HIGH SCHOOL MATHEMATICS

Unit 1.

THE ARITHMETIC OF THE REAL NUMBERS

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

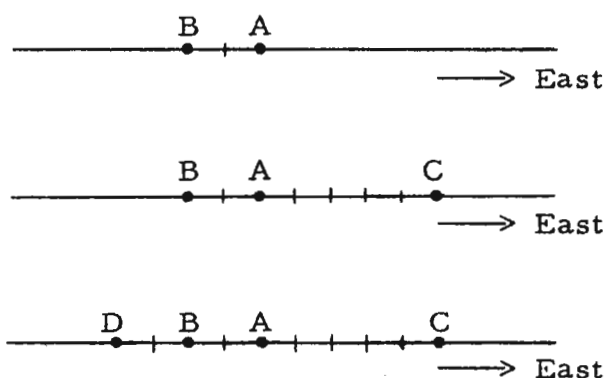MAX BEBERMAN, *Director*

HERBERT E. VAUGHAN, *Editor*

Revised Edition

So, C is 8 units to the right of A. This means that
the ending point of the last trip is 8 units east of A.

The real number which measures the trip from A
to the ending point of the last trip in the succession
is $\vec{8}$. Answer: $\vec{8}$

Sample 2. $\overleftarrow{2}$, $\overrightarrow{7}$, $\overleftarrow{9}$.

Solution.

B   A
●—●
——⟶ East

B   A   C
●—●—————●
——⟶ East

D   B   A   C
●—●—●——————●
——⟶ East

So, the real number which measures the trip
from A to ending point of the last trip in the succession
is $\overleftarrow{4}$. Answer: $\overleftarrow{4}$

[Note: Use drawings for doing these exercises as long as you
need to. You may be able to find short cuts so that you
can give the answer immediately.]

1. $\overrightarrow{2}$, $\overrightarrow{5}$

2. $\overleftarrow{3}$, $\overleftarrow{6}$

3. $\overrightarrow{4}$, $\overleftarrow{3}$

4. $\overrightarrow{6}$, $\overleftarrow{8}$

5. $\overleftarrow{12}$, $\overrightarrow{3}$

6. $\overleftarrow{5}$, $\overrightarrow{8}$

7. $\overrightarrow{3}$, $\overrightarrow{7}$, $\overrightarrow{5}$

8. $\overrightarrow{2}$, $\overrightarrow{9}$, $\overleftarrow{1}$

9. $\overleftarrow{5}$, $\overleftarrow{3}$, $\overleftarrow{10}$

10. $\overrightarrow{5}$, $\overleftarrow{7}$, $\overrightarrow{8}$

11. $\overrightarrow{2}$, $\overleftarrow{9}$, $\overrightarrow{2}$

12. $\overleftarrow{7}$, $\overrightarrow{4}$, $\overrightarrow{8}$

13. $\overrightarrow{2}$, $\overleftarrow{10}$, $\overrightarrow{3}$, $\overleftarrow{5}$

14. $\overleftarrow{6}$, $\overleftarrow{5}$, $\overleftarrow{15}$, $\overleftarrow{20}$

15. $\overrightarrow{3}$, $\overleftarrow{5}$, $\overleftarrow{7}$, $\overleftarrow{21}$

16. $\overrightarrow{78}$, $\overleftarrow{83}$

17. $\overleftarrow{95}$, $\overrightarrow{107}$

18. $\overrightarrow{107}$, $\overleftarrow{95}$

19. $\overleftarrow{50}$, $\overrightarrow{98}$, $\overleftarrow{97}$

20. $\overleftarrow{74}$, $\overrightarrow{10}$, $\overrightarrow{75}$

21. $\overleftarrow{41.5}$, $\overrightarrow{57}$, $\overrightarrow{44.5}$

HIGH SCHOOL MATHEMATICS

Teachers' Edition
UNITS 1 THROUGH 4

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

MAX BEBERMAN, *Director*

HERBERT E. VAUGHAN, *Editor*

Revised Edition

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Instructions for integrating the green pages with the white pages

The green sheets of the COMMENTARY occur in blocks [each consist-ing of one or more sheets] each of which contains discussions referring to a single page or a group of consecutive pages of the text (white sheets). The pages of the text to which a block of green sheets refers are listed in brackets at the foot of each sheet. Thus, 'TC[1-A, B, C]' indicates that this sheet refers to pages 1-A, 1-B, and 1-C. And, 'TC[1-J]a', 'TC[1-J]b', and 'TC[1-J]c' indicate members of a block of green sheets all of which refer to page 1-J. Each block should be inserted among the pages of the text so that the printed side of its first sheet faces the first text page to which it refers. For example, the sheet labeled 'TC[1-A, B, C]' should face page 1-A of the text. And, the sheet labeled 'TC[1-J]a' should face page 1-J of the text, and be backed up by the sheets labeled 'TC[1-J]b' and 'TC[1-J]c'. The final result should be that, on turning to a page of the text, one has opposite it the first of the green pages [if any] which refer to it. And, if there are several such pages, one obtains a view of the second by turning the first, etc.

An exception to these instructions is that the introductory green pages of each unit are to follow the title page of that unit.

For your convenience, we list the first several pages of the completely integrated Teachers' Edition.

[Title page, blank], [1/1, blank], [1/2, blank], [1/3, blank],
[1/4, blank], [1/5, blank], [i, ii], [iii, iv], [blank, 1/6],
[1-A, 1-B], [1-C, 1-D], [1/7, blank], [1-E, 1-F], [1-G, 1-H],
[1-I, 1-J], [1/8, blank], [1/9, blank], [1/10, blank], [blank, 1/12],
[blank, 1/11], [1-K, 1-L], [1/13, blank], [1/14, blank],
[blank, 1/17], [blank, 1/16], [blank, 1/15], [1-M, 1-N],
[1/18, blank], [blank, 1/20], [blank, 1/19], [1-O, blank],
[blank, 1/22], [blank, 1/21], [1-1, 1-2], [1/23, blank],
[1-3, 1-4], [1/24, blank], [blank, 1/25], ...

Note: The numerals '1/1', '1/2', ... are useful for assembling the green pages in order for those who do not want to integrate them with the white pages.

as if it were just another of the bits of drudgery that are involved in the process of growing up. We believe firmly that the learning of mathematics should be a delightful experience for youngsters, and that this delight is the reward for hard work. We also believe that the amount of pleasure a student derives from his learning of mathematics depends on the extent to which he finds opportunity for creative activity.

So, one of the ground rules you will have to establish early in the teaching of UICSM courses is that the student is expected to contribute ideas, principles, and rules. It may be necessary to overcome some initial prejudices which students have toward mathematics. At first, they may insist that they be given a formula or a rule for doing a problem. Since the text does not give it to them, and since you will not give it to them, they may turn to their parents for such help. Hence, it is important that parents understand the kind of attitude we are trying to create. It has been our experience that as soon as a student encounters success in formulating his own rules and short cuts, the initial prejudice is removed. Students who were demanding rules at the beginning of the year turn into students who literally beg the teacher to give them "more time to work it out".

Our notion that the learning of mathematics ought to be an enjoyable experience has been interpreted by some people to mean that the mathematics classroom is a place where parties are held, where everybody does as he pleases, and where no one is really expected to learn anything! Needless to say, this interpretation is made by people who have had no experience in teaching UICSM courses. They may have forgotten that there is such a thing as joy in intellectual activity, and that to smile in a mathematics classroom does not result in destruction of character.

Our experience in teaching the "early adolescent" has shown us that he is interested in many things which the adult considers frivolous. So-called "real life" applications for the adult may not be real life applications for the young learner. Also, many of the things in which the young learner is interested are hardly in accord with the adult's view of real life. Since we believe that interest is a necessary condition for learning, we have tried to set the development of mathematical ideas in situations which are inherently interesting to young people. Thus, one of our standard devices when approaching a new idea is to create a fanciful situation which embodies or illustrates it. Even though a student is aware of the fact that these situations are fanciful, he can easily imagine them. And, the very fact that they are fanciful appeals to the youngster's interest in the make-believe and the fantastic.

"The Transfer Value of Given and Individually Derived Principles" by G. M. Haslerud and Shirley Meyers in the December 1958 issue of The Journal of Educational Psychology.] Later in the course, when students have developed considerable linguistic facility, it will be found that they can give precise statements of discoveries, including those which they made earlier in the course.

Another practice which facilitates discovery is that of using precise language in exposition. Whenever we talk about something directly [rather than by giving examples], we want our exposition to be precise. [Unless we say precisely what an equation is, it is foolhardy to expect students to discover procedures for solving equations.] But, note carefully, this precision in exposition is something we expect of the textbook and the teacher, rather than of the learner. Precise communication is a characteristic of a good textbook and a good teacher; correct action is a characteristic of the good learner. It is important that you observe this distinction as you go through the course. You can begin to expect more precision from the student only after he has been exposed to careful language for a considerable period of time. [For a recent discussion on precise language in the teaching of mathematics, see Chapter 8 of the 24th Yearbook of the National Council of Teachers of Mathematics, The Growth of Mathematical Ideas (Washington, D. C.: The Council, 1959).]

Style and format of the COMMENTARY

The COMMENTARY pages have been designed so that they can be integrated with the white pages of the student-edition. Each green page contains a discussion relevant to one or more white pages. For example, the green page TC[1-A, B, C] contains a discussion concerning the content of the white pages 1-A, 1-B, and 1-C. Since the first of these white pages is a righthand page, the green page TC[1-A, B, C] has holes punched along its right margin so that it will face the white page 1-A. On the other hand, green page TC[1-D, E, F] has holes punched along its left margin so that it will face white page 1-D which is a lefthand page.

You may find it helpful in getting ready to teach a unit to read all of the white pages fairly rapidly, and then read the white pages together with the accompanying green pages. The white pages will give you some notion of what is expected of the student. The green pages will describe these expectations more carefully, and will contain mathematical background for you as well as pedagogical suggestions for accomplishing the goals set for the student. Naturally, you will want to read smaller sections of the green and white pages each day as you prepare for your classes.

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A teacher in one of our pilot schools suggested that the Solution for
Sample 2 would be more effective if pictured like this:
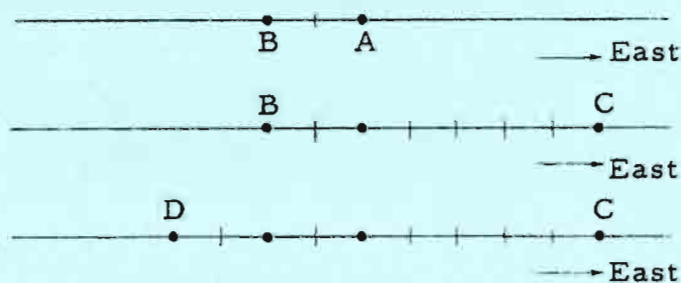
You may want to use this idea for diagramming other trips on the
blackboard.

*

Please note that none of these exercises and none of the exercises on
pages 1-9 and 1-10 involve the real number 0. This is a tricky notion
which requires careful explanation. It is introduced on pages 1-13
and 1-14. If a student raises the question of 0 at this time, compli-
ment him on asking a good question, and tell him that the matter will
be taken up in a day or so.

*

Answers for Part D [on pages 1-5 and 1-6].

1. $\overrightarrow{7}$ 2. $\overleftarrow{9}$ 3. $\overrightarrow{1}$ 4. $\overleftarrow{2}$ 5. $\overleftarrow{9}$

6. $\overrightarrow{3}$ 7. $\overrightarrow{15}$ 8. $\overrightarrow{10}$ 9. $\overleftarrow{18}$ 10. $\overrightarrow{6}$

11. $\overleftarrow{5}$ 12. $\overrightarrow{5}$ 13. $\overrightarrow{10}$ 14. $\overleftarrow{16}$ 15. $\overleftarrow{16}$

16. $\overleftarrow{5}$ 17. $\overrightarrow{12}$ 18. $\overrightarrow{12}$ 19. $\overleftrightarrow{49}$ 20. $\overrightarrow{11}$

21. $\overrightarrow{60}$

1/5

Acknowledgments

The UICSM Project Staff thanks the teachers in cooperating schools for their detailed written reports, for the suggestions and criticisms they gave us in conversations and in training conferences, and for the ideas that came to us as we worked with the students in their classes during our visits.

Special recognition is due Miss Eleanor McCoy, Associate Teacher Coordinator in the Project, for the constructive criticisms she made of the text materials for this edition (and also for those made in the early years of the program when she taught at Pekin Community High School), for the immense amount of help she has given in the preparation of this and the last editions in the tedious jobs of proofreading, page-fitting, and indexing, and for her even more important assistance in making tests, constructing exercises, and writing developments for the COMMENTARY and the text.

We also owe much to a succession of highly proficient typists: Mrs. Naomi Konecky, Mrs. Helen M. Sakamoto, Mrs. E. June Harris, and Mrs. A. Ruth Fifer, and to the staff artists: Mr. Roy Nagao and Mr. Roy Yamachi.

We take this occasion to list former members of the Project Staff who contributed significantly to earlier versions of the text: Professors Daniel S. Babb, Gerhard P. Hochschild, Bruce E. Meserve, Eugene D. Nichols, David A. Page, Robert E. Pingry, and Charles R. Stegmeir.

And, finally, we thank the Carnegie Corporation of New York for its generous support.

> Max Beberman, Director
> Herbert E. Vaughan, Editor
> University of Illinois Committee
> on School Mathematics.

May 1, 1959
Urbana, Illinois

Pages 1-A through 1-E are intended as an introduction to an extremely important pedagogical idea which will come up time and again in teaching UICSM materials. The idea is developed further in pages 1-E through 1-O. But it may be possible to get at the heart of the matter through the humorous interchange between Al and his "tutor" Stan.

❊

The Project Staff has expressed its point of view concerning the importance of distinguishing between symbols and their referents in the article " Words, 'Words', "Words" " which appeared in the March 1957 issue of The Mathematics Teacher. The 1958 Inglis Lecture, An Emerging Program of Secondary School Mathematics, contains a more detailed rationale for our making this distinction. [The Inglis Lecture series is published by the Harvard University Press.] An article which treats this topic from the point of view of the elementary school is Frank Wolf's " "1" and "1" is "11" " which appeared in the April 1958 issue of The Arithmetic Teacher. The monograph Gödel's Proof by Ernest Nagel and James R. Newman (New York: New York University Press, 1958) contains several interesting sections on distinguishing between symbols and their referents [In particular, see pages 28-31 and footnote 24 on pages 82 ff.].

❊

We suggest that you select a student who has some flair for dramatics to read aloud the exchange of correspondence between Al and Stan. [You might find it necessary to read the letters aloud yourself.] Undoubtedly, you will find students who disagree with Al's justifications and with Stan's justifications. This disagreement should be encouraged even if it comes close to splitting hairs.

❊

Students ought to see that Al is perfectly consistent in the way he has responded to Stan's questions. Let the students tell why Al answered as he did for each item before they read his response to Stan.

Be graphic in describing why '3' goes into '8' twice. You might want to bring a paper '8' into class and actually cut it into two '3's! Let the students devise other questions and answer them as Al would, even questions outside of mathematics such as:

Is MARY bigger than Mary?

TC[1-A, B, C]

Arithmetic by mail. --Stan Brown had a pen pal, Al Moore, who lived in Alaska. Stan and Al corresponded quite frequently. Stan liked to receive letters from Al because he wrote about interesting things like hunting and fishing and prospecting for gold. Al enjoyed hearing about the things Stan did, especially about school, for Al had had very little opportunity to attend school. One day, Al wrote to ask if Stan would mind teaching him some arithmetic. Stan agreed but decided he needed to know how much Al already knew. So, in his next letter to Al he included a simple test, and asked Al to write in the answers and to return the test to him. Al sent the test back immediately; he said it was very easy and asked Stan to send some harder questions next time.

Turn the page to see what Al's test looked like when he returned it.

1.	Take 2 away from 21.	.....*1*......
2.	What is half of 3?	*?* ....
3.	Add 5 to 7.	....*57*......
4.	Does $2 \times 4\frac{1}{2}$ equal 9?	....*no*......
5.	Which is larger, .000065 or .25?	..*.000065*...
6.	How many times does 3 go into 8?	..*twice*....
7.	How many times does 9 go into 99?	..*twice*....
8.	Which is larger, 3 or 23?	...*23*......
9.	What is a number smaller than 4?	.....*4*......
10.	What is a number larger than 4?	.....*4*......

Stan was flabbergasted when he looked at Al's answers. Was this a joke? But Al had seemed so serious about wanting to learn arithmetic. Stan decided that Al needed a lot of help. He would start by telling Al about the errors he had made.

    Dear Al,

        You sure have some funny ideas about numbers. But it won't take long to straighten you out (I hope).

        I've enclosed your test with this letter so that you could follow my explanation. Look at the first question. I can see how you thought that the answer was 1 because if you do take 2 away from 21 you are left with 1. But when you take 2 away from 21, you get 19. See? Take the second question. Your answer isn't even a number. What you wrote is half of 3 but half of 3 isn't that, it's $1\frac{1}{2}$. In the third question, you put 5 and 7 together, but adding numbers isn't putting them together. It's, er... well, I'll explain that in

your next lesson. In the fourth question, I can see that you thought that $2 \times 4\frac{1}{2}$ was not the same as 9, but $2 \times 4\frac{1}{2}$ is equal to 9, so your answer is wrong.

Now, in question five, it's true that .000065 is bigger than .25, but really .25 is larger than .000065. You see, .25 is a bigger number than .000065 even though .25 looks smaller than .000065.

I guess I can give you credit for question six, although you really should have said that 3 goes into 8 twice with 2 left over. I guess you just made a careless mistake in dividing in question seven, because 9 goes into 99 eleven times. Question eight you did correctly, so I guess you know how to count.

You made the same kind of mistake in question nine as you did in the very last question. Even though what you wrote for question nine is smaller than 4, I asked for a number smaller than 4 and the one you picked is the same size as 4. Same idea for question ten.

I hope that my explanations of what you did wrong are helpful to you. Let me know when you want your next lesson. Make it soon because we have lots of work to do.

<div style="text-align: right">Your pal,</div>

<div style="text-align: right">Stan</div>

Al was a little disappointed with Stan's letter. He wrote the following in return.

Dear Stan,

I sure appreciate what you are trying to do for me, but I don't think that <u>you</u> can help me at all. Are you sure <u>you</u> understand this stuff?

Sure I got questions 6 and 8 right. I did them just as I did the others. Anyone can see

that 3 goes into 8 twice, and pretty neatly, too,
without any 2 left over, either. You put 3 into
8 the regular way and then you turn another 3
around and put it in on the other side of the 8.

In question 8, you can just see that 23 is
larger than 3 because 23 already has a 3 in it
and a 2 added on in front. You don't need to
know how to count to tell that!

I really laughed at what you said about
question 7. If there's one thing I do know it's
how to count, and eleven 9's make 99999999999
and not 99. You said that .000065 is bigger
than .25. I knew that because I even checked
with a ruler. Then you cross yourself up and
say that .000065 is really smaller. And in
question 4 you don't even need a ruler to tell
that $2 \times 4\frac{1}{2}$ is different from 9.

There's no use in trying to learn arithmetic,
I guess. I think I'll stick to hunting.

Your friend,

Al

EXERCISES

A. Do you think Stan did a good job explaining Al's errors? What
seemed to be Stan's difficulty in doing so? Can you defend
what Al did?

B. In certain print shops the printing press is prepared for opera-
tion by picking individual type pieces out of boxes and placing
the pieces into the bed of the press. Such a print shop has type
of various styles and sizes. Here are two examples of remarks
you might hear a printer make to his helper:

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By careful questioning, try to elicit from the students the notion
that what Stan needs is a writing system which will permit him to
indicate when he is talking about things and when he is talking
about the names of things. The students should notice that Stan
was not attempting to teach arithmetic to Al in his letter. He is
just trying to point out to Al the nature of Al's mistakes. But Al
could not understand this just because Stan kept confusing numbers
with their names.

*

In connection with Part B, here are further amusing statements
which might be overheard in a print shop:

> Find a two bigger than this eight.
>
> Bring me 0 type.
>
> 0.00001 is certainly bigger than 14.

*

When you emphasize that the marks are not the numbers themselves,
perceptive students may ask what the numbers are. They write the
name 'Mickey Mantle' and know what this name stands for because
they know this New York Yankee baseball player. On the other hand,
when they write the name '2', they cannot see or touch the number 2.
The question 'What is the number 2?' is a very profound one, and is
one which has come to be answered in a satisfactory way only recently.
The number 2, just like justice, is an abstraction; it is not something
which can be pointed at.

An excellent book which deals with the question of what numbers are
is Hugh A. Thurston's The Number-System (London: Blackie, 1956).
Another is Irving Adler's The New Mathematics (New York: John
Day, 1958).

> Bill, this seven isn't big enough; get me a bigger
> seven.
>
> I asked you to bring me two threes and you brought
> me three twos. I don't care what you learned
> in school, three twos are not the same as two
> threes.

Make up more examples of confusing statements you might
hear in this print shop.

<u>Things</u> <u>and</u> <u>the</u> <u>names</u> <u>of</u> <u>things</u>. --It is easy to see what Al had in
mind when he took Stan's test. Al was confusing numbers with the
marks that were written on the paper. And even though Stan may
have realized this, he certainly didn't get the idea across to Al. It
might have helped Al if Stan started by saying something like this:

> Look, Al, if I asked you to tell me something
> about Alaska, you wouldn't tell me that it started
> and ended with the same letter, would you? You
> might tell me how big it was, or how many people
> lived there, or what its capital city was. So, when
> I ask you a question about numbers [for example,
> to take 2 away from 21], I don't expect you to tell
> me about the marks on the paper. Instead, I expect
> you to do something with the numbers those marks
> stand for. You see, those marks aren't numbers.
> They just stand for numbers. They're really <u>names</u>
> of numbers, just as the word:
>
> Alaska
>
> is the name of the place in which you live.

Whenever a person writes about numbers, he puts marks on paper;
if he talks about numbers, he makes certain sounds. But the marks
on paper and the sounds are not the numbers themselves. The marks
and the sounds are names of numbers. In order to write <u>about</u> a
thing, you must write a <u>name</u> of the thing.

In everyday conversation, one hardly ever confuses a thing with a name of the thing, because in so many cases you can point to the thing and point to its name, and you can see that they are different. But, there are some things which we talk about but which we can't point to. Take justice, for example. You would have a hard time trying to point to something, and saying that that thing is justice. Also, if you were having a serious discussion about justice, you would get pretty annoyed with someone who claimed he could show you what justice is and did it by handing you a piece of paper like this:

Most people would agree that justice isn't something you can touch or see or hear. Numbers are like that, also. Someone can show you 2 apples or 2 toys or a hat and coat, but none of these pairs of things is the number 2 itself. But, just as people can think about justice even though they can't see or touch it, they can also think about the number 2. And, in talking about and working with the ideas of justice and 2, they use marks such as:

$$justice \qquad\qquad 2$$

but they don't [at least they shouldn't] think that the marks themselves are justice and the number 2.

Now, in reading a mathematics textbook or in listening to someone who is trying to teach you mathematics, it is very important that you be able to tell when the book or person is talking about numbers and when the book or person is talking about the marks which are names of numbers. Al had trouble in making sense out of Stan's explanations because Al couldn't tell when

Stan was talking about marks on paper and when he was talking about
the numbers which the marks named.   And <u>you</u> wouldn't have been
able to tell this from Stan's letter either if you didn't already know a
lot about arithmetic.   Al's trouble with Stan's explanation was that

(1)   in writing out the test, Stan had used the mark:

2

and the mark:

21

as names for numbers [so, his first question was about the
number two and the number twenty-one], but

(2)   Al thought that Stan's first question was about the marks
themselves, and

(3)   in trying to explain Al's errors to him, Stan began by using
the marks as names for themselves ["if you do take 2 away
from 21 you are left with 1"], and then used them again as
names for numbers ["when you take 2 away from 21 you get
19"].

No wonder Al was confused!   <u>In</u> <u>order</u> <u>to</u> <u>write</u> <u>about</u> <u>the</u> <u>marks</u> <u>Stan</u>
<u>needed</u> <u>names</u> <u>for</u> <u>them.</u>   He might have written:

Al, let's use:

Tom

as a name for the mark:

2,

and use:

Dick

and:

Harry

as names for the mark:

21

and the mark:

1.

Now, if you do take Tom away from Dick, you
are left with Harry.   But when you take 2 away
from 21, you get 19.

Or, he might have told Al that he would write a name for a mark by drawing a loop around it. When Al saw a loop, he would know that it, together with the mark inside it, was a word, and that this word was a name for the mark. So, Stan might have written:

> I can see how you thought that the answer was
> (1) because if you do take (2) away from (21),
> you are left with (1). But, when you take 2
> away from 21, you get 19.

Of course, this statement wouldn't teach Al how to subtract, but it would at least tell him that Stan realized that Al was confusing numbers with names of numbers.

Can you think of other devices you could use to write names for marks?

In this book, the device we shall use most frequently to write names for marks is to enclose marks in single quotes ['...']. When you see a mark [such as a word or a sentence, for example] with single quotes around it, you are seeing a name for the mark which is written between the single quotes.

So, the sentence:

> '3' is a name for a number

is about the <u>mark</u>:

> 3

which is <u>named</u> in the sentence. On the other hand, the sentence:

> 3 is an odd number

is about the <u>number</u> which the same mark <u>names</u>.

To decide whether or not to put single quotes around a mark, ask yourself:

> Am I talking about the <u>mark,</u> itself?

> or

> Am I talking about the <u>thing</u> to which the mark refers?

When you are talking with a person, you can sometimes get along without using names for the things you are talking about. You do this by pointing at a thing and saying 'this'. For instance, Stan could have made his explanation to Al, if they had been together, by pointing, first at the '2' in the '21', then at the '21', and then at the '1', and saying:

If you take this away from this, you are left with this.

Sometimes in writing we shall avoid using a name for a mark by a kind of pointing at the mark. [In fact, we have already done so.]

We can point at a mark by displaying it on a separate line, and putting a colon at the end of the preceding line. Think of the colon as the writer's finger pointing at the mark which follows it.

Sometimes we want to point at a word just to call special attention to it, either because it's especially important or because it's a new word.

(1)   'John' is a <u>word</u>, but John is a <u>boy</u>.

(2)   Names of numbers are called <u>numerals</u>.

(3)   Notice that in '$[8 \times (3 + 2)] \times 5$' we use both parentheses and <u>brackets</u>.

In sentence (1), the words 'word' and 'boy' are underlined merely because we wish to emphasize a contrast. In sentence (2), the word 'numeral' is a new word, and we want to call your attention to it. Actually, the sentence is about this word and, so, should be written:

Names of numbers are called 'numerals'.

But, because the word is underlined in (2), and follows 'are called', our carelessness in omitting the single quotes is not likely to cause confusion. In sentence (3), the word 'brackets' is underlined because it is a word which has not been used before. But [unlike the word 'numeral' in (2)] the sentence isn't about this word, so it would be incorrect to enclose it in single quotes.

Read the following paragraphs carefully and note the use of various signals to tell you when names and marks are being discussed.

This morning a girl named <u>Ruth</u> received four letters. Four friends had written to Ruth, and each of them wrote 'Ruth' on the envelope. Each letter started with:

Dear Ruth,

Notice that Ruth has four letters and that 'Ruth' has four letters.

Ruth sometimes wished her name were longer so that she could give herself a short nickname. Her friend Margaret was often called <u>Peg</u>. 'Peg' is short even though Peg is tall. Ruth often wondered how you get 'Peg' out of 'Margaret'.

EXERCISES

A. Some of the following sentences make sense and some do not. In the case of each sentence which does not, put single quotes around some of the words in the sentence so that the resulting sentence does make sense.

1. Bill has a dog.

2. Bill found a dog in a book.

3. I have trouble with my pen when I make a 3.

4. Mary is a part of Maryland.

5. He erased the 5 and put a 4 in its place.

6. John has four letters.

7. Ada is taller than Penelope but Ada is shorter than Penelope.

8. Mr. is an abbreviation for Mister.

9. 6 is a number and 6 is a place holder. Similarly, 0 is a number and 0 is a place holder.

1/8

You may want to agree upon some device with your class for pro-
nouncing single quotation marks. For example, we on the Project
Staff would read the second sentence in the displayed paragraph as:

> Four friends had written to Ruth, and each of
> them wrote squote Ruth unsquote on the envelope.

However, the purpose of reading this paragraph is to see how it is
written [in addition to what it says], and the purpose of reading it
aloud is to focus the student's attention on how the paragraph is
written. Ordinarily, in contrast to this, the purpose of reading
aloud is to focus attention on what the written language says. One
is then translating written language into spoken language rather
than, as here, using spoken language to describe written language.
If one wished to use spoken language to convey the meaning of the
second (written) sentence in the displayed paragraph, one would make
the sounds indicated by:

> four friends had written to ruth and each of
> them wrote the word ruth on the envelope.

[Here, the underlining indicates stress.] This has implications (for
your vocal activities in the classroom) which are discussed on
TC[1-L].

✳

Single quotation marks are used to write names of marks. Obviously,
they can be used in writing names of names of marks, etc.

> John is a boy,
>   'John' is a boy's name,
>   ' 'John' ' is a name for a boy's name.

There are some excellent exercises of this type in Alfred Tarski's
Introduction to Logic (New York: Oxford University Press, 1954).
Incidentally, you will find much material in this book which will be
helpful in handling the ideas in the UICSM program.

✳

1/9

Answers for Part A.

1. Bill has a dog.

2. Bill found a 'dog' in a book.

3. I have trouble with my pen when I make a '3'.

4. 'Mary' is a part of 'Maryland'.

5. He erased the '5' and put a '4' in its place.

6. John has four letters. [or: 'John' has four letters.]

7. Ada is taller than Penelope but 'Ada' is shorter than 'Penelope'.

8. 'Mr.' is an abbreviation for 'Mister'. [An abbreviation of a mark is a second mark which is used in place of the first mark because it is convenient to do so. In most cases the convenience is due to the fact that the abbreviation is shorter than the mark which it abbreviates. But the former need not be a piece of the latter. Example: 'won't' for 'will not'.]

9. 6 is a number and '6' is a place holder. Similarly, 0 is a number and '0' is a place holder.

Exercise 9 may cause some difficulty. In fact, there may even be students who have not heard the expression 'place holder'. But, many students have been taught that the number 0 is not really a number at all but just a place holder in a place value system for writing numerals. This is utter nonsense, and is undoubtedly a cause of many of the so-called zero-difficulties that students encounter in arithmetic. There is a number 0 and there is a symbol '0'. The symbol '0' [sometimes called 'cipher'] is a name for the number 0. In discussions of the decimal system of numeration the symbols '0', '1', '2', ..., '9' are called <u>digits</u>. None of them is a number, but each is a decimal name for a number. The rules for the decimal system of naming numbers tell one how to combine digits to obtain decimal names for numbers. This process is analogous to that of combining letters of the alphabet to obtain nouns in the English language. [The analogy is not too strict, because the digits are themselves nouns, while the letters of the alphabet are not nouns; and, while the meaning of a decimal numeral depends on the meanings of the digits of which it is composed (and on the order in which they

occur in the numeral), letters of the alphabet have no meanings which contribute to the meaning of an English word built out of them.] So, while it makes sense to say that the digit '0' (or any other digit) is a place holder in a decimal numeral, the number 0 is no more a place holder than is the number 6 or the number 84.75. [Incidentally, while it makes sense to say that '0' is a place holder in the numeral '809', this is no more helpful than is saying that 'e' is a place holder in the word 'net'. What may be helpful in explaining decimal notation is to say that in '809' the '9' holds (better: occurs in) the units place, the '0' holds the tens place, and the '8' holds the hundreds place.]

1/27

Students who have been exposed to positive and negative numbers in the past may point out that they used plus and minus signs in the conventional position. After their experience with the arrow-numerals, they should be prepared to accept the idea that the formation of numerals is an arbitrary process, and that this text chooses to do it this way rather than the other way. The discussion starting on page 1-80 will give you the rationale for our selection of the superscript signs in place of the conventional signs. You will have no difficulty in training yourself to use the superscript signs. In fact, you will probably find yourself using the superscript signs in your conventional classes. [On the other hand, you may find it difficult to train yourself to say, for example, 'negative three' instead of 'minus three'. Just keep trying!]

Be sure students understand the procedure for forming a numeral for a real number. As mentioned on TC[1-2, 3], we have taken pains to give a careful description to forestall the student's tendency to say that "a real number is a number of arithmetic with a plus or a minus sign attached to it". If more explanation is needed, point out to students that Mrs. Tom Jones is not Tom Jones with a 'Mrs.' attached to him. [Even though 'Mrs. Tom Jones' is 'Tom Jones' with 'Mrs.' attached to it.] [See page 1-108.]

✳

The Exploration Exercises at the bottom of this page provide a transition from the exercises on page 1-6 to the work on addition on page 1-8. A student can translate to the arrow-numerals and proceed as on page 1-6. After a few such translations, he will be able to do the exercise without translating.

You will find sets of exploration exercises scattered throughout the text. The purpose of exploration exercises is to help students build a concept prior to the discussion which follows the exploration exercises. We want students to keep saying to themselves, as they read the text discussion, things like:

Of course! I knew that long ago.

✳

Answers for Exploration Exercises.

1. $^+9$ 2. $^-12$ 3. $^-3$ 4. $^-3$ 5. $^+9$ 6. $^+9$

Here is a punctuated version of the paragraph in Part B.

> I always put milk on my cereal. Otherwise, it is too dry to eat. But, I have never put 'milk' on my cereal. It would be much easier to put 'milk' on cake frosting than to put 'milk' on corn flakes.

As an extra-credit follow-up to Part B you may ask students to punctuate Stan's letter to Al.

※

This display of numerals for 4 is the beginning of our campaign to get students to understand that a numeral such as '3 + 1' is not a command to add 3 and 1 but is merely a name for 4. In fact, the symbol '3 + 1' is a name of the sum of 3 and 1. We think that students who understand that numbers have many names will not make contradictory statements like:

> The sum of two real numbers is a real number, but you can't add 2 and $\sqrt{2}$,

and:

> The sum of two rational numbers is a rational number, but you can't add $\frac{1}{3}$ and $\frac{1}{5}$.

Students in conventional algebra courses seem to have a hard time understanding that '2 + $\sqrt{2}$' is a name for the sum of 2 and the $\sqrt{2}$, and that we do not have a shorter name for this number. Similarly, '$\frac{1}{3} + \frac{1}{5}$' is a name for the sum of $\frac{1}{3}$ and $\frac{1}{5}$; a shorter name for this sum is '$\frac{8}{15}$'.

※

We included the numeral '4 × 0' in the list in order to get students to look at each of these numerals and assure themselves that the numerals did stand for 4. Be prepared for possible displays of gross misunderstanding concerning "zero facts". For example, one teacher reported to us that a student in his class said:

> My parents told me that 4 × 0 is 4, and that 0 × 4 is 0; you can tell which answer is which by looking at the number that comes first.

_B_.   Use single quotes in punctuating the following paragraph in
order to make sense out of it.

> I always put milk on my cereal. Otherwise,
> it is too dry to eat. But, I have never put milk
> on my cereal. It would be much easier to put
> milk on cake frosting than to put milk on corn
> flakes.

[More exercises are in Part _A_, Supplementary Exercises.]

Numbers and numerals.--We have said that numbers are things
that we can think about, talk about, and write about. In order to
talk and write about numbers, we use names for them. Names
of numbers are called numerals or, sometimes, numerical ex-
pressions. Here are a few of the numerals which are names of
the number 4. [There is one numeral among these which doesn't
name 4; find it!]

IV

four

$8 - 4$

$2 + 2$

$7 - 3$ 4×0

4

2×2

$\dfrac{1842 - 1834}{2}$

$(1 + 1) + (1 + 1)$ $\dfrac{6 + 2}{2}$ $3 \times 1\frac{1}{3}$ IIII

$3 + 1$ 4

quatre 1×4 4×1 $628.424 \div 157.106$

$4 + 0$

$72 \div 18$ the product of 12 and $\frac{1}{3}$

$1 + 3$

A number has many, many numerals. Some of a number's
numerals are simpler looking than others. For example, '4' is
probably the simplest looking of all of 4's numerals. Certainly,
it is simpler looking than the numeral '628.424 ÷ 157.106'. Yet,

both of the numerals:

and:

$$628.424 \div 157.106$$

are names of the same number, the number 4.

A short way of saying that each of two numerals names the same number is to write an equality sign between the numerals. Thus, when we write the sentence '5 + 2 = 6 + 1', we are saying that 5 + 2 is the same number as 6 + 1. Our sentence is true because '5 + 2' and '6 + 1' are numerals for the same number. If we write the sentence '9 + 3 = 4 + 7', our sentence is false because '9 + 3' and '4 + 7' are numerals for different numbers. A short way of saying that '9 + 3' and '4 + 7' are numerals for different numbers is to write an '$\neq$' between the numerals. [Pronounce '$\neq$' as you would pronounce 'is not equal to'.] Thus, the sentence '9 + 3 $\neq$ 4 + 7' is true.

Some of the following true sentences are about numbers, and some are about names of numbers. Be sure that you see that each sentence is true.

(1)   4 + 8 = 9 + 3.

(2)   4 is an even number.

(3)   '4' is a numeral for 4.

(4)   '4' is not a number.

(5)   '8 - 3' and '10 $\div$ 2' are numerals for 5.

(6)   '2 + 2' is a name for 4.

(7)   '2 + 2' is a name for 2 + 2.

(8)   '2 + 2' is a name for 3 + 1.

(9)   2 + 2 is the sum of 2 and 2.

(10)   2 + 2 is the sum of 3 and 1.

1/13

If one reads aloud the paragraph beginning 'A short way', the pur-
pose should be to direct attention to the meaning conveyed by the
written words, rather than to the written language itself. So, in
reading the paragraph one should make the sounds indicated by:

> a short way of saying that each of two numerals names the
> same number is to write an equality sign between the
> numerals. thus, when we write the sentence five-plus-two-
> equals-six-plus-one, we are saying that the number five-
> plus-two is the same number as six-plus-one. our sentence
> is true because the numeral five-plus-two and the numeral
> six-plus-one are numerals for the same number. if we write
> the sentence nine-plus-three-equals-four-plus-seven, our
> sentence is false because the numeral nine-plus-three and
> the numeral four-plus-seven are numerals for different num-
> bers. a short way of saying that the numeral nine-plus-three
> and the numeral four-plus-seven are numerals for different
> numbers is to write an equals-with-a-slash-through-it
> between the numerals. pronounce an equals-with-a-slash-
> through-it as you would pronounce the phrase is-not-equal-to.
> thus, the sentence nine-plus-three-is-not-equal-to-four-plus-
> seven is true.

Questions concerning the written sentences (1) - (15) will be easily
settled if these sentences are translated in the above manner into the
spoken language.

<p style="text-align:center">✳</p>

A correct, and sometimes enlightening, way to read '9 + 3 = 4 + 7'
is as you would read '9 + 3 is 4 + 7'. Similarly, '9 + 3 $\neq$ 4 + 7' may
be read as '9 + 3 is not 4 + 7', or as '9 + 3 is different from 4 + 7'.
Note that when the word 'is' occurs between two proper nouns, such
as '9 + 3' and '4 + 7', it expresses identity. But, contrast this with
the use of 'is' in sentences (2), (3), and (4) on page 1-L. Here, and
in some other sentences in this group, the form is:

<p style="text-align:center">[proper noun] is [common noun],</p>

and the word 'is' expresses being one of a kind. In the first usage
the word 'is' is sometimes called the 'is' of identity; in the second
usage it is called the 'is' of predication. One logician has claimed
that these are just two of 27 different 'is's!

<p style="text-align:center">✳</p>

1/14

The discussion on true and false sentences is quite important. A sentence is something which you look at, something which is written on paper or on the blackboard. Sentences such as:

$$5 + 2 = 6 + 1, \qquad \text{and:} \qquad 9 + 3 = 4 + 7$$

convey information to the reader. The information conveyed to the reader by the first sentence is correct information. In such a case, we say that the sentence is true. The second sentence conveys incorrect information, and we say that such a sentence is false. Later in the course--much later--we shall call sentences with equality signs in them 'equations'. Some equations are true and some equations are false. [The students will also learn about sentences which are neither true nor false. These are sentences like 'x + 5 = 9' and 'x + y = y + x'.] After going through this discussion with the class you may want to proceed immediately to the exercises in Parts D and E on page 1-O.

The words 'true' and 'false' are commonly used with many meanings. In this course, we try to use them with just one of these meanings. So, as above, we say of a <u>sentence</u> that it is true, or that it is false, or that it is neither true nor false. For example, we might write:

'5 + 2 = 6 + 1' is true,

or, for that matter:

'5 + 2 = 6 + 1' is false.

Each of <u>these</u> sentences is about the sentence '5 + 2 = 6 + 1'. The first conveys correct information concerning this <u>sentence</u>, and the second conveys incorrect information. The sentence '5 + 2 = 6 + 1', itself, is about certain numbers and the operation of addition. It conveys correct information concerning these numbers and this operation. One may, for emphasis, write [in place of '5 + 2 = 6 + 1']:

it is the case that 5 + 2 = 6 + 1

or:

it is a fact that 5 + 2 = 6 + 1.

But, if one writes:

it is true that 5 + 2 = 6 + 1,

or:

5 + 2 = 6 + 1 is true,

one is using 'true' in the rather confusing sense which the word has in the sentence 'this fact is true' [confusing, because a fact just <u>is</u>].

TC[1-L]b

Or, if you're lucky, it might even look like this:

> 46
> Miss Jones wrote a
> numeral on the blackboard.

*

In technical works on logic, the notion of distinguishing in written language between symbols and their referents is often called 'distinguishing between use and mention'. [See, for example, Willard V. O. Quine, Methods of Logic (New York: Henry Holt and Company, Inc., 1950), pp. 37-38.] In order to mention something you must use a symbol which names it. For example, in the sentence:

> '4' is pronounced as 'for',

the referents mentioned are two symbols, and the symbols used are names of the symbols which are mentioned. Students who have a good understanding of the use and mention distinction will have no trouble in seeing that the following sentence is true:

> 'California' does not have single quotation marks around it.

This example is very much like Part B of the exercises. To see that this sentence is true, do the following. Take a piece of paper and write on it so that the paper looks like this:

> California

Then, write on the same sheet of paper a sentence which describes

Part B is probably the most difficult exercise in the Introduction.
We included it in order to catch the very kind of error referred to
in the exercise. When you look at the correct paper, what you see
is a numeral for the number 4; you <u>don't</u> see the number 4 itself.
When you look at the incorrect paper, what you see is a name of a
numeral for 4. The symbol on the incorrect paper is made up of
three parts--two single quotation marks and a numeral for 4. This
three-part symbol is a name of a name for 4.
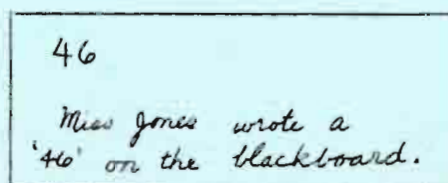
What may confuse you here is that you are not accustomed to stare
at names. You are more accustomed to <u>use</u> names in order to
<u>write about</u> the things they name. You might use the symbol you
see on the incorrect paper in a sentence in which you wished to write
about the numeral which is named by this symbol. [For examples,
consider sentences (3) and (4) on page 1-L, and sentences (12) and
(15) on page 1-M.] In other words, you might use ' '4' ' in a sentence
in which you wished to write about '4', because ' '4' ' is a name for
this numeral. On the other hand, you might use the symbol you see
on the correct paper in a sentence in which you wished to write about
the number which it names. [For example, consider sentences (2),
(3), and (6) on page 1-L, and sentences (11), (13), and (15) on page
1-M.] In other words, you might use '4' in a sentence in which you
wished to write about 4, because '4' is a name for this number.

One way of clarifying this matter in class is to do the following.
Write on the blackboard a numeral for 46. The blackboard should
look like this:

46

Then, ask a student to go to the board and write a sentence which
describes what you just did. The blackboard might then look like this:

46

Miss Jones wrote a
'46' on the blackboard.

(11) $4 + 1 \neq 4 \times 1$.

(12)   You can put a '4' on paper.

(13)   You can't put 4 on paper.

(14)   '$\dfrac{6 + 2}{4}$' and '$\dfrac{8}{4}$' and '2' are numerals for tne same number, but '2' is the simplest of these numerals.

(15)   You can't write 4 but you can write a '4'.

EXERCISES

<u>A</u>.   Write 5 numerals for 6 and 5 numerals for 0.

<u>B</u>.   A student was asked to write on a sheet of paper a numeral for 4. This is what his paper looked like:
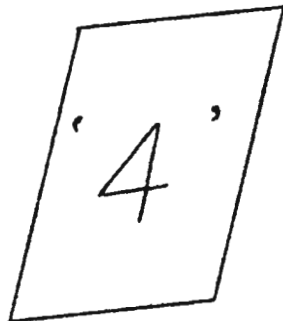
He was wrong. His paper could have looked like this and he would have been correct:
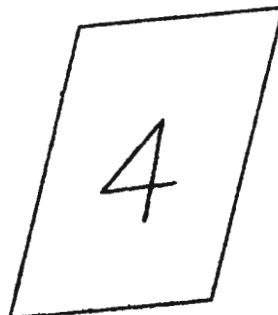
Explain. [Did you make this error in Part A?]

C. The following is a long <u>list</u> of numbers. Although the list con-
tains many numerals, only three numbers are listed. Rearrange
the list into three columns such that all of the numerals in a
column stand for the same number.

[Notice how the word 'list' is used. If someone asks you for
a list of the students in your mathematics class, he doesn't ex-
pect you to collect all the students and to march them to him.
What he hopes is that you will write down the names of the stu-
dents, and that you will then hand him a sheet of paper with names
written on it. This sheet of paper is a list of the students. And
if you wanted to make yourself unpopular, you could write down
nicknames for each student as well as their given names. You
would still be listing the students, but the person reading the
list might think there were many more students in the class. So,
it makes sense to say that although the following list contains many
numerals, only three numbers are listed.]

9×2 $\qquad$ $\dfrac{9 - 6}{10 - 5}$ $\qquad$ $3 \times (7 - 1)$ $\qquad$ $\dfrac{4}{3} + \dfrac{50}{3}$ $\qquad$ $\dfrac{3 - 3}{842}$

$12 \times .05$ $\qquad$ $\dfrac{7 + 3}{15 - 5} - \dfrac{52}{26 \times 2}$ $\qquad$ $29 - 11$ $\qquad$ $\dfrac{7}{5} - \dfrac{4}{5}$ $\qquad$ $15 + 3$

$\left(\dfrac{1}{50} + \dfrac{3}{25}\right) + \left(\dfrac{2}{5} + \dfrac{3}{50}\right)$ $\qquad$ $\dfrac{60}{5} + \dfrac{30}{5}$ $\qquad$ $\dfrac{0}{18}$ $\qquad$ $(3 + 5) + 10$ $\qquad$ $0 + 18$

$\dfrac{6}{10}$ $\qquad$ $8 - 8$ $\qquad$ $(74 - 70) - 4$ $\qquad$ $35 - (10 + 7)$ $\qquad$ $\dfrac{6 + 0}{10 + 0}$

$9.72 + 8.28$ $\qquad$ $\dfrac{3 \times 978}{5 \times 978}$ $\qquad$ $\dfrac{2}{3} - \dfrac{58}{87}$ $\qquad$ $(4 \times 2) + (5 \times 2)$

$\dfrac{(9 + 3) - (2 \times 4.5)}{(18 - 3) \div (7 - 4)}$ $\qquad$ $18 - (3 \times 6)$ $\qquad$ $\left(90 \times \dfrac{4}{9}\right) \times (0.3 \times 1.5)$

1/18

This kind of exercise occurs frequently throughout the UICSM text materials. We felt that it was necessary to explain carefully how we were using the word 'list'. This exercise tends to reinforce the idea that a number has many names. You may be surprised by the fact that students have little difficulty in interpreting the grouping symbols in these numerals. Certainly, they will have more trouble in doing the calculations than in interpreting the grouping symbols.

If a student points to the '9' in '9 × 2' and claims that the number 9 is listed, you might ask if John Jones is listed in a list which contains the entry:

Mrs. John Jones.

✳

Here are the three columns for Part C.

9×2

$3 \times (7 - 1)$

$\dfrac{4}{3} + \dfrac{50}{3}$

$29 - 11$

$15 + 3$

$\dfrac{60}{5} + \dfrac{30}{5}$

$(3 + 5) + 10$

$0 + 18$

$35 - (10 + 7)$

$9.72 + 8.28$

$(4 \times 2) + (5 \times 2)$

$\left(90 \times \dfrac{4}{9}\right) \times (0.3 \times 1.5)$

$\dfrac{9 - 6}{10 - 5}$

$12 \times .05$

$\dfrac{7}{5} - \dfrac{4}{5}$

$\left(\dfrac{1}{50} + \dfrac{3}{25}\right) + \left(\dfrac{2}{5} + \dfrac{3}{50}\right)$

$\dfrac{6}{10}$

$\dfrac{6 + 0}{10 + 0}$

$\dfrac{3 \times 978}{5 \times 978}$

$\dfrac{(9 + 3) - (2 \times 4.5)}{(18 - 3) \div (7 - 4)}$

$\dfrac{3 - 3}{842}$

$\dfrac{7 + 3}{15 - 5} - \dfrac{52}{26 \times 2}$

$\dfrac{0}{18}$

$8 - 8$

$(74 - 70) - 4$

$\dfrac{2}{3} - \dfrac{58}{87}$

$18 - (3 \times 6)$

C. The following is a long <u>list</u> of numbers. Although the list con-
tains many numerals, only three numbers are listed. Rearrange
the list into three columns such that all of the numerals in a
column stand for the same number.

    [Notice how the word 'list' is used. If someone asks you for
a list of the students in your mathematics class, he doesn't ex-
pect you to collect all the students and to march them to him.
What he hopes is that you will write down the names of the stu-
dents, and that you will then hand him a sheet of paper with names
written on it. This sheet of paper is a list of the students. And
if you wanted to make yourself unpopular, you could write down
nicknames for each student as well as their given names. You
would still be listing the students, but the person reading the
list might think there were many more students in the class. So,
it makes sense to say that although the following list contains many
numerals, only three numbers are listed.]

9×2	$\dfrac{9-6}{10-5}$	$3 \times (7-1)$	$\dfrac{4}{3} + \dfrac{50}{3}$	$\dfrac{3-3}{842}$
$12 \times .05$	$\dfrac{7+3}{15-5} - \dfrac{52}{26 \times 2}$		$29 - 11$	$\dfrac{7}{5} - \dfrac{4}{5}$	$15 + 3$
$\left(\dfrac{1}{50} + \dfrac{3}{25}\right) + \left(\dfrac{2}{5} + \dfrac{3}{50}\right)$	$\dfrac{60}{5} + \dfrac{30}{5}$	$\dfrac{0}{18}$	$(3+5) + 10$	$0 + 18$
$\dfrac{6}{10}$	$8 - 8$	$(74 - 70) - 4$	$35 - (10 + 7)$	$\dfrac{6+0}{10+0}$
$9.72 + 8.28$	$\dfrac{3 \times 978}{5 \times 978}$	$\dfrac{2}{3} - \dfrac{58}{87}$	$(4 \times 2) + (5 \times 2)$	
$\dfrac{(9+3) - (2 \times 4.5)}{(18-3) \div (7-4)}$	$18 - (3 \times 6)$	$\left(90 \times \dfrac{4}{9}\right) \times (0.3 \times 1.5)$		

The exercises in Part D illustrate several important ideas. We have found elementary school students [grade 4] who felt that the sentence in Exercise 7 was false because "you just wouldn't write something like that". This illustrates a common misconception that the relation of equality is not symmetric. These grade school children felt that the equality sign acted like an arrow which pointed to the correct answer. You can readily see the importance of removing this misconception.

The sequence of Exercises 9, 10, 11, and 12 is important. A student will say 'true' for Exercise 9, and jump to the conclusion that the sentence in 10 is true also. What is happening here is that the student thinks he is looking at instances of the commutativity principles in these exercises. When he finds that the sentence in Exercise 10 is false, he then attacks the sentence in Exercise 11 with more care, and may be surprised to find that it is true, after all. By the time he gets through with Exercise 12, he is willing to go back over all four exercises and take another look, this time a more careful one.

✻

Here are answers for Part D.

1.	F	2.	T	3.	F	4.	F	5.	T	6.	F
7.	T	8.	F	9.	T	10.	F	11.	T	12.	F

✻

The exercises in Part E are typical of the kinds of completion exercises which occur throughout the text. The purposes of these exercises are to build the concept of equation-solving, and to establish awareness at a nonverbal level of some of the basic principles of arithmetic. [Please do not mention these principles at this time.] Exercises 5 and 6 illustrate this purpose. After students have protested the "unfairness" of Exercise 6(b), point out to them that they will find many such exercises in this text. We expect students to be alert when they do exercises. A correct answer for Exercise 6(b) is:

No numeral will convert this into a false sentence.

[Some teachers have reported that students submitted sentences

D. True or false?

1. $7 + 9 = 4 \times 5$ 2. $3 + 2 = 4 + 1$

3. $8 + 7 = 8 - 7$ 4. $9 \times 5 \ne 40 + 5$

5. $6 \times 2 \ne 10 \times 2$ 6. $8 + 5 = 5 + 3$

7. $8 = (2 + 2) + (2 + 2)$ 8. $2 + (2 + 2) = 2 \times 2$

9. $52 + 68 = 58 + 62$ 10. $52 \times 68 = 58 \times 62$

11. $73 + 92 = 92 + 73$ 12. $73 \times 92 \ne 92 \times 73$

E. Each of the following exercises contains a pair of sentences, and each sentence has a blank space in it. For the first of each pair of sentences, write a numeral in the blank space so that the resulting sentence is true. For the second of the pair, write a numeral in the blank space so that the resulting sentence is false.

> Sample 1. (a) $9 + \underline{\quad} = 7 + 8$
>   (b) $9 + \underline{\quad} = 7 + 8$
>
> Solution. (a) $9 + \underline{6} = 7 + 8$ *True*
>   (b) $9 + \underline{79} = 7 + 8$ *False*
>
> Sample 2. (a) $8 \times \underline{\quad} \ne 12 \times 2$
>   (b) $8 \times \underline{\quad} \ne 12 \times 2$
>
> Solution. (a) $8 \times \underline{5} \ne 12 \times 2$ *True*
>   (b) $8 \times \underline{3} \ne 12 \times 2$ *False*

1. (a) $3 + \underline{\quad} = 5 + 6$ 2. (a) $7 - \underline{\quad} = 3 + 2$
    (b) $3 + \underline{\quad} = 5 + 6$ (b) $7 - \underline{\quad} = 3 + 2$

3. (a) $11 + \underline{\quad} \ne 20 + 7$ 4. (a) $6 \div \underline{\quad} \ne 15 - 12$
    (b) $11 + \underline{\quad} \ne 20 + 7$ (b) $6 \div \underline{\quad} \ne 15 - 12$

5. (a) $\underline{\quad} + 984 = 984 + 793$ 6. (a) $\underline{\quad} \times 0 = 0$
    (b) $\underline{\quad} + 984 = 984 + 793$ (b) $\underline{\quad} \times 0 = 0$

We cannot here <u>define</u> the numbers of arithmetic. We can only hope
that students have previously become acquainted with the numbers of
arithmetic through measurements of magnitudes such as lengths,
areas, volumes, weights, speeds, etc. The numbers of arithmetic
are precisely the numbers which one uses as measures [with respect
to appropriate units] of such magnitudes. For example, the number
2 of arithmetic is

> the inch-measure of a certain
>   length--the length 2 inches,
>
> the foot-measure of another
>   length--the length 24 inches, and
>
> the square-foot-measure of a certain
>   area--the area of a rectangular
>   region of length 2 feet and width 1 foot.

The number $\sqrt{2}$ of arithmetic is the inch-measure of a certain length--
the length of a diagonal of a square of side 1 inch. The number π of
arithmetic is, among other things, the foot-measure of a certain
length--the length of a circle whose diameter has the length 1 foot.
[The number 0 of arithmetic is, perhaps, an exception to the state-
ment that each number of arithmetic is used as a measure of magni-
tudes. Whether it is depends on one's concept of magnitude.]

<center>✳</center>

We introduce real numbers as numbers which can be used to measure
trips, that is, to measure directed changes. Again, this is an attempt
to acquaint students with a type of number through a physical applica-
tion rather than by definition.

However, for <u>your</u> <u>own</u> information, we shall sketch definitions of,
say, the real numbers $^+3$ and $^-2$ in terms of the numbers of arithmetic.
The basis of the definitions is the fact that one can compare two mag-
nitudes of the same kind by comparing their measures with respect to
a chosen unit.

For example, we say that the length 5 feet is longer than the length
24 inches because, comparing their foot-measures, 5 is greater than
2. The relation <u>greater</u> <u>than</u>, holds between numbers a and b of arith-
metic when there is a number c of arithmetic [other than 0] such that
a = b + c. In the example above, 5 is greater than 2 because 5 = 2 + 3
[and 3 $\neq$ 0]. We could express this more precise comparison of the

1.01 <u>Distance</u> <u>and</u> <u>direction</u>.--Imagine an east-west road which has
markers placed one mile apart at the side of the road. The markers
are labeled with letters. If you ride a bicycle from A to G, you say
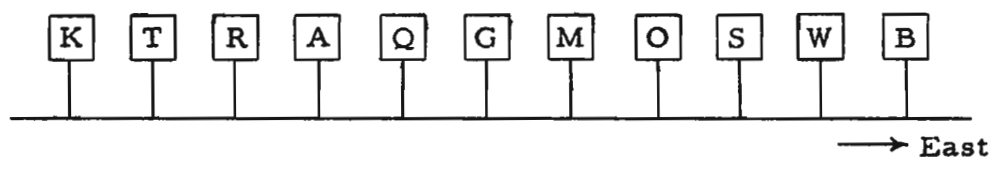
that you have made a trip of 2 miles to the east; if you ride from S
to B, you again say that you have made a trip of 2 miles to the east.
Describe three other 2-miles-to-the-east trips on this road. Describe
three 2-miles-to-the-west trips on this road.

The 2-miles-to-the-east trips and the 2-miles-to-the-west trips
are alike in one important way. The length of each trip is 2 miles
[or, for each trip, the <u>distance</u> in miles between starting and ending
points is 2]. But, the trips are also different in an important way.
The trips to the east are made in a <u>direction</u> opposite to that of the
trips to the west.

Suppose you and a friend are at the point G on this road. Each
of you decides to make a trip that will take you two miles from G.
How many miles apart will you be at the end of your trips? It is
easy to see that you cannot give a definite answer to this question.
It is not enough to know just the distances for the trips; you also have
to know something about the directions.

The numbers with which you have been working in school since
the first grade can be used in measuring distances for trips along
this road. But, using these numbers [let's call them <u>the</u> <u>numbers</u>
<u>of</u> <u>arithmetic</u>] doesn't tell whether the trips have been made in the
same direction or in opposite directions. If two people start at G
and make 2-mile trips, we don't know whether they will be together
at the end of their trips or whether they will be four miles apart.
In order to measure trips which are made in one of two opposite
directions, we need numbers which will take into account both distance

POSITIVE AND NEGATIVE REAL NUMBERS

In the preceding exercises you worked with real numbers, using them to measure trips. We invented numerals for these numbers. We could continue to use these numerals throughout all of our work with real numbers, but it is important that you become familiar with the more standard ones.

