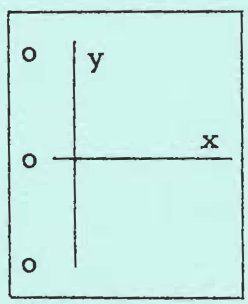
$$\mathfrak{D}_B = \{x\colon \exists_y\ y^2 = x + 1\},$$

and that

$$\mathfrak{R}_B = \{x\colon \exists_y\ x^2 = y + 1\}.$$

But, doing this is akin to answering: $\{x\colon\ x^2 - 5x + 6 = 0\}$, when asked for the solution set of '$x^2 - 5x + 6 = 0$'!]

A foreknowledge of the domain and range of a relation is useful in drawing a graph. Such knowledge helps you select the scales for the axes. If you were graphing the relation in Sample 2 on a sheet on cross-section paper, you might draw the axes like this:



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Answers for Part A [on pages 5-37 and 5-38].

1. $\mathfrak{D}_M = \{-2, 0, 5\}$, $\mathfrak{R}_M = \{-2, 0, 1, 5\} = \mathfrak{F}_M$

2. $\mathfrak{D}_N = \{3, 4\} = \mathfrak{R}_N = \mathfrak{F}_N$

3. $\mathfrak{D}_T = \{x \in I\colon\ 1 \le x \le 10\} = \mathfrak{R}_T = \mathfrak{F}_T$

The notion of the field of a relation makes it easy for us to give a precise description of the property of reflexiveness. The statement that the field of R is the smallest subset of S whose cartesian square contains all the members of R may be a little hard for students to understand without having had the experience of doing the exercises in Part A. Don't push this statement, but instead raise these questions which students should think about as they do Part A:

(1) Is each member of $\mathfrak{D}_R$ a member of $\mathfrak{F}_R$? [Yes.]

(2) Is each member of $\mathfrak{R}_R$ a member of $\mathfrak{F}_R$? [Yes.]

(3) Is each member of $\mathfrak{F}_R$ a member of $\mathfrak{D}_R$? [Not necessarily.]

(4) Is each member of $\mathfrak{F}_R$ a member of $\mathfrak{R}_R$? [Not necessarily.]

(5) Can $\mathfrak{D}_R = \mathfrak{F}_R$? [Yes, provided that $\mathfrak{R}_R \subseteq \mathfrak{D}_R$.]

(6) If either $\mathfrak{D}_R$ or $\mathfrak{R}_R$ is S, does $\mathfrak{F}_R = S$? [Yes.]

Follow through by asking for answers when they have completed Part A.

A person who has an obsession both for drawing graphs of relations on square charts and for using as small an area as practicable has probably discovered the notion of a field. He knows that the domain and range of a relation are subsets of the field. So, he prepares his chart to accommodate the cartesian square of the field. This assures him that there will be no point of the relation whose graph will not fit on the chart.

*

Here are pedagogical suggestions for handling Sample 2. The relation B is a relation among the real numbers. We want to find out what real numbers belong to $\mathfrak{D}_B$. So, we pick some real number, say 7, and ask if $7 \in \mathfrak{D}_B$. This is equivalent to asking:

Is there a real number y such that $y^2 = 7 + 1$?

The answer to this question is 'yes' because $\sqrt{8}$ is such a real number. [$-\sqrt{8}$ is another real number whose square is 7 + 1, but the fact that there are two is irrelevant.] Does $-5 \in \mathfrak{D}_B$? That is:

Is there a real number y such that $y^2 = -5 + 1$?

The answer is 'no' because the square of each real number is non-negative. The smallest number in the domain is -1.

Does $12 \in \mathfrak{R}_B$? That is:

Is there a real number x such that $12^2 = x + 1$?

Yes, 143 is such a real number. In fact, each real number is a

FIELD OF A RELATION

We need one additional notion, that of <u>the field of R</u>.

$$\mathfrak{F}_R = \mathfrak{D}_R \cup \mathfrak{R}_R.$$

Roughly speaking, $\mathfrak{F}_R$ consists of the members of S which "get into the act". That is, $\mathfrak{F}_R$ is the smallest subset of S whose cartesian square contains all the members of R. So, for example, $\mathfrak{F}_U$ is the set of all people who have or are uncles.

EXERCISES

A. For each relation described below, give its domain, range, and field.

<u>Sample 1</u>. The relation A where

$$A = \{(0, 6), (3, 5), (3, 7), (4, 2), (5, 2), (6, 0)\}.$$

<u>Solution.</u> $\mathfrak{D}_A = \{0, 3, 4, 5, 6\}$

$\mathfrak{R}_A = \{0, 2, 5, 6, 7\}$

$\mathfrak{F}_A = \{0, 2, 3, 4, 5, 6, 7\}$

<u>Sample 2</u>. The relation B where $B = \{(x, y): y^2 = x + 1\}$.

<u>Solution.</u> To find the domain one needs to use the fact that a real number is a square if and only if it is nonnegative. So,

$$\mathfrak{D}_B = \{x: x \geq -1\}.$$

To find the range one needs to use the fact that each real number is the sum of a real number and 1. It follows that

$\mathfrak{R}_B$ is the set of real numbers.

Hence, $\mathfrak{F}_B$ is the set of real numbers

1. $M = \{(-2, 5), (-2, -2), (0, 1), (0, 5), (5, 0), (5, -2)\}$

2. $N = \{(3, 4), (4, 3)\}$

3. T, whose graph is on page 5-B.

4. $R = \{(7, 2), (7, 9), (7, 6), (7, 7)\}$

5. $S = \{(8, 1), (6, 1), (4, 1), (12, 1), (2, 1)\}$

6. $C = \{(x, y): y^2 = 2x - 3\}$

7. $D = \{(x, y): x^2 + y^2 = 25\}$

8. $E = \{(x, y) \in I \times I: |x| + |y| \leq 10\}$

9. $F = \{(x, y): y = |x - 2| + 4\}$

10. $G = \{(x, y): y + x = x + y + 3\}$

11. $H = \{(x, y): xy = yx\}$

12. $J = \{(x, y): x^2 - y^2 = 25\}$

13. $K = \{(x, y): 9x^2 + 25y^2 = 225\}$

[Supplementary exercises are in Part E, page 5-242.]

B. True or false?

Sample. $\exists_x\ x^2 - 5x + 6 = 0.$

Solution. True. [For instance, $3^2 - 5 \cdot 3 + 6 = 0.$]

1. $\exists_y\ 3y + 7 = 18$

2. $\exists_k\ k = k + 1$

3. $\exists_t\ t^2 - 1 = 0$

4. $\exists_m\ 3 + m = m + 3$

5. $\exists_x\ (2x - 5 = 0 \text{ and } 5x - 2 = 0)$

6. $\exists_x\ (2x - 5 = 0 \text{ or } 5x - 2 = 0)$

7. $\forall_x\ 2x - 3 = 12$

8. $\exists_x\ 2x - 3 = 12$

9. $\exists_{x \in I}\ 12 = 3x$

10. $\exists_{x \in I}\ 12 = 7x$

11. $\exists_y\ y^5 - 3y^4 = y(y - 1)(y + 9)$

12. $\forall_x\ (\exists_y\ x + y = 3)$

13. $\exists_y\ (\forall_x\ x + y = 3)$

14. $\forall_x\ (\exists_y\ xy = 0)$

15. $\exists_y\ (\forall_x\ xy = 0)$

C. If P and Q are relations then $P \cup Q$ and $P \cap Q$ are relations. [Why?] Can you compute the domain of $P \cup Q$ if you know $\mathcal{D}_P$ and $\mathcal{D}_Q$? How about the domain of $P \cap Q$? How about ranges and fields?

4. $\mathcal{D}_R = \{7\}$, $\mathcal{R}_R = \{2, 6, 7, 9\} = \mathcal{F}_R$

5. $\mathcal{D}_S = \{2, 4, 6, 8, 12\}$, $\mathcal{R}_S = \{1\}$, $\mathcal{F}_S = \{1, 2, 4, 6, 8, 12\}$

6. $\mathcal{D}_C = \{x\colon\ x \geq 3/2\}$, $\mathcal{R}_C$ = the set of all real numbers = $\mathcal{F}_C$

7. $\mathcal{D}_D = \{x\colon\ -5 \leq x \leq 5\} = \mathcal{R}_D = \mathcal{F}_D$

8. $\mathcal{D}_E = \{x \in I\colon\ -10 \leq x \leq 10\} = \mathcal{R}_E = \mathcal{F}_E$

9. $\mathcal{D}_F$ = the set of all real numbers = $\mathcal{F}_F$, $\mathcal{R}_F = \{x\colon\ x \geq 4\}$

10. $\mathcal{D}_G = \emptyset = \mathcal{R}_G = \mathcal{F}_G$

11. $\mathcal{D}_H$ = the set of all real numbers = $\mathcal{R}_H = \mathcal{F}_H$

12. $\mathcal{D}_J = \{x\colon\ |x| \geq 5\}$, $\mathcal{R}_J$ = the set of all real numbers = $\mathcal{F}_J$

13. $\mathcal{D}_K = \{x\colon\ |x| \leq 5\} = \mathcal{F}_K$, $\mathcal{R}_K = \{x\colon\ |x| \leq 3\}$

(12–13: Have students make graphs.)

*

See Exercise 3 of Part E, Supplementary Exercises [page 5-242], for possible classroom discussion questions at this point. Also, see TC[5-37]a.

*

Answers for Part B.

1. True $[3(11/3) + 7 = 18]$

2. False $[\{x\colon\ x = x + 1\} = \emptyset]$

3. True $[(-1)^2 - 1 = 0]$

4. True $[3 + 8 = 8 + 3]$

5. False [Note, however, that the conjunction '$\exists_x\ 2x - 5 = 0$ and $\exists_x\ 5x - 2 = 0$' is true.] Also, '$\exists_x\ (2x - 5 = -7$ and $5x - 2 = -7)$' is true.]

6. True ['$2(5/2) - 5 = 0$ or $5(5/2) - 2 = 0$' is true.]

7. False $[2 \cdot 1 - 3 \neq 12]$

8. True $[2(15/2) - 3 = 12]$

9. True $[12 = 3 \cdot 4$, and $4 \in I]$

10. False $[\{x\colon\ 12 = 7x\} = \{12/7\}$, and $12/7 \notin I]$

11. True $[0^5 - 3 \cdot 0^4 = 0(0 - 1)(0 + 9)]$

12. True $[\forall_x \; x + (3 - x) = 3]$

13. False $[\forall_y \exists_x \; x + y \neq 3]$

[Note the difference between Exercises 12 and 13. Exercise 12 says that, for each first number there is a second number whose sum with the first is 3; Exercise 13 says that, there is a number such that, no matter what number you add it to, the sum is 3.]

14. True $[\forall_x \; x0 = 0]$

15. True $[\forall_x \; x0 = 0]$

[For a discussion of sentences like those in Exercises 12 through 15, see TC[5-34]b and c.]

*

Answers for Part C [on page 5-38].

[$P \cup Q$ and $P \cap Q$ are relations because the union and intersection of sets of ordered pairs are sets of ordered pairs.]

$$\mathcal{D}_{P \cup Q} = \mathcal{D}_P \cup \mathcal{D}_Q.$$

$\mathcal{D}_{P \cap Q} \subseteq \mathcal{D}_P \cap \mathcal{D}_Q$. However, $\mathcal{D}_{P \cap Q}$ may be a proper subset of $\mathcal{D}_P \cap \mathcal{D}_Q$. For example, consider the relations in Exercises 2 and 3 of Part A. $N \cap T$ is $\{(3, 4)\}$, so $\mathcal{D}_{N \cap T}$ is $\{3\}$. But, $\mathcal{D}_N \cap \mathcal{D}_T$ is $\{3, 4\}$. [Similar remarks apply to ranges and fields.]

*

It is interesting to note that if $\mathcal{D}_R \subseteq S$ and $\mathcal{R}_R \subseteq T$ then, although the domain and range of $\tilde{R}$ (with respect to $S \times T$) are subsets of S and T, respectively, nothing more can be said about $\mathcal{D}_{\tilde{R}}$ and $\mathcal{R}_{\tilde{R}}$ unless one has additional information about R. In particular, it is generally not the case that $\mathcal{D}_{\tilde{R}} = \widetilde{\mathcal{D}_R}$ or that $\mathcal{R}_{\tilde{R}} = \widetilde{\mathcal{R}_R}$.

Quiz.

Suppose that

R = {(2, 3), (3, 3), (7, 3), (−2, 3), (0, 3), (−8, 3)},

T = {(3, 0), (3, 5), (3, −8), (3, 7), (3, −10), (3, 2), (3, 3), (3, −2)},

S = {(0, 5, −8, 7, −10, 2, 3, −2},

U = {2, 3, 7, −2, 0, −8},

and V = {3}.

Complete each of the following to a true sentence with one of the letters 'R', 'T', 'S', 'U', or 'V'.

1. The domain of R is _____.
2. The domain of T is _____.
3. The range of R is _____.
4. The range of T is _____.
5. The field of R is _____.
6. The field of T is _____.
7. The domain of the converse of R is _____.
8. The range of the converse of R is _____.
9. The field of the converse of R is _____.
10. The domain of the converse of T is _____.
11. The range of the converse of T is _____.
12. The field of the converse of T is _____.

*

Answers for Quiz.

1. U	2. V	3. V	4. S	5. U	6. S
7. V	8. U	9. U	10. S	11. V	12. S

4. (a) $\{(2, 6), (9, 9), (3, 8)\}$

   (b) $\{(x, y): x = 3y\}$ [or: $\{(x, y): y = \frac{1}{3}x\}$] [Evoke both answers.]

   (c) $\{(x, y): x = y\}$

   (d) $\{(x, y): y = 3x\}$ [or: $\{(x, y): x = \frac{1}{3}y\}$]

   (e) $\{(x, y): y^2 + x^2 = 25\}$

   (f) $\{(x, y): y = 2\}$

   (g) the relation of being a parent of

   (h) the relation of being a cousin of

   (i) the relation < [or: $\{(x, y): y < x\}$]

5. $\mathfrak{D}_S = K$, $\mathfrak{R}_S = J$, $\mathfrak{F}_S = J \cup K$ [Ask the class 'What relation is the converse of S?' The answer is, of course, 'R'.]

6. The converse of a relation R is the relation whose members are obtained by reversing the order of the components of the members of R. [More briefly (and less confusingly): the converse of R is $\{(x, y) \in \mathfrak{R}_R \times \mathfrak{D}_R : y \, R \, x\}$.]

*

In some textbooks, the converse of a relation R is denoted by '$\breve{R}$'.

*

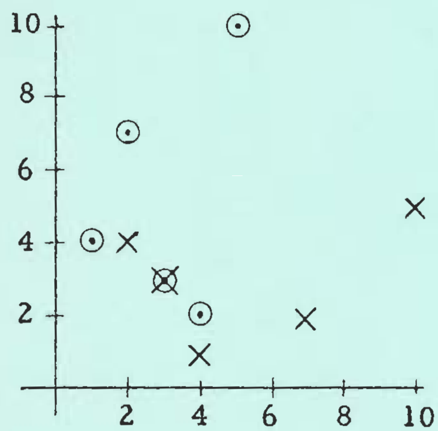
In speaking of the converse of a relation one is referring to the converse of a <u>set</u> of ordered pairs. The word 'converse' is also used in speaking of individual ordered pairs: for each a and b, the converse of (a, b) is (b, a). So, one may say that the converse of a relation R is the relation whose members are the converses of the members of R.

*

Correction. In Exercise 5, delete '(a)'.

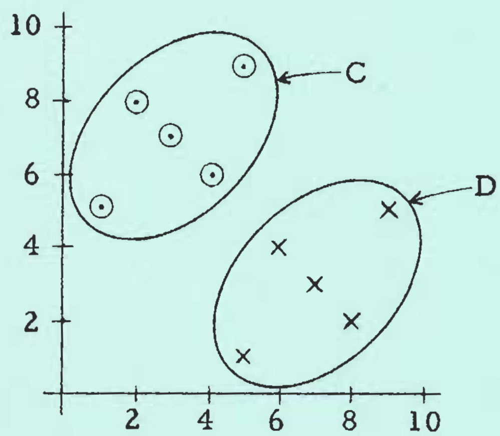
Answers for Part D.

1. (a)

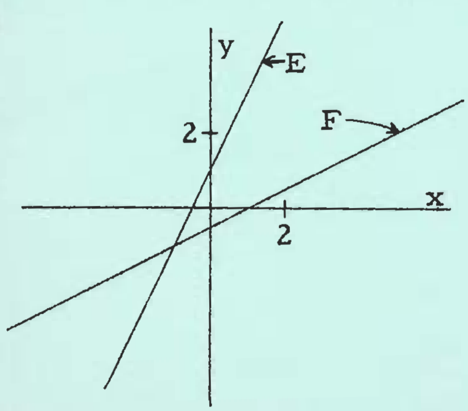


(b) The members of B are obtained by reversing the order of the components of the members of A.

2.



3.



E = {(x, y): y - 2x = 1}

F = {(x, y): x - 2y = 1}

Ask students for a brace-notation name for F.

D. 1. (a) Use the same chart to draw the graphs of the relations A and B where

$$A = \{(1, 4), (2, 7), (3, 3), (4, 2), (5, 10)\}$$

and $B = \{(4, 1), (3, 3), (7, 2), (10, 5), (2, 4)\}$.

Draw little circles around the dots of the graph of A, and draw little crosses through the dots of the graph of B.

(b) Do you see how the members of B are related to those of A?

2. (a) Draw the graph of C where

$$C = \{(1, 5), (3, 7), (4, 6), (2, 8), (5, 9)\}.$$

(b) On the same chart draw the graph of the relation D whose members are related to those of C in the same way that those of B are related to those of A.

3. (a) Draw the graph of E where $E = \{(x, y) : y - 2x = 1\}$.

(b) Draw the graph of the relation F whose members are the ordered pairs obtained by "reversing" the components of the ordered pairs in E.

4. The relation B of Exercise 1 is called *the converse of the relation A*. Also, D is the converse of C, and F is the converse of E. For each of the relations listed below, write a name for the relation which is its converse.

(a) $\{(6, 2), (9, 9), (8, 3)\}$ (b) $\{(x, y) : y = 3x\}$

(c) $\{(x, y) : x = y\}$ (d) $\{(x, y) : y = \frac{1}{3}x\}$

(e) $\{(x, y) : x^2 + y^2 = 25\}$ (f) $\{(x, y) : x = 2\}$

(g) the relation of being a child of

(h) the relation of being a cousin of (i) the relation >

5. (a) Suppose the relation S is the converse of the relation R. If $\mathcal{D}_R = J$ and $\mathcal{R}_R = K$, what are $\mathcal{D}_S$, $\mathcal{R}_S$, and $\mathfrak{F}_S$?

6. Give a definition of the converse of a relation. [*Hint.* The converse of R is $\{(x, y) \in$ ____ : ________________ $\}$.]

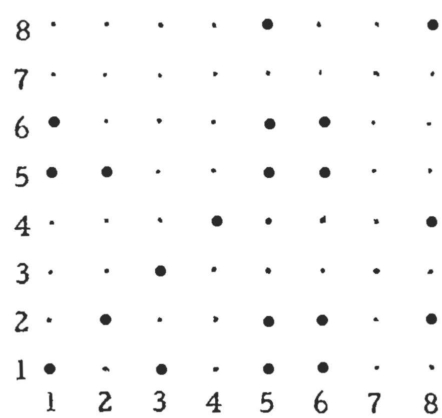
[Supplementary exercises are in Part F, page 5-243.]

REFLEXIVE RELATIONS

In an earlier section we noted that the relation of the inch-measure of one side to the inch-measure of another side of a triangle whose third side is 2 inches long is not a reflexive relation. And, the reason it is not reflexive is that it does not contain ordered pairs like (1, 1) and (0.5, 0.5). These are ordered pairs with equal components which are in the cartesian square of the field of the relation but are not in the relation itself.

A relation R is reflexive if and only if,
for each $x \in \mathfrak{F}_R$, $x \mathrel{R} x$ [that is, $(x, x) \in R$].

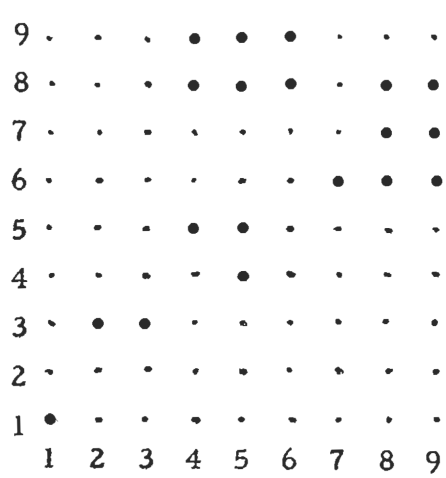
Here is a graph of a relation. Is it reflexive? Justify your answer.



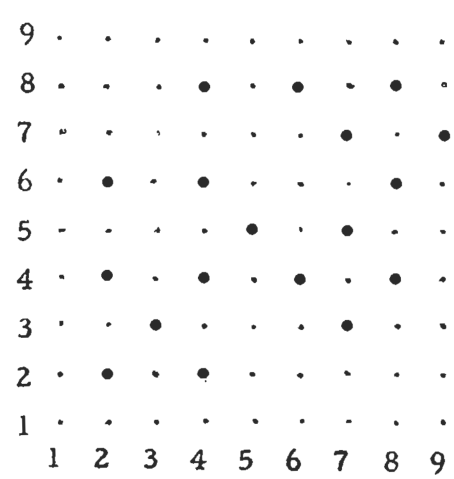
EXERCISES

A. Each exercise contains the graph of a relation. What additional ordered pairs must you include in the relation in order to obtain a reflexive one?

(1)



(2)



The remainder of this section deals with two of the more important properties which a relation may have. Besides furnishing a good background for the study of functions, it provides an opportunity to bring up interesting combinatorial problems. [Parts D and E on pages 5-43 and 5-44, and Part B on page 5-46].

*

The relation whose graph appears in the middle of page 5-40 is reflexive. Its field is $\{1, 2, 3, 4, 5, 6, 8\}$, and each of the ordered pairs (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), and (8, 8) belongs to the relation.

*

Answers for Part A.

The field of the relation whose graph is (1) is $\{x \in I: 1 \leq x \leq 9\}$. To obtain a reflexive relation one <u>must</u> adjoin to the given relation the pairs (2, 2), (4, 4), (6, 6), (7, 7), and (9, 9). [In addition to these one may adjoin others, for example, (2, 6), or (John, 7). But, if one does include, say, (John, 7) then the relation so obtained has John in its field and, to obtain a reflexive relation, one must then adjoin (John, John).]

The field of the relation whose graph is (2) is $\{x \in I: 2 \leq x \leq 9\}$. To obtain a reflexive relation one must adjoin to the given relation the pairs (6, 6) and (9, 9).

*

After doing Exercise 1, students may jump to the conclusion that a reflexive relation is one whose graph includes "the diagonal". That this is not necessarily the case is seen in Exercise 2, and in the chart in the middle of the page. The point (1, 1) does not have to belong to the relation in order that it be reflexive. It does only if 1 is in the field. Suppose you are preparing to graph a relation which is reflexive, and you decide to make a chart which will just accommodate the graph, that is, it will have no unnecessary columns or rows. The chart will then picture a cartesian square, and the relation is a subset of this square. Furthermore, if the rows are named "going up" in the same order that the columns are named "going to the right" <u>then</u> the graph of the relation will contain all the diagonal points. Moreover, any relation which meets these conditions is a reflexive one.

Exercise 15: Each ordered pair of real numbers with equal components satisfies the set selector because $0^2 + 0^4 = 0$. [Strange as it may look, the relation in Exercise 15 is the same as that in Exercise 3. Since squares and fourth powers of real numbers are nonnegative and since the sum is 0, each addend must be 0.]

*

In later exercises [for example, those in Part D on page 5-43] questions may arise concerning the empty set. For example; Is $\emptyset$ a relation? If one recalls that each set all of whose members are ordered pairs is a relation, he sees that the answer to this question is 'yes'. For, one who claims that $\emptyset$ is <u>not</u> a relation must be prepared to exhibit a member of $\emptyset$ which is not an ordered pair. Since $\emptyset$ has no members, it is impossible that he should be able to do this. Note that

$$\mathfrak{D}_\emptyset = \{x: \exists_y\ (x, y) \in \emptyset\} = \emptyset \quad \text{and} \quad \mathfrak{R}_\emptyset = \{x: \exists_y\ (y, x) \in \emptyset\} = \emptyset.$$

Also, $\emptyset$ is reflexive. For, $\mathfrak{D}_\emptyset = \emptyset$ and $\forall_{x \in \emptyset}\ (x, x) \in \emptyset$. [Here, again, one who claims that $\emptyset$ is not reflexive faces the impossible task of exhibiting a member of $\emptyset$--this time, an $x \in \emptyset$ such that $(x, x) \notin \emptyset$.]

Finally, $\emptyset$ is [see page 5-45] symmetric. For, there is no ordered pair (x, y) in $\emptyset$ such that $(y, x) \notin \emptyset$.

Answers for Part B.

1. (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) 2. $\mathfrak{R}_R = \{3, 4, 5\}$

3. If R is a reflexive relation then $\mathfrak{D}_R = \mathfrak{R}_R$. [If R is reflexive and $x \in \mathfrak{F}_R$ then, since x R x, $x \in \mathfrak{D}_R$. So, $\mathfrak{F}_R \subseteq \mathfrak{D}_R$. Since, in any case, $\mathfrak{D}_R \subseteq \mathfrak{F}_R$, it follows that if R is reflexive then $\mathfrak{D}_R = \mathfrak{F}_R$. Similarly, if R is reflexive then $\mathfrak{R}_R = \mathfrak{F}_R$.]

4. Yes. [A relation and its converse have the same field.]

*

Answers for Part C.

The relations in Exercises 1, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, and 15 are reflexive; the others are not.

Exercises 3, 4, 5, 6: Draw the graphs.

Exercise 7: Since $1 \in I^+$, each member of I^+ is a factor of itself. [Suppose G is the set of positive integers greater than 1. Is $\{(x, y) \in G \times G\colon$ y is a factor of x with respect to G$\}$ a reflexive relation? Answer: No. Note that the domain of this relation is the set of composite positive integers.]

Exercise 8: The members of this relation are all the ordered pairs for which the difference of the first component from the second is an integral multiple of 5. For each k in the field of this relation, (k, k) belongs to the relation because k - k = 0 and 0 is an integral multiple of 5. [It may help students if they first attempt to graph this relation.]

Exercise 9: Each person has the same parents as himself.

Exercise 12: One is not one's own sister.

Exercise 13: Notice that to name an ordered pair belonging to this relation one would write, for example:

$$(\text{'}3x + 5 = 4 - 7x + 1\text{'}, \text{'}z^2 + 6 = 8 - 2\text{'})$$

That is, one must use <u>names</u> of the components in naming the pair.

B. 1. Suppose R is a reflexive relation and $\mathfrak{J}_R$ = {1, 2, 3, 4, 5}. What ordered pairs are you sure belong to R?

2. Suppose R is a reflexive relation and $\mathfrak{D}_R$ = {3, 4, 5}. Can you tell what $\mathfrak{R}_R$ is?

3. If you know that a relation is reflexive, what can you say about its domain and range?

4. Is the converse of a reflexive relation reflexive?

C. Which of these relations are reflexive?

1. $\{(3, 7), (8, 2), (8, 8), (3, 3), (2, 8), (2, 2), (7, 7)\}$

2. $\{(4, 1), (1, 1), (6, 4), (6, 6)\}$

3. $\{(x, y): x = y\}$

4. $\{(x, y): y \leq x\}$

5. $\{(x, y): |x| \leq 5 \text{ and } |y| \leq 5\}$

6. $\{(x, y): x^2 + y^2 \leq 25\}$

7. $\{(x, y) \in I^+ \times I^+$: y is a factor of x with respect to $I^+\}$

8. $\{(x, y) \in I \times I: \exists_{q \in I}\ y - x = 5q\}$

9. $\{(x, y) \in P \times P$: y has the same parents as x}, where P is the set of all people.

10. $\{(x, y) \in T \times T$: y has the same perimeter as x}, where T is the set of all triangles.

11. $\{(x, y) \in T \times T$: y has the same shape as x}

12. $\{(x, y) \in P \times P$: y is a sister of x}

13. $\{(x, y) \in Q \times Q$: y has the same roots as x}, where Q is the set of all equations in one variable [equations like '3p - 5 = 2 - 7p' but not like '3a + 6b - 7 = 8a'].

14. $\{(x, y) \in P \times P$: y has the same uncles as x}

15. $\{(x, y): (x - y)^2 + (y - x)^4 = 0\}$

* * *

Given a set S which contains 3 elements, how many subsets does S have? One way of answering this question is just to list the subsets of S and count them. For example:

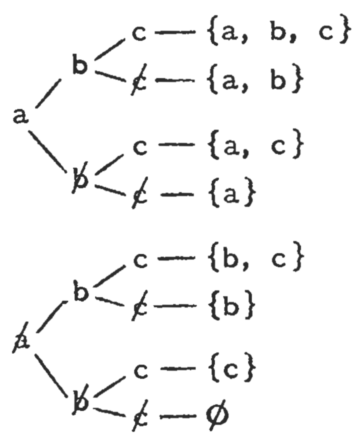
if $$S = \{a, b, c\}$$

then the subsets of S are

Subsets	Count
$\{a, b, c\}$	1
$\{b, c\}$, $\{c, a\}$, $\{a, b\}$	3
$\{a\}$, $\{b\}$, $\{c\}$,	3
and $\emptyset$.	1
	8 ⟵ Total

But, this would be a tedious method if you wanted to find the number of subsets of a set containing, say, 25 elements.

Let's use another method which is easily generalized. Choosing a subset of S amounts to making a sequence of choices, one for each element of S. One decides for each element if it is to be included in the subset or not. There are two possible outcomes of the first choice. Then, for each of these, there are two possible outcomes of the second choice, etc. For our set S, here is a diagram of the procedure:



Each of the first three columns corresponds to a choice; for example, 'a' indicates that a is chosen, 'a̸' that a is rejected. The fourth column lists the subsets obtained by various sequences of choices. Notice that the first column has 2 entries, the second column has 2×2 entries, and the third column has $(2 \times 2) \times 2$ entries. Each column of entries after the first has twice as many entries as the preceding column. So, the third column has 2^3 entries; hence, a set with 3 elements has 2^3 subsets.

So, all together, there are $2^{12} + 4 \cdot 2^6 + 6 \cdot 2^2 + 4 \cdot 2^0 + 2^0$, or 4381, reflexive relations among the members of a given set of four elements.]

Students may be surprised that only about one fifteenth of the relations among a given set of four members are reflexive.

[You may recognize the numbers 1, 4, 6, 4, 1 as the successive coefficients in the expansion of, say, '$(a + b)^4$'. They are sometimes designated by '$\binom{4}{0}$', '$\binom{4}{1}$', '$\binom{4}{2}$', '$\binom{4}{3}$', and '$\binom{4}{4}$', and are called 'binomial coefficients'. So, the number of reflexive relations among the members of a given set of 4 elements is

$$\binom{4}{0}2^{0^2 - 0} + \binom{4}{1}2^{1^2 - 1} + \binom{4}{2}2^{2^2 - 2} + \binom{4}{3}2^{3^2 - 3} + \binom{4}{4}2^{4^2 - 4}$$

or, for short,

$$\sum_{k=0}^{4}\binom{4}{k}2^{k^2 - k} \quad [\text{'}\textstyle\sum\text{' for 'sum'}].$$

For each n, the number of reflexive relations among the members of a given set of n elements is

$$\sum_{k=0}^{n}\binom{n}{k}2^{k^2 - k}.]$$

☆ 7. 50,625 [The second components of those members of a relation R which have a given member of $\mathfrak{D}_R$ as first component form a nonempty subset of $\mathfrak{R}_R$. If $\mathfrak{R}_R$ is to be a subset of a given 4-member set, then there are $2^4 - 1$ nonempty subsets to choose from. If the domain is a given 4-member set then we must make four such choices in succession. So, the number of relations whose domain is a given 4-member set and whose range is a subset of this set is $(2^4 - 1)^4$. In fact, for each m, for each n, the number of relations whose domain is a given n-member set and whose range is a subset of a given m-member set is $(2^m - 1)^n$.]

> Correction. In the third line of Exercise ☆6, change 'Exercise 4' to 'Exercise 5'.

Answers for Part D.

1. 16; 32; 2^{25} [or: 33, 554, 432]

2. 2^{16} [Answer for Hint: 16]

[For each n there are 2^{n^2} relations among the members of a given set of n elements.]

3. 2^{14}

4. $4 \le n(R) \le 16$

5. 2^{12} [Both the domain and the range of such a relation is a set of four elements. Each relation with this domain and range is a subset of a 16-member cartesian square. If such a relation is reflexive then it is the union of two sets, a first set consisting of the 4 "diagonal" ordered pairs, and a second set whose members (if any) are among the remaining 12 pairs. So, the question in Exercise 5 amounts to asking how many subsets a set of 12 elements has.]

[For each n, there are $2^{n^2 - n}$ reflexive relations whose field is a given set of n elements.]

☆6. There are 4381 reflexive relations among the members of a given set of four elements. [We begin by classifying the relations in question according to their fields. Then, compute the number of reflexive relations which have a given field, and finally, add the results. There are as many possible fields as there are subsets of the given set of 4 elements. These subsets are (1) the set itself, (2) 4 subsets each of which has 3 members, (3) 6 subsets each of which has 2 members, (4) 4 subsets each of which has 1 member, and (5) the empty set. We have seen in Exercise 5 that there are

$2^{4^2 - 4}$ reflexive relations which have a given 4-element set as field,

$2^{3^2 - 3}$ reflexive relations which have a given 3-element set as field,

$2^{2^2 - 2}$ reflexive relations which have a given 2-element set as field,

$2^{1^2 - 1}$ reflexive relations which have a given 1-element set as field,

and $2^{0^2 - 0}$ reflexive relations which have $\emptyset$ as field.

* * *

D. 1. How many subsets has a set of four elements? A set of five elements? A set of 25 elements?

2. How many relations are there among the members of a set of four elements? [Hint. How many ordered pairs belong to the cartesian square of a set of four elements?]

3. How many relations among the members of {1, 2, 3, 4} contain both the ordered pairs (2, 4) and (4, 1)?

4. If R is a reflexive relation whose field consists of four elements, what can you say about the number of elements in R?

5. How many reflexive relations are there whose field is a given set of four elements?

☆ 6. How many reflexive relations are there among the members of a given set of four elements? [Hint. Besides those you have counted in Exercise 4, you must take account of reflexive relations whose fields are 3-membered subsets of the given set, 2-membered subsets, etc.]

☆ 7. How many relations among the members of a given set of four elements have this set as their domain?

E. If you solved Exercise 6 of Part D, you found that a 4-membered set has

	1	0-membered subset [$\emptyset$],
	4	1-membered subsets,
	6	2-membered subsets,
	4	3-membered subsets,
and	1	4-membered subset [the set itself].

Perhaps you did this by considering a particular 4-membered set, say {a, b, c, d}, listing its subsets, and counting them. Just as there is an easy method for finding the total number of subsets of a given set, there are also easy methods for finding the number of those subsets [of a given set] which have a given number of elements.

For example, there are easy methods for finding the number of 6-membered subsets of a 10-membered set, the number of 13-membered subsets of an 18-membered set, etc. You will learn one such method in the exercises which follow.

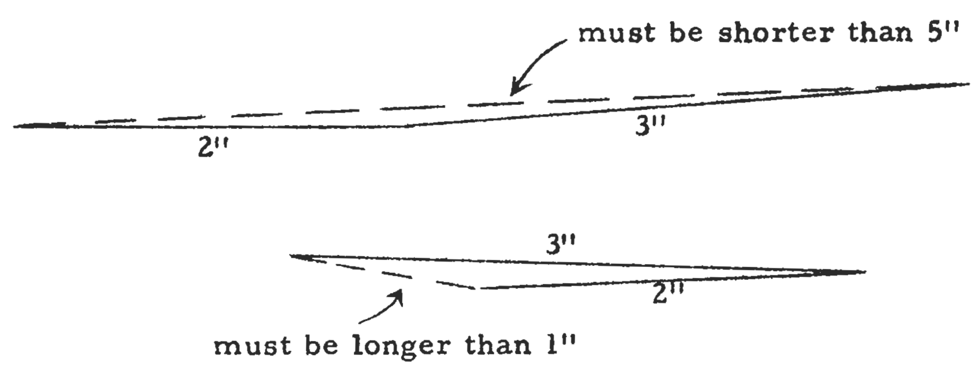
1. Complete the rows labeled '2', '3', '5', and '6' in the table below. [The row labeled '4' lists, for each whole number k, the number of k-membered subsets of a 4-membered set.]

n \ k	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0
2									
3									
4	1	4	6	4	1	0	0	0	0
5									
6				20					
7									
8									

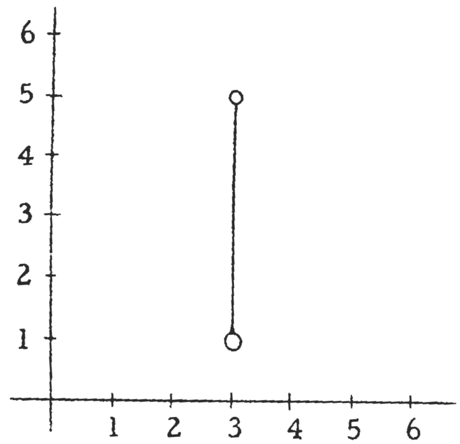
2. Study the table carefully. Do you see a quick way of getting the numbers listed in, say, the 5-row, from the numbers listed in the 4-row? Can you get those in the 4-row from those in the 3-row? Those in the 6-row from those in the 5-row? Use this quick way, if you find it, to fill the 7-row and the 8-row. [Clearly, the table can be extended indefinitely. If you disregard the '0'-entries in such a table, the remaining entries form a triangular array called Pascal's Triangle. Almost every encyclopedia gives some account of the fascinating life of the seventeenth century French mathematician Blaise Pascal.]

pair. That is, pick an ordered pair, say, (4, 7), and see if you can draw a triangle whose sides have inch-measures 4, 7, and 2, going <u>counterclockwise</u>. If you can, (4, 7) belongs to the relation. If you can't, you know that (4, 7) doesn't belong. Now, obviously, you can't test all of the ordered pairs since there are infinitely many of them. Also, some of them have such large components that it wouldn't be practical to test them by making a drawing. So, we shall have to be clever about this, use our imagination a great deal, search for patterns, and perhaps make just a few drawings to test some crucial hypotheses.

Let's start by deciding to describe the relation by making a graph of it. Since the relation is a subset of the cartesian square of the set of nonzero numbers of arithmetic, we shall graph the relation on a picture of the first quadrant of the number plane. [Recall that the positive real numbers behave just like the nonzero numbers of arithmetic. So, instead of an ordered pair of nonzero numbers of arithmetic we can think of the corresponding ordered pair of positive real numbers.] Now, what points shall we plot? Consider ordered pairs with first component 3. That is, consider triangles whose first and third sides have inch-measures 3 and 2. What are the possible inch-measures for the other side? Here are pictures showing two rather extreme positions for the 3-inch side.

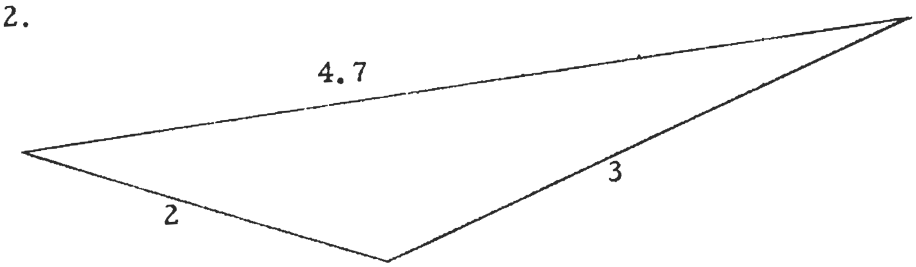


It seems clear that the inch-measure of the second side cannot exceed 5 or be less than 1. In fact, it can't be as large as 5 or as small as 1. [Why?] The second side must have an inch-measure between 1 and 5, and it seems reasonable that there are triangles with sides whose inch-measures are 2, 3, and any number between 1 and 5. So, a subset of the relation we are looking for is the interval whose end points are (3, 1) and (3, 5). Let's graph this interval.

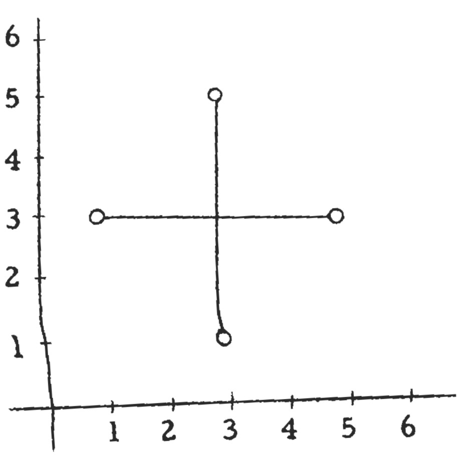


[The hollow dots mean that the corresponding points are not in the relation.]

Now, let's use a bit more imagination to get additional points of the relation. Notice that (3, 4.7) belongs to the relation. Does (4.7, 3) also belong? Here is a picture of a triangle whose sides measure 3, 4.7, and 2.



Can you draw a triangle whose sides measure 4.7, 3, and 2? Do so. [You can hold an edge of a mirror next to the picture of the triangle shown above and look at its reflection. The reflection is a triangle whose sides measure 4.7, 3, and 2.] So, for each point of the interval graphed above, the point with components in the opposite order also belong to the relation. This gives us more points to graph.



After establishing that the relation in question is symmetric, we next try to find out if the relation is reflexive. This one is not because it does not contain pairs such as (1, 1), (0.5, 0.5), and (0.28, 0.28). The properties of symmetry and reflexivity are discussed in greater detail in section 5.04. The reference to them on page 5-27 is just an appetite whetter.

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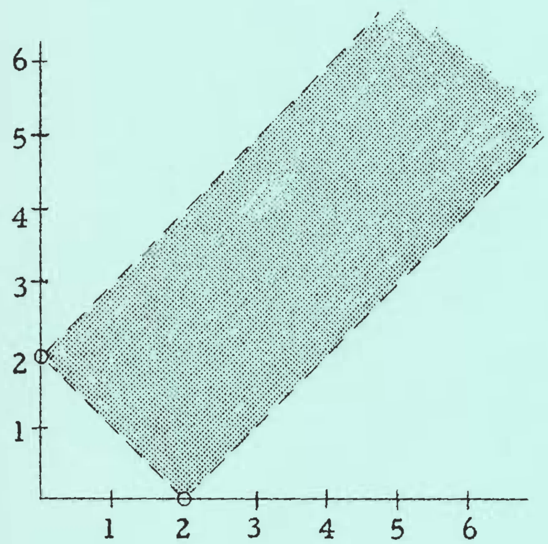
The completed graph should look like this:



[Compare this with the answer for the exercise in Part B on page 5-12.]

*

Answers for Part A [on pages 5-28 and 5-29].

1. [The graph is like the one given above, but with the scale changed so that the corners are at (0, 5) and (5, 0).]

2. $\{(x, y): |x - 5| < y < x + 5\}$

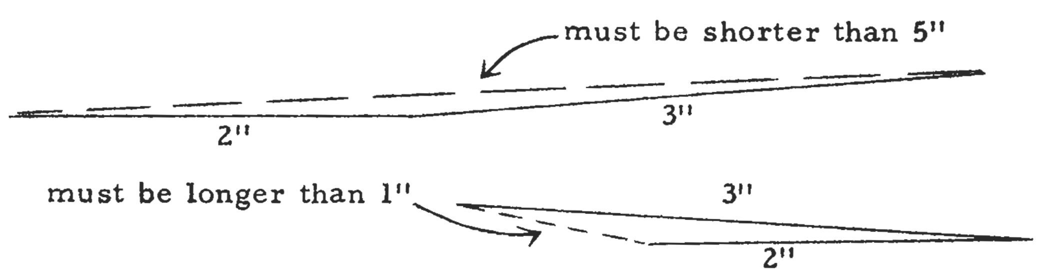
We have discovered an important property of this relation, the property of being symmetric. This is a useful discovery because it helps us find more points in the relation. For example, suppose we know that (101, 100) belongs to the relation. From this, and the fact that the relation is symmetric, we can deduce that what other point belongs to the relation?

There are some points in the relation for which the property of symmetry gives us no additional information. What points are these? What special name do you have for triangles whose side-measures are the components of such points? Is (75, 75) such a point? How about (5, 5)? (2, 2)? Does (0.5, 0.5) belong to the relation? What about (1, 1)? Add more points to the picture on the preceding page by graphing those points in the relation which have equal components. [What special name do we give to triangles with a side of measure 2 whose other side-measures are the components of (2, 2)?]

[In the next section you will learn that a relation is said to be reflexive if and only if it contains all those ordered pairs with equal components which belong to the smallest cartesian square of which the relation is a subset. Although the relation we are now investigating is symmetric, it is not reflexive. Tell why.]

You should now complete the graph of the relation.

Although you have a graph of this relation, it may also be helpful to have a brace-notation name for it. To get such a name, we need a sentence containing two variables, say 'x' and 'y', whose solution set is the relation. Recall the situation discussed on page 5-25. There you discovered that if y is the inch-measure of one side of a triangle

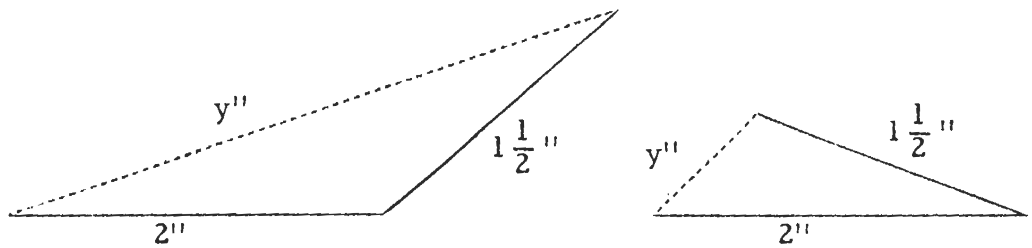


whose other sides are 2 inches and 3 inches long, then

$$y < 3 + 2 \quad \text{and} \quad y > 3 - 2.$$

It also seemed reasonable that for any number which satisfied these inequations, there is such a triangle.

Now, consider this situation:



Here we see that a number can be the inch-measure of a side of a triangle whose other sides are 2 inches and $1\frac{1}{2}$ inches long if and only if it satisfies the sentence:

$$y < 1\frac{1}{2} + 2 \quad \text{and} \quad y > 2 - 1\frac{1}{2}$$

So, in general, y and x can be inch-measures of two sides of a triangle whose third side is 2 inches long if and only if

$$(*) \quad y < x + 2 \quad \text{and} \quad \begin{cases} \text{either } (x \geq 2 \text{ and } y > x - 2) \\ \text{or } (x < 2 \text{ and } y > 2 - x). \end{cases}$$

We can simplify this considerably by noting that

$$\text{if } x \geq 2 \text{ then } x - 2 = {}^{+}|x - 2|$$

and

$$\text{if } x < 2 \text{ then } 2 - x = {}^{+}|x - 2|.$$

Since either $x \geq 2$ or $x < 2$, (*) boils down to:

$$y < x + 2 \text{ and } y > {}^{+}|x - 2|,$$

or, for short:

$${}^{+}|x - 2| < y < x + 2$$

Thus, the relation we have been discussing is

$$\{(x, y): \; |x - 2| < y < x + 2\}.$$

EXERCISES

A. 1. Make a quick sketch of a graph of the relation of the inch-measure of one side of a triangle to the inch-measure of another if the third side is 5 inches long.

2. Write a brace-notation name for this relation.

Answers for Part D [on page 5-29].

The maximum distance obtainable between P and Q is 10, and this occurs when just P, R, and Q are collinear. [In that case, the figure is a triangle, not a quadrilateral.] The minimum distance obtainable between P and Q is 2. [In that case, Q, P, R, and S are collinear.] Thus, the measure of the segment $\overleftrightarrow{PQ}$ can be d if and only if $2 \leq d \leq 10$. [Challenge students to answer these questions when the measure of $\overleftrightarrow{PR}$ is, say, 3.]

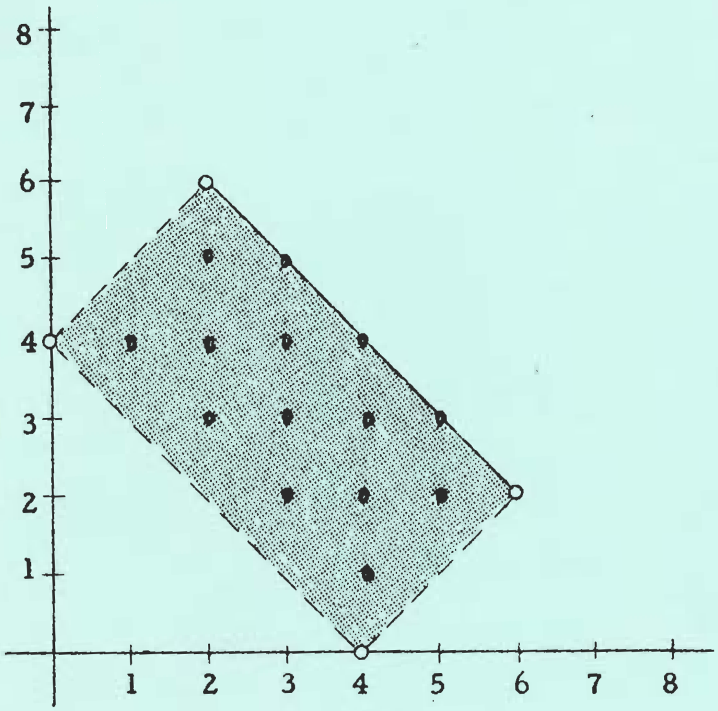
*

Answers for Part E [on page 5-29].

1. No; $2 + 5 \not> 8$	2. No; $9 + 11 \not> 20$	3. No; $8 + 8 \not> 16$
4. Yes	5. Yes	6. Yes
7. No; $1 + 2 \not> 3$	8. Yes	9. No; $6 + 4 \not> 12$

Answers for Part ☆C [on page 5-29].

1.



2. The graphs of points corresponding with the triangles which have inch-perimeter not exceeding 12, one side of inch-measure 4, and the other sides having whole numbers as inch-measures are shown on the figure for Exercise 1. However, not all such triangles are differently-shaped. Those which have the same set of side-measures are considered to have the same shape. Hence, there are just eight triangles which meet the conditions of Exercise 2. They are the triangles whose side-measures are the components of the ordered triples (3, 2, 4), (3, 3, 4), (4, 1, 4), (4, 2, 4), (4, 3, 4), (4, 4, 4), (5, 2, 4), and (5, 3, 4). Notice that symmetry helps in eliminating duplications.

A brace-notation name for the relation referred to in Exercise 1 is:

$$\{(x, y): |x - y| < 4 < x + y \leq 12 - 4\}$$

*

3. (b) $2 < n < 12$

(c) $14 < q < 18$ [or: $|16 - q| < 2 < 16 + q$]

(d) $40 < r < 80$ [or: $|20 - r| < 60 < 20 + r$]

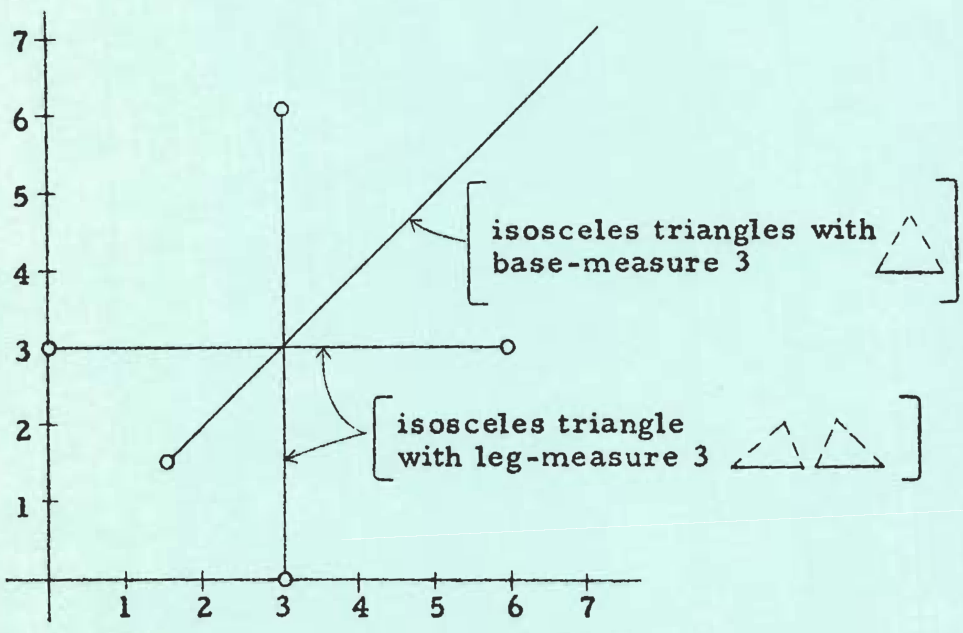
(e) $|a - b| < 11 < a + b$ [or: $|11 - b| < a < 11 + b$]

(f) $|x - z| < y < x + z$

[There are, of course, at least three correct answers for each part of Exercise 3.]

*

Answer for Part ☆B.



Note that this relation is a subset of the relation of the measure of one side to the measure of a second side of a triangle whose third side measures 3. Clearly, the set of all isosceles triangles each of which has a side of measure 3 is a subset of the set of all triangles each of which has a side of measure 3. Note also that the intersection of the two intervals and the half-line contains the point which corresponds with the equilateral triangle of side-measure 3.

*

3. The inch-measures of the sides of some triangle are

   (a) 6, m, and 2 if and only if $4 < m < 8$,

   (b) 7, n, and 5 if and only if ______,

   (c) 16, 2, and q if and only if ______,

   (d) 20, 60, and r if and only if ______,

   (e) 11, a, and b if and only if ______,

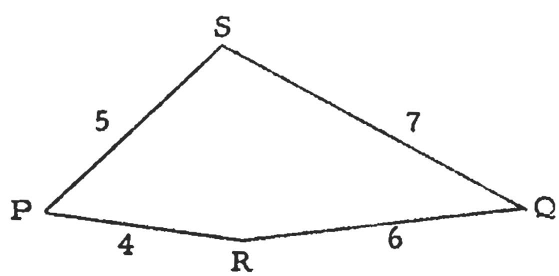
   (f) x, y, and z if and only if ______.

☆ B. Draw a graph of the relation of the inch-measure of one side to the inch-measure of another side of an isosceles triangle whose third side is 3 inches long.

☆ C. 1. Draw a graph of the relation of the inch-measure of one side to the inch-measure of another side of a triangle whose third side is 4 inches long and whose inch-perimeter does not exceed 12.

2. How many such differently-shaped triangles are there whose sides have whole numbers for inch-measures?

D. Four sticks are fastened at their ends to form a quadrilateral as shown.



The quadrilateral is not fixed. That is, it is possible to change its shape by moving, for example, P towards Q, without bending the sticks. How far apart can you move P from Q? How close together can you bring P and Q?

E. For each ordered triple listed below, tell whether its components can be inch-measures of the sides of a triangle.

1. (2, 5, 8)	2. (9, 11, 20)	3. (8, 8, 16)
4. (16, 16, 8)	5. (3, 3, 3)	6. (3, 4, 5)
7. (1, 2, 3)	8. (10, 20, 29)	9. (6, 12, 4)

F. Here is another question dealing with a relation between measures of parts of a geometric figure.

> What is the relation of the degree-measure of one angle to the degree-measure of another angle of a triangle whose third angle is an angle of 50°?

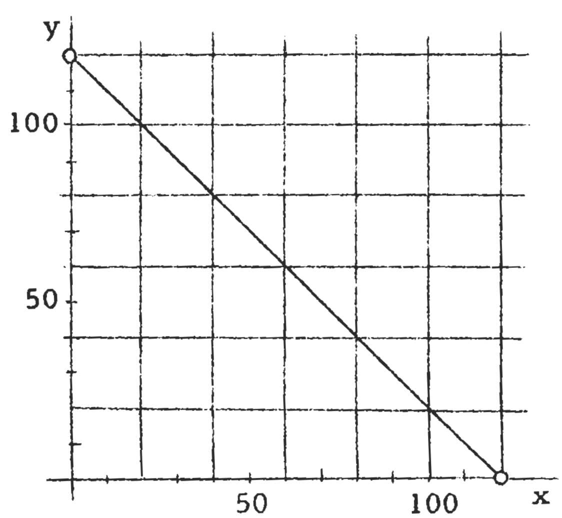
1. Draw at least six triangles each with an angle of 50°, find the degree-measure of each of the other two angles, and plot the corresponding ordered pairs.

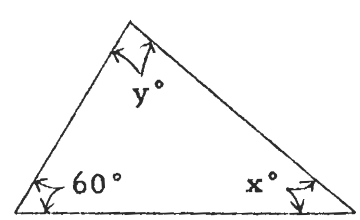
2. Are the six ordered pairs you plotted in Exercise 1 enough to show you the pattern for the rest of the relation? If not, draw more triangles with a 50° angle, measure each of the other two angles, and plot ordered pairs until you do see the pattern of the relation. Is the relation symmetric?

3. Complete the following to get a brace-notation name for the relation of the degree-measure of one angle to the degree-measure of another angle of a triangle whose third angle is an angle of 50°.

$$\{(x, y): \ x > 0, \ y > 0, \ \text{and } x + y = \quad \}.$$

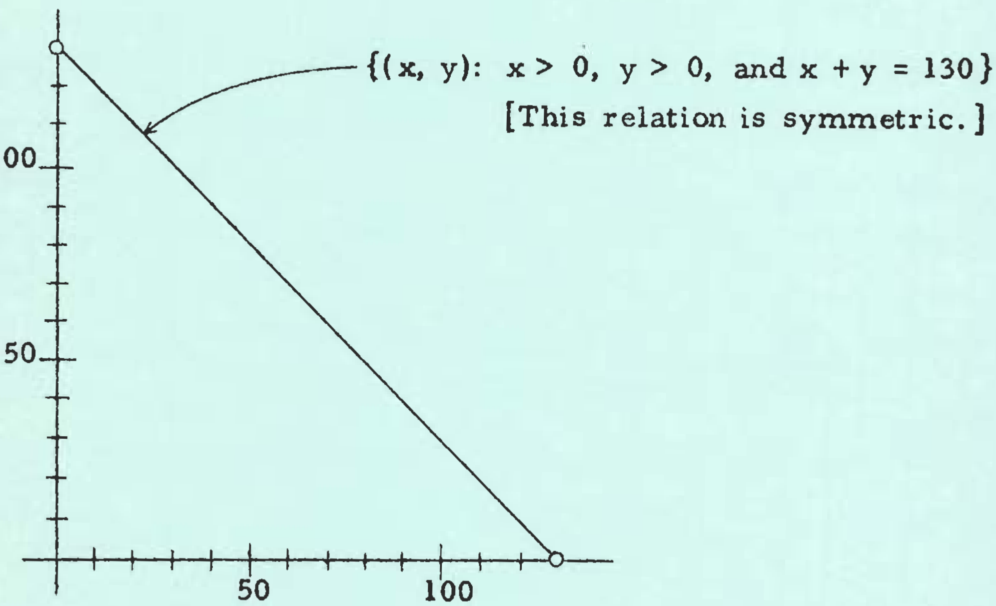
G. Here is a graph of the relation between the degree-measure of one angle to the degree-measure of another angle of a triangle whose third angle is an angle of 60°.





Answers for Part F.

[Students are supposed to discover that the sum of the degree-measures of the other two angles is 130. The fact that students will get the same relation regardless of the side-lengths shows that the sum of the angle-measures is independent of the lengths of the sides.]



Answers for Part G [which begins on page 5-30].

1. $\{(x, y): x > 0, y > 0, \text{ and } x + y = 120\}$

2. (a) 100° (b) 20° (c) 90° (d) 10°
   (e) 60° (f) 81° (g) 80° (h) not a triangle

3. Students should sketch the interval whose end points are (0, 100) and (100, 0). A brace-notation name for this relation is: $\{(x, y): x > 0, y > 0, \text{ and } x + y = 100\}$

4. (a) (50, 50), (80, 20), (20, 80); (60, 60) [Students are supposed to be aware of the fact that two angles of a triangle have the same measure if and only if the triangle is isosceles. [Avoid saying 'equal angles'.] The fact that there is only one point of the 60°-relation which corresponds to an isosceles triangle shows that if one angle of an isosceles triangle is an angle of 60° then so are all of them.]

   (b) (60, 60) [This illustrates the fact that a triangle is equilateral if and only if each of its angles is an angle of 60°.]

5. (a) $(33\frac{1}{3}, 66\frac{2}{3})$, $(66\frac{2}{3}, 33\frac{1}{3})$, (40, 60), (60, 40);

   $(33\frac{1}{3}, 66\frac{2}{3}, 80)$, (40, 60, 80)

   (b) (40, 80), (80, 40), (30, 90), (90, 30);

   (40, 60, 80), (30, 60, 90)

6. (a) 30° (b) 60° (c) 28° (d) 29°

1. Write a brace-notation name for this relation.

2. Use the graph or the name of the relation to complete the following table. [∠A, ∠B, and ∠C are the three angles of a triangle.]

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
∠A	60°	60°	60°			60°	60°	60°
∠B	20°	100°		60°	60°	39°	40°	120°
∠C			30°	110°	60°			

3. Sketch on the chart on page 5-30 a graph of the relation of the degree-measure of one angle to the degree-measure of another angle of a triangle whose third angle is one of 80°. Write its name.

4. (a) Which points of the 80°-relation and of the 60°-relation correspond with isosceles triangles?

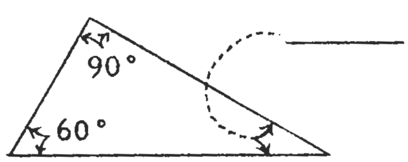
   (b) Which points of the 60°-relation and of the 80°-relation correspond with equilateral triangles?

5. (a) Which points of the 80°-relation correspond with triangles in which the degree-measure of one angle is twice the degree-measure of another? What are the degree-measures of the three angles of such triangles?

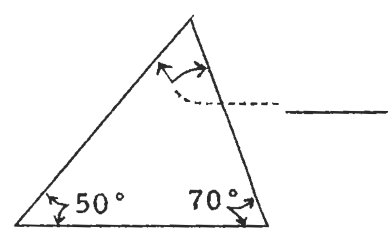
   (b) Repeat for the 60°-relation.

6. Predict the missing angle-measures, and use a protractor to check your predictions.

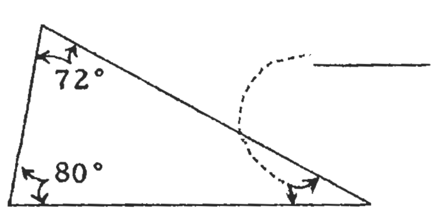
(a)



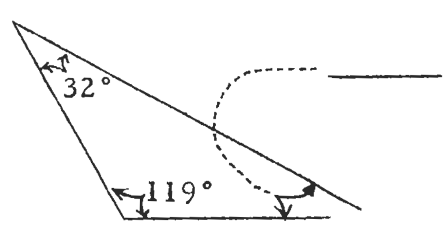
(b)



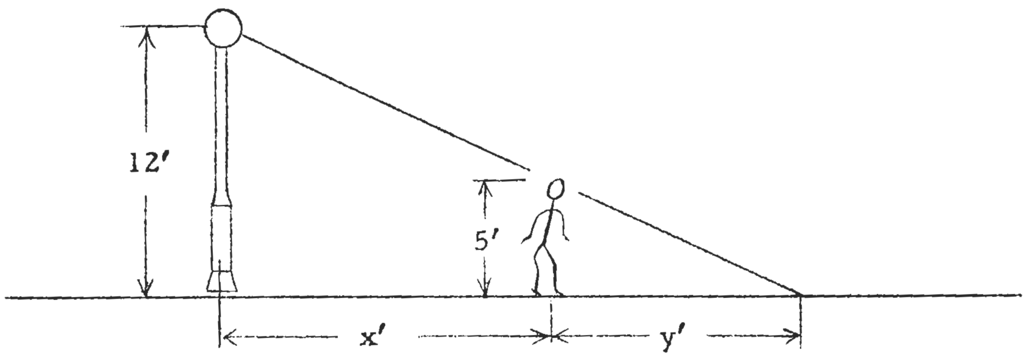
(c)



(d)



☆H. You may have had the experience of walking away from a lamppost and watching your shadow lengthen as you get farther from the post.

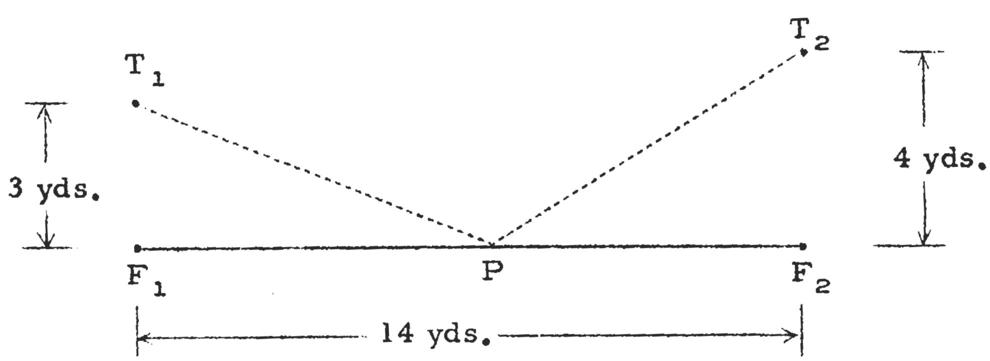


1. Make a graph of the relation of the foot-measure of the shadow of a 5-foot walker to his foot-distance from the 12-foot lamppost.

2. How far is the walker from the post when his shadow is as long as he is tall?

3. How far is the walker from the post when his shadow is twice as long as he is tall?

☆I. Here is a map showing a fence and two trees. If you walk from tree



T_1 to a point P on the fence and from there to tree T_2, the distance walked is the sum of the distances from T_1 to P and from P to T_2.

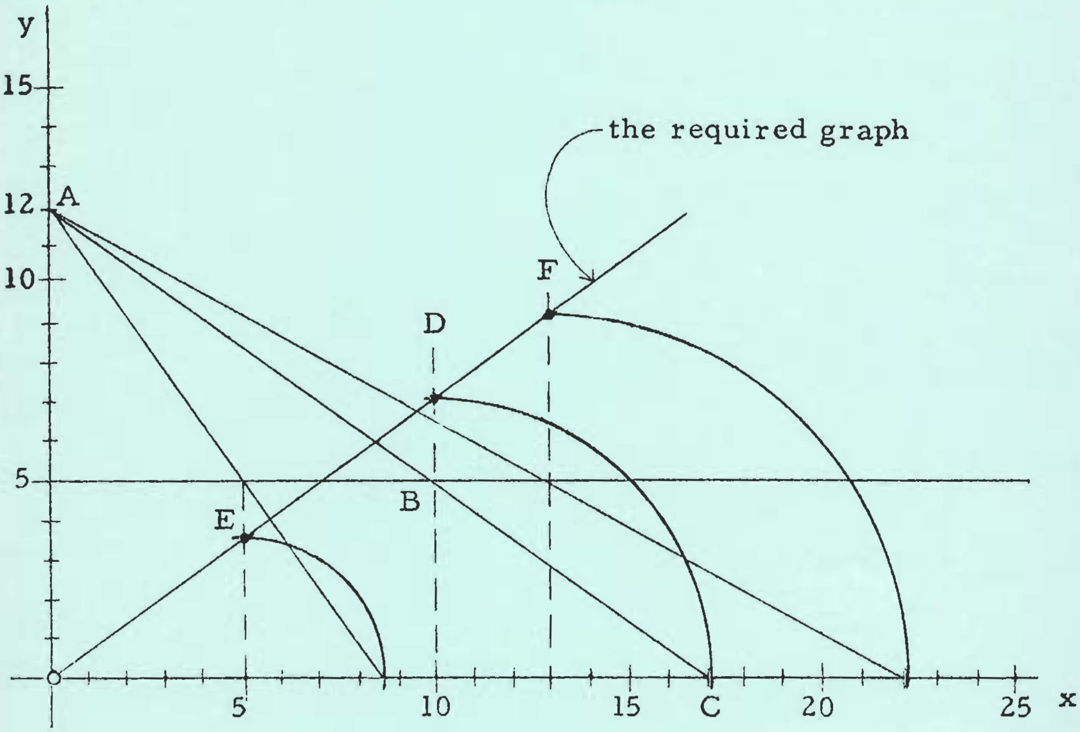
1. Make a graph of the relation of the distance walked to the distance between F_1 and P.

2. Use the graph to tell that location of P for which the distance walked is the smallest.

[Supplementary exercises are in Part D, pages 5-241 through 5-242.]

Answers for Part ☆H.

1. Students may find it helpful to develop a graphical technique for finding points which belong to this relation, a technique which does not require them to make measurements in order to find the components of the points. One such technique is shown in the diagram. [Of course, students should use cross-section paper.]



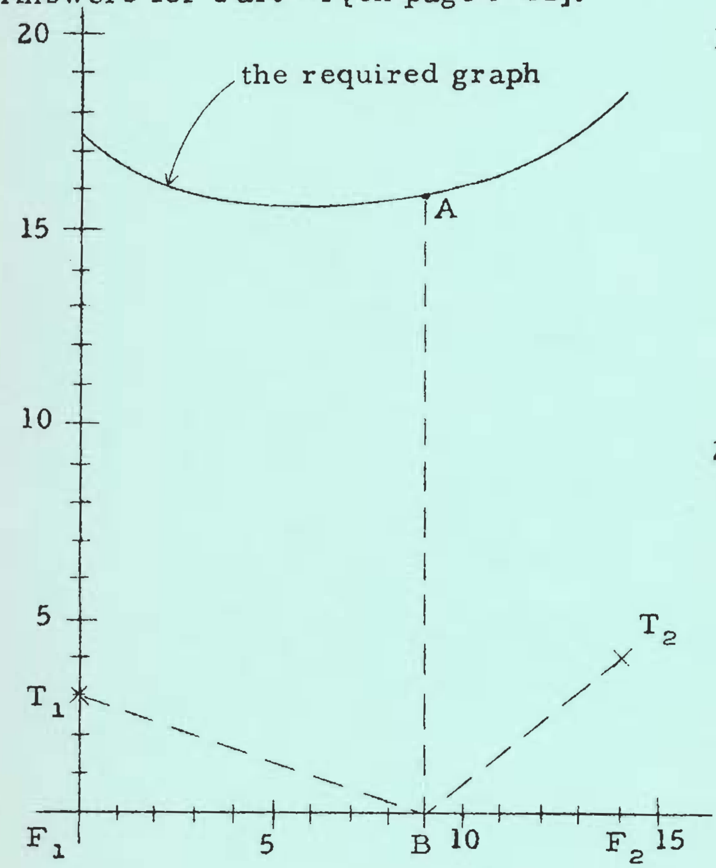
Here is how point D of the graph was located. First, the ray $\overrightarrow{AB}$ was drawn intersecting the x-axis in the point C. [Note that A is 12 units above the horizontal and B is 5 units above the horizontal.] The abscissa of B is the distance between the post and the walker. So, D is a point on the vertical line through B. The point C is the far end of the shadow. So, the distance between the projection of B on the x-axis and point C is the measure [illegible] the shadow, that is, it is the ordinate of D. Use a compass to fi[illegible] point D. [The diagram also shows how points E and F were located.]

[Students who have done some work with similar triangles may recognise that, for each (x, y) which satisfies the conditions of the problem, $y/(x + y) = 5/12$, that is, $y = 5x/7$. So, the relation in question is $\{(x, y): x > 0 \text{ and } y = 5x/7\}$.]

2. This question, in effect, asks for the first components of that point in the relation whose second component is 5. That is the point (7, 5). [Actually, geometric intuition alone will tell most students that the far end of the 5-foot shadow ought to be 12 feet from the foot of the post. So, the walker is 7 feet from the post. It is rewarding to find that the graph of the relation is consistent with intuition.

3. 14 feet.

Answers for Part ☆I [on page 5-32].



1. The diagram illustrates a helpful graphical technique for locating points in this relation. [The distance between A and B is the sum of the distance between T_1 and B and between B and T_2.] The relation is $\{(x, y): 0 \leq x \leq 14 \text{ and } y = \sqrt{9 + x^2} + \sqrt{16 + (14 - x)^2}\}$.

2. We seek the first component of the ordered pair whose graph is the lowest point. This ordered pair is $(6, \sqrt{45} + \sqrt{80})$. So, the required location of P is 6 yards from F_1. Another way of solving Exercise 2 [without using differential calculus] is to reflect T_2 in the line through F_1 and F_2. Let T_3 be the image of T_2. The sum of the distances from T_1 to P to T_3 is the same as the sum of the distances from T_1 to P to T_2. But, the smallest of these sums is obtained when T_1, P, and T_3 are collinear. [This is a consequence of the fact that the measure of one side of a triangle is less than the sum of the measures of the other two sides. Avoid saying, in seriousness, that the shortest distance between two points is a straight line. A straight line is not a distance!] By similar triangles we find that this minimizing location of P is such that the distance between F_1 and P is 6.

The purpose of these Exploration Exercises is to prepare students for the notions of the domain, range, and field of a relation. In answering (a) for each exercise of Part A, they name the domain of the given relation; in answering (b) they name its range. The union of the domain and range of a relation is its field, and in Part B on page 5-34, students are asked to name the fields of the relations given in the exercises of Part A. The exercises of Part C prepare students for a new kind of generalization sentence--existential generalization sentences. Such sentences appear as set-selectors in the definitions of domain and range on page 5-35. They have also been discussed, along with universal generalization sentences, in the COMMENTARY for page 2-27 in Unit 2.

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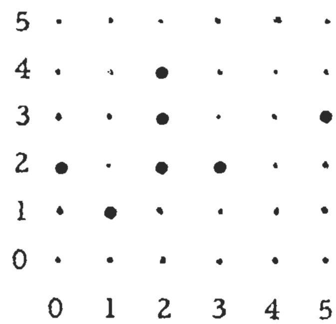
Answers for Part A [on pages 5-33 and 5-34].

1. (a) $\{6, 7, 8\}$ (b) $\{6, 7, 8\}$
2. (a) $\{2, 3, 4, 5\}$ (b) $\{6, 7, 8, 9\}$
3. (a) $\{-2, -1, 0, 1, 2\}$ (b) $\{5\}$
4. (a) the set of real numbers (b) $\{x: 1 \leq x \leq 2\}$
5. (a) $\{4\}$ (b) the set of real numbers
6. (a) the set of real numbers (b) $\{x: x \geq 0\}$
7. (a) $\{x: |x| < 2\}$ (b) $\{x: |x| < 4\}$
8. (a) $\{x: x \geq 2\}$ (b) $\{x: x \leq -1\}$
9. (a) $\{x: |x| \leq 5\}$ (b) $\{x: |x| \leq 5\}$
10. (a) $\{x: |x| \leq 6\}$ (b) $\{x: |x| \leq 6\}$
11. (a) $\{x: |x| \leq 6\}$ (b) $\{x: 0 \leq x \leq 6\}$
12. (a) $\{x: 0 \leq x \leq 6\}$ (b) $\{x: |x| \leq 6\}$
13. (a) $\{x: 0 < |x| < 6\}$ (b) $\{x: 0 < |x| < 6\}$
14. (a) $\{x: |x| \leq 6\}$ (b) $\{x: |x| \leq 6\}$

EXPLORATION EXERCISES

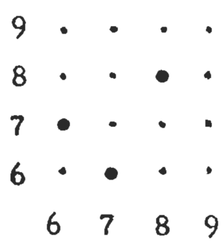
A. Here are graphs or brace-notation names of several relations. For each relation, (a) describe the set of things which are first components, and (b) the set of things which are second components of the members of the relation.

Sample.

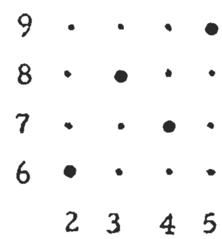


Solution. (a) first components: $\{0, 1, 2, 3, 5\}$
(b) second components: $\{1, 2, 3, 4\}$

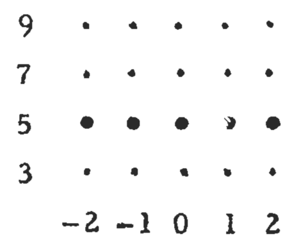
1.



2.



3.



4. $\{(x, y): 1 \leq y \leq 2\}$ 5. $\{(x, y): x = 4\}$ 6. $\{(x, y): y = x^2\}$

7. $\{(x, y): |x| < 2 \text{ and } |y| < 4\}$

8. $\{(x, y): y = 1 - x \text{ and } x \geq 2\}$

9. $\{(a, b): a^2 + b^2 = 25\}$

10. $\{(a, b): a^2 + b^2 = 36\}$

11. $\{(a, b): a^2 + b^2 = 36 \text{ and } b \geq 0\}$

12. $\{(a, b): a^2 + b^2 = 36 \text{ and } a \geq 0\}$

13. $\{(a, b): a^2 + b^2 = 36 \text{ and } ab > 0\}$

14. $\{(a, b): a^2 + b^2 = 36 \text{ and } ab \geq 0\}$

15. $\{(x, y) \in U \times C\colon$ y is a city in x$\}$, where U is the set of all states in the United States and C is the set of all cities in the United States.

16. $\{(x, y) \in U \times C\colon$ y is the capital of x$\}$

17. $\{(x, y) \in P \times P\colon$ y is an uncle of x$\}$, where P is the set of all people.

B. For each of the relations given in Part A, describe the set of things which are either first or second components of the members of the relation.

C. True or false?

1. There is a real number x such that $3 + 2x = 15$.
2. There is a real number y such that $y^2 + 4^2 = 5^2$.
3. There is a real number y such that $y^2 + 6^2 = 5^2$.
4. There is a real number y such that $y + 1 = y + 2$.
5. There is a real number y such that $y + 1 = 1 + y$.
6. There is an integer q such that $22 - 7 = 5q$.
7. There is an integer q such that $22 - 8 = 5q$.
8. There is a real number y such that, for each real number x, $xy = 0$.
9. There is a real number y such that, for each real number $x \neq 0$, $xy = 1$.
10. For each real number $x \neq 0$, there is a real number y such that $xy = 1$.
11. For each real number x, there is a real number y such that $xy = 0$.
12. There is a real number y such that, for each real number x, $x + y = 0$.
13. For each real number x, there is a real number y such that $x + y = 0$.

15. (a) the set of states in the United States

(b) the set of cities in the United States other than Washington, D.C.

16. (a) the set of states in the United States

(b) the set of state capitals in the United States

17. (a) the set of people who have uncles [Note that this is not the set of people who are either nephews or nieces: some of these have only aunts!]

(b) the set of people who are uncles [or: the set of male people who have a nephew or niece]

*

Answers for Part B.

1. $\{6, 7, 8\}$	2. $\{2, 3, 4, 5, 6, 7, 8, 9\}$
3. $\{-2, -1, 0, 1, 2, 5\}$	4. the set of real numbers
5. the set of real numbers	6. the set of real numbers
7. $\{x: \|x\| < 4\}$	8. $\{x: x \leq -1 \text{ or } x \geq 2\}$
9. $\{x: \|x\| \leq 5\}$	10. $\{x: \|x\| \leq 6\}$
11. $\{x: \|x\| \leq 6\}$	12. $\{x: \|x\| \leq 6\}$
13. $\{x: 0 < \|x\| < 6\}$	14. $\{x: \|x\| \leq 6\}$

15. the set of all political organizations in the United States which are either cities or states, other than Washington, D. C.

16. the set of all political organizations in the United States which are either states or state capitals

17. the set of all people who have or are uncles

*

Answers for Part C.

1. T	2. T	3. F	4. F	5. T	6. T	7. F
8. T	9. F	10. T	11. T	12. F	13. T	

Students may, at first, experience some difficulty in distinguishing between the meanings of the sentences in Exercises 8 and 11 [or those in Exercises 9 and 10, or those in Exercises 12 and 13]. Using the existential quantifier '$\exists$' introduced in the COMMENTARY for Unit 2 on TC[2-27]p and, now, in the text, on page 5-35, these exercises can be written:

$$\left.\begin{array}{l} 8.\ \exists_y(\forall_x\ xy = 0) \\ 11.\ \forall_x(\exists_y\ xy = 0) \end{array}\right\} \qquad \left.\begin{array}{l} 9.\ \exists_y(\forall_{x \neq 0}\ xy = 1) \\ 10.\ \forall_{x \neq 0}(\exists_y\ xy = 1) \end{array}\right\} \qquad \left.\begin{array}{l} 12.\ \exists_y(\forall_x\ x + y = 0) \\ 13.\ \forall_x(\exists_y\ x + y = 0) \end{array}\right\}$$

[The parentheses are unnecessary, but they may make it easier to grasp the sense of the sentences.]

A similar pair of sentences is considered on TC[2-27]i. When rewritten they are:

$$\left.\begin{array}{l} (3')\ \exists_y(\forall_x\ x \leq y) \\ (4')\ \forall_x(\exists_y\ x \leq y) \end{array}\right\}$$

The first says that there is a greatest real number [and, so, is false]; the second says that for each real number there is a real number greater than or equal to it [and, so, is true--each number is less than or equal to itself.].

Of each such pair of sentences, the first makes a stronger claim than the second--more precisely, the first implies the second. The first sentence of such a pair asserts that there is a number which bears a certain relation to each number [or, in the case of Exercise 9, to each nonzero number.] The second sentence asserts only that, for each number, there is a number which bears the relation in question to it. So, for example, since the product of each number by 0 is 0, the sentence of Exercise 8 is true--there is a number [0] such that the product of each number by this single number is 0. Consequently, the sentence of Exercise 11 is also true--for each number there is a number [0] such that the product of the first by the second is 0. The sentence of Exercise 9 is false--there is no single number such that the result of multiplying each nonzero number by it is 1. However, the sentence of Exercise 10 is true--for each nonzero number, there is a number [the reciprocal of the first] such that the product of the first by the second is 1. The sentence of Exercise 12 is false--there

is no single number such that the result of adding it to each real number is 0. On the other hand, the sentence of Exercise 13 is true--for each real number, there is a number [the opposite of the first] such that the sum of the first and the second is 0.

Since, for each pair of sentences of the kind being considered here, the second sentence is a consequence of the first, if the first sentence of such a pair is true then the second must be true. And, if the first is false then, still, the second may be true. That, in this case, the second may also be false is shown by the pair:

$$\left.\begin{array}{l} \exists_y (\forall_x\ xy = 1) \\ \forall_x (\exists_y\ xy = 1) \end{array}\right\}$$

*

If you discussed the imagined graph of U as suggested on TC[5-35, 36]a, you might ask students how they could use the graph to tell if Mr. Adams belonged to the range of U. Find Mr. Adams' name in the vertical list and draw a horizontal line through it. If this line passes through a graph of a point in U, he is in the range. Otherwise, he is not. Next, ask students to consider the set of all points in U whose graphs are on this horizontal line. [This could be the empty set.] What can be said about the first components of these points? They are the nieces and nephews of Mr. Adams. To find out if Bill Smith is in the domain of U, draw a vertical line through his name in the horizontal list. If it hits the graph of the relation, Bill Smith has an uncle. The subset of U whose members have graphs on this vertical line are ordered pairs whose second components are the uncles of Bill Smith. Extend the work a bit by asking students to imagine a vertical line being drawn through the name of Alice Smith, Bill's sister. Will her line hit the same points as Bill's? No, but the second components of the points whose graphs her line contains will be the same as those for Bill's line, if we may assume that brothers and sisters have the same uncles. Ask students to consider the lines they would draw to determine if Mr. Adams belonged to the range and to the domain. Of course, these lines cross each other, but do they cross in the graph of a point which belongs to U? Ask if U contains any point with equal components, but avoid hassles over marriage customs!

*

Some excellent references dealing with relations and their properties are

Tarski, Introduction to Logic (New York: Oxford University Press, 1956),

Suppes, Introduction to Logic (New York: Van Nostrand, 1957),

Huntington, The Continuum (New York: Dover, 1957),

Cogan et al., Modern Mathematical Methods and Models, Volume II (Buffalo, New York: Mathematical Association of America, University of Buffalo, 1958), and

Luce, Some Basic Mathematical Concepts (New Haven: School Mathematics Study Group, 1959).

Some pedagogical suggestions concerning the development of the descriptions (1) and (2) of $\mathcal{D}_R$ and $\mathcal{R}_R$ may be in order since they involve a new type of sentence, existential generalizations. We have a relation R among the members of a set S. We wish to know which members of S are involved in the relation, and, in particular which members of S are first components and which are second components of ordered pairs of R. Let us pick a member of S, say x, and ask if x belongs to the domain of R. To answer this question, we search among the members of S. If we find a member of S, say y, such that (x, y) belongs to R, we say:

Yes, $x \in \mathcal{D}_R$ because there is a $y \in S$ such that $(x, y) \in R$.

Similarly, to find the range of R, we start by picking a member of S, say x, and ask if x belongs to $\mathcal{R}_R$. To answer this question, we search among the members of S for a member, say y, such that (y, x) belongs to R. If we are successful in the search, we say:

Yes, $x \in \mathcal{R}_R$ because there is a $y \in S$ such that $(y, x) \in R$.

*

The symbol '$\exists_{y \in S}$' is pronounced as 'there is a y in S such that'. [For a detailed discussion of universal and existential generalizations, see the essay which begins on TC[2-27]e. The "backwards 'E' " reminds one of the first letter of 'exists' in the phrase 'there exists ... '.]

*

In studying the diagram on page 5-36, students should understand that the graphs of $\mathcal{D}_R$ and $\mathcal{R}_R$ are graphs of subsets of S, and that the graph of R is a graph of a set of ordered pairs whose first components belong to $\mathcal{D}_R$ and whose second components belong to $\mathcal{R}_R$. The graphs of $\mathcal{D}_R$ and $\mathcal{R}_R$ are not graphs of axes. [See page 5-62ff.]

*

Answers for questions in the text [on page 5-36].

Bill Smith is in $\mathcal{D}_U$ if and only if the sentence: (Bill Smith, x) $\in U$ has at least one solution. Mr. Adams is in $\mathcal{D}_U$ if and only if there is at least one solution of: (Mr. Adams, x) $\in U$

whose domain is P, the set of all people. Similarly, if M is the set of all male human beings, the sentence:

$$\forall_{x \epsilon M} (\exists_{y \epsilon P}\ x \text{ is an uncle of } y)$$

asserts that each male human being is an uncle of some person. Here, 'ϵ M' indicates that 'x' is a variable whose domain is M, and 'ϵ P' indicates that 'y' is a variable whose domain is P. When the domain of a variable is not so indicated it is to be understood that the domain is the set of real numbers [or, if a restriction is attached, a subset of this set]. [Sometimes, as in section 5.02, we may adopt, for a time, some other convention.]

Now, although, as has just been pointed out, questions concerning the domain of a variable are really questions about the use of language, questions concerning the domain [and range] of a relation are questions about the subject matter. Thus, the domain of the <u>relation</u>

$$\{(x, y) \epsilon P \times P:\ y \text{ is an uncle of } x\}$$

is the set of all people who have uncles. This is a proper subset of the domain, P, of the <u>variable</u> 'x'.

*

Note the script letters for domain and range. We want abbreviations for 'the domain of R' and 'the range of R'. ['$\mathscr{D}_R$' is read as 'the domain of R' and '$\mathscr{R}_R$' is read as 'the range of R'.] The script letters and the subscripts serve this purpose. Students should be cautioned about the importance of using script letters instead of Roman letters in these cases. Naturally, they need not make copies of the particular script letters we use in the text. All they need do is to make letters which are clearly distinguishable from the block upper case letters. For example:

$\mathscr{D}$ $\mathscr{D}$ $\mathscr{R}$ $\mathscr{R}$

In the case of the field of a relation [page 5-37], letters like:

$\mathscr{F}$ $\mathscr{F}$

will do. Students will get practice in making these letters when they do the exercises in Part A on page 5-37.

*

It is important to distinguish between the two uses of the word 'domain'. Students have learned that the domain of a variable is the set of entities which can serve as values of the variable, that is, the set of entities whose names can be used to replace the variable in an expression or in a sentence in which the variable occurs. When one builds a language for the purpose of talking about a particular subject matter, he usually specifies at the outset which symbols he will use as variables and what the domains of these variables are. To ask about the domain of a variable is to ask about one of the ground rules used in setting up the language. In Units 1 through 4, one of our ground rules is that the domain of each variable is a set of real numbers and, unless otherwise specified, is the set of [all] real numbers. Ways of indicating a restriction of the domain are illustrated by:

$\forall_{x \neq 0}\ \frac{0}{x} = 0$	TC[2-84]a
$\{x \neq 0:\ \frac{xx}{x} = \frac{x}{x}\}$	TC[3-27]c
$3x \cdot x = 21 \cdot x,\ [x \neq 0]$	TC[3-45, 46, 47]a
$\{(x, y),\ x \text{ and } y \text{ integers}:\ x = y - 1\}$	4-9
$\{x \in D:\ (x, 3) \in T\}$	TC[5-H]a
$\{(x, y) \in S \times S:\ y < x\}$	5-3

In Unit 5 we have a variety of subject matters--real numbers, people, sets, geometric figures, measures, relations .... One way of making clear what is the subject matter of any particular discussion is to choose, in each case, special symbols as variables. For example, we might decide, once and for all, to use upper case Roman letters as variables whose domain is the set of all people, lower case Greek letters as variables whose domain is the set of all geometric figures, etc. However, this is typographically impractical, and even if it were not, would hardly be worth the trouble. Instead, when we need variables whose domain is not the set of real numbers we shall, as in Exercises 7(ℓ) through 7(p) of Part A on pages 5-8 and 5-9, assign an arbitrary name, say 'D', to the domain and use phrases like '$\in$ D' to indicate that a letter is being used as a variable with this domain. Thus in:

$$\{(x, y) \in P \times P:\ y \text{ is an uncle of } x\},$$

'$\in P \times P$' indicates that both 'x' and 'y' are here being used as variables

The relation U may seem a bit strange since the relations worked with in detail thus far have been numerical ones. It may help to ask students what the ordered pairs are which belong to U. For one thing, they are ordered pairs of people. The first component is a person, and so is the second component. Since these are ordered pairs which belong to the relation of being an uncle of [or: unclehood], the second component of each such ordered pair is an uncle, and the first component is one of his nieces or nephews.

Students may wonder if you can graph the relation U. The answer is 'yes', although, practically speaking, it is impossible. However, it is instructive to discuss the steps you would follow in making such a graph. First, you might make a picture of the cartesian square of the set of all people. One way to do this is to make two lists--one vertical and the other horizontal--of all the people in the world. Then, just as one graphs an ordered pair of numbers, you could graph an ordered pair of people. If you wanted to graph the ordered pair

(Al Brown, Stan Moore),

you would look for Al Brown's name in the horizontal list and draw a vertical line through it; then, look for Stan Moore's name in the vertical list and draw a horizontal line through it. The dot in which the two lines cross is the graph of (Al Brown, Stan Moore). There is also a graph of (Stan Moore, Al Brown) which is different from the graph of (Al Brown, Stan Moore) [assuming Al Brown $\neq$ Stan Moore].

Then, to graph the relation U you would have to select from the cartesian square of P and mark on the picture just those ordered pairs for which the second component is an uncle of the first component.

A brace-notation name for U is: $\{(x, y) \in P \times P: \text{ y is an uncle of x}\}$

*

Answers for questions in the text.

Each person who is not an uncle [in particular, each female person] is an example of a member of P who is not the second component of any member of U.

A nephew who is so only by virtue of having an aunt is not the first component of any member of U.

The domain of U is the set of people who have uncles.

The range of U is the set of people who are uncles [that is, the set of male persons who have nephews or nieces].

5.04 Properties of relations. --In the preceding section we mentioned two of the properties [symmetry and reflexiveness] which relations can have. We want to give precise descriptions of these properties, and in order to do so, we need more terminology.

DOMAIN AND RANGE OF A RELATION

We have said that a relation R among the elements of a set S is a subset of the cartesian square $S \times S$. It may not be the case that each member of S is the first component of a member of R, nor that each member of S is the second component of a member of R. For example, consider the relation U of being-an-uncle-of among the members of the set P of all people. There are people who are not uncles, that is, there are members of P who are not second components of members of U. [Give an example.] Also, there are people who are neither nephews nor nieces. [Is it possible that a nephew not be the first component of a member of U?]

Given a relation R among the members of a set S, it is often convenient to talk about the set of those members of S which are first components of members of R. This set is called the domain of R. [What is the domain of U?] The set of members of S which are second components of members of R is called the range of R. [What is the range of U?] Concisely,

(1) $\mathfrak{D}_R = \{x \in S\colon \text{ there is a } y \in S \text{ such that } (x, y) \in R\}$,

(2) $\mathfrak{R}_R = \{x \in S\colon \text{ there is a } y \in S \text{ such that } (y, x) \in R\}$.

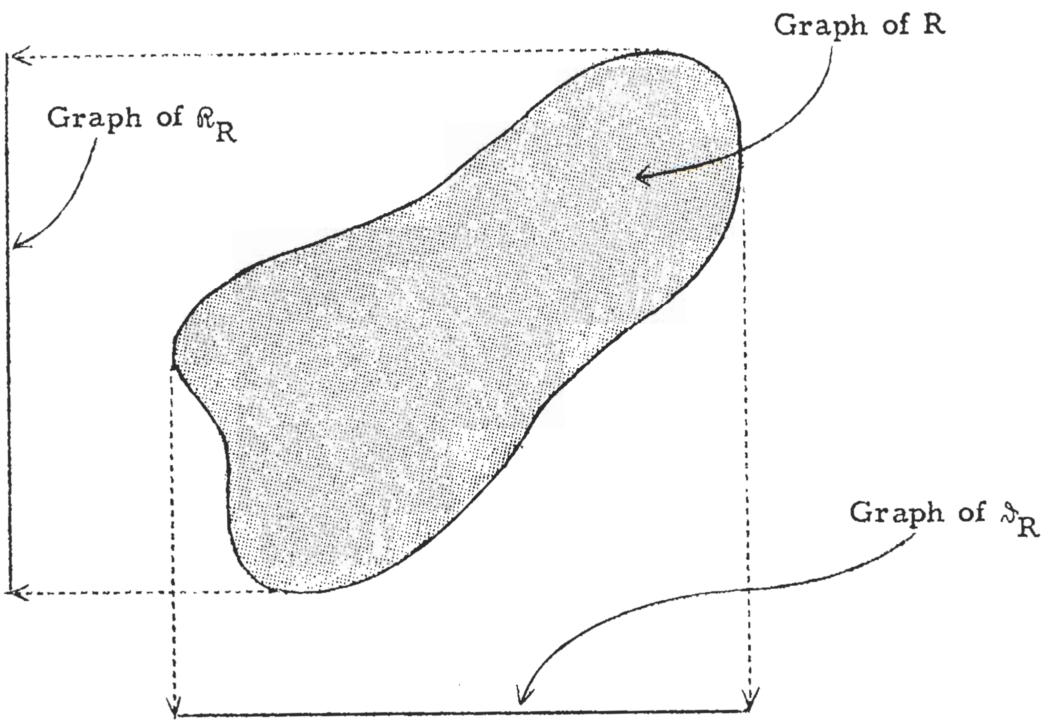
Notice that 'there is a $y \in S$ such that' is a short way of saying 'there is at least one $y \in S$ such that', and it is customary to abbreviate both of these expressions by:

$$\exists_{y \in S}$$

[The symbol '$\exists$' is called an existential quantifier. You are already acquainted with the universal quantifier '$\forall$'.]

So, $\mathfrak{D}_R = \{x \in S\colon \exists_{y \in S}\, (x, y) \in R\}$,

and, $\mathfrak{R}_R = \{x \in S\colon \exists_{y \in S}\, (y, x) \in R\}$.



In the case of the relation U,

$$\mathfrak{D}_U = \{x \in P: \exists_{y \in P} (x, y) \in U\},$$

and

$$\mathfrak{R}_U = \{x \in P: \exists_{y \in P} (y, x) \in U\}.$$

The domain of U is that subset of P which consists of all people who have uncles. The range of U is that subset of P which consists of all people who are uncles. How do we tell if a given element of P, say, Mr. Adams, belongs to $\mathfrak{R}_U$? By definition, Mr. Adams $\in \mathfrak{R}_U$ if and only if the sentence:

$$\exists_{y \in P} (y, \text{Mr. Adams}) \in U$$

is true. And, this sentence is true if and only if the sentence:

$$(y, \text{Mr. Adams}) \in U$$

has at least one solution. [It may be the case that this last sentence has more than one solution. This additional information may be of interest, but it is irrelevant to the question of whether Mr. Adams belongs to $\mathfrak{R}_U$.] What sentence must have at least one solution if Bill Smith is in $\mathfrak{D}_U$? If Mr. Adams is in $\mathfrak{D}_U$?

Skill Quiz.

A. Simplify.

1. $\frac{\frac{3}{4}}{2}$ 2. $\frac{\frac{3}{4}+\frac{1}{2}}{\frac{3}{7}}$ 3. $\frac{\frac{1}{5}}{\frac{1}{8}-\frac{2}{9}}$ 4. $\frac{\frac{5}{3}}{\frac{8}{9}+3}$ 5. $\frac{\frac{4}{7}-1}{\frac{4}{7}+1}$

B. Simplify.

1. $3(2a - b) + (b - a) - 2(6 + c)$ 2. $x(x - y) + y(x - y) - 7(xy) + y^2$

3. $4(x - y)(x + y) - 7(x - y)(x - y) + 6(x - y)^2$

4. $2ab(a + c) - 2bc(a + b)$ 5. $10\left(\frac{x - y}{5}\right) + (x + y)(2 + 3y)$

C. Factor.

1. $x^2 + 5x - 14$ 2. $x^2 - 1$ 3. $3x^2 - 24x + 45$

4. $x^2 + 4xy + 4y^2$ 5. $5 - x^2 + 4x$

D. Solve. [In the case of inequations, give the solution set, using the simplest sentence possible as set selector.]

1. $8x - 14 + 3x = 7$ 2. $5x + 16 < 2x + 9$

3. $x^2 + x = 12$ 4. $8y + 5 < 6y - 7$

5. $x^2 + 4 = 4x$

*

Answers for Quiz.

A. 1. $\frac{3}{8}$ 2. $\frac{35}{12}$ 3. $-\frac{72}{35}$ 4. $\frac{3}{7}$ 5. $-\frac{3}{11}$

B. 1. $5a - 2b - 2c - 12$ 2. $x^2 - 7xy$ 3. $3x^2 + 2xy - 5y^2$

4. $2a^2b - 2b^2c$ 5. $4x + 3xy + 3y^2$

C. 1. $(x + 7)(x - 2)$ 2. $(x - 1)(x + 1)$ 3. $3(x - 5)(x - 3)$

4. $(x + 2y)(x + 2y)$ 5. $(5 - x)(1 + x)$

D. 1. $\frac{21}{11}$ 2. $\{x: x < -\frac{7}{3}\}$ 3. $-4, 3$

4. $\{y: y < -6\}$ 5. 2

member of the range because adding 1 has an inverse.

[One <u>can</u> solve Sample 2 by saying that

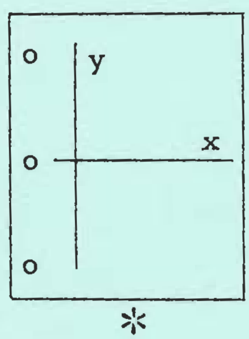
$$\mathfrak{D}_B = \{x: \exists_y\, y^2 = x + 1\},$$

and that

$$\mathfrak{R}_B = \{x: \exists_y\, x^2 = y + 1\}.$$

But, doing this is akin to answering: $\{x: x^2 - 5x + 6 = 0\}$, when asked for the solution set of '$x^2 - 5x + 6 = 0$'!]

A foreknowledge of the domain and range of a relation is useful in drawing a graph. Such knowledge helps you select the scales for the axes. If you were graphing the relation in Sample 2 on a sheet on cross-section paper, you might draw the axes like this:



Answers for Part A [on pages 5-37 and 5-38].

1. $\mathfrak{D}_M = \{-2, 0, 5\}$, $\mathfrak{R}_M = \{-2, 0, 1, 5\} = \mathfrak{G}_M$

2. $\mathfrak{D}_N = \{3, 4\} = \mathfrak{R}_N = \mathfrak{G}_N$

3. $\mathfrak{D}_T = \{x \in I: 1 \le x \le 10\} = \mathfrak{R}_T = \mathfrak{G}_T$

The notion of the field of a relation makes it easy for us to give a precise description of the property of reflexiveness. This statement that the field of R is the smallest subset of S which contains [illegible] all the members of R only is a little hard for students to understand without having had the experience of [illegible] Part B. Don't press this statement, but instead raise these questions which students should think about as they do Part B.

(1) Is each domain of R_1 a member of R_2? [Yes.]

(2) Is each converse of R_1 a member of R_2? [Yes.]

(3) Is each argument of R_2 a member of R_1? [Not necessarily.]

(4) Is each converse of R_2 a member of R_1? [Not necessarily.]

(5) Can $R_2 = R_1$? [Yes, provided that $R_2 \subseteq R_1$.]

(6) If either R_1 or R_2 is S, does $R_1 = R_2$? [Yes.]

Follow through by asking for examples which they have considered Part A.

A person who has an [illegible] for drawing graphs of relations [illegible] and [illegible] as small as [illegible] the [illegible] of a field. [illegible] the field of the relation [illegible]. [illegible] the [illegible] of the field. [illegible] with [illegible] of the relation [illegible] on the [illegible].

There are a couple of suggestions for handling Exercise 4. The exercise is a relation among the real numbers. We want to find out what real numbers belong to R_4. So, we pick some real number, say 1, and ask is $1 \in R_4$? This is equivalent to asking:

Is there a real number y such that $y^2 = 1 + y$?

The answer to this question is 'yes' because [illegible] is such a real number. [illegible] whose square is [illegible]. [illegible] $\in R_4$? [illegible]

Is there a real number y such that $y^2 = -1 + 1$?

The answer is 'no' because the square of each real number is non-negative. [illegible] the domain is [illegible].

Does $-1 \in R_4$? Thus [illegible]

Is there a real number y such that $y^2 = x + 1$?

Yes, [illegible] is such a real number. In fact, each real number is a [illegible]

The notion of the field of a relation makes it easy for us to give a precise description of the property of reflexiveness. The statement that the field of R is the smallest subset of S whose cartesian square contains all the members of R may be a little hard for students to understand without having had the experience of doing the exercises in Part A. Don't push this statement, but instead raise these questions which students should think about as they do Part A:

(1) Is each member of $\mathfrak{D}_R$ a member of $\mathfrak{F}_R$? [Yes.]

(2) Is each member of $\mathfrak{R}_R$ a member of $\mathfrak{F}_R$? [Yes.]

(3) Is each member of $\mathfrak{F}_R$ a member of $\mathfrak{D}_R$? [Not necessarily.]

(4) Is each member of $\mathfrak{F}_R$ a member of $\mathfrak{R}_R$? [Not necessarily.]

(5) Can $\mathfrak{D}_R = \mathfrak{F}_R$? [Yes, provided that $\mathfrak{R}_R \subseteq \mathfrak{D}_R$.]

(6) If either $\mathfrak{D}_R$ or $\mathfrak{R}_R$ is S, does $\mathfrak{F}_R = S$? [Yes.]

Follow through by asking for answers when they have completed Part A.

A person who has an obsession both for drawing graphs of relations on square charts and for using as small an area as practicable has probably discovered the notion of a field. He knows that the domain and range of a relation are subsets of the field. So, he prepares his chart to accommodate the cartesian square of the field. This assures him that there will be no point of the relation whose graph will not fit on the chart.

*

Here are pedagogical suggestions for handling Sample 2. The relation B is a relation among the real numbers. We want to find out what real numbers belong to $\mathfrak{D}_B$. So, we pick some real number, say 7, and ask if $7 \in \mathfrak{D}_B$. This is equivalent to asking:

Is there a real number y such that $y^2 = 7 + 1$?

The answer to this question is 'yes' because $\sqrt{8}$ is such a real number. [$-\sqrt{8}$ is another real number whose square is 7 + 1, but the fact that there are <u>two</u> is irrelevant.] Does $-5 \in \mathfrak{D}_B$? That is:

Is there a real number y such that $y^2 = -5 + 1$?

The answer is 'no' because the square of each real number is non-negative. The smallest number in the domain is -1.

Does $12 \in \mathfrak{R}_B$? That is:

Is there a real number x such that $12^2 = x + 1$?

Yes, 143 is such a real number. In fact, each real number is a

FIELD OF A RELATION

We need one additional notion, that of <u>the field of R</u>.

$$\mathfrak{F}_R = \mathfrak{D}_R \cup \mathfrak{R}_R.$$

Roughly speaking, $\mathfrak{F}_R$ consists of the members of S which "get into the act". That is, $\mathfrak{F}_R$ is the smallest subset of S whose cartesian square contains all the members of R. So, for example, $\mathfrak{F}_U$ is the set of all people who have or are uncles.

EXERCISES

A. For each relation described below, give its domain, range, and field.

Sample 1. The relation A where

$$A = \{(0, 6), (3, 5), (3, 7), (4, 2), (5, 2), (6, 0)\}.$$

Solution. $\mathfrak{D}_A = \{0, 3, 4, 5, 6\}$

$\mathfrak{R}_A = \{0, 2, 5, 6, 7\}$

$\mathfrak{F}_A = \{0, 2, 3, 4, 5, 6, 7\}$

Sample 2. The relation B where $B = \{(x, y): y^2 = x + 1\}$.

Solution. To find the domain one needs to use the fact that a real number is a square if and only if it is nonnegative. So,

$$\mathfrak{D}_B = \{x: x \geq -1\}.$$

To find the range one needs to use the fact that each real number is the sum of a real number and 1. It follows that

$\mathfrak{R}_B$ is the set of real numbers.

Hence, $\mathfrak{F}_B$ is the set of real numbers

1. $M = \{(-2, 5), (-2, -2), (0, 1), (0, 5), (5, 0), (5, -2)\}$
2. $N = \{(3, 4), (4, 3)\}$
3. T, whose graph is on page 5-B.

4. $R = \{(7, 2), (7, 9), (7, 6), (7, 7)\}$

5. $S = \{(8, 1), (6, 1), (4, 1), (12, 1), (2, 1)\}$

6. $C = \{(x, y): y^2 = 2x - 3\}$

7. $D = \{(x, y): x^2 + y^2 = 25\}$

8. $E = \{(x, y) \in I \times I: |x| + |y| \leq 10\}$

9. $F = \{(x, y): y = |x - 2| + 4\}$

10. $G = \{(x, y): y + x = x + y + 3\}$

11. $H = \{(x, y): xy = yx\}$

12. $J = \{(x, y): x^2 - y^2 = 25\}$

13. $K = \{(x, y): 9x^2 + 25y^2 = 225\}$

[Supplementary exercises are in Part E, page 5-242.]

__B.__ True or false?

__Sample.__ $\exists_x\ x^2 - 5x + 6 = 0.$

__Solution.__ True. [For instance, $3^2 - 5 \cdot 3 + 6 = 0.$]

1. $\exists_y\ 3y + 7 = 18$
2. $\exists_k\ k = k + 1$
3. $\exists_t\ t^2 - 1 = 0$
4. $\exists_m\ 3 + m = m + 3$
5. $\exists_x\ (2x - 5 = 0 \text{ and } 5x - 2 = 0)$
6. $\exists_x\ (2x - 5 = 0 \text{ or } 5x - 2 = 0)$
7. $\forall_x\ 2x - 3 = 12$
8. $\exists_x\ 2x - 3 = 12$
9. $\exists_{x \in I}\ 12 = 3x$
10. $\exists_{x \in I}\ 12 = 7x$
11. $\exists_y\ y^5 - 3y^4 = y(y - 1)(y + 9)$
12. $\forall_x\ (\exists_y\ x + y = 3)$
13. $\exists_y\ (\forall_x\ x + y = 3)$
14. $\forall_x\ (\exists_y\ xy = 0)$
15. $\exists_y\ (\forall_x\ xy = 0)$

__C.__ If P and Q are relations then $P \cup Q$ and $P \cap Q$ are relations. [Why?] Can you compute the domain of $P \cup Q$ if you know $\mathcal{D}_P$ and $\mathcal{D}_Q$? How about the domain of $P \cap Q$? How about ranges and fields?

4. $\mathcal{D}_R = \{7\}$, $\mathcal{R}_R = \{2, 6, 7, 9\} = \mathfrak{F}_R$

5. $\mathcal{D}_S = \{2, 4, 6, 8, 12\}$, $\mathcal{R}_S = \{1\}$, $\mathfrak{F}_S = \{1, 2, 4, 6, 8, 12\}$

6. $\mathcal{D}_C = \{x\colon\ x \geq 3/2\}$, $\mathcal{R}_C$ = the set of all real numbers = $\mathfrak{F}_C$

7. $\mathcal{D}_D = \{x\colon\ -5 \leq x \leq 5\} = \mathcal{R}_D = \mathfrak{F}_D$

8. $\mathcal{D}_E = \{x \in I\colon\ -10 \leq x \leq 10\} = \mathcal{R}_E = \mathfrak{F}_E$

9. $\mathcal{D}_F$ = the set of all real numbers = $\mathfrak{F}_F$, $\mathcal{R}_F = \{x\colon\ x \geq 4\}$

10. $\mathcal{D}_G = \emptyset = \mathcal{R}_G = \mathfrak{F}_G$

11. $\mathcal{D}_H$ = the set of all real numbers = $\mathcal{R}_H = \mathfrak{F}_H$

12. $\mathcal{D}_J = \{x\colon\ |x| \geq 5\}$, $\mathcal{R}_J$ = the set of all real numbers = $\mathfrak{F}_J$

13. $\mathcal{D}_K = \{x\colon\ |x| \leq 5\} = \mathfrak{F}_K$, $\mathcal{R}_K = \{x\colon\ |x| \leq 3\}$

} Have students make graphs.

*

See Exercise 3 of Part E, Supplementary Exercises [page 5-242], for possible classroom discussion questions at this point. Also, see TC[5-37]a.

*

Answers for Part B.

1. True $[3(11/3) + 7 = 18]$

2. False $[\{x\colon\ x = x + 1\} = \emptyset]$

3. True $[(-1)^2 - 1 = 0]$

4. True $[3 + 8 = 8 + 3]$

5. False [Note, however, that the conjunction '$\exists_x\ 2x - 5 = 0$ and $\exists_x\ 5x - 2 = 0$' is true.] Also, '$\exists_x\ (2x - 5 = -7$ and $5x - 2 = -7)$' is true.]

6. True ['$2(5/2) - 5 = 0$ or $5(5/2) - 2 = 0$' is true.]

7. False $[2 \cdot 1 - 3 \neq 12]$

8. True $[2(15/2) - 3 = 12]$

9. True $[12 = 3 \cdot 4$, and $4 \in I]$

10. False $[\{x\colon\ 12 = 7x\} = \{12/7\}$, and $12/7 \notin I]$

11. True $[0^5 - 3 \cdot 0^4 = 0(0 - 1)(0 + 9)]$

12. True $[\forall_x \; x + (3 - x) = 3]$ 13. False $[\forall_y \exists_x \; x + y \neq 3]$

[Note the difference between Exercises 12 and 13. Exercise 12 says that, for each first number there is a second number whose sum with the first is 3; Exercise 13 says that, there is a number such that, no matter what number you add it to, the sum is 3.]

14. True $[\forall_x \; x0 = 0]$ 15. True $[\forall_x \; x0 = 0]$

[For a discussion of sentences like those in Exercises 12 through 15, see TC[5-34]b and c.]

*

Answers for Part C [on page 5-38].

[$P \cup Q$ and $P \cap Q$ are relations because the union and intersection of sets of ordered pairs are sets of ordered pairs.]

$$\mathcal{D}_{P \cup Q} = \mathcal{D}_P \cup \mathcal{D}_Q.$$

$\mathcal{D}_{P \cap Q} \subseteq \mathcal{D}_P \cap \mathcal{D}_Q$. However, $\mathcal{D}_{P \cap Q}$ may be a proper subset of $\mathcal{D}_P \cap \mathcal{D}_Q$. For example, consider the relations in Exercises 2 and 3 of Part A. $N \cap T$ is $\{(3, 4)\}$, so $\mathcal{D}_{N \cap T}$ is $\{3\}$. But, $\mathcal{D}_N \cap \mathcal{D}_T$ is $\{3, 4\}$. [Similar remarks apply to ranges and fields.]

*

It is interesting to note that if $\mathcal{D}_R \subseteq S$ and $\mathcal{R}_R \subseteq T$ then, although the domain and range of $\tilde{R}$ (with respect to $S \times T$) are subsets of S and T, respectively, nothing more can be said about $\mathcal{D}_{\tilde{R}}$ and $\mathcal{R}_{\tilde{R}}$ unless one has additional information about R. In particular, it is generally not the case that $\mathcal{D}_{\tilde{R}} = \widetilde{\mathcal{D}_R}$ or that $\mathcal{R}_{\tilde{R}} = \widetilde{\mathcal{R}_R}$.

[illegible]

R = [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Complete each of the following ... [illegible]

1. The domain of R is ______
2. The domain of T is ______
3. The range of R is ______
4. The range of T is ______
5. The field of R is ______
6. The field of T is ______
7. The domain of the converse of R is ______
8. The range of the converse of R is ______
9. The field of the converse of R is ______
10. The domain of the converse of T is ______
11. The range of the converse of T is ______
12. The field of the converse of T is ______

Answers

[illegible]

[illegible]

Quiz.

Suppose that

R = {(2, 3), (3, 3), (7, 3), (-2, 3), (0, 3), (-8, 3)},

T = {(3, 0), (3, 5), (3, -8), (3, 7), (3, -10), (3, 2), (3, 3), (3, -2)},

S = {(0, 5, -8, 7, -10, 2, 3, -2},

U = {2, 3, 7, -2, 0, -8},

and V = {3}.

Complete each of the following to a true sentence with one of the letters 'R', 'T', 'S', 'U', or 'V'.

1. The domain of R is _____.
2. The domain of T is _____.
3. The range of R is _____.
4. The range of T is _____.
5. The field of R is _____.
6. The field of T is _____.
7. The domain of the converse of R is _____.
8. The range of the converse of R is _____.
9. The field of the converse of R is _____.
10. The domain of the converse of T is _____.
11. The range of the converse of T is _____.
12. The field of the converse of T is _____.

*

Answers for Quiz.

1. U	2. V	3. V	4. S	5. U	6. S
7. V	8. U	9. U	10. S	11. V	12. S

4. (a) $\{(2, 6), (9, 9), (3, 8)\}$

(b) $\{(x, y): x = 3y\}$ [or: $\{(x, y): y = \frac{1}{3}x\}$] [Evoke both answers.]

(c) $\{(x, y): x = y\}$

(d) $\{(x, y): y = 3x\}$ [or: $\{(x, y): x = \frac{1}{3}y\}$]

(e) $\{(x, y): y^2 + x^2 = 25\}$

(f) $\{(x, y): y = 2\}$

(g) the relation of being a parent of

(h) the relation of being a cousin of

(i) the relation $<$ [or: $\{(x, y): y < x\}$]

5. $\mathcal{D}_S = K$, $\mathcal{R}_S = J$, $\mathcal{F}_S = J \cup K$ [Ask the class 'What relation is the converse of S?' The answer is, of course, 'R'.]

6. The converse of a relation R is the relation whose members are obtained by reversing the order of the components of the members of R. [More briefly (and less confusingly): the converse of R is $\{(x, y) \in \mathcal{R}_R \times \mathcal{D}_R: y \; R \; x\}$.]

*

In some textbooks, the converse of a relation R is denoted by '$\breve{R}$'.

*

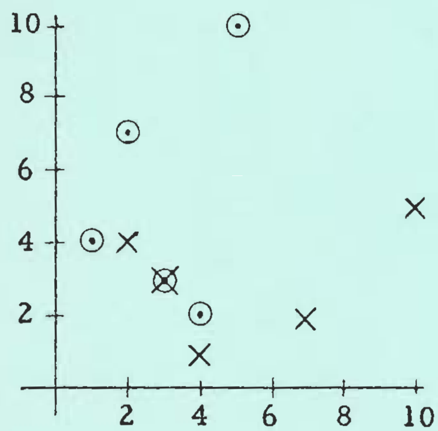
In speaking of the converse of a relation one is referring to the converse of a set of ordered pairs. The word 'converse' is also used in speaking of individual ordered pairs: for each a and b, the converse of (a, b) is (b, a). So, one may say that the converse of a relation R is the relation whose members are the converses of the members of R.

*

Correction. In Exercise 5, delete '(a)'.

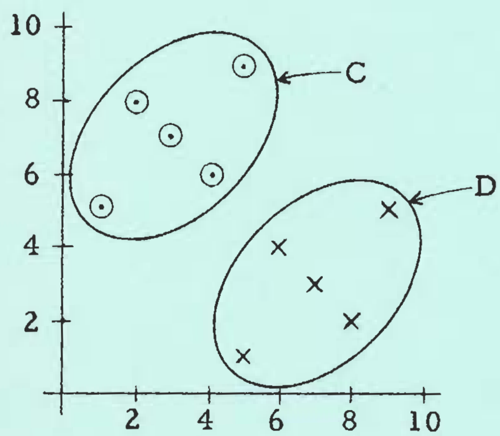
Answers for Part D.

1. (a)

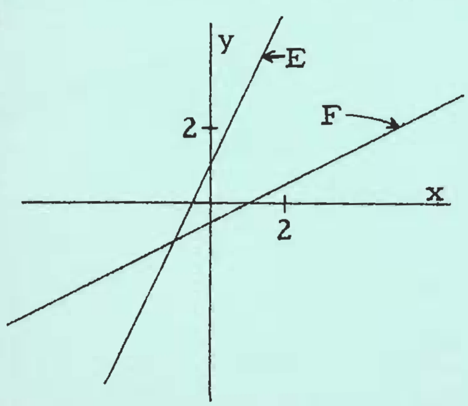


(b) The members of B are obtained by reversing the order of the components of the members of A.

2.



3.



$E = \{(x, y): y - 2x = 1\}$

$F = \{(x, y): x - 2y = 1\}$

Ask students for a brace-notation name for F.

D. 1. (a) Use the same chart to draw the graphs of the relations A and B where

$$A = \{(1, 4), (2, 7), (3, 3), (4, 2), (5, 10)\}$$

and $B = \{(4, 1), (3, 3), (7, 2), (10, 5), (2, 4)\}$.

Draw little circles around the dots of the graph of A, and draw little crosses through the dots of the graph of B.

(b) Do you see how the members of B are related to those of A?

2. (a) Draw the graph of C where

$$C = \{(1, 5), (3, 7), (4, 6), (2, 8), (5, 9)\}.$$

(b) On the same chart draw the graph of the relation D whose members are related to those of C in the same way that those of B are related to those of A.

3. (a) Draw the graph of E where $E = \{(x, y): y - 2x = 1\}$.

(b) Draw the graph of the relation F whose members are the ordered pairs obtained by "reversing" the components of the ordered pairs in E.

4. The relation B of Exercise 1 is called *the converse of the relation A.* Also, D is the converse of C, and F is the converse of E. For each of the relations listed below, write a name for the relation which is its converse.

(a) $\{(6, 2), (9, 9), (8, 3)\}$ (b) $\{(x, y): y = 3x\}$

(c) $\{(x, y): x = y\}$ (d) $\{(x, y): y = \frac{1}{3}x\}$

(e) $\{(x, y): x^2 + y^2 = 25\}$ (f) $\{(x, y): x = 2\}$

(g) the relation of being a child of

(h) the relation of being a cousin of (i) the relation >

5. (a) Suppose the relation S is the converse of the relation R. If $\mathcal{D}_R = J$ and $\mathcal{R}_R = K$, what are $\mathcal{D}_S$, $\mathcal{R}_S$, and $\mathfrak{F}_S$?

6. Give a definition of the converse of a relation. [*Hint.* The converse of R is $\{(x, y) \in$ ____ : ______________ $\}$.]

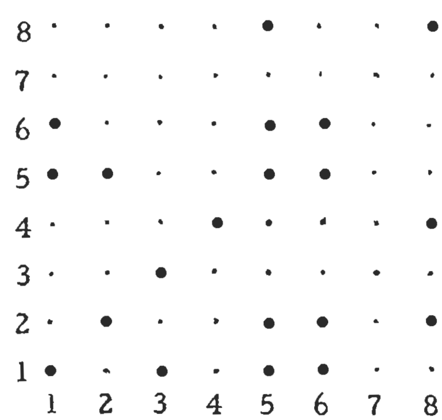
[Supplementary exercises are in Part F, page 5-243.]

REFLEXIVE RELATIONS

In an earlier section we noted that the relation of the inch-measure of one side to the inch-measure of another side of a triangle whose third side is 2 inches long is not a reflexive relation. And, the reason it is not reflexive is that it does not contain ordered pairs like (1, 1) and (0.5, 0.5). These are ordered pairs with equal components which are in the cartesian square of the field of the relation but are not in the relation itself.

> A relation R is reflexive if and only if,
> for each $x \in \mathfrak{F}_R$, $x \mathrel{R} x$ [that is, $(x, x) \in R$].

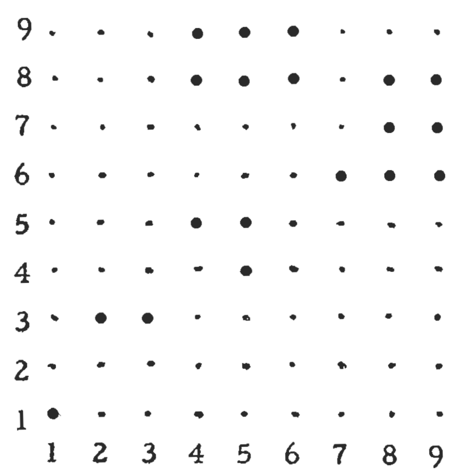
Here is a graph of a relation. Is it reflexive? Justify your answer.



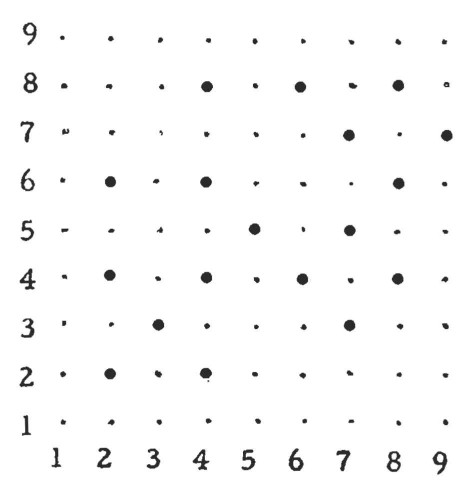
EXERCISES

A. Each exercise contains the graph of a relation. What additional ordered pairs <u>must</u> you include in the relation in order to obtain a reflexive one?

(1)



(2)



The remainder of this section deals with two of the more important properties which a relation may have. Besides furnishing a good background for the study of functions, it provides an opportunity to bring up interesting combinatorial problems. [Parts D and E on pages 5-43 and 5-44, and Part B on page 5-46].

*

The relation whose graph appears in the middle of page 5-40 is reflexive. Its field is $\{1, 2, 3, 4, 5, 6, 8\}$, and each of the ordered pairs (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), and (8, 8) belongs to the relation.

*

Answers for Part A.

The field of the relation whose graph is (1) is $\{x \in I: 1 \leq x \leq 9\}$. To obtain a reflexive relation one must adjoin to the given relation the pairs (2, 2), (4, 4), (6, 6), (7, 7), and (9, 9). [In addition to these one may adjoin others, for example, (2, 6), or (John, 7). But, if one does include, say, (John, 7) then the relation so obtained has John in its field and, to obtain a reflexive relation, one must then adjoin (John, John).]

The field of the relation whose graph is (2) is $\{x \in I: 2 \leq x \leq 9\}$. To obtain a reflexive relation one must adjoin to the given relation the pairs (6, 6) and (9, 9).

*

After doing Exercise 1, students may jump to the conclusion that a reflexive relation is one whose graph includes "the diagonal". That this is not necessarily the case is seen in Exercise 2, and in the chart in the middle of the page. The point (1, 1) does not have to belong to the relation in order that it be reflexive. It does only if 1 is in the field. Suppose you are preparing to graph a relation which is reflexive, and you decide to make a chart which will just accommodate the graph, that is, it will have no unnecessary columns or rows. The chart will then picture a cartesian square, and the relation is a subset of this square. Furthermore, if the rows are named "going up" in the same order that the columns are named "going to the right" then the graph of the relation will contain all the diagonal points. Moreover, any relation which meets these conditions is a reflexive one.

Exercise 15: Each ordered pair of real numbers with equal components satisfies the set selector because $0^2 + 0^4 = 0$. [Strange as it may look, the relation in Exercise 15 is the same as that in Exercise 3. Since squares and fourth powers of real numbers are nonnegative and since the sum is 0, each addend must be 0.]

*

In later exercises [for example, those in Part D on page 5-43] questions may arise concerning the empty set. For example; Is $\emptyset$ a relation? If one recalls that each set all of whose members are ordered pairs is a relation, he sees that the answer to this question is 'yes'. For, one who claims that $\emptyset$ is not a relation must be prepared to exhibit a member of $\emptyset$ which is not an ordered pair. Since $\emptyset$ has no members, it is impossible that he should be able to do this. Note that

$$\mathfrak{D}_{\emptyset} = \{x: \exists_y\ (x, y) \in \emptyset\} = \emptyset \quad \text{and} \quad \mathfrak{R}_{\emptyset} = \{x: \exists_y\ (y, x) \in \emptyset\} = \emptyset.$$

Also, $\emptyset$ is reflexive. For, $\mathfrak{D}_{\emptyset} = \emptyset$ and $\forall_{x \in \emptyset}\ (x, x) \in \emptyset$. [Here, again, one who claims that $\emptyset$ is not reflexive faces the impossible task of exhibiting a member of $\emptyset$--this time, an $x \in \emptyset$ such that $(x, x) \notin \emptyset$.]

Finally, $\emptyset$ is [see page 5-45] symmetric. For, there is no ordered pair (x, y) in $\emptyset$ such that $(y, x) \notin \emptyset$.

Answers for Part B.

1. (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)   2. $\mathfrak{R}_R = \{3, 4, 5\}$

3. If R is a reflexive relation then $\mathfrak{D}_R = \mathfrak{R}_R$. [If R is reflexive and $x \in \mathfrak{F}_R$ then, since $x\, R\, x$, $x \in \mathfrak{D}_R$. So, $\mathfrak{F}_R \subseteq \mathfrak{D}_R$. Since, in any case, $\mathfrak{D}_R \subseteq \mathfrak{F}_R$, it follows that if R is reflexive then $\mathfrak{D}_R = \mathfrak{F}_R$. Similarly, if R is reflexive then $\mathfrak{R}_R = \mathfrak{F}_R$.]

4. Yes. [A relation and its converse have the same field.]

*

Answers for Part C.

The relations in Exercises 1, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, and 15 are reflexive; the others are not.

Exercises 3, 4, 5, 6: Draw the graphs.

Exercise 7: Since $1 \in I^+$, each member of I^+ is a factor of itself. [Suppose G is the set of positive integers greater than 1. Is $\{(x, y) \in G \times G$: y is a factor of x with respect to G$\}$ a reflexive relation? Answer: No. Note that the domain of this relation is the set of composite positive integers.]

Exercise 8: The members of this relation are all the ordered pairs for which the difference of the first component from the second is an integral multiple of 5. For each k in the field of this relation, (k, k) belongs to the relation because $k - k = 0$ and 0 is an integral multiple of 5. [It may help students if they first attempt to graph this relation.]

Exercise 9: Each person has the same parents as himself.

Exercise 12: One is not one's own sister.

Exercise 13: Notice that to name an ordered pair belonging to this relation one would write, for example:

$$(\text{'}3x + 5 = 4 - 7x + 1\text{'},\ \text{'}z^2 + 6 = 8 - 2\text{'})$$

That is, one must use <u>names</u> of the components in naming the pair.

B. 1. Suppose R is a reflexive relation and $\mathfrak{F}_R = \{1, 2, 3, 4, 5\}$. What ordered pairs are you sure belong to R?

2. Suppose R is a reflexive relation and $\mathfrak{D}_R = \{3, 4, 5\}$. Can you tell what $\mathfrak{R}_R$ is?

3. If you know that a relation is reflexive, what can you say about its domain and range?

4. Is the converse of a reflexive relation reflexive?

C. Which of these relations are reflexive?

1. $\{(3, 7), (8, 2), (8, 8), (3, 3), (2, 8), (2, 2), (7, 7)\}$
2. $\{(4, 1), (1, 1), (6, 4), (6, 6)\}$
3. $\{(x, y): x = y\}$
4. $\{(x, y): y \leq x\}$
5. $\{(x, y): |x| \leq 5 \text{ and } |y| \leq 5\}$
6. $\{(x, y): x^2 + y^2 \leq 25\}$
7. $\{(x, y) \in I^+ \times I^+:$ y is a factor of x with respect to $I^+\}$
8. $\{(x, y) \in I \times I: \exists_{q \in I}\ y - x = 5q\}$
9. $\{(x, y) \in P \times P:$ y has the same parents as $x\}$, where P is the set of all people.
10. $\{(x, y) \in T \times T:$ y has the same perimeter as $x\}$, where T is the set of all triangles.
11. $\{(x, y) \in T \times T:$ y has the same shape as $x\}$
12. $\{(x, y) \in P \times P:$ y is a sister of $x\}$
13. $\{(x, y) \in Q \times Q:$ y has the same roots as $x\}$, where Q is the set of all equations in one variable [equations like '$3p - 5 = 2 - 7p$' but not like '$3a + 6b - 7 = 8a$'].
14. $\{(x, y) \in P \times P:$ y has the same uncles as $x\}$
15. $\{(x, y): (x - y)^2 + (y - x)^4 = 0\}$

* * *

Given a set S which contains 3 elements, how many subsets does S have? One way of answering this question is just to list the subsets of S and count them. For example:

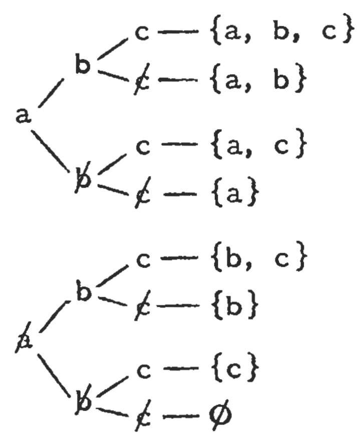
if $$S = \{a, b, c\}$$

then the subsets of S are

	$\{a, b, c\}$	1
	$\{b, c\}, \{c, a\}, \{a, b\}$	3
	$\{a\}, \{b\}, \{c\},$	3
and	$\emptyset.$	1
		8 ⟵ Total

But, this would be a tedious method if you wanted to find the number of subsets of a set containing, say, 25 elements.

Let's use another method which is easily generalized. Choosing a subset of S amounts to making a sequence of choices, one for each element of S. One decides for each element if it is to be included in the subset or not. There are two possible outcomes of the first choice. Then, for each of these, there are two possible outcomes of the second choice, etc. For our set S, here is a diagram of the procedure:



Each of the first three columns corresponds to a choice; for example, 'a' indicates that a is chosen, 'a̸' that a is rejected. The fourth column lists the subsets obtained by various sequences of choices. Notice that the first column has 2 entries, the second column has 2×2 entries, and the third column has $(2 \times 2) \times 2$ entries. Each column of entries after the first has twice as many entries as the preceding column. So, the third column has 2^3 entries; hence, a set with 3 elements has 2^3 subsets.

So, all together, there are $2^{12} + 4 \cdot 2^6 + 6 \cdot 2^2 + 4 \cdot 2^0 + 2^0$, or 4381, reflexive relations among the members of a given set of four elements.]

Students may be surprised that only about one fifteenth of the relations among a given set of four members are reflexive.

[You may recognize the numbers 1, 4, 6, 4, 1 as the successive coefficients in the expansion of, say, '$(a + b)^4$'. They are sometimes designated by '$\binom{4}{0}$', '$\binom{4}{1}$', '$\binom{4}{2}$', '$\binom{4}{3}$', and '$\binom{4}{4}$', and are called 'binomial coefficients'. So, the number of reflexive relations among the members of a given set of 4 elements is

$$\binom{4}{0}2^{0^2 - 0} + \binom{4}{1}2^{1^2 - 1} + \binom{4}{2}2^{2^2 - 2} + \binom{4}{3}2^{3^2 - 3} + \binom{4}{4}2^{4^2 - 4}$$

or, for short,

$$\sum_{k=0}^{4}\binom{4}{k}2^{k^2 - k} \quad [\text{'}\textstyle\sum\text{' for 'sum'}].$$

For each n, the number of reflexive relations among the members of a given set of n elements is

$$\sum_{k=0}^{n}\binom{n}{k}2^{k^2 - k}.]$$

☆ 7. 50,625 [The second components of those members of a relation R which have a given member of $\mathfrak{D}_R$ as first component form a nonempty subset of $\mathfrak{R}_R$. If $\mathfrak{R}_R$ is to be a subset of a given 4-member set, then there are $2^4 - 1$ nonempty subsets to choose from. If the domain is a given 4-member set then we must make four such choices in succession. So, the number of relations whose domain is a given 4-member set and whose range is a subset of this set is $(2^4 - 1)^4$. In fact, for each m, for each n, the number of relations whose domain is a given n-member set and whose range is a subset of a given m-member set is $(2^m - 1)^n$.]

Correction. In the third line of Exercise ☆6, change 'Exercise 4' to 'Exercise 5'.

Answers for Part D.

1. 16; 32; 2^{25} [or: 33, 554, 432]

2. 2^{16} [Answer for Hint: 16]

[For each n there are 2^{n^2} relations among the members of a given set of n elements.]

3. 2^{14}

4. $4 \leq n(R) \leq 16$

5. 2^{12} [Both the domain and the range of such a relation is a set of four elements. Each relation with this domain and range is a subset of a 16-member cartesian square. If such a relation is reflexive then it is the union of two sets, a first set consisting of the 4 "diagonal" ordered pairs, and a second set whose members (if any) are among the remaining 12 pairs. So, the question in Exercise 5 amounts to asking how many subsets a set of 12 elements has.]

[For each n, there are $2^{n^2 - n}$ reflexive relations whose field is a given set of n elements.]

☆6. There are 4381 reflexive relations among the members of a given set of four elements. [We begin by classifying the relations in question according to their fields. Then, compute the number of reflexive relations which have a given field, and finally, add the results. There are as many possible fields as there are subsets of the given set of 4 elements. These subsets are (1) the set itself, (2) 4 subsets each of which has 3 members, (3) 6 subsets each of which has 2 members, (4) 4 subsets each of which has 1 member, and (5) the empty set. We have seen in Exercise 5 that there are

$2^{4^2 - 4}$ reflexive relations which have a given 4-element set as field,

$2^{3^2 - 3}$ reflexive relations which have a given 3-element set as field,

$2^{2^2 - 2}$ reflexive relations which have a given 2-element set as field,

$2^{1^2 - 1}$ reflexive relations which have a given 1-element set as field,

and $2^{0^2 - 0}$ reflexive relations which have $\emptyset$ as field.

* * *

D. 1. How many subsets has a set of four elements? A set of five elements? A set of 25 elements?

2. How many relations are there among the members of a set of four elements? [Hint. How many ordered pairs belong to the cartesian square of a set of four elements?]

3. How many relations among the members of {1, 2, 3, 4} contain both the ordered pairs (2, 4) and (4, 1)?

4. If R is a reflexive relation whose field consists of four elements, what can you say about the number of elements in R?

5. How many reflexive relations are there whose field is a given set of four elements?

☆ 6. How many reflexive relations are there among the members of a given set of four elements? [Hint. Besides those you have counted in Exercise 4, you must take account of reflexive relations whose fields are 3-membered subsets of the given set, 2-membered subsets, etc.]

☆ 7. How many relations among the members of a given set of four elements have this set as their domain?

E. If you solved Exercise 6 of Part D, you found that a 4-membered set has

	1	0-membered subset [∅],
	4	1-membered subsets,
	6	2-membered subsets,
	4	3-membered subsets,
and	1	4-membered subset [the set itself].

Perhaps you did this by considering a particular 4-membered set, say {a, b, c, d}, listing its subsets, and counting them. Just as there is an easy method for finding the total number of subsets of a given set, there are also easy methods for finding the number of those subsets [of a given set] which have a given number of elements.

For example, there are easy methods for finding the number of 6-membered subsets of a 10-membered set, the number of 13-membered subsets of an 18-membered set, etc. You will learn one such method in the exercises which follow.

