HIGH SCHOOL MATHEMATICS

Unit 2.

GENERALIZATIONS AND ALGEBRAIC MANIPULATION

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

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TEACHERS COMMENTARY

This unit treats one of the most important ideas in mathematics -- the idea of a variable. Since this is the first time the student will receive formal instruction in the use of variables, the instruction needs to be given with considerable care. Too often students come to their study of algebra with misconceptions about variables, and leave their study with many of these misconceptions still intact.

One of the innovations we have made in the teaching of this concept is a change in terminology. In the early years of the UICSM program, we used the term 'general number' to denote a variable. And, we noticed that this usage seemed to have unfortunate results. For example, many of our students felt [during a final examination] that the sentence:

$$-x \text{ is a negative number}$$

was a true one. [It is neither true nor false. See page 2-14.] This led us to believe that our students thought not only that numbers were marks on paper but that letters of the alphabet could be numbers. And, if one goes about using the words 'general number', 'literal number', or 'unknown number', to refer to the letters he uses in equations, inequations, and other sentences, it is understandable that the novice will regard these letters as numbers. Even the word 'variable' itself carries an unfortunate connotation, that of change. But, in an equation, say, '$x^2 + 5 = 9$' the use of the variable 'x' implies no notion of change.

We went to the mathematical logician in order to clarify for ourselves, and so for our students, the precise role that letters play in mathematical sentences. For one thing, we learned that a variable is a mark on paper. We do not regard a variable as something which is <u>denoted</u> by a mark as, for example, a number is denoted by a numeral. A variable is not a fuzzy thing which "jumps all over the place". In the equation '$x^2 + 5 = 9$' the letter 'x' <u>is</u> a variable; it is not the case that the letter 'x' stands for a variable.

A second thing we learned is that a variable does not have a referent. A variable, although it is a mark, is not a name. In fact, a variable is nothing more than a blank in an expression. For example, the blank in:

$$\underline{\qquad} + 2 = 9$$

is a variable. It is a symbol which holds a place for a name of an

object. In the sentence:

$$\_\_\_\_\_ + 2 = 9,$$

you can think of the blank as holding a place for names of real numbers. When the blank is replaced by a numeral for a real number, the sentence is converted into either a true sentence or a false sentence. The only essential difference between the sentences:

$$\_\_\_\_\_ + 2 = 9 \quad \text{and:} \quad x + 2 = 9$$

is that the former uses a blank as a variable, while the latter uses a letter as a variable.

The similarity between the role of variables in mathematical sentences and the role of pronouns in English sentences led us to coin the word 'pronumeral', and to use it instead of 'variable'. [Actually, 'pronumeral' is used instead of 'numerical variable'. A variable can also hold a place for names of objects other than numbers. Later in the UICSM program [see Unit 5] when variables are used in sentences which, for example, talk about sets, or points, or triangles rather than numbers, we introduce the word 'variable', and note at that time that 'pronumeral' refers to a rather special kind of variable.]

In order to combat the misconceptions students have about the role of letters, we introduce them to variables by using such marks as: □, ○, △, ______, and: ⬡. The transition from frames to letters is readily made.

Thus, there are three things about pronumerals which we stress:

(1) a pronumeral is a mark,

(2) a pronumeral is not a numeral, and

(3) a pronumeral is a mark which holds a place for numerals.

*

Another feature of Unit 2 which is novel and which deserves mention here is the matter of deductive proof. One of the criticisms sometimes heard about the UICSM program is that it is too formal in that it attempts to develop algebra "axiomatically". [Once in a while we are even told that we are teaching "groups, rings, and fields" to ninth graders!] We are somewhat at a loss in trying to understand this in view of the discovery aspects of Unit 1, but we

think the critics are claiming that we give the students a set of axioms about the real numbers and then ask them to deduce theorems from these axioms.

Actually, a major purpose [after teaching students how to use variables] is to help students to develop the manipulative skills which are customarily taught in beginning algebra courses. We think that the acquisition of these skills is a necessary condition for further study of mathematics. However, we also believe that the manner in which the skills are taught is an item of some importance. We know that students can acquire skill in manipulation even if they are told that '3a + 7a' is equivalent to '10a' because 3 apples and 7 apples make 10 apples, and that '3a + 7b' can't be simplified because you can't add apples and bananas [even though you can multiply them]! If students are given an abundance of illustrative examples, they can learn through imitation and ignore such improprieties. But, we think students ought to begin the acquisition of skills with the exercise of intelligence, and that explanations ought to be things which are based on mathematical principles, rather than be sheer memory devices. UICSM students discover that '3a + 7a' is equivalent to '10a' on the basis of numerical examples [both in Unit 1 and in Unit 2] and justify the equivalence by means of deductive proof in which they show that the generalization:

$$\text{For each a,} \quad 3a + 7a = 10a$$

is a consequence of the distributive principle for multiplication over addition and the computing fact that $3 + 7 = 10$. A student does not have to be told that to simplify '3a + 7a' he should add 3 and 7, write a '10', and write an 'a' next to it. This is a short cut which he will discover. It is important that the student become proficient in applying the short cut, and it is also important that he understand that the short cut is a consequence of certain basic principles about real numbers.

One sequence of content in Unit 2, then, is directed toward proficiency in manipulation. The proficiency is attained through drill in the use of short cuts. [A short cut is being applied when a student recognizes that he can compute $(2 \cdot 7)(5 \cdot 9)$ just by computing $10(7 \cdot 9)$.] And, the short cuts are justified by theorems which are deducible from the basic principles. Naturally, it is not necessary that a student actually deduce all of the theorems which justify the short cuts he will use. Even if it were possible to bring all students to the intellectual level necessary for such an accomplishment, you would not have the time to do it. The very best students in the class will be able to deduce all of the theorems and all students should be able to deduce some of them.

We open this unit on the use of letters in mathematical sentences with a carefully worked-out "gimmick" designed both to entertain the student and to help him build correct ideas. Although our development is completely different from that found in existing textbooks in high school mathematics, it is in line with the concept of the role of 'x', 'y', etc. most widely accepted by logicians. We hope you will find time to look into Tarski's Introduction to Logic and compare his treatment of variables with our development. Another book which you may find useful in this context is the 24th Yearbook of the National Council of Teachers of Mathematics, The Growth of Mathematical Ideas. The chapters "Language and Symbolism in Mathematics", by Fouch and Nichols, and "Relations and Functions", by May and Van Engen, are particularly relevant here.

*

We do not anticipate that the student will have trouble with this unit; however, you may have some difficulty in eliminating the traditional explanations and terminology from your own thinking and conversation. If you are also teaching traditional classes from a conventional textbook, your problem is even more difficult. We trust that you will bear with us even though it may mean temporary schizophrenia for you.

*

The colored insert behind page 2-B serves to make the holes stand out more clearly. The student should not make any marks on page 2-B. He will refrain from doing this if he reads carefully the instructions in the class exercise on page 2-C.

*

The student will need to read the material at the top of page 2-C in order to understand completely Mr. Jones' way of giving True-False tests. The class exercise on page 2-C should clarify the procedure for the student.

<u>True</u> <u>or</u> <u>False</u>.--Mr. Jones who teaches mathematics in Zabranchburg Junior High School has an interesting way of preparing True-False tests. First, he duplicates one sheet of items for each student in his class. Then he uses a paper punch to make holes at various spots on these pages.

Turn to page 2-B to see the first page of Mr. Jones' test.

Name ______________

Class ______________

Date ______________

TRUE - FALSE TEST

<u>Instructions:</u> Write 'T' in the space to the left of an item if the statement is true. Write 'F' in this space if the statement is false.

______ 1. $3 + 7 = 10$

______ 2. $8 - 5 = 12$

______ 3. $5 + \quad = 17$

______ 4. $4 + -3 =$

______ 5. $\quad \times 4 = 24$

______ 6. $\quad \div 3 = 15$

______ 7. $\quad \times 5 = 40$

______ 8. $-2 \times \quad = -8$

______ 9. $\quad + 9 = 19$

______ 10. $\quad \times 0 = 0$

Instruct the student to use the colored insert page as the second page. If he makes his numerals small, it will be easier for his neighbor to align the second page with the first. Suggest to the students that they make easy problems, rather than difficult ones, since the purpose of this exercise is not to test ability to do arithmetic computations. Also, they should take pains to see that about half the completed sentences are false, since there is a tendency to convert all of them into true ones. Check this by walking among them as they compose the second page.

*

Answers for Part D.

1. Yes

2. Yes

3. This exercise is a fooler. Actually, there is no single answer which every student should have for this item. Some students will have composed false items and some students will have composed true ones. The students should be able to foresee this situation. But, if some of the students do not understand that the answers may differ, you can ask for a show of hands of those who have 'T' and for a show of hands of those who have 'F'.

4. For items 1, 2, 10.

5. Some students will have used a numeral which makes the sentence true, and others will have used a numeral which makes the sentence false.

*

Some students may assert that the sentence in the tenth question on page 2-B is true or "always true". In that case, you must point out that what the student means is that no matter what numeral appears in the hole, the resulting sentence is true. But, the sentence:

$$\bigcirc \times 0 = 0$$

is neither true nor false. [See page 2-14.]

*

In reading page 2-D, you may want to encourage some discussion about why the students in Mr. Jones' class were unable to answer most of the items when Mr. Edwards was supervising the test. Elicit from the students the idea that a sentence with a hole in it is neither true nor false. [Later in the unit we call such sentences 'open sentences'.]

Next, he duplicates several different second pages. Each second page can be slid under the first page, and has numerals in positions matching the holes in the first page. When a student takes this test, Mr. Jones gives him a copy of the first page and a copy of one of the second pages. Then the student fastens the pages together and is ready to work on the test.

CLASS EXERCISE

A. Make up your own second page for the True-False test on page 2-B. Choose your numerals for the second page so that about half of the items on the test are false. Now, take the test and record your answers ['T' or 'F'] in a column on another sheet of paper.

B. Exchange second pages with your neighbor. Take this new test and record your answers in a column alongside the answers to the first test.

C. Repeat Part B by exchanging the second page you now have for that of another neighbor.

D. Look at your three columns of answers.

1. Is there a 'T' for item 1 in each column? Should every student in the class have written 'T' for item 1?
2. Is there an 'F' for item 2 in each column? Should every student have written 'F' for item 2?
3. What answer should every student have for item 3? Explain.
4. For what items on the test should every student have the same answer?
5. Explain why it is unlikely for every student to have the same answer for, say, item 8?

TRUE OR FALSE ???!!!

Mr. Edwards, principal of Zabranchburg Junior High, took over Mr. Jones' class one day when Mr. Jones was ill. Mr. Jones had sent instructions for Mr. Edwards to give the class the True-False test which was in the top drawer of the desk. But, Mr. Edwards was not told to give out the second page, also. At the beginning of the class period Mr. Edwards distributed the first page of the test to the class, and told them he would collect papers at the end of five minutes. In the meantime,

NO TALKING!

AND NO QUESTIONS!

Of course, no student had any trouble telling that the first statement was true and the second statement was false. Item 3 puzzled the students. They wished they could ask Mr. Edwards for the second page, but they remembered:

NO TALKING ! NO QUESTIONS !

Why were the students puzzled? Why could they answer item 2 but <u>not</u> item 6, for example? Think carefully about <u>why</u> the students were unable to answer most of the items.

In each of the exercises on pages 2-E, 2-F, and 2-G, the students must find a number which satisfies all of the sentences in the exercise. They will probably begin by looking for a numeral which when written in all the blanks yields true sentences. For example, in the case of the Sample, the student might choose the numeral '5' because he recalls that '5 + 3 = 8' is true. In this case he would fill in the blanks like this:

5 is an odd number. If I add 3 to 5 the sum is 8.

Another student might recognize that '8 - 3' is an appropriate numeral and would write:

8-3 is an odd number. If I add 3 to 8-3, the sum is 8.

Any other numeral for 5, say, '10 ÷ 2', or '4 + 1', might correctly be used in filling the blanks in the Sample. A student might even fill the blanks like this:

4+1 is an odd number. If I add 3 to 10÷2, the sum is 8.

Such a student should realize that, while he has fulfilled the instructions that he write in each blank a numeral for the same number, in order to know that he has done so, he has to know that 4 + 1 = 10 ÷ 2. The student who writes '5' in each blank, or who writes '10 ÷ 2' in each blank does not need such additional information. In any case, the number which the student should find is 5, whether he calls it '5', 'V', '$\frac{15}{3}$', or '57 - 52'.

*

These exercises should be handled on a very informal basis. Needless to say, this is not the place to introduce formal notions of equation-solving! You should work several exercises with the students to be sure they know how to proceed.

Give the students complete freedom in devising their own methods for discovering correct answers. You may have to caution students against getting help from home on these exercises.

*

Answers for Exercises [on pages 2-E, 2-F, and 2-G].

1. 6 2. $^{+}1$ 3. $^{-}1$ 4. $^{-}24$ 5. 20 6. 3

EXERCISES

Below are several exercises each having blanks. For each exercise find a number such that when a numeral for it is put in <u>all</u> of the blanks in that exercise, the sentences in the exercise become true. In some exercises there may be another number that will do the job. [In some there may not be any.]

<u>Sample</u>.

◯ is an odd number. If I add 3 to ◯, the sum is 8.

<u>Solution</u>. Pick a number at random, say, 9. Write a simple name for it in each blank.

(9) is an odd number. If I add 3 to (9), the sum is 8.

The last sentence is false, so try another number. Keep trying until you get true sentences.

Try 5.

(5) is a number. If I add 3 to (5), the sum is 8.

For 5, both sentences become true.

1. ◯ is an even number. If I multiply ◯ by 7, I get 42.

2. ◯ is a real number. If I subtract -3 from ◯, I get $+4$.

3. ◯ is a number. If I add 2 to ◯, the sum is 1.

4. If I divide the number ◯ by -3, the quotient is 8.

5. ◯ is an even number. If I divide ◯ by 4, I get 5.

6. If I add ◯ to ◯, I get 6, and if I add ◯ to 6, I get 9.

(continued on next page)

7. If I add ◯ to ◯ , the sum is ◯ . [Do you have a numeral for the same number in all three blanks?]

8. ◯ is a real number. If I multiply ◯ by ◯ , I get 100. [There are two numbers which will work!]

9. ◯ is a positive number. If I multiply ◯ by ◯ , the product is 81.

10. If I multiply ◯ by ◯ , the product is 25, and if I add ◯ to 7, the sum is 2.

11. If I multiply ◯ by 2 and add 4, I get 17.

12. If I add ◯ to 8 and divide by 3, I get 8.

13. If I add ◯ to 7, I get $2 \times$ ◯ .

14. If I subtract 5 from ◯ and multiply by 5, I get 0.

15. If I add ◯ to 12, I get $0 \times$ ◯ .

16. If I multiply ◯ by 4, the product is ◯ .

17. If I add ◯ to 4, the sum is ◯ .

18. ◯ is an odd number, ◯ > 2, and ◯ < 9.

7. 0 [You may find a tendency among some students to refer to two numbers in order to get true sentences. Thus, we have added the bracketed caution note.]