The real numbers which we have called 'right real numbers' and 'left real numbers' are commonly called <u>positive numbers</u> and <u>negative numbers</u>, respectively. Positive and negative numbers come in pairs. Each pair contains a positive number and a negative number, and both numbers correspond with the same number of arithmetic. The numbers in each pair can be used to measure trips over the <u>same distance</u> but in <u>opposite directions</u>. The direction of trips measured by positive numbers is called <u>the positive direction</u>, and the direction opposite to the positive direction is called <u>the negative direction</u>.

In naming positive numbers we shall use a ' $+$ ' instead of the ' $\rightarrow$ '. For example, we shall write

$$\text{'} {}^+3 \text{'} \text{ instead of '} \overrightarrow{3} \text{'.}$$

In naming negative numbers, we shall use a ' $-$ ' instead of the ' $\leftarrow$ '. So, we shall write

$$\text{'} {}^-3 \text{'} \text{ instead of '} \overleftarrow{3} \text{'.}$$

We form a numeral for a real number by prefixing a ' $+$ ' or a ' $-$ ' to a numeral for the corresponding number of arithmetic. [The numerals ' $+3$ ' and ' -3 ' are pronounced as 'positive three' and 'negative three', respectively.]

EXPLORATION EXERCISES

Each of the following exercises gives the measures of successive trips. Find a real number which measures the direct trip from the starting point of the first trip to the ending point of the second trip.

1. $^+7$, $^+2$ 2. $^-3$, $^-9$ 3. $^+6$, $^-9$

4. $^-10$, $^+7$ 5. $^-11$, $^+20$ 6. $^+20$, $^-11$

and direction. There are such numbers. They are usually called real numbers.* In this unit, you will learn many things about real numbers--you will learn how to compute with them, you will learn how to use them in solving problems, and you will learn that some of them "act like" our familiar numbers of arithmetic.

NUMERALS FOR REAL NUMBERS

In order to work with real numbers we shall need some system of naming them. We must have numerals for them so that we can talk and write about them. Let us make up a system of numerals which no one we know of has ever used before. After we have seen that it is possible to work with numerals which we have invented, we'll switch to a kind of numerals for real numbers which most other people use. Our ideas of real numbers will not change in switching from one system of numerals to another--only the names will change.

We want to use real numbers to measure trips. Since one important aspect of a trip is the distance between starting and ending points, and since our "old" numbers of arithmetic do serve quite well in measuring distances, we shall use as part of each numeral for a real number a numeral for a number of arithmetic. Then, because we need to include direction in measuring trips, we shall complete the numeral for the real number by including an arrow.

| K | T | R | A | Q | G | M | O | S | W | B |

$\longrightarrow$ East
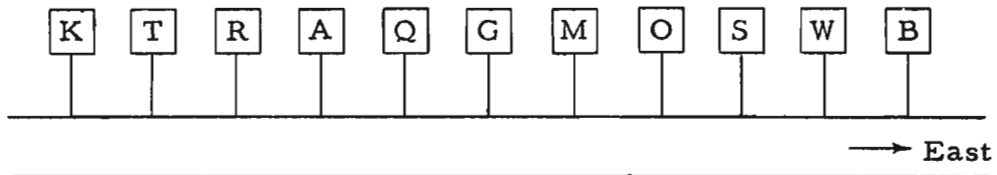
Suppose, again, that you and your friend decide to take 2-mile trips starting at G. If your trip is measured by the real number $\overrightarrow{2}$ and

---

*Real numbers are not any more (or less) real than other kinds of numbers. In a later course [when you learn about complex numbers] you will see why, historically, the word 'real' came to be used. Sometimes real numbers are called directed numbers or signed numbers.

Note the careful description of how numerals for real numbers are formed. We do not want the student to get the notion that "a real number is a number of arithmetic with a plus sign or a minus sign stuck in front of it". We can't stick a plus sign in front of a number since a number is an abstraction and has neither front nor back nor top nor bottom! It is for this reason that <u>we</u> do not use the term 'signed numbers' when referring to the <u>real</u> numbers.

✳

As soon as the student encounters the numeral '$\overrightarrow{2}$', he will want to pronounce it. This will certainly be the case if he is reading aloud. Let the class agree on a pronunciation of this numeral as well as the numeral which includes the arrow pointing to the left at the top of page 1-3. We give a pronunciation just before the exercises on page 1-3. If the students did not suggest a pronunciation which is in agreement with the one we gave, they can use their own or switch to ours. This is a good opportunity to point out the arbitrariness of pronunciations of mathematical symbols. Since these numerals will be dropped after a page or so, it is not important which pronunciation is used. However, for a symbol for which there is a "standard" pronunciation, the students should see that it is usually better to adopt it than one of their own making.

✳

Answers for Part A [on page 1-3].

1. (a) $\overrightarrow{2}$ (b) $\overrightarrow{3}$ (c) $\overleftarrow{2}$ (d) $\overleftarrow{6}$ (e) $\overrightarrow{6}$ (f) $\overrightarrow{3}$

2. (a) K to S, T to W, R to B   (b) B to R, W to T, S to K

    (c) K to M, T to O, R to S, etc. (d) B to G, W to Q, S to A, etc.

    (e) K to T, etc.   (f) K to the point halfway
                                           between R and A, etc.

your friend's by the real number $\overleftarrow{2}$, you can be sure that you will be 4 miles apart at the end of your trips. How far apart will you be if each of you makes a trip of $\overrightarrow{2}$ miles? If each makes a trip of $\overleftarrow{2}$ miles? Suppose you take a trip of $\overleftarrow{3}$ miles and your friend takes one of $\overrightarrow{5}$ miles, starting from the same point. What real number measures the trip which your friend will have to take to get from where he is to where you are?

Notice that we have not yet said whether the real number $\overrightarrow{2}$ measures a trip-to-the-east or a trip-to-the-west. This is something which should be decided for each problem in which you want to measure trips made in one of two opposite directions.

In working with numbers like $\overrightarrow{3}$ and $\overleftarrow{9}$, you will want to talk about them as well as write about them. So, you need to decide upon the pronunciation of their numerals. Let's agree to pronounce '$\overrightarrow{3}$' as you would pronounce 'right three' and '$\overleftarrow{9}$' as 'left nine'.

EXERCISES

A. Let us agree that the number $\overrightarrow{2}$ measures a 2-miles-to-the-east trip. Then $\overleftarrow{2}$ measures a 2-mile trip in the opposite direction.
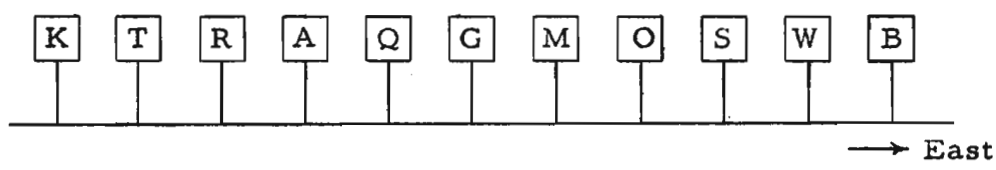
1.   Give the real numbers which measure the trips listed.

   (a)   K to R   (b)   R to G   (c)   W to O

   (d)   B to Q   (e)   Q to B   (f)   G to S

2.   List three trips which are measured by each of the given real numbers.

   (a) $\overrightarrow{8}$ (b) $\overleftarrow{8}$ (c) $\overrightarrow{6}$

   (d) $\overleftarrow{5}$ (e) $\overrightarrow{1}$ (f) $\overrightarrow{2.5}$

<u>B.</u>   Consider taking trips along a north-south
road. Let $\overrightarrow{2}$ measure a 2-miles-to-the-
south trip. Then $\overleftarrow{2}$ measures a 2-mile
trip in the opposite direction.

N
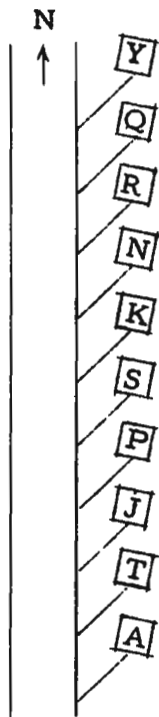
1.   Give the real numbers which measure
these trips.

(a)   T to K   (b)   Y to R

(c)   P to A   (d)   A to P

2.   List three trips which are measured
by each of these real numbers.

(a) $\overrightarrow{3}$ (b) $\overleftarrow{3}$ (c) $\overleftarrow{3.5}$

Markers from top to bottom: Y, Q, R, N, K, S, P, J, T, A

<u>C.</u>   Imagine trips taken along an east-west road, and trips to the
east measured by right real numbers. Each of the following
statements gives the starting and ending points of a trip and
the real number which measures the trip. Use them to fill in
the markers on the map.

Markers left to right, middle one labeled: A

⟶ East
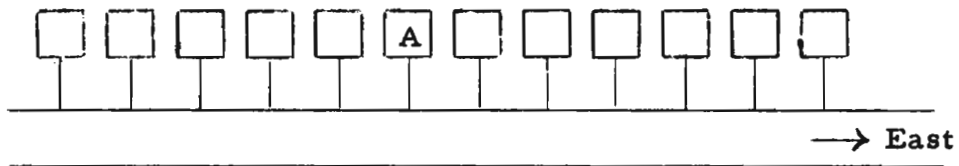
(1)   a trip from A to G is measured by $\overleftarrow{4}$.

(2)   a trip from J to G is measured by $\overleftarrow{2}$.

(3)   J to B, $\overleftarrow{3}$ (4)   I to E, $\overleftarrow{2}$

(5)   F to C, $\overleftarrow{2}$ (6)   G to D, $\overrightarrow{5}$

(7)   L to H, $\overleftarrow{2}$ (8)   B to A, $\overrightarrow{5}$

(9)   A to F, $\overrightarrow{5}$ (10)   E to C, $\overrightarrow{6}$

(11)   L to F, $\overrightarrow{1}$ (12)   K to E, $\overleftarrow{9}$

✳ ✳ ✳

1/24

The main purpose of Part B is to provide variety in the use of real numbers, and to emphasize the direction-distance aspect of trips. The use of the arrow-numerals might lead students to believe that the maps used had to be oriented left to right.

✳

Answers for Part B.

1. (a) $\overleftarrow{4}$ (b) $\overrightarrow{2}$ (c) $\overrightarrow{3}$ (d) $\overleftarrow{3}$

2. (a) Y to N, Q to K, R to S, etc.

    (b) A to P, T to S, J to K, etc.

    (c) A to the point halfway between P and S,

         T to the point halfway between S and K,

         J to the point halfway between K and N, etc.

✳

Part C emphasizes that each trip has an initial point and a terminal point. The real number used to measure the trip tells you the direction of the terminal point from the initial point as well as the distance between the initial and terminal points. A thought-provoking question to ask when students have finished Part C is: How would you fill in the markers if trips to the east were measured by left real numbers?

✳

Answer for Part C.

When the markers are correctly filled in, the map will look like this.

| B | G | E | J | I | A | D | H | C | L | F | K |

——→ East
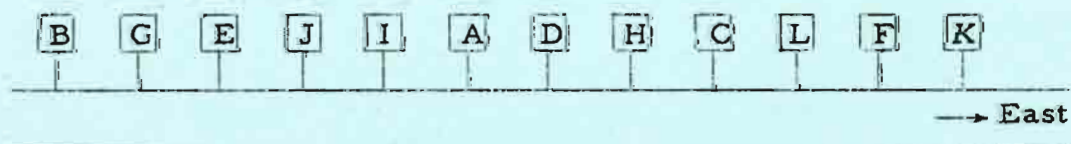
1/25

The purpose of Part D is to develop the concept of addition of real numbers. Do not tell students the purpose of this exercise since it is better pedagogy to have the students fully aware of the concept before naming it. Certainly, it will not increase their effectiveness to be told that they are actually adding real numbers; in fact, you might detract from this effectiveness since they may bring to the solution of the exercises misleading connotations of the word 'addition'. By the time students have done a quarter of the exercises on page 1-6, they will have developed their own rules for addition. Some students may want to tell you and the class the short cut they have discovered. Do not allow them to make a public statement. Announce that you expect students to find short cuts, but that you don't want the game spoiled for others in the class by having a student tell what his short cut is. If a student is overcome with the desire to let you know that he can tell you the rule, then go to his desk, lean over, and have him whisper the rule to you. You will, of course, hear some kind of jumble of words in which the ideas of numeral and number are unfortunately mixed, but do not try to get the student to formulate his rule precisely. It is best to avoid any verbalization at all at this time because it is much too difficult a task for the student. [The rules are given formally in Unit 2.]

※

Suppose, in checking answers to these exercises, a student gives a wrong answer for, say, Exercise 16. If you want to work this exercise for the class, you must use the same physical interpretation as was given in the samples. In other words, it should always be possible for a student to find answers to these exercises by using the physical interpretation. Theoretically, he does not need the short cut addition rules. However, no student will be satisfied with the long, drawn out method of solving these problems. He will start an active search for a short cut. The trick is to get him to discover this short cut by himself. This is probably the first opportunity in the UICSM program for a student to use his own resources in finding a short cut, and the problem is easy enough so that each student will meet with success. If someone else gives the game away [teacher, fellow student, or parent], then you have lost an opportunity to build the attitude in the student that mathematics is a subject which makes sense because it "comes from" the student himself.

One of the important things you learned in Part C was that
if you knew the direction of trips measured by right real num-
bers, you could then tell the location of a second point with
respect to a first point just by knowing the real number which
measures the trip from the first point to the second point.

<p align="center">✳ ✳ ✳</p>

D. Each of the following exercises is a list of real numbers. These
are measures of successive trips along an east-west road. [Right
real numbers are used to measure trips-to-the-east.] The first
real number in each list measures a trip starting at a point A.
The second measures a trip whose starting point is the ending
point of the trip from A. The third measures a trip whose start-
ing point is the ending point of the second trip. Etc.

Your job in each exercise is to tell the location of the ending
point of the last trip with respect to A by giving the real number
which measures the trip from A to that ending point.

Sample 1. $\overrightarrow{3}$, $\overrightarrow{5}$.

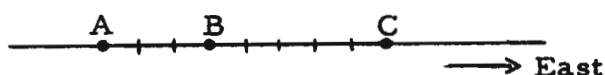Solution. One way to solve this problem is to make a
sketch of an east-west road and mark a point A.

A
⟶ East

Then mark the ending point of the trip measured by $\overrightarrow{3}$.
This is a point 3 units to the right of A.

A   B
⟶ East

Now mark the ending point of a $\overrightarrow{5}$-unit trip whose starting
point is B. This is a point 5 units to the right of B.

A   B   C
⟶ East

1.02 <u>Addition</u> <u>of</u> <u>real</u> <u>numbers</u>. --In the Exploration Exercises
you practiced finding the measure of a trip made from the start-
ing point of the first of two successive trips to the ending point
of the second. For example, consider a $^+3$-mile trip followed by
a $^+5$-mile trip. The direct trip which takes you from the starting
point of the $^+3$-mile trip to the ending point of the $^+5$-mile trip is a
trip of $^+8$ miles. And, similarly, a direct trip which takes you
from the starting point of a $^-9$-mile trip to the ending point of the
$^-10$-mile trip which follows it is a trip of $^-19$ miles.

The idea of a pair of trips one of which is tacked onto the other
suggests the notion of "adding". And, this suggests that what you
did with the measures of two such trips to find the measure of the
direct trip should be called <u>addition</u> <u>of</u> <u>real</u> <u>numbers</u>. The examples
above, then, show that

$$^+3 + {}^+5 = {}^+8$$

and that $$^-9 + {}^-10 = {}^-19.$$

Take still another example.

$$^+9 + {}^-5 = ?$$

We have decided that the sum of a first real number and a second
real number is the measure of a direct trip from the starting point
of one trip to the ending point of a following trip, the trips being
measured by the first and second real numbers, respectively. So,
$^+9 + {}^-5$ is the measure of a direct trip which takes us from a starting
point to that point which would be reached by a $^+9$-mile trip followed
by a $^-5$-mile trip. Do you see that such a direct trip is a $^+4$-mile
trip? We say that

$$^+9 + {}^-5 = {}^+4.$$

Can you solve the problem:

$$^-7 + {}^+2 = ?$$

One of the important things you learned in Part C was that
if you knew the direction of trips measured by right real num-
bers, you could then tell the location of a second point with
respect to a first point just by knowing the real number which
measures the trip from the first point to the second point.

* * *

D. Each of the following exercises is a list of real numbers. These
are measures of successive trips along an east-west road. [Right
real numbers are used to measure trips-to-the-east.] The first
real number in each list measures a trip starting at a point A.
The second measures a trip whose starting point is the ending
point of the trip from A. The third measures a trip whose start-
ing point is the ending point of the second trip. Etc.

   Your job in each exercise is to tell the location of the ending
point of the last trip with respect to A by giving the real number
which measures the trip from A to that ending point.
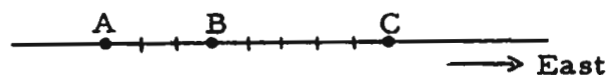
   Sample 1. $\vec{3}, \vec{5}$.

   Solution. One way to solve this problem is to make a
            sketch of an east-west road and mark a point A.

   Then mark the ending point of the trip measured by $\vec{3}$.
   This is a point 3 units to the right of A.

   Now mark the ending point of a $\vec{5}$-unit trip whose starting
   point is B. This is a point 5 units to the right of B.

The use of the word 'addition' in naming an operation with real numbers is the cause of a confusion which is not usually recognized in conventional textbooks. The first confusion of this type occurred when we used the word 'numbers' in connection with real numbers. The word 'numbers' is used by a student in elementary school, first, with reference to natural numbers. Then the word 'numbers' is used to refer to rational numbers, which are different entities from natural numbers. Then the word 'numbers' is used to refer to entities such as π and $\sqrt{2}$ and other irrational numbers, entities of still another kind. Finally, we use the word 'numbers' to refer to real numbers, things which are different from all the other things a student has learned to call 'numbers'. [And, in a later unit, he will use the word 'numbers' for still different things, the complex numbers.] So, the student is continually "enlarging" the meaning of the word 'numbers'. Now, this is a perfectly natural process and is a concomitant of growing up. However, the teacher needs to realize that the student may have difficulties in these successive enlargements of the meaning of a word. [An indication of the reluctance of children to make this enlargement is seen in the use of the word 'fractions' to designate rational numbers. You may find some students who will claim that a rational number is not a number at all. These students still think that 'numbers' refers only to the natural numbers.]

A similar enlargement of meaning takes place with the word 'addition'. We have the operation of addition of natural numbers, the operation of addition of rational numbers, and the operation of addition of numbers of arithmetic. These are distinct operations. * But the word which names the operation is the same in each case. The elementary school student does not seem confused by this enlargement of meaning because in all three cases the notion of addition involves the notion of increasing. [On the other hand, there is much confusion in connection with the word 'multiplication'. The student first learns this word in connection with an operation with natural numbers. In that case the operation is closely related to the operation of addition, and 'multiplication' has the connotation of increasing. But, students frequently

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*An operation is a set of ordered pairs. [See pages 1-67 and 1-107.] The operation adding 2 defined on the set of natural numbers contains such pairs as $(1, 3)$, $(2, 4)$, $(3, 5)$, etc. The operation adding 2 defined on the set of rational numbers contains $(0, 2)$, $(1/2, 5/2)$, $(1, 3)$, $(5/3, 11/3)$, etc. These are evidently different sets of ordered pairs, and so are two operations, even though both go by the same name, adding 2.

1/30

Answers for Part A [on pages 1-9 and 1-10].

1. $^+4$	2. $^+11$	3. $^-5$	4. $^-1$	5. $^-2$
6. $^+6$	7. $^+7$	8. $^-10$	9. $^+1$	10. $^+10$
11. $^+4$	12. $^+8$	13. $^+1$	14. $^+35$	15. $^-1$
16. $^-35$	17. $^+10.2$	18. $^+5.6$	19. $^-6.9$	20. $^-21$
21. $^-419$	22. $^+9$	23. $^+5$	24. $^+1$	25. $^-1$
26. $^+6$	27. $^+15$	28. $^+15$	29. $^+27$	30. $^+7$
31. $^-8$	32. $^+4$	33. $^-4$	34. $^+6$	35. $^+1$
36. $^+5$	37. $^+\frac{1}{5}$	38. $^+\frac{5}{6}$	39. $^-8$	40. $^+4$
41. $^-2$	42. $^+1$	43. $^-1$	44. $^+4$	45. $^-4$
46. $^+1$	47. $^-1$	48. $^+6$	49. $^+1$	50. $^-74$
51. $^+36$	52. $^-106$	53. $^+106$	54. $^-\frac{2}{5}$	55. $^-\frac{5}{7}$
56. $^+\frac{1}{4}$	57. $^-4\frac{3}{4}$	58. $^+1.4$	59. $^-5\frac{5}{6}$	60. $^+1444$
61. $^+1444$	62. $^+6$	63. $^-5$	64. $^+15$	65. $^-129$
66. $^+187$	67. $^-1482$	68. $^+3$		

*

[Note of warning! In all parts of the Miscellaneous Exercises except Parts I and J, we have used the oppositing symbol '−' when naming negative numbers. Therefore, you should not assign exercises from the Miscellaneous Exercises section of the book until after the "New Names for Negative Numbers" subsection [pages 1-86 ff.] has been studied.]

EXERCISES

<u>A</u>. Simplify each of the following numerals.

      <u>Sample</u>. $^+3$ + $^+4$

       <u>Solution</u>. $^+7$

1. $^+6$ + $^-2$ 2. $^+3$ + $^+8$ 3. $^-2$ + $^-3$

4. $^+3$ + $^-4$ 5. $^+3$ + $^-5$ 6. $^+8$ + $^-2$

7. $^+6$ + $^+1$ 8. $^-2$ + $^-8$ 9. $^+9$ + $^-8$

10. $^+3$ + $^+7$ 11. $^+11$ + $^-7$ 12. $^+3$ + $^+5$

13. $^+18$ + $^-17$ 14. $^+18$ + $^+17$ 15. $^-18$ + $^+17$

16. $^-18$ + $^-17$ 17. $^+4.3$ + $^+5.9$ 18. $^-2.7$ + $^+8.3$

19. $^+12.4$ + $^-19.3$ 20. $^+81$ + $^-102$ 21. $^-765$ + $^+346$

      22. $(^+6 + ^+7) + ^-4$ 23. $(^-3 + ^+6) + ^+2$

24. $^+4$ + $^-3$ 25. $^+7$ + $^-8$ 26. $^+9$ + $^-3$

27. $^+5$ + $^+10$ 28. $^+3$ + $^+12$ 29. $^+12$ + $^+15$

30. $^+2$ + $^+5$ 31. $^-4$ + $^-4$ 32. $^-3$ + $^+7$

33. $^-7$ + $^+3$ 34. $^+2.5$ + $^+3.5$ 35. $^-1$ + $^+2$

36. $\dfrac{^+9}{3} + \dfrac{^+6}{3}$ 37. $\dfrac{^+3}{5} + \dfrac{^-2}{5}$ 38. $\dfrac{^+1}{3} + \dfrac{^+1}{2}$

39. $^-3$ + $^-5$ 40. $^+1$ + $^+3$ 41. $^+10$ + $^-12$

42. $^+2$ + $^-1$ 43. $^-2$ + $^+1$ 44. $^+7$ + $^-3$

45. $^+3$ + $^-7$ 46. $^-6$ + $^+7$ 47. $^+8$ + $^-9$

48. $^+21$ + $^-15$ 49. $^-12$ + $^+13$ 50. $^-32$ + $^-42$

51. $^+17$ + $^+19$ 52. $^-181$ + $^+75$ 53. $^+181$ + $^-75$

54. $\dfrac{^+1}{5} + \dfrac{^-3}{5}$ 55. $\dfrac{^-2}{7} + \dfrac{^-3}{7}$ 56. $\dfrac{^-1}{4} + \dfrac{^+1}{2}$

57. $^-1\frac{1}{2} + {}^-3\frac{1}{4}$ 58. $^+4.6 + {}^-3.2$ 59. $^-9\frac{1}{6} + {}^+3\frac{1}{3}$

60. $^+3875 + {}^-2431$ 61. $^-2431 + {}^+3875$ 62. $(^+4 + {}^-3) + {}^+5$

63. $(^-15 + {}^+3) + {}^+7$ 64. $(^-15 + {}^+28) + {}^+2$

65. $(^-29 + {}^-98) + {}^-2$ 66. $(^+87 + {}^+64) + {}^+36$

67. $(^-997 + {}^-482) + {}^-3$ 68. $(^+3 + {}^-7) + (^+9 + {}^-2)$

[More exercises are in Part B, Supplementary Exercises.]

B. Bill's father gave him $3 to start and operate a flower business
for one week. His father told him to use the $3 to buy flowers
the first day and to sell as much as he could each day. He also
told him to use all of the money he collected on one day to buy
flowers the next morning. Although spending all of his money
each morning might not be the best business procedure, his father
wanted to see how far up he could "run" the $3.

Here is a record of his week's business.

	Expenses	Sales	Outcome
Monday	3.00	4.00	
Tuesday	4.00	5.20	
Wednesday	5.20	4.80	
Thursday	4.80	4.90	
Friday	4.90	4.70	
Saturday	4.70	6.80	

On Monday, Bill's assets changed by $1, and this change was
an increase in his assets. That is, the outcome of business on
Monday was a profit of $1. The outcome of business on Friday
was a loss of $0.20.

The outcome of a day's business is a <u>change</u> in assets. This may be, for example, a $⁺1.00 change [read as 'a positive one dollar change'], or a $⁻0.20 change. The <u>measures</u> of the outcomes are real numbers, ⁺1.00 and ⁻0.20. [Strictly, these are the dollar-measures of the outcomes. The cent-measures would be ⁺100 and ⁻20, respectively. This is entirely analogous to the case of a length, say, the length 1 foot, which has the number 1 of arithmetic as its foot-measure and the number 12 of arithmetic as its inch-measure.] In the table on page 1-10 we have listed the measures of the expenses and sales, and expect the students to list the measures of the outcomes. We shall, as in the table headings, usually avoid the awkward phrase 'measure of the outcome'. By 'outcome' we are referring to the real number which is the dollar-measure of outcome.

✳

The word 'predict' in Exercise 2 of Part B does not mean guess or estimate. [Some students have incorrectly interpreted the word in that manner.] When one has become aware [verbally or nonverbally] of a generalization, he uses it as the basis for prediction. This is one of the ways in which we determine that a student has achieved a nonverbal awareness of a generalization.

✳

Answers for Part B [on pages 1-10 and 1-11].

1. Outcomes: Monday, ⁺1.00; Tuesday, ⁺1.20; Wednesday, ⁻0.40; Thursday, ⁺0.10; Friday, ⁻0.20; Saturday, ⁺2.10.

2. Sum of outcomes: ⁺3.80.

✳

Answers for Part C [on pages 1-11, 1-12, and 1-13].
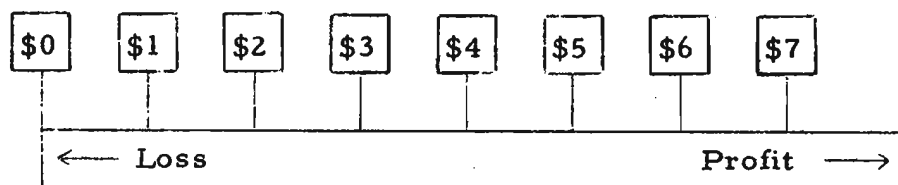
1. If we decide to use positive numbers to measure profits, then

the measure of a $3 profit is ⁺3,
the measure of a $6 loss is ⁻6, and
the measure of a $5 profit is ⁺5.

The dollar-measure of the total outcome is $(⁺3 + ⁻6) + ⁺5$, that is, ⁺2. So the total outcome was $⁺2.00, a profit of $2.00.

2. $[(⁺6.80 + ⁺2.55) + ⁻5.42] + ⁺1.53 = ⁺5.46$. $5.46 profit, or $⁺5.46.

The outcome of a day's business for Bill can be regarded as a trip, for it involves "distance" [difference in assets between the opening and closing of the business day] and "direction" [increase or decrease in assets]. So, the outcome of a day's business can be measured by a real number. If we decide that a profit is to be measured by a positive number then the outcome of Monday's business is $^{+}1.00$, and the outcome of Friday's business is $^{-}0.20$. In dollars, the measures of the two outcomes are $^{+}1.00$ and $^{-}0.20$.

1.   Complete the table given on page 1-10 by listing the [measure of the] outcome of each day's business.

2.   Bill started with $3.00 on Monday morning and ran it up to $6.80 by Saturday evening. <u>Predict</u> the sum of the outcomes for the six days of business, and check your prediction by actually adding the real numbers listed in the outcome column.

<u>C</u>.   Solve the following problems. Use real numbers wherever they apply.

1.   Phil made 3 dollars profit the first day of business, lost 6 dollars the second day, and made 5 dollars profit the third day. What was the outcome of the total business for the three days?

2.   Ed made $6.80 profit the first day of business, made $2.55 profit the second day, lost $5.42 the third day, and made $1.53 the fourth day. What was the outcome of the four days of business?

3. Zabranchburg High's football team gained 3 yards the first
    down, lost 4 yards the second down, gained 5 yards the
    third down, and gained 7 yards the fourth down. Did they
    make a first down?

4. John and Fred are playing a game. John wins 3 points in
    the first round, loses 4 points in the second round, and
    wins 5 points in the third round. What is his score at the
    end of the third round?

☆5. A department store has 6 floors above the ground floor and
    2 floors below the ground floor. The first floor above the
    ground floor is called 'mezzanine', the next floor above the
    mezzanine is called 'first floor', the floor next above the
    first floor is called 'second floor', etc. The first floor below
    the ground floor is called 'first basement' and the floor below
    that one is called 'second basement'. An operator makes the
    following trip: ground floor to mezzanine to second floor to
    first floor to third floor to fourth floor to mezzanine to first
    basement to third floor to second basement to first floor.

    (a) If you use real numbers to measure the separate trips,
         what is the sum of these real numbers? Could you have
         predicted your answer without adding?

    (b) If the floors are 17 feet apart, how many feet did the
         operator travel during the entire trip?

    (c) Which traveled the greater distance, the operator's head
         or his feet?

☆6. Two cyclists start from the same home at the same time.
    John travels 4 miles east, then 2 miles west, then 3 miles
    east. He then travels west until he meets Walt. Walt
    starts by traveling 3 miles west, then 1 mile east, then 3
   miles west, then east until he meets John.

·om

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3. If we decide to use positive numbers to measure gains, then the measures of the successive "trips" are $^+3$, $^-4$, $^+5$, and $^+7$. The measure of the net result is $[(^+3 + ^-4) + ^+5] + ^+7$, that is, $^+11$. So, there was a net gain of 11 yards. Since the requirement for a first down is a gain of at least 10 yards, the team <u>did</u> make a first down. [If we decide to use negative numbers to measure gains, then the measures of the successive trips are $^-3$, $^+4$, $^-5$, and $^-7$. The measure of the net result is $^-11$, and the result is still, of course, a gain of 11 yards.]

4. If we use positive numbers to count points won, then the results of the successive rounds are $^+3$, $^-4$, and $^+5$. John's score at the end of the third round is $(^+3 + ^-4) + ^+5$, or $^+4$. Students may want to say that John's score is "4 points to the good".

☆5. (a) If we use $^+1$ as the measure of a trip upward from one floor to the next, the description of the successive trips as given in the problem could be represented by:

$$^+1, \quad ^+2, \quad ^-1, \quad ^+2, \quad ^+1, \quad ^-4, \quad ^-2, \quad ^+5, \quad ^-6, \quad ^+4.$$

The sum of these real numbers is $^+2$. Students should be able to predict this, since the last trip terminated 2 floors above the starting point of the first trip.

(b) To determine the distance traveled, the student must consider how many times a trip of 17 feet was made. Thus:

ground floor to mezzanine:	1 time,
mezzanine to second floor:	2 times,
second floor to first floor:	1 time,
first floor to third floor:	2 times,
third floor to fourth floor:	1 time,
fourth floor to mezzanine:	4 times,
mezzanine to first basement:	2 times,
first basement to third floor:	5 times,
third floor to second basement:	6 times,
second basement to first floor:	4 times,

or a total of 28 times that a trip of 17 feet was made. So, the operator traveled a total of 476 feet.

(c) The operator's head and feet both traveled the same distance! [You may want to ask students whether they could illustrate this graphically. One of our participating teachers reported that a seventh grade boy in her class did this (to convince others of the correct answer) by using a pencil and a yard-stick. He slid the pencil up the yardstick a given number of

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John was 5 miles east of home, after these 2 miles of travel.

Finally, to show the part of the road traveled by each boy until they meet:

place where they meet
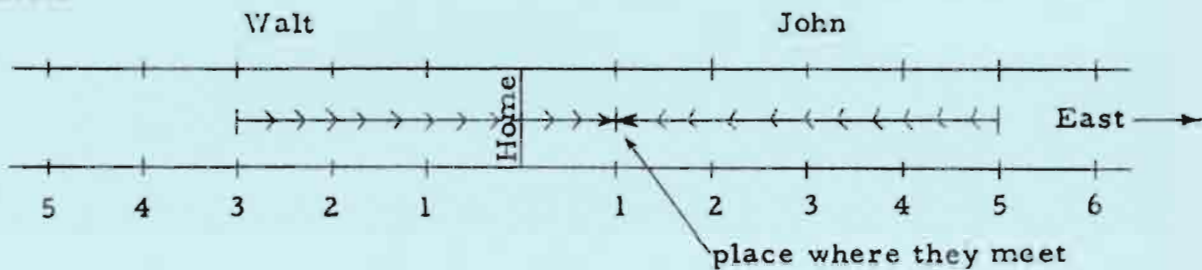
So, the answers to the questions are:

(a)   1 mile   (b)   west   (c)   14

✳

Notice that each of Exercises 5 and 6 is marked by a five-pointed star.   This is our indication of an optional problem [or subsection], a problem which can be omitted without destroying the total develop-ment.   It is usually used with a problem that presents a genuine challenge to the very best students.

    (a) If both cyclists travel at the same speed, how far from home do they meet?

    (b) In which direction must they travel if they head directly for home together?

    (c) How many miles has each cyclist traveled by the time they reach home?

[More exercises are in Part C, Supplementary Exercises.]

TRIPS OF DISTANCE 0

The problems in adding real numbers up to now have not included such problems as finding the sum of $^+3$ and $^-3$, or finding the sum of $^+7$ and $^-7$. Just as it is possible to find the sum of any pair of numbers of arithmetic, we would like it to be the case that there is a sum for every pair of real numbers. Our problem, then, is to give an interpretation of, say, '$^+3 + {}^-3$' so that '$^+3 + {}^-3$' is a numeral for a real number. Let us try our trip interpretation. Using the picture on page 1-1, '$^+3 + {}^-3$' should be a numeral for a real number that measures the trip whose starting point is that of the trip, say, from A to M and which has the same ending point as the trip from M to A. In other words, $^+3 + {}^-3$ should measure the trip from A to A. This is an unusual trip! For even though it involves the distance aspect, it does not involve a direction. In fact, we wonder if we can even consider this as a trip at all. If we don't consider it as a trip, and if we wish to continue thinking of real numbers as numbers which measure trips, then we must admit that collections of marks such as

$$'^+7 + {}^-7'\qquad\text{and}\qquad'^+3 + {}^-3'\qquad\text{and}\qquad'^-4 + {}^+4'$$

are not numerals for real numbers. In other words, we would have to admit that there are pairs of real numbers which do not have sums. Rather than do this, we prefer to stretch our imaginations a bit and regard a "trip from A to A" as a trip which can be measured by a real number. Naturally, the real number to be used in measuring such a trip is neither a positive number nor a negative number. So, it must be one which we have not yet discussed.

A simple numeral which we shall use for this real number is a
numeral which is also used for a number of arithmetic. It is the
numeral '0'. Thus, each of the numerals '$^+7 + ^-7$', '$^+3 + ^-3$', and
'$^-4 + ^+4$' can be simplified to '0'. Each of these numerals names
the <u>real number</u> 0.

   Now, just as we do not confuse the positive and negative real
numbers with numbers of arithmetic [we don't think, for example,
that the real number $^-3$ is the same as the number 3 of arithmetic],
so we must not confuse the real number 0 with the number 0 of
arithmetic. It may seem difficult to avoid this confusion since both
numbers have the same name, and you may wonder why we did not
invent a new numeral for this real number. We could have used
' θ ' or ' ϕ ' or ' $\cancel{X}$ ' or even ' ξ ', but everyone else uses '0', so
we shall also. Actually, it will not be too hard to keep the idea of
the number 0 of arithmetic separate from the idea of the real
number 0, since any problem in which either number is to be used
will tell you which meaning to give to the numeral '0'. For example,
the real number 0 measures the outcome of a day's business in which
expenses were the same as sales, whereas the number 0 of arith-
metic measures the content of the money-box when the box is empty.

<div align="center">EXERCISES</div>

<u>A</u>. Simplify.

   1. $^+9 + ^-9$ 2. $^-9 + ^+9$ 3. $^+18 + ^-18$

   4. $^+3 + 0$ 5. $0 + ^+3$ 6. $0 + ^-3$

   7. $^-3 + 0$ 8. $^+20 + ^+30$ 9. $^+30 + ^-50$

  10. $(^+9 + ^-7) + ^+7$ 11. $(^+19 + ^+12) + ^-12$

  12. $(^-843 + ^+726) + ^-726$ 13. $(^+487 + ^-851) + ^-487$

[More exercises are in <u>Part D</u>, <u>Supplementary Exercises</u>.]

Answers for Part B [on pages 1-22 and 1-23].

1. $^+10$ 2. $^+18$ 3. $^+68$ 4. $^+56$ 5. $^-12$

6. $^-12$ 7. $^-35$ 8. $^-64$ 9. $^-90$ 10. $^-120$

11. $^+56$ 12. $^+45$ 13. $^+1$ 14. $^+96$ 15. 0

16. 0   17. 0   18. 0   19. $^+4$ 20. $^-250$

21. $^-\dfrac{3}{2}$ 22. $^-45$ 23. $^-84$ 24. $^+24$ 25. $^-24$

26. $^+6$ 27. $^-34$ 28. $^-2726$ 29. $^-1755$ 30. $^+10575$

31. $^+6450$ 32. $^+3.3123$ 33. $^-72.182$

34. $^+24$ 35. $^+42$ 36. $^-60$ 37. $^+36$ 38. $^+84$

39. $^+60$ 40. 0   41. 0

✳

Answers for Part C.

1. $^-14$ 2. $^-18$ 3. $^-29$ 4. $^-3$ 5. $^+3$

6. $^+2$ 7. $^-3$ 8. $^-1$ 9. $^-22$ 10. $^-53$

11. $^+120$ 12. $^-64$

✳

Answers for Part D.

1. $^-4$ 2. $^-6$ 3. $^+3$ 4. $^+3$ 5. $^+12$

6. $^-3$ 7. $^-16$ 8. $^-1$ 9. $^+\dfrac{1}{3}$ 10. $^-3$

11. 0   12. $^-2$ 13. 2   14. $^-2$

✳

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We have now introduced the real number 0, and have extended the meaning of 'addition' so that the sum of two opposite real numbers is 0 and that the sum of an ordered pair of real numbers, one component of which is 0, is the other. We can now properly speak of addition as an operation because each ordered pair of real numbers has a unique sum which is itself a real number. The fact that the sum of a real number and a real number is a real number is sometimes expressed by saying that the set of real numbers is <u>closed</u> under addition. [The fact that we use the expression '<u>the</u> sum' indicates our belief that no pair of real numbers has two sums. This is confusingly expressed by 'equal numbers added to equal numbers give equal numbers'. Of course, there are no such things as equal numbers.]

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Answers for Part A.

1. 0	2. 0	3. 0	4. $^+3$	5. $^+3$
6. $^-3$	7. $^-3$	8. $^+50$	9. $^-20$	10. $^+9$
11. $^+19$	12. $^-843$	13. $^-851$		

[Some students may not use or see short cuts in doing Exercises 10 - 13. They will have ample opportunity to discover such short cuts later; it is not necessary to give more exercises than those in Part A on page 1-14 and in Part D of the Supplementary Exercises at this time.]

II. Fill in the blanks to make true sentences.

1. $^+6 + \,^-9 =$ ____ 2. $^-5 +$ ____ $= \,^+7$ 3. $^+4 +$ ____ $= \,^-8$

4. ____ $+ 0 = \,^+7$ 5. ____ $+ \,^-8 = 0$ 6. $^-11 +$ ____ $= \,^-10$

7. $(^+9 + \,^-2) +$ ____ $= \,^-15$ 8. $(^-3 +$ ____ $) + \,^+9 = \,^-20$

III. Multiple-choice. Draw a loop around the correct answer.

1. What is the sum of a positive number and a positive number?

(A) a positive number (B) 0 (C) a negative number

2. If the sum of a real number and a real number is a positive number and one of them is a negative number, what is the other?

(A) a positive number (B) 0 (C) a negative number

3. If the sum of a nonpositive real number and a nonpositive real number is neither a positive number nor a negative number then each of them must be _____.

(A) a positive number (B) 0 (C) a negative number

☀

Answers for quiz.

I. 1. $^+3$ 2. $^-11$ 3. $^+2$ 4. $^+70$ 5. $^-4$

6. 0 7. $^-2$ 8. $^-19$ 9. $^+1.5$ 10. $^+2.8$

II. 1. $^-3$ 2. $^+12$ 3. $^-12$ 4. $^+7$ 5. $^+8$

6. $^+1$ 7. $^-22$ 8. $^-26$

III. 1. a positive number 2. a positive number 3. 0

☀

Other short quizzes are given on the pages listed below.

Answers for Part B.

1. ⁻8	2. 0	3. ⁺9	4. ⁻3	5. 0
6. ⁺15	7. ⁺6	8. ⁻171	9. ⁻2	10. ⁺10

11. ⁻31   12. a numeral for any number in the first blank, and a copy of this numeral in the second blank.

[One purpose of Exercises 13-20 is to drive home the point that one obtains a numeral for a real number whenever one prefixes a '⁺' or a '⁻' to any numeral for a number of arithmetic. For example [Exercise 13], '⁺(7 + 5)' is a numeral for the real number ⁺12, and consists of the numeral '(7 + 5)' for the number 12 of arithmetic together with a prefixed '⁺'. Note also that '⁺5' is <u>not</u> a correct answer for Exercise 13. [See page 1-108.]]

13. 5	14. 2	15. ⁻3	16. 2	17. 4
18. ⁻6	19. 0	20. ⁺3	21. ⁻937	

✳

Answers for Part C.

[In discussing these exercises, no special attention should be given to the word 'subset'. If a student asks what a subset is, you may tell him it is a set made up of things which belong to the other set. We doubt that any student will raise this question.]

1. The set of nonnegative real numbers is the set which consists of the positive numbers and zero.

2. The set of nonpositive real numbers is the set which consists of the negative numbers and zero.

3. Yes, 0. [Note well that 0 is neither positive nor negative, and that 'positive' is <u>not</u> a synonym for 'nonnegative'!]

✳

Here is a quiz which covers the ideas of addition of real numbers.

I. Simplify.

1. ⁺5 + ⁻2	2. ⁻4 + ⁻7	3. ⁻6 + ⁺8	4. ⁺50 + ⁺20
5. ⁺8 + ⁻12	6. ⁻9 + ⁺9	7. ⁺6 + ⁻8	8. ⁻20 + ⁺1
9. ⁻5.3 + ⁺6.8		10. ⁺21.3 + ⁻18.5	

B. Fill in the blanks to make true sentences.

1. $^+8$ + _____ = 0   2. $^-3$ + _____ = $^-3$

3. $^-3$ + _____ = $^+6$ 4. $^-3$ + _____ = $^-6$

5. 0 + _____ = 0   6. _____ + $^+2$ = $^+17$

7. $^-3$ + _____ = $^+3$ 8. _____ + $^+71$ = $^-100$

9. _____ + $^+1$ = $^-1$ 10. _____ + $^+3$ = $^+13$

11. _____ + $^+52$ = $^+52$ + $^-31$ 12. $^+87$ + _____ = _____ + $^+87$

13. $^+7$ + $^+5$ = $^+(7 + \underline{\;\;})$ 14. $^-2$ + $^-11$ = $^-(\underline{\;\;} + 11)$

15. $^+8$ + _____ = $^+(8 - 3)$ 16. $^-(15 - 2)$ = $^-15$ + $^+$_____

17. $^+4$ + $^-7$ = $^-(7 - \underline{\;\;})$ 18. $^-(6 - 3)$ = $^+3$ + _____

19. $^-(8 - 4)$ = _____ + $^-4$ 20. $^-[(5 + 2) - 3]$ = $(^-5 + ^-2)$ + _____

21. $(^+351 + \underline{\;\;})$ + $^-284$ = $^+351$ + $(^-937 + ^-284)$

C. You have learned about three kinds of real numbers--positive real numbers, negative real numbers, and 0. Each real number is either positive, negative, or 0. And no real number is of two of these kinds. So, the real numbers can be classified into three subsets:

> the set consisting of the positive real numbers,
> the set consisting of the negative real numbers, and
> the set consisting of the real number 0.

1. Another way of classifying the real numbers is to note that each real number is either a negative real number or a nonnegative real number. Describe the set of nonnegative real numbers.

2. Describe the set of nonpositive real numbers.

3. Does the set of nonnegative real numbers have any numbers in common with the set of nonpositive real numbers?

EXPLORATION EXERCISES

A. Suppose that a pump <u>fills</u> a tank with water at a rate of 3 gallons
    per minute. What will be the <u>increase</u> (gallons) in the volume
    of water in the tank

    1. 1 minute from now?   2. 4 minutes from now?

    3. $10\frac{1}{2}$ minutes from now?   4. 0 minutes from now?

B. Suppose that the tank is full and the pump empties the tank at a
    rate of 4 gallons per minute. What will be the <u>decrease</u> in the
    volume of water in the tank

    1. 1 minute from now?   2. 4 minutes from now?

    3. $10\frac{1}{2}$ minutes from now?   4. 0 minutes from now?

C. Suppose the pump <u>fills</u> the tank at a rate of 5 gallons per minute.
    How many <u>fewer</u> gallons of water were there in the tank

    1. 1 minute <u>ago</u>?   2. 4 minutes ago?

    3. $10\frac{1}{2}$ minutes ago?   4. 0 minutes ago?

D. Suppose a full tank is emptied by a pump at a rate of 3 gallons
    per minute. How many <u>more</u> gallons of water were there in
    the tank

    1. 1 minute ago?   2. 4 minutes ago?

    3. $10\frac{1}{2}$ minutes ago?   4. 0 minutes ago?

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These Exploration Exercises prepare the student for a physical interpretation of multiplication of real numbers. As in the case of addition of real numbers, students come to understand the operation of multiplication through a physical interpretation. The interpretation is so designed as to lead to computing rules which are in accord with accepted procedures for multiplication of real numbers. Since this is basically a question of definition, the physical interpretation must involve a certain element of arbitrariness. Nevertheless, we have tried to make the interpretation as reasonable as possible.

*

Answers for Exploration Exercises. [Solutions are numbers of arithmetic.]

A. 1. 3   2. 12   3. $31\frac{1}{2}$ 4. 0

B. 1. 4   2. 16   3. 42   4. 0

C. 1. 5   2. 20   3. $52\frac{1}{2}$ 4. 0

D. 1. 3   2. 12   3. $31\frac{1}{2}$ 4. 0

[In doing these problems it is interesting to speculate on the capacity of the tank. The minimum capacity is $52\frac{1}{2}$ gallons.]

*

Recall our convention, mentioned on TC[1-8]b, according to which the numeral to the right of a '×' names the multiplier. It is still permissible to read '⁺4 × ⁻2' as 'positive four times negative two', but you should fairly frequently use the more appropriate phrases, 'the product of positive four by negative two', or 'positive four multiplied by negative two'. [A glance at TC[1-51] will show you the importance of establishing, before reaching that point, the convention of "writing the multiplier on the right".]

1.03 <u>Multiplication</u> <u>of</u> <u>real</u> <u>numbers</u>. --In telling what we mean by addition of real numbers, we gave, in terms of trips, interpretations of numerals such as

$$\text{'}^+4 + {}^-7\text{'}, \quad \text{'}^-8 + {}^+9\text{'}, \quad \text{'}^+3 + {}^+10\text{'}, \quad \text{and} \quad \text{'}^-5 + {}^-11\text{'}.$$

In order to explain multiplication of real numbers, we shall give an interpretation of numerals such as

$$\text{'}^+4 \times {}^+3\text{'}, \quad \text{'}^-8 \times {}^+7\text{'}, \quad \text{'}^-2 \times {}^-5\text{'}, \quad \text{and} \quad \text{'}^+4 \times {}^-2\text{'}.$$

This interpretation should help us find the product of each pair of real numbers.

We know that real numbers are numbers which can be used to measure trips. What are the characteristics of trips which make this possible? A trip involves

(1)   a change in position by a certain amount,

and (2)   a change in position in one of two opposite directions.

In general, anything which involves an amount and one of two opposite directions can be measured by a real number. So, in looking for an interpretation of a numeral such as

$$\text{'}^-4 \times {}^+3\text{'},$$

we look for something which involves both an amount and a direction so that it can be measured by a real number.

A PUMP, A TANK, AND A MOVIE

Think of a pump which can pump water either into or out of a tank, and a camera which takes a movie of the tank while the pump is operating. Suppose the pump and camera are turned on together,

Movie Camera   Glass Tank   Pump
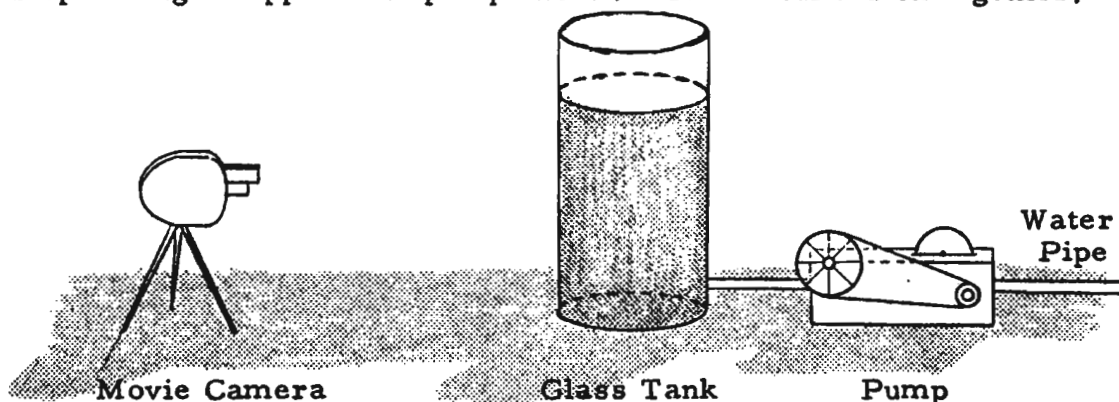
and that 4 gallons of water flow through the pipe each minute.
After the pump and camera have run for 3 minutes, they are
stopped. The film is then developed and projected on a screen.
What change in the water-volume do you observe on the screen?
It is easy to predict that the change observed on the screen will
be a change of 12 gallons. But, will it be an increase? In order
to answer this question, you need to know two more things.

One of the things you need to know is whether the flow
of water was into the tank or out of the tank. Suppose that the
flow was into the tank. Will the picture you see on the screen
show an increase in water-volume? If your answer is 'yes' then
you are probably assuming that the film is being run forward
through the projector. But suppose the film were run backward
through the projector. [Have you ever watched a comedy film
in which a man seems to dive up out of the water and land on a
diving board, or a film of a race horse running backward on a
muddy track, picking up its footprints as it goes?] If the film
were run backward, what change in water-volume would you see
on the screen?

Now, suppose the water was being pumped out of the tank
while the picture was being taken. If the film were run forward
through the projector, what change in water-volume would you
observe on the screen? If the film were run backward, what
change would you see on the screen?

So, in order to predict what change you will observe on the
screen, you need to know

   (a)   the amount of water per minute being pumped into or
         out of the tank, and

   (b)   the number of minutes the film is being run forward or
         backward.

Each of these things involves an amount and a direction, and
therefore can be measured by real numbers. We can use real
numbers to measure the rate at which the water is being pumped,

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When explaining the material on page 1-18, you should be as
dramatic as possible. Ask the students to imagine a pump
which pumps water into the tank, and to think of what happens
to the water level as the pump operates. The kids will tell
you that the water level "goes up". As they say this, you
should move your hands to pantomime this movement upward
from the bottom toward the top of this imaginary tank. Next,
ask them to imagine that someone takes a motion picture of
this tank-filling process. Now ask:

(a) If this film is projected in the normal way, what
would you see on the screen?

and: (b) If this film is projected backward, what would you
see on the screen?

As these questions are being answered, you should again
pantomime the observed changes.

The students need an accurate mental picture of these events.
It should not be difficult for them to acquire one, if you take
a bit of time at the beginning to make the notions clear.

TC[1-18, 19]

16. $0 \times {}^-6$ 17. 0×0 18. ${}^-12 \times 0$

19. ${}^-16 \times \dfrac{{}^-1}{4}$ 20. ${}^-100 \times \dfrac{{}^+5}{2}$ 21. ${}^+5 \times \dfrac{{}^-3}{10}$

22. ${}^+3 \times {}^-15$ 23. ${}^-7 \times {}^+12$ 24. ${}^-3 \times {}^-8$

25. ${}^+6 \times {}^-4$ 26. ${}^-3 \times {}^-2$ 27. ${}^-17 \times {}^+2$

28. ${}^+47 \times {}^-58$ 29. ${}^+27 \times {}^-65$ 30. ${}^+705 \times {}^+15$

31. ${}^-86 \times {}^-75$ 32. ${}^-1.83 \times {}^-1.81$ 33. ${}^+9.65 \times {}^-7.48$

34. $({}^+2 \times {}^-3) \times {}^-4$ 35. $({}^-2 \times {}^+7) \times {}^-3$

36. $({}^+5 \times {}^-3) \times {}^+4$ 37. $({}^+6 \times {}^-2) \times {}^-3$

38. ${}^+4 \times ({}^-3 \times {}^-7)$ 39. ${}^-6 \times ({}^+2 \times {}^-5)$

40. $({}^+73 \times {}^-81) \times 0$ 41. $({}^+5 \times {}^-17) \times ({}^-3 \times 0)$

[More exercises are in <u>Part E</u>, <u>Supplementary Exercises</u>.]

<u>C</u>. Simplify. [Be careful not to confuse addition signs with multi-
plication signs.]

1. $({}^+5 + {}^-3) \times {}^-7$ 2. $({}^+3 \times {}^-4) + {}^-6$

3. $({}^+8 \times {}^-3) + {}^-5$ 4. $({}^+12 + {}^-11) \times {}^-3$

5. $({}^+1 + {}^+1) + {}^+1$ 6. $({}^+1 \times {}^+1) + {}^+1$

7. $({}^-1 + {}^-1) + {}^-1$ 8. $({}^-1 \times {}^-1) \times {}^-1$

9. $({}^-4 \times {}^-2) + ({}^-5 \times {}^+6)$ 10. $({}^-3 \times {}^+7) + ({}^+8 \times {}^-4)$

11. $({}^+71 + {}^-11) \times ({}^+6 + {}^-4)$ 12. $({}^-8 + 0) \times ({}^+4 \times {}^+2)$

<u>D</u>. Fill in the blanks to make true sentences.

1. ${}^+5 \times \underline{\hspace{1cm}} = {}^-20$ 2. ${}^-3 \times \underline{\hspace{1cm}} = {}^+18$

3. ${}^+7 \times \underline{\hspace{1cm}} = {}^+21$ 4. ${}^-7 \times \underline{\hspace{1cm}} = {}^-21$

5. $\underline{\hspace{1cm}} + {}^-3 = {}^+9$ 6. $\underline{\hspace{1cm}} \times {}^-3 = {}^+9$

7. ${}^+8 + \underline{\hspace{1cm}} = {}^-8$ 8. ${}^+8 \times \underline{\hspace{1cm}} = {}^-8$

9. ${}^+3 \times \underline{\hspace{1cm}} = {}^+1$ 10. ${}^+3 + \underline{\hspace{1cm}} = 0$

11. ${}^+3 \times \underline{\hspace{1cm}} = 0$ 12. ${}^+3 + \underline{\hspace{1cm}} = {}^+1$

13. ${}^+8 \times {}^-2 = {}^-(8 \times \underline{\hspace{1cm}})$ 14. ${}^-5 \times \underline{\hspace{1cm}} = {}^+(5 \times 2)$

deciding to use

    positive numbers when water flows into the tank,
and

    negative numbers when water flows out of the tank.

Thus, if 4 gallons of water are being pumped into the tank each
minute, we say that the rate is $^+4$ gallons per minute. Explain
what is meant by saying that the rate is $^-4$ gallons per minute.

    Also, we can use real numbers to measure how long the film
is being projected, deciding to use

    positive numbers when the film is run forward,
and

    negative numbers when the film is run backward.

So, if the film is running for $^+3$ minutes, we know it is being pro-
jected forward (normally) for 3 minutes. Explain what is meant
by saying that the film is running for $^-3$ minutes.

    Now, how does all of this help us in interpreting a numeral
such as

$$\text{`} {}^-4 \times {}^+3\text{'}?$$

We can think of $^-4$ as measuring the rate at which water is being
pumped. We can think of $^+3$ as telling us how long the film is being
projected. And, finally, we can think of the product

$$^-4 \times {}^+3$$

as measuring the change in water-volume which we see on the screen.
[Let's agree to use positive numbers to measure observed increases,
and negative numbers to measure observed decreases.] In this case,
since water is being pumped out of the tank at 4 gallons per minute
and the film is being run forward for 3 minutes, the change in water-
volume observed on the screen is a decrease of 12 gallons. So,
the change is $^-12$ gallons. Since we agreed to think of the product
as a measure of the change, we can say that

$$^-4 \times {}^+3 = {}^-12.$$

Let's take another case:

$$^+9 \times {}^-8 = ?$$

This number, $^+9 \times {}^-8$, measures the change in water-volume which you would observe on a screen. The first number, $^+9$, measures the rate of pumping [is it filling or is it emptying?], and the second number, $^-8$, tells how long the film is being projected [is it being run forward or is it being run backward?]. Does $^+9 \times {}^-8$ measure an observed increase in water-volume or an observed decrease? Since a backward projection of the movie of a tank being filled shows a decrease in water-volume, we can say that

$$^+9 \times {}^-8 = {}^-72.$$

EXERCISES

A. The table below contains problems dealing with the pump-tank-movie interpretation. From each problem you can learn how to multiply a pair of real numbers. We have solved the first problem for you as a sample.