1. Complete the rows labeled '2', '3', '5', and '6' in the table below. [The row labeled '4' lists, for each whole number k, the number of k-membered subsets of a 4-membered set.]

n \ k	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0
2									
3									
4	1	4	6	4	1	0	0	0	0
5									
6				20					
7									
8									

2. Study the table carefully. Do you see a quick way of getting the numbers listed in, say, the 5-row, from the numbers listed in the 4-row? Can you get those in the 4-row from those in the 3-row? Those in the 6-row from those in the 5-row? Use this quick way, if you find it, to fill the 7-row and the 8-row. [Clearly, the table can be extended indefinitely. If you disregard the '0'-entries in such a table, the remaining entries form a triangular array called Pascal's Triangle. Almost every encyclopedia gives some account of the fascinating life of the seventeenth century French mathematician Blaise Pascal.]

Answers for Part E [which begins on page 5-43].

	n \ k	0	1	2	3	4	5	6	7	8
1.	0	1	0	0	0	0	0	0	0	0
	1	1	1	0	0	0	0	0	0	0
	2	1	2	1	0	0	0	0	0	0
	3	1	3	3	1	0	0	0	0	0
	4	1	4	6	4	1	0	0	0	0
	5	1	5	10	10	5	1	0	0	0
	6	1	6	15	20	15	6	1	0	0
2.	7	1	7	21	35	35	21	7	1	0
	8	1	8	28	56	70	56	28	8	1

*

In completing the table, students will be computing the binomial coefficients $\binom{n}{k}$ previously referred to on TC[5-43]b. There are a number of discoveries which they may make while doing so. For example, they should note the symmetry due to the fact that, for each n, for each $k \leq n$, there is the same number of k-member subsets of an n-member set as there are $(n - k)$-member subsets--that is, $\binom{n}{k} = \binom{n}{n-k}$. Also by taking ratios of corresponding numbers listed in successive rows they <u>may</u> discover that $\binom{n-1}{k} / \binom{n}{k} = \frac{n-k}{n}$, for $0 \leq k \leq n$. For example [n = 6], the ratios of numbers listed in the 5-row to those listed in the 6-row are

1/1, 5/6, 10/15, 10/20, 5/15, 1/6, 0/1,

that is, 6/6, 5/6, 4/6, 3/6, 2/6, 1/6, 0/6.

However, the discovery referred to in Exercise 2 is that,

(1) for each n, $\binom{n}{0} = 1$, and

(2) for each $n \geq 1$, for each $k \geq 1$, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

So, for example [n = 7], the numbers listed in the 7-row are

1, 1 + 6, 6 + 15, 15 + 20, 20 + 15, 15 + 6, 6 + 1, 1 + 0, 0 + 0.

Those who discover this will want to know why it works. Here is an explanation:

For each n, $\binom{n}{0} = 1$ because each set has just one 0-member subset, the empty set.

Now suppose that, for a given $n \geq 1$ and a given $k \geq 1$ we want to count the k-member subsets of an n-member set S. We can do this by choosing one member, say e_1, of S and counting, first, the number of k-member subsets of S which contain e_1 and, second, the number of k-member subsets of S which do not contain e_1. Now, the k-member subsets of S which contain e_1 correspond exactly with the (k - 1)-member subsets of the (n - 1)-member set $\widetilde{\{e_1\}}$. So, the number of these is the number of (k - 1)-member subsets of an (n - 1)-member set--$\binom{n-1}{k-1}$. And the k-member subsets of S which do not contain e_1 are precisely the k-member subsets of the (n - 1)-member set $\widetilde{\{e_1\}}$. So, the number of <u>these</u> is the number of k-member subsets of an (n - 1)-member set--$\binom{n-1}{k}$. Hence, altogether, the n-member set S has $\binom{n-1}{n-1} + \binom{n-1}{k}$ k-member subsets.

Students may note that any entry in the table can be obtained mechanically when one has discovered properties (1), (2), and

(3) for each $k \geq 1$, $\binom{0}{k} = 0$

[that is, that the empty set has no nonempty subsets!].

The formula '$\binom{n-1}{k} / \binom{n}{k} = \frac{n-k}{n}$' is, because of property (2), equivalent to:

$$\binom{n-1}{k-1} / \binom{n}{k} = \frac{k}{n},$$

or to:

$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

This last formula can be explained as follows:

> As in the explanation of (2), let S be an n-member set, say, $S = \{e_1, e_2, ---, e_n\}$. We know that there are just $\binom{n-1}{k-1}$ k-member subsets of S which contain e_1, $\binom{n-1}{k-1}$ which contain e_2, ---, and $\binom{n-1}{k-1}$ which contain e_n. If we add these results, getting $n\binom{n-1}{k-1}$, we will have counted each k-member subset of S k times--once for each of its k-members. So, we get $k\binom{n}{k}$.

In the case of any of the relations which are not symmetric, all that a student must do is exhibit an ordered pair (a, b) such that (a, b) belongs to the relation and (b, a) does not. Thus, in Exercise 2, (6, 4) belongs but (4, 6) does not. In Exercise 3, (1, 7) belongs but (7, 1) does not. [In connection with Exercise 3, ask if there is a pair (a, b) such that (a, b) and (b, a) both belong. (-5, -5) is such a pair, and it is the only one.]

Note that the relation in Exercise 4 is the union of $\{(x, y): y = 2x + 5\}$ and $\{(x, y): x = 2y + 5\}$. It is the union of a relation and its converse. Such a union is its own converse, and so it is symmetric. It is instructive to ask if the intersection of the components is symmetric. The answer is 'yes' because the intersection consists just of points on the diagonal [in this case, just the point (-5, -5)]. In general, both the union and the intersection of a relation and its converse are symmetric. [Exercise 6 can be discussed in the same manner.]

Exercise 12 deals with the union of a symmetric relation $[\{(x, y): y + x + 1 = 0\}]$ and a nonsymmetric one $[\{(x, y): y - x + 1 = 0\}]$. In this case, the union is nonsymmetric. But, this is not generally so. For if R is symmetric, T nonsymmetric, and $T \subseteq R$, $R \cup T$ is symmetric.

Exercise 13 provides an example of the intersection of a nonsymmetric relation and its converse. [Recall that '$y - 1 < x < y + 1$' is equivalent to '$y - 1 < x$ and $x < y + 1$'. ['$x < y + 1$' is equivalent to '$x - 1 < y$', which is the "converse" of '$y - 1 < x$'.] And, $\{(x, y): y - 1 < x$ and $x < y + 1\} = \{(x, y): y - 1 < x\} \cap \{(x, y): x < y + 1\}$.]

Although students could do all of these exercises by drawing graphs and searching for symmetry with respect to the diagonal [certainly, a few of them should be done this way], we hope that Exercises 9 and 10 will be done merely by interchanging 'x' and 'y' in the set selectors.

*

As an eleventh exercise for Part B, it would be natural to ask how many symmetric relations there are whose field is a given set of five elements. Notice that this question is related to that of Exercise 8 in the same way as that of Exercise 9 is related to that of Exercise 10 [and not as that of Exercise 10 is related to that of Exercise 9]. The solution of such problems requires a more complicated technique based on extensions of the formula:

$$(*) \qquad n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

We cannot go into this here but, for your information, the answer to the proposed exercise is

$$2^{(5^2+5)/2} - 5 \cdot 2^{(4^2+4)/2} + 10 \cdot 2^{(3^2+3)/2} - 10 \cdot 2^{(2^2+2)/2}$$
$$+ 5 \cdot 2^{(1^2+1)/2} - 2^{(0^2+0)/2},$$

or 28217. One subtracts from the number [2^{15}] of symmetric relations among the members of a given 5-member set S the number of such relations whose fields are proper subsets of S. Each of these relations is a symmetric relation among the members of one of the five 4-member subsets of S. So, if A is the set of symmetric relations among the members of one of these sets, and B, C, D, and E are, respectively, the sets of symmetric relations among the members of the other four 4-member subsets, what we need to calculate is $n(A \cup B \cup C \cup D \cup E)$. For this we need an extension of (*). Using it, and the fact that, for example, $n(A) = 2^{10}$, $n(A \cap B)$ = the number of symmetric relations among the members of a 3-member subset of $S = 2^6$, $n(A \cap B \cap C) = 2^3$, etc., one arrives at the result given above.

*

Answers for Part C [on page 5-46].

The relations in Exercises 1, 4, 5, 6, 7, 8, 11, and 13 are symmetric, the others not. [By the time students have done Exercises 4, 5, 6, 7, 8, 11, and 13, they should see that a relation described as these are is symmetric if, when one interchanges 'x' and 'y' in the given set selector, one obtains an equivalent sentence. Exercises 3, 9, 10, 11, and 12 make the point that this condition is necessary (as well as being sufficient) for symmetry.]

8. 2^{15} [This answer can be obtained by a procedure similar to that used in solving Exercise 5 of Part D on page 5-43. Suppose $S = \{1, 2, 3, 4, 5\}$. A relation among the members of S is a subset of the 25-member set $S \times S$ (and there are 2^{25} such subsets). Each symmetric relation among the members of S can be thought of as resulting from choosing members of $\{(x, y) \in S \times S: x \geq y\}$, and then choosing those members of $S \times S$ "above" the diagonal which are required by symmetry. Since $\{(x, y) \in S \times S: x \geq y\}$ has 15 members $[5 + (5^2 - 5)/2 = 15]$, there are 15 choices to be made. A slight variation is to think of the relation as determined by choosing any subset of the diagonal of $S \times S$ and any subset of the members of $S \times S$ below the diagonal. Since $S \times S$ has 5 diagonal members and $(5^2 - 5)/2$ members below the diagonal, there are 2^5 subsets of the diagonal and 2^{10} subsets whose members are below the diagonal. So, the first choice can be made in 2^5 ways and the second in 2^{10} ways, and there are $2^5 \cdot 2^{10}$ combinations of choices. In general, for each n, there are $2^{(n^2 + n)/2}$ symmetric relations among the members of a given set of n elements.]

9. 2^{10} [This exercise differs from Exercise 8 only in that, here, one has no choice with respect to a diagonal member of $S \times S$: all such ordered pairs must be chosen. So, there remain 10 choices, 2^{10} outcomes. For each n, there are $2^{(n^2 - n)/2}$ relations which are both reflexive and symmetric and whose field is a given set of n elements.]

☆10. 1450 [Exercise 10 is related to Exercise 9 in much the same way as Exercise 6 of Part D on page 5-43 is related to the exercise which precedes it. A relation among the members of a set of 5 elements may have as its field either (1) the empty set, (2) one of 5 singleton sets, (3) one of 10 2-member sets, (4) one of 10 3-member sets, (5) one of 5 4-member sets, or (6) the given set itself. Using the result obtained in Exercise 9, one sees that there are, all together,

$$2^{(0^2 - 0)/2} + 5 \cdot 2^{(1^2 - 1)/2} + 10 \cdot 2^{(2^2 - 2)/2} + 10 \cdot 2^{(3^2 - 3)/2}$$
$$+ 5 \cdot 2^{(4^2 - 4)/2} + 2^{(5^2 - 5)/2},$$

or 1450, relations among 5 elements which are both reflexive and symmetric. For each n, there are

$$\sum_{k=0}^{n} \binom{n}{k} 2^{(k^2 - k)/2}$$

relations among n elements which are both reflexive and symmetric.]

Answers for Part A.

(1) (1, 6), (4, 7), (5, 7), (6, 2), (6, 3), (6, 5), (6, 7), (6, 8), (6, 9), (8, 3), (8, 7), (9, 3), (9, 7)

(2) (1, 2), (1, 6), (2, 3), (2, 6), (3, 1), (3, 6), (4, 5), (4, 6), (5, 6), (7, 4), (8, 9), (9, 7)

(3) None.

(4) (2, 7), (2, 9), (4, 5), (4, 7), (6, 3), (6, 5), (8, 1), (8, 3), (10, 1)

*

Answers for Part B [on page 5-46].

1. Yes; yes; yes; yes. [A symmetric relation among the members of set S must contain an even number (possibly 0) of "nondiagonal" pairs in $S \times S$ and may contain any number of pairs from the diagonal of $S \times S$. So, symmetry puts no restriction on the number of members of a relation.]

2. Yes. [Of course.]

3. Yes. [One example is $\{(1, 1), (2, 2), (3, 3)\}$.]

4. $\mathcal{R}_R = \{3, 4, 5\}$

5. If R is a symmetric relation then $\mathcal{D}_R = \mathcal{R}_R$. [If $x \in \mathcal{D}_R$ then there is a y such that x R y. And, if R is symmetric, y R x. So, $x \in \mathcal{R}_R$. Hence, $\mathcal{D}_R \subseteq \mathcal{R}_R$. Similarly, $\mathcal{R}_R \subseteq \mathcal{D}_R$.]
   If R is symmetric then the converse of R is R itself.

6. A relation which is its own converse is symmetric. [So (Exercise 5), a relation is symmetric if and only if it is its own converse.]

7. We know that there are at least five and no more than twenty-five members in R, but we are not sure of any given one. [Since R is symmetric, $\mathcal{D}_R = \mathfrak{F}_R$. So, each member of $\mathfrak{F}_R$ must be the first component of some member of R. Hence, $5 \leq n(R)$.]

SYMMETRIC RELATIONS

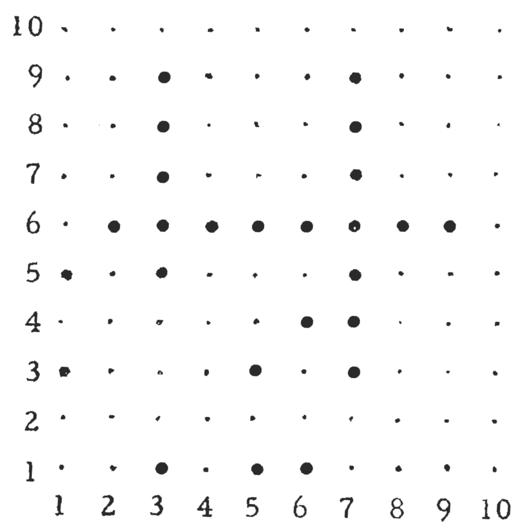
If John is a cousin of Ruth then Ruth is a cousin of John. More generally, we say that the relation of cousinhood is a symmetric relation. Is brotherhood a symmetric relation? [Explain your answer.]

A relation R among the members of a set S is symmetric if and only if, for each $(x, y) \in S \times S$, if $y\ R\ x$ then $x\ R\ y$.

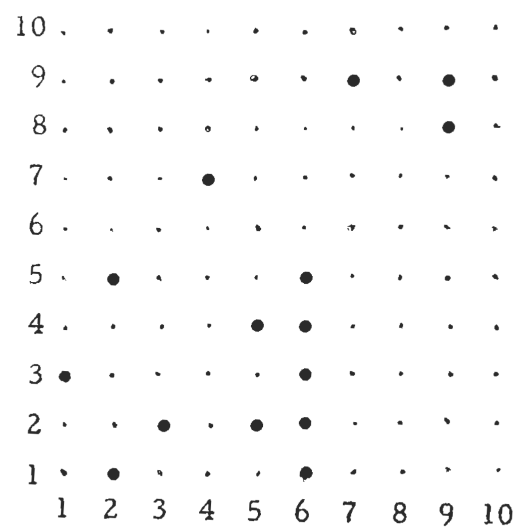
EXERCISES

<u>A</u>. Each exercise contains the graph of a relation. What additional ordered pairs <u>must</u> you include in the relation in order to obtain a symmetric one?

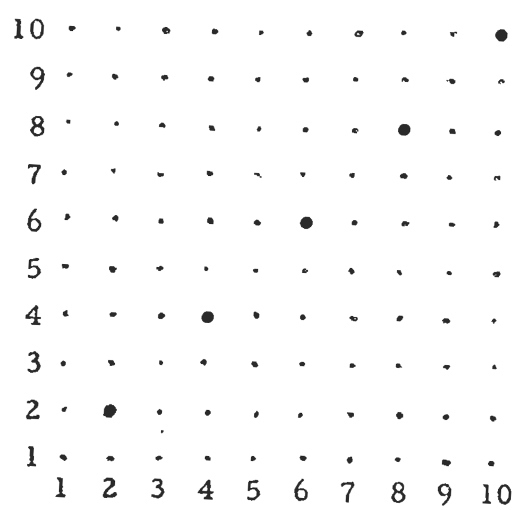
(1)



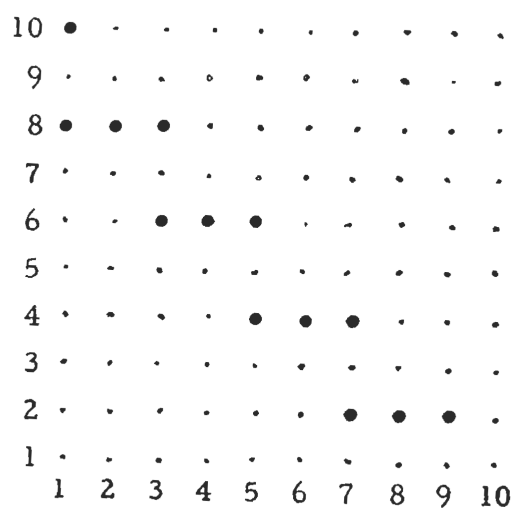
(2)



(3)



(4)



B. 1. Can a symmetric relation consist of 20 ordered pairs? 21? 1? 0?

2. Can a reflexive relation also be symmetric?

3. Can a relation which is both reflexive and symmetric consist of exactly 3 ordered pairs?

4. Suppose R is a symmetric relation and $\mathcal{D}_R = \{3, 4, 5\}$. Can you tell what $\mathcal{R}_R$ is?

5. If you know that a relation is symmetric, what can you say about its domain and range? What about its converse?

6. What do you know about a relation which is its own converse?

7. Suppose R is a symmetric relation and $\mathcal{F}_R = \{1, 2, 3, 4, 5\}$. What ordered pairs are you sure belong to R? What can you say about the number of members of R?

8. How many symmetric relations are there among five elements?

9. How many relations are there which are both symmetric and reflexive, and whose field is a given set of five elements?

☆10. How many relations are there among five elements which are both reflexive and symmetric?

C. Which of these relations are symmetric?

1. $\{(3, 8), (7, 6), (6, 6), (6, 7), (5, 5), (8, 5), (8, 3), (5, 8)\}$

2. $\{(4, 4), (1, 1), (0, 0), (-2, -2), (6, 4), (6, 6), (4, 4)\}$

3. $\{(x, y): y = 2x + 5\}$

4. $\{(x, y): y = 2x + 5 \text{ or } x = 2y + 5\}$

5. $\{(x, y): y + x = 5\}$

6. $\{(x, y): y = 3x - 2 \text{ or } y = \frac{x + 2}{3}\}$

7. $\{(x, y): xy < 0\}$

8. $\{(x, y): x^2 + y^2 = 25\}$

9. $\{(x, y): 4x^2 + 5y^2 = 101\}$

10. $\{(x, y): y = x - 1\}$

11. $\{(x, y): x^2 + xy + y^2 = 10\}$

12. $\{(x, y): y + x + 1 = 0 \text{ or } y - x + 1 = 0\}$

13. $\{(x, y): y - 1 < x < y + 1\}$

*

Quiz.

A. For each relation described below, give its domain, range, and field.

1. $T = \{(-1, 7), (-3, 5), (9, 8), (7, 2), (6, 3), (31, 27)\}$

2. $M = \{(3, 7), (5, 9)\}$

3. $F = \{(x, y): x^2 = 16 - y^2\}$

4. $G = \{(x, y) \in I \times I: |x| < 5 - |y|\}$

5. $K = \{(a, b): 2b = 7a - 15\}$

B. For the relations described below tell whether

(i) the relation is reflexive, (ii) the relation is symmetric.

1. $\{(x, y): x \not< y\}$

2. $\{(x, y): x - y = 0\}$

3. $\{(a, b): |a| - |b| = 0\}$

4. $\{(p, q): p = 5q + 3\}$

5. $\{(x, y): x = y^2 + 2\}$

*

Answers for Quiz.

A. 1. $\mathfrak{D}_T = \{-3, -1, 6, 7, 9, 31\}$
$\mathfrak{R}_T = \{2, 3, 5, 7, 8, 27\}$
$\mathfrak{F}_T = \{-3, -1, 2, 3, 5, 6, 7, 8, 9, 27, 31\}$

2. $\mathfrak{D}_M = \{3, 5\}$
$\mathfrak{R}_M = \{7, 9\}$
$\mathfrak{F}_M = \{3, 5, 7, 9\}$

3. $\mathfrak{D}_F = \{x: -4 \le x \le 4\} = \mathfrak{R}_F = \mathfrak{F}_F$

4. $\mathfrak{D}_G = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\} = \mathfrak{R}_G = \mathfrak{F}_G$

5. $\mathfrak{D}_K$ = the set of all real numbers = $\mathfrak{R}_K = \mathfrak{F}_K$

B. 1. r 2. r, s 3. r, s 4. neither 5. neither

A little help may be needed for Exercise 5. Students do have some kind of concept of right angle. But, to understand the set selector in Exercise 5, they must think of an angle as the union of two noncollinear rays with a common end point. So, an angle is a set of points. The union of a line and a line is also a set of points. And, we are looking for ordered pairs of lines such that the union of the first and the second contains a right angle as a subset. Since these are lines [not rays], the right angle will be a proper subset of the union. In fact, each union which contains a right angle also contains three other right angles.

*

Answers for Part E [on page 5-47].

[To save space, we omit answers for parts (a) of Exercises 1-12.]

1.	(b) yes	(c) yes	(d) congruence (of triangles)
2.	(b) no	(c) yes	(d) parallelism [We assume that a line is not parallel to itself.]
3.	(b) yes	(c) yes	(d) congruence (of circles)
4.	(b) yes	(c) yes	(d) congruence (of segments)
5.	(b) no	(c) yes	(d) perpendicularity
6.	(b) yes	(c) yes	(d) similarity (of triangles)
7.	(b) yes	(c) yes	(d) congruence (of triangles)
8.	(b) yes	(c) yes	(d) similarity (of triangles)
9.	(b) yes	(c) yes	(d) congruence
10.	(b) yes	(c) yes	(d) similarity
11.	(b) yes	(c) yes	(d) [none]
12.	(b) yes	(c) yes	(d) equivalence (of triangles with respect to area)

*

As in the answer for Exercise 9, we use 'congruence' as an abbreviation for 'congruence of plane geometric figures'. Similarly for 'similarity'. The parenthetical phrases in answers for Exercise 1, 3, 4, 6, 7, and 8 point out that the relations in question are subsets of these "larger" relations.

Although there is no "commonly used name" for the relation in Exercise 11, one might use 'equivalence of triangles with respect to perimeter'.

vertices to see whether one triangle can be made to "fit" the other. After a bit of experimentation, you should be able to tell that a pair of triangles will belong to the relation just if they agree in their angles and their sides with respect to at least one matching of their vertices. [For a more formal discussion of triangle-congruence, see Unit 6.]

[Students should depend upon geometric intuition to make their decision about whether one can decide on the truth of (*) just by looking at the sentence. We hope that most of your students can do this without a lot of drawing or model-making. You should be in a fairly good position to judge their ability in view of what they did in section 5.03. You may want to come to class equipped with a dozen cardboard or wire models of triangles, some of which are similar, some with the same area, some congruent, etc. In any event, a correct answer for part (a) consists of two pictures, each of a triangle, such that the triangles agree in their angles and in their sides.]

The relation is reflexive since each triangle has the same size and shape as itself. The relation is symmetric since if a first triangle has the same size and shape as a second triangle, then the second triangle has the same size and shape as the first. [As you can see, the word 'same' is a key one in responding to parts (b) and (c).]

In answering part (d), much depends upon the geometry studied in earlier courses. Undoubtedly, some of your students will know the words 'congruent' or 'congruence'. This is an opportunity to acquaint all of them with it. In view of the other relations in the list to which the word 'congruence' applies, it is best to call the relation in Exercise 1 'triangle-congruence' or 'congruence of triangles'. [It is rather interesting to note that those students who do not know the word 'congruence' still have an awareness of the relation, an awareness at least partially developed by the exercise itself.]

*

Do not insist too strongly on answers for parts (d); otherwise, there may be considerable head-scratching at home for Exercises 11 and 12.

*

> Correction. In the figure on page 5-49, the vertical dotted segment should start at (48, 0) and the horizontal dotted segment should end at [approximately] (0, 50).

Answers for Part D.

1. The relations in Exercises 2 and 13 of Part C are reflexive; the others are not.

2. The only symmetric relations in Part C on page 5-41 are those in Exercises 3, 5, 6, 8, 9, 10, 11, 13, 14, and 15.

*

Before you assign the exercises of Part E, we suggest that you discuss Exercise 1 with the class [or, at least make a start on it]. Ask them to look, first, at the index '$(x, y) \in T \times T$' in the name of the relation. This tells you that the relation in question is a subset of the cartesian square of the set of all triangles; and, this means that the domain of 'x' and of 'y' is the set of all triangles. So, a member of this relation is an ordered pair of triangles. To do part (a) for Exercise 1, then, you will [if the relation is not empty] need to draw pictures of two triangles. But, what kind of triangles?

To answer this question, think about the triangles which belong to this relation. An ordered pair of triangles belongs to the relation if and only if the set selector 'x has the same size and shape as y' is converted into a true sentence by substituting for 'x' a name for the first component and substituting for 'y' a name for the second component. [Notice that we do not put triangles in place of 'x' and 'y' in the set selector; we put names of triangles in those places.] How do we find triangles whose names will convert the set selector into a true sentence?

Let's use a bit of imagination. Pretend that you have access to the set of all triangles [please don't ask us where those triangles are!]. Each triangle has a tag attached to it [please don't ask us how!] and a name of the triangle is printed on the tag. The tags might bear such names as 'ΔJIM', 'ΔLIZ', 'Δ3', 'ΔD', etc. If you choose one, and then another, of those triangles at random, and put a name for the first one chosen in place of 'x' in the set selector, and a name for the second one in place of 'y', can you determine whether the set selector has been converted into a true sentence? You might get a statement like:

(*) ΔJIM has the same size and shape as ΔCAL

Of course, you can not decide whether the sentence is true just by looking at it! You must look at the triangles! You may be able to tell at once that ΔJIM is bigger than ΔCAL. But, the triangles may appear to be about the same size; if so, you will have to try matching the

D. 1. Which of the relations listed in Part C on page 5-46 are reflexive?

2. Which of the relations listed in Part C on page 5-41 are symmetric?

E. Each of the exercises below describes a relation among geometric figures in a plane. For each exercise,

(a) make a picture of the components of an ordered pair which belongs to the relation,

(b) tell whether the relation is reflexive,

(c) tell whether it is symmetric, and

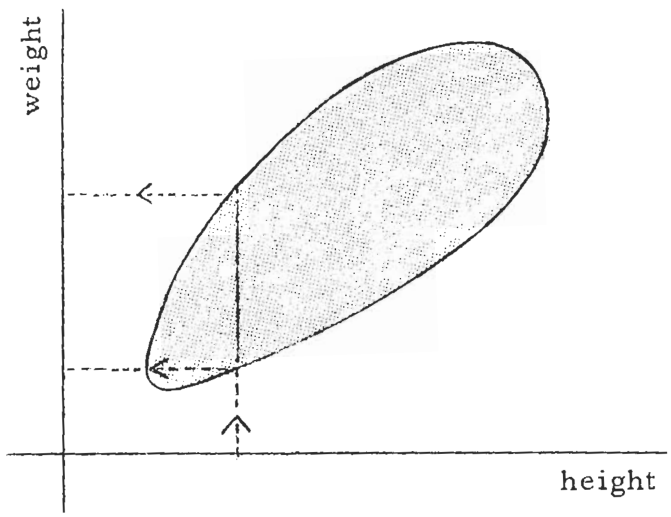
(d) give a commonly used name for the relation.

[T = the set of all triangles; C = the set of all circles;
S = the set of all segments; L = the set of all [straight] lines;
A = the set of all geometric figures]

1. $\{(x, y) \in T \times T$: x has the same size and shape as y$\}$
2. $\{(x, y) \in L \times L$: $x \cap y = \emptyset\}$
3. $\{(x, y) \in C \times C$: x has the same circumference as y$\}$
4. $\{(x, y) \in S \times S$: x has the same length as y$\}$
5. $\{(x, y) \in L \times L$: $x \cup y$ contains a right angle$\}$
6. $\{(x, y) \in T \times T$: x has the same shape as y$\}$
7. $\{(x, y) \in T \times T$: the set of measures of the sides of x is the set of measures of the sides of y$\}$
8. $\{(x, y) \in T \times T$: the set of measures of the angles of x is the set of measures of the angles of y$\}$
9. $\{(x, y) \in A \times A$: x has the same size and shape as y$\}$
10. $\{(x, y) \in A \times A$: x has the same shape as y$\}$
11. $\{(x, y) \in T \times T$: x has the same perimeter as y$\}$
12. $\{(x, y) \in T \times T$: x has the same area as y$\}$

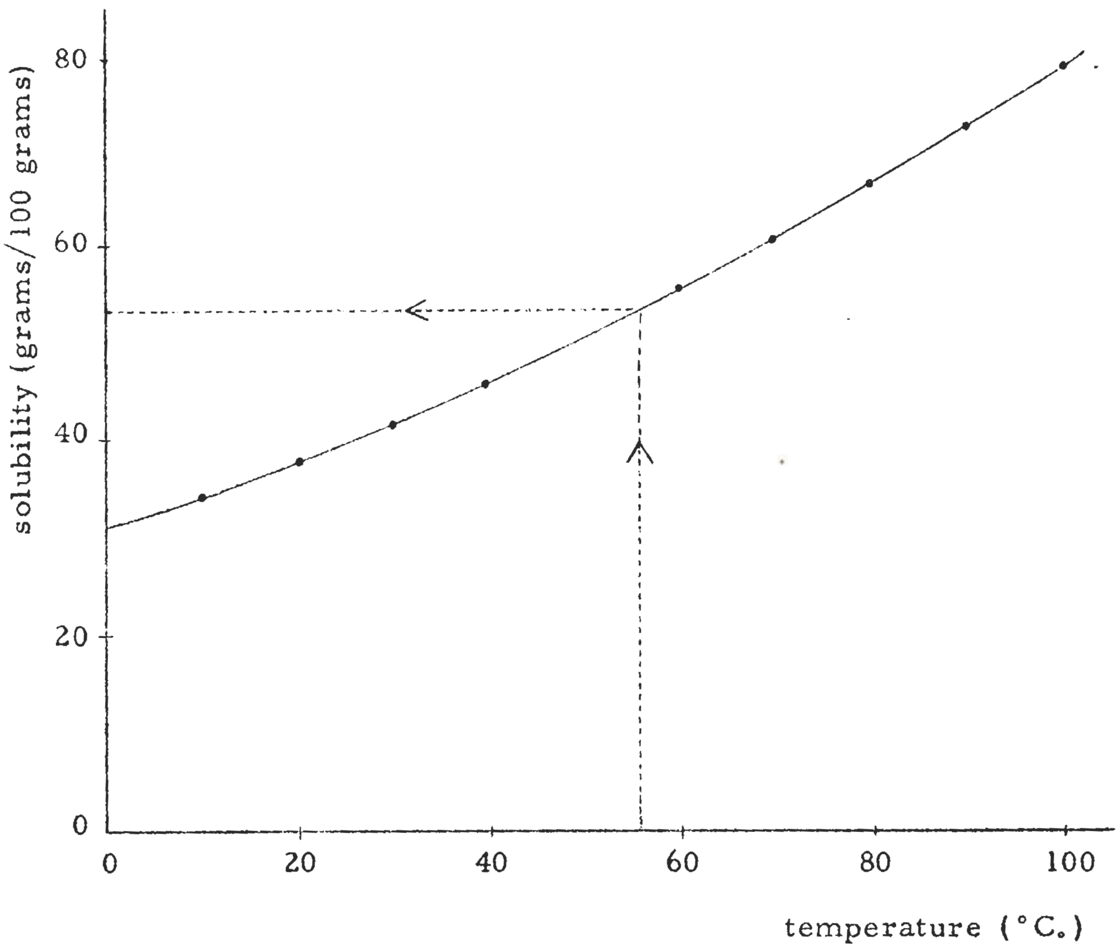
[Supplementary exercises on symmetry and reflexivity of relations are in Part G, pages 5-244 through 5-245, and in Part I, page 5-248. Optional exercises on other properties of relations are in Part H, pages 5-245 through 5-247, and in Part J, pages 5-249 through 5-250.]

5.05 Functions.--We started our discussion of relations by pointing out that scientific investigators often ask questions about the relationship of one thing to another. For example, suppose an anthropologist wants to know the relation of weight to height for a given group of people. After collecting data [ordered pairs], he might get this picture of the relation:



If he has reason to believe that the group he worked with is representative of a larger group, he can use this chart to make certain predictions about individuals in the larger group. For example, given the height of a person, he can use the chart to predict upper and lower bounds for the person's weight.

Another investigator, this time a chemist, wants to know the relation of the solubility of a certain salt to the temperature of the water in which the salt is being dissolved. One way of doing this is to take a known quantity of water, say 100 grams, heat it to a certain temperature and keep it there while slowly adding salt crystals to the water until no more salt can be dissolved. If he uses 100 grams of water then the number of grams of salt which can be dissolved is the solubility of the salt at that temperature. To minimize the effect of experimental error, the chemist would repeat the experiment several times for each temperature and average the results. The ordered pairs for ten temperatures are shown by the heavy dots on the chart on the next page. The chemist makes the reasonable guess that if he draws a smooth curve which follows the pattern of these dots, he will have a picture of the relation of solubility of this salt [ammonium chloride] to temperature.



With this chart, the chemist can predict the solubility of ammonium chloride at any temperature [between 0°C and 100°C]. What solubility would he predict for ammonium chloride at 48°C?

Although both the anthropologist and the chemist discover and graph relations from which predictions can be made, the predictions made by the chemist are of a different kind than those made by the anthropologist. For each temperature, the chemist can predict the corresponding solubility. For each height, the anthropologist can only assign bounds to the set of corresponding weights. The chemist can claim that solubility of ammonium chloride is determined by temperature; the anthropologist cannot claim that weight is determined by height--presumably, other factors are involved.

Relations such as the chemists' in which the value of one quantity determines the corresponding value of another are called <u>functional relations</u>--for short, <u>functions</u>. In graphical terms, a function is a relation such that each vertical line crosses its graph in at most one point. [Vertical lines which do not cross the graph correspond with elements not in the domain of the relation.] In other words:

> A function is a set of ordered pairs no two of which have the same first component.

All functions are relations, but not all relations are functions.

EXERCISES

<u>A.</u> Here are several sets of ordered pairs. All of these sets are relations and some are functions. Tell which are functions.

<u>Sample.</u> $\{(2, 1), (3, 2), (4, 3), (5, 4), (6, 5), (7, 5)\}$

<u>Solution.</u> This relation is not a function because two of its ordered pairs--(7, 5) and (7, 9)--have the same first component.

1. $\{(3, 7), (8, 4), (6, 5), (10, -7), (-4, 2), (6, -4)\}$

2. $\{(4, 3), (6, 3), (-5, 3), (-2, 3), (5, 3), (2, 3)\}$

3. $\{(-1, 6), (-1, 4), (-1, 8), (-1, 8), (-1, 64), (-1, 3)\}$

4. $\{(-2, 4), (-1, 1), (0, 0), (1, 1), (1.5, 2.5), (2, 4), (7, \pi)\}$

5. $\{(3, 2), (2, 3), (-2, 3), (-3, 2), (4, 3), (3, \sqrt{4})\}$

6. $\{(5, 7), (-3, 7), (59, 7), (\pi, 7), (803, 7)\}$

7. $\{(3, 8)\}$

8. $\emptyset$

9. $\{(x, y): y = 2x + 1\}$

10. $\{(x, y): x^2 + y^2 = 5\}$

Correction. In the Sample on page 5-50, insert '(7, 9)' after '(7, 5)'.

When asking whether a given relation is a function, if 'No' is a correct answer, don't accept 'No, it's a relation'. This way of speaking tends to establish the incorrect belief that functions are not relations and that relations are those sets of ordered pairs which are not functions.

*

Answers for Part A [on pages 5-50 and 5-51].

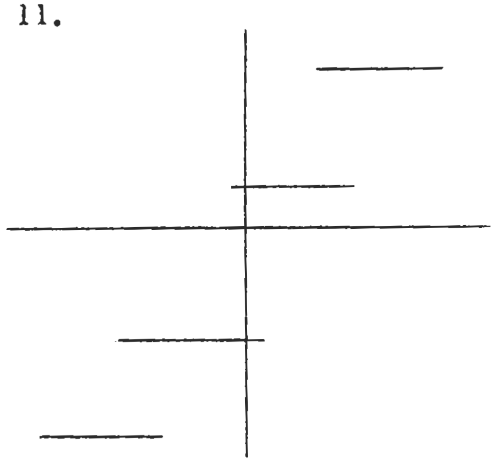
1. Not a function. [It contains two ordered pairs, (6, 5) and (6, −4), which have the same first component.]
2. Function.
3. Not a function. [It contains two ordered pairs, for example, (−1, 6) and (−1, 4), which have the same first component. This relation contains <u>five</u> ordered pairs.]
4. Function.
5. Function. [Since $\sqrt{4} = 2$, '(3, 2)' and '(3, $\sqrt{4}$)' name the same ordered pair.]
6. Function.
7. Function. [Any singleton of ordered pairs is a function. Since there are not two ordered pairs in a singleton, there cannot be two ordered pairs with the same first component.]
8. Function. [An argument like that in Exercise 7 applies here.]
9. Function.
10. Not a function. [It contains two ordered pairs, for example, (0, $\sqrt{5}$) and (0, $-\sqrt{5}$), which have the same first component.]
11. Not a function.
12. Function.
13. Not a function.
14. Not a function.
15. Function.
16. Not a function.

*

Answers for Part B [on page 5-51].

The converses of the relations in Exercises 1, 3, 7, 8, 9, and 13 are functions; the converses of the others are not.

11.



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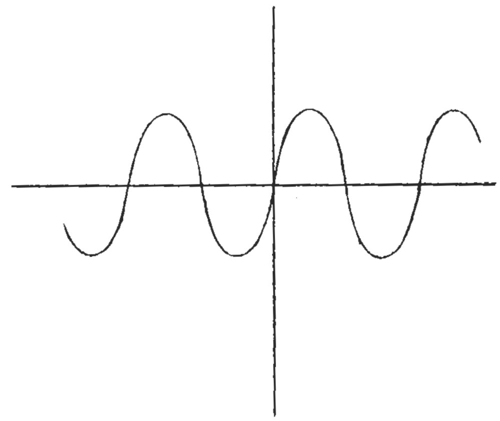

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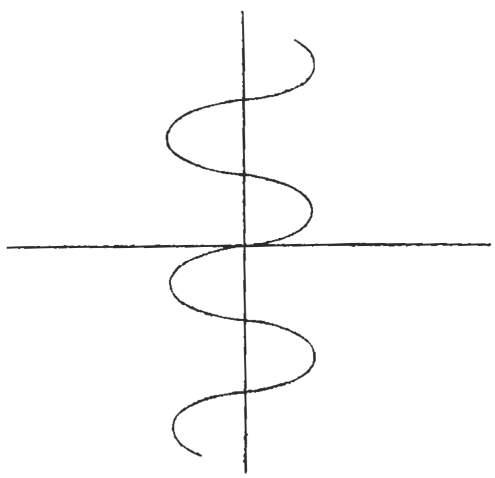

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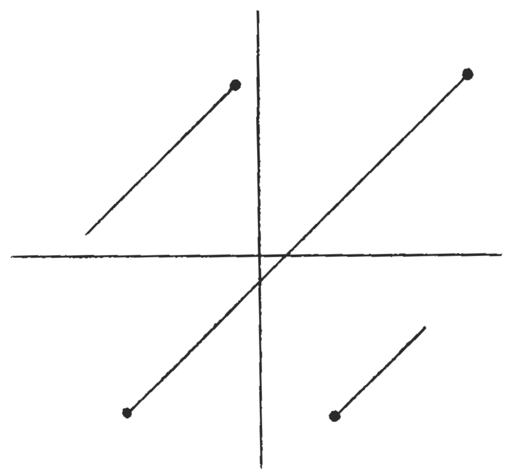

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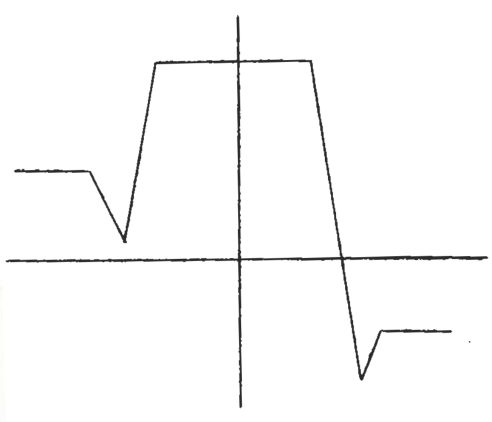
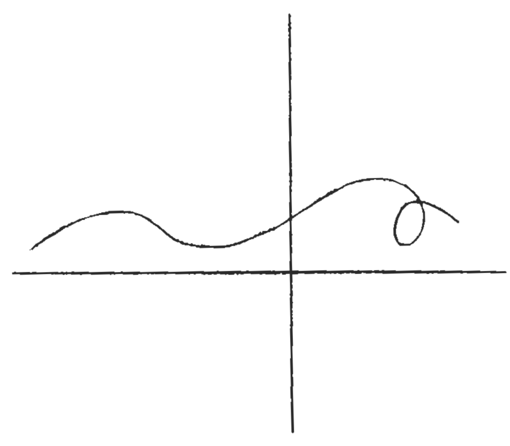


16.



B. Which of the relations in Part A have converses which are functions?

* * *

Consider the functions f and g where

$$f = \{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5)\}$$

and

$$g = \{(2, 3), (3, 4), (4, 5), (5, 6), (6, 7)\}.$$

Is it the case that $f(2) = g(2)$? That $f(3) = g(3)$? That $f(4) = g(4)$?

Is it the case that, for each x which belongs to both $\mathcal{D}_f$ and $\mathcal{D}_g$, $f(x) = g(x)$? [If you answer 'no' to this question, it means that there is an $x \in \mathcal{D}_f \cap \mathcal{D}_g$ such that $f(x) \neq g(x)$. Is this the case?]

Even though f and g have the same value for each of their common arguments, they are different functions. Explain.

* * *

C. For each pair of functions, answer these three questions:

(a) Is it the case that, for each $x \in \mathcal{D}_f \cap \mathcal{D}_g$, $f(x) = g(x)$?

(b) Is it the case that $\mathcal{D}_f = \mathcal{D}_g$?

(c) Is it the case that $f = g$?

1. $f = \{(\text{Tim}, 3), (\text{Bill}, 4), (\text{Ed}, 2), (\text{John}, 4)\}$
   $g = \{(\text{Tim}, 3), (\text{Mary}, 4), (\text{John}, 4), (\text{Cal}, 3)\}$

2. $f = \{(x, y), x = 0, 1, 2, 3, 4\colon y = x + 1\}$
   $g = \{(x, y), x = 2, 3, 4, 5, 6\colon y = x + 1\}$

3. $f = \{(4, 7), (2, 9), (3, 5), (6, 7)\}$
   $g = \{(1, 8), (5, 7), (8, 3), (9, 2)\}$

4. $f = \{(x, y)\colon y = x + 4\}$ $\quad g = \{(x, y), x \neq 0\colon y = x + 4\}$

5. $f = \{(x, y)\colon xy = 1\}$ $\quad g = \{(x, y), x \neq 0\colon y = \frac{1}{x}\}$

6. $f = \{(x, y), 2 \leq x \leq 6\colon y = x^3 - 5x^2 + 7x - 2\}$
   $g = \{(x, y), 1 \leq x \leq 5\colon y = x^3 - 5x^2 + 7x - 2\}$

7. $f = \{(x, y)\colon y = \sqrt{x^2}$ $\quad g = \{(x, y)\colon y = |x|\}$

*

8. If f and g are functions such that $\mathcal{D}_f = \mathcal{D}_g$ and $\mathcal{R}_f = \mathcal{R}_g$, does it follow that $f = g$? Explain.

Correction. In Exercise 7, insert a '}' after '$\sqrt{x^2}$'.

IMPORTANT NOTICE
The material on page 5-52 was inadvertently misplaced. Have your class skip page 5-52, and proceed to a study of pages 5-53 through 5-60. After the exercises on page 5-60 have been studied, and students are acquainted with functional notation, you should assign the material on page 5-52.

The purpose of Part C on page 5-52 is to point out that, given functions f and g, in order to show that f = g it is necessary to show, not only that $f(x) = g(x)$ for each $x \in \mathcal{D}_f \cap \mathcal{D}_g$, but also that $\mathcal{D}_f = \mathcal{D}_g$. Your students, for whom functions are sets of ordered pairs, should have no difficulty in seeing that this is the case. [With some other approaches to this notion of function, the pedagogical problem is more troublesome.]

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Answers for Part C.

1. (a) yes (b) no (c) no
2. (a) yes (b) no (c) no
3. (a) yes (b) no (c) no [A 'no' answer for part (a) requires that one find a common argument of f and g. But, $\mathcal{D}_f \cap \mathcal{D}_g = \emptyset$ (and $f \cap g = \emptyset$).]
4. (a) yes (b) no (c) no [Although $g \neq f$, $g \subseteq f$.]
5. (a) yes (b) yes (c) yes
6. (a) yes (b) no (c) no
7. (a) yes (b) yes (c) yes [Another reminder that $\forall_x \sqrt{x^2} = |x|$.]

*

8. No. For example, both $\{(x, y): y = x\}$ and $\{(x, y): y = x + 1\}$ have the same domain and the same range.

Here is a similar problem which may amuse your students.

> A boy has a drawer which contains 26 blue socks and 26 brown socks. If, without looking, he reaches into the drawer and takes out socks, one at a time, how many must he take out to be sure of having two of the same color?

Many people will give the snap answer: 27 The correct answer, of course, is: 3

☆18. Function. [The domain of 'x' is the set of all rectangles, and the domain of 'y' is the set of all ordered pairs of numbers of arithmetic. If one adopts the convention that the width of a rectangle does not exceed its length, the set selector assigns to each rectangle exactly one ordered pair of dimensions. If one does not adopt some such convention, the relation is not a function.]

9. Function.

10. Function. [In speaking of the area-measure, we assume that some particular unit has previously been specified. A similar remark applies in Exercise 11.]

11. Function.

12. Function.

13. Function. [Each triangle has one and only one circumscribed circle.]

14. Not a function. [Each circle circumscribes many triangles.]

15. Function.

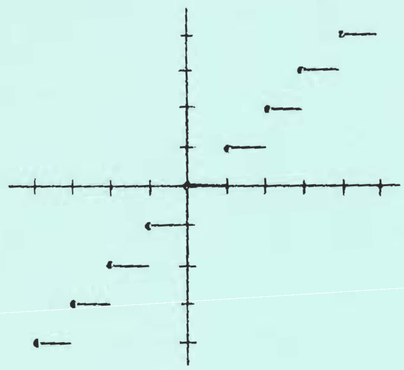
16. [As in Exercises 10 and 11, we assume that units of weight and pressure have been chosen.] (a) Function, (b) Function, (c) Function, (d) Function, (e) Function, (f) The relation is a function if (and only if), for each two days on which Hamster 7's weight is the same, his blood pressure is also the same.

17. (a) Function.

    (b) Not a function. [In addition to the information given in the bracketed sentence you also need to know that there are more than a million and one persons in New York. Hence, if no two of the first 1 000 001 ordered pairs have the same first component, then the 1 000 002nd ordered pair must have the same first component as one of the first 1 000 001. For your information, some authorities claim that no person has more than 140 000 hairs on his head!]

    (c) If there do not exist at least 3 students who have birthdays falling in the same month then at most 2 students have birthdays in each month. If this is the case then, since there are exactly 12 months, the class can have at most 24 students.

6. Function.



7. Function. [This exercise may be difficult for the students. First they must realize that the domain of 'x' is a set of ordered pairs--the set of pairs whose components are the components of points of the unit circle. This set of pairs is $\{(a, b): a^2 + b^2 = 1\}$. So, some of the members of the domain of 'x' are

$$(-1, 0),\ (0, 1),\ (\frac{1}{2}, \frac{\sqrt{3}}{2}),\ (0, -1),\ (1, 0),\ (\frac{1}{2}, -\frac{\sqrt{3}}{2}).$$

After a little thought, students should realize that the domain of $\{(a, b): a^2 + b^2 = 1\}$ is $\{x: -1 \leq x \leq 1\}$. Hence, the first component of a point x must be a number which is equal to or greater than -1 and less than or equal to 1.

Next, students should think about how one finds pairs which belong to the relation. We choose a point x [that is, an ordered pair which is in the domain of 'x']. Suppose we choose (-1. 0). Now, the first component of this point is -1, so, according to the set selector, the pair $((-1, 0), -1)$ belongs to the relation. If we choose another point x, say, the point (1, 0), we get ((1, 0), 1), and this is a pair which is a member of the relation. Still other pairs which belong to the relation are

$$((0, 1), 0),\ ((\frac{1}{2}, \frac{3}{2}), \frac{1}{2}),\ ((0, -1), 0),\ ((\frac{1}{2}, -\frac{\sqrt{3}}{2}), \frac{1}{2}).$$

Be sure students see that, even though the pairs (0, 1) and (0, -1) which are in the domain of 'x' do have the same first component, [C_1 is not a function], they are <u>different</u> pairs. So, when we use them to get pairs which belong to the relation in question [((0, 1), 0) and ((0, -1), 0)] we get ordered pairs with different first components.]

8. Not a function. [For example, (0, (0, 1)) and (0, (0, -1)) both belong to the relation.]

Answers for Part D [on pages 5-53 and 5-54].

[The kind of discussions referred to on TC[5-35, 36]a and TC[5-47]a should pay off in doing these exercises. For example, in Sample 1, students should understand the nature of the domains of 'x' and 'y'. The ordered pairs in this relation are ordered pairs of people. Note that not every member of the domain of 'x' belongs to the domain of the relation. The ordered pairs in the relation in Sample 2 are ordered pairs whose first components are circles and whose second components are points. In this case, the domain of 'x' is the domain of the relation, and the range of the relation is the domain of 'y'.]

1. Not a function. [Since each male has two parents, for each ordered pair in the relation, there is another which has the same first component but a different second component.]

2. Function. [Note that this relation is a subset of the one in Exercise 1. Also, both relations have the same domain.]

3. Function. [In discussing this exercise, you may have some students who will say that the relation is not a function. Their argument will probably be that the relation might contain such pairs as

   (Mr. S. A. Zick, 2), (Mr. I. E. Aye, 1), (Mr. S. A. Zick, 3)

   since, if Mr. Zick owns 3 automobiles it is certainly the case that he owns 2! If this argument arises, point out to the students that, in ordinary usage, when one says 'Mr. Zick owns 3 automobiles', one means that Mr. Zick owns <u>exactly</u> 3 automobiles. This is the convention we are using in Exercise 3 [and, in Exercises 5, 9, 12, 14, and 17(a)]. Students should realize, also, that the range of this function is a subset of the set of whole numbers, and could be determined by analyzing automobile registrations. Certainly, 0, 1, and 2 belong to the range, as do several other small whole numbers. There is surely a largest member of the range. Very likely, not all whole numbers between 0 and this largest number belong to the range... .]

4. Not a function.

5. Function. [Ask students if the converse of this relation is a function. Since it is reasonable to suppose that there were the same number of traffic deaths on some two days, the converse is <u>not</u> a function.]

D. Although many of the functions you will work with in mathematics are sets of ordered pairs of real numbers, functions can be sets of ordered pairs of any kind. For example, the domain of a function can consist of wagons, and its range of horses. Each of the following exercises refers to a relation. Tell if the relation is a function and be prepared to support your answer.

Sample 1. $\{(x, y) \in$ People $\times$ People: x is the father of $y\}$

Solution. Since some fathers have more than one child, this relation is not a function. [Note that 'People' is being used as a name for the set of all human beings.]

Sample 2. $\{(x, y) \in$ Circles $\times$ Points: y is the center of $x\}$

Solution. Since each circle has at most one center, this is a function. [Since each circle has a center, the domain of this function is the set of all circles. Do you think that $\{(x, y) \in$ Points $\times$ Circles: x is the center of $y\}$ is a function?]

1. $\{(x, y) \in$ People $\times$ People: x is a son of $y\}$
2. $\{(x, y) \in$ People $\times$ People: x is a son of y and y is a woman$\}$
3. $\{(x, y) \in$ New Yorkers $\times$ Whole numbers: x owns y automobiles$\}$ [Discuss the range of this relation.]
4. $\{(x, y) \in$ Students $\times$ Students: x and y are in the same grade$\}$
5. $\{(x, y) \in$ Days $\times$ Whole numbers: there were y traffic deaths on $x\}$
6. $\{(x, y) \in$ Real numbers $\times$ Integers: $y \leq x < y + 1\}$ [Graph this relation.]
7. $\{(x, y) \in C_1 \times$ Reals: y is the first component of the point $x\}$ [C_1 is the unit circle, that is, the circle in the number plane with center $(0, 0)$ and radius 1.]
8. $\{(x, y) \in$ Reals $\times C_1$: the point y has x for first component$\}$
9. $\{(x, y) \in$ Houses $\times$ Whole numbers: there are y rooms in $x\}$
10. $\{(x, y) \in$ Squares $\times N$: y is the area-measure of $x\}$ [N is the set of numbers of arithmetic.]

11. $\{(x, y) \in \text{Squares} \times N$: y is the side-measure of x$\}$

12. $\{(x, y) \in \text{Males} \times N$: x is 20 years old and is y inches tall$\}$

13. $\{(x, y) \in \text{Triangles} \times \text{Circles}$: y contains the vertices of x$\}$

14. $\{(x, y) \in \text{Circles} \times \text{Triangles}$: x contains the vertices of y$\}$

15. $\{(x, y) \in \text{Ears of corn} \times \text{Whole numbers}$: x has y rows of kernels$\}$

16. [Suppose a biologist is conducting a week-long nutrition experiment on the members of a set of hamsters. At noon on each day he records the weight and the blood pressure of each hamster. Let H be the set of hamsters and D be the set of days of the experiment.]

    (a) $\{(x, y) \in D \times N$: Hamster 7 weighs y ounces on x$\}$

    (b) $\{(x, y) \in D \times N$: y is the average of the ounce-measures of the weights of the hamsters on x$\}$

    (c) $\{(x, y) \in H \times N$: x weighs y ounces on Thursday$\}$

    (d) $\{(x, y) \in H \times N$: y is the measure of the blood pressure of x on Wednesday$\}$

    (e) $\{(x, y) \in D \times N$: y is the measure of the blood pressure of Hamster 3 on x$\}$

    (f) $\{(x, y) \in N \times N$: $\exists_{z \in D}$ x is the measure of the weight of Hamster 7 on z and y is the measure of its blood pressure on z$\}$

17. [Assume that no person has more than a million hairs on his head.]

    (a) $\{(x, y) \in \text{New Yorkers} \times \text{Whole numbers}$: x has y hairs on his head$\}$

    (b) $\{(x, y) \in \text{Whole numbers} \times \text{New Yorkers}$: y has x hairs on his head$\}$

    (c) Show that in a class of 25 students at least 3 have birthdays falling in the same month.

☆ 18. $\{(x, y) \in \text{Rectangles} \times (N \times N)$: the components of y are the inch-width and the inch-length of x, respectively$\}$

Quiz.

(a) Which of the relations listed below are functions?

(b) Which of the converses of the relations listed below are functions?

1. $\{(5, 0), (0, 0), (-2, 3), (-5, 0)\}$
2. $\{(0, 5), (0, 0), (-2, 3), (-5, 8)\}$
3. $\{(x, y) \in \text{Reals} \times \text{Numbers of arithmetic}: y = |x|\}$
4. $\{(a, b) \in \text{Numbers of arithmetic} \times \text{Reals}: a = |b|\}$
5. $\{(0, 0)\}$
6. $\{(r, s) \in \text{Nonnegative reals} \times \text{Reals}: s = -r \text{ or } s = r\}$
7. $\{(m, n) \in \text{Reals} \times \text{Reals}: n = m\}$
8. $\{(p, q) \in \text{Reals} \times \text{Reals}: q = p\}$
9. $\{(c, d) \in \text{Reals} \times \text{Reals}: c = d\}$
10. $\{(x, y): y^2 = 25 - x^2\}$
11. $\{(x, y), |x| \leq 5: y = \sqrt{25 - x^2}\}$

*

Answers for Quiz.

(a) 1, 3, 5, 6, 7, 8, 9, and 11 are functions.

(b) The converses of 2, 4, 5, 7, 8, and 9 are functions.

9. {(x, y) ∈ Rats × N: x learns the maze in y trials}

10. {(x, y) ∈ Mazes × N: the rat learns x in y trials}

11. {(x, y) ∈ Nonnegative numbers × Nonnegative numbers: $y = \sqrt{x}$}

12. {(x, y) ∈ Reals × Reals: $y = -x$}

13. {(x, y) ∈ Reals × Nonnegative numbers: $y = x^2$}

14. {(x, y) ∈ Positive integers × N: x has y prime factors}

*

On page 5-56, the parentheses in the expression 'F(3)', rather than serving as grouping symbols, indicate application of a function to an argument. In such contexts, '( )' is the translation into mathematical language of the English word 'of'. 'of' generally indicates function-application. For example, the use of the word 'of' in the next-to-last sentence above indicates application of the function whose arguments are English words and whose values are their translations into mathematical symbols. Using 'T' as a name for this function, the sentence in question can be rewritten: '( )' = T ('of'). So, when you see 'of', look for a function. [For example, the two other uses of 'of' in this paragraph suggest that the word 'application' refers to a function. In fact, application is a "function of two variables" [See page 5-109.] One of its ordered pairs is ((F, 3), 5).] In this light, such symbols as '35%' are seen to be names of functions [See TC[1-59].]
35% = multiplying by 0.35; and 35% of 48 = 35% (48) = 48 × 0.35.

*

Answers for Part E.

1. Ex. 9	2. Ex. 3	3. Ex. 6	4. Ex. 13
5. Ex. 7	6. Ex. 10	7. Ex. 5	8. Ex. 17 (a)
9. Ex. 16 (e)	10. Ex. 15	11. Ex. 12	12. Ex. 16 (c)
13. Ex. 18	14. Ex. 11		

*

Answers for Part F [on pages 5-55 and 5-56].

1. $\{(x, y) \in$ Families $\times$ N: there are y children in x$\}$
   [In place of 'N' we could have used a 'W' to refer to the set of whole numbers. A similar remark applies to the index given for Exercises 6, 9, 10, and 14.]

2. $\{(x, y) \in$ Triangles $\times$ N: y is the perimeter of x$\}$

3. $\{(x, y) \in$ Circles $\times$ N: y is the diameter of x$\}$
   [The word 'diameter' is ambiguous. A diameter of a given circle is a segment whose end points belong to the circle and which contains the center of the circle. The diameter of a given circle is the measure of such a segment.]

4. $\{(x, y) \in$ Males $\times$ N: y is the dollar-income of x during his lifetime$\}$

5. $\{(x, y) \in$ Years $\times$ N: y is the dollar-income of John Wilson during x$\}$

6. $\{(x, y) \in$ Zabranchburg High classes $\times$ N: there are y students in x$\}$
   [See comment for Exercise 1.]

7. $\{(x, y) \in$ Years $\times$ N: y inches of rain fell in Portland, Oregon during x$\}$

8. $\{(x, y) \in$ Days $\times$ Reals: y is the temperature of the frog on x$\}$
   [Relative measures, such as centigrade or Fahrenheit temperature-measures, are real numbers. They measure a "directed trip" from the temperature of melting ice [C], or of a melting mixture of ice and salt [F], to the temperature of the observed body. It would be exactly analogous to measure the height of an object as $^{-}40$ if it were 40 feet below the roof level of a given building, and $^{+}30$ if it were 30 feet above the roof level. [Measures of absolute temperature are numbers of arithmetic.]]

E. The relations in Part D which are functions are commonly referred to by noun phrases such as 'the center of a circle' [Sample 2] and 'a male person's mother' [Exercise 2]. We give below several noun phrases which refer to some of the functions given in the exercises of Part D. Name the exercise in the blank following the noun phrase.

1. the number of rooms in a house ________
2. the number of automobiles owned by a New Yorker ________
3. the greatest integer not greater than a real number ________
4. the circle which circumscribes a triangle ________
5. the first component of a point on the unit circle ________
6. the area-measure of a square ________
7. the daily traffic death toll ________
8. the number of hairs on a New Yorker's head ________
9. the daily measure of blood pressure of Hamster 3 ________
10. the number of rows of kernels on an ear of corn ________
11. the inch-height of a 20-year old man ________
12. the ounce-weight of an experimental hamster on Thursday ____
13. the inch-dimensions of a rectangle ________
14. the side-measure of a square ________

F. Write brace-notation names for the functions referred to by the noun phrases.

Sample 1. the radius of a circle

Solution. $\{(x, y) \in \text{Circles} \times N\colon\ y \text{ is the radius of } x\}$

Sample 2. the sum of a real number and 2

Solution. $\{(x, y)\colon\ y = x + 2\}$

1. the number of children in a family
2. the perimeter of a triangle
3. the diameter of a circle
4. the dollar income of a man during his lifetime
5. the annual dollar income of John E. Wilson
6. the number of students in a Zabranchburg High class

7. the number of inches of annual rainfall in Portland, Oregon
8. the daily temperature of a certain frog
9. the number of trials it takes a rat to learn a certain maze
10. the number of trials it takes a certain rat to learn a maze
11. the square root of a nonnegative number
12. the opposite of a real number
13. the square of a number
14. the number of prime factors of a positive integer

FUNCTIONAL NOTATION

Since a function is a relation, a function has a domain, and a range. The members of the domain of a function are often called <u>arguments of the function</u>, and the members of its range are called <u>values of the function</u>. Functions can be named just as one names other relations.

By definition, for each argument x of a function F, there is one and only one value y of F such that $(x, y) \in F$. Suppose F is $\{(x, y):\ y = 2x - 1\}$. Now, 3 is an argument of F, and the y such that $(3, y) \in F$ is $2 \cdot 3 - 1$, or 5. Instead of saying that

$$(3, 5) \in F \quad \text{or that} \quad 5\ F\ 3,$$

it is customary to say that

$$F(3) = 5.$$

[Read this as 'F of 3 is 5'.] So, in this context, the symbol 'F(3)' is a numeral for 5. [Of course, when 'F' is used as a name of a different function, 'F(3)' may be a name for something else, and 'F(3)' will be nonsense if 3 is not an argument of the function.]

Thus, one can form a name for a value of a function by combining a name of the function with a name of a corresponding argument. Such a procedure does not work with relations which are not functions. For example, suppose $R = \{(x, y):\ x^2 + y^2 = 25\}$. In general, it is not possible to specify a member of the range of R merely by specifying a member of its domain. It would be nonsense to say, for example, that 'R(3)' is a name for <u>the</u> member of the range of R which corresponds with 3. Why?

13. (a) 1 (b) -1 (c) $3/2$ (d) $-3/2$ (e) 0
(f) -481 (g) $\{x: x \neq 0\}$

*

Answers for Part C [on page 5-59].

1. $\{2, 6, 24\}$
2. $\{0\}$
3. the set of integers
4. $\emptyset$
5. $\{9\}$

Answers for Part D [on page 5-60].

1. 6
2. $5\frac{1}{2}$
3. 2
4. $1\frac{1}{8}$
5. 7
6. 7
7. $2\frac{1}{2}$
8. $1\frac{1}{2}$
9. No. Since $P(r_3) = P(r_4)$, but $A(r_3) \neq A(r_4)$, the relation contains two ordered pairs $[(7, 2\frac{1}{2})$ and $(7, 1\frac{1}{2})]$ with the same first component.

   No. There are two rectangles [not among those pictured] which have the same area-measures, but different perimeters. For example, suppose r_5 has width-measure 3 and length-measure 5, while r_6 has width-measure 2.5 and length-measure 6. Both have area-measure 15, but $P(r_5) = 16$ while $P(r_6) = 17$.

10. (a) 14; 10 (b) 4; 26 (c) 30
(d) 14 (e) 2; 3

7. (a) 4 (b) 5 (c) -5 (d) -5 (e) 0 (f) -1
(g) [$3.7 \notin \mathscr{R}_G$] (h) [Any numeral for a real number in $\{x\colon\ 4 \le x < 5\}$ is a correct answer.]
(i) the set of real numbers (j) the set of integers

[The function G of Exercise 7 is sometimes called 'the greatest integer function'. Its graph is shown on TC[5-53, 54]b.]

8. (a) 1 (b) -1 (c) 1
(d) [Any numeral for a nonnegative real number is a correct answer.]
(e) [$0 \notin \mathscr{R}_f$]
(f) [Any numeral for a negative real number is a correct answer.]
(g) the set of real numbers (h) $\{1, -1\}$

9. (a) 3 (b) 1 (c) Martin (d) Ruth
(e) [George $\notin \mathscr{D}_A$] (f) 8
(g) {Mary, Ruth, Martin, Alice} (h) $\{1, 3, 5, 8\}$

10. (a) 25 (b) -10 (c) 500 (d) 2500 (e) 21
(f) 2×10^4 (g) 0 (h) -1 (i) the set of real numbers

11. (a) [$1 \notin \mathscr{D}_F$] (b) 2 (c) [$2 \notin \mathscr{D}_F$]
(d) [$-2 \notin \mathscr{D}_F$] (e) 30 (f) 8 (g) [$2.5 \notin \mathscr{D}_F$]
(h) 20.25 (i) [$0 \notin \mathscr{R}_F$] (j) $\{x\colon\ x > 0\}$

12. (a) 12 (b) 2 (c) 22 (d) 2 (e) 2/3
(f) -2 [or: -1] (g) $\{x\colon\ x \ge -1/4\}$

[Students may guess the correct answer to part (g) of Exercise 12 after drawing a graph of the function h. They can obtain a check on the graph by using methods learned in Unit 3 to solve the inequation '$x^2 + 3x + 2 < 0$'. These methods involve factoring '$x^2 + 3x + 2$' and the factored form, '$(x + 2)(x + 1)$', may suggest the equivalent expressions '$[(x + 3/2) + 1/2]\,[(x + 3/2) - 1/2]$' and '$(x + 3/2)^2 - 1/4$'. If so, they can at once verify their guess as to the answer for part (g). This procedure of "completing the square by factoring" is sometimes helpful in studying quadratic functions.]

Answers for Part A.

1. 4   2. 8   3. 4   4. −22

*

Note that part (d) of Sample 1 for Part B cannot be answered by filling the blank. For example, 'g(13) = nonsense' is, itself, nonsense! In the solution given for part (d), the 'is' is not the 'is' of identity [See TC[1-L]a.] This solution could be paraphrased: 'g(13)' ϵ nonsense because $13 \notin \mathcal{D}_g$ [If so paraphrased, 'nonsense' is used as a name for the class of meaningless expressions.] Also, note that equally good answers to parts (e) and (f) are:

(e) $\mathcal{D}_g = \{5, 7, 3, 8\}$ (f) $\mathcal{R}_g = \{9, 3, 7, 4\}$

*

Answers for Part B [on pages 5-57, 5-58, and 5-59].

1. (a) 18   (b) 1   (c) 27   (d) 'f(11)' is nonsense
   (e) $\{1, 2, 3, 4, 5\}$ (f) $\{1, 4, 11, 18, 27\}$

2. (a) 8   (b) [$1.2 \notin \mathcal{D}_F$]   (c) 7   (d) 9
   (e) $\{1.1, 1.3, 1.6, 2\}$ (f) $\{7, 8, 9, 10\}$

3. (a) 16   (b) −5   (c) 10.9   (d) $3\pi + 1$
   (e) the set of real numbers   (f) the set of real numbers

4. (a) 3   (b) 17   (c) [$3.2 \notin \mathcal{D}_f$]   (d) 11   (e) [$14 \notin \mathcal{R}_f$]
   (f) [$3.5 \notin \mathcal{R}_f$]   (g) the set of integers   (h) the set of odd numbers

5. (a) 1/6   (b) 6   (c) −1/2   (d) 2   (e) [$0 \notin \mathcal{D}_g$]
   (f) [$0 \notin \mathcal{R}_g$]   (g) $\{x: x \neq 0\}$ (h) $\{x: x \neq 0\}$

6. (a) 7   (b) 7   (c) 7   (d) 7
   (e) [Any numeral for a real number is a correct answer.]
   (f) [$0 \notin \mathcal{R}_H$]   (g) the set of real numbers   (h) $\mathcal{R}_H = \{7\}$

EXERCISES

A. Each of these functions has a domain which contains the argument 3. Tell the value of the function which corresponds with this argument.

1. $\{(5, 8), (3, 4), (16, 8), (17, 4)\}$
2. $\{(7, 3), (3, 8), (8, 3), (51, 3)\}$
3. $\{(x, y): x + y = 7\}$
4. $\{(x, y): x^3 + y = 5\}$

B. In each of the following exercises you are given a function. Some of the values of the function are given in functional notation. Your job is to use ordinary notation to tell what these values are.

Sample 1. $g = \{(5, 9), (7, 3), (3, 7), (8, 4)\}$

(a) $g(5) =$ ______ (b) $g(7) =$ ______

(c) $g(3) =$ ______ (d) $g(13) =$ ______

(3) $\mathcal{D}_g =$ ____________ (f) $\mathcal{R}_g =$ ____________

Solution. (a) $g(5) = 9$ (b) $g(7) = 3$

(c) $g(3) = 7$

(d) '$g(13)$' is nonsense because 13 is not in the domain of g.

(e) $\mathcal{D}_g = \{3, 5, 7, 8\}$ (f) $\mathcal{R}_g = \{3, 4, 7, 9\}$

1. $f = \{(1, 1), (2, 4), (3, 11), (4, 18), (5, 27)\}$

(a) $f(4) =$ ______ (b) $f(1) =$ ______

(c) $f(5) =$ ______ (d) $f(11) =$ ______

(e) $\mathcal{D}_f =$ ____________ (f) $\mathcal{R}_f =$ ____________

2. $F = \{(1.1, 10), (1.3, 9), (1.6, 8), (2, 7), (1.3, \sqrt{81})\}$

(a) $F(1.6) =$ ______ (b) $F(1.2) =$ ______

(c) $F(2) =$ ______ (d) $F(1.3) =$ ______

(e) $\mathcal{D}_F =$ ____________ (f) $\mathcal{R}_F =$ ____________

3. $G = \{(x, y): y = 3x + 1\}$

(a) $G(5) =$ ______ (b) $G(-2) =$ ______

(c) $G(3.3) =$ ______ (d) $G(\pi) =$ ______

(e) $\mathcal{D}_G =$ ____________ (f) $\mathcal{R}_G =$ ____________

4. $f = \{(x, y): x \text{ is an integer and } y = 2x - 7\}$

(a) f(5) = ______
(b) f(12) = ______
(c) f(3.2) = ______
(d) f(___) = 15
(e) f(___) = 14
(f) f(___) = 3.5
(g) $\mathcal{D}_f$ = ______________
(h) $\mathcal{R}_f$ = ______________

5. $g = \{(x, y): xy = 1\}$

(a) g(6) = ______
(b) g(1/6) = ______
(c) g(−2) = ______
(d) g(___) = 1/2
(e) g(0) = ______
(f) g(___) = 0
(g) $\mathcal{D}_g$ = ______________
(h) $\mathcal{R}_g$ = ______________

6. $H = \{(x, y): y = 7\}$

(a) H(2) = ______
(b) H(3) = ______
(c) H(7) = ______
(d) H(506.5) = ______
(e) H(___) = 7
(f) H(___) = 0
(g) $\mathcal{D}_H$ = ______________
(h) $\mathcal{R}_H$ = ______________

7. $G = \{(x, y): y \text{ is an integer and } y \leq x < y + 1\}$

(a) G(4.3) = ______
(b) G(5) = ______
(c) G(−4.3) = ______
(d) G(−5) = ______
(e) G(0.5) = ______
(f) G(−0.5) = ______
(g) G(___) = 3.7
(h) G(___) = 4
(i) $\mathcal{D}_G$ = ______________
(j) $\mathcal{R}_G$ = ______________

8. $f = \{(x, y): (x \geq 0 \text{ and } y = 1) \text{ or } (x < 0 \text{ and } y = -1)\}$

(a) f(9) = ______
(b) f(−0.003) = ______
(c) f(0) = ______
(d) f(___) = 1
(e) f(___) = 0
(f) f(___) = −1
(g) $\mathcal{D}_f$ = ______________
(h) $\mathcal{R}_f$ = ______________

9. A = {(Mary, 1), (Ruth, 3), (Martin, 5), (Alice, 8)}

(a) A (Ruth) = ______
(b) A (Mary) = ______
(c) A (____) = 5
(d) A (____) = 3
(d) A (George) = ______
(f) A (Alice) = ______
(g) $\mathcal{D}_A$ = ______________
(h) $\mathcal{R}_A$ = ______________

*

Sometimes a function is described by an equation and a statement of the domain of the function. For example, the function f where $f = \{(x, y): y = 3x\}$ is described by: $f(x) = 3x$, $\mathcal{D}_f$ = the set of real numbers.

*

10. $f(x) = 5x$, $\mathcal{D}_f$ = the set of real numbers

(a) $f(5) =$ _____ (b) $f(-2) =$ _____ (c) $f(100) =$ _____
(d) $f(500) =$ _____ (e) $6 + f(3) =$ _____ (f) $1000 \times f(4) =$ _____
(g) $f($____$) = 0$ (h) $f($____$) = -5$ (i) $\mathcal{R}_f =$ ________

11. $F(x) = 4x - 10$, $\mathcal{D}_F = \{x:\ x > 2.5\}$

(a) $F(1) =$ _____ (b) $F(3) =$ _____ (c) $F(2) =$ _____
(d) $F(-2) =$ _____ (e) $F(10) =$ _____ (f) $F(7) - F(5) =$ _____
(g) $F(3) \div F(2.5) =$ _____ (h) $F($____$) = 71$
(i) $F($____$) = 0$ (j) $\mathcal{R}_F =$ ________

12. $h(x) = x^2 + 3x + 2$, $\mathcal{D}_h$ = the set of real numbers

(a) $h(-5) =$ _____ (b) $h(0) =$ _____ (c) $h(3) + h(-3) =$ _____
(d) $h(3 + -3) =$ _____ (e) $h(3) \div h(4) =$ _____ (f) $h($____$) = 0$
(g) $\mathcal{R}_h =$ ________ [Hint. Graph h.]

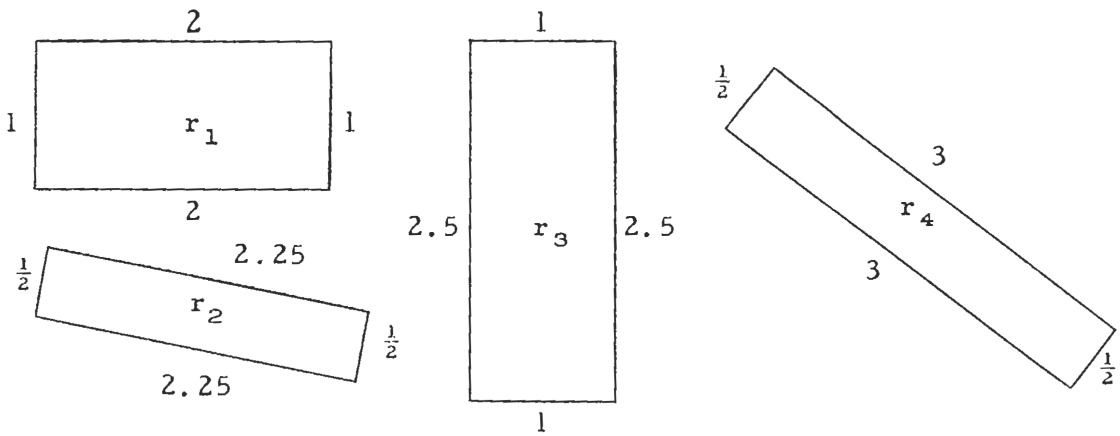
13. $g(x) = \frac{3}{x - 3}$, $\mathcal{D}_g = \{x:\ x \neq 3\}$

(a) $g(6) =$ _____ (b) $g(0) =$ _____ (c) $g(5) =$ _____
(d) $g(1) =$ _____ (e) $g(1003) + g(-997) =$ _____
(f) $g(487) + g($____$) = 0$ (g) $\mathcal{R}_g =$ ________

C. In each of the following exercises you are given the domain of a function and a sentence which can be used to compute the values of the function. Your job is to find the range of the function.

1. $\mathcal{D}_f = \{0, 1, 2\}$, $f(x) = 7x^2 - 3x + 2$
2. $\mathcal{D}_f = \{0\}$, $f(x) = -\sqrt{x}$
3. $\mathcal{D}_f$ = the set of integers, $f(x) = x$
4. $\mathcal{D}_f = \emptyset$, $f(x) = 3x + 2$
5. $\mathcal{D}_f$ = the set of real numbers, $f(x) = 9$

D. Here are some rectangles, r_1, r_2, r_3, and r_4.



Consider the functions A and P, where

$A = \{(x, y) \in R \times N\colon\ y \text{ is the area-measure of } x\}$,

$P = \{(x, y) \in R \times N\colon\ y \text{ is the perimeter of } x\}$,

R is the set of all rectangles, and N is the set of numbers of arithmetic.

1. $P(r_1) =$ ____ 2. $P(r_2) =$ ____ 3. $A(r_1) =$ ____ 4. $A(r_2) =$ ____

5. $P(r_3) =$ ____ 6. $P(r_4) =$ ____ 7. $A(r_3) =$ ____ 8. $A(r_4) =$ ____

9. Consider the relation

$$\{(x, y) \in N \times N\colon\ \exists_{r \in R}\ x = P(r) \text{ and } y = A(r)\}.$$

Is this relation a function? Is its converse a function? Explain your answers.

10. Consider the functions ℓ and w, where
ℓ = the length of a rectangle and w = the width of a rectangle.

(a) Suppose e is a rectangle such that $\ell(e) = 5$ and $w(e) = 2$. Then, $P(e) =$ ____ and $A(e) =$ ____.

(b) If $\ell(e) = 9$ and $A(e) = 36$ then $w(e) =$ ____ and $P(e) =$ ____.

(c) If $\ell(e) = 2w(e)$ and $A(e) = 50$ then $P(e) =$ ____.

☆(d) If $\ell(e) = w(e) + 5$ and $A(e) = 6$ then $P(e) =$ ____.

☆(e) If $A(e) = 6$ and $P(e) = 10$ then $w(e) =$ ____ and $\ell(e) =$ ____.

[Supplementary exercises are in Part K, pages 5-251 through 5-253.]

The word 'mapping' suggests maps of geographical regions. The analogy between "using" a function to determine a correspondence of the members of one set with those of another, and the process of drawing a map of a country, may be helpful. But, like most analyses, it can be misleading if accepted uncritically. The map-drawing process consists of choosing patches of a sheet of paper to correspond with geographical objects <u>and</u> doing some art work on the paper to delineate the chosen patches. The final result is a conventionalized picture of the country, and this is called <u>a map.</u> In such a case, there is a function which determines the chosen correspondence of certain patches of paper with certain geographical objects. Its ordered pairs have geographical objects as first components, and patches of paper as second components. Its domain is a set of geographical objects, and its range is a set of patches of paper. Now, one might be tempted to think of the range of this function as a map. But a map differs from the range of this function in just the way that a graph of an open sentence differs from the solution set of the sentence [see TC[3-11, 12]]. So, when one speaks of a function as determining a mapping, one thinks of a correspondence which might be used in drawing a map, rather than of a map. [Notice that when one graphs functions whose domain and range are sets of real numbers, one is drawing maps of the number plane. Here the number plane plays the role of the country to be mapped, and the functions play the roles of some of the geographical objects. In this case <u>the function which determines the mapping</u> has functions as arguments and patches of graph-paper as values.]

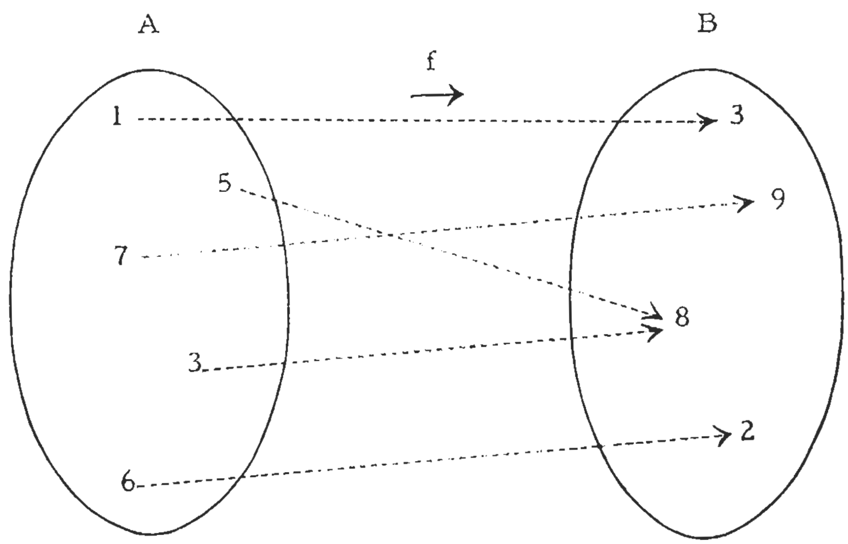
[As stated in this text, a function is said to map its domain <u>on</u> its range. For your information, it is also customary to say that a function maps its domain <u>in</u> any set which includes the range of the function. This terminology being standard, do not allow students to slur 'on's to 'in's.]

*

The answer to the bracketed question is 'No'.

FUNCTIONS AS MAPPINGS

Every function has the property that with each member of its domain, A, there corresponds a unique member of its range, B, and each member of B corresponds with some member of A.



f = {(1, 3), (5, 8), (7, 9), (3, 8), (6, 2)}

Such a correspondence is called a mapping of A on B. It is often convenient to think of a function as determining a mapping of its domain on its range. In the illustration above, we say that f maps {1, 3, 5, 6, 7} on {2, 3, 8, 9}. Also, we call the value of a function for a given argument the image of this argument. So, in the case above, 3 is the image of 1, the image of 7 is 9, 6's image is 2, and 8 is the image of both 3 and 5. [Can an argument of a function have two images?]

If we are given the ordered pairs which belong to a function (either listed between braces, or by means of a table), we can fir l the image of any member of its domain by hunting up the ordered pair whose first component is the given argument and seeing what its second component is. If a function is described as the set of ordered pairs whose components satisfy a given sentence then, to find the image of an argument, we substitute a name for the argument in the sentence and compute to find [a name for] its image.

If a function is described by a graph, we use the familiar "up-or-down and over-or-back" technique.

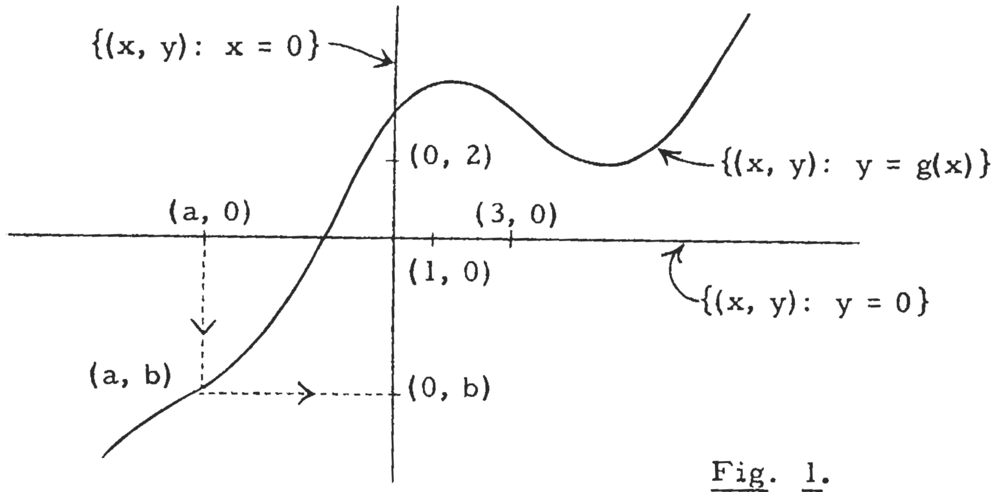


Fig. 1.

As indicated in Figure 1, the function maps the number a of its domain on the number b of its range. In using the graph of g to find the image of a member of the domain, we don't need to view the graphs of g and the axes as pictures of sets of ordered pairs. Instead, we can think of the graph of the x-axis as a picture of the set of real numbers [containing a picture of the domain of g], and we can think of the graph of the y-axis as another picture of the number line [containing a picture of the range of g]. Then, the graph of g is a "mapping line" which we use in finding the image of a member of the domain of g.

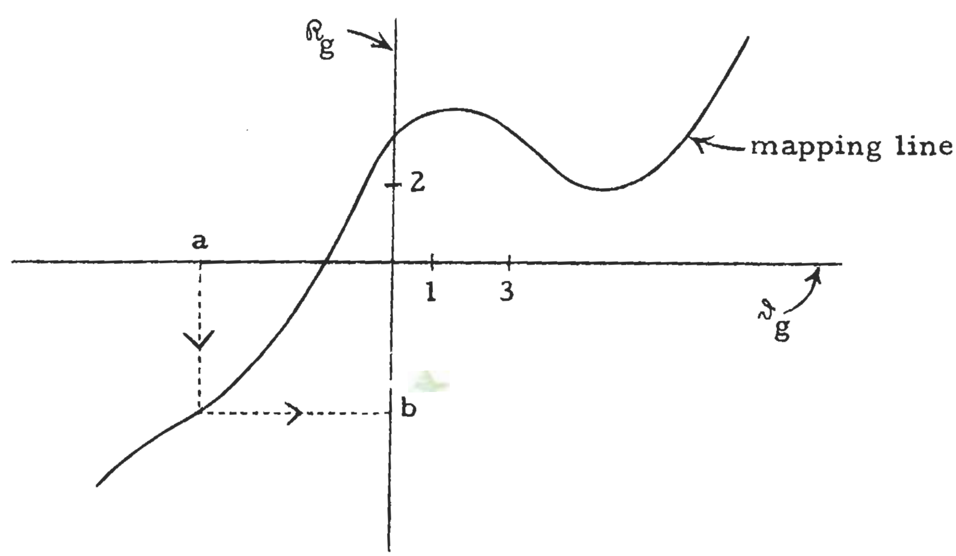


Fig. 2.

Since it is customary to use numerals, rather than names of ordered pairs, in labeling graphs of axes, students may not be awake to the fact that all points on a picture of the number plane, including those on the graphs of the axes, are graphs of ordered pairs. Be sure, in discussing Figure 1, that they are fully aware of this fact before passing to the text below the figure. You might, for example, point out that in using Figure 1 to find the value of g corresponding with a given argument a, one begins by locating the graph of the ordered pair whose first component is a and whose second component is 0. [We don't, <u>in using Figure 1,</u> begin by finding "the graph of a on the graph of the x-axis". But, in Figure 2, we <u>do</u> look for the graph of a on the graph of $\mathfrak{D}_g$.]

*

Compare Figure 3 on page 5-63 with the diagram on page 5-36.

Notice that when we used Figure 1, we thought of the curved line as a picture of a set of ordered pairs, and, for that matter, of each point of the paper as the graph of an ordered pair. In Figure 2, only the horizontal and vertical lines are pictures of sets. The only purpose of the rest of the paper is to hold these lines and the mapping line together. Notice also, that it doesn't matter much where we draw pictures of $\mathcal{D}_g$ and $\mathcal{R}_g$. For example, Figure 3 does as well as Figure 2.

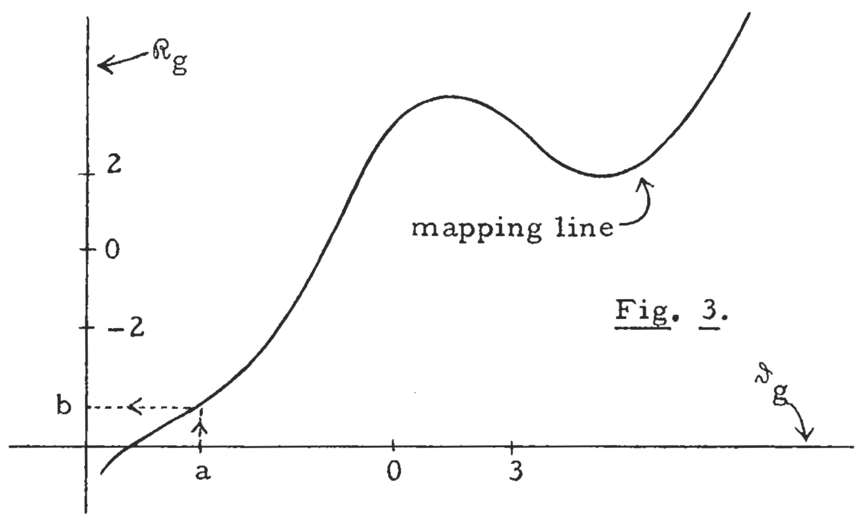


Fig. 3.

Using the graph of a function as a mapping line in determining images of members of the domain of the function is one very handy way of picturing just how the function maps its domain on its range. But there are other ways to do this. For example, Figures 4 and 5 show two ways of picturing the mapping determined by $\{(x, y): y = x + 2\}$.

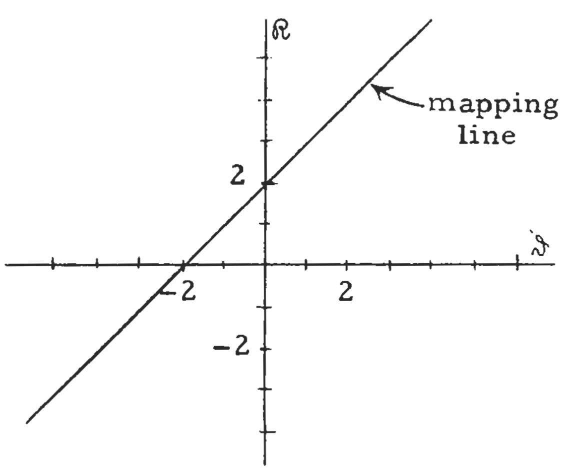


Fig. 4.

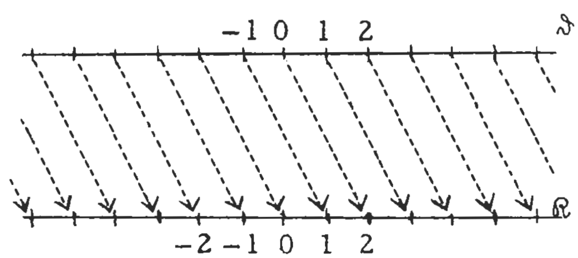
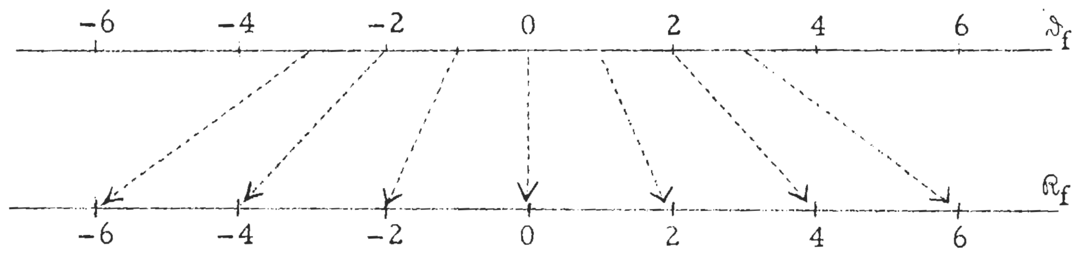
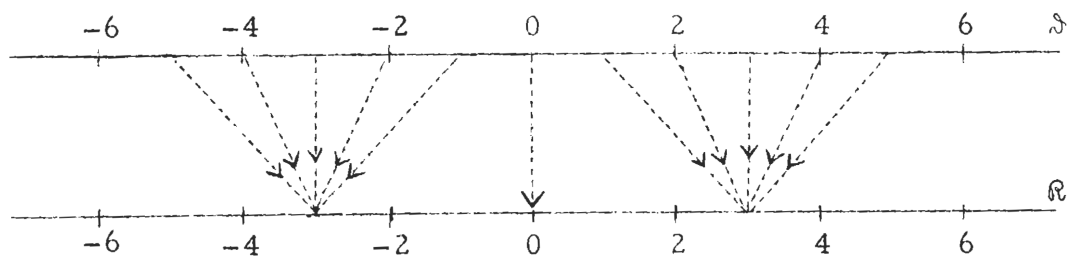


Fig. 5.

Here is a picture of a mapping determined by a function f. Give a brace-notation name for f.



The mapping shown below is determined by the function $\{(x, y): (y = -3 \text{ and } x < 0) \text{ or } (y = x \text{ and } x = 0) \text{ or } (y = 3 \text{ and } x > 0)\}$.



Draw a graph of this function.

Give a brace-notation name for the function which maps the set of real numbers on $\{2\}$. Make a picture like the ones above which shows this mapping. [What do we call such a mapping?]

EXERCISES

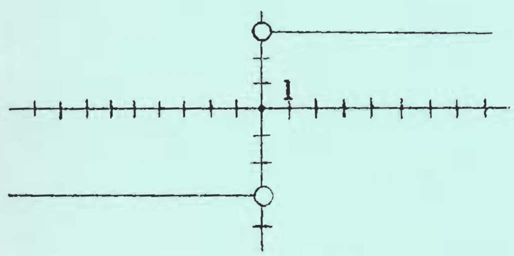
A. Make pictures [like the illustrations immediately above] of the mappings determined by the listed functions.

1. $\{(x, y): x + y = 3\}$
2. $\{(x, y): y = x^2\}$
3. $\{(x, y), x \geq 0: y = \sqrt{x}\}$
4. $\{(x, y): y = |x|\}$
5. $\{(x, y): (x + y = -4 \text{ and } x \leq 0) \text{ or } (x + y = 4 \text{ and } x > 0)\}$

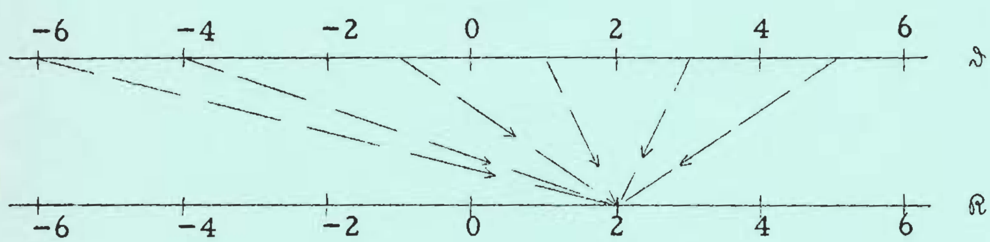
Correction. Delete from line 9b: [What do we call such a mapping?]

Answers to questions in the text on page 5-64.

$f = \{(x, y): y = 2x\}$



$\{(x, y): y = 2\}$



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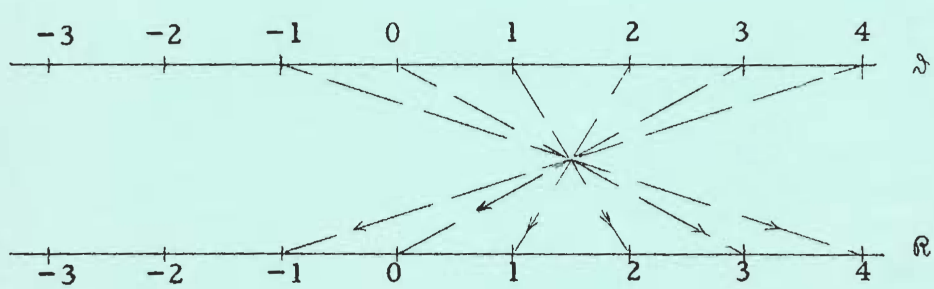
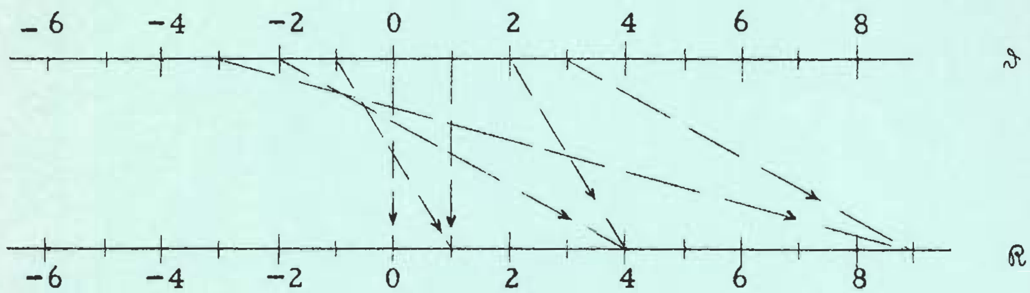
Answers for Part A.

1.

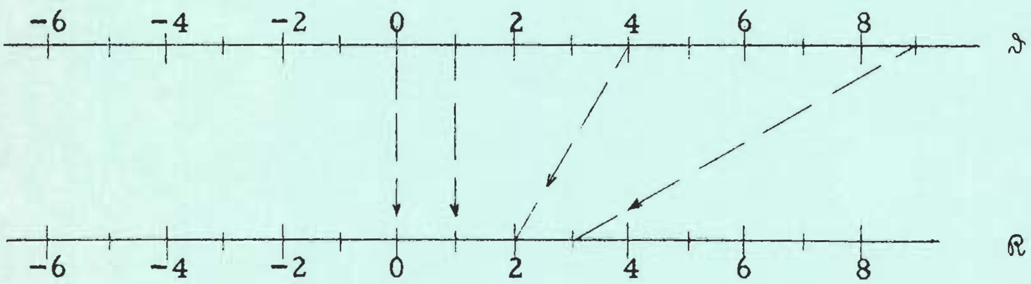


2.



-6 -4 -2 0 2 4 6 8
𝒟
ℛ
-6 -4 -2 0 2 4 6 8

3.



-6 -4 -2 0 2 4 6 8
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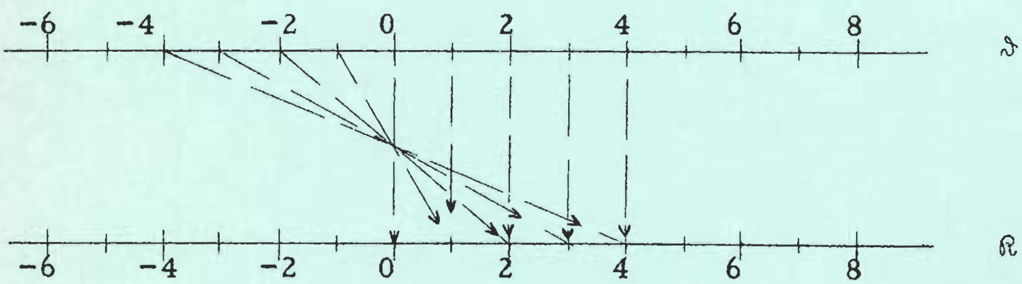
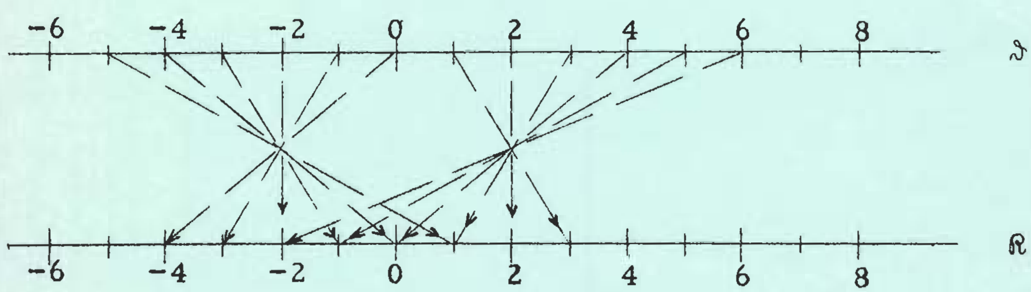
-6 -4 -2 0 2 4 6 8

4.



-6 -4 -2 0 2 4 6 8
𝒟
ℛ
-6 -4 -2 0 2 4 6 8

5.



-6 -4 -2 0 2 4 6 8
𝒟
ℛ
-6 -4 -2 0 2 4 6 8

Answers for Part B [on pages 5-65 and 5-66].

1. (a) 5 (b) 8 (c) 6

2. (a) 8 (b) 3 (c) 1, 8

3. (a) 12 (b) 4 (c) [$3 \notin \mathcal{D}_h$]

   [The fact that 3 is listed in the left loop along with 1, 2, 4, and 5 tells you only that the domain of the mapping is a subset of $\{1, 2, 3, 4, 5\}$. In the case of finite mappings which are pictured this way, we depend on the arrows to tell us which elements are members of the domain. In this connection, note the mapping f pictured at the top of page 5-70. In that mapping, the domain of f is $\{5, 6, 7, 8\}$.]

4. (a) -7 (b) π (c) π

5. (a) 9 (b) [$-2 \notin \mathcal{D}_g$] (c) $^+\sqrt{2}$

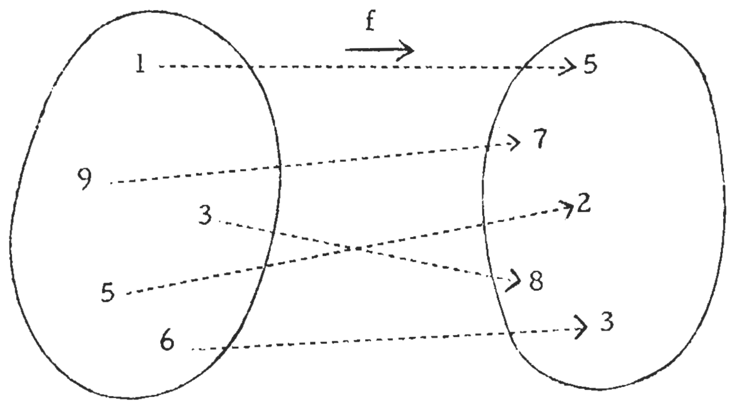
6. (a) 2 (b) [$7 \notin \mathcal{R}_g$] (c) 14
   (d) [$3.5 \notin \mathcal{D}_h$] (e) -10 (f) 0

7. (a) 2.5 (b) -3.5 (c) 0
   (d) 18.6 (e) -12.2 (f) 4/3

8. (a) 3 (b) 3 (c) -4
   (d) 0 (e) -1 (f) each number in $\{x: 8 \leq x < 9\}$

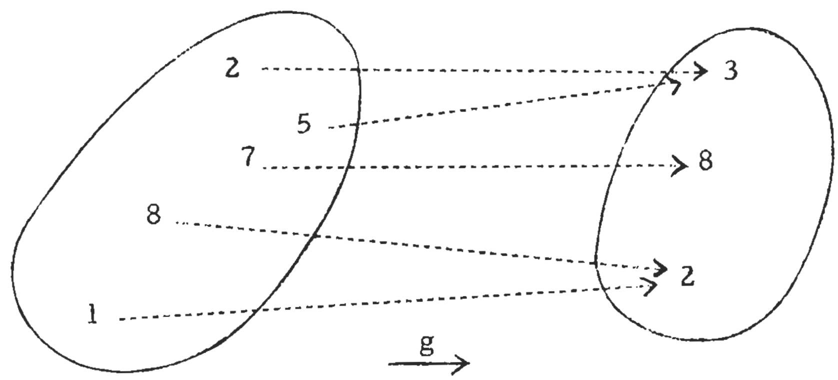
B. Each of the following exercises describes a mapping. Fill in the blanks.

1.



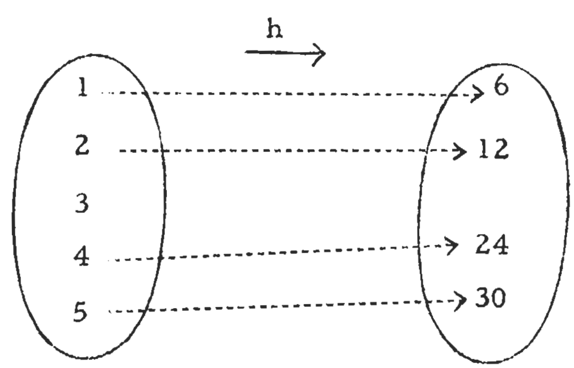
(a) f(1) = ____ (b) f(3) = ____ (c) f(____) = 3

2.



(a) g(7) = ____ (b) g(2) = ____ (c) g(____) = 2

3.



(a) h(2) = ____ (b) h(____) = 24 (c) h(3) = ____

4. f is the function which maps each real number on its opposite.

   (a) $f(7) =$ ____ (b) $f(-\pi) =$ ____ (c) $f($____$) = -\pi$

5. g is the function which maps each <u>positive</u> real number on its absolute value.

   (a) $g(9) =$ ____ (b) $g(-2) =$ ____ (c) $g($____$) = \sqrt{2}$

6. h is the function which maps each <u>integer</u> on its double.

   (a) $h(1) =$ ____ (b) $h($____$) = 7$ (c) $h(7) =$ ____

   (d) $h(3.5) =$ ____ (e) $h(-5) =$ ____ (f) $h($____$) = 0$

7. m maps each real number on its average with 0.

   (a) $m(5) =$ ____ (b) $m(-7) =$ ____ (c) $m($____$) = 0$

   (d) $m($____$) = 9.3$ (e) $m($____$) = -6.1$ (f) $m($____$) = \frac{2}{3}$

8. G maps each real number on the greatest integer less than or equal to the real number.

   (a) $G(3.7) =$ ____ (b) $G(3.4) =$ ____ (c) $G(-3.4) =$ ____

   (d) $G(\frac{1}{2}) =$ ____ (e) $G(-\frac{1}{2}) =$ ____ (f) $G($____$) = 8$

WAYS OF REFERRING TO FUNCTIONS

Mathematicians have been working with functions for at least 300 years, and during this period people developed different notions of what functions are as well as different ways of talking about them. One of the more recent ways of regarding functions is to see them as sets of ordered pairs, and this is the point of view we have adopted in this course. We shall, of course, maintain our point of view throughout the course, but we shall occasionally use other ways of writing about functions just to prepare you to understand these other ways when you come upon them in books or in talking with other students. It turns out that the various concepts of functions and the various ways of referring to them can be interpreted to make sense from our point of view.

For example, people sometimes use phrases like:

(1) the function x^2

(2) the function $y = x^2$

(3) the function $f(x)$ where $f(x) = x^2$

These can all be interpreted as names for $\{(x, y): y = x^2\}$.

Correction. On page 5-67, in line 3b, change '$h \geq 0$' to '$y \geq 0$'.

If one uses, for example, 'x^2' as a name for the squaring function, he introduces a convention which is not consistent with the use of 'x' as a variable. 'x^2' is already a pronumeral expression which has values but does not name anything, and to use it also as a name breeds confusion. For example, if one uses 'x^2' as a name for $\{(x, y): y = x^2\}$ then, since, for example, $\{(x, y): y = x^2\} = \{(y, x): x = y^2\}$, there is no apparent reason for not using 'y^2' as a name for the same function. And, if 'x^2' and 'y^2' are used as names for the same thing, one must admit '$x^2 = y^2$' as a true statement. But, in other connections, one needs to consider '$x^2 = y^2$' to be an open sentence, neither true nor false. A similar objection applies to the use, say, of '$f(x)$' as a name for a function f.

An objection to '$y = x^2$' as a name for the squaring function is that it involves using a sentence as a noun. And, as above, it would lead one to accept '$y = x^2 = x = y^2$' as a true statement in which the middle '=' is a verb while the other two '='s are "letters" occurring in two nouns. This is confusing in itself and also conflicts with a well-established and useful convention according to which '$y = x^2 = x = y^2$' is an abbreviation for the open sentence '$y = x^2$ and $x^2 = x$ and $x = y^2$'.

One might attempt to meet these objections by saying that the phrase 'the function x^2' is not the expression 'x^2' and, for example, the sentence 'the function x^2 = the function y^2' need lead to no confusion. But, in practice, the use of languages like (1) [on page 5-66] usually leads to statements such as 'x^2 is an example of a function' and 'Now, we shall study such functions as $y = x^2$, $y = 3x^2 - 5$, and $2 = x^2 + 2y - 3$'. In both of these, pronumeral expressions or sentences are evidently being used as nouns. The modes of expression illustrated in (4), (5), and (6) [on page 5-67] do not so readily foster illegitimate abbreviations as do (1), (2), and (3).

Answers for questions at bottom of page 5-67.

f_1 is not defined at 2

f_2 is not defined at 1 nor at -1

f_3 is defined neither at 1 nor at -2

f_4 is undefined at any negative number

f_5 is not defined at any negative number

f_6 is defined only for nonpositive real numbers

f_7 is defined only for nonnegative real numbers

Sometimes people refer to this function by saying that it is

(4) the function defined by the expression 'x^2', or

(5) the function defined by the equation '$y = x^2$', or

(6) the function f defined by: $f(x) = x^2$.

This last is very much like the way we described functions in Exercises 10-13 of Part B on page 5-59. The difference is that in (6) the domain of the function is not specified. In accord with a common convention, when a function is described by a phrase like (6), its domain is understood to be the set of those real numbers for which the right side of the equation has values. So, for example, the domain of the function described by (6) is understood to be the set of all real numbers. Also, the domain of the function described by:

$$\text{the function g defined by: } g(x) = \frac{2}{1 - x}$$

is understood to be the set of real numbers different from 1. We shall adopt this convention. [Similar conventions are used by people who employ the other methods of referring to functions.]

Sometimes we talk about "places" where a function is defined, or where it isn't. Consider the function h where $h = \{(x, y): xy = 1\}$. Since 2 is an argument of h, but 0 is not, we say that h is defined at 2 but not at 0. [This corresponds to saying that '$h(2)$' names a member of the range of h but that '$h(0)$' is nonsense.] At which numbers are the functions described below not defined?

$$f_1(x) = \frac{1}{x - 2} \qquad f_2(x) = \frac{1}{x^2 - 1}$$

$$f_3 = \{(x, y): y(x - 1)(x + 2) = 1\}$$

$$f_4 = \{(x, y), h \geq 0: y^2 = x\}$$

$$f_5(x) = \sqrt{x} \qquad f_6(x) = \sqrt{-x}$$

$$f_7 = \{(x, y), x \geq 0: y = x^2\}$$

EXERCISES

A. Each exercise refers to a function. Use brace-notation to name the function.

Sample 1. $q(x) = \frac{5}{2 - x}$

Solution. This refers to the function

$$\{(x, y), x \neq 2: y = \frac{5}{2 - x}\},$$

and introduces the name 'q' for this function.
[Another brace-notation name for q is: $\{(x, y): y(2 - x) = 5\}$]

1. the function $y = 5 - x$

2. the function $1 + 2x^2$

3. the function $y = 3x + 4$, for $x > 0$ [This means that the domain is the set of positive numbers.]

4. $f(x) = 1 + \sqrt{x}$

Sample 2. $f(x) = \begin{cases} 2, & \text{for } x \geq 0 \\ -2, & \text{for } x < 0 \end{cases}$

Solution. $\{(x, y): (y = 2 \text{ and } x \geq 0) \text{ or } (y = -2 \text{ and } x < 0)\}$

[An alternative is:
$\{(x, y): (\text{if } x \geq 0 \text{ then } y = 2) \text{ and } (\text{if } x < 0 \text{ then } y = -2)\}$]

5. $t(x) = \begin{cases} x, & \text{for } x \geq 1 \\ 2 - x, & \text{for } x < 1 \end{cases}$

6. $f(x) = 3$, for $x > 3$
$= x$, for $-3 \leq x \leq 3$
$= -3$, for $x < -3$

7. $g(x) = x + 3$, for $-2 \leq x \leq 2$

8. $d(x) = \begin{cases} -1, & \text{for } x \text{ a rational number} \\ 1, & \text{for } x \text{ an irrational number} \end{cases}$

B. Graph the ten functions given in Part A.

[Supplementary exercises are in Parts L and M, pages 5-254 to 5-255.]

Answers for Part A.

1. $\{(x, y): y = 5 - x\}$

2. $\{(x, y): y = 1 + 2x^2\}$

3. $\{(x, y), x > 0: y = 3x + 4\}$, or: $\{(x, y): x > 0 \text{ and } y = 3x + 4\}$

4. $\{(x, y), x \geq 0: y = 1 + \sqrt{x}\}$, or: $\{(x, y): y \geq 1 \text{ and } (y - 1)^2 = x\}$

*

When discussing Sample 2, do not allow students to speak as though f were two functions. A single function, f, is described by stating a rule for finding its value for each of its arguments. The rule has two parts, but f is one function. If students have difficulty with this, point out that, for example, in Exercise 15 on page 5-51 they had no doubt that the graph shown there pictures a single function, although a written rule for finding its values would require several parts. Also point out that the function of Exercise 1 in Part A on page 5-57 can be described, calling it 'g', by:

$$g(x) = \begin{cases} 4, & \text{for } x = 3 \text{ or } 17 \ [\text{i.e., } x = 3 \text{ or } x = 17] \\ 8, & \text{for } x = 5 \text{ or } 16 \end{cases}$$

On the other hand, point out that the function f of Sample 2 is the union of two other functions. In fact, if $f_1 = \{(x, y): x \geq 0 \text{ and } y = 2\}$ and $f_2 = \{(x, y): x < 0 \text{ and } y = -2\}$ then $f = f_1 \cup f_2$. But the same applies to any function [other than $\emptyset$]. For example,

$$\{(x, y): y = x^2\} = \{(x, y): x \leq 3 \text{ and } y = x^2\} \cup \{(x, y): x > 3 \text{ and } y = x^2\}.$$

*

5. $\{(x, y): (x \geq 1 \text{ and } y = x) \text{ or } (x < 1 \text{ and } y = 2 - x)\}$

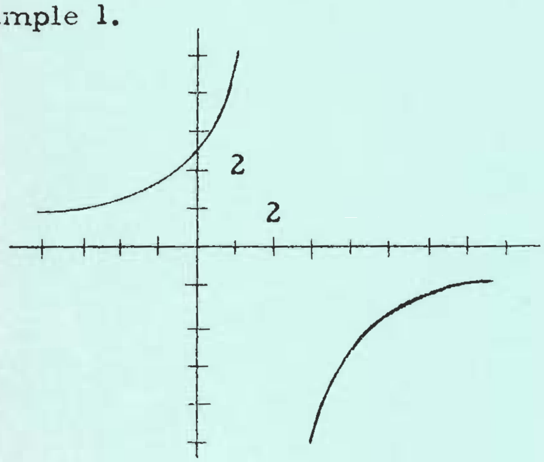
6. $\{(x, y): (x > 3 \text{ and } y = 3) \text{ or } (-3 \leq x \leq 3 \text{ and } y = x) \text{ or } (x < -3 \text{ and } y = -3)\}$

7. $\{(x, y): -2 \leq x \leq 2 \text{ and } y = x + 3\}$

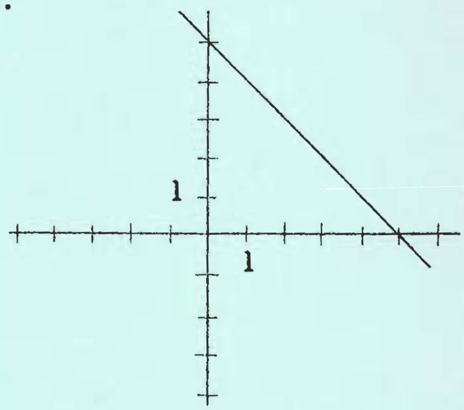
8. $\{(x, y): (x \text{ is rational and } y = -1) \text{ or } (x \text{ is irrational and } y = 1)\}$

Answers for Part B [on page 5-68].

Sample 1.



1.



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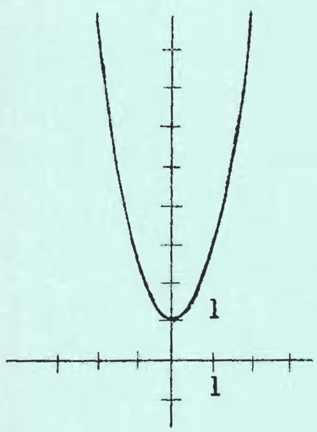
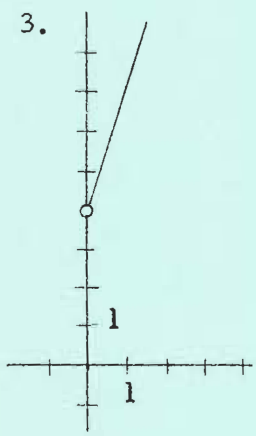


3.



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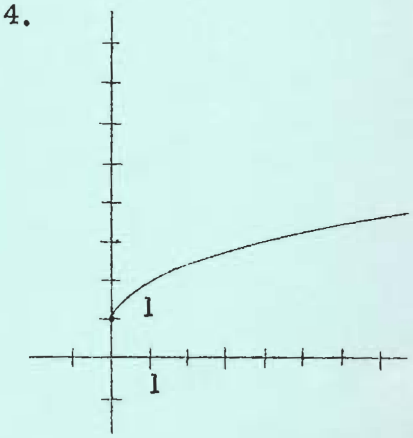
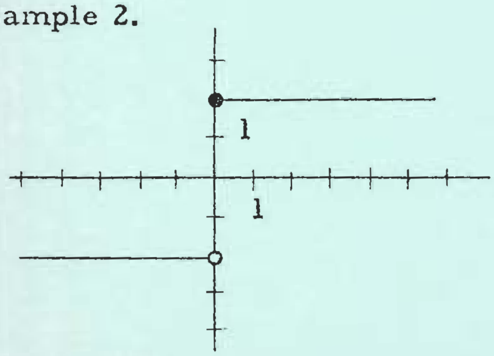


Sample 2.



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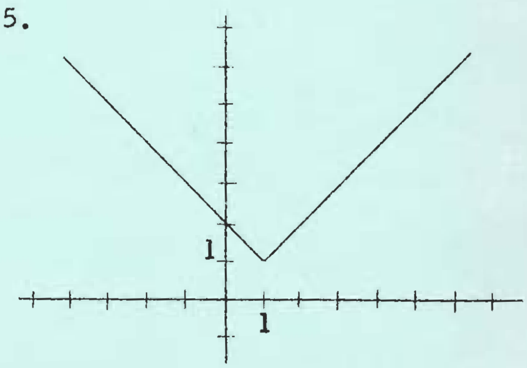

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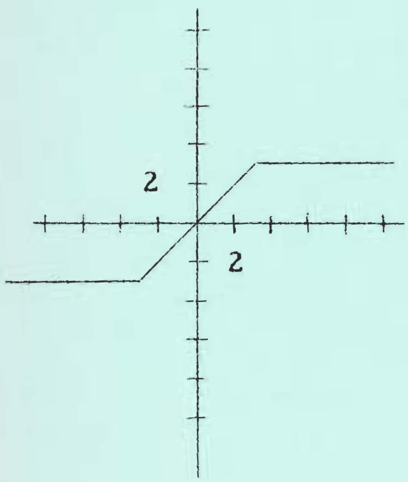

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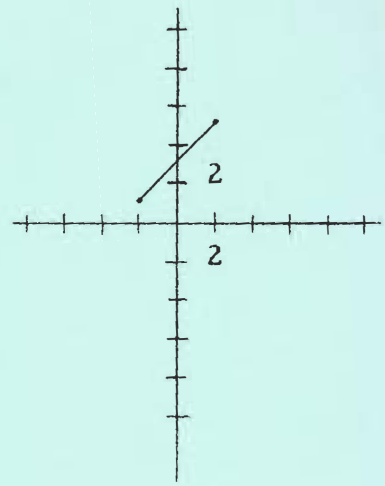


[The function of Exercise 8 cannot be graphed. If you tried, you would get a pair of parallel "lines", with each "line" full of holes.]

Answers for Exploration Exercises [on pages 5-69, 5-70, and 5-71].

1. (a) 4 (b) 12 (c) 4 (d) 12 (e) $[F_2(4) \notin \mathfrak{D}_{F_1}]$
   (f) 3 (g) $[F_1(3) \notin \mathfrak{D}_{F_1}]$ (h) 4

2. (a) -3 (b) -3 (c) 1 (d) 1 (e) -3
   (f) $[-3 \notin \mathfrak{D}_f]$ (g) 5 (h) 5 (i) 6 (j) $[f(5) \notin \mathfrak{D}_f]$

3. (a) 12 (b) 87 (c) -7 (d) 5
   (e) 7 (f) 29 (g) -13 (h) 2.7

4. (a) 19 (b) 3 (c) 33 (d) 5 (e) 100
   (f) 1002 (g), (h), (i), (j) the set of real numbers

   [While solving Exercise 4, students should discover that, for each real number x, $H(G(x)) = x = G(H(x))$. Exercise 4 helps prepare for the work on inverses of functions which begins on page 5-79.]

5. (a) 8 (b) 6 (c) 7 (d) $[8 \notin \mathfrak{D}_g]$ (e) 3
   (f) $[1 \notin \mathfrak{R}_f]$ (g) 5 (h) 3 (i) 3, 5 (j) $[7 \notin \mathfrak{D}_g]$
   (k) 9 (ℓ) $[1 \notin \mathfrak{R}_f]$

6. (a) (1, 7), (2, 7), (3, 9), (4, 6), (5, 4)
   (b) (7, 5), (9, 5), (4, 8), (6, 6) (c) {4, 6, 7, 9}; {4, 6, 7, 9}
   (d) (1, 5), (2, 5), (3, 5), (4, 6), (5, 8)

7. (a) (1, 3), (2, 7), (3, 7), (4, 5), (5, 8)
   (b) (3, 2), (7, 7), (9, 4), (5, 4), (8, 1), (2, 3)
   (c) {3, 5, 7, 8}; {2, 3, 5, 7, 8, 9}
   (d) (1, 2), (2, 7), (3, 7), (4, 4), (5, 1)

8. (a) (−1, 1), (−2, 1), (−3, 2), (3, 3), (2, 4), (1, 8)
   (b) (1, 10), (2, 11), (3, 11), (5, 12), (6, 13)
   (c) {1, 2, 3, 4, 8}; {1, 2, 3, 5, 6}
   (d) (−1, 10), (−2, 10), (−3, 11), (3, 11)

EXPLORATION EXERCISES

Fill the blanks.

<u>Sample.</u> F = {(−1, 5), (1, 9), (2, 6), (−3, 7), (−4, 8)}
G = {(6, 14), (5, 16), (9, 18), (8, 20), (7, 22)}
(a) G(F(1)) = ________ (b) F(G(9)) = ________

<u>Solution.</u> (a) G(F(1)) = G(9) = 18
(b) G(9) = 18, but 'F(18)' is nonsense; hence, so is 'F(G(9))'.

1. F_1 = {(1, 2), (2, 4), (3, 7), (4, 10)}
   F_2 = {(2, 12), (7, 2), (10, 3), (4, 12)}

   (a) $F_1(2)$ = _____ (b) $F_2(F_1(2))$ = _____ (c) $F_1(F_2(7))$ = _____

   (d) $F_2(F_2(7))$ = _____ (e) $F_1(F_2(4))$ = _____ (f) $F_2(F_1(4))$ = _____

   (g) $F_1(F_1(3))$ = _____ (h) $F_1(F_2(F_1(3)))$ = _____

2. f = {(5, −3), (7, −3), (−2, −3), (1, −3)}
   g = {(4, 5), (2, 7), (−3, 1), (6, 6), (5, 4)}

   (a) f(5) = _____ (b) f(1) = _____ (c) g(f(5)) = _____

   (d) g(f(−2)) = _____ (e) f(g(−3)) = _____ (f) g(f(−3)) = _____

   (g) g(g(5)) = _____ (h) g(g(g(g(5)))) = _____

   (i) g(g(g(6))) = _____ (j) f(f(5)) = _____

3. b = {(x, y): y = 5x + 2} and g = {(x, y): y = 2x - 5}

   (a) b(2) = _____ (b) b(b(3)) = _____ (c) g(−1) = _____

   (d) g(g(g(5))) = _____ (e) b(g(3)) = _____ (f) g(b(3)) = _____

   (g) b(b(g(b(0)))) = _____ (h) b(g(____)) = 4

4. $G(x) = 7x - 2$ and $H(x) = \frac{x+2}{7}$

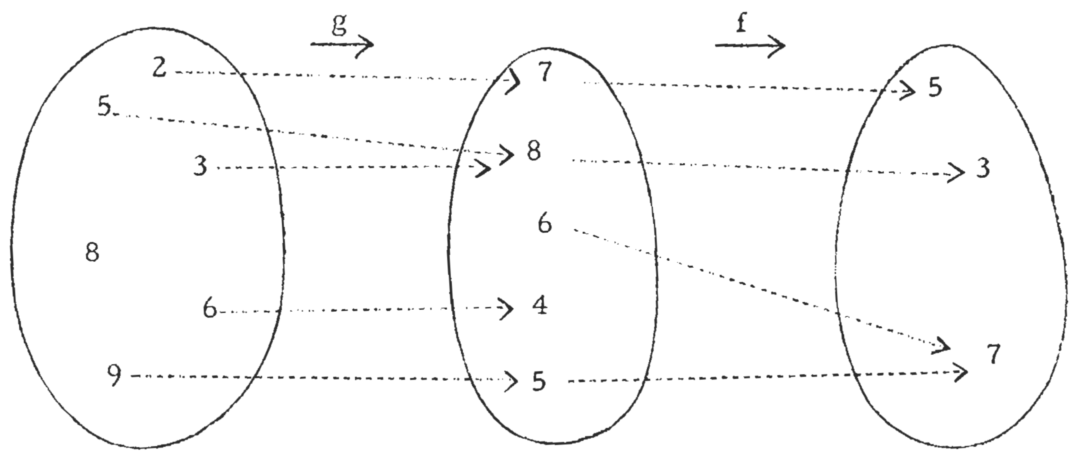
   (a) G(3) = _____ (b) H(19) = _____ (c) G(5) = _____

   (d) H(G(5)) = _____ (e) H(G(100)) = _____ (f) G(H(1002) = _____

   (g) $\mathcal{D}_G$ = ____________________ (h) $\mathcal{R}_G$ = ____________________

   (i) $\mathcal{D}_H$ = ____________________ (j) $\mathcal{R}_H$ = ____________________

5.



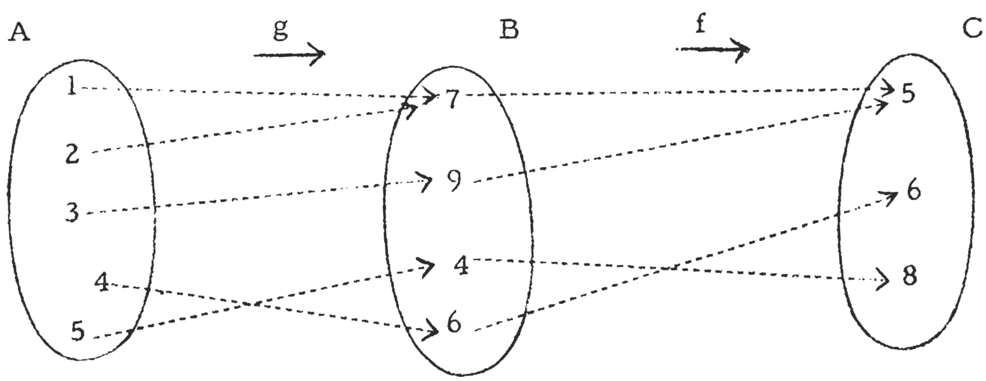
(a) $g(5) =$ ____ (b) $g($____$) = 4$ (c) $f(5) =$ ____

(d) $g(8) =$ ____ (e) $f(8) =$ ____ (f) $f($____$) = 1$

(g) $f(g(2)) =$ ____ (h) $f(g(3)) =$ ____ (i) $f(g($____$)) = 3$

(j) $f(g(7)) =$ ____ (k) $f(g($____$)) = 7$ (ℓ) $f(g($____$)) = 1$

6. Here is a picture which shows a mapping g of A on B, and a mapping f of B on C.



(a) g = {____, ____, ____, ____, ____}

(b) f = {____, ____, ____, ____}

(c) $\mathcal{R}_g =$ __________, $\mathcal{D}_f =$ __________

(d) The picture suggests a mapping h of A on C.

h = {____, ____, ____, ____, ____}

Quiz.

In each of the following exercises you are given a function. Fill in the blanks to make true sentences.

1. $f = \{(1, 9), (-2, 5), (7, 3), (0, 1), (3, 7)\}$

   (a) $f(1) =$ ______ (b) $f(3) =$ ______ (c) $f(f(3)) =$ ______

   (d) $f($___$) = 5$ (e) $f($___$) = 1$ (f) $f(f($___$)) = 9$

2. $g = \{(x, y): y = 3x - 4\}$

   (a) $g(1) =$ ______ (b) $g(5) =$ ______ (c) $g($___$) = 8$

   (d) $g($___$) = -10$ (e) $g(0) =$ ______ (f) $g(g(0)) =$ ______

3. $h(x) = x^2 + 3$, $\mathcal{D}_h = \{-3, -2, -1, 0, 1, 3\}$

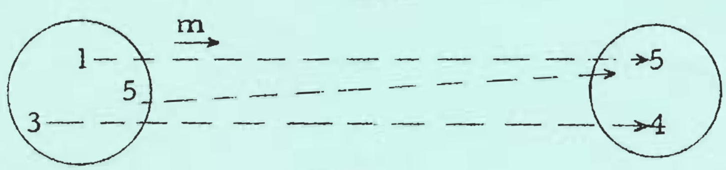
   (a) $h(3) =$ ______ (b) $h(-3) =$ ______ (c) $h($___$) = 3$

   (d) $h($___$) = 7$ (e) $h($___$) = -3$ (f) $h(h(0)) =$ ______

4. k is the function which maps each real number on its triple.

   (a) $k(8) =$ ______ (b) $k(7.2) =$ ______ (c) $k(-3.3) =$ ______

   (d) $k($___$) = 81$ (e) $k(k($___$)) = 81$ (f) $k(k(k($___$))) = 81$

5. 

   (a) $m(1) =$ ______ (b) $m(5) =$ ______ (c) $m($___$) = 4$

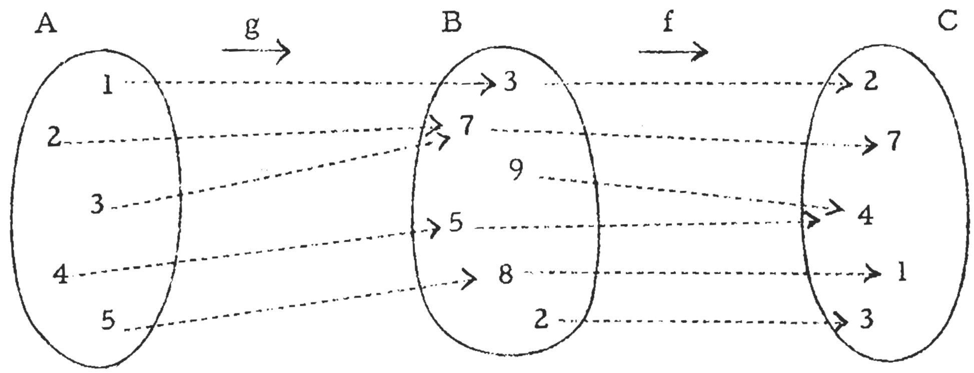
   (d) $m(m(1)) =$ ______ (e) $m($___$) = 5$ (f) $m($___$) = 3$

*

Answers for Quiz.

1.	(a) 9	(b) 7	(c) 3	(d) -2	(e) 0	(f) 0
2.	(a) -1	(b) 11	(c) 4	(d) -2	(e) -4	(f) -16
3.	(a) 12	(b) 12	(c) 0	(d) -2	(e) $[-3 \notin \mathcal{R}_h]$	(f) 12
4.	(a) 24	(b) 21.6	(c) -9.9	(d) 27	(e) 9	(f) 3
5.	(a) 5	(b) 5	(c) 3	(d) 5	(e) 1 and 5	(f) $[3 \notin \mathcal{R}_m]$

7. Here is a picture which shows a mapping g of A on a subset of B, and a mapping f of B on C.



(a) g = {____, ____, ____, ____, ____}

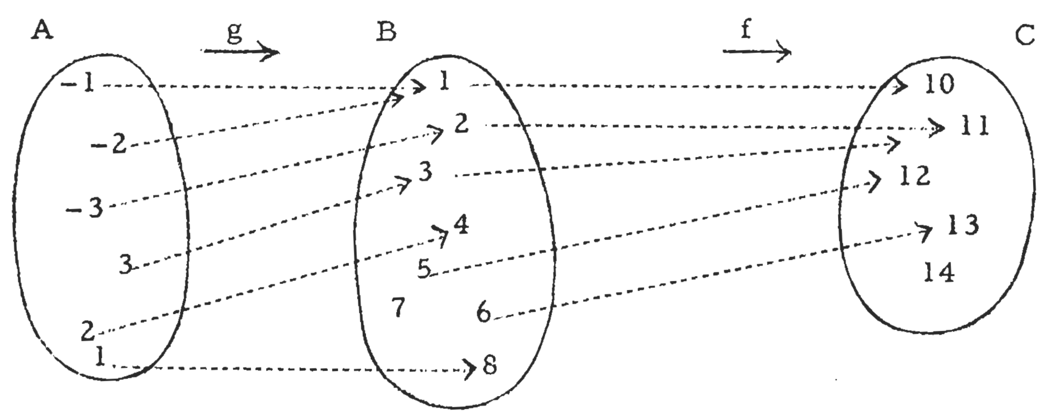
(b) f = {____, ____, ____, ____, ____, ____}

(c) $\mathcal{R}_g$ = ____________, $\mathcal{D}_f$ = ____________

(d) The picture suggests a mapping h of A on a subset of C.

h = {____, ____, ____, ____, ____}

8. Here is another picture of two mappings f and g.



(a) g = {____, ____, ____, ____, ____, ____}

(b) f = {____, ____, ____, ____, ____}

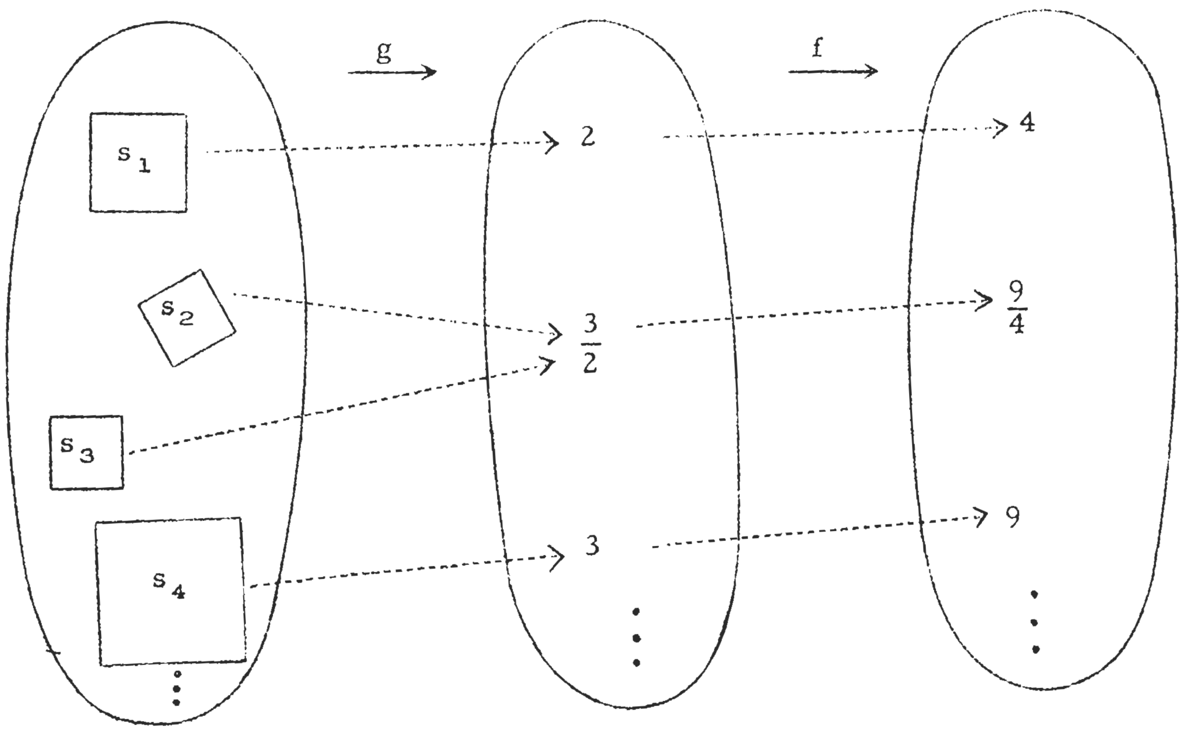
(c) $\mathcal{R}_g$ = ____________, $\mathcal{D}_f$ = ____________

(d) This picture does <u>not</u> suggest a mapping of A on a subset of C because although g maps 2 on 4 and 1 on 8, neither 4 nor 8 is in the domain of f. However, the picture does suggest a mapping, h, of the <u>subset</u> {-1, -2, -3, 3} of A on a subset of C.

h = {____, ____, ____, ____}

COMPOSING FUNCTIONS

Suppose g is the function which maps each square on its side-measure. [Sometimes we call g 'the side-measure of a square'.] And, suppose f is the function $\{(x, y) \in N \times N: y = x^2\}$ which maps each number of arithmetic on the square of itself. Does the picture suggest a mapping of squares on their area-measures?



Do you see that, for each square s, f(g(s)) is the area-measure of s?