8. $^-10$ and $^+10$ [You may want to ask the class what answer they would give for Exercise 8 if '100' is interpreted as a numeral for a number of arithmetic rather than as a short name of $^+100$.]

9. $^+9$ 10. $^-5$ 11. $6\frac{1}{2}$ 12. 16

13. 7 14. 5 15. $^-12$

16. 0 [As in Exercise 7, you may find that some students hesitate to select 0. Use this opportunity to stress that 0 is a perfectly respectable number.]

17. There is no number which will work. [Some students may object to this exercise because there is no number whose name will convert the sentence to a true one. Others may suggest that 0 is the required number, since "nothing works and 0 is nothing". Point out to students that in this course they should expect to find problems whose solutions are somewhat unconventional.]

18. Students should supply all three of the required numbers: 3, 5, and 7.

*

Short quizzes are given on the pages listed below.

TC[2-G, H, I]	TC[2-22, 23]a, b	TC[2-31, 32]d
TC[2-37]c	TC[2-38]e	TC[2-48, 49, 50]b
TC[2-51]d	TC[2-76]a	TC[2-88]c
TC[2-90]b	TC[2-99]b	TC[2-111]

19. $^{-}7$, $^{+}7$ 20. Each number will work. 21. Each number will work.

22. 0, $^{+}1$ 23. 11, 22, 33, 44, 55, 66, 77, 88, 99

*

We think that Sammy's attitude is a common one among beginning students. You might ask the students if they also believe as Sammy does that algebra is arithmetic with letters. If one thinks of algebra as arithmetic with letters, it is perfectly reasonable to wonder, "Does a + b = c?" or, "Does x - y = w?" and to feel that once one learns the correct combinations of letters, one can do algebra just as one did arithmetic. It is one of the major purposes of this unit to reveal to students the role that letters actually play in algebra.

*

The essence of the concept of variables is contained in Fred's discovery [described on page 2-I] of the similarity between letters and the holes in the paper. Both the letter and the hole in the paper are marks which hold places for numerals [but <u>not</u> for numbers]. Try to get the students to point out the similarity in function between the letters and the holes in the paper.

*

Here is a quiz for maintaining skill in using the principles for short cuts. You should ask the students to write answers as quickly as possible.

Simplify.

1. $3.26 \times 82 + 3.26 \times -182$
2. $453 + 39 + 7$
3. $\frac{3}{5} \times -793 \times \frac{5}{3}$
4. $-12 + 876 + 512$
5. $1875 \times 82 \times 0 \times 367.59$
6. $\frac{8}{11} + (64 + -\frac{19}{11})$
7. $-3 \times 15789.6 + 15789.6 \times 13$
8. $15 \times .98$
9. $57 \times 68 + -18 \times 57$
10. $63 \times 52 + 37 \times 52$

*

Answers for Quiz.

1.	−326	2.	499	3.	−793	4.	1376	5.	0
6.	63	7.	157896	8.	14.7	9.	2850	10.	5200

19. The absolute value of $\bigcirc$ is 7.

20. The sum of $\bigcirc$ and 2 is the sum of 2 and $\bigcirc$.

21. $\bigcirc \times 3 = 3 \times \bigcirc$. 22. $\bigcirc \times \bigcirc = \bigcirc$.

23. A name of $\bigcirc$ is a two-digit numeral for a positive whole number. If the digits in the numeral are interchanged, the result is a two-digit numeral which is still a name of $\bigcirc$.

SAMMY'S PROBLEM

Sammy, a student in Mr. Jones' class, heard his older brother say that algebra was a lot different from arithmetic. "In algebra you do problems with letters as well as with numbers." Sammy was puzzled by his brother's remark because he didn't know, for example, how to add letters and numbers. He wondered what a + b could be. Was it c? How could you add 3 and x? Since Sammy was hardly an expert in mathematics, he supposed that a few of the "whizzes" in his class might know. But, he didn't want to approach them directly with his problem.

The next day he thought he found a chance to learn some arithmetic with letters. Mr. Jones gave out the first page of a new True-False test. He told the students that he was not going to hand out second pages this time but, instead, he wanted each student to make up his own second page [as you did on page 2-C] and hand it to his neighbor. In this way, each student made a test for his neighbor. Can you guess the kind of second page Sammy wrote?

Sammy exchanged second pages with Fred. [Fred's nickname was 'The Brain'.] Fred answered the first two items and then stopped in bewilderment. Turn the page to see his test.

Name Fred

Class Math. I

Date October 20

TRUE - FALSE TEST

Instructions: Write a 'T' in the space to the left of an item if the statement is true. Write 'F' in this space if the statement is false.

F 1. $4 + 12 = 17$

T 2. $9 \times 6 = 54$

___ 3. $3 + x = 2$

___ 4. $5 \times b = 25$

___ 5. $8 - 9 = -1$

___ 6. $7 \times x = 15$

___ 7. $w + 2 = 6$

___ 8. $3 \times 7 = 34$

___ 9. $8 + y = 10$

___ 10. $m \times 7 = 34$

Fred raised his hand and said, "Mr. Jones, I don't know how to do this test." Mr. Jones was very surprised, and so was the rest of the class, for there never seemed to be a problem that Fred couldn't solve. Sammy thought to himself that now he would <u>never</u> find out how to do arithmetic with letters. Fred said that Sammy had put letters instead of numerals on the second page. He said, "It's impossible to tell whether '3 + x = 2' is true or false because . . ." Suddenly, Fred stopped talking. He seemed to be thinking very hard. Then he said,

> "I know. The thing wrong with this test is just what was wrong with the test when Mr. Edwards was here."

Fred was right. Can you tell why?

In connection with the discussion of true sentences and false sentences, it may be helpful to recall to the students the discussion on page 1-L of Unit 1. A sentence which consists of an equality sign flanked by a pair of numerals is a true sentence if the numerals are names of the same number; it is a false sentence if the numerals are names of different numbers.

*

In Part A of the exercises we introduce other kinds of "holes". This corresponds to introducing several letters as variables in a conventional course.

*

Pages 2-2 through 2-7 are so designed that students can complete the exercises with a minimum of writing. We suggest that you do not compel the students to copy the given sentences. This is extremely time-consuming, and may make students resentful. All they need do is write numerals in the frames and decide upon the truth or falsity of the resulting sentence. If you want to make a personal check of each student's book, you can have him remove the pages from the text and submit them to you; they can be corrected, returned, and replaced in the text.

2.01 Sentences.--When Mr. Edwards gave out only the first page of the True-False test, the students felt that most of the questions were silly. How could you ask "True or false?" about a sentence with a hole in it? A sentence such as:

$$9 + \bigcirc = 15$$

is neither true nor false. The sentence:

$$9 + 6 = 15$$

is true because '9 + 6' and '15' are numerals for the same number, and the sentence:

$$9 + 7 = 15$$

is false because '9 + 7' and '15' are numerals for different numbers. But, since a hole is not a numeral,

'$9 + \bigcirc$' is not a numeral,

and so, the sentence with the hole in it is neither true nor false.

You can convert a sentence which has a hole in it into a sentence which is either true or false by putting a numeral in the hole.

EXERCISES

A. Each of the following exercises contains a sentence with one or more "holes" in it. Frames like:

are used to show you where the holes are. Your job is to put a numeral in each hole in the sentence as instructed, and then tell whether the new sentence you get is true or false.

Sample 1. (a) Write a '7' in each frame.

$$\square + \square + 3 = \square - 2$$

(b) Write a '-5' in each frame.

$$\square + \square + 3 = \square - 2$$

(c) Write a '5' in each frame.

$$\square + \square + 3 = \square - 2$$

Solution.

(a) $$\boxed{7} + \boxed{7} + 3 = \boxed{7} - 2$$

This sentence is false because
$7 + 7 + 3$ is 17, $7 - 2$ is 5, and $17 \neq 5$.

(b) $$\boxed{-5} + \boxed{-5} + 3 = \boxed{-5} - 2$$

Since $-5 + -5 + 3 = -7$ and $-5 - 2 = -7$,
this sentence is true.

(c) $$\underbrace{\boxed{5} + \boxed{5} + 3}_{13} = \underbrace{\boxed{5} - 2}_{3}$$

Since $13 \neq 3$, this sentence is false.

1. (a) Write a '3' in each frame.

$$2 \times \square + 5 = 7 \times \square - 5$$

(b) Write a '-2' in each frame.

$$2 \times \square + 5 = 7 \times \square - 5$$

(c) Write a '2' in each frame.

$$2 \times \square + 5 = 7 \times \square - 5$$

Answers for Part A [which begins on page 2-1 and continues through page 2-5].

1. (a) F (b) F (c) T

2. (a) T (b) T (c) T

[In Exercise 2 some students may be able to predict that the completed sentences will be true ones. The basis for this prediction, of course, is the fact that the completed sentences are consequences of the distributive principle and the fact that $3 + 2 = 5$. A student may be tempted to assert that the open sentence itself is true. Point out to him that what he really means is that you get a true sentence however you fill the holes with numerals.]

3. (a) F (b) F (c) F

4. (a) T (b) F (c) F

5. (a) F (b) T (c) F

6. (a) T (b) T (c) T

[In connection with Exercise 6, you may want to ask the students if they could ever convert the given sentence into a false one. This type of question will make all students aware of the fact that the converted sentences are instances of the commutative principle for addition.]

7. (a) F (b) F (c) F

8. (a) T (b) F (c) F

9. (a) F (b) T (c) T

[In connection with Exercise 9, there are two possible interpretations [See page 1-110.]. The exercise becomes trivial if the left sides of the sentences obtained in (a), (b), and (c) are interpreted as numerals for numbers of arithmetic, for the right sides name real numbers. [This is the case because absolute valuing is an operation on real numbers; hence the '5', '-3', and '2' which are to be used as replacements for '$\bigcirc$' [in sentences (a), (b), and (c), respectively] must be interpreted as numerals for real numbers.] With such an interpretation, each sentence is false. [In (c), since the '2' which replaces '$\bigcirc$' is a numeral for the real number $^{+}2$, the expression '$2 - 2$' which occurs in the "completed" sentence must name the real number 0.] However, if we interpret '$|\bigcirc - 2|$' as an abbreviation for '$^{+}|\bigcirc - 2|$', then, while the sentence obtained in (a) is again false, those obtained in (b) and (c) are true. In general, '$|\ldots\ldots|$' should be interpreted as an abbreviation for '$^{+}|\ldots\ldots|$' whenever not doing so would render an expression nonsense [Example: $|5| + {}^{-}3$], and whenever not doing

2. (a) Write a '4' in each frame.

$$3 \times \text{⬡} + 2 \times \text{⬡} = 5 \times \text{⬡}$$

(b) Write a '0' in each frame.

$$3 \times \text{⬡} + 2 \times \text{⬡} = 5 \times \text{⬡}$$

(c) Write a '−5' in each frame.

$$3 \times \text{⬡} + 2 \times \text{⬡} = 5 \times \text{⬡}$$

3. (a) Write a '−2' in each frame.

$$9 \times \text{⬡} + 7 = 4 - 2 \times \text{⬡}$$

(b) Write a '0' in each frame.

$$9 \times \text{⬡} + 7 = 4 - 2 \times \text{⬡}$$

(c) Write a '1' in each frame.

$$9 \times \text{⬡} + 7 = 4 - 2 \times \text{⬡}$$

4. (a) '−1' in each frame.

$$\square + 8 = 6 - \square$$

(b) '6' in each frame.

$$\square + 8 = 6 - \square$$

(c) '2' in each frame.

$$\square + 8 = 6 - \square$$

5. (a) '$\frac{16}{3}$' in each frame.

$$\bigcirc + 3 \times \bigcirc = 25$$

(b) '$\frac{25}{4}$' in each frame.

$$\bigcirc + 3 \times \bigcirc = 25$$

(c) '8' in each frame.

$$\bigcirc + 3 \times \bigcirc = 25$$

(continued on next page)

6. (a) '3' in each frame.

$\bigcirc + 2 = 2 + \bigcirc$

(b) '$\frac{5}{2}$' in each frame.

$\bigcirc + 2 = 2 + \bigcirc$

(c) '−12' in each frame.

$\bigcirc + 2 = 2 + \bigcirc$

7. (a) '3' in each frame.

$2 \times \square + 2 = 2 \times \square - 2$

(b) '−2' in each frame.

$2 \times \square + 2 = 2 \times \square - 2$

(c) '8' in each frame.

$2 \times \square + 2 = 2 \times \square - 2$

8. (a) '3' in each frame.

$3 \times \hexagon > 2 \times \hexagon + 1$

(b) '0' in each frame.

$3 \times \hexagon > 2 \times \hexagon + 1$

(c) '−2' in each frame.

$3 \times \hexagon > 2 \times \hexagon + 1$

9. (a) '5' in each frame.

$|\bigcirc - 2| = 2 - \bigcirc$

(b) '−3' in each frame.

$|\bigcirc - 2| = 2 - \bigcirc$

(c) '2' in each frame.

$|\bigcirc - 2| = 2 - \bigcirc$

10. (a) '4' in each frame.

$5 \times \square + 1 < 6 \times \square$

(b) '−3' in each frame.

$5 \times \square + 1 < 6 \times \square$

(c) '1' in each frame.

$5 \times \square + 1 < 6 \times \square$

11. (a) '2' in each frame.

$7 - \bigcirc < {}^{+}|\bigcirc - 7|$

(b) '−3' in each frame.

$7 - \bigcirc < {}^{+}|\bigcirc - 7|$

(c) '2.5' in each frame.

$7 - \bigcirc < {}^{+}|\bigcirc - 7|$

so would render a sentence true or false for the sole reason that numbers of arithmetic are different from real numbers. For example, we interpret the sentence '$|5| = {}^{-}5$' as an abbreviation for '${}^{+}|5| = {}^{-}5$' and thus declare it false because ${}^{+}5 \neq {}^{-}5$. We do not say that '$|5| = {}^{-}5$' is false because the number 5 of arithmetic is different from the real number ${}^{-}5$. Similarly, we interpret '$|5| \not< {}^{+}3$' as an abbreviation for '${}^{+}|5| \not< {}^{+}3$' and declare it true because ${}^{+}5 > {}^{+}3$.]

10.	(a)	T	(b)	F	(c)	F
11.	(a)	F	(b)	F	(c)	F
12.	(a)	F	(b)	T	(c)	T
13.	(a)	F	(b)	F	(c)	F

[In connection with Exercises 12 and 13 on page 2-5, caution students to follow the instructions precisely as they are given. For example, a student may want to write a '16' in each rectangle in Sample 2. Of course, he will come out with the same decision as if he had followed the instructions. However, we are trying to lay the groundwork for substituting complicated pronumeral expressions for pronumerals. Also, a student might feel that he could omit the grouping symbols when making these replacements since the frames themselves serve as grouping symbols. However, when we come to use letters instead of frames, the grouping symbols included in the expression to be substituted will often have to be retained. [In fact, in some cases, the student will need to enclose in grouping symbols the expression to be substituted. You could tell students that when they make the replacement of the long expression for the frame, they should pretend that the frame has disappeared. In fact, if the frame were actually a hole, there would be no sign of the hole when a replacement was made. In that event, if one substituted '9 + 7' instead of '(9 + 7)' in Sample 2, he would come out with the wrong decision about the converted sentence.]]