In this problem you are told that a pump is filling the tank at the rate of 4 gallons per minute. Therefore, a '$^+4$' is written in the column headed 'Pump'. You learn from the second column that the movie has been run backward for 2 minutes. Therefore, you write a '$^-2$' in this column. Now, we ask about the change in water-volume that would be observed on the screen. Since the pump is filling the tank (as indicated by the '$^+4$'), and since the film is run backward (as indicated by the '$^-2$'), the volume of water appears to be decreasing. So, we observe on the screen a decrease in volume of 8 gallons. The number $^-8$ measures this observed change. Finally, we write the corresponding multiplication statement in the last row.

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Answers for Part A [on pages 1-21 and 1-22].

2. decrease of 8 gallons; ⁻4, ⁺2, ⁻8; ⁻4 × ⁺2 = ⁻8.

3. increase of 8 gallons; ⁺4, ⁺2, ⁺8; ⁺4 × ⁺2 = ⁺8.

4. increase of 8 gallons; ⁻4, ⁻2, ⁺8; ⁻4 × ⁻2 = ⁺8.

5. increase of 24 gallons; ⁺8, ⁺3, ⁺24; ⁺8 × ⁺3 = ⁺24.

6. decrease of 24 gallons; ⁻8, ⁺3, ⁻24; ⁻8 × ⁺3 = ⁻24.

7. increase of 24 gallons; ⁻8, ⁻3, ⁺24; ⁻8 × ⁻3 = ⁺24.

8. Emptying 5 gallons per minute, Running backward 6 minutes, increase of 30 gallons; ⁺30; ⁺30.

9. Filling 7 gallons per minute, Running backward 3 minutes, decrease of 21 gallons; ⁻21; ⁻21.

10. Emptying 8 gallons per minute, Not running, no change; 0; 0.

11. Emptying $6\frac{1}{2}$ gallons per minute, Running backward 4 minutes, increase of 26 gallons; ⁺26; ⁺26.

⁂

By the time the students have worked the exercises in Part A on pages 1-21 and 1-22, they should have formulated their own rules for finding products of real numbers. Remember, if they have not formulated such rules, they should continue to use the pump-tank-film interpretation. As in the case of addition of real numbers, no student wants to continue using the long procedure, when he feels sure there must be a short cut for it. Again, it is the student's job to find the short cut.

⁂

As in the case of addition, the physical interpretation which leads to the rules for multiplying real numbers does not prove that the rules are correct. The rules themselves are consequences of the definitions of real numbers and the operations, and these definitions are implicit in the physical interpretation.

⁂

Answers for Part B are on TC[1-23].

Complete the table.

	Pump	Movie	Observed Change in Volume
1.	Filling 4 gal. per minute	Running backward 2 minutes	*decrease of 8 gallons*
	$^{+}4$	$^{-}2$	$^{-}8$
	Corresponding multiplication statement:		$^{+}4 \times ^{-}2 = ^{-}8$
2.	Emptying 4 gal. per minute	Running forward 2 minutes	
	Corresponding multiplication statement:		
3.	Filling 4 gal. per minute	Running forward 2 minutes	
	Corresponding multiplication statement:		
4.	Emptying 4 gal. per minute	Running backward 2 minutes	
	Corresponding multiplication statement:		
5.	Filling 8 gal. per minute	Running forward 3 minutes	
	Corresponding multiplication statement:		
6.	Emptying 8 gal. per minute	Running forward 3 minutes	
	Corresponding multiplication statement:		
7.	Emptying 8 gal. per minute	Running backward 3 minutes	
	Corresponding multiplication statement:		

[Note: In the rest of the problems you are given real numbers and you should fill in the corresponding blanks.]

	Pump	Movie	Observed Change in Volume
8.	⁻5	⁻6	

Corresponding multiplication statement: ⁻5 × ⁻6 =

	Pump	Movie	Observed Change in Volume
9.	⁺7	⁻3	

Corresponding multiplication statement: ⁺7 × ⁻3 =

	Pump	Movie	Observed Change in Volume
10.	⁻8	0	

Corresponding multiplication statement: ⁻8 × 0 =

	Pump	Movie	Observed Change in Volume
11.	⁻6$\frac{1}{2}$	⁻4	

Corresponding multiplication statement: ⁻6$\frac{1}{2}$ × ⁻4 =

B. Simplify. Use the pump-tank-film interpretation as long as you need to, but try to find a short cut.

1. ⁺5 × ⁺2
2. ⁺6 × ⁺3
3. ⁺8$\frac{1}{2}$ × ⁺8

4. $\frac{⁺28}{3}$ × ⁺6
5. ⁺6 × ⁻2
6. ⁻2 × ⁺6

7. ⁻5 × ⁺7
8. ⁺8 × ⁻8
9. ⁻9 × ⁺10

10. ⁺12 × ⁻10
11. ⁻7 × ⁻8
12. ⁻15 × ⁻3

13. ⁻1 × ⁻1
14. ⁻8 × ⁻12
15. ⁺7 × 0

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The purpose of these Exploration Exercises is to get the students to become aware of the concept of isomorphism. We want the student to have this awareness as he reads pages 1-29 through 1-32. We think that if he went immediately to this discussion, it would be quite meaningless for him. In the discussion which starts on page 1-29 we are trying to get across the point that the system of numbers of arithmetic and the system of nonnegative real numbers are different systems which have the same structure. By 'system of numbers' we mean a set of things which we call 'numbers' together with one or more operations. Thus, when we talk about the system of numbers of arithmetic in Section 1.04, we mean the set of numbers of arithmetic together with the operations of addition and multiplication (of numbers of arithmetic). Also, the system of nonnegative real numbers is the set of nonnegative real numbers together with the operations of addition and multiplication (of nonnegative real numbers). To say that one of these number systems is isomorphic to the other means that there is a one-to-one correspondence between the numbers in one system and the numbers in the other such that the sum (product) of two numbers in one system is the "mate" of the sum (product) of the mates in the other system.

Let us consider the example in Part A. Suppose we take as one number system the set of whole numbers 1, 2, 3, ..., [only some of these numbers are listed in the right-hand column] together with the operation of addition of these numbers. The second number system consists of the set of numbers 89, 178, 267, ..., together with the operation of addition of these numbers. Then, these two number systems are isomorphic to each other. The numbers in the right-hand column act' like the numbers in the left-hand column when you add them. But, the fact that these two systems of numbers have the same structure is hardly sufficient reason to say that the numbers in one system are the same as the numbers in the other system. Similarly, since the system of the numbers of arithmetic is isomorphic to the system of nonnegative real numbers, the two systems have the same structure; but this is no reason for saying that the numbers in one system are the same as the numbers in the other. The fact that two number systems can have the same structure without anyone wanting to claim that the numbers are the same is one of the points we regard as most important in these exercises.

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Exercise 7 of Part A points out that even though the two sets of
numbers act like each other with respect to addition, they do not
act like each other with respect to multiplication. If we call the
numbers listed in the right-hand column and all others like them
'set A' and those in the left-hand column 'set B', then we would
say that the system consisting of set A together with addition is
isomorphic to the system consisting of set B together with addition,
but the system consisting of set A together with multiplication is
not isomorphic to the system consisting of set B together with
multiplication.

※

Answers for Part A.

1. 890   2. 1691   3. 1335   4. 1602

5. 356   6. 979   7. No.

※

Answers for Part B [on page 1-25].

1. .034290   2. $24\frac{1}{2}$ 3. $40\frac{5}{6}$ 4. .041148

5. $81\frac{2}{3}$ 6. .068580

※

We suggest that pages 1-24 through 1-28 be covered in not more than
one class period plus one homework assignment.

B. Here is another table in which
    you can add some pairs of numbers
    listed in one column by adding the
    corresponding numbers of the other
    column. For example:

$$+\begin{array}{l}.027432 \ldots \ldots \\ .041148 \ldots \ldots \end{array} \quad +\begin{array}{l}32\frac{2}{3} \\ 49 \end{array}$$

$$\overline{ \quad ? \quad \overline{81\frac{2}{3}}}$$

$81\frac{2}{3}$ corresponds with .068580. So,
.068580 is the sum of .027432 and
.041148. [Check this by adding.]

.030861	$36\frac{3}{4}$
.013716	$16\frac{1}{3}$
.041148	49
.020574	$24\frac{1}{2}$
.006858	$8\frac{1}{6}$
.024003	$28\frac{7}{12}$
.037719	$44\frac{11}{12}$
.027432	$32\frac{2}{3}$
.034290	$40\frac{5}{6}$
.068580	$81\frac{2}{3}$

Use the table to simplify each of the following. Check your
results by direct simplification. [Note that Exercise 2 requires
that you work with numbers listed in the lefthand column.]

1. .013716 + .020574

2. $8\frac{1}{6} + 16\frac{1}{3}$

3. $16\frac{1}{3} + 24\frac{1}{2}$

4. .020574 + .020574

5. $40\frac{5}{6} + 40\frac{5}{6}$

6. .030861 + .037719

C. Here is a table which can be used
to find products of some pairs of
numbers listed in one column by
computing products of the corre-
sponding numbers of the other
column. For example:

.5	....	2
.25	....	4
.2	....	5
.125	....	8
.1	....	10
.0625	....	16
.05	....	20
.04	....	25
.03125	....	32
.025	....	40

$$\begin{array}{r} .5 \ldots\ldots 2 \\ \times \underline{.25 \ldots\ldots} \times \underline{4} \\ ? \qquad 8 \end{array}$$

8 corresponds with .125, and
.125 is the product of .5 and .25.

   Use the table to simplify each of the following, and check
by direct simplification.

1. .5 × .2   2. 8 × 5   3. .0625 × .5

4. 8 × 4   5. .125 × .25   6. .2 × .2

                              ✳

7. Can you use this table for addition?

D. Here is another table which can be used for multiplication.

$^+2$	$^+5$	$^+\frac{1}{3}$	$^+\frac{1}{5}$	$^+\frac{1}{9}$	$^+\frac{2}{3}$	$^+1$	$^+\frac{5}{9}$	$^+\frac{10}{27}$	$^+\frac{20}{81}$
$\frac{1}{2}$	$\frac{1}{5}$	3	5	9	$1\frac{1}{2}$	1	$1\frac{4}{5}$	2.7	4.05

Use the table to find the products, and check by direct
simplification.

1. $3 \times \frac{1}{2}$ 2. $^+\frac{1}{3} \times ^+2$ 3. $9 \times \frac{1}{5}$

4. $^+\frac{1}{9} \times ^+5$ 5. $1\frac{1}{2} \times 2.7$ 6. $^+\frac{2}{3} \times ^+\frac{10}{27}$

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Part C deals with two sets of numbers which act like each other with respect to multiplication but not with respect to addition.

＊

Answers for Part C.

1.	.1	2.	40	3.	.03125	4.	32
5.	.03125	6.	.04	7.	No.		

＊

Answers for Part D.

1. $1\frac{1}{2}$ 2. $^+\frac{2}{3}$ 3. $1\frac{4}{5}$ 4. $^+\frac{5}{9}$

5. 4.05 6. $^+\frac{20}{81}$

＊

Part E emphasizes the fact that an essential item in establishing an isomorphism between two systems is that the numbers in the systems be placed in one-to-one correspondence. The "weird" arrangement in Part E points out that a two-column table is nothing more than a listing of a set of ordered pairs. The individual pairs of cells show the pairs, and the difference in shading shows the order. The table in Part E works for addition.

★E. Here is a table which can be used just as you used the other tables. Figure out how to use it, and see if it works for multiplication or for addition.

Tiles shown:

- -2 | $+10\frac{1}{2}$
- $+\frac{1}{5}$ | $-1\frac{1}{20}$
- -4 | $+21$
- $+\frac{1}{2}$ | $-2\frac{5}{8}$
- $-15\frac{3}{4}$ | $+3$
- $-5\frac{1}{4}$ / $+1$
- $+6$ / $-31\frac{1}{2}$
- 0 | 0
- $+8$ | -42
- $+15\frac{3}{4}$ / -3
- $-36\frac{3}{4}$ | $+7$
- $+5\frac{1}{4}$ | -1
- $+\frac{3}{10}$ | $-1\frac{23}{40}$
- $+4$ | -21
- $+2$ | $-10\frac{1}{2}$

F. For each table, see if it can be used for finding sums or products of some pairs of numbers listed in one row by computing sums or products of the corresponding numbers listed in the other row.

1.

4	9	16	25	36	64	100	144	225	324	400	576
⁺2	⁺3	⁺4	⁺5	⁺6	⁺8	⁺10	⁺12	⁺15	⁺18	⁺20	⁺24

2.

⁻3	⁻2	⁻1	0	$^{+}\frac{1}{4}$	$^{+}\frac{1}{2}$	$^{+}\frac{3}{4}$	⁺1	⁺2	⁺3	⁺4	⁺6	⁺8
⁺9	⁺6	⁺3	0	$^{-}\frac{3}{4}$	$-1\frac{1}{2}$	$-2\frac{1}{4}$	⁻3	⁻6	⁻9	⁻12	⁻18	⁻24

3.

0	1	2	3	4	5	6	7	8	9	12	18	35	54
0	1	2	3	4	5	6	7	8	9	12	18	35	54

4.

1	2	4	8	16	32	64	128	256	1024	2048	4096
0	1	2	3	4	5	6	7	8	10	11	12

5.

0	⁻1	⁻2	⁻3	⁻4	⁻5	⁻6	⁻7	⁻8	⁻10	⁻12	⁻13	⁻20	⁻42
0	1	2	3	4	5	6	7	8	10	12	13	20	42

6.

0	⁺1	⁺2	⁺3	⁺4	⁺5	⁺6	⁺7	⁺8	⁺10	⁺12	⁺13	⁺20	⁺42
0	1	2	3	4	5	6	7	8	10	12	13	20	42

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The exercises in Part F have been carefully selected to lead up to the climax in Exercise 6. Exercise 1 shows that the system consisting of the ''perfect square'' numbers of arithmetic together with the operation of multiplication is isomorphic to the system consisting of the nonnegative integers together with multiplication. Exercise 2 shows that the system consisting of real numbers together with the operation of addition is isomorphic to the system consisting of the real multiples of ‾3 together with the operation of addition.

By the time students reach Exercise 3 they are quite eager to find a table which ''works'' for two operations. Exercise 3 provides such an example. But, as many students will point out, the example is a trivial one. What the example does show is that every number system is isomorphic to itself.

Exercise 4 provides an example of an isomorphism between one number system which consists of the integral powers of 2 together with the operation of multiplication and a second number system which consists of the whole numbers of arithmetic together with the operation of addition. The fact that the operations are named by different words has nothing to do with the fact that the structure of the systems is the same. You will notice, of course, that this is the kind of isomorphism which is used in carrying out computations by means of logarithms.

In Exercise 5 we have an isomorphism between two systems in which one system consists of the set of nonpositive real numbers together with addition, and the other system consists of the numbers of arithmetic together with addition. If we check this table, we find that the isomorphism breaks down if we try to bring multiplication into the systems. By this time, students should understand that although the nonpositive real numbers act like the numbers of arithmetic with respect to addition, this is no reason to say that the nonpositive numbers are the same as the numbers of arithmetic.

Finally, in Exercise 6 we find the case in which the isomorphic systems include both multiplication and addition. And, as we have noted repeatedly, although the structure of the two systems is the same, the elements of the systems are not the same.

※

On page 1-30 you may want to tell the students that mathematicians use the phrase 'are isomorphic to' instead of 'act like'.

1.04 <u>Numbers</u> <u>of</u> <u>arithmetic</u> <u>and</u> <u>real</u> <u>numbers</u>. --Think carefully about what you do when you compute the sum of a pair of nonnegative or a pair of nonpositive real numbers. For example, what do you think about when you do the following problems?

$$^+7 + {}^+12 = ? \quad \text{and} \quad {}^-7 + {}^-12 = ?$$

Most likely, you first simplify:

$$7 + 12$$

and get: 19,

and then you say that

$$^+7 + {}^+12 = {}^+19 \quad \text{and} \quad {}^-7 + {}^-12 = {}^-19.$$

When you do this, you no doubt imagine a table which links up each nonpositive number and each nonnegative number with a number of arithmetic.

0 . . . . . . 0 0
$^-\dfrac{1}{2}$ . . . . . . $\dfrac{1}{2}$ . . . . . $^+\dfrac{1}{2}$
$^-1$ . . . . . 1 . . . . . $^+1$
$^-1\dfrac{1}{2}$ . . . . . $1\dfrac{1}{2}$ . . . . . $^+1\dfrac{1}{2}$
$^-2$ . . . . . . 2 . . . . . $^+2$
$^-3$ . . . . . 3 . $_r$ . . . . $^+3$
$^-4$ . . . . . 4 . . . . . $^+4$
$^-5$ . . . . . 5 . . . . . $^+5$
$^-6$ . . . . . 6 . . . . . $^+6$
.   .   .
.   .   .
.   .   .

In this table, each real number corresponds with that number of arithmetic which gives the distance part of a trip measured by the real number. Thus, in order to find sums of pairs of nonpositive or nonnegative real numbers, you should begin by finding the arithmetic numbers which correspond with the real numbers, and then add the arithmetic numbers. We can summarize these remarks by saying that

> the nonpositive real numbers and
> the nonnegative real numbers <u>act</u>
> <u>like</u> the numbers of arithmetic
> with respect to addition.

You probably discovered this idea when you were looking for short cuts in computing sums.

What about multiplication? Do the nonpositive real numbers act like the corresponding numbers of arithmetic with respect to multiplication? A quick check of the table above shows that they do not.

$$
\begin{array}{ccc}
^-2 \dots\dots & \quad & 2 \\
\times\ ^-3 \dots\dots & \quad & \times\ 3 \\
\hline
? & \quad & 6
\end{array}
$$

6 corresponds with $^-6$ but $^-2 \times\ ^-3 \neq\ ^-6$.

Do the nonnegative real numbers act like the corresponding numbers of arithmetic with respect to multiplication? The answer is 'yes'. So,

> the nonnegative real numbers <u>act</u>
> <u>like</u> the numbers of arithmetic with
> respect to <u>both</u> addition and multi-
> plication.

SHORTER NAMES FOR POSITIVE NUMBERS

In view of the fact that the nonnegative real numbers act like
the numbers of arithmetic with respect to both addition and multi-
plication, it will cause no trouble if we use the <u>names</u> of numbers
of arithmetic as <u>names</u> of the nonnegative real numbers. When we
want to state a fact about real numbers, for example, that

$$(1) \qquad {}^+9 + {}^-3 = {}^+6,$$

we can just write:

$$(2) \qquad 9 + {}^-3 = 6.$$

Anyone who looks at sentence (2) and believes that it should make
sense must conclude that '9' and '6' are numerals for positive
numbers rather than numerals for numbers of arithmetic. [He
would conclude this because it wouldn't make sense to add a nega-
tive real number to a number of arithmetic.] So, when you look
at (2) you "see" it as (1). And writing (2) instead of (1) saves you
~~the trouble~~ of writing the little plus signs.

Consider another example.

$$(3) \qquad 7 \times 4 = 28.$$

Is this a statement about numbers of arithmetic or is it a state-
ment about real numbers? Unless you know the problem which
led to someone's writing (3), you are free to interpret it either
way.

Numerals such as those in sentence (3) which name more than
one number are said to be <u>ambiguous</u>. We have already seen an
ambiguous numeral in the case of '0'. '0' names the number 0 of
arithmetic and it names the real number 0. We are now preparing
to deal with many more cases of ambiguous numerals.

Ambiguous words or names may cause confusion. For example,
suppose there are two students in your class each having the name
'Ann Brown'. If this message is sent from the principal's office:

> Ann Brown is to report to the principal's
> office at 3:30 for a conference

there is likely to be confusion since 'Ann Brown' refers to either
of these students, and the teacher   would not know which student
should get the message.   On the other hand, suppose the following
message is sent:

> Ann Brown is to report to the principal's office
> at 3:30 for a conference with the other fresh-
> man representatives to the Student Council.

It is likely that the ambiguity of 'Ann Brown' in this message would
cause no trouble since the rest of the message makes clear which
Ann Brown is intended.

Similarly, although the '9' in each of the sentences:

$$(4) \qquad 9 \times 4 = 36$$

and:

$$(5) \qquad 9 \times {}^-7 = {}^-63$$

is ambiguous, we know from the rest of sentence (5) that it refers
to the real number ${}^+9$ and not to the number 9 of arithmetic.

EXERCISES

A. Each of the following sentences contains at least one ambiguous
numeral. In which sentences are you unable to tell whether
real numbers or numbers of arithmetic are intended?

1. $2 \times {}^-3 = {}^-6$ 2. $8 + 4 = 3 + 9$

3. $5 + {}^-5 = 0$ 4. $4 \times 7 = 14 \times {}^+2$

5. $8 \times 3 = 6 \times 4$ 6. $10 \times 5 = {}^-25 \times {}^-2$

B. Simplify.

1. $3 \times {}^-4$ 2. ${}^-7 \times 5$ 3. ${}^-8 \times (6 + 4)$

4. ${}^-2 \times {}^-5$ 5. $15 \times {}^-3$ 6. ${}^-2 \times {}^-7$

7. $3 + {}^-4$ 8. $7 + {}^-5$ 9. ${}^-15 + 12$

10. $(17 + {}^-8) + 3$ 11. $({}^-21 + 5) + {}^-5$

12. $({}^-53 \times 6) \times \frac{1}{6}$ 13. $({}^-751 \times 7) \times \frac{{}^-1}{7}$

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There is a very subtle issue which arose during one of our summer
conferences in connection with Exercise 6 of Part A. There is a
possibility that it will be brought up by some of your very perceptive
students. The issue is this. Although the numeral '10 × 5' must
stand for a real number in order that the sentence make sense, it
is not necessarily the case that the component numerals '10' and
'5' must stand for real numbers. They could stand for numbers
of arithmetic in which case the numeral '10 × 5' stands for a number
of arithmetic. Then, by our convention concerning the abbreviation
of numerals for positive real numbers, we could regard such a
numeral as an abbreviation for '$^+$(10 × 5)'. The ambiguity in this
problem centers around the multiplication sign. As pointed out
earlier, the operation of multiplication of real numbers is different
from the operation of multiplication of numbers of arithmetic. Yet,
the same sign is used to denote both of these operations. If
different signs were used, there would be no ambiguity. For
example, if '$\times_A$' denotes the operation of multiplication of numbers
of arithmetic, and '$\times_R$' denotes the operation of multiplication of
real numbers then in

$$\text{'} 10 \times_A 5 \text{'}$$

the numerals '10' and '5' stand for numbers of arithmetic, whereas
the numerals '10' and '5' in

$$\text{'} 10 \times_R 5 \text{'}$$

stand for real numbers. [Similar remarks apply to Exercise 4.]

✻

Answers for Part A.

In Exercises 2 and 5 you cannot tell whether real numbers or
numbers of arithmetic are intended.

✻

Answers for Part B.

1. $^-$12	2. $^-$35	3. $^-$80	4. 10 [or: $^+$10]
5. $^-$45	6. 14 [or: $^+$14]	7. $^-$1	8. 2 [or: $^+$2]
9. $^-$3	10. 12 [or: $^+$12]	11. $^-$21	12. $^-$53

13. 751 [or: $^+$751]

[Do not make an issue of the use of short cuts in Exercises 10-13.]

<u>C</u>. Each of the following sentences contains ambiguous words or
phrases. Be prepared to give two interpretations for each
sentence.

1. There will be little change in ladies' pocketbooks next year.
2. Charles ran after Henry.
3. Please leave the table.
4. One of the horses was scratched.
5. This is a fine day for the race.
6. Look at the scales.
7. Mr. Blattingham gave his address.
8. What does that ring mean?

9.

SLOW
CHILDREN
AT PLAY

10.

SAVE RAGS
AND
WASTE PAPER

1.05 <u>Punctuating numerical expressions</u>.--The pictures below show
the seat arrangements of two classrooms in a school building. Under
each picture someone has tried to indicate the number of seats in the
room and, at the same time, show his method of computing that number.

6 x 5 + 2

6 x 5 + 2

It is easy to see that the room shown on the left has 32 seats and that the room shown on the right has 42 seats. If the collection of marks:

$$6 \times 5 + 2$$

is a numeral, it would seem to be the case that this numeral stands for 32 and for 42! Without the pictures, the numeral '$6 \times 5 + 2$' is of little help.   This is a case of ambiguity which could cause trouble. What can be done to remove the ambiguity?

Here is a case of an English sentence which is ambiguous:

(1)   John will play and Bill will sing or Mary will sing.

This sentence could mean that

(2)   Either John will play and Bill will sing or Mary will sing.

Or, it could mean that

(3)   John will play and either Bill will sing or Mary will sing.

To make an English sentence unambiguous you usually have to rewrite it and use grouping words like 'either...or...'.

In mathematics we can remove ambiguity from expressions by punctuating them with parentheses. We could use parentheses to punctuate sentence (1) to show the first meaning. We would write:

(2′)   (John will play and Bill will sing) or Mary will sing.

To show the other meaning we would write:

(3′)   John will play and (Bill will sing or Mary will sing).

Similarly, we can punctuate '$6 \times 5 + 2$' this way:

$$(6 \times 5) + 2$$

when we mean 32, and like this:

$$6 \times (5 + 2)$$

when we mean 42. [To read '$(6 \times 5) + 2$' aloud, say 'the sum of 6×5 and 2' or 'the quantity 6×5, plus 2' or 'parenthesis 6×5 close parentheses, plus 2'. How do you read aloud '$6 \times (5 + 2)$'?]

Here is another expression which is ambiguous:

$$8 \times 3 + 2 \times 5.$$

Give some numbers which it might name. Here are some of the ways in which this expression can be punctuated to make it unambiguous.

If we want to name 34, we can write:
$$(8 \times 3) + (2 \times 5).$$
To name 130, we can write:
$$[(8 \times 3) + 2] \times 5.$$
If we mean 200 then we should write:
$$[8 \times (3 + 2)] \times 5.$$
Is there another possibility?

Notice that when we want to show a grouping inside of another grouping, we use both parentheses and brackets. [Why don't we just use two pairs of parentheses?] Here is a punctuated expression in which parentheses, brackets, and braces are used:
$$\{[(7 \times 2) + 6] \times 5\} + 7.$$
This expression is a name for a number. Often it is helpful to have a simpler looking name for such a number. [For example, if someone asked you how much your new cap cost, and you replied that it cost $\{[(7 \times 2) + 6] \times 5\} + 7$ cents, he would have left before you had finished speaking.] To find a simpler looking expression equivalent to:
$$\{[(7 \times 2) + 6] \times 5\} + 7$$
[that is, to find a simpler expression which names the same number], we must find a simpler expression equivalent to:
$$\{[(7 \times 2) + 6] \times 5\};$$
this will be easier to do if we find a simpler one equivalent to:
$$[(7 \times 2) + 6],$$
and this is done by first finding a simpler name for the number named by:
$$(7 \times 2).$$
Hence, to simplify our original expression we would think through the following steps to obtain equivalent expressions.

$$\{[(7 \times 2) + 6] \times 5\} + 7$$
$$\{[\quad 14 \quad + 6] \times 5\} + 7$$
$$\{\qquad\quad 20 \ \times 5\} + 7$$
$$\qquad\qquad 100 \ + 7$$
$$\qquad\qquad\quad 107$$

Most people would agree that '107' is a simpler looking name for 107 than is '$\{[(7 \times 2) + 6] \times 5\} + 7$'.

EXERCISES

A. Simplify. [Be careful not to confuse ' + 's with ' × 's.]

1. (3 + 4) × 2   2. 3 + (4 × 2)   3. (7 × 5) + 4

4. (8 + ⁻3) × ⁻2   5. ⁻1 × (⁻6 + 4)   6. (⁻1 × ⁻6) + 4

7. (9 ÷ 3) + 3   8. 9 ÷ (3 + 3)   9. (8 ÷ 2) × 4

10. 8 ÷ (2 × 4)   11. (6 × 5) ÷ 5   12. 6 × (5 ÷ 5)

13. (8 + 2) + 5   14. 8 + (2 + 5)   15. (2 × 5) × 3

16. [(8 + 2) × 3] + 5   17. [(⁻3 + ⁻2) × 6] + ⁻7

18. 5 + [⁻1 × (8 + ⁻2)]   19. (5 + ⁻1) × (8 + ⁻2)

20. (8 × 7) + (2 × 7)   21. (8 + 2) × 7

22. (3 × ⁻6) + (7 × ⁻6)   23. (3 + 7) × ⁻6

24. (⁻2 × ⁻12) + (2 × ⁻12)   25. (⁻2 + 2) × ⁻12

26. 2 × {[(3 + 2) × 4] + [(5 + 9) × 4]}

27. ⁻3 × {[⁻2 × (8 + 12)] + [(9 + 31) × ⁻4]}

28. {[5 × (8 + ⁻1)] + 3} × {⁻2 + [4 × (3 + ⁻3)]}

29. $\frac{5 + 21}{2 \times 13}$ [Note: When a bar is used to indicate division,
it also acts as a grouping symbol. So, '$\frac{5 + 21}{2 \times 13}$' means
(5 + 21) ÷ (2 × 13).]

30. $\frac{28 - 3}{100 - 50}$ 31. $\frac{15 + 4}{(7 \times 5) + 3}$ 32. $\frac{17 + (6 \times 2)}{17 - (6 \times 2)}$

33. $\frac{[(9 \times 3) + (5 \times 2)] + 3}{[8 + (5 \times 5)] - 13}$ 34. $\frac{[(5 \times 6) - (4 \times 2)] + [(3 \times 4) - 3]}{2 + \{5 + [3 + 2 \times (27 - 1)]\}}$

1. Joey and Jane went to the candy store; Joey bought 5 candy
bars at 6 cents each and a 10-cent package of bubble gum.
Jane bought 6 suckers that were 5 cents each and 10 sacks
of peanuts that were also 5 cents each. What single expres-
sion could you write which (if no grouping symbols were used)
might name the cost of either Joey's or Jane's purchases?

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Here is an interesting set of exercises which you may want to give
to the class before they work on Part A. They were suggested to us
by similar exercises found in certain European arithmetic textbooks.
The object in each exercise is to insert grouping symbols and opera-
tion signs to make true sentences.

1.   5   4   2   = 22   2.   5   4   2   = 30

3.   5   4   2   = 18   4.   5   4   2   = 13

5.   5   4   2   = 10   6.   5   4   2   = 0.625

7.   5   4   2   = 2.5   8.   5   4   2   = 40

9.   5   4   2   = 11   10.   5   4   2   = 7

11. 5   4   2   = 3   12.   5   4   2   = 3.25

*

Answers for Part A.

1.	14	2.	11	3.	39	4.	⁻10	5.	2
6.	10	7.	6	8.	$\frac{3}{2}$	9.	16	10.	1
11.	6	12.	6	13.	15	14.	15	15.	30
16.	35	17.	⁻37	18.	⁻1	19.	24	20.	70
21.	70	22.	⁻60	23.	⁻60	24.	0	25.	0
26.	152	27.	600	28.	⁻76	29.	1	30.	$\frac{1}{2}$
31.	$\frac{1}{2}$	32.	$\frac{29}{5}$	33.	2	34.	[See TC[1-36, 37]b.]		

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Notice that there is an ambiguity in the denominator of the fraction
in Exercise 34. Although this was originally an oversight on our
part, it turned out to be useful in at least two of the demonstration
classes which were taught at institutes during the summer of 1958.
Students should express some puzzlement over the exercise itself,
and if they are able to plow through the arithmetic, they should

assert that this fraction stands for either of the two numbers, $\frac{31}{137}$
and $\frac{1}{2}$.

*

Answers for Part B [on pages 1-36 and 1-37].

1. $5 \times 6 + 10$ 2.   yes, yes

[In some parts of the country, students may object to our use of
the word 'sacks' in Exercise 1, and say that we should have used
the word 'bag' [or even 'poke']. In the Middle West the words 'sack'
and 'bag' are used interchangeably.   One of our teachers (having
helped make gardens in her youth) wrote us that in Mrs. Plantin's
garden [Exercise 2] we should have had 3 hills of potato plants in
each of 7 rows.   She explains that a hill of potatoes contains several
potato plants! Of course, if we said 'hills' of potato plants, our
problem might not be quite as effective. We would have to change
the question to:

    Could the expression above which indicates the number of
    plants set out by Mrs. Gardner also be used to indicate
    the number of green pepper plants and the number of hills
    of potatoes in Mrs. Plantin's garden?

and that puts a little different "twist" on it.]

2.   Mrs. Gardner and Mrs. Plantin were making vegetable
     gardens. Mrs. Gardner said she had planted 3 rows of
     cabbages with 7 plants in a row, and 5 rows of tomatoes
     with 3 plants in each row. <u>Could</u> the expression '3 × 7 + 5'
     represent the number of cabbage and tomato plants in
     Mrs. Gardner's garden?

     Her friend, Mrs. Plantin, explained that she had set
     out a row containing 5 green pepper plants; also, she had
     planted potatoes so that she would have 3 potato plants in
     each of 7 rows. <u>Could</u> the expression above which indicates
     the number of plants set out by Mrs. Gardner also be used
     to indicate the number of green pepper and potato plants in
     Mrs. Plantin's garden?

CONVENTIONS FOR OMITTING GROUPING SYMBOLS

You have seen how parentheses and other grouping symbols can
be used to remove ambiguity from numerical expressions. In fact,
there would never be an ambiguous expression if everyone followed
the rule that each operation sign [+, ×, −, ÷] required a pair of
grouping symbols. You would have expressions like these:

$$(5 + 4),   [(6 - 2) + 9],   [3 \times (6 \times 5)],$$

$$\{[(4 + 1) + 3] + (6 - 5)\},   \{[(3 \times 4) + (6 - 2)] \div (12 \div 3)\}.$$

In the expression '(5 + 4)', the parentheses go with the ' +'.

In the expression '[3 × (6 × 5)]', the parentheses go with the second
' ×', and the brackets go with the first ' ×'.

In the expression '{[(4 + 1) + 3] + (6 - 5)}', the braces go with the
third ' +', the brackets go with the second ' +', the first pair of
parentheses goes with the first ' +', and the second pair goes with
the ' − '.

Tell which pairs of grouping symbols go with which operation signs
in the other two expressions written above. Make up three expressions
each of which contains at least two operation signs and a pair of
grouping symbols which go with each sign.

If an expression contains many operation signs, this rule requires
that the expression contain just as many pairs of grouping symbols.

Such expressions are frequently hard to read and look quite compli-
cated. So, people follow certain agreements [or <u>conventions</u>] which
permit them to omit some grouping symbols and still avoid ambiguity.
Under such conventions, an expression like:

$$9 + 2 \times 4 - 4 \div 2$$

is unambiguous. Our job now is to learn what these conventions are
so that when we come upon an expression such as the one above, there
will be no doubt what number is intended by the person who followed
our conventions in writing it.

One convention [which we have been using throughout this book] is
to omit the outermost grouping symbols. For example, we would write

'5 + 4' as an abbreviation for '(5 + 4)',

and   '(6 - 2) + 9' as an abbreviation for '[(6 - 2) + 9]'.

This convention was followed in all of the expressions in Part A on
page 1-36. Turn to that page now and, for the first ten expressions,
tell which operation sign goes with the omitted grouping symbols.
When we use such abbreviated expressions to form larger expressions,
we often have to <u>unabbreviate</u> them by replacing the omitted grouping
symbols. For example, we name the product of 3 + 2 and 4 + 5 by
'(3 + 2) × (4 + 5)'. Although '3 + 2' and '4 + 5' are not ambiguous,
'3 + 2 × 4 + 5' is ambiguous.

Another convention which we shall adopt is illustrated in these
examples.

'3 + 5 + 6' is an abbreviation for '(3 + 5) + 6'.

'5 × 2 × 7' is an abbreviation for '(5 × 2) × 7'.

'4 + 3 + 2 + 9' is an abbreviation for '[(4 + 3) + 2] + 9'

'12 - 2 - 3' is an abbreviation for '(12 - 2) - 3'.

'18 ÷ 3 ÷ 2' is an abbreviation for '(18 ÷ 3) ÷ 2'.

'2 + (3 + 5) + 6' is an abbreviation for '[2 + (3 + 5)] + 6'.

'2 × 3 × (5 × 6)' is an abbreviation for '(2 × 3) × (5 × 6)'.

Notice that each of the abbreviated expressions contains only one kind of
operation sign. The expressions which contain no grouping symbols are
unabbreviated by introducing a pair of grouping symbols for each operation

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Here are the unabbreviated expressions which correspond with those in Exercises 1-10 of Part A on page 1-36. Notice that each operation sign is linked with a pair of grouping symbols. [At the blackboard you can use, say, yellow chalk for one pair of grouping symbols and the corresponding operation sign, and blue chalk for the other pair and its corresponding sign.]

1. $[(3 + 4) \times 2]$ 2. $[3 + (4 \times 2)]$ 3. $[(7 \times 5) + 4]$

4. $[(8 \div {}^-3) \times {}^-2]$ 5. $[{}^-1 \times ({}^-6 + 4)]$ 6. $[({}^-1 \times {}^-6) \div 4]$

7. $[(9 \div 3) + 3]$ 8. $[9 \div (3 + 3)]$ 9. $[(8 \div 2) \times 4]$

10. $[8 \div (2 \times 4)]$

✳

Example (4) on page 1-39 may contradict conventions given in other texts which tell you "to do all multiplications first, then divisions next". Please follow the convention given here. The parallelism between addition-subtraction and multiplication-division is too neat to want to give up.

✳

Do you see that the procedure for unabbreviating an expression is tantamount to a procedure for deciding upon order of operations? In the example at the bottom of page 1-39 students should see that each operation sign [except the last one] has a pair of grouping symbols.

sign, proceeding from left to right. So, we unabbreviate:

$$4 + 3 + 2 + 9$$

by first writing:

$$(4 + 3) + 2 + 9,$$

and then writing:

$$[(4 + 3) + 2] + 9.$$

[As before, we don't include the outermost grouping symbols.] As
in the step from '(4 + 3) + 2 + 9' to '[(4 + 3) + 2] + 9', when some
grouping symbols are already present in an expression to be unabbre-
viated, we bring in grouping symbols for those operation signs which
do not already have them, again proceeding from left to right.
Examples:

(1) $2 + (3 + 5) + 6$ || (2) $2 \times 3 \times (5 \times 6)$
         $\downarrow$ $\downarrow$
       $[2 + (3 + 5)] + 6$ $(2 \times 3) \times (5 \times 6)$

This same convention is used for unabbreviating expressions in
which the operation signs which lack grouping symbols are either all
addition and subtraction signs or all multiplication and division signs.

(3) $4 + 3 - 2 + 7$ || (4) $8 \times 2 \div 4 \times 3$
         $\downarrow$ $\downarrow$
       $(4 + 3) - 2 + 7$ $(8 \times 2) \div 4 \times 3$
         $\downarrow$ $\downarrow$
       $[(4 + 3) - 2] + 7$ $[(8 \times 2) \div 4] \times 3$

(5) $9 - (3 \times 2) + 4 - 5$ || (6) $10 \div 5 \div (2 + 2) \times 2$
         $\downarrow$ $\downarrow$
       $[9 - (3 \times 2)] + 4 - 5$ $(10 \div 5) \div (2 + 2) \times 2$
         $\downarrow$ $\downarrow$
       $\{[9 - (3 \times 2)] + 4\} - 5$ $[(10 \div 5) \div (2 + 2)] \times 2$

Consider this expression:

$$12 + [3 \times (5 + 6) \div 2] - 5.$$

To unabbreviate it, we first attack the expression in the brackets and get:

$$[\{3 \times (5 + 6)\} \div 2].$$

And, now we finish unabbreviating the original expression, getting:

$$\{12 + [\{3 \times (5 + 6)\} \div 2]\} - 5.$$

Here are some more examples of this.

(7) $4 \times \quad (3 \times 2 \times 5 \times 3)$ (8) $38 + [\quad 3 \times 4 \div 6 \times (2 + 4)]$

$4 \times \quad ([3 \times 2] \times 5 \times 3)$ $38 + [\quad (3 \times 4) \div 6 \times (2 + 4)]$

$4 \times (\{[3 \times 2] \times 5\} \times 3)$ $38 + [\{(3 \times 4) \div 6\} \times (2 + 4)]$

EXERCISES

A. On a separate sheet of paper, rewrite each of these expressions in unabbreviated form. [You need not put in outermost grouping symbols.]

Sample. $6 + 4 + 3 + 9$

Solution. $[(6 + 4) + 3] + 9$

1. $2 + 8 + 3$ 2. $7 \times 5 \times 3$

3. $9 - 5 - 3$ 4. $6 \div 2 \div 3$

5. $8 + 3 + 5 + 4$ 6. $9 \times 2 \times 3 \times 5$

7. $^+5 + {}^+2 + {}^+3 + {}^+7$ 8. $^+3 \times {}^+2 \times {}^+6 \times {}^+7$

9. $3 + 8 - 2 + 5$ 10. $15 - 7 + 5 - 9$

11. $5 \times 6 \div 3 \times 4$ 12. $3 \times 4 \div 2 \div 3$

13. $15 - 8 - 3 - 1$ 14. $24 \div 3 \div 2 \div 4$

15. $5 + 9 + (6 + 8)$ 16. $7 + (9 + 5) + 8$

17. $2 \times (5 \times 3) \times 4$ 18. $6 \times 8 \times (3 \times 7) \times 4$

19. $5 + (3 \times 2 \times 8)$ 20. $6 \times (4 + 9 + 3 - 8)$

21. $8 + [3 \times (7 + 2) \times 5] - 6 + 7 - 5$

22. $(5 \times 3) + (4 \times 7) - (8 \times 2) + (8 - 2)$

23. $6 + [5 \times (3 + 7)] + [4 \times (8 - 5)]$

24. $4 \times [5 - 3 - 1] \times [7 + (2 \times 4)]$

B. According to the conventions we have discussed, each of the expressions in Part A is unambiguous. You should be able to simplify each expression without rewriting it in unabbreviated form. Do so.

✳ ✳ ✳

Answers for Part A.

1. $(2 + 8) + 3$ 2. $(7 \times 5) \times 3$

3. $(9 - 5) - 3$ 4. $(6 \div 2) \div 3$

5. $[(8 + 3) + 5] + 4$ 6. $[(9 \times 2) \times 3] \times 5$

7. $[({}^{+}5 \div {}^{+}2) + {}^{+}3] \div {}^{+}7$ 8. $[({}^{+}3 \times {}^{+}2) \times {}^{+}6] \times {}^{+}7$

9. $[(3 + 8) - 2] + 5$ 10. $[(15 - 7) + 5] - 9$

11. $[(5 \times 6) \div 3] \times 4$ 12. $[(3 \times 4) \div 2] \div 3$

13. $[(15 - 8) - 3] - 1$ 14. $[(24 \div 3) \div 2] \div 4$

15. $(5 + 9) + (6 + 8)$ 16. $[7 + (9 + 5)] + 8$

17. $[2 \times (5 \times 3)] \times 4$ 18. $[(6 \times 8) \times (3 \times 7)] \times 4$

19. $5 + ([3 \times 2] \times 8)$ 20. $6 \times (\{[4 + 9] + 3\} - 8)$

21. $\{\{(8 + [\{3 \times (7 + 2)\} \times 5]) - 6\} \div 7] - 5$

22. $\{[(5 \times 3) + (4 \times 7)] - (8 \times 2)\} + (8 - 2)$

23. $\{6 + [5 \times (3 + 7)]\} + [4 \times (8 - 5)]$

24. $\{4 \times [(5 - 3) - 1]\} \times [7 + (2 \times 4)]$

✳

Answers for Part B.

1. 13	2. 105	3. 1	4. 1	5. 20
6. 270	7. ⁺17	8. ⁺252	9. 14	10. 4
11. 40	12. 2	13. 3	14. 1	15. 28
16. 29	17. 120	18. 4032	19. 53	20. 48
21. 139	22. 33	23. 68	24. 60	

In simplifying abbreviated expressions which contain multi-plication or division signs <u>and</u> addition or subtraction signs such as:

$$9 + 2 \times 4 - 4 \div 2,$$

you first do the multiplications and divisions, working from left to right, and then the additions and subtractions in the same order. In particular, the expression:

$$9 + 2 \times 4 - 4 \div 2$$

is unabbreviated in two steps:

$$9 + (2 \times 4) - (4 \div 2),$$

and then:

$$[9 + (2 \times 4)] - (4 \div 2).$$

So, the given expression is, in view of our conventions, completely unambiguous. It stands for 15.

Example 9.   Simplify by first unabbreviating:

$$5 \times 9 + 3 - 8 \div 2 \times 3.$$

Solution.   Unabbreviate the expression.

$$5 \times 9 + 3 - 8 \div 2 \times 3$$
$$= (5 \times 9) + 3 - [(8 \div 2) \times 3]$$
$$= [(5 \times 9) + 3] - [(8 \div 2) \times 3].$$

Now simplify it.

$$[(5 \times 9) + 3] - [(8 \div 2) \times 3]$$
$$= [\ 45\ + 3] - [\ 4\ \times 3]$$
$$=\ \ \ \ \ \ 48\ \ \ \ \ -\ \ \ \ \ \ 12$$
$$=\ \ 36.$$

Example 10.   Simplify by first unabbreviating:

$$3 + (6 + 2 \times 5) - (17 - 4 \times 3).$$

Solution.   Unabbreviate first.

$$3 + (6 + 2 \times 5) - (17 - 4 \times 3)$$
$$=\ \ 3 + (6 + [2 \times 5]) - (17 - [4 \times 3])$$
$$= \{3 + (6 + [2 \times 5])\} - (17 - [4 \times 3]).$$

Simplify next.

$$\{3 + (6 + [2 \times 5])\} - (17 - [4 \times 3])$$

$$= \{3 + (6 + \quad 10 \quad)\} - (17 - \quad 12 \quad)$$

$$= \{3 + \quad\quad 16 \quad\quad\} - 5$$

$$= \quad\quad 19 \quad\quad\quad\quad - 5$$

$$= \quad 14.$$

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C. Rewrite each of the following expressions in unabbreviated form, and then simplify it.

1. $3 + 5 \times 10$ 2. $9 \times 2 + 4$ 3. $8 \div 2 + 5$

4. $7 \div 2 \times 5$ 5. $4 \times 3 - 1$ 6. $2 + 5 \times 6$

7. $4 \times 5 + 3 \times 7 + 2 \times 3$ 8. $6 \div 2 + 15 \div 3 + 20 \div 10$

9. $3 \times 7 - 2 \times 5 + 4 \div 8$ 10. $10 \times 5 \div 2 \times 4 \div 100$

11. $^+5 \times {}^-4 + {}^-3 \times {}^-2$ 12. $4 + {}^-3 \times 7 + {}^-2 \times {}^-4$

13. $8 + {}^-2 + {}^-3 \times 6$ 14. $8 + {}^-2 \times {}^-5 + 10 \times 3$

15. $7 \times (8 + 3) \times (6 \div 2)$ 16. $12 \times (4 + {}^-2) + (20 + {}^-3)$

17. $3 + (16 \div 2 \div 4) + (5 \times 2 - 3 \times 3)$

18. $5 \times 3 - 6 \div 2 + [(4 + 5) \times 3 + 2 + 4 \times 2]$

19. $(5 + 3) \times (7 + 1) - 8 \times 2 \times (3 - 1) + 7 \times 5 + 1$

20. $6 + 3 + 8 \times (6 - 3 \times 2 + 5 + 4 \div 2) + (4 - 3)$

D. Simplify.

1. $3 \times 6 - 4 \times 2$ 2. $5 \times 4 - 3 \times 2$

3. $7 \times 4 \div 2 + 3 \times 7$ 4. $6 \times 5 \div 2 + 4 \times 9 \div 2$

5. $18 - 7 + 2 - 8 \div 2$ 6. $12 - (5 + 3) \div 2 + 6$

7. $18 - (7 + 2) - 8 \div 2$ 8. $12 - (5 + 3) \div (2 + 6)$

9. $84 - \{5 \times [5 \times 2 - 3 + (6 \times 4 - 5) - 7] - (6 \div 2 + 8)\}$

10. $16 + 3 \times [(3 + 5) \times 2 - (2 + 1) \times 4] + 6 \times (8 - 3)$

[More exercises are in Part F, Supplementary Exercises.]

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The rewriting for Part C and [if any is necessary] for Part D should be done on a separate sheet of paper.

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Answers for Part C.

1. $3 + (5 \times 10)$; 53

2. $(9 \times 2) + 4$; 22

3. $(8 \div 2) + 5$; 9

4. $(7 \div 2) \times 5$; 17.5

5. $(4 \times 3) - 1$; 11

6. $2 + (5 \times 6)$; 32

7. $[(4 \times 5) + (3 \times 7)] + (2 \times 3)$; 47

8. $[(6 \div 2) + (15 \div 3)] + (20 \div 10)$; 10

9. $[(3 \times 7) - (2 \times 5)] + (4 \div 8)$; $11\frac{1}{2}$

10. $\{[(10 \times 5) \div 2] \times 4\} \div 100$; 1

11. $(^+5 \times {}^-4) + (^-3 \times {}^-2)$; $^-14$

12. $[4 + (^-3 \times 7)] + (^-2 \times {}^-4)$; $^-9$

13. $(8 + {}^-2) + (^-3 \times 6)$; $^-12$

14. $[8 + (^-2 \times {}^-5)] + (10 \times 3)$; 48

15. $[7 \times (8 + 3)] \times (6 \div 2)$; 231

16. $[12 \times (4 + {}^-2)] + (20 + {}^-3)$; 41

17. $\{3 + ([16 \div 2] \div 4)\} + ([5 \times 2] - [3 \times 3])$; 6

18. $[(5 \times 3) - (6 \div 2)] + [(\{(4 + 5) \times 3\} + 2) + (4 \times 2)]$; 49

19. $[\{[(5 + 3) \times (7 + 1)] - [(8 \times 2) \times (3 - 1)]\} + (7 \times 5)] + 1$; 68

20. $[(6 + 3) + \{8 \times ([\{6 - [3 \times 2]\} + 5] + [4 \div 2])\}] + (4 - 3)$; 66

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Answers for Part D.

1. 10	2. 14	3. 35	4. 33	5. 9
6. 14	7. 5	8. 11	9. 0	10. 58

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Here is a quiz which covers some of the ideas related to the conventions for omitting grouping symbols.

I. Rewrite each of the following expressions in unabbreviated form.

1. $3 \times 8 + 3 \times 9$ 2. $8 \times 2 + (3 + 9)$

3. $3 \times 8 \times (7 + 6)$ 4. $[3 \times 8 + 7] \times 6$

5. $2 \times (9 + 1) \times 3$ 6. $2 \times [9 + 1 + 3]$

II. Simplify.

1. $3 + 7 \times 5$ 2. $3 \times 7 + 5$

3. $8 \times 2 + 3 \times 9$ 4. $8 \times 2 + (3 + 9)$

5. $5 \times {}^-2 + 5 \times {}^-3$ 6. $5 \times ({}^-2 + 5) \times {}^-3$

7. $2 \times 3 + 4 + {}^-4$ 8. $5 + 3 \times {}^-10 + {}^+80$

9. $3 \times 5 \div 2 \times 8$ 10. $3 \times 5 \div (2 \times 8)$

✳

Answers for quiz.

I. 1. $(3 \times 8) + (3 \times 9)$ 2. $(8 \times 2) + (3 + 9)$

3. $(3 \times 8) \times (7 + 6)$ 4. $[(3 \times 8) + 7] \times 6$

5. $[2 \times (9 + 1)] \times 3$ 6. $2 \times [(9 + 1) + 3]$

II. 1. 38   2. 26   3. 43   4. 28

5. ${}^-25$ 6. ${}^-45$ 7. 6   8. ${}^+55$

9. 60   10. 15/16

These exercises should be worked without your pointing out short cuts. That is, each exercise in these two parts can be worked the long way. The student who doesn't discover the short cuts will be able to do the problems anyway. If he tries to do them mentally, he will be compelled to search for short cuts.

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In Part A, we think the students will see that in Exercise 1, for example, the given number is divisible by 7 and by 8. Some may also suggest that 2, 4, and 14 are also divisors. The answers below indicate the obvious divisors [which we hope the students will discover], and also [for your convenience] suggest others by giving a simpler name for the number being considered. Avoid, if possible, emphasizing the other factors.

2. 9, 10, other whole number factors of 90.

3. 13, 20, other whole number factors of 260.

4. 31, 7.

5. 6, 100, other whole number factors of 600.

6. 593, 5.

7. 19, 15, other whole number factors of 285.

8. 7, 13.

9. 41, 138, other whole number factors of 5658.

10. 87, 107, other whole number factors of 9309.

11. 547, 2728, other whole number factors of 1492216.

12. 3163, 7680, other whole number factors of 24291840.

※

Note carefully that in expressions in Part B like:

$$4 + 5 + 6, \quad \text{and:} \quad 4 \times 5 \times 6,$$

grouping symbols have been omitted by convention. Thus,

EXPLORATION EXERCISES

A. For each number listed below, give a whole number (other than
    itself and 1) which divides it. ['divides it' means the same thing
    as 'divides it exactly'. For example, 2, 3, 4, and 6 each divides
    12, but 5 does not divide 12.]

    Sample. $3 \times 5 + 7 \times 5$

        Solution.   This simplifies to '50'. So, 5 is a number which
                    divides $3 \times 5 + 7 \times 5$. [Other numbers are 2, 10,
                    and 25.]

    1. $3 \times 7 + 5 \times 7$ 2. $8 \times 9 + 2 \times 9$

    3. $2 \times 13 + 18 \times 13$ 4. $5 \times 31 + 2 \times 31$

    5. $93 \times 6 + 7 \times 6$ 6. $3 \times 593 + 2 \times 593$

                               ❊

    For each number listed below, give two numbers which divide it.

    7. $3 \times 19 + 12 \times 19$ 8. $6 \times 7 + 7 \times 7$

    9. $67 \times 41 + 71 \times 41$ 10. $51 \times 87 + 56 \times 87$

    11. $1319 \times 547 + 1409 \times 547$ 12. $3163 \times 3833 + 3163 \times 3847$

B. Simplify mentally.

    1. $(387 + 9) + 1$ 2. $(7452 + 75) + 25$

    3. $583 + 92 + 8$ 4. $927 + 152 + {}^-52$

    5. $(37 \times 5) \times 2$ 6. $(98 \times 25) \times 4$

    7. $487 \times 25 \times 40$ 8. $6871 \times 20 \times 5$

    9. $(82 + 47) + 53$ 10. $13 + (987 + 5426)$

    11. $894 + 751 + {}^-94$ 12. ${}^-6341 + {}^-275 + {}^-659$

    13. $(12 \times 15) \times 2$ 14. $(5 \times 84) \times 20$

    15. $55 \times 7 \times 2$ 16. $40 \times 812 \times 50$

    17. $98 + 76 + {}^-76$ 18. ${}^-583 + {}^-9624 + 583$

    19. $634 \times 5 \times \frac{1}{5}$ 20. $97 \times \frac{1}{17} \times 17$

    21. ${}^-\frac{1}{48} \times {}^-79 \times {}^-48$ 22. ${}^-384 \times 5627 \times {}^-\frac{1}{384}$

1.06 Principles for the numbers of arithmetic. --The numbers of
arithmetic have certain properties which you make use of time and
again as you do problems with these numbers.

(I) Here is a start on a multiplication table. Your job is to
fill in the empty spaces.

×	$\frac{2}{3}$	1200	$\frac{3}{4}$	87	21
21	14		$\frac{63}{4}$	1827	441
87	58	104,400		7569	
$\frac{2}{3}$	$\frac{4}{9}$		$\frac{1}{2}$		
$\frac{3}{4}$		900	$\frac{9}{16}$	$\frac{261}{4}$	
1200	800	1,440,000			25,200

There may be some students who are not acquainted with this type of multiplication table. Point out to them that the numbers to be multiplied are listed in the top row and in the left-hand column, and that the products are listed in the cells. Do not point out to them that they are supposed to look for a short cut in this exercise. The very fact that there is a short cut is something which must be discovered by the student.

Similarly, the sorting exercise at the top of page 1-45 calls for a discovery on the part of the student that there is a short cut. The purpose of Examples I and II is to build an awareness of the commutative principle for multiplication. It is the case, however, that students have been aware of this principle for a long time. What we are trying to do in these exercises is to shock them into thinking about the principle. Another way of getting them to think about the commutative principle is to have them fill in the blanks of the following sentences to make true ones out of them:

(a) $9 \times \underline{\hspace{1cm}} = 63$

(b) $\underline{\hspace{1cm}} \times 12 = 3 \times 20$

(c) $9 \times \underline{\hspace{1cm}} = 4 \times 9$

(d) $5 \times \underline{\hspace{1cm}} = 32 \times 5$

(e) $\underline{\hspace{1cm}} \times 95 = 95 \times 142$

(f) $583 \times 684 = 684 \times \underline{\hspace{1cm}}$

Notice that the first two exercises in this list involve multiplication and division computations whereas the other exercises require only an awareness of the commutative principle for multiplication. It may be that a few students will be trapped into carrying out laborious computations. For these students, the recognition that the commutative principle eliminates the need for such computations comes as a genuine surprise.

Once students have become aware of the commutative principle of multiplication, they will be ready to tell you that there is also such a principle for addition.

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Even though the title of this section is 'Principles for the numbers of arithmetic', you may have to remind the students every once in a while that we are now talking about numbers of arithmetic and not real numbers. A later section deals with analogous principles for real numbers.

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Here is the sorting for (II) on page 1-45.

21×17 $\frac{2}{3} \times 96$ $1984 \times \frac{3}{4}$ 657×891 27×31

17×21 $96 \times \frac{2}{3}$ $\frac{3}{4} \times 1984$ 891×657 93×9

※

Verbalizations of the principles [in terms of variables, or "pronum-erals"] are given in Unit 2. As far as Unit 1 is concerned, all we want is that the student know the names of the principles and that he be able to give instances. As the student practices giving and identi-fying instances, he becomes aware of patterns. It is this awareness that we count on in Unit 2 when we come to state the principles. With the statement of the principle at hand, the giving of instances is a mechanical task. We want students to do some thinking before they become mechanically proficient. Here is a way to develop a feeling for the principles without actually stating them.

Ask each of students A and B to think of a number of arithmetic. Then have B tell his number to A and ask A to multiply his num-ber by B's, and to record the product. Then have A tell his number to B and ask B to multiply his number by A's, again re-cording the product. The class should then be able to predict that both students will get the same product. This prediction is based on the student's acceptance of the commutative principle for multiplication. You can obtain the relevant instance of the commutative principle for multiplication in this case as follows. Ask A for his number. Suppose it is 7. When he tells it to you, write the following on the board:

$$7 \times \quad = \quad \times 7.$$

Then ask B for his number. Suppose it is 4. Then fill in the empty spaces to get:

$$7 \times 4 = 4 \times 7.$$

[Using three colors of chalk is helpful--white for the times signs and equality sign, blue for the '7's, and yellow for the '4's.] The completed sentence is an instance of the commutative prin-ciple for multiplication.

To illustrate the generality of the associative principle for addition, select three students, A, B, and C. Ask each to think of a number. Then B tells his number to A, and A adds B's number to his. Then

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C tells his number to A, and A adds C's number to the sum he just obtained, recording the total. Now, C tells his number to B, B adds C's number to his, B tells the sum to A, A adds this sum to his number, and records the total. The class should be able to predict that the totals are the same. To produce the relevant instance, first write this form on the board:

$$(\quad + \quad) + \quad = \quad + (\quad + \quad),$$

remarking that in each case two additions are to be carried out. Then ask for A's number.

$$(7 + \quad) + \quad = 7 + (\quad + \quad).$$

Then for B's number.

$$(7 + 4) + \quad = 7 + (4 + \quad).$$

Then for C's number.

$$(7 + 4) + 9 = 7 + (4 + 9).$$

(II) Sort the expressions below into pairs of numerals for the same number.