What we have just done is to describe what someone does when he computes the area-measure of a square. First, he uses g to find the side-measure. [That is, he measures the side of the square.] Then, he uses f to find the area-measure. [That is, he multiplies the side-measure by itself.]

This is also an example of a way of combining two functions to get a third function. This operation on functions is called composition. In the example above, the third function is the area-measure of a square. It was obtained by composing f with g. We call the function obtained by composing f with g: $f \circ g$. [Read '$f \circ g$' as 'f composed with g' or 'f of g' or 'f circle g'.]

The Exploration Exercises have suggested that, to each ordered pair (f, g) of functions there corresponds a function h defined by:

$$h(x) = f(g(x)), \text{ for each } x \in \mathfrak{D}_g \text{ such that } g(x) \in \mathfrak{D}_f$$

The function h so defined is called <u>the</u> <u>composition</u> [sometimes: the compositum] <u>of</u> <u>f</u> <u>with</u> <u>g</u>, and is usually denoted by '$f \circ g$'. Note that $\mathfrak{D}_{f \circ g} = \{x \in \mathfrak{D}_g : g(x) \in \mathfrak{D}_f\}$ and, so, is always a subset of $\mathfrak{D}_g$. Exercise 8 on page 5-71 shows that $\mathfrak{D}_{f \circ g}$ may be a proper subset of $\mathfrak{D}_g$. This occurs precisely when there are values of g which are not arguments of f. That is, $\mathfrak{D}_{f \circ g} = \mathfrak{D}_g$ if and only if $\mathfrak{R}_g \subseteq \mathfrak{D}_f$. It may even happen [it is easy to make up examples] that no value of g is an argument of f. In this case $\mathfrak{D}_{f \circ g} = \emptyset$, and, also, $f \circ g = \emptyset$.

Composition of functions is a binary operation on functions, just as multiplication of numbers is a binary operation on numbers. As is shown by the example on page 5-73, composition is not commutative. However, as will be seen shortly [page 5-78], composition is associative. Other similarities and differences with multiplication of numbers will come up later in the unit.

The operation of composing functions is a special case of an operation on relations in general. For example, x is a grandson of z if and only if there is a person y such that x is a son of y and y is a child of z. So, the relation of being-a-grandson-of is the result of "composing" the relation of being-a-son-of with the relation of being-a-child-of. This more general kind of composition is called 'relative multiplication'. As another example, the relation of being-a-son-in-law-of is the result of composing the relation of being-the-husband-of with that of being-a-daughter-of.

*

Up to now, names of functions have been, for the most part, uncomplicated symbols such as 'F', 'g', and 'h_2'. Now that more complex symbols, like '$f \circ g$', are being used as function names it sometimes makes reading simpler if one encloses such function names in brackets. This has been done in the last line of the second paragraph on page 5-73 where we have the sentence '$[q \circ p](3) = 5$'. Brackets are also used in the definition at the top of page 5-74, in the Solution to the Sample on page 5-75, and in many of the exercises in later pages.

One way of explaining the brackets in such expressions as '$[q \circ p](3)$' on page 5-73 is to point out that there <u>might</u> be someone who regarded,

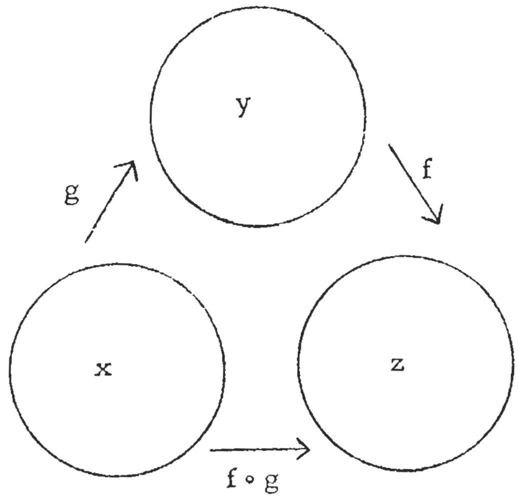
say, the symbol 'q ∘ p (3)' as ambiguous.
The symbol:

[_ _ _](___)

is a more adequate symbol to use in indicating application of a function to an argument than is the customary symbol:

_ _ _ (___)

We shall continue to use the more familiar abbreviated notation '_ _ _ (___)' in contexts where the function name is not a combination of symbols.



If g maps x on y, and f maps y on z, then f ∘ g maps x on z.

Let's take another example. Consider the functions p and q where

$$p = \{(0, 4), (1, 9), (2, -9), (3, 25), (4, -16)\}$$

and

$$q = \{(x, y), x \geq 0: y = \sqrt{x}\}.$$

What is q ∘ p?

The function p maps 0 on 4, and q maps 4 on 2. So, q ∘ p is a function which maps 0 on 2. Similarly, q ∘ p maps 1 on 3. However, although p maps 2 on −9, −9 does not belong to the domain of q. So, q ∘ p is not defined at 2. Similarly, q ∘ p is not defined at 4. But, $[q \circ p](3) = 5$. Thus,

$$q \circ p = \{(0, 2), (1, 3), (3, 5)\}$$

What is the function obtained by composing p with q? That is, what is p ∘ q? To find the ordered pairs in p ∘ q, we could begin by searching among the ordered pairs in q for those whose second components belong to $\mathcal{D}_p$. One such ordered pair in q is (0, 0). The others are (1, 1), (4, 2), (9, 3), and (16, 4). Now, since

$$\begin{array}{ccccc} & \xrightarrow{q} & & \xrightarrow{p} & \\ 0 & \dashrightarrow & 0 & \dashrightarrow & 4 \\ 1 & \dashrightarrow & 1 & \dashrightarrow & 9 \\ 4 & \dashrightarrow & 2 & \dashrightarrow & -9 \\ 9 & \dashrightarrow & 3 & \dashrightarrow & 25 \\ 16 & \dashrightarrow & 4 & \dashrightarrow & -16 \end{array}$$

it follows that

$$p \circ q = \{(0, 4), (1, 9), (4, -9), (9, 25), (16, -16)\}.$$

Here is a <u>definition</u> of the operation of composition:

> For each function f, for each function g,
>
> f∘g is the function such that
>
> (1) $[f \circ g](x) = f(g(x))$, for each $x \in \mathcal{D}_g$ such that $g(x) \in \mathcal{D}_f$,
>
> and
>
> (2) $\mathcal{D}_{f \circ g} = \{x \in \mathcal{D}_g: g(x) \in \mathcal{D}_f\}$.

[Notice that $\mathcal{D}_{f \circ g} \subseteq \mathcal{D}_g$. Under what condition does $\mathcal{D}_{f \circ g} = \mathcal{D}_g$?]

EXERCISES

<u>A.</u> Fill in the blanks.

1. $g = \{(6, 2), (9, 4), (12, -1), (5, 3)\}$
   $f = \{(x, y): y = x^2\}$
   $f \circ g = \{$ ________________________ $\}$

2. $a = \{(\text{John}, 7), (\text{Bill}, 5), (\text{Emma}, 8)\}$
   $b = \{(x, y), x > 6: y = 3x + 1\}$
   $b \circ a = \{$ ________________________ $\}$

3. $s = \{(2, 1), (3, 2), (5, 1), (6, 0), (8, 2)\}$
   $t = \{(0, 2), (1, 3), (2, 6), (3, 8), (4, 1), (5, 0)\}$
   (a) $t \circ s = \{$ ______________________________ $\}$
   (b) $s \circ t = \{$ ______________________________ $\}$

4. $m = \{(0, 4), (1, 2), (2, 3), (3, 8), (4, 10)\}$
   $n = \{(0, 1), (2, 3), (4, 5), (6, 7), (8, 9), (10, 11)\}$
   (a) $n \circ m = \{$ ______________________________ $\}$
   (b) $m \circ n = \{$ ______________________________ $\}$

5. s is the function which maps each equilateral triangle on its side-measure, and t i· the tripling function for numbers of arithmetic.

   t∘s is ______ ______________________________

Answer for bracketed question: $\mathcal{D}_{f \circ g} = \mathcal{D}_g$ if and only if $\mathcal{R}_g \subseteq \mathcal{D}_f$
[For, $\{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\} = \mathcal{D}_g$ if and only if, for each $x \in \mathcal{D}_g$, $g(x) \in \mathcal{D}_f$.]

*

Answers for Part A [on pages 5-74, 5-75, and 5-76].

1. (6, 4), (9, 16), (12, 1), (5, 9)   2. (John, 22), (Emma, 25)

3. (a) (2, 3), (3, 6), (5, 3), (6, 2), (8, 6)
   (b) (0, 1), (1, 2), (2, 0), (3, 2)

4. (a) (0, 5), (1, 3), (3, 9), (4, 11)
   (b) (0, 2), (2, 8)

5. $t \circ s$ is the perimeter of an equilateral triangle. That is, $t \circ s$ is the function which maps each equilateral triangle on its perimeter. Your students may give a brace-notation name, such as:

   $\{(x, y) \in T \times N$: y is the triple of the side-measure of x$\}$, where T is the set of equilateral triangles

   or:

   $\{(x, y) \in T \times N$: y is the perimeter of x$\}$, where T is the set of equilateral triangles

   Both of these are correct. However, the latter is preferable, because it indicates that the student understands the meaning of the word 'perimeter' as it applies to equilateral triangles.

Correction. In the first line of the solution to the sample, insert a comma after '$\mathcal{D}_g$'.

6. (a) $\{(x, y): y = 81x^4\}$
(b) $\{(x, y): y = 3x^4\}$

7. (a) $\{(x, y): y = |x - 5|\}$
(b) $\{(x, y): y = |x| - 5\}$

8. (a) $(x + 3)^2$
(b) $x^2 + 3$

9. (a) x
(b) x

10. (a) 0 (b) 16 (c) 1 (d) 1 (e) 5, −5 (f) $-4 \notin \mathcal{R}_{q \circ p}$

11. (a) 36 (b) 8 (c) 0 (d) 32 (e) 2, −4 (f) 2

12. (a) $2t^2 + 1$
(b) $(2t - 1)^2 + 1$

13. (a) $2t^2 + 1$
(b) $\begin{cases} (2t - 1)^2 + 1, & \text{for } t \geq 1 \\ (7 - t)^2 + 1, & \text{for } t < 1 \end{cases}$

14. (a) 7
(b) 11

☆15. (a) $\{(x, y), x > 0: y = -\sqrt{2x}\}$
(b) $\emptyset$

16. (a) $\{(4, 4), (5, 5), (6, 6)\}$ [or: $\{(x, y), x \in \mathcal{D}_g: y = x\}$]
(b) $\{(3, 3), (7, 7), (-1, -1)\}$ [or: $\{(x, y), x \in \mathcal{D}_f: y = x\}$]

17. (a) $\{(x, y): y = x\}$
(b) $\{(x, y): y = x\}$

18. (a) $\{(x, y): y = 9x + 28\}$
(b) $\{(x, y): y = 27x + 91\}$

19. (a) $\{(x, y): y = 4x - 3\}$
(b) $\{(x, y): y = -8x + 9\}$

20. the perimeter of x [or: $\{(x, y) \in S \times N$: y is the perimeter of x$\}$]; the set of all squares

☆21. (a) the father-in-law of x; the set of married people
(b) the mother of x; the set of married people

Sample. $g = \{(x, y):\ y = 2x\}$
$f = \{(x, y):\ y = x^3\}$

(a) $f \circ g =$ __________

(b) $g \circ f =$ __________

Solution. Since $\mathcal{R}_g \subseteq \mathcal{D}_f$, the domain of $f \circ g$ is $\mathcal{D}_g$ the set of real numbers. For each real number x,

$$[f \circ g](x) = f(g(x)) = f(2x) = (2x)^3 = 8x^3.$$

So, $f \circ g = \{(x, y):\ y = 8x^3\}$.

Similarly, since $\mathcal{R}_f \subseteq \mathcal{D}_g$, the domain of $g \circ f$ is $\mathcal{D}_f$, the set of real numbers. For each real number x,

$$[g \circ f](x) = g(f(x)) = g(x^3) = 2x^3.$$

So, $g \circ f = \{(x, y):\ y = 2x^3\}$.

6. $k = \{(x, y):\ y = 3x\}$
$q = \{(x, y):\ y = x^4\}$

(a) $q \circ k =$ ________________

(b) $k \circ q =$ ________________

7. $c = \{(x, y):\ y = x - 5\}$
$d = \{(x, y):\ y = |x|\}$

(a) $d \circ c =$ ______________

(b) $c \circ d =$ ______________

8. $g(x) = x + 3$
$f(x) = x^2$

(a) $[f \circ g](x) =$ __________

(b) $[g \circ f](x) =$ __________

9. $r(x) = x - 3$
$s(x) = x + 3$

(a) $[s \circ r](x) =$ __________

(b) $[r \circ s](x) =$ __________

10. $p(x) = x - 3,\ q(x) = x^2$

(a) $[q \circ p](3) =$ _____ (b) $[q \circ p](7) =$ _____ (c) $[q \circ p](2) =$ _____

(d) $[p \circ q](2) =$ _____ (e) $[p \circ q]($_____$) = 22$ (f) $[q \circ p]($_____$) = -4$

11. $s(m) = 3m + 5,\ t(m) = (m - 2)^2$

(a) $[t \circ s](1) =$ _____ (b) $[s \circ t](1) =$ _____ (c) $[t \circ s](-1) =$ _____

(d) $[s \circ t](-1) =$ _____ (e) $[t \circ s]($_____$) = 81$ (f) $[s \circ t]($_____$) = 5$

12. $k(t) = t^2 + 1$

$j(t) = 2t - 1$

(a) $[j \circ k](t) =$ ______________

(b) $[k \circ j](t) =$ ______________

13. $k(t) = t^2 + 1$

$$j(t) = \begin{cases} 2t - 1, & \text{for } t \geq 1 \\ 7 - t, & \text{for } t < 1 \end{cases}$$

(a) $[j \circ k](t) =$ ______________

(b) $[k \circ j](t) =$ ______________

14. $g(x) = 2x - 3$

$f(x) = 7$

(a) $[f \circ g](x) =$ ______________

(b) $[g \circ f](x) =$ ______________

☆15. $g = \{(x, y), x > 0: y = 2x\}$

$f = \{(x, y), x > 0: y = -\sqrt{x}\}$

(a) $f \circ g =$ ______________

(b) $g \circ f =$ ______________

16. $g = \{(4, 3), (5, 7), (6, -1)\}$

$f = \{(3, 4), (7, 5), (-1, 6)\}$

(a) $f \circ g =$ ______________

(b) $g \circ f =$ ______________

17. $g = \{(x, y): y = 2x + 5\}$

$f = \{(x, y): y = \frac{x - 5}{2}\}$

(a) $f \circ g =$ ______________

(b) $g \circ f =$ ______________

18. $g = \{(x, y): y = 3x + 7\}$

(a) $g \circ g =$ ______________

(b) $g \circ [g \circ g] =$ ______________

19. $g = \{(x, y): 2x + y = 3\}$

(a) $g \circ g =$ ______________

(b) $[g \circ g] \circ g =$ ______________

20. $g(x)$ = the side-measure of square x, $\mathscr{D}_g$ = the set of all squares.

$f(x) = 4x$, $\mathscr{D}_f$ = the set of numbers of arithmetic.

$[f \circ g](x) =$ ______________, $\mathscr{D}_{f \circ g} =$ ______________

☆21. $g(x)$ = the husband or wife of x, $\mathscr{D}_g$ = the set of married people.

$f(x)$ = the father of x, $\mathscr{D}_f$ = the set of married people.

(a) $[f \circ g](x) =$ ______________, $\mathscr{D}_{f \circ g} =$ ______________

(b) $[g \circ f](x) =$ ______________, $\mathscr{D}_{g \circ f} =$ ______________

[Supplementary exercises are in Part N, pages 5-255 through 5-257.]

(c) To show that the sentence 'x belongs to the domain of $f \circ [g \circ h]$' is equivalent to (*):

By definition,

$$x \text{ belongs to the domain of } f \circ [g \circ h]$$
$$\text{if and only if}$$
$$x \in \mathcal{D}_{g \circ h} \text{ and } [g \circ h](x) \in \mathcal{D}_f.$$

But, again by definition,

$$x \in \mathcal{D}_{g \circ h}$$
$$\text{if and only if}$$
$$x \in \mathcal{D}_h \text{ and } h(x) \in \mathcal{D}_g.$$

Also, by definition, for each $x \in \mathcal{D}_h$ such that $h(x) \in \mathcal{D}_g$,
$$[g \circ h](x) = g(h(x)).$$

So,

$$x \text{ belongs to the domain of } f \circ [g \circ h]$$
$$\text{if and only if}$$
$$[x \in \mathcal{D}_h \text{ and } h(x) \in \mathcal{D}_g] \text{ and } g(h(x)) \in \mathcal{D}_f.$$

Therefore, since conjunction is associative, it follows that the sentence 'x belongs to the domain of $f \circ [g \circ h]$' is equivalent to (*).

Correction. In the last line of part (b) on page 5-78, change

'$[f \circ]g \circ h]](x)$' to '$[f \circ [g \circ h]](x)$'.

↑

Answers for Part B.

In each of the four blanks: $\{(1, 2), (2, 0), (3, 4)\}$

$f \circ g = f_1 \circ g$ if and only if $f_3 \subseteq f$

There are infinitely many such functions.

The only such function whose domain is $\mathcal{R}_g$ is f_3.

*

Answers for Part C [on pages 5-77 and 5-78].

1. The example on page 5-73 has already shown that composition is not commutative. Also, the Sample on page 5-75 and each of the exercises of Part A [on pages 5-74ff.], with the exception of Exercises 9 and 17, show that the result of composing two functions may depend on which function is composed with which.

   There are some interesting special cases:

   If $f = \{(x, y): y = ax + b\}$ and $g = \{(x, y): y = cx + d\}$, then $f \circ g = g \circ f$ if and only if $(a - 1)d = (c - 1)b$.

   If $f = \{(x, y): y = x^2\}$ and $g = \{(x, y): y = x^3\}$, then $f \circ g = g \circ f$. [Similarly for other "power functions".]

2. (a) 4 (b) 18 (c) 18 (d) 18 (e) 13 (f) 13

   (g) 26 (h) 26 (i) 22 (j) 22 (k) $2(x + 5)$

   (ℓ) $2[(x - 3) + 5]$ (m) $(x - 3) + 5$ (n) $2[(x - 3) + 5]$

3. (a) $f(g(h(x)))$; $f(g(h(x)))$ (b) $g(h(x))$; $f(g(h(x)))$

B. Compose each of the functions f_1, f_2, f_3, and f_4 with g where

$$g = \{(1, 1), (2, 0), (3, 2)\},$$

and

$$f_1 = \{(x, y): y = 2x\}, \quad f_2 = \{(x, y), x \geq 0: y = 2x\},$$

$$f_3 = \{(1, 2), (0, 0), (2, 4)\}, \quad f_4 = \{(x, y): y = x^3 - 3x^2 + 4x\}.$$

$f_1 \circ g$ = ________________ $f_2 \circ g$ = ________________

$f_3 \circ g$ = ________________ $f_4 \circ g$ = ________________

What must you know about a function f to predict that $f \circ g = f_1 \circ g$? How many such functions f are there? How many are there such that $\mathfrak{D}_f = \mathfrak{R}_g$?

C. 1. Show that the operation composition-of-functions is not commutative. That is, show that '$\forall_f \forall_g\ f \circ g = g \circ f$' is false.

2. Fill in the blanks.

$f(x) = 2x$, $g(x) = x + 5$, and $h(x) = x - 3$.

(a) $h(7)$ = ______ (b) $[f \circ g](4)$ = ______

(c) $[f \circ g](h(7))$ = ______ (d) $[[f \circ g] \circ h](7)$ = ______

(e) $g(h(11))$ = ______ (f) $[g \circ h](11)$ = ______

(g) $f([g \circ h](11))$ = ______ (h) $[f \circ [g \circ h]](11)$ = ______

(i) $[[f \circ g] \circ h](9)$ = ______ (j) $[f \circ [g \circ h]](9)$ = ______

(k) $[f \circ g](x)$ = ______ (ℓ) $[[f \circ g] \circ h](x)$ = ______

(m) $[g \circ h](x)$ = ______ (n) $[f \circ [g \circ h]](x)$ = ______

3. The preceding exercise suggests that the operation composition-of-functions is associative, that is, that

$$\forall_f \forall_g \forall_h\ [f \circ g] \circ h = f \circ [g \circ h].$$

Let's try to prove this.

(a) Suppose x belongs to the domain of $[f \circ g] \circ h$. Then, by the definition on page 5-74, $x \in \mathcal{D}_h$ and $h(x) \in \mathcal{D}_{f \circ g}$. Also,

$$[[f \circ g] \circ h](x) = [f \circ g](h(x)).$$

Since $h(x) \in \mathcal{D}_{f \circ g}$, it follows from the same definition that

$$[f \circ g](h(x)) = \underline{\hspace{4cm}}.$$

So, if x belongs to the domain of $[f \circ g] \circ h$,

$$[[f \circ g] \circ h](x) = \underline{\hspace{4cm}}.$$

(b) Now, suppose x belongs to the domain of $f \circ [g \circ h]$. Again, by definition, $x \in \mathcal{D}_{g \circ h}$ and $[g \circ h](x) \in \mathcal{D}_f$. Also,

$$[f \circ [g \circ h]](x) = f([g \circ h](x)).$$

Since $x \in \mathcal{D}_{g \circ h}$, it follows from the definition that

$$[g \circ h](x) = \underline{\hspace{4cm}}.$$

So, if x belongs to the domain of $f \circ [g \circ h]$,

$$[f \circ]g \circ h]](x) = \underline{\hspace{4cm}}.$$

(c) In parts (a) and (b), you have shown that, for each x which belongs both to the domain of the function $[f \circ g] \circ h$ and to the domain of the function $f \circ [g \circ h]$, the functions have the same value. But, this does not yet prove that they are the same function. To do so, we must prove that the domain of $[f \circ g] \circ h$ is the domain of $f \circ [g \circ h]$. Let's see what it means to say that x belongs to the domain of $[f \circ g] \circ h$. By definition, this is the case if and only if

$$x \in \mathcal{D}_h \text{ and } h(x) \in \mathcal{D}_{f \circ g}.$$

Again, by definition, <u>this</u> is the case if and only if

$$(*) \qquad x \in \mathcal{D}_h \text{ and } [h(x) \in \mathcal{D}_g \text{ and } g(h(x)) \in \mathcal{D}_f].$$

Now what does it mean to say that x belongs to the domain of $f \circ [g \circ h]$? [Use the definition to show that 'x belongs to the domain of $f \circ [g \circ h]$' is equivalent to (*). This will prove that $[f \circ g] \circ h$ and $f \circ [g \circ h]$ have the same domain.] So, since $[f \circ g] \circ h$ and $f \circ [g \circ h]$ have the same domain, and since, for each x in this domain, $[[f \circ g] \circ h](x) = [f \circ [g \circ h]](x)$, it follows that $[f \circ g] \circ h = f \circ [g \circ h]$.

Quiz.

Suppose that

$f = \{(0, 3), (4, 7), (5, 8)\}$, $g = \{(5, 0), (9, 4), (10, 5)\}$,

$h = \{(5, 0), (9, 4), (10, 5), (12, 7)\}$, $j = \{(x, y): y = x - 5\}$,

$k = \{(5, 0), (9, 4), (10, 5), (12, 5)\}$, and $\ell = \{(x, y): y = x + 3\}$.

Fill in the blanks.

1. $[f \circ g](5) =$ _____ 2. $[f \circ g](9) =$ _____ 3. $[f \circ g](10) =$ _____
4. $f \circ g = \{$ ________________ $\}$
5. $[f \circ h](5) =$ _____ 6. $[f \circ h](9) =$ _____ 7. $[f \circ h](10) =$ _____
8. $f \circ h = \{$ ________________ $\}$
9. $f \circ j = \{$ ________________ $\}$
10. $[f \circ k](5) =$ _____ 11. $[f \circ k](9) =$ _____ 12. $[f \circ k](10) =$ _____
13. $[f \circ k](12) =$ _____ 14. $f \circ k = \{$ ________________ $\}$
15. $[\ell \circ j](5) =$ _____ 16. $[\ell \circ j](9) =$ _____ 17. $[\ell \circ j](10) =$ _____
18. $[\ell \circ j](15) =$ _____ 19. $j(a) =$ _____ 20. $\ell(b) =$ _____
21. $\ell(a - 5) =$ _____ 22. $j(r) =$ _____ 23. $[\ell \circ j](r) =$ _____
24. $\ell \circ j = \{$ ________________ $\}$ 25. $g(5) =$ _____
26. $[j \circ g](5) =$ _____ 27. $[\ell \circ [j \circ g]](5) =$ _____ 28. $[[\ell \circ j] \circ g](5) =$ _____

*

Answers for Quiz.

1. 3 2. 7 3. 8 4. (5, 3), (9, 7), (10, 8) 5. 3

6. 7 7. 8 8. (5, 3), (9, 7), (10, 8) 9. (5, 3), (9, 7), (10, 8)

10. 3 11. 7 12. 8 13. 8

14. (5, 3), (9, 7), (10, 8), (12, 8) 15. 3 16. 7 17. 8

18. 13 19. $a - 5$ 20. $b + 3$ 21. $a - 2$ 22. $r - 5$ 23. $r - 2$

24. $(x, y): y = x - 2$ 25. 0 26. -5 27. -2 28. -2

In discussing the questions on page 5-79, it may be helpful to draw graphs of the functions referred to.

(1) (0, −5), (3/2, −7/2), (−3, −8), . . .

(2) (−5, 0), (−7/2, 3/2), (−8, −3), . . .

(3) yes (4) yes (5) yes (6) no

(7) no (8) yes (9) no (10) yes

A function has at most one inverse.

If a function has an inverse then the converse of this inverse is the given function. So, if a function has an inverse then the function is, itself, the inverse of its inverse.

If g is the inverse of f then $\mathfrak{D}_g = \mathfrak{R}_f$ and $\mathfrak{R}_g = \mathfrak{D}_f$.

$\{(x, y): y = x\}$ is its own inverse. So is $\{(x, y): xy = 1\}$.
[In general, a function is its own inverse just if it is a function which is a symmetric relation.]

If g is the inverse of f then $g \cup f$ is a symmetric relation whose domain and range are the set $\mathfrak{D}_f \cup \mathfrak{R}_f$.

THE INVERSE OF A FUNCTION

You will recall from earlier units that operations such as adding 2 are sets of ordered pairs. Other examples of operations are multiplying by -1, absolute valuing, and square rooting. Since an operation is a set of ordered pairs, an operation is a relation. More particularly, an operation is a function. [In fact, you may recall reading (on page 1-107) that an operation is a set of ordered pairs no two of which have the same first component.] You also learned that an operation has an inverse just if the set of ordered pairs obtained by reversing the components of each of its ordered pairs is an operation [that is, just if the converse is an operation]. In general, a function is said to have an inverse if and only if the converse of the function is also a function. And, in this case, the converse is called <u>the inverse of the given function</u>.

(1) Give three ordered pairs which belong to adding -5.
(2) Give three pairs which belong to the inverse of adding -5.
(3) Is the converse of multiplying by 2 a function?
(4) Does multiplying by 2 have an inverse?
(5) Does multiplying by 0 have a converse?
(6) Does multiplying by 0 have an inverse?
(7) Does absolute valuing have an inverse?
(8) Does absolute valuing have a converse?
(9) Does squaring have an inverse?
(10) Does square rooting have an inverse?

How many inverses can a function have? If a function has an inverse, what is <u>its</u> inverse? If g is the inverse of a function f, what are $\mathcal{D}_g$ and $\mathcal{R}_g$? Give an example of a function which is its own inverse. If g is the inverse of a function f, what can you say about the relation $g \cup f$?

EXERCISES

<u>A</u>. For each function which has an inverse, describe its inverse.

<u>Sample 1</u>. $G = \{(3, 1), (2, -5), (4, 3)\}$

<u>Solution</u>. The converse of G is $\{(1, 3), (-5, 2), (3, 4)\}$. This is a function. So, G has this function as its inverse.

Sample 2. $f(x) = 3x - 5$, $\mathcal{D}_f$ = the set of real numbers

Solution. Let g be the converse of f. Since

$$f = \{(x, y): y = 3x - 5\},$$

it follows from the definition of converse that

$$g = \{(y, x): y = 3x - 5\}. \text{ [Explain.]}$$

Is g a function? It is just if there are not two ordered pairs in g with the same first component. Suppose someone tells you the first component of an ordered pair in g. Can you tell him the second component of this ordered pair? Referring back to the brace-notation name for g, it is easy to see that given a first component p, the corresponding second components are just those numbers x such that $p = 3x - 5$. But, $p = 3x - 5$ if and only if $x = \frac{p + 5}{3}$. So, for each first component p there is just one second component q, the number $\frac{p + 5}{3}$. So, g is a function. In fact, we have shown that

$$g = \{(p, q): q = \frac{p + 5}{3}\}$$

$$= \{(x, y): y = \frac{x + 5}{3}\}.$$

Do you see a quick way of getting this last brace-notation name for g from the name for f?

Another description of g is:

$$g(x) = \frac{x + 5}{3}, \quad \mathcal{D}_g = \text{the set of real numbers}$$

Sample 3. $F = \{(x, y): y = x^2\}$

Solution. Suppose G is the converse of F. Then

$$G = \{(x, y): x = y^2\}$$

F has an inverse if and only if G is a function. But, G is not a function because each positive real number is the square of two real numbers. For example, both (4, 2) and (4, −2) belong to G.

1. $f = \{(-5, 6), (-6, 7), (-7, -10), (1/2, 1/3)\}$

2. $F = \{(1, 5), (2, 9), (4, 9)\}$

3. $g = \{(x, y): y = 5\}$

Here is an an alternative solution for Sample 2.

Solution. A function has an inverse if no two of its ordered pairs have the same second component. We shall show that f has an inverse by showing that no two of its ordered pairs have the same second component. That is, we shall show that if $(a, c) \in f$ and $(b, c) \in f$ then $a = b$.

Suppose that $(a, c) \in f$ and $(b, c) \in f$. Then $c = 3a - 5$ and $c = 3b - 5$. So, $3a - 5 = 3b - 5$. Hence, $3a = 3b$, and [since $3 \neq 0$] $a = b$.

Consequently, f has an inverse.

Unlike the solution in the text, this alternative solution does not furnish a description of the inverse of f. However, it illustrates the fundamental procedure generally used to determine whether or not a function does have an inverse. This procedure is used, for example, on TC[3-108, 109]a to show [proof of generalization (ii)] that the function $\{(x, y): x > 0 \text{ and } y = x^2\}$ has an inverse and, so, to justify introducing the function name '$\sqrt{\ }$'. [In further explanation, note that while, by definition, $\{(x, y): y > 0 \text{ and } x = y^2\}$ is the converse of $\{(x, y): x > 0 \text{ and } y = x^2\}$, we were not justified in introducing a function name [$\sqrt{\ }$] for this relation until we had proved generalization (ii) and, so, knew that the relation in question is a function.]

*

Answers for Part A [which begins on page 5-79].

1. Since the converse of f is $\{(6, -5), (7, -6), (-10, -7), (1/3, 1/2)\}$ and since this relation is a function, the function f has an inverse. Or: since no two ordered pairs in the function f have the same second component, f has an inverse.

2. F does not have an inverse since it contains two ordered pairs, (2, 9) and (4, 9), which have the same second component.

3. The converse of g is $\{(x, y): x = 5\}$. This relation is not a function since it contains two ordered pairs, for example, $(5, -4)$ and $(5, \sqrt{2})$, which have the same first component. Hence, g does not have an inverse.

that f is a subset of the converse of g]. Similarly, to say that

$$\text{(ii)} \quad \text{for each } k \in \mathcal{D}_g, \quad [f \circ g](k) = k$$

is to say that g is a subset of the converse of f.

So, (i) and (ii) together say that g is the converse of f. Since g is a function, this amounts to saying that f has an inverse, and that its inverse is g.

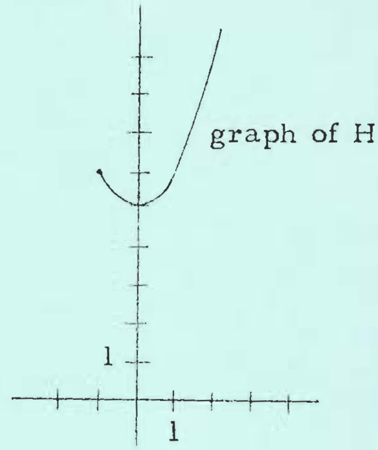
Notice that, since each subset of a function is a function, (i) alone tells us that the converse of f is a function--that is, that f has an inverse. But, to conclude that the inverse of f is g, we need additional information, for example, that $\mathcal{D}_g = \mathcal{R}_f$. Condition (ii), on the other hand, does not ensure that f has an inverse [although, as we have just seen, it does ensure that g does]. For example, (ii) is satisfied by taking for f the squaring function $\{(x, y): y = x^2\}$, and for g the square-rooting function $\{(x, y), x \geq 0: y = \sqrt{x}\}$. $[\forall_{x > 0} (\sqrt{x})^2 = x]$. However, since (ii) says that g is a subset of the converse of f, it follows from (ii) together with '$\mathcal{R}_g = \mathcal{D}_f$' that f has g as inverse. [Returning to (i), we see that if $f = \{(x, y), x \geq 0: y = \sqrt{x}\}$ and $g = \{(x, y): y = x^2\}$ then (i) is satisfied. Now, by definition of '$\sqrt{\ }$', $f = \{(x, y), x \geq 0: y \geq 0 \text{ and } y^2 = x\}$. Hence, the converse of f is $\{(x, y): x \geq 0 \text{ and } x^2 = y\}$, and is a proper subset of g. As predicted, the converse of f is a function. So, f has an inverse, and the inverse of f is a subset of g.]

10. [It may help to draw a graph of H.] H does not have an inverse because its converse contains two ordered pairs [for example, (6, −1) and (6, 1)] which have the same first component. The function h such that

$$h(x) = x^2 + 5 \text{ and } \mathcal{D}_h = \{x: x \geq 0\}$$

is a subset of H which does have an inverse. The inverse of h is the function g such that

$$g(x) = \sqrt{x - 5} \text{ and } \mathcal{D}_g = \{x: x \geq 5\}.$$



[Using brace-notation, $g = \{(x, y), x \geq 5: y = \sqrt{x-5}\}$.]

Another subset of H which has an inverse is the function k such that

$$k(x) = x^2 + 5 \text{ and } \mathcal{D}_k = \{x: -1 \leq x \leq 0\}.$$

[Its inverse is $\{(x, y), x \geq 5: x \leq 6 \text{ and } y = -\sqrt{x-5}\}$.] Each subset of k or of h is also a subset of H which has an inverse. Moreover, the union of a subset of k and a subset of h whose ranges have no member in common is a subset of H which has an inverse [and these are all]. So, one can obtain any subset of H which has an inverse by the following procedure: Choose a subset D_1 of $\{x: -1 \leq x \leq 0\}$ and a subset D_2 of $\{x: x > 0\}$ such that, for no member of D_1 is its opposite in D_2. Then a subset of H of the desired kind is

$$\{(x, y), x \in D_1: y = -\sqrt{x-5}\} \cup \{(x, y), x \in D_2: y = \sqrt{x-5}\}.$$

11. (a) first (b) second

12. (a) $\mathcal{D}_g = \mathcal{R}_f$ and $\mathcal{R}_g = \mathcal{D}_f$ (b) k [in both blanks]

*

<u>Additional remarks about Exercise 12.</u>--If f and g are functions [not necessarily inverse to one another] then, to say that

(i) for each $k \in \mathcal{D}_f$, $[g \circ f](k) = k$

is equivalent to saying that

for each $k \in \mathcal{D}_f$, if $(k, \ell) \in f$ then $(\ell, k) \in g$ [for each $\ell \in \mathcal{R}_f$].

But, this merely says that the converse of f is a subset of g [that is,

Correction. In line 17:
$[g \circ f](k) =$ ____,
↑

4. The function h has an inverse. In fact, h is its own inverse.

5. $f = \{(x, y): y = 4x + 7\}$, and the converse of f is $\{(x, y): x = 4y + 7\}$. Since the latter is the <u>function</u> $\{(x, y): y = (x - 7)/4\}$, f has an inverse. In fact, the inverse of f is the function g such that $g(x) = (x - 7)/4$ and $\mathcal{D}_g$ is the set of real numbers.

*

When a function is described as in Exercise 5 and has an inverse, students should be required to give the same kind of description of the inverse of the given function. For example, to say that the function f of Exercise 5 has an inverse and its inverse is $\{(x, y): x = 4y + 7\}$, while correct, is not acceptable.

*

6. Similar to Exercise 5. The inverse of h is the function g whose domain is the set of real numbers and such that $g(x) = (3 - x)/2$.

7. $G = \{(x, y): x \geq 0 \text{ and } y = (5x + 1)/9\}$. The converse of G is $\{(x, y): y \geq 0 \text{ and } x = (5y + 1)/9\}$. But, this is the <u>function</u> $\{(x, y): y \geq 0 \text{ and } y = (9x - 1)/5\}$. So, G has an inverse. The inverse of G is the function H described by:

$$H(x) = (9x - 1)/5, \quad \mathcal{D}_H = \{x: x \geq 1/9\}$$

$$[\forall_x [(9x - 1)/5 \geq 0 \iff x \geq 1/9]]$$

8. F does not have an inverse since $(2, -1)$ and $(-2, -1)$ both belong to F. [That is, $(-1, 2)$ and $(-1, -2)$ both belong to the converse of F.]

9. G has an inverse. It is the function H such that $H(x) = (10 - x)/2$ [and $\mathcal{D}_H$ = the set of real numbers].

4. $h = \{(1, 3), (3, 1), (6, 2), (2, 6), (0, 0)\}$

5. $f(x) = 4x + 7$, $\mathcal{D}_f$ = the set of real numbers

6. $h(x) = 3 - 2x$, $\mathcal{D}_h$ = the set of real numbers

7. $G(x) = \frac{5x + 1}{9}$, $\mathcal{D}_G = \{x\colon\ x \geq 0\}$

8. $F(x) = |x| - 3$, $\mathcal{D}_F$ = the set of real numbers

9. $G(r) = -2r + 10$, $\mathcal{D}_G$ = the set of real numbers

10. $H(t) = t^2 + 5$, $\mathcal{D}_H = \{x\colon\ x \geq -1\}$ [Is there a subset of H which has an inverse? Is there more than one such subset of H?]

*

11. (a) A relation is a function if and only if no two ordered pairs in the relation have the same ________ component.

    (b) A function has an inverse if and only if no two ordered pairs in the function have the same _________ component.

12. (a) If g is the inverse of a function f then

$$\mathcal{D}_g = \_\_\_\_\_ \text{ and } \mathcal{R}_g = \_\_\_\_\_.$$

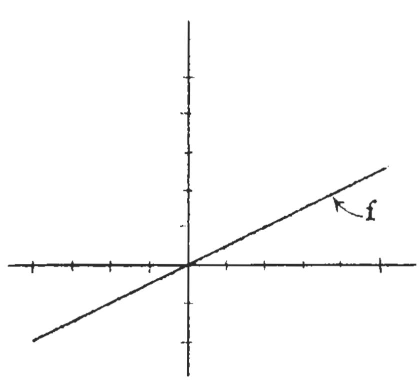
    (b) If g is the inverse of a function f then,

for each $k \in \mathcal{D}_f$, $[g \circ f](k) =$ ________.

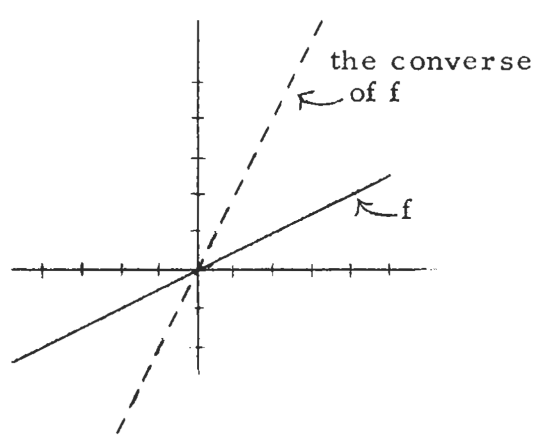
and, for each $k \in \mathcal{D}_g$, $[f \circ g](k) =$ ________.

B. Here are graphs of functions. Sketch the converse of each function, and tell whether the function has an inverse. [Try to predict which functions have inverses before you sketch the converses.]

Sample.

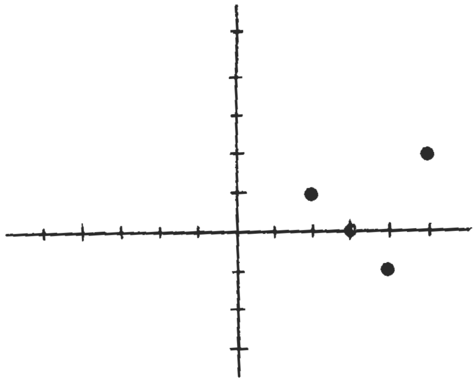


Solution.



The converse of f is a function. So, f has an inverse.

1.



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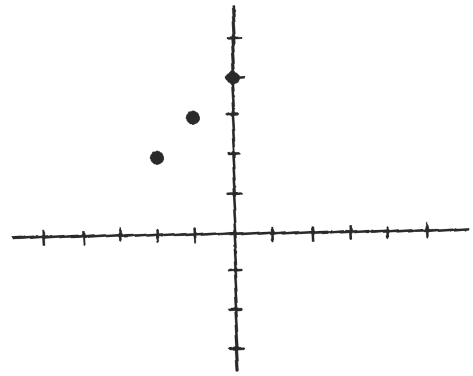

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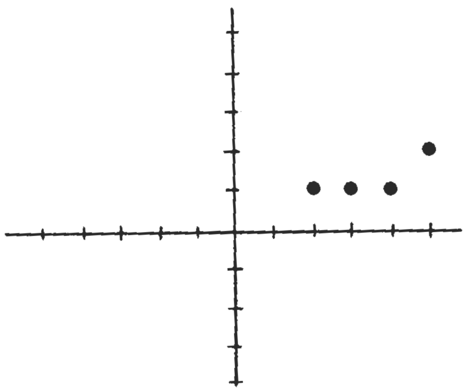

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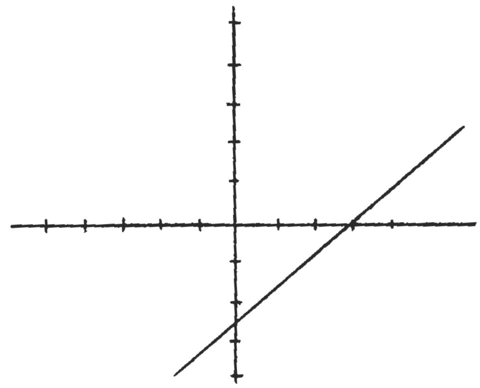

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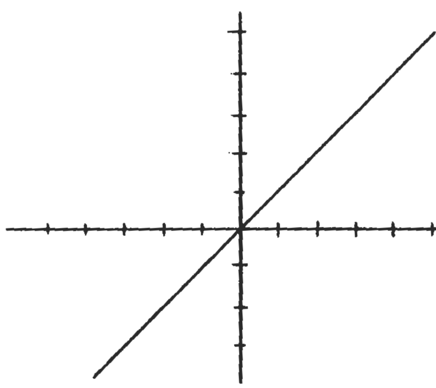

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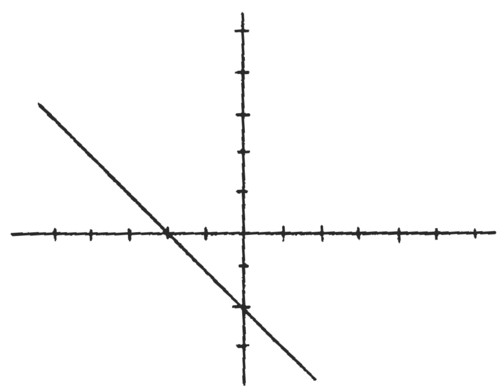
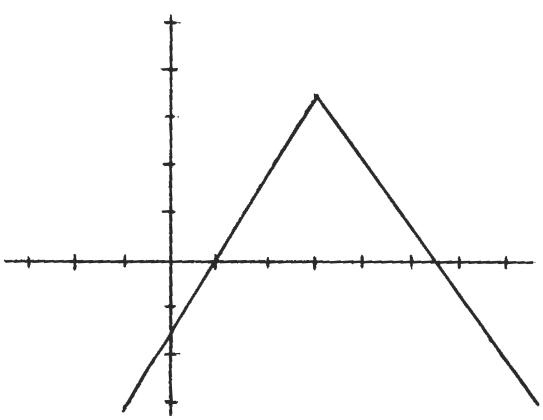
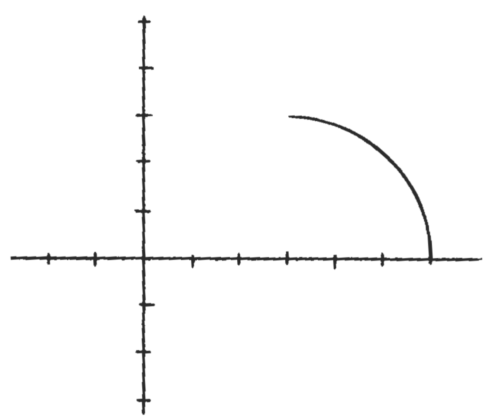


7.

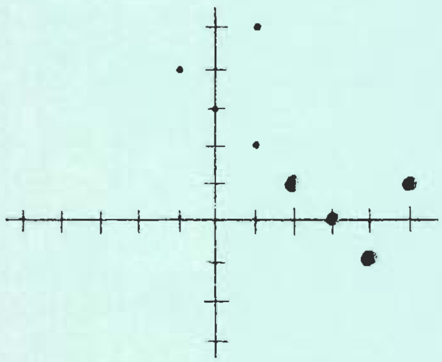


8.



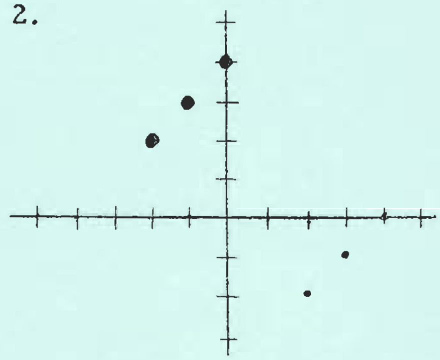
Answers for Part B [which begins on page 5-81].

1.



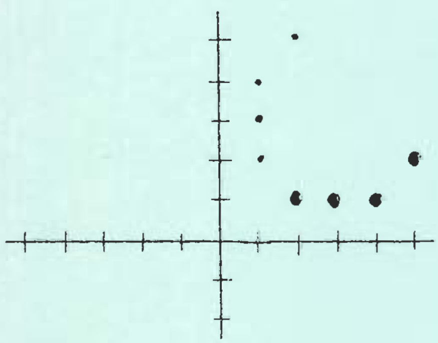
[Yes]

2.



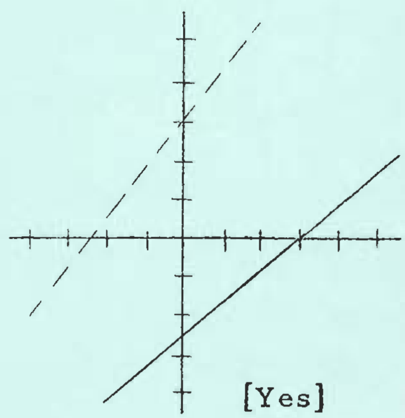
[Yes]

3.



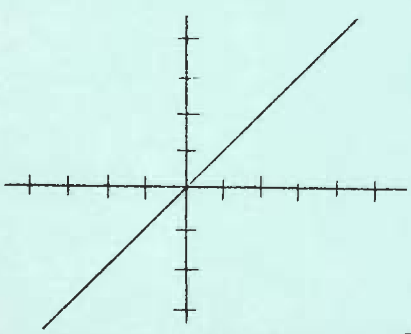
[No inverse]

4.



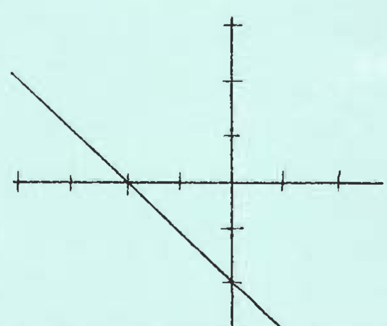
[Yes]

5.



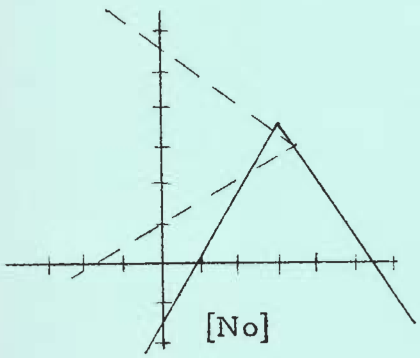
[Yes, it is its own inverse.]

6.



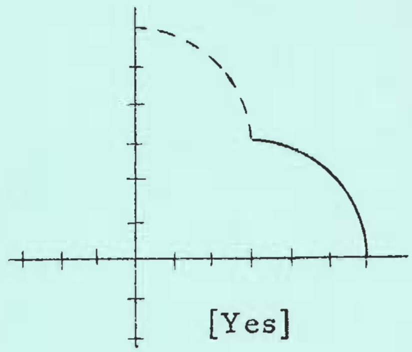
[Yes, it is its own inverse.]

7.



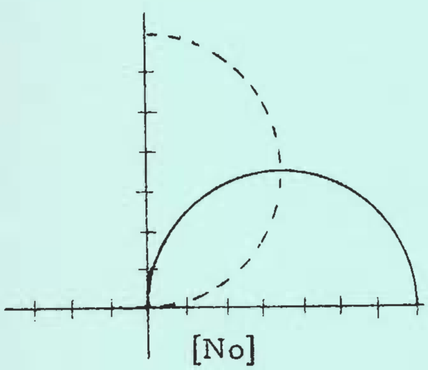
[No]

8.



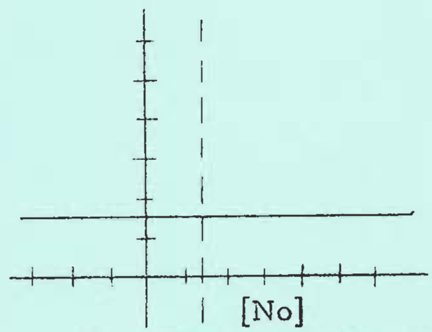
[Yes]

9.



[No]

10.



[No]

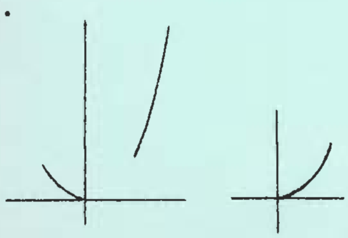
[A constant function does not have an inverse.]

*

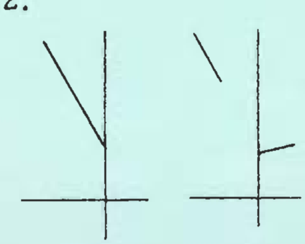
Answers for Part C [on pages 5-83 and 5-84].

For each exercise we show only two of several possible answers. [See TC discussion of Exercise 10 of Part A on page 5-81.] In each case we show one of the subsets whose range is the range of the given function and, except in Exercise 4, one of the subsets whose range is a proper subset of the range of the given function.

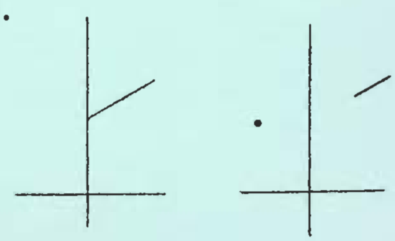
1.



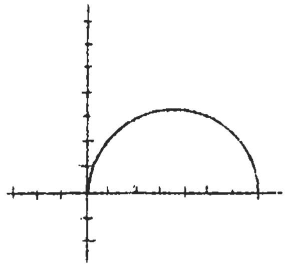
2.



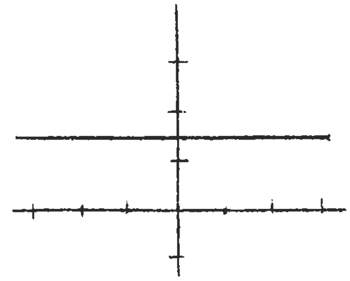
3.



9.



10.

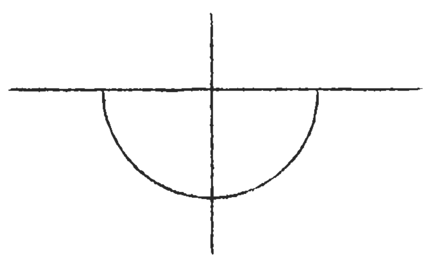


*

Functions such as the one pictured in Exercise 10 [which has the set of real numbers as its domain and whose ordered pairs all have the same second component] are called constant functions. Does a constant function have an inverse?

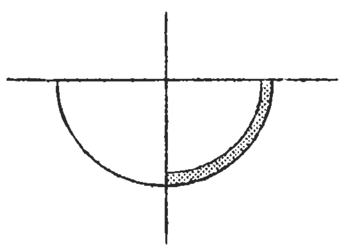
C. Here are pictures of functions which do not have inverses. However, it is the case that every function has subsets which have inverses. For each function pictured below and on page 5-84, choose a subset which has an inverse. Indicate your choice by marking the drawing.

Sample.

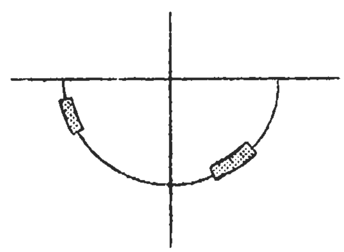


Solution. Here are just two of many possible answers.

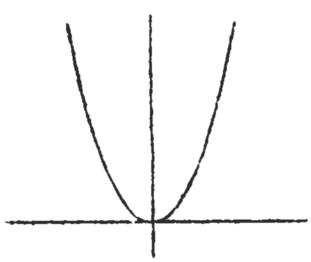
(I)



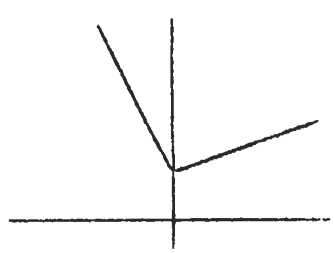
(II)



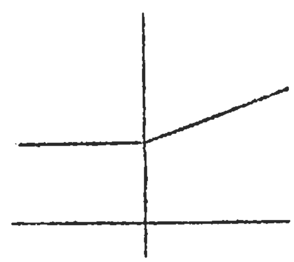
1.

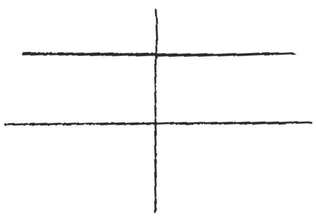


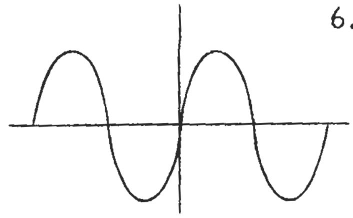
2.

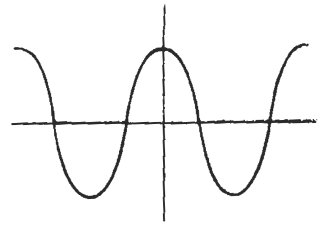


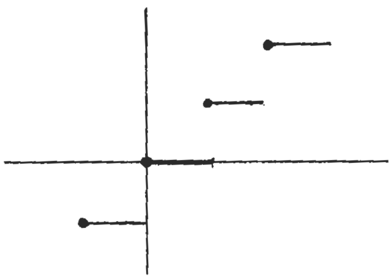
3.

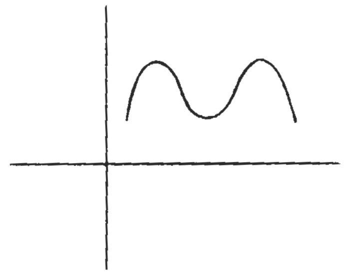


4. 

5. 

6. 

7. 

8. 

*

9. Do you think that each relation has a subset which is a function whose domain is the domain of the relation? Do you think that each function has a subset which has an inverse whose domain is the range of the given function?

D. It is customary to use 'f^{-1}' as an abbreviation for 'the inverse of f'. [You can also read 'f^{-1}' as 'f inverse'.] Practice using this notation in simplifying the expressions given in the exercises. The exercises refer to the functions F_1, F_2, F_3, F_4, and F_5, where the domain of each is the set of real numbers and

$$F_1(x) = 2x + 3, \qquad F_2(x) = \frac{x}{3} - 1, \qquad F_3(x) = -4x$$
$$F_4(x) = x + 2, \qquad F_5(x) = -2x - 4$$

Notice that each of these functions has an inverse.

<u>Sample 1</u>. $F_3^{-1}(F_2(6))$

<u>Solution</u>. $F_2(6) = \frac{6}{3} - 1 = 1.$

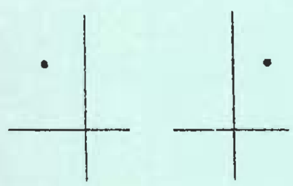
$F_3^{-1}(1) = ?$

To simplify '$F_3^{-1}(1)$' we must find a number x such that $F_3(x) = 1$, that is, such that $-4x = 1$. Such a number is $-1/4$. Therefore, $F_3^{-1}(1) = -1/4$. So,

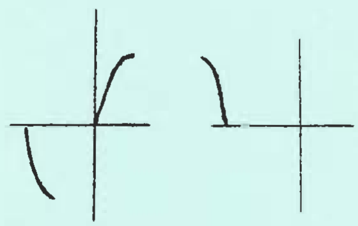
$$F_3^{-1}(F_2(6)) = -\frac{1}{4}.$$

[Note that: $y = F_3^{-1}(x)$, $(x, y) \in F_3^{-1}$, $(y, x) \in F_3$, and: $x = F_3(y)$, all say the same thing.]

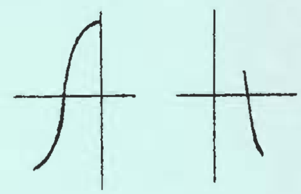
4.



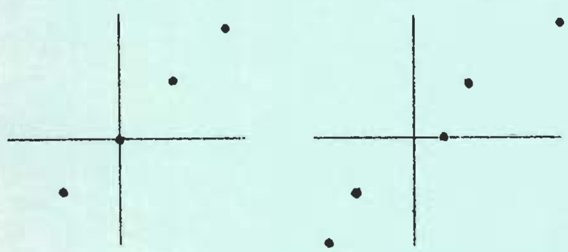
5.



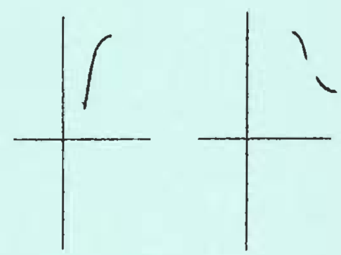
6.



7.



8.



9. Each relation R has a subset which is a function whose domain is the domain of the relation.

<u>Example 1.</u> $R = \{(x, y): y > x\}$ $f = \{(x, y): y = x + 2\}$

$\mathcal{D}_f = \mathcal{D}_R$

<u>Example 2.</u> $R = \{(3, 2), (5, 7), (4, 9), (3, 6), (8, 10)\}$

$f = \{(3, 2), (5, 7), (4, 9), (8, 10)\}$ or:

$\{(5, 7), (4, 9), (3, 6), (8, 10)\}$

$\mathcal{D}_f = \{3, 5, 4, 8\} = \mathcal{D}_R$

<u>Example 3.</u> $R = \{(3, 2), (5, 7), (4, 9)\}$

$f = \{(3, 2), (5, 7), (4, 9)\}$

For any relation R, for each $x \in \mathcal{D}_R$, there is at least one ordered pair in R with x as first component. Pick <u>one</u> of these ordered pairs. The set of such ordered pairs [one for each $x \in \mathcal{D}_R$] is a function which is a subset of R and whose domain is $\mathcal{D}_R$.

It follows that, for each function f, the converse of f has a subset which is a function whose domain is the domain of the converse of f. That is, whose domain is the range of f. The converse of such a subset is a subset of f which has an inverse whose domain is the range of f.

*

Students may need practice in using the inverse-notation in simple cases before they attack the problems in Part D. Use the functions f, h, and G described in Exercises 5, 6, and 7 of Part A on page 5-81.

1. $f^{-1}(2) =$ ______ 2. $G^{-1}(1) =$ ______ 3. $h^{-1}(-3) =$ ______

4. $f(G^{-1}(5)) =$ ______ 5. $G^{-1}(f(2)) =$ ______ 6. $h^{-1}(h(6)) =$ ______

7. $f(f^{-1}(-2)) =$ ______ 8. $G^{-1}(G(\frac{1}{10}) =$ ______ 9. $G(G^{-1}(\frac{1}{10})) =$ ______

10. $h(G^{-1}(f(\frac{1}{6}))) =$ ______ 11. $f^{-1}(f^{-1}(f(2))) =$ ______ 12. $G^{-1}(\pi) =$ ______

<u>Answers.</u>

1. $-\frac{5}{4}$ 2. $\frac{8}{5}$ 3. 3 4. $\frac{211}{5}$

5. $\frac{134}{5}$ 6. 6 7. -2 8. $\frac{1}{10}$

9. $\frac{1}{10} \notin \mathcal{D}_{G^{-1}}$ 10. $-\frac{121}{5}$ 11. $-\frac{5}{4}$ 12. $\frac{9\pi - 1}{5}$

*

The origin of the '$^{-1}$' notation for inverses is the analogy between function composition and multiplication of numbers. Just as, for each number $x \neq 0$, $x^{-1} \cdot x = 1$ and $x \cdot x^{-1} = 1$, so, for each function f which has an inverse, $f^{-1} \circ f = \{(x, y), x \in \mathcal{D}_f: y = x\}$ and $f \circ f^{-1} = \{(x, y): x \in \mathcal{R}_f: y = x\}$. [See Exercise 12 of Part A, page 5-81.] The analogy just referred to is developed further on TC[5-115].

*

Alternate solution for Sample 1 of Part D.

$$F_2(6) = \frac{6}{3} - 1 = 1.$$

Since, for each x, $F_3^{-1}(x) = -\frac{x}{4}$,

it follows that $F_3^{-1}(F_2(6)) = F_3^{-1}(1) = -\frac{1}{4}$.

Quiz.

A. $R = \{(5, 9), (2, 6), (14, 18), (-2, 2), (6, -2)\}$

1. The converse of R is { ______________________ }.

2. Is R a function?

3. Is the converse of R a function?

4. Does R have an inverse?
   If it does, what is it? { ________________________________ }

5. Circle the meaningful expressions in the following list.

   $R(5)$, $R(18)$, $R^{-1}(9)$, $R^{-1}(14)$, $[R \circ R^{-1}](-2)$, $[R \circ R^{-1}](5)$

6. Give a simpler equivalent expression for each expression you circled in Exercise 5.

B. $f = \{(x, y): y = x + 3\}$

1. $f(2) =$ ______	2. $f(5) =$ ______	3. $f(\_\_) = 7$
4. $f^{-1}(7) =$ ______	5. $f(b) =$ ______	6. $f(\_\_) = a$
7. $f^{-1}(a) =$ ______	8. $f^{-1} = \{(a, b): b =$ ________ $\}$	
9. $f^{-1} = \{(x, y): y =$ _______ $\}$		

*

Answers for Quiz.

A. 1. $(9, 5), (6, 2), (18, 14), (2, -2), (-2, 6)$ 2. Yes

3. Yes 4. Yes; $(9, 5), (6, 2), (18, 14), (2, -2), (-2, 6)$

5. $R(5)$, $R^{-1}(9)$, $[R \circ R^{-1}](-2)$ 6. 9, 5, -2

B. 1. 5 2. 8 3. 4 4. 4 5. $b + 3$ 6. $a - 3$
7. $a - 3$ 8. $a - 3$ 9. $x - 3$

Correction. Exercise 15 should read: $5 \cdot F_5(5)$ ↑

Some students may prefer to find the inverse of each function before doing these exercises.

$F_1^{-1}(x) = \frac{x - 3}{2}$, $F_2^{-1}(x) = 3x + 3$, $F_3^{-1}(x) = -\frac{x}{4}$,

$F_4^{-1}(x) = x - 2$, $F_5^{-1}(x) = -\frac{x + 4}{2}$

*

Answers for Part D [which begins on page 5-84].

1. $\frac{1}{4}$	2. $-\frac{7}{3}$	3. $-\frac{8}{3}$	4. $\frac{5}{3}$	5. $\frac{28}{3}$
6. -1	7. $4r + 5$	8. $6t - 6$	9. $-\frac{9}{2}$	10. -3
11. -2	12. -2	13. $-\frac{4}{3}$	14. 5	15. -70
16. $-\frac{21}{2}$	17. $3a + 3$	18. $-\frac{5b + 1}{2}$	19. -1	20. 39
21. 5	22. $7a + 2a - 8$		23. 5	24. 5
25. 843	26. 927			

*

Sample 2. $F_4(F_1(3a))$

Solution. $F_1(3a) = 2(3a) + 3 = 6a + 3,$

$F_4(6a + 3) = (6a + 3) + 2 = 6a + 5.$

So, $F_4(F_1(3a)) = 6a + 5.$

1. $F_3^{-1}(F_1(-2))$
2. $F_1^{-1}(F_2(-2))$
3. $F_2(-2) + F_1(-2)$
4. $F_1(-2) \cdot F_2(-2)$
5. $F_1(3) + F_2(4)$
6. $F_1(4) + F_3(3)$
7. $F_1(2r + 1)$
8. $F_5(1 - 3t)$
9. $F_2^{-1}(F_1^{-1}(-2))$
10. $F_1^{-1}(F_2^{-1}(-2))$
11. $F_1^{-1}(F_1(-2))$
12. $F_2(F_2^{-1}(-2))$
13. $F_3(F_2(4))$
14. $F_4(F_2^{-1}(0))$
15. $5 \cdot F(5)$
16. $5 \cdot F_5^{-1}(1/5)$
17. $F_2^{-1}(a)$
18. $F_5^{-1}(5b - 3)$
19. $F_4^{-1}(F_5(-5/2))$
20. $F_2^{-1}(F_2^{-1}(3))$
21. $F_4^{-1}(F_4(F_4^{-1}(7)))$
22. $F_3^{-1}(F_3(7a^2 + 2a - 8))$
23. $[F_1^{-1} \circ F_1](5)$
24. $[F_1 \circ F_1^{-1}](5)$
25. $[F_2^{-1} \circ F_2](843)$
26. $[F_2 \circ F_2^{-1}](927)$

[Supplementary exercises are in Parts O and P, pages 5-257ff.]

EXPLORATION EXERCISES

In each of the following exercises you are given functions g and h. Your job is to find, if possible, a function f such that $h = f \circ g$.

Sample 1. $g = \{(3, 6), (5, 9), (8, 4), (7, 6)\}$

$f =$ ____________________

$h = \{(3, 9), (5, 12), (8, 7), (7, 9)\}$

Solution. We are looking for a function f such that $f(6) = 9$, $f(9) = 12$, and $f(4) = 7$. So, such a function must contain the ordered pairs (6, 9), (9, 12), and (4, 7). One such function is f where $f = \{(6, 9), (9, 12), (4, 7)\}$. To see that this works, we find the ordered pairs in $f \circ g$:

$f(g(3)) = f(6) = 9$ $\qquad$ $f(g(5)) = f(9) = 12$

$f(g(8)) = f(4) = 7$ $\qquad$ $f(g(7)) = f(6) = 9$

Thus,

$f \circ g = \{(3, 9), (5, 12), (8, 7), (7, 9)\} = h.$ ✓

Sample 2. $g = \{(3, 6), (5, 9), (8, 4), (7, 6)\}$

$f =$ ____________________

$h = \{(3, 9), (5, 12), (8, 4), (7, 6)\}$

Solution. We are looking for a function f such that $f(6) = 9$, $f(9) = 12$, $f(4) = 7$, and $f(6) = 10$. That is, a function which contains the pairs (6, 9), (9, 12), (4, 7), and (6, 10). There is no such function. Why?

1. $g = \{(1, 1), (2, 1), (3, 1), (4, 2), (5, 3), (6, 3), (7, -1)\}$

   $f =$ ____________________

   $h = \{(1, 1), (2, 1), (3, 1), (4, 4), (5, 9), (6, 9), (7, 1)\}$

2. $g = \{(2, 5), (3, 8), (6, 8), (5, 0)\}$

   $f =$ ____________________

   $h = \{(2, 12), (3, 18), (6, 14), (5, 2)\}$

3. $g = \{(0, 1), (1, 5), (2, 9), (3, 8), (4, 9)\}$

   $f =$ ____________________

   $h = \{(1, 7), (2, 12), (4, 12)\}$

4. $g = \{(0, 1), (1, 5), (2, 9)\}$

   $f =$ ____________________

   $h = \{(1, 5), (2, 8), (4, 8)\}$

5. $g = \{(3, 6), (8, 6)\}$

   $f =$ ____________________

   $h = \{(3, 9)\}$

<u>Correction.</u> The third line of Sample 2 should read: $h = \{(3, 9), (5, 12), (8, 7), (7, 10)\}$ ↑ ↑

Answers for Exploration Exercises [on pages 5-86, 5-87, 5-88, and 5-89].

1. $\{(1, 1), (2, 4), (3, 9), (-1, 1)\}$, or any function which contains the members of this set; for example, $\{(x, y): y = x^2\}$, and $\{(x, y), x \in I: y = x^2\}$.

2. Such a function would have to contain (8, 14) and (8, 18). So, there is no such function.

3. $\{(5, 7), (9, 12)\}$, or any function which contains the members of this set <u>and</u> whose domain does not contain 1 or 8. [If, for example, $1 \in \mathcal{D}_f$ then $(0, f(1)) \in f \circ g$. In this case, since $0 \notin \mathcal{D}_h$, $h \neq f \circ g$.]

4. Since $\mathcal{D}_h \not\subseteq \mathcal{D}_g$ [$4 \in \mathcal{D}_h$ and $4 \notin \mathcal{D}_g$], there is no such function.

5. There is no function f such that $f \circ g = h$. For, since $g(3) = 6$ and $h(3) = 9$, (6, 9) must belong to any such function f. So, $[f \circ g](8) = f(g(8)) = f(6) = 9$. Hence $8 \in \mathcal{D}_{f \circ g}$. But, $8 \notin \mathcal{D}_h$.

*

In Samples 1 and 2 and Exercises 1 and 2, $\mathcal{D}_h = \mathcal{D}_g$. <u>When this is the case</u> there is a function f such that $h = f \circ g$ if and only if, whenever g has the same value for two arguments, then h has the same value for these arguments. For, if this condition is satisfied then with each value of g there corresponds just one value of h, and the function f_1 whose members are obtained by pairing such corresponding values is such that $h \subseteq f_1 \circ g$. But $\mathcal{D}_{f_1} = \mathcal{R}_g$, so $\mathcal{D}_{f_1 \circ g} = \mathcal{D}_g = \mathcal{D}_h$. Hence, $h = f_1 \circ g$. Moreover, $f \circ g = h$ if and only if $f_1 \subseteq f$. [See, further, pages 5-89, 90, and 91, and COMMENTARY.] Exercises 3 and 5 illustrate complications which arise when $\mathcal{D}_h \subseteq \mathcal{D}_g$. In Exercise 3 the pairing of corresponding values again [as in Sample 1 and Exercise 1] gives a function f_1 such that $h = f_1 \circ g$. But now, $\mathcal{D}_{f_1} \subset \mathcal{R}_g$. In this case, $f \circ g = h$ if and only if $f_1 \subseteq f$ <u>and</u> $\mathcal{D}_f \cap \mathcal{R}_g = \mathcal{D}_{f_1}$, that is, if and only if $f_1 \subseteq f$ and those members of f which are not in f_1 have first components which are not in $\mathcal{R}_g$. In Exercise 5, although 6 and 9 are [the only] corresponding values of g and h, the function f_1 whose only member is (6, 9) is such that $h \subset f_1 \circ g$, and there is no function f such that $h = f \circ g$.

Corrections.	On page 5-87, in Exercise 7, line 3 should read: $h = \{(x, y) : y = x^2\}$ ↑
	On page 5-89, in Exercise 15, line 3 should read: h(x) = the area-measure of ... ↑

6. $\{(x, y) : y = x^2\}$

7. $\{(x, y) : y = x^2\}$

8. Since, for example, $g(-2) = g(2)$ but $h(-2) \neq h(2)$, there is no function f such that $h = f \circ g$ [the squaring function does not have an inverse].

9. $\{(x, y), x \geq 0 : y = \sqrt{x}\}$ [$\forall_x \sqrt{x^2} = |x|$. Note that $\mathcal{D}_h = \mathcal{D}_g$ and that if $g(x_1) = g(x_2)$ then $h(x_1) = h(x_2)$.]

10. Since $0 \in \mathcal{D}_h$ but $0 \notin \mathcal{D}_g$, there is no function f such that $h = f \circ g$.

11. $\{(x, y) : xy = 1\}$ [The reciprocating function is its own inverse.]

12. $\{(7, 6), (8, 6), (9, 5)\}$

13. Since $g(3) = g(5)$ but $h(3) \neq h(5)$, there is no function f such that $h = f \circ g$.

14. $\{(1, 1), (2, 4), (3, 9), (4, 16)\}$

15. x^2; the set of nonzero numbers of arithmetic [or: the set of numbers of arithmetic]

16. $\{(0, 0), (2, 2), (5, 5), (11, 11)\}$, or any function which contains the members of this set

17. $\sqrt{x}$; the set of nonzero numbers of arithmetic [or: the set of numbers of arithmetic]

☆18. the mother of x; the set of married people

6. $g = \{(x, y): y = x\}$

$f =$ ______________

$h = \{(x, y): y = x^2\}$

7. $g = \{(x, y): y = |x|\}$

$f =$ ______________

$h = \{(x, y): y = x\}$

8. $g = \{(x, y): y = x^2\}$

$f =$ ______________

$h = \{(x, y): y = x\}$

9. $g = \{(x, y): y = x^2\}$

$f =$ ______________

$h = \{(x, y): y = |x|\}$

10. $g = \{(x, y): xy = 1\}$

$f =$ ______________

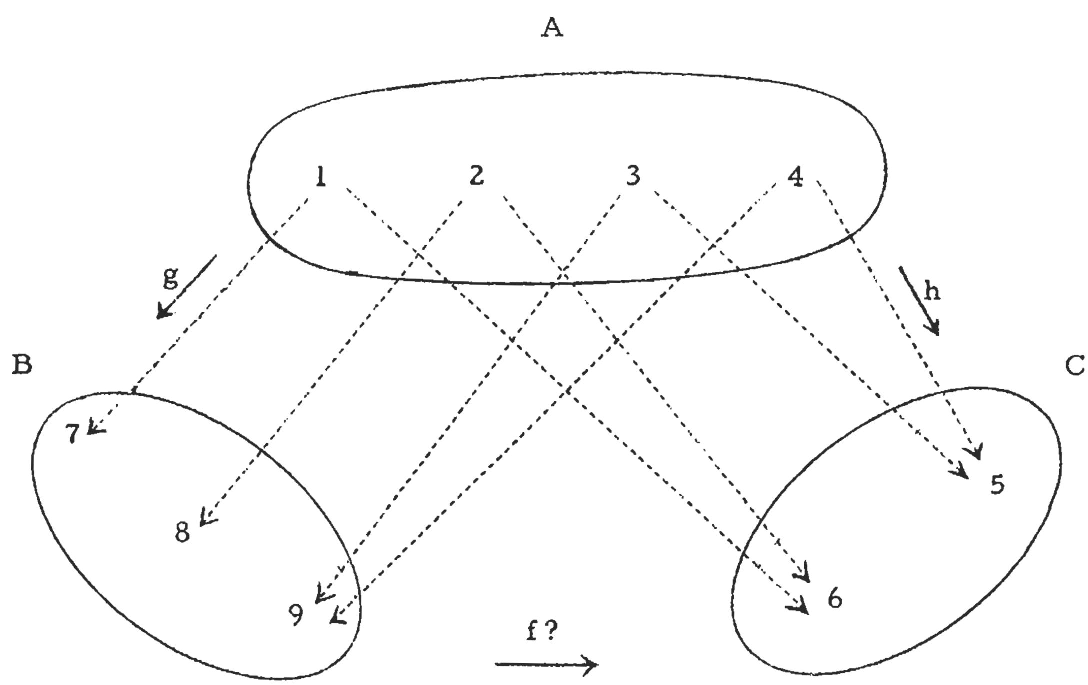
$h = \{(x, y): y = x\}$

11. $g = \{(x, y): xy = 1\}$

$f =$ ______________

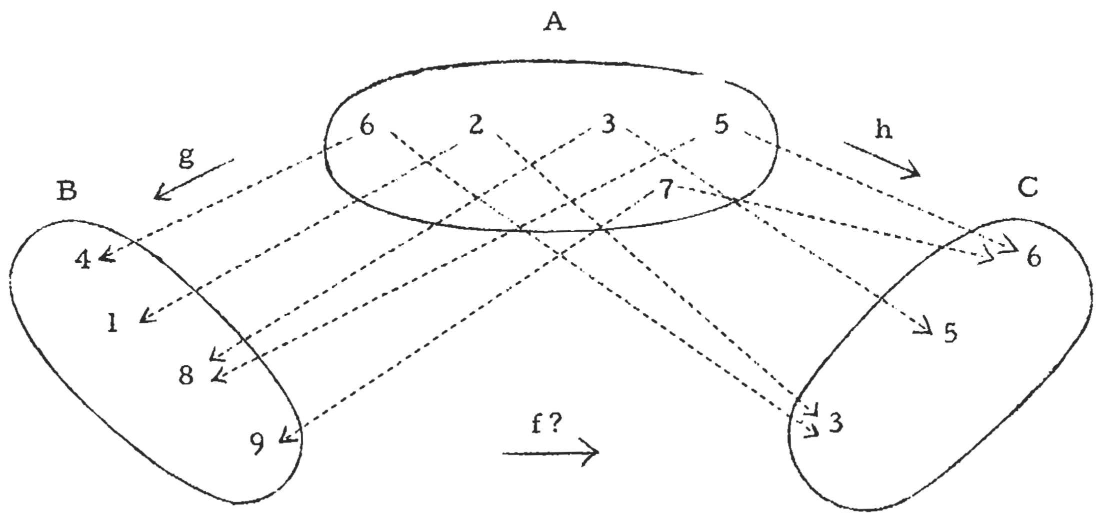
$h = \{(x, y): x \neq 0 \text{ and } y = x\}$

12. Here is a picture which shows a mapping g of A on B, and a mapping h of A on C.



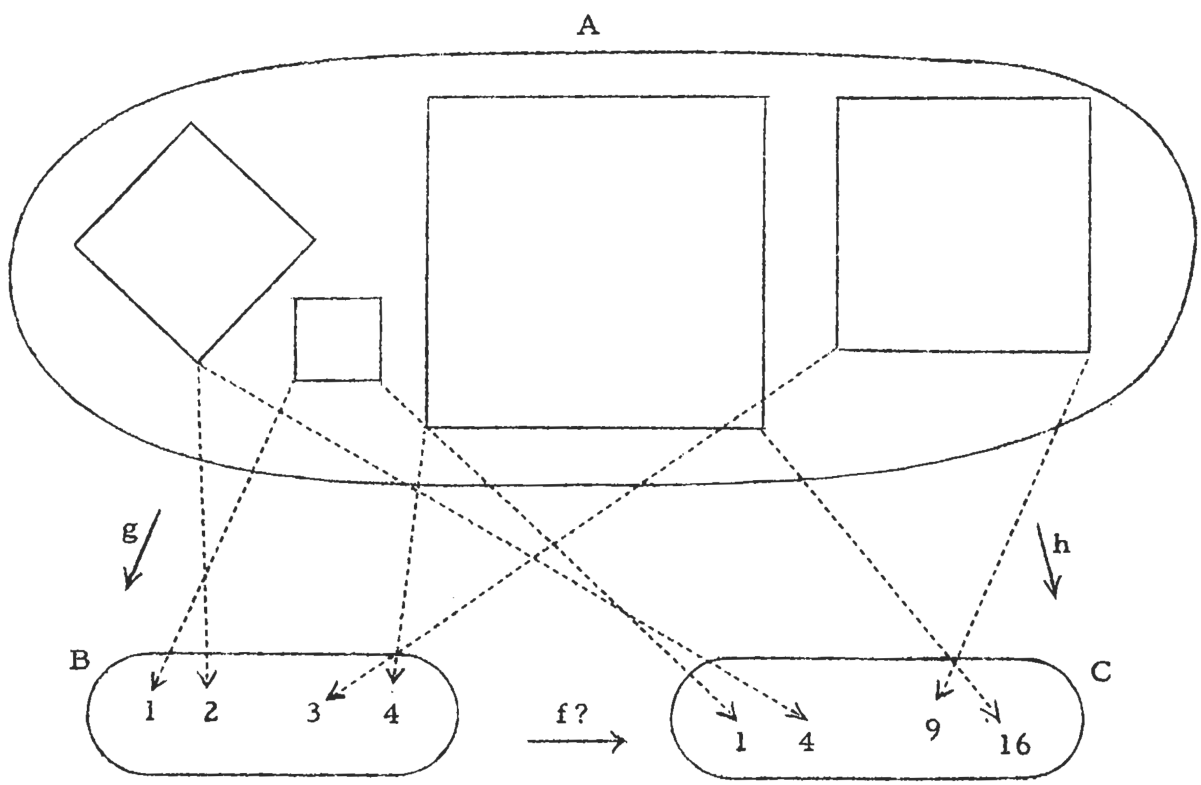
$f =$ ____

13. Here is a picture which shows a mapping g of A on B, and a mapping h of A on C.



f = ________________________________

14. [The members of A are squares.]



f = ________________________________

*

Note that condition (1) [above] is automatically satisfied if $\mathfrak{D}_h = \mathfrak{D}_g$. In this case, to test for functional dependence one merely checks to see whether h has the same value for each two arguments for which g has the same value.

Note that both condition (a) and condition (b) are automatically satisfied if g has an inverse. [See Sample 2 of Part A on page 5-93.] For, in this case, if $g(x_1) = g(x_2)$ then $x_1 = x_2$. To test for functional dependence in this case one merely checks to see whether $\mathfrak{D}_h \subseteq \mathfrak{D}_g$.

[Condition (a) is automatically satisfied if h has the same value for each of its arguments.]

*

The discussion on pages 5-90 through 5-91 is fairly heavy. It is probably the case that most students will accept the theorem at the bottom of page 5-91 just on the basis of the Exploration Exercises on page 5-86.

Correction. On page 5-92, in line 7, change 'belong' to 'belongs'. Also, in the 8th line from the bottom of the page, change '$5 \in \mathcal{D}_h$' to '$5 \notin \mathcal{D}_h$'.

On pages 5-90 and 5-91 we develop a criterion for determining whether, given functions g and h, there is a function f such that $h = f \circ g$.

Since, for any functions f and g, $\mathcal{D}_{f \circ g} \subseteq \mathcal{D}_g$, there cannot be a function f such that $h = f \circ g$ unless $\mathcal{D}_h \subseteq \mathcal{D}_g$. [See, for example, Exercise 4 on page 5-86.] Also, if x_1 and x_2 are arguments of $f \circ g$ such that $g(x_1) = g(x_2)$ then $[f \circ g](x_1) = [f \circ g](x_2)$. So, there cannot be such a function f unless, whenever x_1 and x_2 are arguments of h such that $g(x_1) = g(x_2)$, then $h(x_1) = h(x_2)$. [See Sample 2 on page 5-86.] If this condition is satisfied, and $\mathcal{D}_h \subseteq \mathcal{D}_g$, then with each value of g there corresponds at most one value of h. In this case, the pairs $(g(x), h(x))$, for $x \in \mathcal{D}_h$, are the members of a function f_0 such that, at least, $h \subseteq f_0 \circ g$.

But, suppose that there are arguments, x_1 and x_2 of g such that $g(x_1) = g(x_2) = u$, say, and that, while $h(x_1) = v$, $x_2 \notin \mathcal{D}_h$. In such a case, $(u, v) \in f_0$, so $f_0(g(x_1)) = f_0(u) = v$ and $f_0(g(x_2)) = v$. Hence, both x_1 and x_2 belong to the domain of $f_0 \circ g$ but only one of them, x_1, belongs to $\mathcal{D}_h$. Hence, $h \neq f_0 \circ g$, since h and $f_0 \circ g$ have different domains. [See, for example, Exercise 5 on page 5-86.] So, to ensure that $h = f_0 \circ g$, rather than, merely, that $h \subseteq f_0 \circ g$, we must rule out the possibility that there are arguments x_1 and x_2 of g such that $g(x_1) = g(x_2)$, and $x_1 \in \mathcal{D}_h$, but $x_2 \notin \mathcal{D}_h$.

These three necessary conditions:

(1) $\mathcal{D}_h \subseteq \mathcal{D}_g$ [page 5-90]

(a) if $\{x_1, x_2\} \subseteq \mathcal{D}_h$ and $g(x_1) = g(x_2)$ then $h(x_1) = h(x_2)$ [page 5-91]

(b) if $\{x_1, x_2\} \subseteq \mathcal{D}_g$ and $g(x_1) = g(x_2)$ and $x_1 \in \mathcal{D}_h$ then $x_2 \in \mathcal{D}_h$ [page 5-91]

are also sufficient in order that there be a function f such that $h = f \circ g$. For, if they are satisfied then the set of pairs of corresponding values of g and h is a function f_0 such that $h = f_0 \circ g$.

15. $g(x)$ = the side-measure of square x, $\mathcal{D}_g$ = the set of all squares

$f(x)$ = ____________________ $\mathcal{D}_f$ = ________________________

$h(x)$ = the zrea-measure of square x, $\mathcal{D}_h$ = the set of all squares

16. $g = \{(3, 5), (8, 2), (9, 5), (6, 0), (7, 5), (11, 11)\}$

f = ____

$h = \{(3, 5), (8, 2), (9, 5), (6, 0), (7, 5), (11, 11)\}$

17. $g(x)$ = the area-measure of square x, $\mathcal{D}_g$ = the set of all squares

$f(x)$ = ____________________ $\mathcal{D}_f$ = ________________________

$h(x)$ = the side-measure of square x, $\mathcal{D}_h$ = the set of all squares

☆ 18. $g(x)$ = the husband or wife of x, $\mathcal{D}_g$ = the set of married people

$f(x)$ = ____________________ $\mathcal{D}_f$ = ________________________

$h(x)$ = the mother-in-law of x, $\mathcal{D}_h$ = the set of married people

FUNCTIONAL DEPENDENCE

Given a function g and a function f, you have seen that there is a function $f \circ g$, obtained by composing f with g. The domain of $f \circ g$ consists of those members x of $\mathcal{D}_g$ such that $g(x) \in \mathcal{D}_f$. As you have seen in the preceding Exploration Exercises, given functions g and h, it is sometimes possible to find a function f such that $h = f \circ g$. When there is such a function, we say that h *is functionally related to* g, or, for short, that

h is a function of g.

In such a case, $\mathcal{D}_h \subseteq \mathcal{D}_g$ and the value of h for each of its arguments is determined by the value of g for that argument. So, one sometimes says that h *depends only on* g.

To say that h is a function of g is to claim that there is some function f such that $h = f \circ g$. One way to support such a claim is to discover such a function. [This you did several times in the Exploration Exercises.] Let's find another way.

To say that there is a function f such that $h = f \circ g$ is to say two things:

1. Each argument of h has a g-value [that is, $\mathcal{D}_h \subseteq \mathcal{D}_g$]
2. There is a set f of ordered pairs such that
   (a) f is a function which maps the g-value of each argument of h onto the h-value of this argument [that is, $f(g(x)) = h(x)$ for each $x \in \mathcal{D}_h$], and
   (b) h and $f \circ g$ have the same domain.

Notice, however, that if 2(a) is satisfied then each argument of h is sure to be an argument of $f \circ g$. So, in this case, 2(b) can be replaced by:

(b_1) each argument of $f \circ g$ is an argument of h.

Suppose that g and h satisfy condition 1, and that we have a table which lists, for each argument of h, the corresponding values of g and h.

arguments of h	values of g	values of h
⋮	⋮	⋮
a	g(a)	h(a)
⋮	⋮	⋮
b	g(b)	h(b)
⋮	⋮	⋮

Now, a set of ordered pairs which satisfies 2(a) is just a function which contains each of the ordered pairs of values of g and h listed in the table. If there is such a function then the set f_o of listed ordered pairs will, itself, be such a function. And, if there is such a function which also satisfies 2(b_1), then f_o will satisfy 2(b_1) as well as 2(a). On the other hand, since f_o is the set of listed pairs, f_o is bound to satisfy 2(a) if it is a function.

Summarizing the discussion so far, we see that h is a function of g if and only if

(1) $\mathcal{D}_h \subseteq \mathcal{D}_g$,

(2) f_o [the set of listed ordered pairs] is a function,

and (3) each argument of $f_o \circ g$ is an argument of h.

Now, by the definition of function, f_o is a function if and only if there are not two arguments of h whose g-values are the same but whose h-values are different. [Exercise 2 on page 5-86 is an example of functions g and h which do not meet this condition.] So, condition (2) is satisfied if and only if

(a) for each two arguments x_1 and x_2 of h, if $g(x_1) = g(x_2)$ then $h(x_1) = h(x_2)$.

Exercise 5 on page 5-86 shows that even if conditions (1) and (a) are satisfied, condition (3) may fail to be satisfied. In this exercise, $g = \{(3, 6), (8, 6)\}$ and $h = \{(3, 9)\}$. So, $\mathcal{D}_h = \{3\} \subseteq \{3, 8\} = \mathcal{D}_g$ and, since h has only one argument, (a) is certainly satisfied. However, in this case, $f_o = \{(6, 9)\}$ and $f_o \circ g = \{(3, 9), (8, 9)\}$. So, there is an argument [8] of $f_o \circ g$ which is not an argument of h.

To see what it means for (3) to be satisfied, let's recall what the arguments of $f_o \circ g$ are. The domain of f_o is the set of listed values of g. So, by the definition of function composition, the domain of $f_o \circ g$ is the set of arguments of g whose g-values are listed in the table. A value of g is listed in the table if and only if it is the g-value of some argument of h, that is, if and only if it is $g(x_1)$ for some $x_1 \in \mathcal{D}_h$. So,

x is an argument of $f_o \circ g$

if and only if

$x \in \mathcal{D}_g$ and there is an $x_1 \in \mathcal{D}_h$ such that $g(x_1) = g(x)$.

Thus, to say that

if x is an argument of $f_o \circ g$ then x is an argument of h

amounts to saying that

if $x \in \mathcal{D}_g$ and there is an $x_1 \in \mathcal{D}_h$ such that $g(x_1) = g(x)$ then $x \in \mathcal{D}_h$.

Hence, making use of the fact that $\mathcal{D}_h \subseteq \mathcal{D}_g$, we see that condition (3) is satisfied if and only if

(b) for each two arguments x_1 and x_2 of g, if $g(x_1) = g(x_2)$, and $x_1 \in \mathcal{D}_h$, then $x_2 \in \mathcal{D}_h$.

Finally, (b) and (a) can be combined into:

for all x_1 and x_2 in $\mathcal{D}_g$ such that $g(x_1) = g(x_2)$,
if either x_1 or x_2 belongs to $\mathcal{D}_h$ then both
belong to $\mathcal{D}_h$ and $h(x_1) = h(x_2)$

So, we have established the following theorem:

For each function h, for each function g,
there is a function f such that $h = f \circ g$

if and only if

$\mathcal{D}_h \subseteq \mathcal{D}_g$ and, for all x_1 and x_2 in $\mathcal{D}_g$
such that $g(x_1) = g(x_2)$, if either x_1 or x_2
belongs to $\mathcal{D}_h$ then both belong to $\mathcal{D}_h$ and $h(x_1) = h(x_2)$

Here is an example of the use of this theorem. Suppose

$$g = \{(0, 6), (1, 3), (3, 8), (4, 6), (5, 3), (9, 7)\}$$

and $h = \{(0, 7), (3, 2), (4, 7), (9, 8)\}$. Is h a function of g? Clearly, $\mathcal{D}_h \subseteq \mathcal{D}_g$. So, we look for pairs of arguments of g which have the same g-value. It turns out that the only such pairs are 0 and 4 [$g(0) = 6 = g(4)$] and 1 and 5 [$g(1) = 3 = g(5)$]. Both 0 and 4 belong to $\mathcal{D}_h$ and $h(0) = 7 = h(4)$. Neither 1 nor 5 belong to $\mathcal{D}_h$. So, the conditions of the theorem are satisfied, and h is a function of g. [If we now wish to find a function f such that $h = f \circ g$ we know, without further checking, that f_o, that is, $\{(6, 7), (8, 2), (7, 8)\}$, is such a function. This is the function you would have tried had you been asked to do this example before proving the theorem. At that time you would have had to check to see that $f_o \circ g$ actually is h. In proving the theorem you have shown that, if the conditions given in the theorem are satisfied then $f_o \circ g$ is h, and there is no need for more checking. Also, if the conditions given in the theorem are not satisfied then there is no function f such that $h = f \circ g$.]

Here is a second example. Suppose

$$g = \{(0, 6), (1, 3), (3, 8), (4, 6), (5, 3), (9, 7)\}$$

and $h = \{(0, 7), (1, 10), (3, 2), (4, 7), (9, 8)\}$. Is h a function of g? Again, $\mathcal{D}_h \subseteq \mathcal{D}_g$, and the only pairs of arguments of g which have the same g-value are 0 and 4, and 1 and 5. Both 0 and 4 belong to $\mathcal{D}_h$, and $h(0) = h(4)$. But $1 \in \mathcal{D}_h$ and $5 \notin \mathcal{D}_h$. So, the conditions of the theorem are not satisfied, and h is not a function of g. [In this case, $f_o = \{(6, 7), (3, 10), (8, 2), (7, 8)\}$. Composing f_o with g, we see that

$$[f_o \circ g](0) = 7, \qquad [f_o \circ g](1) = 10, \qquad [f_o \circ g](3) = 2,$$
$$[f_o \circ g](4) = 7, \qquad [f_o \circ g](5) = 10, \qquad [f_o \circ g](9) = 8.$$

So, $f_o \circ g = \{(0, 7), (1, 10), (3, 2), (4, 7), (5, 10), (9, 8)\}$. As is to be expected from the proof, 5 is an argument of $f_o \circ g$. This is so because $g(5) = g(1)$ and $g(1)$ [that is, 3] is an argument of f_o. But, $5 \in \mathcal{D}_h$. Hence, $f_o \circ g \neq h$.]

EXERCISES

A. Use the test for functional dependence to answer the questions in each of the following exercises.

Sample 1. $g = \{(3, 8), (4, 0), (6, 5), (9, 8), (7, 0), (1, 0)\}$
$h = \{(4, 7), (6, 1), (7, 7), (1, 7)\}$

(a) Is g a function of h? (b) Is h a function of g?

8. (a) Yes. [$\mathfrak{s}_A = \mathfrak{s}_P$, and squares with the same perimeter have the same area-measure.]

(b) Yes. [$\mathfrak{s}_P = \mathfrak{s}_A$, and squares with the same area-measure have the same perimeter.]

*

Students should proceed directly to Part B after they finish Exercise 8 of Part A. If you wish to pursue the discussion on pages 5-94 and 5-95, it will be more meaningful to the students if they have pushed through the exercises in Part B.

Corrections. The second line on page 5-93 should read:

(b) Since $\mathscr{D}_h \subseteq \mathscr{D}_g$, we ---

On page 5-95, line 11 should read:

The converse of g is $\{(x, y): x = 3y - 4\}$,

Answers for questions in Solution for Sample 2.

p is not a function of q because $\mathscr{D}_p \not\subseteq \mathscr{D}_q$.

$\{(x, y): x \geq 0 \text{ and } y = 2x + 5\}$ is a function of q because its domain is $\mathscr{D}_q$ and q has an inverse.

*

Answers for Part A [which begins on page 5-92].

1. (a) No. $[k(2) = k(4), \ \ell(2) \neq \ell(4)]$

   (b) No. $[\ell(2) = \ell(5), \ k(2) \neq k(5)]$

   (c) (2, 4) and (4, 3)

   (d) Replace (2, 4) by (2, 3); or replace (4, 3) by (4, 4).

2. (a) No. [$u(5) = u(7)$ and $5 \in \mathscr{D}_v$ but $7 \notin \mathscr{D}_v$]

   (b) No. $[\mathscr{D}_u \not\subseteq \mathscr{D}_v]$

3. Yes. [f has an inverse, and $\mathscr{D}_g = \mathscr{D}_f$.]

4. (a) No. [$t(1) = 3 = t(2)$ but $s(1) \neq s(2)$]

   (b) Yes. [s has an inverse, and $\mathscr{D}_t = \mathscr{D}_s$.]

5. Yes. [$\mathscr{D}_h = \mathscr{D}_g$, and if $g(x_1) = g(x_2)$ then $h(x_1) = h(x_2)$.]

6. (a) Yes. [$\mathscr{D}_r = \mathscr{D}_c$, and circles with the same circumference have the same radius.]

   (b) Yes. [$\mathscr{D}_c = \mathscr{D}_r$, and circles with the same radius have the same circumference.]

7. (a) No. [There are rectangles with the same perimeter but with different area-measures.]

   (b) No. [There are rectangles with the same area-measure but with different perimeters.]

Solution. (a) Since $\mathfrak{D}_g \not\subseteq \mathfrak{D}_h$, g is not a function of h.

(b) Since $\mathfrak{D}_h \not\subseteq \mathfrak{D}_g$, we investigate those arguments of g for which g has the same value. These are 3 and 9, and 4, 7, and 1, because

$$g(3) = g(9) = 8, \text{ and } g(4) = g(7) = g(1) = 0.$$

Now, 3 and 9 do not belong to $\mathfrak{D}_h$. On the other hand, 4, 7, and 1 all belong to $\mathfrak{D}_h$, and $h(4) = 7 = h(7) = h(1)$. So, according to the test for functional dependence, h is a function of g.

Sample 2. $p = \{(x, y): y = 2x + 5\}$, $q = \{(x, y), x \geq 0: y = x^2\}$
Is q a function of p?

Solution. Since $\mathfrak{D}_q \subseteq \mathfrak{D}_p$, q is a function of p unless there are two arguments of p for which p has the same value and such that either one of them belongs to $\mathfrak{D}_q$ and the other does not, or both belong to $\mathfrak{D}_q$ and the values of q for these arguments are different. But, p does not have the same value for any two of its arguments [that is, p has an inverse]. So, q is a function of p.
[Is p a function of q? Is $\{(x, y): x \geq 0 \text{ and } y = 2x + 5\}$ a function of q?]

1. $k = \{(0, 1), (1, 2), (2, 3), (4, 3), (5, 2), (6, 1)\}$
   $\ell = \{(6, 7), (5, 4), (4, 3), (2, 4), (1, 4), (0, 7)\}$
   (a) Is ℓ a function of k? (b) Is k a function of ℓ?
   (c) What two ordered pairs can you remove from ℓ so that the resulting function will be a function of k?
   (d) How can you obtain a function of k by changing the value of ℓ for just one of its arguments?

2. $u = \{(2, 4), (3, 2), (5, 4), (7, 4)\}$
   $v = \{(2, 7), (3, 9), (5, 7)\}$
   (a) Is v a function of u? (b) Is u a function of v?

3. $f = \{(x, y): y = 3x - 4\}$, $g = \{(x, y): y = -2x + 7\}$
   Is g a function of f?

4. $t = \{(x, y): y = 3\}$, $s = \{(x, y): y = 4x + 1\}$
   (a) Is s a function of t? (b) Is t a function of s?

5. $g = \{(x, y), y \geq 0: x^2 + y^2 = 25\}$, $h = \{(x, y), y \leq 0: x^2 + y^2 = 25\}$
Is h a function of g?

6. $c = \{(x, y) \in \text{Circles} \times N:$ y is the circumference of $x\}$
$r = \{(x, y) \in \text{Circles} \times N:$ y is the radius of $x\}$
(a) Is r a function of c? (b) Is c a function of r?

7. $P = \{(x, y) \in \text{Rectangles} \times N:$ y is the perimeter of $x\}$
$A = \{(x, y) \in \text{Rectangles} \times N:$ y is the area-measure of $x\}$
(a) Is A a function of P? (b) Is P a function of A?

8. $P = \{(x, y) \in \text{Squares} \times N:$ y is the perimeter of $x\}$
$A = \{(x, y) \in \text{Squares} \times N:$ y is the area-measure of $x\}$
(a) Is A a function of P? (b) Is P a function of A?

* * *

If g and h are functions for which there exists a function f such that $h = f \circ g$ then there are many such functions f. [Given one such function, you can get others by adding to it ordered pairs whose first components are not values of g.] However, there is a "smallest" such function, the set of all those ordered pairs $(g(x), h(x))$ such that $x \in \mathcal{D}_h$. It is the only function such that $h = f \circ g$ and $\mathcal{D}_f \subseteq \mathcal{R}_g$.

In Sample 2 of Part A on page 5-93 you have seen that if g is a function which has an inverse then, if h is any function such that $\mathcal{D}_h \subseteq \mathcal{D}_g$, h is a function of g. In this case it is particularly easy to find the function f such that $\mathcal{D}_f \subseteq \mathcal{R}_g$ and $h = f \circ g$. To see how, we solve the equation '$h = f \circ g$' for 'f'. To begin with, we compose each side with the inverse of g, and then use the fact that composition is associative.

$$h \circ g^{-1} = [f \circ g] \circ g^{-1} = f \circ [g \circ g^{-1}]$$

But, the domain of $g \circ g^{-1}$ is the domain of g^{-1}, and this is $\mathcal{R}_g$. And, for each $x \in \mathcal{R}_g$, $[g \circ g^{-1}](x) = x$. So, since $\mathcal{D}_f \subseteq \mathcal{R}_g$, the domain of $f \circ [g \circ g^{-1}]$ is $\mathcal{D}_f$ and, for each $x \in \mathcal{D}_f$, $[f \circ [g \circ g^{-1}]](x) = f([g \circ g^{-1}](x)) = f(x)$. Hence, $f \circ [g \circ g^{-1}] = f$. Consequently, [since $f \circ [g \circ g^{-1}] = h \circ g^{-1}$], $f = h \circ g^{-1}$. So, if g has an inverse and $\mathcal{D}_h \subseteq \mathcal{D}_g$, you can find an f such that $h = f \circ g$ merely by composing h with the inverse of g.

Example 1. Consider the function g, where

$$g = \{(3, 6), (5, 9), (7, 5), (8, 4)\}$$

This function g has an inverse, and

$$g^{-1} = \{(6, 3), (9, 5), (5, 7), (4, 8)\}.$$

Now, consider any function h such that $\mathcal{D}_h \subseteq \mathcal{D}_g$, say,

$$h = \{(3, 9), (5, 12), (8, 7)\}.$$

Since

$$h \circ g^{-1} = \{(6, 9), (9, 12), (4, 7)\},$$

$\{(6, 9), (9, 12), (4, 7)\}$ is a function f such that $h = f \circ g$. [It is easy to check that this is the case. Do so now.]

Example 2. Consider the function g, where

$$g = \{(x, y): y = 3x - 4\}.$$

The converse of g is $\{(x, y): x = 3y + 4\}$, that is, $\{(x, y): y = \frac{x + 4}{3}\}$. Since this set of ordered pairs is a function, g has an inverse, and

$$g^{-1} = \{(x, y): y = \frac{x + 4}{3}\}.$$

Now, consider any function h such that $\mathcal{D}_h = \mathcal{D}_g$, say,

$$h = \{(x, y): y = -2x + 7\}.$$

Since $\mathcal{R}_{g^{-1}} = \mathcal{D}_h$, $\mathcal{D}_{h \circ g^{-1}} = \mathcal{D}_{g^{-1}}$, and we can find a description for $h \circ g^{-1}$ merely by substituting. [It was in order for this to work that we chose h so that $\mathcal{D}_h = \mathcal{D}_g$.] Doing so, we find that

$$h \circ g^{-1} = \{(x, y): y = -2(\frac{x + 4}{3}) + 7\}$$
$$= \{(x, y): y = \frac{-2x + 13}{3}\}.$$

To check that this is a function f such that $f \circ g = h$ is to check that $[h \circ g^{-1}] \circ g = h$. Since $\mathcal{R}_{g^{-1}} = \mathcal{D}_g = \mathcal{D}_h$, $\mathcal{D}_{h \circ g^{-1}} = \mathcal{D}_{g^{-1}} = \mathcal{R}_g$. Hence, the domain of $[h \circ g^{-1}] \circ g$ is $\mathcal{D}_g$. So, we can find a description for $[h \circ g^{-1}] \circ g$ merely by substituting. Doing so, we find that

$$[h \circ g^{-1}] \circ g = \{(x, y): y = \frac{-2(3x - 4) + 13}{3}\}$$
$$= \{(x, y): y = -2x + 7\} = h. \quad \checkmark$$

[If we modify Example 2 by choosing for h a function such that $\mathcal{D}_h \subseteq \mathcal{D}_g$ but $\mathcal{D}_h \neq \mathcal{D}_g$, say, the function h where

$$h = \{(x, y), x > 0: y = -2x + 7\},$$

then the desired function f is still $h \circ g^{-1}$. But, in finding a description for this function we must take account of $\mathcal{D}_h$. In this case,

$$h \circ g^{-1} = \{(x, y),\ \frac{x+4}{3} > 0:\ y = -2(\frac{x+4}{3}) + 7\}$$
$$= \{(x, y),\ x > -4:\ y = \frac{-2x+13}{3}\}.]$$

* * *

B. In each of the following exercises you are given functions g and h such that g has an inverse and $\mathcal{D}_h \subseteq \mathcal{D}_g$. Your job is to find $h \circ g$, and to check that it is a function f such that $f \circ g = h$.

1. $g = \{(4, 8), (3, 7), (2, 6), (1, 5)\}$
   $g^{-1} =$ ________________
   $h = \{(3, 9), (2, 14), (1, 2)\}$
   $f = h \circ g^{-1} =$ ________________
   $f \circ g =$ ________________

2. $g = \{(x, y):\ y = 2x + 9\}$
   $g^{-1} =$ ____________
   $h = \{(x, y):\ y = 4x - 11\}$
   $f = h \circ g^{-1} = \{(x, y):$ __________ $\}$
   $f \circ g = \{(x, y):$ __________ $\}$

3. $g = \{(x, y):\ 3x + 2y = 6\}$
   $g^{-1} =$ ____________
   $h = \{(x, y),\ x > 4:\ y = 6x - 10\}$
   $f = h \circ g^{-1} =$ ________________
   $f \circ g =$ ________________

4. $g = \{(x, y):\ y = x + 1\}$
   $g^{-1} =$ ____________
   $h = \{(x, y):\ y \geq 0 \text{ and } x^2 + y^2 = 1\}$
   $f = h \circ g^{-1} =$ ________________
   $f \circ g =$ ________________

☆5. $g = \{(x, y):\ y \geq 0 \text{ and } y^2 = x\}$
   $g^{-1} =$ ________________
   $h = \{(x, y),\ x \geq 0:\ y = \sqrt{x}\}$
   $f = h \circ g^{-1} =$ ______________
   $f \circ g =$ ______________

[Supplementary exercises are in Part Q, page 5-265.]

Correction. At the end of the second line of the directions for Part B, the expression should be '$h \circ g^{-1}$' .

Answers for Part B.

1. $\{(8, 4), (7, 3), (6, 2), (5, 1)\}$; $\{(7, 9), (6, 14), (5, 2)\}$; $\{(3, 9), (2, 14), (1, 2)\}$

2. $\{(x, y): y = \frac{x - 9}{2}\}$; $y = 2x - 29$; $y = 4x - 11$

3. $\{(x, y): y = \frac{6 - 2x}{3}\}$; $\{(x, y), x < -3: y = 2 - 4x\}$; $\{(x, y), x > 4: y = 6x - 10\}$

4. $\{(x, y): y = x - 1\}$; $\{(x, y): y \geq 0 \text{ and } (x - 1)^2 + y^2 = 1\}$; $\{(x, y): y \geq 0 \text{ and } x^2 + y^2 = 1\}$

☆5. $\{(x, y): x \geq 0 \text{ and } y = x^2\}$; $\{(x, y): x \geq 0 \text{ and } y = x\}$; $\{(x, y), x \geq 0: y = \sqrt{x}\}$

*

A procedure very similar to that explained on page 5-94 works to find an f such that $f \circ g = h$ whenever there is such a function. [Warning. The procedure will yield a function f even when h is not a function of g. But, of course, it will not then be the case that $f \circ g = h$.] Suppose that h is a function of g and that f is the function such that $\mathcal{D}_f \subseteq \mathcal{R}_g$ and $h = f \circ g$. Now, even if g does not have an inverse there are subsets of g which have the same range as g and which do have inverses. Suppose that g_1 is such a function. That is, suppose that $g_1 \subseteq g$, $\mathcal{R}_{g_1} = \mathcal{R}_g$, and g_1 has an inverse. Then

$$h \circ g_1^{-1} = [f \circ g] \circ g_1^{-1} = f \circ [g \circ g_1^{-1}].$$

The domain of $g \circ g_1^{-1}$ is the domain of g_1^{-1}, and this is $\mathcal{R}_g$. For each $x \in \mathcal{R}_g$, $[g \circ g_1^{-1}](x) = x$. So, since $\mathcal{D}_f \subseteq \mathcal{R}_g$, the domain of $f \circ [g \circ g_1^{-1}]$ is $\mathcal{D}_f$ and, for each $x \in \mathcal{D}_f$, $[f \circ [g \circ g_1^{-1}]](x) = f([g \circ g_1^{-1}](x)) = f(x)$. Hence, $f \circ [g \circ g_1^{-1}] = f$. Consequently, $h \circ g_1^{-1} = f$. [Notice that the preceding argument is just like that on page 5-94. If g has an inverse, then $g_1 = g$. If g does not have an inverse then there will be more than one choice for g_1. Whichever choice you make, as long as $\mathcal{D}_f \subseteq \mathcal{R}_g$, $f \circ [g \circ g_1^{-1}] = f$. If $h = f \circ g$ then $h \circ g_1^{-1} = f \circ [g \circ g_1^{-1}] = f$. So, $[h \circ g_1^{-1}] \circ g = h$.

In general, the domain of $g_1^{-1} \circ g$ is $\mathcal{D}_g$, and its range is $\mathcal{D}_{g_1}$, a subset of $\mathcal{D}_g$. For each $x \in \mathcal{D}_{g_1}$, $[g_1^{-1} \circ g](x) = x$ but, in general, if $x \in \mathcal{D}_g$, one can only say that $g([g_1^{-1} \circ g](x)) = g(x)$. If the test for functional dependence is satisfied [and only then], one can conclude that, for each $x \in \mathcal{D}_h$, $h([g_1^{-1} \circ g](x)) = h(x)$. So, in this case, $h \circ [g_1^{-1} \circ g] = h$ and, by the associativity of function composition, $h \circ g_1^{-1}$ is a function f such that $f \circ g = h$.]

As an example, consider the functions g and h, where

$$g = \{(3, 6), (5, 9), (8, 4), (7, 6)\}$$

and

$$h = \{(3, 9), (5, 12), (8, 7), (7, 9)\}.$$

$\mathcal{D}_h = \mathcal{D}_g$ and 3 and 7 are the only arguments for which g has the same value. Since $h(3) = h(7)$, h is a function of g. For this function there are just two appropriate choices for g_1.

$g_1 = \{(3, 6), (5, 9), (8, 4)\}$	$g_1 = \{(5, 9), (8, 4), (7, 6)\}$
$g_1^{-1} = \{(6, 3), (9, 5), (4, 8)\}$	$g_1^{-1} = \{(9, 5), (4, 8), (6, 7)\}$
$h \circ g_1^{-1} = \{(6, 9), (9, 12), (4, 7)\}$	$h \circ g_1^{-1} = \{(9, 12), (4, 7), (6, 9)\}$

Note that with either choice, one obtains the same function $h \circ g_1^{-1}$ and that on composing this function with g one obtains h.

As another example, consider the functions g and h, where

$$g = \{(x, y): \ y = x^2\}$$

and

$$h = \{(x, y): \ x^2 \leq 1 \text{ and } y = |x|\}.$$

$\mathcal{D}_h \subseteq \mathcal{D}_g$ and, for two arguments x_1 and x_2 of g, $g(x_1) = g(x_2)$ if and only if x_1 and x_2 are opposites. In this case, either both or neither of them belong to $\mathcal{D}_h$, and if both do, $h(x_1) = h(x_2)$. So, h is a function of g. A

suitable function g_1 is $\{(x, y): x \geq 0 \text{ and } y = x^2\}$. With this choice,

$$g_1^{-1} = \{(x, y): y \geq 0 \text{ and } y^2 = x\}$$
$$= \{(x, y), x \geq 0: y = \sqrt{x}\}, \text{ and}$$
$$h \circ g_1^{-1} = \{(x, y), x \geq 0: (\sqrt{x})^2 \leq 1 \text{ and } y = |\sqrt{x}|\}$$
$$= \{(x, y), x \geq 0: x \leq 1 \text{ and } y = \sqrt{x}\}.$$

Checking,

$$[h \circ g_1^{-1}] \circ g = \{(x, y), x^2 \geq 0: x^2 \leq 1 \text{ and } y = \sqrt{x^2}\}$$
$$= \{(x, y): x^2 \leq 1 \text{ and } y = |x|\}$$
$$= h.$$

An alternative choice for g_1 is $\{(x, y): x \leq 0 \text{ and } y = x^2\}$. With this choice,

$$g_1^{-1} = \{(x, y): y \leq 0 \text{ and } y^2 = x\}$$
$$= \{(x, y), x \geq 0: y = -\sqrt{x}\}, \text{ and}$$
$$h \circ g_1^{-1} = \{(x, y), x \geq 0: (-\sqrt{x})^2 \leq 1 \text{ and } y = |-\sqrt{x}|\}$$
$$= \{(x, y), x \geq 0: x \leq 1 \text{ and } y = \sqrt{x}\}.$$

So, with either choice for g_1 we obtain the same function $h \circ g_1^{-1}$.

*

Quiz.

A. $g = \{(3, 5), (2, 9), (1, 4), (0, 7)\}$

$f = \{(4, 2), (5, 1), (3, 8), (9, 11), (7, 7)\}$

1. $g^{-1} = \{$____________________$\}$ 2. $f^{-1} = \{$____________________$\}$

3. $g(3) + g^{-1}(9) =$ _____ 4. $f(4) + f(3) - f^{-1}(7) =$ _____

5. $f(g(3)) =$ _____ 6. $[f \circ g](2) =$ _____ 7. $f \circ g = \{$_____$\}$

8. $g \circ g^{-1} = \{$____________________$\}$ 9. $f \circ [g \circ g^{-1}] = \{$____________________$\}$

B. 1. $a = \{(3, 8), (4, 0), (6, 5), (9, 8), (7, 9), (1, 3)\}$

$b = \{(4, 7), (6, 1), (7, 9), (1, 11)\}$

Tell why a is not a function of b.

2. $c = \{(5, 8), (4, 2), (9, 8), (10, 8)\}$ $d = \{(5, 7), (4, 9), (9, 7)\}$

Tell why d is not a function of c.

3. $e = \{(1, 2), (2, 3), (3, 4), (5, 4), (6, 3), (7, 2)\}$

$f = \{(7, 8), (6, 5), (5, 4), (3, 5), (2, 5), (1, 8)\}$

Tell why f is not a function of e.

C. Suppose that $g = \{(x, y): y = 2x + 9\}$ and $h = \{(x, y): y = 4x - 11\}$. Find a function f such that $h = f \circ g$.

D. Suppose that d is the function that maps each real number on its double, and t is the function that maps each real number on its triple.

1. Is t a function of d? If not, tell why. If so, describe a function f such that $t = f \circ d$.

2. Is d a function of t? If not, tell why. If so, describe a function g such that $d = g \circ t$.

*

Answers for Quiz.

A. 1. (5, 3), (9, 2), (4, 1), (7, 0)   2. (2, 4), (1, 5), (8, 3), (11, 9), (7, 7)

3. 7   4. 3   5. 1   6. 11

7. (3, 1), (2, 11), (1, 2), (0, 7)   8. (5, 5), (9, 9), (4, 4), (7, 7)

9. (5, 1), (9, 11), (4, 2), (7, 7)

B. 1. $\mathfrak{D}_a \not\subseteq \mathfrak{D}_b$

2. $c(9) = c(10)$ and $9 \in \mathfrak{D}_d$ but $10 \notin \mathfrak{D}_d$

3. $e(3) = e(5)$ but $f(3) \neq f(5)$

C. $f = h \circ g^{-1} = \{(x, y): y = 2x - 29\}$

D. 1. Yes; $f = \{(x, y): y = \frac{3}{2}x\}$ [f is the function that maps each real number on the real number $\frac{3}{2}$ times as large.]

2. Yes; $g = \{(x, y): y = \frac{2}{3}x\}$ [g is the function that maps each real number on the real number $\frac{2}{3}$ as large.]

The importance of variable quantities, and of distinguishing between variables [pronumerals] and names of variable quantities, has been stressed recently by Karl Menger [for example, in "Why Johnny Hates Math--", The Mathematics Teacher, December, 1956]. [This distinction is often lost sight of in dealing with formulas, such as '$A = s^2$', or '$v^2 = 2gh$', in which, properly speaking, 'A', 's', 'v', and 'h' are names of variable quantities, rather than pronumerals.]

Among variable quantities we include both those functions whose ranges are sets of numbers of arithmetic, and those whose ranges are sets of real numbers. [As previously, 'N' is a name for the set of numbers of arithmetic.]

*

The following exercises on pages 5-53 et seq. refer to variable quantities:

Part D, Exercises 3, 5, 6, 7, 9, 10, 11, 12, 15, 16 (a), (b), (c), (d), (e), 17 (a).

Part E, Exercises 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14.

Part F, Exercises 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14.

*

Answer to question on page 5-98, line 4.

The pair (0, 0) does not belong to the second function [but does belong to the first].

*

Remark concerning bracketed sentence on page 5-98 preceding exercises.

If f has an inverse and $A = f \circ s$ then $f^{-1} \circ A = f^{-1} \circ [f \circ s] = [f^{-1} \circ f] \circ s$.

Since $\mathcal{D}_A = \mathcal{D}_s$, $\mathcal{R}_s \subseteq \mathcal{D}_f$. So, $[f^{-1} \circ f] \circ s = s$.

5.06 Variable quantities. --While the range of a function may be any set whatever, many of the more useful functions are numerical-valued, that is, have ranges which consist of numbers. Examples of such functions are:

$\{(x, y) \in \text{Circles} \times N\colon\ y \text{ is the radius of } x\}$
the area-measure of a square
the number of people in a family on December 31, 1959
the double of a number

Turn, now, to page 5-53 and, in Parts D, E, and F, tell which of the functions listed there have numerical values.

Numerical-valued functions are sometimes called variable quantities. [The function which is called 'the area-measure of a square' is a variable quantity. (Area is a quantity which "varies" from square to square). The domain of this variable quantity is the set of all squares. Its range is the set of nonzero numbers of arithmetic.]

DEPENDENCE

Consider the variable quantities A and s where

$$A = \{(x, y) \in \text{Squares} \times N\colon\ y \text{ is the area-measure of } x\},$$

and

$$s = \{(x, y) \in \text{Squares} \times N\colon\ y \text{ is the side-measure of } x\}.$$

The domain of both these variable quantities is the set of all squares. Since two squares have the same area-measure if and only if they are congruent, and since they have the same side-measure if and only if they are congruent, each of the variable quantities A and s is a function of the other [Explain].

These results are not surprising, for you have known for a long time that to find the value of A for a given square, all you do is multiply the corresponding value of s by itself. Also, to find the value of s for a given square, you compute the square root of the corresponding value of A. Thus, one function f such that $A = f \circ s$ is the squaring function for numbers of arithmetic,

$$\{(u, v) \in N \times N\colon\ v = u^2\}.$$

Since, for each square x, $s(x) > 0$, another function which will work just as well is

$$\{(u, v) \in N \times N: u > 0 \text{ and } v = u^2\}.$$

[How does this second function differ from the first?] Similarly, two functions, g, such that $s = g \circ A$ are the square rooting function for numbers of arithmetic,

$$\{(u, v) \in N \times N: v = \sqrt{u}\},$$

and [since for each square x, $A(x) > 0$] the function

$$\{(u, v) \in N \times N: u > 0 \text{ and } v = \sqrt{u}\}.$$

[Do you see that if f is a function such that $A = f \circ s$ and f has an inverse then $s = f^{-1} \circ A$? Hint. Solve '$A = f \circ s$' for 's'.]

EXERCISES

A. Given the variable quantities c and d where

$$c = \{(x, y) \in \text{Circles} \times N: y \text{ is the circumference of } x\},$$

and

$$d = \{(x, y) \in \text{Circles} \times N: y \text{ is the diameter of } x\}.$$

1. Is c a function of d? If not, tell why. If c is a function of d, complete the following:

   $c = f \circ d$ where $f = \{(u, v) \in N \times N:$ _____ $\}$

2. Is d a function of c? If not, tell why. If d is a function of c, complete the following:

   $d = g \circ c$ where $g = \{(u, v) \in N \times N:$ _____ $\}$

B. Repeat Part A for the variable quantities A and ℓ where

$$A = \{(x, y) \in \text{Rectangles} \times N: y \text{ is the area-measure of } x\},$$

and

$$\ell = \{(x, y) \in \text{Rectangles} \times N: y \text{ is the length-measure of } x\}.$$

Answers for Part A.

1. Yes; $v = \pi u$

2. Yes; $v = u/\pi$

*

Answers for Part B.

1. No; you can have two rectangles with the same length-measure but different area-measures.

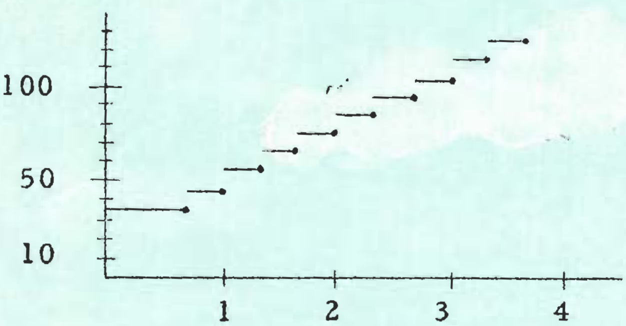
2. No; you can have two rectangles with the same area-measure but different length-measures.

*

Answers for Part C [on page 5-99].

1. Yes; here is a graph of a function g such that, for each trip x,

$$c(x) = g(m(x)).$$



2. No; trips of differing lengths may cost the same.

[Here is a description of the function g whose graph is given in the answer for Exercise 1 of Part C:

$$g(x) = \begin{cases} 35, \text{ for } 0 < x \le 2/3 \\ 35 + 10\,(-[\![-(3x - 2)]\!]), \text{ for } x > 2/3 \end{cases}$$

[The formula in the second line can, of course, be simplified to: $35 - 10\,[\![2 - 3x]\!]$.] See Exercise 5 of Part L of the Supplementary Exercises for a definition of '$[\![\ldots]\!]$'.]

Note concerning the word 'singulary'. The Latin distributives from which such words as 'binary', 'ternary', 'quaternary', etc. derive are 'singuli', 'bini', 'terni', 'quaterni', etc.

*

The essence of the discussion on pages 5-99 and 5-100 is that each operation on numbers can be used to induce an operation on functions. In the case of a singulary operation on numbers, the corresponding singulary operation on functions is defined in terms of the given operation on numbers just as, on page 5-100, we define the oppositing operation on functions in terms of the oppositing operation on numbers.

To bring out the fact that we are actually dealing with two operations--one on numbers and the other on functions--it might be helpful to use, at first, a bold-face '**—**' for the operation on functions. But, since one can tell from context which of the two operations is meant, there is no harm in using the same symbol, '—', for both.

The oppositing operation on functions is of little interest except as applied to functions whose values are real numbers. So, we might, in the definition on page 5-100, have replaced 'functions', both times, by 'real-valued functions', and, since each real number has an opposite, have omitted 'such that g(x) has an opposite'. Our reason for not doing so is that we want to provide a pattern for students to follow in solving Exercise 1 of Part B on page 5-101.

*

Answers for Part A [on page 5-100].

1. (a) -18 (b) 0 (c) -12 (d) -84 (e) -84

   (f) $\{(x, y): y = -(3x + 12)\}$

2. $\{(x, y), x \geq 0: y = -\sqrt{x}\}$ 3. 0 4. $\frac{1}{5}$

☆5. $\{0\}$

☆6. the only real number in $\mathfrak{R}_f$ is 0

C. A taxicab company charges 35 cents for the first two thirds of a mile, or fraction thereof, and 10 cents for each succeeding third of a mile or fraction thereof. Consider the variable quantities c and m where

$$c = \{(x, y) \in \text{Trips} \times N\text{: y cents is the charge for trip x}\},$$

and

$$m = \{(x, y) \in \text{Trips} \times N\text{: x is a trip of y miles}\}.$$

1. Is c a function of m? If not, tell why. If c is a function of m, draw a graph of such a function.

2. Is m a function of c? If not, tell why. If m is a function of c, draw a graph of such a function.

OPERATIONS ON VARIABLE QUANTITIES

In earlier units you became acquainted with various operations on numbers. Some of the operations are oppositing, squaring, absolute valuing, and reciprocating. Others are addition and multiplication. Operations like those in the first list are called singulary operations; such operations are applied to single numbers. Those in the second list are called binary operations; they are applied to ordered pairs of numbers. Since whenever an operation can be applied to a number or to an ordered pair of numbers, the result is unique [otherwise it wouldn't be called 'an operation'!], an operation is simply a function. So, for example, we could use 'op' as a name for the oppositing function and use functional notation to write things like:

$$op(^{+}2) = {}^{-}2, \qquad op(^{-}5) = {}^{+}5, \qquad op(x+y) = op(x) + op(y),$$

$$[op \circ op](^{-}8), = {}^{-}8, \qquad op \circ [op \circ op] = op$$

However, we use instead an operator '−' and write things like:

$$-{}^{+}2 = {}^{-}2, \qquad -(^{-}5) = {}^{+}5, \qquad -(x + y) = -x + -y$$

$$- -{}^{-}8 = {}^{-}8, \qquad \forall_x - - -x = -x$$

In a similar way, composing a function f with a function g can be thought of as operating on g. For example, suppose

$$g = \{(\text{Al}, {}^{+}5), (\text{Ned}, {}^{+}4), (\text{Charles}, 0)\}$$

and f is the oppositing function, op. Then

$$op \circ g = \{(\text{Al}, {}^{-}5), (\text{Ned}, {}^{-}4), (\text{Charles}, 0)\}.$$

Instead of writing 'op ∘ g', we shall write '−g'. Note that when we do this, we are using the operator '−' in a new way. Previously, it named an operation on real numbers; now, we are using it to name an operation on functions. This is a case of ambiguity which is usually resolved by the context. For example, referring to the function g given above, we note that

$$[-g](\text{Al}) = -g(\text{Al}) = -{}^{+}5.$$

The first '−' stands for the operation on functions. The second and third '−'s stand for the operation on real numbers.

We can now define the oppositing operation on functions.

> For each function g, −g is the function such that
>
> $$[-g](x) = -g(x),$$
>
> for each $x \in \mathcal{D}_g$ such that g(x) has an opposite.

EXERCISES

A. Fill the blanks.

1. If $g = \{(x, y): y = 3x + 12\}$ then

   (a) $[-g](2) =$ ____ (b) $[-g](-4) =$ ____ (c) $[-g](0) =$ ____

   (d) $[[-g] \circ g](4) =$ ____ (e) $[-[g \circ g]](4) =$ ____ (f) $-g =$ ____

2. If $t = \{(x, y), x \geq 0: y = \sqrt{x}\}$ then $-t =$ ____________.

3. If $u = \{(x, y): y = 5x - 7\}$ then $u(79) + [-u](79) =$ ____________.

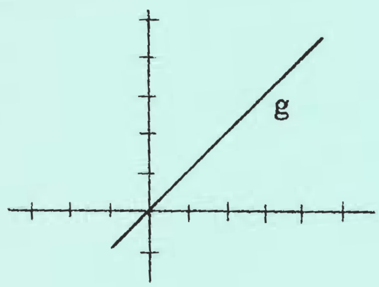
4. If $k = \{(x, y): y = 10x - 2\}$ then the solution of '$k(a) = [-k](a)$' is ______.

☆5. If $\mathcal{R}_f$ consists of real numbers then $f = -f$ if and only if $\mathcal{R}_f =$ ____.

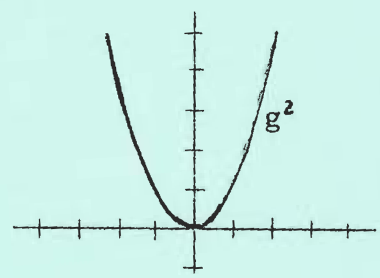
☆6. $f = -f$ if and only if ____________________.

Answers for Part D [on page 5-101].

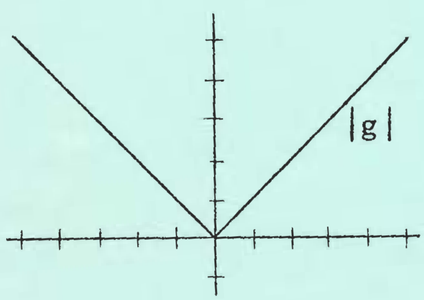
1.



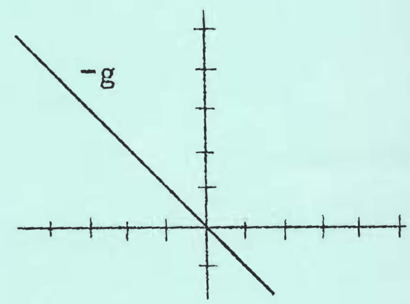
2.



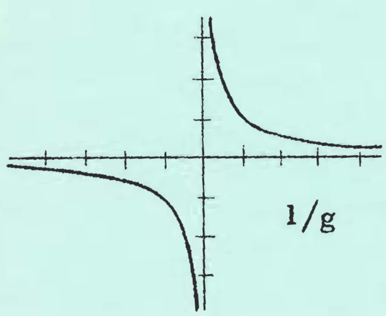
3.



4.



5.



6. $\mathcal{D}_g$ = the set of real numbers $\mathcal{R}_g$ = the set of real numbers

$\mathcal{D}_{g^2}$ = the set of real numbers $\mathcal{R}_{g^2}$ = $\{x: x \geq 0\}$

$\mathcal{D}_{|g|}$ = the set of real numbers $\mathcal{R}_{|g|}$ = $\{x: x \geq 0\}$ [if '$|\ldots|$' abbreviates '$^+|\ldots|$']

$\mathcal{D}_{-g}$ = the set of real numbers $\mathcal{R}_{-g}$ = the set of real numbers

$\mathcal{D}_{1/g}$ = $\{x: x \neq 0\}$ $\mathcal{R}_{1/g}$ = $\{x: x \neq 0\}$

Students will probably follow the pattern of the definition on page 5-100, and give the answers displayed above for Exercise 1 of Part C. You might then suggest the following simpler definitions which cover all cases of interest:

(a) For each variable quantity g, g^2 is the variable quantity such that

$$g^2(x) = (g(x))^2, \text{ for each } x \in \mathcal{D}_g.$$

(b) For each real-valued variable quantity g, $|g|$ is the variable quantity such that

$$|g|(x) = |g(x)|, \text{ for each } x \in \mathcal{D}_g.$$

(c) For each variable quantity g, $\frac{1}{g}$ is the variable quantity such that

$$[\frac{1}{g}](x) = \frac{1}{g(x)},$$

for each $x \in \mathcal{D}_g$ such that $g(x) \neq 0$.

2. (a) 121 (b) 100 (c) $1, -\frac{1}{7}$ (d) 46

(e) 24 (f) $4, -\frac{22}{7}$ (g) $\frac{1}{4}$ (h) $-\frac{1}{3}$

(i) $[0 \notin \mathcal{R}_{1/g}]$ (j) $[\frac{3}{7} \notin \mathcal{D}_{1/g}]$

(k) $\{(x, y), x \neq \frac{3}{7} : y = \frac{1}{(7x - 3)^2}\}$ [or: $\{(x, y) : y(7x - 3)^2 = 1\}$]

*

Answers for Part B.

1. $\sqrt{g(x)}$; $x \in \mathcal{D}_g$ such that $g(x)$ has a square root

2. (a) 2 (b) 14 (c) 0 (d) $[4 \notin \mathcal{D}_{\sqrt{g}}]$

   (e) $[-2 \notin \mathcal{R}_{\sqrt{g}}]$ (f) $\sqrt{g} = \{(x, y), x \geq 5 : y = \sqrt{x - 5}\}$

3. $\{(x, y) : y = |x|\}$

4. s

5. (a) 1 (b) 1 (c) 5 (d) $[-5 \notin \mathcal{D}_{-\sqrt{t}}]$

   (e) $[4 \notin \mathcal{D}_{\sqrt{-t}}]$ (f) $[3 \notin \mathcal{D}_{-\sqrt{t}}]$ (g) $\{(x, y), x \geq \frac{10}{3} : y = -\sqrt{3x - 10}\}$

   (h) $\{(x, y), x \leq \frac{10}{3} : y = \sqrt{-(3x - 10)}\}$

*

Answers for Part C.

1. (a) For each function g, g^2 is the function such that

$$g^2(x) = (g(x))^2,$$

   for each $x \in \mathcal{D}_g$ such that $g(x)$ has a square.

   (b) For each function g, $|g|$ is the function such that

$$|g|(x) = |g(x)|,$$

   for each $x \in \mathcal{D}_g$ such that $g(x)$ has an absolute value.

   (c) For each function g, $\frac{1}{g}$ is the function such that

$$[\frac{1}{g}](x) = \frac{1}{g(x)},$$

   for each $x \in \mathcal{D}_g$ such that $g(x)$ has a reciprocal.

B. 1. Follow the pattern of the definition of oppositing for functions, and complete the following definition of the square-rooting operation:

For each function g, $\sqrt{g}$ is the function such that

$\sqrt{g}\,(x) =$ __________,

for each ____________________________.

Now, fill the blanks in the following exercises.

2. If $g = \{(x, y): y = x - 5\}$ then

(a) $\sqrt{g}(9) =$ ____ (b) $\sqrt{g}\,($___$) = 3$ (c) $\sqrt{g}\,(5) =$ ____

(d) $\sqrt{g}(4) =$ ____ (e) $\sqrt{g}\,($___$) = -2$ (f) $\sqrt{g} =$ ______

3. If $k = \{(x, y): y = x^2\}$ then $\sqrt{k} =$ ____________________

4. If A is the area-measure of a square and s is the side-measure of a square then $\sqrt{A} =$ _____.

5. If $t = \{(x, y): y = 3x - 10\}$ then

(a) $[-t](3) =$ ____ (b) $\sqrt{-t}\,(3) =$ ____ (c) $\sqrt{-t}\,(-5) =$ ____

(d) $[-\sqrt{t}](-5) =$ ____ (e) $\sqrt{-t}\,(4) =$ ____ (f) $[-\sqrt{t}](3) =$ ____

(g) $-\sqrt{t} =$ __________ (h) $\sqrt{-t} =$ _______

C. 1. Write definitions of the operations on functions listed below.

(a) squaring (b) absolute valuing (c) reciprocating

Now, fill the blanks in the following exercises.

2. If $g = \{(x, y): y = 7x - 3\}$ then

(a) $g^2(2) =$ ____ (b) $g^2(-1) =$ ____ (c) $g^2($___$) = 16$

(d) $|g|(7) =$ ____ (e) $|g|(-3) \doteq$ ____ (f) $|g|($___$) = 25$

(g) $\frac{1}{g}(1) =$ ____ (h) $[1/g](0) =$ ____ (i) $[1/g]($___$) = 0$

(j) $[1/g](\frac{3}{7}) =$ ____ (k) $1/|g^2| =$ ____________________

D. Suppose $g = \{(x, y): y = x\}$. Graph the functions listed below.

1. g 2. g^2 3. $|g|$ 4. $-g$ 5. $1/g$

6. What are the domains and ranges of these five functions?

ADDING AND MULTIPLYING VARIABLE QUANTITIES

As an example of the occurrence of operations on functions, consider the formula '$A = s^2$' which you use in finding the area-measure of a given square when you know its side-measure.

$$A = \{(x, y) \in \text{Squares} \times N: \text{ y is the area-measure of x}\},$$

and

$$s = \{(x, y) \in \text{Squares} \times N: \text{ y is the side-measure of x}\}.$$

When you use the formula to find the area-measure of a given square x, you find the value, s(x), of s for this square, and then multiply this number, s(x), by itself. Since the variable quantity s^2 is

$$\{(x, y) \in \text{Squares} \times N: y = (s(x))^2\},$$

this amounts to finding the value, $[s^2](x)$, of s^2 for this given square x. Since the formula tells you that the variable quantity A <u>is</u> the variable quantity s^2, you know that by computing $[s^2](x)$, you have found the area-measure A(x) of square x. Similarly, when given the area-measure of a square, you use the variable quantity $\sqrt{A}$ in order to find the side-measure of the square. The formula '$s = \sqrt{A}$' tells you that this is a correct procedure.