Sample 2. Write a '(9 + 7)' in each frame.

$$3 \times \boxed{} - 7 = \boxed{} + 25$$

Solution.

$$\underbrace{\underbrace{3 \times \boxed{(9 + 7)} - 7}_{3 \times 16 - 7}}_{41} = \underbrace{\underbrace{\boxed{(9 + 7)} + 25}_{16 + 25}}_{41}$$

The new sentence is true.

12. (a) '(3 × 9 + 13)' in each frame.

$$\boxed{} + 5 = 3 \times \boxed{} - 7$$

(b) '(10 × 11 - 104)' in each frame.

$$\boxed{} + 5 = 3 \times \boxed{} - 7$$

(c) '$\left[\frac{8}{3} + \frac{2}{3} \times (16 - 11)\right]$' in each frame.

$$\boxed{} + 5 = 3 \times \boxed{} - 7$$

13. (a) '(15 - 3)' in each frame.

$$2 \times \boxed{} + 2 = 43 - \boxed{}$$

(b) '(⁻8 + 23)' in each frame.

$$2 \times \boxed{} + 2 = 43 - \boxed{}$$

(c) '(7 - 20)' in each frame.

$$2 \times \boxed{} + 2 = 43 - \boxed{}$$

B. The exercises below are like those in Part A, except that several types of frames are used in the same sentence. Follow the instructions for putting numerals in the frames, and tell whether each new sentence is true or false.

1. (a) Write a '4' in each '$\bigcirc$' and a '3' in each '$\square$'.

$$2 \times \bigcirc + \square = 7 - \square + \bigcirc$$

(b) Write a '3' in each '$\bigcirc$' and a '4' in each '$\square$'.

$$2 \times \bigcirc + \square = 7 - \square + \bigcirc$$

(c) Write a '−5' in each '$\bigcirc$' and a '6' in each '$\square$'.

$$2 \times \bigcirc + \square = 7 - \square + \bigcirc$$

2. (a) '−7' in each '$\square$' and '−3' in each '⬡'.

$$4 \times \square - 2 \times \text{⬡} = \square + 5 \times \text{⬡}$$

(b) '5' in each '$\square$' and '2' in each '⬡'.

$$4 \times \square - 2 \times \text{⬡} = \square + 5 \times \text{⬡}$$

(c) '0' in each '$\square$' and '0' in each '⬡'.

$$4 \times \square - 2 \times \text{⬡} = \square + 5 \times \text{⬡}$$

3. (a) '5' in each '$\square$', '3' in each '$\bigcirc$', and '4' in each '⬡'.

$$(\square + \bigcirc) + \text{⬡} = \square + (\bigcirc + \text{⬡})$$

(b) '−4' in each '$\square$', '2' in each '$\bigcirc$', and '−4' in each '⬡'.

$$(\square + \bigcirc) + \text{⬡} = \square + (\bigcirc + \text{⬡})$$

(c) '873' in each '$\square$', '−9384' in each '$\bigcirc$', and '−76.2' in each '⬡'.

$$(\square + \bigcirc) + \text{⬡} = \square + (\bigcirc + \text{⬡})$$

The sentences in Part B contain more than one variable. You may need to emphasize the instructions that copies of the same numeral are to be written in similarly shaped frames in a given exercise. When discussing these exercises, avoid saying or writing things like:

○ = 1, □ = 3, and: ⬡ = 2.

Say, instead: '1' for '○', or: '1' replaces '○', or: a '1' in each '○'.

*

Answers for Part B [on pages 2-6 and 2-7].

1. (a) F (b) F (c) T 2. (a) T (b) F (c) T

[Notice that in Exercise 2(c) the students are asked to put copies of the same numeral in all of the frames. Some students may feel that differently shaped holes should get different numerals, but this exercise is designed to eliminate such a misconception.]

3. (a) T (b) T (c) T [We hope that many of your students will be able to predict that each completed sentence in Exercise 3 will be a true sentence because it is an instance of the associative principle for addition. As before, there may be a tendency for the students to declare that the given sentence is a true one. Again, as before, point out that sentences with holes in them are neither true nor false, and what the student really means is that all "completed" sentences following that pattern are true sentences. The fact that we refer to such unwieldy numbers in Exercise 3(c) is a sign that we want the student to look for a short way of doing the problem. In this case, we want him to recognize an instance of the associative principle for addition.]

4. (a) F (b) T (c) T

5. (a) T (b) T (c) T [As in Exercise 3, we hope that students will be able to predict that each "completed" sentence in Exercise 5 will be a true sentence; a "completed" sentence in Exercise 5 will be an instance of the commutative principle for multiplication.]

6. (a) T (b) T (c) T [In discussing Exercise 6, avoid reading it aloud as 'negative box ...'. We use the word 'negative' when talking about numbers, but never with pronumerals. If you need to read aloud the sentence in Exercise 6, read it as 'the opposite of box, plus 3 times hexagon, equals the opposite of the quantity, box minus 3 times hexagon'. We sound this note of warning now so that later you will not find yourself saying 'negative x' when you mean 'the opposite of x'.]

4. (a) '8' in each '○', '2' in each '⬡', and '20' in each '□'.

$$5 \times \bigcirc \times ⬡ = \square$$

(b) '−1' in each '○', '−2' in each '⬡', and '10' in each '□'.

$$5 \times \bigcirc \times ⬡ = \square$$

(c) '0' in each '○', '0' in each '⬡', and '0' in each '□'.

$$5 \times \bigcirc \times ⬡ = \square$$

5. (a) '5' in each '○' and '$\frac{16}{5}$' in each '□'.

$$\bigcirc \times \square = \square \times \bigcirc$$

(b) '2.4' in each '○' and '−2.4' in each '□'.

$$\bigcirc \times \square = \square \times \bigcirc$$

(c) '783' in each '○' and '9359' in each '□'.

$$\bigcirc \times \square = \square \times \bigcirc$$

6. (a) '4' in each '□' and '6' in each '⬡'.

$$-\square + 3 \times ⬡ = -(\square - 3 \times ⬡)$$

(b) '−3' in each '□' and '−5' in each '⬡'.

$$-\square + 3 \times ⬡ = -(\square - 3 \times ⬡)$$

(c) '$-\frac{2}{3}$' in each '□' and '$\frac{5}{7}$' in each '⬡'.

$$-\square + 3 \times ⬡ = -(\square - 3 \times ⬡)$$

SUBSTITUTION

Consider the sentence:

$$(*) \qquad 3 \times \square + 4 \times \bigcirc = 17 + \square .$$

As you have seen, this sentence can be used as a pattern for writing true-or-false sentences. One way to do this is to write numerals in the frames, observing the rule that we write copies of the same numeral in all frames of the same shape. For example, here is a sentence which follows the pattern of (*):

$$3 \times \boxed{2} + 4 \times \textcircled{5} = 17 + \boxed{2} .$$

Another way of following the pattern is to write numerals in place of the frames:

$$3 \times 2 + 4 \times 5 = 17 + 2 .$$

Of course, we replace all frames of a given shape by copies of the same numeral. In the example above, we replaced each '$\square$' in (*) by a '2' and each '$\bigcirc$' by a '5'. For short, we say that we substituted '2' for '$\square$' and '5' for '$\bigcirc$' in (*). Here are some more sentences which follow the pattern of (*). Tell what substitutions were made in (*) to obtain these sentences.

(1) $3 \times 51 + 4 \times 9 = 17 + 51$

(2) $3 \times -6 + 4 \times 8 = 17 + -6$

(3) $3 \times 7 + 4 \times 7 = 17 + 7$

(4) $3 \times (5 + 2) + 4 \times 7 = 17 + (5 + 2)$

(5) $3 \times (2 \times 6) + 4 \times 3 = 17 + 2 \times 6$

(6) $3 \times (4 + 2 \times 9) + 4 \times (17 - 3 \times 5) = 17 + (4 + 2 \times 9)$

Here are other examples of sentences with frames in them, and of sentences obtained by substituting for the frames.

$5 \times \square \times \triangle > 7 - \triangle$	$6 \times (\square - \triangle) = 8 \times \square \times (7 - \triangle)$
$5 \times 3 \times 4 > 7 - 4$	$6 \times (5 - 2) = 8 \times 5 \times (7 - 2)$
$5 \times 9 \times {}^{-}5 > 7 - {}^{-}5$	$6 \times (3 - 3) = 8 \times 3 \times (7 - 3)$

We give here a precise description of what is meant by 'substitution'. To substitute a numeral for a frame in a sentence or an expression is to replace each occurrence of the frame by copies of the numeral. We also make the point that a sentence with frames in it gives us a pattern for writing other sentences. Naturally, we assume that the student has made this point for himself in the preceding exercises.

Be very careful not to say things like 'substitute $\triangle = 1$'. This is not only grammatically indefensible, but will set up serious blocks against your students' learning to carry out substitutions. As a very brief indication of the sort of difficulty that we want to avoid, note that:

substitute '$x + y$' for 'x' and '$x - y$' for 'y' in '$xy = yx$'

makes perfect sense and is easy to do [the sentence which results is '$(x + y)(x - y) = (x - y)(x + y)$']. But:

substitute $x = x + y$ and $y = x - y$ in '$xy = yx$'

is not only nonsense but very confusing nonsense. [We recognize that colloquialisms like 'substitute $x = 3$' do appear in conventional textbooks, but such colloquialisms are completely indefensible. They do not even save time. UICSM students will be able to divine the meaning if they see them in conventional textbooks or on standardized examinations.]

Notice that in the exercises in Part A we have reduced the size of the frames. This is to forestall attempts on the part of the student to write numerals in the frames. In substituting numerals for frames, the frames are replaced by the numerals. When the student makes the substitutions, all he need do is rewrite the sentence, using numerals instead of frames. A good way to demonstrate this at the blackboard is to take the sentence in Exercise 1 and write it on the board. Then, erase the two occurrences of '$\triangle$' and write '2's in the resulting spaces. Next, erase both occurrences of '$\square$', and write '7's in the resulting spaces. Of course, students will not be able to carry out this erasing procedure on their own papers, but your demonstration will make clear what is expected of them.

*

Sentence (1) on page 2-8 is obtained by substituting, in (*), '51' for '□' and '9' for '○'; (2) by substituting '−6' for '□' and '8' for '○'; (3), '7' for '□' and '7' for '○'; (4), '(5 ÷ 2)' for '□' and '7' for '○'; (5), '(2 × 6)' for '□' and '3' for '○'; (6), '(4 + 2 × 9)' for '□' and '(17 - 3 × 5)' for '○'. Notice that the right side of (5) has been abbreviated by omitting unnecessary parentheses. Whenever a sentence follows some pattern we shall also say, somewhat loosely, that an abbreviation of the sentence follows the same pattern. Also, we shall allow ourselves to say that, for example, sentence (5) is obtained from (*) by substituting '2 × 6' for '□' and '3' for '○', although, in replacing one of the '□'s, we must write the unabbreviated expression '(2 × 6)', rather than merely '2 × 6'. As another example, (4) may be said to be obtained from (*) by substituting '5 + 2' for '□' and '7' for '○'. Exercises 4, 5, and 6 of Part A, and Exercise 2 of Part B, on page 2-9, are cases in which an expression to be substituted for a frame must sometimes be unabbreviated to the extent of being enclosed in grouping symbols.

In case there is any question as to the result of making a substitution, the rule is that the expression substituted should first be enclosed in grouping symbols. Then, after the substitution has been made, one may investigate to see whether our conventions for <u>omitting</u> grouping symbols apply.

Answers for Part A.

1. $9 \times 2 + 3 \times 7 = 15 \times 2 \times 7$
2. $3 \times (-9 + 5) - 2 \times (5 + -9) = -9 + 5$
3. $(4 + 1) \times [(3 + 5) \times (9 + 2)] = (4 + 1) \times (3 + 5) \times (9 + 2)$
4. $3 \times (8 + 3) + 5 \times (8 + 3) = 88$
5. $6 + 3 \times 5 = 2 \times (3 \times 5) - 9$
6. $2 \times (15 - 3 \times 2) - 5 \times -7 = 12 \times (15 - 3 \times 2 - -7)$

*

Answers for Part B [on pages 2-9 and 2-10].

1. $(6 + 9) \times 5 = 6 \times 5 + 9 \times 5$. True, because the new sentence is an instance of the dpma. Check: $75 = 75$.

2. $10 \times (3 + 2 \times 5) = 3 + 2 \times 5 + 117$. True: Check: $130 = 130$.

3. $6 \times (15 \times 2 - 5 \times 2) + 5 = 135$. False. Check: $125 \neq 135$.

4. $5 \times 5 + 12 \times 12 = 13 \times 13$. True. Check: $169 = 169$.

5. $(2 \times 9 - 7) \times (-7 + 7) = 0$. True, because the new sentence is a consequence of the commutative principle for addition, the principle of opposites, and the principle for multiplying by 0. Check: $0 = 0$.

6. $(4 + 2 \times 7) \times (6 \times 3 - 2 \times 5) = (6 \times 3 - 2 \times 5) \times (4 + 2 \times 7)$. True, because the new sentence is an instance of the cpm. Check: $144 = 144$.

7. $(3 \times 5 - 7 \times 2) \times (3 \times 5 - 7 \times 2) = (7 \times 2 - 3 \times 5) \times (7 \times 2 - 3 \times 5)$. True, because $7 \times 2 - 3 \times 5$ is the opposite of $3 \times 5 - 7 \times 2$, and the product of a number by itself is the product of its opposite by its opposite. Check: $1 = 1$.

8. $61 + 37 + 93 = 61 + (37 + 93)$. True, because the new sentence is an instance of the apa. Check: $191 = 191$.

9. $7 \times 3 + 2 \times 2 = 9 \times 3 \times 2$. False. Check: $25 \neq 54$.

10. $(5 + \frac{1}{2}) \times (5 + \frac{1}{2}) = 5 \times (5 + 1) + \frac{1}{4}$. True. Check: $30\frac{1}{4} = 30\frac{1}{4}$. [You may want to anticipate Exercise 13 on page 2-38 by asking students to make other substitutions and check them.]

EXERCISES

A. For each of the following sentences, write the sentence you get by making the indicated substitutions.

1. Substitute '2' for '△' and '7' for '□' in:

$$9 \times \triangle + 3 \times \square = 15 \times \triangle \times \square .$$

2. Substitute '−9' for '□' and '5' for '△' in:

$$3 \times (\square + \triangle) - 2 \times (\triangle + \square) = \square + \triangle .$$

3. Substitute '(4 + 1)' for '□', '(3 + 5)' for '○', and '(9 + 2)' for '△' in:

$$\square \times (\bigcirc \times \triangle) = \square \times \bigcirc \times \triangle .$$

4. Substitute '8 + 3' for '□' in:

$$3 \times \square + 5 \times \square = 88.$$

5. Substitute '3 × 5' for '□' in:

$$6 + \square = 2 \times \square - 9$$

6. Substitute '15 - 3 × 2' for '△' and '−7' for '□' in:

$$2 \times \triangle - 5 \times \square = 12 \times (\triangle - \square).$$

B. Substitute, and tell whether the resulting sentence is true or false. Try to predict the answer before substituting and computing, but be sure to check your prediction.