21×17	657×891	$1984 \times \frac{3}{4}$	$\frac{2}{3} \times 96$
$\frac{3}{4} \times 1984$	17×21	$\frac{396}{59} \times 243$	891×657
$96 \times \frac{2}{3}$	27×31	$243 \times \frac{396}{59}$	93×9

How many products did you need to compute to make your list of pairs? How many products did you need to compute to fill the table in the first problem?

If you had to do more than two computations for problem (II) and any for problem (I), you failed to recognize places to make use of an important property of the numbers of arithmetic. This is that if you multiply a pair of numbers, you get the same answer no matter what order you use in multiplying. This is the commutative principle for multiplication. Instances of this principle are:

$$5 \times 7 = 7 \times 5,$$
$$21 \times \frac{3}{4} = \frac{3}{4} \times 21.$$

Notice that you don't have to simplify the numerals connected by the equality signs in these two sentences in order to know that the sentences are true. If you believe that the commutative principle for multiplication is true then you believe that each instance of it is true, also.

Is there a corresponding property of the numbers of arithmetic for addition? If Mr. Brown has 12 Black Angus cows and 8 Holstein cows on his farm, how many cows of each kind must he buy to have twice as many cows, and the same number of Black Angus as Holstein?

To solve this problem quickly you need to recognize that

$$12 + 8 = 8 + 12,$$

and this is an instance of the commutative principle for addition. Other instances are:

$$986 + 724 = 724 + 986,$$
$$16.357 + 5.009 = 5.009 + 16.357.$$

Notice, again, that you don't feel any urge to compute to see if the last two sentences are true. As long as you believe that the commutative principle for addition is true, you also believe that each instance of it is true.

There are still other properties of the numbers of arithmetic which are useful in solving problems, especially in finding short cuts.

For example, suppose you were trying to find the total number of raffle tickets sold on three consecutive days.

First day....47,   Second day....75,   Third day....25.
One way of doing this is to find the total for the first and second days, and then add to this total the number sold on the third day. So,

$$(47 + 75) + 25 = 122 + 25 = 147.$$

But a much easier way, which most likely has already occurred to you, is to do the problem this way:

$$47 + (75 + 25) = 47 + 100 = 147.$$

There was probably no doubt in your mind that you would get the same sum in doing the problem the second way as you would in doing it the first way. You feel sure about this because you feel sure about another property of the numbers of arithmetic, a property which is expressed by the associative principle for addition. Other instances of this principle are:

$$(8 + 5) + 19 = 8 + (5 + 19)$$

$$23 + 91 + 9 = 23 + (91 + 9),$$

$$15 + (85 + 38) = 15 + 85 + 38.$$

And, as you have probably guessed by now, there is also the associative principle for multiplication. Notice how it gives you short cuts.

$$(27 \times 5) \times 2 = ?$$

$(27 \times 5) \times 2 = 27 \times (5 \times 2).$ So, $(27 \times 5) \times 2 = 27 \times 10 = 270.$

$$(897 \times 4) \times 25 = ?$$

$(897 \times 4) \times 25 = 897 \times (4 \times 25).$ So, $(897 \times 4) \times 25 = 89700.$

$$50 \times (2 \times 68) = ?$$

$50 \times (2 \times 68) = (50 \times 2) \times 68.$ So, $50 \times (2 \times 68) = 6800.$

Sometimes you use short cuts which depend upon more than just one of these principles. For example, suppose you want to find the total number of points you made in a test which had three parts:

Part I 27   Part II 39   Part III 23

The straight-forward procedure would be:

$$(27 + 39) + 23 = 66 + 23 = 89.$$

But a short cut might involve <u>thinking through</u> these steps:

$$(27 + 39) + 23 = 27 + (39 + 23),$$

because of the associative principle for addition, and

$$27 + (39 + 23) = 27 + (23 + 39),$$

because the commutative principle for addition tells us that

$$39 + 23 = 23 + 39.$$

Finally,

$$27 + (23 + 39) = (27 + 23) + 39,$$

because of the associative principle for addition. So, we know that $(27 + 39) + 23 = (27 + 23) + 39$. Since $27 + 23$ is 50, and $50 + 39$ is 89, we know that

$$(27 + 39) + 23 = 89.$$

[Did you, without realizing it, use the associative and commutative principles for addition in finding that $27 + 23$ is 50?]

Suppose you were asked to do long "column addition" as in these examples. Do them.

5	2	5	14
3	9	7	25
5	1	2	32
7	8	3	51
6	9	3	98

Did you skip around to find the easy combinations? Do you believe that you can get a correct total this way? That you do get a correct total even though you skip around is a consequence of the associative and commutative principles for addition.

15. (57 + 24) - ___ = 57   16. (68 + 25) - ___ = 25

17. (6 + ___) - 7 = 6   18. (93 + ___) - 15 = 93

19. (19 + ___) - 3 = 19   20. (72 + ___) - 72 = 72

21. (9 + ___) - ___ = 9   22. (34 + ___) - ___ = 34

23. (53 + ___) - ___ = 53   24. (117 + ___) - ___ = 117

25. (___ + 5) - 5 = ___   26. (___ + 71) - 71 = ___

27. (___ + 48) - 48 = ___   28. (___ + $5\frac{1}{4}$) - $5\frac{1}{4}$ = ___

29. If you add 7 to a number, you can get back the number by
     subtracting ___ from the sum.

     <u>Example 1</u>. 91 + 7 = 98 and 98 - ___ = 91.

     <u>Example 2</u>. (43 + 7) - ___ = 43.

30. If you add 192 to a number, you can get back the number
     by _____ from the sum.

     <u>Example</u>.   (4375 + 192) - ___ = 4375.

31. If you want to undo the result of adding 15, subtract ___
     from the sum.

     <u>Example</u>.   (869 + 15) - ___ = 869.

32. Subtracting ___ undoes what adding 97 did.

33. _____ undoes what adding 54 did.

34. _____ undoes what adding 71 did.

<u>C</u>. Fill in the blanks to make true sentences.

1. (___ × 5) × $\frac{1}{5}$ = 3   2. (___ × 7) × $\frac{1}{7}$ = 4

3. (___ × 9) × $\frac{1}{9}$ = 2   4. (___ × 35) × $\frac{1}{35}$ = 86

5. (___ × $\frac{1}{4}$) × 4 = 12   6. (___ × $\frac{1}{7}$) × 7 = 42

7. (___ × $\frac{1}{8}$) × 8 = 40   8. (___ × $\frac{1}{11}$) × 11 = 99

EXERCISES

A. Each of the following sentences is an instance of one of the four principles you have just learned. Tell which principle.

1. $9 + 7 = 7 + 9$

2. $3 \times 5 = 5 \times 3$

3. $61 + 17 = 17 + 61$

4. $97 \times 816 = 816 \times 97$

5. $81 + (9 + 13) = (81 + 9) + 13$

6. $(93 \times 5) \times 100 = 93 \times (5 \times 100)$

7. $71 \times (51 + 47) = (51 + 47) \times 71$

8. $523 + 43 + 79 = 523 + (43 + 79)$

9. $657 \times 982 \times 539 = 657 \times (982 \times 539)$

10. $(841 + 56) + (75 + 37) = (75 + 37) + (841 + 56)$

11. $[(72 + 45) + 63] + 85 = (72 + 45) + (63 + 85)$

12. $72 + [(45 + 63) + 85] = [72 + (45 + 63)] + 85$

13. $72 + 45 + (63 + 85) = 72 + [45 + (63 + 85)]$

14. $72 + (45 + 63) + 85 = 85 + [72 + (45 + 63)]$

15. $(81 + 37) + (92 + 54) = (92 + 54) + (81 + 37)$

16. $7 + 3\frac{1}{4} = (7 + 3) + \frac{1}{4}$ ['$3\frac{1}{4}$' is an abbreviation for '$(3 + \frac{1}{4})$'.]

17. $85\frac{1}{5} \times (25 + 48) = (25 + 48) \times 85\frac{1}{5}$

18. $72 + (45 + 63) + 85 = 72 + [45 + 63 + 85]$

19. $(72 + 45 + 63) + 85 = (72 + 45) + (63 + 85)$

20. $(3 \times 7 \times 2 \times 5) \times (8 + 4 + 3 + 2) = (8 + 4 + 3 + 2) \times (3 \times 7 \times 2 \times 5)$

B. None of the following sentences is an instance of any of the four principles you have just learned. However, some of the sentences are consequences of the principles. Tell which sentences are consequences of which principles.

Sample 1. $5 + (7 \times 2) = 5 + (2 \times 7)$.

Solution. Since '$7 \times 2 = 2 \times 7$' is an instance of the commutative principle for multiplication, the given sentence is a consequence of that principle.

Answers for Part A.

1.	cpa	2.	cpm	3.	cpa	4.	cpm
5.	apa	6.	apm	7.	cpm	8.	apa
9.	apm	10.	cpa	11.	apa	12.	apa
13.	apa	14.	cpa	15.	cpa	16.	apa
17.	cpm	18.	apa	19.	apa	20.	cpm

[Notice, in Exercise 16, the use of the apa in justifying simplification of expressions containing "mixed numbers".]

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Note on the associative principles.--Recall [page 1-38] that, for example,

'72 + 45 + 63' is an abbreviation for '(72 + 45) + 63'.

Consequently,

'72 + 45 + 63 + 85' is an abbreviation for '(72 + 45) + 63 + 85',

and, similarly,

'(72 + 45) + 63 + 85' is an abbreviation for '[(72 + 45) + 63] + 85'.

So, for example:

$$72 + 45 + 63 + 85 = [(72 + 45) + 63] + 85$$

is merely an abbreviation of the trivial statement:

$$[(72 + 45) + 63] + 85 = [(72 + 45) + 63] + 85.$$

On the other hand:

$$72 + 45 + 63 + 85 = (72 + 45) + (63 + 85)$$

is an abbreviation of:

$$[(72 + 45) + 63] + 85 = (72 + 45) + (63 + 85),$$

which is an instance of the associative principle for addition, and, so, is by no means trivial. Other examples of "trivial" sentences:

$$72 + (45 + 63 + 85) = 72 + [(45 + 63) + 85]$$
$$72 + 45 + (63 + 85) = (72 + 45) + (63 + 85)$$
$$72 + (45 + 63) + 85 = [72 + (45 + 63)] + 85$$
$$(72 + 45) + 63 + 85 = [(72 + 45) + 63] + 85$$
$$(72 + 45 + 63) + 85 = [(72 + 45) + 63] + 85$$
$$(72 + 45) + 63 + 85 = (72 + 45 + 63) + 85$$

Answers for Part B [which begins on page 1-48].

[In addition to the answers we list the instances of principles from which the given sentences follow. None of the given sentences is itself an instance of cpa, cpm, apa, or apm.]

1. cpm; '6 × 5 = 5 × 6'

2. cpa; '4 + 7 = 7 + 4'

3. cpa; '7 + 3 = 3 + 7'

4. cpa; '9 + 3 = 3 + 9'

5. convention, or: abbreviation

6. convention

7. cpa; '6 + $\frac{4}{5}$ = $\frac{4}{5}$ + 6'

8. convention

9. cpm and cpa; '(6 + 5) × 3 = 3 × (6 + 5)' and '6 + 5 = 5 + 6' [or: '6 + 5 = 5 + 6' and '(5 + 6) × 3 = 3 × (5 + 6)'.]

10. apa and cpa; '8 + 5 + 3 = 8 + (5 + 3)', '5 + 3 = 3 + 5', and '8 + (3 + 5) = 3 + 5 + 8' [or: '8 + 5 + 3 = 3 + (8 + 5)', '8 + 5 = 5 + 8', and '3 + (5 + 8) = 3 + 5 + 8'.]

11. cpa; '4 + 7 = 7 + 4', '3 + 8 = 8 + 3', and '(7 + 4) + (8 + 3) = (8 + 3) + (7 + 4)' [or: ... .]

12. cpm; '10 × 3 = 3 × 10' and '15 × 3 = 3 × 15'.

*

Answers for Part C.

1. 72

2. 59

3. a numeral for any number of arithmetic in the first blank and a copy of this numeral in the second blank.

4. Same as for Ex. 3
5. 10
6. 9

7. Same as for Ex. 3
8. Same as for Ex. 3
9. 7

10. 7
11. 10
12. 84

13. 12
14. 13
15. 1/2

16. 8
17. 859
18. 9

*

Answers for Part D.

1. No. 7 - 5 = 2, but '5 - 7' is nonsense; 6 ÷ 2 = 3 but 2 ÷ 6 = 1/3.

2. No. (8 - 5) - 2 ≠ 8 - (5 - 2); (12 ÷ 4) ÷ 2 ≠ 12 ÷ (4 ÷ 2).

Sample 2.   72 + (45 + 63) + 85 = [72 + (45 + 63)] + 85

Solution.   The expression on the left of the equality sign is merely an abbreviation for the expression on the right. So, none of the four principles is required in showing that the given sentence is true.

1. $6 \times 5 \times 3 = 5 \times 6 \times 3$ 2. $4 + 7 + 3 = 7 + 4 + 3$

3. $4 + (7 + 3) = 4 + (3 + 7)$ 4. $(9 + 3) \times 5 = (3 + 9) \times 5$

5. $7 + 4 \times 3 = 7 + (4 \times 3)$ 6. $8 + 2 + 9 = (8 + 2) + 9$

7. $9\frac{1}{5} + 6\frac{4}{5} = 9\frac{1}{5} + (\frac{4}{5} + 6)$ 8. $14 \times 2\frac{1}{7} = 14 \times (2 + \frac{1}{7})$

9. $(6 + 5) \times 3 = 3 \times (5 + 6)$ 10. $8 + 5 + 3 = 3 + 5 + 8$

11. $(4 + 7) + (3 + 8) = (8 + 3) + (7 + 4)$

12. $(10 \times 3) + (15 \times 3) = (3 \times 10) + (3 \times 15)$

C. Fill the blanks to make true sentences.

1. $19 + \underline{\quad} = 72 + 19$ 2. $31 \times \underline{\quad} = 59 \times 31$

3. $6 \times \underline{\quad} \times 9 = 6 \times (\underline{\quad} \times 9)$ 4. $85 + 97 \times \underline{\quad} = 85 + \underline{\quad} \times 97$

5. $10 + 7 + 3 = 3 + \underline{\quad} + 7$ 6. $(8 + 5) \times 2 = 8 + \underline{\quad} \times 2$

7. $9 + \underline{\quad} + 4 = 9 + (\underline{\quad} + 4)$ 8. $(\underline{\quad} + 5) + 7 = 7 + 5 + \underline{\quad}$

9. $\frac{3 \times 7}{7 \times 5} = \frac{3 \times 7}{5 \times \underline{\quad}}$ 10. $\frac{1}{3} \times (7 \times \frac{1}{5}) = \frac{1}{15} \times \underline{\quad}$

11. $3 \times 5 \times 3\frac{1}{3} = \underline{\quad} \times 5$ 12. $61 + (\underline{\quad} + 39) = 184$

13. $6 \div 3 = \underline{\quad} \div 6$ 14. $9 - 5 = \underline{\quad} - 9$

15. $24 \div 4 \div 2 = 24 \div (4 \div \underline{\quad})$

16. $15 - 6 - 1 = 15 - (\underline{\quad} - 1)$

17. $107 \times \underline{\quad} + 372 \times 76 = 76 \times 372 + 859 \times 107$

18. $(8 + 7) \times (9 + 16) = (16 + \underline{\quad}) \times (7 + 8)$

D.   1.   Are subtraction and division commutative operations? Give examples to justify your answers.

     2.   Are subtraction and division associative? Give examples.

ANOTHER PRINCIPLE

Perhaps you have found short cuts for some problems which involve both multiplication and addition.

$$7 \times 11 + 3 \times 11 = ?$$

Do you see a short way of solving this problem? If you don't, you may see it after you have filled in the blanks in the following sentences.

$$4 \times 15 + 6 \times 15 = \underline{60} + \underline{90} = \underline{150} = \underline{10} \times 15$$

$$8 \times 29 + 2 \times 29 = \underline{} + \underline{} = \underline{} = \underline{} \times 29$$

$$13 \times 21 + 17 \times 21 = \underline{} + \underline{} = \underline{} = \underline{} \times 21$$

$$5 \times 9 + 6 \times 9 = \underline{} + \underline{} = \underline{} = \underline{} \times 9$$

$$21 \times 8 + 19 \times 8 = \underline{} + \underline{} = \underline{} = \underline{} \times 8$$

$$\tfrac{2}{5} \times 7 + \tfrac{3}{5} \times 7 = \underline{} + \underline{} = \underline{} = \underline{} \times 7$$

Can you do these problems by a short cut?

$$8 \times 7 + 3 \times 7 = ?$$

$$6 \times 582 + 4 \times 582 = ?$$

The same short cut can be used in the following problem.

Suppose you have two vacation jobs, one of which pays 85 cents an hour, and the other $1.15 an hour. How much have you earned if you worked 35 hours on each job? Some people may solve this problem the hard way by first multiplying to tell how much was earned on each job, and then adding the results.

$$.85 \times 35 + 1.15 \times 35 = 29.75 + 40.25 = 70.00.$$

Do you see an easy way? First, add the rates of pay, and then multiply.

$$(.85 + 1.15) \times 35 = 2.00 \times 35 = 70.00$$

The blanks should be filled in like this:

$$= \underline{232} + \underline{58} = \underline{290} = \underline{10} \times 29$$
$$= \underline{273} + \underline{357} = \underline{630} = \underline{30} \times 21$$
$$= \underline{45} + \underline{64} = \underline{99} = \underline{11} \times 9$$
$$= \underline{168} + \underline{152} = \underline{320} = \underline{40} \times 8$$
$$= \underline{\frac{14}{5}} + \underline{\frac{21}{5}} = \underline{\frac{35}{5}} = \underline{1} \times 7$$

✳

On TC[1-44, 45]b and c, we describe a classroom technique for developing a feeling for the commutative and associative principles. Here is a similar technique for helping students feel "comfortable" with the distributive principle. This principle is the one which is most surprising to students, probably because it "mixes" both addition and multiplication. Again select students ., B, and C. B tells his number to A, and A adds B's number to his. Then A tells this sum to C who multiplies it by his number, recording the product. Now, A tells his number to C, and C multiplies it by his number. Similarly, B tells his number to C, and C again multiplies it by his number, adds this product to the one just obtained, and records the sum. The class should predict that the recorded product is the same as the recorded sum. The relevant instance is constructed in these steps:

$$(\quad + \quad) \times \quad = (\quad \times \quad) + (\quad \times \quad)$$

[A sum and a product in one case, two products and a sum in the other.]

$$(7 + \quad) \times \quad = (7 \times \quad) + (\quad \times \quad)$$
$$(7 + 4) \times \quad = (7 \times \quad) + (4 \times \quad)$$
$$(7 + 4) \times 9 = (7 \times 9) + (4 \times 9).$$

By convention, we can omit the grouping symbols on the right side.

$$(7 + 4) \times 9 = 7 \times 9 + 4 \times 9$$

Whenever questions arise concerning the proper identification of a sentence as an instance of a certain principle, the proper "pattern" sentence should be written [that is, the one with operation signs and grouping symbols only], and the empty spaces filled, one at a time.

Here, in identifying the distributive principle for multiplication over addition, we depart from the usual convention. Most other texts would say that, for example, '$3 \times (5 + 8) = 3 \times 5 + 3 \times 8$' is an instance of the dpma and that '$(5 + 8) \times 3 = 5 \times 3 + 8 \times 3$' is an instance of the right (-hand) distributive principle for multiplication over addition. But, to be consistent with the convention according to which we write a name for the multiplier to the right of the multiplication sign, one should, as we do, say that $(5 + 8) \times 3 = 5 \times 3 + 8 \times 3$ because multiplication distributes over addition and, so, say that '$(5 + 8) \times 3 = 5 \times 3 + 8 \times 3$' is an instance of the dpma. On page 1-52 we say that '$3 \times (5 + 8) = 3 \times 5 + 3 \times 8$' is an instance of the left dpma. When correctly filled in, the sentences in Exercises 9, 10, and 11 of Part A are instances of the ℓdpma.

※

Answers for Part A.

| 1. 3 | 2. 8 | 3. 9 | 4. 8 | 5. 3 | 6. 3 |
| 7. 2 | 8. 8 | 9. 2 | 10. 5 [for both blanks] | 11. 9 |

Sentences 1, 2, 3, 4, and 6 are instances of the dpma. [The sentence in Exercise 10 can also be completed to a true sentence by writing a '1' in the first blank and a '7' in the second. If something like this occurs to your students, tell them that it's clever of them to have thought of it but that it will be helpful later if they establish the habit of writing copies of the same numeral in similarly shaped blanks.]

※

You may want to ask your class of what principles the completed sentences in Exercises 5 and 8 are instances. [Exercise 5: apa; Exercise 8: cpm]

The fact that you are sure that this easy way will give you the total earned is a consequence of your belief in a principle called the distributive principle for multiplication over addition. The instance of it which we just used is:

$$.85 \times 35 + 1.15 \times 35 = (.85 + 1.15) \times 35.$$

Here are other examples of how this principle makes computations easier.

Example 1. $8\frac{1}{6} \times 5 = ?$

Solution. $(8 + \frac{1}{6}) \times 5 = 8 \times 5 + \frac{1}{6} \times 5$

$$= 40 + \frac{5}{6}$$

$$= 40\frac{5}{6}.$$

Example 2. $91 \times 61 + 9 \times 61 = ?$

Solution. $91 \times 61 + 9 \times 61 = (91 + 9) \times 61$

$$= 100 \times 61$$

$$= 6100.$$

EXERCISES

A.   Fill in the blanks to make true sentences, and then tell which are instances of the distributive principle for multiplication over addition.

1. $5 \times 7 + 3 \times 7 = (5 + \underline{}) \times 7$

2. $8 \times 5 + 2 \times 5 = (\underline{} + 2) \times 5$

3. $7 \times 9 + 3 \times 9 = (7 + 3) \times \underline{}$

4. $(2 + 8) \times 5 = 2 \times 5 + \underline{} \times 5$

5. $(6 + 3) + 2 = 6 + (\underline{} + 2)$

6. $2 \times 7 + \underline{} \times 7 = (2 + 3) \times 7$

7. $10 + 3 \times 5 = \underline{} \times 5 + 3 \times 5$

8. $(6 + 5) \times \underline{} = 8 \times (6 + 5)$

9. $7 \times (3 + 2) = 7 \times 3 + 7 \times \underline{}$

10. $5 \times (4 + 8) = \underline{} \times 4 + \underline{} \times 8$

11. $9 \times 7 + 9 \times 3 = \underline{} \times (7 + 3)$

✳ ✳ ✳

If you wrote a '9' in the blank in Exercise 3 of Part A, you obtained the true sentence:

$$7 \times 9 + 3 \times 9 = (7 + 3) \times 9,$$

and this sentence is an instance of the distributive principle for multiplication over addition. An instance of this principle tells you a way of computing the sum of two products which have the <u>same</u> multiplier.

If you wrote a '9' in the blank in Exercise 11 of Part A, you obtained the true sentence:

$$9 \times 7 + 9 \times 3 = 9 \times (7 + 3).$$

This sentence tells you a way of computing the sum of two products which have <u>different</u> multipliers. So, this sentence is <u>not</u> an instance of the distributive principle for multiplication over addition. However, it is an instance of another principle which is called <u>the left distributive principle for multiplication over addition</u>. Here are some other instances of this new principle.

$$12 \times 3 + 12 \times 97 = 12 \times (3 + 97)$$

$$20 \times (4 + \frac{1}{2}) = 20 \times 4 + 20 \times \frac{1}{2}.$$

Write three instances of the left distributive principle for multiplication over addition, and three instances of the distributive principle for multiplication over addition.

✳ ✳ ✳

B.   Fill in the blanks to make true sentences, and then tell what principles they are instances of.

1. $4 \times 9 + 7 \times 9 = (4 + \underline{\hspace{1em}}) \times 9$

2. $9 \times 4 + 9 \times 7 = 9 \times (\underline{\hspace{1em}} + 7)$

3. $6 \times 8\frac{1}{3} = 6 \times 8 + \underline{\hspace{1em}} \times \frac{1}{3}$

4. $4 \times 7 \times 3 = 4 \times (7 \times \underline{\hspace{1em}})$

5. $(8 + 2) \times (9 + 3) = (9 + 3) \times (8 + \underline{\hspace{1em}})$

6. $(6 + 5) \times 9 + (7 + 3) \times 9 = [(6 + 5) + (\underline{\hspace{1em}} + 3)] \times 9$

7. $(8 + 5) \times (6 + 2) = (8 + 5) \times \underline{\hspace{1em}} + (8 + 5) \times 2$

8. $(8 + 5) \times (6 + 2) = 8 \times (6 + \underline{\hspace{1em}}) + 5 \times (6 + 2)$

Three instances of the left distributive principle for multiplication over addition:

(1) $5 \times (3 + 2) = 5 \times 3 + 5 \times 2$

(2) $3\frac{1}{2} \times (4 + 5) = 3\frac{1}{2} \times 4 + 3\frac{1}{2} \times 5$

(3) $7 \times 8 + 7 \times 9 = 7 \times (8 + 9)$

Three instances of the distributive principle for multiplication over addition:

(1) $(3 + 2) \times 5 = 3 \times 5 + 2 \times 5$

(2) $(4 + 5) \times 3\frac{1}{2} = 4 \times 3\frac{1}{2} + 5 \times 3\frac{1}{2}$

(3) $8 \times 7 + 9 \times 7 = (8 + 9) \times 7$

※

Answers for Part B.

1.	7,	dpma	2.	4,	ℓdpma	3.	6,	ℓdpma
4.	3,	apm	5.	2,	cpm	6.	7,	dpma
7.	6,	ℓdpma	8.	2,	dpma			

Of course, other principles besides the pm1 are needed to justify the conclusion stated at the bottom of page 1-53. For example, one needs to know that 7/7 = 1, that 5/5 = 1, and how to multiply rational numbers. However, you should at present overlook this point unless your students insist on it. [In Unit 2, students will learn to derive theorems about real numbers from basic principles and one of these theorems will have as an instance the sentence $\dfrac{`2}{7} + \dfrac{3}{5} = \dfrac{2 \cdot 5 + 3 \cdot 7'}{7 \cdot 5}$ in which the numerals name real numbers, rather than, as on page 1-53, numbers of arithmetic.]

*

For <u>your</u> information, principles for real numbers, analagous to the principles for numbers of arithmetic which are illustrated on page 1-54, are illustrated on page 1-61.

MORE PRINCIPLES

We have mentioned and illustrated six principles which express certain properties of operations with the numbers of arithmetic. There are also certain numbers of arithmetic which have interesting properties. For example, there is a number of arithmetic which when added to any number of arithmetic gives you back that number. In each of the blanks below, write a numeral which makes the sentence true.

$$4 + \underline{\hspace{1cm}} = 4$$

$$5\frac{1}{2} + \underline{\hspace{1cm}} = 5\frac{1}{2}$$

$$\frac{23}{18} + \underline{\hspace{1cm}} = \frac{23}{18}$$

Is there a number of arithmetic which does a similar thing for multiplication? What must you write in each blank below to make the sentence true?

$$4 \times \underline{\hspace{1cm}} = 4$$

$$5\frac{1}{2} \times \underline{\hspace{1cm}} = 5\frac{1}{2}$$

$$\frac{23}{18} \times \underline{\hspace{1cm}} = \frac{23}{18}$$

These properties of 0 and 1 are expressed by <u>the principle for adding 0</u> and <u>the principle for multiplying by 1</u>. [Can you guess what <u>the principle for multiplying by 0</u> is? Give some instances of it.]

The principle for multiplying by 1 is used in working with fractions. For example, suppose you want to add $\frac{2}{7}$ and $\frac{3}{5}$. In order to simplify the expression:

$$\frac{2}{7} + \frac{3}{5},$$

we need to find names for $\frac{2}{7}$ and $\frac{3}{5}$ which have the same denominator. The procedure usually followed is to multiply $\frac{2}{7}$ by $\frac{5}{5}$ to get $\frac{10}{35}$, and to multiply $\frac{3}{5}$ by $\frac{7}{7}$ to get $\frac{21}{35}$. Then we write:

$$\frac{10}{35} + \frac{21}{35} = \frac{31}{35}.$$

So,

$$\frac{2}{7} + \frac{3}{5} = \frac{31}{35}.$$

Although we know that $\frac{31}{35}$ is the sum of $\frac{10}{35}$ and $\frac{21}{35}$, how do we know

it is the sum of $\frac{2}{7}$ and $\frac{3}{5}$? The principle for multiplying by 1 assures

us that since $\frac{5}{5}$ is 1, the product of $\frac{2}{7}$ and $\frac{5}{5}$ is $\frac{2}{7}$. So, $\frac{10}{35}$ is $\frac{2}{7}$.

Similarly, the principle for multiplying by 1 assures us that $\frac{21}{35}$ is

$\frac{3}{5}$. So, the sum of $\frac{10}{35}$ and $\frac{21}{35}$ is the sum of $\frac{2}{7}$ and $\frac{3}{5}$.

Here is a summary of the principles we have discussed for the
numbers of arithmetic.

The Commutative Principle for Addition [cpa]

$7 + 29 = 29 + 7$

$$6\frac{1}{2} = \frac{1}{2} + 6$$

$$9.34 + 5 = 5 + 9.34$$

The Associative Principle for Addition [apa]

$(9 \div 5) + 15 = 9 + (5 + 15)$

$$6\frac{3}{5} + (7\frac{2}{5} + 8\frac{1}{3}) = (6\frac{3}{5} + 7\frac{2}{5}) + 8\frac{1}{3}$$

$$87 + 9 + 91 = 87 + (9 + 91)$$

The Commutative Principle for Multiplication [cpm]

$590 \times 2 = 2 \times 590$

$$81.3 \times 17.7 = 17.7 \times 81.3$$

$$4\frac{1}{2} \times 6\frac{1}{4} = 6\frac{1}{4} \times 4\frac{1}{2}$$

The Associative Principle for Multiplication [apm]

$(84 \times 5) \times 20 = 84 \times (5 \times 20)$

$$16\frac{2}{3} \times (6 \times 59) = (16\frac{2}{3} \times 6) \times 59$$

$$97 \times 25 \times 4 = 97 \times (25 \times 4)$$

The Distributive Principle for Multiplication over Addition [dpma]

$5 \times 98 + 95 \times 98 = (5 + 95) \times 98$

$$(\frac{1}{3} + \frac{1}{2}) \times 12 = \frac{1}{3} \times 12 + \frac{1}{2} \times 12$$

$$\frac{5}{7} \times 59 + \frac{2}{7} \times 59 = (\frac{5}{7} + \frac{2}{7}) \times 59$$

Alth

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Answers for Part A.

1.	pa0	2.	pm1	3.	cpa	4.	cpm
5.	pm0	6.	pm1	7.	cpm	8.	dpma
9.	apm	10.	ℓdpma	11.	dpma	12.	ℓdpma
13.	dpma	14.	cpm	15.	cpm	16.	dpma

[Students may need to be reminded, in connection with Exercise 9, that '3.59 × 8.61 × 7.32' is an abbreviation for '(3.59 × 8.61) × 7.32'.]

✳

We suggest that you assign Part G of the Supplementary Exercises at this time. Have students do the problems orally. [See the discussion in T C[1-44, 45] and on TC[1-50] on identifying instances.]

✳

Here is a quiz which tests the ability to apply the principles listed on pages 1-54 and 1-55, and the ability to identify the principles used.

Fill in the blanks to make true sentences, and write the name of the principle of which the true sentence is a consequence.

1. $85 + \underline{\quad} = 85$ [_____]
2. $\underline{\quad} \times 1 = 798$ [_____]
3. $846 + \underline{\quad} = 982 + 846$ [_____]
4. $68 + (73 + 15) = 68 + \underline{\quad} + 15$ [_____]
5. $91 \times (84 + 12) = 91 \times (12 + \underline{\quad})$ [_____]
6. $105 + (31 \times 76) = (31 \times \underline{\quad}) + 105$ [_____]
7. $(35 + 64) \times 75 = 35 \times 75 + \underline{\quad} \times 75$ [_____]
8. $61 \times (7 + 9) = (7 + \underline{\quad}) \times 61$ [_____]
9. $1 \times (62 + 37) = 1 \times 62 + \underline{\quad} \times 37$ [_____]
10. $[3 + (7 + 9)] \times 5 = [(7 + 9) + 3] \times \underline{\quad}$ [_____]

✳

Answers for quiz.

1.	0, pa0	2.	798, pm1	3.	982, cpa	4.	73, apa
5.	84, cpa	6.	76, cpa	7.	64, dpma	8.	9, cpm
		9.	1, ℓdpma			10.	5, cpa

The <u>Left</u> <u>Distributive</u> <u>Principle</u> <u>for</u> <u>Multiplication</u> <u>over</u> <u>Addition</u> [ℓdpma]

$$5 \times 8 + 5 \times 12 = 5 \times (8 + 12)$$

$$7 \times 3\frac{1}{7} = 7 \times 3 + 7 \times \frac{1}{7}$$

The <u>Principle</u> <u>for</u> <u>Adding</u> <u>0</u> [pa 0]

$$6 + 0 = 6$$

$$87\frac{1}{3} + 0 = 87\frac{1}{3}$$

The <u>Principle</u> <u>for</u> <u>Multiplying</u> <u>by</u> <u>1</u> [pm 1]

$$19 \times 1 = 19$$

$$86.73 = 86.73 \times 1$$

The <u>Principle</u> <u>for</u> <u>Multiplying</u> <u>by</u> <u>0</u> [pm 0]

$$7 \times 0 = 0$$

$$0 = 318 \times 0$$

EXERCISES

<u>A</u>. Each of the following sentences is an instance of one of the prin-
ciples for the numbers of arithmetic. Tell which principle.

1. $7 + 0 = 7$ 2. $3 \times 1 = 3$ 3. $4 + 1 = 1 + 4$

4. $5 \times 8 = 8 \times 5$ 5. $0 = 9 \times 0$ 6. $17 = 17 \times 1$

7. $4 \times (5 + 8) = (5 + 8) \times 4$ 8. $(5 + 8) \times 4 = (5 \times 4) + (8 \times 4)$

9. $3.59 \times 8.61 \times 7.32 = 3.59 \times (8.61 \times 7.32)$

10. $758 \times (321 + 684) = 758 \times 321 + 758 \times 684$

11. $67 \times 531 + 33 \times 531 = (67 + 33) \times 531$

12. $(85 + 3) \times (17 + 12) = (85 + 3) \times 17 + (85 + 3) \times 12$

13. $8 \times (17 \times 9) + 12 \times (17 \times 9) = (8 + 12) \times (17 \times 9)$

14. $(97 + 35) \times (9 + 2) = (9 + 2) \times (97 + 35)$

15. $\frac{94}{35} \times 1 = 1 \times \frac{94}{35}$ 16. $(\frac{2}{3} + \frac{1}{2}) \times 6 = \frac{2}{3} \times 6 + \frac{1}{2} \times 6$

[More exercises are in <u>Part</u> <u>G</u>, <u>Supplementary</u> <u>Exercises</u>.]

* * *

As we have said earlier, the principles for the numbers of arithmetic are very useful in computing because they give you different ways of reaching the same result. This means that you have different ways of carrying out a given computational task. With practice you will learn how to pick the easiest way. Here are some examples.

Example 1.   493 + 39 + 7 = ?

The uninspired way of simplifying '493 + 39 + 7' consists of two steps. First, add 39 to 493; second, add 7 to this sum.

Hard way

$$493 + 39 + 7 = 532 + 7$$

$$= 539.$$

A more sensible approach is to observe that it is easier to add 7 to 493 than it is to add 39 to 493, and that once you've done this, it is easy to add 39 to the sum.

Easy way

$$493 + 7 + 39 = 500 + 39$$

$$= 539.$$

But, wait a minute! How were you able to predict that you would get the same answer the easy way? You could have based your prediction on the principles. For example:

$$493 + 39 + 7 = 493 + (39 + 7) \text{[apa]}$$

$$493 + (39 + 7) = 493 + (7 + 39) \text{[cpa]}$$

$$493 + (7 + 39) = 493 + 7 + 39 \text{[apa]}.$$

Or, you could have used the principles like this:

$$493 + 39 + 7 = 7 + (493 + 39) \text{[cpa]}$$

$$7 + (493 + 39) = 7 + 493 + 39 \text{[apa]}$$

$$7 + 493 + 39 = 493 + 7 + 39 \text{[cpa]},$$

But, is it always the case [as here] that when one reverses the pairs in an operation, the new set of pairs is also an operation? Let's try it with the operation absolute valuing. Here is a list of some of the pairs in the new set.

$$(7, ^-7) \quad (7, ^+7) \quad (0, 0) \quad (5, ^+5)$$
$$(5, ^-5)$$
$$\cdot \;\; \cdot \;\; \cdot$$

Is this set of pairs an operation? No, because it contains pairs with the same first number but different second numbers. Since this set of reversed pairs is not an operation, we say that absolute valuing does not have an inverse. Can you think of another operation which does not have an inverse? In general, an operation has an inverse just if the set of its reversed pairs is an operation.

Let's look again at the set of pairs we get by reversing the pairs in absolute valuing. Even though this set of pairs is not itself an operation, we can "split" the set into two sets each of which is an operation.

$$(7, ^-7) \quad (5, ^-5) \quad (6, ^-6) \qquad\qquad (7, ^+7) \quad (5, ^+5) \quad (6, ^+6)$$
$$(104, ^-104) \quad (1, ^-1) \quad (0, 0) \qquad\qquad (104, ^+104) \quad (1, ^+1) \quad (0, 0)$$
$$\cdot \;\; \cdot \;\; \cdot \qquad\qquad\qquad\qquad \cdot \;\; \cdot \;\; \cdot$$

Notice that the operation listed on the left is an operation on the set of numbers of arithmetic. So is the one listed on the right. [If (0, 0) were not included in one of these sets, that set would still be an operation, but not on the set of numbers of arithmetic. Why?] When the first operation operates on a number of arithmetic, the result is the corresponding nonpositive real number. What is the result of operating on numbers of arithmetic with the second operation? Do you recognize these operations? You learned about them when we first mentioned real numbers. These operations do not have standard names, but we have been using two signs almost as though they were signs for the operations. These are the signs '$^-$' and '$^+$'. If you apply the operation $^-$ to the number 7 of arithmetic, you get the corresponding nonpositive real number $^-7$. If you apply the operation $^+$ to the number 58 of arithmetic, you get the corresponding nonnegative real number $^+58$.

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In abbreviated form, the formal argument looks like this.

$$C = D \quad \frac{B = C \quad A = B}{A = C}$$
$$\frac{}{A = D}$$

This argument contains two applications of the substitution rule.

   (1) $B = C$, $A = B$, therefore $A = C$.

   (2) $C = D$, $A = C$, therefore $A = D$.

Since lines 7b and 5b are consequences of the apa, and line 6b is a consequence of the cpa, this argument shows that '$493 + 39 + 7 = 493 + 7 + 39$' is a consequence of these two principles.

The argument displayed above may be interpreted as consisting of two applications of a logical principle called the transitivity of equality ["If $A = B$ and $B = C$ then $A = C$."] Notice that this principle is justified by the substitution procedure which we have been discussing [and the latter is valid by virtue of our interpretation of '$=$']. Another logical principle called the symmetry of equality ["If $A = B$ then $B = A$."] is also justified by the same substitution procedure, taken together with the reflexivity of equality ["$A = A$." or "A thing is itself."] This is how:

$$\frac{A = B \quad A = A}{B = A}$$

or even like this:

$$493 + 39 + 7 = 39 + 493 + 7 \quad [cpa]$$

$$39 + 493 + 7 = 39 + (493 + 7) \quad [apa]$$

$$39 + (493 + 7) = 493 + 7 + 39 \quad [cpa].$$

In any case the principles assure you that

$$493 + 39 + 7 = 493 + 7 + 39.$$

Example 2. $987 \times 593 + 593 \times 13 = ?$

Hard way

$$987 \times 593 + 593 \times 13 = 585291 + 7709$$

$$= 593000.$$

Easy way

$$593 \times (987 + 13) = 593 \times 1000$$

$$= 593000.$$

The principles justify the easy way by enabling us to show that

$$987 \times 593 + 593 \times 13 = 593 \times (987 + 13).$$

Here is one way of showing this:

$$987 \times 593 + 593 \times 13 = 987 \times 593 + 13 \times 593 \quad [cpm]$$

$$987 \times 593 + 13 \times 593 = (987 + 13) \times 593 \quad\quad [dpma]$$

$$(987 + 13) \times 593 = 593 \times (987 + 13) \quad\quad [cpm].$$

And, here is another way:

$$987 \times 593 + 593 \times 13 = 593 \times 987 + 593 \times 13 \quad [cpm]$$

$$593 \times 987 + 593 \times 13 = 593 \times (987 + 13) \quad\quad [\ell dpma].$$

These examples and the exercises which follow help you to learn how to use the principles to check short cuts. With practice, the checking procedure becomes almost automatic, and even suggests short cuts.

<center>✳</center>

B. Each sentence below suggests a short cut in carrying out a computation. Your job is to justify the short cut by showing that the sentence is a consequence of the principles. In connection with Example 1 above, we have given three samples of how to show that the sentence:

$$493 + 39 + 7 = 493 + 7 + 39$$

is a consequence of the principles. In Example 2, we gave two samples of how to derive the sentence:

$$987 \times 593 + 593 \times 13 = 593 \times (987 + 13)$$

from the principles. For each of the following sentences, it is enough if you give just one derivation of the sentence from the principles.

1. $5 \times (9 \times \frac{3}{5}) = \frac{3}{5} \times 5 \times 9$

2. $43 \times 31 + 31 \times 57 = 31 \times (43 + 57)$

3. $8\frac{2}{7} + \frac{3}{7} = 8 + (\frac{2}{7} + \frac{3}{7})$

4. $799 + (58 + 1) = 799 + 1 + 58$

5. $(85 + 85) \times \frac{1}{2} = [85 \times (1 + 1)] \times \frac{1}{2}$

6. $75 \times 9 + 75 = 75 \times (9 + 1)$

7. $5\frac{1}{5} + 3\frac{2}{5} = (5 + 3) + (\frac{1}{5} + \frac{2}{5})$

8. $9 \times 38 + 70 \times 38 + 21 \times 38 = (21 + 9 + 70) \times 38$

9. $27 + 13 = (2 + 1) \times 10 + (7 + 3)$
    [Recall that '13' is an abbreviation for '$1 \times 10 + 3$'.]

10. $25 + 3 \times (25 + 7) = (1 + 3) \times 25 + 3 \times 7$

✩11. $27 \times 13 = (2 \times 1) \times (10 \times 10) + (2 \times 3 + 7 \times 1) \times 10 + 7 \times 3$

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Unless you have a class of very able students, do not expect all of them to be able to give derivations for each of the exercises in Part B. Every student should be able to do Exercises 1-6; some will not be able to do Exercises 7 and 8 without some hints; there may be just a few who will do Exercises 9 and 10 satisfactorily. Exercise 11 should not be assigned except as "honor" work. --It is good practice to discuss the more involved derivations in class, so that students will begin to get a "feeling" for what constitutes a correct application of the principles. The exercises in Part C on page 1-84 will provide another opportunity for using the principles to establish the truth of a sentence.

<div align="center">✳</div>

Answers for Part B. [Your students may suggest other correct derivations.]

1. $5 \times (9 \times \frac{3}{5}) = (5 \times 9) \times \frac{3}{5}$ [apm]

   $(5 \times 9) \times \frac{3}{5} = \frac{3}{5} \times (5 \times 9)$ [cpm]

   $\frac{3}{5} \times (5 \times 9) = \frac{3}{5} \times 5 \times 9$ [apm]

2. $43 \times 31 + 31 \times 57 = 31 \times 43 + 31 \times 57$ [cpm]

   $31 \times 43 + 31 \times 57 = 31 \times (43 + 57)$ [ℓdpma]

3. $8\frac{2}{7} + \frac{3}{7} = 8 + (\frac{2}{7} + \frac{3}{7})$ [apa]

   ['$8\frac{2}{7}$' is an abbreviation for '$(8 + \frac{2}{7})$'.]

4. $799 + (58 + 1) = 799 + (1 + 58)$ [cpa]

   $799 + (1 + 58) = 799 + 1 + 58$ [apa]

5. $(85 + 85) \times \frac{1}{2} = (85 \times 1 + 85 \times 1) \times \frac{1}{2}$ [pm1]

   $(85 \times 1 + 85 \times 1) \times \frac{1}{2} = [85 \times (1 + 1)] \times \frac{1}{2}$ [ℓdpma]

6. $75 \times 9 + 75 = 75 \times 9 + 75 \times 1$ [pml]

       $75 \times 9 + 75 \times 1 = 75 \times (9 + 1)$ [ℓdpma]

7. $5\frac{1}{5} + 3\frac{2}{5} = (5\frac{1}{5} + 3) + \frac{2}{5}$ [apa]

       $(5\frac{1}{5} + 3) + \frac{2}{5} = [5 + (\frac{1}{5} + 3)] + \frac{2}{5}$ [apa]

       $[5 + (\frac{1}{5} + 3)] + \frac{2}{5} = [5 + (3 + \frac{1}{5})] + \frac{2}{5}$ [cpa]

       $[5 + (3 + \frac{1}{5})] + \frac{2}{5} = [(5 + 3) + \frac{1}{5}] + \frac{2}{5}$ [apa]

       $[(5 + 3) + \frac{1}{5}] + \frac{2}{5} = (5 + 3) + (\frac{1}{5} + \frac{2}{5})$ [apa]

8. $9 \times 38 + 70 \times 38 + 21 \times 38 = (9 + 70) \times 38 + 21 \times 38$ [dpma]

       $(9 + 70) \times 38 + 21 \times 38 = [(9 + 70) + 21] \times 38$ [dpma]

       $[(9 + 70) + 21] \times 38 = [21 + (9 + 70)] \times 38$ [cpa]

       $[21 + (9 + 70)] \times 38 = (21 + 9 + 70) \times 38$ [apa]

9. $27 + 13 = [(2 \times 10 + 7) + 1 \times 10] + 3$ [apa]

       $[(2 \times 10 + 7) + 1 \times 10] + 3 = [2 \times 10 + (7 + 1 \times 10)] + 3$ [apa]

       $[2 \times 10 + (7 + 1 \times 10)] + 3 = [2 \times 10 + (1 \times 10 + 7)] + 3$ [cpa]

       $[2 \times 10 + (1 \times 10 + 7)] + 3 = [(2 \times 10 + 1 \times 10) + 7] + 3$ [apa]

       $[(2 \times 10 + 1 \times 10) + 7] + 3 = (2 \times 10 + 1 \times 10) + (7 + 3)$ [apa]

       $(2 \times 10 + 1 \times 10) + (7 + 3) = (2 + 1) \times 10 + (7 + 3)$ [dpma]

10. $25 + 3 \times (25 + 7) = 25 \times 1 + 3 \times (25 + 7)$ [pml]

       $25 \times 1 + 3 \times (25 + 7) = 25 \times 1 + (3 \times 25 + 3 \times 7)$ [ℓdpma]

       $25 \times 1 + (3 \times 25 + 3 \times 7) = 1 \times 25 + (3 \times 25 + 3 \times 7)$ [cpm]

       $1 \times 25 + (3 \times 25 + 3 \times 7) = 1 \times 25 + 3 \times 25 + 3 \times 7$ [apa]

       $1 \times 25 + 3 \times 25 + 3 \times 7 = (1 + 3) \times 25 + 3 \times 7$ [dpma]

*11. $[27 \times 13 = (2 \times 10 + 7) \times (1 \times 10 + 3)]$ [abbreviation]

$(2 \times 10 + 7) \times (1 \times 10 + 3)$

$\qquad = 2 \times 10 \times (1 \times 10 + 3) + 7 \times (1 \times 10 + 3)$ [dpma]

$2 \times 10 \times (1 \times 10 + 3) + 7 \times (1 \times 10 + 3)$

$\qquad = [2 \times 10 \times (1 \times 10) + 2 \times 10 \times 3] + [7 \times (1 \times 10) + 7 \times 3]$ [dpma]

$[2 \times 10 \times (1 \times 10) + 2 \times 10 \times 3] + [7 \times (1 \times 10) + 7 \times 3]$

$\qquad = 2 \times 10 \times (1 \times 10) + 2 \times 10 \times 3 + 7 \times (1 \times 10) + 7 \times 3$ [apa]

$2 \times 10 \times (1 \times 10) + 2 \times 10 \times 3 + 7 \times (1 \times 10) + 7 \times 3$

$\qquad = 2 \times 10 \times (1 \times 10) + [2 \times 10 \times 3 + 7 \times (1 \times 10)] + 7 \times 3$ [apa]

$2 \times 10 \times (1 \times 10) + [2 \times 10 \times 3 + 7 \times (1 \times 10)] + 7 \times 3$

$\qquad = 2 \times 10 \times 1 \times 10 + [2 \times (10 \times 3) + 7 \times 1 \times 10] + 7 \times 3$ [apm]

$2 \times 10 \times 1 \times 10 + [2 \times (10 \times 3) + 7 \times 1 \times 10] + 7 \times 3$

$\qquad = 2 \times (10 \times 1) \times 10 + [2 \times (10 \times 3) + 7 \times 1 \times 10] + 7 \times 3$ [apm]

$2 \times (10 \times 1) \times 10 + [2 \times (10 \times 3) + 7 \times 1 \times 10] + 7 \times 3$

$\qquad = 2 \times (1 \times 10) \times 10 + [2 \times (3 \times 10) + 7 \times 1 \times 10] + 7 \times 3$ [cpm]

$2 \times (1 \times 10) \times 10 + [2 \times (3 \times 10) + 7 \times 1 \times 10] + 7 \times 3$

$\qquad = 2 \times 1 \times 10 \times 10 + [2 \times 3 \times 10 + 7 \times 1 \times 10] + 7 \times 3$ [apm]

$2 \times 1 \times 10 \times 10 + [2 \times 3 \times 10 + 7 \times 1 \times 10] + 7 \times 3$

$\qquad = 2 \times 1 \times (10 \times 10) + [2 \times 3 \times 10 + 7 \times 1 \times 10] + 7 \times 3$ [apm]

$2 \times 1 \times (10 \times 10) + [2 \times 3 \times 10 + 7 \times 1 \times 10] + 7 \times 3$

$\qquad = (2 \times 1) \times (10 \times 10) + (2 \times 3 + 7 \times 1) \times 10 + 7 \times 3$ [dpma]

[You will notice that in some steps of the above proof, a given principle was applied more than once. A separate step for each application would have required too much space.]

10. $|^+7 - ^+5| + |^+5 - ^+2|$ $|^+7 - ^+2|$

11. $|^+4 - ^+6| + |^+6 - ^+3|$ $|^+4 - ^+3|$

12. $|^+4 - ^-2| + |^-2 - ^+8|$ $|^+4 - ^+8|$

13. $|^-2 - ^+1| + |^+1 - ^+3|$ $|^-2 - ^+3|$

14. $|^+2| \times |^-3|$ $|^+2 \times ^-3|$ 15. $|^-3| \times |^-4|$ $|^-3 \times ^-4|$

16. $|^+7 - ^-3|$ $|^+7| - |^-3|$ 17. $|^+9 + ^-2|$ $|^+9| + |^-2|$

DOES ABSOLUTE VALUING HAVE AN INVERSE?

(I)   I am thinking of a number. If I add $^+2$ to it, I get $^+12$. What number am I thinking of?

(II)   I am thinking of a number. The absolute value of this number is 7. What number am I thinking of?

Do you see a difference between these problems? You can tell what number was thought of in the first problem because the operation adding $^+2$ has an inverse. In the second problem, although you know that the number must be either $^+7$ or $^-7$, you can't tell which. The operation absolute valuing does not have an inverse. Let's look into what this means.

To begin with, we ought to be clear on what an operation is. The sets of pairs which we have been calling 'operations' have this important property--none of them contains two pairs that have the same first number [but different second numbers]. That is, an operation operates on a number to produce a unique result. [When the operation adding $^+2$ operates on $^-8$, the result is $^-6$, and nothing else; when the operation absolute valuing operates on $^-4$, the result is 4, and nothing else.] This notion of uniqueness of result is the essence of the idea of operation. So, let us agree that a set of pairs is an operation just if no two pairs in the set have the same first number.

Now, from any operation you can get a second set of pairs by reversing each pair which belongs to the given operation. For example, when you reverse each pair in the operation adding $^+7$, you get a set of pairs. If you think a moment, you will see that this second set of pairs is an operation. In fact, it is the operation we call 'subtracting $^+7$'.

Answers for Part C.

1.	32	2.	16	3.	47	4.	118	5.	2100
6.	1925	7.	250	8.	170	9.	1000	10.	750
11.	34000	12.	42000	13.	2500	14.	0	15.	30.6
16.	.02	17.	9	18.	1/27	19.	1	20.	1

✻

After discussing Part D, ask: Could someone who didn't know the dpma get along by knowing the £dpma [and the cpm]?

✻

Answer for Part D.

$$39 \times 83 + 39 \times 17 = 83 \times 39 + 17 \times 39 \qquad \text{[cpm]}$$
$$83 \times 39 + 17 \times 39 = (83 + 17) \times 39 \qquad \text{[dpma]}$$
$$(83 + 17) \times 39 = 39 \times (83 + 17) \qquad \text{[cpm]}$$

✻

Exercise 15 may cause some questioning on the part of the students as to how one interprets a symbol such as '102%'. We interpret '102%' as a name for an operation, and '102% of 30' as a numeral, analogous say, to 'the square of 4', 'the (common) logarithm of 2', 'half of 8', and 'the square root of 9'. In each case we are naming the number which results when a certain operation is applied to a certain number. The result of applying the square rooting operation to the number 9 is the square root of 9 [another name for this number is '3']; the result of halving 8 is half of 8 [another name is '4']; the value of the logarithm function for the argument 2 is the logarithm of 2 [a name for an approximation to this number is '0.3010']; the result of squaring 4 is the square of 4; and the result of applying the operation 102% to the number 30 is 102% of 30. In describing an operation on numbers one must tell how to find the result of applying it to a number. For example, one finds the square of a number by multiplying the number by itself; and one finds the half of a number by dividing the number by 2. In particular, one finds 102% of a number by multiplying the number by 102/100. Note that 'of' is not just an alternative to '×'. Rather, it refers to applying an operation to a number. For example, to find the square of 4 does not mean to multiply square by 4; to find the logarithm of 2 does not mean to multiply logarithm by 2. Similarly, to find 102% of 30 does not mean to multiply 102% by 30. [It does mean to multiply 30 by 1.02.]

TC[1-59]

C. Simplify, using as many short cuts and doing as little writing as
    possible. [Be prepared to justify your short cuts on the basis of
    the principles.]

1. $7 \times (8 \times \frac{4}{7})$ 2. $9 \times (\frac{2}{9} \times 8)$

3. $16 + (27 + 4)$ 4. $75 + (18 + 25)$

5. $88 \times 21 + 21 \times 12$ 6. $19\frac{1}{4} \times 12 + 19\frac{1}{4} \times 88$

7. $15 \times (10 + 1) + 85$ 8. $65 + 7 \times (10 + 5)$

9. $30 \times 31 + 70$ 10. $29 \times 25 + 25$

11. $68 \times (\frac{1}{2} \times 1000)$ 12. 84×500

13. $67 \times 25 + 33 \times 25$ 14. $(18 \times 75) \times (93 \times 0)$

15. 102% of 30   16. $0.77 \times 0.01 + 1.23 \times \frac{1}{100}$

17. $\dfrac{3 \times 9 + 8 \times 9}{11}$ 18. $\dfrac{7}{7 \times 15 + 12 \times 7}$

19. $\dfrac{5 \times 7 + 4 \times 7}{9 \times 3 + 4 \times 9}$ 20. $\dfrac{6 \times 5 + 7 \times 5}{5 \times 8 + 25}$

[More exercises are in Part H, Supplementary Exercises.]

D. Suppose you want to solve the problem:

$$39 \times 83 + 39 \times 17 = ?$$

You know a short cut which is based on the left distributive principle
for multiplication over addition. Remember Stan and Al? Suppose
Stan hasn't told Al about the left distributive principle. Could Al
justify the short cut on the basis of the other principles? In other
words, could Al derive the sentence:

$$39 \times 83 + 39 \times 17 = 39 \times (83 + 17)$$

from just the principles mentioned on page 1-54? How would he do it?

1.07 Principles for the real numbers. --The principles you have learned
about in the last section refer to the system of numbers of arithmetic.
It is natural to ask if there are similar principles for the system of real
numbers. If we asked this question about just the nonnegative real num-
bers, the answer would be 'yes' [Why?]. But, the system of real num-
bers includes the negative numbers as well; so, for example, the question
whether multiplying by a real number distributes over adding requires
some investigation.

(I) For each sentence below, simplify both sides and label the sentence 'True' or 'False'.

(a) $^+3 + {}^-2 = {}^-2 + {}^+3$

(b) $(^+6 + {}^-4) + {}^+7 = {}^+6 + ({}^-4 + {}^+7)$

(c) $(^+7 + {}^-12) \times {}^-3 = {}^+7 \times {}^-3 + {}^-12 \times {}^-3$

(d) $(^+6 \times {}^-4) \times {}^+7 = {}^+6 \times ({}^-4 \times {}^+7)$

(e) $^+3 \times {}^-2 = {}^-2 \times {}^+3$

(f) $(^-3 \times {}^-5) + {}^+4 = ({}^-3 + {}^+4) \times ({}^-5 + {}^+4)$

(g) $^-5 + {}^-6 = {}^-6 + {}^-5$

(h) $^-3 + ({}^-12 + {}^-5) = ({}^-3 + {}^-12) + {}^-5$

(i) $^-6 \times {}^+4 + {}^-8 \times {}^+4 = ({}^-6 + {}^-8) \times {}^+4$

(j) $(^+9 + {}^-3) \times {}^-5 = {}^+9 + ({}^-3 \times {}^-5)$

(k) $^-3 \times {}^-12 \times {}^-5 = {}^-3 \times ({}^-12 \times {}^-5)$

(l) $^-5 \times {}^-6 = {}^-6 \times {}^-5$

(m) $^-2 + {}^+4 + {}^-6 = {}^-2 + ({}^+4 + {}^-6)$

(n) $^-2 \times ({}^+4 \times {}^-6) = ({}^-2 \times {}^+4) \times {}^-6$

(II) Classify as many as possible of the true sentences in (I) as instances of these five principles <u>for real numbers</u>.

Commutative principle for addition

Commutative principle for multiplication

Associative principle for addition

Associative principle for multiplication

Distributive principle for multiplication over addition

(III) Make up one more instance of each principle mentioned in (II) for real numbers. How many of your five new sentences are true?