Formulas such as '$A = \ell w$' and '$P = 2(\ell + w)$', for the area-measure and perimeter of a rectangle, indicate that variable quantities can be multiplied and added. The definitions of these operations are obvious enough.

For all variable quantities g and h,

$$g + h = \{(x, y), x \in \mathcal{D}_g \cap \mathcal{D}_h: y = g(x) + h(x)\},$$

and

$$gh = \{(x, y), x \in \mathcal{D}_g \cap \mathcal{D}_h: y = g(x) \cdot h(x)\}.$$

In finding the area-measure of a rectangle x, you multiply $\ell(x)$ by w(x). This is nothing more than computing the value, $[\ell w](x)$, of the variable quantity ℓw for its argument x.

<u>Example 1</u>. If f = {(3, 5), (7, 8), (5, 9), (8, 5), (2, 0)} and g = {(3, 9), (7, 2), (4, 3), (8, 16), (5, 7)}, what are f + g and fg?

<u>Solution</u>. According to the definitions, f + g and fg are variable

quantities whose domain is the intersection of the domains of f and g. Applying the definition we see that

$$f + g = \{(3,\ 5 + 9),\ (7,\ 8 + 2),\ (5,\ 9 + 7),\ (8,\ 5 + 16)\}$$
$$= \{(3,\ 14),\ (7,\ 10),\ (5,\ 16),\ (8,\ 21)\},$$

and

$$fg = \{(3,\ 45),\ (7,\ 16),\ (5,\ 63),\ (8,\ 80)\}.$$

<u>Example 2</u>. If $f = \{(x, y),\ x > 0:\ y = 2x\}$ and
$g = \{(x, y),\ x < 5:\ y = x - 4\}$,
what are f + g and fg?

<u>Solution</u>. The intersection of the domains of f and g is
$\{x:\ x > 0\} \cap \{x:\ x < 5\}$,

that is,

$$\{x:\ 0 < x < 5\}.$$

So,

$$f + g = \{(x, y),\ 0 < x < 5:\ y = (2x) + (x - 4)\}$$
$$= \{(x, y),\ 0 < x < 5:\ y = 3x - 4\},$$

and

$$fg = \{(x, y),\ 0 < x < 5:\ y = 2x^2 - 8x\}.$$

<u>Example 3</u>. Given f and g of Example 2, complete these sentences.

(a) $[f + g](3) =$ ______ (b) $[fg](3) =$ ______

(c) $[f + g](\frac{3}{4}) =$ ______ (d) $[fg](\sqrt{2}) =$ ______

(e) $[f + g](-\frac{1}{2}) =$ ______ (f) $[fg](6) =$ ______

<u>Solution</u>. (a) $[f + g](3) = 3 \cdot 3 - 4 = 5$ (b) $[fg](3) = 2 \cdot 3^2 - 8 \cdot 3 = -6$

(c) $[f + g](\frac{3}{4}) = 3 \cdot \frac{3}{4} - 4 = -\frac{7}{4}$

(d) $[fg](\sqrt{2}) = 2(\sqrt{2})^2 - 8\sqrt{2} = 4 - 8\sqrt{2}$

(e) and (f) '$[f + g](-\frac{1}{2})$' and '$[fg](6)$' are nonsense. [Why?]

* * *

Consider, again, the formula '$P = 2(\ell + w)$'. When using this formula, you obtain values of P by multiplying the number 2 by corresponding values of $\ell + w$. This is just what you would do if you were finding values of the product of the variable quantity

$$\{(x, y),\ x \in \mathcal{D}_{\ell + w}:\ y = 2\}$$

by the variable quantity $\ell + w$. So, we could replace the formula

$$P = 2(\ell + w)$$

by the formula:

$$P = \{(x, y),\ x \in \mathcal{D}_{\ell + w}:\ y = 2\} \cdot (\ell + w)$$

This amounts to interpreting the symbol '2' as a name for the variable quantity $\{(x, y),\ x \in \mathcal{D}_{\ell + w}:\ y = 2\}$. If we do so, operations on real-valued variable quantities satisfy principles just like our basic principles for real numbers, and we can manipulate formulas according to rules just like those which we use in manipulating numerical equations. So, whenever we have a formula in which both numerals and names of variable quantities occur, we shall interpret each numeral as standing for a variable quantity whose domain is that of the other variable quantities and whose range is the set consisting of the number named by the numeral. [Variable quantities which are named by numerals are sometimes called constant variable quantities, or for short, constants.]

* * *

Example 4. If $f = \{(x, y):\ y = 3x + 7\}$, what are $9f$ and $f + 9$?

Solution. We interpret '9' as standing for a variable quantity whose range is $\{9\}$. Since $\mathcal{D}_f$ is the set of real numbers, we say that 9 is $\{(x, y):\ y = 9\}$. Then, by the definitions of multiplication and addition for variable quantities,

$$\begin{aligned} 9f &= \{(x, y):\ y = 9\} \cdot \{(x, y):\ y = 3x + 7\} \\ &= \{(x, y):\ y = 9(3x + 7)\} \\ &= \{(x, y):\ y = 27x + 63\}, \end{aligned}$$

and

$$\begin{aligned} f + 9 &= \{(x, y):\ y = 3x + 7\} + \{(x, y):\ y = 9\} \\ &= \{(x, y):\ y = (3x + 7) + 9\} \\ &= \{(x, y):\ y = 3x + 16\}. \end{aligned}$$

Instead of reinterpreting '2' as a name for a variable quantity, we might define, for each number z and each variable quantity f, $zf = \{(x, y) : y = z \cdot f(x)\}$, thus introducing a third kind of multiplication in addition to those already introduced; to wit, multiplication of numbers and multiplication of variable quantities. This procedure can be carried through, but gives rise to annoying complications. For example, in dealing with the variable quantities circumference and diameter of a circle [c and d], we want to be able to interpret not only '$c = \pi d$' [which we can do whether we consider 'π' as a numeral or as a name for a variable quantity], but also '$c/d = \pi$'. The latter makes sense only if 'π' is a name for the variable quantity c/d, i.e. for the variable quantity whose domain is the set of all circles and whose range consists of the single number π. With this re-interpretation of numerals as being names of variable quantities, it turns out that one can manipulate formulas according to what are, essentially, the same rules as one uses in manipulating arithmetical sentences. How this comes about is discussed on pages 5-112 through 5-115.

The ambiguity resulting from the fact that each numeral is, now, not only a name for a number but also a name for any function whose range consists of this number, does not present a serious problem. As in the case of numerals for numbers of arithmetic, which may on occasion be used as names for corresponding nonnegative real numbers, either the proper interpretation is clear from context, or it is immaterial which interpretation is adopted.

*

Note that there is some conflict between the terms 'constant variable quantity' and 'constant function' [see page 5-83]. Although a variable quantity is a function, a constant variable quantity may not be a constant function! A constant function has the set of real numbers as domain.

[The word 'constant' is also sometimes used to refer to symbols which function as nouns [for example, numerals] in contrast to symbols, called 'variables', which function as pronouns [for example, pronumerals].]

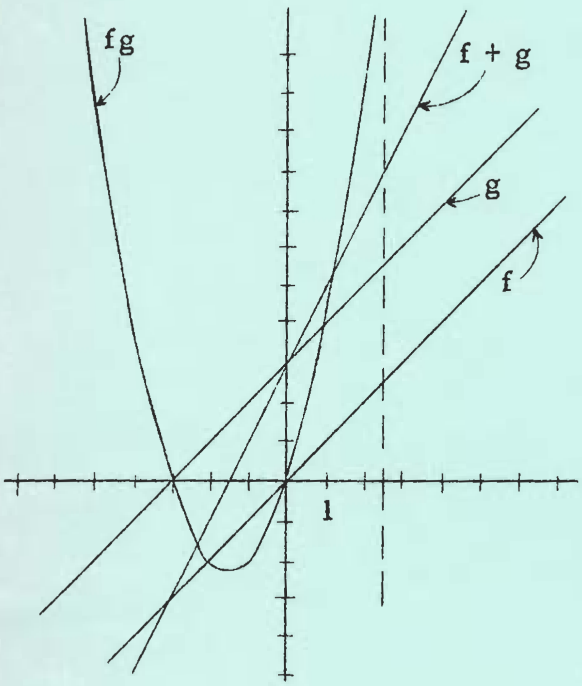
(ii) Point out that the parentheses in Exercise 3 are not punctuation, but, rather, indicate application of a function to an argument [see TC[5-55, 56]b]; and that the value of the variable quantity 3 at any of its arguments is the number 3.

(iii) Show that the incorrect answer '31' is the correct answer to another exercise: $[f^2 + 3j](2) =$ ____, where $j = \{(x, y): y = x\}$.

Similar remarks are appropriate if students give answers: 26, 51, 7, 11, 77, or: 18 for Exercises 5, 6, 7, 8, 9, 10, respectively.

Answers for Part C [on page 5-105].

1, 2, 3.



A clever way to do Exercise 2 is to lay a strip of paper vertically on the drawing [as indicated by the dotted line], mark on an edge of the strip the points where it crosses the graphs of the x-axis and g, and then shift the strip vertically until the mark which was on the graph of the x-axis is on the graph of f. Then, the other mark is on the graph of f + g. Repeat this process until you have obtained enough points on the graph of f + g. [This method of graphing is called 'addition of ordinates'. It can be a great time-saver especially in situations more complicated than that illustrated here.]

*

Answers for Part D [on page 5-105].

1. 25	2. 30	3. 28	4. 45	5. 23
6. 48	7. 6	8. 8	9. 48	10. 14

In solving Exercise 3 of Part D, students may obtain '31' as an answer rather than the correct answer: 28. In this case, apparently the closeness of the '3' and the '2' suggested multiplication. There are several ways of correcting this error.

(i) Have students show that $f^2 = \{(x, y): y = (7x - 9)^2\}$ and that $f^2 + 3 = \{(x, y): y = (7x - 9)^2 + 3\}$, so that $[f^2 + 3](2) = (7 \cdot 2 - 9)^2 + 3$.

Correction. In line 5, change the exercise numeral from '2' to '3'.

Answers for Part A.

1. $h + j = \{(7, 1), (5, 2), (8, 18)\}$
2. $hj = \{(7, -12), (5, -48), (8, 65)\}$
3. $j + h = h + j$
4. $jh = hj$
5. $7 + h = \{(3, 9), (7, 11), (0, 12), (5, 1), (8, 20)\}$
6. $3j = \{(5, 24), (8, 15), (7, -9), (2, 9), (6, 0), (9, 54)\}$
7. $h^2 = \{(3, 4), (7, 16), (0, 25), (5, 36), (8, 169)\}$
8. 1
9. −48
10. 31
11. 64
12. 23
13. 5
14. $(7, 1) \in h + j$, but $(7, 1) \notin h \cup j$ $[f + g = f \cup g \Longleftrightarrow f = g \text{ and } \mathcal{R}_f = \{0\}]$

*

Answers for Part B.

1. $c + d = \{(x, y): y = 5x + 2\}$
2. $cd = \{(x, y): y = (2x + 7)(3x - 5)\}$
3. 12
4. 170
5. 31/3
6. 31/3

[Notice, for Exercises 5 and 6, that $c + (-\frac{2}{3})d$ is a constant variable quantity.]

7. $c^2 = \{(x, y): y = (2x + 7)^2\}$
8. $d^2 = \{(x, y): y = (3x - 5)^2\}$

*

EXERCISES

A. Given $h = \{(3, 2), (7, 4), (0, 5), (5, -6), (8, 13)\}$
and $j = \{(5, 8), (8, 5), (7, -3), (2, 3), (6, 0), (9, 18)\}$.

1. What is $h + j$?
2. What is hj?
2. What is $j + h$?
4. What is jh?
5. What is $7 + h$?
6. What is $3j$?
7. What is h^2?
8. $[j + h](7) =$ ___________
9. $[hj](5) =$ ___________
10. $[j + 2h](8) =$ ___________
11. $[jj](5) =$ ___________
12. $[2j + h](8) =$ ___________
13. $j(j(5)) =$ ___________
14. Prove that $h + j \neq h \cup j$.

B. Given $c = \{(x, y): y = 2x + 7\}$ and $d = \{(u, v): v = 3u - 5\}$.

1. What is $c + d$?
2. What is cd?
3. $[c + d](2) =$ ___________
4. $[cd](5) =$ ___________
5. $[c + (-\frac{2}{3})d](9) =$ ___________
6. $[c + (-\frac{2}{3})d](183) =$ ___________
7. What is c^2?
8. What is d^2?

C. Suppose $f = \{(x, y): y = x\}$ and $g = \{(x, y): y = x + 3\}$.

1. Graph f and g on the same chart.
2. Graph $f + g$ on this chart. [Be lazy about it!]
3. Graph fg on this chart.

D. Suppose $f = \{(x, y): y = 7x - 9\}$.

1. $[f^2](2) =$ ___________
2. $[f^2 + f](2) =$ ___________
3. $[f^2 + 3](2) =$ ___________
4. $[f^2 + 4f](2) =$ ___________
5. $[4f + 3](2) =$ ___________
6. $[f^2 + 4f + 3](2) =$ ___________
7. $[f + 1](2) =$ ___________
8. $[f + 3](2) =$ ___________
9. $[(f + 1)(f + 3)](2) =$ ___________
10. $[(f + 1) + (f + 3)](2) =$ ___________

*

Here are definitions for subtracting and dividing variable quantities:

For all variable quantities g and h,

$$g - h = \{(x, y),\ x \in \mathcal{D}_g \cap \mathcal{D}_h :\ y = g(x) - h(x)\},$$

and $$\frac{g}{h} = \{(x, y),\ x \in \mathcal{D}_g \cap \mathcal{D}_h \text{ and } h(x) \neq 0 :\ y = \frac{g(x)}{h(x)}\}.$$

*

E. Suppose $f = \{(1, 4), (2, 9), (3, 6), (4, 7)\}$

and $g = \{(1, 2), (2, 3), (3, 2), (4, 0)\}$.

1. What is $f - g$?
2. What is $g - f$?
3. What is $\frac{g}{f}$? $\frac{f}{g}$?
4. What are the domain and range of f? Of g? Of $\frac{g}{f}$? Of $\frac{f}{g}$?

F. Suppose $f = \{(x, y),\ x > 0:\ y = 6x\}$ and $g = \{(x, y),\ x > 0:\ y = 2x\}$.

1. $[f - g](5) =$ __________
2. $[f + g](5) =$ __________
3. $[f - 3g](5) =$ __________
4. $[f - 3g](78) =$ __________
5. What is $f - 3g$?
6. What is $\frac{1}{3}f - g$?
7. $[\frac{f}{g}](5) =$ __________
8. $[\frac{f}{g}](27) =$ __________
9. What is $\frac{f}{g}$?
10. What is $\frac{g}{f}$?

G. Consider the constants 2 and 3.

1. If x_1 and x_2 are two arguments in a domain of 2, what are the values corresponding to these arguments? That is, what are $2(x_1)$ and $2(x_2)$?
2. Repeat Exercise 1 for the constant 3.
3. Suppose 2 and 3 are constants with a common domain. Does it follow that $2 \cdot 3$ is a constant? How about $2 + 3$? $2 - 3$? $2 \div 3$?
4. If x_1 and x_2 are two arguments in a common domain of 2 and 3, what are the values of $2 + 3$ corresponding to these arguments? Does $[2 \cdot 3](x_1) = [2 \cdot 3](x_2)$?

Answers for Part E.

1. $f - g = \{(1, 2), (2, 6), (3, 4), (4, 7)\}$

2. $g - f = \{(1, -2), (2, -6), (3, -4), (4, -7)\} = -(f - g)$

3. $g/f = \{(1, 1/2), (2, 1/3), (3, 1/3), (4, 0)\}$;
   $f/g = \{(1, 2), (2, 3), (3, 3)\}$

4. $\mathcal{D}_f = \{1, 2, 3, 4\} = \mathcal{D}_g$, $\mathcal{R}_f = \{4, 6, 7, 9\}$, $\mathcal{R}_g = \{0, 2, 3\}$,
   $\mathcal{D}_{g/f} = \{1, 2, 3, 4\}$, $\mathcal{D}_{f/g} = \{1, 2, 3\}$, $\mathcal{R}_{g/f} = \{1/2, 1/3, 0\}$,
   $\mathcal{R}_{f/g} = \{2, 3\}$

*

Answers for Part F.

1. 20
2. 40
3. 0
4. 0

5. $f - 3g = \{(x, y), x > 0: y = 0\}$. It is the constant whose domain is $\{x: x > 0\}$ and whose range is $\{0\}$. It is one of the constants which we denote by '0'.

6. $\frac{1}{3}f - g = f - 3g$
7. 3
8. 3

9. $f/g = \{(x, y), x > 0: y = 3\}$. It is the constant whose domain is $\{x: x > 0\}$ and whose range is $\{3\}$.

10. $g/f = \{(x, y), x > 0: y = 1/3\}$

*

Answers for Part G.

1. $2(x_1) = 2$; $2(x_2) = 2$
2. $3(x_1) = 3$; $3(x_2) = 3$
3. yes, yes, yes, yes
4. $[2 + 3](x_1) = 5$; $[2 + 3](x_2) = 5$; yes

*

Quiz.

A. Suppose that M and N are variable quantities where

$M = \{(\text{Hans}, 1), (\text{Joe}, 5), (\text{Lila}, 4), (\text{Eva}, 9)\}$

and $N = \{(\text{Hans}, 4), (\text{Joe}, 16), (\text{Lila}, 13)\}$

1. $M(\text{Joe}) + N(\text{Joe}) =$ ______
2. $[M + N](\text{Lila}) =$ ______
3. $[-M](\text{Hans}) =$ ______
4. $[M^2](\text{Eva}) =$ ______
5. $\sqrt{N}(\text{Joe}) =$ ______
6. $MN = \{$______________$\}$
7. $[M - N](\text{Joe}) =$ ______
8. $[M + N](\text{Eva}) =$ ______

B. $f = \{(x, y): y = 3x - 5\}$, $g = \{(x, y): y = 5x + 3\}$

1. $[f + g](3) =$ ______
2. $[f + g](-3) =$ ______
3. $[f - g](5) =$ ______
4. $[f - g](\_\_) = 0$
5. $[fg](4) =$ ______
6. $[f \circ g](4) =$ ______
7. $[g \circ f](4) =$ ______
8. $[gf](4) =$ ______
9. $f(-4) = g(\_\_)$
10. $[fg](x) =$ ______
11. $[f^2](x) =$ ______
12. $[g^2](x) =$ ______
13. $[f^2 + g^2](x) =$ ______
14. $[f^2 + g](x) =$ ______
15. $[f - g^2](x) =$ ______
16. $[f/g](1) =$ ______
17. $[g/f](1) =$ ______
18. $[f/g](\_\_) = 7/23$
19. $[f/g](-3/5) =$ ______
20. $[f/g](\_\_) = 3/5$

*

Answers for Quiz.

A. 1. 21   2. 17   3. -1 4. 81   5. 4

6. (Hans, 4), (Joe, 80), (Lila, 52)   7. -11 8. $[\text{Eva} \notin \mathcal{D}_{M+N}]$

B. 1. 22   2. -26 3. -18 4. -4 5. 161

6. 64   7. 38   8. 161   9. -4 10. $15x^2 - 16x - 15$

11. $9x^2 - 30x + 25$ 12. $25x^2 + 30x + 9$ 13. $34x^2 + 34$

14. $9x^2 - 25x + 28$ 15. $-25x^2 - 27x - 14$ 16. $-\frac{1}{4}$

17. -4 18. 4   19. $[-\frac{3}{5} \notin \mathcal{D}_{f/g}]$ 20. $[\frac{3}{5} \notin \mathcal{R}_{f/g}]$

In reading the formula '$A = \frac{1}{2}h(b_1 + b_2)$' students should say 'one-half times h times the sum of b_1 and b_2'. [Since the arguments of h are not numerical, the parentheses do not indicate that h operates on $b_1 + b_2$, as is usual in formulas; the parentheses are merely grouping symbols.]

The common domain of A, $\frac{1}{2}$, h, b_1, and b_2 is the set of all trapezoids.

In the context '$V = \frac{1}{3}\pi r^2 h$', the common domain of V, $\frac{1}{3}$, r, h, π, r^2, and $\frac{1}{3}r$ is the set of all circular conical solids.

In the context '$c = \pi d$', the common domain of c, π, and d, is the set of all circles, and, c(x) is the number c which has this domain.

To assert '$c = \pi d$' is to say that c and πd are the same variable quantity —that is, that $D_c = D_{\pi d}$ and that, for each member x of this domain, $c(x) = (\pi d)(x)$.

If $d(x_1) = d(x_2)$, the circles x_1 and x_2 have the same diameters, are the same size, and are congruent. Two such circles also have the same circumference—that is, $c(x_1) = c(x_2)$.

In reading the formula '$A = \frac{1}{2}h(b_1 + b_2)$' students should say 'one-half times aitch times bee one plus bee two'. [Since the arguments of h are not numerical, the parentheses do not indicate that h operates on $b_1 + b_2$. As is usual in formulas, the parentheses are merely grouping symbols.]

The common domain of A, $\frac{1}{2}$, h, b_1, and b_2 is the set of all trapezoids.

In the context '$V = \frac{1}{3}\pi r^2 h$', the common domain of V, $\frac{1}{3}$, π, r, h, r^2, and $\frac{1}{3}\pi$ is the set of all circular conical solids.

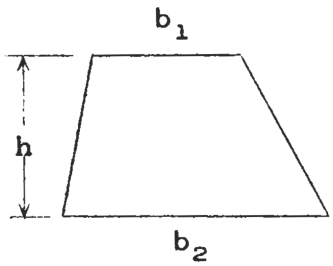
In the context '$c = \pi d$', the common domain of c, π, and d, is the set of all circles. And, c/d is the constant π which has this domain.

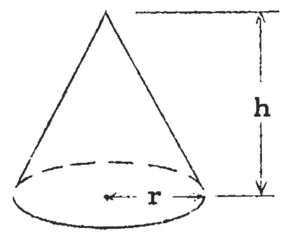
To assert '$c = \pi d$' is to say that c and πd are the same variable quantity --that is, that $\mathcal{D}_c = \mathcal{D}_{\pi d}$ and that, for each member x of this domain, c(x) is $[\pi d](x)$.

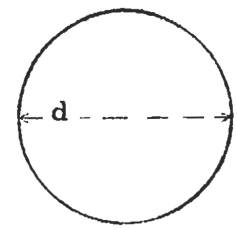
If $d(x_1) = d(x_2)$, the circles x_1 and x_2 have the same diameter, or: are the same size, or: are congruent. Two such circles also have the same circumference--that is, $c(x_1) = c(x_2)$.

FORMULAS

A formula is a statement about variable quantities. Read the formulas in the diagrams.







$A = \frac{1}{2}h(b_1 + b_2)$ $V = \frac{1}{3}\pi r^2 h$ $c = \pi d$

The first formula tells you in very concise form that the area-measure of a trapezoid is one half the product of its height-measure and the sum of its base-measures. The variable quantity A <u>is</u> the variable quantity $\frac{1}{2}h(b_1 + b_2)$. What are the domains of A, $\frac{1}{2}$, h, b_1, and b_2?

The second formula tells you that the variable quantity which is the volume-measure of a circular cone is the variable quantity $\frac{1}{3}\pi r^2 h$. What are the domains of the variable quantities V, r, and h? What is the domain of the variable quantity r^2? What are the domains of the constants $\frac{1}{3}$, π, and $\frac{1}{3}\pi$?

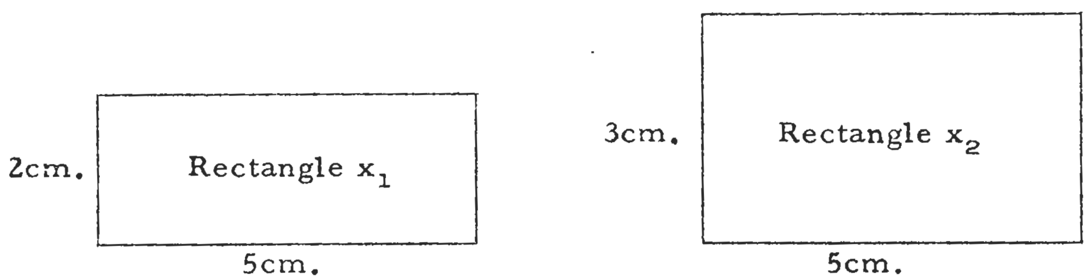
The third formula tells you that c is πd, that is, the variable quantity which is the circumference of a circle is the same variable quantity as the product of the constant π by the variable quantity which is the diameter of the circle. What is the quotient of the variable quantity c by the variable quantity d?

The formula '$c = \pi d$' tells you that, for each circle x, $c(x) = [\pi d](x)$. By the rule for multiplying variable quantities, we know that, for each circle x, $[\pi d]x = \pi(x) \cdot d(x)$. Since π is a constant, we know that, for each x, $\pi(x) = \pi$. So, for each circle x, $c(x) = \pi \cdot d(x)$. Consider two arguments in the domain of d, say, circle x_1 and circle x_2, such that $d(x_1) = d(x_2)$. [What do you know about the circles x_1 and x_2?] Does it follow that $c(x_1) = c(x_2)$? In that case, we have met the conditions [page 5-91]. which tell us that the variable quantity c is a function of the variable quantity d. So, there is a function f such that $c = f \circ d$. One such function f is $\{(u, v) \in N \times N: v = \pi u\}$.

We can use the formula '$c = \pi d$' both to discover that c is a function of d and to find a function f such that $c = f \circ d$. Since c is a function of d, we say that c depends only on d. We also say that c is the dependent variable quantity when d is the independent variable quantity.

From the formula for the area-measure of a square we learn that A is a function of s, so that A is the dependent variable quantity when s is the independent variable quantity. From the formula for the side-measure of a square ['$s = \sqrt{A}$'] we see that s depends only on A. So, s is the dependent variable quantity when A is the independent variable quantity.

Now, consider the formula which tells you that P is $2(\ell + w)$. We know that, for each rectangle x, $P(x) = 2(\ell(x) + w(x))$. Take two arguments in the common domain of P, ℓ, and w, say, rectangles x_1 and x_2.



Here $\ell(x_1) = \ell(x_2)$. Does it follow that $P(x_1) = P(x_2)$? Hence, does it follow that P is a function of ℓ? Do you think that P is a function of w? [Justify your answer.]

Suppose you have two rectangles, x_3 and x_4, such that $\ell(x_3) = \ell(x_4)$ and $w(x_3) = w(x_4)$. [What can you say about these rectangles?] Does it follow that $P(x_3) = P(x_4)$? In this case [in analogy with the case of one variable quantity being a function of another], we say that P is a function of (ℓ, w). This means that there is a function f such that, for each rectangle, x, $P(x) = f((\ell(x), w(x)))$. The range of such a function includes the range of P, and the domain of such a function contains the ordered pairs whose first and second components are the values of ℓ and w, respectively, for arguments of P. One such function f is

$$\{((u, v), w) \in (N \times N) \times N: w = 2(u + v)\}.$$

What is $f((5, 2))$? $f((5, 3))$? $f((9, 6)) = ?$ $f((3, ?)) = 10$ $f((?, 7)) = 34$

Note that the choice of which of a set of variable quantities is to be taken as the dependent one [the others, then, being called 'independent'] is largely arbitrary. For example, reference to the formula '$c = \pi d$' usually means that the speaker has chosen to consider c as the dependent variable quantity, but reference to the equivalent formula '$d = c/\pi$' indicates that he has chosen to think of d as dependent on c.

*

In the case of the rectangles x_1 and x_2, it of course does not follow from the fact that $\ell(x_1) = \ell(x_2)$ that $P(x_1) = P(x_2)$. In fact, for the given rectangles, $P(x_1) \neq P(x_2)$. And, since $\ell(x_1) = \ell(x_2)$ but $P(x_1) \neq P(x_2)$, it follows that P is not a function of ℓ. A similar example shows that P is not a function of w.

Two rectangles, x_3 and x_4, such that $\ell(x_3) = \ell(x_4)$ and $w(x_3) = w(x_4)$ are congruent. So, in that case, $P(x_3) = P(x_4)$.

$f((5, 2)) = 14$; $f((5, 3)) = 16$; $f((9, 6)) = 30$; $f((3, 2)) = 10$; $f((10, 7)) = 34$

*

The phrases [on page 5-109] 'functions of two variables' and 'functions of one variable' do not, of course, make literal sense. A function is not "of" anything; like any other entity, a function just "is"! But these phrases are the ones conventionally used to distinguish between functions whose arguments are ordered pairs of numbers and those whose arguments are numbers.

*

Answers for questions on page 5-109.

V is not a function of h because there exist cones x_1 and x_2 such that $h(x_1) = h(x_2)$ but $V(x_1) \neq V(x_2)$.

V is not a function of π because π is a constant variable quantity and V is not. [Similarly, V is not a function of 1/3.]

V is a function of (h, r) [as well as of (r, h)]. One function g such that $V = g((h, r))$ is

$$\{((u, v), w) \in (N \times N) \times N: w = \frac{1}{3}\pi uv^2\}.$$

It is different from the function f which V is of (r, h).

Note that the choice of which of a set of variable quantities is to be taken as the dependent one [the others, then, being called 'independent'] is largely arbitrary. For example, reference to the formula 'c = ad' usually means that the speaker has chosen to consider c as the dependent variable quantity, but reference to the equivalent formula 'd = c/a' indicates that he has chosen to think of d as dependent on c.

*

In the case of the rectangles x_1 and x_2, it of course does not follow from the fact that $A(x_1) = A(x_2)$ that $P(x_1) = P(x_2)$. In fact, for the given rectangles, $P(x_1) \neq P(x_2)$. And, since $A(x_1) = A(x_2)$ but $P(x_1) \neq P(x_2)$, it follows that P is not a function of A. A similar example shows that P is not a function of w.

Take rectangles, x_3 and x_4, such that $l(x_3) = l(x_4)$ and $w(x_3) = w(x_4)$ are congruent. So, in that case, $P(x_3) = P(x_4)$.

$H((2, 5)) = 14$; $H((5, 2)) = 14$; $H((3, 4)) = 14$; $H((1, 5)) = 10$; $H((10, 7)) = 34$

*

The phrases [on page 5-104] 'functions of two variables' and 'functions of one variable' do not, of course, make literal sense. A function is not 'of' anything, like any other entity, a function just 'is'. But these phrases are the ones conventionally used to distinguish between functions whose arguments are ordered pairs of numbers and those whose arguments are numbers.

*

Answers for questions on page 5-108.

V is not a function of h because there exist cones x_1 and x_2 such that $h(x_1) = h(x_2)$ and $V(x_1) \neq V(x_2)$.

V is not a function of π because π is a constant variable quantity and V is not. [Similarly, V is not a function of 1/3.]

V is a function of (h, r) [as well as of (r, h)]. One function g such that V = g((h, r)) is

$$\{((u, v), w) \in (N \times N) \times N : w = \frac{1}{3}\pi u v^2\}.$$

g is different from the function f which V is of (r, h).

In the case of a variable quantity being a function of another, we say that the first is the dependent variable quantity and the second is the independent variable quantity. When a variable quantity is a function of an ordered pair of variable quantities, we say that the first is the dependent variable quantity and that the components of the pair are the independent variable quantities.

So, the formula '$P = 2(\ell + w)$' tells us that P depends only on ℓ and w, that is, that there is a function which can be applied to pairs of corresponding values of ℓ and w to produce the corresponding value of P. Functions like these whose domains consist of ordered pairs are sometimes called functions of two variables to contrast them with the functions we have been dealing with up to now which are called functions of one variable.

Consider the formula '$V = \frac{1}{3}\pi r^2 h$' for the volume of a circular cone. Is V a function of r? The answer is 'no' because there are cones, x_1 and x_2, such that $r(x_1) = r(x_2)$ and $V(x_1) \neq V(x_2)$. Can you describe two such cones? Is V a function of h? [Tell why.] Is V a function of π? Is V a function of $\frac{1}{3}$? Is V a function of (r, h)? Let's see. Suppose x_1 and x_2 are cones such that $(r(x_1), h(x_1)) = (r(x_2), h(x_2))$. Now,

$$V(x_1) = [\tfrac{1}{3}\pi](x_1) \cdot [r^2](x_1) \cdot h(x_1),$$

and

$$V(x_2) = [\tfrac{1}{3}\pi](x_2) \cdot [r^2](x_2) \cdot h(x_2).$$

Since $\frac{1}{3}\pi$ is a constant, $[\frac{1}{3}\pi](x_1) = [\frac{1}{3}\pi](x_2)$. Since r^2 is a function of r, and $r(x_1) = r(x_2)$, it follows that $[r^2](x_1) = [r^2](x_2)$. So, since $h(x_1) = h(x_2)$, it follows that $V(x_1) = V(x_2)$. Hence, V is a function of (r, h), that is, there is a function f such that, for each cone x, $V(x) = f(r(x), h(x))$. Such a function f is a function of two variables. One possible function is

$$\{((u, v), w) \in (N \times N) \times N: \ w = \tfrac{1}{3}\pi u^2 v\}.$$

Is V a function of (h, r)? What function? If there is such a function is it different from the function f mentioned above?

EXERCISES

A. Consider the variable quantity P where

$$P = \{(Al, 9), (Bob, 6), (Cora, 8)\}.$$

1. If the variable quantity K is 2P + 1, what is the domain of K?

2. (a) K(Al) = ______ (b) K(Bob) = ______ (c) K(Cora) = ______

3. (a) $[K^2](Al)$ = _____ (b) $[K^2 + 3K + 2](Bob)$ = ______

B. Consider the variable quantity M where

$$M = \{(1, 8), (2, 7), (3, 4), (4, 3)\}.$$

1. If the variable quantity N is the variable quantity 3M + 4,
   (a) N(2) = __________ (b) N(4) = __________
   (c) $[N^2](3)$ = __________ (d) $[N^2 + 2N](1)$ = __________

2. If K = 7M - 1, list the ordered pairs in K.

3. If J = 5M - 2, is there a function f such that $J = f \circ M$? Find such an f.

4. (a) Suppose $A = f \circ M$ where $f = \{(x, y): y = 3x^2 + 2x + 1\}$. List the ordered pairs which belong to A.
   (b) Is there a formula for A in terms of M? If there is, write one.

5. (a) Suppose T = {(1, 64), (2, 49), (3, 16), (4, 9)}. Is T a function of M? [That is, is there an f such that $T = f \circ M$?] What function? [That is, describe one such f.]
   (b) Can you write a formula for T in terms of M? Try to.

6. (a) Suppose S = {(1, 15), (2, 13), (3, 7), (4, 5)}. Is S a function of M? What function?
   (b) Can you write a formula for S in terms of M? Try to.

7. (a) Suppose Q = {(1, 98), (2, 347), (3, 629), (4, −708)}. Is Q a function of M? What function?
   ☆ (b) Can you write a formula for Q in terms of M? Try to, but don't try too hard.

8. (a) If R = 3M + 5, what function is R of M?
   (b) Is M a function of R? What function?
   (c) Can you write a formula for M in terms of R? Try to.

Answers for Part A.

1. $\mathcal{S}_K$ = {Al, Bob, Cora}

2. (a) 19 (b) 13 (c) 17 3. (a) 361 (b) 210

*

Answers for Part B [on pages 5-110 and 5-111].

1. (a) 25 (b) 13 (c) 256 (d) 840

2. (1, 55), (2, 48), (3, 27), (4, 20)

3. Yes; $\{(x, y): y = 5x - 2\}$, or: {(3, 13), (4, 18), (7, 33), (8, 38)}

4. (a) (1, 209), (2, 162), (3, 57), (4, 34)

   (b) $A = 3M^2 + 2M + 1$ ['in terms of M' means 'referring to no variable quantities other than constants and M'.]

5. (a) Yes; {(8, 64), (7, 49), (4, 16), (3, 9)} [or any more inclusive function such as $\{(x, y): y = x^2\}$]

   (b) Yes; $T = M^2$

6. (a) Yes; {(8, 15), (7, 13), (4, 7), (3, 5)} or: $\{(x, y): y = 2x - 1\}$

   (b) Yes; $S = 2M - 1$

7. (a) Yes; {(8, 98), (7, 347), (4, 629), (3, −708)} [or any more inclusive function]

   ☆(b) Yes; $Q = 63.8M^3 - 1250.95M^2 + 7733.05M - 14371.2$

There are two procedures for obtaining a function f, whose domain is the set of all real numbers, such that $Q = f \circ M$. One way begins by assuming that there are numbers a, b, c, and d such that $f(x) = a + bx + cx^2 + dx^3$, drawing the conclusion that, since $f(8) = 98$, $f(7) = 347$, $f(4) = 629$, and $f(3) = -708$,

$$a + 8b + 64c + 512d = 98,$$

$$a + 7b + 49c + 343d = 347,$$

$$a + 4b + 16c + 64d = 629,$$

and

$$a + 3b + 9c + 27d = -708.$$

It is clear that a solution (a, b, c, d) of the above equations will yield a function f which does the job. The system can be solved by standard elimination procedures, and it is seen that

$$f(x) = -14371.2 + 7733.05x - 1250.95x^2 + 63.8x^3.$$

A second procedure is to use the Lagrange interpolation formula which, for a function whose values, for four arguments x_1, x_2, x_3, and x_4, are specified to be y_1, y_2, y_3, and y_4, respectively, is:

$$f(x) = \frac{(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} y_1 + \frac{(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} y_2$$

$$+ \frac{(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} y_3 + \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} y_4$$

In the present case, this yields:

$$f(x) = \frac{(x - 7)(x - 4)(x - 3)}{(8 - 7)(8 - 4)(8 - 3)} \cdot 98 + \frac{(x - 8)(x - 4)(x - 3)}{(7 - 8)(7 - 4)(7 - 3)} \cdot 347$$

$$+ \frac{(x - 8)(x - 7)(x - 3)}{(4 - 8)(4 - 7)(4 - 3)} \cdot 629 + \frac{(x - 8)(x - 7)(x - 4)}{(3 - 8)(3 - 7)(3 - 4)} \cdot -708$$

To see how this works, note that, according to the above formula,

$$f(8) = 1 \cdot 98 + 0 \cdot 347 + 0 \cdot 629 + 0 \cdot -708 = 98,$$

$$f(7) = 0 \cdot 98 + 1 \cdot 347 + 0 \cdot 629 + 0 \cdot -708 = 347,$$

$$f(4) = 0 \cdot 98 + 0 \cdot 347 + 1 \cdot 629 + 0 \cdot -708 = 347,$$

and $f(3) = 0 \cdot 98 + 0 \cdot 347 + 0 \cdot 629 + 1 \cdot -708 = -708.$

[There are, of course, many other functions which can be used in place of f.]

*

8. (a) $\{(x, y): y = 3x + 5\}$, or: $\{(3, 14), (4, 17), (7, 26), (8, 29)\}$

   (b) Yes; $\{(x, y): y = (x - 5)/3\}$

   (c) $M = (R - 5)/3$

Correction. On page 5-112, the last part of line 17 should be 'quantity as fh + gh'.
↑

9. (a) Yes; $\{(x, y): y = x - 5\}$; $E = M - 5$
   (b) Yes; $M = E + 5$
   (c) (1, 3/8), (2, 2/7), (3, −1/4), (4, −2/3)
   (d) Yes; $\{(x, y): xy = x - 5\}$, or: $\{(x, y), x \neq 0 : y = 1 - 5/x\}$
   (e) $E/M = 1 - 5/M$

10. (a) $B = M$ (b) $C = 3$, where 3 is the variable quantity $\{(1, 3), (2, 3), (3, 3), (4, 3)\}$.

☆11. (a) That constant 2 whose domain is {1, 2, 3, 4} is a function of M. [A constant is a function of each variable quantity which has the same domain.]
   (b) No; [$2(1) = 2(2)$, but $M(1) \neq M(2)$]

*

Answers for Part C.

1. (a) 43 (b) 51 (c) −21 (d) 29 (e) 1849 (f) 836

2. (a) (1, −1), (2, −1), (3, 23), (4, −23)
   (b) (1, −96), (2, −133), (3, −85), (4, −144)
   (c) (1, −96), (2, −133), (3, −85), (4, −144)

*

Answers for Part D.

1. $A(x_2) = A(x_1) + 15$
2. $A(x_2) = 7 \cdot A(x_1)$

9. (a) Suppose that E is a variable quantity whose domain is that of M, and that each value of E is 5 less than the corresponding value of M. Is E a function of M? What function? Write (if you can) a formula for E in terms of M.

(b) Is M a function of E? Write (if you can) a formula for M in terms of E.

(c) List the ordered pairs in the variable quantity $\frac{E}{M}$.

(d) Is $\frac{E}{M}$ a function of M? What function?

(e) Write (if you can) a formula for $\frac{E}{M}$ in terms of M.

10. (a) Suppose $B = f \circ M$ where $f = \{(x, y): y = x\}$. Write a formula for B in terms of M.

(b) Suppose $C = g \circ M$ where $g = \{(x, y): y = 3\}$. Write a formula for C in terms of M.

☆ 11. (a) Is the constant 2 a function of M?

(b) Is M a function of the constant 2?

C. Suppose A and B are variable quantities where

$$A = \{(1, 5), (2, 6), (3, 6), (4, -5)\}.$$

and

$$B = \{(1, 11), (2, 13), (3, -11), (4, 13)\}.$$

1. If $C = 2A + 3B$ then

(a) $C(1) =$ ______________ (b) $C(2) =$ ______________

(c) $C(3) =$ ______________ (d) $C(4) =$ ______________

(e) $[C^2](1) =$ ______________ (f) $[C^2 - 5](4) =$ ______________

2. (a) If $D = 2A - B$, what are the ordered pairs in D?

(b) If $D = (A - B)(A + B)$, what are the ordered pairs in D?

(c) If $D = A^2 - B^2$, what are the ordered pairs in D?

D. Suppose A and B are variable quantities such that $A = 3B$.

1. If x_1 and x_2 belong to the common domain of A and B, and if $B(x_2)$ is $B(x_1) + 5$, what can you say about $A(x_2)$ and $A(x_1)$?

2. If $B(x_2)$ is $7 \cdot B(x_1)$, what can you say about $A(x_2)$ and $A(x_1)$?

[Supplementary exercises are in Part R, page 5-266.]

TRANSFORMING FORMULAS

Earlier we said that operations on real-valued variable quantities satisfy principles just like our basic principles for real numbers, and that this meant that you could manipulate formulas just like you manipulate numerical equations. Let's check up on these assertions. To do so, choose some set D and consider all real-valued variable quantities whose domain is D. Also, use '0' and '1' as names for the variable quantities whose domain is D and whose value for each argument is the real number 0, or the real number 1, respectively.

One of our basic principles for real numbers is the dpma:

$$\forall_x \forall_y \forall_z\ (x + y)z = xz + yz$$

So, one thing we want to check is the distributive principle for multiplication of variable quantities with respect to addition of variable quantities. This is:

$$\forall_f \forall_g \forall_h\ (f + g)h = fh + gh$$

We want to show that if f, g, and h are real-valued variable quantities with domain D, then (f + g)h is the same variable quantitiy as fh + gh. This is easy. By the definitions of addition and multiplication of variable quantities, the domain of f + g is $\mathcal{D}_f \cap \mathcal{D}_g$, and that of (f + g)h is $\mathcal{D}_{f+g} \cap \mathcal{D}_h$. Since $\mathcal{D}_f = \mathcal{D}_g = \mathcal{D}_h = D$, the domain of (f + g)h is also D. Similarly, the domain of fh + gh is D. But, if $x \in D$ then

$$\begin{aligned} &[(f + g)h](x) \\ &= [f + g](x) \cdot h(x) && \text{definition of multiplication} \\ &= \big(f(x) + g(x)\big) \cdot h(x) && \text{definition of addition} \\ &= f(x) \cdot h(x) + g(x) \cdot h(x), && \text{dpma for real numbers} \end{aligned}$$

and

$$\begin{aligned} &[fh + gh](x) \\ &= [fh](x) + [gh](x) && \text{definition of addition} \\ &= f(x) \cdot h(x) + g(x) \cdot h(x), && \text{definition of multiplication} \end{aligned}$$

also. So, since (f + g)h and fh + gh have the same domain and have the same value for each argument, (f + g)h = fh + gh.

In the same way, you can show that the commutative and associative principles for addition and multiplication of real numbers imply the corresponding principles for addition and multiplication of real-valued variable quantities with domain D. So, we have:

$$\forall_f \forall_g\ f + g = g + f \qquad \forall_f \forall_g\ fg = gf$$

$$\forall_f \forall_g \forall_h\ f + g + h = f + (g + h) \qquad \forall_f \forall_g \forall_h\ fgh = f(gh)$$

For the same reason [that is, because operations on variable quantities amount to operations on their values], we also have principles for the variable quantities 0 and 1:

$$\forall_f\ f + 0 = f$$

$$\forall_f\ f \cdot 1 = f$$

and a principle of opposites and a principle for subtraction:

$$\forall_f\ f + -f = 0 \qquad \forall_f \forall_g\ f - g = f + -g$$

Finally, if, for each $x \in D$, $g(x) \neq 0$ then the domain of $f \div g$ is also D, and, for each $x \in D$,

$$\begin{aligned} &[(f \div g)g](x) \\ &= [f \div g](x) \cdot g(x) && \text{[Why?]} \\ &= \big(f(x) \div g(x)\big) \cdot g(x) && \text{[Why?]} \\ &= f(x). && \text{[Why?]} \end{aligned}$$

So, the principle of quotients for real-valued variable quantities with domain D is:

For each f, for each g such that 0 is not a value of g,
$(f \div g)g = f$.

This corresponds with the last of our basic principles for real numbers.

Our checking of the basic principles now pays us a big dividend. Each of the theorems about real numbers which you derived in Unit 2 from the basic principles for real numbers [there were 78 of them!] can be translated into a statement about real-valued variable quantities with domain D. <u>And</u>, you know that each of these statements is true, because it can be derived from the principles for variable quantities in just the same way that the corresponding theorem about real numbers is derived from the ten basic principles for real numbers.

For example, the cancellation principle for addition [for real numbers] translates into:

$$\forall_f \forall_g \forall_h \text{ if } f + h = g + h \text{ then } f = g$$

And, the test-pattern for the cancellation principle translates into:

Suppose that	$f + h = g + h$	
Then	$f + h + -h = g + h + -h,$	[uniqueness principle]
	$f + (h + -h) = g + (h + -h),$	[apa for variable quantities]
	$f + 0 = g + 0$	[po for variable quantities]
and	$f = g.$	[pa0 for variable quantities]
Hence,	if $f + h = g + h$ then $f = g$.	

As another example of translation, the division theorem [for real numbers] translates into:

(*) For each f, for each g such that 0 is not a value of g, for each h,

if $hg = f$ then $h = f \div g$.

On translating a proof of the division theorem, one obtains a proof of the displayed theorem concerning variable quantities.

Although we have been discussing real-valued variable quantities, all those theorems of Unit 2 which do not involve subtraction or opposition can be translated into theorems about variable quantities, with a given domain D, whose values are numbers of arithmetic. This is because all the basic principles for real numbers except the po and the ps hold for numbers of arithmetic and, so, the translations of them given above hold for variable quantities whose values are numbers of arithmetic. Thus, for example, (*) shows that the formula '$\pi = c \div d$' can be derived from the formula '$\pi d = c$'. [Notice that the first of these formulas would be nonsense if 'π' did not name a variable quantity.] Using the principle of quotients for variable quantities [together with the fact that no circle has diameter zero], one can derive the second formula from the first. So, the two formulas are equivalent.

On the basis of the results arrived at, one can use analogues of the equation transformation principles to solve a formula for one of

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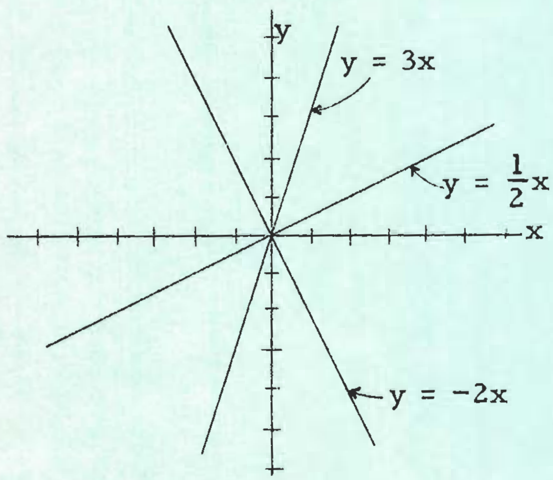
Answers for Exploration Exercises.

Answers for Part A [on page 5-115].

1.



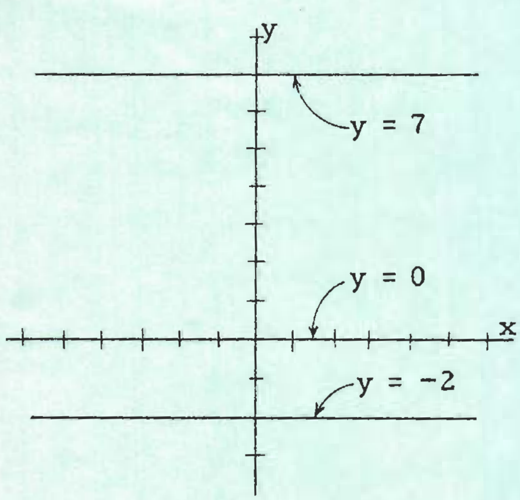
2. {(0, 0)}

3. {(0, 0)}

*

Answers for Part B [on page 5-115].

1.



2. ∅

3. ∅

Answers for Quiz

A. 1. $(e_1, 13), (e_2, 10), (e_3, 7), (e_4, 16)$

2. $(e_1, -1), (e_2, -1), (e_3, -4), (e_4, -1)$

3. $(e_1, 77), (e_2, 98), (e_3, 49), (e_4, 196)$

4. Yes, $f = \{(x, y): y = 2x + 1\}$ [or any function which contains as a subset $\{(-1, 3), (3, 7), (7, 15), 11)\}$]

B. 1. 45 2. 51 3. 77 4. 96 5. 4225 6. 4356

7. Yes, $f = \{(x, y): y = 3x + 4\}$ [or any function which contains as a subset $\{(5, 69), (1, 51), (3, 37), (11, 79)\}$]

8. Yes, $g = \{(x, y): y = 14x + 9\}$ [or any function which contains as a subset $\{(4, 65), (3, 51), (2, 37), (5, 79)\}$]

9. $M = 14A + 9$ [since $M = 7B + 2$ and $B = 2A + 1$]

C. 1. 100π 2. 64π 3. 2 4. 3/4 5. 1/9

*

Answers for Quiz.

A. 1. $(e_1, 13), (e_2, 10), (e_3, 7), (e_4, 16)$

2. $(e_1, -1), (e_2, -1), (e_3, -1), (e_4, -1)$

3. $(e_1, 97), (e_2, 58), (e_3, 29), (e_4, 146)$

4. Yes; $f = \{(x, y): y = 2x + 1\}$ [or any function which contains as a subset $\{(4, 9), (3, 7), (2, 5), (5, 11)\}$]

B. 1. 65   2. 51   3. 79   4. 66   5. 4225   6. 4356

7. Yes; $f = \{(x, y): y = 7x + 2\}$ [or any function which contains as a subset $\{(9, 65), (7, 51), (5, 37), (11, 79)\}$]

8. Yes, $g = \{(x, y): y = 14x + 9\}$ [or any function which contains as a subset $\{(4, 65), (3, 51), (2, 37), (5, 79)\}$]

9. $M = 14A + 9$ [since $M = 7B + 2$ and $B = 2A + 1$]

C. 1. 100π 2. 84π 3. 2   4. 1/4   5. 1/9

*

Quiz.

A. Suppose that A and B are variable quantities where

$A = \{(e_1, 4), (e_2, 3), (e_3, 2), (e_4, 5)\}$

and $B = \{(e_1, 9), (e_2, 7), (e_3, 5), (e_4, 11)\}$.

1. If $C = A + B$ then $C = \{$______________$\}$.
2. If $D = 2A - B$ then $D = \{$______________$\}$.
3. If $E = A^2 + B^2$ then $E = \{$______________$\}$.
4. Is B a function of A? If not, tell why. If so, describe a function f such that $B = f \circ A$.

B. Suppose that A and B are the variable quantities described in Part A, and that M is the variable quantity 7B + 2.

1. $M(e_1) =$ _____ 2. $M(e_2) =$ _____ 3. $M(e_4) =$ _____
4. $[M + 1](e_1) =$ _____ 5. $[M^2](e_1) =$ _____
6. $[M^2 + 2M + 1](e_1) =$ _____
7. Is M a function of B? If not, tell why. If so, describe a function f such that $M = f \circ B$.
8. Is M a function of A? If not, tell why. If so, describe a function g such that $M = g \circ A$.
9. Write a formula for M in terms of A.

C. The formula '$V = \frac{1}{3}\pi h r^2$' tells you that the volume of a circular cone is $\frac{1}{3}\pi$ times the height of the cone times the square of the radius of the base.

1. If $h(c_1) = 12$ and $r(c_1) = 5$ then $V(c_1) =$ _____.
2. If $h(c_2) = 9/7$ and $r(c_2) = 14$ then $V(c_2) =$ _____.
3. If $r(c_3) = r(c_4)$ and $h(c_3) = 2[h(c_4)]$ then $V(c_3) \div V(c_4) =$ _____.
4. If $h(c_5) = h(c_6)$ and $r(c_5) = 1/2[r(c_6)]$ then $V(c_5) \div V(c_6) =$ _____.
5. If $V(c_7) = V(c_8)$ and $r(c_7) = 3[r(c_8)]$ then $h(c_7) \div h(c_8) =$ _____.

Answers for Exercises.

1. $b = P - a - c$
2. $a = \frac{P - b}{2}$
3. $h = \frac{3V}{B}$
4. $c = 2s - a - b$
5. $h = \frac{3V}{\pi r^2}$
6. $b_1 = 2m - b_2$
7. $b_1 = \frac{2A}{h} - b_2$
8. $a = \frac{2S}{n} - \ell$
9. $n = \frac{2S}{a + \ell}$
10. $h = \frac{T - 2\pi r^2}{2\pi r}$
11. $r = \frac{E}{C} - R$
12. $r = 1 - \frac{a}{S}$
13. $a = \frac{S(r - 1)}{rn - 1}$
14. $V = \frac{KT}{P}$
15. $d = \frac{rS}{2\pi r^2 - S}$
16. $M = \frac{6V - (B_1 + B_2)h}{4h}$

*

There is another analogy between operations on functions and operations on numbers which you may wish to point out to your students. Consider all functions whose arguments and values are real numbers. When one adds or composes such functions he obtains functions of the same kind--that is, this set of functions is closed under the operations of addition and composition. As we have seen, addition of functions is commutative and associative, and function composition is associative, but not commutative. Moreover, it is easy to see that function composition is distributive with respect to addition in the sense that the generalization:

$$\forall_f \forall_g \forall_h \; [f + g] \circ h = f \circ h + g \circ h$$

holds. [That a left-hand distributive principle does not hold is shown by, among other examples, the well-known fact that the square of a sum is not the sum of the squares.] If one restricts one's consideration to real-valued functions whose domain is the set of all real numbers then addition of functions is a commutative <u>group</u> operation [see TC[5-264]]. In this case, including composition gives an example of an important kind of mathematical system called <u>a ring.</u> Since, for each f, $f \circ 1 = f = 1 \circ f$, it is <u>a ring with unit</u>.

the letters occurring in it. For example, we can solve '$P = 2(\ell + w)$' for 'w'.

$$P = 2(\ell + w)$$

$$P = 2\ell + 2w$$

addition transformation principle

$$P - 2\ell = 2w$$

multiplication transformation principle

$$w = \frac{P - 2\ell}{2}$$

EXERCISES

Solve each of the following formulas for the indicated letter.

1. $P = a + b + c$; b
2. $P = 2a + b$; a
3. $V = \frac{1}{3}Bh$; h
4. $s = \frac{1}{2}(a + b + c)$; c
5. $V = \frac{1}{3}\pi r^2 h$; h
6. $m = \frac{1}{2}(b_1 + b_2)$; b_1
7. $A = \frac{1}{2}h(b_1 + b_2)$; b_1
8. $S = \frac{n}{2}(a + \ell)$; a
9. $S = \frac{n}{2}(a + \ell)$; n
10. $T = 2\pi rh + 2\pi r^2$; h
11. $C = \frac{E}{R + r}$; r
12. $S = \frac{a}{1 - r}$; r
13. $S = \frac{ar^n - a}{r - 1}$; a
14. $K = \frac{PV}{T}$; V
15. $S = \frac{2\pi dr^2}{d + r}$; d
16. $V = \frac{h}{6}(B_1 + B_2 + 4M)$; M

EXPLORATION EXERCISES

A. 1. Graph on the same chart the functions defined by these equations.

(a) $y = 3x$ (b) $y = \frac{1}{2}x$ (c) $y = -2x$

2. Give the set of ordered pairs which is the intersection of the three functions of Exercise 1.

3. Give the set of ordered pairs which is the intersection of the functions $\{(x, y): y = ax\}$, for all a.

B. 1. Graph on the same chart the functions defined by these equations.

(a) $y = -2$ (b) $y = 7$ (c) $y = 0$

2. Give the set of ordered pairs which is the intersection of the three functions of Exercise 1.

3. Give the set of ordered pairs which is the intersection of any two of the functions $\{(x, y): y = b\}$, for all b.

C. 1. Graph the function $g_1 + g_2$ where $g_1 = \{(x, y): y = 3x\}$ and $g_2 = \{(x, y): y = 2\}$.

2. Graph $\{(x, y): y = 3x + 2\}$.

D. 1. Graph on the same chart the functions defined by these equations.

(a) $y = 4x + 2$ (b) $y = -3x + 2$ (c) $y = \frac{2}{3}x + 2$

2. Repeat Exercise 1 for these equations.

(a) $y = 4x + 0$ (b) $y = -3x + 0$ (c) $y = -\frac{1}{2}x + 0$

3. Repeat Exercise 1 for these equations.

(a) $y = 4x - 5$ (b) $y = -6x - 5$ (c) $y = \frac{3}{5}x - 5$

E. In each of the following exercises you are given a pair of equations each of which defines a function. For each exercise, give the set of ordered pairs which is the intersection of the functions.

1. $y = 2x + 5$
   $y = 3x + 5$

2. $y = -3x - 7$
   $y = 6x - 7$

3. $y = 571x + 9$
   $y = -35x + 9$

4. $y = 84x$
   $y = -3x$

5. $y = 7x + 2$
   $y = 7x + 8$

6. $y = -5x + 3$
   $y = -5x - 7$

F. Give numerals for the frames to make true sentences.

1. $\{(x, y): y = 3x + \bigcirc\}$ contains the point (0, 7)

2. $(0, 83) \in \{(x, y): y = 2x + \bigcirc\}$

3. The graph of $\{(x, y): y = -5x + \bigcirc\}$ crosses the graph of the y-axis 12 units above the graph of the origin.

4. The graph of $\{(x, y): y = -\frac{1}{2}x + \bigcirc\}$ contains the graph of the origin.

5. $\{(x, y): y = \square x + 0\}$ contains the point (1, 5)

6. $\{(x, y): y = \square x + 0\}$ contains the point (1, 52)

7. $\{(x, y): y = \square x + 0\}$ contains the point (5, 1)

8. $(3, 6) \in \{(x, y): y = \square x\}$

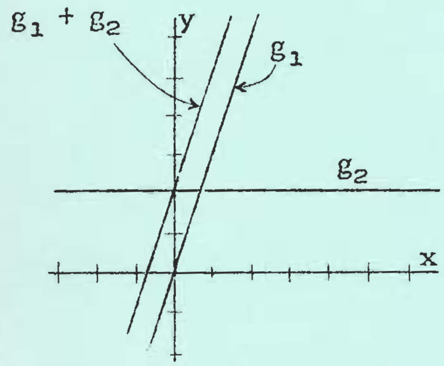
9. $(6, 3) \in \{(x, y): y = \square x\}$

10. $(2, 5) \in \{(x, y): y = \square x\}$

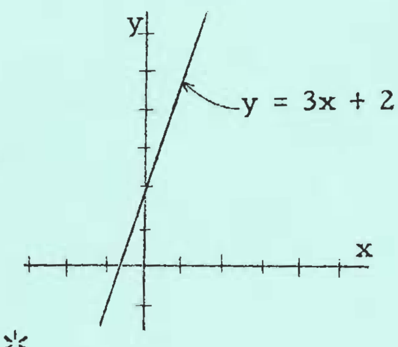
11. $(2, -1) \in \{(x, y): y = \square x\}$

Answers for Part C.

1.



2.



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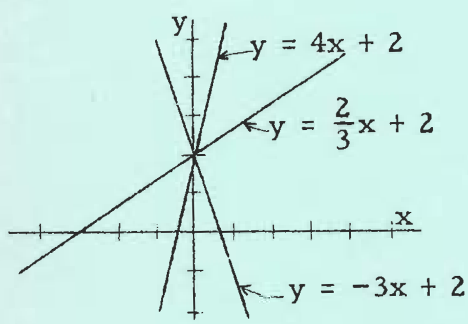
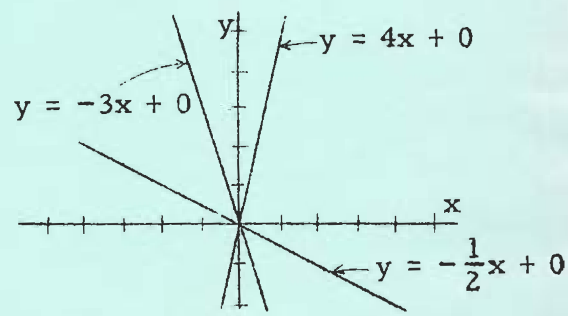
Answers for Part D.

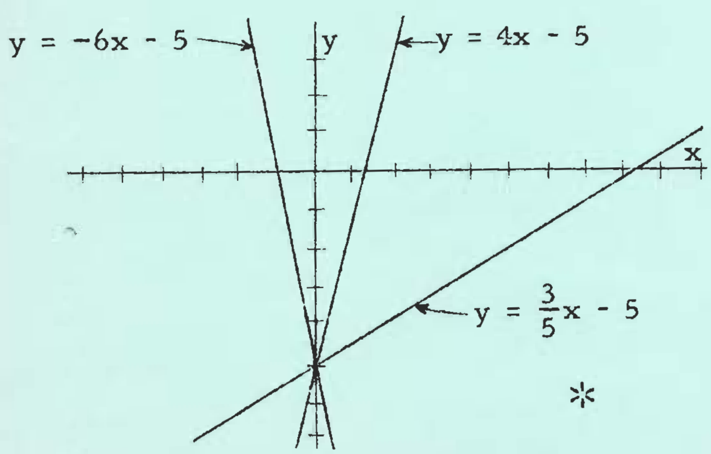
1.



2.



3.



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Answers for Part E [on page 5-116].

[Students should be allowed to use whatever algebraic techniques they can discover in doing these problems.]

1. $\{(0, 5)\}$ 2. $\{(0, -7)\}$ 3. $\{(0, 9)\}$ 4. $\{(0, 0)\}$ 5. $\emptyset$ 6. $\emptyset$

*

Answers for Part F [on pages 5-116 and 5-117].

1. 7	2. 83	3. 12	4. 0	5. 5	6. 52
7. 1/5	8. 2	9. 1/2	10. 5/2	11. −1/2	

IV. 1. (a) $\{1, 3, 5\}$ (b) $\{2, 4, 6\}$ (c) $\{1, 2, 3, 4, 5, 6\}$

(d) $\{(2, 1), (4, 3), (6, 5)\}$

2. (a) the set of real numbers (b) the set of nonnegative real numbers

(c) the set of real numbers (d) $\{(x, y) : y = \sqrt{x} \text{ or } y = -\sqrt{x}\}$

V. 1. (a) $\frac{(x - 3)}{4}$ (b) $2(x - 5)$ (c) 2 (d) 2 (e) $\frac{39}{8}$ (f) $\frac{-9}{4}$

2. (a) the set of all real numbers (b) $\{x: x \geq 1\}$

(c) 13 (d) 13 (e) 169 (f) -13 (g) 1 (h) 1

(i) (13, 2) and (13, -2) both belong to the converse of f; so, the converse of f is not a function.

(j) $\{(x, y), x \geq 0: y = 3x^2 + 1\}$ [There are many others.]

3. (a) (1, 5), (2, 6), (3, 6), (4, 4) (b) (5, 2), (6, 1), (4, 2)

(c) 2 (d) 2 (e) 2 and 3 (f) (1, 2), (2, 1), (3, 1), (4, 2)

VI. 1. $f = \{(x, y) : y = x^2\}$ 2. $f = \{(x, y) : y = \frac{1}{2}x + 4\}$

3. No; $g(1) = 3 = g(2)$ but $h(1) \neq h(2)$.

VII. 1. Union of two straight lines, one containing the points (0, 8) and (8, 0) and the other containing the points (0, 0) and (1, 3). Both lines contain the point (2, 6).

2. The set consisting of just the point (2, 6).

3. The union of the first and third quadrants.

4. The circular region with radius 3 and center (0, 0) including the boundary.

5. A right angle with vertex (3, 0) and symmetric with respect to the straight line containing (3, 0) and parallel to the second component axis. The angle opens upward.

6. A parabola opening upward with extreme point (-3, 0) and containing the points (0, 9) and (-6, 9).

VI. For each pair of functions, g and h, tell whether h is a function of g. If not, tell why. If so, describe a function f such that $h = f \circ g$.

1. $g = \{(x, y): y = |x|\}$, $h = \{(x, y): y = x^2\}$
2. $g = \{(x, y): y = 2x\}$, $h = \{(x, y): y = x + 4\}$
3. $g = \{(x, y): y = 3\}$, $h = \{(x, y): y = 3 - x\}$

VII. Graph each of the relations listed below.

1. $\{(x, y): x + y = 8\} \cup \{(x, y): y = 3x\}$
2. $\{(x, y): x + y = 8\} \cap \{(x, y): y = 3x\}$
3. $\{(x, y): xy > 0\}$
4. $\{(x, y): x^2 + y^2 \le 9\}$
5. $\{(x, y): y = |x - 3|\}$
6. $\{(x, y): y = (x + 3)^2\}$

*

Answers for Quiz.

I.

1. T	2. F	3. T	4. T	5. T	6. T	7. F
8. T	9. T	10. F	11. T	12. T	13. F	14. T
15. T	16. T	17. F	18. F	19. T	20. T	21. F
22. F	23. T	24. T	25. F			

II.
1. (1, 1) (3, 3), (5, 5), (7, 7), (9, 9), (11, 11), (13, 13)
2. (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)
3. (9, 9), (9, 10), (9, 11), (10, 9), (10, 10), (10, 11), (11, 9), (11, 10), (11, 11)
4. $6 < x < 14$
5. $x = 110$
6. $\{2, 4, 6, 8\}$ [or: $\mathcal{R}_K$]
7. 4
8. 1/3
9. $-2x + 8$
10. quadrupling function [or: 4-times function, or: the function that maps each real number on the real number 4 times as large]

III.
1. C
2. S and C
3. (a) 4 (b) 21 (c) 9 (d) 16 (e) 21 (f) 21 (g) 4
(h) 4 (i) 12 (j) 9 (k) 0 (ℓ) 25 (m) 13 (n) 4
4. (c)
5. (b)

IV. For each of the two relations given below, tell its

(a) domain (b) range (c) field (d) converse

1. $H = \{(1, 2), (3, 4), (5, 6)\}$ 2. $J = \{(x, y) : y = x^2\}$

V. 1. Suppose that $f = \{(x, y) : y = 4x + 3\}$ and $g(x) = \frac{1}{2}x + 5$.

(a) $f^{-1} = \{(x, y) : y =$ ________$\}$ (b) $g^{-1}(x) =$ ____________

(c) $f^{-1}(f(2)) =$ ______________ (d) $[f \circ f^{-1}](2) =$ ________

(e) $g(f^{-1}(2)) =$ ______________ (f) $[f^{-1} \circ g^{-1}](2) =$ ______

2. Suppose that $f = \{(x, y) : y = 3x^2 + 1\}$

(a) $\mathcal{D}_f =$ ____________ (b) $\mathcal{R}_f =$ ____________

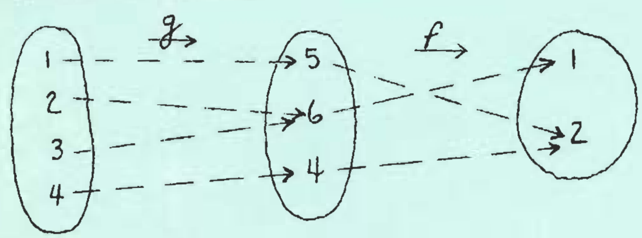
(c) $f(2) =$ _____ (d) $f(-2) =$ _____ (e) $[f^2](2) =$ _______

(f) $[-f](2) =$ ___ (g) $\sqrt{f}(0) =$ _____ (h) $[1/f](0) =$ _____

(i) Prove that f does not have an inverse.

(j) Describe a subset of f that does have an inverse.

3. The mappings g and f are shown below.

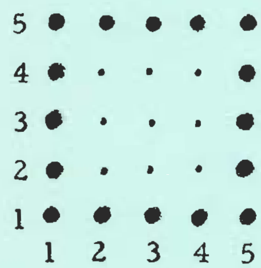


(a) $g = \{$________________$\}$ (b) $f = \{$________________$\}$

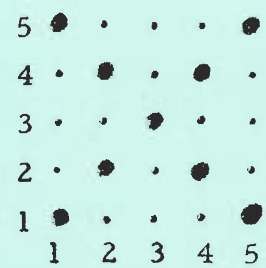
(c) $f(g(1)) =$ ____ (d) $f(g(4)) =$ ____ (e) $f(g($__$)) = 1$

(f) $f \circ g = \{$____________________$\}$

III. Suppose $V = \{1, 2, 3, 4, 5\}$ and S and C are relations among the members of V. Here are graphs of S and C.



Graph of S



Graph of C

1. Which of these relations are reflexive?
2. Which of these relations are symmetric?
3. How many elements are in each of the sets listed below? [Complements are with respect to $V \times V$.]

   (a) $S \cap C$ (b) $S \cup C$ (c) $\tilde{S}$ (d) $\tilde{C}$

   (e) $\tilde{S} \cup \tilde{C}$ (f) $\widetilde{S \cap C}$ (g) $\tilde{S} \cap \tilde{C}$ (h) $\widetilde{S \cup C}$

   (i) $S \cap (\tilde{S} \cup \tilde{C})$ (j) $C \cup (S \cap C)$ (k) $S \cap (\tilde{S} \cap \tilde{C})$

   (ℓ) $S \cup (\tilde{S} \cup \tilde{C})$ (m) $(S \cap C) \cup \tilde{S}$ (n) $(S \cup C) \cap (S \cap C)$

4. Which of the following describes S?

   (a) $\{(x, y) \in V \times V: (x < 2 \text{ or } x > 4) \text{ and } (y < 2 \text{ or } y > 4)\}$

   (b) $\{(x, y) \in V \times V: |x - 3| > 1 \text{ and } |y - 3| > 1\}$

   (c) $\{(x, y) \in V \times V: x = 1 \text{ or } x = 5 \text{ or } y = 1 \text{ or } y = 5\}$

   (d) $\{(x, y) \in V \times V: 1 < x < 5 \text{ and } 1 < y < 5\}$

   (e) $\{(x, y) \in V \times V: 1 < x < 5 \text{ or } 1 < y < 5\}$

5. Which of the following describes C?

   (a) $\{(x, y) \in V \times V: x = y \text{ or } x = -y\}$

   (b) $\{(x, y) \in V \times V: |x - 3| = |y - 3|\}$

   (c) $\{(x, y) \in V \times V: |x| = |y|\}$

   (d) $\{(x, y) \in V \times V: x - 3 = y - 3\}$

   (e) $\{(x, y) \in V \times V: 3 - y = x - 3\}$

22. For all variable quantities M and N, the domain of $M + N$ is the union of the domains of M and of N.

23. For each real-valued function f, the range of the function $-f$ consists of the opposites of the values of f.

24. For each real-valued function f and for each argument x of f such that $f(x) \geq 0$, $\sqrt{f}(x) = \sqrt{f(x)}$.

25. If $a \in \mathcal{D}_R$ but $(a, b) \notin R$, it follows that $b \notin \mathcal{R}_R$.

II. Fill the blanks.

1. The reflexive relation whose field is $\{1, 3, 5, 7, 9, 11, 13\}$ and which has the fewest members is $\{$ ____________________ $\}$.

2. The intersection of all reflexive relations whose field is $\{1, 2, 3, 4, 5\}$ is $\{$ ____________________ $\}$.

3. The union of all reflexive relations whose field is $\{9, 10, 11\}$ is $\{$ ________________ $\}$.

4. The three sides of a triangle are 4 inches, 10 inches and x inches long if and only if ________________.

5. The degree-measures of the angles of a triangle are 20, 50 and x if and only if __________________.

6. If K is a symmetric relation and $\mathcal{R}_K = \{2, 4, 6, 8\}$ then $\mathcal{D}_K =$ ______________________.

7. $(5,$ ____ $) \in \{(x, y) \in I^+ \times I^+ : 4x - 2 < y + 15 < 3x + 5\}$

8. If $(3, 8) \in \{(x, y) : y = ax + 7\}$ then $a =$ ______.

9. If $f = \{(x, y) : x + 2y = 8\}$ then $f^{-1} = \{(x, y) : y =$ ______ $\}$.

10. If d is the doubling function [that is, the function that maps each real number on its double] then $d \circ d$ is the _________.

Examination [for pages 5-A through 5-115].

I. True or false?

1. A variable quantity is a numerical valued function.
2. Every relation is a function.
3. Every function is a relation.
4. Intersecting is distributive with respect to unioning.
5. For all subsets x and y of a given set S, $x \cup (x \cap y) = x$.
6. For all subsets x and y of a given set S, $y \cap (x \cup y) = y$.
7. $\forall_x\ x + 3 = 5$
8. $\exists_x\ x + 7 = 12$
9. $\forall_x\ (\exists_y\ x + y = 10)$
10. $\exists_x\ (\forall_y\ x + y = 10)$
11. There is a unique value corresponding to each argument of a function.
12. For each function f and for all arguments x_1 and x_2 of f, if $f(x_1) \neq f(x_2)$ then $x_1 \neq x_2$.
13. The relation $\{(x, y) : y < x\}$ is symmetric.
14. The relation $\{(x, y) : y \geq x\}$ is reflexive.
15. The converse of a symmetric relation is symmetric.
16. The converse of a reflexive relation is reflexive.
17. For all relations R_1 and R_2, if $R_1 \cup R_2$ is symmetric and R_1 is symmetric then R_2 is symmetric.
18. The perimeter of a rectangle is a function of its length-measure.
19. The area-measure of an equilateral triangle is a function of its side-measure.
20. The degree-measure of an angle of a triangle is a function of the sum of the degree-measures of the other two angles.
21. The measure of a side of a triangle is a function of the sum of the measures of the other two sides.

12. 2	13. 3	
14. ○2 , △5	15. ○2 , □3	16. ○2 , □7
17. ○2 , □77	18. ○2 , △17	19. ○2 , □7
20. ○2 , □4	21. ○2 , □−1	22. ○8 , □2
23. ○−4 , □3	24. △5 , ▽12	25. △3 , ▽21
26. ○329 , □210	27. ○4 , □13 , (26)	28. ○4 , □−13 , (−26)

*

Answers for Part G [on pages 5-117 and 5-118].

1. 19k   2. −29k   3. ak   4. m/k   5. b   6. b, 7 + b

7. b, 21 + b   8. b, a + b, 2a + b   9. b, ap

10. 9 + b, 9p   11. 14 + b, −2p   12. a3 + b, ap

13. aq + b, ap   14. ap

Note that a constant function of a real variable is a constant--that is, a variable quantity whose range is a singleton. But, of course, not every constant is a constant function of a real variable.

*

Answer for question in the last paragraph on page 5-118.

The domain, and the range, of each of the given functions is the set of real numbers.

*

12. $(-1, -2) \in \{(x, y): y = \square x\}$ 13. $(-3, -9) \in \{(x, y): y = \square x\}$

14. $\{(x, y): y = 3x + \bigcirc\}$ contains the points (0, 2) and $(1, \triangle)$

15. $\{(x, y): y = \square x + \bigcirc\}$ contains the points (0, 2) and (1, 5)

16. $\{(x, y): y = \square x + \bigcirc\}$ contains the points (0, 2) and (1, 9)

17. $\{(0, 2), (1, 79)\} \subseteq \{(x, y): y = \square x + \bigcirc\}$

18. $\{(0, 2), (3, \triangle)\} \subseteq \{(x, y): y = 5x + \bigcirc\}$

19. $\{(0, 2), (3, 23)\} \subseteq \{(x, y): y = \square x + \bigcirc\}$

20. $\{(0, 2), (5, 22)\} \subseteq \{(x, y): y = \square x + \bigcirc\}$

21. $\{(0, 2), (5, -3)\} \subseteq \{(x, y): y = \square x + \bigcirc\}$

22. $\{(0, 8), (1, 10)\} \subseteq \{(x, y): y = \square x + \bigcirc\}$

23. $\{(0, -4), (2, 2)\} \subseteq \{(x, y): y = \square x + \bigcirc\}$

24. $\{(0, \triangle), (1, \triangledown)\} \subseteq \{(x, y): y = 7x + 5\}$

25. $\{(0, \triangle), (2, \triangledown)\} \subseteq \{(x, y): y = 9x + 3\}$

26. $\{(0, 329), (5, \square + 329)\} \subseteq \{(x, y): y = 42x + \bigcirc\}$

27. $\{(0, 4), (1, 4 + \square), (2, 4 + \bigcirc)\} \subseteq \{(x, y): y = 13x + \bigcirc\}$

28. $\{(0, 4), (1, 4 + \square), (2, 4 + \bigcirc)\} \subseteq \{(x, y): y = -13x + \bigcirc\}$

G. Complete to make true generalizations.

1. $\forall_k \{(0, 0), (k, \_\_\_)\} \subseteq \{(x, y): y = 19x\}$

2. $\forall_k \{(0, 0), (k, \_\_\_)\} \subseteq \{(x, y): y = -29x\}$

3. $\forall_a \forall_k \{(0, 0), (k, \_\_\_)\} \subseteq \{(x, y): y = ax\}$

4. $\forall_{k \neq 0} \forall_m \{(0, 0), (k, m)\} \subseteq \{(x, y): y = \_\_\_ \cdot x\}$

5. $\forall_a \forall_b (0, \_\_\_) \in \{(x, y): y = ax + b\}$

6. $\forall_b \{(0, \_\_\_), (1, \_\_\_\_\_)\} \subseteq \{(x, y): y = 7x + b\}$

7. $\forall_b \{(0, \_\_\_), (3, \_\_\_\_\_)\} \subseteq \{(x, y): y = 7x + b\}$

8. $\forall_a \forall_b \{(0, \_\_\_), (1, \_\_\_\_), (2, \_\_\_\_)\} \subseteq \{(x, y): y = ax + b\}$

9. $\forall_a \forall_b \forall_p \{(0, \_\_\_), (p, b + \_\_\_\_)\} \subseteq \{(x, y): y = ax + b\}$

10. $\forall_b \forall_p \{(1, \_\_\_\_), (1 + p, 9 + b + \_\_\_\_)\} \subseteq \{(x, y): y = 9x + b\}$

11. $\forall_b \forall_p \{(-7, \_\_\_\_), (-7 + p, 14 + b + \_\_\_\_)\} \subseteq \{(x, y): y = -2x + b\}$

12. $\forall_a \forall_b \forall_p \{(3, \_\_\_\_), (3 + p, a3 + b + \_\_\_\_)\} \subseteq \{(x, y): y = ax + b\}$

13. $\forall_a \forall_b \forall_p \forall_q \{(q, \_\_\_\_), (q + p, aq + b + \_\_\_\_)\} \subseteq \{(x, y): y = ax + b\}$

14. $\forall_a \forall_b \forall_s \forall_t$ if $(s, t) \in \{(x, y): y = ax + b\}$

then $\forall_p\ (s + p, t + \_\_\_) \in \{(x, y): y = ax + b\}$

5.07 Linear functions.--The domain of functions such as

$$\{(x, y): y = 2\}, \quad \{(x, y): y = -3\}, \quad \text{and} \quad \{(x, y): y = 43.5\},$$

is the set of real numbers. The range of each such function consists of a single real number. These functions are called constant functions of a real variable. We say 'of a real variable' because the domain is the set of real numbers. There are also constant functions of two [or more] real variables such as $\{((x, y), z): z = 3\}$. But, since we shall not discuss them in this section, it will not cause confusion if we abbreviate 'constant function of a real variable' to 'constant function'.

The graphs of constant functions are horizontal lines, that is, lines perpendicular to the graph of the y-axis.

Now, consider functions of a real variable such as

$$\{(x, y): y = 3x - 2\}, \quad \{(x, y): y = -2x + 0\}, \quad \text{and} \quad \{(x, y): y = \tfrac{1}{2}x + 5\}.$$

What are the domain and range of these functions? Notice that each of these functions is the sum of two functions, one of which is a constant function. For example,

$$\{(x, y): y = 3x - 2\} = \{(x, y): y = 3x\} + \{(x, y): y = -2\}.$$

Figure 1 shows the graph of $\{(x, y): y = 3x\}$; Figure 2 shows the graph

Answers for Part A [on pages 5-120 and 5-121].

1. Yes; $a = -5$, $b = 3$
2. Yes; $a = 3/2$, $b = 5/2$
3. Yes; $a = 1/3$, $b = -7/3$
4. Yes; $a = 5$, $b = 11$
5. Yes; $a = -1$, $b = 9$
6. No; $\mathcal{D}_f = \{0, 1, 2\} \neq$ the set of real numbers [However, f is a <u>subset</u> of the linear function $\{(x, y): y = x + 9\}$.]

If the graphs of Figure 1 and Figure 2 [on page 5-119] were drawn on the same chart then that of Figure 3 could be formed by the method of addition of ordinates. [See discussion on TC[5-105]b.]

*

Answers for questions in first paragraph on page 5-119.

For the value 0 of 'a', and each value of 'b', $\{(x, y): y = ax + b\}$ is a constant function, and its graph is a horizontal line.

None of the graphs are vertical lines. [After all, the sets are functions.]

For each nonzero value of 'a', and each value of 'b', the graph is an oblique straight line.

*

The explanation called for in the last paragraph on page 5-119 might run as follows: In the preceding paragraph of the text, linear functions have been identified as functions whose graphs are oblique straight lines. Now, $\{(x, y): y = 7\}$ is a function whose graph is a straight line, but not an oblique one; and $\{(x, y): x = 4\}$ is not even a function [although its graph is a straight line]. Hence, neither of these sets of ordered pairs is a linear function.

Another interesting function to consider is the function g defined by:

$$g(x) = \frac{x^2 - 4}{x - 2}$$

Its graph is an oblique straight line [$y = x + 2$] with a hole at (2, 4). It is not a linear function because its domain is not the set of all real numbers.

*

Answers for questions on page 5-120.

If $f(x) = ax + b$, then $f(0) = b$ and $f(1) = a + b$. Hence, if $a \neq 0$, $f(0) \neq f(1)$. Consequently, no linear function is a constant function.

The domain of each linear function is the set of all real numbers.

The domain of $\{(x, y): y = 7x - 5 \text{ and } |y| \leq 1000000\}$ is not the set of real numbers. For example, 1000000 does not belong to the domain. [The largest member of the domain is $(10^6 + 5)/7$, and the smallest is $(-10^6 + 5)/7$.] So, the function in question is not a linear function.

*

$\{(x, y): C = 0\}$, and is either the cartesian square of the set of real numbers, or $\emptyset$, according as $C = 0$ or $C \neq 0$. So, if $B = 0$ then $\{(x, y): Ax + By + C = 0\}$ is not a function unless, also, $A = 0$ and $C \neq 0$. In this case it is not a linear function.

Having treated all cases, we see that $\{(x, y): Ax + By + C = 0\}$ is a linear function if and only if neither A nor B is 0.

*

Answer to question at foot of page 5-122.

$(3, -5) \notin g$ because g, being a linear function, has an inverse, and $(0, -5) \in g$.

First, we need a definition of 'linear function'.

f is a linear function [of a real variable]
if and only if
there are real numbers $a \neq 0$ and b such that
$$f = \{(x, y):\ y = ax + b\}.$$

This definition tells us, for example, that $\{(x, y):\ y = 7x - 5\}$ is a linear function since 7 and −5 are real numbers, and $7 \neq 0$. How does the definition rule out constant functions? What is the domain of a linear function? How does the definition rule out the function

$$\{(x, y):\ y = 7x - 5 \text{ and } |y| \leq 1000000\},$$

a large portion of whose graph looks like a straight line?

Now, let's attack the problem at hand. The relation g is $\{(x, y):\ 2y - 6x + 7 = 0\}$. To claim that g is a linear function is to claim that there are real numbers $a \neq 0$ and b such that

$$\{(x, y):\ 2y - 6x + 7 = 0\} = \{(x, y):\ y = ax + b\}.$$

And, to support the claim, all you must do is tell what these numbers are. How do you discover them? Just solve the set selector '$2y - 6x + 7 = 0$' for 'y'.

$$2y - 6x + 7 = 0$$
$$2y = 6x - 7$$
$$y = 3x - \frac{7}{2}$$

This last equation is equivalent to the given set selector. Therefore,

$$g = \{(x, y):\ y = 3x - \frac{7}{2}\}.$$

So, since there are numbers $a \neq 0$ and b [3 and −7/2, respectively] such that $g = \{(x, y):\ y = ax + b\}$, the definition assures us that the relation g is a linear function.

EXERCISES

A ~~For~~ each exercise, tell whether the given relation f is a linear

$- 3x - 5 = 0\}$

$a = 5b + 11\}$

TC[5-121, 122]c

$), (1, 10), (2, 11)\}$

of $\{(x, y): y = -2\}$; and Figure 3 shows the graph of their sum.

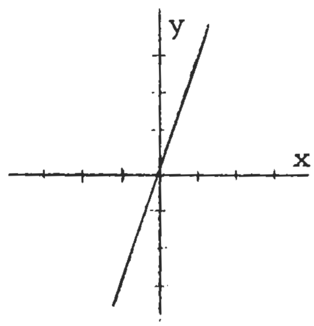


Fig. 1.

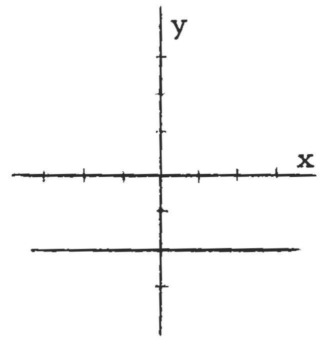


Fig. 2.

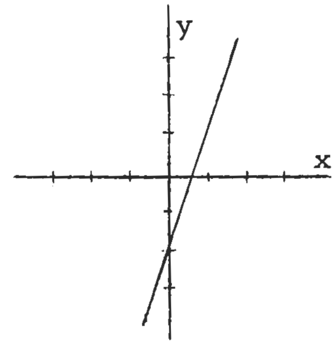


Fig. 3.

You have seen in the Exploration Exercises that a graph of each of the functions $\{(x, y): y = ax\}$, for any a, is a straight line containing the graph of the origin. Also, a graph of each of the functions $\{(x, y): y = b\}$, for any b, is a horizontal line. Now, consider graphs of the functions $\{(x, y): y = ax + b\}$, for all a and b. Are any of these graphs horizontal lines? For what values of 'a' and of 'b' will you get a horizontal line? Are any of these graphs vertical lines, that is, lines perpendicular to the graph of the x-axis? Are any of these graphs oblique straight lines, that is, lines which are neither horizontal nor vertical? For what values of 'a' and of 'b' will you get an oblique straight line?

Some of the functions $\{(x, y): y = ax+b\}$, for all a and b, are constant functions. These are the functions $\{(x, y): y = ax+b\}$, for $a = 0$ and any b. The others [that is, those which are not constant functions] are the functions $\{(x, y): y = ax+b\}$, for $a \neq 0$ and any b. These are called <u>linear functions</u> [actually, 'linear functions of a real variable', but we are abbreviating as we did in the case of constant functions].

Given a relation g, how can we tell whether it is a linear function? Suppose $g = \{(x, y): 2y - 6x + 7 = 0\}$. Is g a linear function? If we graphed it, we would find that the graph appeared to be a straight line. [Graphs of $\{(x, y): y = 7\}$ and $\{(x, y): x = 4\}$ are straight lines, but these are not linear functions. Explain.] Since the graph seems to be an oblique straight line, we are pretty sure that g is a linear function. But, graphing can sometimes be tedious as well as inaccurate. So, let's find an easier test.

☆3. If f and g are linear functions then there are numbers $a \neq 0$ and b such that $f = \{(x, y): y = ax + b\}$, and numbers $c \neq 0$ and d such that $g = \{(x, y): y = cx + d\}$. So, $f \circ g = \{(x, y): y = (ac)x + (ad + b)\}$. Since $ac \neq 0$, $f \circ g$ is a linear function. [Note that if either f or g is a linear function and the other is a constant function then $f \circ g$ is a constant function.]

*

Answers for Part C [on page 5-121].

1. Yes
2. No [The product of each function whose domain is the set of real numbers by the constant function 0 is also this constant function and, so, no such product is a linear function.]
3. No [The sum of the linear functions $\{(x, y): y = x\}$ and $\{(x, y): y = -x\}$ is the constant function 0.]

[In extension of Exercises 2 and 3 of Part C, note that the product of a linear function by any constant function other than 0 is a linear function; and that the sum of a linear function and a linear function is either a linear function or a constant function.]

*

Answers for Part ☆D [on page 5-121].

1. Yes [All such relations for which $AB \neq 0$.]
2. constant functions; no functions; linear functions containing (0, 0); the function $\emptyset$; not a function, just the set of all ordered pairs of real numbers
3. Suppose $B \neq 0$. Then

$$\{(x, y): Ax + By + C = 0\} = \{(x, y): y = (-A/B)x - C/B\}.$$

If, in addition, $A \neq 0$ then $-A/B \neq 0$. Hence, if neither A nor B is 0, the function in question is a linear function with $a = -A/B$ and $b = -C/B$. Also, if $A = 0$ and $B \neq 0$, the function is a constant function and so is not a linear function.

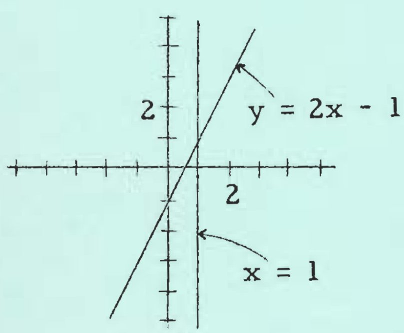
It remains to be shown that if $B = 0$ then $\{(x, y): Ax + By + C = 0\}$ is not a linear function. Now, if $A \neq 0$, this set of ordered pairs is $\{(x, y): x = -C/A\}$, and is not a function. And, if $A = 0$, it is

Corrections. Exercises 2 and 3 of Part B should be starred.

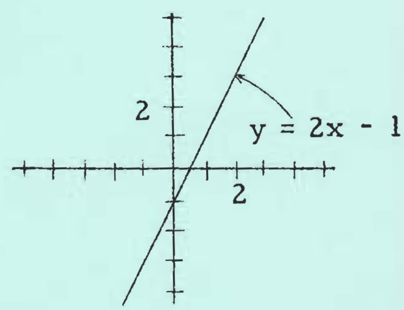
On page 5-122, in the last line, insert a ']' after 'isn't?'.

7. Yes; $a = 14/29$, $b = -37/29$
8. No; $f = \{(x, y): x = 3\}$, and this is not a function.
9. No; f is a constant function.
10. Yes; $a = 7$, $b = 0$
11. No; $\mathfrak{D}_f = \{x: x \geq 0\} \neq$ the set of real numbers
12. Yes; $a = -1$, $b = 0$
13. No; $\mathfrak{D}_f = I \neq$ the set of real numbers
14. No; f is not a function.
15. Yes; $a = -2$, $b = 11$
16. Yes; $a = -5/3$, $b = 5$
17. Yes; $a = -5/7$, $b = 31/7$
18. Yes; $a = -2/3$, $b = 13/3$
19. Yes; $a = -3$, $b = 0$
20. No; f is not a function.
21. No; $\mathfrak{D}_f = \{0\} \neq$ the set of real numbers

☆22. No; f is not a function.



☆23. Yes; $a = 2$, $b = -1$ [Note that $\forall_x\ x^2 + 1 \neq 0$.]



[Try: $f = \{(x, y): x^2 - 2xy + y^2 = 0\}$ as an extra exercise. This is, of course, just $\{(x, y): y = x\}$.]

*

Answers for Part B.

1. Yes [$f^{-1} = \{(x, y): y = (x + 5)/2\}$]; yes [$a = 1/2$, $b = 5/2$]

☆2. If f is a linear function then there are numbers $a \neq 0$ and b such that $f = \{(x, y): y = ax + b\}$. The converse of f is $\{(x, y): x = ay + b\}$ -- that is, since $a \neq 0$, $\{(x, y): y = (1/a)x - (b/a)\}$. So, if f is a linear function then so is its converse. Hence, each linear function has an inverse, and this inverse is a linear function.

7. $f = \{(x, y): 2(8 - 3x) - 9(2 - 3y) + 7 = 8(x - 5) + 2(4 - y)\}$.

8. $f = \{(x, y): x + y = 3 + y\}$

9. $f = \{(x, y): y + 2x = 2(x - 7)\}$

10. $f = \{(x, y): y = 7x\}$

11. $f = \{(x, y): y = 7x \text{ and } y \geq 0\}$

12. $f = \{(r, s): r = -s\}$

13. $f = \{(x, y) \in I \times I: y = 2x + 5\}$

14. $f = \{(x, y): y \geq 2x + 1\}$

15. $f = \{(x, y): y - 3 + 2(x - 4) = 0\}$

16. $f = \{(x, y): \frac{x}{3} + \frac{y}{5} = 1\}$

17. $f = \{(x, y): 5(x - 2) + 7(y - 3) = 0\}$

18. $f = \{(x, y): 3y + 2x - 5 = 8\}$

19. $f = \{(x, y): y = x^2 - x(3 + x)\}$

20. $f = \{(x, y): y = 2x \text{ or } y = 3x\}$

21. $f = \{(x, y): y = 2x \text{ and } y = 3x\}$

☆22. $f = \{(x, y): (2x - 1 - y)(x - 1) = 0\}$

☆23. $f = \{(x, y): (2x - 1 - y)(x^2 + 1) = 0\}$

B. Suppose $f = \{(x, y): y = 2x - 5\}$.

1. Does f have an inverse? If it does, is f^{-1} a linear function?

2. Prove that each linear function has an inverse, and that its inverse is a linear function.

3. Prove that if f and g are linear functions then so is $f \circ g$.

C. 1. Is each sum of a linear function and a constant function a linear function?

2. Is each product of a linear function by a constant function a linear function?

3. Is each sum of a linear function and a linear function a linear function? [Careful!]

☆D. Consider the relations $\{(x, y): Ax + By + C = 0\}$, for all real numbers A, B, and C.

1. Are any of these relations linear functions?

2. What kind of functions do we have in case $A = 0$ and $B \neq 0$? In case $B = 0$ and $A \neq 0$? In case $C = 0$ and $A \neq 0$ and $B \neq 0$? In case both A and B are 0 and $C \neq 0$? In case A, B, and C are 0?

3. Prove that, for all A, B, and C, $\{(x, y): Ax + By + C = 0\}$ is a linear function if and only if neither A nor B is 0. [Hint. Use the definition of a linear function.]

GRAPHING A LINEAR FUNCTION

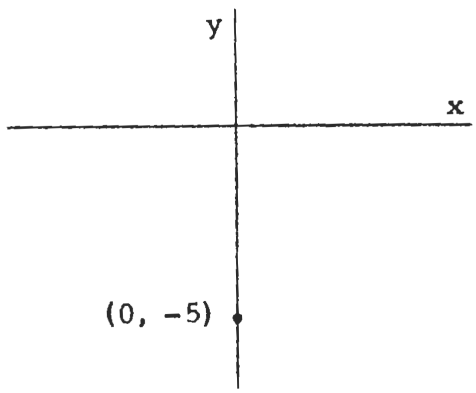
The fact that for each linear function there is a defining equation in 'y' and 'x' of the form:

$$(*) \qquad y = ax + b$$

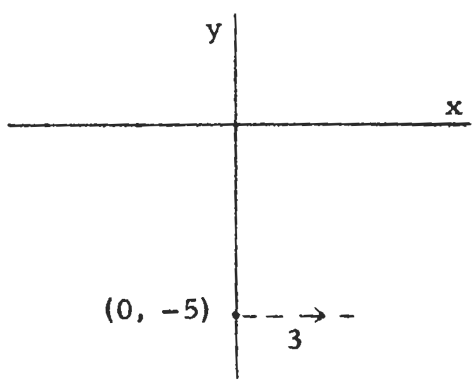
gives us a very quick way of graphing linear functions. For example, consider the linear function g where g is $\{(x, y): 2x - y - 5 = 0\}$. Since g is a linear function, we can transform the set selector to an equation of the form (*):

$$y = 2x - 5$$

Since the domain of any linear function is the set of real numbers, we know that g is defined at 0. The defining equation '$y = 2x - 5$' tells us that (0, −5) is a member of g. The graph of (0, −5) is on the graph of the y-axis.



To find the graph of another member of g, place your pencil point on the graph of (0, −5), and move the pencil a certain distance to the right. Say you move it a distance 3 to the right. [This brings you to the graph of (3, −5), and (3, −5) is <u>not</u> a member of g. How do you know that it isn't?



Quiz.

Each of the relations described below is a linear function. For each of them, write the defining equation in the form 'y = ax + b'.

Sample. $\{(x, y):\ 2x - 3y - 7 = 0\}$; y = ___ x + ___

Solution. y = $\frac{2}{3}$ x + $-\frac{7}{3}$

1. $\{(x, y):\ 5x - y = 10\}$; y = ___ x + ___

2. $\{(x, y):\ 16x = 3 - 4y\}$; y = ___ x + ___

3. the function which maps each real number on its double; y = ___ x + ___

4. the function which maps each real number on its opposite; y = ___ x + ___

5. the sum of $\{(x, y):\ x = 2y - 5\}$ and $\{(x, y):\ y = 2x - 5\}$; y = ___ x + ___

6. the composition of $\{(x, y):\ x = 2y - 5\}$ with $\{(x, y):\ y = 2x - 5\}$; y = ___ x + ___

7. the inverse of $\{(x, y):\ 3x + 4y + 2 = 0\}$; y = ___ x + ___

*

Answers for Quiz.

1. 5, −10
2. −4, $\frac{3}{4}$
3. 2, 0
4. −1, 0
5. $\frac{5}{2}$, $-\frac{5}{2}$
6. 1, 0
7. $-\frac{4}{3}$, $-\frac{2}{3}$

Notice the inequations in lines 15 and 16 of page 5-125. You can, when going over this material in class, insert a very brief review of procedures for transforming inequations [See Unit 3, page 3-100.]:

$$\begin{aligned} ak &> 0 \\ ak + q &> 0 + q \quad \Big\} \text{ atpi} \\ q + ak &> q \quad \Big\} \text{ cpa, pa0} \end{aligned}$$

*

Answers to questions on page 5-125, lines 19ff.

If a is positive and k is negative, the graph of (p + k, q + ak) is below and to the left of the graph of (p, q). Yes, the graph of the function rises to the right.

When a is negative and k is either positive or negative, the graph rises to the left. If $a < 0$ and $k < 0$ then $ak > 0$. So, $q + ak > q$.

When the intercept is positive, the graph of f crosses the vertical axis above the horizontal axis; when the intercept is negative, the graph crosses below the horizontal axis; when the intercept is 0, the graph crosses the horizontal axis at the graph of (0, 0).

*

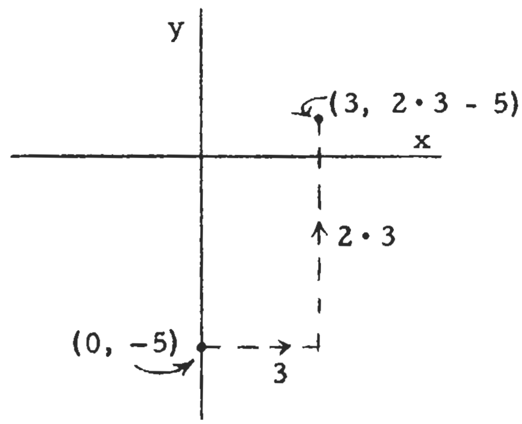
Answers to questions at top of page 5-126.

The graph rises to the right if the slope is positive, to the left if the slope is negative. No, the slope cannot be 0.

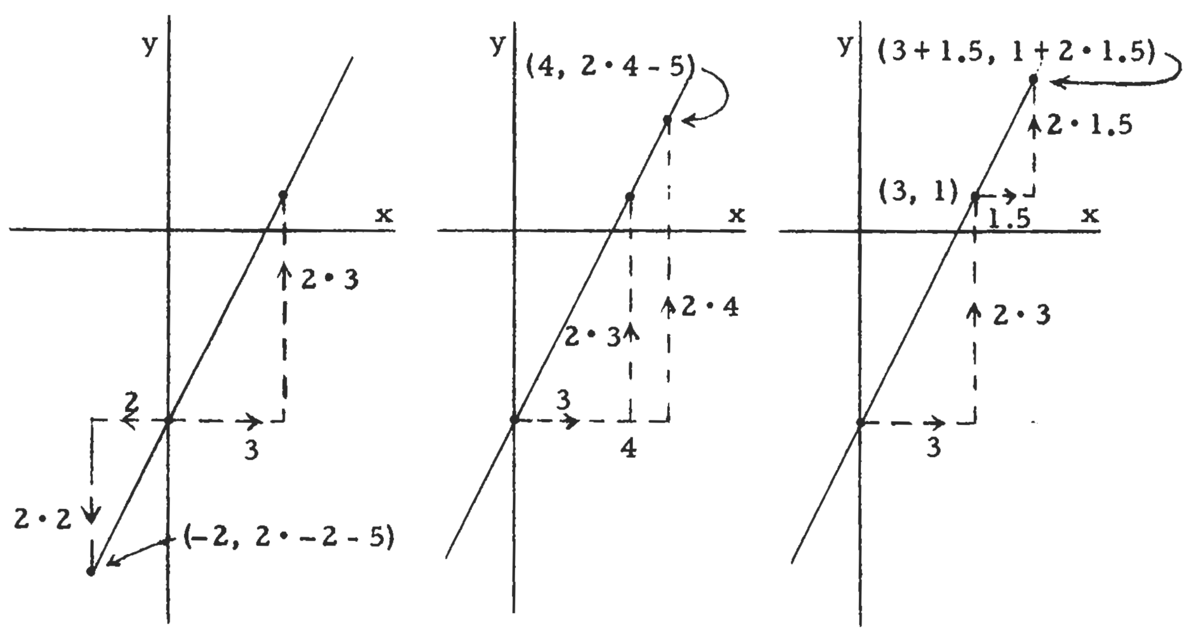
A valuable kind of classroom exercise consists in drawing numerous oblique lines on the chalk board and asking students to estimate the slopes of the linear functions whose graphs they are. Disputes should be resolved by taking rough measurements [perhaps using the eraser length as unit], and taking account of the direction of rise.

Notice that, although we have chosen to use pronumerals 'a' and 'b' in discussing linear functions, it would have been very natural to have introduced the variable quantities *the slope of a linear function* and *the intercept of a linear function*. Had we done so, and used 'a' and 'b' as names for these variable quantities, then, in place of 'y = ax + b' as the type of defining equation for a linear function, we should have used 'y = a(f) • x + b(f)'. Here, instead of the numerical variables 'a' and 'b' we have the linear functional variable 'f'.

Now, since g is defined at 3, the vertical line through the graph of (3, −5) must contain the graph of a member of g. The x-coordinate of this graph is 3, and we use the defining equation to tell us the y-coordinate. It is 2 • 3 - 5. You reach the graph of (3, 2 • 3 - 5) by moving your pencil point up the distance 2 • 3.

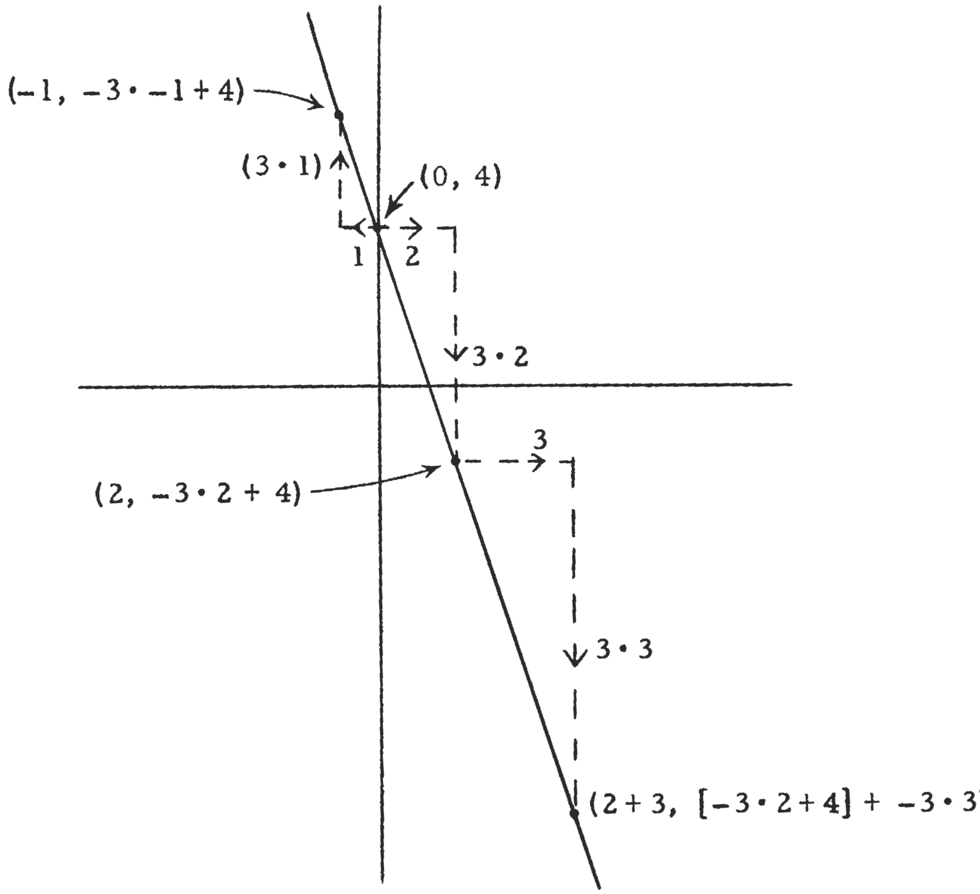


The graphs of these two members of g are all you need in order to draw the straight line which is the graph of g. However, it is good practice to graph a third member of g to catch possible plotting or computing errors. There are many ways in which you can use the ordered pairs already plotted to find the graph of a third member of g. Three such ways are shown in the diagrams below.



[Before reading any further, graph the function defined by the equation 'y = 3x - 2' using the method illustrated above.]

Here is a diagram showing how the graphs of three more members of {(x, y): y = −3x + 4} were obtained after graphing (0, 4).



[Study the diagram carefully, and then apply what you have learned to the job of graphing the function defined by the equation 'y = −2x + 4'.]

SLOPE AND INTERCEPT OF A LINEAR FUNCTION

In graphing linear functions you must have noticed the important roles played by the values of 'a' and of 'b' given by the defining equation 'y = ax + b'. The value of 'b' tells you where the graph of the function crosses the graph of the y-axis. The crossing point is the graph of what ordered pair? What does the value of 'a' tell you? Well, suppose you have already graphed one of the members of the function. Usually this is (0, b), especially if b is an integer, but let's say that you have graphed (p, q), and that you want to proceed from there to the graph of another member of the function. Move your pencil point from the graph of (p, q) along the horizontal grid line which contains the

Answers for Part A [on pages 5-127 and 5-128].

(1) $\frac{2}{3}$, -2, $y = \frac{2}{3}x - 2$

(2) $-\frac{2}{3}$, -2, $y = -\frac{2}{3}x - 2$

(3) 2, 0, $y = 2x$

(4) $\frac{3}{2}$, -4, $y = \frac{3}{2}x - 4$

(5) $-\frac{1}{4}$, 1, $y = -\frac{1}{4}x + 1$

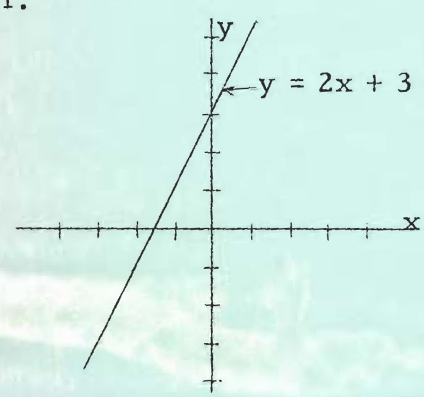
(6) 2, 1, $y = 2x + 1$

In finding the intercept of the function graphed in Exercise 6, students may argue, as on page 5-125, that, since a = 2, if k = 2 then ak = 4, and the intercept, b, is $-3 + 4$.

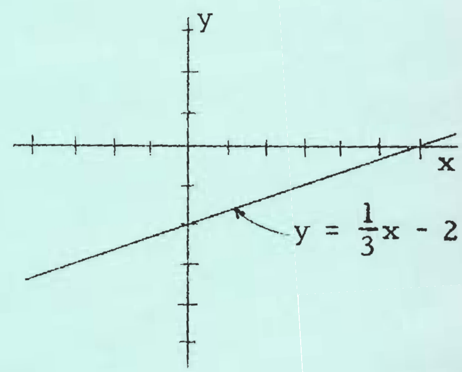
*

Answers for Part B [on page 5-128].

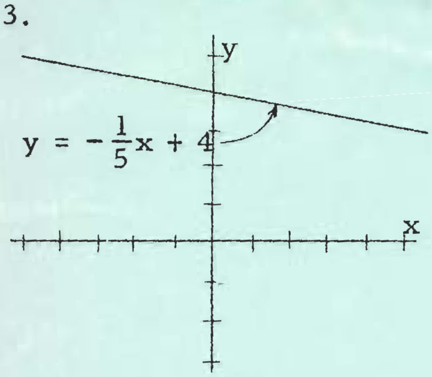
1.



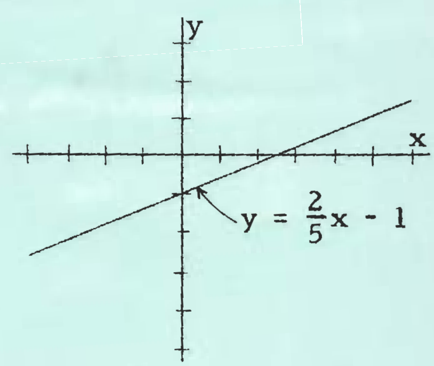
2.



3.



4.



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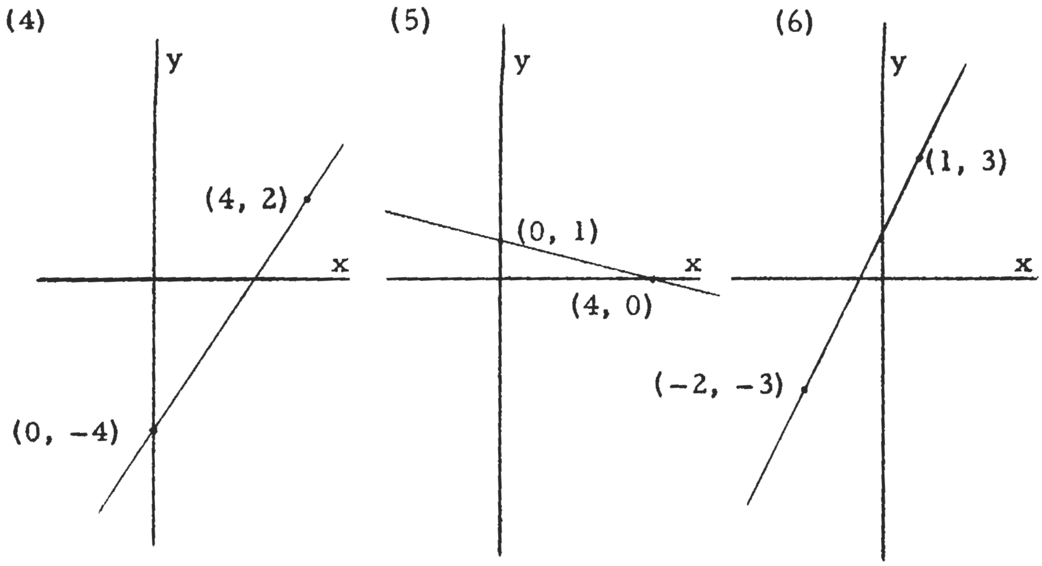


B. In each of the following exercises you are given the intercept and the slope of a linear function. Draw its graph and write its defining equation.

1. intercept, 3; slope, 2
2. intercept, −2; slope, $\frac{1}{3}$
3. intercept, 4; slope, $-\frac{1}{5}$
4. intercept, −1; slope, $\frac{2}{5}$

C. Is there a linear function whose slope is 0? If you were to define 'slope' for constant functions, how would you define it? How about 'intercept' for constant functions?

D. 1. Draw graphs of two linear functions with the same slope, say, $\{(x, y): y = 2x + 3\}$ and $\{(x, y): y = 2x - 4\}$. What is the set of ordered pairs which is the intersection of these functions?

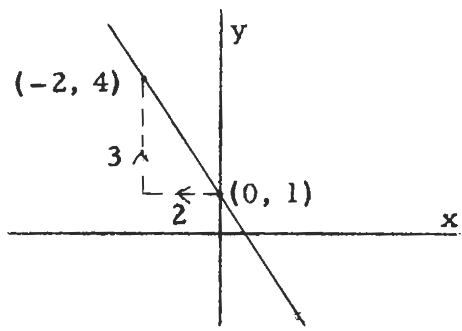
☆ 2. Prove that the intersection of two linear functions with the same slope is the empty set. [Hint. Suppose f_1 and f_2 are linear functions and that their defining equations are '$y = a_1x + b_1$' and '$y = a_2x + b_2$'. Now, each ordered pair in $f_1 \cap f_2$ satisfies both of the defining equations. Suppose (p, q) is such an ordered pair. Then $q = a_1p + b_1$ and $q = a_2p + b_2$. That is, $a_1p + b_1 = a_2p + b_2$. Or, $(a_1 - a_2)p = b_2 - b_1$, ...]

3. Show that if f_1 is a linear function and f_2 is a linear function and f_1 and f_2 have the same slope, it is not necessarily the case that $f_1 \cap f_2 = \emptyset$.

EXERCISES

A. Each of the following exercises contains a graph of a linear function. In each exercise, find the slope and the intercept of the function, and write its defining equation. [Check your work by seeing if the coordinates of the two given points satisfy the defining equation.]

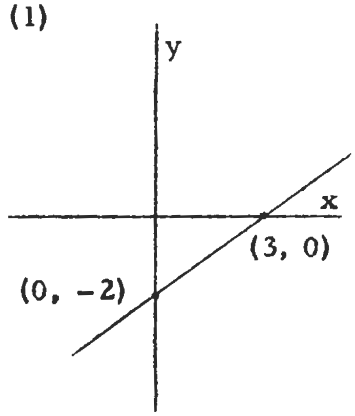
Sample.



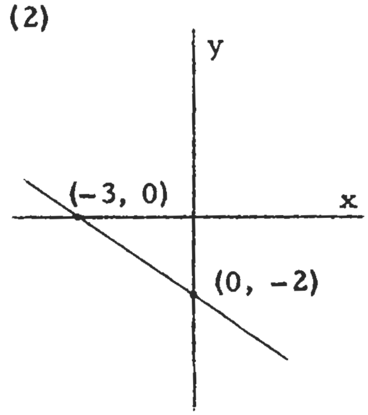
Solution. The graph rises to the left; so, the slope is negative. To get from the graph of (0, 1) to the graph of (−2, 4), you move a distance 2 in the horizontal direction and a distance 3 in the vertical direction. So, to get the vertical move you multiply the horizontal move by 3/2. Hence, the slope is −3/2. The intercept is the second component of the point in which the function intersects the y-axis. This point is (0, 1); so, the intercept is 1. The defining equation is: $y = -\frac{3}{2}x + 1$

Check.	
$4 = -\frac{3}{2} \cdot -2 + 1$?	$1 = -\frac{3}{2} \cdot 0 + 1$?
$3 + 1$	$0 + 1$
4 ✓	1 ✓

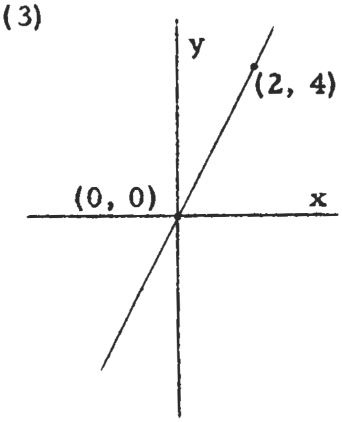
(1)



(2)



(3)



Answers for Part E.

1. 2   2. $\frac{4}{3}$ 3. $\frac{3}{4}$ 4. $-\frac{7}{5}$ 5. -1 6. -1 7. $-\frac{4}{13}$

8. No linear function contains both points.

9. No function [linear or otherwise] contains both points.

☆10. $\frac{y_2 - y_1}{x_2 - x_1}$

*

Answers for Part F.

These exercises foreshadow the discussion starting on page 5-134.

1. 2   2. 2   3. 2

4. Yes [$\{(x, y): y = 2x\}$]. This illustrates the fact that if the linear function containing a first point and a second point has the same slope as the linear function containing the second point and a third point then they are the same linear function.

   No. [A graph shows this quite readily, and, obviously, there are no numbers $a \neq 0$ and b such that $2 = 1a + b$ and $2 = 2a + b$. Or, since a linear function has an inverse, (1, 2) and (2, 2) cannot belong to the same linear function.] This illustrates the fact that the linear function containing two points may differ from the linear function containing two other points even though the functions have the same slope.

*

Explanation asked for on page 5-130, line 8.

There are as many linear functions containing (5, 7) as there are nonzero real numbers because there are as many such functions as there are possible slopes.

*

A quick way to obtain a defining equation for any linear function which contains (5, 7) is to substitute for 'a' in '$y - 7 = a(x - 5)$'.

* * *

Given the point (5, 7), how many constant functions contain this point? From the graphical point of view, this is like asking how many horizontal lines contain the graph of (5, 7). Clearly, there is just one constant function which contains the graph of (5, 7). What is it?

How many linear functions contain the point (5, 7)? Again, from the point of view of graphs, infinitely many oblique lines pass through the graph of (5, 7). [There are just as many as there are nonzero real numbers. Explain.] So, one would expect that infinitely many linear functions contain (5, 7). Let's find a few. Each linear function is defined by an equation of the form:

$$y = ax + b$$

So, a linear function contains (5, 7) if and only if the values of 'a' and 'b' in its defining equation satisfy:

$$(*) \qquad 7 = a5 + b \text{ and } a \neq 0$$

There are many pairs of numbers (a, b) which satisfy (*). For example, (1, 2) does. So does (3, −8). And, so does (80, −393). Thus, some of the linear functions which contain (5, 7) are

$$\{(x, y)\colon y = x + 2\},\ \{(x, y)\colon y = 3x - 8\},\text{ and } \{(x, y)\colon y = 80x - 393\}.$$

How many linear functions contain the points (5, 7) and (3, 11)? From our knowledge of graphs we would say 'just one'. Let's find it. This time we seek values of 'a' and 'b' which satisfy both:

$$(1) \qquad 7 = a5 + b$$

and:

$$(2) \qquad 11 = a3 + b$$

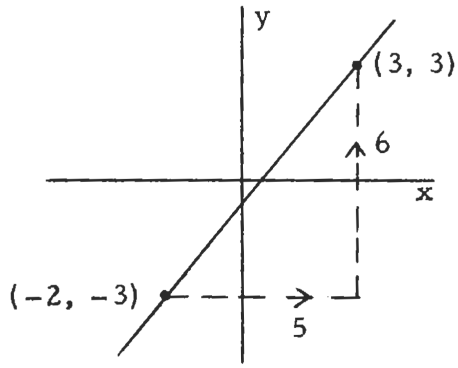
We could find such values by graphing (1) and (2) and estimating the coordinates of the graph of the point in the intersection of the solution sets in (a, b) of (1) and (2). But, there are easier ways. [Before reading any further, try to discover at least one of these easier ways.]

Suppose there is a linear function which contains (5, 7) and (3, 11). From (1) we learn that the intercept of this function is 7 - a5, and from (2) we learn that the intercept is 11 - a3. Since a linear function has

E. In each of the following exercises you are given two points. Find the slope of a linear function [if there is one] which contains these points.

Sample. (-2, -3), (3, 3)

Solution. Let's graph the points and draw the straight line which is determined by the graphs.



We note that the graph rises to the right. So, the slope is positive. To get from (-2, -3) to (3, 3), we move a distance 5 horizontally, and a distance $\frac{6}{5} \cdot 5$ vertically. So, the slope is $\frac{6}{5}$.

1. (3, 4), (5, 8)
2. (0, 0), (3, 4)
3. (0, 0), (4, 3)
4. (2, 1), (-3, 8)
5. (4, 5), (5, 4)
6. (0, 4), (4, 0)
7. $(\frac{1}{2}, 3)$, (-6, 5)
8. (3, 8), (4, 8)
9. (3, 5), (3, 6)

☆ 10. (x_1, y_1), (x_2, y_2), $[x_1 \neq x_2,\ y_1 \neq y_2]$

F. 1. What is the slope of a linear function which contains the points (1, 2) and (3, 6)?

2. What is the slope of a linear function which contains the points (3, 6) and (7, 14)?

3. What is the slope of a linear function which contains the points (2, 2) and (4, 6)?

4. Is there a linear function which contains both the pair of points mentioned in Exercise 1 and the pair mentioned in Exercise 2? Both the pair mentioned in Exercise 1 and the pair mentioned in Exercise 3?

* * *

G. Each exercise lists two ordered pairs of numbers. Find the linear function [if there is one] which contains them.

Sample. (2, 5), (−3, 9)

Solution.

$$\begin{aligned} 5 &= 2a + b \\ 9 &= -3a + b \\ \hline -4 &= 5a \\ -\frac{4}{5} &= a \\ 5 &= 2 \cdot -\frac{4}{5} + b \\ \frac{33}{5} &= b. \end{aligned}$$

$$\{(x, y): \ y = -\frac{4}{5}x + \frac{33}{5}\}$$

1. (7, 2), (13, 5)
2. (8, 6), (0, 0)
3. (4, 1), (−3, 8)
4. (2, 11), (5, 20)
5. (1, −10), (−3, 10)
6. $(\frac{1}{2}, 6)$, (3, 36)
7. (6, 9), (8, 9)
8. (0, 5), (5, 0)
9. (4, 0), (0, 3)
10. (−4, 0), (0, 3)
11. (3, 5), (3, 7)

☆12. $(x_1, y_1), (x_2, y_2)$

H. For each exercise, find the intersection of the given linear functions without drawing their graphs.

Sample. {(x, y): y = 3x + 7}, {(x, y): y = 2x + 5}

Solution. We are looking for numbers x and y which satisfy both:

(1) y = 3x + 7

and: (2) y = 2x + 5.

This problem can be solved by methods used in Part G. For example, if the intersection is not ∅ then there are numbers x and y which satisfy (1) and (2). Such numbers must satisfy: 0 = (3x + 7) - (2x + 5). Etc.
Or, if the intersection is not empty, it contains an ordered pair (p, q) such that q = 3p + 7 and q = 2p + 5. Hence, 3p + 7 = 2p + 5. Etc.

1. {(x, y): y = 4x - 2}, {(x, y): y = 3x + 8}

exactly one intercept [How do you know this?], it must be the case that

$$(3) \qquad 7 - a5 = 11 - a3.$$

From (3) we find that [if there is a linear function which contains (5, 7) and (3, 11)] the slope is −2. But, for a = −2, both (1) and (2) are satisfied by the same value, 17, of 'b'. So, there is a linear function which contains (5, 7) and (3, 11). It is the function

$$\{(x, y): y = -2x + 17\}.$$

Let's consider another way of finding numbers a and b which satisfy: (1) 7 = a5 + b and (2) 11 = a3 + b. Suppose there are such numbers. Then, using the principle that

$$\forall_x \forall_y \forall_u \forall_v \text{ if } x = y \text{ and } u = v \text{ then } x - u = y - v,$$

it follows that these numbers a and b satisfy:

$$(3') \qquad 7 - 11 = (a5 + b) - (a3 + b),$$

that is, they satisfy: −4 = 2a. In other words, if there are numbers a and b such that (1) and (2) then a is −2. But, −2 satisfies (1) and (2) just if b = 17. So, there are numbers a and b which satisfy (1) and (2), and they are −2 and 17, respectively.

There is a third way of solving this problem. We find the possible values of 'a' as before. Then, we multiply to transform (1) and (2) to:

$$(1') \quad 21 = a15 + 3b \qquad \text{and} \qquad (2') \quad 55 = a15 + 5b$$

Next, we use the principle stated above to conclude that if there are numbers a and b such that (1′) and (2′) [or, equivalently, such that (1) and (2)] then these numbers satisfy:

$$(3'') \qquad 21 - 55 = (a15 + 3b) - (a15 + 5b),$$

that is, they satisfy: −34 = −2b. So, if there are numbers a and b which satisfy (1) and (2) they must be −2 and 17, respectively. Substituting in (1) and (2) shows that they do.

The methods illustrated on the preceding pages are methods for solving a system of two equations in two variables. You will learn more about such methods later in this unit.

The displayed principle in the second paragraph on page 5-131 is analogous to that given in Exercise 6 on page 2-66 of Unit 2. Here is a proof.

Suppose that	$x = y$ and $u = v$.
Since	$x - u = x - u$,
it follows that	$x - u = y - u$,
and, hence, that	$x - u = y - v$.
Therefore,	if $x = y$ and $u = v$ then $x - u = y - u$.

Notice that no mathematical principles are involved in the proof. Only logical rules and logical principles are needed. [Compare with the solution for Exercise 6 on TC[2-66]b.]

*

[One purpose of the exercises in Parts G and H is to introduce students to methods of solving two linear equations in two variables [See pages 5-200 ff.]. So, students should use the procedure given in the Sample.]

Answers for Part G [on page 5-132].

1. $\{(x, y): y = \frac{1}{2}x - \frac{3}{2}\}$ 2. $\{(x, y): y = \frac{3}{4}x\}$ 3. $\{(x, y): y = -x + 5\}$

4. $\{(x, y): y = 3x + 5\}$ 5. $\{(x, y): y = -5x - 5\}$ 6. $\{(x, y): y = 12x\}$

7. No linear function contains both ordered pairs.

8. $\{(x, y): y = -x + 5\}$ 9. $\{(x, y): y = -\frac{3}{4}x + 3\}$

10. $\{(x, y): y = \frac{3}{4}x + 3\}$ 11. No function contains both (3, 5) and (3, 7).

☆12. If $x_1 \neq x_2$ and $y_1 \neq y_2$ then $\{(x, y): y = \frac{y_1 - y_2}{x_1 - x_2} \cdot x + (y_1 - x_1 \cdot \frac{y_1 - y_2}{x_1 - x_2})\}$ is a linear function which contains both ordered pairs.
[Simpler descriptions are:

$$\{(x, y): y = \frac{y_1 - y_2}{x_1 - x_2} \cdot x + \frac{x_1y_2 - x_2y_1}{x_1 - x_2}\},$$

and:

$$\{(x, y): \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}\}.]$$

If $x_1 = x_2$ and $y_1 = y_2$ [so that, contrary to the instructions, there are not two ordered pairs listed] then, for each $a \neq 0$, $\{(x, y): y = ax + (y_1 - ax_1)\}$ is a linear function which contains the given ordered pair. In any other case [$x_1 = x_2$ and $y_1 \neq y_2$, or $x_1 \neq x_2$ and $y_1 = y_2$], either no function or, at least no linear function contains both ordered pairs.

Actually, the slope of the linear function which contains the points (0, 0) and (2, 5) is less than that of the linear function which contains (2, 5) and (5, 13). So, the rectangle is not completely covered by the pieces of the square. The part not covered is [enclosed by] a parallelogram, one of whose diagonals is a diagonal of the rectangle, and whose area-measure is 1.

*

Answers for Part J [on page 5-133].

1. Yes [{(x, y): y = x}]
2. Yes [{(x, y): y = 2x + 2}]
3. Yes [{(x, y): $y = \frac{1}{2}x + \frac{17}{2}$ }]
4. Yes [{(x, y): y = −x + 6}]
5. No
6. Yes [{(x, y): $y = -\frac{1}{5}x + 1$}]
7. No
8. No

*

<u>Quiz.</u>

1. What are the slope and the intercept of {(x, y): 2x - 4y = 5}?
2. Write the defining equation of the linear function which contains the points (3, 8) and (−2, 4). [y = ___x + ___]
3. Find the ordered pair which belongs to the functions {(x, y): 4x + y = 7} and {(x, y): y = 2x - 5}.
4. Write the defining equation of the linear function whose slope is 7 and which contains the point (−2, 8).
5. Find two ordered pairs which belong to the inverse of the linear function which contains (0, 8) and (8, 0).
6. Write a brace-notation name for the constant function which contains the point (−3, −2).

*

Answers for Quiz.

1. $\frac{1}{2}$; $-\frac{5}{4}$ 2. $y = \frac{4}{5}x + \frac{28}{5}$ 3. (2, −1) 4. y = 7x + 22

5. (0, 8) and (8, 0) [or any other two points in the linear function {(x, y): y + x = 8}]

6. {(x, y): y = −2}

Answers for Part H [which begins on page 5-132].

1. $\{(10, 38)\}$ 2. $\{(-1, -4)\}$ 3. $\{(18, 2)\}$ 4. $\emptyset$

5. $\{(-3, -3)\}$ 6. $\{(1, 1)\}$ 7. $\{(1, 5)\}$ 8. $\{(6, 2)\}$

9. $\{(-3, 9)\}$ 10. $\emptyset$ 11. $\{(x, y): y = -\frac{1}{2}x + \frac{9}{2}\}$

[Students are expected to invent techniques for Exercises 6 - 11.]

*

Answers for Part I.

1. $\{(x, y): y = \frac{4}{3}x + \frac{13}{3}\}$ 2. There is no such linear function.

3. The slope of the linear function containing (-89, -93) and (67, -8) is 85/156, and that of the linear function containing (67, -8) and (15, -37) is 87/156. So, no linear function contains all three ordered pairs. However, each of the sets of three ordered pairs listed below is a subset of some linear function.

$\{(-89, -93), (67, -8), (15, -109/3)\}$

$\{(-89, -95), (67, -8), (15, -37)\}$

$\{(-89, -93), (67, -9), (15, -37)\}$

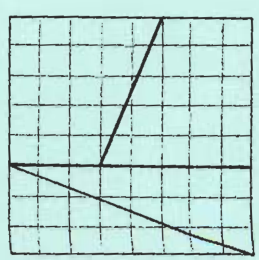
$\{(-89, -93), (67, -8), (1171/85, -37)\}$

$\{(-89, -93), (482/7, -8), (15, -37)\}$

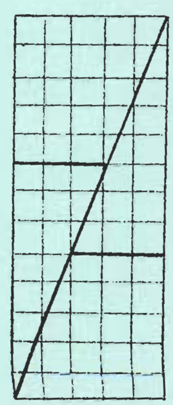
$\{(-89, -2269/29), (67, -8), (15, -37)\}$

*

Here is a familiar puzzle whose solution is like that of Exercise 3 of Part I. Apparently, a square can be cut into 4 pieces which, when reassembled, make up a rectangle whose area-measure is greater than that of the original square. [Solution on next page.]



$8 \times 8 = 64$



$5 \times 13 = 65$

* * *

You will recall from the definition on page 5-91 that a variable quantity h is a function of a variable quantity g if and only if there is a function f such that $h = f \circ g$. We say that h is a <u>linear</u> <u>function</u> of g if one such function f is a linear function. Since the domain of each linear function f is the set of <u>all</u> real numbers, if h is a linear function of g and $\mathcal{R}_g$ consists only of real numbers then $\mathcal{D}_h = \mathcal{D}_g$. So, if g is a real-valued function, h is a linear function of g if and only if $\mathcal{D}_h = \mathcal{D}_g$ and there are real numbers $a \neq 0$ and b such that, for each $e \in \mathcal{D}_g$, $h(e) = ag(e) + b$. Another way of putting it is to say that h is a linear function of a real-valued function g if and only if $\mathcal{D}_h = \mathcal{D}_g$ and the set of all ordered pairs $(g(e), h(e))$, for $e \in \mathcal{D}_g$, is a subset of some linear function.

<u>Example.</u> Suppose $g = \{(1, 12), (2, 15), (3, 18), (4, 17)\}$
and $h = \{(1, 25), (2, 31), (3, 37), (4, 35)\}$.
Is h a linear function of g?

Since the set of ordered pairs $(g(e), h(e))$, for all $e \in \mathcal{D}_g$, is

$$\{(12, 25), (15, 31), (18, 37), (17, 35)\}$$

the question really asks whether there is a linear function of which this set of ordered pairs is a subset. One way of getting a clue to the answer is to see whether an oblique straight line can be drawn through the graphs of these ordered pairs. Another way which does not involve the inaccuracies of graphing is to try to find numbers $a \neq 0$ and b such that all four ordered pairs satisfy the equation:

$$y = ax + b.$$

If you are successful, you will not only know that h is a linear function of g but you will also have a defining equation for the linear function in question. [If you can prove that there are no numbers $a \neq 0$ and b which fit the conditions then you will know that h is not a linear function of g. Would that mean that h is not a function of g?]

There is another way of answering this question which does not require finding the linear function, itself. To understand this other way, we need to look at an important property of linear functions.

Suppose f is a linear function which contains the two points (x_1, y_1) and (x_2, y_2). [Since these are two points, and since f is a function, it

2. $\{(x, y): y = -3x - 7\}$, $\{(x, y): y = 4x\}$
3. $\{(x, y): y = \frac{x - 2}{8}\}$, $\{(x, y): y = \frac{2x - 30}{3}\}$
4. $\{(x, y): y = 2 + 5x\}$, $\{(x, y): y = 5x + 1\}$
5. $\{(x, y): y - 3x = 6\}$, $\{(x, y): y - 2x = 3\}$
6. $\{(x, y): 3x + y = 4\}$, $\{(x, y): y = 2x - 1\}$
7. $\{(x, y): 3y - 6x = 9\}$, $\{(x, y): y = x + 4\}$
8. $\{(x, y): 3x - 5y = 8\}$, $\{(x, y): 8y - x = 10\}$
9. $\{(x, y): 5y + 4x = 33\}$, $\{(x, y): 3y - 7x = 48\}$
10. $\{(x, y): 4y - 3x = 12\}$, $\{(x, y): 6x = 15 + 8y\}$
11. $\{(x, y): 2y + x = 9\}$, $\{(x, y): 3y + x = 18 - y - x\}$

<u>I.</u> Each exercise lists three ordered pairs of numbers. Find the linear function [if there is one] which contains them.

1. (2, 7), (5, 11), (8, 15) 2. (3, 5), (8, 4), (14, 2)

*

3. Predict whether a straight line can be drawn through the graphs of (−89, −93), (67, −8), and (15, −37). If your prediction is that it cannot, change a component of just one of the ordered pairs so that a straight line can be drawn through their graphs.

<u>J.</u> Each exercise lists a set of ordered pairs. In each case, tell whether there is a linear function of which the given set is a subset.

1. {(1, 1), (2, 2), (3, 3), (4, 4)}
2. {(0, 2), (1, 4), (2, 6), (3, 8)}
3. {(−3, 7), (1, 9), (5, 11), (9, 13)}
4. {(3, 3), (2, 4), (0, 6), (−5, 11)}
5. {(1, 8), (2, 11), (3, 13), (4, 16)}
6. {(10, −1), (15, −2), (25, −4), (30, −5)}
7. $\{(2, \frac{1}{2}), (3, \frac{1}{3}), (4, \frac{1}{4}), (5, \frac{1}{5})\}$
8. {(4, 6), (5, 8), (7, 12), (9, 8)}

Suppose we know that $\frac{y_5 - y_4}{x_5 - x_4}$ = the slope of f. Does it follow that (x_4, y_4) and (x_5, y_5) belong to f? Explain.]