1. '5' for '□', '6' for '△', and '9' for '○' in:

$$(\triangle + \bigcirc) \times \square = \triangle \times \square + \bigcirc \times \square .$$

2. '3 + 2 × 5' for '▭' in:

$$10 \times \boxed{\quad\quad} = \boxed{\quad\quad} + 117.$$

3. '15 × 2 - 5 × 2' for '⬭' in:

$$6 \times ⬭ + 5 = 135.$$

4. '5' for '△', '12' for '□', and '13' for '⬡' in:

$$\triangle \times \triangle + \square \times \square = ⬡ \times ⬡ .$$

5. '9' for '△' and '−7' for '□' in:

$$(2 \times \triangle - 7) \times (\square + 7) = 0.$$

6. '4 + 2 × 7' for '▭' and '6 × 3 - 2 × 5' for '⬭' in:

▭ × ⬭ = ⬭ × ▭ .

7. '3 × 5 - 7 × 2' for '△' and '7 × 2 - 3 × 5' for '⬡' in:

△ × △ = ⬡ × ⬡ .

8. '61' for '⬡', '37' for '□', and '93' for '△' in:

⬡ + □ + △ = ⬡ + (□ + △).

9. '3' for '△' and '2' for '□' in:

7 × △ + 2 × □ = 9 × △ × □ .

10. '5' for '□' in: (□ + $\frac{1}{2}$) × (□ + $\frac{1}{2}$) = □ × (□ + 1) + $\frac{1}{4}$.

C. Sometimes instead of substituting numerals for frames in a sentence, we substitute expressions which themselves contain frames. This gives us a new sentence which contains frames.

Sample. Write the sentence you get when you substitute

'3 × □' for '▭' and '4 + ⬡' for '⬭' in:

▭ + ⬭ = ⬭ + ▭ .

Solution. 3 × □ + (4 + ⬡) = 4 + ⬡ + 3 × □ .

[Why do we need to use only one pair of parentheses?]

1. Substitute '6 - 2 × ▭' for '⬭' in:

5 × ▭ + 3 × ⬭ = 17.

2. Substitute '△ - 2 × □' for '▭' and '3 × □ + −△' for '⬭' in:

▭ × 3 - 6 = 11 + ⬭ × 4.

3. (a) Substitute '3 + □' for '▭' and '2 - △' for '⬭' in:

▭ + ⬭ = ⬭ + ▭ .

(b) In the sentence you obtained in (a), substitute '5' for '□' and '9' for '△'. Is the resulting sentence true? Could you have predicted your answer before doing any computations?

Answers for Part C [on pages 2-10 and 2-11].

1. $5 \times \boxed{} + 3 \times (6 - 2 \times \boxed{}) = 17$

2. $(\triangle - 2 \times \square) \times 3 - 6 = 11 + (3 \times \square + - \triangle) \times 4$

3. (a) $3 + \square + (2 - \triangle) = 2 - \triangle + (3 + \square)$

   (b) $3 + 5 + (2 - 9) = 2 - 9 + (3 + 5)$. True, because the new sentence is an instance of the cpa.

[Bring out that each result of substituting numerals for '$\boxed{}$' and '$\bigcirc$' in:

$$\boxed{} + \bigcirc = \bigcirc + \boxed{}$$

is an instance of the cpa. And that consequently each result of substituting numerals for '$\square$' and '$\triangle$' in the sentence obtained in answer to 3(a) is also an instance of the cpa.]

4. (a) $\square \times [3 + \triangle + (2 + \square)] = \square \times (3 + \triangle) + \square \times (2 + \square)$

   (b) $-3 \times [3 + 2 + (2 + -3)] = -3 \times (3 + 2) + -3 \times (2 + -3)$. True; the new sentence in (b) is an instance of the ℓdpma.

   (c) $-3 \times [3 + 2 + (2 + -3)] = -3 \times (3 + 2) + -3 \times (2 + -3)$. True; the new sentence in (c) is an instance of the ℓdpma.

5. (a) $-\square = - - - \square$

   (b) $-{}^{-}2 = - - - {}^{-}2$. This sentence is true. $-{}^{-}2 = {}^{+}2$, and $- - - {}^{-}2$ is $- - {}^{+}2$, and $- - {}^{+}2 = - {}^{-}2$ which is ${}^{+}2$.

   (c) $\square - \triangle = - - (\square - \triangle)$

   (d) ${}^{-}3 - {}^{-}3 = - - ({}^{-}3 - {}^{-}3)$. This sentence is true. ${}^{-}3 - {}^{-}3$ is 0, and $- -0 = -0 = 0$.

Answers for Part D.

[We suggest that these be handled orally.]

	True	False
	A numeral for	A numeral for any number
1.	8	different from 8
2.	6	different from 6
3.	50	different from 50
4.	6	different from 6
5.	1	different from 1
6.	4	different from 4
7.	any number	[No numeral]
8.	0	different from 0
9.	any number	[No numeral]
10.	0 or for 1	different from 0 and different from 1

4. (a) Substitute '$3 + \triangle$' for '$\triangle$' and '$2 + \square$' for '$\bigcirc$' in:

$$\square \times (\triangle + \bigcirc) = \square \times \triangle + \square \times \bigcirc .$$

(b) Substitute '-3' for '$\square$' and '2' for '$\triangle$' in the sentence you obtained in (a). Is the resulting sentence true?

(c) In the sentence originally given in (a), substitute '-3' for '$\square$', '$3 + 2$' for '$\triangle$', and '$2 + -3$' for '$\bigcirc$'. Is the resulting sentence true?

5. (a) Substitute '$-\square$' for '$\triangle$' in:

$$\triangle = --\triangle .$$

(b) Substitute '$^{-}2$' for '$\square$' in the sentence you obtained in (a). Is the resulting sentence true?

(c) Substitute '$\square - \triangle$' for '$\triangle$' in the sentence originally given in (a).

(d) Substitute '$^{-}3$' for '$\square$' and '$^{-}3$' for '$\triangle$' in the sentence you obtained in (c). Is the resulting sentence true?

<u>D</u>. For each of the following sentences, find a substitution which will make it true, and a substitution which will make it false.

<u>Sample</u>. $4 \times \square = 80$

<u>Solution</u>. (a) Substituting '20' for '$\square$' you get:

$$4 \times 20 = 80,$$

which is a true sentence.

(b) Substituting '11' for '$\square$' you get:

$$4 \times 11 = 80,$$

which is a false sentence.

1. $7 \times \square = 56$
2. $3 \times \square + 2 = 20$
3. $\square + \square = 100$
4. $2 \times \square + 8 \times \square = 60$
5. $3 \times \square + 17 \times \square = 20$
6. $3 \times \square + 17 \times \square = 80$
7. $5 \times \square + 11 \times \square = 16 \times \square$
8. $13 \times (\square + 2) = 26$
9. $7 \times (\square \times 5) = 35 \times \square$
10. $6 \times \square \times 4 \times \square = 24 \times \square$

E. In the preceding exercises you have been using sentences with frames as patterns for writing other sentences which follow the patterns. Now see if you can reverse the process by writing for each exercise below a sentence with frames which serves as a pattern for the sentences given in the exercise.

1.

$$3 + 9 = 9 + 3$$
$$^-8 + 0 = 0 + {}^-8$$
$$1 + 1 = 1 + 1$$
$$2 \times 3 + 6 \times 5 = 6 \times 5 + 2 \times 3$$

Pattern sentence: ______________________

2.

$$4 + 7 = 8 + 3$$
$$4 + 2 = 8 + 5$$
$$4 + \square = 8 + 7$$
$$4 + 3 \times 5 = 8 + (6 - \triangle)$$

Pattern sentence: ______________________

3.

$$5 + 9 < 5 - 9$$
$$6 \times 5 + {}^-3 < 6 \times 5 - {}^-3$$
$$5 + \square + 2 < 5 + \square - 2$$
$$\square + \triangle + (\bigcirc - \triangle) < \square + \triangle - (\bigcirc - \triangle)$$

Pattern sentence: ______________________

4.

$$2 + 3 \times 7 = (2 + 3) \times 7$$
$$5 + 9 \times 7 = (5 + 9) \times 7$$
$$1 + 0 \times 7 = (1 + 0) \times 7$$
$$0 + 1 \times 7 = (0 + 1) \times 7$$
$$5 + \triangle + (9 + \square) \times 7 = [5 + \triangle + (9 + \square)] \times 7$$

Pattern sentence: ______________________

In answering the exercises of Part E on pages 2-12 and 2-13, students may come up with any number of correct answers. For example, each of the following is an equally good answer for Exercise 1 of Part E:

$$\square + \triangle = \triangle + \square,\ \triangle + \square = \square + \triangle,\ \bigcirc + \square = \square + \bigcirc,\ \triangle + \bigcirc = \bigcirc + \triangle.$$

Moreover, the sentence '$\square = \triangle$' is a correct [though unwanted] answer for each exercise of Part E other than Exercise 3. It may even be helpful, in preparing students for finding the "best" pattern, to point out [before they attempt many exercises of Part E] that each sentence in Exercises 1 and 2 follows the pattern of '$\square = \triangle$', but also follows the finer [as opposed to coarser] pattern of '$\square + \triangle = \bigcirc + \bigcirc$'. The sentences of Exercise 1 follow the still finer pattern of '$\square + \triangle = \triangle + \square$', while those of Exercise 2 follow the pattern of '$4 + \triangle = 8 + \bigcirc$'. [One pattern is finer than a second if each sentence which follows the first pattern also follows the second pattern. Consider these patterns.

(1) $\square + \triangle = \triangle + \bigcirc$ (2) $\square + \triangle = \bigcirc + \bigcirc$

The sentence:

(a) $3 + {}^{-}5 = {}^{-}5 + 7$

follows from (1), and also could be obtained from (2). However, the sentence:

(b) $3 + {}^{-}5 = {}^{-}9 + 7$

follows from (2); but (b) could not be obtained from (1), according to our convention for substitution. Hence, we say that pattern (1) is finer than pattern (2).]

*

Answers for Part E [on pages 2-12 and 2-13].

1. $\square + \triangle = \triangle + \square$
2. $4 + \square = 8 + \triangle$
3. $\square + \triangle < \square - \triangle$
4. $\square + \triangle \times 7 = (\square + \triangle) \times 7$

[Be sure that students notice that the first three sentences in Exercise 4 are false.]

5. $\square \times \triangle = \triangle \times \square$
6. $\square \times (\triangle + \bigcirc) = \square \times \triangle + \square \times \bigcirc$
7. $(\square + \triangle) \times \bigcirc = \square \times \bigcirc + \triangle \times \bigcirc$
8. $\square + \triangle + \bigcirc = \square + (\triangle + \bigcirc)$
9. $\square \times 1 = \square$
10. $1 \times \square = \square$
11. $(\square + \triangle) \times (\square - \triangle) = \square \times \square - \triangle \times \triangle$

5. All the instances of the commutative principle for multiplication.

Pattern sentence: ____________________________

6.

$$6 \times (3 + 4) = 6 \times 3 + 6 \times 4$$

$$^-8 \times (^-5 + {}^-6) = {}^-8 \times {}^-5 + {}^-8 \times {}^-6$$

$$(4 + 9) \times (7 + 3) = (4 + 9) \times 7 + (4 + 9) \times 3$$

$$15 \times (4 + 7 + 9) = 15 \times (4 + 7) + 15 \times 9$$

$$^-1 \times (6 - \triangle + \square) = {}^-1 \times (6 - \triangle) + {}^-1 \times \square$$

Pattern sentence: ____________________________

7.

$$(2 + 8) \times 3 = 2 \times 3 + 8 \times 3$$

$$(6 + 5 + 7) \times 8 = (6 + 5) \times 8 + 7 \times 8$$

$$[7 + (2 - 3)] \times \square = 7 \times \square + (2 - 3) \times \square$$

$$(1 \times 2 + 3) \times (4 + 5 + 6) = (1 \times 2) \times (4 + 5 + 6) + 3 \times (4 + 5 + 6)$$

Pattern sentence: ____________________________

8. All the instances of the associative principle for addition.

Pattern sentence: ____________________________

9. All the instances of the principle for multiplying by 1.

Pattern sentence: ____________________________

10.

$$1 \times 57 = 57$$

$$1 \times (8 + 3) = 8 + 3$$

$$1 \times (\square \times \triangle) = \square \times \triangle$$

$$1 \times -(3 - 7) = -(3 - 7)$$

Pattern sentence: ____________________________

11.

$$(7 + 4) \times (7 - 4) = 7 \times 7 - 4 \times 4$$

$$(5 + 2 + 1) \times (5 + 2 - 1) = (5 + 2) \times (5 + 2) - 1 \times 1$$

$$[1 + 3 + (7 \times 5)] \times [1 + 3 - (7 \times 5)] = (1 + 3) \times (1 + 3) - (7 \times 5) \times (7 \times 5)$$

$$(8 + 3 \times \square) \times (8 - 3 \times \square) = 8 \times 8 - (3 \times \square) \times (3 \times \square)$$

Pattern sentence: ____________________________

2.02 Pronouns.--Suppose someone challenges you to say 'True' or 'False' about the following sentence:

He was a president of the United States.

You don't need to know anything about presidents, or even about the United States, in order to know that it would not be correct to answer one way or the other. The sentence is neither true nor false. [If you put a man's name in place of 'He' in the given sentence then the new sentence would be either true or false.]

Trying to answer 'True' or 'False' about the sentence:

He was a president of the United States

is just like trying to answer 'True' or 'False' about the sentence:

$$9 + \bigcirc = 15.$$

Both sentences have "holes" in them. [You have seen that a frame such as '$\bigcirc$' serves as a hole. So does the word 'He'.] Neither sentence is true, and neither sentence is false.

A sentence which is either true or false is called a statement. For example, '9 + 6 = 15' and 'Albert Einstein was a president of the United States' are statements. Sentences which can be turned into statements by filling holes with names are called open sentences. An open sentence is neither true nor false.

Since the holes in an open sentence hold places for nouns, they are called pronouns. The pronouns in the mathematical sentences we have been working with hold places for those nouns which are names of numbers, that is, they hold places for numerals. And so, we shall call such pronouns pronumerals.

> In general,
> pronouns in an open sentence
> hold places for nouns, and
> in particular,
> pronumerals in an open sentence
> hold places for numerals.

We hope that students understand that nouns are words and not things for which the words stand. So, a pronoun is a word which holds a place in a sentence [or in an expression] for nouns. A pronoun does not hold a place for things for which nouns stand. The word 'He' in the sentence:

He was the president of the United States

holds a place for names of men, and not a place for men. It would make sense to replace the word 'He' by 'Abraham Lincoln' but not by Abraham Lincoln. Similarly, the mathematical pronouns such as '$\bigcirc$' and '$\square$' hold places for numerals rather than for numbers.

*

The thing that needs stressing in these pages is that sentences containing frames or letter pronumerals show patterns. The letters show the patterns just as easily as do the frames. The advantage of the frame is that it is more readily seen to be a mark whose purpose is to hold a place for numerals.