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It is important that the students actually simplify both sides of the
sentences in (I). They are here investigating whether the principles
apply to operations on real numbers. Of course, the principles do
apply, and the reason that they do can be found in the nature of the
physical interpretations we gave for the operations. The interpre-
tations were so selected as to lead to definitions of addition and
multiplication consistent with the principles.

*

Here are answers for (I).

(a) $^+1 = {}^+1$ True   (b) $^+2 \div {}^+7 = {}^+6 \div {}^+3$

                                         $^+9 = {}^+9$ True

(c) $^-5 \times {}^-3 = {}^-21 \div {}^+36$ (d) $^-24 \times {}^+7 = {}^+6 \times {}^-28$

      $^+15 = {}^+15$ True $^-168 = {}^-168$ True

(e) $^-6 = {}^-6$ True   (f) $^+15 + {}^+4 = {}^+1 \times {}^-1$

                                   $^+19 = {}^-1$ False

(g) $^-11 = {}^-11$ True   (h) $^-3 + {}^-17 = {}^-15 + {}^-5$

                                   $^-20 = {}^-20$ True

(i) $^-24 + {}^-32 = {}^-14 \times {}^+4$ (j) $^+6 \times {}^-5 = {}^+9 \div {}^+15$

       $^-56 = {}^-56$ True $^-30 = {}^+24$ False

(k) $^+36 \times {}^-5 = {}^-3 \times {}^+60$ (l) $^+30 = {}^+30$ True

      $^-180 = {}^-180$ True

(m) $^+2 + {}^-6 = {}^-2 \div {}^-2$ (n) $^-2 \times {}^-24 = {}^-8 \times {}^-6$

      $^-4 = {}^-4$ True $^+48 = {}^+48$ True

*

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[In the same way one can establish any such equations as:

$$^-3 \times {}^-1 = {}^+3, \quad {}^+2 \times {}^-1 = {}^-2, \quad \text{etc.}$$

So, if one wishes to define multiplication of real numbers in such a way that it will be associative as well as satisfying the ℓdpma and the pm$^+$1, it must, for example, be the case that

$$
\begin{aligned}
^+3 \times {}^-2 &= {}^+3 \times ({}^+2 \times {}^-1) \\
&= ({}^+3 \times {}^+2) \times {}^-1 \\
&= [{}^+3 \times ({}^+1 + {}^+1)] \times {}^-1 \\
&= ({}^+3 + {}^+3) \times {}^-1 \\
&= {}^+6 \times {}^-1 \\
&= {}^-6.
\end{aligned}
$$

And if, further, multiplication is to be commutative, it must be the case that $^-2 \times {}^+3 = {}^-6.$]

✳

The answer to both questions in Exercise 3 of Part A on page 1-61 is: Yes. The reason for the second answer is that, just as is suggested in Part D on page 1-59 in the case of numbers of arithmetic, so here, in dealing with real numbers, each instance of the ℓdpma for real numbers is a consequence of an instance of the dpma and three instances of the cpm. Students should illustrate this in class by justifying, say, '$^-3 \times ({}^+7 + {}^-12) = {}^-3 \times {}^+7 + {}^-3 \times {}^-12$' by application of the cpm and dpma.

Correct lists for Exercise II, page 1-60, might have appeared as follows.

The Commutative Principles for Addition and Multiplication

$^+3 + {}^-2 = {}^-2 + {}^+3$ $\qquad\qquad$ $^+3 \times {}^-2 = {}^-2 \times {}^+3$

$^-5 + {}^-6 = {}^-6 + {}^-5$ $\qquad\qquad$ $^-5 \times {}^-6 = {}^-6 \times {}^-5$

The Associative Principles for Addition and Multiplication

$(^+6 + {}^-4) + {}^+7 = {}^+6 + ({}^-4 + {}^+7)$ $\qquad$ $(^+6 \times {}^-4) \times {}^+7 = {}^+6 \times ({}^-4 \times {}^+7)$

$^-3 + ({}^-12 + {}^-5) = ({}^-3 + {}^-12) + {}^-5$ $\qquad$ $^-3 \times {}^-12 \times {}^-5 = {}^-3 \times ({}^-12 \times {}^-5)$

$^-2 + {}^+4 + {}^-6 = {}^-2 + ({}^+4 + {}^-6)$ $\qquad$ $^-2 \times ({}^+4 \times {}^-6) = ({}^-2 \times {}^+4) \times {}^-6$

The Distributive Principle for Multiplication over Addition

$(^+7 + {}^-12) \times {}^-3 = {}^+7 \times {}^-3 + {}^-12 \times {}^-3$

$^-6 \times {}^+4 + {}^-8 \times {}^+4 = ({}^-6 + {}^-8) \times {}^+4$

EXERCISES

A. 1. Make up two more instances of each of the five principles and check each of your ten sentences.

2. The fact that the real numbers have these same five properties as the numbers of arithmetic is important because they give us short cuts in working with the real numbers. The other properties of the numbers of arithmetic--those expressed by the principle for adding 0 and the principles for multiplying by 1 and by 0--also hold for the real numbers. That is, there is a principle for adding the real number 0, a principle for multiplying by the real number $^+1$, and a principle for multiplying by the real number 0. Make up three instances of each of these three real number principles and check each of the nine sentences.

3. Is there a left distributive principle for multiplication over addition for the system of real numbers? What must be your answer if you accept the five principles mentioned at the top of this page?

B. For each of the following sentences fill in the blank to make it true, <u>and</u> tell what principle the true sentence illustrates.

   1. $^+5 \times$ _____ $= ^-12 \times {}^+5$ 2. $^-3 \times {}^-7 =$ _____ $\times {}^-3$

   3. $(^-3 + {}^+7) \times {}^+5 = {}^-3 \times {}^+5 +$ _____ $\times {}^+5$

   4. $^-6 \times ({}^+8 \times$ ___$) = ({}^-6 \times {}^+8) \times {}^+15$

   5. $^+9 + {}^-3 + {}^+5 = {}^+9 + ($ ___ $+ {}^+5)$

   6. $^-2 \times {}^-7 + {}^-2 \times$ ___ $= {}^-2 \times ({}^-7 + {}^+17)$

   7. $^+9 \times$ ___ $= 0$ 8. $^-3 \times$ ___ $= {}^-3$

   9. $^-5 +$ ___ $= {}^-5$ 10. $^+5 \times {}^-7 \times$ _____ $= 0$

  11. $0 =$ _____ $\times 0$ 12.   _____ $+ 0 = 0$

  13. $4 \times (7 +$ ___$) = 4 \times 7 + 4 \times$ _____

  14. $^-25 =$ _____ $+ 0$ 15. $^-25 =$ _____ $\times 0$

C. Use the principles for the real numbers to help you simplify the following numerical expressions. Do as little writing as you can.

   1. $(^-18\tfrac{1}{2} + 85) + 3\tfrac{1}{2}$ 2. $(212 + {}^-473) + 473$

   3. $7 \times 8 + 8 \times {}^-9$ 4. $^-7 \times 8 + 2 \times {}^-7$

   5. $(^-185 \times \tfrac{1}{3}) \times 3$ 6. $(\tfrac{^-1}{5} \times 5.792) \times {}^-5$

   7. $(^-793 \times \tfrac{3}{5}) \times \tfrac{5}{3}$ 8. $^-12 + 876 + 512$

   9. $^-18 \times 57 + 57 \times 68$ 10. $^-3 \times 15789.6 + 15789.6 \times 13$

  11. $^-16\tfrac{1}{2} + 86 + {}^-3\tfrac{1}{2}$ 12. $^-972.75 \times {}^-37 + {}^-27.25 \times {}^-37$

  13. $(892 \times \tfrac{1}{2}) \times ({}^-37 \times 0) \times 18$ 14. $27 \times {}^-117 + 17 \times 27$

  15. $^-8 + ({}^+3 + {}^+8)$ 16. $(^-3 + {}^-7) + ({}^+3 + {}^+7)$

  17. $^-453 + ({}^+624 + {}^+453)$ 18. $(^-587 + {}^-426) + ({}^+587 + {}^+426)$

  19. $\tfrac{1}{16} \times 45.678 \times 16 + {}^-45.678$

  20. $891.23 \times 386.9 + (\tfrac{1}{7} \times {}^-891.23) \times (7 \times {}^+386.9)$

1/77

Answers for Part B.

1. ⁻12, cpm 2. ⁻7, cpm 3. ⁺7, dpma 4. ⁺15, apm

5. ⁻3, apa 6. ⁺17, ℓdpma 7. 0, pm0 8. ⁺1, pm⁺1

9. 0, pa0 10. 0, pm0

11. A numeral for any number may be written in the blank, pm0

12. 0, pa0

13. A numeral for any number may be written in the blanks, ℓdpma

14. ⁻25, pa0 15. It is not possible to make the sentence true.

✳

Answers for Part C.

1. 70 2. 212 3. ⁻16 4. ⁻70

5. ⁻185 6. 5.792 7. ⁻793 8. 1376

9. 2850 10. 157,896 11. 66 12. 37000

13. 0 14. ⁻2700 15. ⁺3 16. 0

17. ⁺624 18. 0 19. 0 20. 0

1/78

The purpose of the Exploration Exercises on pages 1-63 through 1-66 is to prepare the student for the work on inverse operations in Section 1.08. Students should go through these exercises rapidly. Note that the Exploration Exercises refer to operations with numbers of arithmetic and not to operations with real numbers. We are trying to develop the general notion of inverse operations. It will be applied to real numbers in Section 1.09.

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Answers for Part A.

1.	3	2.	9	3.	7	4.	40	5.	8
6.	4	7.	10	8.	4	9.	20	10.	45
11.	283	12.	$57\frac{1}{8}$	13.	253	14.	185.9		

�֍

Answers for Part B [on pages 1-63 and 1-64].

1.	7	2.	20	3.	12	4.	900	5.	9
6.	8423	7.	7	8.	19	9.	29	10.	876
11.	82	12.	2431	13.	8	14.	49	15.	24
16.	68	17.	7	18.	15	19.	3	20.	72

In each of the exercises from 21 through 28 a numeral for any number can be used in the first blank, and a copy of this numeral must be written in the second blank of the exercise.

29.	7	30.	subtracting 192	31.	15
32.	97	33.	subtracting 54	34.	subtracting 71

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Answers for Part C [on page 1-64].

1.	3	2.	4	3.	2	4.	86	5.	12
6.	42	7.	40	8.	99				

EXPLORATION EXERCISES

A. Guess the number.

1. Multiply it by 8, and you get 24.

2. Double it, and you get 18.

3. Add 10 to it, and you get 17.

4. Add 23 to it, subtract 3 from the sum, and you get 60.

5. Add 12 to it, subtract 12 from the sum, and you get 8.

6. Multiply it by 5, multiply the product by 2, and you get 40.

7. Multiply it by 7, multiply the product by $\frac{1}{7}$, and you get 10.

8. Multiply it by 9, multiply the product by $\frac{1}{9}$, and you get 4.

9. Multiply it by $\frac{1}{5}$, multiply the product by 5, and you get 20.

10. Add 98 to it, subtract 98 from the sum, and you get 45.

11. Add 974 to it, subtract 974 from the sum, and you get 283.

12. Add $81\frac{2}{3}$ to it, subtract $81\frac{2}{3}$ from the sum, and you get $57\frac{1}{8}$.

13. Multiply it by $\frac{1}{78}$, multiply the product by 78, and you get 253.

14. Add 2.7 to it, subtract 2.7 from the sum, and you get 185.9.

B. Fill in the blanks to make true sentences.

1. (___ + 4) - 4 = 7   2. (___ + 9) - 9 = 20

3. (___ + 8) - 8 = 12   4. (___ + 5) - 5 = 900

5. (___ + 51) - 51 = 9   6. (___ + 79) - 79 = 8423

7. (7 + 5) - 5 = ____   8. (19 + 3) - 3 = ____

9. (29 + 2) - 2 = ____   10. (876 + 15) - 15 = ____

11. (82 + 769) - 769 = ____   12. (2431 + 1893) - 1893 = ____

13. (9 + 8) - ___ = 9   14. (73 + 49) - ___ = 73

Answers for Part C [on pages 1-65 and 1-66].

9. 6	10. 7	11. 8	12. 173	13. 15
14. 36	15. 8	16. 92	17. 2	18. 8
19. 7	20. 13	21. $\frac{1}{9}$	22. $\frac{1}{7}$	23. $\frac{1}{11}$

24. $\frac{1}{77}$ 25. $\frac{1}{17}$ 26. [none]   27. $\frac{1}{3}$

28. $\frac{1}{19}$ 29. 8   30. 19

In Exercises 31 and 32 a numeral for any number can be used in the first blank, and a numeral for the reciprocal of that number must be written in the second blank. [In connection with Exercises 31 and 32, the teacher might ask whether there is a number such that a numeral for it could be written in both blanks of the exercise and make the sentence true.]

In each of the Exercises 33 and 34, a numeral for any number may be written in the first blank, and a copy of this numeral should be written in the second blank. [In each of these Exercises (33 and 34), a student could get true sentences by writing different numerals for the same number in the two blanks; however, we want to prepare the students for the convention in Unit 2 that blanks of the same shape must be filled by copies of the same numeral.]

35. $\frac{1}{7}$ 36. 6

In the bracketed paragraph, answers to successive questions are:
$\frac{1}{9}$; 9; $\frac{100}{37}$; $\frac{3}{2}$; 1; yes; 0 has no reciprocal; there is no number such that 0 times that number is 1.

[In connection with the bracketed paragraph, you may find it desirable to ask other similar questions. For example, 'What is the reciprocal of 0.25?' and 'What is the reciprocal of 2983/2983?' [Answers: 4, 1.] Since you are here discussing numbers of arithmetic, you will, of course, not ask 'What is the reciprocal of ¯2?'.]

37. 53	38. $\frac{1}{12}$	39. 17	40. 17	41. $\frac{1}{9}$
42. 9	43. 7	44. $\frac{1}{7}$	45. reciprocal, 7	

46.   a numeral for any number can be used in the first blank, and a copy of this numeral should be written in the second blank.

9. $(6 \times 3) \times \frac{1}{3} =$ ____

10. $(7 \times 2) \times \frac{1}{2} =$ ____

11. $(8 \times 5) \times \frac{1}{5} =$ ____

12. $(173 \times 84) \times \frac{1}{84} =$ ____

13. $(15 \times \frac{1}{3}) \times 3 =$ ____

14. $(36 \times \frac{1}{9}) \times 9 =$ ____

15. $(8 \times \frac{1}{9}) \times 9 =$ ____

16. $(92 \times \frac{1}{79}) \times 79 =$ ____

17. $(6 \times 0.5) \times$ ____ $= 6$

18. $(72 \times 0.125) \times$ ____ $= 72$

19. $(63 \times \frac{1}{7}) \times$ ____ $= 63$

20. $(56 \times \frac{1}{13}) \times$ ____ $= 56$

21. $(51 \times 9) \times$ ____ $= 51$

22. $(38 \times 7) \times$ ____ $= 38$

23. $(85 \times 11) \times$ ____ $= 85$

24. $(77 \times 6) \times$ ____ $= 6$

25. $(29 \times 17) \times$ ____ $= 29$

26. $(583 \times 0) \times$ ____ $= 583$

27. $(81 \times$ ____ $) \times 3 = 81$

28. $(751 \times$ ____ $) \times 19 = 751$

29. $(52 \times$ ____ $) \times \frac{1}{8} = 52$

30. $(847 \times$ ____ $) \times \frac{1}{19} = 847$

31. $(48 \times$ ____ $) \times$ ____ $= 48$

32. $(31 \times$ ____ $) \times$ ____ $= 31$

33. $($ ____ $\times 7) \times \frac{1}{7} =$ ____

34. $($ ____ $\times \frac{1}{8}) \times 8 =$ ____

35. If you multiply a number by 7, you can get back the number by multiplying the product by ____ .

   Example 1. $5 \times 7 = 35$ and $35 \times$ ____ $= 5$.

   Example 2. $(31 \times 7) \times$ ____ $= 31$.

36. If you multiply a number by $\frac{1}{6}$, you can get back the number by multiplying the product by ____ .

   Example 1. $12 \times \frac{1}{6} = 2$ and $2 \times$ ____ $= 12$.

   Example 2. $(54 \times \frac{1}{6}) \times$ ____ $= 54$.

[Note: Since $2 \times 0.5 = 1$, 0.5 is called the reciprocal of 2. Since $0.5 \times 2 = 1$, 2 is called the reciprocal of 0.5. In general, pairs of numbers whose product is 1 are called reciprocals, and each is the reciprocal of the other. What is the reciprocal of 9? Of $\frac{1}{9}$? Of 0.37? Of $\frac{2}{3}$? Of 1? Is there a number of arithmetic which does not have a reciprocal? What is the reciprocal of 0? $0 \times ? = 1$.]

37. If you multiply a number by ____, you can get back the number by multiplying the product by the reciprocal of 53.

38. If you multiply a number by 12, you can get back the number by multiplying the product by _____.

39. If you want to undo the result of multiplying by _____, multiply the product by the reciprocal of 17.

   Example. $(93 \times$ ____$) \times \frac{1}{17} = 93.$

40. Multiplying by ____ undoes what multiplying by the reciprocal of 17 did.

   Example. $(42 \times \frac{1}{17}) \times$ _____ $= 42.$

41. Multiplying by _____ undoes what multiplying by 9 did.

42. Multiplying by _____ undoes what multiplying by the reciprocal of 9 did.

43. Dividing by _____ undoes what multiplying by 7 did.

44. Multiplying by _____ undoes what multiplying by 7 did.

45. Multiplying by the _____ of _____ undoes what multiplying by 7 did.

46. Dividing by _____ does what multiplying by the reciprocal of _____ does.

1.08 Inverse operations. --Here is a grade school subtraction problem:

$$13 - 4 = ?$$

Everyone knows that the result is 9. But, let's look a bit more deeply into how we know that this is the correct answer.

   In Part B of the preceding Exploration Exercises you learned that

   subtracting 4 undoes what adding 4 did.

A classroom device for introducing the notion of an operation as a
set of ordered pairs is the following. Ask students for examples of
adding 4. Then record the examples in a column like this:

$$3 + 4 = 7$$
$$2 + 4 = 6$$
$$6 + 4 = 10$$
$$12 + 4 = 16$$
$$0 + 4 = 4$$
$$8 + 4 = 12.$$

Next, erase the '+ 4 = ' from each sentence, leaving a list of pairs.
Thus, the pair (3, 7) corresponds with an example of adding 4. If
we think of the operation adding 4 as somehow made up of all its
"examples", it is easy to think of the operation as the set of pairs
corresponding with these examples.

<p style="text-align:center">✳</p>

Here is a classroom device which helps develop the idea of the
inverse of an operation. Tell students you are going to list the pairs
in multiplying by 5. Start the list as follows.

<p style="text-align:center">(3, 15), (7, 35), (4, 20)</p>

Then, list the first number of a pair and ask the students to give the
second. Also, list the second number and ask them for the first.

1/80

Here we introduce a device which should be quite helpful in getting students to understand the idea of operations and their inverses. In particular, it should make very clear what happens when they come to subtraction of real numbers in Section 1.09.

We regard an operation such as adding 4 as a set of ordered pairs in which the second component in each ordered pair is the sum of the first component and 4. Thus, the expression 'adding 4' is a noun whose referent is an operation. Notice that the student must begin developing the concept of an ordered pair, and this is a painless way to do it. For example, he knows that the pair (3, 7) belongs to the operation adding 4, but that the pair (7, 3) does not. So, it must be the case that (3, 7) is not equal to (7, 3). The operation subtracting 4 [some of the pairs in this operation are listed on page 1-68] has a very close connection to the operation adding 4. In fact, if you were to interchange the components of the ordered pairs in one operation, you would get the ordered pairs in the other. We say that the inverse of adding 4 is subtracting 4. This use of the word 'inverse' is precisely the use we make of the word in connection with functions and their inverses. [See Unit 5.] [Also, see pages 1-103 through 1-109.]

✳

If the subsection "Operations" is read aloud then, for example, the sentence which begins on line 3 of this subsection should be read as:

it says that the phrase subtracting-four and the phrase the-inverse-of-adding-four are names of the same thing.

[See T C[1-L]a.]

✳

Answers for Part A [on page 1-68].

Pairs that belong to the inverse: $(11\frac{3}{5}, 3\frac{1}{5})$ $(17\frac{4}{5}, 9\frac{2}{5})$ $(9\frac{1}{5}, \frac{4}{5})$ $(19\frac{1}{2}, 11\frac{1}{10})$ $(15\frac{11}{15}, 7\frac{1}{3})$ $(109\frac{13}{20}, 101\frac{1}{4})$ $(44.3, 35.9)$

The operations are: adding $8\frac{2}{5}$, and subtracting $8\frac{2}{5}$.

✳

This statement tells us what we mean by 'subtracting 4'. Subtracting 4 is just the operation you carry out to undo what adding 4 did. So, if you wish to subtract 4 from 13, you must try to find the number to which 4 was added to get 13. This number is 9, since 9 + 4 = 13. So, we say that 13 - 4 = 9.

Let us rewrite the subtraction statement:

$$13 - 4 = 9$$

as:

$$(9 + 4) - 4 = 9.$$

Do you see from this last statement that subtracting 4 undoes what adding 4 did? A shorter way of expressing this idea is to say that

subtracting 4 is the inverse of adding 4.

Similarly, we can say that

$$(3582 + 649) - 649 = 3582$$

because

subtracting 649 is the inverse of adding 649.

OPERATIONS

Let us examine a statement such as:

subtracting 4 is the inverse of adding 4

and see what it tells us. It says that 'subtracting 4' and 'the inverse of adding 4' are names of the same thing. What is this thing? 'subtracting 4' names an operation, and so does 'adding 4'. Let's look at these operations.

When you add 4, you start with a number and get to a number. For example, you may start with 3 and end with 7, start with 0 and end with 4, start with 15 and end with 19, start with 937 and end with 937 + 4. You can think of the operation adding 4 as the whole set of such pairs of numbers. Here is a list of just some of these pairs. Would it be possible to list all of them? Could the pair (9, 13)

(3, 7)   (0, 4)   (15, 19)   (937, 937 + 4)

(19, 23)   (81, 85)   (52, 56)   (45, 49)

$(1\frac{2}{3}, 5\frac{2}{3})$ (7.5, 11.5)   (9.03, 9.03 + 4)   (7, 11)

. . .

$$-\,-\,\cdot-\,-\,-\,-\quad-\,\cdot\,-$$

$$13 - 4 = 9$$

as:

$$(9 + 4) - 4 = 9.$$

Do you see from this last statement that subtracting 4 undoes what adding 4 did? A shorter way of expressing this idea is to say that

subtracting 4 is the inverse of adding 4.

Similarly, we can say that

$$(3582 + 649) - 649 = 3582$$

because

subtracting 649 is the inverse of adding 649.

OPERATIONS

Let us examine a statement such as:

subtracting 4 is the inverse of adding 4

and see what it tells us. It says that 'subtracting 4' and 'the inverse of adding 4' are names of the same thing. What is this thing? 'subtracting 4' names an operation, and so does 'adding 4'. Let's look at these operations.

When you add 4, you start with a number and get to a number. For example, you may start with 3 and end with 7, start with 0 and end with 4, start with 15 and end with 19, start with 937 and end with 937 + 4. You can think of the operation adding 4 as the whole set of such pairs of numbers. Here is a list of just some of these pairs. Would it be possible to list all of them? Could the pair (9, 13)

(3, 7)	(0, 4)	(15, 19)	(937, 937 + 4)
(19, 23)	(81, 85)	(52, 56)	(45, 49)
$(1\frac{2}{3}, 5\frac{2}{3})$	(7.5, 11.5)	(9.03, 9.03 + 4)	(7, 11)

. . .

This statement tells us what we mean by 'subtracting 4'. Subtracting 4 is just the operation you carry out to undo what adding 4 did. So, if you wish to subtract 4 from 13, you must try to find the number to which 4 was added to get 13. This number is 9, since 9 + 4 = 13. So, we say that 13 - 4 = 9.

Let us rewrite the subtraction statement:

$$13 - 4 = 9$$

as:

$$(9 + 4) - 4 = 9.$$

Do you see from this last statement that subtracting 4 undoes what adding 4 did? A shorter way of expressing this idea is to say that

subtracting 4 is the inverse of adding 4.

Similarly, we can say that

$$(3582 + 649) - 649 = 3582$$

because

subtracting 649 is the inverse of adding 649.

OPERATIONS

Let us examine a statement such as:

subtracting 4 is the inverse of adding 4

and see what it tells us. It says that 'subtracting 4' and 'the inverse of adding 4' are names of the same thing. What is this thing? 'subtracting 4' names an operation, and so does 'adding 4'. Let's look at these operations.

When you add 4, you start with a number and get to a number. For example, you may start with 3 and end with 7, start with 0 and end with 4, start with 15 and end with 19, start with 937 and end with 937 + 4. You can think of the operation adding 4 as the whole set of such pairs of numbers. Here is a list of just some of these pairs. Would it be possible to list all of them? Could the pair (9, 13)

(3, 7)   (0, 4)   (15, 19)   (937, 937 + 4)

(19, 23)   (81, 85)   (52, 56)   (45, 49)

$(1\frac{2}{3}, 5\frac{2}{3})$ (7.5, 11.5)   (9.03, 9.03 + 4)   (7, 11)

· · ·

be included in this list? How about (40, 45)? How about (11, 7)?

Notice that (11, 7) could not be included in this list because
11 + 4 ≠ 7. But, there is a related operation to which (11, 7) <u>does</u>
belong. What is it?

Here is a list of some of the pairs which belong to the operation
subtracting 4. [As before, it would not be possible to list all the
pairs which belong to the operation.]

<pre>
   (11, 7)   (7, 3)   (4, 0)   (19, 15)   (937 + 4, 937)

       (23, 19)   (85, 81)   (56, 52)   (49, 45)

    (5⅔, 1⅔)   (11.5, 7.5)   (9.03 + 4, 9.03)

                              · · ·
</pre>

Examine the two lists we have given. Suppose someone put
another pair into the first list. Would that suggest a pair which
you could put into the second list? Give three more pairs which
you could put into the list for adding 4. Give three which you
could put into the list for subtracting 4.

EXERCISES

<u>A</u>. Here is a list of some of the pairs of numbers which belong to
a certain operation. Alongside it, make a list of some of the
pairs which belong to the <u>inverse</u> of that operation.

<pre>
   (3⅕, 11⅗)   (9⅖, 17⅘)

          (⅘, 9⅕)

  (11 1/10, 19½)   (7⅓, 15 11/15)

  (101¼, 109 13/20)   (35.9, 44.3)
                 ·
                 ·
                 ·
</pre>

Can you guess what these two operations are called?

be ir

Answers for Part B.

Adding 7: (5, 12) (3, 10) (15, 22) (35, 42) (96, 103) $(10\frac{1}{2}, 17\frac{1}{2})$
    (31.2, 38.2) ... .

Subtracting 7: (12, 5) (10, 3) (22, 15) (42, 35) (103, 96) $(17\frac{1}{2}, 10\frac{1}{2})$
    (38.2, 31.2) ... .

✳

Answers for Part C.

1.   Multiplying by 5.
2.   (35, 7) (10, 2) (5, 1) (11.5, 2.3) (0, 0) (15, 3) (25, 5) (45, 9)
     (5000, 1000) (20, 4) (30, 6)
3.   They are inverse operations.
4.   Same list as for Exercise 2.
5.   The operations dividing by 5 and multiplying by the reciprocal
     of 5 produce the same result; hence 'dividing by 5' and
     'multiplying by the reciprocal of 5' must be names for the same
     operation.

B. Make two lists of pairs of numbers which illustrate that

    subtracting 7 is the inverse of adding 7.

C. 1. Here is a list of eleven pairs which belong to a certain
        operation. Can you guess what operation this is?

(7, 35)   (2, 10)   (1, 5)

(2.3, 11.5)   (0, 0)   (3, 15)

(5, 25)   (9, 45)   (1000, 5000)   (4, 20)

(6, 30)

• • •

2. Make a list of eleven pairs which belong to the operation
    dividing by 5.

3. What do your answers to Exercises 1 and 2 suggest about
    dividing by 5 and the operation referred to in Exercise 1?

4. Make a list of eleven pairs which belong to the operation
    multiplying by the reciprocal of 5.

5. What do your answers to Exercises 2 and 4 suggest about
    the operations dividing by 5 and multiplying by the recip-
    rocal of 5?

❋ ❋ ❋

Here are two lists of pairs which belong to operations. Note that the second operation is the inverse of the first.

adding 5

(0, 5)   (15, 20)

(31, 36)   ($4\frac{1}{2}$, $9\frac{1}{2}$)

(7, 12)   (91, 96)

(40, 45)   (27, 32)

(22, 27)   (10, 15)

(2.41, 7.41)

(36, 41)   (96, 101)

(1004, 1009)

(2, 7)   (45, 50)

subtracting 5

(5, 0)   (20, 15)

(36, 31)   ($9\frac{1}{2}$, $4\frac{1}{2}$)

(12, 7)   (96, 91)

(45, 40)   (32, 27)

(27, 22)   (15, 10)

(7.41, 2.41)

(41, 36)   (101, 96)

(1009, 1004)

(7, 2)   (50, 45)
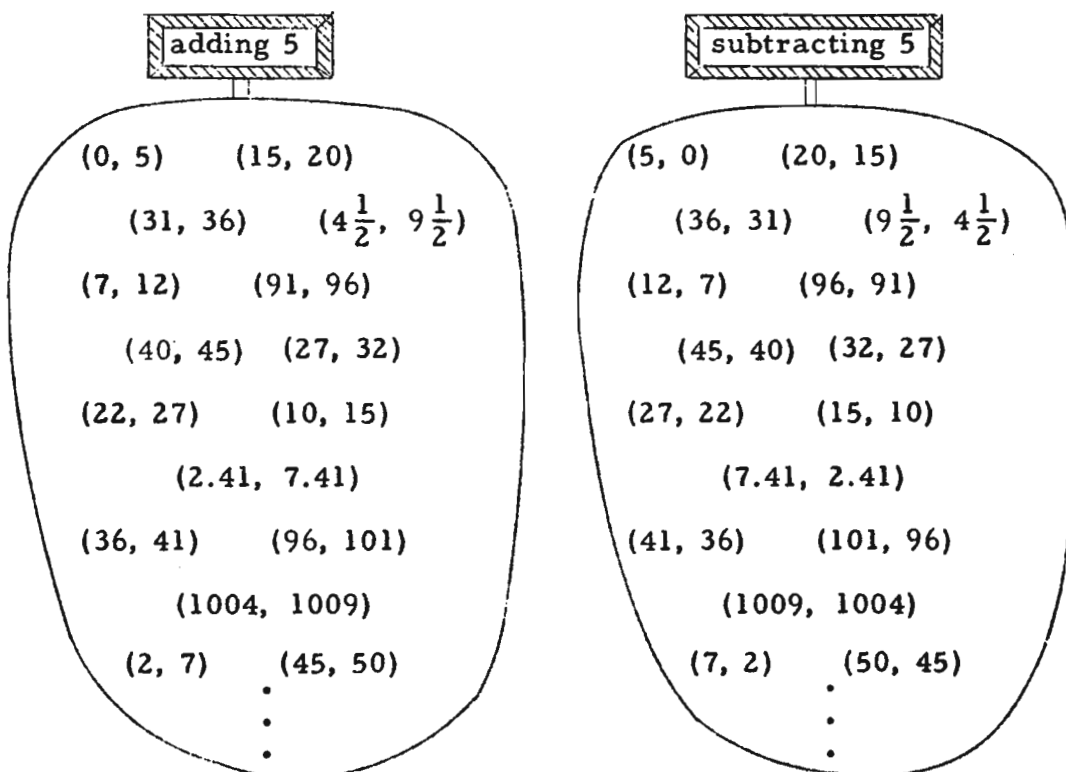
How could we use them to do a problem? Consider the problem:

$$91 + 5 = ?$$

This is a problem in adding 5. Go to the adding 5 list, and look for the pair whose first number is 91. What is the second number in this pair? Is it the result of adding 5 to 91?

Now, try a subtraction problem:

$$45 - 5 = ?$$

This is a problem in subtracting 5. Go to the subtracting 5 list, and look for the pair whose first number is 45. What is the second number in the pair? Is it what you get when you subtract 5 from 45?

Answers for quiz.

I. 1. (7, 9)   2. (0, 0)   3. adding 10   4. (8, 1.6)

    5. (70, 64)   6. (0, 0)   7. adding 10

II. 1. (2, 4)   2. (1, 4)   3. (0, 0)   4. $(2\frac{1}{2}, 7\frac{1}{2})$

❋

For recreation you can introduce more "complicated" operations.
For example, you can ask for pairs which belong to multiplying by
2 and then adding 3. Such pairs are (1, 5), (7, 17), (0, 3), and
(100, 203). Or, if you wish, you can give some pairs and ask for
a name of an operation which contains such pairs.

❋

Answers for Part A of the Exploration Exercises [on page 1-72].

1. $^{+}4$ 2. $^{+}6$ 3. $^{-}8$ 4. $^{+}4$ 5. $^{-}1$

6. $^{-}6$ 7. $^{-}9$ 8. 0   9. $^{-}173$ 10. $^{+}286$

✷

Here is a quiz which covers the idea of an operation as a set of pairs and the idea of the inverse of an operation.

I. **Multiple-choice.** Draw a loop around the correct answer.

1. The operation adding 2 contains _____.

(A) (3, 6)   (B) (5, 9)   (C) (7, 9)   (D) (1/2, 1/4)

2. The operation multiplying by 2 contains _____.

(A) (8, 10)   (B) (30, 15)   (C) (0, 0)   (D) (5, 3)

3. (5, 15) belongs to _____.

(A) adding 5   (B) dividing by 3
(C) multiplying by 1/3   (D) adding 10

4. Dividing by 5 contains _____.

(A) (8, 1.6)   (B) (10, 5)   (C) (12, 2.2)   (D) (0, 5)

5. The inverse of adding 6 contains _____.

(A) (6, 1)   (B) (7, 13)   (C) (50, 46)   (D) (70, 64)

6. The inverse of multiplying by 7 contains _____.

(A) (14, 21)   (B) (0, 0)   (C) (7, 0)   (D) (5, 35)

7. The inverse of adding 10 is _____.

(A) multiplying by 1/10   (B) subtracting 10
(C) adding 10   (D) adding 1/10

II. 1. What pair belongs both to adding 2 and to multiplying by 2?

2. What pair belongs both to adding 3 and to multiplying by 4?

3. What pair belongs both to multiplying by 7 and to the inverse of multiplying by 7?

✷4. What pair belongs both to multiplying by 3 and to adding 5?

✷

He-

Answers for exercises at the top of page 1-71.

(1) 20   (2) 7.41   (3) 91   (4) 22   (5) 12   (6) 31

*

Answers for Part D.

1. 24   2. 162   3. 7   4. 8.4

*

Answers for Part E [on page 1-71].

1. 5843.52   2. 2238.25   3. 6824.37   4. 3205.91

*

Answers for Part F [on page 1-72].

1. 743   2. 156   3. 1   4. 59

*

Part G on page 1-72 presents an interesting consequence of regard-
ing operations as sets of pairs. Make this a class exercise and
handle it as follows. Ask students to name the operation some of
whose pairs are

        (1, 2), (3, 6), (15, 30), (70, 140), ...

[Describe the operation further by stating that each second number
is the double of the first.] Students will respond that this operation
is called 'multiplying by 2'. Then remark that this is not the name
you had in mind. [Some student might suggest 'doubling', which is
an appropriate name also.] Try to elicit the name 'dividing by 1/2'.
Then point out that the names 'multiplying by 2' and 'dividing by 1/2'
are synonyms. They stand for the same thing. Now, proceed with
Part G. The operation described therein has several names--
'multiplying by 1', 'dividing by 1', 'adding 0', 'subtracting 0'. [A
name which we shall introduce later [page 1-88] is 'sameing'. A
technical name for this operation is 'the identity operation'.] The
point of this exploratory work is to develop the idea that an operation
can have many names. This is exceedingly important in connection
with subtracting real numbers, for the major point there will be that,
for example, the operation which we call 'subtracting $^+4$' is also
called 'adding the opposite of $^+4$'. Since we know how to add the
opposite of $^+4$ [that is, since we know the pairs which belong to add-
ing the opposite of $^+4$], we must also know how to subtract $^+4$.

TC[1-71, 72]a

Use these lists to solve the following problems [even though you can do the problems without the lists]:

(1)   15 + 5 = ?

(2)   2.41 + 5 = ?

(3)   96 - 5 = ?

(4)   27 - 5 = ?

(5)   7 + 5 = ?

(6)   36 - 5 = ?

✳ ✳ ✳

D. Here is a list of pairs which belong to the operation adding 19.

(3, 22)   (5, 24)   (7, 26)   (2, 21)   (0, 19)   (26, 45)

(21, 40)   (6.2, 25.2)   (8.4, 27.4) $(2\frac{1}{2}, 21\frac{1}{2})$ (19, 38)

(181, 200)   (25.2, 44.2)   (162, 181)   (27.4, 46.4)

• • •

Use just this list to solve the following problems.

1.   5 + 19 = ?   2.   181 - 19 = ?

3.   26 - 19 = ?   4.   27.4 - 19 = ?

E. Here is a list of pairs which belong to the operation adding 9734.62.

(15578.14, 25312.76)   (3205.91, 12940.53)

(5843.52, 15578.14) (12940.53, 22675.15) (6824.37, 16558.99)

(11972.87, 21707.49)   (2238.25, 11972.87)

(16558.99, 26293.61)

• • •

Use just this list to solve the following problems.

1. 15578.14 - 9734.62 = ?   2. 11972.87 - 9734.62 = ?

3. 16558.99 - 9734.62 = ?   4. 12940.53 - 9734.62 = ?

F. Here is a list of pairs which belong to the operation multiplying
    by 789.

(59, 46551)   (0, 0)   (743, 586227)   (46551, 36728739)

(586227, 462533103)   (156, 123084)   (1, 789)

(123084, 97113276)   (789, 622521)

· · ·

Use this list to solve the following problems.

1. 586227 ÷ 789 = ?   2. $123084 \times \frac{1}{789}$ = ?

3. 789 ÷ 789 = ?   4. 46551 × the reciprocal of 789 = ?

G. Think of the set of all pairs of numbers of arithmetic in which the
    first number is the same as the second number. What operation
    is this?

EXPLORATION EXERCISES

A. Guess the number.

1. Add $^+3$ to it, and you get $^+7$.

2. Add $^-4$ to it, and you get $^+2$.

3. Add $^+5$ to it, and you get $^-3$.

4. Add $^-7$ to it, add $^+7$ to the sum, and you get $^+4$.

5. Add $^-9$ to it, add $^+9$ to the sum, and you get $^-1$.

6. Add $^+5$ to it, add $^-5$ to the sum, and you get $^-6$.

7. Add $^-2$ to it, add $^+2$ to the sum, and you get $^-9$.

8. Add $^+3$ to it, add $^-3$ to the sum, and you get 0.

9. Add $^-10$ to it, add $^+10$ to the sum, and you get $^-173$.

10. Add 0 to it, add 0 to the sum, and you get $^+286$.

F.   He·

1/86

Answers for Part B [on pages 1-73 and 1-74].

1.	$^+5$	2.	$^-1$	3.	$^-8$	4.	$^+4$	5.	$^-73$
6.	$^+824$	7.	$^+7$	8.	$^+12$	9.	$^-3$	10.	$^-7$
11.	$^+9.3$	12.	$^-58$	13.	$^+35$	14.	$^-21$	15.	$^-65$
16.	$^-72$								

In Exercises 17 and 18, a numeral for any number may be written in the first blank, and a numeral for the opposite of this number must be written in the second blank. In each of the exercises from 19 through 22, a numeral for any number can be used in the first blank, and a copy of this numeral must be written in the second blank of the exercise.

23.   '$^-8$' in each blank   24.   '$^+5$' in each blank

[In the bracketed paragraph, the answer to the question is: 0.]

25.	$^+3$	26.	$^-3$	27.	$^-9$	28.	$^+9$

29.   '$^-7$' in each blank   30.   '$^+5$' in each blank

31.	$^-5$	32.	$^+11$	33.	$^-17$	34.	$^+7$

35.   the opposite of $^-2$ 36.   '$^+7$' in each blank

37.   '$^-24$' in each blank   38.   its opposite

[In answering Exercise 36, an alternative is to write 'the opposite of $^-7$' in the first blank, and '$^+7$' in the second blank. Similarly, for Exercise 37.]

✳

Section 1.09 [page 1-75] on subtraction of real numbers provides the payoff for regarding operations as sets of ordered pairs. If we know what the operation adding $^-5$ is [that is, if we know the ordered pairs which belong to it], then we know what the inverse of this operation is. It is simply the set of ordered pairs which are obtained by inter-changing the components of each ordered pair in adding $^-5$. We use the expression 'subtracting $^-5$' as a name for the operation which is the inverse of adding $^-5$. But, if we examine the pairs in the operation subtracting $^-5$ we find that this operation is the operation adding $^+5$. So, subtracting $^-5$ is precisely the same operation as adding $^+5$.

TC[1-73, 74, 75]

B.   Fill in the blanks to make true sentences.

1.   (___ + ⁺2) + ⁻2 = ⁺5   2.   (___ + ⁺3) + ⁻3 = ⁻1

3.   (___ + ⁻4) + ⁺4 = ⁻8   4.   (___ + ⁻8) + ⁺8 = ⁺4

5.   (___ + ⁺6) + ⁻6 = ⁻73   6.   (___ + ⁻51) + ⁺51 = ⁺824

7.   (⁺7 + ⁻3) + ⁺3 = ____   8.   (⁺12 + ⁺5) + ⁻5 = ____

9.   (⁻3 + ⁺8) + ⁻8 = ____   10. (⁻7 + ⁻15) + ⁺15 = ____

11. (⁺9.3 + ⁻2.1) + ⁺2.1 = ____   12. (⁻73 + ⁻58) + ⁺73 = ____

13. (⁺14 + ⁻35) + ____ = ⁺14   14. (⁻17 + ⁺21) + ____ = ⁻17

15. (⁻31 + ⁺65) + ____ = ⁻31   16. (⁺72 + ⁺100) + ____ = ⁺100

17. (⁺19 + ____) + ____ = ⁺19   18. (⁻35 + ____) + ____ = ⁻35

19. (___ + ⁺72) + ⁻72 = ____   20. (___ + ⁻18) + ⁺18 = ____

21. (___ + ⁻57) + ⁺57 = ____   22. (___ + 0) + 0 = ____

23. If you add ⁺8 to a number, you can get back the number by
     adding _____ to the sum.

          Example 1.   ⁻2 + ⁺8 = ⁺6 and ⁺6 + ___ = ⁻2.

          Example 2.   (⁻384 + ⁺8) + ____ = ⁻384.

24. If you add ⁻5 to a number, you can get back the number by
     adding ____ to the sum.

          Example 1.   ⁻11 + ⁻5 = ⁻16 and ⁻16 + ____ = ⁻11.

          Example 2.   (⁺384 + ⁻5) + ____ = ⁺384.

[Note: Since ⁺73 + ⁻73 = 0, ⁻73 is called the opposite of ⁺73. Since
     ⁻73 + ⁺73 = 0, ⁺73 is called the opposite of ⁻73. In general,
     pairs of real numbers whose sum is 0 are called opposites,
     and each is the opposite of the other. How do we know that
     ⁻10 is the opposite of ⁺10? Because our rule for addition
     tells us that ⁺10 + ⁻10 = 0. Each real number has an oppo-
     site. What is the opposite of 0?]

                    (continued on next page)

25. If you add ____ to a number, you can get back the number by adding the opposite of $^+3$ to the sum.

26. If you add ____ to a number, you can get back the number by adding the opposite of $^-3$ to the sum.

27. If you add $^-9$ to a number, you can get back the number by adding the opposite of ____ to the sum.

28. If you add $^+9$ to a number, you can get back the number by adding the opposite of ____ to the sum.

29. If you want to undo the result of adding ____, add the opposite of $^-7$ to the sum.

  Example 1. $(^+36 + ____) + ^+7 = ^+36$.

  Example 2. $(^+36 + ____) +$ the opposite of $^-7 = ^+36$.

30. Adding ____ undoes what adding $^-5$ did.

  Example. $(^+42 + ^-5) + ____ = ^+42$.

31. Adding the opposite of ____ undoes what adding $^-5$ did.

32. Adding the opposite of ____ undoes what adding $^+11$ did.

33. Adding the opposite of ____ is the inverse of adding $^-17$.

34. Adding the opposite of ____ is the inverse of adding $^+7$.

35. Adding _____ is the inverse of adding $^-2$.

36. The inverse of adding $^-7$ is adding _____.

  Example. $(^-382 + ^-7) + ____ = ^-382$.

37. The inverse of adding $^+24$ is adding _____.

  Example. $(^+15 + ^+24) + ____ = ^+15$.

38. The inverse of adding a real number is adding _____.

1.09 Subtraction of real numbers. --As with numbers of arithmetic, we shall use the word 'subtracting' in naming the operation which is the inverse of adding a given real number. So, for example, we shall say that

subtracting $^-5$ is the inverse of adding $^-5$

and that

subtracting $^+4$ is the inverse of adding $^+4$.

Now, let us solve a subtraction problem:

$$^+9 - {}^-5 = ?$$

This is a problem in subtracting $^-5$. We can solve this problem by thinking in terms of the operation adding $^-5$ and its inverse.

adding $^-5$		subtracting $^-5$	
$(^+7, {}^+2)$	$(^+8, {}^+3)$	$(^+2, {}^+7)$	$(^+3, {}^+8)$
$(^-2, {}^-7)$	$(^-3, {}^-8)$	$(^-7, {}^-2)$	$(^-8, {}^-3)$
$(^+4, {}^-1)$	$(^+10, {}^+5)$	$(^-1, {}^+4)$	$(^+5, {}^+10)$
$(^+12, {}^+7)$	$(^+14, {}^+9)$	$(^+7, {}^+12)$	$(^+9, {}^+14)$
$(^+27, {}^+22)$	$(^+1, {}^-4)$	$(^+22, {}^+27)$	$(^-4, {}^+1)$
$(^-7, {}^-12)$	$(^+9, {}^+4)$	$(^-12, {}^-7)$	$(^+4, {}^+9)$
$(^+22, {}^+17)$	$(^+2, {}^-3)$	$(^+17, {}^+22)$	$(^-3, {}^+2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
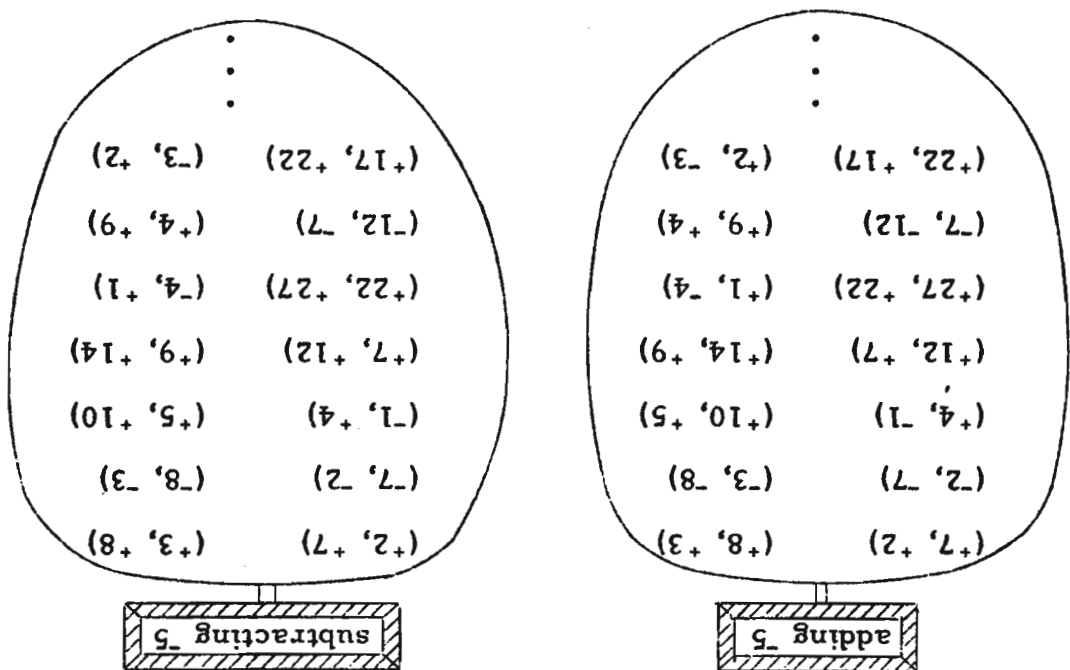
We go to the list for subtracting $^-5$, and look for the pair whose first number is $^+9$. Its second number is $^+14$. So,

$$^+9 - {}^-5 = {}^+14.$$

Check these subtraction problems.

(a) $^+22 - {}^-5 = {}^+27$ (b) $^-7 - {}^-5 = {}^-2$

(c) $^-3 - {}^-5 = {}^+2$ (d) $^+4 - {}^-5 = {}^+9$

EXERCISES

A. 1. Make a list of ten pairs which belong to the operation adding ⁻3.

2. Use these ten pairs to make a list of pairs which belong to the inverse of adding ⁻3, that is, to the operation subtracting ⁻3.

3. Make a list of ten pairs which belong to the operation adding ⁺3.

4. Examine the lists you get in Exercises 2 and 3. What do they suggest about the operations of subtracting ⁻3 and adding ⁺3? Could you have predicted this without making the lists?

B. 1. Here is a list of pairs which belong to the operation adding ⁺7. Use it to construct a list of pairs which belong to the operation adding ⁻7.


```
 adding ⁺7              adding ⁻7

(⁺9, ⁺16)    (⁺2, ⁺9)

(⁺1, ⁺8)     (⁻3, ⁺4)

(⁻8, ⁻1)     (⁻10, ⁻3)

(0, ⁺7)      (⁻7, 0)

(⁺4, ⁺11)    (⁻17, ⁻10)

(⁻1, ⁺6)     (⁺7, ⁺14)

      (⁻4, ⁺3)
         .
         .
         .
```
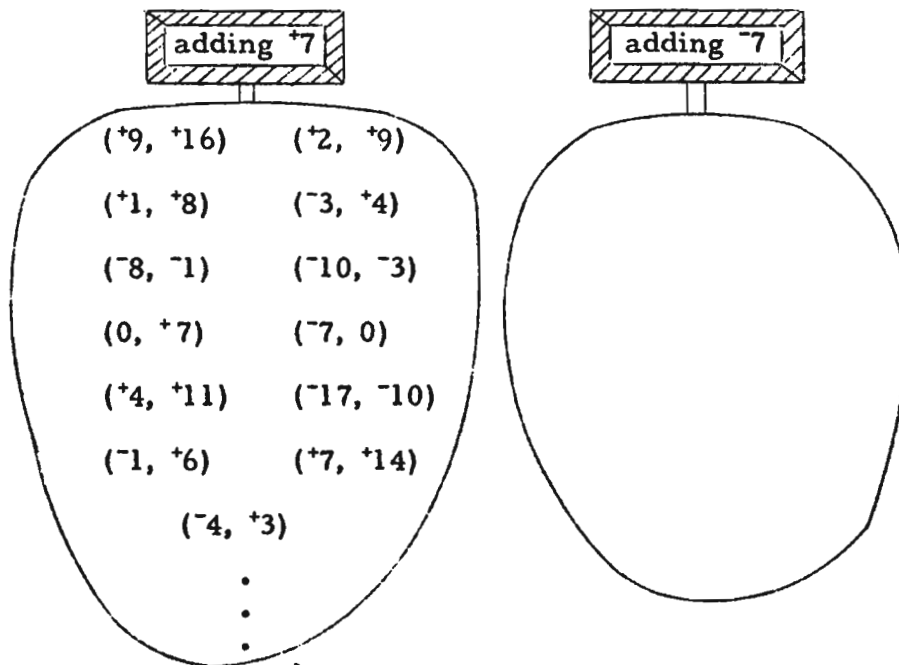
2. Solve each problem by using the list for adding ⁺7. Then, solve each problem again by using the list for adding ⁻7.

(a) ⁺4 - ⁺7 = ?   (b) 0 - ⁺7 = ?

(c) ⁻1 - ⁺7 = ?   (d) ⁺7 - ⁺7 = ?

Answers for Part A.

1. $(^+10, \,^+7)\ (^+13, \,^+10)\ (^-8, \,^-11)\ (^+28, \,^+25)\ (^-2, \,^-5)\ (^+2, \,^-1)$ etc.

2. $(^+7, \,^+10)\ (^+10, \,^+13)\ (^-11, \,^-8)\ (^+25, \,^+28)\ (^-5, \,^-2)\ (^-1, \,^+2)$ etc.

3. Same list as for Exercise 2 may be used.

4. They suggest that these two operations are the same. This could have been predicted because the apa, the fact that $^-3 + \,^+3 = 0$, and the pa0 tell one that adding $^+3$ undoes what adding $^-3$ did [e. g., $^+10 + \,^-3 + \,^+3 = \,^+10$]--that is, that adding $^+3$ is the inverse of adding $^-3$. So, adding $^+3$ is subtracting $^-3$.

✳

Answers for Part B [on pages 1-76 and 1-77].

1. Here is a list of pairs that belong to the operation adding $^-7$:

$(^+16, \,^+9),\ (^+9, \,^+2),\ (^+8, \,^+1),\ (^+4, \,^-3),\ (^-1, \,^-8),\ (^-3, \,^-10),\ (^+7, \,0),$

$(0, \,^-7),\ (^+11, \,^+4),\ (^-10, \,^-17),\ (^+6, \,^-1),\ (^+14, \,^+7),\ (^+3, \,^-4).$

2. [In Exercise 2 of Part B students are to find the answer to each question in two ways; first using the list for adding $^+7$, then using the list for adding $^-7$. Sample: To find $^+4 - \,^+7$, since subtracting $^+7$ is the inverse of adding $^+7$, we look for an ordered pair belonging to adding $^+7$ whose <u>second</u> component is $^+4$. The only such pair is $(^-3, \,^+4)$. So, $^+4 - \,^+7 = \,^-3$. Alternatively, since subtracting $^+7$ is adding $^-7$, we look for an ordered pair in adding $^-7$ whose <u>first</u> component is $^+4$. The only such pair is $(^+4, \,^-3)$. So, $^+4 - \,^+7 = \,^-3.$]

(a) $^-3$	(b) $^-7$	(c) $^-8$	(d) 0

3.

(a) $^+5$	(b) $^+8$	(c) $^+93$	(d) $^-14$
(e) $^-28$	(f) $^+36$	(g) $^-107$	(h) $^-22$

3. Now solve these problems all of which involve <u>subtracting $^{+}7$</u>.

   (a) $^{+}12 - ^{+}7 = ?$ (b) $^{+}15 - ^{+}7 = ?$

   (c) $^{+}100 - ^{+}7 = ?$ (d) $^{-}7 - ^{+}7 = ?$

   (e) $^{-}21 - ^{+}7 = ?$ (f) $^{+}43 - ^{+}7 = ?$

   (g) $^{-}100 - ^{+}7 = ?$ (h) $^{-}15 - ^{+}7 = ?$

* * *

The problem:

$$^{-}9 - ^{+}8 = ?$$

involves subtracting $^{+}8$. Subtracting $^{+}8$ is the inverse of adding $^{+}8$.
And, as you have seen, the inverse of adding $^{+}8$ is adding $^{-}8$, that
is,

   the inverse of adding $^{+}8$ is <u>adding the opposite of</u> $^{+}8$.
So,

   subtracting $^{+}8$ is <u>the same as</u> adding the opposite of $^{+}8$.

Hence, the subtraction problem:

$$^{-}9 - ^{+}8 = ?$$

can be <u>converted</u> into the addition problem:

$$^{-}9 + ^{-}8 = ?$$

Therefore,

$$^{-}9 - ^{+}8 = ^{-}9 + ^{-}8 = ^{-}17.$$

* * *

Solve these subtraction problems.

   <u>Sample</u>. $^{+}8 - ^{-}3 = ?$

      <u>Solution</u>. Since subtracting $^{-}3$ and adding $^{+}3$ are both the
               inverse of adding $^{-}3$, subtracting $^{-}3$ is the
               same as adding $^{+}3$. So,
                  $^{+}8 - ^{-}3 = ^{+}8 + ^{+}3 = ^{+}11.$

1. ${}^{+}12 - {}^{-}3 = ?$ 2. ${}^{-}5 - {}^{+}4 = ?$ 3. ${}^{-}11 - {}^{+}12 = ?$

4. ${}^{-}6 - {}^{-}9 = ?$ 5. ${}^{+}8 - {}^{+}10 = ?$ 6. $8 - 10 = ?$

7. $15 - 3 = ?$ 8. $15 - {}^{-}3 = ?$ 9. ${}^{-}15 - 3 = ?$

D. Simplify as quickly as possible.

1. ${}^{+}3 - {}^{+}2$ 2. ${}^{+}12 - {}^{+}13$ 3. ${}^{+}4 - {}^{+}17$

4. ${}^{+}8 - {}^{+}8$ 5. ${}^{+}3 - {}^{+}8$ 6. $4 - 17$

7. ${}^{+}5 - {}^{-}2$ 8. ${}^{+}12 - {}^{-}10$ 9. ${}^{+}12 - {}^{-}12$

10. ${}^{-}3 - {}^{+}4$ 11. ${}^{-}3 - 4$ 12. ${}^{-}5 - {}^{+}7$

13. ${}^{-}7 - {}^{-}4$ 14. ${}^{-}9 - {}^{-}11$ 15. ${}^{-}12 - {}^{-}12$

16. ${}^{-}14 - {}^{-}7$ 17. ${}^{-}14 - 6$ 18. $6 - {}^{-}14$

19. $8 - 2$ 20. $8 - 1$ 21. $8 - 0$

22. $8 - {}^{-}1$ 23. $8 - {}^{-}2$ 24. $8 - {}^{-}4$

25. ${}^{-}5 - {}^{-}3$ 26. ${}^{-}5 - {}^{-}2$ 27. ${}^{-}5 - 1$

28. ${}^{-}5 - 0$ 29. ${}^{-}5 - 1$ 30. ${}^{-}5 - 2$

31. $8 - {}^{-}3$ 32. $5 - {}^{-}4$ 33. $12 - {}^{-}8$

34. $7 - 9$ 35. $11 - 7$ 36. $13 - {}^{-}5$

37. ${}^{-}17 - {}^{-}2$ 38. $7 - {}^{-}3$ 39. $9 - 18$

40. ${}^{-}5 - 6$ 41. ${}^{-}5 - {}^{-}7$ 42. ${}^{-}7 - 5$

E. 1. Suppose '9' and '12' are numerals for numbers of arith-
metic. Which of the following is a numeral?

(a) $12 - 9$ (b) $9 - 12$

2. Suppose '9' and '12' are numerals for real numbers. Which
of the following is a numeral?

(a) $12 - 9$ (b) $9 - 12$

Answers for Part C [which begins on page 1-77].