We have shown that if f is a linear function and (x_1, y_1) and (x_2, y_2) are two ordered pairs such that $\{(x_1, y_1), (x_2, y_2)\} \subseteq f$, a third ordered pair $(x_3, y_3) \in f$ $[x_1 \neq x_2 \neq x_3]$ if and only if

$$\frac{y_3 - y_2}{x_3 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

This result provides us with a quick test for determining whether [or not] a given set of ordered pairs is a subset of a linear function. Let's take the example we started with on page 5-134. Is there a linear function f such that

$$\{(12, 25), (15, 31), (18, 37), (17, 35)\} \subseteq f?$$

Compute the y-difference and the x-difference for the first two ordered pairs, (12, 25) and (15, 31).

$$y_2 - y_1 = 31 - 25 = 6$$

$$x_2 - x_1 = 15 - 12 = 3$$

Then compute the y-difference and the x-difference for the second and third ordered pairs, (15, 31) and (18, 37).

$$y_3 - y_2 = 37 - 31 = 6$$

$$x_3 - x_2 = 18 - 15 = 3$$

Since the ratio of the y-difference to the x-difference in the first case is the same as the ratio of the y-difference to the x-difference in the second case, we know that there is a linear function, say, p, which contains the first three ordered pairs. Now, we find the <u>difference-ratio</u> for the third and fourth ordered pairs.

$$y_4 - y_3 = 35 - 37 = -2$$

$$x_4 - x_3 = 17 - 18 = -1$$

$$\frac{y_4 - y_3}{x_4 - x_3} = \frac{-2}{-1}$$

Is the difference-ratio here the same as the one in the case of the second and third ordered pairs? In other words, does $6/3 = -2/-1$? Yes! So, there is a linear function, say, q, which contains the second third, and fourth ordered pairs, But, there is only one linear function follows that $x_1 \neq x_2$. Since f is a linear function, and since $x_1 \neq x_2$, it follows that $y_1 \neq y_2$.] Now, if a and b are the slope and intercept of f, then

$$y_2 = ax_2 + b.$$

and

$$y_1 = ax_1 + b.$$

From this it follows that

$$\begin{aligned} y_2 - y_1 &= (ax_2 + b) - (ax_1 + b) \\ &= ax_2 - ax_1 \\ &= a(x_2 - x_1). \end{aligned}$$

So,

$$\frac{y_2 - y_1}{x_2 - x_1} = a.$$

In other words, if a linear function f contains the two points (x_1, y_1) and (x_2, y_2) then

$$\frac{y_2 - y_1}{x_2 - x_1} = \text{the slope of f.}$$

[Did you discover this result when you worked the exercises in Part E?]

Now, suppose (x_3, y_3) is a third point such that

$$\frac{y_3 - y_2}{x_3 - x_2} = \text{the slope of f.}$$

Does it follow that $(x_3, y_3) \in f$? The answer is 'yes'. Let's see why.

$$\frac{y_3 - y_2}{x_3 - x_2} = a$$

if and only if $\quad y_3 - y_2 = a(x_3 - x_2). \quad [x_3 \neq x_2]$

that is, if and only if $\quad y_3 = a(x_3 - x_2) + y_2.$

Since $(x_2, y_2) \in f$, we know that $y_2 = ax_2 + b$. So,

$$y_3 = a(x_3 - x_2) + y_2$$

if and only if

$$\begin{aligned} y_3 &= a(x_3 - x_2) + (ax_2 + b) \\ &= ax_3 - ax_2 + ax_2 + b \\ &= ax_3 + b, \end{aligned}$$

that is, if and only if $\quad (x_3, y_3) \in f.$

So, we have shown not only that $(x_3, y_3) \in f$ if $\frac{y_3 - y_2}{x_3 - x_2} = a$, but also that $(x_3, y_3) \in f$ <u>only</u> $\frac{y_3 - y_2}{x_3 - x_2} = a$.

Corrections. In the last line at the bottom of page 5-135, insert an 'if' after 'only'.

In the first line on page 5-136, insert a '[' in front of 'suppose'.

Answer to bracketed question at top of page 5-136: No.

[Ask students to compare Exercises 2 and 10 of Part G on page 5-132. If the pairs given in Exercise 2 are used for (x_4, y_4) and (x_5, y_5) respectively, the slope is the same as if the pairs given in Exercise 10 are used. Yet, the pairs given in Exercise 10 do not belong to the same linear function as do the pairs given in Exercise 2. Also, recall Exercise 4 at bottom of page 5-129.]

Answers for Part K [on pages 5-137 and 5-138].

1. Yes 2. Yes 3. No 4. No 5. Yes 6. No

*

Answers for Part L [on page 5-138].

Explanation: If g and h are linear functions then their common domain is the set of all real numbers; so, $\mathcal{D}_h = \mathcal{D}_g$. Also, by Exercise 2 of Part B on page 5-121, g has an inverse, and g^{-1} is a linear function. By Exercise 3 [same Part], since both h and g^{-1} are linear functions, so is $h \circ g^{-1}$.

1. $f = \{(x, y): y = -2x - 9\}$
2. $f = \{(x, y): 9x + 4y = 13\}$
3. $f = \{(x, y): x = 15y + 2\}$
4. $f = \{(x, y): 2x + 15y = 23\}$

Here are two ways of solving Exercise 3, above. [Each of the other three exercises can be solved in the same two ways.]

Since $g = \{(x, y): y = 3x - 4\}$, $g^{-1} = \{(x, y): y = \frac{x + 4}{3}\}$.

Hence, $h \circ g^{-1} = \{(x, y): \frac{x + 4}{3} = 5y + 2\}$.

This function, $\{(x, y): x = 15y + 2\}$, is a linear function f such that $h = f \circ g$.

An <u>alternate method</u> of solution makes use of the fact that, since g and h are linear functions, there is a linear function f such that $h = f \circ g$. That is, there are numbers $a \neq 0$ and b such that, for each x, $h(x) = a \cdot g(x) + b$; that is, such that

$$\forall_x \ \frac{x}{5} - \frac{2}{5} = a(3x - 4) + b.$$

By inspection, $a = \frac{1}{15}$ and, since, in that case, $-\frac{2}{5} = -\frac{4}{15} + b$, it follows that $b = -\frac{2}{15}$. So,

$$f = \{(x, y): y = \frac{1}{15}x - \frac{2}{15}\} = \{(x, y): x = 15y + 2\}.$$

4. $g = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$,
$h = \{(1, 4), (2, 9), (3, 16), (4, 25)\}$.

5. $g = \{(x, y): y = x^2 + 3\}$
$h = \{(x, y): y = x^2\}$

6. $g = \{(x, y): y = x + 3\}$
$h = \{(x, y): y = x^2\}$

L. You have seen [page 5-93] that if g has an inverse and $\mathcal{D}_h \subseteq \mathcal{D}_g$ then there is a function f such that $h = f \circ g$. In fact, $h \circ g^{-1}$ is one such function. Now, if g and h are linear functions, g has an inverse, $\mathcal{D}_h = \mathcal{D}_g$, and $h \circ g^{-1}$ is a linear function. [Explain.] Hence:

If g and h are linear functions then h is a linear function of g.

For each of Exercises 1-4, find a linear function f such that $h = f \circ g$.

1. $g = \{(x, y): y = x - 7\}$
$h = \{(x, y): y = 5 - 2x\}$

2. $g = \{(x, y): x + 3y = 6\}$
$h = \{(x, y): 3x - 4y = 5\}$

3. $g = \{(x, y): y = 3x - 4\}$
$h = \{(x, y): x = 5y + 2\}$

4. $g = \{(x, y): y = 5x - 6\}$
$h = \{(x, y): 2x + 3y = 7\}$

*

5. If f is a function which has an inverse then

$$f^{-1} \circ f = \{(x, y) \in \mathcal{D}_f \times \mathcal{D}_f: y = x\}.$$

Explain the remark just made, and use it in proving that if a function h is a linear function of a real-valued function g, then g is a linear function of h.

* * *

Strictly speaking [as, up to now, we have been], since the values of each linear function are real numbers, a function h which is a linear function of a function g must also be real-valued. However, it is convenient to say that a function h whose values are numbers of arithmetic is a linear function of another such function g when $\mathcal{D}_h = \mathcal{D}_g$ and there exist numbers $a \neq 0$ and b such that, for each $e \in \mathcal{D}_g$, $h(e) = ag(e) + b$. This amounts to pretending that each value of h or g is the corresponding nonnegative real number.

* * *

which contains the second and third ordered pairs. Hence, p = q, and there is a linear function of which the given set of four pairs is a subset.

* * *

K. Each exercise gives a variable quantity g and a variable quantity h. In each case, tell whether h is a linear function of g.

Sample 1. g = {(a, 5), (b, −2), (c, −9), (d, −10), (e, −13)}
h = {(a, 18), (b, 4), (c, −10), (d, −12), (e, −18)}

Solution. $\mathcal{D}_h = \mathcal{D}_g$. So, far, so good. Now, consider the set of ordered pairs of corresponding values of h and g, and see whether this set is a subset of a linear function. Let's list the ordered pairs of corresponding values in a table and then compute the g(e)-differences and the h(e)-differences.

	g(e)	h(e)	
	5	18	
−7			−14
	−2	4	
−7			−14
	−9	−10	
−1			−2
	−10	−12	
−3			−6
	−13	−18	

Now, compute the difference-ratio for each pair of ordered pairs. It is 2 in each case. So, the set of ordered pairs (g(e), h(e)), for all $e \in \mathcal{D}_g$, is a subset of a linear function. Thus, h is a linear function of g.

[g is also a linear function of h because the difference-ratio is the same for the pairs of ordered pairs (h(e), g(e)). What is the difference-ratio in this case?]

1. g = {(Al, 9), (Bill, 4), (Carl, 6), (Dora, 12)},
   h = {(Al, 40), (Bill, 15), (Carl, 25), (Dora, 55)}.

2. g = {(1, 3), (2, 4), (3, 5), (4, 6), (5, 7)},
   h = {(1, 1), (2, −1), (3, −3), (4, −5), (5, −7)}.

3. g = {(1, 7), (2, 5), (3, 8), (4, 9)},
   h = {(1, 22), (2, 16), (3, 26), (4, 28)}.

Correction. On page 5-139, part (c) of Exercise 2 should be starred.

5. If f has an inverse then, for each $x \in \mathcal{D}_f$, $(f(x), x) \in f^{-1}$. So, for each $x \in \mathcal{D}_f$, $[f^{-1} \circ f](x) = f^{-1}(f(x)) = x$. In particular, $\mathcal{D}_f \subseteq \mathcal{D}_{f^{-1} \circ f}$ and since, for each function g, $\mathcal{D}_{g \circ f} \subseteq \mathcal{D}_f$, $\mathcal{D}_{f^{-1} \circ f} = \mathcal{D}_f$. Summarizing, for each $x \in \mathcal{D}_{f^{-1} \circ f}$, $[f^{-1} \circ f](x) = x$. So, $f^{-1} \circ f = \{(x, y) \in \mathcal{D}_f \times \mathcal{D}_f : y = x\}$.

Now, if g is a real-valued function, f is a linear function, and $h = f \circ g$, then $f^{-1} \circ h = f^{-1} \circ [f \circ g] = [f^{-1} \circ f] \circ g = g$, by the result just proved. Since f^{-1} is a linear function, g is a linear function of h.

[Essentially, we have proved a more general result: If f has an inverse, $h = f \circ g$, and $\mathcal{R}_g \subseteq \mathcal{D}_f$, then $g = f^{-1} \circ h$.]

*

The remark at the foot of page 5-138 is placed there because of its application to Exercises 1 and 2 of Part M on page 5-139. In general, measures are numbers of arithmetic and to be able to say that variable quantities whose values are measures are linear functions of one another, one must confuse numbers of arithmetic with the corresponding nonnegative real numbers. We shall consistently do this in the next section, which begins on page 5-140. The justification for doing so is discussed in the COMMENTARY for Unit 3 on TC[3-55]. Briefly, doing so simplifies the algebraic manipulations [one does not need to restrict subtraction to the case where the minuend exceeds the subtrahend]; and, the fact that the system of numbers of arithmetic is isomorphic, with respect to addition and multiplication, to the system of nonnegative real numbers, ensures that the results obtained by this procedure can be properly interpreted.]

*

Answers for Part M [on page 5-139].

1. (a) Yes (b) $\{(x, y): y = \pi x\}$

2. (a) No (b) ___ ☆(c) Yes; the linear function in question is $\{(x, y): y = x\}$

3. (a) $A = 2B - 1$ (b) $A = 5B - 1$ (c) $A = (B - 1)/3$
   (d) $A = -B$ (e) $A = (B - 6)/5$ (f) $A = (B - 6)/5$
   (g) $A = -B + 9$ (h) $A = B^2$ (i) $A = 24/B$

*

Quiz.

1. Is the perimeter (P) of an equilateral triangle a linear function of its side-measure (s)? If not, tell why. If so, describe the linear function f such that $P = f \circ s$.

2. Suppose that A and B are variable quantities where

$$A = \{(e_1, 5), (e_2, 6), (e_3, 7), (e_4, 8)\}$$

and $B = \{(e_1, 9), (e_2, 12), (e_3, 15), (e_4, 18)\}$.

Write a formula for B in terms of A.

3. If f is a linear function which contains (−2, 5) and (8, 7) then f also contains (13, ___?___).

4. Describe the linear function r such that $p = r \circ q$ where $q = \{(x, y): x + 2y = 6\}$ and $p = \{(x, y): 3x - y = 5\}$.

*

Answers for Quiz.

1. Yes; $f = \{(x, y): y = 3x\}$

2. $B = 3A - 6$ 3. 8 4. $r = \{(x, y): y = -6x + 13\}$

5.08 Applications of linear functions.--One very common application of linear functions involves the notion of variable quantities which are proportional to each other. The word 'proportional' is used a great deal in everyday life.

> The city of Zabranchburg will increase its educational budget by 20% in the next school year. However, local school taxes will not increase in the same proportion because most of the increase in expenses will be taken care of by state funds.
>
> The elongation of a stretched spring is proportional to the stretching force.
>
> Most of these recipes can be used in preparing food for a larger number of persons than indicated just by increasing the ingredients a proportional amount.

The word 'proportional' has been given a very precise meaning in mathematics. Before we state a definition, investigate your present understanding of the word by reading each of the following statements and deciding whether it is true or whether it is false.

True	False	
.....	.....	1. The perimeter of a square is proportional to the measure of a side.
.....	.....	2. The weight of a boy is proportional to his height.
.....	.....	3. The volume of a gas sample under a given pressure is proportional to the temperature.
.....	.....	4. The weight of a steel beam of a given cross sectional area is proportional to the length of the beam.
.....	.....	5. The amount of juice in an orange is proportional to the radius of the orange.
.....	.....	6. The distance traveled by a car operating at a given speed is proportional to the time of travel.

M. 1. (a) Is the circumference of a circle a linear function of its diameter?

(b) If it is, find the linear function in question.

2. (a) Is the area-measure of a square a linear function of its side-measure?

(b) If it is, find the linear function in question.

(c) Is the area-measure of a square a linear function of the square of its side-measure?

3. The table in each exercise lists the pairs of corresponding values of the variable quantities A and B. Write a formula for A in terms of B.

(a)

B(e)	A(e)
3	5
4	7
5	9
6	11
7	13

(b)

B(e)	A(e)
−2	−11
0	−1
2	9
4	19
6	29

(c)

A(e)	B(e)
5	16
7	22
10	31
14	43
19	58

(d)

B(e)	A(e)
12	−12
−15	15
−12	12
15	−15
0	0

(e)

A(e)	B(e)
10	56
9	51
8	46
7	41
6	36

(f)

A(e)	B(e)
0.1	6.5
0.2	7
0.3	7.5
0.4	8
0.5	8.5

(g)

B(e)	A(e)
2	7
5	4
8	1
9	0
12	−3

(h)

B(e)	A(e)
3	9
4	16
6	36
8	64
12	144

(i)

B(e)	A(e)
3	8
4	6
6	4
8	3
12	2

[Supplementary exercises are in Part S, pages 5-267 through 5-269.]

Answers for True-False questions on pages 5-140 and 5-141.

1. T   2. F   3. T [if 'temperature' refers to absolute temperature; otherwise, not.]

4. T   5. F   6. T   7. F [if he has a fixed charge for making a call, in addition to a charge depending on time worked.]

8. F   9. T   10. F

Some students may feel that one variable quantity is proportional to another if they "increase together"--that is, if "when one increases, so does the other". The only answer to such feelings is to say that people who use words properly don't use this word so loosely. The matter should be cleared up by the definition on page 5-141, so, don't spend more time than absolutely necessary discussing questions 1 - 10.

*

Notice that, in the definition of proportionality, 'k' is a pronumeral, and the factor of proportionality is a number. This is consistent with the equation '$P(e) = kQ(e)$'. On the other hand, if one writes '$P(e) = [kQ](e)$' or, as we later do, '$P = kQ$', then, in each of these equations 'k' names the constant variable quantity whose domain is the common domain of P and Q and whose range consists of the factor of proportionality.

*

The perimeter of a rectangle whose length-measure is 5 is given by the formula:

$$P = 10 + 2w$$

If the width-measure of such a rectangle is 1 then its perimeter is 12; if the width-measure is 2, the perimeter is 14. Since, if $12 = k \cdot 1$ then $k = 12$, and if $14 = k \cdot 2$ then $k = 7$, and, since $12 \neq 7$, there is no number k such that, for each rectangle r, $P(r) = k \cdot w(r)$. So, the perimeter of a rectangle whose length-measure is 5 is not proportional to its width-measure. [However, P is proportional to $w + 5$. Compare with True-False question 3. The volume, V, of a gas sample is not proportional to its centigrade temperature, t; but, V is proportional to $t + 273$. In general, if a variable quantity P is a linear function of some variable quantity Q then P is proportional to the sum of Q and some constant.]

*

Answers for Part A [on page 5-142].

1. Yes; 3   2. No   3. Yes; −3   4. No

5. Yes; 5/3   6. Yes; 1   7. Yes; 3/8   8. No

EXERCISES

A. In each of the following exercises there is a table which lists the pairs of corresponding values of variable quantities P and Q. In each case, tell whether P is proportional to Q, and if it is, give the factor of proportionality.

Sample.

P(e)	11	8	7	5	9	6
Q(e)	22	16	14	10	18	13

Solution. We note that each value of P is one half the corresponding value of Q, except for the value 6 of P. So, P is not proportional to Q.

Answer. P is not proportional to Q because $9 = \frac{1}{2} \cdot 18$, but $6 \neq \frac{1}{2} \cdot 13$.

1.

P(e)	Q(e)
9	3
12	4
15	5
-21	-7
0	0
-9	-3

2.

Q(e)	P(e)
3	5
4	6
5	7
-7	-5
0	2
-3	-1

3.

Q(e)	P(e)
-3	9
-2	6
-1	3
0	0
1	-3
2	-6

4.

P(e)	Q(e)
4	-2
1	-1
0	0
1	1
4	2
9	3

5.

Q(e)	P(e)
3	5
4	20/3
6	10
-6	-10
0	0
-12	-20

6.

Q(e)	P(e)
11	11
6	6
-2	-2
5	5
1	1
3	3

7.

P(e)	3	3	3	3	3	3	3	3	3
Q(e)	8	8	8	8	8	8	8	8	8

8.

Q(e)	2	3	4	5	6	7	8	9	10
P(e)	6	9	12	15	18	22	24	27	30

.... 7. A television repairman's labor charge is proportional to the time he works.

.... 8. The compound interest on a sum of money deposited in a bank is proportional to the length of time the sum of money is left on deposit.

.... 9. The annual simple interest on a loan of a given principal is proportional to the interest rate.

.... 10. The number of years a student has been in school is proportional to the number of years in his age.

Take the first of these statements, the one which tells you that the perimeter of a square is proportional to the measure of a side. In brief, it says that the variable quantity P is proportional to the variable quantity s. According to the definition we shall soon give, this statement is true. If we double the side-measure of a square, we get a new square whose perimeter is double the perimeter of the given square. If we increase the side-measure by adding 3 to it, the perimeter increases proportionately by 12. On the other hand, the last statement in the list is false. A student who is 8 years old has been in school, say, 2 years. But, a student who is only twice as old will have been in school five times as long.

For variable quantities P and Q,
P is proportional to Q
if and only if
$\mathcal{D}_P = \mathcal{D}_Q$, and there is a number $k \neq 0$ such that,
for each $e \in \mathcal{D}_Q$, $P(e) = kQ(e)$.
[We say that P is proportional to Q with <u>factor of proportionality k</u>.]

Illustrate this definition in the case of P and s for squares. Show that P is proportional to s and tell the factor of proportionality. Also, show that s is proportional to P, and tell the factor of proportionality in that case.

Use the definition to show that the perimeter of a rectangle whose length-measure is 5 is <u>not</u> proportional to the width-measure of the rectangle.

Answers for Part B.

1. $c = \{(x, y) \in \text{Circles} \times N\colon \text{ y is the circumference of x}\}$,

   $r = \{(x, y) \in \text{Circles} \times N\colon \text{ y is the radius of x}\}$; so, $\mathcal{D}_c = \mathcal{D}_r$.

   From geometry, we know that there is a number $2\pi \neq 0$ such that, for each $e \in \mathcal{D}_r$, $c(e) = 2\pi \cdot r(e)$. Hence, the circumference of a circle is proportional to its radius, and the factor of proportionality is 2π.

2. Yes. [The factor of proportionality is 6.]

3. Let Q be $\{(e, s) \in \text{Squares} \times N\colon \text{ s is the side-measure of e}\}$, and P be $\{(e, A) \in \text{Squares} \times N\colon \text{ A is the area-measure of e}\}$.

   Suppose that e_1 is a square whose side-measure is 1, and e_2 is a square whose side-measure is 2. If P <u>is</u> proportional to Q, then there must exist a number $k \neq 0$ such that $P(e_1) = kQ(e_1)$ and $P(e_2) = kQ(e_2)$. If this is so, then,

   since $P(e_1) = 1$ and $Q(e_1) = 1$, $1 = k \cdot 1$ and $k = 1$,

   and, since $P(e_2) = 4$ and $Q(e_2) = 2$, $4 = k \cdot 2$ and $k = 2$.

   Since $1 \neq 2$, it follows that there exists no number $k \neq 0$ such that, for each $e \in \mathcal{D}_Q$, $P(e) = kQ(e)$.

4. $V = \{(x, y) \in \text{Cones} \times N\colon$ x is a circular cone whose base has radius-measure 3, and y is the volume-measure of x$\}$,

   $h = \{(x, y) \in \text{Cones} \times N\colon$ x is a circular cone whose base has radius-measure 3, and y is the height-measure of x$\}$;

   so, $\mathcal{D}_V = \mathcal{D}_h$.

   From geometry, we know that there is a number $3\pi \neq 0$ such that, for each $e \in \mathcal{D}_h$,

   $$V(e) = \frac{1}{3}\pi \cdot 3^2 \cdot h(e) = 3\pi \cdot h(e).$$

   Hence, V is proportional to h, with factor of proportionality 3π.

*

Answers for Part C.

1. Yes 2. No 3. Yes 4. No 5. No 6. Yes

D. Complete these sentences.

1. If P is proportional to Q with factor of proportionality k then Q ________ ['is' or 'is not'] proportional to P. [If Q is proportional to P, what is the factor of proportionality?]

2. If P is proportional to Q then P ________ ['is' or 'is not'] a linear function of Q. [If P is a linear function of Q, what is the slope and what is the intercept of the linear function?]

3. If P is proportional to Q with factor of proportionality k then, for each $e \in \mathcal{D}_Q$ such that $Q(e) \neq 0$,

$$\frac{P(e)}{Q(e)} = \_\_\_\_\_ \text{ and } \frac{Q(e)}{P(e)} = \_\_\_\_\_.$$

4. (a) If P is proportional to Q with factor of proportionality k then, for each $e \in \mathcal{D}_Q$ such that $Q(e) \neq 0$, $[\frac{P}{Q}](e) =$ ________.

   (b) Is $\frac{P}{Q}$ a variable quantity?

E. Suppose Y is proportional to X and that none of the values of X is 0. Suppose, further, that y_1 and x_1 are corresponding values of Y and X. So, if k is the factor of proportionality then $y_1 = kx_1$. Similarly, if y_2 and x_2 are other corresponding values of Y and X then $y_2 = kx_2$.

1. If $y_1 = 8$ and $x_1 = 2$ then $k =$ ________.
2. If $y_1 = 12$ and $x_1 = 10$ then $k =$ ________.
3. If $x_2 = 3.5$ and $k = 4$ then $y_2 =$ ________.
4. If $y_2 = 360$ and $k = 12$ then $x_2 =$ ________.
5. If $y_1 = 16$, $x_1 = 4$, and $y_2 = 28$ then $x_2 =$ ________.
6. If $y_1 = 3$, $x_1 = 7$, and $x_2 = 21$ then $y_2 =$ ________.
7. If $y_2 = 35$, $y_1 = 5$, and $x_2 = 14$ then $x_1 =$ ________.
8. Prove that none of the values of Y is 0.
9. If $\frac{y_1}{x_1} = 19$, $\frac{y_2}{x_2} =$ ________.
10. Prove that $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.
11. Prove that $\frac{x_1}{y_1} = \frac{x_2}{y_2}$.

B. 1. Show that the circumference of a circle is proportional to its radius, and give the factor of proportionality.

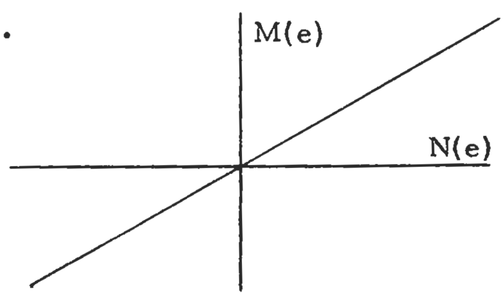
2. Is the perimeter of a regular hexagon proportional to its side-measure?

3. Show that the area-measure of a square is not proportional to its side-measure.

4. You know that the volume-measure of a circular cone is $\frac{1}{3}\pi r^2 h$. Show that the volume-measure of a circular cone whose base has radius-measure 3 is proportional to the height-measure of the cone, and give the factor of proportionality.

C. Here are some charts which show the graphs of sets of ordered pairs of corresponding values of variable quantities M and N. [Assume that, in each case, the entire set has been graphed.] For each exercise, tell whether M is proportional to N.

1.



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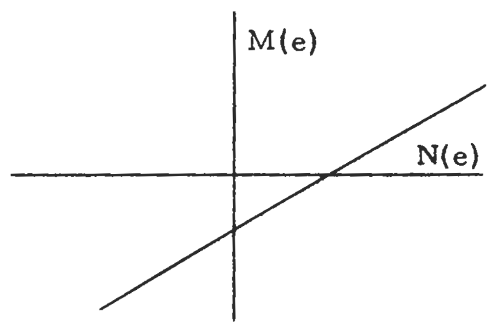

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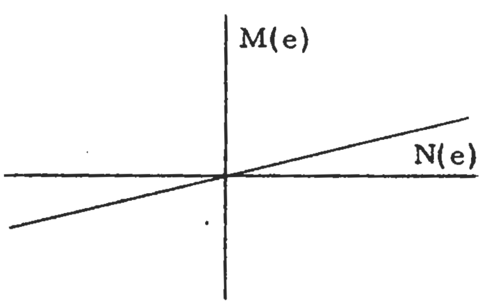

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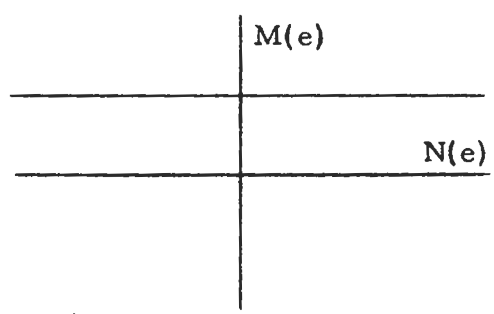
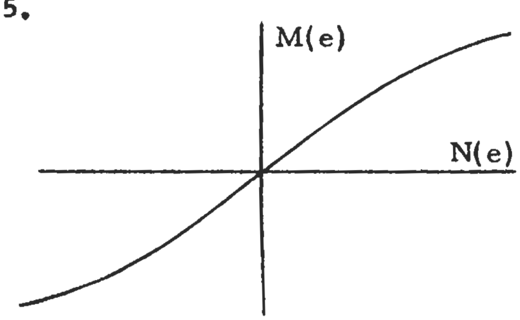
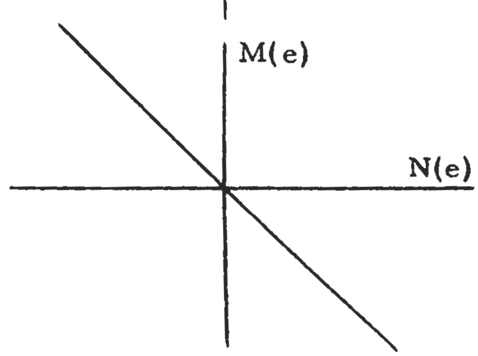


5.



6.



Correction. On page 5-145, in line 4 of Exercise 19, change 'm + q' to 'm + p'.

Answers for Part D.

1. is $[\frac{1}{k}]$ 2. is [slope = factor of proportionality, intercept = 0]

3. k; $\frac{1}{k}$ 4. (a) k (b) Yes, a constant variable quantity

*

Answers for Part E [on pages 5-144 and 5-145].

1. 4 2. 1.2 3. 14 4. 30 5. 7 6. 9 7. 2

8. If, for some $e \in \mathcal{D}_Y$, $Y(e) = 0$, then, for this e, $X(e) = \frac{0}{k} = 0$. But, [by assumption] 0 is not a value of X. Hence, there is no $e \in \mathcal{D}_Y$ such that $Y(e) = 0$--that is, 0 is not a value of Y.

   [An alternative proof. We are given that Y is proportional to X, and that no value of X is 0. So, there exists a $k \neq 0$ such that,

   $$\text{for each } e \in \mathcal{D}_X,\ Y(e) = kX(e).$$

   By the 0-product theorem, if $k \neq 0$ and $X(e) \neq 0$ then $kX(e) \neq 0$. Hence, $Y(e) \neq 0$.]

9. 19 10. $\frac{y_1}{x_1} = k = \frac{y_2}{x_2}$ 11. $\frac{x_1}{y_1} = \frac{1}{k} = \frac{x_2}{y_2}$ 12. 3

13. By Exercise 11, $\frac{x_1}{y_1} = \frac{x_2}{y_2}$. So, [by the MTP, etc.] $x_1y_2 = x_2y_1$. ["Product of the means equals product of the extremes."]

14. $\frac{3}{2}$ 15. 9 16. $\frac{17}{3}$

17. By Exercise 10, $\frac{y_1}{x_1} = \frac{y_2}{x_2}$. 18. $\frac{3}{8}$

   So, $\frac{y_1}{x_1} + 1 = \frac{y_2}{x_2} + 1$. [ATP]

   Hence, $\frac{y_1 + x_1}{x_1} = \frac{y_2 + x_2}{x_2}$.

★19. (a) $y_1 = kx_1$ and $y_2 = kx_2$

So, $y_1 + y_2 = kx_1 + kx_2$,

that is, $y_1 + y_2 = k(x_1 + x_2)$.

Hence, $\frac{y_1 + y_2}{x_1 + x_2} = k$.

But, $\frac{y_1}{x_1} = k$.

Therefore, $\frac{y_1 + y_2}{x_1 + x_2} = \frac{y_1}{x_1}$.

(b) Suppose that, for some $a \neq 0$, (m, n) and (p, q) belong to $\{(x, y): y = ax\}$. Then,

$n = am$ and $q = ap$.

So, $n + q = am + ap$

and $n + q = a(m + p)$.

Hence, $(m + p, n + q) \in \{(x, y): y = ax\}$.

*

On page 5-146, the answer to the bracketed question preceding 'EXERCISES' is 'Yes'.

*

Answers for Part A [on pages 5-146 and 5-147].

1. 15	2. $\frac{27}{7}$	3. 10	4. 8
5. $\frac{45}{2}$	6. 40	7. $\frac{4}{15}$	8. $\frac{15}{7}$
9. 0.75	10. 6, −6	11. 4, −4	12. 8, −8

*

Quiz

1. Suppose that P and Q are variable quantities such that

$$P = \{(p_1, 8), (p_2, 7), (p_3, 7), (p_4, 9)\}$$ [illegible]

and $Q = \{(q_1, 4), (q_2, 9), (q_3, 12), (q_4, 10)\}$. [illegible]

(a) Is Q a linear function of P? If yes, tell why. If no, determine the linear function f such that Q = f(P).

(b) Is Q proportional to P? If yes, tell why. If no, give the factor of proportionality.

2. Suppose that A is proportional to B and that (a_1, b_1) and (a_2, b_2) are ordered pairs of corresponding values of A and B. If $a_1 = 12$, $b_1 = 16$, and $b_2 = 4$, what is a_2?

3. Prove that the area-function (A) of a rectangle is not proportional to its height-function (h).

Answers for Quiz.

1. (a) [illegible]

[illegible]

2. [illegible]

3. For each rectangle r, A(r) = h(r) · b(r). Consider the rectangles r_1 and r_2 where r_1 is a 3 by 4 rectangle and r_2 is a 3 by 10 rectangle. Then $A(r_1) = 3 \cdot 4$ and $A(r_2) = 3 \cdot 10$ [illegible]

Quiz.

1. Suppose that P and Q are variable quantities such that

$$P = \{(e_1, 3), (e_2, 5), (e_3, 7), (e_4, 9)\}$$

and $Q = \{(e_1, 6), (e_2, 9), (e_3, 12), (e_4, 15)\}$.

   (a) Is Q a linear function of P? If not, tell why. If so, describe the linear function f such that $Q = f \circ P$.

   (b) Is Q proportional to P? If not, tell why. If so, give the factor of proportionaltiy.

2. Suppose that A is proportional to B and that (a_1, b_1) and (a_2, b_2) are ordered pairs of corresponding values of A and B. If $a_1 = 7$, $b_1 = 12$, and $a_2 = 3$, what is b_2?

3. Prove that the area-measure (A) of a rectangle is not proportional to its length-measure (ℓ).

*

Answers for Quiz.

1. (a) Yes; $f = \{(x, y): y = \frac{3}{2}x + \frac{3}{2}\}$

   (b) No; $Q(e_1) = 2 \cdot P(e_1)$, $Q(e_2) = \frac{9}{5} \cdot P(e_2)$, and $2 \neq \frac{9}{5}$. [So, there is no number $k \neq 0$ such that for each $e \in \mathcal{D}_P$, $Q(e) = kP(e)$.]

2. 36/7

3. For each rectangle r, $A(r) = w(r) \cdot \ell(r)$. Consider the rectangles r_1 and r_2 where r_1 is a 3 by 5 rectangle and r_2 is a 4 by 10 rectangle. Then, $A(r_1) = 3 \cdot \ell(r_1)$ and $A(r_2) = 4 \cdot \ell(r_2)$, and $3 \neq 4$.

computed by dividing X-values by Y-values. Sometimes we eliminate the fixed ratio from consideration by deriving from (3) the equation:

$$(4) \qquad \frac{y_1}{x_1} = \frac{y_2}{x_2}.$$

Sentence (4) is also called a proportion. Since we shall have no need to use sentences like (2), our use of the word 'proportion' will be restricted to sentences like (4). Thus, a proportion is an equation each of whose sides is a fraction.

Suppose Y is proportional to X and y_1, y_2, y_3, ... are values of Y which correspond with the nonzero values x_1, x_2, x_3, ... of X. Then, we can write:

$$\frac{y_1}{x_1} = \frac{y_2}{x_2}, \quad \frac{y_2}{x_2} = \frac{y_3}{x_3}, \quad \ldots .$$

This conjunction of proportions is usually abbreviated to:

$$(5) \qquad \frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} = \ldots$$

[Does it follow from (5) that $\frac{y_1}{x_1} = \frac{y_3}{x_3}$?]

EXERCISES

A. Solve these proportions.

Sample. $\frac{3}{a} = \frac{5}{7}$

Solution.

$$\frac{3}{a} = \frac{5}{7}$$

$$(7a)\frac{3}{a} = \frac{5}{7}(7a)$$

$$21 = 5a$$

$$4.2 = a$$

The root of the proportion is 4.2.

1. $\frac{6}{x} = \frac{2}{5}$
2. $\frac{3}{y} = \frac{7}{9}$
3. $\frac{8}{5} = \frac{16}{b}$
4. $\frac{2}{1} = \frac{x}{4}$
5. $\frac{x}{9} = \frac{5}{2}$
6. $\frac{8}{x} = \frac{1}{5}$
7. $\frac{5a}{3} = \frac{4}{9}$
8. $\frac{6}{7b} = \frac{2}{5}$
9. $\frac{4.01}{2x} = \frac{8.02}{3}$
10. $\frac{3}{x} = \frac{x}{12}$
11. $\frac{y}{2} = \frac{8}{y}$
12. $\frac{4}{z} = \frac{z}{16}$

12. If $x_1 y_2 = 12$ and $x_2 = 4$ then $y_1 =$ ______.

13. Prove that $x_1 y_2 = x_2 y_1$.

14. If $\frac{x_1}{x_2} = \frac{2}{3}$ then $\frac{y_2}{y_1} =$ ______.

15. If $x_1 = 6x_2$ and $y_1 = 54$ then $y_2 =$ ______.

16. If $y_1 + x_1 = 17$, $x_1 = 3$, and $x_2 = 6$ then $\frac{y_2 + x_2}{x_2} =$ ______.

17. Prove that $\frac{y_1 + x_1}{x_1} = \frac{y_2 + x_2}{x_2}$. [<u>Hint</u>. Prove, first, that $\frac{y_1}{x_1} + 1 = \frac{y_2}{x_2} + 1$.]

18. If $\frac{y_1}{x_1} = \frac{3}{8}$, $y_1 = 12$, and $x_2 = 40$ then $\frac{y_1 + y_2}{x_1 + x_2} =$ ______.

☆ 19. (a) Prove that $\frac{y_1 + y_2}{x_1 + x_2} = \frac{y_1}{x_1}$ [provided $x_1 \neq -x_2$].

(b) Prove that, for all m, n, p, and q, if (m, n) and (p, q) belong to a linear function whose intercept is 0 then so does (m + q, n + q).

PROPORTIONS

Suppose the variable quantity Y is proportional to the variable quantity X with factor of proportionality k, and that y_1 and x_1 are corresponding values and that y_2 and x_2 are other corresponding values. It is customary to say, in this case, that

(1) y_1, x_1, y_2, and x_2 are <u>in proportion</u>.

The ancient Greeks expressed (1) by writing:

(2) $$y_1 : x_1 :: y_2 : x_2.$$

[which is read as 'y_1 is to x_1 as y_2 is to x_2']. Sentence (2) is sometimes called <u>a proportion</u>. Since $y_1 = kx_1$ and $y_2 = kx_2$, it follows that if neither x_1 nor x_2 is 0 then

(3) $$\frac{y_1}{x_1} = k \text{ and } \frac{y_2}{x_2} = k.$$

This latter fact is often expressed by saying that if Y is proportional to X with factor of proportionality k then all pairs of corresponding nonzero values have the same ratio [or: a <u>fixed</u> ratio], the ratio being k if it is computed by dividing Y-values by X-values, or $\frac{1}{k}$ if it is

13. $\frac{29}{4}$ 14. $-\frac{5}{2}$ 15. 10

16. (4, 9) [You might remind students of the convention mentioned in Unit 4 regarding the solution of sentences which contain the pronumerals 'x' and 'y'. It is customary, when giving solutions to sentences involving more than one variable, to list the components of the solution in the order which corresponds to the alphabetical order of the variables.]

17. (2, 10) 18. $(\frac{32}{3}, 3, \frac{20}{3})$

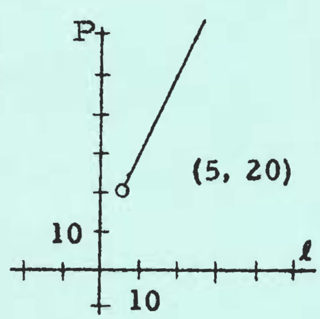
*

Answers for Part B [on pages 5-147, 5-148, 5-149].

1. $\frac{P_1}{s_1} = \frac{P_2}{s_2}$, or: $\frac{P_1}{P_2} = \frac{s_1}{s_2}$, or: $\frac{s_1}{P_1} = \frac{s_2}{P_2}$, or: $\frac{P_2}{P_1} = \frac{s_2}{s_1}$

2. A = 5B 3. Yes; the factor of proportionality is $2\sqrt{2}$, so '2.8', or even '3' would be an acceptable answer.

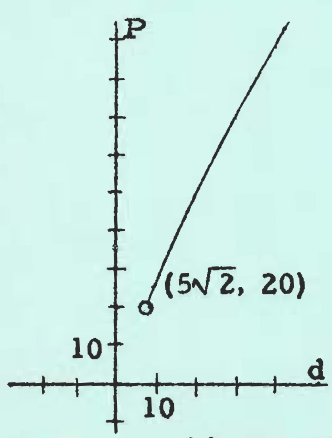
4. (a)



The appropriate formula for part (a) is, of course, '$P = 10 + 2\ell$', and the graphs of the ordered pairs found by students should fit somewhere on the half-line in the accompanying sketch.

(b) Yes (c) No

(d)



The appropriate formula for part (d) is '$P = 10 + 2\sqrt{d^2 - 25}$' and the graphs of the ordered pairs found by students should fit somewhere on the piece of a hyperbola shown in the accompanying sketch. Students' graphs are very likely to suggest that P is a linear function of d, and to leave it somewhat in doubt whether P is proportional to d. In fact, P is not a linear function of d [see formula above], and so, in particular, is not proportional to d. This can be shown by graphing the students' data, using a larger scale on the horizontal axis than on the vertical axis.

(e) No; no (f) No; no

5. Draw five rectangles such that the length of each is twice its width. Make measurements and computations and record the data in a table like this:

w(r)	ℓ(r)	d(r)	P(r)

(a) Graph the ordered pairs (w(r), d(r)), for all of the five rectangles r.

(b) Does the graph suggest that d is a linear function of w? That d is proportional to w?

(c) Do you think that P is a linear function of w? That P is proportional to w? [If the latter, what is the factor of proportionality?]

(d) Do you think that P is a linear function of d? That P is proportional to d? [If the latter, what is the factor of proportionality?]

(e) Is the domain of the variable quantities in Exercise 5 the same as the domain of those in Exercise 4?

6. Imagine an automobile traveling on a highway at a constant [or: fixed] rate of 50 miles per hour for a period of 10 hours. Consider the variable quantities d and t where d is the distance traveled and t is the time which has elapsed since the beginning of the period. [You might think of the domain of d and t as a set of observations. For the first observation, o_1, $t(o_1) = 0$ and $d(o_1) = 0$. The second observation, o_2, might be made 1.5 hours later in which case $t(o_2) = 1.5$ and $d(o_2) = 75$.]

(a) Make as many observations as you wish [after all, this is only pretending]. For each observation, o, plot the ordered pair (t(o), d(o)). [A common way of saying this is: <u>plot d against t</u>.]

(b) Now, imagine the same situation as before except that the constant rate is 40 miles per hour. On the same chart as in (a), plot d against t.

13. $\frac{x - 5}{2} = \frac{9}{8}$ 14. $\frac{5}{3 - x} = \frac{10}{11}$ 15. $\frac{7}{3} = \frac{2x + 1}{9}$

16. $\frac{2}{x} = \frac{6}{12} = \frac{y}{18}$ 17. $\frac{x}{3} = \frac{y}{15} = \frac{18}{27}$ 18. $\frac{8}{x} = \frac{y}{4} = \frac{6}{8} = \frac{5}{z}$

B. 1. The perimeter (P) of an equilateral triangle is proportional to its side-measure (s). Suppose t_1 and t_2 are two equilateral triangles, and that $P(t_1) = P_1$, $s(t_1) = s_1$, $P(t_2) = P_2$, and $s(t_2) = s_2$. Write a proportion involving 'P_1', 's_1', 'P_2', and 's_2'.

2. Suppose the variable quantity A is proportional to the variable quantity B, and that, for some x in the domain of B, $A(x) = 10$ and $B(x) = 2$. Write a formula for A in terms of B.

3. On squared paper make drawings of five squares, compute the perimeter, and measure the diagonal of each. Plot the five pairs (d_1, P_1), (d_2, P_2), ..., (d_5, P_5). Is P proportional to d? If so, use the graph to estimate the factor of proportionality.

4. On squared paper make drawings of five rectangles each with width-measure 5. Measure the length and the diagonal, and compute the perimeter of each, recording your data in a table like this:

$\ell(r)$	$d(r)$	$P(r)$

(a) Graph the ordered pairs $(\ell(r), P(r))$, for all of the five rectangles r.

(b) Does the graph suggest that P is a linear function of ℓ?

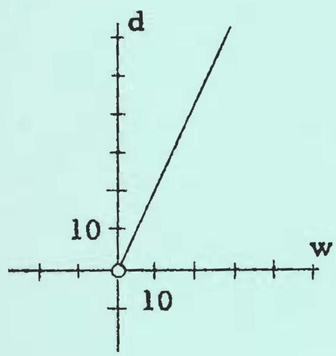
(c) Does the graph suggest that P is proportional to ℓ? [If so, estimate the factor of proportionality.]

(d) Graph the ordered pairs $(d(r), P(r))$, for all of the five rectangles r.

(e) Does the graph suggest that P is a linear function of d? That P is proportional to d?

(f) Do you think that d is a linear function of ℓ? That d is proportional to ℓ?

5. (a)



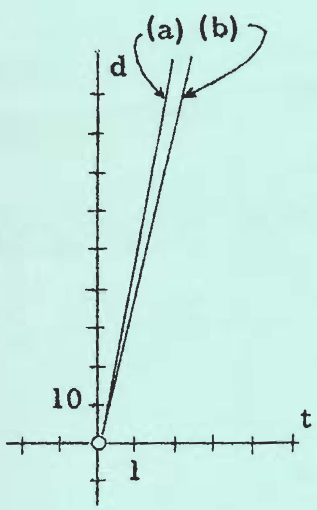
Since $\ell = 2w$, and $d = \sqrt{\ell^2 + w^2}$, the appropriate formula for part (a) is '$d = \sqrt{5}w$'. The graphs of students' ordered pairs should fit somewhere on the half-line shown in the accompanying sketch.

(b) Yes; yes (c) Yes; yes; [6]

(d) Yes; yes; [$\frac{6}{\sqrt{5}}$, approximately $\frac{8}{3}$]

(e) No

6.



The appropriate formulas for parts (a) and (b) are, of course, '$d = 50t$' and '$d = 40t$', respectively.

[Note that the axes are graphed with different scales.]

(c) In the case of the more rapidly moving car, the graph is steeper.

(d) 260 miles

(e) Yes; in part (a), the factor of proportionality is 50, and in part (b), 40.

*

Answers for Part C [on page 5-149].

1. $3 + \frac{2}{B}$; not proportional 2. 5; proportional; 5

[Note that in the answer for Exercise 1, '3' and '2' name constants. In the answer for Exercise 2, the first '5' names a constant, but the second '5' names a number.]

3. π; proportional; π 4. πx; not proportional

5. 9×10^{16}; proportional; 9×10^{16} 6. $\frac{3}{Q^2}$; not proportional

[Notice that if M and N are variable quantities, with the same domain,

rence in

w

whose ratio is a nonzero constant k, then, <u>if 0 is not a value of N</u>, M is proportional to N. For, in this case, it follows from the principle of quotients that M = kN. But, if N(e), say, is 0, the formula '$\frac{M}{N} = k$' tells us nothing about M(e). In order, in this case, to be sure that M is proportional to N we must also know that M has the value 0 for each argument for which N has the value 0.]

*

<u>Quiz.</u>

1. Solve these proportions.

   (a) $\frac{x}{15} = \frac{9}{5}$ (b) $\frac{12}{x - 2} = \frac{4}{7}$ (c) $\frac{2}{x} = \frac{x}{18}$

2. For each of the following formulas, tell whether A is proportional to B. [Assume that all other letters in the formulas stand for nonzero constants.]

   (a) $A = kB$ (b) $A = g^2B$ (c) $A = kB + m$

   (d) $B = sA$ (e) $A = k\sqrt{B}$ (f) $p^2A = q^2B$

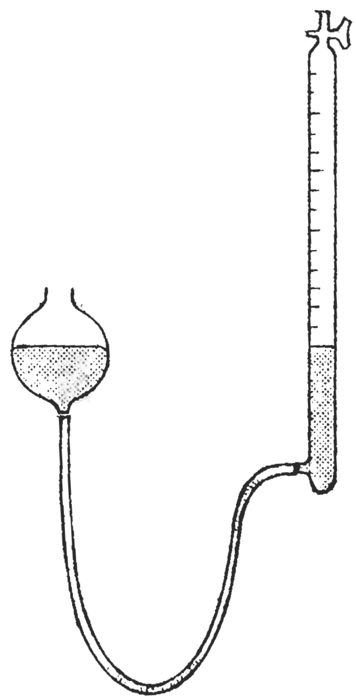
*

Answers for Quiz.

1. (a) 27 (b) 23 (c) 6, −6
2. (a) Yes (b) Yes (c) No (d) Yes (e) No (f) Yes

INVERSE PROPORTIONALITY

Robert Boyle, a seventeenth century English scientist, studied the effect of change of pressure on the volume of a sample of gas which is kept at a constant temperature. A type of experiment he may have carried out is illustrated in the diagram. A sample of air is confined



in a glass tube, and the pressure on this sample of air is changed by lowering or raising the bulb of mercury. It seems clear that as the pressure is increased by raising the bulb, the volume occupied by the sample of air is decreased. Boyle noticed that all gases behaved alike when subjected to changes in pressure, and his experiments gave him reason to state the following principle. [The principle is known as <u>Boyle's law</u>.]

> The volume occupied by a sample of gas at constant temperature <u>is inversely proportional to</u> the pressure.

At the top of the next page is a table which lists pairs of corresponding values of the variable quantities P and V for a sample of helium at 0° centigrade.

Correction. In line 19
. . . a physics or chemistry test-

Students should see that, since $20.65 < 22.37$ and $1.0852 > 1.0020$, $20.65/1.0852 < 22.37/1.0020$. So, they should see, without computing, that the ratios for the first two pairs of values listed in the table are different.

*

When completed, the table at the bottom of page 5-151 should look like the following:

b(r)	24	20	16	12	10	8	6	5	4	3	2	1
h(r)	1	1.2	1.5	2	2.4	3	4	4.8	6	8	12	24

*

Line 6 from bottom of page 5-152. This principle can be proved in an entirely analogous manner to that in which the principle on page 5-131 is proved. See the COMMENTARY for 5-131.

P	V
1.0852	20.65
1.0020	22.37
0.8067	27.78
0.6847	32.73
0.5387	41.61
0.3550	63.10
0.1937	115.65

What does 'is inversely proportional to' mean? It certainly doesn't mean what 'is proportional to' does. You can see that if you check the ratios for the first two pairs of values. Is 20.65/1.0852 the same as 22.37/1.0020? [Try to answer this question without doing any computing.] But, if, instead of considering the ratios for pairs of corresponding values, we consider the products, we find that each product is 22.41 correct to the nearest hundredth. So, Boyle's law tells us that, for a fixed temperature, the product of corresponding values of the variable quantities P and V is fixed. That is, PV is a constant. [Actually, Boyle's law does not hold at very high pressures or at very low temperatures. See an encyclopedia or a physics of chemistry textbook for more information about Boyle's law.]

We now give a definition of inverse proportionality.

For variable quantities M and N,
M is inversely proportional to N
if and only if
$\mathscr{D}_M = \mathscr{D}_N$, and there is a number $k \neq 0$ such that,
for each $e \in \mathscr{D}_N$, $M(e) \cdot N(e) = k$.

[k is the factor of inverse proportionality.]

There are many examples of variable quantities which are inversely proportional. For instance, consider the set of all rectangles r whose area-measure is 24, and the variable quantities b and h [base and height] whose domain is this set of rectangles. Complete the table below which lists just a few of the pairs of corresponding values of b and h.

b(r)	24			12		8	6	5	4	3	2	1
h(r)		1.2	1.5		2.4							

In order to complete the table, you probably used the idea that, for each rectangle r in the domain of b and h, $b(r) \cdot w(r) = 24$. And, this is the same as saying that b is inversely proportional to h [and, also, that h is inversely proportional to b].

Another example of inverse proportionality is the situation involving the set of all annual loans which bear \$60 in interest. The variable quantities are the principal (p) loaned and the annual interest rate (r). A principal of \$1000 would require a rate of 6% to yield \$60 in interest in one year. But, a principal of \$2000 would require a rate of 3%. So, there is a $k \neq 0$ such that, for each loan ℓ in the domain of p and r, $p(\ell) \cdot r(\ell) = k$. In this case, $k = 60$.

You may wonder how the notion of proportion which deals with fixed ratios is tied up with that of inverse proportionality which deals with fixed products. Recall how proportions came into the picture when we discussed <u>direct</u> proportionality. [We shall say 'direct proportionality' now instead of just 'proportionality' since we have two kinds of proportionality to keep in mind.] We said that M is [directly] proportional to N if and only if $\mathcal{D}_M = \mathcal{D}_N$, and there is a $k \neq 0$ such that, for each $e \in \mathcal{D}_N$, $M(e) = kN(e)$. From this it follows that if m_1, n_1, and m_2, n_2 are pairs of corresponding values of M and N, then

$$(1) \qquad m_1 = kn_1 \text{ and } m_2 = kn_2.$$

From (1) [assuming that neither n_1 nor n_2 is 0] we derive the proportion:

$$\frac{m_1}{n_1} = \frac{m_2}{n_2}$$

We can derive other proportions from (1) and the assumption about n_1 and n_2. In particular, by the principle:

$$\forall_x \forall_y \forall_{u \neq 0} \forall_{v \neq 0} \text{ if } x = y \text{ and } u = v \text{ then } \frac{x}{u} = \frac{y}{v},$$

it follows that

$$\frac{m_1}{m_2} = \frac{kn_1}{kn_2},$$

or, more simply, that

$$(1') \qquad \frac{m_1}{m_2} = \frac{n_1}{n_2}.$$

Now, consider the case of inverse proportionality. By definition, M is inversely proportional to N if and only if $\mathcal{D}_M = \mathcal{D}_N$, and there is a $k \neq 0$ such that, for each $e \in \mathcal{D}_N$, $M(e) \cdot N(e) = k$. As before, if m_1, n_1, and m_2, n_2 are corresponding values of M and N [none of the values of M and N is 0; explain], then

(2) $$m_1 n_1 = k \text{ and } m_2 n_2 = k.$$

From this and the principle for real numbers mentioned on page 5-152 it follows that

$$\frac{m_1 n_1}{m_2 n_2} = \frac{k}{k}.$$

This can be transformed to:

$$\frac{m_1 n_1}{m_2 n_2} \cdot \frac{n_2}{n_1} = \frac{k}{k} \cdot \frac{n_2}{n_1}$$

Simplifying, we get a proportion:

(2′) $$\frac{m_1}{m_2} = \frac{n_2}{n_1}$$

Compare (1′) and (2′).

In the exercises which follow we sometimes use the phrases <u>varies directly as</u> and <u>varies inversely as</u>. These mean precisely the same things as 'is directly proportional to' and 'is inversely proportional to'. Similarly, cases of direct proportionality are sometimes referred to as cases of <u>direct variation</u>, and cases of inverse proportionality as cases of <u>inverse variation</u>.

EXERCISES

<u>A.</u> Suppose Y and X are variable quantities such that Y is inversely proportional to X. $y_1, y_2, y_3, \ldots$ are values of Y which correspond with the values $x_1, x_2, x_3, \ldots$ of X. Fill the blanks.

<u>Sample.</u> If $y_1 = 5$, $x_1 = 8$, and $y_2 = 4$ then $x_2 =$ ______.

<u>Solution.</u> There are several ways of handling this problem. One way is to find k, the factor of inverse proportionality. k is the product of each pair of corresponding values of Y and X. So,

$$k = y_1 x_1 = 5 \cdot 8 = 40.$$

Therefore,

$$y_2x_2 = 40$$

$$4x_2 = 40$$

$$x_2 = 10.$$

A second way is to use the fact that

$$y_1x_1 = y_2x_2.$$

Substituting gives us: $5 \cdot 8 = 4x_2$

We transform to get: $x_2 = 10$

A third way is to set up a proportion:

$$\frac{y_1}{y_2} = \frac{x_2}{x_1},$$

substitute in it:

$$\frac{5}{4} = \frac{x_2}{8},$$

and then solve it.

1. If $y_1 = 8$ and $x_1 = 2$ then $k =$ ________.
2. If $y_1x_1 = 48$ and $x_2 = 3$ then $y_2 =$ ________.
3. If $y_1 = 5$, $x_1 = 9$, and $y_2 = 3$ then $x_2 =$ ________.
4. If $y_1 = 2$, $x_1 = 10$, and $x_2 = 5$ then $y_2 =$ ________.
5. If $x_1 = 3x_2$ and $y_2 = 12$ then $y_1 =$ ________.
6. If $\frac{x_1}{x_2} = \frac{3}{4}$ then $\frac{y_1}{y_2} =$ ________.
7. If $y_1 = 3.2$, $y_2 = 0.32$, and $x_1 = 2.8$ then $x_2 =$ ________.

__B.__ 1. Turn to the table you completed on page 5-151, and plot b against h using as many more ordered pairs (h(r), b(r)) as you need to get a "smooth" graph.

2. From the appearance of your graph and from what you know about these variable quantities, would you say that b is a function of h? If so, give a function f such that $b = f \circ h$.

3. Do you think that b is a linear function of h?

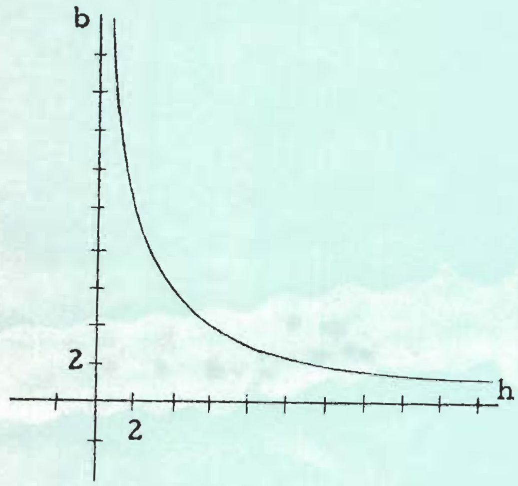
Answers for Part A [which begins on page 5-153].

1. 16 2. 16 3. 15 4. 4 5. 4 6. $\frac{4}{3}$ 7. 28

*

Answers for Part B [on pages 5-154 and 5-155].

1.



2. Yes; $f = \{(x, y): \; y = \frac{24}{x}\}$

3. No

5. If M varies directly as N and $0 \notin \mathcal{R}_N$ then M varies inversely as 1/N, and the factor of inverse variation is the same as the factor of direct variation. [If $0 \in \mathcal{R}_N$, then $\mathcal{D}_{1/N} \neq \mathcal{D}_N$, so $\mathcal{D}_{1/N} \neq \mathcal{D}_M$. Hence, in this case, M is not inversely proportional to 1/N because M and 1/N do not have the same domain.]

6. If M varies inversely as N then $\mathcal{D}_M = \mathcal{D}_N$, and there is a constant variable quantity $k \neq 0$ such that $MN = k$. In this case, $0 \notin \mathcal{R}_N$, so $\mathcal{D}_{1/N} = \mathcal{D}_N = \mathcal{D}_M$, and $M = k \cdot \frac{1}{N}$. So, by definition, M varies directly as 1/N.

*

Answers for Part D [on pages 5-155 and 5-156].

1. Since F varies directly as W, there is a number $k \neq 0$ such that, for each body e, $F(e) = k \cdot W(e)$. For a body e_o such that $W(e_o) = 10$, $F(e_o) = 3$. So, $3 = k \cdot 10$, and $k = 0.3$. Hence, if $W(e) = 25$ then $F(e) = 0.3 \cdot 25 = 7.5$. The frictional force on a 25-lb. block is 7.5 pounds.

[<u>Alternative</u> <u>solution</u>. Since F varies directly as W and $0 \notin \mathcal{R}_W$, for any bodies e_1 and e_2,

$$\frac{F(e_2)}{W(e_2)} = \frac{F(e_1)}{W(e_1)}.$$

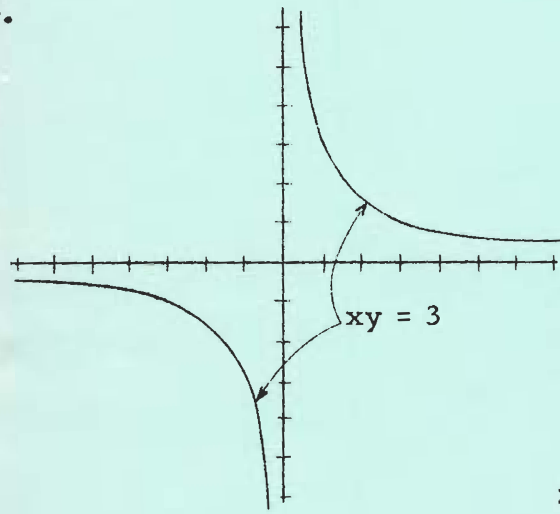
If $W(e_1) = 10$ then $F(e_1) = 3$. So, if $W(e_2) = 25$,

$$\frac{F(e_2)}{25} = \frac{3}{10}.$$

Hence, if $W(e_2) = 25$ then $F(e_2) = 7.5$.]

☆ 4. Three ordered pairs which belong to each function which b is of h are (1, 24), (2, 12), and (3, 8). There is just one linear function which contains the first two of these three points, and its slope is −12. The linear function which contains the second and third points has slope −4. So, no linear function contains all three points, and b is not a linear function of h.

5.



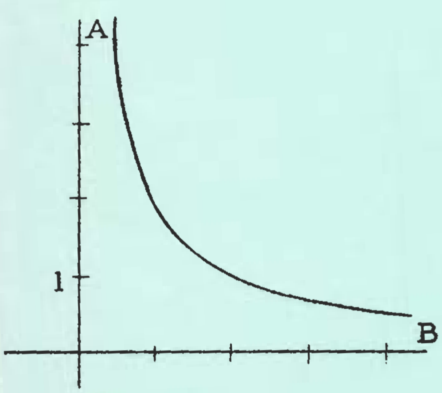
The domain and range of this function are the set of nonzero real numbers.

*

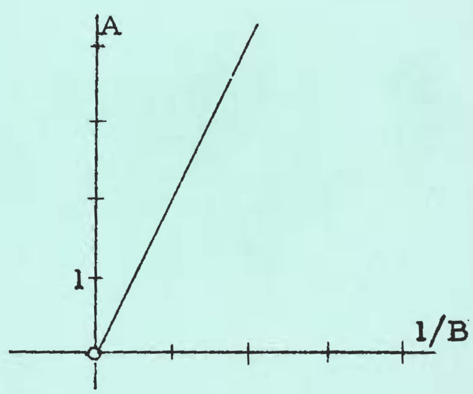
Answers for Part C.

1. $A = \frac{2}{B}$

2.



3.



4. Yes; $\{(x, y): y = 2x\}$

☆4. If your answer to Exercise 3 is 'yes', give the linear function in question. If your answer is 'no', prove that there is no linear function g such that $b = g \circ h$.

5. Graph the function $\{(x, y): xy = 3\}$. What are the domain and range of this function? This function is called <u>an hyperbola.</u> The graph you drew in Exercise 1 is a picture of a <u>branch</u> of an hyperbola. Just as the oblique straight line through the origin is characteristic of direct proportionality [or: direct variation], so the hyperbola whose branches come arbitrarily close to the axes is characteristic of inverse proportionality [or: inverse variation].

<u>C</u>. Suppose the variable quantity A varies inversely as the variable quantity B, that their common range is the set of positive real numbers, and that, for some $e \in \mathcal{D}_B$, $A(e) = 1$ and $B(e) = 2$.

1. Write a formula for A in terms of B.

2. Plot A against B.

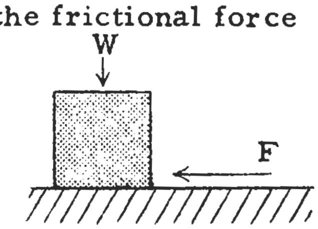
3. Plot A against the variable quantity $\frac{1}{B}$.

4. Does the graph in Exercise 3 suggest that A is a linear function of $\frac{1}{B}$? If so, what linear function?

5. Is it the case that if M varies directly as N then M varies inversely as $\frac{1}{N}$? If it is and if the factor of direct variation is k, what is the factor of inverse variation?

6. Show how each case of inverse variation can be viewed as a case of direct variation.

<u>D</u>. 1. The frictional force on a body sliding on a horizontal surface varies directly as the weight of the body. If the frictional force on a 10-lb. block of wood sliding on a horizontal surface is 3 pounds, what would be the frictional force on a 25-lb block of wood sliding on the same surface?



2. Since F varies directly as W, what is the factor of direct variation? [In the case of frictional forces and sliding bodies, the factor of direct variation is technically called the coefficient of sliding friction.]

3. Write a formula for F in terms of W, and use the formula to tell the frictional force on a 100-lb. block of wood sliding on the same horizontal surface mentioned in Exercise 1.

E. 1. A bookstore is running a sale on its nonfiction books. If the sale price (S) of a book varies directly as its list price (L), and a $4.50-book [that is, a book whose list price is $4.50] is on sale for $4.05, what should be the sale price on a $6.80-book?

2. What is the list price of a book which is on sale for $6.93?

3. Write a formula for S in terms of L. What is the discount rate for the sale?

F. Here is a list of stations along the Illinois Central Railroad from Chicago to Centralia with the distances and the 1959 fares from Chicago to the various stations.

Miles	Stations	Fare from Chicago
0	Chicago	
54.5	Kankakee	$1.51
79.8	Gilman	2.21
112.4	Rantoul	3.11
126.5	Champaign	3.52
171.0	Mattoon	4.77
197.9	Effingham	5.51
251.1	Centralia	6.97

1. Do these data support the statement that the cost of a trip by rail varies directly as the distance traveled?

Correction. On page 5-158, in line 12 change 'quantities ℓ and W' to 'quantities ℓ and w'.

2. The factor of direct variation [in Exercise 1] is 0.3.

3. F = 0.3W; 30 pounds

*

Answers for Part E.

1. \$6.12 2. \$7.70 3. S = 0.9L 4. 10% [or: $\frac{1}{10}$]

*

Answers for Part F [on pages 5-156 and 5-157].

1. Yes [more or less] 2. 0.0278; c = 0.0278d

3. Yes [factor of proportionality = 1.1×0.0278]

4. No [$\frac{52}{32.6} \neq \frac{27}{14.1}$]

*

Answers for Part G [on page 5-157].

1. (a) $3\frac{3}{4}$ hours (b) $H = \frac{30}{M}$ 2. (a) $H = \frac{32}{M}$ (b) 4 hours

*

Answers for Part H [on page 5-157].

1. $\frac{340}{31}$ inches 2. $A = \frac{31}{2}w$ 3. $w = \frac{2}{31}A$ 4. $\frac{31}{2}$; $\frac{2}{31}$

*

Quiz.

1. Suppose that the variable quantity Y is inversely proportional to the variable quantity X. If x_1 and y_1 are corresponding values of X and Y, and $x_1 = 27$ and $y_1 = 3$, what is the factor of inverse proportionality?

2. Grade I apples cost 1.5 times as much as Grade II apples. If you can buy p pounds of Grade I apples for c cents, how many pounds of Grade II apples can you buy for c cents?

3. (a) A varies directly as B and B varies inversely as C. If the factor of direct variation is k_1 and the factor of inverse variation is k_2, write a formula for A in terms of C.

   (b) Does C vary directly as A or inversely as A?

*

Answers for Quiz.

1. 81 2. 3p/2 3. (a) $A = \frac{k_1 k_2}{C}$ (b) inversely

2. If the cost (c) of travel from Chicago toward Centralia is proportional to the distance (d) traveled, what is the factor of proportionality? Write a formula for c in terms of d.

3. The fares listed in the table do not include a 10% Federal tax on transportation. Suppose the tax were added to the fares. If the fare without tax varies directly as distance traveled, does the fare including tax vary directly as the distance traveled?

4. Train No. 25 leaves Gilman at 2:31 a.m., arrives at Rantoul at 3:23 a.m., and stays there five minutes; it arrives at Champaign at 3:55 a.m.. Do these data support the statement that the time it takes to travel from one station to the next is proportional to the distance between stations?

G. Suppose the number (H) of hours to do a certain job varies inversely as the number (M) of machines working.

1. (a) Suppose 10 machines can complete a job in 3 hours. How long will it take 8 machines to complete the job?

   (b) Write a formula for H in terms of M.

2. (a) Suppose 8 machines can complete a job in 4 hours. Write a formula for H in terms of M.

   (b) If 2 machines break down after 1 hour of work, how long will it take the other machines to complete the job?

H. 1. The area-measure of a rectangle having a fixed length-measure varies directly as the width-measure. If the area of one such rectangle is 186 square inches and the rectangle is 12 inches wide, how wide is such a rectangle if its area is 170 square inches?

2. Write a formula for the area-measure (A) of such a rectangle in terms of its width-measure (w).

3. Write a formula for w in terms of A.

4. Compute the ratio of A to w. Of w to A.

JOINT VARIATION

Consider the rectangle formula:

$$P = 2(\ell + w)$$

Does P vary directly as ℓ? To answer this question we compute the ratio of P to ℓ.

$$\frac{P}{\ell} = \frac{2(\ell + w)}{\ell} = 2(1 + \frac{w}{\ell}).$$

If $\frac{w}{\ell}$ is not a constant, $\frac{P}{\ell}$ is not a constant. Hence, P does not vary directly as ℓ. Similarly, P does not vary directly as w.

Now, consider two rectangles, r_1 and r_2, for which

$$\ell_2 + w_2 = 5(\ell_1 + w_1).$$

What can you say about P_2 and P_1? Do you see that the variable quantity P varies directly as the sum of the variable quantities ℓ and W? What is the ratio of P to $\ell + w$?

Another case like this occurs in connection with circles. Since $A = \pi r^2$, A does not vary directly as r [the ratio $\frac{A}{r}$ is πr which is not a constant.] But, A does vary directly as the square of r because $\frac{A}{r^2} = \pi$, and π is a constant. Similarly, for rectangles, A varies directly as the product of ℓ and w. [What is the factor of variation?]

Suppose P varies directly as the product of Q and R. This means that there is a constant $k \neq 0$ such that

$$(*) \qquad P = kQR.$$

A case of direct variation like this involving the product of two variable quantities is sometimes called joint variation. One says that P varies jointly as Q and R. Now, if 0 is not a value of Q, it follows from (*) that

$$R = \frac{1}{k} \cdot \frac{P}{Q},$$

or, using 'k′' as a name for the constant $\frac{1}{k}$,

$$(**) \qquad R = k' \cdot \frac{P}{Q}.$$

This tells us that R varies directly as the ratio of P to Q. And, this situation is usually described by saying that

R varies directly as P and inversely as Q.

<u>Example 1.</u> Suppose z varies jointly as x and y. If $z_1 = 4$ when $x_1 = 5$ and $y_1 = 8$, what is z_2 when $x_2 = 6$ and $y_2 = 15$? The expression 'z varies jointly as x and y' can be translated to 'z varies directly as the product of x and y'. So, there is a constant $k \neq 0$ such that

$$z = kxy.$$

We can solve the given problem by first finding the fixed value of k:

$$4 = k \cdot 5 \cdot 8$$

$$k = \frac{1}{10},$$

and then using the formula '$z = \frac{1}{10}xy$':

$$z_2 = \frac{1}{10} \cdot 6 \cdot 15$$

$$z_2 = 9$$

Or, we can set up a proportion:

$$\frac{z_1}{z_2} = \frac{x_1 y_1}{x_2 y_2},$$

and substitute:

$$\frac{4}{z^2} = \frac{5 \cdot 8}{6 \cdot 15},$$

$$z_2 = 4 \cdot \frac{6 \cdot 15}{5 \cdot 8} = 9$$

<u>Example 2.</u> Suppose z varies directly as x and inversely as y. If $z_1 = 3$ when $x_1 = 9$ and $y_1 = 4$, what is z_2 when $x_2 = 7$ and $y_2 = 5$? The expression 'z varies directly as x and inversely as y' can be translated to 'z varies directly as the ratio of x to y'. So, there is a constant $k \neq 0$ such that

$$z = k\frac{x}{y}.$$

We can find the fixed value of k as follows:

$$3 = k\frac{9}{4},$$

$$k = \frac{4}{3}$$

Then, $$z_2 = \frac{4}{3} \cdot \frac{7}{5} = \frac{28}{15}.$$

We can also set up the proportion:

$$\frac{z_1}{z_2} = \frac{\frac{x_1}{y_1}}{\frac{x_2}{y_2}}$$

and derive from it:

$$\frac{z_1}{z_2} = \frac{x_1}{y_1} \cdot \frac{y_2}{x_2}$$

So,

$$\frac{z_1}{z_2} = \frac{x_1}{x_2} \cdot \frac{y_2}{y_1}$$

Substituting and solving, we get:

$$\frac{3}{z_2} = \frac{9}{7} \cdot \frac{5}{4},$$

$$z_2 = 3 \cdot \frac{7}{9} \cdot \frac{4}{5} = \frac{28}{15}$$

EXERCISES

A. For each of the following cases of variation, write a formula which expresses one of the variable quantities in terms of the others. In each case, use 'k' to name the constant whose value is the factor of variation.

Sample. y varies inversely as the square of x.

Solution. By the definition of inverse variation,

$$yx^2 = k.$$

So, the required formula is:

$$y = \frac{k}{x^2}$$

1. z varies jointly as x and the square of y.
2. z varies directly as the cube of x and inversely as y.
3. z varies jointly as x and y and inversely as w.
4. The volume-measure (V) of a circular cone varies jointly as the square of the radius (r) of its base and its height-measure (h). [What is the value of k?]

Answers for Part A [which begins on page 5-160].

1. $z = kxy^2$
2. $z = kx^3/y$
3. $z = kxy/w$
4. $V = kr^2h[\pi/3]$
5. $V = kr^3 [k = 4\pi/3]$
6. $F = km_1m/d^2$
7. $I = kE/R$
8. $V = ks^2h$
9. (a) $v = kt$ (b) $s = kt^2$
10. (a) $w = kbd^2/\ell$ (b) $D = kw\ell^3/(bd^3)$

*

Answers for Part B [on pages 5-161 and 5-162].

1. (a) 14 (b) 80
2. (a) 6/5 (b) 100
3. (a) 1/2

   (b) Suppose k_1 and k_2 are nonzero numbers such that $N = k_1R^2$ and $P = k_2R$. Then, since $M = \frac{1}{2} \cdot \frac{N^2}{P^3}$, it follows that

$$M = \frac{1}{2} \cdot \frac{{k_1}^2R^4}{{k_2}^3R^3} = \frac{{k_1}^2}{2{k_2}^3} \cdot R .$$

   This last result tells us that M varies directly as R.

   (c) Suppose k_3 and k_4 are nonzero numbers such that $N = k_3\sqrt{S}$ and $P = \frac{k_4}{\sqrt[3]{S}}$. Then, since $M = \frac{1}{2} \cdot \frac{N^2}{P^3}$, it follows that

$$M = \frac{1}{2} \cdot \frac{{k_3}^2S}{\frac{{k_4}^3}{S}} = \frac{{k_3}^2}{2{k_4}^3} \cdot S^2.$$

   This last result tells us that M varies directly as the square of S.

4. (a) 3375 pounds (b) $\frac{2205\pi}{64}$ pounds

☆5. about 687

*

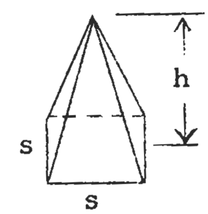
[We suggest that students do Part O of the Miscellaneous Exercises on pages 5-229 and 5-230 as preparation for the exercises on pages 5-163, 5-164, and 5-165.]

5. The volume-measure (V) of a sphere is directly proportional to the cube of its radius (r)

6. The force (F) of gravitational attraction between two bodies of masses m_1 and m_2 varies jointly as their masses and inversely as the square of the distance (d) between them.

7. The number (I) of amperes of current in an electric circuit varies directly as the number (E) of volts in the electromotive force and inversely as the number (R) of ohms of resistance.

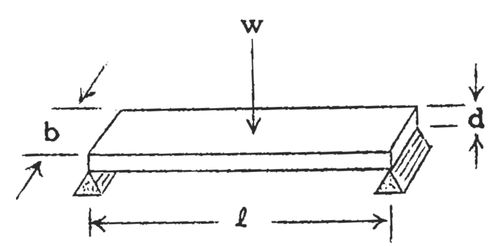
8. The volume-measure (V) of a square pyramid varies directly as the height-measure (h) and the square of the side-measure (s) of the base.



9. (a) The velocity (v) of a body falling from rest varies directly as the number (t) of seconds of fall.

   (b) The distance (s) a body falls from rest varies directly as the square of the number (t) of seconds of fall.

10. (a) The safe load (w) for a horizontal rectangular beam supported at the ends varies jointly as the breadth (b) and the square of the depth (d), and inversely as the distance (ℓ) between supports.



   (b) The deflection (D) of the middle of the beam varies jointly as the load (w) and the cube of the distance (ℓ) between supports, and inversely as the product of the breadth (b) and the cube of the depth (d).

B. 1. w varies directly as u and inversely as v. The value of w is 7 when the value of u is 10 and the value of v is 5.

   (a) What is the value of w when the value of u is 8 and the value of v is 2?

   (b) What is the value of u when the value of w is 28 and the value of v is 10?

2. A varies directly as B and inversely as the square of C.

   (a) If $A_1 = 5$ and $B_1 = 8$ and $C_1 = 4$, what is A_2 if $B_2 = 12$ and $C_2 = 10$?

   (b) If $A_1 = 3$ when $B_1 = 9$ and $C_1 = \sqrt{3}$, what is B_2 if $A_2 = 4$ and $C_2 = 5$?

3. M is proportional to the square of N and inversely proportional to the cube of P.

   (a) If the value of M is 1 when the value of N is 4 and the value of P is 2, what is the factor of variation?

   (b) Suppose N varies directly as the square of R, and P varies directly as R. Write a formula for M in terms of R.

   (c) Suppose N varies directly as the square root of S, and P varies inversely as the cube root of S. Write a formula for M in terms of S.

4. The force (F) of wind blowing on a flat surface [such as a billboard or a sail] and at right angles to it, is jointly proportional to the area (A) of the surface and the square of the speed (r) of the wind. The pressure of the wind is 5 pounds per square foot when the wind is blowing at 20 miles per hour.

   (a) What is the total force of a 30-mile per hour wind on a billboard 10 feet by 30 feet?

   (b) What is the total force of a 30-mile per hour wind on a bass drum 3.5 feet in diameter?

☆ 5. Kepler's third law of planetary motion states that the time (t) of one revolution of a planet around the sun is proportional to the square root of the cube of the distance (d) between the planet and the sun. If the distance between Mars and the sun is about 1.5237 times the distance between Earth and the sun, how many Earth days are there in a Martian year? [An Earth year is 365.26 Earth days.]

* * *

*

Quiz.

1. Suppose that A varies directly as B and inversely as the square of C.

   (a) Write a formula which expresses A in terms of B and C. [Use 'k' to name the constant whose value is the factor of variation.]

   (b) If $A(e_1) = 3$, $B(e_1) = 6$, and $C(e_1) = 2$ then $k =$ __?__.

   (c) Find $B(e_2)$ if $A(e_2) = 7$ and $C(e_2) = 4$.

2. What change takes place in the perimeter of an equilateral triangle if you increase the side-measure by 50%?

3. The height-measure of a circular cone varies directly as the volume-measure and inversely as the square of the radius of the base. If you increase the volume-measure by 300% but leave the height-measure unchanged, what change takes place in the radius of the base?

*

Answers for Quiz.

1. (a) $A = \frac{kB}{C^2}$ (b) 2 (c) 56

2. increases by 50%

3. increases by 100%

Answers for Part C.

1. triples
2. decreases by 50%
3. increases by 80%
☆4. increases by x%

*

Answers for Part D [on page 5-164].

1. quadruples
2. increases by 96%
3. decreases by 51%
☆4. increases by $(2x + \frac{x^2}{100})\%$

*

Answers for Part E [on page 5-164].

1. quadruples
2. decreases by 25%
3. nothing
4. nothing
5. increases by 60%
6. decreases by 37.5%

☆7. increases by $(x + y + \frac{xy}{100})\%$

*

Answers for Part F [on pages 5-164 and 5-165].

1. $V = kr^2h$
2. doubles
3. quadruples
4. decreases by $77\frac{7}{9}\%$
☆5. increases by $(2x + \frac{x^2}{100})\%$

*

Answers for Part G [on page 5-165].

1. $\ell_2 = \ell_1/3$
2. halved
3. increases by 237.5%
4. increases by 12.5%
5. decrease it by $33\frac{1}{3}\%$

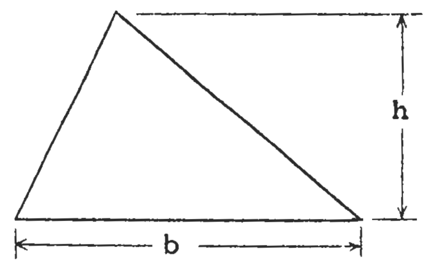
☆6. increases by $(p + q + r + \frac{pq + qr + rp}{100} + \frac{pqr}{10000})\%$

*

Answers for Part H [on page 5-165].

1. decreases by 75%
2. none

The area-measure of a triangle varies jointly as its height-measure and its base-measure. If k is a constant whose value is the factor of variation then a formula which expresses A in terms of h and b is:



$$A = khb$$

[Do you recall the value of k from an earlier course?]

Now, consider a triangle for which the values of A, h, and b are A_1, h_1, and b_1, and a second triangle for which the values are A_2, h_2, and b_2. If $h_2 = 2h_1$ and $b_2 = 3b_1$, what can you say about A_2 and A_1? [In other words, if you double the height and triple the base of a triangle, what happens to the area?] Since $A_2 = kh_2b_2$, it follows that $A_2 = k(2h_1)(3b_1) = 6(kh_1b_1)$. Since $A_1 = kh_1b_1$, we see that $A_2 = 6A_1$. [So, the area is multiplied by 6.]

What happens to the area of a triangle if you increase the height by 50% and decrease the base by 25%? Here, $h_2 = h_1 + 0.5h_1 = 1.5h_1$, and $b_2 = b_1 - 0.25b_1 = 0.75b_1$. So,

$$\begin{aligned} A_2 = kh_2b_2 &= k(1.5h_1)(0.75b_1) \\ &= 1.125(kh_1b_1) \\ &= 1.125A_1 \\ &= A_1 + 0.125A_1. \end{aligned}$$

Thus, the area increases by 12.5% [or, by $\frac{1}{8}$].

* * *

C. The perimeter (P) of a square varies directly as its side-measure (s).

1. What change takes place in the perimeter if you triple the side?
2. What change takes place in the perimeter if you decrease the side by 50%?
3. What change takes place in the side if you increase the perimeter by 80%?
4. ☆ What change takes place in the perimeter if you increase the side by x%?

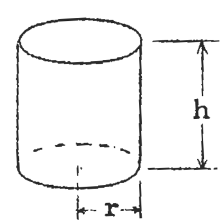
D. The area-measure of a square varies directly as the square of its side-measure.

1. What change takes place in the area if you double the side?
2. What happens to the area if you increase the side by 40%?
3. What happens to the area if you decrease the side by 30%?
4. ☆ What change takes place in the area if you increase the side by x%?

E. The area-measure of a rectangle varies jointly as the length-measure and the width-measure.

1. What happens to the area if you double each dimension?
2. What happens to the area if you increase the length by 50% and decrease the width by 50%?
3. What happens to the area if you increase the length by a third and decrease the width by a fourth?
4. What happens to the area if you increase the length by 100% and decrease the width by 50%?
5. What change takes place in the width if you double the area and increase the length by 25%?
6. What change takes place in one of the dimensions if the other is increased by 60% and the area remains the same?
7. ☆ If you increase one dimension by x% and the other by y%, what change takes place in the area?

F. The volume-measure (V) of a circular cylinder varies jointly as its height-measure (h) and the square of the radius (r) of its base.



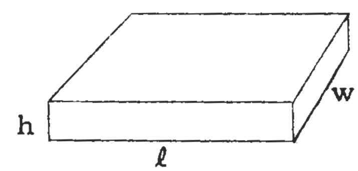
1. Write a formula for V in terms of h and r and using 'k' as the name of the constant whose value is the factor of variation.
2. If you double the height but do not change the radius, what change takes place in the volume?

3. If you double the radius but do not change the height, what change takes place in the volume?

4. If you increase the radius by 50% and decrease the volume by 50%, what change takes place in the height?

☆5. If you increase the radius by x% and leave the height unchanged, what happens to the volume?

G. The length-measure (ℓ) of a rectangular block varies directly as the volume-measure (V) and inversely as the product of the width-measure (w) and the height-measure (h).



1. If $V_2 = 3V_1$, $w_2 = 3w_1$, and $h_2 = 3h_1$, what can you say about ℓ_2 and ℓ_1?

2. If you double the volume, the width, and the height, what change takes place in the length?

3. If you increase each of the dimensions by 50%, what change takes place in the volume?

4. If you increase both the length and the width by 50%, and decrease the height by 50%, what change takes place in the volume?

5. If you decrease the height by 25% and double the width, what change must you make in the length to keep the volume unchanged?

☆6. If you increase the length by p%, the width by q%, and the height by r%, what change takes place in the volume?

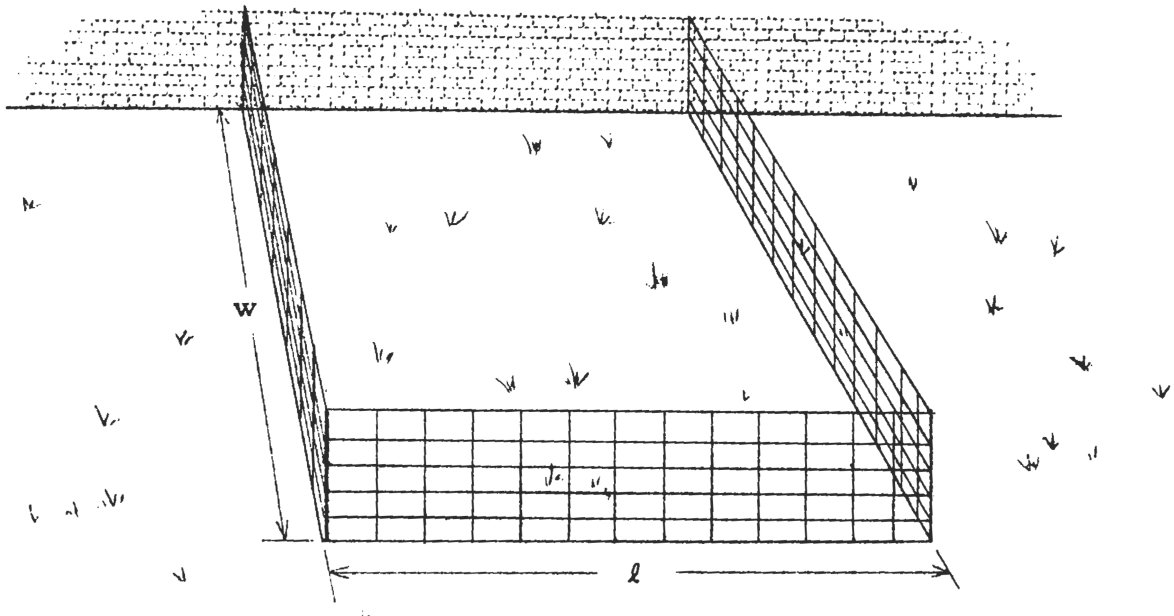
H. Suppose A varies jointly as B and C and inversely as the square of D.

1. If you double D and leave B and C unchanged what happens to A?

2. If you triple each of B, C, and D, what change takes place in A?

[Supplementary exercises are in Part T, pages 5-269 through 5-271.]

5.09 Quadratic functions.--A farmer wants to fence off a rectangular pen using part of an existing brick wall for one of the sides. He has 120 feet of fencing which he can use for this purpose.



What should the dimensions be if the fence is to enclose as large an area as possible?

Here is a table listing some of the possible dimensions and the corresponding area-measures.

w	ℓ	A
1	118	118
2	116	232
3	114	342
10	100	1000
20	80	1600
50	20	1000
58	4	232

It seems reasonable that there are values of ℓ and w which correspond with a value of A which is larger than any other value of A. Our problem is to find these values of ℓ and w.

We know that

(1) $$A = \ell w.$$

We also know that the 120 feet of fencing must be used for one length and two widths of the rectangular pen. So, for the set of all rectangular pens of this design,

$$(2) \qquad \ell + 2w = 120.$$

We transform (2) to get:

$$(3) \qquad \ell = 120 - 2w$$

So, (1) tells us that A is a function of (ℓ, w), and (3) tells us that ℓ is a function of w. Given a value of w, the value of ℓ is determined, and so is the value of A. Hence, A is a function of w. We can find out what function A is of w by substituting (3) into (1):

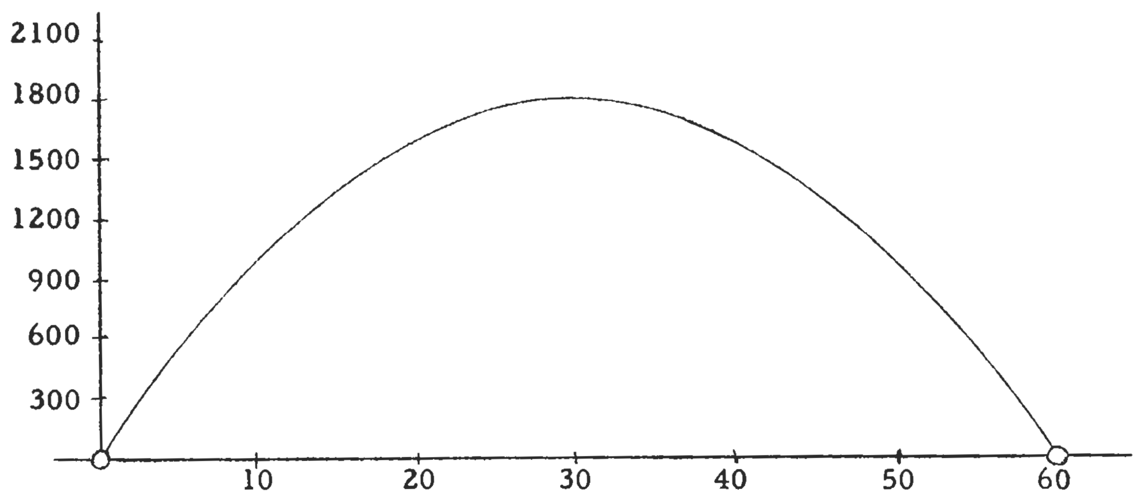
$$(4) \qquad A = (120 - 2w)w$$

Thus, the ordered pairs of corresponding values of w and A are the ordered pairs in

$$\{(x, y),\ 0 < x < 60:\ y = (120 - 2x)x\}.$$

Call this function 'f'. Now, our job is to find that argument of f for which f has its largest value. This argument will give us the required width, and then we can use (3) to find the required length.

Let's make a graph of f.



An examination of the graph suggests that the maximum value of f corresponds with the argument 30. So, it seems that the pen of maximum area is the one which is 30 feet wide and 60 feet long. [What is this maximum area?]

In this section you will learn a quicker and surer way to solve a problem like this.

QUADRATIC FUNCTIONS OF ONE REAL VARIABLE

Up to now we have considered two special classes of functions--the constant functions and the linear functions.

f is a <u>constant</u> <u>function</u> if and only if
there is a number a such that $f = \{(x, y): y = a\}$.

f is a <u>linear</u> <u>function</u> if and only if
there are numbers $a \neq 0$ and b such that $f = \{(x, y): y = ax + b\}$.

We shall now consider another class of functions which are called <u>quadratic functions of one real variable</u>, or, for short, <u>quadratic functions</u>.

f is a <u>quadratic</u> <u>function</u>
if and only if
there are numbers $a \neq 0$, b, and c such that
$f = \{(x, y): y = ax^2 + bx + c\}$.

You have already seen some examples of quadratic functions. For example, the area-measure of a square is a quadratic function of its side-measure. The quadratic function in question is $\{(x, y): y = x^2\}$, which we have called the <u>squaring</u> <u>function</u>. In the fencing problem discussed earlier, the area-measure of the pen is a quadratic function of the width-measure, the quadratic function in question being $\{(x, y): y = (120 - 2x)x\}$.

How can you tell that $\{(x, y): y = (120 - 2x)x\}$ is a quadratic function? Just use the definition. If you can produce numbers $a \neq 0$, b, and c such that

$$\{(x, y): y = (120 - 2x)x\} = \{(x, y): y = ax^2 + bx + c\},$$

then the given function is a quadratic function. As in the case of linear functions, you obtain these numbers by transforming the set selector.

$$y = (120 - 2x)x$$
$$y = 120x - 2x^2$$
$$y = -2x^2 + 120x + 0$$

Hence, $a = -2 \neq 0$, $b = 120$, and $c = 0$. So, $\{(x, y): y = (120 - 2x)x\}$ is a quadratic function.

*

Quiz.

1. Each of the relations described below is a quadratic function. Give the defining equation for each.

   (a) $\{(x, y)\colon\ y = 6 - 2x + 3x^2\}$; $y =$ ___ $x^2 +$ ___ $x +$ ___

   (b) $\{(x, y)\colon\ y = (x - 3)^2\}$; $y =$ ___ $x^2 +$ ___ $x +$ ___

   (c) $\{(x, y)\colon\ y = 5 - (2x + 1)^2\}$; $y =$ ___ $x^2 +$ ___ $x +$ ___

   (d) the function which maps each real number on 3 more than its square; $y =$ ___ $x^2 +$ ___ $x +$ ___

   (e) $\{(x, y)\colon\ y = (50 - 2x)x\}$; $y =$ ___ $x^2 +$ ___ $x +$ ___

2. Find the ordered pair whose graph is the lowest point in the graph of $\{(x, y)\colon\ y = (x - 2)^2 + 5\}$.

*

Answers for Quiz.

1. (a) 3, −2, 6 (b) 1, −6, 9 (c) −4, −4, 4

   (d) 1, 0, 3 (e) −2, 50, 0

2. (2, 5)

Answers for Part A.

1. linear 2. quadratic 3. quadratic 4. none 5. quadratic

6. Neither constant, linear, nor quadratic; but, a subset of a quadratic function.

7. quadratic 8. quadratic 9. constant $[\{(x, y): y = 14\}]$

*

[In the answers which follow, the parabolas are all congruent to the graph of $\{(x, y): y = x^2\}$. Of course, this assumes that the graphs are drawn with respect to axes which have the same scale for all problems. We identify the parabola just by giving the extreme point and telling whether the parabola opens upward or downward.]

*

Answers for Part B.

1. f [(0, 0), up]; g [45°-line];

   h [parallel to and 1 unit above the horizontal axis]

2. (a) (0, 1), up (b) (0, 2), up (c) (0, −5), up
   (d) $(-\frac{1}{2}, -\frac{1}{4})$, up (e) (−1, −1), up (f) (3, −9), up
   (g) (0, 0), down (h) (0, −2), down (i) (−2, 4), down

*

Answers for Part C.

1. (0, 0), up 2. (3, 0), up 3. (−1, 0), up
4. (0, 1), up 5. (3, 1), up 6. (−1, −5), up

Answers for Part D. [Use '$\forall_{x \neq 0}\ x^2 > 0$' to justify answers.]

1. 0 2. 3 3. −1 4. 0 5. 3 6. −1

*

Answers for Part E.

1. (3, 5), up 2. (5, −9), up 3. (0, 0), down
4. (3, 0), down 5. (−7, 4), down 6. (1, 0), up

*

EXERCISES

A. For each set listed, tell whether it is a constant function, a linear function, a quadratic function, or none of these.

1. $\{(x, y): y = 3(2 - x)\}$
2. $\{(x, y): y = 6 - x^2\}$
3. $\{(x, y): y + 2x^2 = 5 - 3x\}$
4. $\{(x, y): 3(x - 5) = x(x - 1)\}$
5. $\{(x, y): y = (x - 2)^2\}$
6. $\{(x, y), x > 0: y = 3x^2 + 2x - 7\}$
7. $\{(x, y): y = -(x + 3)^2 - 5\}$
8. $\{(x, y): y - 3x = (x - 3)^2\}$
9. $\{(x, y): y = (x + 2)(x - 3) - (x - 5)(x + 4)\}$

B. Consider the functions f, g, and h where

$$f = \{(x, y): y = x^2\},\ g = \{(x, y): y = x\},\ \text{and } h = \{(x, y): y = 1\}.$$

1. Graph f, g, and h on the same chart.
2. Use the graphs you drew in Exercise 1 to help you make quick sketches of the graphs of the functions listed below.

(a) $f + h$ (b) $f + 2h$ (c) $f - 5h$

(d) $f + g$ (e) $f + 2g$ (f) $f - 6g$

(g) $-f$ (h) $-(f + 2h)$ (i) $-(f + 4g)$

C. Sketch the graphs of the functions defined by these equations.

1. $y = x^2$
2. $y = (x - 3)^2$
3. $y = (x + 1)^2$
4. $y = x^2 + 1$
5. $y = (x - 3)^2 + 1$
6. $y = (x + 1)^2 - 5$

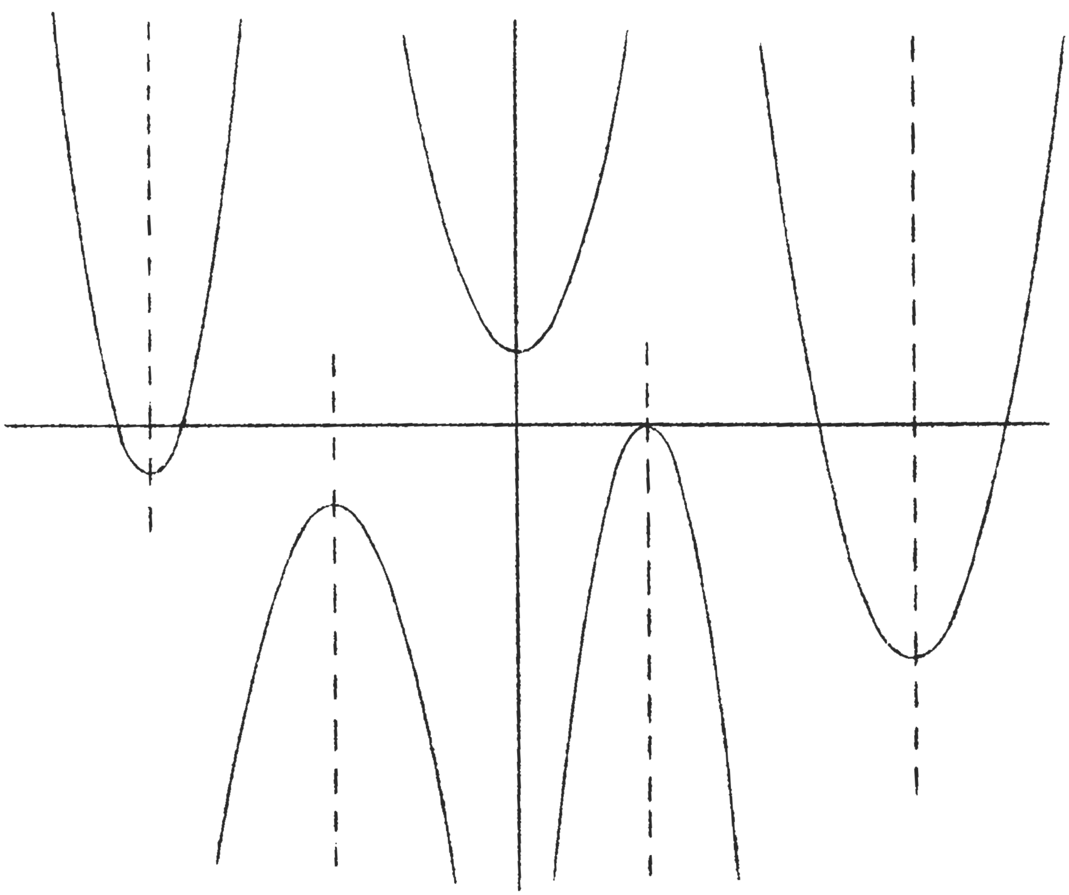
D. Find the argument which corresponds with the minimum value for each of the quadratic functions you graphed in Part C.

E. Find the argument which corresponds with the extreme value [maximum or minimum] of each function listed below. Then, graph the function.

1. $\{(x, y): y = (x - 3)^2 + 5\}$
2. $\{(x, y): y = (x - 5)^2 - 9\}$
3. $\{(x, y): y = -x^2\}$
4. $\{(x, y): y = -(x - 3)^2\}$
5. $\{(x, y): y = -(x + 7)^2 + 4\}$
6. $\{(x, y): y = x^2 - 2x + 1\}$

GRAPHING A QUADRATIC FUNCTION

In the preceding exercises you practiced graphing quadratic functions. You may have noted that all graphs of quadratic functions have the same shape. In fact, they are all examples of curves called parabolas. A parabola which is the graph of a quadratic function is symmetric with respect to some vertical line. Here are examples of graphs of quadratic functions. Note the graph of the axis of symmetry in each case. Also, notice that each quadratic function has an "extreme point" which is the point in the intersection of the quadratic function and



its axis of symmetry. The graph of the extreme point is the vertex of the parabola.

Just as in graphing linear functions it was helpful to use the notions of slope and intercept, so in graphing quadratic functions it would be helpful to know the function's axis of symmetry, its extreme point, and whether the parabola opens upward or downward. It would also be helpful to know the intersection of the function with the y-axis and the

Answers for questions on page 5-171.

The intersection of a quadratic function and the y-axis consists of a single point, (0, c).

Knowing the axis of symmetry is useful because, knowing this, we can find the extreme point, and also halve the labor of graphing the function.

$\{(x, y): y = 9\} \cap \{(x, y): y = x^2\} = \{(3, 9), (-3, 9)\}$

The length of the interval $\overline{(3, 9)(-3, 9)}$ is 6, and its midpoint is (0, 9).

The set of all such midpoints is $\{(x, y): x = 0 \text{ and } y \geq 0\}$.

Consider some pairs of arguments of the squaring functions which have a constant difference, say, 0 and $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{3}{4}$, $\frac{1}{2}$ and 1, $\frac{3}{4}$ and $\frac{5}{4}$, and 1 and $\frac{3}{2}$. For each pair, compute the difference of the values of the squaring function for the two arguments.

$$(\tfrac{1}{2})^2 - 0^2 = \tfrac{1}{4}, \quad (\tfrac{3}{4})^2 - (\tfrac{1}{4})^2 = \tfrac{1}{2}, \quad 1^2 - (\tfrac{1}{2})^2 = \tfrac{3}{4}$$

$$(\tfrac{5}{4})^2 - (\tfrac{3}{4})^2 = 1, \quad (\tfrac{3}{2})^2 - 1^2 = \tfrac{5}{4}$$

Note that these differences increase. The larger the first argument is, the larger is the difference. This is so because, for each x,

$$(x + \tfrac{1}{2})^2 - x^2 = x + \tfrac{1}{4}.$$

And, for each difference h, for each x, $(x + h)^2 - x^2 = 2hx + h^2$. So, [if $h > 0$,] the larger the first argument, the larger the difference. [Another way of seeing this is by noting that, for each x_1 and x_2, $x_2^{\,2} - x_1^{\,2} = (x_2 + x_1)(x_2 - x_1)$.]

*

It will pay to draw, on the board and using a large scale, a graph of the squaring function. By graphing such ordered pairs as (1/2, 1/4), (1/4, 1/16), and (−1/4, 1/16), bring out the "flatness" of the function at its extreme point. Graphs of quadratic functions should look like the one on the right:

and not like the one below:

intersection with the x-axis. Since a quadratic function is determined by the values of 'a', 'b', and 'c' in the defining equation:

$$y = ax^2 + bx + c,$$

we should expect to be able to use these numbers to give all of the information mentioned above. [One item of information which is available immediately is the intersection of the function and the y-axis. How many points are there in this intersection? What are the components?]

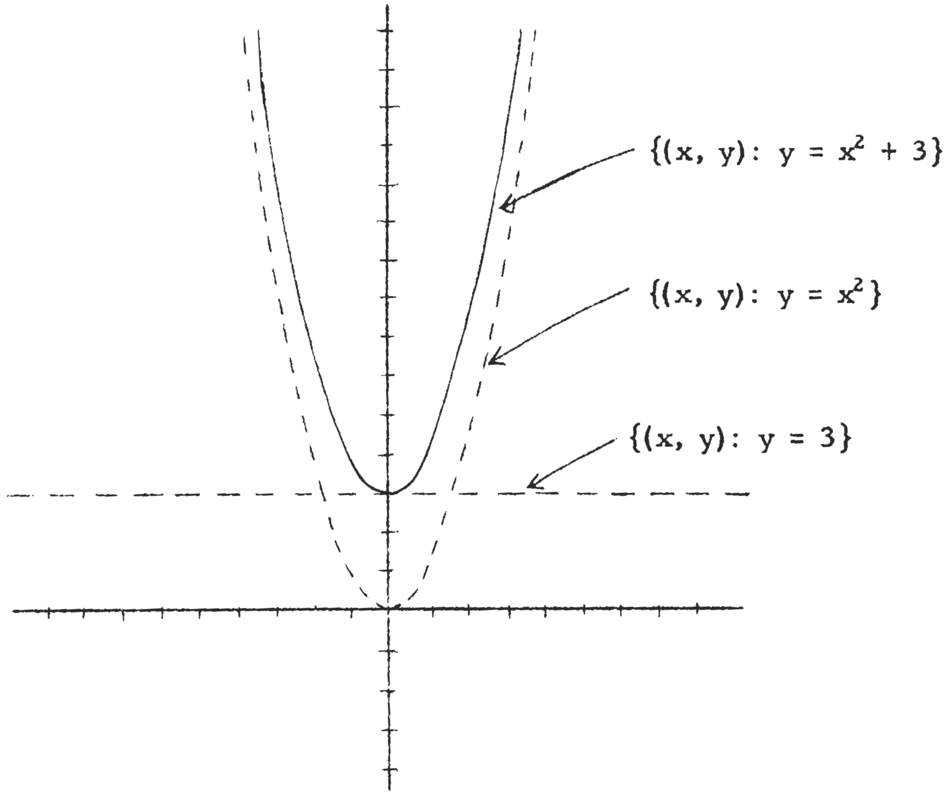
Let's explore a bit to find an easy way of discovering the axis of symmetry. [Why is it helpful to know the axis of symmetry in graphing a quadratic function?] Let's take the simplest quadratic function, the squaring function, which is

$$\{(x, y): y = x^2\}.$$

Think of all the horizontal lines which cross the graph of the squaring function. The lowest of these lines contains the graph of (0, 0). Hence, (0, 0) is the extreme point of the squaring function, and its graph is the vertex of the parabola. Each of the other horizontal lines crosses the graph in two places. Pick some horizontal line, say, the one which is 9 units above the horizontal axis. What are the components of the points in which $\{(x, y): y = 9\}$ intersects the squaring function? How long is the interval whose end points are these points? What is the midpoint of this interval? What is the set of all such midpoints? This set is contained in the axis of symmetry of the squaring function. If you hold the edge of a mirror along the graph of the y-axis with the mirror facing the graph of the first quadrant, you can see the image of the graph of the first quadrant portion of the squaring function. And, this image is exactly the graph of the second quadrant portion of the function. An advantage in knowing the axis of symmetry of a quadratic function is that in graphing the function you need plot points on only one side of the graph of the axis of symmetry. For each point so plotted, you can get a second one by reflecting it in the axis of symmetry.

Another point of interest concerning the squaring function is that the graph rises slowly to the right and to the left from the graph of (0, 0). The rate of rise increases as soon as you move away from the vertex and the curve gets steeper and steeper as you continue to move. Can you explain this in terms of the properties of squaring?

Now, consider the quadratic function $\{(x, y): y = x^2 + 3\}$. This is the sum of the squaring function and the constant function $\{(x, y): y = 3\}$. The graph of the sum of these functions has precisely the same shape as



the graph of the squaring function. The only difference is the position of the vertex. The vertex is the graph of (0, 3) instead of the graph of (0, 0). The axis of symmetry is still $\{(x, y): x = 0\}$.

What is the axis of symmetry of $\{(x, y): y = x^2 - 5\}$? What is the extreme point of this function?

Let's consider the opposite of the squaring function, $\{(x, y): y = -x^2\}$. What is its axis of symmetry? What is its extreme point? Answer these questions for the quadratic functions $\{(x, y): y = -x^2 + 5\}$ and $\{(x, y): y = -x^2 - 4\}$. Answer them for $\{(x, y): y = -(x^2 + 4)\}$.

In general, for each q, the axis of symmetry of the quadratic function $\{(x, y): y = x^2 + q\}$ is $\{(x, y):$ __________$\}$, the extreme point is ________, and its graph opens ______________; the axis of symmetry of $\{(x, y): y = -x^2 + q\}$ is $\{(x, y):$ __________$\}$, the extreme point is ____________, and its graph opens _______________.

The purpose of the discussion on pages 5-172, 5-173, and 5-174 is to bring out the fact that the graphs of all quadratic functions for which a = 1 or a = −1 are congruent. They can all be drawn [to the same scale] by using a single parabolic ruler.

*

Answers for questions on page 5-172.

Function	Axis of symmetry	Extreme point
$\{(x, y): y = x^2 - 5\}$	the y-axis	(0, −5)
$\{(x, y): y = -x^2\}$	the y-axis	(0, 0)
$\{(x, y): y = -x^2 + 5\}$	the y-axis	(0, 5)
$\{(x, y): y = -x^2 - 4\}$	the y-axis	(0, −4)

For $\{(x, y): y = -(x^2 + 4)\}$, the questions have already been answered, since $\{(x, y): y = -(x^2 + 4)\} = \{(x, y): y = -x^2 - 4\}$.

*

Fill-ins for the six blanks at the bottom of page 5-172.

x = 0; (0, q); upward; x = 0; (0, q); downward

*

Answers for questions on page 5-173.

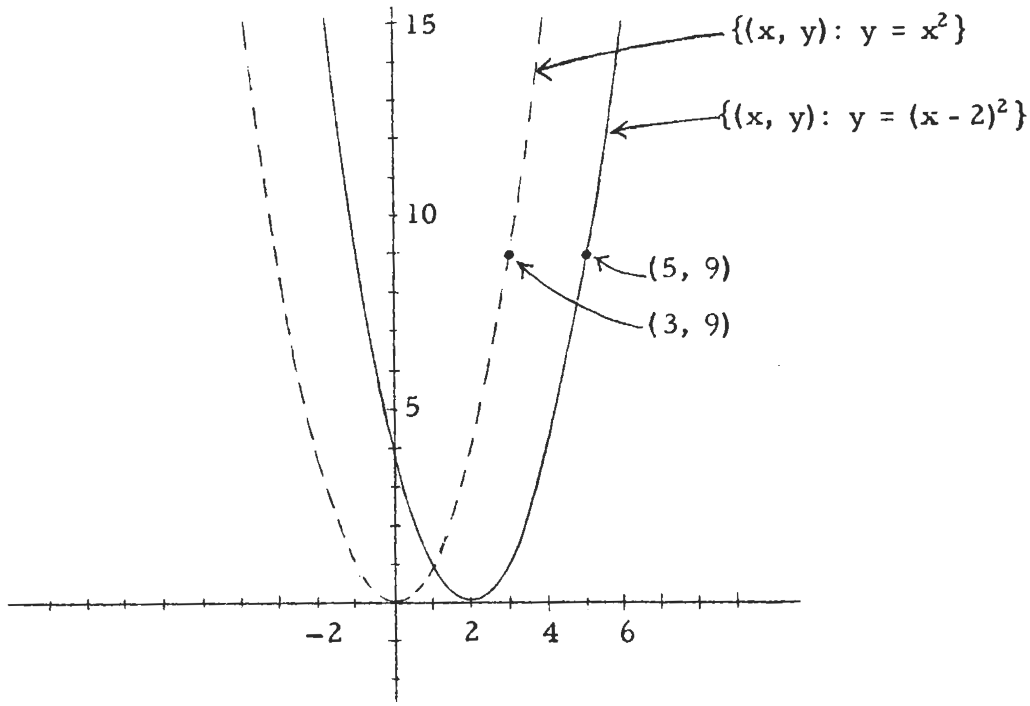
Function	Axis of symmetry	Extreme point
$\{(x, y): y = (x - 2)^2\}$	$\{(x, y): x = 2\}$	(2, 0)
$\{(x, y): y = -(x - 2)^2\}$	$\{(x, y): x = 2\}$	(2, 0)
$\{(x, y): y = (x + 3)^2\}$	$\{(x, y): x = -3\}$	(−3, 0)
$\{(x, y): y = -(x + 3)^2\}$	$\{(x, y): x = -3\}$	(−3, 0)
$\{(x, y): y = (x - 2)^2 + 5\}$	$\{(x, y): x = 2\}$	(2, 5)

*

Fill-ins for the six blanks at the bottom of page 5-173.

x = p; (p, 0); upward; x = p; (p, 0); downward

Now, let's consider the quadratic function $\{(x, y): y = (x - 2)^2\}$. This function may remind us of the squaring function. In fact we see that (5, 9) belongs to this function just because (5 - 2, 9) belongs to the squaring function. Turning this around: Because (3, 9) belongs to the squaring function, we know that $(3 + 2, 9) \in \{(x, y): y = (x - 2)^2\}$. So, we can obtain a graph of $\{(x, y): y = (x - 2)^2\}$ just by shifting a graph of the squaring function 2 units to the right.



[Another way to get a graph of $\{(x, y): y = (x - 2)^2\}$ is to draw a graph of the squaring function and then re-draw the graph of the vertical axis 2 units to the left.] What is the axis of symmetry of $\{(x, y): y = (x - 2)^2\}$? What is its extreme point? Answer these questions for the quadratic functions $\{(x, y): y = -(x - 2)^2\}$, $\{(x, y): y = (x + 3)^2\}$, $\{(x, y): y = -(x + 3)^2\}$, and $\{(x, y): y = (x - 2)^2 + 5\}$.

In general, for each p, the axis of symmetry of the quadratic function $\{(x, y): y = (x - p)^2\}$ is $\{(x, y):$ ____________$\}$, the extreme point is _________, and its graph opens _________; the axis of symmetry of $\{(x, y): y = -(x - p)^2\}$ is $\{(x, y):$ ____________$\}$, the extreme point is _________, and its graph opens _________.

For each p and q, the axis of symmetry of the quadratic function $\{(x, y): y = (x - p)^2 + q\}$ is $\{(x, y):$ __________$\}$, the extreme point is ______, and its graph opens __________; the axis of symmetry of $\{(x, y): y = -(x - p)^2 + q)\}$ is $\{(x, y):$ __________$\}$, the extreme point is _____, and its graph opens __________.

Explain how to get a graph of $\{(x, y): y = (x - 3)^2 + 5\}$ by shifting a graph of the squaring function.

Consider, now, the quadratic function

$$\{(x, y): y = (x - 3)^2 - 3^2\}$$

or, more simply, $\{(x, y): y = x^2 - 6x\}$. What is its axis of symmetry? Its extreme point? For each p, the axis of symmetry of

$$\{(x, y): y = (x - p)^2 - p^2\},$$

or, more simply, $\{(x, y): y = x^2 - 2px\}$ is $\{(x, y):$ ____________$\}$.

What are the axis of symmetry and the extreme point of each of the quadratic functions listed below?

(a) $\{(x, y): y = (x - 2)^2 - 2^2\}$ (b) $\{(x, y): y = (x + 2)^2 - 2^2\}$

(c) $\{(x, y): y = x^2 - 4x\}$ (d) $\{(x, y): y = x^2 + 4x\}$

(e) $\{(x, y): y = x^2 - 5x\}$ (f) $\{(x, y): y = x^2 + 5x\}$

(g) $\{(x, y): y = x^2 - 5x + 3\}$ (h) $\{(x, y): y = x^2 + 5x - 7\}$

(i) $\{(x, y): y = -(x^2 - 3x)\}$ (j) $\{(x, y): y = -(x^2 + 3x)\}$

(k) $\{(x, y): y = -x^2 + 3x + 4\}$ (ℓ) $\{(x, y): y = -x^2 - 3x - 5\}$

For each b, the axis of symmetry of $\{(x, y): y = x^2 + bx\}$ is $\{(x, y):$ __________$\}$. For each b and c, the axis of symmetry of $\{(x, y): y = x^2 + bx + c\}$ is $\{(x, y):$ __________$\}$, the extreme point is ______, and its graph opens __________.

For each b and c one can obtain a graph of $\{(x, y): y = x^2 + bx + c\}$ by shifting a graph of the squaring function until the graph of its vertex is on the graph of (___, ______).

Up to now we have been discussing quadratic functions whose defining equations are of the form:

$$y = x^2 + bx + c \qquad \text{or:} \qquad y = -x^2 + bx + c$$

You know how to use the numbers b and c to find the axis of symmetry

Fill-ins for the six blanks at the top of page 5-174.

$x = p$; (p, q);   upward; $x = p$; (p, q);   downward

*

A graph of $\{(x, y): y = (x - 3)^2 + 5\}$ can be obtained by shifting a graph of the squaring function 3 units to the right and 5 units up.

*

Answers for questions on page 5-174.

The axis of $\{(x, y): y = x^2 - 6x\}$ is $\{(x, y): x = 3\}$. Its extreme point is $(3, -9)$.

[Fill-in: $x = p$]

(a) $\{(x, y): x = 2\}$; $(2, -4)$

(b) $\{(x, y): x = -2\}$; $(-2, -4)$

(c) [same as for (a)]

(d) [same as for (b)]

(e) $\{(x, y): x = \frac{5}{2}\}$; $(\frac{5}{2}, -\frac{25}{4})$

(f) $\{(x, y): x = -\frac{5}{2}\}$; $(-\frac{5}{2}, -\frac{25}{4})$

(g) $\{(x, y): x = \frac{5}{2}\}$; $(\frac{5}{2}, -\frac{13}{4})$

(h) $\{(x, y): x = -\frac{5}{2}\}$; $(-\frac{5}{2}, -\frac{53}{4})$

(i) $\{(x, y): x = \frac{3}{2}\}$; $(\frac{3}{2}, \frac{9}{4})$

(j) $\{(x, y): x = -\frac{3}{2}\}$; $(-\frac{3}{2}, \frac{9}{4})$

(k) $\{(x, y): x = \frac{3}{2}\}$; $(\frac{3}{2}, \frac{25}{4})$

(ℓ) $\{(x, y): x = -\frac{3}{2}\}$; $(-\frac{3}{2}, -\frac{11}{4})$

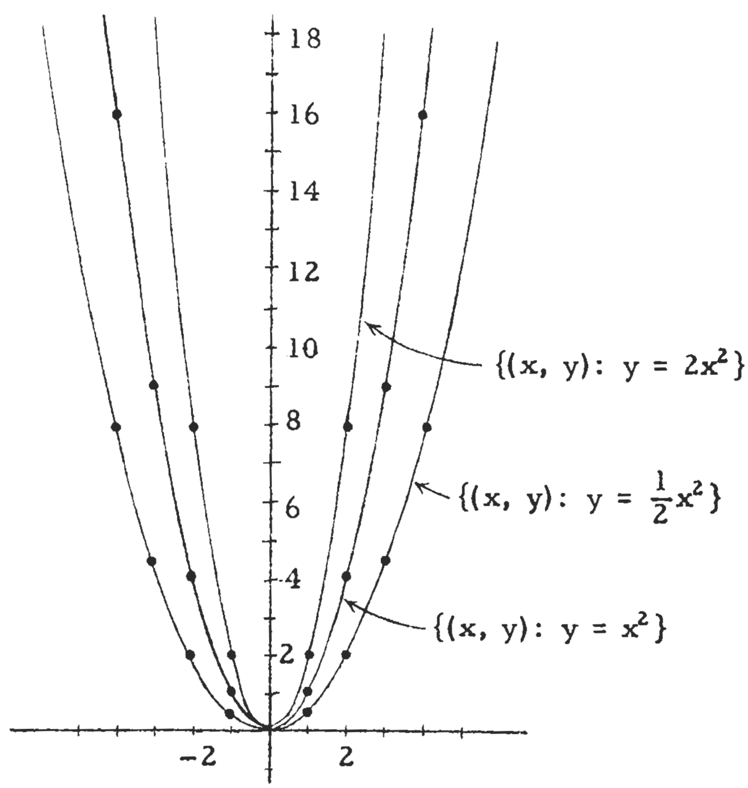
*

Fill-ins for the five blanks at the bottom of page 5-174.

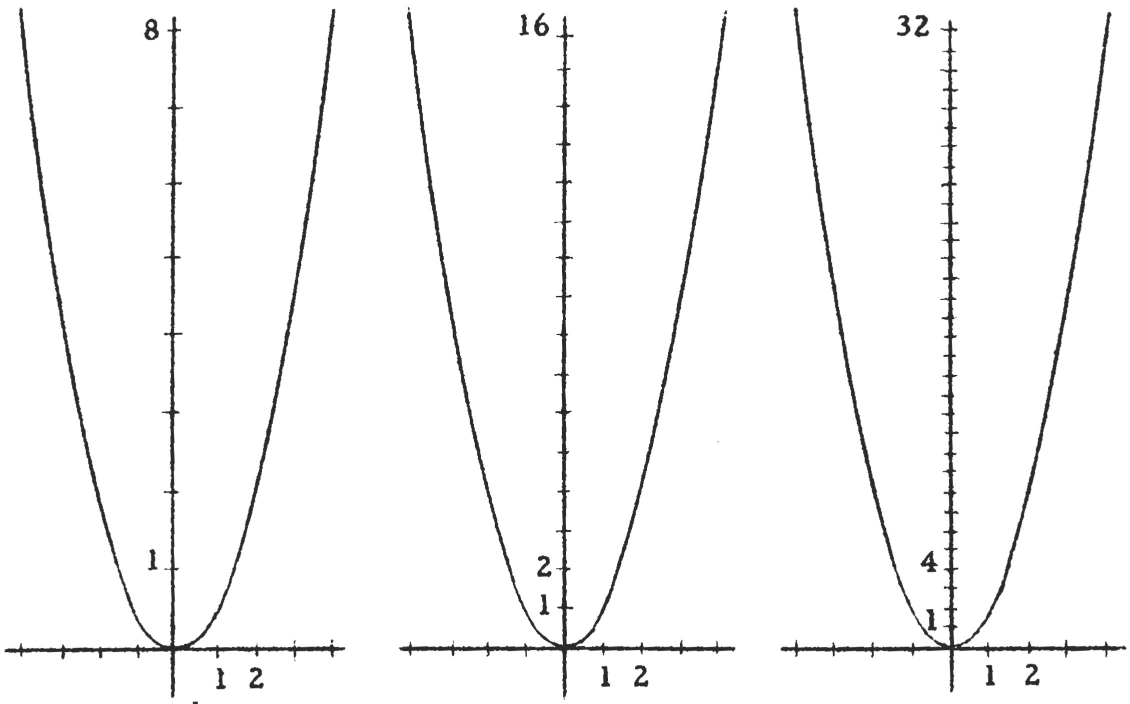
$x = -\frac{b}{2}$, $x = -\frac{b}{2}$; $(-\frac{b}{2}, -\frac{b^2}{4} + c)$;   upward; $(-\frac{b}{2}, -\frac{b^2}{4} + c)$

and the extreme point of any quadratic function whose defining equation is of either of these forms. You also know that a graph of any such quadratic function can be obtained by merely shifting a graph of the squaring function [and, in case the defining equation is of the second form, tipping the graph upside down]. As a practical application of this last piece of knowledge, you can make a "parabolic ruler": Draw carefully a graph of the squaring function, paste your graph paper on heavy cardboard, and cut along the graph. Now you have a ruler which you can use to draw graphs of any quadratic functions whose defining equations are of either of the forms given near the bottom of page 5-174. But, how about other quadratic functions?

Here are graphs of $\{(x, y): y = \frac{1}{2}x^2\}$, $\{(x, y): y = x^2\}$, and $\{(x, y): y = 2x^2\}$:



It is easy to see how, for example, to obtain a graph of $\{(x, y): y = \frac{1}{2}x^2\}$ once one has a graph of the squaring function. Just move each point of the graph of the squaring function half-way to the graph of the x-axis. Instead of moving the points down, you could move the scale markings on the graph of the y-axis up. Here are the graphs of the same three functions:



$y = \frac{1}{2}x^2$ $\qquad$ $y = x^2$ $\qquad$ $y = 2x^2$

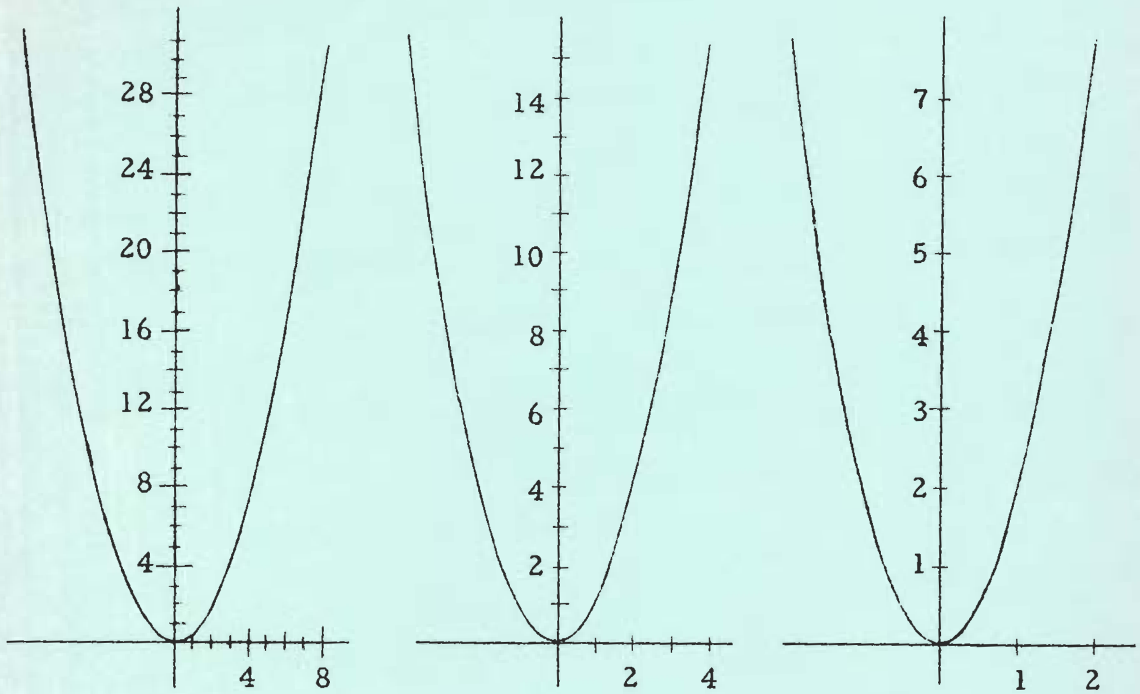
You can now use your parabolic ruler to draw a graph of any quadratic function. Let's try an example.

Graph the quadratic function $\{(x, y): y = 2x^2 - 12x + 22\}$.

First, we transform the defining equation to an equivalent one:

$$y = 2(x^2 - 6x + 11)$$

This shows us that we could get a graph of the given function by first drawing a graph of $\{(x, y): y = x^2 - 6x + 11\}$ and then changing the scale markings on the graph of the y-axis so that the unit length is just half as long as the unit length on the graph of the x-axis. If you used the same scale on the graph of the x-axis that you used in making your parabolic ruler, then you could use the ruler to draw a graph of the given function. Since the axis of symmetry of $\{(x, y): y = x^2 - 6x + 11\}$ is $\{(x, y): x = 3\}$ and the extreme point is $(3, 2)$, the axis of symmetry of $\{(x, y): y = 2(x^2 - 6x + 11)\}$ is $\{(x, y): x = 3\}$ and <u>its</u> extreme point is $(3, 4)$. [Explain]. Also, both graphs open upward. So, you can



$y = \frac{1}{2}x^2$ $y = x^2$ $y = 2x^2$

the graph of (u, v) becomes the graph of $(\frac{u}{a}, \frac{v}{a})$, one has a graph of

$\{(x, y) : y = ax^2 + bx + c\}$. In case $a > 0$, the required scale is, of course, obtained merely by taking a unit a times as long as the old.

[If $a < 0$, changing the graph of (u, v) into the graph of $(\frac{u}{a}, \frac{v}{a})$ involves

reversing the directions of the axes, as well as using a unit $|a|$ times the length of the original unit. This amounts to a reflection through the origin, as well as a change of unit. So, to get the graph of

$$\{(x, y) : y = ax^2 + bx + c\}$$

when $a < 0$, one must reflect the graph of $\{(x, y) : y = x^2 + bx + ac\}$ through the origin as well as change to a new unit which is $|a|$ times as long as the old.]

Answers for questions on page 5-177.

(a) vertex at (2, −1), pointing down

(b) vertex at (2, 7), pointing up

(c) vertex at $(-\frac{3}{2}, -\frac{1}{2})$, pointing down

(d) vertex at $(-\frac{3}{2}, 4)$ pointing up

(e) vertex at (−3, −11), pointing down

(f) vertex at (−3, 7), pointing up

(g) vertex at (−2, 1), pointing down

(h) vertex at (−2, 9), pointing up

*

Fill-ins for the four blanks at the bottom of page 5-177.

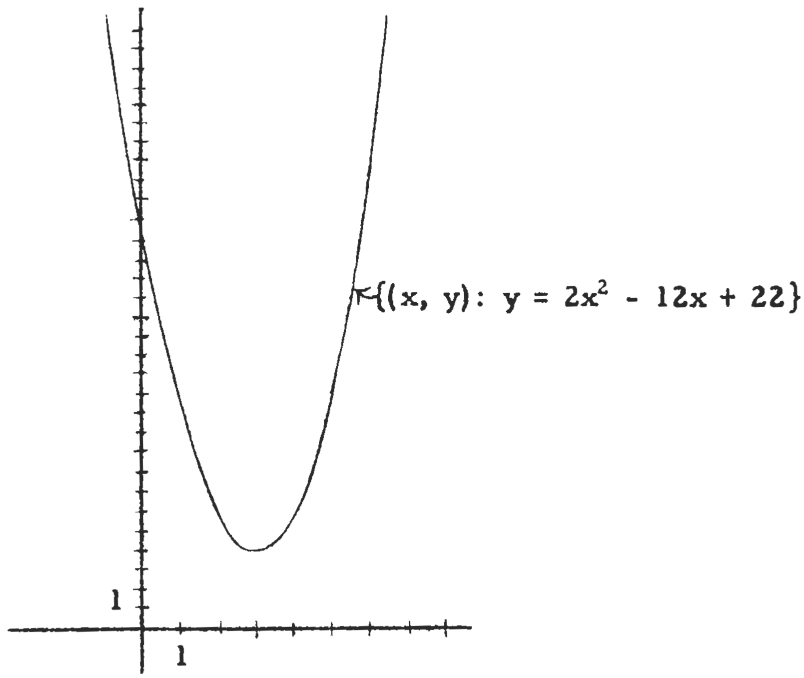
$\{(x, y): x = -\frac{b}{2a}\}$; $(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a})$; $a > 0$; $a < 0$

*

Students should see that if one uses the same scale on both axes in graphing several quadratic functions, with the same values of 'b' and 'c' in each case, but different values for 'a', then the graphs corresponding with values of 'a' of larger absolute value will be the narrower. In particular, the graphs corresponding with values of 'a' greater than 1 [or less than −1] will be narrower and more pointed than the graph of the squaring function, and those corresponding with values of 'a' between −1 and 1 will be broader and flatter than the graph of the squaring function. However, if one keeps the same scale on the x-axis and, for each graph, uses a different unit, $1/|a|$ times as long, on the y-axis, then the graphs will be the same size and shape as the graph of the squaring function. Some students may make a more fundamental discovery than this. It is that if one graphs the squaring function, using the same unit on both axes, and then graphs any quadratic function $\{(x, y): y = ax^2 + bx + c\}$ using, on <u>both</u> axes, a unit $|a|$ times as long as the original one, then the two graphs will have the same size and shape. In geometrical terms, all parabolas are similar geometric figures. Algebraically, the reason for this is that, for $a \neq 0$,

$$\text{`}y = ax^2 + bx + c\text{' and `}ay = (ax)^2 + b(ax) + ac\text{'}$$

are equivalent equations. Now, $(u, v) \in \{(x, y): y = x^2 + bx + ac\}$ if and only if $(\frac{u}{a}, \frac{v}{a}) \in \{(x, y): ay = (ax)^2 + b(ax) + ac\}$; that is, if and only if $(\frac{u}{a}, \frac{v}{a}) \in \{(x, y): y = ax^2 + bx + c\}$. So, if one graphs $\{(x, y): y = x^2 + bx + ac\}$, using the same scale on both axes, and then changes the scales so that draw a graph of $\{(x, y): y = 2x^2 - 12x + 22\}$ by choosing proper scales on the graphs of the axes and placing your parabolic ruler with its vertex on the graph of (3, 4) and its center line on the upper part of the graph of $\{(x, y): x = 3\}$.



How would you place your ruler to draw graphs of the quadratic functions listed below?

(a) $\{(x, y): y = 2x^2 - 8x + 7\}$

(b) $\{(x, y): y = -2x^2 + 8x - 1\}$

(c) $\{(x, y): y = 2x^2 + 6x + 4\}$

(d) $\{(x, y): y = -2x^2 - 6x - \frac{1}{2}\}$

(e) $\{(x, y): y = 2x^2 + 12x + 7\}$

(f) $\{(x, y): y = -2x^2 - 12x - 11\}$

(g) $\{(x, y): y = 2x^2 + 8x + 9\}$

(h) $\{(x, y): y = -2x^2 - 8x + 1\}$

For all $a \neq 0$, b, and c, the axis of symmetry of

$$\{(x, y): y = ax^2 + bx + c\}$$

is ________________; the extreme point is ________________; the graph opens upward if _____, and opens downward if _____.

COMPLETING THE SQUARE

In graphing the quadratic function $\{(x, y): y = 2x^2 - 12x + 22\}$ we found that its axis of symmetry is $\{(x, y): x = 3\}$, that its extreme point is $(3, 4)$, that its graph opens upward, and that, to use the parabolic ruler to draw a graph of this function, we had to use a unit length on the graph of the y-axis half as long as the unit length on the graph of the x-axis. Now, consider the quadratic function $\{(x, y): y = 2(x - 3)^2 + 4\}$. What is its axis of symmetry? Its extreme point? Does its graph open upward or downward? What unit length would you choose on the graph of the y-axis to enable you to use your parabolic ruler in drawing the graph of this function? Are $\{(x, y): y = 2(x - 3)^2 + 4\}$ and $\{(x, y): y = 2x^2 - 12x + 22\}$ different functions?

If the extreme point of $\{(x, y): y = ax^2 + bx + c\}$ is (p, q) then

$$\{(x, y): y = ax^2 + bx + c\} = \{(x, y): y = a(\_\_\_\_\_\_)^2 + \_\_\_\_\}.$$

You now see that each quadratic function has a defining equation of the form:

$$y = a(x - p)^2 + q,$$

and, of course, each such equation for which $a \neq 0$ defines a quadratic function. When you have a defining equation of this form for a quadratic function, it is very easy to describe the important characteristics of the function. Its axis of symmetry is ______________; its extreme point is _______; the function has a minimum value, q, if _______; or a maximum value, ____, if _______; if, in drawing a graph, you use the <u>same</u> unit length on the graphs of the x- and y-axes, then the graph is narrower than the graph of the squaring function if $|a|$ ____, and is broader than the graph of the squaring function if ________.

Evidently, it would be useful to have an easy way of transforming an expression of the form:

$$ax^2 + bx + c$$

to an equivalent expression of the form:

$$a(x - p)^2 + q$$

You may feel that solving quadratic equations by "completing the square" is not a very efficient procedure as compared, say, to the use of the quadratic formula. However, transforming a quadratic expression by completing the square has many important applications, aside from the solution of quadratic equations.

In more advanced courses students will frequently find it necessary to convert an expression such as '$ax^2 + bx + c$' into the simpler form '$au^2 + q$' by completing the square and substituting 'u' for '$x + \frac{b}{2a}$' and 'q' for '$\frac{4ac - b^2}{4a}$'. In fact, they will probably go further and simplify '$2x^2 - 12x + 22$', for example, to '$v^2 + 2^2$', where $v = \sqrt{2}(x - 3)$. It will be important that they know that each quadratic expression can be transformed, in this way, into one of the four forms:

$$v^2 + d^2, \qquad -(v^2 + d^2), \qquad v^2 - d^2, \qquad d^2 - v^2,$$

and that they be skilled in carrying out such transformations. So, it is essential that students now become adept at completing the square.

*

Fill-ins for the nine blanks on page 5-178.

$x - p$; q; $\{(x, y): x = p\}$; (p, q); $a > 0$; q; $a < 0$; > 1; $|a| < 1$

Let us try to transform the expression:

$$x^2 - 12x + 40$$

to one of the form:

$$(x - p)^2 + q$$

We know that, for each p and q,

$$(x - p)^2 + q$$
$$= x^2 - 2px + p^2 + q.$$

Now, compare this last expression with the given one:

$$\begin{array}{c} x^2 - 2px + p^2 + q \\ \updownarrow \quad \updownarrow \quad \updownarrow \quad \searrow\!\swarrow \\ x^2 - 12x + \quad 40 \end{array}$$

If you substitute '6' for 'p' in the top expression, it becomes one which is equivalent to:

$$x^2 - 12x + 36 + q$$

Then, substitute '4' for 'q', simplify, and you get the given expression. So, '$(x - 6)^2 + 4$' is equivalent to '$x^2 - 12x + 40$'. [Check this by expanding in '$(x - 6)^2 + 4$' and simplifying.] We can see by inspection that the axis of symmetry of $\{(x, y): y = (x - 6)^2 + 4\}$ is $\{(x, y): x = 6\}$, and that the extreme point is (6, 4).