*

There may be some objection by grammarians to our assertion at the bottom of page 2-16 that 'first boy' is a pronoun. In any event, expressions such as this one function as pronouns.

*

In discussing the first paragraph on page 2-18, you should continue to avoid such phrases as 'substitute a = 75'. Instead, say: substitute '75' for 'a'. You will appreciate the advantages of having been careful when you come to say: substitute '9 + a + 5 × b' for 'a', rather than 'substitute a = 9 + a + 5 × b'.

*

The students' practice in substituting numerals for frames should leave them no room for doubt that a pronumeral is merely a symbol which can be replaced by a numeral, that is, by a name for a number. However, it is still possible for them to become confused, and this may occur if you happen to speak carelessly about pronumerals.

Consider the following statements.

(1) A pronumeral is a symbol which one may replace by a name for any number.

(2) Pronumerals stand for names of numbers.

(3) Pronumerals stand for numbers.

(4) A pronumeral is a symbol which is a name for any number.

Of these, (1) is correct, and is not misleading. One can argue that, properly interpreted, (2) is correct [stand for = stand in place of = hold places for]. But, it is clearly misleading [stand for = symbolize = name]. Statement (3) is incorrect. Here 'stand for' cannot mean the same as 'stand in place of', for numbers can't occupy places; so, 'stand for' must be interpreted as 'name', and pronumerals are <u>not</u> names of anything. This same criticism applies to (4).

*

There is another use of pronouns in English which must be distinguished from the use which we are now stressing. Two illustrations are:

<u>He</u> who hesitates is lost.

and:

<u>One</u> must be 21 years of age if one is to vote.

In these sentences the underlined pronouns are being used to express generality. A more explicit statement of the fact asserted by the second of these examples is:

<u>Each</u> <u>one</u> must be 21 years of age if one [or: he] is to vote.

It is to avoid confusions which can arise through these two uses of pronouns that, in our mathematics language, we insist [see pages 2-27 ff.] on the use of quantifying phrases such as 'for each x' in statements of generalizations. For example, in 'xy = yx', the pronouns 'x' and 'y' are place holders, while in:

For each number <u>x</u>, for each number <u>y</u>, xy = yx

they occur in quantifying phrases. The sentence 'xy = yx' is neither true nor false; but the displayed generalization sentence is true, and is a statement of the commutative principle for multiplication.

TC[2-14, 15, 16, 17, 18]b

Draw a line under each pronoun in the following sentences.

(1) He is Al's father.

(2) $\square$ is an even positive number.

(3) $\square$ + ⬡ $= 17 - 2 \times \square$

(4) She is taller than Mary and she is her elder sister.

Individual pronumerals in an open sentence show you where to write names. But, taken together, they show you more than that. For example, in order to convert sentence (3) into a statement, you can write three names. However, you would not write something like this:

$$79 + 85 = 17 - 2 \times 94,$$

or even something like this:

$$79 + 85 = 17 - 2 \times 85.$$

When substituting for pronumerals, you follow the rule that pronumerals of the same shape are to be replaced by copies of the same numeral. Pronumerals of the same shape "link" places where similar replacements are to be made. In sentence (3), the two '$\square$'s are to be replaced by copies of the same numeral, and the '⬡' can be replaced either by another copy of the same numeral or by a different numeral.

The rules for replacing pronouns in English open sentences cannot be stated so simply. [In fact, they probably just can't be stated!] In sentence (4) we can substitute and get:

(a) Elsie is taller than Mary and Elsie is Mary's elder sister,

or, we can substitute and get:

(b) Elsie is taller than Mary and Mary is Elsie's elder sister.

If you pointed to Elsie and said sentence (4), you would probably have meant (a); but if, in speaking, you emphasized the second 'she', you might very well have meant (b). If you meant (a), you were linking the 'she's. If you meant (b), you were linking the first 'she' and the 'her', and you were *not* linking the two 'she's. [Notice that in (b) you did not even replace the linked pronouns by copies of the same word; you replaced 'she' by 'Elsie' and 'her' by 'Elsie's'.] So, the rule that two occurrences of the same pronoun should be replaced by copies of the same name is not strictly followed in English. And, *different* pronouns are sometimes linked.

You could overcome these difficulties in the English language if you introduced additional pronouns and adopted strict linking rules for them. For example, suppose '⬡' is a feminine pronoun and that always in a sentence all '⬡'s are linked. Then, we could state sentence (4) in such a way that it would be impossible to get (b); and in another way such that it would be impossible to get (a).

To get (a) but not (b), we would write:

> ⬡ is taller than Mary and ⬡ is her elder sister.

To get (b) but not (a), we would write:

> ⬡ is taller than Mary and she is ⬡ 's elder sister.

If in the first of these sentences you substitute 'Elsie' for '⬡' [and 'Mary's' for 'her'], you will get sentence (a). If in the second sentence you again substitute 'Elsie' for '⬡' [and 'Mary' for 'she'], you will get sentence (b).

Actually, there are pronouns in English which obey strict linking rules. Look at the sentence:

> He went because he felt it was polite to do so.

Are the two 'he's linked? You really can't tell. But, here is a restatement in ordinary English using pronouns which *clearly* are not linked.

> A *first* *boy* went because a *second* *boy* felt it was polite to do so.

The English language allows you to manufacture pronouns by prefixing to a noun words like 'first', 'second', 'third', 'former', 'latter', etc.

Do you think that 'he' and 'his' in:

He knew a man who roomed with his cousin in school

are linked? Use the pronouns 'first man', and 'second man', etc. to restate this sentence in two ways, each showing a different linkage.

STANDARD PRONUMERALS

Although we have been using frames as pronumerals, it is more common to use letters. In doing so, we shall use the lower-case letters of the alphabet:

a, b, c, ..., x, y, z

as well as the upper-case letters:

A, B, C, ..., X, Y, Z.

In using these you must be careful to observe the difference between 'a' and 'A', 'b' and 'B', etc. These are just as different as '$\square$' and '$\triangle$' or as '$\square$' and '$\boxed{}$'.

We use letter-pronumerals just as we used frame-pronumerals. You can see a pattern in an open sentence which has letter-pronumerals just as easily as you could with an open sentence which had frame-pronumerals. For example, from the open sentence:

$$c + 7 \times a + b = c - (a + b)$$

we can generate (or make) statements by substituting numerals for the pronumerals 'a', 'b', and 'c'. Here are some of these statements:

$$15 + 7 \times 3 + 9 = 15 - (3 + 9),$$

$$19 + 7 \times 8 + 6 = 19 - (8 + 6),$$

$$-4 + 7 \times 1 + 5 = -4 - (1 + 5).$$

We can also generate <u>open</u> sentences from the given sentence by substituting expressions which contain pronumerals. For example:

$$(9 + x) + 7 \times (3 - y) + (k + m) = (9 + x) - [(3 - y) + (k + m)],$$

$$(y - w) + 7 \times (2 + r) + (s - y) = (y - w) - [(2 + r) + (s - y)].$$

Now, just as an open sentence gives you a pattern for other sentences, an expression like '$9 + a + 5 \times b$' gives you a pattern for other expressions. You can use it to generate numerals, or to generate other expressions which themselves give patterns for generating numerals. For example, from:

$$9 + a + 5 \times b$$

you can generate these <u>numerals</u>:

$$9 + 75 + 5 \times 15$$

$$9 + {}^{-}3 + 5 \times (7 - 5).$$

Also, from '$9 + a + 5 \times b$' you can generate these <u>expressions</u>:

$$9 + (8 + x) + 5 \times (7 - y)$$

$$9 + (3 - c) + 5 \times (2 + k)$$

$$9 + (A + b) + 5 \times (a - B)$$

$$9 + (9 + a + 5 \times b) + 5 \times (9 + a + 5 \times b).$$

As you can see, the expressions and sentences which can be generated from <u>pronumeral</u> <u>expressions</u> and from open sentences can look quite complicated. One small way in which they can be made to look less complicated is to follow the convention of omitting the multiplication signs. For example:

'$3(5 + 9)$' is an abbreviation for '$3 \times (5 + 9)$',

'ab' is an abbreviation for '$a \times b$',

'$q(3a + 4p)$' is an abbreviation for '$q \times (3 \times a + 4 \times p)$',

'$({}^{-}2a + 13)d$' is an abbreviation for '$({}^{-}2 \times a + 13) \times d$',

'$5x$' is an abbreviation for '$5 \times x$',

'$x5$' is an abbreviation for '$x \times 5$',

<u>but</u> '55' is <u>not</u> an abbreviation for '5×5',

and '$a - b$' is <u>not</u> an abbreviation for '$a \times {}^{-}b$'.

Another abbreviation for '$\times$' is obtained by using a '$\cdot$' in place of a '$\times$'. For example, '$3 \cdot 7$' is an abbreviation for '3×7', and '$u \cdot v$' is an abbreviation for '$u \times v$'.

22. $3a + 2 \cdot 5\,(bB - 1)$

$3 \cdot 3 + 2 \cdot 5(5 \cdot -3 - 1)$

$9 + 10(-15 - 1)$

$9 + 10(-16)$

$9 + -160$

-151

23. $7dd - 3d + 1 - (d - 1)(d - 1)$

$7 \cdot -4 \cdot -4 - 3 \cdot -4 + 1 - (-4 - 1)(-4 - 1)$

$112 - -12 + 1 - (-5)(-5)$

$112 + 12 + 1 - 25$

$125 - 25$

100

24. $(A - a)(A - a)(A - a)$

$(7 - 3)(7 - 3)(7 - 3)$

$(4)(4)(4)$

$16 \cdot 4$

64

25. $(3 - 2y)(6 - \frac{1}{2}y) - (y + 3)$

$(3 - 2 \cdot \frac{2}{3})(6 - \frac{1}{2} \cdot \frac{2}{3}) - (\frac{2}{3} + 3)$

$(3 - \frac{4}{3})(6 - \frac{1}{3}) - \frac{11}{3}$

$(\frac{5}{3})(\frac{17}{3}) - \frac{11}{3}$

$\frac{85}{9} - \frac{11}{3}$

$\frac{52}{9}$

16. $2a[d - 3(b + 2)]$

$2 \cdot 3[-4 - 3(5 + 2)]$

$6[-4 - 3 \cdot 7]$

$6[-25]$

-150

17. $(2d - c)[8a - 6x + 2A(3 - 5B)]$

$(2 \cdot -4 - 0)[8 \cdot 3 - 6 \cdot \frac{1}{2} + 2 \cdot 7(3 - 5 \cdot -3)]$

$-8 \quad [24 - 3 + 14(3 - -15)]$

$-8 \quad [\ 21 \quad + 14(18)]$

$-8 \quad (\ 21 \quad + \quad 252)$

$-8 \quad (273)$

-2184

18. $(5 + 2a) \div (3 - 6d)$

$(5 + 2 \cdot 3) \div (3 - 6 \cdot -4)$

$(5 + 6) \div (3 - -24)$

$11 \div 27$

$\frac{11}{27}$

19. $(4x - 7y) \div (9y - 2z)$

$(4 \cdot \frac{1}{2} - 7 \cdot \frac{2}{3}) \div (9 \cdot \frac{2}{3} - 2 \cdot -\frac{5}{4})$

$(2 - \frac{14}{3}) \div (6 - -\frac{5}{2})$

$-\frac{8}{3} \div \frac{17}{2}$

$-\frac{8}{3} \times \frac{2}{17}$

$-\frac{16}{51}$

[Ask students if they could rewrite the expressions in Exercises 18 and 19 without using the '÷'. They should compare such expressions with those in Exercises 13, 14, and 15.]

20. $57 - 5.7 + bA$

$57 - 5.7 + 5 \cdot 7$

$51.3 \quad + \quad 35$

86.3

21. $2 \cdot 4a - 4 \cdot 2A - 3 \cdot 5 \cdot 7y$

$2 \cdot 4 \cdot 3 - 4 \cdot 2 \cdot 7 - 3 \cdot 5 \cdot 7 \cdot \frac{2}{3}$

$24 \quad - \quad 56 \quad - \quad 70$

$-32 \quad - \quad 70$

-102

8. $a + a + b + b$

$3 + 3 + 5 + 5$

16

9. $xx \quad + \; x + x$

$\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$1\frac{1}{4}$

10. $\frac{2a + 5b}{3d}$

$\frac{2 \cdot 3 + 5 \cdot 5}{3 \cdot -4}$

$\frac{6 + 25}{-12}$

$-\frac{31}{12}$ [or: $-2\frac{7}{12}$]

11. $\frac{5B - 3y}{8z}$

$\frac{5 \cdot -3 - 3 \cdot \frac{2}{3}}{8 \cdot -\frac{5}{4}}$

$\frac{-15 - 2}{-10}$

$\frac{17}{10}$ [or: 1.7]

12. $\frac{9a + 7a}{4b}$

$\frac{9 \cdot 3 + 7 \cdot 3}{4 \cdot 5}$

$\frac{27 + 21}{20}$

$\frac{48}{20}$ [or: 2.4]

13. $\frac{7d + A}{3b + A}$

$\frac{7 \cdot -4 + 7}{3 \cdot 5 + 7}$

$\frac{-28 + 7}{15 + 7}$

$-\frac{21}{22}$

14. $\frac{9xy}{3xz}$

$\frac{9 \cdot \frac{1}{2} \cdot \frac{2}{3}}{3 \cdot \frac{1}{2} \cdot -\frac{5}{4}}$

$\frac{3}{-\frac{15}{8}}$

$-\frac{24}{15}$ [or: $-\frac{8}{5}$]

15. $\frac{a(b - c)}{a(b + c)}$

$\frac{3(5 - 0)}{3(5 + 0)}$

$\frac{15}{15}$ [or: 1]

Part A of the Supplementary Exercises contains very easy exercises similar to those on page 2-19, and suitable for oral class work. We suggest that after reading aloud the first paragraph of Part A, you have the class turn to page 2-138 and try some of the exercises orally. [You may want to assign Exercises 41 through 60 for homework.] If the class seems to have had no trouble with the exercises on page 2-138, they should be ready to try these on page 2-19. [More difficult exercises of the same type occur in Part B of the Supplementary Exercises. Others occur in Part H of the Miscellaneous Exercises, page 2-120 through 2-123.]

*

When students do the exercises in Part A they should be urged to write the original expression for each exercise and then write under this the new expression which is obtained by making the substitutions. This procedure is not illustrated in the Solution but is in the answers below.

*

Answers for Part A.