1. $^+12 - {}^-3 = {}^+12 + {}^+3 = {}^+15$ 2. $^-5 - {}^+4 = {}^-5 + {}^-4 = {}^-9$

3. $^-11 - {}^+12 = {}^-11 + {}^-12 = {}^-23$ 4. $^-6 - {}^-9 = {}^-6 + {}^+9 = {}^+3$

5. $^+8 - {}^+10 = {}^+8 + {}^-10 = {}^-2$ 6. $8 - 10 = 8 + {}^-10 = {}^-2$

7. $15 - 3 = 15 + {}^-3 = 12$ 8. $15 - {}^-3 = 15 + {}^+3 = 18$

9. $^-15 - 3 = {}^-15 + {}^-3 = {}^-18$

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Answers for Part D.

1. $^+1$	2. $^-1$	3. $^-13$	4. 0	5. $^-5$
6. $^-13$	7. $^+7$	8. $^+22$	9. $^+24$	10. $^-7$
11. $^-7$	12. $^-12$	13. $^-3$	14. $^+2$	15. 0
16. $^-7$	17. $^-20$	18. 20	19. 6	20. 7
21. 8	22. 9	23. 10	24. 12	25. $^-2$
26. $^-3$	27. $^-4$	28. $^-5$	29. $^-6$	30. $^-7$
31. $^+11$	32. 9	33. 20	34. $^-2$	35. 4
36. 18	37. $^-15$	38. 10	39. $^-9$	40. $^-11$
41. $^+2$	42. $^-2$			

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Answers for Part E.

1. (a)   2. (a), (b)

✖

Here is a quiz which tests the ability to add and subtract real numb

I. Simplify.

1. $^+3 - {}^-2$ 2. $^+7 - {}^+8$ 3. $^-5 + {}^+8$ 4. $^-3 - 0$

5. $6 - {}^-8$ 6. $12 - {}^+10$ 7. $^-2 - {}^+8$ 8. $^+7 - {}^-7$

II. Fill the blanks to make true sentences.

1. $^+9 - \underline{\quad} = {}^+7$ 2. $^-3 + \underline{\quad} = {}^+3$ 3. $\underline{\quad} - {}^+7 = {}^+7$

4. $^-6 - \underline{\quad} = {}^-6$ 5. $^+3 + \underline{\quad} = {}^-2$ 6. $^-7 - \underline{\quad} = {}^+7$

III. <u>Multiple-choice.</u> Draw a loop around the correct answer.

1. If you subtract a positive number <u>from</u> a negative number, the result is _____.

(A) a positive number   (B) 0   (C) a negative number

2. I add a real number to a positive number and get 0 as the sum. If I subtract the real number from the positive number, the result would be _____.

(A) a positive number   (B) 0   (C) a negative number

3. Mary and John each pick a real number. The difference of Mary's number from John's is a negative number. What is the difference of John's number from Mary's?

(A) a positive number   (B) 0   (C) a negative number

✳

Answers for quiz.

I. 1. $^+5$ 2. $^-1$ 3. $^+3$ 4. $^-3$

    5. 14   6. 2   7. $^-10$ 8. $^+14$

II. 1. $^+2$ 2. $^+6$ 3. $^+14$ 4. 0

    5. $^-5$ 6. $^-14$

III. 1. a negative number   2. a positive number

    3. a positive number

In discussing the second part of the Solution in Part F you [or one of your students] should fill in the missing steps:

$$5 + {}^+3 + {}^-8 + 7 + {}^-2$$
$$= (5 + 3) + ({}^-8 + 7) + {}^-2 \quad\left.\right\}\text{apa}$$
$$= (5 + 3) + (7 + {}^-8) + {}^-2 \quad\left.\right\}\text{cpa}$$
$$= (5 + 3 + 7) + {}^-8 + {}^-2 \quad\left.\right\}\text{apa}$$
$$= (5 + 3 + 7) + ({}^-8 + {}^-2) \quad\left.\right\}\text{apa}$$
$$= 5 + (3 + 7) + ({}^-8 + {}^-2). \quad\left.\right\}\text{apa}$$

The second step is an application of the commutative principle for addition and the others are applications of the associative principle for addition. In doing the exercises of Part F, students should <u>not</u> be required to go through a detailed justification like the above.

<p align="center">*</p>

Answers for Part F [on pages 1-79 and 1-80].

1.	⁻3	2.	⁻22	3.	⁻4	4.	15	5.	⁻32
6.	⁻27	7.	15	8.	⁻1	9.	⁻8	10.	2

In Exercises 11-16, students will not know that, for example, $-({}^-2 + {}^+5) = {}^+2 + {}^-5$. [We hope they will know this after they have finished Exercise 13 on page 1-86.] So, students will solve Exercise 15, for example, as follows.

$${}^-2 - ({}^-8 + {}^+2) - ({}^-3 + {}^-2) - ({}^-5 + {}^-8) + ({}^-7 - {}^-9)$$
$$= {}^-2 - {}^-6 - {}^-5 - {}^-13 + {}^+2$$
$$= {}^-2 + {}^+6 + {}^+5 + {}^+13 + {}^+2$$

<p align="center">etc.</p>

11.	⁻12	12.	⁻5	13.	⁻7	14.	⁻1	15.	24

F.   Since subtracting a real number is precisely the same thing as adding its opposite [the principle for subtraction], every subtraction problem can be converted into an addition problem.

> Sample.   5 - ⁻3 + ⁻8 + 7 - ⁺2
>
> Solution.   5 - ⁻3 + ⁻8 + 7 - ⁺2
>
>   = 5 + ⁺3 + ⁻8 + 7 + ⁻2,

At this point you can simplify either by working from left to right:

$$5 + {}^+3 + {}^-8 + 7 + {}^-2$$
$$= \quad 8 + {}^-8 + 7 + {}^-2$$
$$= \qquad 0 + 7 + {}^-2$$
$$= \qquad\quad 7 + {}^-2$$
$$= \qquad\qquad 5,$$

or by using the commutative and associative principles for addition of real numbers to change the order:

$$5 + {}^+3 + {}^-8 + 7 + {}^-2$$
$$= (5 + 3 + 7) + ({}^-8 + {}^-2)$$
$$= \quad 15 \quad + \quad {}^-10$$
$$= \quad 5,$$

Simplify.

1. ⁻3 + 8 - ⁻2 - ⁺7 + ⁻3   2. ⁻5 + ⁻3 - ⁺7 + 9 + ⁻16

3. ⁺2 + ⁻4 + ⁻3 + ⁻5 - ⁻6   4. 8 - ⁻5 + ⁻3 - ⁻6 - ⁻7 - ⁺8

5. 3 - 15 + ⁻2 - 17 + ⁻1   6. ⁻1 + ⁻5 - 6 - 3 - 10 + ⁻2

7. ⁻2 + ⁻3 - ⁻5 - ⁻5 + 10   8. ⁻4 - ⁻6 + ⁻3 - 8 - ⁻9 + ⁻1

9. 11 - 15 + 7 - 9 - 3 + 1   10. 1 - ⁻2 - 6 - ⁻9 - ⁻3 - 7

11. $^-9 - (^-2 + ^+5)$ 12. $^-3 - (^-4 - ^-6)$

13. $(^-1 + ^-3) - (^-5 - ^-8)$ 14. $(6 - ^-12) - (4 - 9)$

15. $^-2 - (^-8 + ^+2) - (^-3 + ^-2) - (^-5 + ^-8) + (^-7 - ^-9)$

[More exercises are in Part J, Supplementary Exercises.]

1.10 Opposites. --In studying the system of numbers of arithmetic
and the system of real numbers, we have noticed certain similarities.
For example, we have seen that the nonnegative numbers act like the
numbers of arithmetic with respect to addition.   What are some other
similarities?

        There is, also, an important difference between the two systems.
It concerns the operation of subtraction.   Can you tell what this dif-
ference is?

        If you pick two numbers of arithmetic, a first number and a
second number, can you subtract the first number from the second
number?   Can you subtract 9 from 11?   Can you subtract 6 from 2?
Notice that there are cases in which you cannot subtract the first
selected number from the second.   So, we say that subtraction is
not always possible in the system of numbers of arithmetic.

        Is subtraction always possible in the system of real numbers?
Can you subtract $^+9$ from $^+11$? $^+6$ from $^+2$? $^-5$ from $^-8$?   To subtract
a real number is to add its opposite.   Since addition is always pos-
sible, and since each real number has an opposite, subtraction is
always possible in the real number system.

        An important fact about the real numbers is that, for each real
number, there is a real number [the opposite of the first] which
when added to the first gives the sum 0.   This fact is implied by
the principle of opposites--a number plus its opposite is 0.   You
will see in Unit 2 that it follows from the principle of opposites
together with other principles that if the sum of a first number and
a second number is 0, the second number is the opposite of the
first.

        What real number is the opposite of $^-4$?   It is $^+4$, because
$^-4 + ^+4 = 0$.   What is the opposite of $^+4$?   [What principle tells you
that if a first number is the opposite of a second then the second
is the opposite of the first?]

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The last two paragraphs on page 1-80 foreshadow the "algebraic" point of view of Unit 2. The following discussion is intended to show you the direction in which we are going, and so enable you to correct any misconceptions on the part of your students.

At this point students may have the idea that the opposite of a positive number is the corresponding negative number, the opposite of a negative number is the corresponding positive number, the opposite of 0 is 0, and that, "consequently" each real number has one, and only one, opposite. This is, of course, correct. But, the point of view which they need to acquire is subtly different from this, and can be expressed by saying [see note at bottom of page 1-73] that

for each real number there is one, and only one, real number which when added to the first gives the sum 0,

and that this second real number is called 'the opposite' of the first.

Part of this point of view is embodied in the principle of opposites which, in the language of Unit 2 can be stated as:

(1)   for each real number x, $x + -x = 0$.

The remainder can be stated as:

(2) for each real number x, (and) for each real number y,
             if $x + y = 0$ then $y = -x$.

These two statements are made, somewhat less formally, in the first of the two paragraphs under discussion. As remarked there, (2) is a consequence of (1) together with the cpa, the apa, and the pa0. [Here is a sketchy proof: If $x + y = 0$ then $x + y + -x = 0 + -x$, so $y + x + -x = -x + 0$, $y + (x + -x) = -x + 0$, $y + 0 = -x + 0$, and $y = -x$. Don't give this to your class. They will get a more adequate treatment in Unit 2 after they have learned to appreciate it. Any attempt to foreshadow this now will be a source of confusion.] It is important that students grasp the content of (2) as preparation for exercises like those in Part C on page 1-84. [But, of course, they are not prepared to see (2) as written above.]

The answers to the questions in the last paragraph are:

       (1) "What is the opposite of ⁺4?"--We have just mentioned that ⁻4 + ⁺4 = 0. So, by the cpa, we know that ⁺4 + ⁻4 = 0. Hence, ⁻4 is the opposite of ⁺4.

       (2) "What principle ... first?"--Basically, the cpa is the principle we need.

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The operation oppositing is a <u>singulary</u> operation; the operations add-
ing and multiplying are <u>binary</u> operations. A binary operation is one
which is applied to an ordered pair of elements in a set; a singulary
operation is one which is applied to a single element in a set. Other
singulary operations are finding the square root of, finding the recip-
rocal of, finding the logarithm of, adding 2, multiplying by 7, etc.

The three examples near the bottom of page 1-81 suggest that the
operation oppositing could also be called 'subtracting <u>from</u> 0'. We
don't call it this because in our development of the real number system
oppositing is more primitive than subtracting. In fact, subtracting is
defined in terms of oppositing [as expressed by the principle for sub-
traction].

THE OPERATION OPPOSITING

Finding the opposite of a real number is an operation just as adding $^+4$ or multiplying by $^-2$ are operations. Here are some pairs of real numbers which belong to the operation <u>oppositing</u>.
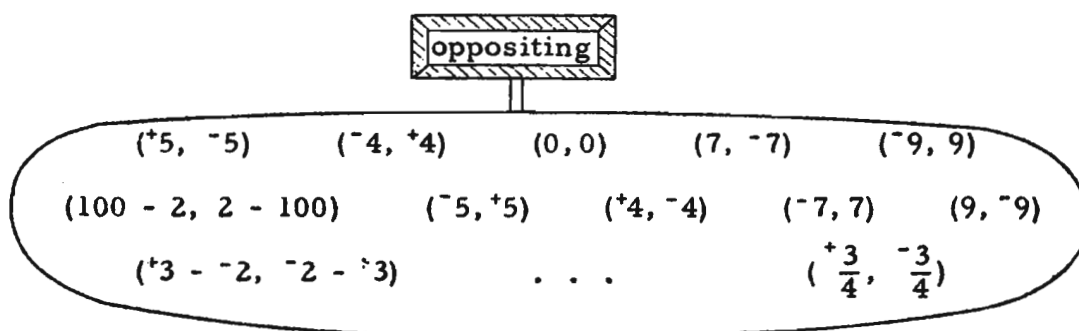
List some pairs which belong to the <u>inverse of oppositing</u>. What can you say about oppositing and its inverse?

Just as we have signs for other operations [×, +, -, ÷], we should like to have a sign for oppositing. We might use a '☆'. Then '☆ $^+4$' would mean the opposite of $^+4$. So, we would have statements like:

$$☆\,^+4 = \,^-4\,,$$

$$☆\,^-11 = \,^+11,$$

$$^+4 - \,^-3 = \,^+4 + ☆\,^-3,$$

$$0 - \,^+5 = ☆\,^+5,$$

$$0 - \,^-12 = ☆\,^-12,$$

$$0 - (^+3 - \,^-8) = ☆\,(^+3 - \,^-8).$$

The last three examples suggest the notation that most people use for oppositing. It is just to write a minus sign. So, following this practice, we shall write '$-\,^+4$' [read as 'the opposite of positive four'], to mean the opposite of $^+4$, and '$-\,^-11$' when we mean the opposite of $^-11$. Similarly, the expression:

$$-\,(^+3 - \,^-8)$$

which is a name for the opposite of ($^+3$ - $^-8$) can be simplified to:

$$-(^+11)$$

which is finally simplified to:

$$^-11.$$

EXERCISES

<u>A.</u>   Use this new notation to write a name for the opposite of each given number.

1. $^+8$ - $^-7$ [Answer: $-(^+8 - ^-7)$]

2. $^-3 + ^+5$ 3. $^+5 - ^-6$ 4. $^-2 \times ^-3 + ^-5$

5.   6 - 2   6.   9 - 15   7.   3 - 5 + 6 × (5 + 3)

<u>B.</u>   Read aloud first, and then simplify.

   <u>Sample.</u> $-(12 - 9) + (9 - 12)$

   <u>Solution.</u>   Read as 'the opposite of the difference of 9 from 12, plus the difference of 12 from 9'.

$$-(12 - 9) + (9 - 12)$$
$$= -^+3 + ^-3$$
$$= ^-3 + ^-3$$
$$= ^-6.$$

1. $-^+8$ 2. $-^-15$ 3. $-(^+5 - ^-7)$

4. $-(^+2 \times ^-3)$ 5. $-(55 - 30)$ 6. $-55 + 30$

7. $-(^-3 \times ^-7)$ 8. $-(10 - 12)$ 9. $-^+10 + 12$

10. $-^+10 - 12$ 11. $--^+7$ 12. $-(5 - 7) + (5 - 7)$

13. $---^-3$ 14. $-^-7 + -^+7$ 15. $--(^-3 - -^+3)$

16. $-(93 - 97)$ 17. $--(84 - 89)$ 18. $--(100 - 101)$

19. $-(^+9 + ^-7)$ 20. $-(^-9 + ^+7)$ 21. $-(^-3 + ^-6)$

   22. $-(983 + 729 - 604) + (983 + 729 - 604)$

   23. $(3572 - 4871) + (4871 - 3572)$

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Answers for Part A.

2. $-(^-3 + ^+5)$ 3. $-(^+5 - ^-6)$ 4. $-(^-2 \times ^-3 + ^-5)$

5. $-(6 - 2)$ 6. $-(9 - 15)$ 7. $-[3 - 5 + 6 \times (5 + 3)]$

*

As a general rule, the "scope" of a symbol for a singulary operation is taken to be as small as makes sense. Thus,

$\quad$ '$-^-2 + ^+3$' means the same as '$(-^-2) + ^+3$'

$\quad$ '$-^-2 \times ^+3$' means the same as '$(-^-2) \times ^+3$'

$\quad$ '$-(^+2 + ^-3) \times ^+5$' means the same as '$[-(^+2 + ^-3)] \times ^+5$'

$\quad$ '$-^+10 - ^-12$' means the same as '$(-^+10) - ^-12$'

$\quad\quad\quad\quad$ etc.

Another example, not yet familiar to your students, is the squaring operation denoted by '^{2}'.

$\quad$ '$^+3 \times ^-2^2$' means the same as '$^+3 \times (^-2^2)$'

$\quad$ '$^-3 + ^+2^2$' means the same as '$^-3 + (^+2^2)$'.

When, in applying two singulary operations, the operator for one is a prefix and the operator for the other is a suffix as in '-3^2', one must either use grouping symbols or adopt a convention. Thus, by convention,

$\quad$ '-3^2' means the same as '$-(3^2)$'

$\quad\quad$ [rather than '$(-3)^2$'].

*

Answers for Part B.

1. the opposite of positive 8; $^-8$

2. the opposite of negative 15; $^+15$

3. the opposite of the difference of negative 7 from positive 5; $^-12$

4. the opposite of the product of positive 2 by negative 3; $^+6$

5. the opposite of the difference of 30 from 55; $^-25$

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6. the sum of the opposite of 55 and 30 [or: the opposite of 55, plus 30]; ⁻25

7. the opposite of the product of negative 3 by negative 7; ⁻21
[A good exercise to follow Exercise 7 of Part B would be:

$$-\,{}^{-}3 \times {}^{-}7.$$

Students should see that this is equivalent to the expression given in Exercise 7, and they should contrast this situation with the one in Exercises 5 and 6.]

8. the opposite of the difference of 12 from 10; ⁺2

9. the sum of the opposite of positive 10 and 12; 2

10. the difference of 12 from the opposite of positive 10; ⁻22

11. the opposite of the opposite of positive 7; ⁺7

12. the sum of the opposite of the difference of 7 from 5, and the difference of 7 from 5; 0
[You may have difficulty getting your students to read the expression as we have suggested here [particularly in view of the Solution of the Sample]. At first attempt a student will probably say 'the opposite of the difference of 7 from 5, plus the difference of 7 from 5'. Accept this, but then ask, 'Is this principally a problem of addition? Or, is it a problem of subtraction?' [In an expression like the one given in Exercise 12, the '+' is considered the principal operator.] The students will probably agree that it is primarily a problem of adding two differences. Then ask, 'What do we call the answer to an addition problem?' After someone replies, 'the sum', you may ask whether someone could read the given expression in such a way as to indicate to the listener that it is a problem in finding a sum. You may need to copy '−(5 - 7) + (5 - 7)' on the board; then help the class read it by pointing to the '+' and saying 'the sum of'. Next point to the ' − ' in ' −(5 - 7)' and say 'the opposite of', etc. [This same technique may be necessary for obtaining a careful reading of the expressions given in Exercises 22 and 23.]]

13. the opposite of the opposite of the opposite of negative 3; ⁺3

14. the sum of the opposite of negative 7 and the opposite of positive 7; 0

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15. the opposite of the opposite of the difference of the opposite of positive 3 from negative 3; 0

16. the opposite of the difference of 97 from 93; 4

17. the opposite of the opposite of the difference of 89 from 84; ⁻5

18. the opposite of the opposite of the difference of 101 from 100; ⁻1

19. the opposite of the sum of positive 9 and negative 7; ⁻2

20. the opposite of the sum of negative 9 and positive 7; ⁺2

21. the opposite of the sum of negative 3 and negative 6; ⁺9

22. the sum of the opposite of the difference of 604 from the sum of 983 and 729, and the difference of 604 from the sum of 983 and 729; 0

23. the sum of the difference of 4871 from 3572 and the difference of 3572 from 4871; 0

✳

It may take a while for you and your students to get used to the readings given in the answers above. However, these readings have the advantage of spelling out the meanings of the expressions. The difference between, say, the conventional reading for Exercise 3:

minus the quantity positive 5 minus negative 7

and the one given in the answer on TC[1-82]a is the difference between "reading the symbols" and reading aloud what the expression says in written form. [A similar distinction, regarding the reading aloud of single-quoted expressions, was called to your attention on TC[1-J]a and, again, on TC[1-L]a.] We have violated our own convictions here by the way in which we stated the Solution of the Sample. Actually, to read aloud what the expression [in the Sample] says in written form, one should say 'the sum of the opposite of the difference of 9 from 12, and the difference of 12 from 9'.

Although we believe that it is worthwhile developing in class the answers given for Part B, it is not necessary to insist on the point made here throughout the course. On this, see the last paragraph on TC[1-88]c.

whic⌐

The examples on page 1-83 illustrate the use of the ideas discussed on TC [1-80] and serve as samples for Part C on page 1-84. The exercises of Part C are all written with an oppositing sign preceding the first numeral, in order to prepare students for the 0-sum theorem of Unit 2. [See page 2-68.] The 0-sum theorem tells us that for each first number and for each second number, if the sum of the first number and the second number is 0, then the opposite of the first number is the second number. Before assigning these exercises, you should work out the second example on page 1-83 [on the board] as follows, showing all the steps.

$$(^-98 + 35) + (98 + ^-35) = [(^-98 + 35) + 98] + ^-35 \qquad \text{[apa]}$$

$$[(^-98 + 35) + 98] + ^-35 = [^-98 + (35 + 98)] + ^-35 \qquad \text{[apa]}$$

$$[^-98 + (35 + 98)] + ^-35 = [^-98 + (98 + 35)] + ^-35 \qquad \text{[cpa]}$$

$$[^-98 + (98 + 35)] + ^-35 = [(^-98 + 98) + 35] + ^-35 \qquad \text{[apa]}$$

$$[(^-98 + 98) + 35] + ^-35 = (^-98 + 98) + (35 + ^-35) \qquad \text{[apa]}$$

$$(^-98 + 98) + (35 + ^-35) = 0 + 0 \qquad \text{[principle of opposites;}$$
$$98 = -^-98; \ ^-35 = -35]$$

$$0 + 0 = 0 \qquad \text{[pa0]}$$

So, since $(^-98 + 35) + (98 + ^-35) = 0$, it follows that $-(^-98 + 35) = 98 + ^-35$.

Answers for representative exercises in Part C [on page 1-84].

1. $(8 + 5) + (^-8 + ^-5) = [(8 + ^-8) + 5] + ^-5$ [apa, cpa]

   $[(8 + ^-8) + 5] + ^-5 = [0 + 5] + ^-5$ [principle of opposites; $^-8 = -8$]

      $[0 + 5] + ^-5 = 0 + (5 + ^-5)$ [apa]

        $0 + (5 + ^-5) = 0 + 0$ [principle of opposites; $^-5 = -5$]

           $0 + 0 = 0$ [pa0]

So, since $(8 + 5) + (^-8 + ^-5) = 0$, it follows that $-(8 + 5) = ^-8 + ^-5$.

3. $(38 - 16) + (^-38 + 16) = (38 + -16) + (^-38 + 16)$ [principle for subtraction]

  $(38 + -16) + (^-38 + 16) = [(38 + ^-38) + -16] + 16$ [apa, cpa]

  $[(38 + ^-38) + -16] + 16 = [0 + -16] + 16$ [principle of opposites; $^-38 = -38$]

      $[0 + -16] + 16 = 0 + (-16 + 16)$ [apa]

      $0 + (-16 + 16) = 0 + (16 + -16)$ [cpa]

      $0 + (16 + -16) = 0 + 0$ [principle of opposites]

           $0 + 0 = 0$ [pa0]

So, since $(38 - 16) + (^-38 + 16) = 0$, it follows that $-(38 - 16) = ^-38 + 16$.

* * *

Consider the numbers

$$^-8 - {}^+4 \qquad \text{and} \qquad {}^+8 - {}^-4.$$

Is the second number, ${}^+8 - {}^-4$, the opposite of the first, ${}^-8 - {}^+4$? One way to find the answer to this question is to add the second number to the first. If the sum is 0, the answer is 'yes'. If the sum is not 0, the answer is 'no'. Let's try it.

$$(^-8 - {}^+4) + ({}^+8 - {}^-4) = (^-8 + {}^-4) + ({}^+8 + {}^+4) \qquad \text{[principle for sub-}$$
$$\text{traction; } -{}^+4 = {}^-4;\ -{}^-4 = {}^+4]$$

$$(^-8 + {}^-4) + ({}^+8 + {}^+4) = [(^-8 + {}^+8) + {}^-4] + {}^+4 \qquad \text{[apa, cpa]}$$

$$[(^-8 + {}^+8) + {}^-4] + {}^+4 = [0 + {}^-4] + {}^+4 \qquad \text{[principle of opposites; } {}^+8 = -{}^-8]$$

$$[0 + {}^-4]\ + {}^+4 \quad = 0 + ({}^-4 + {}^+4) \qquad \text{[apa]}$$

$$0 + ({}^-4 + {}^+4) \quad = 0 + 0 \qquad \text{[principle of opposites; } {}^+4 = -{}^-4]$$

$$0 + 0 \quad = \quad 0 \qquad \text{[principle for adding 0]}$$

So, since

$$(^-8 - {}^+4) + ({}^+8 - {}^-4) = 0,$$

it follows that ${}^+8 - {}^-4$ is the opposite of ${}^-8 - {}^+4$. That is, that

$$^+8 - {}^-4 = -(^-8 - {}^+4).$$

[Is ${}^-8 - {}^+4$ the opposite of ${}^+8 - {}^-4$?]

Consider a second example. Is it the case that

$$-(^-98 + 35) = 98 + {}^-35?$$

To find the answer to this question, we can proceed as before and find out whether $(^-98 + 35) + (98 + {}^-35) = 0$.

$$(^-98 + 35) + (98 + {}^-35) = (^-98 + 98) + (35 + {}^-35) \qquad \text{[Why?]}$$

$$(^-98 + 98) + (35 + {}^-35) = 0 \qquad \text{[Why?]}$$

So, $(98 + {}^-35) = -(^-98 + 35)$.

* * *

C. For each exercise, use the ''adding method'' and the principle of opposites to check each sentence.

  1. $-(8 + 5) = {}^-8 + {}^-5$ [Hint: Show that $(8 + 5) + ({}^-8 + {}^-5) = 0.$]

  2. $-(95 + {}^-27) = {}^-95 + 27$

  3. $-(38 - 16) = {}^-38 + 16$

  4. $-({}^-57 - {}^-9) = 57 - 9$

  5. $-(24 - 30) = 30 - 24$

  6. $-({}^-17 - 15) = 15 - {}^-17$

  7. $-(25 + 20) = -25 + -20$

  8. $-(57 - 12) = {}^-57 + 12$

  9. $-(38 - 10) = 10 - 38$

 10. $-(53 \times {}^-28) = {}^-53 \times {}^-28$

 11. $-74 = 74 \times {}^-1$

★12. $-(38 + {}^-57 - {}^-76) = -38 - {}^-57 + {}^-76$

★13. $-[(725 - 631) - (497 - 985)] = (631 - 725) - (985 - 497)$

D. Multiple-choice. [There may be more than one right answer.]

  1. $12 - 3 = ?$

    (a) $3 + {}^-12$ (b) $^-3 + 12$ (c) $-(12 - 3)$ (d) $-(3 - 12)$

  2. $5 + 4 - 10 = ?$

    (a) $5 + 6$ (b) $9 - 10$ (c) $5 - (10 - 4)$ (d) $5 + (4 - 10)$

  3. $5 - 4 - 10 = ?$

    (a) $5 - (4 - 10)$ (b) $5 - (4 + 10)$

    (c) $5 - (10 - 4)$ (d) $-(10 + 4 - 5)$

  4. $3 \times {}^-2$ is the opposite of ?

    (a) $^-3 \times 2$ (b) $-(3 \times {}^-2)$ (c) $-({}^-3 \times 2)$ (d) $-(3 \times 2)$

7. (25 + 20) + (-25 + -20) = [(25 + -25) + 20] + -20   [apa, cpa]

   [(25 + -25) + 20] + -20 = [0 + 20] + -20   [principle of opposites]

   [0 + 20] + -20 = 0 + (20 + -20)   [apa]

   0 + (20 + -20) = 0 + 0   [principle of opposites]

   0 + 0 = 0   [pao]

So, since (25 + 20) + (-25 + -20) = 0, it follows that -(25 + 20) =
-25 + -20.

10. 53 × 28 + -53 × 28 = (53 + -53) × 28   [dpma]

    (53 + -53) × 28 = 0 × 28   [principle of opposites; -53 = -53]

    0 × 28 = -28 × 0   [cpm]

    -28 × 0 = 0   [pmo]

So, since 53 × 28 + -53 × 28 = 0, it follows that -(53 × 28) =
-53 × 28.

11. 74 + 74 × -1 = 74 × 1 + 74 × -1   [pm1]

    74 × 1 + 74 × -1 = 74 × (1 + -1)   [dpma]

    74 × (1 + -1) = 74 × 0   [principle of opposites; 1 = -1]

    74 × 0 = 0   [pmo]

So, since 74 + 74 × -1 = 0, it follows that -74 = 74 × -1.

12. (38 + -57 - 76) + (-38 - 57 + 76) =
    (38 + -57 + -76) + (-38 + 57 + 76)   [prin. for subtraction]

    (38 + -57 + -76) + (-38 + 57 + 76) =
    (38 + -38) + (-57 + 57) + (76 + -76)   [apa, cpa]

    = [0 + 0] + 0   [principle of opposites]

    [0 + 0] + 0 = 0 + 0   [pao]

    0 + 0 = 0   [pao]

So, since (38 + -57 - 76) + (-38 - 57 + 76) = 0, it follows that
-(38 + -57 - 76) = -38 - 57 + 76.

5. 978 × 357 = ?

   (a) ⁻978 × ⁻357   (b) −(978 × ⁻357)   (c) −(⁻978 × 357)

6. −(596 - 984) is the opposite of ?

   (a) ⁻984 + 596   (b) −596 + 984   (c) ⁻596 + ⁺984   (d) 596 - 984

7. −(19 + 11) = ?

   (a) 19 - 11   (b) −19 + −11   (c) 11 + 19   (d) ⁻19 + ⁻11

8. −(⁻19 + ⁻11) = ?

   (a) −⁻19 + −⁻11 (b) ⁺19 + ⁺11   (c) ⁺19 - ⁻11   (d) ⁻11 - ⁻19

9. −(⁺3 + ⁺5) = ?

   (a) ⁻3 + ⁻5   (b) −⁺3 + −⁺5

   (c) ⁺3 × ⁻1 + ⁺5 × ⁻1   (d) (⁺3 + ⁺5) × ⁻1

10. −(⁻3 + ⁺5) = ?

   (a) ⁺3 + ⁻5   (b) −⁻3 + −⁺5

   (c) ⁻3 × ⁻1 + ⁺5 × ⁻1   (d) (⁻3 + ⁺5) × ⁻1

11. −(⁺3 - ⁻5) = ?

   (a) ⁻3 + ⁻5   (b) −⁺3 - −⁻5

   (c) ⁺3 × ⁻1 - ⁻5 × ⁻1   (d) (⁺3 - ⁻5) × ⁻1

12. A first number is the opposite of a second number

   (a) if the sum of the numbers is 0.

   (b) if the opposite of the first number is the opposite of
        the second number.

   (c) if the first number is the product of the second number
        by ⁻1.

   (d) if the second number is the product of the first number
        by ⁻1.

13. The opposite of the sum of a first number and a second number is

    (a)   the sum of the opposites of the numbers.

    (b)   the first number minus the second number.

    (c)   the second number minus the first number.

    (d)   the opposite of the first number, minus the second number.

    (e)   the product of the sum by $^-1$.

NEW NAMES FOR NEGATIVE NUMBERS

Earlier in this unit we agreed that we could shorten such names as '$^+9$' and '$^+304$' by leaving out the raised plus signs. So, we could write '9' and '304' when we intended the real numbers $^+9$ and $^+304$. Now, to each positive real number there corresponds a real number which is its opposite. And we get a name for this opposite by writing an oppositing sign to the left of a numeral for the positive number. Hence, -9 is the opposite of the real number 9. But the opposite of the positive number 9 is a negative number. Hence, a name for this negative number is '-9'. And this is the number which we have been calling '$^-9$'.

For most of our purposes, we shall use such names as:

$$9, \quad 35, \quad 6\frac{1}{4}, \quad 15.83$$

for positive numbers, and such names as:

$$-7, \quad -58, \quad -3\frac{1}{5}, \quad -49.7$$

for negative numbers. In other cases where we want to stress the notion of oppositing, or the fact that the nonnegative real numbers are different from the numbers of arithmetic, we shall return to our use of names like:

$$^+9, \quad ^-7, \quad ^+35, \quad ^-58, \quad ^+6\frac{1}{4}, \quad ^-3\frac{1}{5}, \quad ^+15.83, \quad ^-49.7.$$

The section on new names for negative numbers is included in order to introduce students to the standard notation for negative numbers. Unfortunately, the standard notation is dangerous in that it leads the student to confuse the opposite of a number with a negative number. We hope that, with the careful build-up we have given to these ideas, the student will be aware of the distinction at all times.

Since $-9 = {}^-9$, it is customary to pronounce '-9' as 'negative 9'. It would be better to avoid this pronounciation at present and stick to 'the opposite of 9'. This will pay off when, in Unit 2, students meet '$-x$'.

※

The answer to the problem posed in the bracketed section at the top of page 1-87 is: $-7 = {}^-7$.

If the raised minus sign were not available, it would not be possible to express the fact that the opposite of 7 is negative 7, without using the words 'negative' and 'opposite'. An instructive technique to use with this question is to write [after the sentence '$-7 = {}^-7$' has been written on the board]:

$$-7 = {}^-\!\!\widehat{7}$$

and ask what kind of number is named by the numeral in the loop. [Answer: a number of arithmetic.] Then write:

$$-\widehat{7} = {}^-7$$

and ask a similar question. [Answer: a real number--the real number ${}^+7$.] These questions will review the notion of how numerals for real numbers are formed, and also strengthen the idea that oppositing is an operation which is applied to real numbers. The students should see that the sentence '$-7 = {}^-7$' is an abbreviation for '$-{}^+7 = {}^-7$'.

※

If we are serious in trying to reach the goal of getting students to understand what they are doing then we should regard the discussion of the three uses of the minus sign as exceedingly important. [Of course, a student who can distinguish between the three uses will not necessarily get higher scores on a skill test than a student who doesn't understand the distinction. That this is the case simply means that the mathematical notation is ambiguous enough to permit people who do not understand what they are doing to get the right answers.] Since we wish students to develop understanding as well as skill, we should not treat this discussion in a superficial manner. A very effective way for getting to the heart of the matter is to invent three signs which differ much more from each other than the three now in use, and to let students use these ad hoc signs for 10 to 15 minutes. [See TC[1-88]a.]

※

[Here is a case in which you would want to use a raised minus sign in naming a negative number: Without using the words 'opposite' and 'negative', write a sentence which states that the opposite of seven is negative seven.]

Using the oppositing sign to name negative numbers, instead of the raised minus sign, means that we are now going to use the same sign in <u>three</u> ways,

(1)   when naming a negative number,

(2)   when naming the opposite of a real number,

and   (3)   when indicating a subtraction problem.

You will seldom make a mistake because of this ambiguity for you will always be able to tell when meaning (3) is intended, and since the opposite of a positive number is a negative number, confusing (1) with (2) will make no difference. Here is an expression in which the minus sign is used with different meanings:

$$-7 - - -8.$$

You can think of this as subtracting the opposite of the opposite of 8 from the opposite of 7

$$-^+7 - - -^+8,$$

in which case the expression simplifies to '$^-15$'. Or, you can think of it as subtracting the opposite of $^-8$ from $^-7$

$$^-7 - - ^-8,$$

in which case, the expression again simplifies to '$^-15$'. Can you give two more ways of thinking of this expression?

Notice how the expression is to be read.

MORE NAMES FOR POSITIVE NUMBERS

In many books [this one, too] you will see numerals like '+7', '+3$\frac{1}{2}$', and '+(4 - 3)', in which the plus sign is used in the same position as the oppositing sign.   One interpretation of such a numeral is that the writer wants to emphasize that he is talking about positive numbers.   [We have done this when we have written, say, '+7' instead of the shorter '7'.]   Another interpretation is that just as the minus sign in '−7' may refer to the operation oppositing, the plus sign in '+3' refers to the operation "sameing".   For example,

$$+^{+}3 = {}^{+}3, \quad +{}^{-}8 = {}^{-}8, \quad +{}-3 = -3, \quad \text{and} \quad +(5 - 9) = 5 - 9.$$

When the plus sign is used in naming a positive number as it is in '+7', you could read it as 'positive', or not pronounce it at all. When it is used as a sameing sign, as it must be in '+(3 - 5)' [why 'must'?], read it as 'plus', or don't read it at all.

We have said that oppositing is the operation which takes you from a real number to its opposite.   Can you describe the operation sameing?   Recall the principles for the real numbers.   Do any of these principles tell you about an operation which is really sameing? Describe the inverse of sameing.

EXERCISES

A.   You have seen that there are several ways of naming real num-
     bers.   It is important that you become familiar with all of these
     ways, and that you develop skill in simplifying expressions
     which contain the various kinds of numerals.   The expressions
     given below should be read and simplified.

     Sample 1.   −8 - 6 + −3

          Solution.   Read as 'the opposite of 8, minus 6, plus the
                      opposite of 3', or as 'negative 8, minus 6,
                      plus negative 3'.
          Simplify, converting the subtraction to addition-of-
          the-opposite:
          $$-8 - 6 + -3 = -8 + -6 + -3$$
          $$= -14 + -3 = -17.$$

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On page 1-88 we take the final step of introducing the student to
standard notation for positive and negative numbers. Just as there
are three meanings which are denoted by the minus sign, there are
three meanings denoted by the plus sign. The "strangest" of these
is that the plus sign denotes the operation sameing. In conventional
courses, we explain this by telling the student that he should pretend
that the plus sign is not there. We say in this course that the
plus sign in such cases stands for the operation of sameing. [The
technical name for such an operation is: the identity operation.]

An answer to the question 'why 'must'?' near the middle of page 1-88
is:

> Because '⁺(3 - 5)' is nonsense since '3 - 5' must be a numeral
> for a real number. There is no number whose name is 'positive
> negative two'.

An answer to the question in the paragraph just before the exercises
on page 1-88 is:

> The operation adding 0, and the operation multiplying by ⁺1,
> and the operation sameing, are all the same operation.

<div align="center">✳</div>

The following exercise was included in the 1958-59 edition and received
mixed reactions. It can be used very successfully in class to point
out the three uses of the minus sign and of the plus sign. If you use
it, we suggest that you introduce the six signs [or, of course, any
other six that you prefer] one at a time, taking time in each case to
fix the meaning of the sign by several examples of its use.

> You have seen that the minus sign is used in three ways, and
> that the plus sign is used in three ways. In order to see if we
> understand these different uses, let's invent six signs for the
> six uses and practice a bit in working with them.
>
> '☆' for the minus sign of subtraction   [7 ☆ 5 is 2.]
>
> '✳' for the minus sign of oppositing   [✳3 is the opposite of 3.]
>
> '~' for the minus sign of direction   [~3 is negative 3.]
>
> '⊕' for the plus sign of addition   [9 ⊕ 5 is 14.]
>
> '♡' for the plus sign of sameing   [♡9 is 9.]
>
> '♀' for the plus sign of direction   [♀7 is positive 7.]

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1. Using these new signs rewrite the following expressions.

Sample 1. ⁺5 + ⁻2 - ⁺8 - -⁻3

Solution. ♀5 ⊕ ~2 ☆ ♀8 ☆ * ~3

(a)   8 - ⁻5 + ⁻6   (b)   +2 - +7 + 9 - -37
(c)   −5 - --9   (d)   9 + ++---9

2. Simplify.

Sample 2.   ~5 ⊕ * ~7

Solution.   Read as 'the sum of negative 5 and the opposite of negative 7'. So,

$$\sim5 \oplus * \sim7 = \sim5 \oplus ♀7 = ♀2.$$

Sample 3.   ~5* ~7

Solution.   Read as 'negative 5, the opposite of negative 7'. This is not the name of a number; it's just nonsense.

(a)   ♀7 ⊕ ~3   (b)   ~3 ☆ ~5   (c)   ♡9 ☆ * ~4
          (d)   ~5 ♡⊕ *6   (e)   ♀10 ☆ ~5 ⊕ * ~7

✳

Here are answers for the preceding exercises.

1. (a)   8 ☆ ~5 ⊕ ~6   (b)   ♡2 ☆ ♡7 ⊕ 9 ☆ *37
   (c)   *5 ☆ **9   (d)   9 ⊕ ♡♡ ***9

2. (a)   Read as 'the sum of positive 7 and negative 3'. So,

$$♀7 \oplus \sim3 = ♀4.$$

   (b)   Read as 'the difference of negative 5 from negative 3'. So,

$$\sim3 ☆ \sim5 = \sim3 \oplus * \sim5 = \sim3 \oplus ♀5 = ♀2.$$

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(c) Read as 'the difference of the opposite of negative 4 from [plus] 9'. So,

♡9 ☆ ✳ ~4 = ♀9 ⊕ ✳✳ ~4 = ♀9 ⊕ ~4 = ♀5.

[See lines 13, 14, page 1-88 in regard to reading the "sameing" sign as used here.]

(d) Read as 'negative 5, plus plus the opposite of 6' or as 'the sum of negative 5 plus and the opposite of 6'. This is not the name of a number; it's just nonsense.

(e) Read as 'the sum of the difference of negative 5 from positive 10 and the opposite of negative 7'. So,

$$♀10 ☆ ~5 ⊕ ✳ ~7 = ♀10 ⊕ ✳ ~5 ⊕ ✳ ~7$$
$$= ♀10 ⊕ ♀5 ⊕ ♀7$$
$$= ♀22.$$

�֎

On TC[1-82]c we asked that you have students practice reading expressions aloud in a way that conveys their meaning. After such practice, there is no harm in students' adopting the more conventional reading of symbols suggested in the Samples for Part A on pages 1-88 and 1-89.

Sample 2.   +8 - —5 + —6 - +12

   Solution.   Read as 'positive 8, minus negative 5, plus
               negative 6, minus positive 12'.

Simplify.

+8 - —5 + —6 - +12 = +8 + +5 + —6 + —12
                   =   +13 + —6 + —12
                   =   +7 + —12
                   =   —5.

Sample 3.   2 - 5 + 6 - 3 + 9 - 6 - 5

   Solution.   Read as '2, minus 5, plus 6, minus 3, plus 9,
               minus 6, minus 5'.

Simplify.
      2 - 5 + 6 - 3 + 9 - 6 - 5
    = 2 + —5 + 6 + —3 + 9 + —6 + —5
    =   —3 + 6 + —3 + 9 + —6 + —5
    =   3 + —3 + 9 + —6 + —5
    =   0 + 9 + —6 + —5
    =   9 + —6 + —5
    =   3 + —5
    =   —2.

   Here is a second method of simplifying. [Note
that the associative and commutative principles for
addition are used in this second method.]

   We see that

      2 - 5 + 6 - 3 + 9 - 6 - 5
    = 2 + —5 + 6 + —3 + 9 + —6 + —5.

So, the given expression:
      2 - 5 + 6 - 3 + 9 - 6 - 5
can be thought of as naming the sum of positive and
negative numbers. The positive numbers are 2, 6, and
9, and the negative numbers are —5, —3, —6, and —5.

Add the positive numbers, add the negative numbers, and then add the two sums.

$$2 - 5 + 6 - 3 + 9 - 6 - 5$$

$$= (2 + 6 + 9) + (-5 + -3 + -6 + -5)$$

$$= \qquad 17 + -19$$

$$= \qquad\quad -2$$

[Note: In doing the following simplifications you may wish to put in more steps than you feel you really need. It is a good idea to do this at the beginning. Then, as you do more problems, you will discover short cuts which will enable you to do many problems without writing more than one or two steps.]

1. 9 - 3 - 7   2. −5 + 6 - 11   3. −12 - 4 + 19

4. −6 + 5 + 9   5. +7 - 3 - 4   6. +8 + 12 + 17

7. +4 - +7 + −3   8. +5 - −3 + −4   9. +5 - −6 - −5

B. Simplify.

1. +12 - +6 - −7 + −8 + −13 - −7

2. −3 + −6 - −4 + −7 - −7 + −13

3. 3 + −7 + 3 - −7 - 5 + 12 + −3

4. −4 + 0 + −8 - −3 + 11 + −17 + 16

5. 7 - 3 + −3 - +8 + 17 - −1 - −6 + 4

6. −10 + 17 + −3 - −10 - −7 - −3 - −1

7. −10 - −10 + −10 + −3 - −3 + 0 + +2 - +2

8. −19 + −3 - −4 + −6 + −17 + −4 - −3 - −19

9. 0 - 7 + −3 + 7 - −8 + 9 - −10

10. 3 - 5 + 6 - 7 + 9 + 8 - 3

11. 0 + 5 - 6 + 8 - 0 + 9 - 5

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Answers for Part A [which begins on page 1-88].

1. −1	2. −10	3. 3	4. 8	5. 0
6. 37	7. −6	8. 4	9. 16	

✻

Answers for Part B [on pages 1-90 and 1-91].

1. −1	2. −18	3. 10	4. 1	5. 21
6. 25	7. −10	8. −23	9. 24	10. 11
11. 11	12. 4	13. 4	14. 18	15. −18
16. 4	17. 13	18. $4\frac{1}{2}$	19. $-\frac{11}{12}$	20. −1.07
21. −14.837				

✻

Answers for Part C [on page 1-91].

1. negative number	2. negative number	3. positive number
4. negative number	5. positive number	6. negative number
7. negative number	8. positive number	9. negative number
10. negative number		

✻

Part D is difficult. The first 18 exercises in Part L of the Supplementary Exercises are of this type but much easier. You may want to assign them first. Another helpful device is to suggest that the students unabbreviate the expressions before trying to simplify.

✻

Answers for Part D [on pages 1-91 and 1-92].

1. −9	2. 17	3. −20	4. 43	5. 46
6. 49	7. −5	8. 4	9. 25	10. 12
11. −19	12. −43	13. −67	14. −11	15. 0
16. 0	17. −10	18. 230	19. −191	

12. 2 - 7 + 8 - 6 + 4 + 3   13. 1 - 1 + 2 - 2 + 3 - 6 + 7

14. 10 - 8 + 7 - 5 + 6 + 8   15. 5 - 15 - 20 + 18 + 2 - 8

16. 4 + 7 - 3 - 8 + 9 - 5   17. 6 + 4 - 3 + 12 + 10 - 16

18. $5\frac{1}{2} + 9 - 2\frac{1}{2} - 7\frac{1}{2}$ 19. $6\frac{1}{3} - 2\frac{1}{4} - 5\frac{1}{2} + 3\frac{1}{6} - 2\frac{2}{3}$

20. 4.83 - 10 - 3.8 + 7.9   21. 16.75 - 11.3 + 40.72 - 61.007

[More exercises are in Part K, Supplementary Exercises.]

C. Examine each of the following expressions, and tell whether it stands for a positive number, a negative number, or 0.

1. $-^+5$ 2. -7 3. $+4$ 4. $+^-5$

5. $+(3 - 2)$ 6. $+(7 - 9)$ 7. $-(8 - 3)$

8. $-(4 - 7)$ 9. $---5$ 10. $+---5$

D. Simplify.

Sample 1. $(3 - 8) \times (7 - 11) - (5 - 12) \times (6 - 10)$

Solution. $(3 - 8) \times (7 - 11) - (5 - 12) \times (6 - 10)$

$= (-5 \times -4) - (-7 \times -4)$

$=\quad\quad 20 \quad - \quad 28$

$=\quad\quad -8.$

Sample 2. $6 - 3 \times (4 - 7)$

Solution. $6 - 3 \times (4 - 7)$

$= 6 - (3 \times -3)$

$= 6 - -9$

$= 15.$

1. $5 + 7 \times (3 - 5)$ 2. $-8 + -5 \times (-2 + -3)$

3. $-9 \times (5 + -2) + 7$ 4. $-3 \times (-7 - 8) + -2$

5. $6 \times (3 - -8) - 2 \times (5 - -5)$

6. $(-2 - 3) \times (-7 - 5) + (-3 + -8) \times (-2 + 3)$

(continued on next page)

7. $(5 - 8) \times (9 - 12) - (6 - 13) \times (7 - 9)$

8. $(4 - 3) \times (2 - 7) - (12 - 15) \times (8 - 5)$

9. $(6 - 1) \times (7 - 2) - (12 + 1) \times (3 - 3)$

10. $7 - 2 \times (5 - 8) - 3 \times (4 - 2) - 5 \times (6 - 7)$

11. $-2 - 6 - (3 + 5) - (7 + 2) - (8 - 3) - (-9 - 2)$

12. $+6 + -2 - 3 - 8 - 7 \times (5 - 2 - 1) + 2 \times (-3 - 6 - 2)$

13. $-2 \times [5 + -(3 - 4)] - 5 + [(7 + 3) - 6 \times (2 + 8)]$

14. $[(8 - 2) - (7 - 5)] \times +4 - [(2 - 7) - 7 \times (3 - 5)] \times 3$

15. $[(9 - 5) + (7 - 24)] \times 83 + [(24 - 7) + (5 - 9)] \times 83$

16. $(18 - 27) \times (53 - 41) + (27 - 18) \times (53 - 41)$

17. $17 - \{5 - [2 - 3 \times (4 + 1)] - 6 \times (7 - 5) + 21\}$

18. $-7 - 3 \times \{4 + 2 \times (3 - 5) - 3 \times [7 - 2 \times (4 - 10) + 5] - 7\}$

19. $5 \times (3 + -9) - 7 \times [-6 - 3 \times (2 - 3) + 4 \times (3 + 4) - 2]$

[More exercises are in Part L, Supplementary Exercises.]

1.11 Division of real numbers. --In grade school you learned that to do the division problem:

$$24 \div 6 = ?$$

you had to search for a number which when multiplied by 6 gave 24. This number is 4 because $4 \times 6 = 24$. Look at the following sentence:

$$(4 \times 6) \div 6 = 4.$$

Does it illustrate the idea that dividing by 6 is the inverse of multiplying by 6?

On the following page, at the left, is a list of pairs which belong to the operation multiplying by ¯3. The list on the right is made by reversing each pair in the list on the left.

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In Unit 2 we give a much more extensive treatment of division. Here we wish only to bring out the fact that, just as subtracting ⁻3 is the operation which is inverse to adding ⁻3, dividing by ⁻3 is the operation which is inverse to multiplying by ⁻3.

The ordered pairs which are listed in the upper right quadrant of page 1-93 belong, of course, to the operation dividing by ⁻3.

<center>⁎</center>

Answers for exercises in the middle of page 1-93.

(1) ⁻5   (2) ⁻3   (3) ⁺6   (4) ⁺4   (5) ⁻7   (6) 0

<center>⁎</center>

Let the student solve the division problems listed at the bottom of page 1-93 and at the top of page 1-94 by thinking in terms of inverse operations. In short order, he will develop the rules for himself.

<center>⁎</center>

Answers for exercises at bottom of page 1-93, and top of page 1-94.

1. ⁺6	2. ⁻6	3. ⁻6	4. ⁺6	5. ⁻7
6. ⁺2	7. ⁻3	8. ⁺3	9. ⁻9	10. 3
11. ⁻3	12. ⁻8	13. ⁺8	14. 8	15. ⁻8
16. ⁻9	17. 0	18. 0		
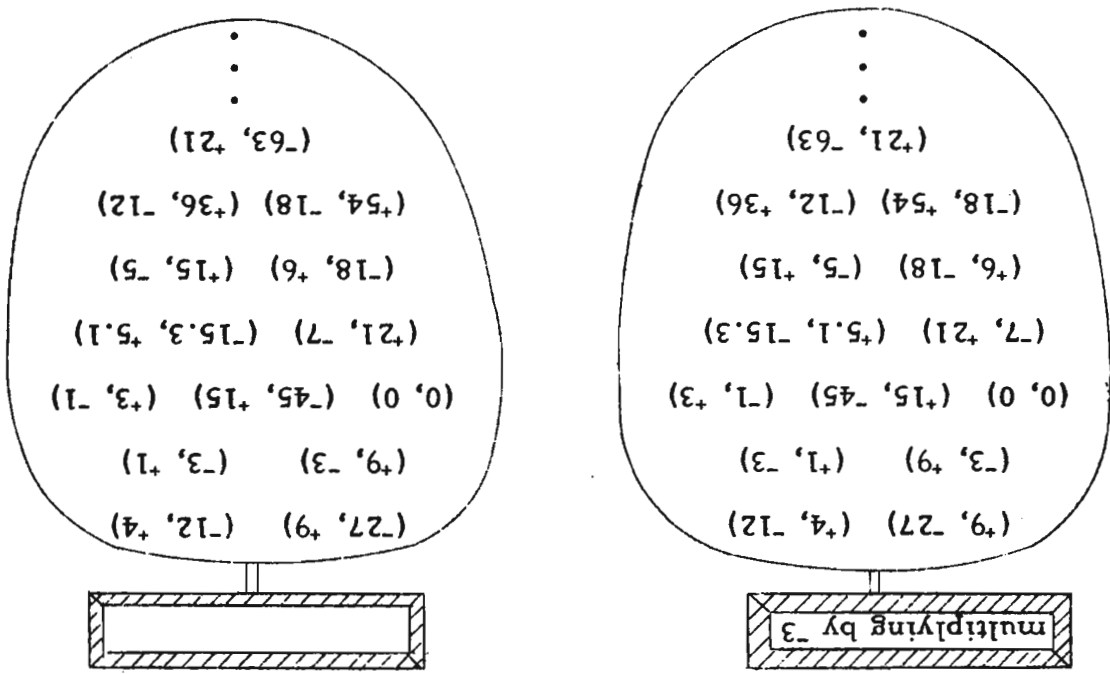
Can you give a name for the operation to which the pairs in the right list belong? This operation is the inverse of multiplying by ‾3.

Use the lists to solve the following problems in division.

(1) +15 ÷ ‾3 = ?   (2) +9 ÷ ‾3 = ?

(3) ‾18 ÷ ‾3 = ?   (4) ‾12 ÷ ‾3 = ?

(5) +21 ÷ ‾3 = ?   (6) 0 ÷ ‾3 = ?

Now, suppose you wanted to solve the division problem:

‾63 ÷ +7 = ?

To solve this problem you could use a list for the operation dividing by ‾7, or a list for the operation multiplying by +7. But, even without such lists, you could still use your knowledge of multiplication to solve this division problem. You could imagine yourself searching through a long multiplying by +7 list for a pair in which the second number was ‾63. The first number in the pair gives you the answer to the division problem. What is the answer?

Do these division problems.

1. +18 ÷ +3 = ?   2. +18 ÷ ‾3 = ?   3. ‾18 ÷ +3 = ?

4. ‾18 ÷ ‾3 = ?   5. ‾14 ÷ +2 = ?   6. ‾2 ÷ ‾1 = ?

7. ⁻9 ÷ ⁺3 = ?   8. ⁻12 ÷ ⁻4 = ?   9. ⁺27 ÷ ⁻3 = ?

10. 18 ÷ 6 = ?   11. 18 ÷ -6 = ?   12. 24 ÷ -3 = ?

13. -24 ÷ -3 = ?   14. 24 ÷ 3 = ?   15. -24 ÷ 3 = ?

16. 36 ÷ -4 = ?   17. 0 ÷ 5 = ?   18. 0 ÷ -3 = ?

WAYS OF NAMING A QUOTIENT

The quotient of a first number (dividend) by a second number (divisor) may be named by putting a divide-by sign between numerals for the numbers, the dividend numeral being placed on the left. So, the quotient of 8 by -2 is named by:

$$8 \div -2.$$

A simpler name for this quotient is:

$$-4.$$

In grade school you learned that a fraction can also be used to name a quotient. Thus, the quotient of 8 by -2 is named by:

$$8/-2 \text{ or by: } \frac{8}{-2}.$$

The part of the fraction which names the dividend is called <u>the numerator of the fraction</u>, and the part which names the divisor is called <u>the denominator of the fraction</u>. In the fraction '$\frac{8}{-2}$', the numerator is '8' and the denominator is '-2'.

EXERCISES

<u>A</u>. Simplify.

1. 12 ÷ 3   2. -17 ÷ 1   3. -6 ÷ -2

4. 8 ÷ -2   5. 10 ÷ -1   6. -7 ÷ -1

7. 0 ÷ -3   8. 9 ÷ 3   9. 16 ÷ -4

10. 17 ÷ -1   11. -27 ÷ -3   12. 9 ÷ -9

13. $\frac{-16}{8}$ 14. $\frac{-21}{-7}$ 15. $\frac{-33}{3}$

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Notice that on page 1-94 we state that a fraction is a type of numeral. The fraction has three parts, numerator, fraction bar, and denominator. If we want to refer to the number named by the numerator or the number named by the denominator, we use expressions such as 'numerator-number' and 'denominator-number'.

There are good reasons for using the word 'fraction' to denote a numeral rather than a number. For example, we would want to say that '$\frac{1}{2}$' is a different fraction from '$\frac{2}{4}$'; however, $\frac{1}{2}$ is the same number as $\frac{2}{4}$. Thus, each number has an infinite number of fraction names. [The expressions '$\frac{2}{4}$' and '2/4' are abbreviations for '$(2 \div 4)$'. See page 2-87.]

❋

In the article in The Arithmetic Teacher previously referred to [See TC[1-A, B, C].], Frank L. Wolf asks the reader to tell what is wrong with the following reasoning. [This should give you another reason why we wish to regard a fraction as a numeral rather than as a number].

> Since 5 is a divisor of 10 we know that 5 is a divisor of the numerator of $\frac{10}{15}$. Hence, because $\frac{10}{15} = \frac{2}{3}$ and because we may substitute equals for equals, 5 is a divisor of the numerator of $\frac{2}{3}$. Therefore, 5 is a divisor of 2.

❋

Answers for Part A [on pages 1-94 and 1-95].

1.	4	2.	−17	3.	3	4.	−4	5.	−10
6.	7	7.	0	8.	3	9.	−4	10.	−17
11.	9	12.	−1	13.	−2	14.	3	15.	−11
16.	0	17.	−2	18.	3	19.	−6	20.	6
21.	−6	22.	4	23.	4	24.	−4	25.	3
26.	−3	27.	3	28.	4	29.	4	30.	−2
31.	−7	32.	0	33.	−5	34.	$\frac{1}{2}$	35.	$-\frac{1}{3}$
36.	$-\frac{3}{2}$								

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After the very brief introduction to the notion of the impossibility of division by zero on page 1-65 in connection with the question: What is the reciprocal of 0?, the student is now ready for a somewhat more formal discussion of this problem in Part B on page 1-95. The student will discover when he tries to do the three "division problems" in Exercise 3 that the list he made in Exercise 2 is a "good-for-nothing" list. You should not engage in a formal dis-cussion at this point; all one needs to say here is that you cannot do problems which involve division by zero. In Unit 2 we treat this matter in much greater detail.

✳

Answers for Part B [on page 1-95].

1. ($^+$10, 0), ($^-$8, 0), ($1\frac{1}{2}$, 0), ($\frac{4}{5}$, 0), (.732, 0), ($^-$.001, 0), (6, 0), ($^-$35.5, 0), ($\frac{7}{3}$, 0), (19827, 0), (0, 0) ... .