The process of transforming a quadratic expression in 'x', that is, an expression of the form:

$$ax^2 + bx + c \; [a \neq 0]$$

to one of the form:

$$a(x - p)^2 + q$$

is called completing the square. It is important that you become skillful at completing the square. Not only is this skill useful when you need to graph quadratic functions and [as you will see later] in solving quadratic equations, but it is essential for solving many problems which involve quadratic functions.

On the next page we give some quadratic expressions, and ones equivalent to them obtained by completing the square. You should study them with care.

$x^2 + 6x + 12 \ldots\ldots (x + 3)^2 + 3$

$x^2 - 8x - 5 \ldots\ldots (x - 4)^2 - 21$

$x^2 + 3x + 2 \ldots\ldots (x + \frac{3}{2})^2 - \frac{1}{4}$

$x^2 - x + 7 \ldots\ldots (x - \frac{1}{2})^2 + \frac{27}{4}$

$3x^2 + 12x + 5 \ldots\ldots 3(x + 2)^2 - 7$

$-x^2 + 5x + 4 \ldots\ldots -(x - \frac{5}{2})^2 + \frac{41}{4}$

$15x^2 + 7x - 2 \ldots\ldots 15(x + \frac{7}{30})^2 - \frac{169}{60}$

EXERCISES

A. Recall from Unit 3 the work you did in expanding expressions like '$(x - 7)^2$' and '$(y + 3)^2$'. As in all such manipulations, you gain skill by looking for a short cut and then practicing the short cut until you can use it almost without thinking. Take the expression '$(y + 3)^2$'. Let's expand it the long way:

$$\begin{aligned}
&(y + 3)^2 \\
&= (y + 3)(y + 3) && \text{[Why?]} \\
&= y(y + 3) + 3(y + 3) && \text{[Why?]} \\
&= y^2 + y3 + (3y + 9) && \text{[Why?]} \\
&= y^2 + (3y + 3y) + 9 && \text{[Why?]} \\
&= y^2 + (3 + 3)y + 9 && \text{[Why?]} \\
&= y^2 + 6y + 9
\end{aligned}$$

Now, expand '$(y + 5)^2$', but do it all in one step. Similarly, expand '$(y - 7)^2$' in one step.

Expand.

1. $(x - 2)^2$ 2. $(y + 4)^2$ 3. $(z - 8)^2$ 4. $(x - 1)^2$

5. $(p - 7)^2$ 6. $(y + 9)^2$ 7. $3(x + 5)^2$ 8. $-5(x - 4)^2$

9. $-(x - 4)^2$ 10. $(x - \frac{3}{2})^2$ [Answer. $x^2 - 3x + \frac{9}{4}$]

11. $(x - \frac{1}{2})^2$ 12. $(x + \frac{3}{2})^2$ 13. $(y + \frac{5}{2})^2$ 14. $(z - \frac{7}{2})^2$

Correction. In line 12, delete the colon after 'expression'.

Answers for the five 'Why?'s on page 5-180.

dpma; ℓdpma; cpm and apa; dpma; $3 + 3 = 6$

*

Answers for Part A [on pages 5-180 and 5-181].

1. $x^2 - 4x + 4$
2. $y^2 + 8y + 16$
3. $z^2 - 16z + 64$
4. $x^2 - 2x + 1$
5. $p^2 - 14p + 49$
6. $y^2 + 18y + 81$
7. $3x^2 + 30x + 75$
8. $-5x^2 + 40x - 80$
9. $-x^2 + 8x - 16$

[The answer for Exercise 10 is given in the text.]

11. $x^2 - x + \frac{1}{4}$
12. $x^2 + 3x + \frac{9}{4}$
13. $y^2 + 5y + \frac{25}{4}$
14. $z^2 - 7z + \frac{49}{4}$
15. $x^2 - \frac{x}{2} + \frac{1}{16}$
16. $x^2 + \frac{5x}{2} + \frac{25}{16}$
17. $x^2 - kx + \frac{k^2}{4}$
18. $y^2 + my + \frac{m^2}{4}$
19. $x^2 - \frac{2mx}{n} + \frac{m^2}{n^2}$
20. $x^2 - \frac{6x}{m} + \frac{9}{m^2}$
21. $y^2 - \frac{5y}{n} + \frac{25}{4n^2}$
22. $x^2 + \frac{bx}{a} + \frac{b^2}{4a^2}$

[Note that we have omitted restrictions against dividing by 0 in Exercises 19 - 22. Just make a brief mention of this restriction.]

*

In line 5 on page 5-181, we say "most of those in Part A" because the expressions obtained by expanding those in Exercises 8 and 9 of Part A are not perfect squares. [The expression in Exercise 7 is equivalent to '$(\sqrt{3}x + 5\sqrt{3})^2$'.]

*

Answers for Part B [on page 5-181].

1. Yes, $(x+3)^2$
2. No
3. Yes, $(x-2)^2$
4. No
5. Yes, $(x-10)^2$
6. Yes, $(x-6)^2$
7. Yes, $(x+\frac{5}{2})^2$
8. No
9. Yes, $(x-\frac{3}{2})^2$
10. No
11. Yes, $(x+a)^2$
12. No
13. Yes, $(x-4c)^2$
14. Yes, $(x+\frac{9m}{2})^2$
15. No
16. Yes, $(x+\frac{b}{2a})^2$

Answers for Part C [on pages 5-181 and 5-182].

1. $x^2 + 8x \; \boxed{+ 4^2}$; $\forall_x \; (x + 4)^2 = x^2 + 8x + 4^2$

2. $x^2 + 2x \; \boxed{+ 1^2}$; $\forall_x \; (x + 1)^2 = x^2 + 2x + 1^2$

3. $x^2 - 6x \; \boxed{+ 3^2}$; $\forall_x \; (x - 3)^2 = x^2 - 6x + 3^2$

4. $x^2 - 12x \; \boxed{+ 6^2}$; $\forall_x \; (x - 6)^2 = x^2 - 12x + 6^2$

5. $x^2 + 20x \; \boxed{+ 10^2}$; $\forall_x \; (x + 10)^2 = x^2 + 20x + 10^2$

6. $x^2 + 1000x \; \boxed{+ 500^2}$; $\forall_x \; (x + 500)^2 = x^2 + 1000x + (500)^2$

7. $x^2 - 500x \; \boxed{+ 250^2}$; $\forall_x \; (x - 250)^2 = x^2 - 500x + (250)^2$

8. $x^2 + 4bx \; \boxed{+ (2b)^2}$; $\forall_b \forall_x \; (x + 2b)^2 = x^2 + 4bx + (2b)^2$

9. $x^2 - 6kx \; \boxed{+ (3k)^2}$; $\forall_k \forall_x \; (x - 3k)^2 = x^2 - 6kx + (3k)^2$

10. $x^2 + 3x \; \boxed{+ (\frac{3}{2})^2}$; $\forall_x \; (x + \frac{3}{2})^2 = x^2 + 3x + (\frac{3}{2})^2$

11. $x^2 + 9x \; \boxed{+ (\frac{9}{2})^2}$; $\forall_x \; (x + \frac{9}{2})^2 = x^2 + 9x + (\frac{9}{2})^2$

12. $x^2 - 7x \; \boxed{+ (\frac{7}{2})^2}$; $\forall_x \; (x - \frac{7}{2})^2 = x^2 - 7x + (\frac{7}{2})^2$

13. $x^2 - \frac{x}{2} \; \boxed{+ (\frac{1}{4})^2}$; $\forall_x \; (x - \frac{1}{4})^2 = x^2 - \frac{x}{2} + (\frac{1}{4})^2$

14. $x^2 - kx \; \boxed{+ (\frac{k}{2})^2}$; $\forall_k \forall_x \; (x - \frac{k}{2})^2 = x^2 - kx + (\frac{k}{2})^2$

15. $x - \frac{x}{k} \; \boxed{+ (\frac{1}{2k})^2}$; $\forall_{k \neq 0} \forall_x \; (x - \frac{1}{2k})^2 = x^2 - \frac{x}{k} + (\frac{1}{2k})^2$

16. $x^2 - \frac{4x}{k} \; \boxed{+ (\frac{2}{k})^2}$; $\forall_{k \neq 0} \forall_x \; (x - \frac{2}{k})^2 = x^2 - \frac{4x}{k} + (\frac{2}{k})^2$

17. $x^2 - \frac{3x}{k} \; \boxed{+ (\frac{3}{2k})^2}$; $\forall_{k \neq 0} \forall_x \; (x - \frac{3}{2k})^2 = x^2 - \frac{3x}{k} + (\frac{3}{2k})^2$

18. $x^2 + \frac{bx}{a} \; \boxed{+ (\frac{b}{2a})^2}$; $\forall_{a \neq 0} \forall_b \forall_x \; (x + \frac{b}{2a})^2 = x^2 + \frac{bx}{a} + (\frac{b}{2a})^2$

15. $(x - \frac{1}{4})^2$ 16. $(x + \frac{5}{\text{[illegible]}})^2$ 17. $(x - \frac{k}{2})^2$ 18. $(y + \frac{m}{2})^2$

19. $(x - \frac{m}{n})^2$ 20. $(x - \frac{3}{m})^2$ 21. $(y - \frac{5}{2n})^2$ 22. $(x + \frac{b}{2a})^2$

B. Each of the following is a quadratic expression in 'x'. Tell which of them are <u>perfect squares</u>. [A perfect square is the expression you get when you expand an expression like most of those in Part A.]

<u>Sample 1.</u> $x^2 - 8x + 16$

<u>Solution.</u> This is a perfect square because

$$\forall_x \ (x - 4)^2 = x^2 - 8x + 16.$$

<u>Sample 2.</u> $x^2 - 3x + (\frac{3}{2})^2$

<u>Solution.</u> This is a perfect square because

$$\forall_x \ (x - \frac{3}{2})^2 = x^2 - 3x + (\frac{3}{2})^2.$$

1. $x^2 + 6x + 9$
2. $x^2 + 4x + 16$
3. $x^2 - 4x + 4$
4. $x^2 - 10x - 25$
5. $x^2 - 20x + 100$
6. $x^2 - 12x + 36$
7. $x^2 + 5x + (\frac{5}{2})^2$
8. $x^2 + 7x + \frac{7}{2}$
9. $x^2 - 3x + \frac{9}{4}$
10. $x^2 - 11x - \frac{121}{4}$
11. $x^2 + 2ax + a^2$
12. $x^2 - 4bx + 2b^2$
13. $x^2 - 8cx + 16c^2$
14. $x^2 + 9mx + (\frac{9m}{2})^2$
15. $x^2 + \frac{m}{n}x + \frac{m^2}{n^2}$
16. $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$

C. Each of the following is a quadratic expression in 'x'. Your job is to tack something onto it so that the resulting expression is a perfect square, and to prove that it is a perfect square.

<u>Sample.</u> $x^2 + 11x$

<u>Solution.</u> $x^2 + 11x \boxed{+ (\frac{11}{2})^2}$

<u>Proof.</u> $\forall_x \ (x + \frac{11}{2})^2 = x^2 + 11x + (\frac{11}{2})^2$

1. $x^2 + 8x$
2. $x^2 + 2x$
3. $x^2 - 6x$
4. $x^2 - 12x$
5. $x^2 + 20x$
6. $x^2 + 1000x$
7. $x^2 - 500x$
8. $x^2 + 4bx$

9. $x^2 - 6kx$ 10. $x^2 + 3x$ 11. $x^2 + 9x$ 12. $x^2 - 7x$

13. $x^2 - \frac{1}{2}x$ 14. $x^2 - kx$ 15. $x^2 - \frac{1}{k}x$ 16. $x^2 - \frac{4x}{k}$

17. $x^2 - \frac{3x}{k}$ 18. $x^2 + \frac{bx}{a}$

D. Transform each of the following quadratic expressions in 'x' by completing the square, and simplifying.

Sample 1. $x^2 + 3x + 7$

Solution. '$x^2 + 3x$' can be made into a perfect square by tacking on a '$+ (\frac{3}{2})^2$'. Suppose we just write:

$$x^2 + 3x + (\frac{3}{2})^2 + 7$$

This expression is not equivalent to the given one. But, we can get an equivalent expression by writing:

$$x^2 + 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2 + 7$$

This last expression is equivalent to:

$$(x + \frac{3}{2})^2 - (\frac{3}{2})^2 + 7$$

And, this is the expression in which the square has been completed. We simplify it:

$$(x + \frac{3}{2})^2 - \frac{9}{4} + 7$$

$$= (x + \frac{3}{2})^2 - \frac{9}{4} + \frac{28}{4}$$

$$= (x + \frac{3}{2})^2 + \frac{19}{4}$$

Answer. $(x + \frac{3}{2})^2 + \frac{19}{4}$

1. $x^2 + 6x + 2$ 2. $x^2 + 12x - 3$ 3. $x^2 - 8x + 9$

4. $x^2 - 16x + 30$ 5. $x^2 + 10x + 11$ 6. $x^2 - 4x + 2$

7. $x^2 + x - 7$ 8. $x^2 - x$ 9. $x^2 + 5x + 4$

10. $x^2 - 7x - 2$ 11. $x^2 + \frac{1}{2}x + 5$ 12. $x^2 - \frac{1}{3}x + 1$

13. $x^2 + 2mx + n$ 14. $x^2 - 4bx + d$ 15. $x^2 - \frac{x}{k} + \frac{c}{k}$

16. $x^2 + \frac{bx}{a} + \frac{c}{a}$

Answers for Part D [on pages 5-182 and 5-183].

1. $(x+3)^2 - 7$

2. $(x+6)^2 - 39$

3. $(x-4)^2 - 7$

4. $(x-8)^2 - 34$

5. $(x+5)^2 - 14$

6. $(x-2)^2 - 2$

7. $(x+\frac{1}{2})^2 - \frac{29}{4}$

8. $(x-\frac{1}{2})^2 - \frac{1}{4}$

9. $(x+\frac{5}{2})^2 - \frac{9}{4}$

10. $(x-\frac{7}{2})^2 - \frac{57}{4}$

11. $(x+\frac{1}{4})^2 + \frac{79}{16}$

12. $(x-\frac{1}{6})^2 + \frac{35}{36}$

13. $(x+m)^2 + (n-m^2)$

14. $(x-2b)^2 + (d-4b^2)$

15. $(x-\frac{1}{2k})^2 + \frac{4ck-1}{4k^2}$

16. $(x+\frac{b}{2a})^2 + \frac{4ac-b^2}{4a^2}$

17. $2(x-5)^2 - 43$

18. $5(x+2)^2 - 24$

19. $3(x+2)^2 - 13$

20. $2(x+\frac{1}{4})^2 - \frac{73}{8}$

21. $2(x-\frac{3}{4})^2 + \frac{23}{8}$

22. $-5(x-\frac{1}{2})^2 - \frac{11}{4}$

23. $a(x+\frac{5}{2})^2 + \frac{28-25a}{4}$

24. $a(x+\frac{b}{2a})^2 + \frac{4ac-b^2}{4a}$

Quiz.

1. Transform each of the following quadratic expressions in 'x' by completing the square and simplifying.

Sample. $x^2 + 7x + 2$

Solution. $(x + \frac{7}{2})^2 - \frac{41}{4}$

(a) $x^2 + 10x + 1$ (b) $x^2 - 18x - 3$

(c) $x^2 + x + 3$ (d) $-x^2 + 5x + 7$

(e) $x^2 + tx$ (f) $2x^2 - 12x + 5$

2. (a) Find the ordered pair of the function $\{(x, y): y = x^2 + 9x - 8\}$ whose second component is the extreme value of the function.

(b) Write an equation of the axis of symmetry of the function $\{(x, y): y = 8 + x(5 - x)\}$.

*

Answers for Quiz.

1. (a) $(x + 5)^2 - 24$ (b) $(x - 9)^2 - 84$ (c) $(x + \frac{1}{2})^2 + \frac{11}{4}$

(d) $-(x - \frac{5}{2})^2 + \frac{53}{4}$ (e) $(x + \frac{t}{2})^2 - \frac{t^2}{4}$ (f) $2(x - 3)^2 - 13$

2. (a) $(-\frac{9}{2}, -\frac{113}{4})$ (b) $x = \frac{5}{2}$

Answers for Part E.

1. $(-\frac{9}{2}, -\frac{113}{4})$; min

2. $(-\frac{9}{2}, \frac{113}{4})$; max

3. $(\frac{1}{5}, \frac{34}{5})$; min

4. $(\frac{5}{14}, \frac{277}{28})$; max

5. $(-\frac{1}{6}, \frac{11}{12})$; min

6. (1, 0); min

7. (2, 13); min

8. [There is no extreme point.]

9. There are two extreme points, (-1/2, 27/4) and (2, 13). The minimum value of the function is 27/4; the maximum value of this function is 13. [Exercises 7, 8, and 9 of Part E deal with functions which, while not themselves quadratic functions, are subsets of the quadratic function $\{(x, y) : y = x^2 + x + 7\}$. Graphing this quadratic function will help in explaining the answers for these three exercises. For Exercise 7, note that (2, 13) belongs to the function in question and that the graph of this point is the lowest point on the graph of the function. The range of the function is $\{y : y \geq 13\}$. For Exercise 8, note these three things:

   (1) (2, 13) does not belong to the graph of the function in question,

   (2) although there are values of the function as close to 13 as one wishes, all values of the function are greater than 13,

   (3) the function has no least value.

   The range of the function in Exercise 8 is $\{y : y > 13\}$, and the range of the function in Exercise 9 is $\{y : 27/4 \leq y \leq 13\}$.]

*

Answers for Part F [on pages 5-183 and 5-184].

1. $\forall_x (120 - 2x)x = -2(x - 30)^2 + 1800$. So, the extreme point of the function is (30, 1800). Hence, the pen should be 30 feet wide [and 60 feet long].

2. 25 yards by 50 yards

*

<u>Sample 2.</u> $3x^2 + 6x + 5$

<u>Solution.</u>

$$\begin{aligned} & 3x^2 + 6x + 5 \\ &= 3(x^2 + 2x) + 5 \\ &= 3(x^2 + 2x + 1 - 1) + 5 \\ &= 3[(x + 1)^2 - 1] + 5 \\ &= 3(x + 1)^2 - 3 + 5 \\ &= 3(x + 1)^2 + 2 \end{aligned}$$

17. $2x^2 - 20x + 7$
18. $5x^2 + 20x - 4$
19. $3x^2 + 12x - 1$
20. $2x^2 + x - 9$
21. $2x^2 - 3x + 4$
22. $-5x^2 + 5x - 4$
23. $ax^2 + 5ax + 7$
24. $ax^2 + bx + c$

<u>E.</u> For each function, find the ordered pair in it whose second component is the extreme value of the function, and tell whether the extreme value is a minimum or a maximum.

<u>Sample.</u> $\{(x, y): y = 3(x - 4)^2 + 7\}$

<u>Solution.</u> (4, 7); minimum

1. $\{(x, y): y = x^2 + 9x - 8\}$
2. $\{(x, y): y = -x^2 - 9x + 8\}$
3. $\{(x, y): y = 5x^2 - 2x + 7\}$
4. $\{(x, y): y = 9 + 5x - 7x^2\}$
5. $\{(a, b): b = 1 + a + 3a^2\}$
6. $\{(u, v): u^2 - v = 2u - 1\}$
7. $\{(x, y): x \geq 2 \text{ and } y = x^2 + x + 7\}$
8. $\{(x, y): x > 2 \text{ and } y = x^2 + x + 7\}$
9. $\{(x, y): -1 \leq x \leq 2 \text{ and } y = x^2 + x + 7\}$

<u>F.</u> 1. Solve the fencing problem given on page 5-166, without drawing a graph, by finding the extreme point of the quadratic function

$$\{(x, y): y = (120 - 2x)x\}.$$

2. A rectangular field is to be fenced off beside a river. No fence is needed along the river bank. What are the dimensions of the field of largest area which can be fenced off using 100 yards of fencing?

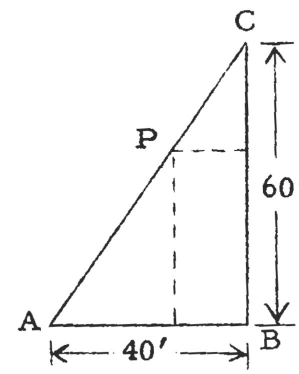
3. A man has 600 yards of fencing which he is going to use to enclose a rectangular field and to divide it down the middle with a fence parallel to one side. What are the dimensions of the largest field he can enclose?

4. A long rectangular sheet of metal 10 inches wide is to be made into a trough by bending up the sides. How deep should the trough be to carry the most water? [Cross section: ]

5. A small orchard now has 60 trees; it yields, on the average, 400 apples per tree. For each additional tree planted in this orchard the average yield per tree will be reduced by approximately 6 apples. How many trees will give the largest crop of apples for this orchard?

6. Prove that, of all rectangles with a given perimeter p, those which are squares have the greatest area. [Hint. If the width-measure of a rectangle of perimeter p is x then its length-measure is ______.]

7. A man wants to build a house, having a rectangular foundation, on a lot whose shape is such that, taking account of building restrictions, the foundation must not extend outside a right triangular region, as shown:



If one corner of the foundation is at B, how far should each of the adjacent corners be from B to make the area enclosed by the foundation as large as possible?
[Hint. If you think of A as the origin of the number plane and of B as a point on the positive half of the x-axis, what can you say about the slopes of segments $\overline{AP}$ and $\overline{AC}$?]

8. A potato grower wishes to ship as early as possible in the season in order to sell at the best price. If he ships July 1, he can ship 6 tons at a profit of $2.00 per ton. He estimates that by waiting he can add 3 tons per week to his shipment, but that the profit will be reduced $\frac{1}{3}$ dollar per ton per week. When should he ship for maximum profit?

[Supplementary exercises are in Part U, page 5-272.]

Corrections. [for page 5-185]
line 1: change 'a gardener' to 'A gardener'
line 6b: insert 'than' between 'sytematic' and 'the'

3. 100 yards by 150 yards

4. $2\frac{1}{2}$ inches

5. 63 [If one plots total yield against number of trees, one obtains points of the graph of $\{(x, y): y = x[400 - (x - 60)6]\}$. Since the maximum of this quadratic function corresponds with the argument $63\frac{1}{3}$ [and since $63\frac{1}{3} \geq 60$], 63 trees will produce the largest crop.]

6. In terms of its width-measure x, the length-measure of a rectangle of perimeter p is $\frac{p}{2} - x$, and its area-measure is $(\frac{p}{2} - x)x$. The argument corresponding with the maximum of $\{(x, y): y = \frac{px}{2} - x^2\}$ is $\frac{p}{4}$. So, the rectangles of perimeter p which have the greatest area are those whose width-measure is $\frac{p}{4}$--that is, those which are squares.

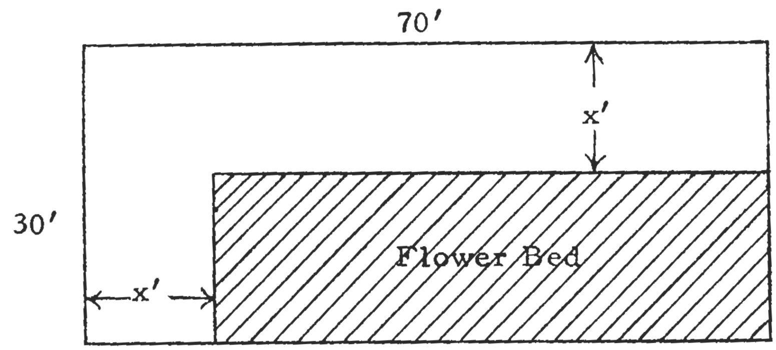
7. Since $\overleftrightarrow{AP}$ and $\overleftrightarrow{AC}$ have the same slope [or: by similar triangles], if x is the width-measure of the foundation and y is its length-measure, $\frac{y}{40 - x} = \frac{60}{40}$. So, the width-measure of the desired foundation is the argument of the extreme point of $\{(x, A): A = \frac{3}{2}(40 - x)x\}$. This argument is 20. So, the corner on $\overleftrightarrow{AB}$ should be 20 feet from B and the corner on $\overleftrightarrow{BC}$ should be 30 feet from B.

8. 2 weeks after July 1 [The argument of the extreme point of $\{(x, y): y = (6 + 3x)(2 - \frac{1}{3}x)\}$ is 2.]

*

We suggest that students work on Parts L, M, and N of the Miscellaneous Exercises [pages 5-225ff.] before they do the work on quadratic equations in section 5.10.

5.10 Quadratic equations.--a gardener is planning a flower bed which is to take up half the area of a 30′ by 70′ rectangular plot of land. The flower bed is to be rectangular in shape and situated in one corner of the plot in such a way that the rest of the plot can be paved to provide a sidewalk of uniform width on both sides of the bed.
How wide should the sidewalk be?



If the sidewalk is x feet wide, the dimensions of the flower bed are 70 - x feet and 30 - x feet, and its area is (70 - x)(30 - x) square feet. So, we are looking for a number x between 0 and 30 such that

$$(70 - x)(30 - x) = \frac{1}{2} \cdot 2100.$$

Let's try to solve this equation. Expand the left side.

$$2100 - 100x + x^2 = 1050$$

$$x^2 - 100x + 1050 = 0$$

This is a quadratic equation, and as you may recall from Unit 3, since the right side is '0', we should try to factor the left side. This turns out to be a difficult task. Try it and see.

Let's look for another method to give us factors, a method which is more systematic the usual trial-and-error procedure. The left side is a quadratic expression in 'x'. Let's complete the square.

$$x^2 - 100x + 1050$$

$$= x^2 - 100x + 50^2 - 50^2 + 1050$$

$$= (x - 50)^2 - 2500 + 1050$$

$$= (x - 50)^2 - 1450$$

So, the given equation is equivalent to:

$$(*) \qquad (x - 50)^2 - 1450 = 0$$

Now, we could solve this equation graphically by considering the quadratic function $\{(x, y): y = (x - 50)^2 - 1450\}$, graphing it, and finding the places where the graph crosses the horizontal axis. These are the graphs of points with second component 0. So, their first components are numbers which satisfy (*).

Another way to solve (*) is to recall how you would solve an equation like:

$$y^2 - 4^2 = 0$$

The left side is easily factored. So,

$$(y - 4)(y + 4) = 0,$$

$$y - 4 = 0 \text{ or } y + 4 = 0,$$

$$y = 4 \text{ or } y = -4.$$

Equation (*) can be handled the same way. We note that $(x - 50)^2$ is the square of $x - 50$, and that 1450 is the square of $\sqrt{1450}$. Hence, (*) is equivalent to:

$$(x - 50)^2 - (\sqrt{1450})^2 = 0,$$

$$[(x - 50) - \sqrt{1450}][(x - 50) + \sqrt{1450}] = 0,$$

$$x - 50 - \sqrt{1450} = 0 \text{ or } x - 50 + \sqrt{1450} = 0,$$

$$x = 50 + \sqrt{1450} \text{ or } x = 50 - \sqrt{1450}$$

So, the roots of (*) are $50 + \sqrt{1450}$ and $50 - \sqrt{1450}$. Since $30 < \sqrt{1450} < 40$, the larger root is between 80 and 90, and the smaller one is between 10 and 20. The larger root doesn't meet the conditions of our problem since we are looking for a number between 0 and 30. Hence, the sidewalk should be $50 - \sqrt{1450}$ feet wide.

The numeral '$50 - \sqrt{1450}$' is not very handy for a gardener since tape measures are not marked this way. The gardener would be satisfied with a rational number approximation to $50 - \sqrt{1450}$. First, we find a rational number approximation to $\sqrt{1450}$. Since 1450 is 14.5×10^2, $\sqrt{1450} = 10\sqrt{14.5}$. The table on page 5-219 gives 3.742 and 3.873 as rational approximations to $\sqrt{14}$ and $\sqrt{15}$. So, $\sqrt{14.5}$ is between 3.742 and 3.873. Hence, $37.42 < \sqrt{1450} < 38.73$. We also see from the table that 38^2 is 1444, which is less than 1450. So, $\sqrt{1450}$ is very close to 38. 50 - 38 = 12. So, the sidewalk should be about 12 feet wide.

The technique of completing the square is one which can be used in solving all quadratic equations in one variable. Of course, some quadratic equations involve quadratic expressions whose factors can be found readily by inspection. For example, we wouldn't take the trouble to complete the square in solving:

$$x^2 - 5x + 6 = 0$$

since it is easier to factor '$x^2 - 5x + 6$' by trial-and-error. [What are the roots?] But, we <u>could</u> find the factors by completing the square:

$$x^2 - 5x + (\frac{5}{2})^2 - (\frac{5}{2})^2 + 6 = 0$$

$$(x - \frac{5}{2})^2 - \frac{25}{4} + 6 = 0$$

$$(x - \frac{5}{2})^2 - \frac{1}{4} = 0$$

$$(x - \frac{5}{2})^2 - (\frac{1}{2})^2 = 0$$

$$[(x - \frac{5}{2}) - \frac{1}{2}][(x - \frac{5}{2}) + \frac{1}{2}] = 0$$

$$x - \frac{5}{2} = \frac{1}{2} \text{ or } x - \frac{5}{2} = -\frac{1}{2}$$

$$x = 3 \text{ or } x = 2$$

Practice using the method of completing the square in solving the quadratic equations '$x^2 - 8x - 20 = 0$' and '$x^2 + 3x + 2 = 0$'.

In solving a quadratic equation it is usually helpful to transform it to one of the form:

$$(*) \qquad ax^2 + bx + c = 0$$

This is called <u>the standard form of a quadratic equation in one variable</u>. An equation in one variable is a quadratic equation if and only if it can be transformed to an equation of the form of (*), in which the value of 'a' is not zero.

Example 1. Solve: $2x(x + 3) = 9(x + 1) - 2$

Solution. First, transform it to an equation of standard form.

$$2x(x + 3) = 9(x + 1) - 2$$
$$2x^2 + 6x = 9x + 9 - 2$$
$$2x^2 - 3x - 7 = 0$$

The factors are not apparent by inspection. So, we complete the square. First, multiply by $\frac{1}{2}$ to make it a little easier to complete the square.

$$\frac{1}{2}(2x^2 - 3x - 7) = 0 \cdot \frac{1}{2}$$
$$x^2 - \frac{3}{2}x - \frac{7}{2} = 0$$

Now, complete the square.

$$x^2 - \frac{3}{2}x + (\frac{3}{4})^2 - (\frac{3}{4})^2 - \frac{7}{2} = 0$$
$$(x - \frac{3}{4})^2 - \frac{9}{16} - \frac{56}{16} = 0$$
$$(x - \frac{3}{4})^2 - \frac{65}{16} = 0$$
$$[(x - \frac{3}{4}) - \sqrt{\frac{65}{16}}\,][(x - \frac{3}{4}) + \sqrt{\frac{65}{16}}\,] = 0$$
$$x - \frac{3}{4} = \sqrt{\frac{65}{16}} \text{ or } x - \frac{3}{4} = -\sqrt{\frac{65}{16}}$$
$$x = \frac{3}{4} + \sqrt{\frac{65}{16}} \text{ or } x = \frac{3}{4} - \sqrt{\frac{65}{16}}$$

The roots are $\frac{3}{4} + \sqrt{\frac{65}{16}}$ and $\frac{3}{4} - \sqrt{\frac{65}{16}}$. Can these expressions for the roots be simplified? Yes, this principle for real numbers:

$$\forall_{x \geq 0} \forall_{y > 0} \sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

assures us that $\frac{3}{4} + \sqrt{\frac{65}{16}} = \frac{3}{4} + \frac{\sqrt{65}}{\sqrt{16}} = \frac{3}{4} + \frac{\sqrt{65}}{4} = \frac{3 + \sqrt{65}}{4}$.

So, the roots are $\frac{3 + \sqrt{65}}{4}$ and $\frac{3 - \sqrt{65}}{4}$.

Quiz.

1. Solve these equations by completing the square. Show all of your work.

   (a) $x^2 + 4x - 1 = 20$ (b) $x^2 - 5x - 1 = 0$

2. Find the points in the intersection of $\{(x, y): y = 3(x - 1)^2\}$ and the x-axis.

*

Answers for Quiz.

1. (a) 3, -7 (b) $\frac{5 + \sqrt{29}}{2}$, $\frac{5 - \sqrt{29}}{2}$

2. (1, 0)

In order to prove:

$$(*) \quad \forall_{x \geq 0} \forall_{y > 0} \sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}},$$

one makes use of the fact that, by definition,

$$\forall_{u \geq 0} \sqrt{u} = \underline{\text{the}}\ v \text{ such that } v \geq 0 \text{ and } v^2 = u.$$

[The logical admissibility of the definition of '$\sqrt{\ }$' is reviewed on TC[5-80] in the discussion for Sample 2.]

<u>Proof of (*):</u>

Since $x \geq 0$ and $y > 0$, it follows that $x \geq 0$ and $y \geq 0$.
So, by definition, since $x \geq 0$, and $y \geq 0$,

$$\sqrt{x} = \text{the } u \text{ such that } u \geq 0 \text{ and } u^2 = x$$

and $$\sqrt{y} = \text{the } v \text{ such that } v \geq 0 \text{ and } v^2 = y.$$

Since $y > 0$, $y \neq 0$. Since $y \neq 0$ and $0^2 = 0$, $\sqrt{y} \neq 0$.
So, $\sqrt{x} \geq 0$ and $\sqrt{y} > 0$. Hence,

$$\frac{\sqrt{x}}{\sqrt{y}} \geq 0.$$

Also $$\left(\frac{\sqrt{x}}{\sqrt{y}}\right)^2 = \frac{(\sqrt{x})^2}{(\sqrt{y})^2} = \frac{x}{y}.$$

Consequently,

$$\frac{\sqrt{x}}{\sqrt{y}} = \text{the } w \text{ such that } w \geq 0 \text{ and } w^2 = \frac{x}{y}.$$

So, by definition, $\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$.

[For additional comments on similar topics, see TC[3-110, 111]a, b, and c.]

Answers for Part A.

1. $-3 + 4\sqrt{2}$, $-3 - 4\sqrt{2}$

2. $-4 + \sqrt{21}$, $-4 - \sqrt{21}$

3. $\frac{-5 + \sqrt{29}}{2}$, $\frac{-5 - \sqrt{29}}{2}$

4. $\frac{9 + \sqrt{65}}{2}$, $\frac{9 - \sqrt{65}}{2}$

5. $1 + 2\sqrt{2}$, $1 - 2\sqrt{2}$

6. 3, -4

7. $\frac{-9 + \sqrt{145}}{4}$, $\frac{-9 - \sqrt{145}}{4}$

8. $\frac{3}{2}$, -1

9. $\frac{7 + \sqrt{61}}{6}$, $\frac{7 - \sqrt{61}}{6}$

10. $4 + 2\sqrt{3}$, $4 - 2\sqrt{3}$

11. $\frac{-1 + \sqrt{19}}{3}$, $\frac{-1 - \sqrt{19}}{3}$

12. $\frac{-1 + \sqrt{61}}{10}$, $\frac{-1 - \sqrt{61}}{10}$

13. -4

14. 5

*

Answers for Part B [on page 5-190].

1. $(-7 + 3\sqrt{21}, 0)$, $(-7 - 3\sqrt{21}, 0)$

2. (12, 0), (8, 0)

3. (4, 0)

4. (5, 0), $(-7, 0)$

5. (5, 0), $(-\frac{3}{2}, 0)$

6. (4, 0)

7. (2, 0), $(-2, 0)$

8. none

9. (5, 0), (1, 0)

10. $(-1, 0)$, (2, 0)

☆11. If $p^2 - 4q \geq 0$, the intersection consists of $(\frac{-p + \sqrt{p^2 - 4q}}{2}, 0)$, and $(\frac{-p - \sqrt{p^2 - 4q}}{2}, 0)$. If $p^2 - 4q < 0$, the intersection is $\emptyset$.

☆12. If $b^2 - 4ac \geq 0$, the intersection consists of $(\frac{-b + \sqrt{b^2 - 4ac}}{2a}, 0)$, and $(\frac{-b - \sqrt{b^2 - 4ac}}{2a}, 0)$. If $b^2 - 4ac < 0$, the intersection is $\emptyset$.

*

B. For each exercise, find the points [if any] in the intersection of the given function with the x-axis.

Sample. $\{(x, y): y = x^2 - 10x + 26\}$

Solution. We could do this problem by graphing the function and estimating the first components of the points in the intersection of the function with the x-axis. But, it is much easier just to find the roots of the equation:

$$x^2 - 10x + 26 = 0$$

We solve by completing the square.

$$(x - 5)^2 - 25 + 26 = 0$$
$$(x - 5)^2 + 1 = 0$$

Now, '$(x - 5)^2 + 1$' cannot be factored as in the other cases because it is not an expression of the form '$A^2 - B^2$'. In fact, since $(x - 5)^2$ is nonnegative for each x, there is no real number x such that $(x - 5)^2 + 1 = 0$. Hence, the quadratic equation has no roots, and the intersection of the function with the x-axis is $\emptyset$.

1. $\{(x, y): y = x^2 + 14x - 140\}$
2. $\{(x, y): y = x^2 - 20x + 96\}$
3. $\{(x, y): y = 3x - 12\}$
4. $\{(x, y): y = (x - 5)(x + 7)\}$
5. $\{(x, y): y = 2x^2 - 7x - 15\}$
6. $\{(x, y): y = x^2 - 8x + 16\}$
7. $\{(x, y): y = 5x^2 - 20\}$
8. $\{(x, y): y = 5x^2 + 20\}$
9. $\{(x, y): y = (x - 3)^2 - 4\}$
10. $\{(x, y): y = (x + 1)(x - 2)^2\}$

☆11. $\{(x, y): y = x^2 + px + q\}$

☆12. $\{(x, y): y = ax^2 + bx + c\}$, $[a \neq 0]$

THE QUADRATIC FORMULA

Since a quadratic equation in one variable can be transformed to standard form:

$$ax^2 + bx + c = 0, \; [a \neq 0]$$

the solution set of the quadratic equation is completely determined by the numbers a, b, and c. Similarly, since each <u>linear equation in one variable</u> can be transformed to standard form:

$$(*) \qquad ax + b = 0, \; [a \neq 0]$$

> <u>Correction.</u> On page 5-192, line 4b:
> ... call this <u>the</u> <u>quadratic</u> <u>formula</u> ...

Note that the discussion on pages 5-191 and 5-192 shows that, under the restriction '$a \neq 0$ and $b^2 - 4ac \geq 0$' the sentences:

$$ax^2 + bx + c = 0$$

and:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are <u>equivalent</u>. For example, subject to the restriction '$a \neq 0$', '$ax^2 + bx + c = 0$' is equivalent to '$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$' because of the principle:

$$\forall_u \forall_{v \neq 0} [u = 0 \Longleftrightarrow u/v = 0]$$ [Th. 53 and Th. 54 on TC[2-61]g],

and the sentences in lines 4 and 5 on page 5-192 are equivalent, subject to the restriction '$a \neq 0$ and $b^2 - 4ac \geq 0$' because of the principle:

$$\forall_u \forall_v [uv = 0 \Longleftrightarrow (u = 0 \text{ or } v = 0)]$$ [The cpm, the pm0, and Th. 56 on TC[2-61]g].

Establishing equivalence is important in justifying the use of the quadratic formula in solving quadratic equations. For, if one were to note only that each of the sentences in question is a consequence of the preceding sentence, then all he would be justified in concluding is that if a quadratic equation [for which the restrictions are satisfied] has a root then it is one of the numbers "given" by the quadratic formula. In this case he would have no reason to believe that a number given by the formula was in fact a root until he had checked by substituting in the given equation. But, once one has seen that the equation and the formula are equivalent, there is no need to check roots given by the formula [except for the purpose of discovering errors in arithmetic].

*

The '$|2a|$' [here, of course, an abbreviation for '$^+|2a|$'] can, as is done in the text, be replaced by '$2a$' for the following reason. The restriction '$a \neq 0$' is equivalent to '$a > 0$ or $a < 0$', giving us two cases. If $a > 0$ then $^+|2a| = 2a$; so, the replacement is justified in this case. If $a < 0$ then $^+|2a| = -2a$; so, in this case, '$^+|2a|$' may be replaced by '$-2a$'. If this is done, one obtains [after a little simplification] the same formulas for the roots as in the first case.

Example 2. Solve: $5x - \frac{2}{x} + 10 = 0$

Solution. Transform to an equation in standard form.

$$5x - \frac{2}{x} + 10 = 0$$

$$x(5x - \frac{2}{x} + 10) = 0 \cdot x$$

$$5x^2 - 2 + 10x = 0$$

$$5x^2 + 10x - 2 = 0$$

[All of the roots of the given equation are roots of this last equation. However, if the last equation has 0 as a root, this root is not a root of the given equation. Why?] Now, complete the square. First, multiply by $\frac{1}{5}$.

$$x^2 + 2x - \frac{2}{5} = 0$$

$$(x + 1)^2 - 1 - \frac{2}{5} = 0$$

$$(x + 1)^2 - \frac{7}{5} = 0$$

$$[(x + 1) - \sqrt{\frac{7}{5}}][(x + 1) + \sqrt{\frac{7}{5}}] = 0$$

$$x + 1 = \sqrt{\frac{7}{5}} \text{ or } x + 1 = -\sqrt{\frac{7}{5}}$$

$$x = -1 + \sqrt{\frac{7}{5}} \text{ or } x = -1 - \sqrt{\frac{7}{5}}$$

The roots are $-1 + \sqrt{\frac{7}{5}}$ and $-1 - \sqrt{\frac{7}{5}}$.

EXERCISES

A. Solve these equations by completing the square.

1. $x^2 + 6x - 23 = 0$
2. $x^2 + 8x - 5 = 0$
3. $x^2 + 5x - 1 = 0$
4. $x^2 - 9x + 4 = 0$
5. $x^2 - 2x = 7$
6. $x^2 + x - 9 = 3$
7. $2x^2 + 9x - 8 = 0$
8. $2x^2 - x - 3 = 0$
9. $3x^2 - 2x - 1 = 5x$
10. $x^2 + 4 = 8x$
11. $2(3 - x) = 3x^2$
12. $4x^2 + x + 8 = 11 - x^2$
13. $x^2 + 8x + 16 = 0$
14. $x^2 - 10x + 25 = 0$

nonnegative. Thus, we can factor if we suppose that $b^2 - 4ac \geq 0$. Let's suppose this, and continue solving for 'x'.

$$(x + \frac{b}{2a})^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2 = 0$$

$$\left[(x + \frac{b}{2a}) - \sqrt{\frac{b^2 - 4ac}{4a^2}}\right]\left[(x + \frac{b}{2a}) + \sqrt{\frac{b^2 - 4ac}{4a^2}}\right] = 0$$

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \text{ or } x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{4a^2}} \text{ or } x = -\frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Now, since $b^2 - 4ac \geq 0$ and $4a^2 > 0$,

$$\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{|2a|}.$$

Thus, the roots are

$$-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{|2a|} \text{ and } -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{|2a|}.$$

[If a is positive then $|2a| = 2a$, and if a is negative, $|2a| = -2a$.]

So, we have proved the theorem:

$$\forall_{a \neq 0} \forall_b \forall_c \text{ such that } b^2 - 4ac \geq 0,$$

$$\{x: ax^2 + bx + c = 0\} = \{\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}\}$$

[What happened to the '$|2a|$'?]

Sometimes, in order to be brief, we write:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and call this the quadratic formula. The symbol '±' [read as 'plus or minus'] reminds you that each of the numbers

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

is a root of the equation '$ax^2 + bx + c = 0$'.

EXERCISES

A. Use the quadratic formula to solve each equation. If no real number satisfies it, say so.

1. $2x^2 - 7x - 5 = 0$
2. $3x^2 + x - 7 = 0$
3. $5x^2 + x = 1$
4. $2x(2x + 7) = 2x - 3$
5. $6x - 8x^2 + 7 = 0$
6. $8 + 3y - 5y^2 = 0$
7. $3n(2 - n) = 5n + 6$
8. $2x(5 + x) = x^2 + x(4 + x)$
9. $5p^2 + 2p = 0$
10. $5 - 11x^2 = 0$
11. $2x + \frac{3}{x} + 6 = 0$
12. $5x + \frac{7}{x} = 0$
13. $(1 - x)^2 + 3(1 - x) - 5 = 0$
14. $3(\frac{1}{x})^2 + \frac{2}{x} - 15 = 0$

☆15. $(\sqrt{x})^2 + 2\sqrt{x} - 4 = 0$

☆16. $\sqrt{x} - x - 3 = 0$

B. For each quadratic equation, tell whether it has roots. If it does, tell whether it has two roots or one root.

Sample. $x^2 + 6x + 9 = 0$

Solution. By the quadratic formula,

$$x = \frac{-6 \pm \sqrt{36 - 4 \cdot 1 \cdot 9}}{2}$$

$$= \frac{-6 \pm \sqrt{0}}{2}.$$

Since $-6 + 0 = -6$ and $-6 - 0 = -6$, the equation has exactly one root, the real number -3. [Do you see a quick way to determine whether a quadratic equation has 0, 1, or 2 real number roots?]

1. $x^2 - 8x + 16 = 0$
2. $x^2 - x - 3 = 0$
3. $x^2 - 7 = 0$
4. $8x^2 + 24x + 9 = 0$
5. $8x^2 - 24x + 9 = 0$
6. $(x - 3)^2 - 5 = 0$
7. $(x - 3)^2 = 0$
8. $(x - 3)^2 + 5 = 0$
9. $5 - (3 - x)^2 = 0$

[A quadratic equation like the one in the Sample is sometimes said to have a double root. Then, if you count a double root as two roots, you can say that each quadratic equation whose solution set is not $\emptyset$ has two roots.]

the solution set of a linear equation is completely determined by the numbers a and b. In fact, for each $a \neq 0$, for each b,

$$(**) \qquad \{x: ax + b = 0\} = \{-\frac{b}{a}\}.$$

It would be helpful if we could find a similar result for quadratic equations. This would then give us a quick way of finding the solution set of a quadratic equation without bothering to hunt for factors or to complete the square.

Equation (**) was derived from (*) simply by solving (*) for 'x'. So, let's try to solve:

$$ax^2 + bx + c = 0$$

for 'x'.

We solve by the method of completing the square.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + (-\frac{b^2}{4a^2} + \frac{c}{a}) = 0$$

$$(x + \frac{b}{2a})^2 + \frac{-b^2 + 4ac}{4a^2} = 0$$

$$(x + \frac{b}{2a})^2 + \frac{-(b^2 - 4ac)}{4a^2} = 0$$

$$(x + \frac{b}{2a})^2 + -\frac{b^2 - 4ac}{4a^2} = 0$$

$$(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

Now, in order to be able to factor, we must be sure that $\frac{b^2 - 4ac}{4a^2}$ is the square of a real number. Hence, it must be nonnegative. [Or, to put it another way, if $\frac{b^2 - 4ac}{4a^2}$ is negative, since $(x + \frac{b}{2a})^2$ is nonnegative, there is no number x such that $(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2} = 0$.] We know that $4a^2$ is positive. So, $\frac{b^2 - 4ac}{4a^2}$ is nonnegative if and only if $b^2 - 4ac$ is

Let's use the quadratic formula in solving the equation:

$$x(2x + 3) = 3(1 - x)$$

First, we transform this equation into one of standard form.

$$2x^2 + 3x = 3 - 3x$$

$$2x^2 + 6x - 3 = 0$$

Here, a = 2, b = 6, and c = −3. So, the roots can be obtained from the quadratic formula:

$$\frac{-6 \pm \sqrt{6^2 - 4 \cdot 2 \cdot -3}}{2 \cdot 2}$$

$$= \frac{-6 \pm \sqrt{60}}{4}$$

$$= \frac{-6 \pm \sqrt{4 \cdot 15}}{4}$$

$$= \frac{-6 \pm \sqrt{4} \cdot \sqrt{15}}{4}$$

$$= \frac{-6 \pm 2\sqrt{15}}{4}$$

$$= \frac{2(-3 \pm \sqrt{15})}{4}$$

$$= \frac{-3 \pm \sqrt{15}}{2}$$

The roots are $\frac{-3 + \sqrt{15}}{2}$ and $\frac{-3 - \sqrt{15}}{2}$.

Let's try another quadratic equation.

$$x^2 - 5x + 7 = 0$$

This equation is in standard form, and a = 1, b = −5, and c = 7. If we apply the quadratic formula and simplify, we get:

$$\frac{5 \pm \sqrt{-3}}{2}$$

But, '$\sqrt{-3}$' doesn't stand for a real number. Recall, from our derivation of the quadratic formula, that the given equation has real number roots just if $b^2 - 4ac \geq 0$. In the case at hand, $b^2 - 4ac = -3$ and $-3 \not\geq 0$. So, no real number satisfies the given equation.

In discussing the exercises of Part A it will be wise to ask the class whether an equation such as '$2a^2 - 7a - 5 = 0$' is equivalent to the equation of Exercise 1. Students need to recognize that, even though the quadratic formula was derived from a quadratic equation of the form '$ax^2 + bx + c = 0$', it may be used to solve any quadratic equation in one variable. Often students think that they cannot use the quadratic formula to solve a quadratic equation for which the variable is 'a' [or 'y', or 'n', or ...].

Also, even though the first example on page 5-193 points out that, in order to use the quadratic formula, it is helpful to first transform a given equation into standard form [if it isn't] and then identify the values to be used for 'a', 'b', and 'c' in the formula, these points will probably need further emphasis. In fact, it may be advisable to assign first the task of transforming to standard form each of the given equations which is not in standard form. Then, ask students to name the values to be used for 'a', 'b', and 'c' in each case. After students become skillful in doing these things, the task of using the formula to obtain roots is relatively easy.

*

Answers for Part A.

1. $\frac{7 + \sqrt{89}}{4}, \frac{7 - \sqrt{89}}{4}$
2. $\frac{-1 + \sqrt{85}}{6}, \frac{-1 - \sqrt{85}}{6}$
3. $\frac{-1 + \sqrt{21}}{10}, \frac{-1 - \sqrt{21}}{10}$
4. $\frac{-3 + \sqrt{6}}{2}, \frac{-3 - \sqrt{6}}{2}$
5. $\frac{3 + \sqrt{65}}{8}, \frac{3 - \sqrt{65}}{8}$
6. 1.6, −1
7. [no real roots]
8. 0 [linear equation]
9. 0, −0.4
10. $\frac{\sqrt{55}}{11}, -\frac{\sqrt{55}}{11}$
11. $\frac{-3 + \sqrt{3}}{2}, \frac{-3 - \sqrt{3}}{2}$
12. [no real roots]
13. $\frac{5 + \sqrt{29}}{2}, \frac{5 - \sqrt{29}}{2}$

[In solving the equation of Exercise 13, it may help the students to suggest that, if '□' were substituted for '(1 - x)', the equation would become:

$$\square^2 + 3\square - 5 = 0$$

and, by the quadratic formula, this equation is equivalent to the sentence:

$$\square = \frac{-3 + \sqrt{9 + 20}}{2} \quad \text{or} \quad \square = \frac{-3 - \sqrt{9 + 20}}{2}$$

Hence, the given equation is equivalent to:

$$1 - x = \frac{-3 + \sqrt{9 + 20}}{2} \quad \text{or} \quad 1 - x = \frac{-3 - \sqrt{9 + 20}}{2}$$

which can be simplified to: $x = \frac{5 - \sqrt{29}}{2}$ or $x = \frac{5 + \sqrt{29}}{2}$.]

14. $\frac{1 + \sqrt{46}}{15}$, $\frac{1 - \sqrt{46}}{15}$

[This exercise should be handled in a manner similar to that suggested for Exercise 13. If we use '$\square$' for '$(\frac{1}{x})$', the equation becomes:

$$3\square^2 + 2\square - 15 = 0$$

and, by the quadratic formula, is equivalent to:

$$\square = \frac{-2 + \sqrt{4 + 180}}{6} \quad \text{or} \quad \square = \frac{-2 - \sqrt{4 + 180}}{6}$$

Hence, the given equation is equivalent to:

$$\frac{1}{x} = \frac{-2 + \sqrt{184}}{6} \quad \text{or} \quad \frac{1}{x} = \frac{-2 - \sqrt{184}}{6}$$

which can be simplified to '$x = \frac{3}{-1 + \sqrt{46}}$ or $x = \frac{3}{-1 - \sqrt{46}}$'.

You may wish at this time to show students how, by "rationalizing denominators" this last sentence can be transformed to:

$$x = \frac{1 + \sqrt{46}}{15} \quad \text{or} \quad x = \frac{1 - \sqrt{46}}{15},$$

thereby making it easier to find rational approximations. An alternate procedure is to transform the given equation into '$15x^2 - 2x - 3 = 0$', and apply the quadratic formula.]

☆15. $6 - \sqrt{20}$ [only one root]

[In solving the equation of Exercise 15, use a procedure similar to that for Exercises 13 and 14 to show that, by the quadratic formula, this equation is equivalent to:

$$\sqrt{x} = \frac{-2 + \sqrt{4 + 16}}{2} \quad \text{or} \quad \sqrt{x} = \frac{-2 - \sqrt{4 + 16}}{2}$$

that is, to the sentence '$\sqrt{x} = -1 + \sqrt{5}$ or $\sqrt{x} = -1 - \sqrt{5}$'.

approximation is at most 0.0005/4, or 0.000125. For Exercise 2, the rounding-off error for the larger root is 0; so, this approximation is in error by at most 0.0005/6, or $0.00008\overline{3}$. But, the rounding-off error for the smaller root is $0.0000\overline{3}$; so, the best we can say is that the approximation is in error by at most $0.00008\overline{3} + 0.0000\overline{3}$, or $0.0001\overline{16}$.]

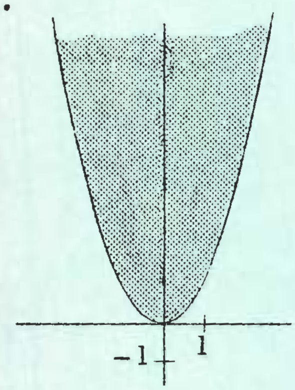
Rational approximations to the roots of certain equations in Part A [on page 5-194].

1. 4.1085, −0.6085 [0.00013]
2. 1.3700, −1.7033 [0.00012]
3. 0.3583, −0.5583 [0.00005]
4. −0.2755, −2.7245 [0.00025]
5. 1.3828, −0.6328 [0.00012]
10. 0.6742, −0.6742 [0.00007]
11. −0.6340, −2.3660 [0.00025]
13. 5.1925, −0.1925 [0.00025]
14. 0.5188, −0.3855 [0.00007]
15. 1.528 [0.0005]

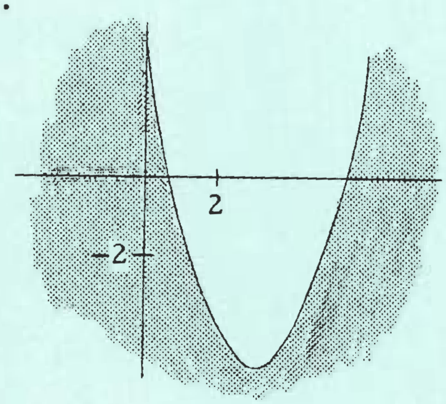
*

Answers for Part ☆F [on page 5-195].

1.



2.



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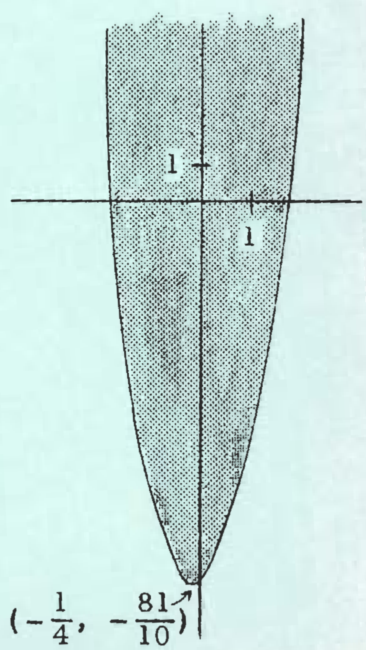
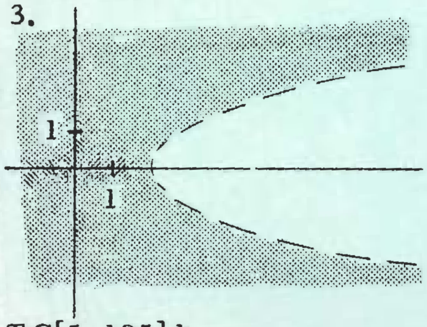
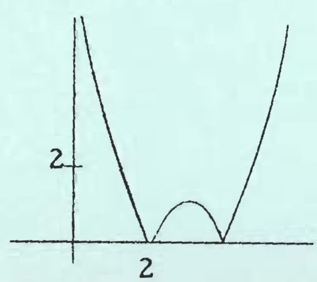


3.



5.



Since a principal square root is never negative, and since $-1 - \sqrt{5} < 0$, this last sentence is equivalent to '$\sqrt{x} = -1 + \sqrt{5}$'. Since $-1 + \sqrt{5} \geq 0$, this equation is equivalent to '$x = (-1 + \sqrt{5})^2$'. [In computing an approximation to $(-1 + \sqrt{5})$ it is more advantageous to use the form '$6 - \sqrt{20}$', rather than '$6 - 2\sqrt{5}$'. This is the case because the approximation obtained from the table for either $\sqrt{20}$ or $\sqrt{5}$ may be in error by as much as 0.0005, and multiplying the latter approximation by -2 yields an approximation to $\sqrt{20}$ which may be in error by as much as 2×0.0005.].]

☆16. [no real roots]

[The equation in Exercise 16 can be transformed like this:

$$\sqrt{x} - x - 3 = 0$$

} ps, cpa

$$-x + \sqrt{x} - 3 = 0$$

} MTP $[-1 \neq 0]$

$$x - \sqrt{x} + 3 = 0$$

} $\forall_{u \geq 0}\ \sqrt{u}$ is the $v > 0$ such that $v^2 = u$

$$(\sqrt{x})^2 - \sqrt{x} + 3 = 0$$

Since, for this "quadratic equation in '$\sqrt{x}$'", $b^2 - 4ac = 1 - 12 < 0$, there are no values of '$\sqrt{x}$' which satisfy the equation. Hence, there are no values of 'x' which satisfy it.]

*

Answers for Part B [on page 5-194].

1. one	2. two	3. two	4. two	5. two
6. two	7. one	8. none	9. two	

*

It is, of course, incorrect [though commonly done] to say that, for example, '$x^2 - 6x + 9 = 0$' has two roots, each of which is 3. Rather than comparing the <u>number</u> of roots of a polynomial equation with its degree, one should consider the <u>sum of the multiplicities</u> of its roots. For example, '$(x - 3)^2(x + 5)(x - 2)^3 = 0$' is an equation of degree 6 which has, not 6, but only 3 roots. However, one of these is a root of multiplicity 2, a second is of multiplicity 1, and the third is of multiplicity 3; and $2 + 1 + 3 = 6$, the degree of the equation. The notion of multiplicity of roots will be discussed in a later unit.

3. 1, −1.7 [1 and $-\frac{5}{3}$ are the exact roots.]

4. 1.3, −0.3 [1.264 and −0.264 are correct to the nearest 0.001. In fact, each is in error by at most 0.00025.]

5. 3.3, −3.3 [3.317 and −3.317 are correct to the nearest 0.001.]

6. 6.5, 1.9 [6.482 and 1.851 are correct to the nearest 0.001. The error in the first is at most $0.00008\overline{3}$; the error in the second is at most $0.00041\overline{6}$.]

*

Answers for Part ☆D [on page 5-195].

1. $x^2 - 7x + 12 = 0$ 2. $x^2 - 10x + 25 = 0$ 3. $x^2 = 0$ 4. $x^2 - 10 = 0$

5. $x^2 - 64 = 0$ 6. $x^2 + \frac{1}{12}x - \frac{1}{2} = 0$, [or: $12x^2 + x - 6 = 0$]

7. $x^2 - (r_1 + r_2)x + r_1 r_2 = 0$ 8. $x^2 - 4x - 1 = 0$

[After solving Exercise 7 of Part D, students should see a short cut for solving Exercise 8, and may find it interesting to use this short cut to check their answers for the earlier exercises.]

*

Answers for Part ☆E [on page 5-195].

1. $\sqrt{2}$, $-2\sqrt{2}$ 2. $\frac{\sqrt{3} + \sqrt{6}}{2}$, $\frac{\sqrt{3} - \sqrt{6}}{2}$

*

If you would like to give your students more practice in finding rational approximations to the roots of quadratic equations whose roots are irrational, you might have them find such approximations for the equations of Exercises 1 - 5, 10, 11, 13, 14, and 15 in Part A on page 5-194. For your convenience, we list approximations to the roots which are correct to the nearest 0.0001. [These approximations were obtained by using the table on page 5-219.] We also give an estimate of the error in these approximations.

[In estimating errors we have used, in each case, estimates of the actual rounding-off error. This is never more than 0.00005, and is usually less. Thus, for Exercise 1, the rounding-off error in the approximations to both roots is 0. So, the total error in each

However, if we round off, and take either 3.608 or 3.609 as an approximation to $\frac{5+\sqrt{89}}{4}$, then there is an additional error of [in this case] precisely 0.0005. So, the best we can say is that 3.608 [and 3.609] is in error by at most 0.000625.

In general, to estimate the error in a rational approximation to a root obtained by using the quadratic formula, one must add an estimate of the error made in approximating the square root and an estimate of the rounding-off error. If one rounds off the result of the division to 3 decimal places [thus, approximating the quotient correct to the nearest 0.001], the rounding-off error is never more than 0.0005. So, the error in the root is at most $0.0005 + \frac{0.0005}{|2a|}$, or $(1 + \frac{1}{|2a|}) \times 0.0005$. [But, as pointed out in examples which follow, one can usually find a better estimate of the rounding-off error, and so decrease one's estimate of the error in the root.]

If $|a| \geq 1$ then this estimate is at most $\frac{3}{2} \times 0.0005$, or 0.00075. If a = 2, the estimate is, as in the example above, $\frac{5}{4} \times 0.0005$, or 0.000625. If one rounds off the division to 4 decimal places [that is, correct to the nearest 0.0001] then the error in the root is at most $0.00005 + \frac{0.0005}{|2a|}$, or $(\frac{1}{10} + \frac{1}{|2a|}) \times 0.0005$. In this case, if $|a| \geq 1$, the estimate is <u>at most</u> $\frac{6}{10} \times 0.0005$, or 0.0003. If $|a| \geq 5$, the estimate is <u>at most</u> 0.0001. These results show that it is worthwhile to round off the division to 4 decimal places [correct to the nearest 0.0001] even when using a 3-place table of square roots. Although the approximations obtained will not in general be correct to the fourth decimal place, they will be more accurate than those obtained by rounding off to 3 decimal places.

*

Answers for Part C [on page 5-195].

1. 6.1, −1.1 [6.14 and −1.14 are correct to the nearest 0.001. In fact, each is in error by at most 0.00025.]

2. 1.3, −6.3 [1.275 and −6.275 are correct to the nearest 0.001. In fact, each is in error by at most 0.00025.]

It may be fruitful to have the class spend some time considering the possible error in the rational approximations to irrational roots of quadratic equations. We know that, for any quadratic equation in one variable, if $b^2 - 4ac \geq 0$, the roots are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Let's consider the problem of approximating the first of these roots. Suppose y is a rational approximation to $\sqrt{b^2 - 4ac}$ correct to the nearest 0.001. [Such approximations are given in the table on page 5-219.] We know then that

$$-0.0005 < \sqrt{b^2 - 4ac} - y < 0.0005.$$

So,

$$-0.0005 \leq (-b + \sqrt{b^2 - 4ac}) - (-b + y) < 0.0005.$$

Hence,

$$|(-b + \sqrt{b^2 - 4ac}) - (-b + y)| \leq 0.0005,$$

and

$$\left| \frac{-b + \sqrt{b^2 - 4ac}}{2a} - \frac{-b + y}{2a} \right| \leq \frac{0.0005}{|2a|}.$$

Therefore, the absolute error introduced by using an approximation which is correct to the nearest 0.001 is at most $\frac{0.0005}{|2a|}$. [You can easily see that this is also the absolute error introduced by using the same approximation to $\sqrt{b^2 - 4ac}$ in finding a rational approximation to the other root of the equation.] For example, the larger root of the equation of the Sample is $\frac{5 + \sqrt{89}}{4}$. And, as noted in the solution, the table on page 5-219 gives 9.434 as the approximation to $\sqrt{89}$ correct to the nearest 0.001. So, the absolute error in using $\frac{5 + 9.434}{4}$, as an approximation to the root, is $\frac{0.0005}{4}$, or 0.000125. Now,

$$\frac{5 + 9.434}{4} = \frac{14.434}{4}$$

$$= 3.6085.$$

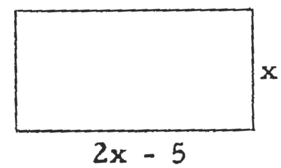
So, 3.6085 differs from the larger root of the equation by at most 0.000125--that is,

$$3.608375 \leq \frac{5 + \sqrt{89}}{4} \leq 3.608625.$$

G. In solving the following problems you may find it convenient to use quadratic equations.

Sample 1. The length of a certain rectangle is 5 inches less than twice its width. The area is 75 square inches. Find the number of inches in the width and the number of inches in the length.

Solution. Suppose the width is x inches. Then, the length is 2x - 5 inches and, since the area is 75 square inches,



$$(*) \qquad x(2x - 5) = 75.$$

Now, $x(2x - 5) = 75$ if and only if $2x^2 - 5x - 75 = 0$. And, $2x^2 - 5x - 75 = 0$ if and only if $x = \frac{5 \pm \sqrt{25 + 600}}{4}$.

$$\frac{5 \pm \sqrt{25 + 600}}{4} = \frac{5 \pm \sqrt{625}}{4} = \frac{5 \pm 25}{4}$$

Hence, the roots of (*) are 7.5 and −5. So, the width is 7.5 inches [Why not −5?] and the length is $2 \cdot 7.5 - 5$, or 10 inches. [Notice that (*) does not completely characterize the measure of the width of the rectangle. You also need the sentence '$x > 5/2$'. Why?]

Check. $10 = 2 \cdot 7.5 - 5$; $10 \cdot 7.5 = 75$

Sample 2. A class of students wanted to buy a $12 gift for the school principal [Zabranchburg High, where else?]. They decided on an amount to assess each class member. Six persons in another class heard of the plan and wanted to join in buying the gift. The assessment for each person was then reduced by 10 cents. How many persons were in the class?

Solution. Suppose there were x persons in the class. Then each of these x persons was originally assessed $\frac{12}{x}$ dollars. When they were joined by the other 6, the assessment dropped to $\frac{12}{x} - \frac{1}{10}$ dollars. So, x + 6 people pay a total of $(x + 6)(\frac{12}{x} - \frac{1}{10})$ dollars. Thus, we seek a number x [a positive integer] such that

$$(x + 6)(\frac{12}{x} - \frac{1}{10}) = 12.$$

C. Solve these equations. [If the roots are irrational numbers, give rational approximations to them which are correct to the nearest tenth.]

Sample. $2x^2 - 5x - 8 = 0$

Solution. $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 2 \cdot -8}}{2 \cdot 2}$$

$$\frac{5 \pm \sqrt{25 + 64}}{4}$$

The roots are $\frac{5 + \sqrt{89}}{4}$ and $\frac{5 - \sqrt{89}}{4}$.

The table on page 5-219 tells us that $\sqrt{89}$ is 9.434 correct to the nearest thousandth.

$$\frac{5 + 9.434}{4} = \frac{14.434}{4} = 3.6085$$

$$\frac{5 - 9.434}{4} = \frac{-4.434}{4} = -1.1085$$

So, rational approximations correct to the nearest tenth are 3.6 and −1.1.

1. $x^2 - 5x - 7 = 0$
2. $8 = 5x + x^2$
3. $3x^2 + 2x - 5 = 0$
4. $3x(x - 1) = 1$
5. $11 - x^2 = 0$
6. $3(x - 2)^2 = 13(x - 2) + 2$

☆ D. Write a quadratic equation in 'x' in standard form whose solution set is

1. $\{3, 4\}$
2. $\{5\}$
3. $\{0\}$
4. $\{\sqrt{10}, -\sqrt{10}\}$
5. $\{8, -8\}$
6. $\{\frac{2}{3}, -\frac{3}{4}\}$
7. $\{r_1, r_2\}$
8. $\{2 - \sqrt{5}, 2 + \sqrt{5}\}$

☆ E. Solve these equations.

1. $x^2 + x\sqrt{2} - 4 = 0$
2. $x^2 - x\sqrt{3} - \frac{3}{4} = 0$

☆ F. Graph these sentences.

1. $y \geq x^2$
2. $y \leq (x - 3)^2 - 5$
3. $x < y^2 - y + 2$
4. $y \geq (x - 2)(2x + 5)$
5. $y = |(x - 2)(x - 4)|$

Quiz.

1. Use the quadratic formula to solve each equation. [Show your work.] If no real number satisfies it, say so.

   (a) $2x^2 - 9x - 6 = 0$ (b) $3t(2 - t) - 6(t + 1) + t = 0$

   (c) $x = 7 + \frac{2}{x}$

2. Find three consecutive positive odd numbers the sum of whose squares is 515.

*

Answers for Quiz.

1. (a) $\frac{9 + \sqrt{129}}{4}$, $\frac{9 - \sqrt{129}}{4}$ (b) no real roots

   (c) $\frac{7 + \sqrt{57}}{2}$, $\frac{7 - \sqrt{57}}{2}$

2. 11, 13, 15

The numbers 5 and 75 referred to in Sample 1 are, of course, numbers of arithmetic, and the domain of the variable 'x' in equation (*) is the set of numbers of arithmetic. But, as in Unit 3 [See TC[3-55], also TC[5-138, 139]], in solving (*) we pretend that the domain of 'x' is the set of real numbers. Then, the numbers of arithmetic corresponding with such nonnegative roots as are obtained are solutions of (*) in its original interpretation [where the domain of 'x' is the set of numbers of arithmetic].

It need not be the case that all numbers of arithmetic which merely satisfy (*) furnish acceptable solutions of the stated problem. For, since a rectangle cannot have 0 as the measure of either its width or its length, acceptable values of 'x' must also satisfy the restrictions '$x > 0$' and '$x > 5/2$'. So, a complete algebraic statement of the problem consists of two sentences, (*) and '$x > 5/2$'.

*

Remarks somewhat similar to the preceding apply to the Solution for Sample 2. Here, the domain of 'x' is, actually, the set of whole numbers of arithmetic, but, in solving the equation at the top of page 5-197, we pretend that the domain of 'x' is the set of real numbers [See TC[3-64, 65, 66, 67]b]. And, again, in order to get a complete statement of the problem one must adjoin to the equation a restriction: x is a nonzero whole number of arithmetic [or, when we reinterpret the domain of 'x': x is a positive integer].

3. If the width of the rectangle is x inches then the length is 19 - x inches. So, we look for a root of '$x(19 - x) = 78$' [which, since the width of a rectangle is not 0 inches and is smaller than its length, should also satisfy '$0 < x < 19 - x$', or '$0 < x < 19/2$']. The roots of the equation are 13 and 6. So, the width is 6 inches and the length is 13 inches.

4. If the width is x feet then the length is 500 - x feet and the area is $x(500 - x)$ square feet. So, we need to find the first component of the extreme point of the quadratic function $\{(x, y): y = x(500 - x)\}$. This is 250. Since $500 - 250 = 250$, the largest rectangular field whose foot-perimeter is 1000 is a square whose side-length is 250 feet.

5. For some x, the numbers in question are $x - 1/2$ and $x + 1/2$. So, we wish to solve the equation:

$$\frac{1}{x - \frac{1}{2}} + \frac{1}{x + \frac{1}{2}} = \frac{40}{21},$$

or: $20x^2 - 21x - 5 = 0$. The roots are 5/4 and −1/5. Hence the problem has two solutions, (3/4, 7/4) and (−7/10, 3/10).

6. If x is the inch-measure of a side of such a square, then $x^2 + 12 = 4x$, and $x > 0$. But, the quadratic equation has no solution $[(-4)^2 - 4 \cdot 1 \cdot 12 < 0]$. So, there is no such square.

7. If the hiker walked x miles per hour, he walked for 6/x hours. $6/(x + 2)$ hours is 1/2 hour less. So, $6/x - 6/(x + 2) = 1/2$. This equation simplifies to '$x^2 + 2x - 24 = 0$' whose roots are −6 and 4. So, the hiker walked 4 miles per hour for 3/2 hours.

8. If x is the number named by the numerator, then

$$\frac{x}{x - 2} - \frac{x - 2}{x} = \frac{24}{35}.$$

Reducing to standard form, one obtains '$6x^2 - 47x + 35 = 0$'. Its roots are 7 and 5/6. Hence, the fraction is either '7/5' or '(5/6)/−(7/6)'. [Note that although $(5/6)/-(7/6) = -5/7$, '−5/7' is not a solution. '(5/6)/−(7/6)' ≠ '−5/7'. See TC[1-94, 95]a.]

*

Answers for Part G [which begins on page 5-196].

1. If x is the "middle one" of three consecutive integers, then the integers are $x - 1$, x, and $x + 1$. So, it suffices to find a real integer x such that

$$(x - 1)^2 + x^2 + (x + 1)^2 = 302.$$

This equation simplifies to '$x^2 = 100$', whose roots are 10 and -10. Since both are integers, the problem has two solutions, (9, 10, 11), and (-11, -10, -9). [It is instructive to compare this solution with one beginning: If x is the smallest of three consecutive integers... This leads, eventually, to the equation '$x^2 + 2x - 99 = 0$', which is more difficult to solve than is '$x^2 = 100$'. The moral is: When setting up a problem for algebraic solution, make use of any symmetries which are available.]

2. For each four consecutive odd integers, there is an even integer x such that the four odd integers are $x - 3$, $x - 1$, $x + 1$, and $x + 3$. So, we look for an even integer x such that

$$(x - 3)^2 + (x - 1)^2 + (x + 1)^2 + (x + 3)^2 = 36$$

--that is, such that $4x^2 + 20 = 36$. The roots of this equation are 2 and -2, so, since both are even integers, the problem has two solutions, (-1, 1, 3, 5), and (-5, -3, -1, 1).

[Ask if there is a solution for the problem obtained by replacing 'real integers' by 'positive integers'. The answer is, of course, 'no'. Another similar problem, but which has a solution [but just one solution] can be obtained by adding to the exercise 'and the sum of the numbers themselves is positive'. Consideration of such problems should increase students' understanding that a word problem can seldom be completely expressed by one equation. For the given problem, one needs the additional stipulation 'x is an even integer'. For the first suggested modification, one must add 'x is an even integer and $x > 3$'. For the second suggested modification, one needs the additional restriction 'x is a positive even integer'.]

[There are, of course, other ways of setting up the given problem. One may say that there is an integer x such that the consecutive odd integers are $2x - 3$, $2x - 1$, $2x + 1$, and $2x + 3$. Or, one may say that there is an odd integer x such that the four consecutive odd integers are x, $x + 2$, $x + 4$, and $x + 6$ [or: $x - 2$, x, $x + 2$, and $x + 4$]. All these lead to more complicated equations than the one arrived at in the solution given above.]

EXPLORATION EXERCISES

A. 1. Graph these equations on the same chart.

(1) $5x - 6y + 30 = 0$

(2) $10x - 3y - 30 = 0$

2. Graph these equations on the same chart used in Exercise 1.

(3) $6[5x - 6y + 30] - 2[10x - 3y - 30] = 0$

(4) $4[5x - 6y + 30] - 3[10x - 3y - 30] = 0$

3. Graph these equations on the same chart used in Exercise 1.

(5) $2[5x - 6y + 30] - 1[10x - 3y - 30] = 0$

(6) $1[5x - 6y + 30] - 2[10x - 3y - 30] = 0$

4. What ordered pair (x, y) satisfies both (1) and (2)? Both (3) and (4)? Both (5) and (6)?

5. For each equation given below, find one ordered pair (x, y) which satisfies it.

(a) $9[5x - 6y + 30] - 7[10x - 3y - 30] = 0$

(b) $9[5x - 6y + 30] + 7[10x - 3y - 30] = 0$

(c) $84[5x - 6y + 30] - 257[10x - 3y - 30] = 0$

(d) $-34[5x - 6y + 30] - 2172[10x - 3y - 30] = 0$

(e) $\frac{1}{2}[5x - 6y + 30] + \frac{1}{3}[10x - 3y - 30] = 0$

B. Pick out those relations from the list below whose graphs are either vertical or horizontal straight lines, and tell which.

1. $\{(x, y): 3[5x + 7y - 8] - 5[3x - 2y + 12] = 0\}$

2. $\{(x, y): 9[2x - y + 3] + [57x + 9y - 81] = 0\}$

3. $\{(a, b): 7[a - 3b - 12] - [7a + 15b - 78] = 0\}$

4. $\{(a, b): 15[2a - 6b - 9] + 7[3a - 5b + 2] = 0\}$

5. $\{(m, n): 85[3m - 17n + 71] - 3[85m + 52n - 62] = 0\}$

6. $\{(p, q): [p - 3q + 5] + [q - p + 18] = 0\}$

$$(x + 6)(\frac{120 - x}{10x}) = 12$$

$$(x + 6)(120 - x) = 120x$$

$$720 + 114x - x^2 = 120x$$

$$x^2 + 6x - 720 = 0$$

$$(x - 24)(x + 30) = 0$$

$$x = 24 \text{ or } x = -30$$

So, there were 24 members in the class.

Check. Each of 24 students contributes 50 cents to collect \$12, and each of 30 students contributes 40 cents to collect the same amount.

1. Find three consecutive real integers such that the sum of their squares is 302. $[(x - 1)^2 + x^2 + (x + 1)^2 = 302]$

2. Find four consecutive odd real integers such that the sum of their squares is 36.

3. The inch-perimeter of a given rectangle is 38, and the area of the rectangle is 78 square inches. Find the width and the length.

4. Find the dimensions of the largest rectangular field which can be fenced with 1000 feet of fencing.

5. Find two numbers whose difference is 1, and the sum of whose reciprocals is $\frac{40}{21}$.

6. If you add 12 to the number of square inches in the area of a square, you get the number of inches in the perimeter. Find the length of a side of the square.

7. A hiker walked 6 miles. To have walked this distance in $\frac{1}{2}$ hour less time, he would have had to walk 2 miles faster per hour. Find his rate and how long he hiked.

8. The number named by the numerator of a certain fraction exceeds the denominator-number by 2, and the fraction stands for a number which exceeds its reciprocal by $\frac{24}{35}$. Find the fraction.

[Supplementary exercises are in Part V, pages 5-273 through 5-274.]

Answers for Part C.

1. $\boxed{7}$, $\textcircled{5}$ 2. $\boxed{-3}$, $\textcircled{7}$ 3. $\boxed{-3}$, $\textcircled{2}$

4. $\boxed{5}$, $\textcircled{-2}$ 5. $\boxed{1}$, $\textcircled{1}$ 6. $\boxed{1}$, $\textcircled{-1}$

[Other correct answers may be obtained by taking any multiple of the numbers listed. For example, here are three more correct answers for Exercise 2 of Part C:

$\boxed{3}$, $\textcircled{-7}$; $\boxed{-6}$, $\textcircled{14}$; $\boxed{-1}$, $\textcircled{7/3}$

A similar remark applies to Part D.]

*

Answers for Part D.

1. $\boxed{-2}$, $\textcircled{3}$ 2. $\boxed{5}$, $\textcircled{-1}$ 3. $\boxed{8}$, $\textcircled{-1}$

4. $\boxed{4}$, $\textcircled{7}$ 5. $\boxed{9}$, $\textcircled{-5}$ 6. $\boxed{9}$, $\textcircled{-5}$

*

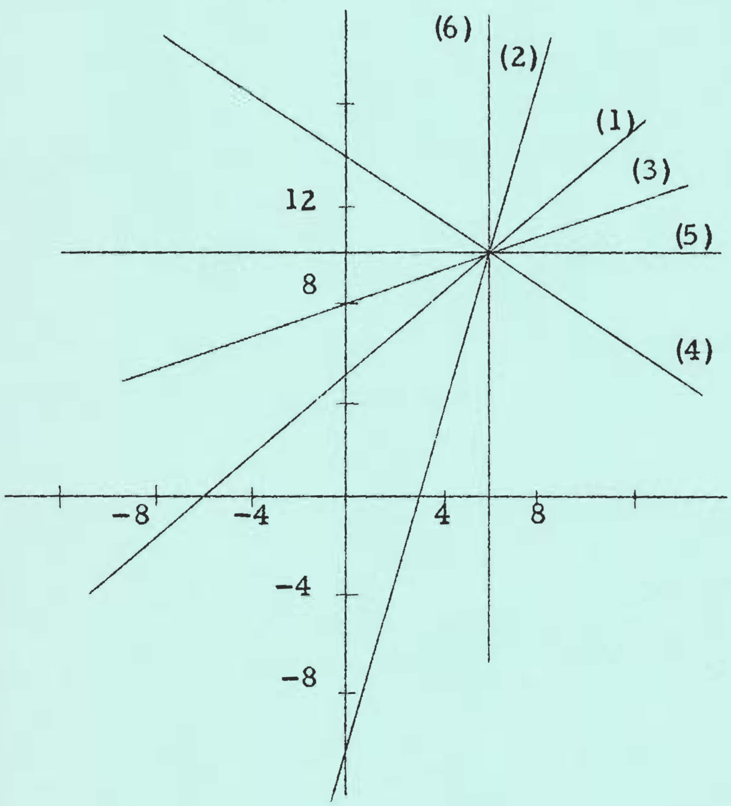
Answers for Part E.

1. $(-\frac{3}{31}, \frac{23}{31})$ 2. $(\frac{5}{32}, \frac{13}{32})$ 3. $(12, 5)$

4. $(\frac{31}{43}, -\frac{44}{43})$ 5. $(-\frac{1}{7}, \frac{15}{7})$ 6. $(2, \frac{15}{2})$

Answers for Part A.

1., 2., 3.



4. (6, 10); (6, 10); (6, 10)

5. [Students may, correctly, give many answers for each part of this exercise. It is to be hoped, however, that they will see that '(6, 10)' is a correct answer for each part.]

*

Answers for Part B.

1. horizontal
2. vertical
3. horizontal
4. neither
5. horizontal
6. horizontal

*

5.11 Systems of equations.--The preceding Exploration Exercises dealt with the problem of solving a system of two linear equations in two variables. In an earlier discussion [page 5-191] we said that a linear equation in one variable 'x' is an equation which can be transformed to one of the [standard] form:

$$ax + b = 0, \quad [a \neq 0]$$

A linear equation in two variables 'x' and 'y' is one which can be transformed into an equation of the [standard] form:

$$ax + by + c = 0, \quad [a \neq 0 \text{ or } b \neq 0]$$

[Do you see that the condition '$a \neq 0$ or $b \neq 0$' permits you to consider a linear equation in one variable as a linear equation in two variables? For example, the equation '$2y + 7 = 0$' can be thought of as an abbreviation for '$0x + 2y + 7 = 0$'. But, you must be careful to note that the solution set of such an equation, when you think of it as an equation in two variables, is a set of ordered pairs and so is different from the solution set of the same equation when you consider it as an equation in one variable. For example, $\{(x, y): 2y + 7 = 0\}$ is the set of all ordered pairs of real numbers which have $-\frac{7}{2}$ for second component. But, $\{y: 2y + 7 = 0\} = \{-\frac{7}{2}\}$.]

A pair of linear equations in two variables 'x' and 'y' is called a system of two linear equations in two variables 'x' and 'y'. To solve a system of two equations is to find the ordered pairs (x, y) which satisfy both equations in the system. So, to solve the system of linear equations in 'x' and 'y':

$$\left.\begin{aligned} ax + by + c &= 0 \\ a'x + b'y + c' &= 0 \end{aligned}\right\}$$

is to find the members of

$$\{(x, y): ax + by + c = 0\} \cap \{(x, y): a'x + b'y + c' = 0\},$$

or, in other words, to find the members of:

$$\{(x, y): ax + by + c = 0 \text{ and } a'x + b'y + c' = 0\}$$

The latter is the solution set of the system, and each member of it is called a solution of the system.

Correction. On page 5-202, in line 20, change: '3k - 6m' is 0 to: '-3k - 6m' is 0

Explanation asked for in line 13 on page 5-201.

For each real number x, there is precisely one ordered pair, $(x, -3x + 14)$, whose first component is x and which belongs to the function defined by (1′). And, there is precisely one ordered pair, $(x, 5x - 18)$, whose first component is x and which belongs to the function defined by (2′). So, a real number x is the first component of an ordered pair which belongs to both functions if and only if

$$(x, -3x + 14) = (x, 5x - 18),$$

that is, if and only if $-3x + 14 = 5x - 18$.

*

Answers for questions at the bottom of page 5-201.

The pm0 tells us that $\forall_k\ k \cdot 0 = 0$.

By the pa0, $0 + 0 = 0$.

*

Answer for question 'Why 0?' in line 7 on page 5-202.

Because this choice will give us an equation equivalent to one which contains only one variable.

*

Notice that the system consisting of equations (3) and (6) is equivalent to that consisting of (3) and (4). [And, so is the system consisting of (4) and (6).] This follows from the easily-proved theorem:

$$\forall_k \forall_{m \neq 0} \forall_p \forall_q\ [(p = 0 \text{ and } q = 0) \text{ if and only if } (p = 0 \text{ and } kp + mq = 0)]$$

So, having found the value of 'y' which satisfies (6), one can use either (3) or (4) to find the corresponding value of 'x'. And, there is no need to check by substituting into the other of these two equations except to catch errors in arithmetic.

C. For each exercise, put numerals in the frames so that the resulting expression names a relation whose graph is a vertical straight line.

1. $\{(x, y): \square[3x - 5y + 4] + \bigcirc[2x + 7y - 5] = 0\}$

2. $\{(x, y): \square[x + 7y - 3] + \bigcirc[5x + 3y - 2] = 0\}$

3. $\{(x, y): \square[\frac{1}{4}x - 2y + 7] + \bigcirc[2x - 3y - 9] = 0\}$

4. $\{(x, y): \square[2y + 7x - 3] + \bigcirc[8 + 5y - 4x] = 0\}$

5. $\{(x, y): \square[5x - 2y + 5] + \bigcirc[9x + 2y - 3] = 0\}$

6. $\{(x, y): \square[5x - 2y + 5] + \bigcirc[9x - 2y - 3] = 0\}$

D. Repeat Part C, but get horizontal straight lines.

E. 1. Find numbers x and y such that

$$3x - 5y + 4 = 0$$

and

$$2x + 7y - 5 = 0.$$

[Refer to Exercise 1 of Part C and Exercise 1 of Part D.]

2. Find numbers x and y such that

$$x + 7y - 3 = 0$$

and

$$5x + 3y - 2 = 0.$$

[Refer to Exercise 2 of Parts C and D.]

3. Find an ordered pair (x, y) which satisfies both of the equations:

$$\frac{1}{4}x - 2y + 7 = 0$$

and:

$$2x - 3y - 9 = 0$$

4. Find a member of the intersection of

$$\{(x, y): 2y + 7x - 3 = 0\} \text{ and } \{(x, y): 8 + 5y - 4x = 0\}.$$

5. $\{(x, y): 5x - 2y + 5 = 0\} \cap \{(x, y): 9x + 2y - 3 = 0\} = \{\_\_\_\_\_\_\}$

6. Find an ordered pair which belongs to

$$\{(x, y): 5x - 2y + 5 = 0 \text{ and } 9x - 2y - 3 = 0\}.$$

That is, then

$$(5) \qquad (5k + 7m)x + (-3k - 6m)y + (-11k - 10m) = 0.$$

Now, this last equation looks worse than the equations we started with! But, observe that (5) holds for all k and m. So, we can convert (5) to a simpler equation by picking values for 'k' and for 'm' in a clever way. A good choice would be one for which either '5k + 7m' or '-3k - 6m' has the value 0 [Why 0?]. For example, suppose we choose 7 for 'k' and -5 for 'm'. Then, the value of '5k + 7m' is 0, and (5) is converted into:

$$(6) \qquad (35 - 35)x + (-21 + 30)y + (-77 + 50) = 0,$$

which is equivalent to:

$$(7) \qquad 9y - 27 = 0$$

So, if there is an ordered pair (x, y) which satisfies both (3) and (4) then its second component is 3. Each of (3) and (4) tells us that its first component must be 4. So, if the system of (3) and (4) has a nonempty solution set, its only member is (4, 3). Since (4, 3) satisfies both (3) and (4) [as we discover by checking], the system has a nonempty solution set. So, its solution set is {(4, 3)}.

We could have selected values for 'k' and 'm' such that the value of '3k - 6m' is 0. If we take 2 for 'k' and -1 for 'm', (5) is converted into:

$$(8) \qquad (10 - 7)x + (-6 + 6)y + (-22 + 10) = 0,$$

which is equivalent to '3x - 12 = 0', and to 'x = 4'.

The foregoing is, of course, largely a discussion of a procedure for solving a system of linear equations in two variables. In practice, you might [with a different example] proceed as follows.

$$\begin{array}{ll} (1) & 2x + 5y - 31 = 0 \\ (2) & 3x - 7y + 26 = 0 \end{array}\Bigg\}$$

$$\begin{array}{rl} 6x + 15y - 93 = 0 & \text{[Multiply (1) by 3.]} \\ \underline{-6x + 14y - 52 = 0} & \text{[Multiply (2) by -2.]} \\ 29y - 145 = 0 & \text{[Add.]} \\ y = 5 & \end{array}$$

Check in (2):

$$4 \cdot -\frac{7}{19} \stackrel{?}{=} 3 - 3 \cdot \frac{85}{57}$$

$$-\frac{28}{19} \quad \Bigg\| \quad 3 - \frac{85}{19}$$

$$\frac{57}{19} - \frac{85}{19}$$

$$-\frac{28}{19} \quad \checkmark$$

Notice that we check in (2) rather than in (2''). Do you see why?

Answer. The solution set is $\{(\frac{85}{57}, -\frac{7}{19})\}$.

EXERCISES

A. Solve these systems of equations.

1. $\left.\begin{aligned} 4x + 3y &= -1 \\ 7x + 4y &= -2 \end{aligned}\right\}$

2. $\left.\begin{aligned} 6y + 2x &= 34 \\ 5y + 3x &= 35 \end{aligned}\right\}$

3. $\left.\begin{aligned} 2r + 3s &= -9 \\ r - 2s &= 27 \end{aligned}\right\}$

4. $\left.\begin{aligned} 10x - 7y &= 0 \\ 3x + 4y &= 0 \end{aligned}\right\}$

5. $\left.\begin{aligned} x &= 12 - 3y \\ x &= y + 7 \end{aligned}\right\}$

6. $\left.\begin{aligned} y &= 5 - 2x \\ x &= 3 - 4y \end{aligned}\right\}$

7. $\left.\begin{aligned} 2x + 7y - 24 &= 0 \\ 3x - 5y - 5 &= 0 \end{aligned}\right\}$

8. $\left.\begin{aligned} 7x + 4y - 17 &= 0 \\ 3x - 2y - 11 &= 0 \end{aligned}\right\}$

9. $\left.\begin{aligned} 4x - 5y &= 22 \\ 4 &= y + 2x \end{aligned}\right\}$

10. $\left.\begin{aligned} 6x + 7(2y + 3) &= 0 \\ 5x &= 6y - 44 \end{aligned}\right\}$

11. $\left.\begin{aligned} x &= 4y + 2 \\ 2x + 3y &= 15 \end{aligned}\right\}$

12. $\left.\begin{aligned} 3x - 5y &= 5 \\ x &= 9 - 2y \end{aligned}\right\}$

13. $\left.\begin{aligned} y &= 17 - 2x \\ x &= y + 5 \end{aligned}\right\}$

14. $\left.\begin{aligned} y &= 3x - 2 \\ y &= 7x + 5 \end{aligned}\right\}$

15. $\left.\begin{aligned} 3y &= 9x + 2 \\ 7x + 2y &= 15 \end{aligned}\right\}$

16. $\left.\begin{aligned} \frac{x - y}{3} &= 5 \\ \frac{x + y}{5} &= 3 \end{aligned}\right\}$

You know that, for each a and b, $\{(x, y): ax + by + c = 0\}$ is a linear function if and only if $a \neq 0 \neq b$. So, you can use your knowledge of linear functions in solving many systems of linear equations in two variables. For example, consider the following system:

$$\left.\begin{array}{ll}(1) & 3x + y - 14 = 0 \\ (2) & 5x - y - 18 = 0\end{array}\right\}$$

Solving each equation for 'y' we obtain the equivalent equations:

$$\begin{array}{ll}(1') & y = -3x + 14 \\ (2') & y = 5x - 18,\end{array}$$

which are clearly defining equations for linear functions. So, to solve the system is to find ordered pairs which belong to both functions. A number x is the first component of such an ordered pair if and only if

$$-3x + 14 = 5x - 18 \qquad \text{[Explain].}$$

The root of this equation is 4, and substituting '4' for 'x' in any of the equations (1), (2), (1'), and (2') gives 2 as the corresponding value of 'y'. So, the single solution of the system consisting of (1) and (2) is (4, 2). So, the solution set is $\{(4, 2)\}$.

Now, the foregoing is an excellent technique for solving a system whose equations are easy to solve for 'y' [or for 'x']. But, consider the following system.

$$\left.\begin{array}{ll}(3) & 5x - 3y - 11 = 0 \\ (4) & 7x - 6y - 10 = 0\end{array}\right\}$$

Solving these equations for 'y' [or for 'x'] involves messy fractions. There is a better way to solve the system, and this way was illustrated in the Exploration Exercises. As we discuss this better way, keep the graphical interpretation of linear equations in mind.

Suppose there is an ordered pair (x, y) such that (3) and (4). If $5x - 3y - 11 = 0$, then for each k, $k(5x - 3y - 11) = 0$. [What principle for real numbers justifies this?] Similarly, if $7x - 6y - 10 = 0$, then for each m, $m(7x - 6y - 10) = 0$. But, if $k(5x - 3y - 11)$ is 0 and $m(7x - 6y - 10)$ is 0 then

$$k(5x - 3y - 11) + m(7x - 6y - 10) = 0. \qquad \text{[Why?]}$$

Answer for question on page 5-204.

The only need for checking is to catch errors in arithmetic, and checking in (2) may catch errors made in transforming (2) into (2''), as well as any that are made at later stages in the solution.

*

Answers for Part A [on pages 5-204 and 5-205].

[We give, in each case, the solution set.]

1. $\{(-\frac{2}{5}, \frac{1}{5})\}$	2. $\{(5, 4)\}$	3. $\{(9, -9)\}$
4. $\{(0, 0)\}$	5. $\{(\frac{33}{4}, \frac{5}{4})\}$	6. $\{(\frac{17}{7}, \frac{1}{7})\}$
7. $\{(5, 2)\}$	8. $\{(3, -1)\}$	9. $\{(3, -2)\}$
10. $\{(-7, \frac{3}{2})\}$	11. $\{(6, 1)\}$	12. $\{(5, 2)\}$
13. $\{(\frac{22}{3}, \frac{7}{3})\}$	14. $\{(-\frac{7}{4}, -\frac{29}{4})\}$	15. $\{(\frac{41}{39}, \frac{149}{39})\}$
16. $\{(15, 0)\}$	17. $\{(5, 1)\}$	18. $\{(\frac{11}{9}, -\frac{7}{3})\}$
19. $\emptyset$	20. $\{(x, y): 3x + 8y = -2\}$	21. $\{(-\frac{23}{34}, -\frac{39}{17})\}$
22. $\{(\frac{143}{37}, -\frac{58}{37})\}$	23. $\{(-13, 14)\}$	24. $\{(4, 4)\}$

Answers for Part B.

[Reference to $\{(x, y): ax + by + c = 0\}$ and $\{(x, y): a'x + b'y + c' = 0\}$ as linear functions implies that a, b, a', and b' are not 0.]

1. The slopes are $-a/b$ and $-a'/b'$. So, the functions have the same slope if and only if $a/b = a'/b'$--that is [see page 5-145] if and only if a, b, a', b' are in proportion.

2. In this case, there is only <u>one</u> linear function in question.

3. In this case, there are two functions, but they have the same slope.

*

Call to the attention of the students the fact that, in the discussion of independent, dependent, and inconsistent equations, the text deals only with the case of equations which define linear functions. Some of your students may wish to check the fact that the same test for independence [that $ab' \neq a'b$] works for any system of two linear equations in two variables. They can do so by solving Part ☆B on page 5-208.

*

The following diagram may help to sort out the notions of consistency, inconsistency, dependence, and independence.

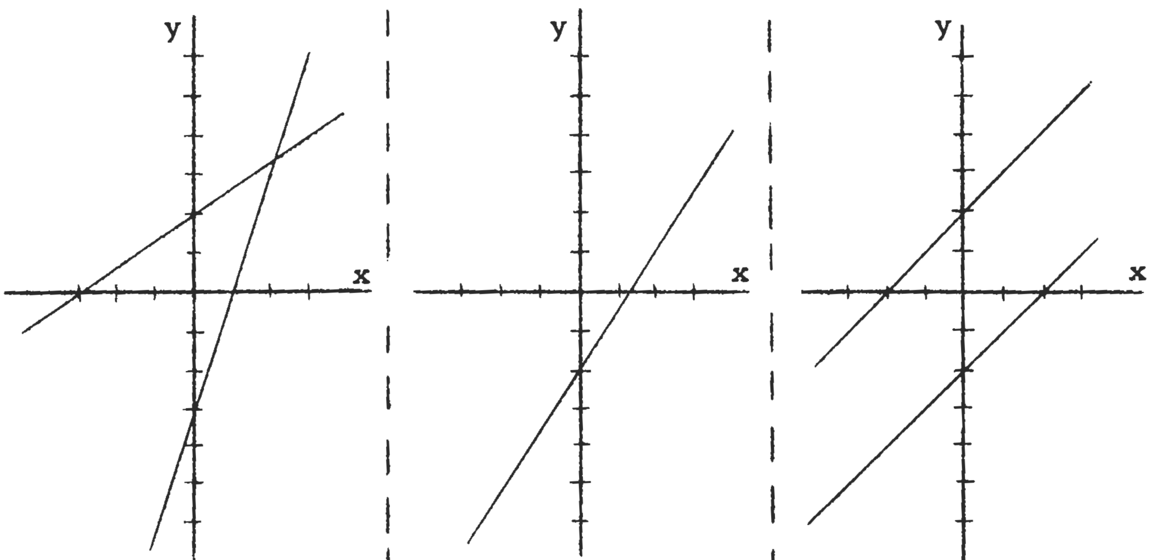
- consistent
  - independent [$ab' \neq a'b$]
  - dependent } [$ab' = a'b$]
- inconsistent } [$ab' = a'b$]

Each system of linear equations in two variables is either consistent or inconsistent [but not both]. The equations in a consistent system are either dependent or independent [but not both]. [As is suggested by Exercises 2 and 3 of Part B on page 5-205, one can distinguish between dependent and inconsistent equations [those for which $ab' = a'b$] by evaluating '$bc' - b'c$' and '$ca' - c'a$'. If [when $ab' = a'b$] $bc' = b'c$ and $ca' = c'a$ then the equations are dependent; otherwise, they are inconsistent.]

A system of equations which has at least one solution is called a consistent system of equations; one which has no solution is called an inconsistent system of equations.

A consistent system of two linear equations in two variables either has precisely one solution or consists of two equations which have the same graph. In the first case [precisely one solution], the equations are said to be independent equations; in the second case [same graph], they are said to be dependent equations.

Two linear equations in two variables are independent [and, so, have precisely one solution] if and only if $ab' \neq a'b$. So, if $ab' = a'b$, the equations are either dependent or inconsistent. [Equations such that $aba'b' \neq 0$ are dependent if and only if $ab' = a'b$ and $bc' = b'c$; they are inconsistent if and only if $ab' = a'b$ and $bc' \neq b'c$.]



Independent Equations	Dependent Equations	Inconsistent Equations
$ab' \neq a'b$	$ab' = a'b$, $bc' = b'c$	$ab' = a'b$, $bc' \neq b'c$

EXERCISES

A. For each system, tell whether it is a system of independent linear equations, a system of dependent linear equations, or a system of inconsistent linear equations. If you claim that the equations are independent, find the ordered pair which is a solution.

Sample. $\left.\begin{array}{l} 2(x - 4y) + 1 = 3(x + 3y) - (2x - y) + 6 \\ 7(x - 2y + 7) = 4(x - 5y + 6) \end{array}\right\}$

17. $\left.\begin{array}{l} x - 13y = -3 - x \\ 3(x + 2y) = 23 - 2y \end{array}\right\}$

18. $\left.\begin{array}{r} 1 + 2y + 3x = 0 \\ 9(x - 1) = 2 \end{array}\right\}$

19. $\left.\begin{array}{l} 5y + 7 = 2x \\ 2(3x - 5y) = 3(x - 4) - x \end{array}\right\}$

20. $\left.\begin{array}{l} 3x + 2(4y + 1) = 0 \\ 5(2x + 5y) - x = y - 6 \end{array}\right\}$

21. $\left.\begin{array}{r} 6(x - 3) + 7(2 - y) = 8 \\ 12(x + 4) - 3(6 - y) = 15 \end{array}\right\}$

22. $\left.\begin{array}{l} 2(x - 3y + 4) = 5(3x + y - 5) \\ x = 2y + 7 \end{array}\right\}$

23. $\left.\begin{array}{l} 3x - 2(x - y) = 15 \\ 5x - 5(x - y) = 70 \end{array}\right\}$

24. $\left.\begin{array}{l} 3(x - 2) - 2(3 - y) = 2y \\ 3(y - 2) - 2(3 - x) = 2x \end{array}\right\}$

B. Suppose that $\{(x, y): ax + by + c = 0\}$ and $\{(x, y): a'x + b'y + c' = 0\}$ are linear functions, f and g. [This means, of course, that $aba'b' \neq 0$.]

1. Prove that f and g have the same slope if and only if a, b, a', and b' are in proportion.

2. What can you say about f and g if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$?

3. What can you say about f and g if $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$?

INDEPENDENT, DEPENDENT, AND INCONSISTENT EQUATIONS

Your work in Part B may have suggested that a system of two linear equations in two variables 'x' and 'y':

$$\left.\begin{array}{l} ax + by + c = 0 \\ a'x + b'y + c' = 0 \end{array}\right\} aba'b' \neq 0$$

has precisely one solution if and only if $ab' \neq a'b$. If $ab' \neq a'b$, it follows that $\frac{a}{b} \neq \frac{a'}{b'}$, and $-\frac{a}{b} \neq -\frac{a'}{b'}$. But, $-\frac{a}{b}$ and $-\frac{a'}{b'}$ are the slopes of the linear functions corresponding to the given equations. So, if $ab' \neq a'b$, the corresponding linear functions have different slopes. Hence, their intersection consists of just one point. Now, if the slopes are the same, either the functions are the same or the functions intersect in the empty set. So, if the slopes are the same, the intersection does not consist of precisely one point. Therefore, the intersection consists of just one point if and only if the slopes are different. And, from this it follows that the intersection consists of just one point if and only if $ab' \neq a'b$.

solved system can be carried out only so far as to yield the system:

$$\left.\begin{aligned}(ab' - a'b)x &= c'b - cb' \\ (ba' - b'a)y &= c'a - ca'\end{aligned}\right\}$$

If $ab' = a'b$ then this system has no solution unless $c'b - cb' = c'a - ca' = 0$. So, if $ab' = a'b$ then the given system has no solution [that is, is inconsistent] unless $c'b = cb'$ and $c'a = ca'$. Finally, if $ab' = a'b$, $bc' = b'c$, and $ca' = c'a$ then the given equations are dependent. [One can see this very easily in case $aba'b' \neq 0$--that is, in case each of the given equations defines a linear function. For, if neither b nor b' is 0, and $ab' = a'b$ and $bc' = b'c$, then $-a'/b' = -a/b$ and $-c'/b' = -c/b$. Hence, in this case, the linear function defined by the second equation has the same slope and intercept as does the linear function defined by the first equation. So, the two equations define the same function. Hence, they have the same solution set--that is, they are dependent.] If $ab' = a'b$, $bc' = b'c$, and $ca' = c'a$ then, for all x and y,

(1) $\quad a(a'x + b'y + c') = a'(ax + by + c)$

and (2) $\quad b(a'x + b'y + c') = b'(ax + by + c)$.

If $a \neq 0$ then equation (1) shows that if $ax + by + c = 0$ then $a'x + b'y + c' = 0$, while, if $b \neq 0$, this fact follows from equation (2). Since either $a \neq 0$ or $b \neq 0$, it follows that each solution of the first equation of the given system is a solution of the second equation of the given system. A similar argument, based on the fact that either $a' \neq 0$ or $b' \neq 0$, shows that each solution of the second of the given equations is a solution of the first of them. So, as was to be proved, if $ab' = a'b$, $bc' = b'c$, and $ca' = c'a$ then the given equations have the same solution set--that is, are dependent.

*

Quiz.

Solve these systems of equations.

1. $\left.\begin{aligned}m + n &= -6 \\ m - n &= -10\end{aligned}\right\}$

2. $\left.\begin{aligned}3x - 4y &= 8 \\ x + 2y &= 1\end{aligned}\right\}$

3. $\left.\begin{aligned}6x + 7y &= 3(5 - y) \\ 26 + 10y &= -6x\end{aligned}\right\}$

*

Answers for Quiz.

1. $(-8, 2)$
2. $(2, -\frac{1}{2})$
3. inconsistent

Correction. On page 5-208, in Exercise 15, change '7(x - r)' to '7(s - r)'. ↑

Answers for Part A [which begins on page 5-206].

1. independent $[5 \cdot -7 \neq 6 \cdot 3]$; $(\frac{96}{53}, \frac{52}{53})$
2. independent $[9 \cdot 2 \neq 3 \cdot 7]$; $(\frac{4}{3}, 0)$
3. inconsistent [Since $8 \cdot 6 = -16 \cdot -3$, the equations are either inconsistent or dependent. On transforming the system into:

$$\left.\begin{aligned} s &= \frac{8}{3}t - \frac{7}{3} \\ s &= \frac{8}{3}t - \frac{5}{2} \end{aligned}\right\}$$

it becomes clear that the equations are inconsistent.]

4. independent $[1 \cdot -2 \neq 3 \cdot 1]$; (0, 0)
5. independent; $(\frac{1}{3}, \frac{1}{3})$
6. inconsistent
7. inconsistent
8. inconsistent
9. independent; (5, 0)
10. dependent
11. inconsistent
12. independent; $(-\frac{53}{10}, -\frac{26}{5})$
13. inconsistent
14. independent; (3, 0)
15. independent; $(-\frac{81}{109}, -\frac{36}{109})$
16. dependent
17. independent; $(-\frac{71}{2}, -\frac{3}{8})$

*

Answers for Part ☆B [on page 5-208].

$$\left.\begin{aligned} x &= \frac{c'b - cb'}{ab' - a'b} \\ y &= \frac{c'a - ca'}{ba' - b'a} \end{aligned}\right\}$$

These equations are, of course, not applicable unless $ab' \neq a'b$. If $ab' \neq a'b$ then this "solved" system is equivalent to the given system, and the former, like the latter, has the unique solution

$$\left(\frac{c'b - cb'}{ab' - a'b}, \frac{c'a - ca'}{ba' - b'a}\right).$$

If $ab' = a'b$ then the elimination process which otherwise leads to the

11. $\left.\begin{aligned} 2x - 3y &= 5 \\ 4x - 6y &= 11 \end{aligned}\right\}$

12. $\left.\begin{aligned} 2x - 3y &= 5 \\ 4y - 6x &= 11 \end{aligned}\right\}$

13. $\left.\begin{aligned} \frac{35}{12}A - \frac{5}{4}B &= 1 \\ \frac{14}{15}A - \frac{2}{5}B &= 2 \end{aligned}\right\}$

14. $\left.\begin{aligned} y &= \frac{6 - 2x}{7} \\ x &= \frac{9 + 10.5y}{3} \end{aligned}\right\}$

15. $\left.\begin{aligned} r + 5s - 6(r - 3s) + 7(x - r) &= 3s \\ 3(2r - 7s) + 5(r + 2s + 3) &= 6(1 - r) \end{aligned}\right\}$

16. $\left.\begin{aligned} 7a + 3(a - 3b + 1) &= 1 + 2(2a + b + 8) - 3b \\ 7(2a - 3b - 5) + a(2b + 1) &= b(2a - 1) \end{aligned}\right\}$

17. $\left.\begin{aligned} (u - 5)^2 - (v + 4)^2 + 10u &= 12 + (u - v)(u + v) \\ (v - 8)^2 - (u + v)^2 &= -2uv - (u + 1)^2 \end{aligned}\right\}$

☆ B. Solve this system for 'x' and 'y'. [That is, express 'x' and 'y' in terms of 'a', 'b', 'c', 'a′', 'b′', and 'c′'.]

$$\left.\begin{aligned} ax + by + c &= 0 \\ a'x + b'y + c' &= 0 \end{aligned}\right\}$$

How does the result show the cases in which the equations are inconsistent or dependent?

SYSTEMS OF THREE EQUATIONS IN THREE VARIABLES

The ideas involved in solving systems of two linear equations in two variables can be extended to solving systems of three linear equations in three variables. In general, the procedure followed in solving a system of two linear equations in two variables is to obtain a single equation in one variable by eliminating the other variable from the given equations. You already know how to solve the resulting equation in one variable, and you have seen how doing so enables you to solve the given system.

When you are faced with the problem of solving a system of three linear equations in three variables, say:

$$\left.\begin{aligned} 5x + 2y + 3z - 8 &= 0 \\ 3x - 4y - z + 14 &= 0 \\ 2x + 7y - 3z - 3 &= 0 \end{aligned}\right\},$$

Solution. First, we transform these equations into standard linear form.

$$(1) \qquad x - 18y - 5 = 0$$

$$(2) \qquad 3x + 6y + 25 = 0$$

Since $1 \cdot 6 \neq 3 \cdot -18$, or, equivalently, since $\frac{1}{3} \neq \frac{-18}{6}$, the equations are independent. So, there is one and only one ordered pair (x, y) which satisfies them. Let's find it. From equation (1) we know that

$$x = 18y + 5.$$

Substituting '18y + 5' for 'x' in (2), we get:

$$3(18y + 5) + 6y + 25 = 0,$$

$$54y + 15 + 6y + 25 = 0,$$

$$60y = -40,$$

$$y = -\frac{2}{3}$$

$$x = 18 \cdot -\frac{2}{3} + 5 = -7$$

Check. $$3 \cdot -7 + 6 \cdot -\frac{2}{3} + 25 \overset{?}{=} 0$$

$$-21 + -4 + 25 = 0 \quad \checkmark$$

The unique solution is $(-7, -\frac{2}{3})$.

1. $\left.\begin{array}{l} 5x + 3y - 12 = 0 \\ 6x - 7y - 4 = 0 \end{array}\right\}$

2. $\left.\begin{array}{l} 9x = 12 - 7y \\ 3x = 4 - 2y \end{array}\right\}$

3. $\left.\begin{array}{l} 8t = 3s + 7 \\ 6s = 16t - 15 \end{array}\right\}$

4. $\left.\begin{array}{l} x + y = 0 \\ 3x - 2y = 0 \end{array}\right\}$

5. $\left.\begin{array}{l} a + 8b = 3 \\ 8a + b = 3 \end{array}\right\}$

6. $\left.\begin{array}{l} 7t + 3a = -6 \\ -7t - 3a = 12 \end{array}\right\}$

7. $\left.\begin{array}{l} 4x - 2y = 9 \\ y = 2x + 4 \end{array}\right\}$

8. $\left.\begin{array}{l} 8k - 7 = m \\ 8k + 7 = m \end{array}\right\}$

9. $\left.\begin{array}{l} 2y + x = 5 - y \\ 4y - x = y - 5 \end{array}\right\}$

10. $\left.\begin{array}{l} 6(x + z) - z = 3(x + 1) + 1 \\ 4(2x + 4z - 3) = z - x \end{array}\right\}$

Answers for questions in lower third of page 5-209.

Any solution (x, y, z) of the given system is such that

$$5x + 2y + 3z - 8 = 0$$

$$3x - 4y - z + 14 = 0,$$

and $$2x + 7y - 3z - 3 = 0.$$

Hence, [by the pm0 and the pa0],

$$[5x + 2y + 3z - 8] + 3[3x - 4y - z + 14] = 0$$

and $$[5x + 2y + 3z - 8] + [2x + 7y - 3z - 3] = 0.$$

So, if (x, y, z) is any solution of the given system then (x, y) is a solution of the system consisting of (a) and (b).

That (−1, 2, 3) is a solution of the second equation can be seen by substitution. However, such a substitution-check is unnecessary. For, since (−1, 2, 3) is a solution both of equation (a) [considered, now as an equation in 3 variables] and the first equation, it is a solution of:

$$[14x - 10y + 34] - [5x + 2y + 3z - 8] = 0,$$

that is, it is a solution of:

$$9x - 12y - 3z + 42 = 0,$$

which is equivalent to the second equation. Similar remarks, with reference to equation (b) rather than (a), show that (−1, 2, 3) is a solution of the third equation.

We have seen [answer, above, to first question, and because the only solution of the system consisting of (a) and (b) is (−1, 2)] that if (x, y, z) is a solution of the given system then $x = -1$ and $y = 2$. Since the only solution (x, y, z) of the first equation for which $x = -1$ and $y = 2$ is (−1, 2, 3), no other triple can be a solution of the given system.

*

Answers for Exercises on page 5-210.

1. $\{(1, 2, 3)\}$
2. $\{(3, 1, 4)\}$
3. $\{(5, -7, 3)\}$
4. $\emptyset$ [The equations are inconsistent.]
5. $\{(3, -3, \frac{1}{3})\}$
6. $\{(\frac{31}{4}, \frac{7}{4}, -\frac{9}{4})\}$
7. $\{(-6, -1, -7, 2)\}$
8. $\{(5, -7, 2, -3)\}$

☆9. $\{(x, y, z) : z = 1 \text{ and } x + 2y = -1\}$. [The equations are dependent.]

☆10. $\{(x, y, z) : x + y + 4z = 7\}$. [The equations are dependent.]

Another procedure for eliminating variables is by substitution. Reconsider the system dealt with on the preceding page:

$$\left.\begin{array}{ll} (1) & 5x + 2y + 3z - 8 = 0 \\ (2) & 3x - 4y - z + 14 = 0 \\ (3) & 2x + 7y - 3z - 3 = 0 \end{array}\right\}$$

Solve (2) for 'z': $z = 3x - 4y + 14$

Then, substitute in (1) and in (3):

$$\left.\begin{array}{ll} (1') & 5x + 2y + 3(3x - 4y + 14) - 8 = 0 \\ (2') & 2x + 7y - 3(3x - 4y + 14) - 3 = 0 \end{array}\right\}$$

This last system is one of two equations in two variables. Complete the solution.

EXERCISES

Solve these systems.

1. $\left.\begin{array}{l} 2x - 3y + 5z - 11 = 0 \\ 7x + y - 3z = 0 \\ 3x - 2y + 7z = 20 \end{array}\right\}$

2. $\left.\begin{array}{l} 5x = 12 - y + z \\ 2y = x - z + 3 \\ z = x + y \end{array}\right\}$

3. $\left.\begin{array}{l} x + y + z = 1 \\ 2x + z = 13 \\ z - y = 10 \end{array}\right\}$

4. $\left.\begin{array}{l} x + 2y + 3z = 4 \\ 7 - z = 2y + 3x \\ 3y + 4z = 1 - 2x \end{array}\right\}$

5. $\left.\begin{array}{l} 4(3x - y) = 3(17 - 3z) \\ x + y - 1 = 1 - 6z \\ 2(x - 10) = 7(y + 3z) \end{array}\right\}$

6. $\left.\begin{array}{l} x + y = 5 - 2z \\ x + 2y = 9 - z \\ y + 2x = 15 - z \end{array}\right\}$

7. $\left.\begin{array}{l} x + y = w - z \\ 3x + y - z = 2w \\ x = y - w \\ w + 2z = x - 1 \end{array}\right\}$

8. $\left.\begin{array}{l} x + y + u = 4 \\ y + u + v = -5 \\ u + v + x = 0 \\ v + x + y = -8 \end{array}\right\}$

☆9. $\left.\begin{array}{l} 4x + 1 + 3z + 8y = 0 \\ z + 10y + 4 + 5x = 0 \\ 14y + 5z + 7x + 2 = 0 \end{array}\right\}$

☆10. $\left.\begin{array}{l} x + y + 4z = 7 \\ 2x + 2y + 8z = 14 \\ 3x + 3y + 12z = 21 \end{array}\right\}$

a natural procedure is to try to "reduce" this problem to one of solving a system of two linear equations in two variables [just as you "reduced" the problem of solving a system of two linear equations in two variables to one of solving one equation in one variable]. This can be done by choosing one of the variables and eliminating it, first from one pair of the given equations, and then from another pair. For example, if you want to eliminate 'z', take the first two equations of the system:

$$5x + 2y + 3z - 8 = 0,$$
$$3x - 4y - z + 14 = 0$$

Then, transform them and add, thus eliminating 'z'.

$$\begin{array}{lr} & 5x + 2y + 3z - 8 = 0 \\ & 9x - 12y - 3z + 42 = 0 \\ \hline \text{(a)} & 14x - 10y \qquad + 34 = 0 \end{array}$$

Then, eliminate 'z' from, say, the first and third equations of the given system.

$$\begin{array}{lr} & 5x + 2y + 3z - 8 = 0 \\ & 2x + 7y - 3z - 3 = 0 \\ \hline \text{(b)} & 7x + 9y \qquad - 11 = 0 \end{array}$$

Equations (a) and (b) make a system of two linear equations in two variables.

$$\left.\begin{array}{lr} \text{(a)} & 14x - 10y + 34 = 0 \\ \text{(b)} & 7x + 9y - 11 = 0 \end{array}\right\}$$

If x, y, and z satisfy the equations in the given system, x and y must satisfy the two equations in this last system [Why?]. We solve this system of two equations in two variables and find that the solution set is $\{(-1, 2)\}$.

Substitute in, say, the first of the three equations in the given system:

$$5 \cdot -1 + 2 \cdot 2 + 3z - 8 = 0,$$
$$z = 3$$

So, $(-1, 2, 3)$ is a solution of the first equation. Is it a solution of both the second and third equations? Is it the only solution of the given system?

$$3x^2 - 2 \cdot 9 + 6 = 0$$

$$x^2 = 4$$

$$x = 2 \text{ or } x = -2$$

<u>Check.</u> $2 \cdot 4 + 5 \cdot 9 - 53 \stackrel{?}{=} 0$

$$8 + 45 - 53 = 0 \quad \checkmark$$

The solution set is $\{(2, 3), (2, -3), (-2, 3), (-2, -3)\}$.

EXERCISES

<u>A</u>. Solve these systems.

1. $\left.\begin{array}{l} \frac{1}{x} + \frac{1}{y} = 10 \\ \frac{1}{x} - \frac{1}{y} = 3 \end{array}\right\}$

2. $\left.\begin{array}{l} \frac{3}{x} - \frac{5}{y} = 6 \\ \frac{4}{x} + \frac{3}{y} = 37 \end{array}\right\}$

3. $\left.\begin{array}{l} x + 2y = 6 \\ \frac{x}{4} + \frac{y}{5} = 2 \end{array}\right\}$

4. $\left.\begin{array}{l} 3x^2 - \sqrt{y} = 9 \\ 2x^2 + \sqrt{y} = 11 \end{array}\right\}$

5. $\left.\begin{array}{l} 6x - \frac{13}{y} = 2 \\ 5x - \frac{12}{y} = 4 \end{array}\right\}$

6. $\left.\begin{array}{l} 2x^2 = y^2 + 14 \\ 2y^2 = x^2 + 47 \end{array}\right\}$

7. $\left.\begin{array}{l} 3x^2 - 5y^2 = 4 \\ y^2 - 4 = 0 \end{array}\right\}$

8. $\left.\begin{array}{l} 3x^2 + 2y = 27 \\ x^2 - y = 4 \end{array}\right\}$

9. $\left.\begin{array}{l} \frac{1}{2x} + \frac{2}{3y} = 18 \\ \frac{3}{4x} + \frac{4}{5y} = 21 \end{array}\right\}$

10. $\left.\begin{array}{l} \frac{x+y}{2} - \frac{x-y}{3} = 8 \\ \frac{x+y}{3} + \frac{x-y}{4} = 11 \end{array}\right\}$

☆ 11. $\left.\begin{array}{l} x + y^2 + \frac{1}{z} = 8 \\ 3x + y^2 - \frac{1}{z} = 4 \\ 2x - 3y^2 + \frac{3}{z} = -1 \end{array}\right\}$

☆ 12. $\left.\begin{array}{l} \frac{1}{x} = \frac{1}{z} - \frac{2}{y} \\ 2(\frac{1}{y} - \frac{1}{x}) = \frac{1}{z} \\ \frac{3}{y} - \frac{5}{x} - 1 = \frac{1}{z} \end{array}\right\}$

<u>B</u>. 1. Find an equation whose graph is a straight line which contains the graphs of (7, 3) and (−6, 2).

SYSTEMS OF NONLINEAR EQUATIONS "IN LINEAR FORM"

Procedures used in handling systems of linear equations can be applied to systems of nonlinear equations which are <u>linear in form.</u>

<u>Example 1.</u>

$$\left.\begin{aligned} 2\left(\frac{1}{x}\right) - 3\left(\frac{1}{y}\right) - 17 &= 0 \\ 5\left(\frac{1}{x}\right) + 6\left(\frac{1}{y}\right) - 2 &= 0 \end{aligned}\right\}$$

[These equations are linear in '$\frac{1}{x}$' and '$\frac{1}{y}$'.]

<u>Solution.</u>

$$\begin{aligned} 4\left(\frac{1}{x}\right) - 6\left(\frac{1}{y}\right) - 34 &= 0 \\ 5\left(\frac{1}{x}\right) + 6\left(\frac{1}{y}\right) - 2 &= 0 \\ \hline 9\left(\frac{1}{x}\right) \qquad - 36 &= 0 \\ \frac{1}{x} &= 4 \\ x &= \frac{1}{4} \\ 2(4) - 3\left(\frac{1}{y}\right) - 17 &= 0 \\ -3\left(\frac{1}{y}\right) &= 9 \\ \frac{1}{y} &= -3 \\ y &= -\frac{1}{3} \end{aligned}$$

<u>Check.</u>

$$5\left(\frac{1}{\frac{1}{4}}\right) + 6\left(\frac{1}{-\frac{1}{3}}\right) - 2 \stackrel{?}{=} 0$$

$$20 + -18 - 2 = 0$$

The solution set is $\{(\frac{1}{4}, -\frac{1}{3})\}$.

<u>Example 2.</u>

$$\left.\begin{aligned} 3x^2 - 2y^2 + 6 &= 0 \\ 2x^2 + 5y^2 - 53 &= 0 \end{aligned}\right\}$$

[These equations are linear in 'x^2' and 'y^2'.]

<u>Solution.</u>

$$\begin{aligned} 6x^2 - 4y^2 + 12 &= 0 \\ 6x^2 + 15y^2 - 159 &= 0 \\ \hline -19y^2 + 171 &= 0 \\ y^2 &= 9 \\ y = 3 \text{ or } y &= -3 \end{aligned}$$

Correction. On page 5-213, third line from the bottom, change '6w = 202' to '6w = 162'.

Answers for Part A.

1. $\{(\frac{2}{13}, \frac{2}{7})\}$ 2. $\{(\frac{1}{7}, \frac{1}{3})\}$ 3. $\{(\frac{28}{3}, -\frac{5}{3})\}$ 4. $\{(2, 9), (-2, 9)\}$

5. $\{(-4, -\frac{1}{2})\}$ 6. $\{(5, 6), (5, -6), (-5, 6), (-5, -6)\}$

7. $\{(2\sqrt{2}, 2), (2\sqrt{2}, -2), (-2\sqrt{2}, 2), (-2\sqrt{2}, -2)\}$

8. $\{(\sqrt{7}, 3), (-\sqrt{7}, 3)\}$ 9. $\{(-\frac{1}{4}, \frac{1}{30})\}$ 10. $\{(18, 6)\}$

☆11. $\{(1, 2, \frac{1}{3}), (1, -2, \frac{1}{3})\}$ ☆12. $\emptyset$

*

Answers for Part B [on pages 5-212 and 5-213].

1. We must find an equation of the form '$y = ax + b$' which is satisfied by (7, 3) and (−6, 2). In order to do this, we solve the system of equations:

$$\left.\begin{aligned} 3 &= 7a + b \\ 2 &= -6a + b \end{aligned}\right\}$$

The solution set is $\{(\frac{1}{13}, \frac{32}{13})\}$. Hence '$y = \frac{1}{13}x + \frac{32}{13}$' is an equation whose graph is a straight line which contains the graphs of (7, 3) and (−6, 2).

2. $y = 3x^2 + 2x - 5$

☆3. $x = -4y^2 + 3y + 2$

☆4. $\{(4, 29), (2, 3)\}$

*

We suggest assigning Parts Q and S of the Miscellaneous Exercises at this time [5-232 and 5-234ff.] to prepare students for the subsection on problems which starts on page 5-213.

So, the cashier has 27 one-dollar bills and 35 five-dollar bills.

Check. 35 is 8 more than 27. The 35 five-dollar bills are worth 175 dollars. 175 + 27 = 202.

Example 2. A grocer wishes to blend two grades of coffee to obtain a mixture of 100 pounds which he can sell for 89 cents per pound. He has 85-cent coffee and 95-cent coffee. How much of each grade of coffee should he use in the mixture?

Solution. Suppose he mixes c pounds of 85-cent coffee with n pounds of 95-cent coffee. Then, $c + n = 100$. Also, this mixture is worth $85c + 95n$ cents. So, $85c + 95n = 89 \cdot 100$. Hence, we are looking for numbers, c and n, such that

$$\left.\begin{array}{lr} (1) & c + n = 100 \\ \text{and} \quad (2) & 85c + 95n = 8900. \end{array}\right\}$$

Solve this system and check.

Example 3. I am thinking of an integer between 9 and 100. I interchange the digits in its decimal numeral. This gives me a number which is 3 less than 4 times the original number. Find the original number if the digits stand for numbers whose sum is 7.

Solution. The decimal numeral contains two digits [Why?]. Suppose the tens digit is t and the units digit is u. Then, the original number is $10t + u$. The new number is $10u + t$. So, $10u + t = 4(10t + u) - 3$. Also, $t + u = 7$. Hence, we want numbers t and u such that

$$\left.\begin{array}{lr} (1) & 10u + t = 4(10t + u) - 3 \\ \text{and} \quad (2) & t + u = 7. \end{array}\right\}$$

We solve equation (2) for 't':

$$t = 7 - u,$$

and then substitute for 't' in equation (1).

2. Find an equation whose graph is a parabola with a vertical axis of symmetry and which contains the points (4, 51), (−3, 16), and (1, 0).

☆ 3. Find an equation whose graph is a parabola with a horizontal axis of symmetry and which contains the points (1, 1), (−25, 3), and (2, 0).

☆ 4. $\{(x, y): y = 3x^2 - 5x + 1\} \cap \{(x, y): y = 13x - 23\} = ?$

USING SYSTEMS TO SOLVE PROBLEMS

Many of the problems you solved in earlier units by using one equation in one variable can be solved by using a system of two equations in two variables. Frequently, it is easier to "set up" a system of equations than it is to set up a single equation. Of course, you should get the correct answer to the problem in either case.

Example 1. A cashier has 8 more five-dollar bills than she has one-dollar bills. The total value of these bills is $202. How many bills of each denomination does she have?

Solution. Suppose the cashier has w one-dollar bills and f five-dollar bills. Then, $f = w + 8$. Moreover, since the five-dollar bills are worth 5f dollars and the one-dollar bills are worth w dollars, the total value of these bills is $5f + w$ dollars. So, $5f + w = 202$. Hence, we are looking for numbers, f and w, such that

$$\left.\begin{array}{lll} & (1) & f = w + 8 \\ \text{and} & (2) & 5f + w = 202. \end{array}\right\}$$

We solve this system by substitution.

$$5(w + 8) + w = 202$$

$$5w + 40 + w = 202$$

$$6w = 202$$

$$w = 27$$

$$f = 27 + 8 = 35$$

Answers for Exercises [on pages 5-215 through 5-218].

1. 30 dimes, 90 nickels [$d + n = 120$, $10d + 5n = 750$]

2. 67 three-cent stamps, 37 two-cent stamps [r... 3-cent stamps, t... 2-cent stamps; $r + t = 104$, $3r + 2t = 275$]

3. 12 lbs. of 52¢ tea, 8 lbs. of 47¢ tea [x... lbs of 52¢ tea, y... lbs. of 47¢ tea; $x + y = 20$, $52x + 47y = 1000$]

4. The data are inconsistent; the cheapest milk obtainable is 6¢ per quart, so it is impossible to make a mixture costing only 5¢ per quart.

5. 250 cups at 10¢, 600 cups at 5¢ [$f + t = 850$, $5f + 10t = 5500$]

6. 45, 25 [$\ell - s = 20$, $5\ell + 7s = 400$]

7. .60 ['60' is acceptable if one regards '06' as a decimal numeral.] [f... first digit, s... second digit; $f + s = 6$, $0.1(10f + s) = f + 10s$]

8. \$20,000 at 4%, \$5,000 at 3% [x... dollars at 4%, y... dollars at 3%; $x + y = 25000$, $.04x + 03y = 950$]

9. \$3,000 at 6.5%, \$2,000 at 5.5% [n... dollars at 6.5%, d... dollars at 5.5%; $n + d = 5000$, $.065n + .055d = 305$]

10. 72 [$8(t + u) = 10t + u$, $t + 10u = 4t - 1$]

11. \$1,500 at 4%, \$2,000 at 3% [n... dollars at 4%, d... dollars at 3%; $n + 500 = d$, $.04n = .03d$]

12. $7\frac{2}{3}$, $17\frac{1}{3}$ [$x + y = 25$, $y = 2x + 2$]

13. 12, 14 [$\frac{s + \ell}{2} = 13$, $\ell = s + 2$]

14. 80 feet wide, 100 feet long [$2\ell + 2w = 360$, $\ell = w + 20$]

15. 5¢ for cork, \$1.05 for bottle [$c + b = 110$, $b - 100 = c$]

16. (86, −28) [$f + s = 58$, $f - s = 114$]

$$10u + (7 - u) = 4[10(7 - u) + u] - 3$$
$$10u + 7 - u = 4[70 - 10u + u] - 3$$
$$10u + 7 - u = 280 - 40u + 4u - 3$$
$$9u + 7 = 277 - 36u$$
$$45u = 270$$
$$u = 6$$
$$t = 7 - 6 = 1$$

The original number is 16.

Check. 61 is $4 \cdot 16 - 3$, and $1 + 6$ is 7.

EXERCISES

1. A jar of dimes and nickels contains 120 coins worth a total of \$7.50. How many dimes are there in the jar? How many nickels?

2. Pat found an envelope containing 104 stamps. Some were 3-cent stamps and the rest 2-cent stamps. The stamps were worth \$2.75. How many 3-cent and how many 2-cent stamps did Pat find.

3. An importer blends 20 pounds of tea by mixing 47-cent tea and 52-cent tea. The mixture is to be sold at 50 cents a pound. How many pounds of each kind of tea must be mixed?

4. A housewife decides to save money by mixing powdered milk and whole milk. Whole milk costs 24 cents a quart and skim milk made from powder will cost 6 cents a quart. If the family uses 90 quarts of milk per month and if the housewife decided to spend 5 cents per quart, how many quarts of whole milk must she use each month?

5. Two boys run a lemonade stand. They sold 850 cups of lemonade on a certain day, some at 5 cents per cup, the rest at 10 cents per cup. Their total receipts were \$55. How many cups were sold at each price?

6. Find two numbers whose difference is 20 and such that seven percent of the smaller added to five percent of the larger gives the sum 4.

7. The individual digits in a two-digit decimal numeral stand for numbers whose sum is 6. When the digits are interchanged, the new numeral stands for a number which is 1/10 of the original number. Find the original number.

8. A man invests a total of $25,000 in two enterprises. He invests a part of this money at 3% and the remainder at 4%. How much does he invest at each rate if his annual income is $950?

9. Mr. Adams invested $5,000 in two enterprises, one bearing interest at 5.5% and the other at 6.5%. If his total annual income from these investments is $305, how much did he invest in each enterprise?

10. Eight times the sum of the numbers named by the individual digits in a two-digit decimal numeral is equal to the number named by the two-digit numeral. Interchanging the digits of the numeral gives the name of a new number which is 1 less than 4 times the number named by the tens digit of the given numeral. Find the original number.

11. Mrs. Charles invested $500 more at 3% than she invested at 4%. If she received the same income from both investments, how much did she invest at each rate?

12. The sum of two numbers is 25. One number is 2 more than twice the other. Find the two numbers.

13. The average of two numbers is 13. The larger is 2 more than the smaller. Find the numbers.

14. The foot-perimeter of a rectangular lot is 360. Its length is 20 feet more than the width. What are its dimensions?

15. A bottle and a cork together cost $1.10. If the bottle costs $1.00 more than the cork, what is the cost of each?

16. Find an ordered pair of numbers whose sum if 58 and whose difference, second from first, is 114.

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Quiz.

1. Solve this system of equations.

$$\left.\begin{aligned} 3x + y - z &= 10 \\ 2x - 3y - 5z &= -3 \\ 4x + 2y + 7z &= 0 \end{aligned}\right\}$$

2. Three cans of carrots and five cans of peas cost \$1.55. Two of these cans of carrots and three cans of the peas cost \$.96. What is the price of one can of carrots?

*

Answers for Quiz.

1. (1, 5, −2)

2. 15 cents

Correction. On page 5-218, in the third line of Exercise 29, change 'threr' to 'there'.

17. The data are insufficient. [$3f + 2s = 50$, $.06f + .04s = 1$; since by the MTP the second equation can be transformed into an equation which is a copy of the first, the equations as given by the data are dependent.]

18. 0.5, 5.5[$\ell - s = 5$, $\ell = 3s + 4$]

19. 9 dimes, 28 nickels [$d + n = 37$, $10d + 5n = 230$]

20. $6\frac{1}{4}$, $1\frac{1}{4}$[$\ell - s = 5$, $\ell = 5s$]

21. Andy is $\frac{1}{10}$ year, Bill is $1\frac{3}{5}$ years [$B = 6A + 1$, $2B - 3 = 2A$]

22. Red, 2 for 9¢; green, 2 for 5¢ [$2r + 4g = 19$, $4r + 2g = 23$]

23. \$8,500 at 4.5%, \$14,500 at 3% [n... dollars at 4.5%, d... dollars at 3%; $n + d = 23000$, $.045n + .03d = 817.50$]

24. 304, 896 [$s + \ell = 1200$, $3s + 2\ell = 2704$]

25. 5 quart jars, 9 pint jars [$q + p = 14$, $2q + p = 19$]

26. The data are inconsistent. [Since the cheaper grade of rice costs 27 cents per pound, it is impossible to make a mixture which costs only 24 cents per pound.]

27. 16 units wide, 24 units long [$2\ell + 2w = 80$, $w = \frac{2}{3}\ell$]

28. The data are inconsistent. [m... Mary's age now, j... John's age now; $m - 3 = 3(j - 3)$, $m + 3 = 4(j + 3)$. The solution of this system is $(-15, -51)$. But, we are seeking numbers of arithmetic; and, since there is no pair of positive numbers which is a solution of the system, there is no pair of numbers of arithmetic which satisfies the conditions of the problem.]

29. 12 five-dollar bills, 15 one-dollar bills [$f + 3 = o$, $5f + o = 75$]

30. Marilyn, 12 hours, Ruth, 6 hours [$\frac{1}{r} + \frac{1}{m} = \frac{1}{4}$, $\frac{3}{r} + \frac{6}{m} = 1$]

31. $\frac{3}{4}$[$\frac{n + 3}{d + 3} = \frac{6}{7}$, $\frac{n - 2}{d - 2} = \frac{1}{2}$]

32. '567' [$h + t + u = 3h + 3$, $u = h + 2$,
$100t + 10h + u = 7.3[(100t + 10h + u) - (100h + 10t + u)]$]

33. $23\frac{17}{21}$ feet [$\pi d(\frac{4}{5}d) = 500\pi = \pi(d - 4)h$]

17. The sum of three times one number and twice the second number is 50. Four percent of the second number added to six percent of the first number gives the sum 1. Find the two numbers.

18. The difference between two numbers is 5. The larger number is 4 more than 3 times the smaller. What are the numbers?

19. Thirty-seven coins consisting of nickels and dimes are worth $2.30. How many coins of each kind are there?

20. The difference between two numbers is 5. If the larger number is 5 times the smaller, what are the numbers?

21. Bill's present age in years is 1 more than six times Andy's age in years. If both boys were twice as old as they actually are, then Andy would be 3 years younger than Bill. How old is each boy now?

22. Milton bought 2 red apples and 4 green ones for 19 cents. Oswald bought 4 of the red apples and 2 green ones for 23 cents. What was the price of each kind?