1.
$$7a + 3d$$
$$7 \cdot 3 + 3 \cdot -4$$
$$21 + -12$$
$$9$$

2.
$$8d - 5c$$
$$8 \cdot -4 - 5 \cdot 0$$
$$-32 - 0$$
$$-32$$

3.
$$ab + 3y$$
$$3 \cdot 5 + 3 \cdot \frac{2}{3}$$
$$15 + 2$$
$$17$$

4.
$$2ax - 3ay$$
$$2 \cdot 3 \cdot \frac{1}{2} - 3 \cdot 3 \cdot \frac{2}{3}$$
$$3 - 6$$
$$-3$$

5.
$$6xy + 4z$$
$$6 \cdot \frac{1}{2} \cdot \frac{2}{3} + 4 \cdot -\frac{5}{4}$$
$$2 + -5$$
$$-3$$

6.
$$(8B - 3b)\ (8B - 3b)$$
$$(8 \cdot -3 - 3 \cdot 5)(8 \cdot -3 - 3 \cdot 5)$$
$$(-24 - 15)\ (-24 - 15)$$
$$-39 \cdot -39$$
$$1521$$

7.
$$aa + bb$$
$$3 \cdot 3 + 5 \cdot 5$$
$$9 + 25$$
$$34$$

EXERCISES

A. A value of a pronumeral is a number whose name may be substituted for the pronumeral. A pronumeral expression also has values. These are the numbers whose names can be obtained by substituting numerals for the pronumerals in the expression. For example, the value of '$a + 2b$' for the value 3 of 'a' and 5 of 'b' is $3 + 2 \cdot 5$, or 13.

Find the value of each of the following pronumeral expressions for the given values of the pronumerals.

Pronumeral	'a'	'A'	'b'	'B'	'c'	'd'	'x'	'y'	'z'
Value	3	7	5	-3	0	-4	$\frac{1}{2}$	$\frac{2}{3}$	$-\frac{5}{4}$

Sample. $(8a + 2b)(A + B)$

Solution. $(8 \cdot 3 + 2 \cdot 5)(7 + -3)$
$= (24 + 10)4$
$= 34 \cdot 4$
$= 136$.

[Note: You may be tempted to skip steps. Don't, until you are sure it won't lead to errors.]

1. $7a + 3d$
2. $8d - 5c$
3. $ab + 3y$
4. $2ax - 3ay$
5. $6xy + 4z$
6. $(8B - 3b)(8B - 3b)$
7. $aa + bb$
8. $a + a + b + b$
9. $xx + x + x$
10. $\frac{2a + 5b}{3d}$
11. $\frac{5B - 3y}{8z}$
12. $\frac{9a + 7a}{4b}$
13. $\frac{7d + A}{3b + A}$
14. $\frac{9xy}{3xz}$
15. $\frac{a(b - c)}{a(b + c)}$
16. $2a[d - 3(b + 2)]$
17. $(2d - c)[8a - 6x + 2A(3 - 5B)]$
18. $(5 + 2a) \div (3 - 6d)$
19. $(4x - 7y) \div (9y - 2z)$
20. $57 - 5.7 + bA$
21. $2 \cdot 4a - 4 \cdot 2A - 3 \cdot 5 \cdot 7y$
22. $3a + 2 \cdot 5(bB - 1)$
23. $7dd - 3d + 1 - (d - 1)(d - 1)$
24. $(A - a)(A - a)(A - a)$
25. $(3 - 2y)(6 - \frac{1}{2}y) - (y + 3)$

[More exercises are in Parts A and B, Supplementary Exercises.]

B. Complete these tables by filling in the blanks with numerals for the values of the pronumeral expressions [or: of the pronumeral] which correspond with the given values.

1.

x	1	2	-4	3.5	
3x + 4					25

2.

y	-2	-1	0	1	2
1 - 2y					

3.

z	-2	-1	0	1	
zz + 1					10

4.

k	6.2	-3.7		9.3
3(2k + 4)			0	

5.

t	5	30		17
8t + 12t			200	

6.

s	4	6	28	
3(5s) + 5(17s)				900

7.

m	6	-3	78	2/3
10m ÷ (2m)				

8.

d	2	-3	6.7	-4.3
d(d + 1) - dd				

9.

x	2	3	5		
y	7	-2		6	
8x + 5y			50	90	26

10.

p	2	3	-2	-3	
q	3	2	-3	-2	6
p - q					15

C. Use each of the following open sentences to generate a statement. Tell whether the statement is true or false. Try to generate some true statements and some false statements.

1. $5x + 4 = 14$

2. $5 - 8y = -3$

3. $A + 4 = 5 + 6A$

4. $7t - 1 = 1 + t$

5. $3z = z + 8$

6. $10 - 2s = s + 5$

7. $2B + 4 \neq 2(2 + B)$

8. $8 - 3c = 1\frac{1}{2}(5\frac{1}{3} - 2c)$

9. $x + 1 = x$

10. $2z - 1 = 1 + 2z$

11. $\frac{5r - 1}{5r + 1}$

12. $\frac{3m + 2}{8 - 3m} = \frac{6m + 5}{2 - 4m}$

13. $6j - j > 2$

14. $k - 3k < 8$

15. $10g - 8\frac{1}{2}g = 1.5g$

16. $12f + 7\frac{1}{3}f = 19\frac{1}{3}f$

17. $3xx - 12 = 0$

18. $uu = u + 20$

19. $(v - 7)(v + 7) = vv - 49$

20. $(w - 3)(2 + w) = ww - w - 6$

[More exercises are in Part C, Supplementary Exercises.]

Answers for Part B.

1.

x	1	2	−4	3.5	7
3x + 4	7	10	−8	14.5	25

2.

y	−2	−1	0	1	2
1 - 2y	5	3	1	−1	−3

3.

z	−2	−1	0	1	3 [or: −3]
zz + 1	5	2	1	2	10

4.

k	6.2	−3.7	−2	9.3
3(2k + 4)	49.2	−10.2	0	67.8

5.

t	5	30	10	17
8t + 12t	100	600	200	340

6.

s	4	6	28	9
3(5s) + 5(17s)	400	600	2800	900

7.

m	6	−3	78	2/3
10m ÷ (2m)	5	5	5	5

8.

d	2	−3	6.7	−4.3
d(d + 1) - dd	2	−3	6.7	−4.3

9.

x	2	3	5	15/2	
y	7	−2	2	6	
8x + 5y	51	14	50	90	26

[unlimited possibilities for the blanks above '26']

10.

p	2	3	−2	−3	21
q	3	2	−3	−2	6
p - q	−1	1	1	−1	15

[We trust that students will discover short cuts in doing Exercises 4, 5, 6, 7, and 8.]

*

The students are not solving the equations in Part C. Caution them that some of the statements they obtain should be true and some should be false. Of course, when they obtain a true statement, they have solved the equation; but avoid this terminology at this time. These matters will come up in Unit 3. However, you can keep your able students busy and happy by challenging them to find replacements which will make true statements [except, perhaps, for Exercise 12]. [Do not give this kind of extra assignment as homework because the urge to get help from parents is too strong.] Also, you may want to use the trick of asking students to find a number which will give a <u>false</u> statement in such exercises as 16, 19, and 20.

*

Answers for Part C.

[For Exercises 1-6, 17, and 18, we simply list the numbers which will give true statements.]

1. 2 2. 1 3. $-(1/5)$ 4. $1/3$ 5. 4 6. $5/3$

7. No number will give a true statement--that is, there is no number such that, when you substitute a numeral for it for 'B', you get a true statement. [Each statement obtained is the denial of a consequence of the ℓdpma and the computing fact that $2 \cdot 2 = 4$.]

8. Each number will give a true statement. [Each statement is a consequence of the ℓdpma and two computing facts.]

9. No number will give a true statement.

10. No number will give a true statement.

11. This exercise is interesting. It does not contain a typographical error. The students should recognize that the expression in Exercise 11 is not a sentence. Therefore, one cannot form a true statement nor can he form a false statement by replacing the pronumerals by numerals.

12. 6.72 and -0.89 are numbers that are approximations to the numbers which will give true statements. We doubt that any of your students will be able to determine this, and you may not even want to suggest these approximations. The students will find many replacements that will give false statements.

13. Each number greater than $2/5$ will give a true statement, and these are the only numbers which do.

14. Each number greater than -4 will give a true statement, and these are the only numbers which do.

15. Each number will give a true statement.

16. Each number will give a true statement.

[In Exercises 17 through 20, do not introduce exponent notation. If some students already know how to use this notation, let them use it on their own papers. [We bring in exponents in Units 3 and 4.] The student's symbolism load is heavy enough at this time without introducing another symbol which is frequently the cause of mechanical errors. We think a student is less likely to confuse 'x^2' with '$2x$' if he works with 'xx' for a while before using exponent symbols.]

17. 2, -2 18. 5, -4

19. Each number will give a true statement.

20. Each number will give a true statement.

20. (b) ['x + y' for 'a', 'u + v' for 'b']
    (f) ['x + y' for 'x', 'u' for 'y', 'v' for 'z']
    (h) ['x' for 'm', 'y' for 'n', 'u + v' for 'p']

21. (b) ['x + y' for 'a', 'x + y' for 'b']
    (f) ['x + y' for 'x', 'x' for 'y', 'y' for 'z']
    (h) ['x' for 'm', 'y' for 'n', 'x + y' for 'p']

22. (a) ['7' for 'x', '2k' for 'y']
    (c) ['7' for 'y', '2k' for 'x']
    (e) ['7' for 'a', '2' for 'b', 'k' for 'c']

23. (b) ['7 + 2' for 'a', 'k' for 'b']
    (h) ['7' for 'm', '2' for 'n', 'k' for 'p']

24. (a) ['7k' for 'x', '2k' for 'y']
    (c) ['7k' for 'y', '2k' for 'x']
    (d) ['7' for 'x', 'k' for 'y', '2k' for 'z']
    (e) ['7k' for 'a', '2' for 'b', 'k' for 'c']
    (i) ['7' for 'u', 'k' for 'w', '2' for 'v']

12. (b) ['2a + 3b' for 'a', 'c' for 'b']
    (h) ['2a' for 'm', '3b' for 'n', 'c' for 'p']

13. (b) ['(2a + 3b)5' for 'a', 'c' for 'b']
    (o) ['2a + 3b' for 'x', '5' for 'y', 'c' for 'z']
    [Note that, since '(2a + 3b)5c' is an abbreviation for '[(2a + 3b)5]c', it cannot be generated from '(m + n)p'.]

14. (b) ['7 · 5' for 'a', 'x' for 'b']
    (o) ['7' for 'x', '5' for 'y', 'x' for 'z']

15. (b) ['7' for 'a', '5x' for 'b']
    (j) ['7' for 'P', '5' for 'Q', 'x' for 'R']

16. (a) ['5x' for 'x', '7x' for 'y']
    (c) ['5x' for 'y', '7x' for 'x']
    (d) ['5' for 'x', 'x' for 'y', '7x' for 'z']
    (e) ['5x' for 'a', '7' for 'b', 'x' for 'c']
    (i) ['5' for 'u', 'x' for 'w', '7' for 'v']

17. (a) ['x5' for 'x', 'x7' for 'y']
    (c) ['x5' for 'y', 'x7' for 'x']
    (d) ['x' for 'x', '5' for 'y', 'x7' for 'z']
    (e) ['x5' for 'a', 'x' for 'b', '7' for 'c']
    (g) ['x' for 'a', '5' for 'b', '7' for 'c']

18. (b) ['1' for 'a', '1' for 'b']
    (m) ['1' for 'x']
    (n) ['1' for 'x']

19. (a) ['x' for 'x', 'y' for 'y']
    (c) ['x' for 'y', 'y' for 'x']

5. (a) ['2 + 8' for 'x', '3 + 5' for 'y']
   (c) ['2 + 8' for 'y', '3 + 5' for 'x']
   (k) ['2 + 8' for 'x', '3' for 'y', '5' for 'z']
   (ℓ) ['2' for 'a', '8' for 'b', '3 + 5' for 'c']

6. (a) ['2 + (8 + 3)' for 'x', '5' for 'y']
   (c) ['2 + (8 + 3)' for 'y', '5' for 'x']
   (ℓ) ['2' for 'a', '8 + 3' for 'b', '5' for 'c']

7. (b) ['5 + 2 · 3' for 'a', '1' for 'b']
   (h) ['5' for 'm', '2 · 3' for 'n', '1' for 'p']
   (m) ['5 + 2 · 3' for 'x']

8. (a) ['9 · 1' for 'x', '3 · 1' for 'y']
   (c) ['9 · 1' for 'y', '3 · 1' for 'x']
   (d) ['9' for 'x', '1' for 'y', '3 · 1' for 'z']
   (e) ['9 · 1' for 'a', '3' for 'b', '1' for 'c']
   (i) ['9' for 'u', '1' for 'w', '3' for 'v']

9. (a) ['x · 1' for 'x', 'x · 1' for 'y']
   (c) ['x · 1' for 'y', 'x · 1' for 'x']
   (d) ['x' for 'x', '1' for 'y', 'x · 1' for 'z']
   (e) ['x · 1' for 'a', 'x' for 'b', '1' for 'c']
   (g) ['x' for 'a', '1' for 'b', '1' for 'c']
   (i) ['x' for 'u', '1' for 'w', 'x' for 'v']

10. (b) ['2a' for 'a', 'b' for 'b']
    (o) ['2' for 'x', 'a' for 'y', 'b' for 'z']

11. (b) ['2a' for 'a', '3b' for 'b']
    (j) ['2a' for 'P', '3' for 'Q', 'b' for 'R']
    (o) ['2' for 'x', 'a' for 'y', '3b' for 'z']

The exercises in Parts D and E are designed to give students practice in recognizing the form of an expression. Notice that students are asked to <u>be prepared</u> to give substitutions, that is, to be able to expand their answers by adding comments like the bracketed ones in the Solution for Sample 1. But students should not be required to write such expanded answers on their homework papers. A student's written answer for Sample 1 should be: (a), (c), (d), (e), (i). In discussing these problems in class it is helpful to use arrows in showing the substitutions. For example, to show that the expression in Sample 1 can be generated from expression (d), you might write the given expression and (d) on the board with arrows and braces as follows:

$$\underbrace{(4 + 5)}_{x}\underbrace{2}_{y} + \underbrace{3 \cdot 2}_{z}$$

$$xy + z$$

*

Answers for Part D.

1. (a) [‘9’ for ‘x’, ‘5 · 6’ for ‘y’]
   (c) [‘9’ for ‘y’, ‘5 · 6’ for ‘x’]
   (e) [‘9’ for ‘a’, ‘5’ for ‘b’, ‘6’ for ‘c’]

2. (a) [‘3 · 5’ for ‘x’, ‘2’ for ‘y’]
   (c) [‘3 · 5’ for ‘y’, ‘2’ for ‘x’]
   (d) [‘3’ for ‘x’, ‘5’ for ‘y’, ‘2’ for ‘z’]

3. (a) [‘3a’ for ‘x’, ‘3b’ for ‘y’]
   (c) [‘3a’ for ‘y’, ‘3b’ for ‘x’]
   (d) [‘3’ for ‘x’, ‘a’ for ‘y’, ‘3b’ for ‘z’]
   (e) [‘3a’ for ‘a’, ‘3’ for ‘b’, ‘b’ for ‘c’]
   (g) [‘3’ for ‘a’, ‘a’ for ‘b’, ‘b’ for ‘c’]

4. (a) [‘2 + 8 + 3’ for ‘x’, ‘5’ for ‘y’]
   (c) [‘2 + 8 + 3’ for ‘y’, ‘5’ for ‘x’]
   (ℓ) [‘2 + 8’ for ‘a’, ‘3’ for ‘b’, ‘5’ for ‘c’]

D. Here are fifteen pronumeral expressions. By substituting in any of these expressions you can generate other expressions which have the same pattern. Each of the exercises is an expression which can be generated from one or more of the fifteen expressions. For each exercise, tell which of the expressions it can be generated from, and be prepared to give the substitutions.