2. (0, $^+$10), (0, $^-$8), (0, $1\frac{1}{2}$), (0, $\frac{4}{5}$), (0, .732), (0, $^-$.001), (0, 6), (0, $^-$35.5), (0, $\frac{7}{3}$), (0, 19827), (0, 0) ... .

The list of pairs a student makes in answer to Exercise 1 may be captioned 'multiplying by 0', and his list for Exercise 2 may be captioned 'dividing by 0'. But, if so, point out to him that the set of ordered pairs which he is calling dividing by 0 is not an operation on the set of all real numbers. So, since all the similar phrases 'dividing by $^+$1', 'dividing by $^-$3.5', etc. do name operations, the caption 'dividing by 0' may be misleading. And this is a good reason for deciding against using it.

3. Division by 0 is impossible.

✳

The answer to the bracketed question at the bottom of page 1-95 is that if you find that, in comparing a first number of arithmetic with a second number of arithmetic, you can get the second number by add-ing 0 to the first number, you are comparing just one number with itself.

✳

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✳

Answers for quiz.

I. 1. ⁻4   2. ⁺20   3. ⁻1   4. 0
    5. ⁻1   6. 0   7. ⁻10   8. 0
    9. ⁻5   10. ⁺4   11. ⁻2   12. ⁺40
   13. ⁻1   14. ⁻1   15. ⁺60   16. ⁻11

II. 1. ⁺3   2. ⁺4   3. ⁻5   4. ⁻15
    5. ⁺5   6. ⁺4   7. ⁺4

III. 1. a positive number   2. a positive number
     3. a negative number

16. 0/−17   17. 34/−17   18. −18/−6

19. $\frac{18}{-3}$ 20. $\frac{18}{3}$ 21. $\frac{-18}{3}$

22. $\frac{4}{1}$ 23. $\frac{-4}{-1}$ 24. $\frac{-4}{1}$

25. $-\frac{6}{-2}$ 26. $-\frac{-9}{-3}$ 27. $-\frac{-12}{4}$

28. +4/+1   29. 4/+1   30. 6/−3

31. $\frac{-7}{1}$ 32. $\frac{0}{-8}$ 33. $\frac{-15}{3}$

34. $^+1 \div {}^+2$ 35. $^+1 \div {}^-3$ 36. $^+1 \div \frac{^-2}{3}$

[More exercises are in Part M, Supplementary Exercises.]

B. 1.   Construct a list of ten pairs which belong to the operation
         multiplying by 0.

    2.   Make another list of pairs by reversing the pairs in the
         first list.

    3.   Use the lists to try to solve these problems.

         (a) 6 ÷ 0 = ?   (b) $\frac{-8}{0}$ = ?   (c) 0 ÷ 0 = ?

1.12 Comparing numbers. --Who has more stamps, Richard with
2350 or Al with 1820? This is an easy question. Richard does,
because 2350 is a larger number than 1820. And, how do you know
2350 is larger than 1820? You may have decided this by remember-
ing that in counting 1, 2, 3, etc., 1820 comes before 2350.

    Another way of telling that 2350 is larger than 1820 is to recog-
nize that you can get 2350 by adding a number [not 0] to 1820. And,
as long as we are talking about numbers of arithmetic, this is a
perfectly good method to use. To decide which of two numbers is
the greater, just decide which of the two numbers you would have
to add to in order to get the other. The one you have to add to is the
smaller, and the other is the larger. [If you find that you can get
one of the numbers of arithmetic by adding 0 to the other, how many
numbers were you comparing in the first place?]

Answers for Part A.

1. T   2. T   3. T   4. F   5. T   6. T

※

Answers for Part B.

1. 9 is greater than 4 [or: 4 is less than 9]

2. 16 is less than 61 [or: 61 is greater than 16]

3. 10,002 is less than 100,002 [or: 100,002 is greater than 10,002]

4. $3\frac{4}{5}$ is greater than $3\frac{3}{4}$ [or: $3\frac{3}{4}$ is less than $3\frac{4}{5}$]

5. $9\frac{1}{8}$ is less than $9\frac{1}{7}$ [or: $9\frac{1}{7}$ is greater than $9\frac{1}{8}$]

6. 0.10002 is greater than 0.100002 [or: 0.100002 is less than 0.10002]

7. 21 > 17   8. 93 > 39   9.   74 > 73.5

10. $\frac{3}{17} < \frac{5}{19}$ 11. $\frac{9}{24} = \frac{30}{80}$ 12.   .304 > .208

※

Answers for Part C.

1. A numeral for any number greater than 19 will make the sentence true [e.g., 19.5].

2. A numeral for any number smaller than 34 [e.g., 33].

3. A numeral for any number greater than 55.1 [e.g., 55.2].

4. A numeral for any number greater than $6\frac{1}{8}$ [e.g., $6\frac{1}{4}$].

5. A numeral for any number smaller than $\frac{9}{73}$ [e.g., $\frac{8}{73}$].

6. A numeral for any number smaller than .0003 [e.g., .0002].

Here is another case in which we use a familiar phrase but with a new meaning. The students know the meaning of ' > ' with respect to numbers of arithmetic. We now want to use this same symbol to denote a relation among real numbers. Clearly, this relation for real numbers is different from the one for numbers of arithmetic, just as multiplication as an operation for real numbers is different from multiplication as an operation for numbers of arithmetic. So, the only way in which we come to use the symbol ' > ' with correctness in connection with real numbers is to have a definition which tells its meaning. We give this definition on pages 1-97 and 1-98, trying to show some of the desirable characteristics of the definition. But, it is an arbitrary matter just as the definitions of addition and multiplication for real numbers are arbitrary matters.

❋

Answers for Part A [on page 1-98].

1. $^+8 < {}^+11$	2. $^+3 > {}^-1$	3. $^-10 < {}^+7$
4. $^-5 > {}^-30$	5. $^+15 > {}^+3$	6. $-7 < -4$
7. $-98 < 2$	8. $5 > 2$	9. $5 > 1$
10. $5 > 0$	11. $5 > -1$	12. $5 > -2$
13. $5 > -5$	14. $5 > -7$	15. $-7 = -7$
16. $-7 < -6$	17. $-7 > -8$	18. $-7 > -9$
19. $-20 < 19$	20. $-19 < 20$	

❋

Answers for Part B [on page 1-98].

1. A numeral for any number greater than -19 will make the sentence true [e.g., -18.9].

2. A numeral for any number greater than 19 [e.g., 20].

3. A numeral for any number less than 19 [e.g., 18.9].

4. A numeral for any number less than -23 [e.g., -24].

5. A numeral for any number less than 0 [e.g., $-.5$].

6. A numeral for any number greater than 0 [e.g., .25].

7. A numeral for any number less than -30 [e.g., -60].

8. A numeral for any number greater than -30 [e.g., -15].

9. A numeral for any number less than $-\frac{1}{7}$ [e.g., $-\frac{1}{6}$].

COMPARING REAL NUMBERS

How shall we proceed in comparing real numbers? Is $^+5 < {}^+17$? Is $^+4 > {}^-2$? Is $^-3 < {}^-2$? Is $^-100 > {}^+1$? There are some clues around which can help us understand why mathematicians do decide as they do about which of two real numbers is the larger.

Consider the sentence:

$$4 < 19.$$

Is this a sentence about numbers of arithmetic, or is it a sentence about real numbers? Actually, we can't tell, because the numerals '4' and '19' are ambiguous. Up to now, this ambiguity did not bother us because no matter what interpretation we used, sentences such as '5 + 7 = 12' and '6 × 4 = 24' made sense. So, in order to keep this freedom of interpretation, let's agree that if we see the sentence:

$$4 < 19,$$

it could be telling us either that the number 4 of arithmetic is less than the number 19 of arithmetic, or that $^+4$ is less than $^+19$. So, which is smaller, $^+7$ or $^+20$? $^+15$ or $^+2$? $^+1000$ or 0?

Let's review how we answer questions like these for numbers of arithmetic. All we do is find out which number you have to add to to get the other. The one you add to is the smaller.

But, does this simple test work for real numbers? Take the pair of real numbers $^+2$ and $^+15$. Is there a number you can add to $^+2$ to get $^+15$? Is there a number you can add to $^+15$ to get $^+2$? The answer to <u>both</u> questions is 'yes'. So, the simple adding test is no help. How could we change the test so that it would help? The accepted answer is: find out which number you would have to add a <u>positive</u> number to to get the other; the one you have to add a positive number to is the smaller. Use this test to check the following sentences:

$$^+5 < {}^+17, \qquad ^+101 > {}^+100, \qquad 0 < {}^+6.9.$$

Now we have a test for deciding which is the larger of two non-negative numbers. And this test is such that when we ask the question 'Is 4 < 19?', we get the same answer whether we are thinking of numbers of arithmetic or of real numbers.

What about dealing with the rest of the real numbers? Once again, the accepted answer is that the same test is to be used for all pairs of real numbers. The number of the pair to which you can add a positive number to get the other is the smaller. For example, ⁻3 < ⁺7 because if you add ⁺10 to ⁻3, you get ⁺7. The sentence '⁻12 < ⁻1' is true because we can add ⁺11 to ⁻12 to get ⁻1. Use this test of adding a positive number to check each of the following true sentences.

⁻5 < ⁺12   ⁻11 > ⁻13   ⁻100 < ⁺1   ⁻73 < 0

EXERCISES

A. Use one of the signs ' > ', ' < ', and ' = ', and write a true sentence which compares the given real numbers of each pair.

    Sample.   (⁻15, ⁻19)

        Solution.   To which of these numbers can I add a positive number to get the other? Since

$$⁻19 + ⁺4 = ⁻15,$$

⁻19 is the smaller. So, I write:

$$⁻15 > ⁻19.$$

[Could you write something else?]

1. (⁺8, ⁺11)   2. (⁺3, ⁻1)   3. (⁻10, ⁺7)   4. (⁻5, ⁻30)

5. (⁺15, ⁺3)   6. (−7, −4)   7. (−98, 2)   8. (5, 2)

9. (5, 1)   10. (5, 0)   11. (5, −1)   12. (5, −2)

13. (5, −5)   14. (5, −7)   15. (−7, −7)   16. (−7, −6)

17. (−7, −8)   18. (−7, −9)   19. (−20, 19)   20. (−19, 20)

B. Fill in the blanks to make true sentences.

1. −19 < _____   2. 19 < _____   3. 19 > _____

4. −23 > _____   5. 0 > _____   6. _____ > 0

7. _____ < −30   8. _____ > −30   9. _____ < −$\frac{1}{7}$

On the other hand, one could think of blobs of chalk (or of ink),
organized into infinitely long streaks, as forming an approximate
model for geometry. So, it would be appropriate to refer (in the
less precise sense) to such streaks and blobs as lines and points.
['would be' because infinitely long streaks of chalk don't exist.]
And, it is customary to speak of any moderately straight and
reasonably long streak as a line, and to speak of a small bit of it
as a point. [Incidentally, this use of 'point' and 'line' in the
teaching of geometry leads to serious misconceptions for students.]
We shall, as stated on page 1-99, use such streaks as pictures of
parts of the number line, and use bits of them as pictures of points.
To avoid confusion with 'point of the number line' [which we shall
ordinarily abbreviate to 'point'], we shall use 'dot' when referring
to a ''point'' of a ''streak line''. [On occasion, we shall use 'point'
and 'line' to refer to the physical objects when the content makes
clear what is going on.]

Notice that many conventional texts use 'number line' for what we
call a 'picture of part of the number line'. In our terminology it
doesn't make sense to say:

> Draw a number line and mark the numbers 0 and 1 on it.

Rather, we would say:

> Draw a picture of the number line and mark dots on it
> corresponding with the numbers 0 and 1.

※

Answers for questions in the text.

(1)

0    $1\frac{1}{2}$ 2   5   8
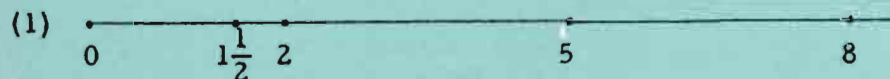
(2) Yes   (3) because $0 < 1\frac{1}{2} < 2$

(4) 2 is ''to the left of'' 5   (5) 8 is ''to the right of'' 1

(6) 2 is ''to the right of'' -4 (7) -2 is ''to the left of'' -1

(8) because $-1 < -\frac{3}{4} < 0$

※

Note that the answer to (1), above, is a picture, not of a line, but of
a ray.

Notice that the appropriateness of the <u>picture</u> on page 1-99 of the
number line is a consequence of the way in which the relation > is
defined for real numbers. While doing the exercises on page 1-98,
students will probably construct similar "mental pictures" for them-
selves.

Although you will probably find that your students have no difficulty
with the terminology introduced on pages 1-99 and 1-103, you may
yourself experience some confusion due to conflicts with your pre-
vious use of the terms 'number line' and 'point'. The words 'line'
[or, more explicitly, 'straight line'] and 'point' are used in many
ways. In abstract deductive geometry these words are <u>primitive</u>
<u>terms</u> and, properly, neither is assigned a referent. When one does
assign appropriate referents to these words [and to the other primi-
tive terms--for example, 'congruent'] one obtains a model of abstract
deductive geometry. Because of this it is customary to use 'line' and
'point' to refer to the corresponding entities in any model of geome-
try; and also, to use these words in a less precise sense, to refer to
similar entities which occur in what are, in some sense, "approxi-
mate" models of geometry. And, stemming from such uses, there
are metaphorical usages like these in 'Line up!' and 'What point are
you driving at?'.

Now, from the point of view of elementary geometry, an important
characteristic of a "line" is that it consist of "points" which are
"ordered" in such a way that it is possible to set up a one-to-one
correspondence between them and the real numbers so that a point
is "between" two others just if the real number which corresponds
with it is greater than the number which corresponds with one of the
two points and less than that which corresponds with the other. Since
the set of real numbers, when ordered by > [or by <], certainly,
itself, satisfies this requirement, it is appropriate to refer to its
members, the real numbers, as points. For definiteness, we call
the ordered set of real numbers <u>the number</u> line, and speak of each
real number as <u>a point of the number line</u>. The main advantage in
using this geometric language is that doing so helps one to apply
one's geometric intuition to the organization of relationships among
real numbers. [In fact, mathematicians are likely to refer to the
members of any set as points whenever this cue to geometric intui-
tion seems likely to be helpful.]

1.13 The number line. --Many people think of the numbers of arithmetic as being "lined up in order", starting with the smallest number, 0, and with larger numbers to the "right" of smaller numbers.

(1) Draw a picture in the space above to show this arrangement. Make dots on the picture for the numbers 0, $1\frac{1}{2}$, and 2.

(2) Is the dot for $1\frac{1}{2}$ between the dots for 0 and 2?

(3) Why should the dot for $1\frac{1}{2}$ be between the dots for 0 and 2?

(4) Make a dot for the number 5. How does your picture show that 2 < 5?

(5) Make a dot for the number 8. How does your picture show that 8 > 1?

One of the advantages of our agreement for comparing real numbers is that we can also think of all the real numbers as lined up in order.

$$\longleftarrow\!\!\!\!\!-\!\!\!\!\!-\!\!\!\!\!-\!\!\!\!\!-\!\!\!\!\!-\!\!\!\!\!-\!\!\!\!\!-\!\!\!\!\!-\!\!\!\!\!-\!\!\!\!\!\longrightarrow$$

$^-5 \quad ^-4 \quad ^-3 \quad ^-2 \quad ^-1 \quad 0 \quad ^+1 \quad ^+2 \quad ^+3 \quad ^+4$
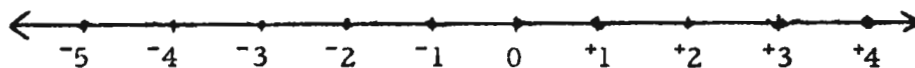
(6) How does this picture show that 2 > —4?

(7) How does this picture show that —2 < —1?

(8) Make a dot on the picture for $-\frac{3}{4}$. Why should this dot be placed between the dots for 0 and —1?

In comparing real numbers it is helpful to think of them as being lined up. So, we often call the ordered set of real numbers the line of real numbers, or just the number line, for short. When you draw a picture like the one just above question (6), you have a picture (of part) of the number line.

EXERCISES

A. Here is a picture of the number line.
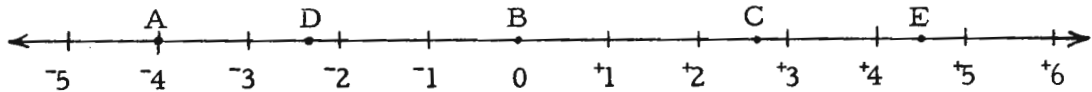
Give the real number which corresponds with each labeled dot.

A: B: C: D: E:

B. Here is another picture of the number line. Mark dots on the picture which correspond with the listed real numbers.
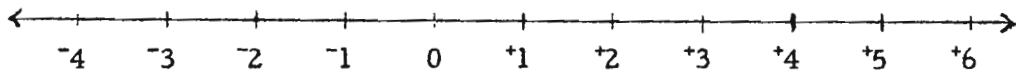
A: 5   B: 2   C: −3   D: $\frac{1}{2}$ E: $-2\frac{1}{3}$ F: −3.2   H: $-\frac{1}{4}$

C. A mental picture of the number line makes it easy for you to compare real numbers. Write a comparison sentence about each pair.

1. (−3, +17)   2. (+5.5, +3)

3. (−6, −5)   4. (−152, −2,176)

5. (+.0012, −.0138)   6. (+.001, +.0001)

7. (−6.382, +$\frac{1}{2}$)   8. ($\frac{11}{3}$, $\frac{21}{6}$)

9. (−1,428, +.0052)   10. (−.00016, −43,213)

[More exercises are in Part N, Supplementary Exercises.]

D. True or false? [Just as '≠' is used for 'is not equal to', so are '$\not<$' used for 'is not less than' and '$\not>$' for 'is not greater than'.]

1. 5 $\not>$ 10   2. 4 $\not<$ 0   3. −10 ≠ −20

4. −10 ≠ −10   5. .5 $\not>$ .75   6. 2 $\not>$ 2

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Answers for Part A.

A: −4   B: 0   C: $2\frac{2}{3}$ D: $-2\frac{1}{3}$ E: $4\frac{1}{2}$

⁂

Answers for Part B.

⁂

Answers for Part C.

1. −3 < +17 [or: +17 > −3]   2. +5.5 > +3 [or: +3 < +5.5]

3. −6 < −5 [or: −5 > −6]   4. −152 > −2,176 [or: −2,176 < −152]

5. +.0012 > −.0138 [or: 6. +.001 > +.0001 [or:
    −.0138 < +.0012]   +.0001 < +.001]

7. −6.382 < $+\frac{1}{2}$ [or: 8. $\frac{11}{3} > \frac{21}{6}$ [or: $\frac{21}{6} < \frac{11}{3}$]
    $+\frac{1}{2} > -6.382$]

9. −1428 < +.0052 [or: 10. −.00016 > −43213 [or:
    +.0052 > −1428]   −43213 < −.00016]

⁂

Answers for Part D [on pages 1-100 and 1-101].

1. T   2. T   3. T   4. F   5. T
6. T   7. T   8. T   9. F   10. T
11. F   12. T   13. No

⁂

Answers for Part E [on page 1-101].

1. T   2. F   3. T   4. T   5. F
6. T   7. F   8. F   9. T   10. T

7. $5 \not> 10 \div 2$ 8. $-3 \not> 0$ 9. $-13 \not< 5/3$

10. $5 = 5$ 11. $5 < 5$ 12. $5 \not> 5$

<center>✳</center>

13. Do '$\not>$' and '$<$' tell you the same thing? Compare
     Exercises 11 and 12.

<center>✳ ✳ ✳</center>

Suppose each of two students, Alice and Rachel, picks a real
number and each whispers her choice to Ned. Ned tells you that
Alice's number is not less than Rachel's. Can you conclude that
Alice's is greater than Rachel's?

When you write the comparison sign '$\not<$' between numerals and
have a true sentence, the numeral on the left can name a number
which is <u>greater</u> <u>than</u> <u>or</u> <u>equal</u> <u>to</u> the number named by the numeral
on the right. So, sometimes, instead of using '$\not<$' we use '$\geq$', a
combination of '$>$' and '$=$'. Similarly, we sometimes use '$\leq$'
instead of '$\not>$'. [Read '$\geq$' as 'is greater than <u>or</u> equal to' and '$\leq$'
as 'is less than <u>or</u> equal to'.]

So, for example, the sentence:

(1) $5 \not> 6$

says the same thing as does the sentence:

(2) $5 < 6$ or $5 = 6$,

and (1) and (2) say the same thing as does:

(3) $5 \leq 6$.

<center>✳ ✳ ✳</center>

<u>E.</u> True or false?

1. $5 > -6$ 2. $10 < -26$ 3. $4 \leq 4$

4. $\frac{1}{2} \geq \frac{1}{2}$ 5. $-3 \leq -\frac{21}{2}$ 6. $\frac{1}{5} \not< \frac{1}{6}$

7. $1.53 \leq 1.053$ 8. $-1.542 \geq +0.001$

9. $-\frac{11}{7} \leq -\frac{21}{14}$ 10. $972 - 846 \geq -(972 - 846)$

F. For each of the following exercises, make as many true sent-
    ences as you can by inserting the signs ' = ', ' $\neq$ ', ' < ', ' $\not<$ ',
    ' > ', ' $\not>$ ', ' $\leq$ ', and ' $\geq$ '.

   Sample.   6   4

      Solution.   6 > 4,   6 $\neq$ 4,   6 $\geq$ 4,   6 $\not<$ 4

1. 5   3   2. −4   −4   3. 6   −3

4. −10   −9   5. $-\frac{10}{3}$ 2   6. $-\frac{1}{782}$ 0

7. 0   0   8. $\frac{3}{7}$ $\frac{-3}{-7}$ 9. $\frac{8-2}{3-7}$ $\frac{2-8}{3-7}$

※

10. Suppose we abbreviated:

         is less than or is equal to or is greater than

     by:

                     $\frac{\leq}{>}$.

     In which of the Exercises 1-9 above could you use this sign
     to make a true sentence? Do you think anybody uses a
     sign like this?

[More exercises are in Part Q, Supplementary Exercises.]

※ ※ ※

     None of the pictures of the number line which we have drawn
looked like this.

This picture would be perfectly all right if we were interested
only in using it to check a sentence like ' $^+3 > ^-1$ '. But, we
can draw our pictures so that they also enable us to check a
sentence like ' 4 - 2 = $^-1$ - $^-3$ ' just by looking at the picture.
The picture above doesn't help in this case. Why not?

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Answers for Part F.

1. $5 \neq 3$, $5 \not< 3$, $5 > 3$, $5 \geq 3$

2. $-4 = -4$, $-4 \not< -4$, $-4 \not> -4$, $-4 \geq -4$, $-4 \leq -4$

3. $6 \neq -3$, $6 > -3$, $6 \geq -3$, $6 \not< -3$

4. $-10 \neq -9$, $-10 < -9$, $-10 \leq -9$, $-10 \not> -9$

5. $-\frac{10}{3} \neq 2$, $-\frac{10}{3} < 2$, $-\frac{10}{3} \leq 2$, $-\frac{10}{3} \not> 2$

6. $-\frac{1}{782} \neq 0$, $-\frac{1}{782} < 0$, $-\frac{1}{782} \leq 0$, $-\frac{1}{782} \not> 0$

7. $0 = 0$, $0 \not< 0$, $0 \not> 0$, $0 \leq 0$, $0 \geq 0$

8. $\frac{3}{7} = \frac{-3}{-7}$, $\frac{3}{7} \not< \frac{-3}{-7}$, $\frac{3}{7} \not> \frac{-3}{-7}$, $\frac{3}{7} \leq \frac{-3}{-7}$, $\frac{3}{7} \geq \frac{-3}{-7}$

9. $\frac{8-2}{3-7} \neq \frac{2-8}{3-7}$, $\frac{8-2}{3-7} < \frac{2-8}{3-7}$, $\frac{8-2}{3-7} \leq \frac{2-8}{3-7}$, $\frac{8-2}{3-7} \not> \frac{2-8}{3-7}$

10. The symbol '$\leq$' could be used to make a true sentence in each of the Exercises 1-9.

✳

Answer to question on last line of text:

The distance, on paper, between the dots for 2 and 4 is not the same as the distance between the dots for ¯3 and ¯4.

✳

You can make a true-false quiz on Part F by taking the sentences given above and making slight modifications in some of them. [For example, use '$5 \neq 3$' as it is, but change '$5 \not< 3$' to '$3 \not< 5$' or to '$5 \not> 3$'.]

F. F

We have found it convenient to use geometric terminology in talking about the system consisting of the set of real numbers together with the relation >. Thus, we call such a system <u>the number line</u>, and we call each element in this system a <u>point of the number line</u>. We extend the use of geometric terminology by introducing the phrases 'the distance between real numbers' and 'the distance between points of the number line'. [See TC[1-99]a.]

<div align="center">✳</div>

Answers for Part A [on pages 1-103 and 1-104].

1.	3	2.	3	3.	10	4.	5	5.	5
6.	32	7.	45	8.	0	9.	21	10.	1313
11.	747	12.	524						

13. In either case, after subtracting the two real numbers the girls should think of the number of arithmetic to which the real number corresponds. This number of arithmetic is the distance between Ruth's number and Rachel's number.

The picture didn't help because the distance between the dots for 2 and 4 is not the same as the distance between the dots for ⁻3 and ⁻1. Pictures of the number line such that sentences like '4 - 2 = ⁻1 - ⁻3' can be checked by comparing distances between dots are said to be drawn with a uniform scale. When you did the exercises in Parts A and B on page 1 - 100, you probably took it for granted that the pictures were drawn with a uniform scale.

THE ABSOLUTE VALUE OPERATION

You have seen that it is helpful to think of the set of real numbers as a line, each real number being a point of the line. When thinking about ordinary lines, we have a notion about what we mean by 'distance between points'. Can we develop such a notion for the number line? That is, can we make sense out of 'the distance between real numbers'?

Think of a picture of the number line which has been drawn with a uniform scale. The length of the segment between the dots for ⁻4 and ⁻1 is 3 units, no matter what unit was used in drawing the picture. Whatever unit was used in drawing the picture, the distance (with respect to this unit) between the dots is 3. So, let us agree that the distance between the real numbers ⁻4 and ⁻1 is 3. [As you saw in our earlier discussion of trips, distances are numbers of arithmetic. What is the distance between ⁻1 and ⁻4?]

EXERCISES

A.   For each of the listed pairs of real numbers, give the number of arithmetic which is the distance between the real numbers.

   1. (⁺7, ⁺10)   2. (⁺10, ⁺7)   3. (⁻5, ⁺5)

   4. (⁻7, ⁻12)   5. (⁺12, ⁺7)   6. (⁻8, ⁺24)

7. (⁻10, ⁺35)   8. (⁻72, ⁻72)   9. (⁻38, ⁻59)

10. (⁻724, ⁺589)   11. (⁻57, ⁻804)   12. (⁺72, ⁺596)

<div align="center">✻</div>

13. Ruth and Rachel each pick a real number. To find the
distance between these real numbers, Paul suggests that
Ruth subtract her number from Rachel's. Ned protests,
and says that Rachel should subtract her number from
Ruth's. Arthur, who understands that a distance is a
number of arithmetic, says that both are right as far as
they go, but each needs to take one more step.

     What is that step?

<div align="center">✻ ✻ ✻</div>

You have seen in the exercises in Part A that you can find the
distance between real numbers, say, ⁻5 and ⁻9, by first subtracting
either from the other.

$$^-5 - {}^-9 = {}^+4 \qquad \text{or} \qquad {}^-9 - {}^-5 = {}^-4$$

Then, the distance between the real numbers is the number of arith-
metic which corresponds with both of these differences.

    4 is the number of arithmetic which corresponds with ⁺4

<div align="center">and</div>

    4 is the number of arithmetic which corresponds with ⁻4.

So,

    the distance between ⁻5 and ⁻9 is 4.

We call the number of arithmetic which corresponds with a real
number the absolute value of the real number. For example,

    the absolute value of ⁻7 is 7,
    the absolute value of ⁺3 is 3,
    the absolute value of 0 is 0,
    the absolute value of −4 is 4,
  and the absolute value of (9 - 15) is 6.

So, the distance between two real numbers is the absolute value of
the difference of either one from the other.

From here, through page 1-110, it is essential to keep clear the distinction between numbers of arithmetic and nonnegative real numbers. So, we have been very careful to avoid creating contexts in which the meaning of a numeral might be ambiguous.

✳

Our definition of 'absolute value of a real number' differs from definitions given, for example, in college algebra textbooks. [Definitions of the latter kind make the absolute value of a real number a nonnegative real number, rather than a number of arithmetic.] We have experimented with both definitions and believe the one given on page 1-104 to be the more satisfactory to use at this level. On page 1-110 we point out that, just as names for numbers of arithmetic such as '5' and '9' can be used ambiguously to name nonnegative real numbers [$^+5$ and $^+9$, respectively], so can names of numbers of arithmetic like '$|^+5|$' and '$|^-9|$' be used ambiguously to denote nonnegative real numbers [$^+|^+5|$ and $^+|^-9|$, respectively].

1/119

Answers to questions on page 1-105.

Top of page: 27, 27, 27; 573, 573, 573; 700, 20, 73, 48, 100, 0.

List of pairs for absolute valuing: ($^-$3, 3), ($^+$3, 3), ($^-$142, 142),

($\frac{^-1}{2}$, $\frac{1}{2}$), ($^-5\frac{2}{3}$, $5\frac{2}{3}$), ($^-$192, 192), ($^-$1.92, 1.92), ($^-$17, 17), ($\frac{^+1}{4}$, $\frac{1}{4}$)

Bottom of page: (1) $^-$2, $^+$2   (2) $^-$3, $^+$3   (3) $^-$527, $^+$527 (4) No

(5) Yes, the absolute value of the real number 0 is the number 0 of arithmetic.

✳

Answers for Part B [on page 1-106].

1.	9	2.	17	3.	12	4.	20	5.	12
6.	142	7.	18	8.	0	9.	3	10.	15
11.	34	12.	15	13.	9	14.	7	15.	7
16.	12	17.	0	18.	219	19.	3	20.	5

✳

Answers for Part C [on page 1-106].

1. $^+$3, or: $^-$3   2. $^+$5, or: $^-$5   3. $^+$5, or: $^-$5

4. $^+$4, or: $^-$4   5. $^+$5, or: $^-$15   6. $^+$12, or: $^+$2

7. $^+$14, or: $^-$6   8. $^+$3

✳

Answers for Part D [on pages 1-106 and 1-107].

1.	>	2.	>	3.	>	4.	>	5.	<	6.	<
7.	>	8.	>	9.	=	10.	=	11.	>	12.	>
13.	=	14.	=	15.	=	16.	>	17.	<		

What is the absolute value of ⁻27? What is the distance between ⁻27 and 0? What is the distance between 0 and ⁻27?

What is the distance between ⁺400 and ⁺973? What is the absolute value of (⁺400 - ⁺973)? Of (⁺973 - ⁺400)?

What is the absolute value of (⁺634 - ⁻66)? Of (⁻5 × ⁺4)? Of ⁺73? Of ⁻48? Of ⁻100? Of 0?

Notice that absolute valuing is an operation. It takes you from each real number to a single corresponding number of arithmetic. We say that it is an operation on the set of real numbers to the set of numbers of arithmetic. Up to now, we have used a list of some of the pairs in an operation in order to "visualize" it. Make such a list [about ten pairs] for absolute valuing.
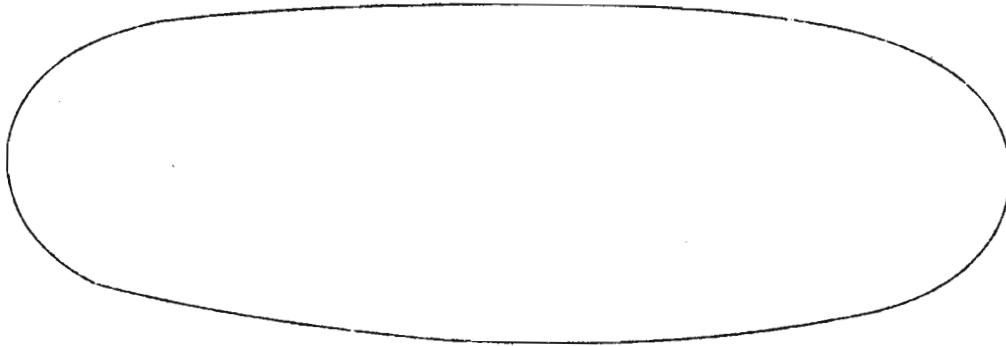
Here is another way to visualize absolute valuing.
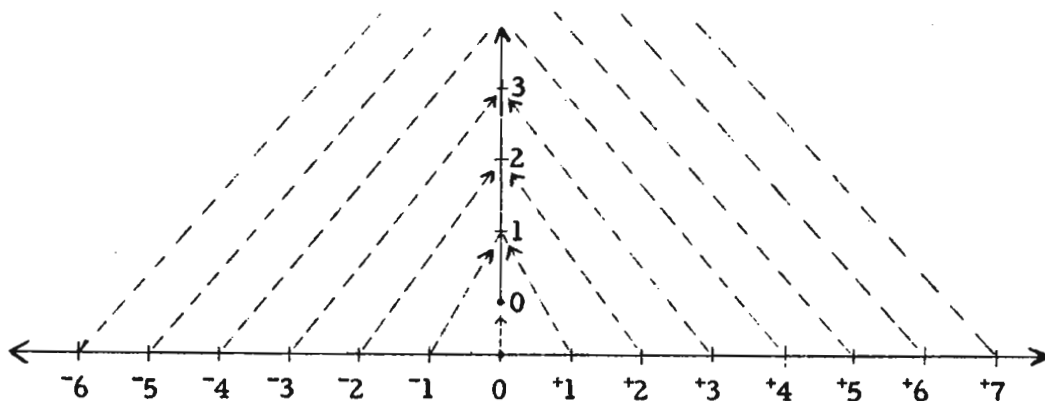
(1)   What two real numbers each have absolute value 2?

(2)   What two real numbers each have absolute value 3?

(3)   What two real numbers each have absolute value 527?

(4)   Does any real number have absolute value −3?

(5)   Does any real number have absolute value 0?

As in the case of other operations, it is convenient to have a sign for absolute valuing. The standard sign consists of two vertical bars. Thus,

$$‘\,|\,{}^-5\,|\,’ \text{ means the absolute value of } {}^-5,$$

and

$$‘\,|\,{}^+17\,|\,’ \text{ means the absolute value of } {}^+17.$$

[We pronounce ‘$|\ldots|$’ as ‘the absolute value of $\ldots$’.]

✳ ✳ ✳

<u>B</u>.   Simplify.

    <u>Sample</u>. $|\,{}^-3\,| + |\,{}^+7\,|$

    <u>Solution</u>. $|\,{}^-3\,| + |\,{}^+7\,| = 3 + 7 = 10$

1. $|\,{}^+7\,| + |\,{}^-2\,|$ 2. $|\,{}^-5\,| + |\,{}^+12\,|$ 3. $|\,{}^-8\,| + |\,{}^-4\,|$

4. $|\,{}^-9\,| + |\,{}^-11\,|$ 5. $|\,{}^+2\,| + |\,{}^+10\,|$ 6. $|\,{}^-71\,| + |\,{}^+71\,|$

7. $|\,{}^+21\,| - |\,{}^-3\,|$ 8. $|\,{}^-15\,| - |\,{}^+15\,|$ 9. $|\,{}^-103\,| - |\,{}^+100\,|$

10. $|\,{}^+5\,| \times |\,{}^-3\,|$ 11. $|\,{}^-2\,| \times |\,{}^-17\,|$ 12. $|\,{}^-5\,| \times |\,{}^+3\,|$

13. $|\,{}^+6 + {}^+3\,|$ 14. $|\,{}^+9 - {}^+2\,|$ 15. $|\,{}^+2 - {}^+9\,|$

16. $|\,{}^-5 + {}^-7\,|$ 17. $|\,{}^-5 + {}^+5\,|$ 18. $|\,{}^-102 - {}^+117\,|$

19. $|\,{}^+4 - {}^-3\,| + |\,{}^-5 - {}^-7\,| - |\,{}^+4 - {}^-2\,|$

20. $|\,{}^-2 + {}^-3\,| \times |\,{}^+4 - {}^-5\,| - |\,{}^+6 - {}^-2\,| \times |\,{}^+5 - {}^-5\,|$

<u>C</u>.   Fill in the blanks to make true sentences.

1. $7 + |\,\underline{}\,| = 10$ 2. $|\,\underline{}\,| + 4 = 9$

3. $8 - |\,\underline{}\,| = 3$ 4. $9 \times |\,\underline{}\,| = 36$

5. $|\,\underline{} + {}^+5\,| = 10$ 6. $|\,\underline{} + {}^-7\,| = 5$

7. $|\,\underline{} - {}^+4\,| = 10$ 8. $|\,\underline{} - {}^+3\,| = 0$

<u>D</u>.   Complete each to a true sentence by inserting one of the signs ‘ = ’, ‘ < ’, or ‘ > ’.

1. $|\,{}^+5\,| \quad |\,{}^+3\,|$ 2. $|\,{}^-5\,| \quad |\,{}^+3\,|$ 3. $|\,{}^+7\,| \quad |\,{}^+2\,|$

4. $|\,{-}^+7\,| \quad |\,{-}^+2\,|$ 5. $|\,0\,| \quad |\,{}^-7\,|$ 6. $|\,0\,| \quad 7$

7. $|\,{}^+\frac{6}{10}\,| \quad |\,{}^-\frac{2}{5}\,|$ 8. $|\,{}^-\frac{1}{2}\,| \quad |\,{}^+\frac{1}{3}\,|$ 9. $|\,{}^-\frac{8}{12}\,| \quad |\,{}^+\frac{2}{3}\,|$

Notice that (0, 0) belongs to each of these operations. To use '⁻' and
'⁺' as signs for the operations, we must define '⁻0' and '⁺0' to be
numerals for the real number 0. [Warning: Even though we are
claiming that ⁻0 = 0 and ⁺0 = 0, 0 is still neither negative nor positive.
But, it is both nonpositive and nonnegative.]

EXERCISES

A. For each number listed, tell by checking in the appropriate column
    whether it is a number of arithmetic, a positive real number, a
    negative real number, or the real number 0. [Since throughout
    this section on absolute valuing it has been essential to distinguish
    between numbers of arithmetic and real numbers, we have not used
    the convention according to which a numeral for a number of arith-
    metic is used to name the corresponding nonnegative real number.
    We continue this policy in the table.]

Number	Number of Arithmetic	Positive Real Number	Negative Real Number	Real Number 0
1. ⁺(7 ÷ 3)				
2. ⁺8 - ⁺10				
3. $\|⁻173\|$				
4. 45 - 45				
5. ⁺45 - ⁺45				
6. ⁺$\|⁻15\|$				
7. ⁻$\|⁺7 ÷ ⁻5\|$				
8. $\|⁺10 ÷ ⁻10\|$				
9. $\|⁻3\| - \|⁺3\|$				
10. ⁺$\|⁻2 × ⁺5\|$				
11. ⁺$\|⁻\|⁺5\|\|$				

10. $|^+7 - {}^+5| + |^+5 - {}^+2| \quad |^+7 - {}^+2|$

11. $|^+4 - {}^+6| + |^+6 - {}^+3| \quad |^+4 - {}^+3|$

12. $|^+4 - {}^-2| + |^-2 - {}^+8| \quad |^+4 - {}^+8|$

13. $|^-2 - {}^+1| + |^+1 - {}^+3| \quad |^-2 - {}^+3|$

14. $|^+2| \times |^-3| \quad |^+2 \times {}^-3| \qquad$ 15. $|^-3| \times |^-4| \quad |^-3 \times {}^-4|$

16. $|^+7 - {}^-3| \quad |^+7| - |^-3| \qquad$ 17. $|^+9 + {}^-2| \quad |^+9| + |^-2|$

DOES ABSOLUTE VALUING HAVE AN INVERSE?

(I) I am thinking of a number. If I add $^+2$ to it, I get $^+12$. What number am I thinking of?

(II) I am thinking of a number. The absolute value of this number is 7. What number am I thinking of?

Do you see a difference between these problems? You can tell what number was thought of in the first problem because the operation adding $^+2$ has an inverse. In the second problem, although you know that the number must be either $^+7$ or $^-7$, you can't tell which. The operation absolute valuing does not have an inverse. Let's look into what this means.

To begin with, we ought to be clear on what an operation is. The sets of pairs which we have been calling 'operations' have this impor- tant property--none of them contains two pairs that have the same first number [but different second numbers]. That is, an operation operates on a number to produce a unique result. [When the operation adding $^+2$ operates on $^-8$, the result is $^-6$, and nothing else; when the operation absolute valuing operates on $^-4$, the result is 4, and nothing else.] This notion of uniqueness of result is the essence of the idea of opera- tion. So, let us agree that a set of pairs is an operation just if no two pairs in the set have the same first number.

Now, from any operation you can get a second set of pairs by reversing each pair which belongs to the given operation. For example, when you reverse each pair in the operation adding $^+7$, you get a set of pairs. If you think a moment, you will see that this second set of pairs is an operation. In fact, it is the operation we call 'subtracting $^+7$'.

17. $|1| - |-2| \leq 1 - -2$

18. $|-2| \times |-3| = -2 \times -3$

19. $-2 \times {}^+3 \not< -2 \times {}^+3$

20. $|{}^+2| \times |{}^+3| = {}^+2 \times {}^+3$

21. $|4 - 7| = {}^-|7 - 4|$

22. $|{}^+2 - -13| = |-13 - {}^+2|$

23. $|-5 - -3| = |5 - 3|$

24. $|-15 - 2| = |2 + 15|$

C. When we think of the real numbers as points of a line, the number line, we can also think of trips on the number line. We have already decided how to find the distance between two real numbers, so we can also use real numbers to measure trips on the number line once we have chosen a positive direction. It helps in understanding many problems to translate them into problems concerning trips on the number line. For this purpose it is most convenient to choose the direction from 0 to ${}^+1$ as the positive direction. [For example, the measure of a trip from ${}^+7$ to ${}^+1$ is ${}^-6$.] We shall use this choice in each of the following exercises.

1. What is the measure of each of the following trips?

    (a) from 0 to ${}^+1$

    (b) from ${}^+6$ to ${}^+15$

    (c) from ${}^+15$ to ${}^+6$

    (d) from ${}^-3$ to ${}^+7$

    (e) from ${}^-6$ to ${}^-8$

    (f) from ${}^+5.7$ to ${}^+38.2$

    (g) from ${}^-7.6$ to ${}^-5.4$

    (h) from ${}^-127$ to ${}^-239$

2. [Draw a picture of the number line and label some of its points to help you with this problem.] A trip from point M to point N is measured by ${}^+2$, a trip from N to P is measured by -6, and a trip from P to D is measured by -2. If the point N is ${}^+5$, what real number is point D?

3. A trip from W to Q is measured by ${}^+9$, and a trip from Q to A is measured by ${}^+7$. If point A is ${}^+4$, what real numbers are the points W and Q?

D. 1. Using the positive direction for a profit, the answer is $^+2.75$ dollars.

2. Using the positive direction for points gained, the player's score at the beginning of a round could be represented by '$^+5$', so the change is measured by $^-7$, and his score at end of final round was $^-2$.

3. $^-12°$

4. $^-10$

5. Using the positive direction for temperatures above zero, the answer is $^+7°$.

6. 26,100 dollars.

7. $^-16°$

8. Using the positive direction for an increase in water volume, the answers are:

(a) $^+17$ gallons

(b) There must have been at least 25 gallons of water in the tank, but there could have been many more. The largest amount would depend, of course, on the size of the tank.

(c) 25 gallons

(d) 33 gallons

4.   The measure of a trip

>   from C to R is 14,   from R to S is —32,
>   from S to A is ⁺10, and from A to C is ⁺8.

Give the real numbers which are the points A, C, and S

(a)   if point R is 3.   (b)   if R is —3.

(c)   if R is 0.   (d)   if R is $\frac{1}{2}$.

(e)   if R is —105. 2.   (f)   if R is 3986. 7.

5.   A trip from A to Z is measured by —5, and a trip from Z to
      C is measured by ⁺9. If point A is —6, what is point C?

D.   Each of the following problems involves changes. [In Exercise 1
      the change is in financial status, in Exercise 2 the change is in
      point standing, etc.] Choose a direction of change for the positive
      direction, and use real numbers to measure changes.

1.   Bill is in business and loses $2.00 on one day and makes a
      profit of $4.75 on the second day. What is the change in his
      financial status over the two-day period?

2.   A player has 5 points at the beginning of a round, loses 3 points
      during that round, gains 7 points the next round, loses 12
      points the next round, and gains 1 point the final round. What
      is the change in his point standing from the beginning of the first
      round mentioned to the end of the final round? What was his
      point standing at the end of the final round?

3.   If the temperature is +15° Fahrenheit and it drops 27°, what
      is the temperature at the end of the drop?

[Note: In doing these problems does it help if you think of "trips"
        along the number line?]

4. Let us say that the main floor in a department store corresponds with the real number 0, that the floor 3 levels below corresponds with the real number +3, and the floor 2 levels above the main floor corresponds with -2. What real number corresponds with the floor at which the elevator stops after moving 11 floors up from floor +1?

5. The highest temperature on January 3 in Chicago was 10° above zero. On January 4 the highest temperature was 2° higher than on January 3, and on January 5 it was 5° lower than January 4. What was the highest temperature on January 5?

6. At the end of one year, a firm had a balance of $-10,700, and at the end of the next year, the balance was $15,400. How much better off was the firm at the end of the second year?

7. The temperature at 7 p.m. on a certain day in New York City was 7°. By 3 a.m. of the next day the temperature had dropped to -9°. What was the change in temperature from 7 p.m. to 3 a.m. the next day?

8. During one day 12 gallons of water was pumped out of a tank, and that night 17 gallons was pumped into the tank. During the next day 30 gallons was pumped out, and 42 gallons was pumped into the tank that night.

    (a) What is the change in volume of water in the tank from the morning of the first day to the morning of the third day?

    (b) How much water was in the tank on the morning of the first day?

    (c) What is the least amount of water which could have been in the tank on the morning of the first day?

    (d) How much water was there in the tank on the morning of the first day if there were 50 gallons in it on the morning of the third day?

E. 1. 21   2. 10   3. -108   4. 1   5. -1

6. 1, -1   7. -2

✲

F. Here are the lists.

| $\frac{1}{2}$ | $-\frac{6}{2}$ | 1 | $\frac{11}{3}$ |

$\frac{31}{62}$ -3 $\frac{12}{12}$ $\frac{5}{11} \times \frac{121}{15}$

0.5 $\frac{9}{3} \times (-1)$ $\frac{-.0042}{-.0042}$ $\frac{40 - -4}{12}$

350% of $\frac{1}{7}$

$-\frac{-2}{4}$ 20% of $(100\%$ of $5)$ $3\frac{2}{3}$

$-\frac{-8}{-16}$ $-2 \times -\frac{1}{2}$

$\frac{-4}{-8}$ $\frac{22}{6}$

$\frac{1}{3} + \frac{1}{6}$ 150% of 2 $\frac{3 + 7}{6 + 4}$

$\frac{1}{38 \times \frac{1}{19}}$ $-1 - -4$

$\frac{16}{32} + 0$

✲

G. 1. 10   2. -2   3. 30   4. 0   5. 2

6. -4   7. 6   8. 12   9. 12   10. -5

11. 25   12. 9   13. 6 [or: -6]   14. 9

15. 9   16. 3   17. -1   18. -1   19. -14

20. oppositing [or: subtracting 14, or: multiplying by -1,
          or: dividing by -1, or: adding -14]

21. 7

E. Guess the number.

1. A certain number is added to -5, and the result is 16.

2. -5 is subtracted from a certain number, and the result is 15.

3. A certain number is divided by 6, and the result is -18.

4. A certain number is added to its reciprocal, and the result is 2.

5. If I add a certain number to its reciprocal, the sum is -2.

6. If I add the opposite of a certain number to its reciprocal the sum is 0.

7. If I add a certain number to $+1$, I get -1.

F. Rearrange this list into columns with all the numerals for the same number in one column.

$\frac{1}{2}$ $-\frac{6}{2}$ $-\frac{-8}{-16}$ 1

                                          $\frac{31}{62}$

150% of 2 $\frac{11}{3}$ $\frac{12}{12}$ 0.5 $\frac{22}{6}$

          $\frac{-.0042}{-.0042}$ 350% of $\frac{1}{7}$ $\frac{5}{11} \times \frac{121}{15}$ -3

$\frac{40 - -4}{12}$ $-\frac{-2}{4}$ $-1 - -4$ 20% of (100% of 5)

          $\frac{-4}{-8}$

                    $\frac{9}{3} \times (-1)$ $3\frac{2}{3}$ $\frac{1}{3} + \frac{1}{6}$

$-2 \times -\frac{1}{2}$ $\frac{3 + 7}{6 + 4}$ $\frac{1}{38 \times \frac{1}{19}}$ $\frac{16}{32} + 0$

<u>G</u>.   Fill in the blanks to make true sentences.

   <u>Sample</u>.   (9, _____) belongs to the operation adding 5.

   <u>Solution</u>.   Since 9 + 5 = 14, we can make a true
                    sentence by writing a '14' in the blank.

1.   (3, _____) belongs to adding 7.

2.   (−8, _____) belongs to adding 6.

3.   (−5, _____) belongs to multiplying by −6.

4.   (−7, _____) belongs to subtracting −7.

5.   (_____, 11) belongs to adding 9.

6.   (_____, 36) belongs to multiplying by −9.

7.   (8, _____) belongs to the <u>inverse</u> of adding 2.

8.   (9, _____) belongs to the inverse of adding ⁻3.

9.   (−12, _____) belongs to oppositing.

10.   (_____, −5) belongs to sameing.

11.   (−5, _____) belongs to squaring.   [To square a number is to
                                          multiply it by itself.]

12.   (+3, _____) belongs to squaring.

13.   (_____, +36) belongs to squaring.

14.   (−27, _____) belongs to dividing by −3.

15.   (−27, _____) belongs to multiplying by the reciprocal of −3.

16.   (−3, _____) belongs to adding the opposite of −6.

17.   (7, −7) belongs to multiplying by _____.

18.   (7, −7) belongs to dividing by _____.

19.   (7, −7) belongs to adding _____.

20.   (7, −7) belongs to _____.

21.   (⁻7, _____) belongs to absolute valuing.

<u>H.</u> 1. −6   2. −28   3. −31   4. 43

5. −67   6. −46   7. −36   8. 105

9. (a) 24.366   (b) 7.7825   (c) $-4\frac{1}{2}$ (d) 1   (e) $-\frac{5}{16}$

10. (a) −5   (b) −6   (c) $-7\frac{5}{12}$ (d) −12   (e) −17

11. 74

<u>I.</u> To complete the table:

Feb. 68.6°,   March 69.0°,   April 70.1°,   May 72.0°,

June 73.4°,   July   74.3°,   Aug. 75.0°,   Sept. 75.7°,

Oct. 74.6°,   Nov.   73.7°,   Dec. 70.3°.

Answers to questions on p. 1-119.

1. 72.13° [Draw a line at '72.13' to simplify Exercise 2.]

2. Jan.   3.33,   Feb. 3.53,   Mar. 3.13,   Apr. 2.03,
    May   0.13,   June ⁻1.27,   July ⁻2.17,   Aug. ⁻2.87,
    Sept. ⁻3.57,   Oct. ⁻2.47,   Nov. ⁻1.57,   Dec. 1.83.

3. The positive differences total ⁺13.9, and the negative dif-
    ferences total ⁻13.9.

4. Yes   5.   Yes!

<u>J.</u> 1. 32°, 18°, 6°, ⁻7°, ⁻15°, ⁻23°.   2. 10°

3. Nome, 3°   4. 11°   5. Ketchikan   6. 1, 12

7. This will need some discussion; we merely wanted students
    to do some speculating about the uses for information of
    this kind.

<u>H.</u> Solve these problems.

1. The average of two numbers is 12. One of the numbers is 30. What is the other number?

2. Four numbers average −7. What is their sum?

3. Two numbers average ‑22. One number is −13. What is the other number?

4. Three numbers average +14. One number is −26; a second number is +25. What is the third number?

5. Two numbers average −25; one number is +17. What is the other number?

6. Two numbers average ‑33; one number is ‑20. What is the other number?

7. There are four numbers whose average is −9. What is their sum?

8. Two numbers average +27; one number is −51. What is the other number?

9. Find the average for each set of numbers.

   (a) {3.5, 6.09, 2.37, 100.12, 9.75}
   (b) {3.89, 12.1, 7.14, 8.0}
   (c) {+5, −14}
   (d) {+3, −101, +105, −3}
   (e) $\{-\frac{3}{8}, +\frac{1}{2}, -\frac{3}{4}, -\frac{15}{24}\}$

10. Fill in the blank so that the average for each set is −2.

   (a) {‑6, −9, +12, ____}
   (b) {−9, ____, 9}
   (c) $\{\frac{1}{2}, $ ____ $, -\frac{1}{3}, -\frac{3}{4}\}$
   (d) {−3.0, −3.4, ____, +2.0, +6.4}
   (e) {2, −2, −1, 0, ____, 3, 1}

11. The average of three numbers is 47. One of the numbers is 67. What is the sum of the other two?

I.   Below is a chart which gives the mean (average) temperature
for each month of a certain year in Hawaii. Use the chart and
complete the following table. Then answer the questions which
are at the top of the next page.

Month	Mean Temperature	Month	Mean Temperature	Month	Mean Temperature
Jan.	68.8°	May		Sept.	
Feb.		June		Oct.	
March		July		Nov.	
April		Aug.		Dec.	

Monthly Mean Temperatures in Hawaii
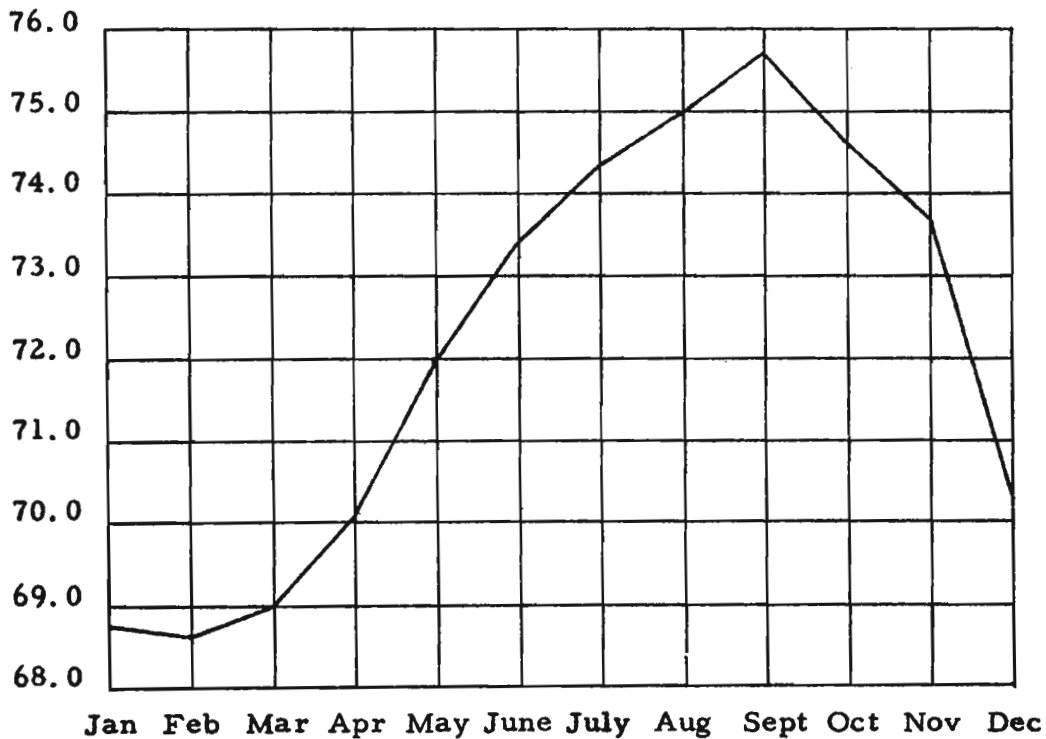
1. Find the mean of the mean monthly temperatures for the year.

2. Subtract the mean temperature for each month from the mean for the year.

3. Add all the differences obtained in Exercise 2.

4. Did you get 0 for Exercise 3?

5. Suppose you subtract the mean of a set of numbers from each of the numbers. The resulting differences are called deviations from the mean. Do you think that, for each set of numbers, the sum of the deviations from the mean is 0?

J. The bar chart on the next page shows mean January temperatures in Fahrenheit degrees computed over a period of years at some of the Alaska weather stations.

   1. Give the mean of the January temperatures at the following weather stations: Ketchikan, Valdez, Susitna, Ruby, Tanana, and Dawson.

   2. According to the graph, how many degrees colder was Anchorage than Seward?

   3. What station had the temperature closest to 0? What was this temperature?

   4. How many degrees warmer was Ruby than Barrow?

   5. What was the warmest station?

   6. How many stations had their mean January temperature above freezing? How many below freezing?

   7. The mean temperature for the thirteen stations is +2.8° What practical use could be made of this fact?

Fahrenheit Temperatures at
Various Alaskan Weather Stations

K.   Punctuate to make sense.

1.   Ray was making a list of the class. First he wrote Jim. Beside
     Jim he wrote 1. Then he wrote Helen and beside Helen he
     wrote 3. He looked at the 3 and realized that he had made an
     error. He erased the 3 and this time put 2 beside Helen.
     After writing 13 names he wrote 14. When he finished the
     list he had 23 names and 23 numerals.

2.   Harry in the sentence in which Chester had had had had had
     had had had had had the approval of the examiners.

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__K.__ Here is one way to punctuate the sentences.

  1. Ray was making a list of the class. First he wrote 'Jim'.
      Beside 'Jim' he wrote '1'. Then he wrote 'Helen' and
      beside 'Helen' he wrote '3'. He looked at the '3' and
      realized that he had made an error. He erased the '3' and
      this time put '2' beside 'Helen'. After writing 13 names
      he wrote '14'. When he had finished the list he had 23
      names and 23 numerals.

  2. Harry, in the sentence in which Chester had had 'had had',
      had had 'had'; 'had had' had had the approval of the exami-
      ners.

Answers for TEST.

  I.   1. −5   2. 2   3. 12   4. −9   5. 3
        6. −11   7. −13   8. 10   9. −5   10. 606
        11. 19.2   12. −14   13. −51   14. 0   15. 0
        16. 14   17. 30   18. −50   19. −48   20. 8

  II.   The book was on the table but it wasn't on a 'table'. If the
        book were on a 'table' that was on a paper on the table then
        you couldn't see that a 'table' was on the paper on the table.