23. Mr. Jones invested a total of $23,000 in two businesses. He receives a 3% return from one business and a 4.5% return from the other. His annual income from these two investments is $817.50. How much did he invest at each rate?

24. The sum of two numbers is 1200. Three times the smaller plus twice the larger equals 2704. What are the numbers?

25. A box contains 14 jars. Some are pint jars, the rest are quart jars. The 14 jars have a combined capacity of 19 pints. How many jars of each size are there in the box?

26. How many pounds of 32-cent rice and how many pounds of 27-cent rice must be mixed to make 50 pounds of 24-cent rice?

27. The width of a rectangle is 2/3 of its length and its perimeter is 80. Find its dimensions.

28. Three years ago Mary was three times as old as John. Three years hence she will be four times as old as John will be. Find the present age of each.

29. A man drew $75.00 in one-dollar bills and five-dollar bills from the bank. There were 3 more one-dollar bills than five-dollar bills. How many bills of each kind were threr?

30. Ruth and Marilyn can do a typing job together in 4 hours. If Ruth works alone on the job for 3 hours, Marilyn can finish the job by herself in 6 hours. How long would it take each typist to do the entire job if she worked alone?

31. If 3 is added to the numerator-number and to the denominator-number of a fraction, the ratio of the first sum to the second is $\frac{6}{7}$. If 2 is subtracted from both numerator-number and denominator-number of the original fraction, the ratio of the first difference to the second is $\frac{1}{2}$. What is the original fraction?

32. The individual digits in a three-digit decimal numeral stand for numbers whose sum is 3 more than 3 times the number named by the hundreds digit. If the tens digit is interchanged with the hundreds digit, the new numeral stands for a number which is 7.3 times the difference of the original number from the new number. In the original numeral, the units digit names a number which is two more than the number named by the hundreds digit. What is the original numeral?

33. The area-measure of the cylindrical surface of a gasoline storage tank is 500π, and its height-measure is $\frac{4}{5}$ of its diameter-measure. What should be the height of a similar tank which has the same area-measure but is 4 feet smaller in diameter? [<u>Hint</u>. A formula for finding the area-measure of a cylindrical surface is '$S = 2\pi rh$'.]

[Supplementary exercises are in Part W, pages 5-275 through 5-278.]

Quiz items covering Unit 5. [These items differ somewhat from the exercises found in the textbook. You may want to use some of them along with others of your own design in a final examination.]

1. If 1 is a root of the quadratic equation in 'x':

$$5x^2 + tx + 1 = 0,$$

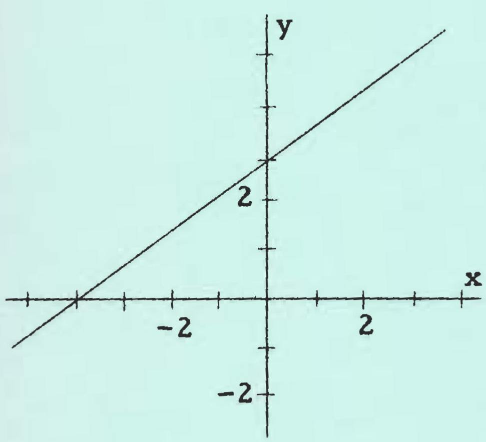
what is the other root?

2. Suppose f is the function such that

$$f(x) = \frac{(x - 3)(x + 5)}{(4x - 1)(x + 7)}.$$

For what real numbers is f not defined?

3.



$f = \{(x, y): y = \underline{\ ?\ } x + \underline{\ ?\ }\}$

4. The graph of the equation '$|x| + |y| = 10$' is __(?)__.

(A) a line (B) two lines (C) two rays

(D) four half-lines (E) a square

5. The dimensions of a 3-inch by 5-inch snapshot are doubled to make an enlargement. The area of the enlarged picture is __?__ times the area of the original snapshot.

6. Consider the set P of all linear functions which contain the ordered pair common to $\{(x, y): x - y - 2 = 0\}$ and $\{(x, y): 2x - y - 1 = 0\}$. Write the defining equation for the linear function in P which has slope 1.

7. Solve for 'M': $R = \frac{S(P - M)}{2s}$

8. If x and y are inversely proportional and if y is 30 when x is 5 then, when x is 25, y is ___?___.

9. $\forall_x \forall_y$ if $x - y = 5$ and $x + y = 20$ then $x^2 - y^2 =$ ___?___.

10. The number of ordered pairs which belong to both $\{(x, y): y = x^2\}$ and $\{(x, y): y = |x|\}$ is ___?___.

11. A boy who has only k half-dollars and d dimes buys n notebooks at 15 cents each. How many cents does he have left?

12. Two variable quantities, P and Q, are so related that 5 times each value of P is twice the corresponding value of Q. Is P proportional to Q? If not, tell why. If so, give the factor of proportionality.

13. [Suppose that 'X'' denotes the complement of the set X with respect to some set S.] $(A \cup B)' =$ ___(?)___.

(A) $A' \cup B'$ (B) $A' \cup B$ (C) $A \cup B'$

(D) $A \cap B$ (E) $A' \cap B'$

14. Suppose the sum of three consecutive integers is T. Then, the smallest of these integers is ___?___.

15. Solve: $3x^2 - 2x = 2$

16. If the graphs of (2, 3), (4, 9), and (6, k) are all contained in a straight line then k = ___?___.

17. Simplify: $\dfrac{1}{1 + \dfrac{1}{1 + a}}$

18. Consider the linear function f defined by the equation '$2x + 4y + 5 = 0$'.

   (a) Find the slope of this function.

   (b) Write an equation which defines the function whose graph is parallel to the graph of f and which contains (0, 0).

   (c) Find the ordered pairs in the intersection of f and the x-axis.

19. Suppose that

$$f = \{(1, 2), (2, 2), (3, 0)\}$$

and

$$g = \{(1, 3), (2, 3), (0, 1)\}.$$

Then, it follows that ___(?)___.

(A) $f \cup g$ is a function

(B) the domain of $f + g = \{1, 2, 3, 0\}$

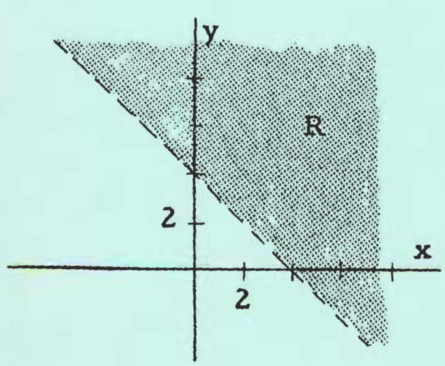
(C) the range of $f + g$ is $\{5\}$

(D) the range of $f + g$ is $\{2, 3\}$

(E) None of these follows.

20. If $(-1, 8)$ belongs to the quadratic function defined by '$y = 2x^2 + 4x + k$' then $k =$ _?_.

21.



$R = \{(x, y):$ ______(?)______ $\}$

(A) $x + y = 4$

(B) $x > 4$

(C) $x + y > 4$

(D) $x - y > 4$

(E) $x + y > 8$

22. If the slope of $\{(x, y): 2y - kx = 4\}$ is -3 then $k =$ _?_.

23. $\{(x, y): x + y - 2 + |x + y - 2| = 0\} = \{(x, y): \underline{\quad (?) \quad}\}$

(A) $x = 1$ and $y = 1$ (B) $x + y - 2 > 0$ (C) $x + y - 2 = 0$

(D) $x + y - 2 < 0$ (E) $x + y - 2 \leq 0$

24. Two trains leave from the same city at the same time and travel in opposite directions. In 3 hours they are 300 miles apart. If one train's rate is 60 miles per hour then the other train's rate is ___?___ miles per hour.

25. If the graphs of the equations '$x^2 + y^2 = 25$' and '$y = x^2$' are drawn on the same chart, what is the total number of points common to the graphs?

26. If the area-measure of a rectangle is $12x - x^2$ and x is the measure of a side then the area-measure is a maximum if $x =$ ___?___.

27. Suppose the sets A and B are represented by circular regions. Then, the shaded region in Figure 1 represents $A \cap B$, and the

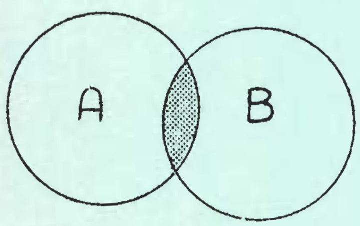


Figure 1.

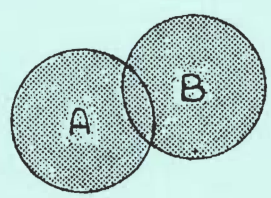


Figure 2.

shaded region in Figure 2 represents $A \cup B$. So, the shaded region in Figure 3 represents ___(?)___.

(A) $A \cup (B \cap C)$

(B) $A \cap (B \cup C)$

(C) $(A \cap B) \cap C$

(D) $A \cup (B \cup C)$

(E) $(A \cup B) \cap C$

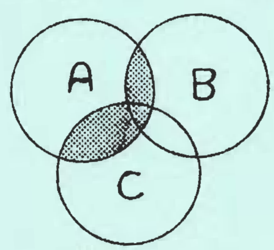


Figure 3.

28. Find the coordinates of the lowest point of the graph of the equation '$y = x^2 - 6x + 9$'.

29. True or false?

(a) $\forall_x \sqrt{x^2 + 9} = x + 3$ (b) $\exists_x \sqrt{x^2 + 9} = x + 3$

(c) $\forall_x x^2 + 1 = 0$ (d) $\exists_x x^2 + 1 = 0$

(e) $\forall_x x^2 - 8x + 16 > 0$ (f) $\exists_x x^2 - 8x + 16 > 0$

30. Suppose that $f = \{(1, 2), (3, 4), (5, 6), (7, 8)\}$. Then, the inverse of f is __(?)__.

(A) $\{(7, 8), (5, 6), (3, 4), (1, 2)\}$

(B) $\{(1, -2), (3, -4), (5, -6), (7, -8)\}$

(C) $\{(2, 1), (4, 3), (6, 5), (8, 7)\}$

(D) $\{(1, \frac{1}{2}), (3, \frac{1}{4}), (5, \frac{1}{2}), (7, \frac{1}{8})\}$

(E) nonexistent

31. If $(2, -1) \in \{(x, y): x = ky + t\} \cap \{(x, y): 3x + ky = t\}$ then k = __?__ and t = __?__.

32. The roots of '$(m - 4)(m^2 + 3m - 10) = 0$' are __?__, __?__, and __?__.

33. A quadratic function has 2 as maximum value. If it contains the ordered pairs $(-\frac{1}{2}, 0)$ and $(\frac{3}{2}, 0)$ then its defining equation is: $y =$ __?__ $x^2 +$ __?__ $x +$ __?__.

34. The graph of '$x^2 + 2xy + y^2 = 1$' is __(?)__.

(A) a straight line (B) two straight lines (C) a circle

(D) a parabola (E) an hyperbola

35. The sum of the area-measures of a rectangle and a square is 68. The rectangle is twice as long as it is wide, and a side of the square is 2 units longer than the width of the rectangle. Find the width-measure and the length-measure of the rectangle.

36. What is the solution set in x of '$4 - 2x < 10$'? [Use brace-notation and the simplest set selector to describe the solution set.]

37. Ten quarts of a solution containing m% alcohol are mixed with twenty quarts of another solution containing n% alcohol. The part of the resulting mixture which is alcohol is _?_ .

38. Bill travels from A to B, a trip of 30 miles, at the rate of 6 miles per hour, and returns immediately from B to A. If his average rate for the entire trip was 5 miles per hour then his return rate was _?_ miles per hour.

39. Consider the equation in 'x':

$$x^3 + px + q = 0$$

If two of its roots are -1 and 3 then p = _?_ and q = _?_ .

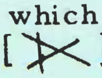
40. If the quadratic equation '$x^2 + 6x - 2 + k = 0$' has just one root then k = _?_ .

41. Write a quadratic equation whose solution set is $\{\frac{1}{3}, -3\}$.

42. The graphs of the equations '$3x + 5y = 9$', '$6x - 10y = 10$', and '$6x + 10y = 11$' are _(?)_ .

(A) three parallel lines [///]

(B) two parallel lines crossed by a third line [-/-/-]

(C) three lines no two of which are parallel but with no point common to all three []

(D) three lines intersecting in a single point []

(E) two lines only

TC[5 - 218]f

43. If y varies inversely as x then the graph obtained by plotting y against x is __(?)__.

(A) a straight line (B) a circle (C) an hyperbola

(D) a parabola (E) an ellipse

44. Suppose that f = {(1, 2), (2, 3), (3, 4), (5, 6)}. Which of the following statements is true?

(A) the domain of f = {1, 2, 3, 4, 5} (B) the range of f = {2, 3, 4, 6}

(C) f(2) = 1 (D) f(f(2)) = 2 (E) None of these is true.

45. The area-measure of a circle varies directly as the square of its radius. The factor of variation is __?__.

46. Suppose f is the function such that $f(x) = \sqrt{2x + 3}$. The domain of f = {x: ____?____ }

47. The expression '$x^2 + \frac{k}{n}x +$ __?__' is a perfect square.

48. Suppose that $t > 0$ and $2t^2 + 5t - 33 = 0$. Then, t = __?__.

49. If x varies directly as y and if x = 12 when y = 8 then x = __?__ when y = 10.

50. Consider a quadratic equation in 'x' of the form:

$$qx^2 + sx + t = 0, \qquad [q > 0]$$

Suppose the roots are r_1 and r_2.

(a) If $r_1 < 0$ and $r_2 > 0$ then __(?)__.

(A) $s > 0$ (B) $s < 0$ (C) $t > 0$ (D) $t < 0$

(b) If $r_1 = r_2$ then __(?)__.

(E) $s^2 = 4qt$ (F) $s^2 = -4qt$ (G) $s^2 = qt$ (H) $s^2 = -qt$

51. Write an equation whose straight line graph contains the graph of (0, −3) and is parallel to the graph of '$y = 2x + 6$'.

52. Suppose that D is the doubling function and S is the squaring function.

(a) $S(-5) =$ ____ (b) $D(-5) =$ ____ (c) $S(0) =$ ____

(d) $D(0) =$ ____ (e) $D^2(3) =$ ____ (f) $S^2(3) =$ ____

(g) $[D \circ S](5) =$ ____ (h) $[S \circ D](5) =$ ____ (i) $D^{-1}(9) =$ ____

*

Answers for quiz items.

1. $\frac{1}{5}$ 2. $\frac{1}{4}$ and −7 3. $\frac{3}{4}$, 3 4. (E) 5. 4

6. $y = x - 2$ 7. $M = \frac{PS - 2sR}{S}$ 8. 6 9. 100

10. 3 11. $50k + 10d - 15n$ 12. Yes; 2/5

13. (E) 14. $\frac{T - 3}{3}$ 15. $\frac{1 + \sqrt{7}}{3}$, $\frac{1 - \sqrt{7}}{3}$ 16. 15

17. $\frac{1 + a}{2 + a}$ 18. (a) $-\frac{1}{2}$ (b) $y = -\frac{1}{2}x$ (c) $(-\frac{5}{2}, 0)$ 19. (C)

20. 10 21. (C) 22. −6 23. (E) 24. 40

25. 2 26 6 27. (B) 28. (3, 0)

29. (a) F (b) T (c) F (d) F (e) F (f) T

30. (C) 31. 2, 4 32. 4, −5, 2 33. −2, 2, 3/2

34. (B) 35. 4, 8 36. $\{x: x > -3\}$

37. $\frac{m + 2n}{300}$ 38. 30/7 39. −7, −6 40. 11

41. $3x^2 + 8x - 3 = 0$		42. (B)	43. (C)	44. (B)	
45. π		46. $x \geq -\frac{3}{2}$	47. $\left(\frac{k}{2n}\right)^2$	48. 3	
49. 15		50. (a) (D)	(b) (E)	51. $y = 2x - 3$	
52. (a) 25	(b) -10	(c) 0	(d) 0	(e) 36	
(f) 81	(g) 50	(h) 100	(i) 4.5		

TABLE OF SQUARES AND SQUARE ROOTS

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$
1	1	1.000	3.162	51	2601	7.141	22.583
2	4	1.414	4.472	52	2704	7.211	22.804
3	9	1.732	5.477	53	2809	7.280	23.022
4	16	2.000	6.325	54	2916	7.348	23.238
5	25	2.236	7.071	55	3025	7.416	23.452
6	36	2.449	7.746	56	3136	7.483	23.664
7	49	2.646	8.367	57	3249	7.550	23.875
8	64	2.828	8.944	58	3364	7.616	24.083
9	81	3.000	9.487	59	3481	7.681	24.290
10	100	3.162	10.000	60	3600	7.746	24.495
11	121	3.317	10.488	61	3721	7.810	24.698
12	144	3.464	10.954	62	3844	7.874	24.900
13	169	3.606	11.402	63	3969	7.937	25.100
14	196	3.742	11.832	64	4096	8.000	25.298
15	225	3.873	12.247	65	4225	8.062	25.495
16	256	4.000	12.649	66	4356	8.124	25.690
17	289	4.123	13.038	67	4489	8.185	25.884
18	324	4.243	13.416	68	4624	8.246	26.077
19	361	4.359	13.784	69	4761	8.307	26.268
20	400	4.472	14.142	70	4900	8.367	26.458
21	441	4.583	14.491	71	5041	8.426	26.646
22	484	4.690	14.832	72	5184	8.485	26.833
23	529	4.796	15.166	73	5329	8.544	27.019
24	576	4.899	15.492	74	5476	8.602	27.203
25	625	5.000	15.811	75	5625	8.660	27.386
26	676	5.099	16.125	76	5776	8.718	27.568
27	729	5.196	16.432	77	5929	8.775	27.749
28	784	5.292	16.733	78	6084	8.832	27.928
29	841	5.385	17.029	79	6241	8.888	28.107
30	900	5.477	17.321	80	6400	8.944	28.284
31	961	5.568	17.607	81	6561	9.000	28.460
32	1024	5.657	17.889	82	6724	9.055	28.636
33	1089	5.745	18.166	83	6889	9.110	28.810
34	1156	5.831	18.439	84	7056	9.165	28.983
35	1225	5.916	18.708	85	7225	9.220	29.155
36	1296	6.000	18.974	86	7396	9.274	29.326
37	1369	6.083	19.235	87	7569	9.327	29.496
38	1444	6.164	19.494	88	7744	9.381	29.665
39	1521	6.245	19.748	89	7921	9.434	29.833
40	1600	6.325	20.000	90	8100	9.487	30.000
41	1681	6.403	20.248	91	8281	9.539	30.166
42	1764	6.481	20.494	92	8464	9.592	30.332
43	1849	6.557	20.736	93	8649	9.644	30.496
44	1936	6.633	20.976	94	8836	9.695	30.659
45	2025	6.708	21.213	95	9025	9.747	30.822
46	2116	6.782	21.448	96	9216	9.798	30.984
47	2209	6.856	21.679	97	9409	9.849	31.145
48	2304	6.928	21.909	98	9604	9.899	31.305
49	2401	7.000	22.136	99	9801	9.950	31.464
50	2500	7.071	22.361	100	10000	10.000	31.623

Answers for MISCELLANEOUS EXERCISES.

A.	1.	$4x^2 - 9$	2.	$8x$	3.	$3a + 7b$
	4.	$3x^3$	5.	$7x/6$	6.	$7/6$
	7.	$-7\sqrt{2}$	8.	$2k$	9.	$2y^2$
	10.	$3x^2 - 7x - 20$	11.	$m - 5n$	12.	$7a + 6b + 9$
	13.	$23x/12$	14.	$x - 3$	15.	$4x^4y$
	16.	$8b + 1$	17.	$4a^2 + 17a + 4$	18.	$16\sqrt{2}$
	19.	$6k^2 - 19k + 10$	20.	$10y$	21.	$2x^4$
	22.	$49s^2 - t^2$	23.	3	24.	$x - 2y$
	25.	$4\sqrt{3}$	26.	$-x^2 + 7x - 3$	27.	$(x + 5)/5$
	28.	$(x + 5y)/6$	29.	$x^2 + 2x + 8$	30.	$m^2 - 7m + 10$
	31.	$16 - x^2$	32.	13		

*

B.	1.	$3m(k + 5)$	2.	$5k^4m(1 + 14k^2)$	3.	$4r(x + 3)$
	4.	$7x^3y(x^2 + 3)$	5.	$(c - 5)(c + 5)$	6.	$(a - 6)(a + 6)$
	7.	$5a^2b^2(a + 3)$	8.	$4a^3b^3(a^2 + 2)$	9.	$(3x - 1)(3x + 1)$
	10.	$(x - 7)(x + 7)$	11.	$3rs^2(rs + 2)$	12.	$4ab(b + 5)$

*

C.	1.	5	2.	4	3.	3, -3	4.	-6	5.	1
	6.	12	7.	12	8.	5	9.	4	10.	10
	11.	6	12.	6	13.	7	14.	3	15.	5
	16.	no roots	17.	14	18.	1	19.	2.5	20.	5
	21.	-1	22.	1	23.	4	24.	4.5	25.	6

	26.	9	27.	3	28.	$\{x: x < 3\}$
	29.	$\{x: x < 6\}$	30.	6	31.	7
	32.	$\{x: x > -3\}$	33.	$\{x: x > 12\}$	34.	$\{y: y > 7\}$

*

D. 1. $\frac{m}{k}$ dollars 2. $x + 3$ 3. 80 4. (c) [about 499]

C. Solve.

1. $3m = 15$
2. $5r - 3 = 17$
3. $2x^2 = 18$
4. $7x + 3 = 5x - 9$
5. $5(t + 4) = 25$
6. $\frac{t}{3} + \frac{t}{4} = 7$
7. $\frac{3}{5} = \frac{b}{20}$
8. $4t = 20$
9. $3x + 2 = 14$
10. $\frac{4}{6} = \frac{x}{15}$
11. $\frac{x}{2} + \frac{2x}{3} = 7$
12. $4(s - 3) = 12$
13. $5x - 3 = 32$
14. $4p + 7 = 19$
15. $5x - 4 = 3x + 6$
16. $6(y+3) = 2(1+3y)$
17. $\frac{m}{7} = \frac{6}{3}$
18. $\frac{x}{3} + \frac{x}{4} = \frac{7}{12}$
19. $6b - 15 = 0$
20. $x^2 - 2x - 15 = 0$ and $x > 0$
21. $6 - 3x = 9$
22. $6y + 13 = 4y + 15$
23. $\frac{m}{2} + \frac{3m}{4} = 5$
24. $\frac{3}{x} = \frac{4}{6}$
25. $2m - 5 = 7$
26. $4(x + 6) + x = 8x - 3$
27. $4 = \frac{12}{b}$
28. $3x + 17 > 5 + 7x$
29. $2(3x - 1) < 5x + 4$
30. $\frac{r}{2} + \frac{r}{3} = 5$
31. $\frac{y + 2}{3} - \frac{y - 2}{5} = 2$
32. $3x + 1 < 8x + 16$
33. $\frac{8}{20} < \frac{x}{30}$
34. $10y - 2(3y + 1) > 26$

D. 1. If k feet of lumber cost m dollars, what is the cost of one foot?

2. For each x, $(x - 3)(\_\_\_\_\_\_) = x^2 - 9$.

3. In a class of 30 students, 6 were absent. What percent of the students were present?

4. The average distance between the sun and the Earth is 92,900,000 miles. If light travels at the rate of 186,300 miles per second, it takes __________ seconds for light from the sun to reach the Earth.

(a) less than 10
(b) between 10 and 100
(c) between 100 and 1000

MISCELLANEOUS EXERCISES

A. Simplify.

1. $(2x + 3)(2x - 3)$
2. $(3x + 1) + (4x - 3) + (x + 2)$
3. $5a + 4b - (2a - 3b)$
4. $12x^4y^3 \div (4xy^3)$
5. $\frac{2x + 3}{3} + \frac{x - 2}{2}$
6. $\frac{2x + 3}{3x} + \frac{x - 2}{2x}$
7. $\sqrt{2} + 3\sqrt{2} - 11\sqrt{2}$
8. $7k - 3m - (5k - 3m)$
9. $12y^3z^2 \div (6yz^2)$
10. $(3x + 5)(x - 4)$
11. $4m - 3n - (3m + 2n)$
12. $7a + 5b + 9 + b$
13. $\frac{5x}{3} + \frac{x}{4}$
14. $\frac{x^2 - 9}{5} \div \frac{2(x + 3)}{10}$
15. $24x^6y^2 \div (6x^2y)$
16. $2b + 3 + (5b - 6) + (b + 4)$
17. $(4a + 1)(4 + a)$
18. $5\sqrt{18} + \sqrt{2}$
19. $(2k - 5)(3k - 2)$
20. $3y + 4 + (5y - 3) + 2y - 1$
21. $28x^5y \div (14xy)$
22. $(7s - t)(7s + t)$
23. $15(x + 1)^3y^4 \div [5(x + 1)^3y^4]$
24. $(3x + 7y) + (5x - 4y) - (7x + 5y)$
25. $\sqrt{27} + \sqrt{3}$
26. $x^2 + 4x - 1 - (2x^2 - 3x + 2)$
27. $\frac{(x + 5)^2}{25} \cdot \frac{5}{x + 5}$
28. $\frac{x + y}{2} + \frac{y - x}{3}$
29. $(x + 2)^2 - 2(x - 2)$
30. $(2 - m)^2 - 3(m - 2)$
31. $(4 + x)(4 - x)$
32. $(4 + \sqrt{3})(4 - \sqrt{3})$

B. Factor.

1. $3km + 15m$ [Answer. $3m(k + 5)$]
2. $5k^4m + 70k^6m$ [Answer. $5k^4m\,(1 + 14k^2)$]
3. $4rx + 12r$
4. $7x^5y + 21x^3y$
5. $c^2 - 25$
6. $a^2 - 36$
7. $5a^3b^2 + 15a^2b^2$
8. $4a^5b^3 + 8a^3b^3$
9. $9x^2 - 1$
10. $x^2 - 49$
11. $3r^2s^3 + 6rs^2$
12. $4ab^2 + 20ab$

5. (0, 2) 6. -11 7. $1/(x - 3)$

*

E.

1. $x^2 - 5x - 36$ 2. $x^2 - 10x + 16$ 3. $x^2 + x - 56$
4. $x^2 + 8x + 16$ 5. $x^2 - 10x + 25$ 6. $9x^2 - 6x + 1$
7. $25 - 20y + 4y^2$ 8. $15 - 2x - x^2$ 9. $6x^2 + 13x - 28$
10. $a^4 - 25$ 11. $m^4 - n^4$ 12. $3x^2 + 39x - 204$
13. $4x^2 - 4x - 15$ 14. $4Q^2 - 4QA + A^2$ 15. $A^2 - 4AQ + 4Q^2$

*

F.

1. $(x + 7)(x + 9)$ 2. $(m - 11)(m + 2)$ 3. $(x - 19)(x + 19)$
4. $(18 - p)(1 - p)$ 5. $(k + 72)(k - 2)$ 6. $(3 - y)(3 + y)(9 + y^2)$
7. $(D + E)^2$ 8. $2(x + 3)(x + 5)$ 9. $7(y - 3)(y + 2)$
10. $5(A - 3)(A - 8)$ 11. $(3F - 2N)(2F - 3N)$ 12. $(d + 13)(d - 7)$

*

G.

1. 0, 7 2. 10, 20 3. 3, -17 4. $\{x: x < 2 \text{ or } x > 3\}$
5. all real numbers 6. no roots

*

H.

1. F [-1] 2. F [-1] 3. F [1/3] 4. F [-1] 5. F ['and']
6. F [$1.414^2 < 2$] 7. F [1/2] 8. F 9. T

*

I.

1. $1\frac{3}{4}$ 2. $n + 2$ 3. 5 4. $x/15$ 5. $x = (y-2)/3$
6. 75 7. $2x - 3$ 8. $(x - 2y)/2$ 9. $5\sqrt{3}$ [$\sqrt{75} > \sqrt{45}$]

*

J.

1. T 2. T 3. T 4. T 5. T
6. F[n(A) = 27, n(B) = 3] 7. F[n(A) = 1, n(B) = 49]

H. True or false?

1. $\forall_m\ |-3m| = 3m$
2. $\forall_x\ \sqrt{36x^2} = 6x$
3. $\forall_a\ 4a^2 \geq 2a$
4. $\forall_n\ \sqrt{n} \cdot \sqrt{n} = n$
5. '$x = 5$ and $x = 7$' is equivalent to '$x^2 - 12x + 35 = 0$'
6. $\sqrt{2} = 1.414$
7. $\forall_x$ 2x is an even number
8. $\exists_x\ 5(x - 1) = 3x + 2(7 + x)$
9. $\forall_x\ 5(3 - x) = 3(5 - x) - 2x$

I. 1. On a scale drawing of a house, $\frac{1}{8}$ inch on the drawing represents 1 foot. So, a 14-foot long room is represented by a line segment ________ inches long.

2. If n is an odd number, the next larger odd number is ________.
3. If $a = 3$ and $b = -2$ then $a - b =$ ________.
4. If the average gasoline consumption of a certain car is 15 miles to the gallon, how many gallons are needed to drive this car x miles?
5. Solve for 'x': $3x + 2 = y$
6. What percent of 80 is 60?
7. What is the average of $x - 8$ and $3x + 2$?
8. Reduce to lowest terms: $\frac{x^2 - 4y^2}{2x + 4y}$
9. Which is larger, $5\sqrt{3}$ or $3\sqrt{5}$?

J. True or false?

1. For each set B, $B \cup B = B$.
2. $\forall_B\ B \cup \emptyset = B$
3. $\forall_B\ B \cap \emptyset = \emptyset$
4. $\forall_B\ B \cap B = B$
5. $\forall_H \forall_K$ if K contains 6 elements $[n(K) = 6]$ and H contains 3 elements $[n(H) = 3]$, then $H \times K$ contains 18 elements $[n(H \times K) = 18]$.
6. $\forall_A \forall_B$ if $n(A \times B) = 81$ then $n(A) = n(B)$
7. $\forall_A \forall_B$ if $n(A \times B) = 49$ then $n(A) = n(B)$

5. Draw the straight line which contains the graphs of (3, 4) and (−6, −2). This line crosses the graph of the y-axis in the graph of what ordered pair?

6. For each x, $x^2 + 4x - 7 = (x + 2)(x + 2) +$ ________.

7. For each $x \neq 3$, $\frac{x^2 - 5x + 7}{x - 3} = x - 2 +$ ________.

E. Expand.

1. $(x + 4)(x - 9)$ [Answer. $x^2 - 5x - 36$]
2. $(x - 8)(x - 2)$
3. $(x - 7)(x + 8)$
4. $(x + 4)^2$
5. $(x - 5)^2$
6. $(3x - 1)^2$
7. $(5 - 2y)^2$
8. $(3 - x)(x + 5)$
9. $(2x + 7)(3x - 4)$
10. $(a^2 + 5)(a^2 - 5)$
11. $(m - n)(m^2 + n^2)(m + n)$
12. $3(x - 4)(x + 17)$
13. $\frac{1}{2}(2x - 5)(4x + 6)$
14. $(2Q - A)(2Q - A)$
15. $(A - 2Q)(A - 2Q)$

F. Factor.

1. $x^2 + 16x + 63$
2. $mm - 9m - 22$
3. $x^2 - 361$
4. $18 + p^2 - 19p$
5. $k^2 + 70k - 144$
6. $81 - y^4$
7. $D^2 + 2DE + E^2$
8. $2x^2 + 16x + 30$
9. $7y^2 - 7y - 42$
10. $5A^2 - 55A + 120$
11. $6F^2 - 13NF + 6N^2$
12. $d^2 + 6d - 91$

G. Solve.

1. $x(x - 7) = 0$
2. $c^2 - 30c + 200 = 0$
3. $2H^2 + 28H - 2 = 100$
4. $x^2 - 5x + 6 > 0$
5. $m^8 + 1728 = 1728 + m^8$
6. $5y^2 + 1 = y(2y + 3y)$

8. T   9. F [$A = \emptyset \neq B$]

10. T [$A \cup (B \cup C) = (A \cup A) \cup (B \cup C)$. Then, by commutativity and associativity, it follows that $(A \cup A) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.]

*

K. 1. $7x$ 2. T 3. 7 [neglecting sawcuts] 4. $2x/3$ dollars 5. $3t + 2$
6. 1 7. \$4.20 8. -2 9. $3c$ 10. $2x - 2y$

*

L. 1. two 2. one 3. one 4. 0 5. 25
6. 9 7. 144 8. 81 9. 36 10. 30
11. 2 12. 783 13. 5163 14. 835 15. 589
16. 347, 698 17. 1873, 6952 18. 1873, 6952
19. 1873, 6952 20. 189, 189 21. 64, 36
22. 64, 36 23. 5, 7 24. 5, 7 25. one 26. 5, 7
27. 18, 2 28. p 29. ab 30. ab 31. ab
32. ab 33. ab 34. one, $\sqrt{ab}$ 35. 25, 3, 5, 3
36. 500, 100, 10 37. 49, 2, 49, 2, 7, 2
38. 28, 4, 7, 4, 7 39. 99, 9, 11, 9, 11
40. 16, 10, 16, 10, 4 41. 36, 3, 36, 3, 3
42. 9, 7, 3, 7 43. 363 44. 32

*

M. 1. $\frac{\sqrt{2}}{4}$ 2. $\frac{3 + \sqrt{3}}{2}$ 3. $\frac{\sqrt{3}}{2}$ 4. $\frac{\sqrt{6}}{4}$
5. $\frac{\sqrt{3}}{4}$ 6. $\frac{2\sqrt{2}}{11}$ 7. $8 + \sqrt{3}$ 8. $7 - \sqrt{7}$
9. $\frac{12 + \sqrt{7}}{5}$ 10. $\frac{-1 - \sqrt{22}}{3}$ 11. 1 12. $\frac{6 - \sqrt{2}}{2}$
13. $5\sqrt{2}$ 14. $5\sqrt{2}$ 15. $8\sqrt{3}$ 16. $2\sqrt{5}$

12. $\sqrt{783} \times \sqrt{783} =$ ______

13. $\sqrt{5163} \times \sqrt{5163} =$ ______

14. $\sqrt{971 \times 835} \times \sqrt{971 \times 835} = 971 \times$ ______

15. $\sqrt{589 \times 762} \times \sqrt{589 \times 762} =$ ______ $\times 762$

16. $\sqrt{347 \times 698} \times \sqrt{698 \times 347} =$ ______ $\times$ ______

17. $\sqrt{1873 \times 6952} \times \sqrt{1873 \times 6952} =$ ______ $\times$ ______

18. $(\sqrt{1873} \times \sqrt{1873})(\sqrt{6952} \times \sqrt{6952}) =$ ______ $\times$ ______

19. $(\sqrt{1873} \times \sqrt{6952})(\sqrt{1873} \times \sqrt{6952}) =$ ______ $\times$ ______

20. $572 \times 189 = (\sqrt{572} \times \sqrt{572})(\sqrt{\_\_\_\_} \times \sqrt{\_\_\_\_})$

21. $64 \times 36 = (\sqrt{\_\_\_\_} \times \sqrt{\_\_\_\_})(\sqrt{64} \times \sqrt{36})$

22. $64 \times 36 = \sqrt{64 \times 36} \times \sqrt{\_\_\_\_ \times \_\_\_\_}$

23. $\sqrt{5 \times 7} \times \sqrt{5 \times 7} =$ ____ $\times$ ____

24. $(\sqrt{5} \times \sqrt{7}) \times (\sqrt{5} \times \sqrt{7}) =$ ____ $\times$ ____

25. How many positive numbers are there whose square is 5×7?

26. $\sqrt{5 \times 7} = \sqrt{\_\_\_\_} \times \sqrt{\_\_\_\_}$

27. $\sqrt{18} \times \sqrt{2} = \sqrt{\_\_\_\_ \times \_\_\_\_}$

28. $\forall_{p \geq 0} \sqrt{p}\sqrt{p} =$ ____

29. $\forall_{a \geq 0} \forall_{b \geq 0} (\sqrt{a}\sqrt{a})(\sqrt{b}\sqrt{b}) =$ ____

30. $\forall_{a \geq 0} \forall_{b \geq 0} (\sqrt{a}\sqrt{b})(\sqrt{a}\sqrt{b}) =$ ____

31. $\forall_{a \geq 0} \forall_{b \geq 0} \sqrt{a}\sqrt{b}$ is <u>the</u> positive number whose square is ____.

32. $\forall_{a \geq 0} \forall_{b \geq 0} \sqrt{ab}\sqrt{ab} =$ ____

33. $\forall_{a \geq 0} \forall_{b \geq 0} \sqrt{\_\_\_\_}$ is <u>the</u> positive number whose square is ab.

34. In view of the results in Exercises 31 and 33, if ab is nonnegative, how many positive numbers are there whose square is ab?
Hence, $\forall_{a \geq 0} \forall_{b \geq 0} \sqrt{a}\sqrt{b} =$ ____.

35. $\sqrt{75} = \sqrt{\_\_\_\_ \times \_\_\_\_} = \sqrt{25} \times \sqrt{3} =$ ____ $\times \sqrt{\_\_\_\_}$

8. $\forall_A \forall_B$ if $A = B$ then $A \times B = B \times A$.

9. $\forall_A \forall_B$ if $A \times B = B \times A$ then $A = B$.

10. Unioning is distributive over unioning.

<u>K.</u> 1. If a room has x rows of desks with 7 desks in a row, how many desks are there in the classroom?

2. True or false: $5 \in \{x: 2x + 8 = x^2 - 7\}$

3. What is the maximum number of pieces 1 foot 4 inches in length that you can cut from a board 9 feet 6 inches long?

4. If a man has x dollars and spends one third of this, how much does he have left?

5. If 12t + 8 is the perimeter of a square, what is the side-measure?

6. If $x = 6$ and $y = 3$ then $\frac{4}{x} + \frac{1}{y} =$ ________.

7. A telephone bill is \$4.40 including a 10% tax. What would the bill be if the tax were 5%?

8. If $x = -2$ and $y = -4$ then $\frac{1}{x} + \frac{6}{y} =$ ________.

9. If it takes 3d days for 3c cats to catch 3r rats, how many cats would it take to catch 21r rats in 21d days?

10. $x - y$ exceeds its opposite by how much?

<u>L.</u> 1. How many numbers are there whose square is 25?

2. How many <u>positive</u> numbers are there whose square is 144?

3. How many positive numbers are there whose square is 701?

4. $\forall_{p \geq 0} \sqrt{p} \geq$ ________

5. $\sqrt{25} \times \sqrt{25} =$ ________

6. $\sqrt{9} \times \sqrt{9} =$ ________

7. $\sqrt{144} \times \sqrt{144} =$ ________

8. $\sqrt{81} \times \sqrt{81} =$ ________

9. $(\sqrt{36})^2 =$ ________

10. $\sqrt{30} \times \sqrt{30} =$ ________

11. $\sqrt{2} \times \sqrt{2} =$ ________

17. 2   18. 2   19. 1   20. 5

21. 7   22. 6   23. $\sqrt{6}$ 24. 5

25. $\sqrt{5}$ 26. 11   27. $\sqrt{11}$ 28. $\sqrt{3}$

29. $\sqrt{3}$ 30. a, a, 1, $\sqrt{b}$, $\sqrt{a}$, $\sqrt{b}$ 31. $2\sqrt{5}$

32. $\frac{\sqrt{3}}{2}$ 33. $\frac{12}{13}$ 34. $\frac{\sqrt{7}}{3}$ 35. 3

36. $\frac{\sqrt{3}}{3}$ 37. $4\sqrt{5}$ 38. $\sqrt{3}$ 39. $\frac{3\sqrt{7}}{2}$

40. $\frac{3\sqrt{3}}{5}$ 41. $\frac{\sqrt{3}}{6}$ 42. $\sqrt{2} + \sqrt{5}$ 43. $16\sqrt{2} + 8\sqrt{5}$

44. 60   45. $\sqrt{1817}$ 46. $7\sqrt{2}$ 47. $7\sqrt{3}$

*

N. 1. $\frac{7 + 3\sqrt{5}}{2}$ 2. $\frac{14 + 5\sqrt{3}}{2}$ 3. $\frac{5 - 2\sqrt{6}}{2}$ 4. $\frac{8 + 3\sqrt{7}}{8}$

5. $\frac{33 + 12\sqrt{6}}{4}$ 6. $\frac{33 - 12\sqrt{6}}{4}$ 7. $\frac{13 + 4\sqrt{3}}{4}$

8. Yes. $[(1 + \sqrt{6})^2 - 2(1 + \sqrt{6}) - 5 = 7 + 2\sqrt{6} - 2 - 2\sqrt{6} - 5 = 0]$

9. $2\left(\frac{5 + \sqrt{89}}{4}\right)^2 - 5\left(\frac{5 + \sqrt{89}}{4}\right) - 8$

$= \frac{114 + 10\sqrt{89}}{8} - \frac{50 + 10\sqrt{89}}{8} - \frac{64}{8} = \frac{0}{8} = 0$

$2\left(\frac{5 - \sqrt{89}}{2}\right)^2 - 5\left(\frac{5 - \sqrt{89}}{4}\right) - 8$

$= \frac{114 - 10\sqrt{89}}{8} - \frac{50 - 10\sqrt{89}}{8} - \frac{64}{8} = \frac{0}{8} = 0$

26. $\left(\frac{\sqrt{121}}{\sqrt{11}}\right)^2$ 27. $\frac{\sqrt{121}}{\sqrt{11}}$ 28. $\frac{\sqrt{93}}{\sqrt{31}}$ 29. $\frac{\sqrt{51}}{\sqrt{17}}$

*

30. $\forall_{a \geq 0} \forall_{b > 0} \sqrt{\frac{a}{b}} = \sqrt{\_\_\_ \cdot \frac{1}{b}} = \sqrt{\_\_\_} \cdot \sqrt{\frac{1}{b}} = \sqrt{a} \cdot \frac{\_\_\_}{\_\_\_} = \frac{\_\_\_}{\_\_\_}$

*

31. $\frac{6\sqrt{55}}{3\sqrt{11}}$ 32. $\frac{7\sqrt{15}}{14\sqrt{5}}$ 33. $\sqrt{\frac{144}{169}}$ 34. $3\sqrt{\frac{7}{81}}$

35. $\frac{3\sqrt{243}}{3\sqrt{27}}$ 36. $\frac{4\sqrt{51}}{12\sqrt{17}}$ 37. $24\sqrt{\frac{10}{72}}$ 38. $\frac{\sqrt{42}}{\sqrt{14}}$

39. $12\sqrt{\frac{14}{128}}$ 40. $\frac{3\sqrt{6}}{5\sqrt{2}}$ 41. $\frac{2\sqrt{21}}{6\sqrt{28}}$ 42. $\frac{\sqrt{6} + \sqrt{15}}{\sqrt{3}}$

43. $2\sqrt{128} + \sqrt{320}$ 44. $5\sqrt{2} \times 2\sqrt{18}$

45. $\frac{1817}{\sqrt{1817}}$ 46. $\frac{91\sqrt{70}}{13\sqrt{35}}$ 47. $\frac{21}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

N. Expand.

1. $\left(\frac{3 + \sqrt{5}}{2}\right)^2$ [Solution. $\frac{9 + 6\sqrt{5} + 5}{4} = \frac{14 + 6\sqrt{5}}{4} = \frac{7 + 3\sqrt{5}}{2}$]

2. $\left(\frac{5 + \sqrt{3}}{2}\right)^2$ 3. $\left(\frac{-2 + \sqrt{6}}{2}\right)^2$ 4. $\left(\frac{-3 - \sqrt{7}}{4}\right)^2$

5. $\left(\frac{3 + 2\sqrt{6}}{2}\right)^2$ 6. $\left(\frac{3 - 2\sqrt{6}}{2}\right)^2$ 7. $\left(\frac{-1 - 2\sqrt{3}}{2}\right)^2$

*

8. Is $1 + \sqrt{6}$ a root of '$x^2 - 2x - 5 = 0$'?

9. Show that $\frac{5 + \sqrt{89}}{4}$ and $\frac{5 - \sqrt{89}}{4}$ satisfy '$2x^2 - 5x - 8 = 0$'.

36. $\sqrt{\_\_} = \sqrt{100 \times 5} = \sqrt{\_\_\_} \times \sqrt{5} = \_\_\_ \times \sqrt{5}$

37. $\sqrt{98} = \sqrt{\_\_\_ \times \_\_\_} = \sqrt{\_\_\_} \times \sqrt{\_\_\_} = \_\_\_\_ \times \sqrt{\_\_\_\_}$

38. $\sqrt{\_\_\_} = \sqrt{\_\_\_ \times \_\_\_} = \sqrt{\_\_\_} \times \sqrt{\_\_\_} = 2\sqrt{7}$

39. $\sqrt{\_\_\_} = \sqrt{\_\_\_ \times \_\_\_} = \sqrt{\_\_\_} \times \sqrt{\_\_\_} = 3\sqrt{11}$

40. $\sqrt{160} = \sqrt{\_\_\_ \times \_\_\_} = \sqrt{\_\_\_} \times \sqrt{\_\_\_} = \_\_\sqrt{10}$

41. $\sqrt{108} = \sqrt{\_\_\_ \times \_\_\_} = \sqrt{\_\_\_} \times \sqrt{\_\_\_} = 6\sqrt{\_\_\_}$

42. $\sqrt{63} = \sqrt{\_\_\_} \times \sqrt{\_\_\_} = \_\_\_ \times \sqrt{\_\_\_}$

43. $\sqrt{\_\_\_} = 11\sqrt{3}$ 44. $\sqrt{\_\_\_} = 4\sqrt{2}$

M. Simplify.

1. $\frac{\sqrt{18}}{12}$ [Solution. $\frac{\sqrt{18}}{12} = \frac{\sqrt{9 \cdot 2}}{12} = \frac{\sqrt{9}\sqrt{2}}{12} = \frac{3\sqrt{2}}{12} = \frac{\sqrt{2}}{4}$]

2. $\frac{6 + \sqrt{12}}{4}$ [Solution. $\frac{6 + \sqrt{12}}{4} = \frac{6 + \sqrt{4}\,\sqrt{3}}{4} = \frac{6 + 2\sqrt{3}}{4} = \frac{2(3 + \sqrt{3})}{4} = \frac{3 + \sqrt{3}}{2}$]

3. $\frac{\sqrt{27}}{6}$ 4. $\frac{\sqrt{24}}{8}$ 5. $\frac{\sqrt{75}}{20}$ 6. $\frac{\sqrt{200}}{55}$

7. $\frac{16 + \sqrt{12}}{2}$ 8. $\frac{14 - \sqrt{28}}{2}$ 9. $\frac{36 + \sqrt{63}}{15}$

10. $\frac{-2 - \sqrt{88}}{6}$ 11. $\frac{-5 + \sqrt{49}}{2}$ 12. $\frac{12 - \sqrt{8}}{4}$

13. $2\sqrt{2} + 3\sqrt{2}$ 14. $2\sqrt{2} + \sqrt{18}$ 15. $5\sqrt{3} + \sqrt{27}$

16. $3\sqrt{20} - 4\sqrt{5}$ 17. $\frac{2\sqrt{3}}{\sqrt{3}}$ 18. $\frac{\sqrt{12}}{\sqrt{3}}$

19. $\frac{\sqrt{45}}{3\sqrt{5}}$ 20. $\frac{2\sqrt{7} + \sqrt{63}}{\sqrt{7}}$ 21. $\frac{\sqrt{48} + \sqrt{27}}{\sqrt{3}}$

22. $\left(\frac{\sqrt{12}}{\sqrt{2}}\right)^2$ 23. $\frac{\sqrt{12}}{\sqrt{2}}$ 24. $\left(\frac{\sqrt{35}}{\sqrt{7}}\right)^2$ 25. $\frac{\sqrt{35}}{\sqrt{7}}$

O. 1. 22 2. 50 3. 150 4. 150

5. 80 6. 150 7. 1.5 H 8. 100 + x

9. m + 0.01 mn [or: m(1 + .01n)] 10. 12.5a

11. 3600/d 12. 4 13. 30 14. 100f/g

15. 0.01jk 16. 100p/r 17. 120 18. 25

19.

List price	50	72	100	170	40	78	60	320
Discount rate	40%	$12\frac{1}{2}$%	25%	30%	$62\frac{1}{2}$%	$37\frac{1}{2}$%	$3\frac{1}{3}$%	$6\frac{1}{4}$%
Purchase price	30	63	75	119	15	48.75	58	300

20. [If the discount rate were 22%, he would pay $5460.]

(a) 63 (b) 380 (c) 380 (d) 802.20

(e) 2730 (f) 600 (g) 680 (h) 30

21. $2000 22. $24 23. He lost $5.

*

P. [Students should be encouraged to consult a dictionary for the meanings of terms which may be new to them.]

1. 200 2. 12 3. 6 4. 160

5. 3 6. 198 7. 7.62 8. 100

9. 30.48 10. 39.37* 11. 0.9144 12. 100000

13. 1.609344 14. 0.3281* 15. 3600 16. 1500

17. 0.2 18. 0.35 19. 91.44 20. 5

21. 0.5 22. 0.3048 23. 0.01 24. 144

25. 0.5 26. 6.4516 27. 495 28. 6.665*

29. 0.0278 30. 200 31. 2700 32. 2

33. 16387.064 34. 5451776000 35. 0.000062 36. d/7

37. 16p 38. p/2000 39. s/3600 40. 63360m

*This is an approximation based on the fact that 1 centimeter is approximately 0.3937 inches. The United States standard inch is defined to be 2.54 centimeters.

20. Wholesale merchants sometimes offer special inducements to get purchasers to buy large quantities of an item. One such inducement involves chain discounts. For example, if one purchases at least one thousand of the items, he gets a discount of 20% but if he purchases more than five thousand, he gets discounts of 20% and 2%. So, if the list price is $1, and the purchaser buys 3000 articles, he pays 3000 - 20%(3000), or 2400, dollars. But, if he buys 7000, the purchase price is computed as follows.

$$7000 - 20\%(7000) = 5600$$
$$5600 - 2\%(5600) = 5488$$

So, he pays $5488. [Is this what he would pay if the discount rate were a straight 22%?]

Complete the following table.

	List price	Chain discount	Purchase price
(a)	100	30%, 10%	______
(b)	500	20%, 5%	______
(c)	500	5%, 20%	______
(d)	1000	16%, 4.5%	______
(e)	3200	$12\frac{1}{2}$%, $2\frac{1}{2}$%	______
(f)	____	10%, 5%	513
(g)	____	15%, 1%	572.22
(h)	800	40%, __%	336

21. Jack bought a current model foreign car for $1600. The list price was discounted 20% because the new model had just come out. The price of the new model was 20% more than the list price of the current model. What was the list price of Jack's car?

22. A storekeeper marked down a $40 portable radio 20%. If he offered you an additional 25% discount from the new price, how much would he want for the radio?

23. A man bought 2 shares of stock and sold them a month later for $60 each. On one share he made a profit of 20%, and on the other he took a loss of 20%. Did he gain, break even, or lose on the transaction? If he didn't break even, how much was his gain or loss?

O. 1. 20 increased by 10% is ______.

[Solution. $20 + 10\%(20) = 20 + 20 \cdot 0.1 = 20 + 2 = 22$]

2. 40 increased by 25% is ______.

3. 75 increased by 100% is ______.

4. 200% of 75 is ______.

5. 16 is ______% of 20.

6. 45 is ______% of 30.

7. H increased by 50% is ______.

8. 100 increased by x% is ______.

9. m increased by n% is ______.

10. a is ______% of 8.

11. 36 is ______% of d.

12. ______ increased by 50% is 6.

13. 40 increased by ______% is 52.

14. f is ______% of g.

15. ______ is j% of k.

16. p is r% of ______.

17. Jim missed 20% of the problems on his math test. He got 96 problems right. How many problems were there on the test?

18. Allan entered the math contest again this year and scored 28% better than he did last year. His score this year was 32. What was Allan's score last year?

19. If the list price of an article is \$50, and the discount [or: discount rate] is 20% then the purchase price of the article is 50 - 20%(50), or 40, dollars. Complete the following table.

List price	50	72	100		40	78	60	320
Discount rate	40%	$12\frac{1}{2}$%		30%		$37\frac{1}{2}$%		
Purchase price			75	119	15		58	300

Q. The number 47 has many names. A few of them are '$9 \times 5 + 2$', '$40 + 7$', and '$2 \times 3 \times 7 + 5$'. The simplest looking of all is '47'. This numeral is called the decimal numeral for 47. It is an abbreviation for '$4 \cdot 10^1 + 7$'. Similarly, the decimal numeral for 603 is '603', and this is an abbreviation for '$6 \cdot 10^2 + 0 \cdot 10^1 + 3$'. The fact that we use powers of ten in developing the name for the number explains the use of the word 'decimal'.

1. If the digits in '43' are reversed, what number is named?

2. If you reverse the digits in the decimal numeral for 79567, what number is named?

3. Repeat Exercise 2 for 68086. For 99.

4. What is the sum of the digits of the decimal numeral for 342? [That is, what is the sum of the numbers named by '3', '4', and '2'?]

5. What is the sum of the digits in that whole number between 71 and 79 which is exactly divisible by 5? [This is a quick way of asking: What is the sum of the numbers named by the digits in the decimal numeral for that whole number between ... ?]

6. The sum of the digits in a two-digit number [a two-digit number is one whose decimal numeral consists of two digits] is 9. If the number is between 50 and 60, what is it?

7. The sum of the digits in a two-digit number is 7. If you reverse the digits and add this number to the original number, what sum do you get?

8. Pick a two-digit number which has different digits. [For example, 47 but not 33.] Reverse the digits to get a second number, and subtract the smaller of the two numbers from the larger. Reverse the digits in the difference, and add this number to the difference.

9. Repeat Exercise 8 for a new two-digit number, and compare the sum you get now with the one you got in Exercise 8.

10. Pick any number such that the sum of its digits is [exactly] divisible by 3. Is the number itself divisible by 3?

P. 1. 2 dollars = ____ cents 2. 4 yards = ____ feet

3. 3 quarts = ____ pints 4. 16 decimeters = ____ centimeters

5. 24 pints = ____ gallons 6. 3.3 minutes = ____ seconds

7. 3 inches = ____ centimeters 8. 254 centimeters = ____ inches

9. 1 foot = ____ centimeters 10. 1 meter = ____ inches

11. 1 yard = ____ meters 12. 1 kilometer = ____ centimeters

13. 1 mile = ____ kilometers 14. 3 decimeters = ____ yards

15. 1 hour = ____ seconds 16. 1.5 kilometers = ____ meters

17. 2 centimeters = ____ decimeters

18. 35 millimeters = ____ decimeters

19. 3 feet = ____ centimeters 20. 2.5 quarts = ____ pints

21. 1 pint = ____ quarts 22. 1 foot = ____ meters

23. 0.1 millimeters = ____ centimeters

24. 1 square foot = ____ square inches

25. 72 square inches = ____ square feet

26. 1 square inch = ____ square centimeters

27. 55 square yards = ____ square feet

28. 43 square centimeters = ____ square inches

29. 278 square centimeters = ____ square meters

30. 2 square meters = ____ square decimeters

31. 100 cubic yards = ____ cubic feet

32. 3456 cubic inches = ____ cubic feet

33. 1000 cubic inches = ____ cubic centimeters

34. 1 cubic mile = ____ cubic yards

35. 62 cubic centimeters = ____ cubic meters

36. d days = ____ weeks 37. p pounds = ____ ounces

38. p pounds = ____ tons 39. s seconds = ____ hours

40. m miles = ____ inches

Q. 1. 34   2. 76597   3. 68086; 99   4. 9   5. 12
6. 54   7. 77   8. 99   9. same   10. Yes

*

R. 1. (a) 3   (b) 0   (c) 31   (d) 23   (e) 11
(f) 11   (g) 20   (h) 20   (i) 20   (j) 12
(k) 19   (l) 19   (m) 17   (n) 17   (o) $16\frac{23}{31}$

2. (a) 7; no   (b) 6   (c) $6\frac{7}{8}$

3. (a) 128   (b) $127\frac{9}{11}$

*

S. 1. 2 and 5   2. 35.5 and 40.5   3. 4 and 15
4. 2, 8; −8, −2   ☆5. 9 and 6   6. Al is 24 and Bill is 8.
7. Kathy is 16 years 4 months and Cheryl is 14 years 8 months.
8. A half-dollar and a dime [Although one of the coins is not a dime, the other is.]
9. $68.25   10. 24 dimes, 96 nickels, 72 pennies
11. 2   12. 2   13. 33; 60; 55; 45
14. 2/3 gallon   15. 0.9 gallon   16. 1/16 quart
17. 12.5 gallons   18. 0.2 gallon   19. [can't be done]
20. $71.24   21. y/x cents; wy/x cents   22. $8\frac{1}{3}$
23. $53\frac{11}{13}$ pounds of creams

3. Pete's bowling scores so far this year are 132, 97, 122, 145, 117, 136, 128, 102, 169, 121, and 137.

   (a) What is his median score?

   (b) What is the arithmetic mean score?

S. 1. The sum of two numbers is 7, and one of them is 3 more than the other. What are they?

2. The difference of one number from another is 5, and their sum is 76. What are the numbers?

3. A number exceeds another by 11 and their sum is 19. What are the numbers?

4. The difference of one number from another is 6, and their product is 16. What are the numbers?

☆ 5. If a first number is added to 51, the sum is 10 times a second number. If the second number is subtracted from 51, the difference is 5 times the first number. What are the numbers?

6. Al is now 3 times as old as his young cousin Bill, and in 8 years he will be twice as old. How old are Al and Bill now?

7. 13 years ago Kathy was twice as old as Cheryl. The square of the sum of their ages now is 961. What are their ages now?

8. An envelope contains 2 coins worth 60¢. However, one of the coins is not a dime. What are the coins?

9. Leslie found a chest containing just quarters, half-dollars, and silver dollars. There were twice as many silver dollars as half-dollars, and twice as many half-dollars as quarters. The chest contained 91 coins. What is the value of all Leslie's coins?

10. Carson has $7.92 worth of pennies, nickels, and dimes in his pocket. He has 3 times as many pennies as dimes and one-fourth as many dimes as nickels. How many of each coin does Carson have in his pocket?

11. A farmer has 2 cows in his north pasture. If he puts 37 sheep in the same pasture, how many cows will then be in the pasture?

R. 1. This chart shows the distribution of scores on a test. For example, two people scored 20 points, three scored 19 points, etc.

Score	Frequency
20	||
19	|||
18	~~||||~~ |
17	~~||||~~ ||||
16	|||
15	~~||||~~
14	
13	||
12	|

(a) How many scored 16?

(b) How many scored 14?

(c) How many people took the test?

(d) How many scored at least 16?

(e) How many scored above 17?

(f) How many scored below 17?

(g) How many scored no better than 17?

(h) How many scored no worse than 17?

(i) If the students scores were arranged in order from the highest to the lowest, which score would be first?

(j) Which would be thirty-first?

(k) Which would be fourth? (ℓ) Which would be fifth?

(m) Which score is the middle score? [This is called the median score.]

(n) Which score was obtained by the largest number of students? [This is called the modal score.]

(o) What is the average for the distribution? [The word 'average' is commonly used to refer to the arithmetic mean [pronounced as 'a-rith-met'-ic mean'] of the scores. You compute it by multiplying each score by its frequency, adding the products, and dividing by the total number of people.]

2. Here is another distribution of test scores.

Score	3	4	5	6	7	8	9	10
Frequency	1	2	0	5	0	4	3	1

(a) What is the median score? Did anyone get this score?

(b) What is the modal score?

(c) What is the arithmetic mean score?

24. Don can paddle his blue canoe down an 8-mile stream in 2 hours, but it takes him 6 hours to paddle back upstream.

    (a) What is his average rate of speed for a round trip?

    (b) What is the rate of flow of the stream?

25. The new pump can fill Ginny's swimming pool in 4 hours, but it takes the old one 12 hours to do the same job. How long would it take both pumps to fill the pool if they operated together?

T. Solve each of the given equations for the indicated variable.

1. $3x + y = 7$; y
2. $y - 5x = 9$; y
3. $2y - 8x = 12$; y
4. $3y + 6x = 10$; y
5. $2a + 5b = 7$; b
6. $tx = s + nx$; x
7. $3m - n - 8 = 0$; n
8. $3m - n - 8 = 0$; m
9. $ax + y + c = 0$; y
10. $ax + by + c = 0$; y
11. $my + ny = k$; y
12. $10r + 7s - 5 = 0$; s
13. $5y = 10z + 5$; y
14. $2y + 8 = 12x - y + 3$; y
15. $5(x - 3) + 7(y - 4) = 0$; y
16. $9(5 - a) - (b - 7) = 0$; b
17. $kp - mp = k^2 - m^2$; p
18. $tx = t^2 - s^2 - sx$; x
19. $y(x - 3) + x(5 - y) = 0$; x
20. $3(2x - 4y + 1) - 2(3x + y - 5) = 0$; y

U. Simplify.

1. $3a + 7 - a$
2. $10x - 8x + 7$
3. $5k + 2m - 3k - 3m$
4. $6y + 7x - 6x - y$
5. $3x^2 + 2x + 5x^2 - 8x$
6. $2a^2 - 3b^2 + 9a^2 + b^2 - a^2$
7. $5ab - 7bc + 2ab - bc$
8. $12xy - 3x + 4xy - 7y$
9. $-mn + 6m^2 - 3n + 2mn$
10. $6p^2 - 3pq + q^2 - 2p^2 + 2q^2$
11. $6(a - 2b - 5) - 3(3a + 4b)$
12. $5(x - 3y - 1) + 7(2x + y - 3)$
13. $7(m + 2n) - (5m - n)$
14. $-(x - 2y - z) - 4(-x + y - 3z)$
15. $x(2x - 7) - 3x(4 - 2x)$
16. $y(1 - 8y) - 2y(1 + 3y) - (3 - y)$

12. A drug manufacturer has a 50-gallon barrel holding 20 gallons of a 10% alcohol solution. If he adds 17.5 gallons of water, how many gallons of alcohol will he then have in the barrel?

13. David has 55 milliliters of a 40% sulphuric acid solution. If he adds 5 ml. of pure acid to his solution, how many ml. of pure water will it contain? How many ml. of solution will there be? What percent of the new solution is water? What percent (of the new solution) is acid?

14. How much ginger ale must be added to 1 gallon of fruit juice to make a punch that is 40% ginger ale?

15. A one gallon solution consists of 5% Lozak and 95% Bazol. How much Lozak must be added to get a new solution which is 50% Bazol?

16. How much Bazol must be added to a quart of a 15% Bazol solution to get one that is a 20% solution?

17. How much water must be added to 50 gallons of a 10% Lozak solution to get one that is an 8% solution?

18. How many gallons of a 20% Lozak solution must be added to 2 quarts of a 6% Lozak solution to get a solution that is 10% Lozak?

19. How many gallons of a 39% Bazol solution must be added to 16 pints of a 51% Bazol solution to give a 60% Bazol solution?

20. If 1 pound of candy sells for $1.37, what is the price of 52 pounds of the same candy?

21. If x pounds of tea costs y cents, what is the price of 1 pound of tea? What is the cost of w pounds of tea?

22. John sold 5 pounds of 65¢-a-pound grass seed for 40¢ a pound. How many pounds of this seed must John sell at 80¢ a pound to make up for his error?

23. A candy factory makes fruit and nut chocolates to sell for $1.28 per pound. Their creams sell for $1.80 per pound. If 100 pounds of a mixture is to be worth $1.56 per pound, how many pounds of creams should it contain?

24. (a) 2 mph (b) $1\frac{1}{3}$ mph 25. 3 hours

<u>T.</u>

1. $y = -3x + 7$
2. $y = 5x + 9$
3. $y = 4x + 6$
4. $y = -2x + \frac{10}{3}$
5. $b = -\frac{2}{5}a + \frac{7}{5}$
6. $x = \frac{s}{t - n}$
7. $n = 3m - 8$
8. $m = \frac{1}{3}n + \frac{8}{3}$
9. $y = -ax - c$
10. $y = -\frac{a}{b}x - \frac{c}{b}$
11. $y = \frac{k}{m + n}$
12. $s = -\frac{10}{7}r + \frac{5}{7}$
13. $y = 2z + 1$
14. $y = 4x - \frac{5}{3}$
15. $y = -\frac{5}{7}x + \frac{43}{7}$
16. $b = -9a + 52$
17. $p = k + m$
18. $x = t - s$
19. $x = \frac{3}{5}y$
20. $y = \frac{13}{14}$

<u>U.</u>

1. $2a + 7$
2. $2x + 7$
3. $2k - m$
4. $x + 5y$
5. $8x^2 - 6x$
6. $10a^2 - 2b^2$
7. $7ab - 8bc$
8. $16xy - 3x - 7y$
9. $6m^2 + mn - 3n$
10. $4p^2 - 3pq + 3q^2$
11. $-3a - 24b - 30$
12. $19x - 8y - 26$
13. $2m + 15n$
14. $3x - 2y + 13z$
15. $8x^2 - 19x$
16. $-14y^2 - 3$

17. $2x^3 - x^2 + 3x$

18. $-3x^2 - 34xy + 15y^2 - 5y$

19. $6a^2 - 13ab + 14b^2$

20. $7x^4 - 16x^3 + 17x^2$

21. $-30abc$

22. $6xyz$

23. $-70abm$

24. $288x^3y^2$

25. $0.88a^3$

26. $6k^4$

27. $-24a^9$

28. $18b^6$

29. $6x^4y^4$

30. $-6a^4b^7$

31. $360m^8n^8$

32. $24a^{13}b^9$

33. $3x^3y^3$

34. $2a^3b^4$

35. $30a^2b^2$

36. $13x^2y^4 - 17x^3y^7$

37. $\frac{1}{2}$

38. $\frac{x}{y}$

39. $\frac{x}{y}$

40. c

41. $\frac{5}{x^2}$

42. $\frac{1}{3xy}$

43. $\frac{x(a + b)}{3}$

44. $c - d$

45. $\frac{11x + 30}{15x}$

46. $\frac{x^2 + 2x - 12}{4x}$

47. $\frac{3xy - 28}{6x}$

48. $\frac{2 - 3x + 4y}{xy}$

49. $\frac{6 - 9x}{2x^2}$

50. $\frac{2xy - 3x + y}{x^2y^2}$

51. $\frac{7x + 19}{(x + 3)(x + 2)}$

52. $\frac{7(y + 2)}{(y - 2)(y + 5)}$

53. $\frac{3z - 11}{(z - 1)(z - 2)}$

54. $\frac{5x + 16}{(x + 3)^2}$

55. $\frac{3x - 2}{(x + 4)(x - 1)}$

56. $\frac{3x - 8}{(x - 5)(x + 1)}$

57. $\frac{8x + 23}{(x + 2)(x + 3)}$

58. $\frac{2(2y + 21)}{y^2 - 16}$

59. $\frac{3a^2 + 9ab + 8b^2 - 6a + 5b}{(a - b)(a + 3b)}$

60. $\frac{z + 18}{(z - 3)(z + 5)}$

61. $\frac{15ab^2y}{14x}$

62. $\frac{b^2x}{6a^4y}$

63. $\frac{7 - y}{9(x - 5)^2}$

64. $\frac{(x - 2)(y + 5)}{(x - 4)(y + 7)}$

65. $\frac{(x - 2)(y + 5)}{(x - 4)(y + 7)}$

66. $\frac{(a + 2)(b + 6)}{(a + 4)(b - 3)}$

67. $\frac{a^2(a - 7)(a + 5)}{a^3 - 2a^2 - 35}$

SUPPLEMENTARY EXERCISES

A. Draw graphs of these relations. [Label the axes in accordance with the conventions adopted in Unit 4.]

1. $\{(x, y):\ y = 3x - 2\}$
2. $\{(x, y):\ y - 3x = 4\}$
3. $\{(a, b):\ a^2 + b^2 = 25\}$
4. $\{(s, t):\ |s| + |t| \leq 10\}$
5. $\{(u, v):\ v = 8\}$
6. $\{(u, v):\ u = 8\}$
7. $\{(a, b):\ a \geq 2 \text{ or } b \leq 3\}$
8. $\{(a, b):\ a \not\geq 2 \text{ and } b \not\leq 3\}$
9. $\{(x, y),\ x \neq 0:\ y = x - \frac{x}{x}\}$
10. $\{(x, y):\ y = x^3\}$
11. $\{(x, y):\ x^2 \neq 25\}$
12. $\{(x, y):\ 2x + y \leq 6\}$
13. $\{(x, y):\ (x \in I \text{ and } y = 3) \text{ or } (x \notin I \text{ and } y = -1)\}$
14. $\{(x, y):\ |3 - x| \leq 2 \text{ and } |y - 4| \leq 2\}$
15. $\{(x, y):\ |x| + |y| \geq |x + y|\}$
16. $\{(x, y):\ xy = 12\}$
17. $\{(x, y):\ x > 5 \text{ and } |y| + x = 3\}$

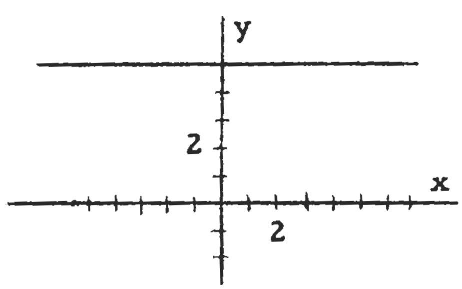
☆18. $\{(x, y):\ (x - 2)(x - 3) \geq 0\}$

☆19. $\{(x, y):\ xy - x^2 = 0\}$

☆20. $\{(x, y):\ x^2 - 7x + 10 \leq y\}$

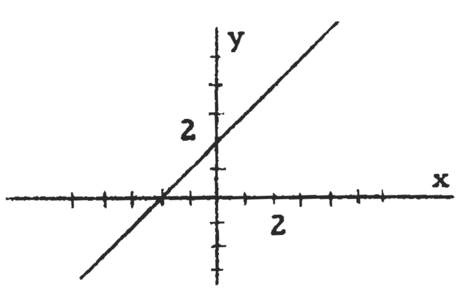
B. In each exercise you are given a graph of a relation and five names of relations. Encircle the letter preceding a name of the pictured relation. <u>There may be more than one correct choice.</u>

1.



(a) $\{(x, y):\ x = 5\}$
(b) $\{(x, y):\ y = 5x\}$
(c) $\{(x, y):\ y = 5\}$
(d) $\{(x, y):\ y = 5 \text{ and } x = 2\}$
(e) $\{(x, y):\ 2y - 2 = 3 + y\}$

2.



(a) $\{(x, y):\ y = x - 2\}$
(b) $\{(x, y):\ y = x + 2\}$
(c) $\{(x, y):\ 2y = x + 2\}$
(d) $\{(x, y):\ 3y = 3x + 6\}$
(e) $\{(x, y):\ y = 5x + 2\}$

17. $x^2(1 - x) + x(3 - 2x + 3x^2)$

18. $3x(2y - x) - 5y(8x - 3y + 1)$

19. $6a(a - b) + 7b(-a + 2b)$

20. $2x^2(x^2 - 3x + 1) - 5x^2(-x^2 + 2x - 3)$

21. $5a(3b)(-2c)$

22. $x(-2y)(-3z)$

23. $5m(-2a)(7b)$

24. $8x(2x)^2(-3y)^2$

25. $0.1a(1.1a)(8a)$

26. $-k(3k)(-2k^2)$

27. $4a^2(3a^3)(-2a^4)$

28. $9b^3(-b^2)(-2b)$

29. $x^2y(3xy^2)(2xy)$

30. $2ab(3a^2b^3)(-ab^3)$

31. $5m(3m^2n)^2(2mn^2)^3$

32. $-3a(-2a^2b)^3(-a^3b^3)^2$

33. $5xy^2(2x^2y) - 7x^3y^3$

34. $2a^2b^2(-3ab^2) - 4ab^3(-2a^2b)$

35. $6a^2b^2(ab + 5) - 3ab^3(2a^2)$

36. $7x^2y^4(-2xy^3 + 1) - 3xy(x^2y^6 - 2xy^3)$

37. $\dfrac{5ab}{10ab}$

38. $\dfrac{3x^2y}{3xy^2}$

39. $\dfrac{x^2y^3z^2}{xy^4z^2}$

40. $\dfrac{(ab^2)(ac)^3}{a^4b^2c^2}$

41. $\dfrac{-5xy}{-x(x^2y)}$

42. $\dfrac{3x(-2y)^2}{(-3x)^2(4y^3)}$

43. $\dfrac{3x^2(a + b)^2}{9x(a + b)}$

44. $\dfrac{(c - d)^3}{(d - c)^2}$

45. $\dfrac{1}{3} + \dfrac{2}{x} + \dfrac{2}{5}$

46. $\dfrac{x}{4} - \dfrac{3}{x} + \dfrac{1}{2}$

47. $\dfrac{y}{2} - \dfrac{5}{x} + \dfrac{1}{3x}$

48. $\dfrac{2}{xy} - \dfrac{3}{y} + \dfrac{4}{x}$

49. $\dfrac{3}{x^2} - \dfrac{7}{x} + \dfrac{5}{2x}$

50. $\dfrac{1}{x^2y} - \dfrac{3}{xy^2} + \dfrac{2}{xy}$

51. $\dfrac{2}{x+3} + \dfrac{5}{x+2}$

52. $\dfrac{4}{y-2} + \dfrac{3}{y+5}$

53. $\dfrac{8}{z-1} - \dfrac{5}{z-2}$

54. $\dfrac{1}{(x+3)^2} + \dfrac{5}{x+3}$

55. $\dfrac{3}{x+4} + \dfrac{1}{(x+4)(x-1)}$

56. $\dfrac{7}{(x-5)(x+1)} + \dfrac{3}{x+1}$

57. $\dfrac{4}{x^2 + 5x + 6} + \dfrac{3}{x + 2} + \dfrac{5}{x + 3}$

58. $\dfrac{7}{y-4} - \dfrac{3}{y+4} + \dfrac{2}{y^2 - 16}$

59. $\dfrac{2a + 3b}{a - b} + \dfrac{a + b}{a + 3b} - \dfrac{6a - 5b}{a^2 + 2ab - 3b^2}$

60. $\dfrac{2}{z-3} + \dfrac{5}{z^2 + 2z - 15} - \dfrac{1}{z + 5}$

61. $\dfrac{3xy^2}{2ab} \cdot \dfrac{5a^2b^3}{7x^2y}$

62. $\dfrac{4a^2b^5}{9xy^3} \cdot \dfrac{3(xy)^2}{8(a^2b)^3}$

63. $\dfrac{2(x-5)^2}{3(y-7)^2} \cdot \dfrac{-(y-7)^3}{6(x-5)^4}$

64. $\dfrac{(x-2)(x+3)}{(y-4)(y+7)} \cdot \dfrac{(y-4)(y+5)}{(x+3)(x-4)}$

65. $\dfrac{x^2 + x - 6}{y^2 + 3y - 28} \cdot \dfrac{y^2 + y - 20}{x^2 - x - 12}$

66. $\dfrac{a^2 - 2a - 8}{b^2 - 8b + 15} \cdot \dfrac{b^2 + b - 30}{a^2 - 16}$

67. $\dfrac{a^3 - 4a^2 - 21a}{a^4 - 2a^3 - 35a} \cdot \dfrac{a^5 + 8a^4 + 15a^3}{a^3 + 6a^2 + 9a}$

In Exercise 9, the index '(x, y), $x \neq 0$' in the brace-notation name tells you that the relation in question is a subset of the set of all ordered pairs of real numbers whose first component is not 0. So, strictly speaking, the graph of the y-axis ought to be eliminated from the picture, perhaps by using a dashed line as we do. The important fact to be noted by the students is that the relation is not a straight line but rather the union of two disjoint and collinear half-lines with a common end point which belongs to neither of them. They should show this on the picture by placing a hollow dot at the graph of the common end point.

Exercise 13 provides an interesting graph. One part of it is a series of discrete dots, and the other is a series of discrete intervals.

The relation in Exercise 11 is the complement of a pair of parallel lines.

Students may like to know that the relation in Exercise 16 is an hyperbola [specifically, an equilateral hyperbola, one whose asymptotes are perpendicular]. [See page 5-155.]

The relation in Exercise 17 is the empty set. There is no ordered pair whose second component is such that the sum of its absolute value and a number greater than 5 is 3.

Students may get a clue for Exercise 19 from Exercise 18. The trick in Exercise 19 is to factor the left member of the set selector.

$$xy - x^2 = 0$$

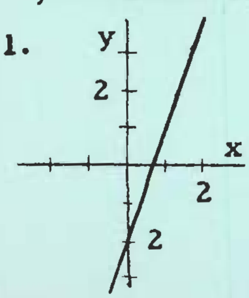
$$x(y - x) = 0$$

$$x = 0 \text{ or } y - x = 0$$

The last sentence is equivalent to the first. The locus is the union of the y-axis and the locus of '$y = x$'.

Students will probably first find the boundary of the graph for Exercise 20 by graphing the equation: $x^2 - 7x + 10 = y$. Then, for each point on the boundary, this point and all points with the same x-coordinate and a larger y-coordinate are in the graph.

1.



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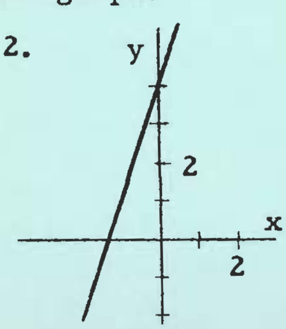

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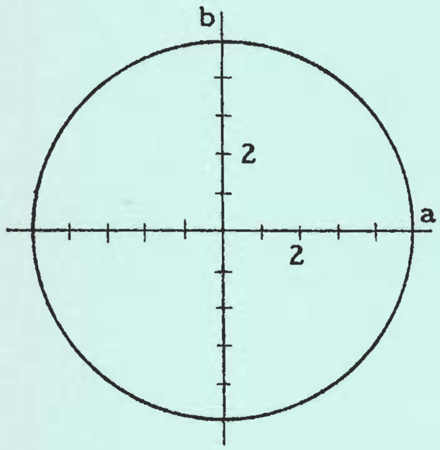
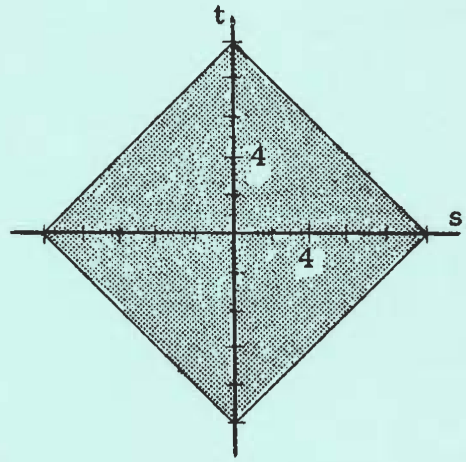


4.



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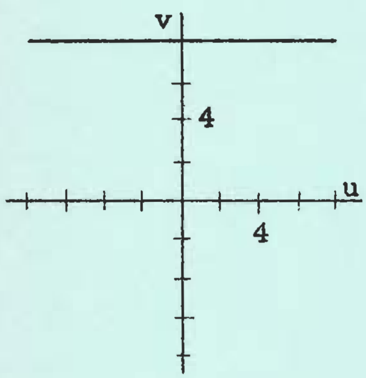

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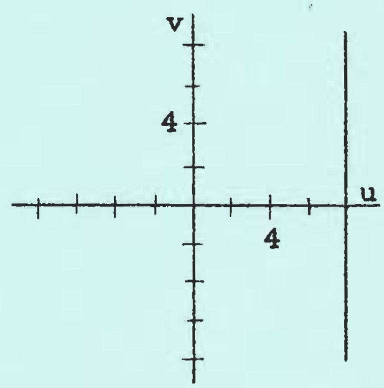

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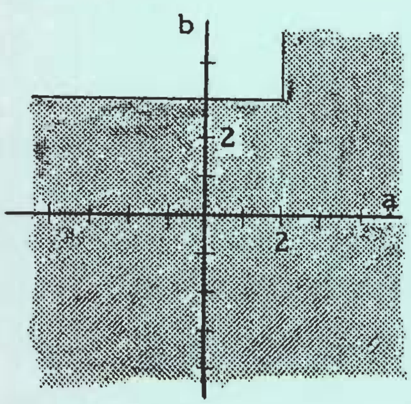

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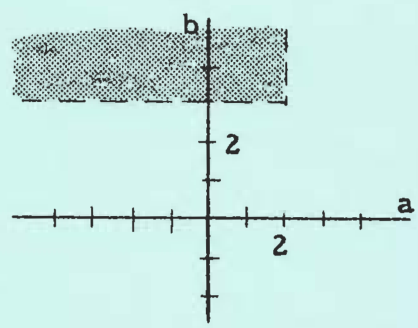

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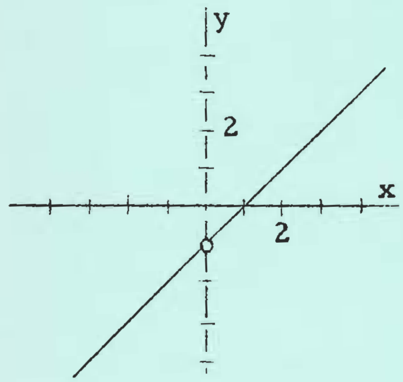

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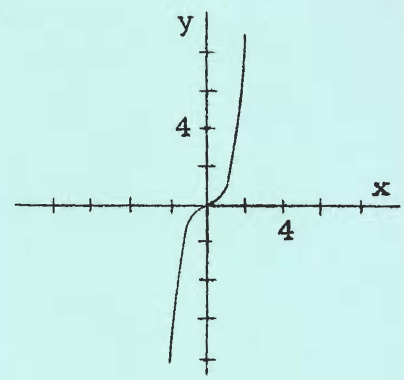

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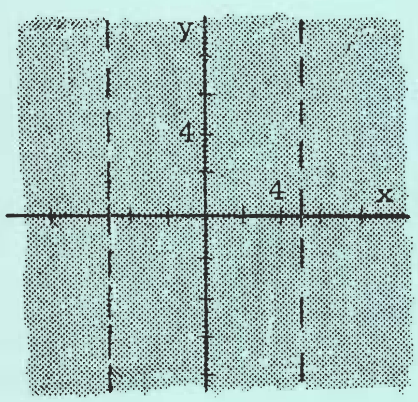

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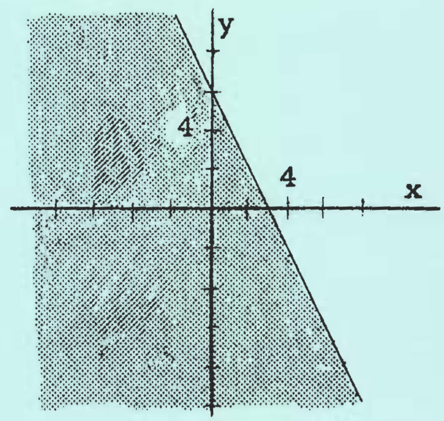

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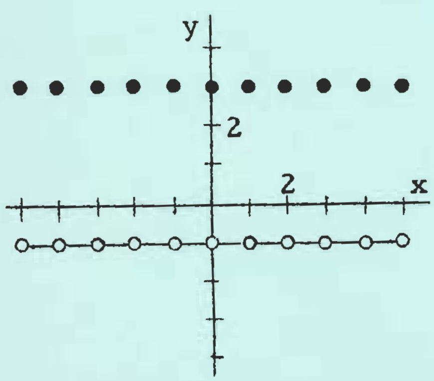

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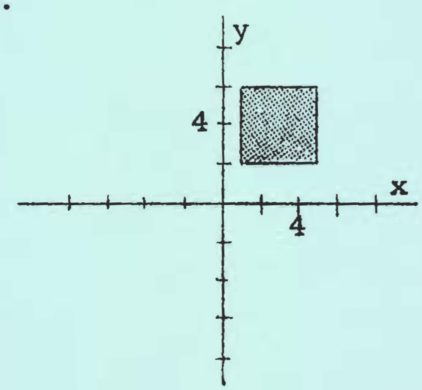

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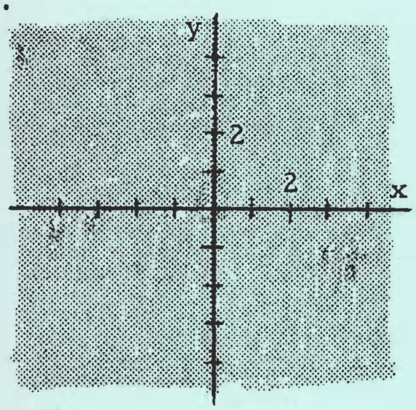
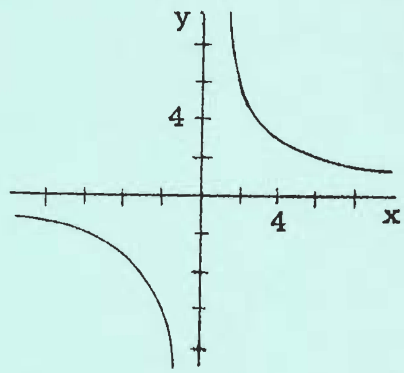
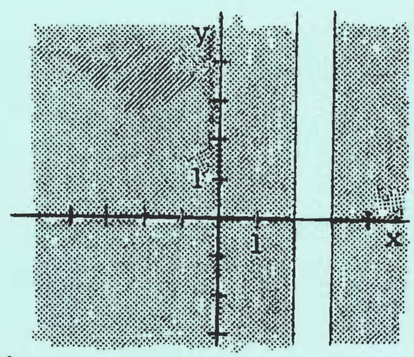


16.



17. [The relation is empty. So, there are no points to graph.]

☆18.



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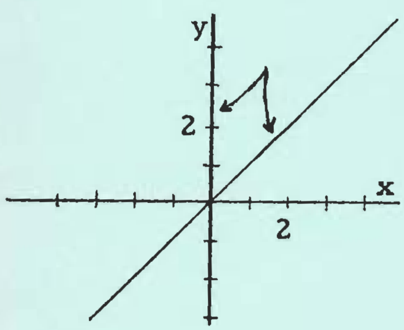
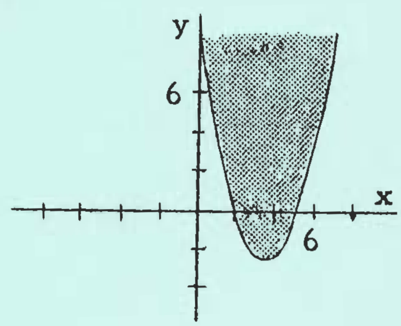


☆20.

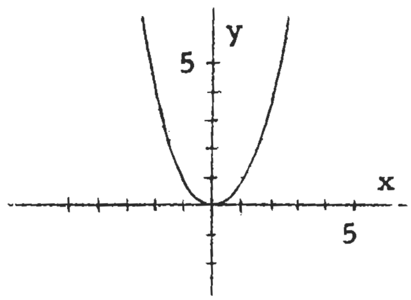


*

B. [on pages 5-238 and 5-239]

1. (c), (e) 2. (b), (d) 3. (a) 4. (b), (d)

5. (b), (c), (e) 6. (b), (c), (e) 7. (c)

3. 

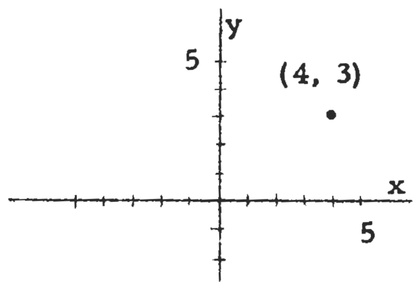
(a) $\{(x, y): y = x^2\}$

(b) $\{(x, y), y \geq 0: \sqrt{y} = x\}$

(c) $\{(x, y): y = |x|\}$

(d) $\{(x, y): y^2 = x\}$

(e) $\{(x, y): y = 2x\}$

4. 

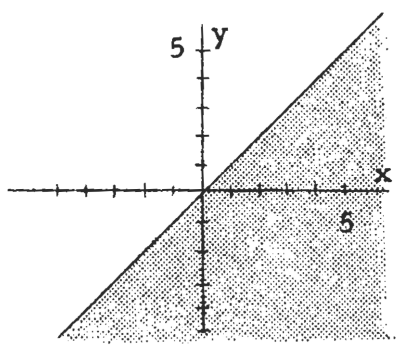
(a) $\{(x, y): x = 4 \text{ or } y = 3\}$

(b) $\{(x, y): y = 3 \text{ and } x = 4\}$

(c) $\{(x, y): y - x = 1 \text{ and } y + 2x = 10\}$

(d) $\{(x, y): y - 2x = -5 \text{ and } 3x - y = 9\}$

(e) $\{(x, y): y = x + 1 \text{ and } x = 3\}$

5. 

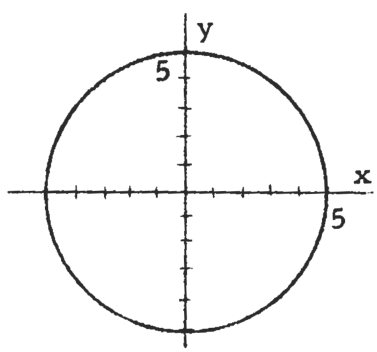
(a) $\{(x, y): y > x\}$

(b) $\{(x, y): y \leq x\}$

(c) $\{(x, y): x \geq y\}$

(d) $\{(x, y): x \geq 0\}$

(e) $\{(x, y): y - x \text{ is nonpositive}\}$

6. 

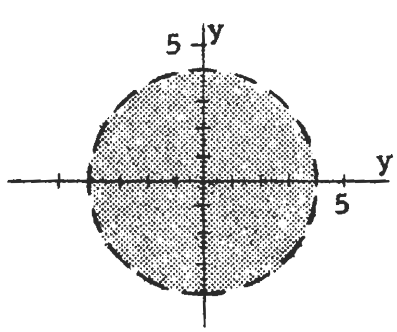
(a) $\{(x, y): x^2 - y^2 = 25\}$

(b) $\{(x, y): x^2 = 25 - y^2\}$

(c) $\{(x, y): \sqrt{2}\, x^2 + \sqrt{2}\, y^2 = \sqrt{1250}\}$

(d) $\{(x, y): |x| + |y| = 5\}$

(e) $\{(x, y): x^2 + y^2 = 25\}$

7. 

(a) $\{(x, y): x^2 + y^2 \leq 16\}$

(b) $\{(x, y): x^2 + y^2 > 16\}$

(c) $\{(x, y): x^2 + y^2 < 16\}$

(d) $\{(x, y): x < 4 \text{ and } y < 4\}$

(e) $\{(x, y): y < 4 \text{ and } x^2 + y^2 = 4\}$

C. 1. Graph the relations A, B, C, D, and E where

$A = \{(x, y): y - x = 5\}$, $B = \{(x, y): y + 2x = 2\}$,

$C = \{(x, y): y - 3x \leq 7\}$, $D = \{(x, y): 1 < x < 5\}$,

$E = \{(x, y): |y| \leq 2\}$.

2. Graph and find the ordered pairs in $A \cap B$.

3. Graph $B \cup C$.

4. Graph $C \cup E$.

5. Graph and list the elements in $\{(x, y) \in I \times I: (x, y) \in D \cap E\}$.

6. Graph $A \cap C$; $B \cap C$; $(A \cap C) \cup (B \cap C)$; $(A \cup B) \cap C$.

7. Graph $B \cap E$; $B \cap D$; $B \cap (D \cap E)$; $B \cap D) \cap E$.

8. Graph $C \cap E$; $C \cap D$.

9. Graph $\tilde{C}$; $\tilde{D}$; $\tilde{E}$; $\tilde{A}$.

10. Graph $\tilde{D} \cap \tilde{E}$; $\tilde{D} \cup \tilde{E}$; $\widetilde{D \cup E}$; $\widetilde{D \cap E}$.

11. Graph $(D \cap E) \cup \tilde{C}$.

12. Graph $[C \cap (\widetilde{D \cap E})] \cap B$.

13. Graph $M \cap N$ where

$M = \{(x, y): x^2 + y^2 \leq 25\}$

$N = \{(x, y): x^2 + y^2 > 9\}$.

14. Graph $\{(x, y): x^2 + y^2 \leq 16\} \cap \{(x, y): |x| > 2\}$.

☆ 15. Graph $\{(x, y): y^2 - x^2 = 0\} \cap \{(x, y): xy > 0\}$.

☆ 16. Graph $A \cup B \cup C \cup D \cup E \cup F \cup G$ where

$A = \{(x, y): x^2 + y^2 = 4\}$,

$B = \{(x, y): x = 0 \text{ and } -9 \leq y \leq -2\}$,

$C = \{(0.5, 1), (-0.5, 1)\}$

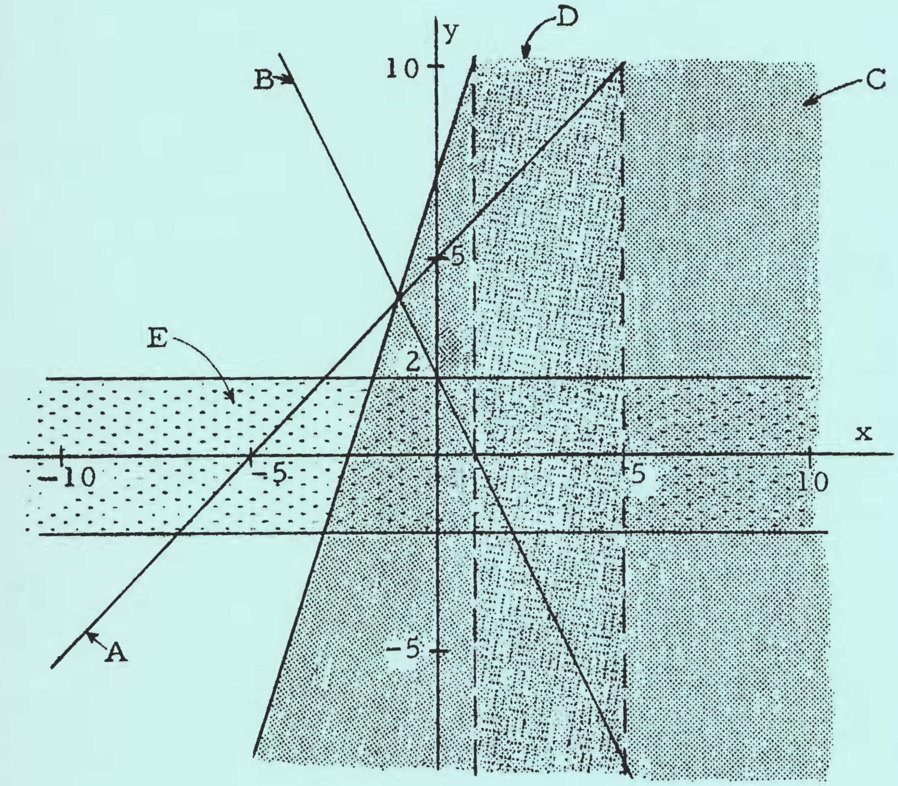
$D = \{(x, y): |x| = 1/16 \text{ and } y = -1/8\}$

$E = \{(x, y): y < -1/2 \text{ and } x^2 + y^2 = 1\}$

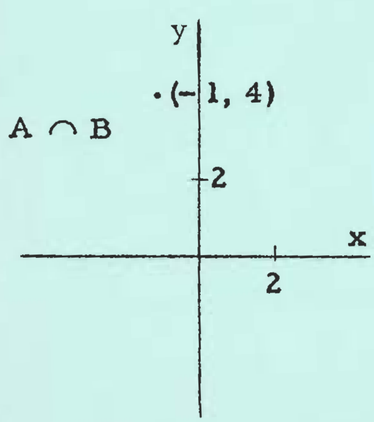
$F = \{(x, y): |x| \leq 2 \text{ and } y = (-3/2)|x| - 9\}$

$G = \{(x, y): y = -4 \text{ and } |x| \leq 4\}$

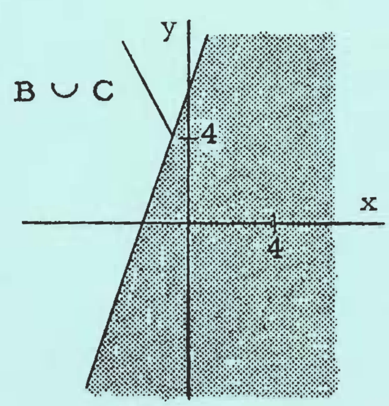
C. 1.



2.



3.



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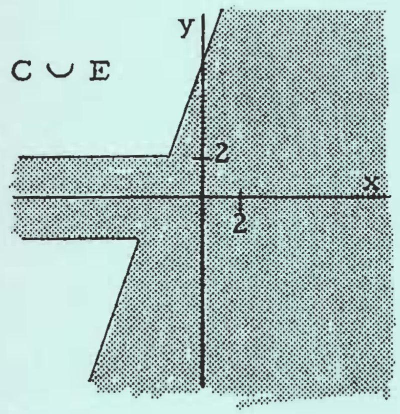
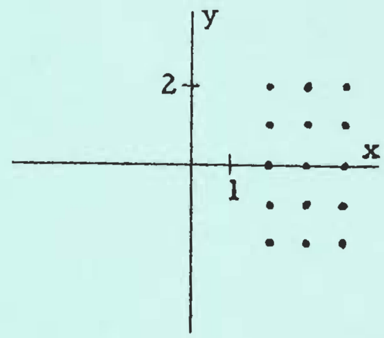
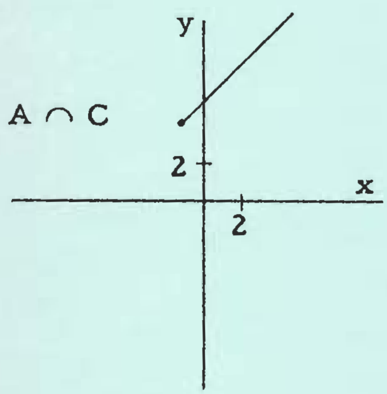


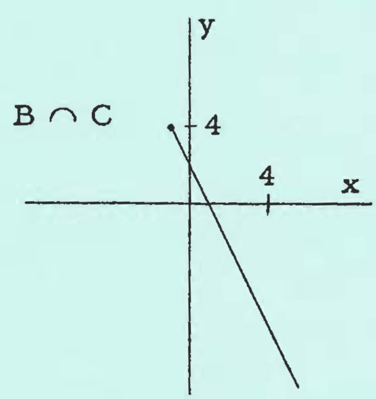
5.

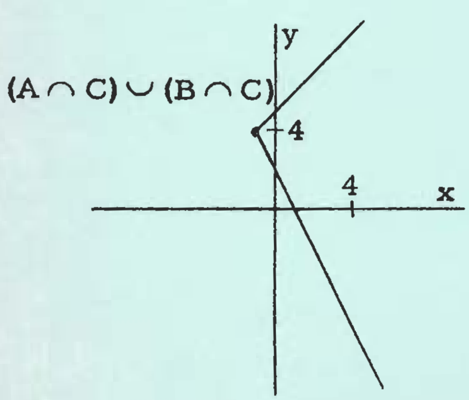


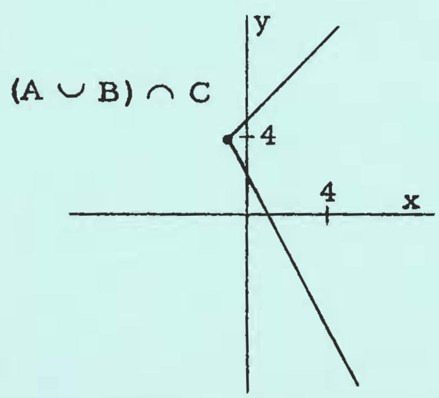
(2, 2), (2, 1), (2, 0), (2, −1), (2, −2)
(3, 2), (3, 1), (3, 0), (3, −1), (3, −2)
(4, 2), (4, 1), (4, 0), (4, −1), (4, −2)

6.



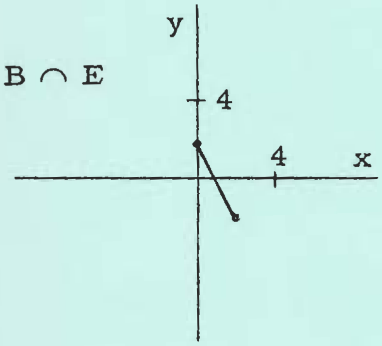




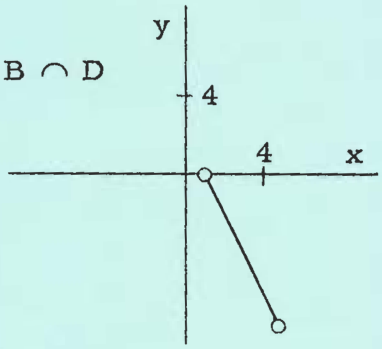


7.

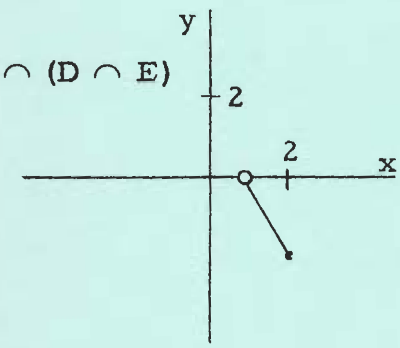
B ∩ E



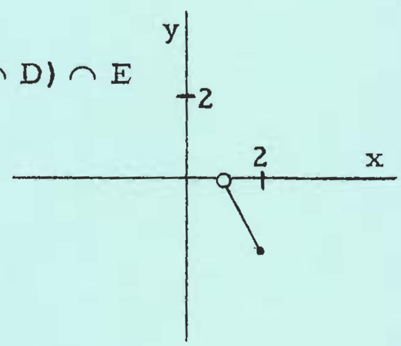
B ∩ D



B ∩ (D ∩ E)

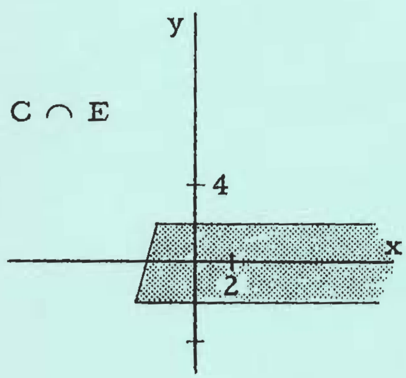


(B ∩ D) ∩ E

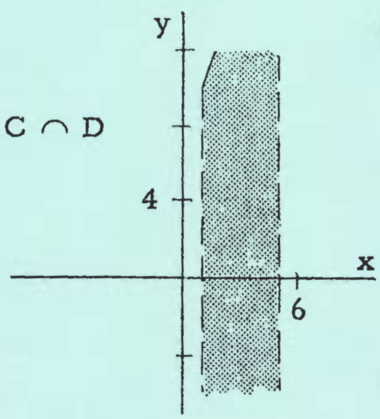


8.

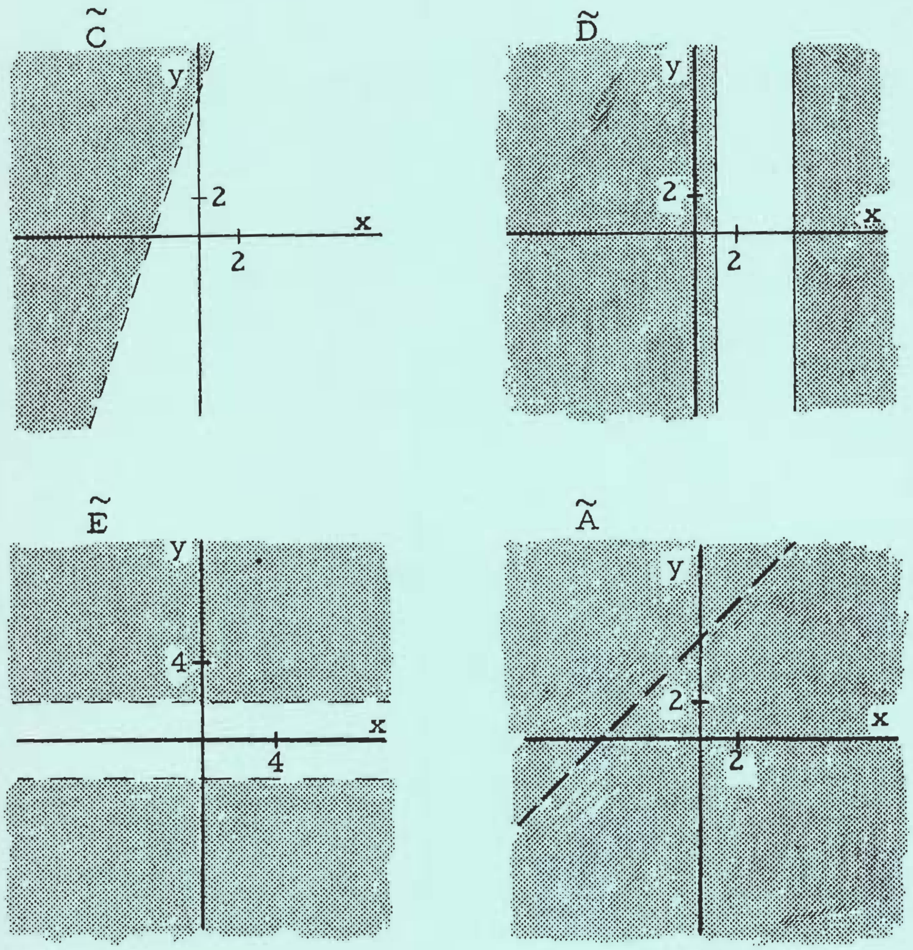
C ∩ E



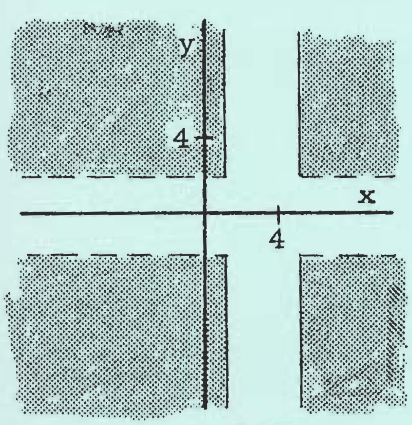
C ∩ D



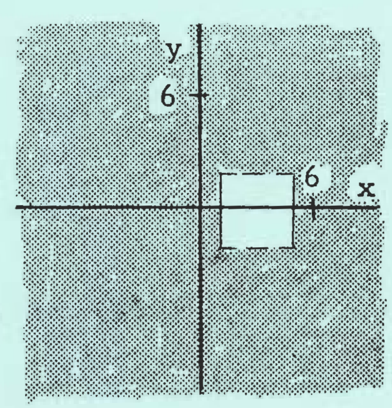
9.



10.

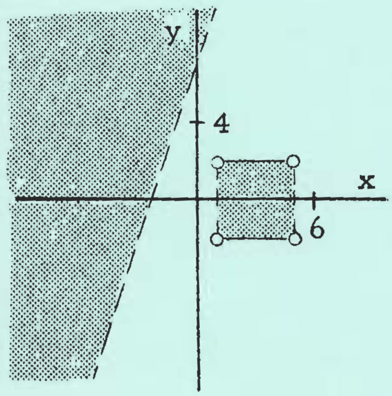


$[\tilde{D} \cap \tilde{E} = \widetilde{D \cup E}]$

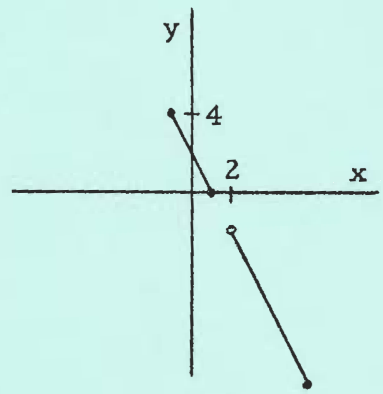


$[\tilde{D} \cup \tilde{E} = \widetilde{D \cap E}]$

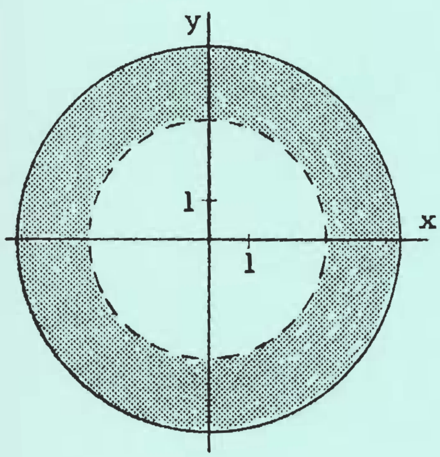
11. $(D \cap E) \cup \tilde{C}$



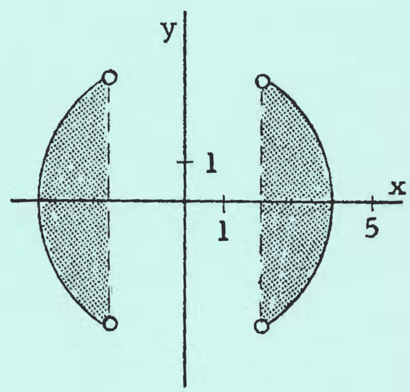
12. $[C \cap \widetilde{(D \cap E)}] \cap B$



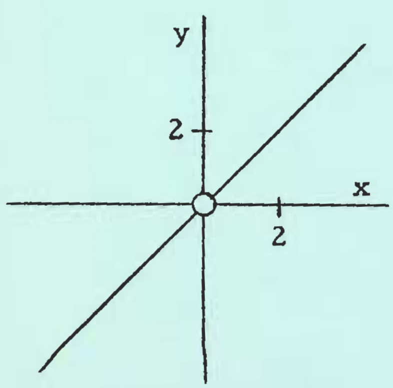
13.



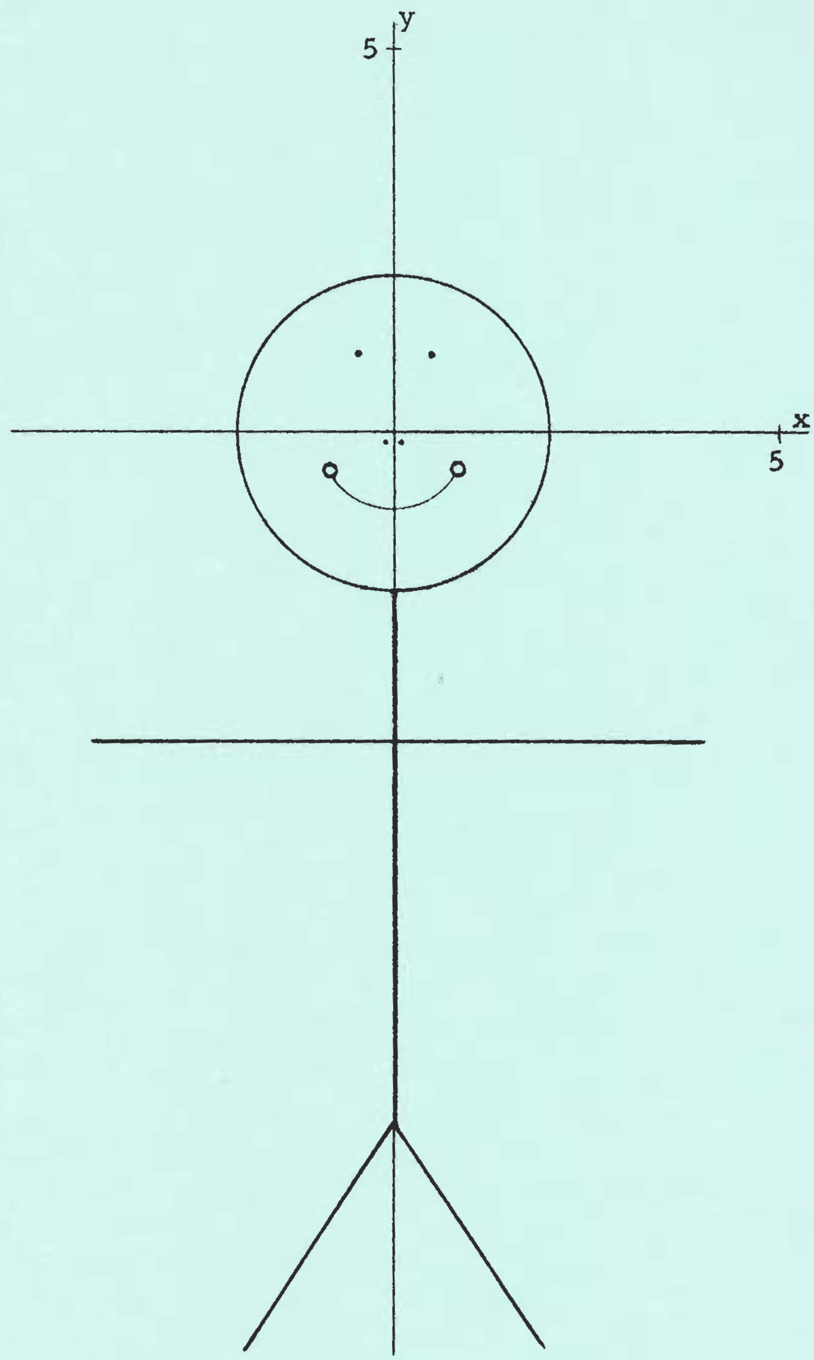
14.



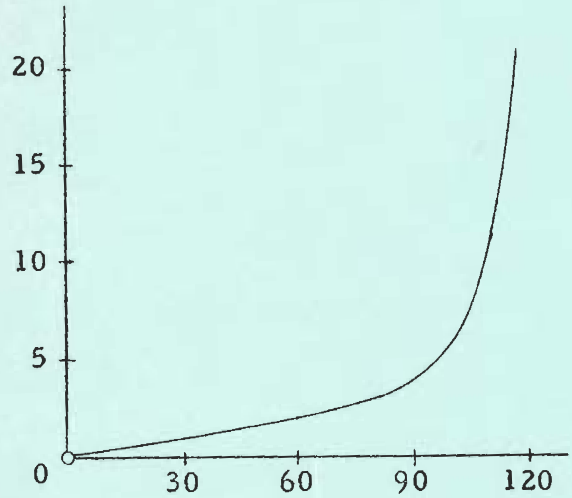
☆15.



☆16.



D. 1.



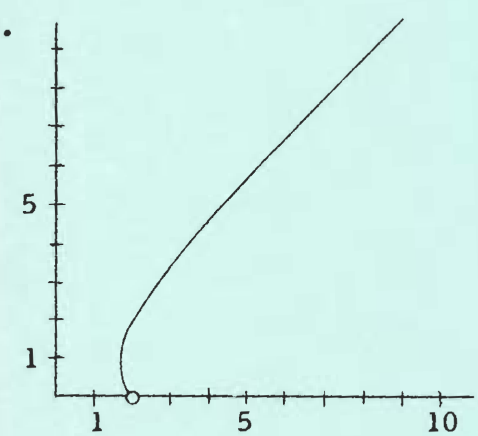
[Function is
$\{(x, y),\ 0 < x < 120:$
$y = \dfrac{2 \sin x^\circ}{\sin(120 - x)^\circ}\}.]$

2. (a) 1 (b) 2 (c) 4
(d) 0; 120

3. (a) The inch-measure of $\overleftrightarrow{AC}$ must be approximately 1.31; that of $\overleftrightarrow{BC}$, 1.76. So, the graph consists of a single dot.

(b) The inch-measure of $\overleftrightarrow{BC}$ must be approximately 2.64, the degree-measure of $\angle B$ is approximately 79. The relation consists of a single ordered pair.

4.



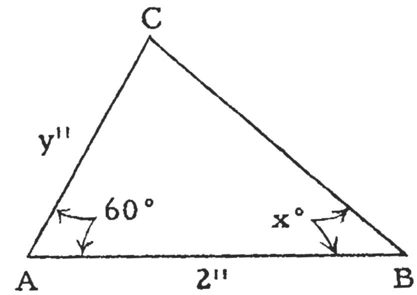
[From the Law of Cosines we find that the relation is $\{(x, y),\ y > 0:\ x^2 - (y - 1)^2 = 3\}.]$

(a) one

(b) two

(c) none

D. 1. Consider a triangle, ΔABC, in which $\angle A$ is an angle of 60° and the inch-measure of $\overline{AB}$ is 2. Draw a graph of the relation of the inch-measure of side $\overline{AC}$ to the degree-measure of $\angle B$.



2. Use the graph to help you complete the following:

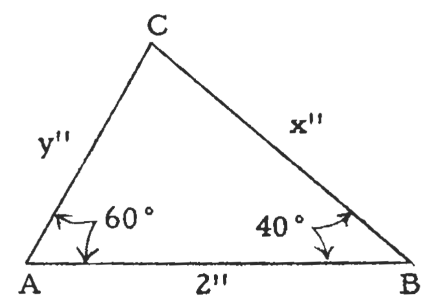
(a) If $\angle B$ is an angle of 30° then the inch-measure of $\overline{AC}$ is ___.

(b) If $\angle B$ is an angle of 60° then the inch-measure of $\overline{AC}$ is ___.

(c) If $\angle C$ is an angle of 30° then the inch-measure of $\overline{AC}$ is ___.

(d) The degree-measure of $\angle B$ must be a number between _____ and _____.

3. (a) Consider a triangle, ΔABC, in which $\angle A$ is an angle of 60°, the inch-measure of $\overline{AB}$ is 2, and $\angle B$ is an angle of 40°. Draw a graph of the relation of the inch-measure of $\overline{AC}$ to the inch-measure of $\overline{BC}$. Interpret your results.



(b) Now consider a triangle, ΔABC, in which $\angle A$ is an angle of 60°, the inch-measure of $\overline{AB}$ is 2, and the inch-measure of $\overline{AC}$ is 3. What can you say about the relation of the inch-measure of $\overline{BC}$ to the degree-measure of $\angle B$? Interpret your results in terms of the variety of triangles you can draw which fit the three conditions given for ΔABC.

4. Consider a triangle, ΔABC, which fits the two conditions given in Exercise 1. Draw a graph of the relation of the inch-measure of $\overline{AC}$ to the inch-measure of $\overline{BC}$. [Be sure you include the graphs of points which correspond with triangles in which the

degree-measure of $\angle B$ is 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, and 110.]

(a) How many differently-shaped triangles ABC can you draw in which $\angle A$ is an angle of 60°, the inch-measure of $\overline{AB}$ is 2, and the inch-measure of $\overline{BC}$ is 4?

(b) In which the inch-measure of $\overline{BC}$ is 1.8?

(c) In which the inch-measure of $\overline{BC}$ is 1?

E. 1. Give the domain and range of each relation in Part A of the Supplementary Exercises.

2. Give the domain, range, and field of each relation described below.

(a) $A = \{(x, y) \in I^+ \times I^+: x > y\}$

(b) $B = \{(x, y) \in I \times I: xy \text{ is odd}\}$

(c) $C = \{(x, y) \in I \times I: xy \text{ is even}\}$

(d) $D = \{(x, y) \in I \times I: x \text{ is odd and } x + y \text{ is odd}\}$

(e) $E = \{(x, y) \in I^+ \times I^+: y = x^2\}$

(f) $F = \{(x, y) \in I \times I: y = 13\}$

(g) $G = \{(x, y) \in I^+ \times I^+: y < 25 \text{ and } y = 36\}$

(h) $H = \{(x, y) \in I^+ \times I^+: x < 30$ and y is the sum of the proper factors of x with respect to $I^+\}$

Note: With respect to I^+, a proper factor of a positive integer is one of its factors which is smaller than it.

3. (a) If you know the domain and the range of a relation, can you use this information alone to tell what the field of the relation is?

(b) If you know the domain and the field of a relation, can you use this information alone to tell what the range of the relation is?

(c) If you know the domain and the range of a relation, can you use this information alone to tell what ordered pairs belong to the relation?

Correction. Part E, Exercise 2(h):
$H = \{(x, y) \in I^+ \times I^+,\ x > 1 : x < 30 \ldots$

E. 1. [In listing solutions we use, here, 'R' for the set of real numbers.]

1. R, R 2. R, R 3. $\{x: -5 \leq x \leq 5\}$, $\{x: -5 \leq x \leq 5\}$

4. $\{x: -10 \leq x \leq 10\}$, $\{x: -10 \leq x \leq 10\}$ 5. R, $\{8\}$

6. $\{8\}$, R 7. R, R 8. $\{x: x < 2\}$, $\{x: x > 3\}$

9. $\{x: x \neq 0\}$, $\{x: x \neq -1\}$ 10. R, R

11. $\{x: -5 \neq x \neq 5\}$, R 12. R, R 13. R, $\{3, -1\}$

14. $\{x: 1 \leq x \leq 5\}$, $\{x: 2 \leq x \leq 6\}$ 15. R, R

16. $\{x: x \neq 0\}$, $\{x: x \neq 0\}$ 17. $\emptyset$, $\emptyset$

18. $\{x: x \leq 2 \text{ or } x \geq 3\}$, R 19. R, R

20. R, $\{x: x \geq -\frac{9}{4}\}$

2. (a) $\{x \in I^+: x \neq 1\}$, I^+, I^+

(b) $\{x \in I: x \text{ is odd}\}$, $\{x \in I: x \text{ is odd}\}$, $\{x \in I: x \text{ is odd}\}$

(c) I, I, I

(d) $\{x \in I: x \text{ is odd}\}$, $\{x \in I: x \text{ is even}\}$, I

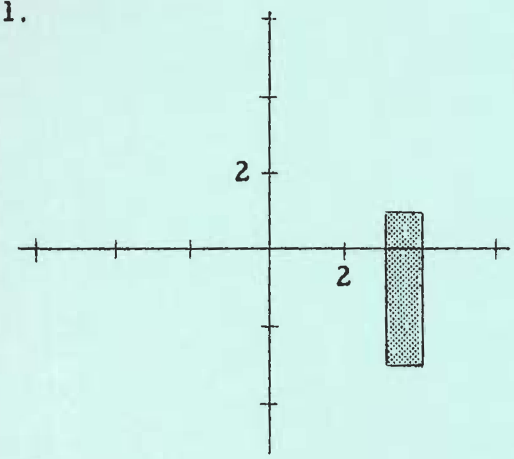
(e) I^+, the set of squares of positive integers, I^+

(f) I, $\{13\}$, I (g) $\emptyset$, $\emptyset$, $\emptyset$

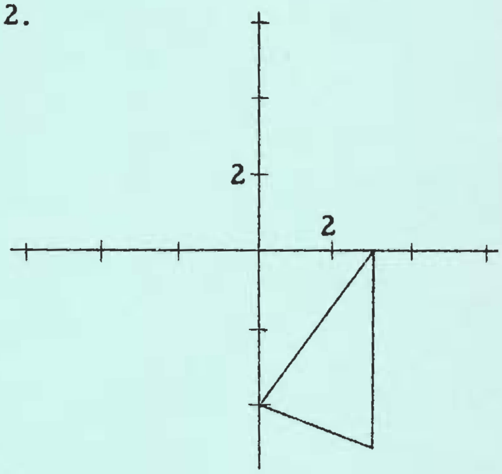
(h) $\{x \in I^+: 1 < x < 30\}$,
$\{1, 3, 4, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 21, 22, 28, 36\}$,
$\{x \in I^+: x < 30 \text{ or } x = 36\}$

3. (a) Yes (b) No (c) No

F. 1.



2.



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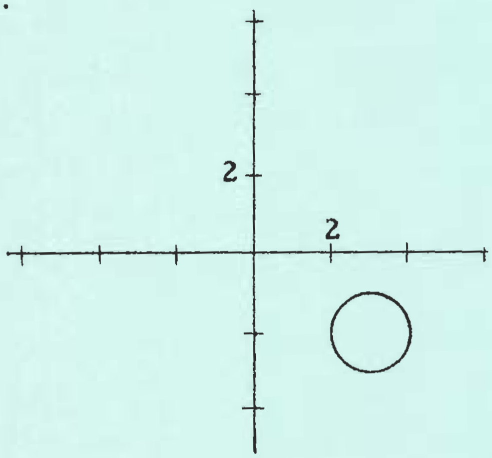
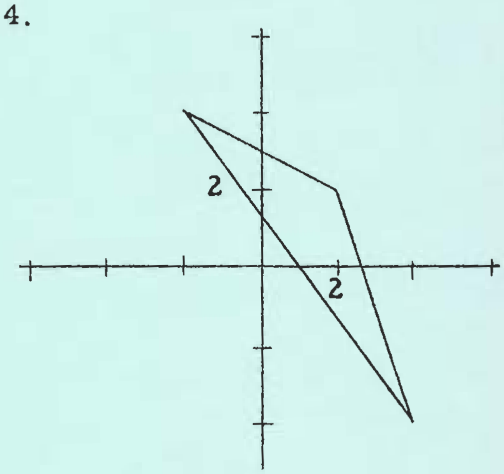


4.



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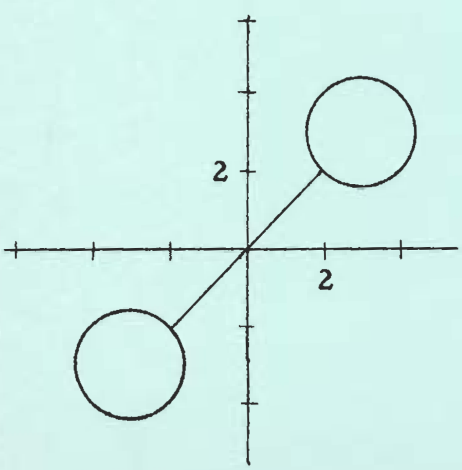
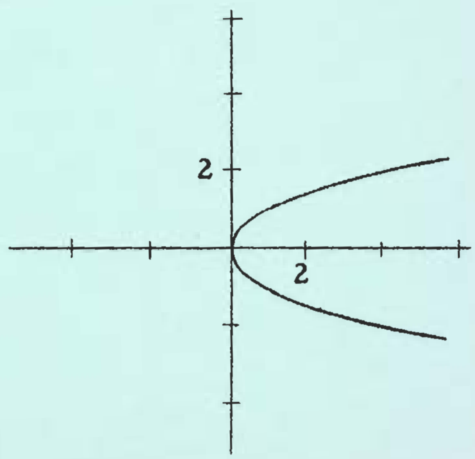
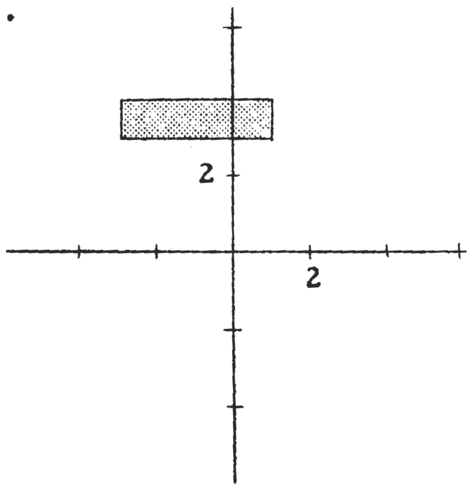


6.



F. Each of the following exercises contains a graph of a relation. Sketch the graph of the converse of the relation.

1.



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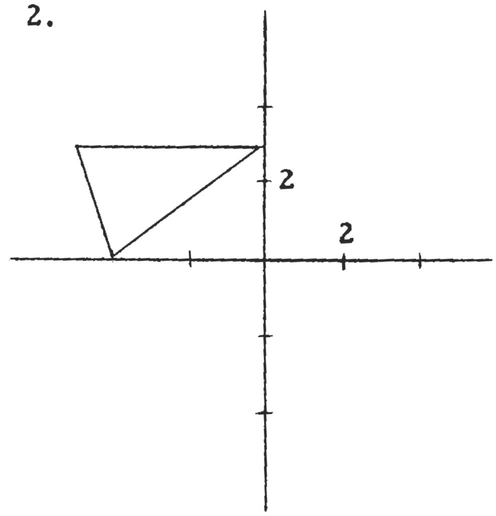

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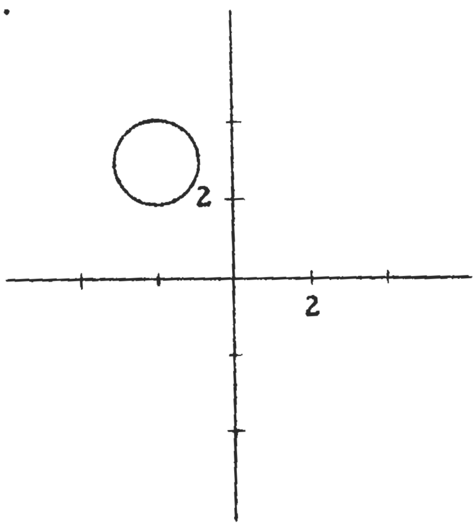

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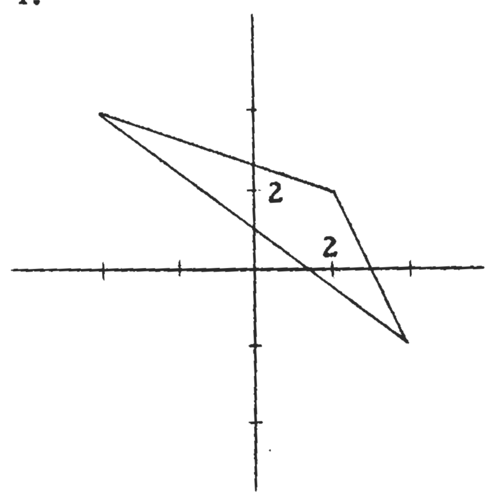

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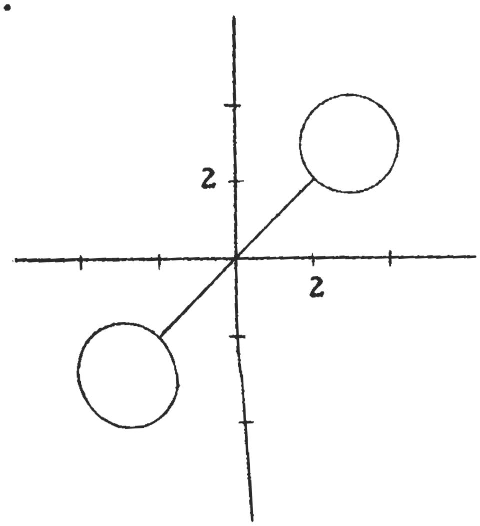
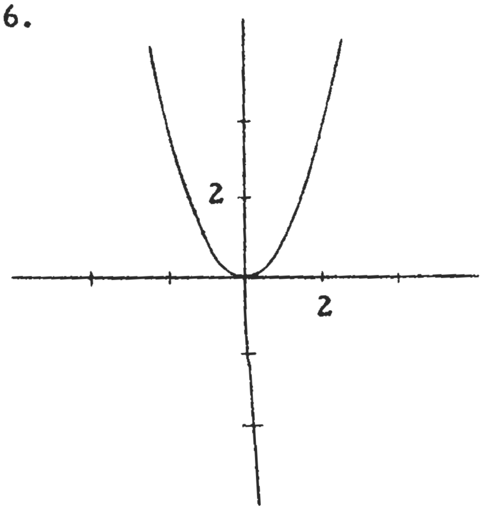


6.



G. 1. Which of the relations given in Part A of the Supplementary Exercises are reflexive? Which are symmetric?

2. Which of the relations listed below are reflexive, and which are symmetric? [C is the set of all students in a Zabranchburg High mathematics class.]

(a) $\{(x, y) \in C \times C$: x is older than y$\}$

(b) $\{(x, y) \in C \times C$: x is in the same chemistry class as y$\}$

(c) $\{(x, y) \in C \times C$: x is a brother or sister of y$\}$

(d) $\{(x, y) \in C \times C$: x lives on the same street as y$\}$

(e) $\{(x, y) \in C \times C$: x has the same birthday as y$\}$

(f) $\{(x, y) \in C \times C$: x knows the address of y$\}$

3. There are many relations whose field is $\{1, 3, 5, 7, 9, 11, 13\}$ and which are reflexive. List the members of that one of these relations which has the fewest members.

4. Suppose R_1 and R_2 are two reflexive relations with the same nonempty field. Which of the relations listed below must be reflexive? [Each complement is relative to the cartesian square of the field.]

(a) $R_1 \cap R_2$ (b) $R_1 \cup R_2$ (c) $\tilde{R}_1$ (d) $\tilde{R}_1 \cap R_2$

(e) $\tilde{R}_1 \cup R_2$ (f) $\tilde{R}_1 \cap \tilde{R}_2$ (g) $\tilde{R}_2$ (h) $\tilde{R}_1 \cup \tilde{R}_2$

5. (a) Describe the intersection of all reflexive relations with $\{1, 2, 3, 4, 5\}$ as field.

(b) Describe the union of all reflexive relations with $\{1, 2, 3, 4, 5\}$ as field.

6. Suppose S_1 and S_2 are two symmetric relations among the members of a set M. Which of the relations listed below are symmetric? [Each complement is relative to $M \times M$.]

(a) $\tilde{S}_1$ (b) $\tilde{S}_2$ (c) $S_1 \cup S_2$ (d) $S_1 \cap S_2$

(e) $\tilde{S}_1 \cup \tilde{S}_2$ (f) $\tilde{S}_1 \cap \tilde{S}_2$ (g) $\widetilde{S_1 \cup S_2}$ (h) $\widetilde{S_1 \cap S_2}$

(i) $\widetilde{\tilde{S}_1 \cup \tilde{S}_2}$ (j) $\widetilde{\tilde{S}_1 \cap \tilde{S}_2}$ (k) $S_1 \cup \tilde{S}_2$ (ℓ) $\tilde{S}_1 \cap S_2$

G. 1. The relations in Exercises 7, 15, 17, and 19 are reflexive; those in Exercises 3, 4, 15, 16, and 17 are symmetric.

2. The relations in parts (b), (d), (e), and (f) are reflexive. Those in parts (b), (c), (d), and (e) are symmetric.

3. (1, 1), (3, 3), (5, 5), (7, 7), (9, 9), (11, 11), (13, 13)

4. The relations in parts (a), (b), and (e) are reflexive.

5. (a) $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$

   (b) $\{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$

6. All the relations listed are symmetric. [Once one has seen that $\widetilde{S}_1$, $S_1 \cup S_2$ and $S_1 \cap S_2$ must be symmetric if S_1 and S_2 are, the remaining parts are trivial. Take part (i), for example. Since S_1 and S_2 are symmetric, their complements are symmetric. Since $\widetilde{S}_1$ and $\widetilde{S}_2$ are symmetric, their union is symmetric. Since $\widetilde{S}_1 \cup \widetilde{S}_2$ is symmetric, so is its complement.]

7. The relations in parts (c), (e), (f), (g), and (h) are symmetric.

8. (a) The set of symmetric relations among the members of a set S is closed under unioning, intersecting, and complementing.

   (b) The set of reflexive relations among the members of a set S is closed under unioning and intersecting [but not under complementing].

Corrections. [page 5-247]

In the first line of Exercise 4, and in the second line of Exercise 5, change 'Exercise 8' to 'Exercise 7'.

In line 16, underline 'antisymmetric relations'.

In Exercise 8, left column of exercise letters, change '(b)' to '(c)' and '(c)' to '(e)'.

☆H. 1. reflexive: (b), (e), (i), (j); irreflexive: (a), (h), (j); neither: (c), (d), (f), (g)

2. reflexive: (b), (c), (e), (f), (h), (j); irreflexive: (a), (d), (g), (i) [For some values of 'R' and 'N', the relation named in part (a), that in part (d), or that in part (i) is $\emptyset$. In such a case, the relation in question is reflexive as well as irreflexive.]

3. (a) (iii) (b) (i) T (ii) F (iii) F

(c) (i) There is an $(x, y) \in P \times P$ such that $y \, F \, x$ and $x \not F \, y$.

(ii) There is an $(x, y) \in P \times P$ such that $y \, B \, x$ and $x \not B \, y$.

(iii) There is an $(x, y) \in P \times P$ such that $y \, M \, x$ and $x \not M \, y$.

4. (a), (e)

5. (a) [There are many correct answers; for example, the relations <, and being a grandchild of, are two such.]

(b) [Any relation R among the members of a set S such that (1) there is an $(x, y) \in S \times S$ such that $(x, y) \in R$ and $(y, x) \notin R$ and (2) there is an $(x, y) \in S \times S$ such that $(x, y) \in R$ and $(y, x) \in R$.]

(c) [Any relation R among the members of a set S such that (1) there is no $(x, y) \in S \times S$ such that $(x, y) \in R$ and $(y, x) \in R$ and (2) there is no $(x, y) \in S \times S$ such that $(x, y) \in R$ and $(y, x) \notin R$--but, the only such relation R is $\emptyset$.]

6. (a)

7. (b), (c), (e)

8. (a) (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)

(b) (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3) [Or, instead of (1, 3), any ordered pair not previously listed.]

(c) (1, 3), (3, 1), (2, 4), (4, 2), (1, 5), (5, 1) [There are other correct answers.]

(d) (1, 3), (2, 4), (1, 5) [There are other correct answers.]

(e) (1, 3), (2, 4), (1, 5) [There are other correct answers.]

(f) [Impossible. An asymmetric relation is irreflexive, and an irreflexive relation is reflexive only if its field is $\emptyset$.]