(a) x + y	(b) ab	(c) y + x
(d) xy + z	(e) a + bc	(f) x(y + z)
(g) ab + ac	(h) (m + n)p	(i) uw + vw
(j) P(QR)	(k) x + (y + z)	(ℓ) a + b + c
(m) x · 1	(n) 1 · x	(o) xyz

Sample 1. (4 + 5)2 + 3 · 2

Solution. (a) ['(4 + 5)2' for 'x', '3 · 2' for 'y']

(c) ['(4 + 5)2' for 'y', '3 · 2' for 'x']

(d) ['4 + 5' for 'x', '2' for 'y', '3 · 2' for 'z']

(e) ['(4 + 5)2' for 'a', '3' for 'b', '2' for 'c']

(i) ['4 + 5' for 'u', '2' for 'w', '3' for 'v']

Sample 2. 2x + 2y + 3z

Solution. (a) ['2x + 2y' for 'x', '3z' for 'y']

(c) ['2x + 2y' for 'y', '3z' for 'x']

(e) ['2x + 2y' for 'a', '3' for 'b', 'z' for 'c']

(ℓ) ['2x' for 'a', '2y' for 'b', '3z' for 'c']

1. 9 + 5 · 6	2. 3 · 5 + 2	3. 3a + 3b
4. 2 + 8 + 3 + 5	5. 2 + 8 + (3 + 5)	6. 2 + (8 + 3) + 5
7. (5 + 2 · 3)1	8. 9 · 1 + 3 · 1	9. x · 1 + x · 1
10. 2ab	11. 2a(3b)	12. (2a + 3b)c
13. (2a + 3b)5c	14. 7 · 5x	15. 7(5x)
16. 5x + 7x	17. x5 + x7	18. 1 · 1
19. x + y	20. (x + y)(u + v)	21. (x + y)(x + y)
22. 7 + 2k	23. (7 + 2)k	24. 7k + 2k

E. Each exercise contains four expressions. Your job is to write a pronumeral expression from which you can generate the first three but not the fourth.

Sample. $\begin{cases} 8 \cdot 5 + 5 \\ 8 \cdot 3 + 3 \\ 8 \cdot 2 + 2 \end{cases}$
$7 \cdot 9 + 9$

Solution. Pronumeral expressions from which the first three can be generated are:

$$x,\ x + y,\ xy + z,$$

$$xy + y,\ 8x + y,\ 8x + x.$$

'$7 \cdot 9 + 9$' can also be generated from 'x', '$x + y$', '$xy + z$', and '$xy + y$', but not from '$8x + y$' or '$8x + x$'.

Answer. $8x + y$

[Another correct answer: $8x + x$. Of course, you could use other pronumerals instead of 'x' and 'y'.]

1. $\begin{cases} 9 + 15 \\ 9 + -41 \\ 9 + \frac{1}{2} \end{cases}$
$7 + 15$

2. $\begin{cases} 3 - 2 + 3 \\ 9 - 2 + 9 \\ 3x - 2 + 3x \end{cases}$
$3 - 5 + 3$

3. $\begin{cases} 5 \cdot 9 + 4 \cdot 8 \\ 5 \cdot 8 + 4 \cdot 8 \\ 5 \cdot -3 + 4 \cdot 12 \end{cases}$
$5 \cdot 7 + 8 \cdot 7$

4. $\begin{cases} 85 \cdot 1 \\ -2 \cdot 1 \\ x \cdot 1 \end{cases}$
$1 \cdot x$

5. $\begin{cases} 3x + 0 \\ 3y + 0 \\ 3z + 0 \end{cases}$
$0 + 0$

6. $\begin{cases} 1 + 2 \cdot 5 \\ 1 + 2x \\ 1 + 2y \end{cases}$
$3a$

7. $\begin{cases} 5 + (9 + 1) \\ a + (b + c) \\ 2x + (3y + z) \end{cases}$
$9 + 5 + 1$

8. $\begin{cases} 5 \cdot 3 \cdot 9 \\ 6 \cdot 2x \\ 8x \cdot 3 \end{cases}$
$9(2x)$

9. $\begin{cases} 3 + (5 + 2) + 7 \\ 6 + \quad 8 \quad + 5 \\ 7 + 4 + (5 + 9) \end{cases}$
$6 + (2 + 8)$

Answers for Part E [on pages 2-22 and 2-23].

1. $9 + x$
2. $x - 2 + x$ [or: $x - 2 + y$]
3. $5x + 4y$ [or: $xy + 4z$]
4. $x \cdot 1$
5. $3x + 0$ [or: $3x + y$]
6. $1 + 2x$ [or: $1 + x$, or: $x + 2y$, or: $x + yz$]
7. $x + (y + z)$
8. xyz
9. $x + y + z$
10. $7(x + y)$ [or: $x(y + z)$, or: $7x$]
11. $6x + 6y$ [or: $xy + 6z$, or: $6x + yz$, or: $xy + xz$]
12. $4(x + y)$ [or: $x(y + z)$]

*

Here is a multiple-choice quiz which covers some of the ideas of Parts D and E.

Directions. In each question you are given an expression and, below it, several pattern-expressions. Draw a loop around that pattern-expression from which the given expression can be obtained by substitution.

1. $5x + 9x$

   (A) $ab + ac$ (B) $xy + zy$

   (C) $(a + b)c$ (D) $x(y + z)$

2. $3a + 2b + 5a$

   (A) $(x + y) + z$ (B) $x + (y + z)$

   (C) $(x + y)z$ (D) $(x + y) + x$

3. $8p + [7r + 8(p + r)]$

(A) $x + y + z$ (B) $x + [y + z + v]$

(C) $x + [y + (z + v)]$ (D) $x + (y + z)$

4. $10(x + 3) + 5\{2x + [3(x + 5) + 6]\}$

(A) $ab + cde$ (B) $x + yx$

(C) $x + y[z + (u + v)]$ (D) $xyz + u[p + (qr + s)]$

5. $x(a + b) + (a + b)y$

(A) $mn + np$ (B) $xy + yx$

(C) $ba + ca$ (D) $(a + b)c$

6. $2k + 7m + 3k + 2(m + 5k)$

(A) $(x + y) + (u + v)$ (B) $x + (y + u) + v$

(C) $[(x + y) + u] + v$ (D) $x + [y + (u + v)]$

7. $(3p + 1)s + (2p + 5)s$

(A) $(m + n)q$ (B) $qx + qy$

(C) $ab + cb$ (D) $x(y + z)$

8. $(2x + 1)5y$

(A) $(ab)c$ (B) $a(bc)$

(C) $(a + b)(cd)$ (D) $(ab)(cd)$

*

Answers for Quiz.

1. (B)	2. (A)	3. (D)	4. (C)
5. (A)	6. (C)	7. (C)	8. (A)

The purpose of the Exploration Exercises is to teach students an adequate form for stating universal generalizations. This is a difficult teaching task, and we hope that the Exploration Exercises will make the job easier. You should select two students who are expressive readers and have them read this dialogue, one student taking the part of Stan, and the other of Al. [If the class objects that it would be unlikely that the two boys could communicate this way by letter, tell them to imagine that they have become ham radio-television operators!] As the two readers proceed through the dialogue, you should summarize from time to time. For example, when the reader finishes Al's remark at the bottom of page 2-23, you should say something like:

> Notice, that although you can get a pretty good idea of what the left distributive principle is from instances, the instances are <u>not</u> the principle.

Notice how difficult it is for Stan to give a precise description of an instance of the left distributive principle. Al is very quick to take advantage of the flaw in this description. Also, notice how much easier it is to use letter pronumerals for this purpose. The fact that letter pronumerals obey a linking rule makes it easy to display the pattern.

Although the open-sentence-business is sufficient to tell you how to get instances of the left distributive principle, it tells you nothing about numbers. The left distributive principle is a principle which asserts something about addition and multiplication of numbers. Al is quite sophisticated in pointing out the distinction between a statement which tells something about numerals and sentences [such as Stan's statement, 'Take the open ... confusion.'] and a statement which tells something about numbers! Stan's next attempt at the top of page 2-25 is pretty close to correct. However, Al splits a hair by pointing out implicitly that '3 + 4' and '4 + 3' each stand for the sum of 3 and 4. The use of the open sentence involving the letter pronumerals makes it unnecessary to use the complicated phraseology which Stan employs at the bottom of page 2-25. And, Al makes this point in his remark at the top of page 2-26.

Notice the phrases:

> for each first real number,
>
> for each second real number,
>
> for each third real number.

It is phrases such as these which tell you that a generalization is being stated. Without such phrases, we simply have an open sentence which itself is neither true nor false. The leading phrases together with the open sentence make up the generalization.

10. $\begin{cases} 7(3 + 5) \\ 7(8 + 2) \\ 7(5 + 3) \end{cases}$
$\quad (4 + 5)7$

11. $\begin{cases} 6 \cdot 9 + 6 \cdot 2 \\ 6 \cdot 3 + 6 \cdot 5 \\ 6 \cdot 4 + 6 \cdot 8 \end{cases}$
$\quad 2 \cdot 6 + 3 \cdot 6$

12. $\begin{cases} 4(5 + x + 3) \\ 4(4 + 8 + y) \\ 4(5 + 9) \end{cases}$
$\quad 4 \cdot 3$

EXPLORATION EXERCISES

A. Remember Al Moore who lived in Alaska and wanted to learn mathematics by correspondence with his pen pal, Stan Brown? Well, it happened that Al really did it. He learned about real numbers and pronumerals and open sentences. Stan had told him about various principles of real numbers which helped Al get short cuts, and also helped him remember lots of computing facts. [For example, as soon as he had memorized the fact that 7×8 is 56, he also knew that 8×7 is 56.] Al now wanted to know more about these principles. So, he wrote Stan and said:

> Just what is the left distributive principle?

Stan replied:

> I thought you knew that. Here are some instances of it.

$$7(5 + 8) = 7 \cdot 5 + 7 \cdot 8$$

$$-9(7 + -3) = -9 \cdot 7 + -9 \cdot -3$$

$$-4(y + 2) = -4 \cdot y + -4 \cdot 2$$

Al wrote back:

> Say, I don't want instances of it. I want to know what the principle is! After all, am I supposed to remember these examples every time I want to use the left distributive principle? And are these all of the instances? Remember, I don't have anyone up here I can ask. How can I tell an instance of the left distributive principle when I meet it?

So, Stan answered:

> Well, of course, I can't write down every instance. That would be impossible. But I can give you a rule for getting any instance or recognizing one when you see it. First, there is a numeral, followed by a left parenthesis, then a numeral, then a plus sign, then a numeral, then a right parenthesis, then an equality sign, then a numeral, then a multiplication dot, then a numeral, then a plus sign, then a numeral, then a multiplication dot, then, finally, a numeral. <u>That's</u> an instance of the left distributive principle.

Al sent an airmail reply:

> Aha! So, I suppose this is an instance of the left distributive principle:
>
> $$5(9 + 17) = 8 \cdot 6 + 41 \cdot 7.$$
>
> I followed your rule!

Stan was quick to answer:

> No, that's not it. I see I didn't make myself clear. I'll try again. Take the open sentence:
>
> $$x(y + z) = xy + xz.$$
>
> This gives you a <u>pattern</u> for generating instances. Just substitute a numeral for 'x', a numeral for 'y', and a numeral for 'z', and put in two multiplication dots on the right side to avoid confusion.

Al thought about this, and his next letter said:

> Well, I think I can follow that rule all right. But, didn't you tell me a long time ago that the left distributive principle said something about numbers? Your rule just tells me about numerals and sentences.

And, in my first lesson I learned that numerals aren't numbers. Tell me what the left distributive principle says about numbers.

Stan tries:

It says that for each real number you take, if you multiply it by the sum of a real number and a real number, you get the same number as you would if you took that first number and multiplied it by one of the other numbers and . . . . Wait, I'll start again.

For each first real number you take, if you multiply it by the sum of your second real number and your third real number, you get the same number as you would if you added the numbers you get by multiplying your first number by your second number and by multiplying your first number by your third number. Whew!!

Al complains:

'Whew' is right! That's complicated. It sounded as though this sentence:

$$5(3 + 4) = 5 \cdot 4 + 5 \cdot 3$$

is an instance of the principle. But, by that pattern business you wrote about in one of your other letters, this sentence wouldn't be an instance. Right?

Wearily, Stan replies:

Darn! I meant that you should add the third number to the second, and that you should add the product of the first number by the third to the product of the first number by the second. Maybe I didn't say it that way; I forget.

Al now tries to be helpful:

> I think maybe you ought to use that pattern-sentence along with the talk about choosing numbers to help keep things straight. How about this?

For each first real number I pick, for each second real number, and for each third real number, it turns out that
the 1st number × (the 2nd number + the 3rd number)
equals
the 1st number × the 2nd number + the 1st number × the 3rd number.

> Is this what you were trying to tell me?

Stan, with glee:

> Yeah, that's it! And you've given me an idea of how to say it in a shorter way. Remember when we talked about phrases like 'the first real number', 'the second real number', and 'the third real number' being like pronouns? Well, let's use pronumerals. Here goes.

For each x, for each y, for each z,

$$x(y + z) = xy + xz.$$

> This is it, isn't it?

Al closes this exchange with:

> I think we've got it now. And, I like the short way you said the same thing I did. But you didn't have to use 'x', 'y', and 'z' in that order did you? Couldn't I say that this is the left distributive principle, just as well?

For each y, for each z, for each x,

$$y(z + x) = yz + yx.$$

> Or, you could even use different letters.

For each a, for each p, for each t,

$$a(p + t) = ap + at.$$

occurs first. And, the property which is said to be generalized is expressed by the predicate which one obtains by dropping this first quantifying phrase [and, strictly, replacing each remaining occurrence of its corresponding pronumeral by '...']. [But, no harm will now result if we consider that the open sentence which remains when the quantifying phrase is dropped "expresses" the property in question.] Thus, one obtains in this way from (3‴) an expression equivalent to (3):

$$\text{for each } y, \ y \leq \ldots,$$

from (4‴) an expression equivalent to (4):

$$\text{there is an } x \text{ such that } \ldots \leq x,$$

and, from (1‴), (2‴), and (5‴) one obtains, respectively, (1), (2), and (5). So, one can lean on one's predilection for the subject-predicate form of sentence for aid in interpreting sentences written in this manner. [As previously noted, the collective strength of 'all' and the distributive strength of 'any' are likely to override the English convention we are here leaning on. For this reason, these words should be avoided as quantifiers. Although the word 'each' is probably the best choice, 'every' will <u>usually</u> do nearly as well.]