  III. (b)   IV. (d)   V. (d)   VI. (a)   VII. (c)

  VIII.   1. >   2. >   3. <   4. >   5. =
          6. >   7. >   8. >   9. =   10. >

IV.   Which of the following is a name for $^-8 + 13$?

(a) $8 + {}^-13$ (b) $8 + 13$ (c) $-8 + {}^-13$

(d) $13 - 8$ (e) $-(-8 + 13)$

V.   Which of the following is an instance of the commutative principle for multiplication?

(a) $10 \times \left(\frac{1}{4} \times \frac{4}{5}\right) = \left(10 \times \frac{1}{4}\right) \times \frac{4}{5}$

(b) $\frac{1}{4} \times -\frac{1}{7} = -\frac{1}{4} \times \frac{1}{7}$ (c) ${}^+7 + {}^-2 = {}^-2 + {}^+7$

(d) $\frac{1}{2} \times -\frac{3}{7} = -\frac{3}{7} \times \frac{1}{2}$ (e) ${}^-12 + (8 \times {}^-3) = ({}^-12 + {}^-3) \times 8$

VI.   Which of the following is an instance of the associative principle for addition?

(a) $(-8 + -9) + 2 = -8 + (-9 + 2)$

(b) ${}^+2 + (-8 + {}^-9) = {}^+2 + ({}^-9 + -8)$

(c) $(-8 + 2) + {}^-9 = (-8 + {}^-9) + 2$

(d) $2 \times (-8 + {}^-9) = 2 \times {}^-8 + 2 \times -9$

(e) $(-8 \times {}^-9) \times 1 = -8 \times {}^-9$

VII.   Simplify.

$${}^+2 \times ({}^+3 \times {}^-2 + {}^-5 \times {}^+4) + {}^-10$$

(a) ${}^-42$ (b) ${}^+42$ (c) ${}^-62$ (d) ${}^-72$ (e) ${}^-18$

VIII.   Between the two numerals given in these exercises insert one of the symbols '>', '<' or '=' so that the resulting sentence is true.

1. $.52 \qquad .498$

2. ${}^-3.45 \qquad {}^-3.72$

3. $\frac{1}{7} \qquad \frac{2}{9}$

4. ${}^-756.3 \qquad {}^-784.5$

5. $.0025 \qquad \frac{1}{400}$

6. $.032 \qquad {}^-.033$

7. $0 \qquad -\frac{4}{5}$

8. $\frac{2}{11} \qquad 0$

9. $\frac{7 + 3}{28 - 8} \qquad \frac{167}{334}$

10. $\frac{1}{.4} \qquad \frac{1}{.5}$

IV.

IX.   1. F   2. F   3. F   4. F   5. T
      6. F   7. F   8. T   9. F   10. T
     11. T   12. F   13. T   14. F   15. F
     16. T   17. T   18. T   19. F   20. F

X.   1. $(0, ^-7)$, $(^-2, ^-9)$, $(3, ^-4)$, $(5, ^-2)$, $(10, 3)$ ... .
      2. $(2, 5)$, $(0, 3)$, $(^-1, ^+2)$, $(^-10, ^-7)$, $(21, 24)$ ... .
      3. $(20, -5)$, $(4, -1)$, $(2, -\frac{1}{2})$, $(-8, +2)$, $(-16, +4)$ ... .
      4. $(^+8, ^-8)$, $(^-6, ^+6)$, $(^-3, ^+3)$, $(^+101, ^-101)$, $(^-2.5, ^+2.5)$ ... .
      5. $(10, -20)$, $(1, -2)$, $(-4, +8)$, $(-10, +20)$, $(-2, +4)$ ... .

XI.   (c), (e)   XII.   (c), (d)

XIII. 1. C   2. F   3. B   4. E   5. H
       6. D   7. I   8. A   9. G   10. D
      11. B   12. A   13. J

XIV.   1. No! [The team had a net gain of only 9 yards.]
       2. $ 1686.68

IX. True or false? [In these exercises, we <u>are</u> using the convention according to which a numeral for a number of arithmetic is used to name the corresponding nonnegative real number.]

1. $-1000 > -2$

2. $-10 > 8$

3. $-\frac{21}{71} > -\frac{20}{71}$

4. $-\frac{14}{3} < -5$

5. $0.016 > 0.0016$

6. $-0.016 > -0.0016$

7. $1\frac{1}{2} < \frac{5}{4}$

8. $\frac{-6}{-3} = 2$

9. $-\frac{16}{3} > 1$

10. $-1425 < 0$

11. $^+|^+7| \not< |^-3|$

12. $|-10001| > |^-10002|$

13. $^-|5.47| \leq |5.469|$

14. $|8 - 13.5| \not> -|2.466|$

15. $|-1278.543| = -|1278.543|$

16. $^-|\frac{37}{15}| \leq ^-|2.466|$

17. $\frac{231}{157} \not< \frac{231}{258}$

18. $\frac{1}{.8} < \frac{1}{.79}$

19. $|87| \geq 92$

20. $.0079 \not> -|.00791|$

X. 1. List 5 pairs of real numbers which belong to the operation adding -7.

2. List 5 pairs of real numbers which belong to the operation which is the inverse of subtracting 3.

3. List 5 pairs of real numbers which belong to the operation dividing by -4.

4. List 5 pairs of real numbers which belong to the operation oppositing.

5. List 5 pairs of real numbers which belong to the operation which is the inverse of dividing by -2.

XI. Which operations are the same as multiplying by -3?

(a) dividing by -3

(b) multiplying by the reciprocal of -3

(c) dividing by the reciprocal of -3

(d) the inverse of multiplying by -3

(e) the inverse of dividing by -3

XII.   Which operations are the same as subtracting 6?

(a)   adding the reciprocal of 6

(b)   the inverse of subtracting 6

(c)   adding the opposite of 6

(d)   the inverse of subtracting −6

(e)   the inverse of adding −6

XIII.   Each of the following sentences is a consequence of one of the
principles you studied in this unit. Below them are the names of
these principles, each being preceded by a letter. In the blank
at the left of each statement write the letter corresponding to the
principle which is illustrated.

Sample: $\underline{I}$ 0.   5 + −5 = 0   [Note: The letter 'I' is placed
alongside the statement because
it illustrates the principle of
opposites.]

_____   1.   3 + (4 + 7) = (3 + 4) + 7

_____   2.   (5 + 0) + 7 = 5 + 7

_____   3.   4 + (7 × 9) = 4 + (9 × 7)

_____   4.   (6 × 2) + (4 × 2) = (6 + 4) × 2

_____   5.   (3 × 4) × 1 = 3 × 4

_____   6.   [(8 × 13) × 7] + 5 = [8 × (13 × 7)] + 5

_____   7.   ⁻7 + −⁻7 = 0

_____   8.   3 + (8 + 5) = (8 + 5) + 3

_____   9.   6 × (3 × 0) = 6 × 0

_____ 10.   (⁻587 × 169) × $\frac{1}{169}$ = ⁻587 × (169 × $\frac{1}{169}$)

_____ 11.   (7 × 0) + 5 = (0 × 7) + 5

_____ 12.   2 × [(8 + 4) + 3] = 2 × [3 + (8 + 4)]

_____ 13.   9 − 5 = 9 + −5

A.   Commutative principle for addition

B.   Commutative principle for multiplication

C.   Associative principle for addition

D.   Associative principle for multiplication

E.   Distributive principle for multiplication over addition

F.   Principle for adding 0

G.   Principle for multiplying by 0

H.   Principle for multiplying by 1

I.   Principle of opposites

J.   Principle for subtraction

XIV.   1.   In a football game, the team from Zilchville High got pos-
              session of the ball and gained 5 yards on the first down.
              They lost 12 yards on the second down, gained 8 yards on
              the third down, and gained 8 yards on the fourth down.
              Did they keep possession of the ball?

        2.   Below is a record of Mr. Sellars' bank account for one
              week.   Determine his balance at the end of the week.

              Monday -- Balance on hand: $1297.58; deposit: $415.00;
                            checks paid: $56.75, $32.19, $77.95.

              Tuesday -- Checks paid: $41.68, $8.92, $13.12, $87.78.

              Wednesday -- Deposits: $219.37, $682.46; checks paid:
                            $486.39, $17.62.

              Thursday -- Checks paid: $63.97, $39.76, $102.96.

              Friday -- Deposits: $57.81, $43.55.

SUPPLEMENTARY EXERCISES

A. Use single quotes in punctuating each of the following paragraphs in order to make sense out of it.

1. Marika, who just arrived in this country from Greece, went to the First National Store to buy a box of Kleenex. She walked up one aisle and down another until she saw a stack of boxes, each one of which had a Kleenex on it. She looked for the price and finally found 2/29 on the end of a box. She wondered if this meant that you could buy 29 boxes for 2 dollars or if it meant that 2 boxes cost 29 dollars. Neither possibility seemed very reasonable to her. She tried cross-ing out the 2s, but she wasn't sure whether she should get 0/9 or 1/19. She also thought about dividing both 2 and 29 by 2; since she wasn't sure, she decided to ask the clerk.

2. John was 8 years old. At his birthday party he had a cake with a small 8 on it. He wrote ate on a piece of paper and put the paper beside the cake. He put ate by the 8, but he didn't have 88. He had 8 ate. He could find the sum of 8 and 8 but he couldn't find the sum of 8 and ate. So, he ate the 8. But, he didn't eat the ate. The ate was left because it hadn't been eaten yet. John tried to eat the ate, but the ate was too big to be eaten. John took the e off the ate [it didn't hurt] and put it in front of the at. Then it spelled eat. So he did. He ate the eat. It was awful.

3. Mary is quite confused. She is Mary but yet she is not Mary. She said, ''If my name is Mary then I must be Mary. But you say I am not Mary. If I am not Mary, why does every-one call me Mary?'' I told Mary that if she wrote Mary on a piece of paper, that was Mary but not Mary. The reason for this is that Mary is Mary but Mary is not Mary. Do you think Mary will ever understand this? I hope so. I know that Mary doesn't understand this because it can't.

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Answers for SUPPLEMENTARY EXERCISES.

A. 1. In line 4, write single quotes about the 'Kleenex'; in line 5, about '2/29'; in line 9, about '2'; in line 10, about '0/9' and '1/19'.

2. In line 2, write single quotes about '8' and 'ate'; in line 3, about 'ate' and '8'; in line 4, about '88' and '8 ate'; in line 5, about '8' and 'ate'; in line 6, about '8' and both 'ate's; in line 7, about 'ate'; in line 8, about both 'ate's and the 'e'; in line 9, about 'at' and 'eat'; in line 10, about 'eat'.

3. In line 1, write single quotes about the third occurrence of 'Mary'; in line 2, about the first occurrence of 'Mary'; in line 3, about both 'Mary's; in line 4, about the first and third 'Mary's; in line 5, about the first 'Mary'. Line 6 may be punctuated:

---'Mary' is 'Mary' but Mary is not 'Mary'

or:

---Mary is Mary but 'Mary' is not Mary

In line 8, write single quotes about 'Mary'.

<u>A.</u>

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<u>B.</u>
1. $^+11$ 2. $^+2$ 3. $^+3$ 4. $^-11$

5. $^-2$ 6. $^-1$ 7. $^-14$ 8. $^+15$

9. $^-14$ 10. $^-8$ 11. $^-22$ 12. $^+15$

13. $^+4$ 14. $^+18$ 15. $^-47$ 16. $^-70$

17. $^-55$ 18. $^+20$ 19. $^+1$ 20. $^-8$

21. $^-53$ 22. $^-15$ 23. $^+46$ 24. $^+11$

25. $^+10$ 26. $^+5$ 27. $^-23$ 28. $^+1$

29. $^-20$ 30. $^+6$ 31. $^-17$ 32. $^-548$

33. $^+3$ 34. $^-\dfrac{7}{6}$ 35. $^+\dfrac{1}{3}$ 36. $^-\dfrac{11}{35}$

37. $^+\dfrac{1}{15}$ 38. $^-\dfrac{1}{8}$ 39. $^+\dfrac{11}{5}$ 40. $^+\dfrac{11}{12}$

41. $^-\dfrac{3}{7}$ 42. $^+\dfrac{31}{6}$ 43. $^+12.32$ 44. $^+2.62$

45. $^-17.07$ 46. $^-0.37$ 47. $^-4.073$ 48. $^+0.93$

49. $^+5.02$ 50. $^-6.73$

B. Simplify.

1. $^+8 + {}^+3$ 2. $^-7 + {}^+9$ 3. $^-3 + {}^+6$

4. $^-2 + {}^-9$ 5. $^+5 + {}^-7$ 6. $^-8 + {}^+7$

7. $^-4 + {}^-10$ 8. $^+12 + {}^+3$ 9. $^-5 + {}^-9$

10. $^-12 + {}^+4$ 11. $^-17 + {}^-5$ 12. $^+18 + {}^-3$

13. $^-17 + {}^+21$ 14. $^+33 + {}^-15$ 15. $^-51 + {}^+4$

16. $^-62 + {}^-8$ 17. $^-5 + {}^-50$ 18. $^+3 + {}^+17$

19. $(^+2 + {}^+3) + {}^-4$ 20. $(^-5 + {}^-8) + {}^+5$

21. $(^-51 + {}^+5) + {}^-7$ 22. $(^-7 + {}^+9) + {}^-17$

23. $(^+41 + {}^-10) + {}^+15$ 24. $(^-5 + {}^-11) + {}^+27$

25. $^+6 + (^-3 + {}^+7)$ 26. $^+8 + (^+9 + {}^-12)$

27. $^-12 + (^-3 + {}^-8)$ 28. $^+10 + (^-4 + {}^-5)$

29. $(^+4 + {}^-7) + (^-5 + {}^-12)$

30. $(^-6 + {}^+9) + (^-9 + {}^+12)$

31. $(^-71 + {}^+40) + (^-35 + {}^+49)$

32. $(^+124 + {}^-72) + (^-584 + {}^-16)$

33. $\dfrac{^+2}{3} + \dfrac{^+7}{3}$ 34. $\dfrac{^-5}{12} + \dfrac{^-9}{12}$ 35. $\dfrac{^+5}{9} + \dfrac{^-2}{9}$

36. $\dfrac{^+2}{7} + \dfrac{^-3}{5}$ 37. $\dfrac{^-1}{3} + \dfrac{^+2}{5}$ 38. $\dfrac{^+5}{8} + \dfrac{^-3}{4}$

39. $\left(\dfrac{^+1}{5} + \dfrac{^+2}{5}\right) + \dfrac{^+8}{5}$ 40. $\left(\dfrac{^-1}{3} + \dfrac{^+1}{2}\right) + \dfrac{^+3}{4}$

41. $\left(\dfrac{^-2}{7} + \dfrac{^-1}{3}\right) + \dfrac{^+4}{21}$ 42. $\left(\dfrac{^+8}{3} + \dfrac{^+10}{3}\right) + \dfrac{^-5}{6}$

43. $^+4.03 + {}^+8.29$ 44. $^-5.21 + {}^+7.83$ 45. $^-7.83 + {}^-9.24$

46. $^-2.1 + {}^+1.73$ 47. $^+3.08 + {}^-7.153$ 48. $^-5.9 + {}^+6.83$

49. $(^+5.93 + {}^-7.12) + {}^+6.21$ 50. $(^-5.08 + {}^+0.35) + {}^-2$

C. Answer each of the following questions.

1. On the New York Stock Exchange, a certain kind of stock whose
    par value is $50 (per share) was listed on Monday at $62\frac{1}{2}$, on
    Tuesday at $61\frac{7}{8}$, on Wednesday at $61\frac{1}{2}$, on Thursday at $61\frac{1}{8}$,
    and on Friday at $61\frac{3}{4}$.

    (a) Use real numbers to list the changes from each day to the
         next day.

    (b) Mr. Brockman sold 250 shares of this stock on Tuesday,
         and Mr. Stockert sold 250 shares of the same kind of stock
         on Friday. Mr. Brockman received how much more money
         than Mr. Stockert?

2. The table below gives the T. V. weatherman's report of the high,
    the low, and the normal mean temperatures in a certain city
    for the first week of June.

Day	High	Low	Mean	Normal Mean
Monday	87	61		72
Tuesday	85	57		70
Wednesday	80	48		68
Thursday	83	53		65
Friday	78	50		66
Saturday	75	47		68
Sunday	77	55		67

    (a) Compute the mean temperature for each day. [Weather-
         men compute the mean temperature by averaging the high
         and low temperatures. This does not give you the exact
         mean but does come close.]

    (b) Use real numbers to indicate the deviation of the mean
         from the normal mean for each day. If the mean for a
         given day is below the normal mean, use a negative real
         number for the deviation; if it is above, use a positive
         real number.

C. 1. (a) $\frac{-5}{8}$, $\frac{-3}{8}$, $\frac{-3}{8}$, $\frac{+5}{8}$.

   (b) $31.25

2. (a) Mon. 74, Tue. 71, Wed. 64, Thu. 68, Fri. 64, Sat. 61, Sun. 66.

   (b) ⁺2, ⁺1, ⁻4, ⁺3, ⁻2, ⁻7, ⁻1

   (c) ⁻2, ⁻5, ⁺3, ⁻5, ⁻3, ⁺2

   (d) ⁻5, ⁻8, ⁺4, ⁻2, ⁻3, ⁺8

3. (a) ⁻.5, ⁺.8, ⁻.9, ⁺.2

   (b) ⁻1.3, ⁻1.2, ⁺.4, ⁻1.1

   (c) 100.9, 100.5, 100.3, 100.2, 99.7

4. ⁺13

5. Yes, 5 units Northwest of the pump. [Bob hikes three times as fast as John.]

6. 5 blocks from Nick's home (going toward Bill's home) [or: 1 block from Bill's home when going toward Nick's home.]

7. 14 blocks, South [Carol traveled 4 times as fast as Sally.]

8. 16 miles, East [Betty hikes 5 times as fast as Jane.]

(c) Use real numbers to indicate the changes in the ''High'' readings from Monday through Sunday.

(d) Use real numbers to indicate the changes in the ''Low'' readings from day to day throughout the week.

3. The temperature chart of a certain hospital patient showed these readings.

	Morning	Noon	Evening
Wednesday	99.6	99.9	103.2
Thursday	99.1	100.2	102.2
Friday	99.9	100.1	100.9
Saturday	99.0	100.2	101.4
Sunday	99.2	99.6	100.3

(a) Use real numbers to indicate changes in the morning temperature readings for the successive days.

(b) Use real numbers to measure the changes in the evening temperatures for the successive days.

(c) Find the average temperature reading for each day.

Bruce was playing a game with Randy. On the first play Bruce gained 10 points. On successive plays he lost 6, gained 8, gained 3, lost 7, gained 9, lost 4. What was his standing after the final play?

←— Southeast   Northwest —→

pump for ⊖ drinking water
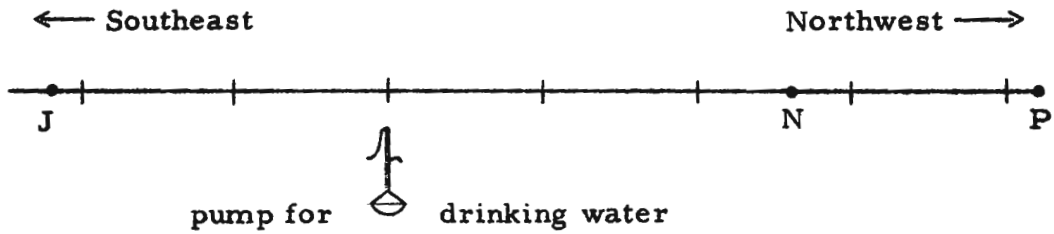
On a hiker's trail, John starts from P which is 21 units north-
west of the spot where pure drinking water may be obtained,
and walks toward the pump. Bob starts at J (which is 11 units
southeast of the pump) at the same time, and also hikes toward
it. They pass each other at N, which is 13 units northwest of
the pump, and continue on their way. However, Bob becomes
thirsty and turns back at P to hike to the pump, and then on
to J. Does Bob overtake John before he arrives at the pump?
How far from the pump is John when Bob overtakes him?
[The boys hike at steady rates.]

6. Nick's home
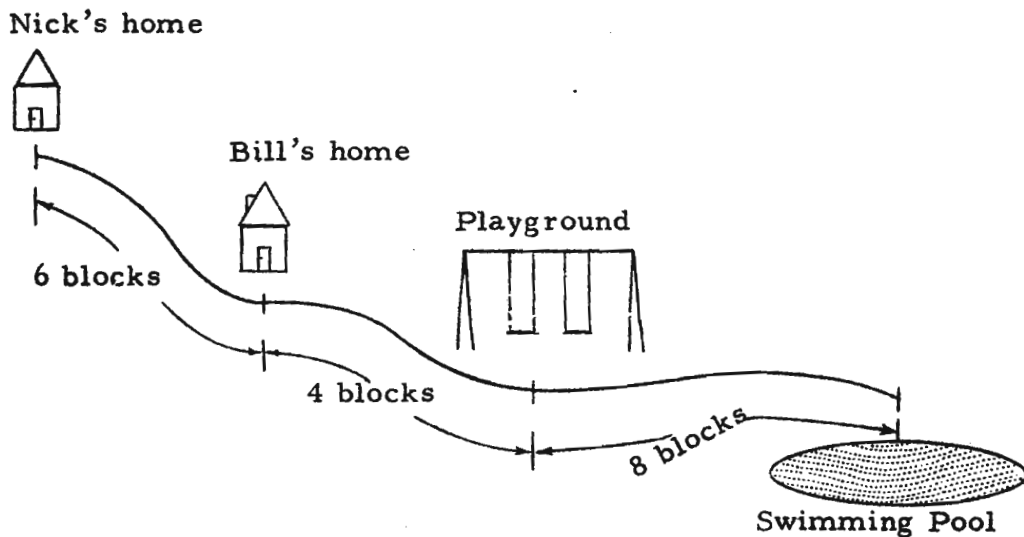
Bill leaves home at 10:00 a.m. to walk to the playground
which is 4 blocks away. His friend Nick leaves his home
at 10:00 a.m. to visit Bill.   When Nick arrives at Bill's
home and finds he isn't there, he has a hunch that Bill may

<u>D.</u>   1.   0

2. $^+10$

3. $^-\frac{1}{6}$

4. $^+8\frac{1}{6}$

5. $^+\frac{3}{8}$

6. $^+0.95$

7. $^+900$

8.   0

9. $^+\frac{3}{4}$

10. $^+7\frac{11}{16}$

11. $^-2.05$

12. $^-2,000,000$

13.   0

14. $^+1$

15. $^-1$

16.   0

17. $^-25$

18. $^-17.4$

19. $^-5.16$

20. $^-27$

21. $^+82$

22. $^+1,098,761$

23. $^-1$

24.   0

25.   0

26. $^+3\frac{5}{7}$

27. $^+19.61$

28. $^-4$

29. $^-\frac{5}{8}$

30. $^+\frac{35}{48}$

31. $^-16\frac{34}{35}$

32. $^-0.22$

33. $^+3\frac{97}{99}$

34. $^-19$

35. $^-17$

36. $^+5.02$

37. $^-0.33$

38. $^+0.099$

39. $^-0.09$

40.   0

41. $^-23$

42. $^-16$

43. $^-32$

44. $^+\frac{1}{8}$

45. $^-20$

46. $^-15$

47.   0

48. $^-150$

49. $^+100$

50. $^+50$

have gone to the playground, so he goes on toward it. Mean-
while, Bill hasn't found anything of interest at the playground,
so he has started on to the swimming pool, which is 8 blocks
farther. When Nick can't see Bill at the playground, he turns
around and starts home; however, when he gets there he
still wants to talk to Bill, and decides to return to Bill's
home. But Bill is on his way toward Nick's home, since he
saw none of his friends at the pool. If Bill and Nick have
walked at the same rate (and you disregard any time lost
while they look around at playground and pool), can you dis-
cover how far from Nick's home the two boys finally meet?

7.   Ann starts from Sally's home and bicycles north. At the
      same time, Carol starts from her own home, which is 22
      blocks south of Sally's, and is also bicycling north. Ann
      travels only 4 blocks when she turns around and starts south.
      Carol is 18 blocks north of Sally's home when she turns back;
      at that time Ann is midway between the point where Carol
      turns back and Carol's home. When Carol gets home and does
      not see Ann, she starts north again and travels until she meets
      her. How far from Sally's home do they meet? In what direc-
      tion from Sally's home is the meeting point?

8.   Jane and Betty both start hiking at the same time, and both
      travel in an easterly direction. Jane starts at M, where a sign
      post pointing west reads '4 miles to Fish Hook'. Betty starts
      at Q; the sign board there points east and reads, '8 miles to
      Fish Hook'.
          Betty passes Jane at the R signboard which points west
      and reads   '7 miles to Fish Hook'. She continues hiking
      east until she reaches a certain apple tree along the road.
      She turns here (after hastily picking an apple to munch on the
      way home!) and starts back to Q. On the way she meets Jane
      at the W signboard. It points west and reads, '10 miles to
      Fish Hook'.
          If both Betty and Jane are hiking at steady rates, what is
      the distance of the apple tree from the village of Fish Hook?
      What direction is the tree from the village?

<u>D</u>. Simplify.

1. $^+5 + {}^-5$

2. $0 + {}^+10$

3. $^-2\frac{1}{2} + {}^+2\frac{1}{3}$

4. $^+3\frac{2}{3} + {}^+4\frac{1}{2}$

5. $^+11\frac{1}{8} + {}^-10\frac{3}{4}$

6. $^+1 + {}^-.05$

7. $^+1001 + {}^-101$

8. $^-1,237,248 + {}^+1,237,248$

9. $\left(0 + {}^+8\frac{5}{16}\right) + {}^-7\frac{9}{16}$

10. $^+3\frac{7}{16} + {}^+4\frac{1}{4}$

11. $^+6.35 + {}^-8\frac{2}{5}$

12. $^-1,000,000 + {}^-1,000,000$

13. $^-2,000,000 + {}^+2,000,000$

14. $^-7 + {}^+8$

15. $^-8 + {}^+7$

16. $({}^-1 + {}^+1) + ({}^-1 + {}^+1)$

17. $^-423 + {}^+398$

18. $^-9.8 + {}^-7.6$

19. $^-2 + {}^-3.16$

20. $({}^-9 + {}^-9) + {}^-9$

21. $({}^+28 + {}^+29) + {}^+25$

22. $^+1,098,762 + {}^-1$

23. $^-17,098 + {}^+17,097$

24. $^+18,607,487 + {}^-18,607,487$

25. $({}^+6.5 + {}^-6.5) + ({}^+7.5 + {}^-7.5)$

26. $^+7 + {}^-3\frac{2}{7}$

27. $^+18.01 + {}^+1.6$

28. $\left({}^-3 + {}^-8\right) + {}^+7$

29. $^-3\frac{1}{2} + {}^+2\frac{7}{8}$

30. $^-8\frac{1}{3} + {}^+9\frac{1}{16}$

31. $^-7\frac{4}{7} + {}^-9\frac{2}{5}$

32. $^+.03 + {}^{\frac{-1}{4}}$

33. $^{\frac{-1}{9}} + {}^+4\frac{1}{11}$

34. $({}^+9 + {}^-5) + {}^-23$

35. $({}^-8 + {}^+3) + {}^-12$

36. $^+5 + {}^+0.02$

37. $^-.05 + {}^-.28$

38. $^+.312 + {}^-.213$

39. $^-.008 + {}^-.082$

40. $({}^+5 + {}^+2) + ({}^-3 + {}^-4)$

41. $({}^-10 + {}^+2) + ({}^-12 + {}^-3)$

42. $({}^-10 + {}^-7) + ({}^+3 + {}^-2)$

43. $({}^+12 + {}^-32) + {}^-12$

44. $\left({}^+\frac{1}{2} + {}^{\frac{-1}{4}}\right) + {}^{\frac{-1}{8}}$

45. $^-18 + ({}^+10 + {}^-12)$

46. $({}^-25 + {}^+5) + ({}^-4 + {}^+9)$

47. $({}^-17 + {}^-8) + ({}^-5 + {}^+30)$

48. $({}^-97 + {}^-3) + ({}^+25 + {}^-75)$

49. $({}^+302 + {}^-201) + ({}^-11 + {}^+10)$

50. $({}^+76 + {}^-16) + ({}^-20 + {}^+10)$

E.   Simplify.

1. $^-7 \times {}^+4$ 2. $^+7 \times {}^-4$ 3. $^+7 \times {}^+4$

4. $^-7 \times {}^-4$ 5. $^+8 \times {}^-5\frac{1}{2}$ 6. $^-8 \times {}^-5\frac{1}{2}$

7. $^-7\frac{1}{2} \times {}^-3\frac{1}{3}$ 8. $^-8 \times 0$ 9. $\frac{{}^-5}{4} \times \frac{{}^-4}{5}$

10. $^-6 \times {}^+4\frac{1}{3}$ 11. $^+10 \times {}^+\frac{1}{8}$ 12. $^+\frac{1}{2} \times {}^+2\frac{1}{3}$

13. $^+9\frac{1}{3} \times {}^-2\frac{1}{2}$ 14. $^-2\frac{1}{2} \times {}^-9\frac{1}{3}$ 15. $^+2\frac{1}{2} \times {}^-8\frac{1}{4}$

16. $^+5 \times {}^+2$ 17. $^+3 \times {}^+4$ 18. $^-7 \times {}^+3$

19. $^-4 \times {}^-9$ 20. $^+5 \times {}^-3$ 21. $^+8 \times {}^-11$

22. $^-12 \times {}^+9$ 23. $^-4 \times {}^+6$ 24. $^-3 \times 0$

25. $0 \times {}^+5$ 26. $^-7 \times {}^-7$ 27. $^-4 \times {}^-3$

28. $^-5 \times {}^+2$ 29. $0 \times {}^-6$ 30. $^+4 \times 0$

31. $(^-5 \times {}^-4) \times {}^+4$ 32. $(^-2 \times {}^-9) \times {}^-4$ 33. $^-7 \times ({}^+8 \times {}^-7)$

34. $(^+6 \times {}^-10) \times {}^-4$ 35. $(^+6 \times {}^-10) \times {}^+4$ 36. $(^-8 \times {}^-12) \times {}^+20$

37. $^+3 \times ({}^-4 \times {}^-5)$ 38. $(^-2 \times {}^-3) \times {}^-8$ 39. $^-10 \times ({}^-20 \times 0)$

40. $\frac{{}^-1}{2} \times (\frac{{}^-1}{2} \times {}^+1)$ 41. $(^-5 \times {}^-5) \times {}^-5$ 42. $(^-40 \times {}^+4) \times {}^-2$

43. $^+30 \times ({}^-2 \times {}^-4)$ 44. $(^-35 \times \frac{{}^-1}{5}) \times {}^+7$ 45. $^-4 \times ({}^-1 \times \frac{{}^-1}{4})$

46. $^-5 \times ({}^+10 \times \frac{{}^-1}{2})$ 47. $(^-10 \times 0) \times {}^-8$ 48. $(^-7 \times {}^-9) \times {}^+\frac{1}{3}$

49. $^+\frac{1}{4} \times ({}^-8 \times {}^-12)$ 50. $(^-12 \times {}^+\frac{1}{3}) \times {}^-9$

F.   Simplify.

1. $1 + 2 \times 7$ 2. $1 + 7 \times 2$ 3. $15 - 3 \times 2$

4. $12 + 4 \div 2$ 5. $6 \times 5 + 3$ 6. $11 \times 4 - 9$

7. $7 + 8 - 3 - 2 + 12$ 8. $12 \times 2 \div 4 + 1 \times 3 + 6$

9. $3 \times 2 \times (5 \times 12)$ 10. $3 \times 6 \times 5 + (1 + 3) \times 4$

11. $13 \times (2 \times 4) \div 8 + 3 \times 5 \times 2 \div 10$

(continued on next page)

12. $\dfrac{12}{5} \times \dfrac{10}{13} \times \dfrac{5}{2}$ 13. $44 \times 2 \div 8 \times 6 \div 3 \times 4\dfrac{1}{2}$

14. $36 \div \dfrac{1}{3} \times \dfrac{1}{6} + 48 \times \dfrac{5}{6} + 2$ 15. $^-2 \times (^-8 + ^-7) \times ^-8$

16. $(5 + ^-4) \times ^-3 + 8$ 17. $^-7 \times ^-3 \times ^-4 \times \dfrac{5}{6}$

18. $8 \times ^-3 + ^-2 \times ^-3$ 19. $13 + ^-26 \times \dfrac{1}{2} + ^-8$

20. $10 + ^-8 + [7 + ^-3 + ^-2] + ^-9 + 12$

21. $^-3 \times ^-8 + 2$ 22. $^-7 \times 3 + 4 \times ^-5$

23. $5 \times ^-7 + 5 \times ^-3$ 24. $^-2 \times 5 + 5 \times 6$

25. $^-82 + (^-3 + 82)$ 26. $^-2 \times (^-20 + 20)$

27. $12 \times ^-3 \times ^-1 \times ^-2$ 28. $^-3 \times 0 + ^-4 \times ^-1$

29. $(^-18 + ^-12) \times (^-5 \times 6)$ 30. $(48 + ^-16 + ^-32) \times (8 \times ^-9)$

G. Each of the following sentences is an instance of one of the principles for the numbers of arithmetic. Tell which principle.

1. $8 \times 8 + 2 \times 8 = (8 + 2) \times 8$ 2. $1 \times (5 + 2) = (5 + 2) \times 1$

3. $(3 + 4) + 5 = 3 + (4 + 5)$ 4. $73 + 0 = 73$

5. $18 + 32 = 32 + 18$ 6. $392 \times 1 = 392$

7. $618 \times 0 = 0$ 8. $17 \times (8 + 3) = 17 \times 8 + 17 \times 3$

9. $(16 \times 15) \times 4 = 16 \times (15 \times 4)$ 10. $397 \times 18 = 18 \times 397$

11. $798.3 + 0 = 798.3$ 12. $62 + 13 + 7 = 62 + (13 + 7)$

13. $3\dfrac{1}{2} + 5\dfrac{1}{4} = 5\dfrac{1}{4} + 3\dfrac{1}{2}$ 14. $357.25 \times 0 = 0$

15. $.7361 = .7361 \times 1$ 16. $14 \times 5 \times 18 = 14 \times (5 \times 18)$

17. $\dfrac{2}{5} \times \dfrac{3}{7} = \dfrac{3}{7} \times \dfrac{2}{5}$ 18. $373.8 = 373.8 + 0$

19. $16 \times 5\dfrac{1}{4} = (16 \times 5) + (16 \times \dfrac{1}{4})$

20. $7.23 + .77 = .77 + 7.23$ 21. $\dfrac{2}{3} \times \dfrac{5}{6} \times \dfrac{3}{5} = \dfrac{2}{3} \times \left(\dfrac{5}{6} \times \dfrac{3}{5} \right)$

22. $0 = 19.3 \times 0$ 23. $\dfrac{13}{17} \times 1 = \dfrac{13}{17}$

24. $\left(\dfrac{1}{7} + \dfrac{3}{14} \right) + \dfrac{11}{42} = \dfrac{1}{7} + \left(\dfrac{3}{14} + \dfrac{11}{42} \right)$

1.5

12. $\frac{60}{13}$ 13. 99   14. 60   15. ⁻240

16. 5   17. ⁻70   18. ⁻18   19. ⁻8

20. 7   21. 26   22. ⁻41   23. ⁻50

24. 20   25. ⁻3   26. 0   27. ⁻72

28. 4   29. ⁺900   30. 0

G.   1. dpma   2. cpm   3. apa   4. pa0

5. cpa   6. pml   7. pm0   8. ℓdpma

9. apm   10. cpm   11. pa0   12. apa

13. cpa   14. pm0   15. pml   16. apm

17. cpm   18. pa0   19. ℓdpma   20. cpa

21. apm   22. pm0   23. pml   24. apa

12. $\frac{12}{5} \times$

14.

1

25. pa0	26. cpa	27. ℓdpma	28. apm
29. pm0	30. pm1	31. apa	32. apm
33. dpma	34. pa0	35. cpm	36. pm1
37. apm	38. cpa	39. pm0	40. apa
41. dpma	42. cpm	43. pm1	44. pa0

H.	1. 120	2. 10,000	3. 620	4. 60
	5. 8000	6. 70	7. 91	8. 210
	9. 0	10. 0	11. 108	12. 2562
	13. 450	14. 1776	15. 720	16. 540

25. $57.3 = 57.3 + 0$ 26. $.32 + .73 = .73 + .32$

27. $35 \times 7 + 35 \times \frac{1}{5} = 35 \times 7\frac{1}{5}$ 28. $.4 \times (.8 \times 1.5) = (.4 \times .8) \times 1.5$

29. $93\frac{1}{7} \times 0 = 0$ 30. $475.8 = 475.8 \times 1$

31. $\left(10\frac{1}{2} + 3\frac{3}{4}\right) + 5\frac{1}{4} = 10\frac{1}{2} + \left(3\frac{3}{4} + 5\frac{1}{4}\right)$

32. $.92 \times .34 \times 5 = .92 \times (.34 \times 5)$

33. $(.90 + .02) \times 1.7 = (.90 \times 1.7) + (.02 \times 1.7)$

34. $27\frac{3}{5} + 0 = 27\frac{3}{5}$ 35. $187 \times 37 = 37 \times 187$

36. $43\frac{1}{3} \times 1 = 43\frac{1}{3}$ 37. $\frac{5}{6} \times \left(\frac{3}{7} \times \frac{2}{3}\right) = \frac{5}{6} \times \frac{3}{7} \times \frac{2}{3}$

38. $\frac{9}{15} + \frac{3}{8} = \frac{3}{8} + \frac{9}{15}$ 39. $0 = 1297.8 \times 0$

40. $827 + (73 + 769) = (827 + 73) + 769$

41. $25\frac{1}{4} \times 12 = 25 \times 12 + \frac{1}{4} \times 12$

42. $\frac{3}{14} \times \frac{2}{21} = \frac{2}{21} \times \frac{3}{14}$ 43. $\frac{9}{17} = \frac{9}{17} \times 1$

44. $\frac{9}{17} = \frac{9}{17} + 0$

H.   Simplify. Do as much of the computation as possible without writing.

1. $2 \times \{[(5 \times 3) + (5 \times 2)] + [(5 \times 5) + (5 \times 2)]\}$

2. $10 \times \{[50 \times 6] + [50 \times 7] + [(50 \times 5) + (50 \times 2)]\}$

3. $.5 \times \{[(62 \times 8) + (62 \times 4)] + [(62 \times 5) + (62 \times 3)]\}$

4. $\frac{1}{3} \times \{[(5 \times \frac{1}{2}) + (\frac{1}{2} \times 7)] + [(340 \times \frac{1}{2}) + (\frac{1}{2} \times 8)]\}$

5. $(8 \times 40) \times 25$ 6. $5 \times (2 \times 7)$

7. $(5 \times 7) + (7 \times 8)$ 8. $(15 \times 7) + (7 \times 15)$

9. 125×0 10. $75 \times 0 \times 9$

11. $(8 + 75) + 25$ 12. $752 + (48 + 1,762)$

13. $(45 \times 5) \times 2$ 14. $\frac{8}{9} \times (9 \times 222)$

15. $(36 \times 17) + (36 \times 3)$ 16. $(18 \times 27) + (3 \times 18)$

17. $(51 \times 1{,}763) + (51 \times 237)$ 18. $17\frac{1}{3} \times 12$

19. 102% of 35   20. $\frac{72}{100} \times 200$

21. $\left(\frac{7}{12} + \frac{5}{6}\right) \times \frac{6}{5}$ 22. $3\frac{1}{10} \times 8$

23. $(972.75 \times 37) + (37 \times 27.25) + 490$

24. $(29 \times 51) + (62 \times 51) + (9 \times 51)$

25. $\frac{1}{2} \times \{[33 \times 18] + [18 \times 7] + [18 \times 60]\} + 100$

I. Simplify.

1. $^-8 - {}^+3$	2. $^-8 - {}^-3$	3. $^+8 - {}^-3$
4. $^+3 - 0$	5. $0 - {}^+3$	6. $^+8 - {}^+3$
7. $^-6 - {}^+14$	8. $^-7 - {}^+8$	9. $^+9 - {}^-2$
10. $0 - {}^-4$	11. $^-6 - {}^-7$	12. $^-13 - {}^-13$
13. $2 - {}^-2$	14. $7 - {}^+2$	15. $^+9 - {}^+15$
16. $^-17 - {}^-8$	17. $^-23 - 31$	18. $17 - 29$
19. $^-4 - 9$	20. $9 - {}^-4$	21. $^-36 - {}^-43$
22. $^-5 - {}^-8$	23. $^-8 - {}^-5$	24. $5 - 8$
25. $8 - 5$	26. $0 - 47$	27. $17 - 17$
28. $638 - 635$	29. $^-638 - {}^-635$	30. $635 - 638$
31. $7 - {}^-31$	32. $^-5 - {}^-21$	33. $^+6 - {}^-31$
34. $8 - 6$	35. $10 - 18$	36. $0 - \frac{1}{8}$
37. $^-2 - {}^-13$	38. $^+8 - {}^-6$	39. $^+10 - {}^+28$
40. $^-15 - 0$	41. $^+5 - {}^+5$	42. $^+16 - {}^+14$
43. $^+\frac{1}{2} - {}^+\frac{1}{3}$	44. $^+2 - {}^+13$	45. $^+5 - {}^-5$
46. $^+16 - {}^-14$	47. $^+3 - {}^+1\frac{1}{4}$	48. $^-2 - {}^+13$
49. $^-5 - {}^+5$	50. $^+10 - {}^+8$	51. $^+3 - 0$
52. $^+2 - {}^-13$	53. $^-5 - {}^-5$	54. $^-16 - {}^+14$
55. $^-10 - {}^+28$	56. $^-.3 - {}^+.5$	57. $^+.02 - {}^-.09$

58. $^+155$ 59. $^-55$ 60. $^-826$ 61. $^+4.8$

62. $^-20.9$ 63. $^+36.4$ 64. $^-492$ 65. $^-46$

66. $^-7.4$ 67. $^+\dfrac{1}{2}$ 68. $^+\dfrac{1}{8}$ 69. $^+\dfrac{19}{15}$

70. $^-\dfrac{101}{63}$ 71. $^+\dfrac{3}{56}$ 72. $^+\dfrac{5}{16}$ 73. $^+9\dfrac{2}{5}$

74. $^-13$ 75. $^+1\dfrac{1}{2}$ 76. $^-20\dfrac{2}{5}$ 77. $^+\dfrac{17}{15}$

78. $^-18.9$

<u>J.</u>   1. $^-17$ 2. $^-7$ 3. $^+1$ 4. $^+2$

5. $^-8$ 6. $^+13$ 7. $^+21$ 8. $^-26$

9. $^-3$ 10. $^-4$ 11. 0 12. $^-19$

13. $^+8$ 14. $^+25$ 15. $^+2$ 16. $^-3$

17. $^-5$ 18. $^-29$ 19. $^-10$ 20. $^-3\dfrac{3}{4}$

<u>K.</u>   1. 17 2. -23 3. -2 4. -26

5. 5 6. -4 7. 0 8. -85

9. -54 10. -39 11. -14 12. -10

58. $^+103 - {}^-52$

59. $^-92 - {}^-37$

60. $^-102 - {}^+724$

61. $^+8.5 - {}^+3.7$

62. $^-9.6 - {}^+11.3$

63. $^+17.1 - {}^-19.3$

64. $983 - 1475$

65. $683 - 729$

66. $75.3 - 82.7$

67. $\dfrac{^+2}{3} - \dfrac{^+1}{6}$

68. $\dfrac{^-3}{4} - \dfrac{^-7}{8}$

69. $\dfrac{^+3}{5} - \dfrac{^-2}{3}$

70. $\dfrac{^-5}{7} - \dfrac{^+8}{9}$

71. $\dfrac{^+3}{7} - \dfrac{^+3}{8}$

72. $\dfrac{^-1}{16} - \dfrac{^-3}{8}$

73. $^+6\dfrac{1}{5} - {}^-3\dfrac{1}{5}$

74. $^-5\dfrac{2}{3} - {}^+7\dfrac{1}{3}$

75. $^-6\dfrac{1}{6} - {}^-7\dfrac{2}{3}$

76. $^-17\dfrac{1}{5} - {}^+3\dfrac{1}{5}$

77. $^-8\dfrac{1}{5} - {}^-9\dfrac{1}{3}$

78. $^-12\dfrac{1}{2} - {}^+6.4$

J. Simplify.

1. $^-8 + {}^-3 - {}^+6$

2. $^-4 - {}^-3 - {}^+6$

3. $^+7 - {}^-3 + {}^-9$

4. $^+5 - {}^-2 + {}^-5$

5. $^-6 - {}^+7 - {}^-8 + {}^-3$

6. $^+10 - {}^-3 - {}^+7 - {}^-7$

7. $^+11 + {}^-12 - {}^-13 + {}^+9$

8. $^-4 - {}^-5 - 12 - 15$

9. $15 - 6 - 3 - 7 - 2$

10. $19 + 5 - 3 - 2 - 23$

11. $(^+4 - {}^-3) + (^-5 + {}^-2)$

12. $(^-7 - {}^+7) + (^-8 - {}^-3)$

13. $(^+7 - {}^-2) - (^-3 - {}^-4)$

14. $(^+10 - {}^-3) - (^-5 + {}^-7)$

15. $(9 - 2) - (8 - 3)$

16. $(12 - 14) + (7 - 8)$

17. $(5 - 12) - (9 - 11)$

18. $(6 - 61) - (17 - 43)$

19. $(^-6.3 + {}^+1.4) - (^-4.8 + {}^+7.7) + (^-5.4 + {}^+3.2)$

20. $\dfrac{^-1}{2} + \left(\dfrac{^-3}{4} - \dfrac{^+1}{4} \right) + \left(^+3\dfrac{1}{4} + {}^-5\dfrac{1}{2} \right)$

K. Simplify.

1. $8 - 3 + 12$

2. $7 - 18 - 12$

3. $-5 + 9 - 6$

4. $-3 - 8 - 15$

5. $-6 + 9 + 2$

6. $-5 + 3 - 2$

7. $-12 - 3 + 15$

8. $-61 + 11 - 35$

9. $-10 - 4 - 40$

10. $-17 + 3 - 25$

11. $10 - 11 - 13$

12. $-9 - 8 + 7$

13. $-7 + 9 + 12 - 8 - 5 - 10 + 15$

14. $+12 + 8 - 9 - 6 - 5 + 7 + 3 - 8$

15. $-25 - 5 - 20 + 42 + 5 - 6 + 4$

16. $+70 - 20 - 7 - 6 + 3 + 4 - 14 - 27$

17. $+13 + 3 - 5 - 3 + 2 - 5 - 6$

18. $-27 - 13 + 32 + 4 - 9 - 3 + 5$

19. $-12 - 19 - 3 + 6 - 2 + 4 + 12 + 8$

20. $-13 - 2 + 9 + 8 + 7 - 10 - 3 + 7$

21. $+19 + 3 - 10 - 8 + 6 - 11 - 4 + 7$

22. $+17 - 4 - 3 + 6 - 8 - 12 + 5$

23. $-23 - 4 - 3 + 12 - 2 + 8 + 14 - 2$

24. $-31 + 27 - 3 - 8 + 2 + 14$

25. $+18 + 9 - 5 - 11 - 8 + 7 - 12$

26. $+17 - 8 - 11 + 9 - 12 - 5 + 7$

27. $-12 + 7 + 16 - 5 + 9 - 8 - 11$

28. $+7 - 8 - 12 + 15 - 5 - 11 + 9$

29. $-11 - 9 + 6 + 14 + 10 - 12 - 4$

30. $-5 + 9 - 8 + 7 + 13 - 11 - 12$

31. $+17 + 4 - 8 - 9 - 3 - 9$

32. $-23 + 19 - 3 + 10 - 5 - 7$

33. $-\frac{1}{2} + \frac{3}{4} - \frac{1}{4} + 3\frac{1}{4} - 5\frac{1}{2}$

34. $+.5 + .3 - .8 - .7 - .2 + 1.2$

35. $-7\frac{1}{2} + 2\frac{1}{4} + 1\frac{1}{2} - 8\frac{1}{4} - 3\frac{3}{4}$

36. $+.12 + .11 - .10 - .02 + .03 - .13 - .01$

37. $-6.3 + 1.4 + 4.8 - 7.7 + 3.2 - 5.4$

1/137

13. 6   14. 2   15. −5   16. 3

17. −1   18. −11   19. −6   20. 3

21. 2   22. 1   23. 0   24. 1

25. −2   26. −3   27. −4   28. −5

29. −6   30. −7   31. −8   32. −9

33. −2$\frac{1}{4}$ 34. .3   35. −15$\frac{3}{4}$ 36. 0

37. −10

13.

L.	1.	-12	2.	14	3.	-25	4.	-20
	5.	54	6.	-39	7.	-11	8.	-13
	9.	-4	10.	60	11.	0	12.	-18
	13.	-24	14.	14	15.	27	16.	-56
	17.	-10	18.	-3	19.	-11	20.	-55
	21.	57	22.	44	23.	-27	24.	-25
	25.	17	26.	0	27.	14	28.	-28
	29.	69	30.	32	31.	166	32.	320
			33.	-12			34.	2500

<u>L</u>. Simplify.

1. −3 × (5 − 1)   2. −7 × (8 − 10)   3. −5 × (7 − 2)

4. 5 × (4 − 8)   5. 6 × (7 + 2)   6. −3 × (8 − −5)

7. 4 + (9 − 12) × 5   8. 5 + 3 × (15 − 21)

9. 6 − 2 × (10 − 5)   10. 11 − 7 × (2 − 9)

11. −9 − 3 × (8 − 11)   12. −12 − 3 × (4 − 2)

13. (2 − 8) × (7 − 3)   14. (12 − 5) × (7 − 5)

15. (6 + 3) × (6 − 3)   16. (5 − 9) × (5 + 9)

17. −3 + (7 − 5) + (6 − 8) + (2 − 5) + (7 − 11)

18. −9 − (4 − 7) + (12 − 3) − (6 − 8) − (15 − 7)

19. −5 + 2 × (8 − 3) + 5 × (6 − 2) + 4 × (8 − 17)

20. −11 − 3 × (4 − 1) − 2 × (9 − 8) − 3 × (9 + 2)

21. (9 − 3) × (7 − 5) + (6 − 1) × (11 − 2)

22. (5 − 7) × (11 − 3) + (12 − 2) × (15 − 9)

23. (6 + 4) × (7 − 2) − (8 − 15) × (9 − 20)

24. (12 − 7) × (9 − 18) − (8 + 2) × (15 − 17)

25. [5 + 3 × (8 − 4)] × [9 − 4 × (8 − 6)]

26. [−6 − 2 × (7 − 10)] × [21 + 5 × (6 − 10)]

27. 11 − [6 + 2 × (7 − 3)] + [5 − 3 × (6 − 10)]

28. −19 + [−3 − 9 × (11 − 12)] − [−15 − 5 × (2 − 8)]

29. 15 − 2 × [3 + 4 × (7 − 2)] − 5 × [8 − 2 × (5 + 9)]

30. −83 − 2 × [−4 − 7 × (9 − 2)] − 3 × [−2 − (9 − 8)]

31. 100 − 6 × {3 − [5 − 2 × (7 − 3)] + [−2 − 3 × (7 − 2)]}

32. [17 − (8 − 3) × (4 − 7)] × [18 − (7 − 9) × (8 − 12)]

33. [51 − (6 + 5) × (7 − 2)] × [−17 − (5 − 9) × (8 − 3)]

34. −26 × (57 − 82) + −31 × (57 − 82) − 43 × (57 − 82)

M. Simplify.

1. $^+7 \div ^+2$ 2. $^-18 \div ^-3$ 3. $^-21 \div ^+7$

4. $^+42 \div ^-7$ 5. $^+8 \div ^-6$ 6. $^-8 \div ^-6$

7. $^-6 \div ^+8$ 8. $^-6 \div ^-8$ 9. $^+10 \div ^-20$

10. $^-20 \div ^+10$ 11. $^-20 \div ^-10$ 12. $^+10 \div ^+15$

13. $^+15 \div ^-3$ 14. $^-15 \div ^+3$ 15. $^+16 \div ^-32$

16. $^+26 \div ^-13$ 17. $^-34 \div ^-17$ 18. $^+27 \div ^-54$

19. $^-.08 \div ^+4$ 20. $^-.106 \div ^-.3$ 21. $^+.873 \div ^-.09$

22. $^-1,000,005 \div ^+15$ 23. $^+9873234 \div ^-1234$ 24. $^+225 \div ^-15$

25. $^-196 \div ^+14$ 26. $^+169 \div ^-13$ 27. $^-256 \div ^-16$

28. $^+289 \div ^-17$ 29. $^+324 \div ^+18$ 30. $^-361 \div ^-19$

31. $\dfrac{^+72}{^-8}$ 32. $\dfrac{^-54}{^+2}$ 33. $\dfrac{^-38}{^-19}$ 34. $98/14$

35. $132/12$ 36. $^-108/9$ 37. $56/^-8$ 38. $^-84/12$

39. $^-91/^-13$ 40. $\dfrac{128}{^-4}$ 41. $\dfrac{^-147}{^-3}$ 42. $\dfrac{^-156}{4}$

43. $\dfrac{999,999}{^-33}$ 44. $\dfrac{^-98762}{^-23}$ 45. $625/^-25$ 46. $\dfrac{63}{^-9}$

47. $\dfrac{^-171}{9}$ 48. $\dfrac{^-112}{^-8}$

N. Write a comparison sentence for each of the pairs of numbers listed.

1. $(^+5, \ ^+3)$ 2. $(^+7, \ ^-4)$ 3. $(^-5, \ ^+4)$

4. $(^-7, \ ^-12)$ 5. $(^-20, \ ^-15)$ 6. $(^+4, \ ^+17)$

7. $(^-17, \ ^-16)$ 8. $(^-16, \ ^-17)$ 9. $(^-16, \ ^+16)$

10. $(^+16, \ ^-16)$ 11. $(^+16, \ ^-17)$ 12. $(^-4, \ ^-4.1)$

13. $(^+5 + ^-7, \ ^-5 + ^+7)$ 14. $(^-2 \times ^-3, \ 2 \times 3)$

15. $(2 + 3\frac{1}{2}, \ 7 - \frac{1}{2})$ 16. $(9 - 17, \ 17 - 9)$

17. $(5 \times (3 - 5), \ ^-5 \times (5 - 3))$ 18. $(-3 - 1\frac{1}{4}, \ 5 - 9\frac{3}{4})$

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M.   1. $\frac{^+7}{2}$ 2. $^+6$ 3. $^-3$ 4. $^-6$

     5. $\frac{^-4}{3}$ 6. $\frac{^+4}{3}$ 7. $\frac{^-3}{4}$ 8. $\frac{^+3}{4}$

     9. $\frac{^-1}{2}$ 10. $^-2$ 11. $^+2$ 12. $\frac{^+2}{3}$

     13. $^-5$ 14. $^-5$ 15. $\frac{^-1}{2}$ 16. $^-2$

     17. $^+2$ 18. $\frac{^-1}{2}$ 19. $^-0.02$ 20. $^+\frac{53}{150}$

     21. $^-9.7$ 22. $^-66,667$ 23. $^-8,001$ 24. $^-15$

     25. $^-14$ 26. $^-13$ 27. $^+16$ 28. $^-17$

     29. $^+18$ 30. $^+19$ 31. $^-9$ 32. $^-27$

     33. $^+2$ 34. 7 35. 11 36. $^-12$

     37. $^-7$ 38. $^-7$ 39. $^+7$ 40. $^-32$

     41. $^+49$ 42. $^-39$ 43. $^-30,303$ 44. $^+4,294$

     45. $^-25$ 46. $^-7$ 47. $^-19$ 48. $^+14$

<div align="center">✳</div>

N.   [There is more than one way in which these comparison sentences could be written. We have listed several, but not all, possibilities for each exercise.]

     1. $^+5 > {}^+3$, $^+5 \not< {}^+3$, $^+5 \neq {}^+3$, $^+5 \geq {}^+3$

     2. $^+7 > {}^-4$, $^+7 \not< {}^-4$, $^+7 \neq {}^-4$, $^+7 \geq {}^-4$

     3. $^-5 < {}^+4$, $^-5 \not> {}^+4$, $^-5 \neq {}^+4$, $^-5 \leq {}^+4$

     4. $^-7 > {}^-12$, $^-7 \not< {}^-12$, $^-7 \neq {}^-12$, $^-7 \geq {}^-12$

     5. $^-20 < {}^-15$, $^-20 \not> {}^-15$, $^-20 \neq {}^-15$, $^-20 \leq {}^-15$

O. 1. <   2. >   3. >   4. =

5. >   6. >   7. >   8. <

9. >   10. <   11. <   12. <

13. <   14. <   15. F   16. F

17. T   18. T   19. F   20. T

21. T   22. T   23. T   24. F

25. F   26. T   27. T   28. F

29. T   30. F   31. F   32. T

33. T   34. T   35. F   36. T

37. F   38. F   39. F   40. T

41. T   42. F   43. F   44. T

45. F   46. F

O. Insert an ' > ', an ' < ', or an ' = ' to get a true statement.

1. $\dfrac{1}{0.9}$ $\dfrac{1}{0.8}$ 2. $7\dfrac{3}{10}$ $4\dfrac{4}{5}$

3. $\dfrac{1}{6}$ $\dfrac{1}{7}$ 4. $\text{-}8$ -8

5. 7 $\text{-}9$ 6. 148 -14

7. $\dfrac{1}{373}$ $\dfrac{1}{377}$ 8. $-5\dfrac{3}{8}$ $4\dfrac{1}{4}$

9. 342 $^{-}342$ 10. $-\dfrac{1}{6}$ $-\dfrac{1}{11}$

11. $^{-}1,999,999$ $^{-}1,999,997$

12. $-\dfrac{20}{31}$ $-\dfrac{20}{32}$ 13. $.082$ $.820$

                14. $^{-}1.001$ $^{-}1.0001$
                         ✳

True or false?

15. $\text{-}99 > -97$ 16. $8 > 8.1$ 17. $-\dfrac{17}{4} = -4.25$

18. $-1.2 \le 0$ 19. $7 \not< 8$ 20. $-10 \not> 0$

21. $-8 \not> \text{-}8$ 22. $-8 \not> 8$ 23. $8 \not> 8$

24. $8 \not> -8$ 25. $(-7 + 7) \not= 0$ 26. $.097 \ge \dfrac{12}{125}$

27. $\dfrac{3}{5} < .6001$ 28. $-\dfrac{1}{7} > -.142$ 29. $.0028 \not< \dfrac{7}{2500}$

30. $-5 \ge -2$ 31. $5 \not> 2$ 32. $5 \not= \text{-}5$

33. $\text{-}5 \not< -5$ 34. $2.3 \not> 2.4$ 35. $-\dfrac{1}{3.8} \ge \dfrac{1}{5.4}$

36. $-\dfrac{1}{200} \le -\dfrac{1}{2000}$ 37. $-13 = 13$ 38. $-.0008 \not< .008$

39. $-\dfrac{.198}{2} \ge -\dfrac{.196}{2}$ 40. $33\dfrac{1}{3} > 33.33$ 41. $^{-}.082 < {}^{-}.0082$

42. $^{+}.091 > {}^{+}.901$ 43. $-\dfrac{1}{5} \ge -\dfrac{1}{6}$ 44. $\dfrac{10000}{29786} > \dfrac{9999}{29787}$

        45. $\dfrac{.084}{.2} < \dfrac{.087}{.3}$ 46. $\dfrac{1}{.08} \le \dfrac{1}{.09}$