7. Which of these relations are symmetric?

(a) motherhood

(b) sisterhood

(c) $\{(x, y): x \neq y\}$

(d) $\{(x, y): x \not> y\}$

(e) $\{(x, y): |x| + |y| < |x + y|\}$

(f) $\{(x, y) \in E \times E$: x is equivalent to y$\}$ where E is the set of all equations

(g) $\{(x, y) \in L \times L$: x is parallel to y$\}$ where L is the set of all straight lines

(h) $\{(x, y) \in L \times L$: x is perpendicular to y$\}$

(i) $\{(x, y) \in S \times S: x \subseteq y\}$ where S is the set of all subsets of $\{1, 2, 3, 4\}$

8. Recall that we say that the set of integers is closed under the operation addition because each sum of integers is an integer.

(a) Consider the set of all symmetric relations among members of a set S. Under which of the operations unioning, intersecting, and complementing is this set of relations closed?

(b) Repeat part (a) for the set of all reflexive relations among the members of a set S.

☆ H. Consider the relation > among the real numbers. The field of this relation is the set of real numbers. This relation is not reflexive [that is, it is nonreflexive] because there is a member $k \in \mathfrak{F}_>$ such that $k \not> k$. [7 is such a member.] In fact, for each member $k \in \mathfrak{F}_>$, $k \not> k$. A relation which has this latter property is said to be an irreflexive relation.

A relation R is irreflexive if and only if

$$\forall_{x \in \mathfrak{F}_R} (x, x) \notin R.$$

1. Classify the relations listed below into three categories--reflexive, irreflexive, and nonreflexive but not irreflexive.

(a) $\{(x, y) \in P \times P$: x is the grandfather of y$\}$

(b) $\{(x, y): x + y = y + x\}$

(c) $\{(x, y): xy = 0\}$

(d) $\{(x, y): x^2 + y^2 = 17\}$ (e) $\{(x, y): x \leq y\}$

(f) $\{(x, y): y = x^2 - 42\}$ (g) $\{(x, y): y = |x|\}$

(h) $\{(x, y) \in L \times L$: x is perpendicular to y$\}$, where L is the set of all straight lines

(i) $\{(x, y) \in I \times I$: x and y have a common integral factor$\}$

(j) $\{(x, y): x^2 + y^2 = -17\}$

2. Suppose R is a reflexive relation and N is an irreflexive relation and $\mathfrak{F}_R = \mathfrak{F}_N \neq \emptyset$. Classify the relations listed below into two categories--reflexive and irreflexive. [Complements are with respect to the cartesian square of the common field.]

(a) $N \cap R$ (b) $N \cup R$ (c) $\tilde{N}$ (d) $\tilde{R}$ (e) $\tilde{N} \cap R$

(f) $\tilde{N} \cup R$ (g) $N \cup \tilde{R}$ (h) $\widetilde{N \cup \tilde{R}}$ (i) $N \cap \tilde{R}$ (j) $\widetilde{N \cap \tilde{R}}$

3. Consider the relations F, B, and M where

$F = \{(x, y) \in P \times P$: y is the father of x$\}$,

$B = \{(x, y) \in P \times P$: y is a brother of x$\}$,

and $M = \{(x, y) \in P \times P$: y is the spouse of x$\}$.

(a) Which of the following statements are true?

(i) For each $(x, y) \in P \times P$, if $y\, F\, x$ then $x\, F\, y$.

(ii) For each $(x, y) \in P \times P$, if $y\, B\, x$ then $x\, B\, y$.

(iii) For each $(x, y) \in P \times P$, if $y\, M\, x$ then $x\, M\, y$.

(b) True or false?

(i) For each $(x, y) \in P \times P$, if $y\, F\, x$ then $x \not{F}\, y$.

(ii) For each $(x, y) \in P \times P$, if $y\, B\, x$ then $x \not{B}\, y$.

(iii) For each $(x, y) \in P \times P$, if $y\, M\, x$ then $x \not{M}\, y$.

(c) Write the denial of each statement given in part (a). [For example, the denial of statement (i) of part (a) is:

There is an $(x, y) \in P \times P$ such that $y\, F\, x$ and $x \not{F}\, y$]

* * *

Relations such as fatherhood are called <u>asymmetric relations</u>.

A relation R among the members of a set S is asymmetric if and only if, for each $(x, y) \in S \times S$, if $y\ R\ x$ then $x \not R\ y$.

* * *

4. Which of the relations given in Exercise 8 of Part G are asymmetric?

5. (a) Give a relation which is asymmetric [but different from those in Exercise 8 of Part G].

   (b) Give a relation which is nonsymmetric but not asymmetric.

   (c) Give a relation which is asymmetric but not nonsymmetric.

6. The relations $\geq$ and B [brotherhood] are neither symmetric nor asymmetric. Which of the following statements are true?

   (a) $\forall_x \forall_y$ if $y \geq x$ and $x \geq y$ then $x = y$

   (b) $\forall_{x \in P} \forall_{y \in P}$ if $y\ B\ x$ and $x\ B\ y$ then $x = y$

* * *

Relations such as $\geq$ are called antisymmetric relations.

R is an antisymmetric relation among the members of a set S if and only if

$\forall_{x \in S} \forall_{y \in S}$ if $y\ R\ x$ and $x\ R\ y$ then $x = y$.

* * *

7. Which of these relations are antisymmetric?

   (a) $\{(x, y) \in L \times L$: y is perpendicular to x$\}$

   (b) $\{(x, y) \in S \times S:\ y \subseteq x\}$ where S is the set of all subsets of $\{1, 2, 3, 4\}$

   (c) $\{(x, y):\ x + 2y = 3\}$ (d) $\{(x, y):\ x^2 = y^2\}$

   (e) $\{(x, y) \in I^+ \times I^+:\ x = \sqrt{y}\}$

8. Describe [list the elements of] a nonempty relation with the least number of elements whose field is $\{1, 2, 3, 4, 5\}$ and which is

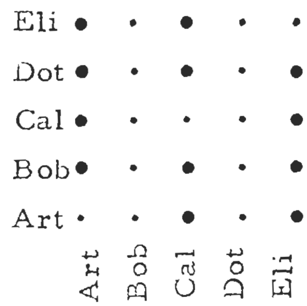
   (a) reflexive and symmetric (b) reflexive and nonsymmetric

   (b) irreflexive and symmetric (d) irreflexive and nonsymmetric

   (c) irreflexive and asymmetric (f) reflexive and asymmetric

I. 1. Here is a picture of a relation whose field is the set of all children in some family.

$C = \{\text{Art, Bob, Cal, Dot, Eli}\}$



(a) Is this relation reflexive? Symmetric?

(b) What is the domain of the relation? Range?

(c) How many more ordered pairs must be included to get a relation which is reflexive and has the same field? Symmetric?

2. Consider the relation $\{(x, y) \in C \times C\colon\ y \text{ is older than } x\}$. Here is a list of the ordered pairs which belong to this relation.

(Bob, Cal), (Bob, Art), (Bob, Eli), (Bob, Dot),
(Cal, Art), (Cal, Eli), (Cal, Dot), (Art, Eli),
(Eli, Dot), (Art, Dot)

(a) Who is older, Cal or Eli?

(b) Who is older, Art or Bob?

(c) Who is the oldest? The youngest?

3. Consider the relation

$$\{(x, y) \in I^+ \times I^+\colon\ \exists_{m \in I^+}\ x = my\}.$$

Another name for this relation is:

$\{(x, y) \in I^+ \times I^+\colon\ y \text{ is a factor of } x \text{ with respect to } I^+\}$

Let's call this relation '$|$'.

(a) Draw a graph of $|$. What is the domain of $|$? Range?

(b) Is $|$ reflexive? Symmetric?

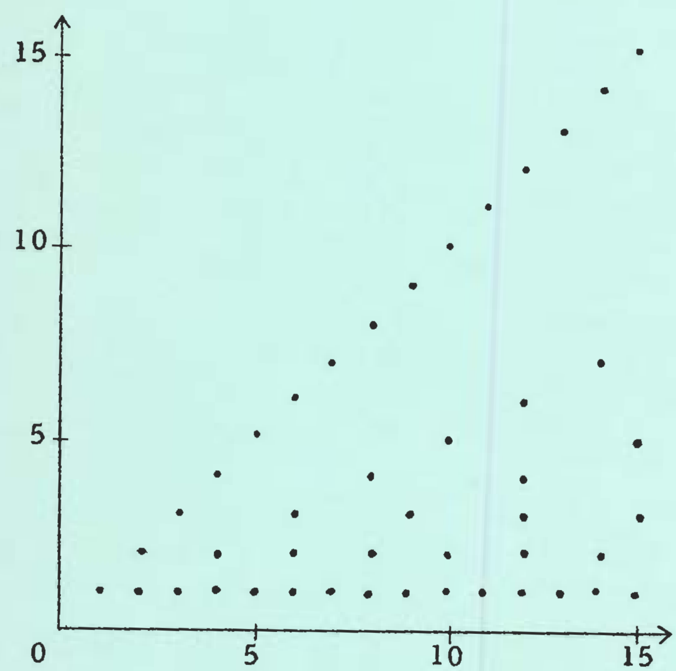
I. 1. (a) The relation is neither reflexive nor symmetric.

(b) The domain of the relation is {Art, Cal, Eli}; its range is C.

(c) Include 5 additional ordered pairs to obtain a reflexive relation; 6 to obtain a symmetric relation.

2. (a) Eli (b) Art (c) Dot; Bob

3. (a)



$\mathcal{D}_{|} = I^{+} = \mathcal{R}_{|}$

(b) | is reflexive, but not symmetric [it is antisymmetric].

☆J.

Exercise	reflexive	irreflexive	symmetric	asymmetric	antisymmetric	transitive	intransitive
1.		✓					
2.	✓		✓			✓	
3.	✓		✓				
4.		✓	✓				
5.	✓		✓			✓	
6.		✓		✓	✓		✓
7.	✓		✓			✓	
8.	✓		✓			✓	
9.	✓		✓			✓	
10.	✓		✓			✓	
11.	✓		✓			✓	
12.	✓				✓	✓	
13.	✓	✓	✓	✓	✓	✓	✓
14.	✓		✓		✓	✓	
15.		✓		✓	✓	✓	
16.	✓				✓	✓	
17.			✓				
18.		✓	✓				✓
19.		✓		✓	✓	✓	
20.		✓		✓	✓		✓
21.			✓				
22.	✓					✓	
23.		✓	✓				
* 24.		✓	✓				✓
25.	✓		✓			✓	

* The set named is a relation among geometric figures in a plane.

☆ J. Here is a summary of the definitions of reflexive, irreflexive, symmetric, asymmetric, and antisymmetric relations, as well as definitions of transitive and intransitive relations.

A relation R among the elements of a set M is

reflexive........ $\forall_{x \in \mathfrak{D}_R}$ $x \, R \, x$

irreflexive...... $\forall_{x \in \mathfrak{D}_R}$ $x \not R \, x$

symmetric...... $\forall_{x \in M} \forall_{y \in M}$ if $y \, R \, x$ then $x \, R \, y$

asymmetric..... $\forall_{x \in M} \forall_{y \in M}$ if $y \, R \, x$ then $x \not R \, y$

antisymmetric... $\forall_{x \in M} \forall_{y \in M}$ if $y \, R \, x$ and $x \, R \, y$ then $x = y$

transitive....... $\forall_{x \in M} \forall_{y \in M} \forall_{z \in M}$ if $y \, R \, x$ and $z \, R \, y$ then $z \, R \, x$

intransitive ..... $\forall_{x \in M} \forall_{y \in M} \forall_{z \in M}$ if $y \, R \, x$ and $z \, R \, y$ then $z \not R \, x$

For each relation given below list the properties which it has.

1. $\{(x, y) \in P \times P$: y is a sister of x$\}$
2. $\{(x, y) \in P \times P$: y goes to the same school as x$\}$
3. $\{(x, y) \in P \times P$: y lives less than a mile from x$\}$
4. $\{(x, y) \in P \times P$: y does not live on the same street as x$\}$
5. $\{(x, y) \in P \times P$: y is in the same clubs as x$\}$
6. $\{(x, y) \in P \times P$: y sits in the row directly behind x at assembly$\}$
7. $\{(x, y) \in I \times I$: x - y or y - x is an even number$\}$
8. $\{(x, y) \in I \times I$: x and y have a common integral factor $\neq 1\}$
9. $\{(x, y) \in E \times E$: x is equivalent to y$\}$ where E is the set of all equations
10. $\{(x, y) \in A \times A$: y is equivalent to x$\}$ where A is the set of all algebraic expressions in 't' and 'u'
11. $\{(x, y) \in E \times E$: y has the same roots as x$\}$
12. $\{(x, y) \in S \times S$: $y \subseteq x\}$ where S is the set of all subsets of $\{1, 2, 3, 4, 5\}$
13. $\{(x, y) \in I \times I$: x is even and $xy = 7\}$

14. $\{(x, y): y = x\}$

15. $\{(x, y): y > x\}$

16. $\{(x, y): y \leq x\}$

17. $\{(x, y): x^2 + y^2 = 1\}$

18. $\{(x, y) \in P \times P$: y is the spouse of x$\}$

19. $\{(x, y) \in P \times P$: y is an ancestor of x$\}$

20. $\{(x, y) \in P \times P$: y is a son of x$\}$

21. $\{(x, y) \in I^+ \times I^+$: y and x have no common factor $\neq 1$ with respect to $I^+\}$

22. $\{(x, y) \in I \times I$: y is a factor of x with respect to I$\}$

23. $\{(x, y) \in L \times L$: y is parallel to x$\}$

24. $\{(x, y) \in L \times L$: y is perpendicular to x$\}$

25. $\{(x, y) \in L \times L$: y is parallel to x or $y = x\}$

☆26. Prove each of the following, or give a counter-example.

(a) For each relation R, if R is symmetric and transitive then R is reflexive.

(b) $\forall_R$ if R is asymmetric then R is irreflexive.

(c) $\forall_R$ if R is irreflexive then R is asymmetric.

(d) $\forall_R$ R is transitive and asymmetric if and only if R is transitive and irreflexive.

(e) $\forall_R$ if R is reflexive then $\widetilde{R}$ [with respect to $\mathfrak{F}_R \times \mathfrak{F}_R$] is irreflexive.

(f) $\forall_R$ if R is symmetric then $\widetilde{R}$ is symmetric.

(g) $\forall_R$ if R is transitive then $\widetilde{R}$ is transitive.

(h) $\forall_R$ if R is asymmetric then R is antisymmetric.

(i) $\forall_R$ if R is antisymmetric then R is asymmetric.

☆26. (a) Suppose that $a \in \mathfrak{D}_R$. Since R is symmetric, $a \in \mathfrak{R}_R$. Hence, there is a $b \in \mathfrak{R}_R$ such that $(a, b) \in R$. Since R is symmetric, $(b, a) \in R$. Since R is transitive and since $(a, b) \in R$ and $(b, a) \in R$, $(a, a) \in R$. So, R is reflexive.

(b) Suppose that $a \in \mathfrak{D}_R$. Since R is asymmetric, not both (a, a) and (a, a) belong to R--that is, $(a, a) \notin R$. So, R is irreflexive.

(c) Each of Exercises 1, 4, 18, 23, and 24 gives a counter-example--that is, a relation which is irreflexive but not asymmetric.

(d) If R is transitive and asymmetric then, by part (b), R is irreflexive. On the other hand, if R is transitive and irreflexive then, if $(a, b) \in R$, $(b, a) \notin R$ [since $(a, a) \notin R$]. So, a transitive and irreflexive relation is asymmetric.

(e) If R is reflexive then, for each $a \in \mathfrak{D}_R$, $(a, a) \in R$. Hence, for each $a \in \mathfrak{D}_R$, $(a, a) \notin \tilde{R}$. So, $\tilde{R}$ is irreflexive.

(f) If $(a, b) \in \tilde{R}$ then $(a, b) \notin R$ and, if R is symmetric, $(b, a) \notin R$--that is, $(b, a) \in \tilde{R}$. Hence, $\tilde{R}$ is symmetric.

(g) Each of Exercises 2, 5, 9, 10, 11, 12, 14, 19, 22, and 25 gives a counter-example.

(h) If R is asymmetric then, if $(a, b) \in R$, $(b, a) \notin R$. Hence, in particular, if $a \neq b$ then not both (a, b) and (b, a) belong to R. So, R is antisymmetric.

(i) Each of Exercises 12, 14, and 16 gives a counter-example.

> Correction. On page 5-252, line 1:
> ...the function in Exercise 3(c) whose...

K. 1. (a), (c), and (f) are functions, the others are not.

2. The converse of (c) is a function, those of the others are not.

3. (a) Yes (b) Yes (c) Yes (d) yes

(e) No [r_2 and r_5 have the same perimeter but different area-measures.]; No [r_3 and r_4 have the same area-measure but different perimeters.]

(f) (1) 12 (2) 10 (3) 24 (4) $\frac{13}{2}$

4. (a) (1) 5 (2) 2 (3) 3 (4) 3

(5) 6 (6) 1 (7) 7 (8) 12

(9) {1, 2, 3, 4, 5, 6} (10) {1, 2, 3, 5, 6}

(b) (1) −2 (2) 28 (3) −12 (4) −27

(5) 8 (6) −17 (7) 28 (8) 3

(9) 5 (10) $\frac{13}{5}$ (11) 10a - 12 (12) 15b - 7

(13) c - 12 (14) d

(c) (1) 10 (2) 16 (3) 25 (4) 37

(5) 91 (6) −65

(7) (i) 4 (ii) 8 (iii) $\frac{13}{3}$ (iv) 1 (v) 1

(vi) $-\frac{1}{5}$ (vii) −1 (viii) The solution set is the set of real numbers.

K. 1. Tell which of the sets listed below are functions.

(a) $\{(9, 5), (4, 3), (8, 9), (6, 5), (7, 1)\}$

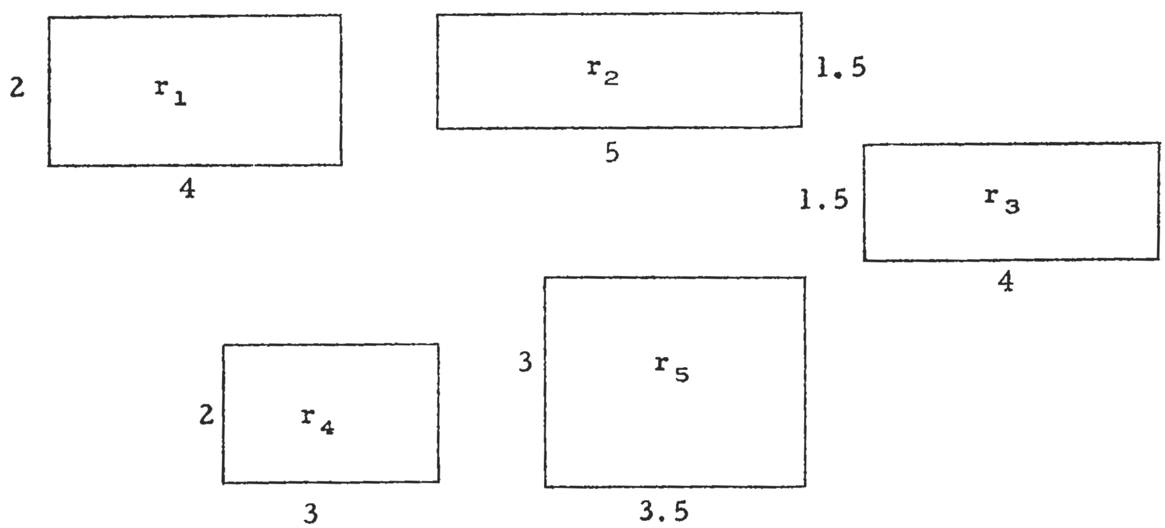
(b) $\{(2, 5), (3, 8), (8, 3), (6, 5), (3, 4)\}$

(c) $\{(5, 17), (82, 61), (0, -3), (4, 2), (-3, 9)\}$

(d) $\{(x, y): x \leq y\}$ (e) $\{(x, y): x \neq y\}$ (f) $\{(x, y): y = (x - 1)^2\}$

2. Which of the relations given in Exercise 1 have converses which are functions?

3. Consider the set S which consists of the five rectangles shown below.



(a) Is $\{(r_1, 4), (r_2, 5), (r_3, 4), (r_4, 3), (r_5, 3.5)\}$ a function?

(b) Is $\{(r_1, 2), (r_2, 1.5), (r_3, 1.5), (r_4, 2), (r_5, 3)\}$ a function?

(c) Consider the set of 5 ordered pairs each of whose first components is one of these five rectangles, and each of whose second components is the corresponding perimeter. Is this set of ordered pairs a function?

(d) Repeat (c) for area-measure.

(e) Consider the set of 5 ordered pairs each of whose first components is the perimeter of one of these five rectangles, and each of whose second components is the corresponding area-measure. Is this a function? Is the converse of this set of ordered pairs a function?

(f) Suppose P is the function whose domain is S and whose range is the set of perimeters of the five rectangles in S. Complete each of the following sentences.

(1) $P(r_1) =$ ________ (2) $P(r_4) =$ ________

(3) $P(r_2) + P(r_3) =$ ________ (4) $\frac{1}{2} P(r_5) =$ ________

4. (a) $f = \{(1, 5), (2, 6), (3, 1), (4, 2), (5, 5), (6, 3)\}$

(1) $f(1) =$ ________ (2) $f(4) =$ ________

(3) $f(6) =$ ________ (4) $f($___$) = 1$

(5) $f(f(4)) =$ ________ (6) $f(f(6)) =$ ________

(7) $f(3) + f(2) =$ ________ (8) $f(4) \cdot f(2) =$ ________

(9) $\mathcal{D}_f =$ ________________ (10) $\mathcal{R}_f =$ ________________

(b) $g = \{(x, y): y = 5x - 12\}$

(1) $g(2) =$ ________ (2) $g(8) =$ ________

(3) $g(0) =$ ________ (4) $g(-3) =$ ________

(5) $g(4) =$ ________ (6) $g(-1) =$ ________

(7) $g(g(4)) =$ ________ (8) $g(g(g(3))) =$ ________

(9) $g($___$) = 13$ (10) $g($________$) = 1$

(11) $g(2a) =$ ________ (12) $g(3b + 1) =$ ________

(13) $g(c/5) =$ ________ (14) $g(\frac{d + 12}{5}) =$ ________

(c) $h = \{(x, y): y = 3x - 2\}$ and $j = \{(x, y): y = 6x + 1\}$

(1) $h(4) =$ ________ (2) $h(6) =$ ________

(3) $j(4) =$ ________ (4) $j(6) =$ ________

(5) $h(j(5)) =$ ________ (6) $j(h(-3)) =$ ________

(7) Solve these equations.

(i) $h(a) = 10$ (ii) $j(b) = 49$

(iii) $h(2a) = 24$ (iv) $j(3b - 5) = -11$

(v) $h(k) = k$ (vi) $j(k) = k$

(vii) $h(p) = j(p)$ (viii) $h(2p + 1) = j(p)$

(d) (1) 2 (2) $\frac{3}{5}$ (3) $\frac{2}{7}$ (4) 3 (5) $\frac{2 - 6a}{7}$

(6) $\frac{3 - 18k}{5}$ (7) $\frac{2 - 42m}{3}$ (8) $\frac{13 - 35n}{6}$ (9) $\frac{11}{27}$

(e) (1) −4 (2) 36 (3) 14 (4) 0 (5) 0

(6) $-\frac{25}{4}$ (7) $4a^2 - 18a + 14$ (8) $25b^2 - 45b + 14$

(9) $4m^2 - 22m + 24$ (10) $9n^2 - 15n$ (11) $p^2 - \frac{25}{4}$ (12) $4k^4 - 18k^2 + 14$

(f) (1) 17 (2) −3 (3) 4 (4) $-\frac{5}{4}$ (5) $4k + 5$

(6) $4(k + h) + 5$ (7) 4b (8) 4d (9) $-\frac{5}{3}$

(g) (1) 0 (2) 0 (3) 4 (4) 3 (5) 3

(6) $[-\frac{1}{4} \notin \mathcal{D}_s]$ (7) $[\frac{5}{2} \notin \mathcal{D}_s]$ (8) $\frac{96}{25}$ (9) 4

☆(10) If $s(t_0) = s(t_1)$ then

$$16t_0 - 16t_0^2 = 16t_1 - 16t_1^2.$$

So,

$$16t_0 - 16t_1 = 16t_0^2 - 16t_1^2,$$

$$16(t_0 - t_1) = 16(t_0^2 - t_1^2),$$

$$16(t_0 - t_1) = 16(t_0 - t_1)(t_0 + t_1),$$

and

$$1 = t_0 + t_1. \quad [t_0 - t_1 \neq 0]$$

(d) $F = \{(x, y): 3x + 7y - 2 = 0\}$ and $G = \{(x, y): 6x + 5y - 3 = 0\}$

(1) $F(-4) =$ _____ (2) $G(0) =$ _____ (3) $F(0) =$ _____

(4) $G(-2) =$ _____ (5) $F(2a) =$ _____ (6) $G(3k) =$ _____

(7) $F($____$) = 6m$ (8) $G($____$) = 7n - 2$

(9) Solve the equation: $F(p) = G(p)$

(e) $H = \{(x, y): y = x^2 - 9x + 14\}$

(1) $H(3) =$ _____ (2) $H(-2) =$ _____ (3) $H(0) =$ _____

(4) $H(7) =$ _____ (5) $H(2) =$ _____ (6) $H(9/2) =$ _____

(7) $H(2a) =$ _______ (8) $H(5b) =$ _______

(9) $H(2m - 1) =$ _______ (10) $H(3n + 2) =$ _______

(11) $H(p + \frac{9}{2}) =$ _______ (12) $H(2k^2) =$ _______

(f) $M(x) = 4x + 5$, $\mathcal{D}_M$ = the set of real numbers

(1) $M(3) =$ _______ (2) $M(-2) =$ _______

(3) $M($____$) = 21$ (4) $M($____$) = 0$

(5) $M(k) =$ _______ (6) $M(k + h) =$ _______

(7) $M(a + b) - M(a) =$ _______ (8) $M(5c + d) - M(5c) =$ _____

(9) Solve: $M(x) = x$

(g) $s(t) = 16t - 16t^2$, $\mathcal{D}_s = \{t: 0 \leq t \leq 1\}$

(1) $s(0) =$ __________ (2) $s(1) =$ __________

(3) $s(\frac{1}{2}) =$ __________ (4) $s(\frac{3}{4}) =$ __________

(5) $s(\frac{1}{4}) =$ __________ (6) $s(-\frac{1}{4}) =$ __________

(7) $s(\frac{5}{2}) =$ __________ (8) $s(\frac{2}{5}) =$ __________

(9) The largest value of s is _____.

☆ (10) Show that if $t_o \neq t_1$ but $s(t_o) = s(t_1)$ then $t_o + t_1 = 1$.

L. 1. Graph the function $y = 8 - 2x$.

2. Graph the doubling function.

3. Graph f where

$$f(x) = \begin{cases} 3, & \text{for } x \geq 1 \\ -x, & \text{for } x < 1. \end{cases}$$

4. Graph the function $\frac{7}{3 - x}$.

5. Graph $f(x) = [\![x]\!]$. ['$[\![x]\!]$' is an abbreviation of 'the greatest integer not greater than x'. So, for example, $[\![5.3]\!] = 5$, $[\![5.9]\!] = 5$, $[\![12]\!] = 12$, $[\![1/2]\!] = 0$, and $[\![-5.3]\!] = -6$.]

6. Graph $[\![x + 1]\!]$.

7. Graph $[\![x]\!] + 1$.

8. Graph $x - [\![x]\!]$.

9. Graph $[\![x]\!] - x$.

10. Graph $\{(x, y): [\![y]\!] = [\![x]\!]\}$. Is this a function?

11. The <u>signum function</u>, sg, is defined as follows:

$$sg(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ -1, & \text{for } x < 0 \end{cases}$$

(a) Graph sg.

(b) Graph $x[sg(x)]$.

(c) Graph $2x[sg(x)]$.

(d) Graph $(2x + 1)[sg(x)]$.

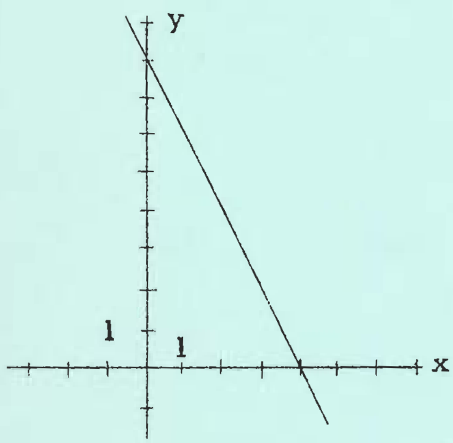
M. 1. Suppose $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{10, 11, 12, 13\}$, and f is the function which maps A on B in such a way that $f(1) = 10$, $f(2) = 11$, $f(3) = f(4) = 12$, and $f(5) = f(6) = 13$. List the ordered pairs in f.

2. Suppose f is a function which maps the set of real numbers on itself in such a way that the image of each real number is its opposite. Write a brace-notation name for f.

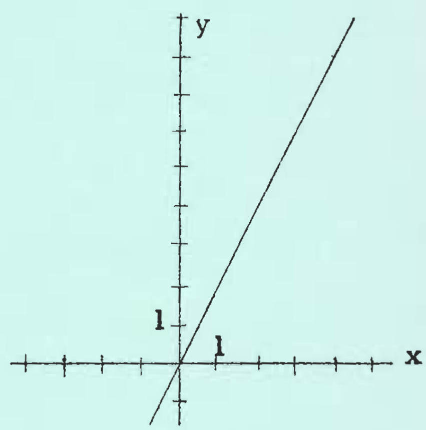
3. If f is the doubling function [with $\mathfrak{D}_f = \mathfrak{R}_f$ = the set of real numbers], what is the image of 8? Of what real number is -13 the image?

4. Write a brace-notation name for the mapping f which takes each real number to 1 more than twice its square.

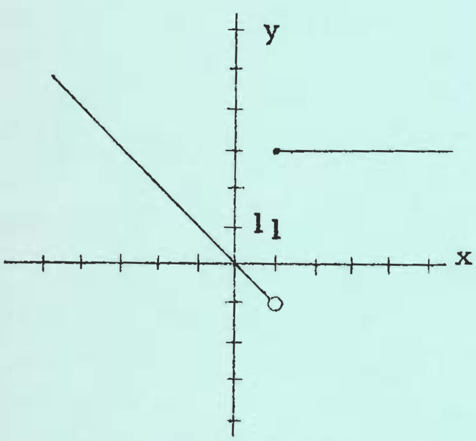
L. 1.



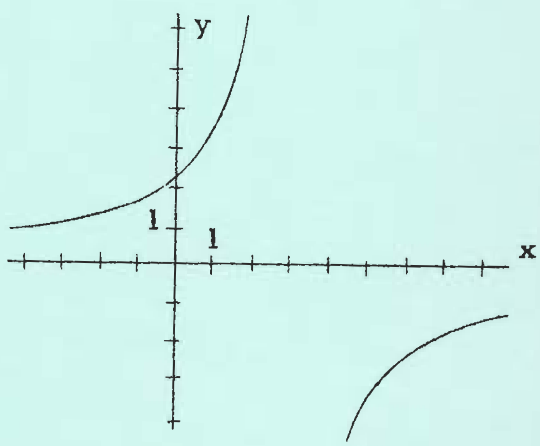
2.



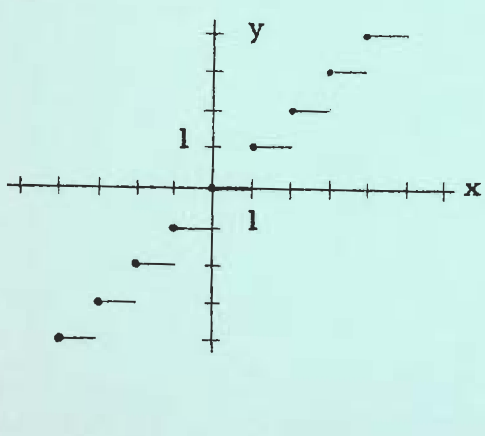
3.



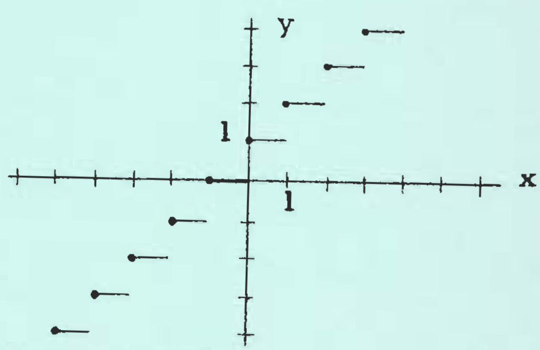
4.



5.



6.

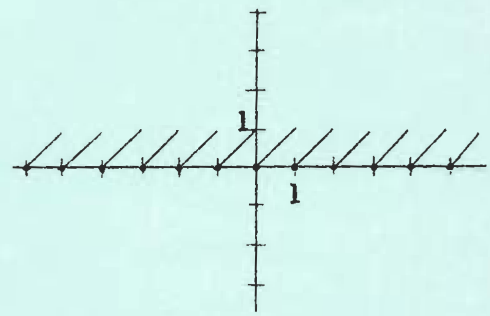


[Note the colloquial way of referring to this function.]

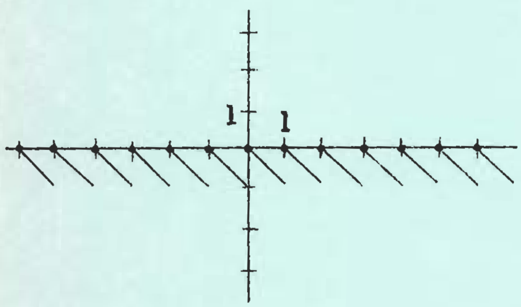
7.

[Same as Exercise 6.]

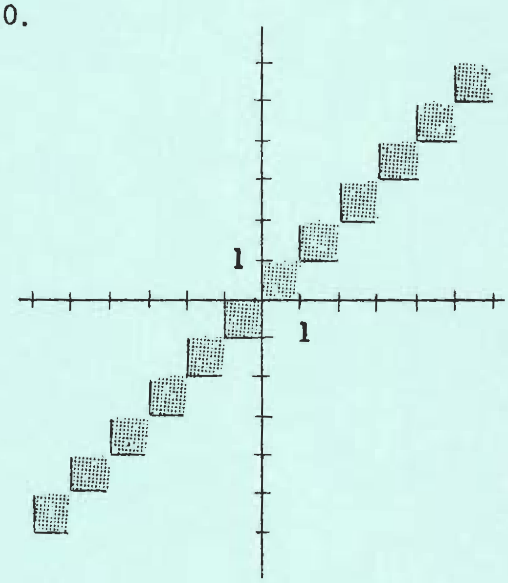
8.



9.

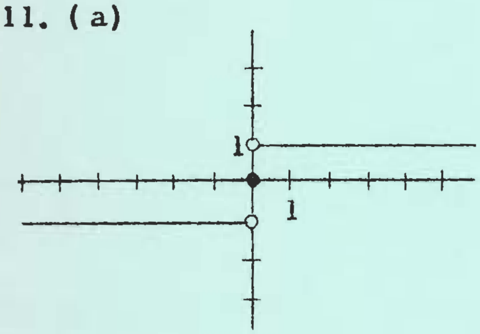


10.

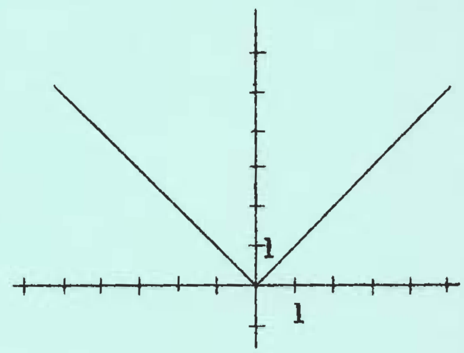


[Not a function]

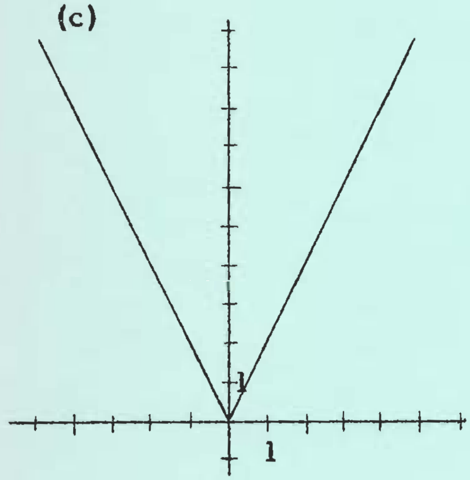
11. (a)



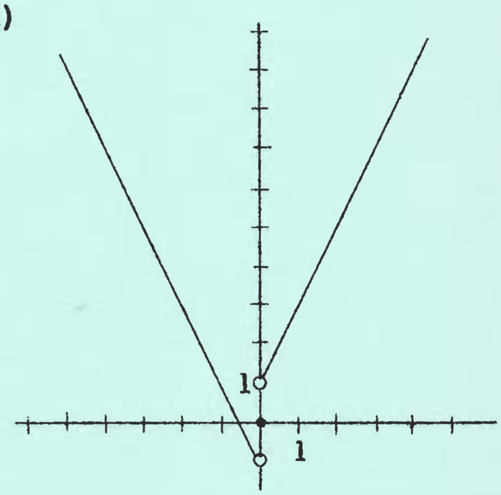
(b)



(c)



(d)



M. 1. (1, 10), (2, 11), (3, 12), (4, 12), (5, 13), (6, 13)

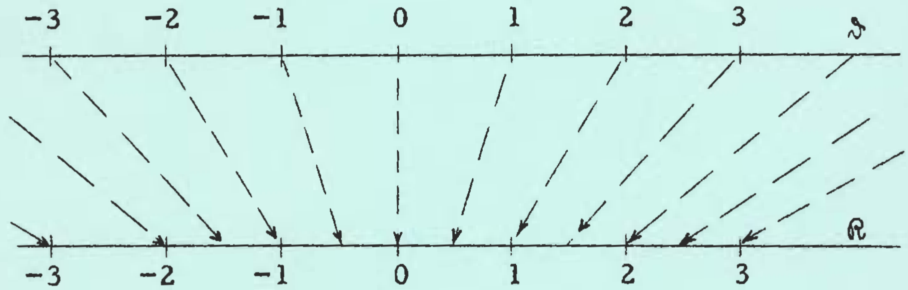
2. $\{(x, y): y = -x\}$

3. 16; $-\frac{13}{2}$

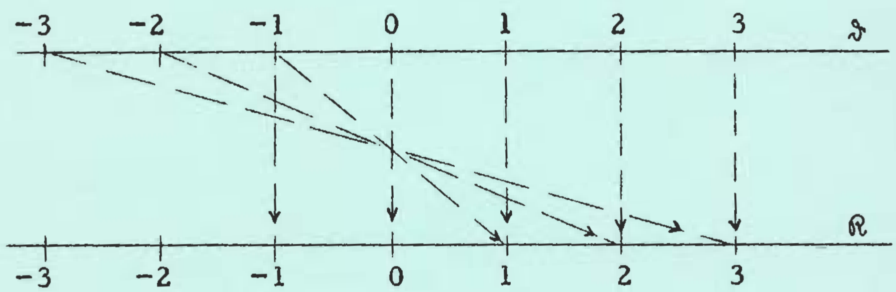
4. $\{(x, y): y = 2x^2 + 1\}$

5. $\{(x, y): y = 3x + 7\}$
[Ex. 5 is on page 5-255.]

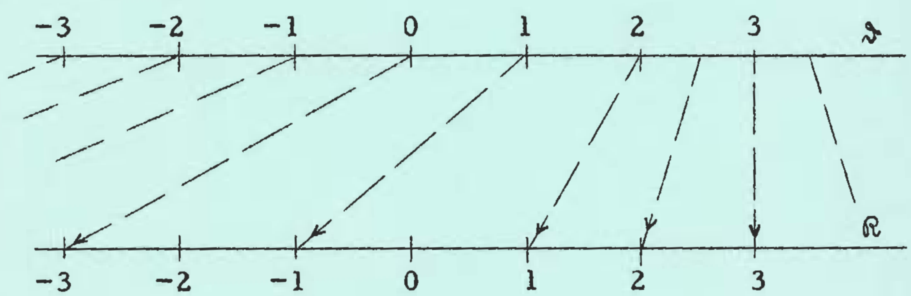
7. (a)


-3 -2 -1 0 1 2 3
𝒟
ℛ
-3 -2 -1 0 1 2 3

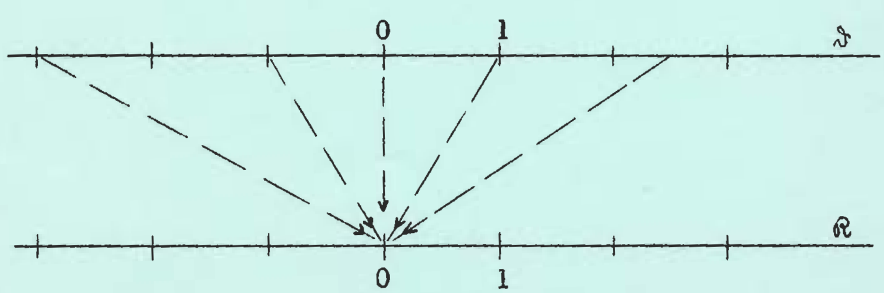
(b)


-3 -2 -1 0 1 2 3
𝒟
ℛ
-3 -2 -1 0 1 2 3

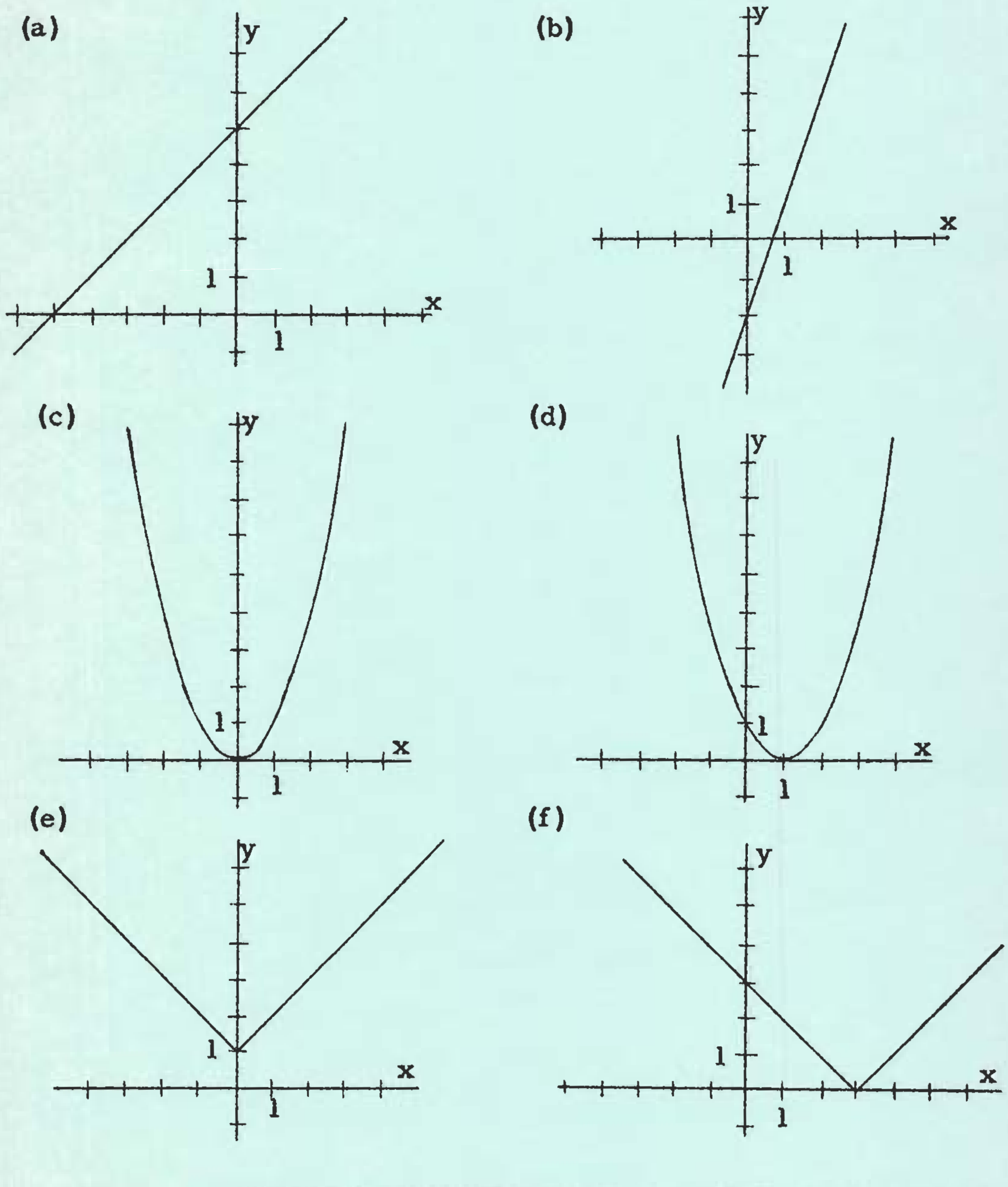
(c)


-3 -2 -1 0 1 2 3
𝒟
ℛ
-3 -2 -1 0 1 2 3

(d)


0 1
𝒟
ℛ
0 1

6.



[On comparing parts (c) and (d) of Exercise 6, and parts (e) and (f), students may discover if, for some function f and some number k, a function g is defined by: $g(x) = f(x - k)$, then the graph of g is the graph of f shifted k units to the right; and that if a function h is defined by: $h(x) = f(x) + k$, then the graph of h is the graph of f shifted k units up.]

5. Write a brace-notation name for the mapping g which takes each real number x to 3x + 7.

6. The mapping described by '$x \to 2x + 1$' is the mapping which takes each real number x to the real number 2x + 1. In other words, it is the function $\{(x, y): y = 2x + 1\}$. Draw a graph of each mapping described below.

(a) $x \to x + 5$ (b) $x \to 3x - 2$

(c) $x \to x^2$ (d) $x \to (x - 1)^2$

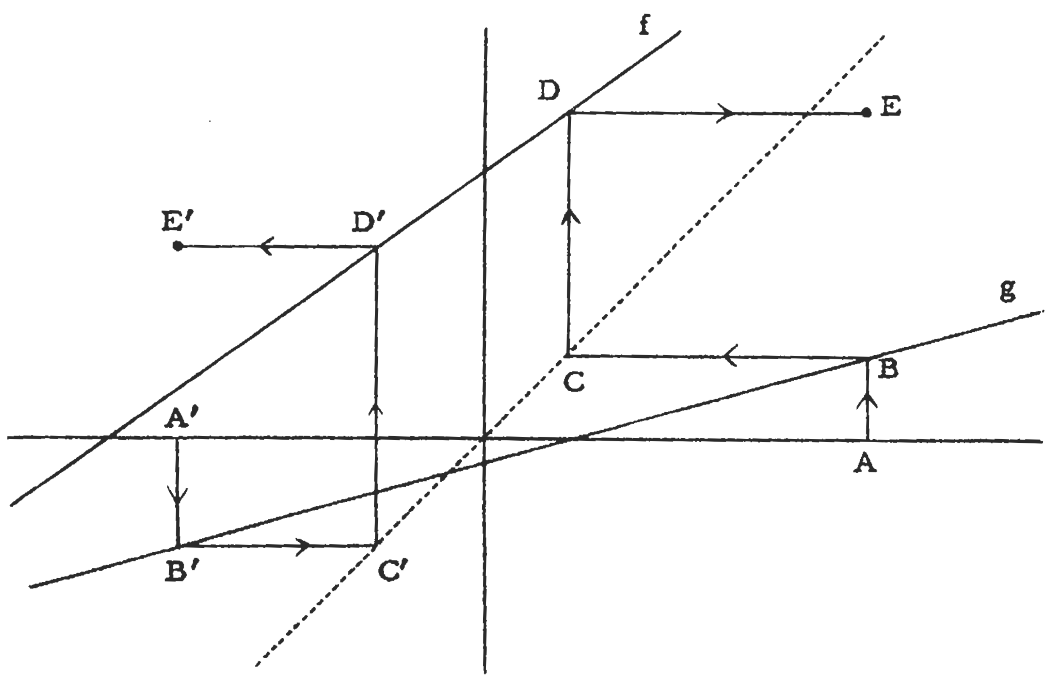
(e) $x \to |x| + 1$ (f) $x \to |x - 3|$

7. Use a diagram like those on page 5-64 to picture each mapping described below.

(a) $x \to \frac{1}{2}x$ (b) $x \to |x|$

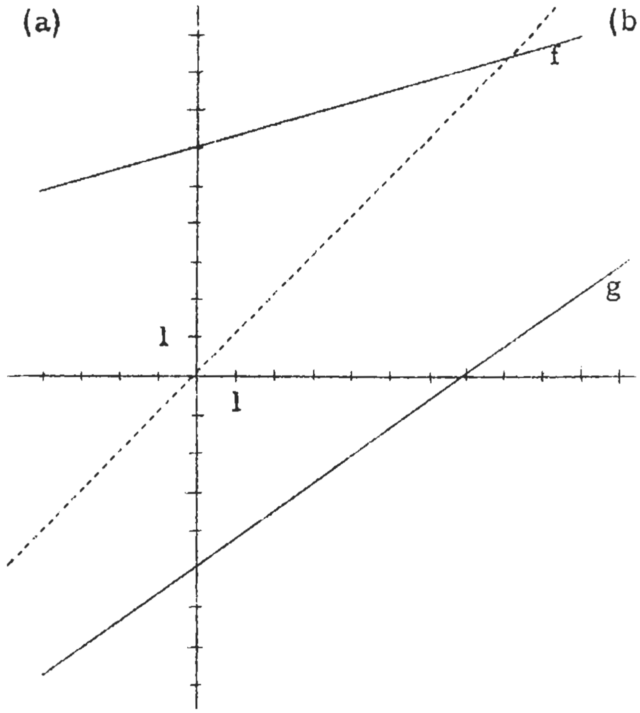
(c) $x \to 2x - 3$ (d) $x \to 0$

N. 1. The diagram below shows how to find points in the graph of $f \circ g$ when you have the graphs of f and g. Figure out how it works. [The dashed line is the graph of $\{(x, y): y = x\}$. The points E and E′ are two points on the graph of $f \circ g$. Find some more, and complete the picture.]

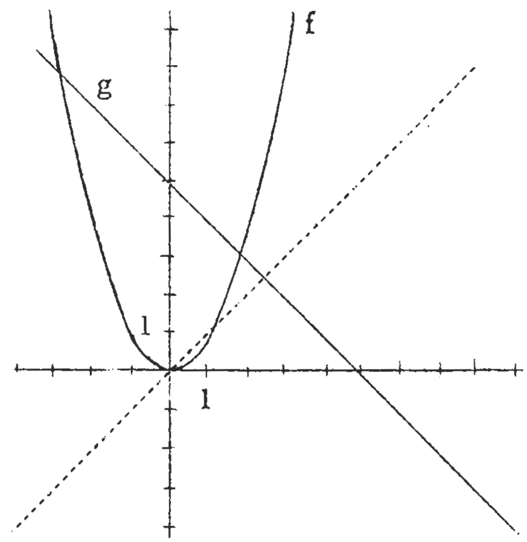


In each of the following exercises, use the graphical method to draw the graph of $f \circ g$.

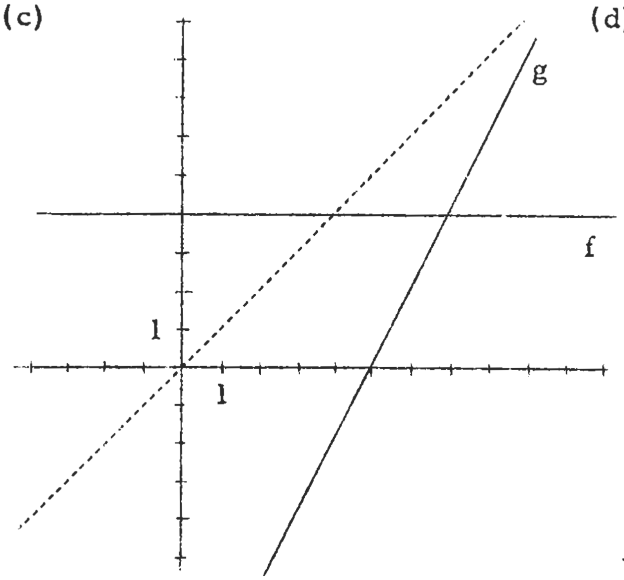
(a)



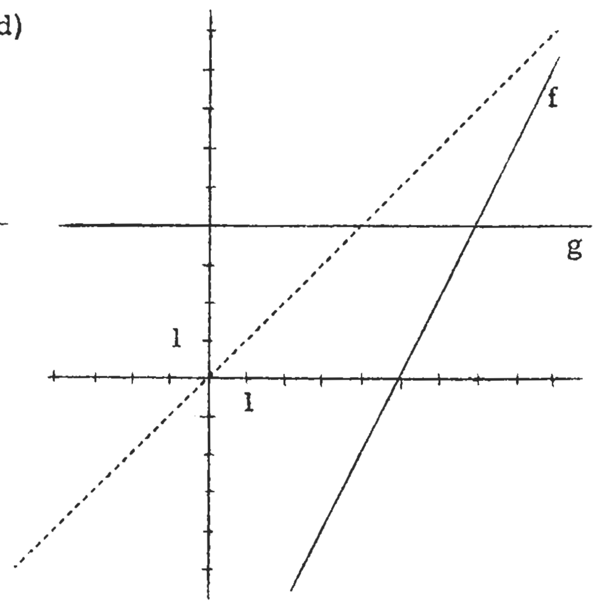
(b)



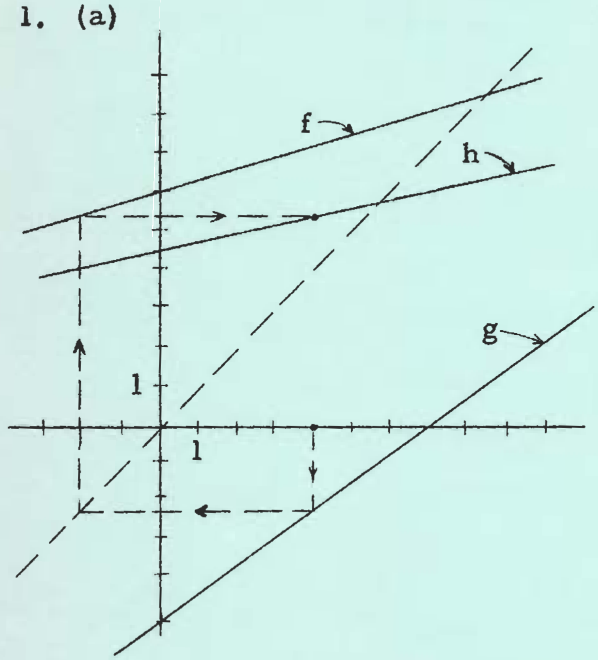
(c)



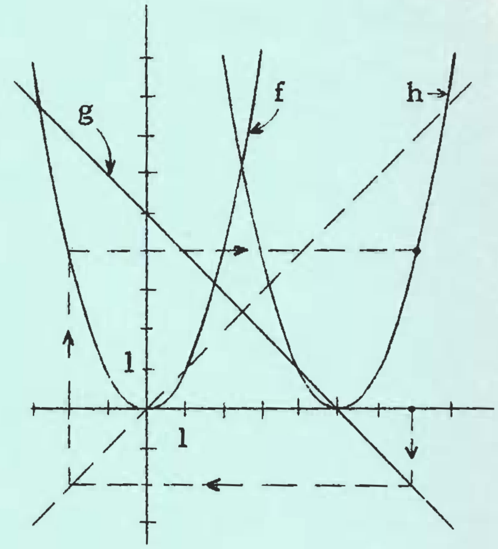
(d)



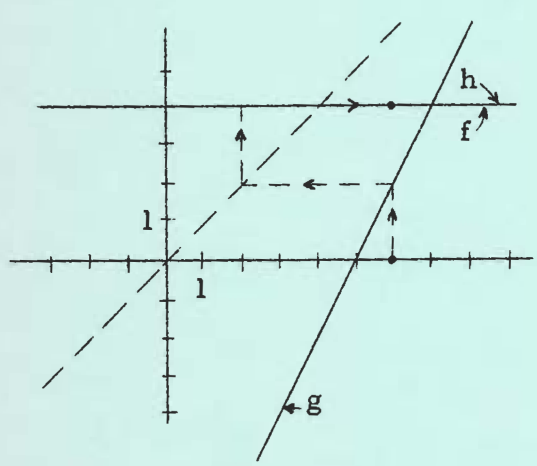
N. 1. (a)



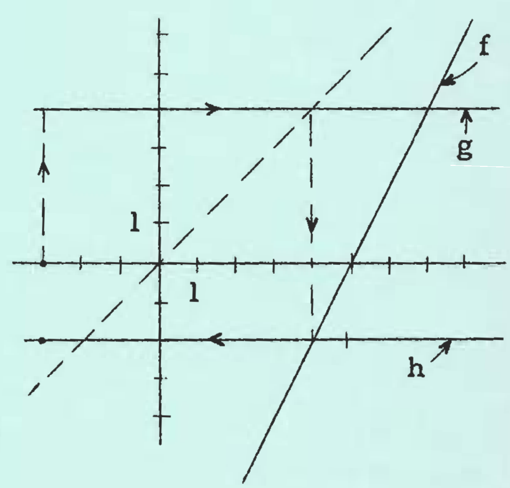
(b)



(c)

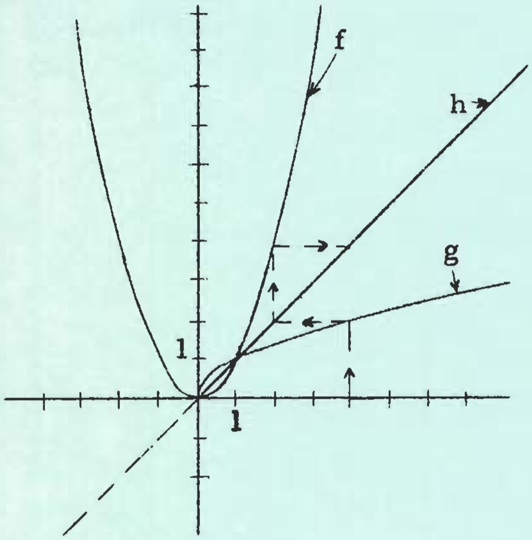


(d)

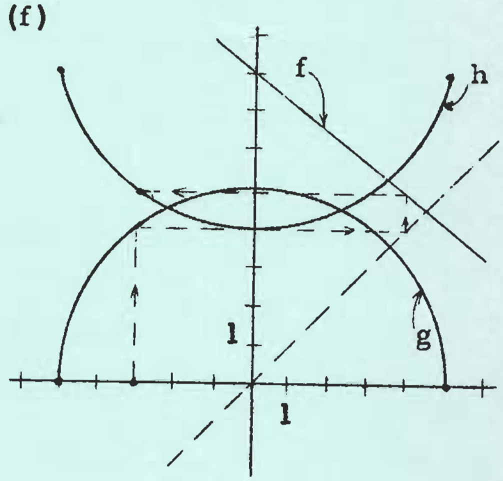


> **Correction.** In the diagram for (e), mark the "vertical parabola" with an 'f' and the "horizontal" one with a 'g'.

(e)



(f)



2. $x \to 6x + 14$

(a) $x \to 7x - 4$ (b) $x \to 4x - 9$ (c) $x \to 3 - 5x$

(d) $x \to x$ (e) $x \to (8x - 4)^2$

(f) $x \to 3(2x - 5)^2 - 2(2x - 5) + 5$ [or: $x \to 12x^2 - 64x + 90$]

<u>O.</u> 1. (a) $\{(x, y) : y = x + 2\}$ (b) $\{(x, y) : y = x7\}$

(c) $\{(x, y) : y = xx\}$ (d) $\{(x, y) : x = y + 2\}$

(e) $\{(x, y) : x = y7\}$

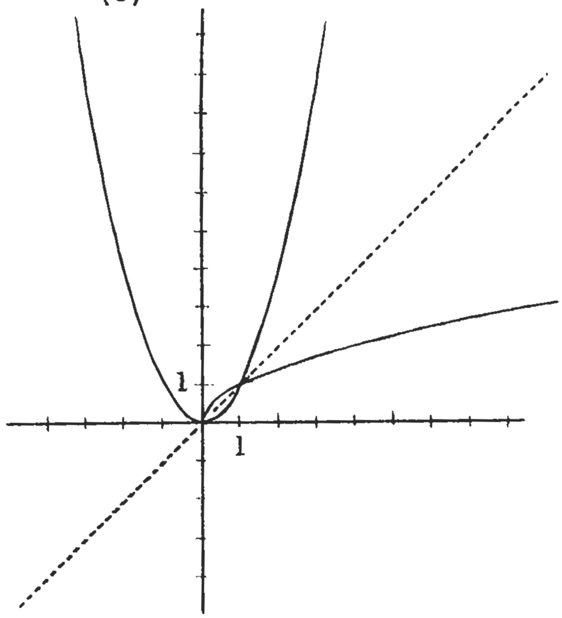
(f) $\{(x, y), y \in N: y = |x|\}$, or: $\{(x, y), y \in N: x = {}^{+}y \text{ or } x = {}^{-}y\}$

(g) $\{(x, y) : x + y = 0\}$ (h) $\{(x, y) : y = x\}$

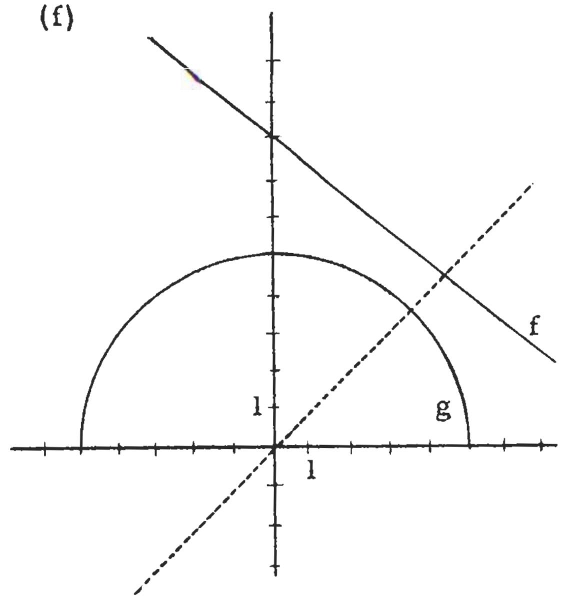
(i) $\{(x, y) : y + x = 0\}$ (j) $\{(x, y) : y = x2\}$

(k) $\{(x, y) : x = y3)$

(e)



(f)



2. Suppose g is the function described by '$x \to 2x + 3$' and f is the function described by '$x \to 3x + 5$'. Describe $f \circ g$.

(a) g: $x \to 7x + 1$; f: $x \to x - 5$

(b) g: $x \to 2x - 3$; f: $x \to 2x - 3$

(c) g: $x \to 3 - 5x$; f: $x \to x$

(d) g: $x \to \frac{1}{2}x - 3$; f: $x \to 2x + 6$

(e) g: $x \to 8x - 4$; f: $x \to x^2$

(f) g: $x \to 2x - 5$; f: $x \to 3x^2 - 2x + 5$

O. 1. Write a brace-notation name for each function listed below.

(a) adding 2 [Answer. $\{(x, y): y = x + 2\}$]

(b) multiplying by 7 (c) squaring

(d) the inverse of adding 2

(e) the inverse of multiplying by 7

(f) absolute valuing (g) oppositing

(h) sameing (i) the inverse of oppositing

(j) doubling (k) the inverse of tripling

2. Tell which of these functions have inverses, and describe the inverse.

(a) $A = \{(4, 7), (8, 9), (12, 11), (16, 13)\}$

(b) $B = \{(5, 6), (6, 7), (8, 9), (10, 5), (11, 6)\}$

(c) $C(x) = x - 1$, $\mathcal{D}_C$ = the set of real numbers

(d) $D(x) = 3x + 5$, $\mathcal{D}_D$ = the set of real numbers

(e) $f(x) = 4x - 5$, $\mathcal{D}_f$ = the set of real numbers

(f) $g(x) = 2x^2$, $\mathcal{D}_g$ = the set of real numbers

(g) $F = \{(x, y): y = 3x - 3\}$

(h) $G = \{(x, y): y = 4 - 9x\}$

(i) $H = \{(x, y): x + 2y = 8\}$

(j) $K = \{(x, y): 3x - 5y - 7 = 0\}$

(k) $M = \{(x, y): 2(y - 7) + 3(x - 5) = 2(x - 2) + x - 1\}$

3. $f = \{(x, y): y = \frac{3x - 2}{5}\}$ and $g = \{(x, y): y = \frac{2x - 4}{7}\}$

(a) $f(4) =$ ________ (b) $g(9) =$ ________

(c) $f^{-1}(5) =$ ________ (d) $g^{-1}(4) =$ ________

(e) $f^{-1}(0) =$ ________ (f) $g^{-1}(0) =$ ________

(g) $f^{-1}(f(14)) =$ ________ (h) $f(f^{-1}(14)) =$ ________

(i) $f^{-1}(3a) =$ ________ (j) $g^{-1}(2b) =$ ________

(k) Solve these equations.

(i) $f(a) = a$ (ii) $g(a) = a$

(iii) $f^{-1}(a) = a$ (iv) $g^{-1}(a) = a$

(v) $f(a) = g(a)$ (vi) $f^{-1}(a) = g^{-1}(a)$

(vii) $f(g(a)) = a$ (viii) $g(f(a)) = a$

(ix) $f^{-1}(g^{-1}(a)) = a$ (x) $g^{-1}(f^{-1}(a)) = a$

2. The functions named in parts (a), (c), (d), (e), (g), (h), (i), and (j) have inverses; that is, functions A, C, D, f, F, G, H, and K have inverses. [Functions B, g, and M do not have inverses because B(5) = B(11), g(1) = g(−1), and M(0) = 12 = M(1).]

$A^{-1} = \{(7, 4), (9, 8), (11, 12), (13, 16)\}$

$C^{-1}(x) = x + 1$, $\mathcal{D}_{C^{-1}}$ = the set of real numbers

$D^{-1}(x) = \frac{x - 5}{3}$, $\mathcal{D}_{D^{-1}}$ = the set of real numbers

$f^{-1}(x) = \frac{x + 5}{4}$, $\mathcal{D}_{f^{-1}}$ = the set of real numbers

$F^{-1} = \{(x, y):\ y = \frac{x + 3}{3}\}$ $\qquad$ $G^{-1} = \{(x, y):\ y = \frac{4 - x}{9}\}$

$H^{-1} = \{(x, y):\ 2x + y = 8\}$ $\qquad$ $K^{-1} = \{(x, y):\ 5x - 3y + 7 = 0\}$

3. (a) 2 (b) 2 (c) 9 (d) 16 (e) $\frac{2}{3}$

(f) 2 (g) 14 (h) 14 (i) $\frac{15a + 2}{3}$ (j) $7b + 2$

(k) (i) -1 (ii) $-\frac{4}{5}$ (iii) -1 (iv) $-\frac{4}{5}$ (v) $-\frac{6}{11}$

(vi) $-\frac{8}{11}$ (vii) $-\frac{26}{29}$ (viii) $-\frac{24}{29}$ (ix) $-\frac{24}{29}$ (x) $-\frac{26}{29}$

[Note that it is not a coincidence that, in part (k), (i) and (iii) have the same answer, as do (ii) and (iv), (vii) and (x), and (viii) and (ix). In each case, the two equations are equivalent by virtue of the fact that, for each function h which has an inverse,

$$\forall_{x \in \mathcal{D}_h}\ h^{-1}(h(x)) = x$$

and

$$\forall_{x \in \mathcal{R}_h}\ h(h^{-1}(x)) = x.$$

So, for example, if $a \in \mathcal{D}_f$ then $f^{-1}(f(a)) = a$; and if, in particular, $f(a) = a$ then $f^{-1}(f(a)) = f^{-1}(a)$. Hence, if $f(a) = a$ then $f^{-1}(a) = a$. Similarly, if $f^{-1}(a) = a$ then $f(a) = a$. Consequently, (i) and (iii) are equivalent.]

4. (b) $x \to \frac{x - 10}{5}$ (c) $x \to \frac{x + 12}{2}$ (d) [no inverse]

(e) $x \to \frac{x + 5}{7}$ (f) [no inverse] (g) $x \to \frac{x - 4}{3}$

5. (a) 3 (b) $-\frac{5}{2}$ (c) 12 (d) [none]

(e) $\frac{5}{6}$ (f) $-\frac{1}{2}$ (g) -2

P. 1. given set = $\{(2, 1), (3, 3), (4, 5)\}$;
image = $\{(5, -4), (6, -2), (7, 0)\}$

2. given set = $\overline{(2, 1), (4, 5)}$; image = $\overline{(5, -4), (7, 0)}$

3. given set = $\{(4, 4), (5, 3), (6, 2), (7, 1)\}$;
image = $\{(4, -4), (5, -3), (6, -2), (7, -1)\}$

4. given set = $\overline{(0, 1), (3, 7)}$; image = $\overline{(0, -1), (3, -7)}$

5. given set = $\{(x, y) : y = 2x + 1\}$; image = $\{(x, y) : y = -2x - 1\}$

6. given set = $\{(x, y) : y = x\}$; image = $\{(x, y) : y = -x\}$

7. given set = $\overline{(3, 0), (3, 4)}$; image = $\overline{(0, 3), (-4, 3)}$

8. given set = $\overline{(0, 3), (3, 3)} \cup \overline{(3, 3), (3, 0)}$;
image = $\overline{(-3, 0), (-3, 3)} \cup \overline{(-3, 3), (0, 3)}$

9. given set = $\overline{(-3, 0), (-3, 3)} \cup \overline{(-3, 3), (0, 3)}$
image = $\overline{(0, 3), (-3, -3)} \cup \overline{(-3, -3), (-3, 0)}$

10. given set = image 11. [on page 5-261] S

4. For each of the mappings described below, describe its inverse if it has one.

   (a) $x \to 3x - 6$ [Answer. $x \to \frac{1}{3}x + 2$]

   (b) $x \to 5x + 10$ (c) $x \to 2x - 12$ (d) $x \to x^2 + 1$

   (e) $x \to 7x - 5$ (f) $x \to |x| - 1$ (g) $x \to 4 + 3x$

5. For each of the mappings listed in Exercise 4, find an argument [if there is one] which is its own image. [Sample. (a) The mapping described by '$x \to 3x - 6$' takes 3 to $3 \cdot 3 - 6$, or 3. So, the argument 3 is its own image.]

P. You may recall the number plane games you worked in Unit 4. You were given a rule, for example:

$$(x, y) \to (x + 1, y - 2)$$

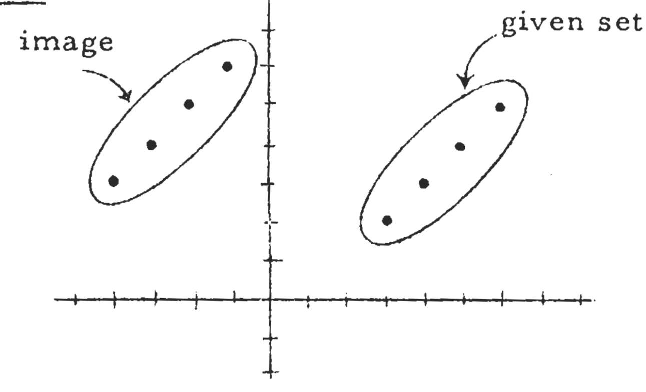
and told to start with a certain point, say, (5, 3), and apply the rule one or more times. Applying the rule once amounts to "jumping" from (5, 3) to (6, 1). Applying it again amounts to jumping from (6, 1) to (7, 4). Etc.

The rule for a number plane game actually defines a function which maps points of the plane onto points of the plane. That is, the domain and range of such a function are subsets of the number plane.

In each of the following exercises, you are given a number-plane-game function and a set in the number plane. Your job is to graph the given set, apply the function to each point in the set, and graph the resulting image.

Sample. Function: $(x, y) \to (x - 7, y + 1)$
Set: $\{(3, 2), (4, 3), (5, 4), (6, 5)\}$

Solution.



1. Function: $(x, y) \to (x + 3, y - 5)$
   Set: $\{(2, 1), (3, 3), (4, 5)\}$

2. Function: $(x, y) \to (x + 3, y - 5)$
   Set: $\{(x, y): y = 2x - 3 \text{ and } 2 \leq x \leq 4\}$

3. Function: $(x, y) \to (x, -y)$
   Set: $\{(4, 4), (5, 3), (6, 2), (7, 1)\}$

4. Function: $(x, y) \to (x, -y)$
   Set: $\{(x, y): y = 2x + 1 \text{ and } 0 \leq x \leq 3\}$

5. Function: $(x, y) \to (x, -y)$
   Set: $\{(x, y): y = 2x + 1\}$

6. Function: $(x, y) \to (-x, y)$
   Set: $\{(x, y): y = x\}$

7. Function: $(x, y) \to (-y, x)$
   Set: $\{(x, y): x = 3 \text{ and } 0 < y < 4\}$

8. Function: $(x, y) \to (-y, x)$
   Set: the union of the segments $\overline{(0, 3)(3, 3)}$ and $\overline{(3, 3)(3, 0)}$

9. Function: $(x, y) \to (-y, x)$
   Set: $\overline{(-3, 0)(-3, 3)} \cup \overline{(-3, 3)(0, 3)}$

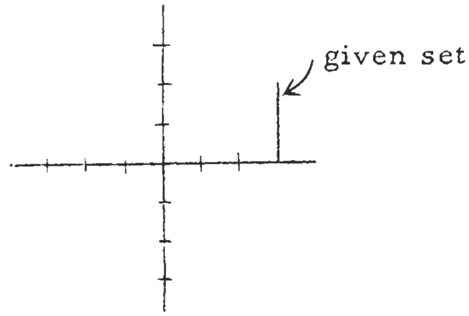
10. Function: $(x, y) \to (-y, x)$
    Set: the square whose vertices are $(3, 3)$, $(-3, 3)$, $(-3, -3)$, and $(3, -3)$

*

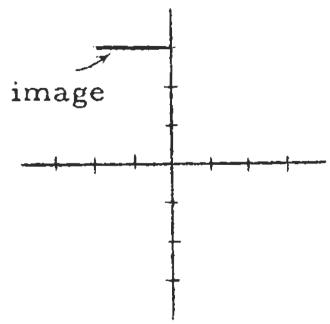
In the preceding exercises you learned something about functions which map points of the number plane onto points of the number plane. You probably discovered easy graphical methods for finding the result of applying a function like the one given by the rule:

$$f: (x, y) \to (-y, x)$$

to a set like the one pictured below.



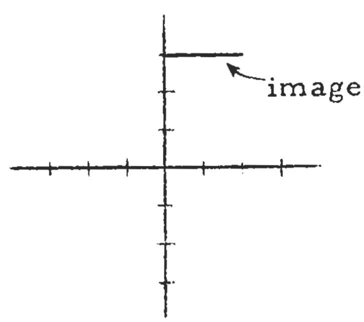
You would get the graph of the image by rotating this picture counter-clockwise a quarter-turn.



If you apply the function given by the rule:

$$g: (x, y) \to (y, x)$$

to the set shown in the first picture, the image is obtained by reflecting the given set in the line $\{(x, y): y = x\}$.



*

11. What image do you obtain if you apply the function f to a square S whose center is the origin and whose sides are parallel to the axes?

12. What is the image of S under g?

13. What is the image of S under $f \circ g$?

14. What is the image of S under $g \circ f$?

15. Suppose R is a rectangle whose vertices are (3, 2), (−3, 2), (−3, −2), and (3, −2). What is the image of R under f?

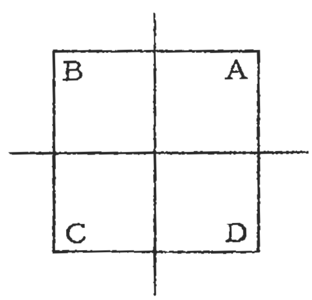
16. Suppose R is the rectangle described in Exercise 15, and T is its image under f. What is the image of R under

(a) g (b) $f \circ f$ (c) $f \circ g$ (d) $g \circ f$ (e) $g \circ g$

17. In view of your answers to (c) and (d) of Exercise 16, should you conclude that $f \circ g = g \circ f$? [Can you find a point whose image under $f \circ g$ is different from its image under $g \circ f$?]

*

In the preceding exercises you have noticed that there are different functions which map a square onto itself. For example, if ABCD is the square S of Exercise 11, then the function $f[(x, y) \to (-y, x)]$ maps



A on B, B on C, C on D, and D on A. On the other hand, the function $g[(x, y) \to (y, x)]$ maps A on A, B on D, C on C, and D on B. There are other such mappings of S on itself, but not very many.

One way of discovering them is to cut out a cardboard square, and label the corners 'A', 'B', 'C', and 'D' on both sides of the cardboard. Put the cardboard square on a sheet of paper, trace around it, and copy the labels written on the corners onto the paper. Lift up the cardboard and see how many ways you can place it on the square in the drawing.

12. S 13. S 14. S

[Of course, the answers for Exercises 13 and 14 are clear once one has answered Exercises 11 and 12.]

15. the rectangle whose vertices are (-2, 3), (-2, -3), (2, -3), and (2, 3)

16. (a) T (b) R (c) R (d) R (e) R

17. No. [Each point other than the origin has different images under $f \circ g$ and $g \circ f$. In fact, $f \circ g: (x, y) \rightarrow (-x, y)$, and $g \circ f: (x, y) \rightarrow (x, -y)$.]

*

To reason out how many ways the cardboard square can be placed on the drawing, note that the corner A can be placed in any one of four positions, after which the corner B, since it is adjacent to A, must be placed at one of the two corners of the drawing adjacent to that chosen for A. Now D must be placed at the other of these two corners, and C at the one remaining corner. So, there are just $4 \cdot 2$ ways of placing this square on the drawing.

*

18. The five remaining rows in the table will be: BADC, CBAD, CDAB, DCBA, and DABC. [These correspond, respectively, to f_8, f_7, f_3, f_6, and f_4 in the list at the foot of page 5-263.]

*

Answers to questions at the bottom of page 5-263.

f_4 is a counter-clockwise rotation through three quarter-turns; so, $f_4 = f \circ f \circ f$.

$f_8 = g \circ f_4 = g \circ f \circ f \circ f$ [Note that, since composition of functions is associative, brackets are unnecessary.]

[Don't forget that one way is to put it right back the way it was. This way is the mapping which takes A onto A, B onto B, C onto C, and D onto D.]

You may find it more interesting to find the total number of ways by doing the following. Ask yourself how many choices there are for the image of A when S is mapped onto itself. Then, for each of these choices, how many choices are there for the image of B. Having chosen images for A and B, how many choices of image are there for C?

One way to keep track of these mappings as you discover them is to fill out a row in the following table for each mapping of the square on itself.

	A	B	C	D
(1)	B	C	D	A
(2)	A	D	C	B
(3)	A	B	C	D
		⋮		

The sample rows filled out in the table correspond with the functions f, g, and the function which leaves each point where it is.

18. Complete this table before reading further.

*

Here is a list of eight number plane mappings which will map a square like S onto itself.

f_1: $(x, y) \to (x, y)$ f_5: $(x, y) \to (y, x)$

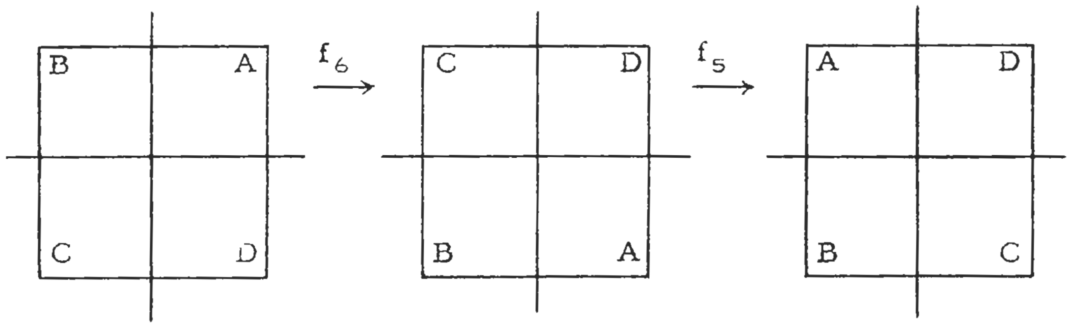
f_2: $(x, y) \to (-y, x)$ f_6: $(x, y) \to (x, -y)$

f_3: $(x, y) \to (-x, -y)$ f_7: $(x, y) \to (-y, -x)$

f_4: $(x, y) \to (y, -x)$ f_8: $(x, y) \to (-x, y)$

Notice that f_1 is the mapping that doesn't "move" anything, that f_2 and f_5 are the mappings f and g referred to earlier, that $f_3 = f \circ f$, that $f_6 = g \circ f$, and that $f_7 = g \circ f \circ f$. What is f_4? f_8?

Suppose we compose one of these mappings with another, say, f_5 with f_6. This gives us the mapping $f_5 \circ f_6$. Let's see it in action.



So, $f_5 \circ f_6$ is the mapping that rotates S counterclockwise a quarter-turn. In other words, $f_5 \circ f_6 = f_2$.

19. Complete the following composition table. [It may help to notice that $f \circ g = g \circ f \circ f \circ f$.]

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
f_1								
f_2								
f_3								
f_4								
f_5						f_2		
f_6								
f_7								
f_8								

20. Which of the eight mappings are rotations? Which are reflections?

21. Which of the eight mappings is f_2^{-1}? f_6^{-1}? $(f_2 \circ f_6)^{-1}$? $f_6^{-1} \circ f_2^{-1}$?

Correction. On page 5-264, in Exercise 21, change '$(f_2 \circ f_6)^{-1}$' to '$[f_2 \circ f_6]^{-1}$'.

19.

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
f_1	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
f_2	f_2	f_3	f_4	f_1	f_8	f_5	f_6	f_7
f_3	f_3	f_4	f_1	f_2	f_7	f_8	f_5	f_6
f_4	f_4	f_1	f_2	f_3	f_6	f_7	f_8	f_5
f_5	f_5	f_6	f_7	f_8	f_1	f_2	f_3	f_4
f_6	f_6	f_7	f_8	f_5	f_4	f_1	f_2	f_3
f_7	f_7	f_8	f_5	f_6	f_3	f_4	f_1	f_2
f_8	f_8	f_5	f_6	f_7	f_2	f_3	f_4	f_1

$(x, y) \to (x, y)$	identity
$(x, y) \to (-y, x)$	f
$(x, y) \to (-x, -y)$	$f \circ f$
$(x, y) \to (y, -x)$	$f \circ f \circ f$
$(x, y) \to (y, x)$	g
$(x, y) \to (x, -y)$	$g \circ f$
$(x, y) \to (-y, -x)$	$g \circ f \circ f$
$(x, y) \to (-x, y)$	$g \circ f \circ f \circ f$

The work of completing the composition table can be carried out in several ways. For example, one may fill out the f_4-row by noticing that f_4 first replaces the first component of a point by its opposite and then interchanges the components. Carrying out this operation on the results of applying $f_1, f_2, \ldots, f_8$ to (x, y) shows that

$$[f_4 \circ f_1]((x, y)) = f_4((x, y)) = (y, -x),$$

$$[f_4 \circ f_2]((x, y)) = f_4((-y, x)) = (x, y),$$

$$[f_4 \circ f_3]((x, y)) = f_4((-x, -y)) = (-y, x),$$

$$[f_4 \circ f_4]((x, y)) = f_4((y, -x)) = (-x, -y),$$

$$[f_4 \circ f_5]((x, y)) = f_4((y, x)) = (x, -y),$$

$$[f_4 \circ f_6]((x, y)) = f_4((x, -y)) = (-y, -x),$$

$$[f_4 \circ f_7]((x, y)) = f_4((-y, -x)) = (-x, y),$$

$$[f_4 \circ f_8]((x, y)) = f_4((-x, y)) = (y, x).$$

So, $f_4 \circ f_1 = f_4$, $f_4 \circ f_2 = f_8$, $f_4 \circ f_3 = f_2$, etc. The other rows can be filled out in a similar manner.

Here is a more interesting way to do it. As noted at the bottom of page 5-263, $f_2 = f$, $f_3 = f \circ f$, $f_4 = f \circ f \circ f$, $f_5 = g$, $f_6 = g \circ f$, $f_7 = g \circ f \circ f$, and $f_8 = g \circ f \circ f \circ f$. If we denote f_1 by 'i' [for 'identity mapping'] then $f \circ f \circ f \circ f = i = g \circ g$. Also, $f \circ i = f = i \circ f$, and $g \circ i = g = i \circ g$. Now,

$$f_4 \circ f_1 = [f \circ f \circ f] \circ i = [f \circ f] \circ [f \circ i] = f \circ f \circ f = f_4,$$

$$f_4 \circ f_2 = [f \circ f \circ f] \circ f = i = f_1,$$

$$f_4 \circ f_3 = [f \circ f \circ f] \circ [f \circ f] = f \circ [f \circ f \circ f \circ f] = f \circ i = f = f_2,$$

$$f_4 \circ f_4 = f_4 \circ [f \circ f \circ f] = f_4 \circ [f_3 \circ f] = [f_4 \circ f_3] \circ f = f_2 \circ f = f_3,$$

$$f_4 \circ f_5 = [f \circ f \circ f] \circ g = \ ?$$

To answer this last question, note that $[f \circ g]((x, y)) = f((y, x)) = (-x, y) = f_8((x, y))$. So, $f \circ g = f_8 = g \circ f \circ f \circ f = g \circ f_4$. So,

$$f_4 \circ f_5 = f \circ f \circ f \circ g = f \circ f \circ [f \circ g] = f \circ f \circ [g \circ f_4] = f \circ [f \circ g] \circ f_4 = f \circ [g \circ f_4] \circ f_4$$

$$= [f \circ g] \circ [f_4 \circ f_4] = [g \circ f_4] \circ [f_4 \circ f_4] = [g \circ f_4] \circ f_3 = g \circ [f_4 \circ f_3]$$

$$= g \circ f_2 = g \circ f = f_6,$$

$$f_4 \circ f_6 = f_4 \circ [f_5 \circ f] = [f_4 \circ f_5] \circ f = f_6 \circ f = f_7,$$

$$f_4 \circ f_7 = f_4 \circ [f_6 \circ f] = [f_4 \circ f_6] \circ f = f_7 \circ f = f_8,$$

$$f_4 \circ f_8 = f_4 \circ [f_7 \circ f] = [f_4 \circ f_7] \circ f = f_8 \circ f = g \circ f \circ f \circ f \circ f = g \circ i = g = f_5.$$

20. f_2, f_3, and f_4 are rotations; f_5, f_6, f_7, and f_8 are reflections.

21. $f_2^{-1} = f_4$; $f_6^{-1} = f_6$; $[f_2^{-1} \circ f_6]^{-1} = f_5^{-1} = f_5$; $f_6^{-1} \circ f_2^{-1} = f_6 \circ f_4 = f_5$

*

The composition table of Exercise 19 gives a good deal of information about the behavior of the functions f_1 through f_8 with respect to the operation of function composition. To see better what information is contained in the table, let's forget how we constructed it and suppose merely that someone has told us that there is a set G of eight things $\{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$ and an operation $\circ$ on them, and that the table defines this operation. That is, if s and t are any members of G then $s \circ t$ is the member of G listed in the s-row and t-column of the table.

One thing that we could learn from the table is that $\circ$ is associative. We could learn this by checking each of 8^3 instances of the associative principle. [For example, according to the table, $[f_4 \circ f_3] \circ f_5 = f_2 \circ f_5 = f_8$, and $f_4 \circ [f_3 \circ f_5] = f_4 \circ f_7 = f_8$. So, $[f_4 \circ f_3] \circ f_5 = f_4 \circ [f_3 \circ f_5]$.]

Another thing we might notice is that each member of G is listed in each row of the table. What this means is that if s and u are members of G then there is a $t \in G$ such that $s \circ t = u$. For, given s and u, we can find u listed somewhere in the s-row. So, we can solve equations like '$f_5 \circ x = f_4$'. [According to the table, the root of this equation is f_8.] Similarly, since each of the eight things is listed in each column, we can solve equations like '$x \circ f_5 = f_4$'. [According to the table, the root of this equation is f_6.]

So, the table tells us that $\circ$ is an associative operation on the members of G, and that equations of the forms '$s \circ x = u$' and '$x \circ s = u$' have roots. More briefly put, what the table tells us is that G is a <u>group</u> with respect to the operation $\circ$

You are already acquainted with many groups. For example, the integers form a group with respect to addition; so do the even integers, the rational numbers, and the real numbers. Also, the positive rational numbers form a group with respect to multiplication; so does the set of all nonzero rational numbers. So do the set of positive real numbers, and the set of all nonzero real numbers. Notice that each of these groups has an additional property which G does not have. The group operation $\circ$ is not commutative, while, as you know, addition and multiplication of real numbers are commutative. Groups for which the group operation is commutative are called <u>commutative groups</u>. Although G is not commutative it has several commutative <u>subgroups</u>--that is, subsets which are themselves groups with respect to the operation $\circ$. One very obvious one is $\{f_1, f_2, f_3, f_4\}$. Another is $\{f_1, g\}$.

Among the most elementary properties of groups are (1) that each group has an <u>identity element</u>--that is, an element i such that, for each member g of the group, $g \circ i = g$, and (2) that each member g of the group has an <u>inverse</u>--that is, an element g^{-1} such that $g \circ g^{-1} = i$. In fact, these two properties together with associativity constitute another characterization of the group concept. For the eight-membered group G, i is f_1, and the inverses of the members of G can be found from the table. Notice that, even though G is not commutative, for each $g \in G$, $g^{-1} \circ g = g \circ g^{-1} = i$. Notice also that, for elements g and h of G, $[g \circ h]^{-1} = h^{-1} \circ g^{-1}$. Both of these properties hold of all groups. For the examples given above in which the group operation is addition of real numbers, i is 0 and the inverse of each element is its opposite. For those in which the group operation is multiplication of real numbers, i is 1 and the inverse of each element is its reciprocal.

The group concept is one of the major unifying concepts in mathematics, and there is an extensive literature on the subject. You may find interesting the article "To Teach Modern Algebra" by Carl H. Denbow in the March, 1959 issue of <u>The Mathematics Teacher</u>, and the book <u>The New Mathematics</u>, by Irving Adler, published by John Day.

Another thing we might notice is that each member of G is listed in each row of the table. What this means is that if a and b are members of G then there is a $c \in G$ such that $a \circ c = b$. Thus, given a and b, we can find a listed somewhere in the a-row, and we can solve equations like [illegible]. (According to the table, the root of this equation is [illegible].) Similarly, since each of the eight things is listed in each column, we can solve equations like [illegible]. (According to the table, the root of this equation is [illegible].)

So, the table tells us that ∘ is an operation on [illegible] the members of G, and that equations of the forms '$a \circ x = b$' and '$x \circ a = b$' have roots. Since this is so, what the table tells us is that G is a group with respect to the operation ∘.

You are already acquainted with many groups. For example, the integers form a group with respect to addition, and so do the even integers, the rational numbers, and the real numbers. Also, the positive rational numbers form a group with respect to multiplication. So does the set of all nonzero rational numbers. So do the set of positive real numbers, and the set of all nonzero real numbers. Notice that each of these groups has an additional property which G does not have. The group operation ∘ is not commutative, while, as you know, addition and multiplication of real numbers are commutative. Groups for which the group operation is commutative are called commutative groups. Although G is not commutative, it has subsets which are commutative subgroups—that is, subsets which are commutative groups with respect to the operation ∘. One very simple one is [illegible]. Another is [illegible].

Among the more elementary properties of groups are (1) that each group has an identity element—that is, an element i such that, for each member g of the group, $g \circ i = g$, and (2) that each member g of the group has an inverse—that is, an element g' such that $g \circ g' = i$. In fact, these two properties together with associativity are [illegible] characteristic [illegible] of the group concept. For the eight-membered group G, i is [illegible], and the inverses of the members of G can be found from the table. Notice that, even though G is not commutative, for each $g \in G$, $g \circ i = g = i \circ g$. Notice also that, for each $g \in G$, $g \circ g' = i = g' \circ g$. Both of these properties hold of all groups. For the examples given above in which the group operation is addition of real numbers, i is 0 and the inverse of each element is its opposite. For those in which the group operation is multiplication of real numbers, i is 1 and the inverse of each element is its reciprocal.

The group concept is one of the major unifying concepts in mathematics, and there is an extensive literature on the subject. You may find interesting the article [illegible] by Carl H. Denbow in the [illegible] issue of *The Mathematics Teacher* and the book *The Lore of Mathematics* [illegible], by Irving Adler, published by [illegible].

Q. 1. (a) Yes; $\{(x, y): y = x - 5\}$ (b) Yes; $\{(x, y): y = -x\}$

(c) Yes; $\{(x, y): y = -x\}$ [or: $\{(x, y); x \leq 2 : y = -x\}$]

(d) No [$\mathcal{D}_h \not\subseteq \mathcal{D}_g$]

(e) Yes; $\{(x, y): y = 3x - 8\}$ [or: $\{(x, y), x \geq \frac{3}{4}: y = 3x - 8\}$]

2. (a) Yes

(b) No [There are two arguments x_1 and x_2 of both g and h such that $g(x_1) = g(x_2)$ but $h(x_1) \neq h(x_2)$.]

(c) No [$\mathcal{D}_h \not\subseteq \mathcal{D}_g$] (d) Yes (e) Yes (f) Yes

R. 1. (a) (Don, 11), (Mike, 20), (Sally, 11)

(b) (Don, 30), (Mike, 99), (Sally, 28)

(c) (Don, 17), (Mike, 29), (Sally, 15)

(d) (Don, 16), (Mike, 31), (Sally, 18)

(e) (Don, 1), (Mike, −2), (Sally, −3)

(f) (Don, 11), (Mike, −40), (Sally, −33)

(g) (Don, 11), (Mike, −40), (Sally, −33)

2. (a) 28 (b) −32 (c) −28 (d) −2

(e) 88 (f) 217 (g) 52 (h) 217

(i) $3(7x + 3) - 5$ [or: $21x + 4$] (j) $(3x - 5)(7x + 3)$

(k) $(3x - 5)^2$ (ℓ) $(7x + 3)^2$

(m) $58x^2 + 12x + 34$ (n) $-40x^2 - 72x + 16$

3. (a) Yes (b) Yes (c) Yes

(d) No [$B(1) \neq f(A(1))$] (e) Yes

(f) No [A(0) is not in the domain of m, so 0 is not in the domain of the function $m \circ A$.]

Q. 1. For each exercise, tell whether h is a function of g, and if it is, give a function f such that $h = f \circ g$.

(a) $g = \{(x, y): y = 7x + 1\}$ and $h = \{(x, y): y = 7x - 4\}$

(b) $g = \{(x, y): y = 2 - 3x\}$ and $h = \{(x, y): y = 3x - 2\}$

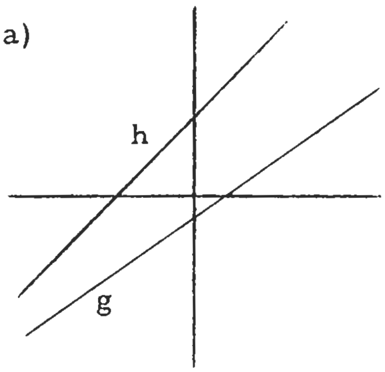
(c) $g = \{(x, y), x \geq 0: y = 2 - 3x\}$ and $h = \{(x, y), x \geq 0: y = 3x - 2\}$

(d) $g = \{(x, y), x < 0: y = 2 - 3x\}$ and $h = \{(x, y), x \geq 0: y = 3x - 2\}$

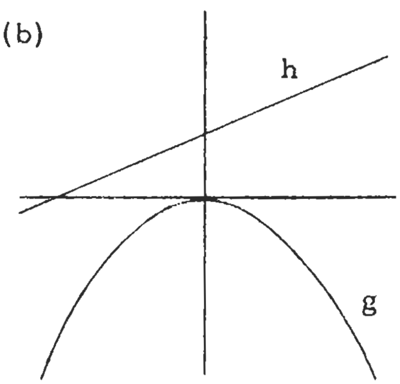
(e) $g = \{(x, y): y = x^2 - x + 1\}$ and $h = \{(x, y): y = 3x^2 - 3x - 5\}$

2. For each exercise, tell whether h is a function of g.

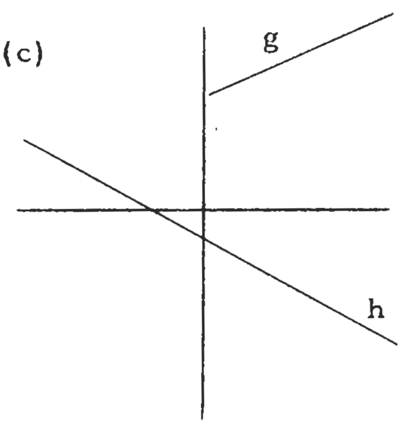
(a)



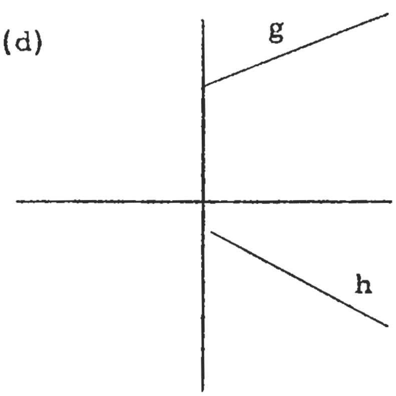
(b)



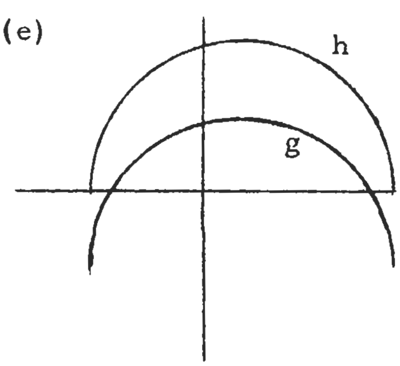
(c)



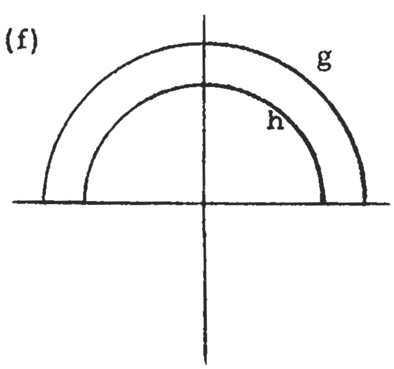
(d)



(e)



(f)



R. 1. Consider the variable quantities M and N where

M = {(Don, 6), (Mike, 9), (Sally, 4)}

and N = {(Don, 5), (Mike, 11), (Sally, 7)}

(a) $M + N$ = {________________________}

(b) MN = {________________________}

(c) $2M + N$ = {________________________}

(d) $M + 2N$ = {________________________}

(e) $M - N$ = {________________________}

(f) $(M + N)(M - N)$ = {________________________}

(g) $M^2 - N^2$ = {________________________}

2. Suppose $f(x) = 3x - 5$ and $g(x) = 7x + 3$.

(a) $[f + g](3)$ = ________

(b) $[f + g](-3)$ = ________

(c) $[f - g](5)$ = ________

(d) $[f - g]$(____) = 0

(e) $[f \circ g](4)$ = ________

(f) $[fg](4)$ = ________

(g) $[g \circ f](4)$ = ________

(h) $[gf](4)$ = ________

(i) $[f \circ g](x)$ = ________

(j) $[fg](x)$ = ________

(k) $[f^2](x)$ = ________

(ℓ) $[g^2](x)$ = ________

(m) $[f^2 + g^2](x)$ = ________

(n) $[f^2 - g^2](x)$ = ________

3. Suppose A = {(0, 5), (1, 10), (2, 15)} and B = {(0, 8), (1, 13), (2, 18)}.

(a) Is B a function of A?

(b) If h = {(x, y): y = x + 3}, does $B = h \circ A$?

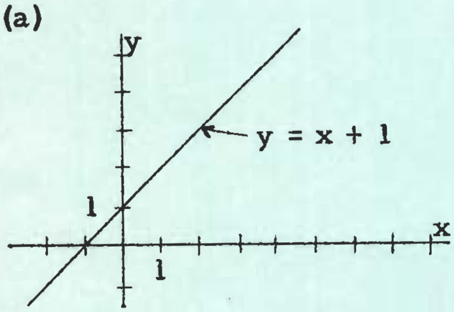
(c) If g = {(3, 4), (5, 8), (6, 7), (9, 11), (10, 13), (15, 18)}, does $B = g \circ A$?

(d) If f = {(5, 8), (6, 9), (10, 12), (12, 14), (15, 18)}, does $B = f \circ A$?

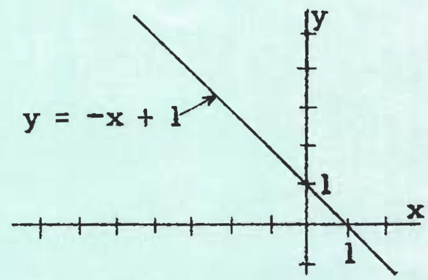
(e) If k = {(x, y), $x \geq 5$: y = x + 3}, does $B = k \circ A$?

(f) If m = {(x, y), $x > 5$: y = x + 3}, does $B = m \circ A$?

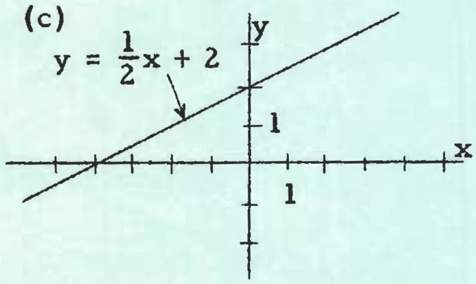
S. 1. (a)



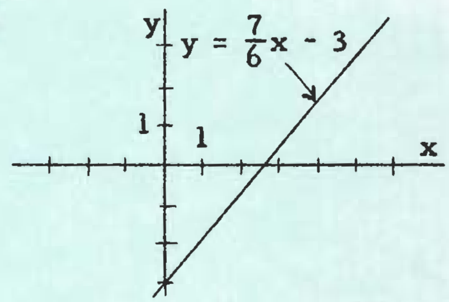
(b)



(c)



(d)



2. (a), (i), (ℓ), (t), (b), (g), (j), (p), (q),
(c), (f), (k), (o), (s), (d), (e), (h), (m), (n), (r)

3. (a) $\{(x, y): y = 7x - 26\}$ (b) $\{(x, y): 5x + 8y = 0\}$
(c) $\{(x, y): x + y = 5\}$ (d) $\{(x, y): 5y = 13x - 1\}$

4. (a) (1, 8) (b) (2, 10) (c) (5, 19) (d) (7, 1)
(e) (2, 6) (f) (3, 11) (g) (5, 20) (h) (4, 3)
(i) (6, 2) (j) $(\frac{3}{23}, \frac{67}{23})$

[In discussing the parts of Exercise 4 bring out that (a) can be most easily solved by first solving '$3x + 5 = 5x + 3$', and that (b), (c), and (d) can be treated similarly. Also, part (e) is easily solved by "adding and subtracting": $2y = 12$, $2x = 4$. (f) and (h) can be treated similarly.]

S. 1. Each exercise gives the slope and intercept of a linear function. Draw its graph and write its defining equation.

(a) intercept, 1; slope, 1
(b) intercept, 1; slope, −1
(c) intercept, 2; slope, $\frac{1}{2}$
(d) intercept, −3; slope, $\frac{7}{6}$

2. The solution set in (x, y) of each of the following equations is a linear function. Which of these functions have the same slope?

(a) $y = 2x + 5$
(b) $y + \frac{1}{2}x + 5 = 0$
(c) $5(y + 2x) = 1$
(d) $2y - x = 7$
(e) $2(y + 1) = x$
(f) $3y + 6x = 7$
(g) $2(y + x) + 2y = 2$
(h) $\frac{4y + 1}{2} = x$
(i) $7y - 14x + 2 = 0$
(j) $8y + 4x - 1 = 0$
(k) $22x + 11y = -3$
(ℓ) $18x - 9y = 2$
(m) $6y - 3x + 5 = 0$
(n) $3y = \frac{3}{2}x + 5$
(o) $2y = 1 - 4x$
(p) $6y + 3x - 5 = 0$
(q) $-5x - 10y + 2 = 0$
(r) $10y = -5(2 - x)$
(s) $x = \frac{y - 4}{-2}$
(t) $x = \frac{3y - 5}{6}$

3. Find the linear function [if there is one] which contains the two ordered pairs.

(a) (4, 2), (3, −5)
(b) (0, 0), (8, −5)
(c) (6, −1), (−1, 6)
(d) (2, 5), (−3, −8)

4. Find the ordered pair (x, y) [if there is one] which satisfies both equations.

(a) $y = 3x + 5$
$y = 5x + 3$

(b) $y = 6x - 2$
$y = x + 8$

(c) $y = 4x - 1$
$y = 6x - 11$

(d) $x = 3y + 4$
$x = 2y + 5$

(e) $y - x = 4$
$y + x = 8$

(f) $y - 2x = 5$
$y + 2x = 17$

(g) $y - 3x = 5$
$y + 2x = 30$

(h) $2y + x = 10$
$2y - x = 2$

(i) $2y + 5x = 34$
$2y - 4x = -20$

(j) $3y + 2x = 9$
$4y - 5x = 11$

5. Each exercise describes a linear function. Find the linear function which fits the description.

<u>Sample.</u> It contains $(5, -3)$, and its slope is -2.

<u>Solution.</u>
$$y = ax + b$$
$$-3 = a5 + b$$
$$-3 = -2 \cdot 5 + b$$
$$7 = b$$

So, the linear function is $\{(x, y): y = -2x + 7\}$.

(a) It contains $(2, 1)$, and its slope is 4.

(b) Its intersection with $\{(x, y): y = 5x - 3\}$ is $\emptyset$, and it contains $(2, 14)$.

(c) Its intersection with $\{(x, y): 6x - 2y + 5 = 0\}$ is $\emptyset$, and its intercept is 9.

(d) For each member of its domain, the corresponding value is twice the corresponding value of $\{(x, y): y = 7x - 1\}$.

(e) $\{(3, 7), (5, 8), (7, 9)\}$ is one of its subsets.

(f) Its intersection with the x-axis is $\{(8, 0)\}$, and its intersection with the y-axis is $\{(0, -2)\}$.

(g) It contains the midpoints of the intervals $\overline{(0, 3), (3, 0)}$ and $\overline{(0, 5), (5, 0)}$.

(h) Its graph is perpendicular to the graph of $\{(x, y): y = 2x\}$ and crosses it at the graph of the origin.

6. Suppose f and g are linear functions and that $f(x) = ax + b$ and $g(x) = cx + d$.

(a) $[f \circ g](x) =$ ________

(b) $[g \circ f](x) =$ ________

(c) $[fg](x) =$ ________

(d) $[gf](x) =$ ________

5. (a) $\{(x, y): y = 4x - 7\}$ (b) $\{(x, y): y = 5x + 4\}$

(c) $\{(x, y): y = 3x + 9\}$ (d) $\{(x, y): y = 2(7x - 1)\}$

(e) $\{(x, y): y = \frac{x}{2} + \frac{11}{2}\}$ (f) $\{(x, y): y = \frac{x}{4} - 2\}$

(g) $\{(x, y): y = x\}$ (h) $\{(x, y): y = -\frac{1}{2}x\}$

[Part (a) of Exercise 5 can be solved like the Sample. Here is another method: For any nonzero value of 'm' the equation '$y - 1 = m(x - 2)$' defines a linear function which contains (2, 1). The value of 'm' is the slope of the linear function. So, an answer for (a) is: $\{(x, y): y - 1 = 4(x - 2)\}$. The same method [or that of the Sample] can be used for part (b), once one sees that he is concerned with a linear function having slope 5. An alternative is to note that '$y - 5x = 14 - 5 \cdot 2$' defines a linear function of slope 5 which contains (2, 14). For part (c), one may find the slope [3] and arrive at once at '$y = 3x + 9$'. For part (h), students may graph '$y = 2x$' on squared paper, draw the line perpendicular [using a protractor, perhaps] to this through the graph of the origin, and notice that this line goes through the graph of, for example, (−2, 1).]

6. (a) $a(cx + d) + b$ (b) $c(ax + b) + d$

(c) $(ax + b)(cx + d)$ (d) $(cx + d)(ax + b)$

7. (a)

3	8
4	11
5	14
6	17
10	29
15	44

(b)

5	7
7	15
7.5	17
8	19
8.3	20.2
10	27

(c)

2	−3
8	−1
9	$-\frac{2}{3}$
13	$\frac{2}{3}$
−5	$-\frac{16}{3}$
−7	−6

> Correction. On page 5-271, in line 13, there should be a '?' after 'second'.

T. 1. 125 2. $\frac{5}{3}$ 3. $\frac{25}{3}$ 4. \$85 5. (c)

6. (a) $\frac{3}{4}$ (b) 20 (c) $\frac{7}{2}$ (d) 144

(e) 35 (f) 72 (g) 9 (h) 15

(i) 4 (j) 20 (k) 3 (ℓ) 7

7. 68¢ 8. $2\frac{7}{9}$ 9. $12\frac{1}{2}$ 10. 24.5 feet 11. (b)*

12. 8 13. 4 14. $\frac{10}{3}$ 15. 25

16. (a) $C = kr$ (b) $s = kP$ (c) $R = k\ell$ (d) $V = kT$

(e) $RA = k$ (f) $nc = k$ (g) $Id^2 = k$ (h) $f = \frac{k\sqrt{F}}{\ell}$

17. r is multiplied by 5 18. 4×402.5 [or: 1610]; 338.1

19. 25.2 20. $\frac{9}{4}$ feet 21. $\frac{pbc}{aq}$ ☆22. 14.5

☆23. $9\sqrt{2}$ centimeters

---

* If $M/N = 4$ and if, for each argument e of N such that $N(e) = 0$, $M(e) = 0$, then M varies directly as N. Compare with Exercises 5 and 6 of Part C on page 5-155.

7. Each table lists ordered pairs which belong to a linear function. Complete the tables.

(a)

3	8
4	11
5	
6	17
	29
15	

(b)

5	7
7	15
7.5	
8	
8.3	
10	

(c)

2	-3
8	-1
9	
13	
-5	
-7	

T. 1. If P is directly proportional to C, and P has the value 35 when C has the value 7, what is the value of P when C has the value 25?

2. If x varies directly as y and x = 12 when y = 4, what is y when x = 5?

3. If x is directly proportional to y and x is 9 when y is 5, what value of y corresponds to the value 15 of x?

4. If 12 hats cost $60, how much will 17 hats of the same kind cost?

5. If A varies directly as B then ____________________.
   (a) AB is a constant variable quantity
   (b) A + B is a constant variable quantity
   (c) A/B is a constant variable quantity

6. Solve these proportions.

(a) $\frac{y}{3} = \frac{5}{20}$ (b) $\frac{5}{4} = \frac{k}{16}$ (c) $\frac{z}{9} = \frac{14}{36}$

(d) $\frac{x}{80} = \frac{36}{20}$ (e) $\frac{25}{x} = \frac{5}{7}$ (f) $\frac{3}{a} = \frac{1}{24}$

(g) $\frac{3}{4} = \frac{x}{12}$ (h) $\frac{5}{3} = \frac{25}{y}$ (i) $\frac{1}{3} = \frac{k}{k + 8}$

(j) $\frac{m + 10}{m} = \frac{9}{6}$ (k) $\frac{y}{12 - y} = \frac{1}{3}$ (ℓ) $\frac{2}{1} = \frac{21 - b}{b}$

7. If 3 pears cost 17 cents, what is the cost of a dozen pears at the same rate?

8. A 3-pound cake requires $1\frac{2}{3}$ cups of sugar. How many cups should be used for a 5-pound cake of the same type?

9. The ratio of cement to sand in a concrete mixture is 1 : 4. How many shovelfuls of cement should be used with 50 shovelfuls of sand?

10. The ratio of the length of a rectangle to its width is 10 : 7. If the rectangle is 35 feet long, what is its width?

11. Which of these formulas expresses the fact that one variable quantity varies directly as another?

(a) $A + B = 100$ (b) $10C = D$ (c) $\frac{M}{N} = 4$ (d) $PQ = 10$

12. If x varies inversely as y, and x is 12 when y is 4, what is y when x is 6?

13. AB is a constant variable quantity. If $A_1 = 8$, $B_1 = 9$, and $B_2 = 18$, what is A_2?

14. U varies inversely as T and U is 80 when T = 0.25. What value of U corresponds with the value 6 of T?

15. If M is inversely proportional to N and $M_1 = 5 = N_1$, what is the factor of inverse proportionality?

16. Write a formula to express one of the variable quantities in terms of the other. In each case, use 'k' to name the constant variable quantity whose value is the factor of variation.

(a) The circumference (C) of a circle varies directly as the radius (r).

(b) The side-measure (s) of an equilateral triangle is directly proportional to the perimeter (P).

(c) The resistance (R) of a piece of copper wire varies directly as its length (ℓ).

(d) The volume (V) of a gas under constant pressure varies directly as its absolute temperature (T).

(e) The resistance (R) of a piece of copper wire varies inversely as the area-measure (A) of a cross-section of the wire.

(f) The number (n) of items that can be bought for a fixed amount of money is inversely proportional to the cost (c) of each item.

(g) The illumination (I) from a source of light varies inversely as the square of the distance (d) from the source.

(h) The number (f) of vibrations made per second by a violin string of a certain diameter is inversely proportional to the length-measure (ℓ) of the string and directly proportional to the square root of the force (F) with which it is stretched.

17. If $A = \pi r^2$, what is the effect on r when the area-measure is multiplied by 25?

18. If a body falls from rest, the distance fallen from the starting point during an interval of elapsed time varies directly as the square of the elapsed time. If a body falls 402.5 feet in 5 seconds, how many feet does it fall in 10 seconds? How many feet does it fall <u>during</u> the eleventh second.

19. A is directly proportional to B and inversely proportional to C. If the value of A is 9 when the value of B is 18 and the value of C is 30, what is the value of A when the value of B is 42 and the value of C is 25?

20. The area-measure of the surface of a cube varies directly as the square of the measure of its edge. If the area of the surface of a cube whose edge is 5/3 feet long is 50/3 square feet, how long is the edge of a cube whose surface area is 243/8 square feet?

21. The volume-measure (V) of a sphere varies jointly as its diameter (d) and the area-measure (S) of its surface. If $V_1 = a$, $d_1 = b$, $S_1 = c$, $V_2 = p$, and $S_2 = q$, what is d_2?

☆ 22. Suppose $A = B + C$ and B varies directly as D^3 and C varies inversely as D. If $A_1 = -53$, $D_1 = -3$, $A_2 = 14.5$, $D_2 = 2$, and $D_3 = 2$, what is A_3?

☆ 23. The volume of a circular disk varies jointly as its thickness and the square of the radius of its face. Two disks which are 5 centimeters and 7 centimeters thick whose faces have radii 60 centimeters and 30 centimeters, respectively, are melted and formed into 50 congruent disks, each 3 centimeters thick. Find the radius of the face of each of the new disks.

U. 1. Transform each of the following quadratic expressions in 'x' by completing the square, and simplifying.

(a) $x^2 + 7x - 5$ $[(x + \frac{7}{2})^2 - \frac{69}{4}]$

(b) $x^2 + 8x - 5$

(c) $x^2 + 9x - 2$

(d) $x^2 - 10x + 1$

(e) $x^2 - 3x + 5$

(f) $x^2 + 2x + 17$

(g) $4x - 7 + x^2$

(h) $2 - 5x + x^2$

(i) $3x^2 - 6x + 9$

(j) $12x^2 - 2x + 5$

2. Find the equation of the axis of symmetry for each quadratic function.

(a) $f(x) = x^2 + 7x - 5$

(b) $f(x) = x^2 + 8x - 5$

(c) $f(x) = 3 - 2x + x^2$

(d) $f(x) = 6 + 5x - x^2$

(e) $f(x) = 2x^2 + x - 7$

(f) $f(x) = -5x^2 + 7x - 3$

3. The graph of the relation $\{(x, y): x^2 + y^2 = 25\}$ is a circle whose center is the graph of (0, 0) and whose radius is 5. The graphs of the relations listed below are circles. For each relation, give the point whose graph is the center of the circle, and give the radius of the circle.

(a) $\{(x, y): x^2 + y^2 = 36\}$

(b) $\{(x, y): x^2 + y^2 = 2\}$

(c) $\{(x, y): (x - 2)^2 + y^2 = 25\}$

(d) $\{(x, y): (x - 6)^2 + y^2 = 25\}$

(e) $\{(x, y): (x + 5)^2 + y^2 = 25\}$

(f) $\{(x, y): x^2 + (y - 1)^2 = 25\}$

(g) $\{(x, y): x^2 + (y + 7)^2 = 25\}$

(h) $\{(x, y): (x - 2)^2 + (y - 3)^2 = 25\}$

(i) $\{(x, y): (x + 4)^2 + (y - 9)^2 = 25\}$

(j) $\{(x, y): x^2 + 8x + 16 + y^2 - 18y + 81 = 25\}$

(k) $\{(x, y): x^2 - 6x + 9 + y^2 + 8y + 16 = 25\}$

(ℓ) $\{(x, y): x^2 - 6x \quad + y^2 + 8y \quad = 0\}$

(m) $\{(x, y): x^2 - 12x \quad + y^2 - 4y \quad = -15\}$

(n) $\{(x, y): x^2 + y^2 - 14x - 2y + 14 = 0\}$

(o) $\{(x, y): x^2 + y^2 = 2(10 - 5x - 2y)\}$

U. 1. (a) $(x + \frac{7}{2})^2 - \frac{69}{4}$ (b) $(x + 4)^2 - 21$ (c) $(x + \frac{9}{2})^2 - \frac{89}{4}$

(d) $(x - 5)^2 - 24$ (e) $(x - \frac{3}{2})^2 + \frac{11}{4}$ (f) $(x + 1)^2 + 16$

(g) $(x + 2)^2 - 11$ (h) $(x - \frac{5}{2})^2 - \frac{17}{4}$ (i) $3(x - 1)^2 + 6$

(j) $12(x - \frac{1}{12})^2 + \frac{59}{12}$

2. (a) $x = -\frac{7}{2}$ (b) $x = -4$ (c) $x = 1$

(d) $x = \frac{5}{2}$ (e) $x = -\frac{1}{4}$ (f) $x = \frac{7}{10}$

3. (a) (0, 0); 6 (b) (0, 0); $\sqrt{2}$ (c) (2, 0); 5

(d) (6, 0); 5 (e) (−5, 0); 5 (f) (0, 1); 5

(g) (0, −7); 5 (h) (2, 3); 5 (i) (−4, 9); 5

(j) (−4, 9); 5 (k) (3, −4); 5 (ℓ) (3, −4); 5

(m) (6, 2); 5 (n) (7, 1); 6 (o) (−5, −2); 7

Correction. Delete Exercise 7(g) on page 5-274. It repeats Exercise 5 on page 5-184.

V. 1. (a) 4, 1 (b) 2, 5 (c) −4, −2 (d) −5, −3

(e) −2, 3 (f) −3, 4 (g) −7, 0 (h) 2, $\frac{5}{2}$

(i) $-\frac{2}{3}$, 3 (j) $\frac{1}{2}$, 1 (k) $\frac{1}{2}$, 2

2. (a) $-1 - \sqrt{5}$, $-1 + \sqrt{5}$ (b) −7, 3 (c) −2, 8

(d) 2, 6 (e) −2, −1 (f) −6, 1

(g) $1 - \sqrt{6}$, $1 + \sqrt{6}$ (h) $-3 - \sqrt{13}$, $-3 + \sqrt{13}$

3. (a) −8, 5 (b) −1, $-\frac{2}{3}$ (c) $5 - \sqrt{10}$, $5 + \sqrt{10}$

(d) $-\frac{3}{2}$, 2 (e) $\frac{2 - \sqrt{14}}{5}$, $\frac{2 + \sqrt{14}}{5}$

(f) $\frac{5 - \sqrt{57}}{4}$, $\frac{5 + \sqrt{57}}{4}$ (g) $\frac{7 - \sqrt{17}}{4}$, $\frac{7 + \sqrt{17}}{4}$

(h) $\frac{4 - \sqrt{6}}{2}$, $\frac{4 + \sqrt{6}}{2}$

4. (a) −4.4, −0.4 (b) −0.6, 2.3

(c) −0.8, 3.8 (d) −2.6, 0.6

5. (a) (0, −4) (b) $\left(\frac{-5 - \sqrt{41}}{2}, 0\right)$, $\left(\frac{-5 + \sqrt{41}}{2}, 0\right)$

(c) $\left(\frac{-5 - 3\sqrt{5}}{2}, 1\right)$, $\left(\frac{-5 + 3\sqrt{5}}{2}, 1\right)$

(d) (−6, 2), (1, 2) (e) (−7, 10), (2, 10)

6. (a) 3 (b) 10 (c) 3 (d) 18 (e) $x^2 + 3x - 10 = 0$

7. (a) 11 and 16 (b) $2\sqrt{269}$ (c) 14" by 14"

(d) 10 and 7, or −7 and −10 (e) 8 and 13, or −13 and −8

(f) 6.25 miles

V. 1. Solve these equations by transforming to standard form and searching for factors.

(a) $x^2 = 5x - 4$

[Solution. $x^2 - 5x + 4 = 0$; $(x - 4)(x - 1) = 0$; $x = 4$ or $x = 1$; the roots are 4 and 1.]

(b) $x^2 + 10 = 7x$ (c) $a^2 + 3(a + 2) + 3a + 2 = 0$

(d) $m^2 + 15 = -8m$ (e) $6 + x = x^2$ (f) $y^2 = 12 + y$

(g) $p^2 = -7p$ (h) $10 - 9y = -2y^2$ (i) $1 = \frac{1}{2}k^2 - \frac{7}{6}k$

(j) $2x^2 - 3x + 1 = 0$ (k) $2x^2 = 5x - 2$

2. Solve these equations by transforming to standard form and completing the square.

(a) $x^2 + 2x - 4 = 0$ (b) $y^2 + 4y - 21 = 0$ (c) $k^2 - 6k = 16$

(d) $x^2 = 4(2x - 3)$ (e) $x(x + 3) + 2 = 0$ (f) $6 = y(5 + y)$

(g) $m^2 = 2m + 5$ (h) $4(1 - m) = 2m(1 + m) - m^2$

3. Solve by using the quadratic formula.

(a) $m^2 + 3m - 40 = 0$ (b) $3k^2 + 5k = -2$ (c) $c^2 - 10c = -15$

(d) $x = -2(3 - x^2)$ (e) $2(1 + 2r) = 5r^2$ (f) $2x^2 - 4 = 5x$

(g) $\frac{2}{y^2} + 1 = \frac{7}{2y}$ (h) $\frac{1}{x - 1} - \frac{3}{x - 2} + 2 = 0$

4. Find rational approximations [correct to the nearest tenth] to the roots of these equations.

(a) $x^2 + 4x - 2 = 0$ (b) $5k + 4 = 3k^2$

(c) $3(x + 1) = x^2$ (d) $3 = 2y(y + 2)$

5. Find the ordered pairs in these intersections.

(a) $\{(x, y): y = x^2 + 5x - 4\} \cap$ the y-axis

(b) $\{(x, y): y = x^2 + 5x - 4\} \cap$ the x-axis

(c) $\{(x, y): y = x^2 + 5x - 4\} \cap \{(x, y): y = 1\}$

(d) $\{(x, y): y = x^2 + 5x - 4\} \cap \{(x, y): y = 2\}$

(e) $\{(x, y): y = x^2 + 5x - 4\} \cap \{(x, y): y = 10\}$

6. (a) If one root of the quadratic equation '$x^2 - 5x + k = 0$' is 2, what is the other root?

   (b) For what value of 'n' is '$x^2 - 7x + n = 0$' satisfied by 2?

   (c) What is the sum of the roots of the equation '$x^2 - 3x - 40 = 0$'?

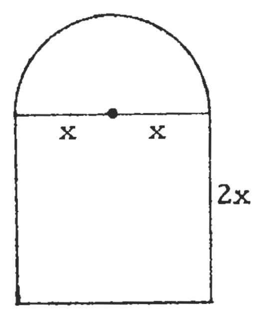
   (d) What is the product of the roots of the quadratic equation '$x^2 - 9x + 18 = 0$'?

   (e) Write a quadratic equation whose roots are 2 and −5.

7. (a) John is 5 years older than Bob. The product of their ages [that is, the product of the number of years in their ages] is 176. Find their ages.

   (b) The length of a rectangle is 3 inches more than its width. If its area is 65 square inches, what is its inch-perimeter?

   (c) The figure shows a square surmounted by a semicircle. If one uses 22/7 as an approximation for π, he obtains 273 as an approximation for the total area. What are the dimensions of the square?



   (d) One number exceeds another by 3. The sum of their squares is 149. What are the numbers?

   (e) One number exceeds another by 5. Their product is 104. What are the numbers?

   (f) A man walks for x hours at x miles per hour. If he had increased his speed by 2 miles per hour, he would have walked 11.25 miles. How far did he actually walk?

   (g) A small orchard has 60 trees and yields, on the average, 400 apples per tree. For each additional tree planted in this orchard, the average yield per tree is reduced by approximately 6 apples. How many trees will give the largest crop of apples for this orchard?

W. 1. (a) $(8, 4)$ (b) $(9, 4)$ (c) $(-8, 2)$

(d) $(5, 1)$ (e) $(-3, 3)$ (f) $(5, 3)$

(g) $(1, -1)$ (h) $(-1, 1)$ (i) $(4, 5)$

(j) $(-1, -3)$ (k) $(-1, 3)$ (ℓ) $(-1, -4)$

(m) $(2, -1)$ (n) $(1, 3)$ (o) $(7, -2)$

(p) $(4, -1)$ (q) $(2, -\frac{1}{2})$ (r) $(3, 5)$

(s) $(-3, 6)$ (t) $(-2, 13)$

2. (a) $(2, -7, -\frac{3}{4})$ (b) $(\frac{1}{2}, -1, 3)$ (c) $(2, -\frac{1}{2}, \frac{1}{3})$

(d) $(2, \frac{1}{2}, -1)$ (e) $(\frac{2}{3}, -3, 3)$ (f) $(\frac{1}{2}, -\frac{1}{3}, \frac{1}{4})$

3. (a) $\{(9, -5), (-3, 1)\}$ (b) $\{(-4, -11), (1, -1)\}$

(c) $\{(0, -1), (2, 3)\}$ (d) $\{(-9, -6), (6, 9)\}$

(e) $\{(2, -3), (-2, 1)\}$ (f) $\{(\frac{3}{2}, \frac{5}{2}), (-1, -5)\}$

(g) $\{(8, -11), (2, 1)\}$ (h) $\{(7, 5), (0, -2)\}$

W. 1. Solve these systems of equations.

(a) $\left.\begin{aligned} a + b &= 12 \\ a - b &= 4 \end{aligned}\right\}$

(b) $\left.\begin{aligned} x + y &= 13 \\ x - y &= 5 \end{aligned}\right\}$

(c) $\left.\begin{aligned} m + n &= -6 \\ m - n &= -10 \end{aligned}\right\}$

(d) $\left.\begin{aligned} 3a + b &= 16 \\ 2a + b &= 11 \end{aligned}\right\}$

(e) $\left.\begin{aligned} x + y &= 0 \\ y - x &= 6 \end{aligned}\right\}$

(f) $\left.\begin{aligned} 4x + 3y &= 29 \\ 2x - 3y &= 1 \end{aligned}\right\}$

(g) $\left.\begin{aligned} 3p + 7q &= -4 \\ 2p + 5q &= -3 \end{aligned}\right\}$

(h) $\left.\begin{aligned} 3x &= 4 - 7y \\ 4x &= 3y - 7 \end{aligned}\right\}$

(i) $\left.\begin{aligned} 5u &= 4v \\ \frac{1}{2}u &= 2(6 - v) \end{aligned}\right\}$

(j) $\left.\begin{aligned} y &= 3x \\ x - y &= 2 \end{aligned}\right\}$

(k) $\left.\begin{aligned} 5a + b + 2 &= 0 \\ a + 2b &= 5 \end{aligned}\right\}$

(ℓ) $\left.\begin{aligned} 3m &= p + 1 \\ 3p + 7 &= 5m \end{aligned}\right\}$

(m) $\left.\begin{aligned} 2x - y &= 5 \\ 2x + 3y &= 1 \end{aligned}\right\}$

(n) $\left.\begin{aligned} x + 2y &= 7 \\ 4x - y &= 1 \end{aligned}\right\}$

(o) $\left.\begin{aligned} 5x + 4y &= 27 \\ x - 2y &= 11 \end{aligned}\right\}$

(p) $\left.\begin{aligned} x + y &= 3 \\ 2x - y &= 9 \end{aligned}\right\}$

(q) $\left.\begin{aligned} 3x - 4y &= 8 \\ x + 2y &= 1 \end{aligned}\right\}$

(r) $\left.\begin{aligned} 3x - y &= 4 \\ 2x + 3y &= 21 \end{aligned}\right\}$

(s) $\left.\begin{aligned} 5x + 3y &= 3 \\ 2x + 3y &= 12 \end{aligned}\right\}$

(t) $\left.\begin{aligned} 4x + y &= 5 \\ 2x + y &= 9 \end{aligned}\right\}$

2. Solve these systems of equations.

(a) $\left.\begin{aligned} x - 2y + 20z &= 1 \\ 3x + y - 4z &= 2 \\ 2x + y - 8z &= 3 \end{aligned}\right\}$

(b) $\left.\begin{aligned} 2x - y + z &= 5 \\ 4x - 3y &= 5 \\ 6x + 2y + 2z &= 7 \end{aligned}\right\}$

(c) $\left.\begin{aligned} 3x + 2y &= 5 \\ 4x - 3z &= 7 \\ 6y - 6z &= -5 \end{aligned}\right\}$

(d) $\left.\begin{aligned} 3x - 2y + z &= 4 \\ 2x + 4y - 3z &= 9 \\ -x + 8y - 2z &= 4 \end{aligned}\right\}$

(e) $\left.\begin{aligned} 3x - 2y - 3z &= -1 \\ 6x + y + 2z &= 7 \\ 9x + 3y + 4z &= 9 \end{aligned}\right\}$

(f) $\left.\begin{aligned} 3x + 3y - 2z &= 0 \\ 4x - 9y + 4z &= 6 \\ 5x - 6y + 6z &= 6 \end{aligned}\right\}$

3. Solve these systems of equations.

Sample. $\left.\begin{array}{l} x^2 - xy + y = 5 \\ 2x + y = 3 \end{array}\right\}$

Solution.

$$y = 3 - 2x$$
$$x^2 - x(3 - 2x) + (3 - 2x) = 5$$
$$x^2 - 3x + 2x^2 + 3 - 2x = 5$$
$$3x^2 - 5x - 2 = 0$$
$$(3x + 1)(x - 2) = 0$$
$$x = -\frac{1}{3} \quad \text{or} \quad x = 2$$

$$\begin{array}{l|l} y = 3 - 2 \cdot -\frac{1}{3} & y = 3 - 2 \cdot 2 \\ \quad = 3 + \frac{2}{3} & \quad = 3 - 4 \\ \quad = \frac{11}{3} & \quad = -1 \end{array}$$

Check. $(-\frac{1}{3})^2 - -\frac{1}{3} \cdot \frac{11}{3} + \frac{11}{3} \overset{?}{=} 5$

$$\frac{1}{9} + \frac{11}{9} + \frac{11}{3} \overset{?}{=} 5$$
$$\frac{45}{9} = 5 \quad \checkmark$$

$$2^2 - 2 \cdot -1 + -1 \overset{?}{=} 5$$
$$4 + 2 + -1 = 5 \quad \checkmark$$

The solution set is $\{(-\frac{1}{3}, \frac{11}{3}), (2, -1)\}$.

(a) $\left.\begin{array}{l} x^2 - 3y^2 = 6 \\ x + 2y = -1 \end{array}\right\}$

(b) $\left.\begin{array}{l} y = 2x - 3 \\ 3x^2 = 4 + xy \end{array}\right\}$

(c) $\left.\begin{array}{l} x^2 + y^2 - 3y = 4 \\ 2x - y = 1 \end{array}\right\}$

(d) $\left.\begin{array}{l} x^2 - xy + y^2 = 63 \\ y - x = 3 \end{array}\right\}$

(e) $\left.\begin{array}{l} x^2 - xy - x = 8 \\ y + x = -1 \end{array}\right\}$

(f) $\left.\begin{array}{l} x^2 + xy = 6 \\ 3x - y = 2 \end{array}\right\}$

(g) $\left.\begin{array}{l} 2x^2 - y^2 = 7 \\ 2x + y = 5 \end{array}\right\}$

(h) $\left.\begin{array}{l} x^2 + y^2 - 10y = 24 \\ y = x - 2 \end{array}\right\}$

4. (a) \$1.40 [$3c + 2p = 69$, $2c + 3q = 71$]

(b) 13¢ [$d + 25r = 63$, $d + 50r = 113$]

(c) 5900 lbs. [$t + c = 8000$, $t + \frac{1}{3}c = 6600$]

(d) 5:00 p.m. [$f = s + 2$, $30f = 40s$]

(e) The data are inconsistent. [t...dollar-cost per square foot for the table top, ℓ...dollar-cost of the table legs; $\ell + 16t = 42$, $\ell + 25t = 72$. The solution of this system is $(-34/3, 10/3)$. But, we are seeking numbers of arithmetic; and, since there is no pair of positive numbers which is a solution of the system, there is no pair of numbers of arithmetic which satisfies the conditions of the problem.]

(f) The data are inconsistent. [To reduce the percentage of salt in the mixture, one should add water, not salt.]

(g) 1 inch [x...inch-height of 1 copy of Unit 1, y...inch-height of 1 copy of Unit 2; $8x + 3y = 12.5$, $4x + 5y = 11.5$]

(h) \$3, \$5 [$35r + 10v = 155$, $30r + 15v = 165$]

(i) 20 children, 10 adults [$25c + 60a = 1100$, $30c + 75a = 1350$]

(j) 17, 22 [$a = b + 5$, $a + b = 39$]

(k) 4 m.p.h., 36 m.p.h. [$3w + 2r = 84$, $2w + 3r = 116$]

(ℓ) 800, older; 900, newer [$3e + 4n = 6000$, $4e + 2n = 5000$]

(m) \$9000 [x...dollars invested at 3%, y...dollars invested at 4%; $.03x + .05y = 370$, $.04x + .04y = 360$. The second equation tells us that $x + y = 9000$. No need to solve the system.]

4. (a) A man bought 3 cans of corn and 2 cans of peas for a total of 69 cents. 2 cans of corn and 3 cans of peas would have cost a total of 71 cents. What would 5 cans of corn and 5 cans of peas cost?

(b) A razor blade dispenser containing 25 razor blades costs 63 cents. The same dispenser with 50 razor blades costs $1.13. What is the cost of the dispenser by itself?

(c) A truck with a full load of coal weighs 8000 pounds. The same truck with a third of a load of coal weighs 6600 pounds. How much does the truck weigh?

(d) At 9 a.m., a man starts on a trip by car; he averages 30 miles per hour. His son starts after him at 11 a.m., and averages 40 miles per hour. When will the son catch up with his father?

(e) A furniture manufacturer makes tables. The cost of the material for the table top is computed on the basis of the area, while the cost of the material for the legs is computed on the basis of length. A table whose top measures 4 feet by 4 feet and whose legs are 3 feet long costs $42 for materials, while a table 5 feet by 5 feet whose legs are also 3 feet long, made of the same materials, costs $72. What is the cost of a square foot of table top? What is the cost of a foot of material for the legs?

(f) How many pounds of pure salt must be added to 50 pounds of a 25% solution of salt and water in order to yield a mixture that will be 3% salt?

(g) A stack containing 8 copies of Unit 1 and 3 copies of Unit 2 is 12.5 inches high. Another stack containing 4 copies of Unit 1 and 5 copies of Unit 2 is 11.5 inches high. How thick is a copy of Unit 1?

(h) A man gets different rates of pay for work done during his regular hours and for work done overtime. He gets $155 for 35 regular hours and 10 hours overtime. For 30 hours regular time and 15 hours overtime he gets $165. What is his regular hourly rate of pay? What is his overtime rate of pay?

(i) Children paid 25 cents each and adults paid 60 cents each for admission to a movie. A total of $11.00 was collected. If children had paid 30 cents each and adults 75 cents each, the total collected would have been $13.50. How many children attended? How many adults?

(j) The intersection of two sets is the empty set. One contains 5 elements more than the other, and their union contains 39 elements. How many elements are there in each set?

(k) A man walks for 3 hours and then rides for 2 hours. He covers a total distance of 84 miles. If he were to walk for 2 hours and ride for 3 hours at the same average rates, he would cover 116 miles. What is his average rate of walking? What is his average rate of riding?

(ℓ) Two machines produce bottle caps. The older machine works for 3 hours and the newer machine for 4 hours. The total number of caps produced is 6000. If the older machine worked for 4 hours and the newer machine for 2 hours, the total produced would be 5000. How many caps does each machine produce in an hour?

(m) A man invests two sums of money, one sum at 3% and one sum at 5%. The total income is $370. If each amount were invested at 4%, the total income would be $360. What is the total amount invested?