*

The quantifying phrases 'for each' and 'there is a ... such that' have convenient abbreviations. One pair of such abbreviations is:

$$\forall \qquad\qquad \exists$$

Thus, we may write (1′), (2′), (3′), (4′), and (5′) as:

$$\forall_x \ x \leq {}^+3$$

$$\exists_x \ {}^+2 \leq x$$

$$\exists_x\forall_y \ y \leq x$$

$$\forall_y\exists_x \ y \leq x$$

and:

$$\forall_x \ x \leq x .$$

And, the left distributive principle may be written:

$$\forall_x\forall_y\forall_z \ x(y+z) = xy + xz .$$

Note that, to avoid additional punctuation, the index directly associated with each quantifier '$\forall$' and '$\exists$' is written as a subscript.

Thus, we obtain:

$$\text{for each } x, \text{ for each } y, \text{ for each } z,$$
$$x(y + z) = xy + xz$$

as a statement of the ℓdpma. It is essential to realize that here again, the pronumerals 'x', 'y' and 'z' perform just two functions. They link quantifiers with argument places, and, by virtue of the convention that their domain is the set of real numbers, each indicates the domain of the quantifier with which it is associated. [An entirely erroneous impression is obtained if one reads 'for each x' as 'for each number, say x'.]

Note that with this convention the universal quantifier is 'for each', rather than merely 'each'. [It would be natural to use 'for some' as the existential quantifier, but experience shows that 'there is a ... such that' is generally easier to interpret.]

With this convention, sentences (1″), (2″), (3″), (4″), and (5″) are replaced by:

(1‴) for each x, $x \leq {}^{+}3$

(2‴) there is an x such that ${}^{+}2 \leq x$

(3‴) there is an x such that, for each y, $y \leq x$

(4‴) for each y, there is an x such that $y \leq x$

and

(5‴) for each x, $x \leq x$.

Of course, since the pronumerals serve merely as indices and to specify the domains of the quantifiers, any pronumerals with the proper domains can be used, provided that different pronumerals are used to index different quantifiers. Thus, in place of (3‴) one might write:

there is a z such that, for each x, $x \leq z$,

or: there is a y such that, for each z, $z \leq y$.

Recall that our first convention was to indicate precedence order for quantifiers by supplying each quantifying phrase in a sentence with a numerical subscript. The first step from here to our later conventions was to move these phrases to the beginning of the sentence, arranging them in order of precedence. So, by our final convention, the principal quantifying phrase in a generalization is the one which

(3″) each $\text{number}_2 \leq$ some number_1 ,

and, for the associated universal generalization [indicated by (4′)]:

(4″) each $\text{number}_1 \leq$ some number_2 .

In place of (5′) one would then write:

(5″) each $\text{number}_1 \leq$ each number_1 .

For consistency, we would replace (1′) and (2′) by:

(1″) each $\text{number}_1 \leq {}^{+}3$

and:

(2″) ${}^{+}2 \leq$ some number_1 .

With this convention a statement of the left distributive principle would be:

$$\text{each number}_1(\text{each number}_2 + \text{each number}_3)$$
$$= \text{each number}_1 \cdot \text{each number}_2 + \text{each number}_1 \cdot \text{each number}_3 .$$

If one reads this expression, word by word, and tries to interpret it as a statement of the ℓdpma, he is likely to be annoyed by the repetition of quantifiers. A way out of this difficulty is to place the quantifying phrases in the numerical order of their subscripts, at the beginning, and write:

$$\text{for each number}_1, \text{ for each number}_2, \text{ for each number}_3,$$
$$\text{number}_1(\text{number}_2 + \text{number}_3) = \text{number}_1 \cdot \text{number}_2 + \text{number}_1 \cdot \text{number}_3,$$

leaving the indexed common nouns 'number_1', 'number_2' and 'number_3' in the argument places to indicate which argument places are governed by each quantifier. The indexed common nouns, both in the quantifying phrases and in the argument places, now serve two <u>and only two</u> functions. As stated, they link quantifiers to argument places [thus solving the linking problem], and they indicate the domain over which the generalization takes place [in the case of each of the three quantifiers above, the set of real numbers]. Now, these two functions are shared by pronouns and common nouns, and it is only natural that we should replace the indexed words by pronumerals.

quantifier which occurs in this phrase is the principal quantifier. We have seen that there are various conventions for determining which quantifying phrase in an English sentence is the principal one. In simple cases, the principal quantifying phrase is the one which occurs first in the sentence, but in more complicated cases other devices are used. [A device not previously mentioned for indicating order among quantifiers is the use of adjectives such as 'first' and 'second' as in 'each first number' and 'some second number'.]

There is a second problem concerning the use of quantifiers which also must be solved by suitable conventions. Suppose that one wants to say that the property expressed by the predicate:

(5) $\ldots \leq \ldots$ [not: the relation expressed by '$\ldots \leq$ ---']

holds universally in the set of real numbers. One might, somewhat naively, write:

each number $\leq$ each number,

but this is at best ambiguous. Here, the English convention is to write:

(5′) each number $\leq$ itself.

But, again, other conventions are needed [as Stan discovered] when one has to deal with sentences containing more complicated predicates. Stan was dealing with the predicate:

$$\ldots(\text{---} + \text{\_\_\_}) = \ldots \cdot \text{---} + \ldots \cdot \text{\_\_\_},$$

and had great difficulty in inserting quantifying phrases [page 2-25] so that the proper argument places would be linked.

What we need is a single, simple, convention which will solve the two problems--(a) the problem of determining the order of precedence among argument places of a predicate [or among quantifiers in a sentence built on the predicate], and (b) the problem of indicating when two argument places are linked.

One solution is to assign numerical subscripts to the quantifying phrases so that numerical order agrees with order of precedence. For example, for the existential generalization [indicated by (3′)] which states that there is a greatest real number, one might write:

But, while this is a universal generalization, it does not say what we wish to say. What it does do is to justify saying that each equation like:

$$^{+}2 - x = 10, \quad ^{-}3 - x = 10, \quad 483 - x = 10, \quad \text{etc.}$$

has a root. But, these are not the equations with which we are here concerned. To say what we wish to say, we must also change the predicate. We can do this by introducing a new operator:

$\overline{\smile}$ [read as 'subtracted from']

defined in such a way that, for example,

'$^{+}2 \overline{\smile} {^{+}12}$' is synonymous with '$^{+}12 - {^{+}2}$',

'$^{-}3 \overline{\smile} {^{+}7}$' is synonymous with '$^{+}7 - {^{-}3}$',

and '$483 \overline{\smile} 493$' is synonymous with '$493 - 483$'.

Then, we replace '$-$' by '$\overline{\smile}$' to get:

each number $\overline{\smile}$ some number = 10.

This is the desired generalization [if you want to live with the convention!] It tells us that each of the equations:

$$^{+}2 \overline{\smile} x = 10, \quad ^{-}3 \overline{\smile} x = 10, \quad 483 \overline{\smile} x = 10, \quad \text{etc.}$$

has a root. But, *by the meaning of* '$\overline{\smile}$', this is the same as saying that:

$$x - {^{+}2} = 10, \quad x - {^{-}3} = 10, \quad x - 483 = 10, \quad \text{etc.}$$

have roots.

It should be clear by now that although it is possible to tolerate in English the convention which has been illustrated, it would be most inconvenient to follow such a convention in mathematical discourse. Shortly, we shall introduce a more satisfactory convention.

The problem which has concerned us up to now is that of establishing the desired order of precedence among quantifying phrases in expressions such as (*) and (**). When we wish (*) to be analyzed as an existential generalization, as suggested by (3′), we shall speak of the 'some number' which occurs in it as *the principal quantifying phrase*; and when we wish (*) analyzed as (4′), we shall say that the 'each number' is the principal quantifying phrase. In general, *the principal quantifying phrase* in a generalization sentence is the one which is thought of as having been written in a blank in the appropriate predicate in order to *complete* the sentence. And we shall say that the

We have seen that the convention according to which the order in which quantifiers occur in a sentence determines what the sentence says, is inadequate, and has to be bolstered by other conventions. [This kind of inadequacy becomes even more apparent if one considers sentences in which three or more quantifiers occur.] A more important inadequacy is a consequence of the fact that dependence on this convention requires a change in the predicate [the use of '$\geq$' in place of '$\leq$', or of 'is loved by' in place of 'is in love with'] when one wishes to state one kind of generalization [say, existential] rather than the other. Such a change is often awkward, and may even require the invention of a new symbol as is illustrated in the following example.

Suppose, for example, that we want to make a statement which justifies claiming that all equations like:

$$x - {}^{+}2 = 10, \quad x - {}^{-}3 = 10, \quad x - 483 = 10, \text{ etc.}$$

have roots. We want to write a universal generalization sentence. We can say what we wish to by saying that for each number you take [${}^{+}2$, ${}^{-}3$, 483, etc.], there is bound to be some number [the root of the equation] which you can subtract it from and get 10. One might be inclined to think that he is asserting this [in concise language] when he writes:

some number - each number = 10.

But, because of the convention, this sentence is actually an existential generalization [the predicate is: ... - each number = 10]. It says that all of the equations in question have a common root, and this is false.

The situation facing us is that facing a person who wants to write the universal generalization (†) rather than the existential generalization (††), or who wants to write (*) rather than (**). The convention requires such a person to interchange the quantifiers. But, in order to say what he wishes to, he must also change the predicates. For example, he must replace 'loved by' in (††) by 'in love with', and '$\geq$' in (**) by '$\leq$'.

In the case at hand, the sentence analagous to (††) and (**) is:

some number - each number = 10.

The convention tells us that in order to get a universal generalization, we must interchange the quantifiers. We do so, and get:

each number - some number = 10.

The convention, according to which one deletes the first of the quantifying phrases from a sentence in order to obtain the predicate of the sentence, can be illustrated by other pairs of sentences, such as:

(†) each person is in love with some person
[a universal generalization]

(††) some person is loved by each person
[an existential generalization]

However, this convention [as you have probably found] is rather a weak one, and in practice it is bolstered up with others. For example, instead of (**), one is likely to write:

some number $\geq$ every number

or, still less ambiguously:

some number $\geq$ all numbers .

The bolstering conventions here are that the quantifier 'every' has the collective [as opposed to the distributive] connotation more strongly than does 'each'; and that 'all' is even "more collective" [and "less distributive"] than is 'every'. In fact, the quantifier 'all' is so strongly collective that when it occurs in a sentence, together with another quantifier, one automatically makes it part of the predicate. As an example of this, note that the sentence:

all numbers $\leq$ some number

is interpreted as an existential generalization, saying just what (**) is meant to, even though the existential quantifier 'some' follows the universal quantifier 'all'. [In contrast to 'every' and 'all', the word 'any' is, in many contexts, much more <u>distributive</u> than is 'each'. For example, the sentence:

some number $\geq$ any number

(especially if, in reading it, one emphasizes the word 'some') can easily be interpreted as a universal generalization, while this is more difficult in the case of (**). However, the distributivity of 'any' is most easily seen if one contrasts a sentence such as:

it is not the case that <u>any</u> pig is purple

with the corresponding sentence:

it is not the case that <u>each</u> pig is purple

(and then compares with one another the sentences obtained from these by deleting the phrase 'it is not the case that').]

Let us now consider the expression:

(*) each number $\leq$ some number .

We may think of (*) as obtained by writing a 'some number' in the blank in (3), as can be indicated more explicitly by writing:

(3′) each number $\leq$ | some number .

Thinking so, we interpret (*) as an existential generalization sentence which says that there is a greatest real number. Under this interpretation, (*) is a false sentence. On the other hand, we may think of (*) as obtained by writing an 'each number' in the blank in (4), as indicated by:

(4′) each number | $\leq$ some number.

Thinking so, we interpret (*) as a universal generalization sentence which says that each number has the property of being <u>less</u> <u>than</u> <u>or</u> <u>equal</u> <u>to</u> <u>some</u> <u>number</u>. Since each number is, in fact, less than or equal to itself, this universal generalization is true and, under this interpretation, (*) is a true sentence. So, the meaning of (*) depends essentially on how we choose to analyze it as a sentence. This is in sharp contrast to our findings in connection with sentence (0), sentence (1′), and sentence (2′).

Clearly, if we are to consider (*) itself as a sentence, we must have some way of deciding for one analysis and against the other. Now, in English, we do have a "rule" for making such decisions. If you read (*) without thinking of the preceding discussion, you probably tend to interpret it as a universal generalization--that is, as suggested by (4′). The reason for this is that subject-predicate sentences are much more common in English than are predicate-subject sentences. So, although (*) is not a subject-predicate sentence ['each number' is not a noun], you are likely to regard it as such and thereby identify (4) as the predicate in (*). [You <u>may</u> be able to interpret (*) as an existential generalization if you emphasize strongly the quantifying phrase 'each number', for this emphasis may link the phrase, for you, with the '$\leq$ ', and so cause you to think of the property expressed by (3). (On the other hand, for some people, this emphasis will have the opposite effect!)] The feeling that subject-predicate sentences are "more respectable" than other kinds of sentences has the result that in order to say that the property expressed by (3) is existentially generalized, --that is, in order to say that there is a greatest number, one is likely to write:

(**) some number $\geq$ each number.

So, according to the first analysis, (1′) is not a subject-predicate sentence but a new kind of sentence,

a universal generalization sentence.

We shall call 'each number' a quantifying phrase. It consists of the quantifier 'each' and the common noun 'number'. More specifically, we call the word 'each' a universal quantifier.

Similar remarks can now be made about the sentence:

(2′) $\quad {}^{+}2 \leq$ some number .

This sentence can be analyzed as referring to the property of being greater than or equal to $^{+}2$, which is expressed by the predicate:

(2) $\quad {}^{+}2 \leq$ --- ,

or it can be analyzed as referring to the property of being less than or equal to some number, which is expressed by the predicate:

(4) $\quad \ldots \leq$ some number .

According to the first of these analyses, (2′) says that the first property holds existentially in [or: is existentially generalized over] the set of real numbers. According to the second analysis, (2′) says that the second property holds of the real number $^{+}2$. According to both analyses, (2′) conveys the same information, but it does so in different ways. According to the second analysis, (2′) is a subject-predicate sentence, but, according to the first, (2′) is

an existential generalization sentence.

Attempts to construe 'some number' as a noun result in assertions that there are "particular, but not completely specified, numbers", and must fail because, for example, the sentences:

some number is both even and odd

and:

some number is even and some number is odd

are not equivalent. We shall call 'some number' a quantifying phrase, and call 'some' an existential quantifier.

Let's concentrate on (1′). We can interpret it as saying that each number has the property of being less than or equal to $^+3$, and this is what we have in mind when we think of (1′) as having been obtained by writing an 'each number' in the blank in (1). Or, we can interpret (1′) as saying that $^+3$ has the property expressed by the predicate:

(3) each number $\leq$ ---,

--that is, the property of being greater than or equal to each number. In this case, we think of (1′) as obtained by filling the blank in (3) with a '$^+3$'. Like the sentence (0), (1′) conveys the same information no matter which way we analyze it; but it does so in one case by referring to the property of being less than or equal to $^+3$ and in the other case by referring to the property of being greater than or equal to each number. According to the first analysis, (1′) says that the first of these properties holds universally in [or: is universally generalized over] the set of real numbers.

According to the second analysis, (1′) says that the second property holds of the real number $^+3$. So, according to the second analysis, we look on (1′) as a predicate-subject sentence; it says that a certain property holds of a certain thing.

If we try to interpret (1′) as a subject-predicate sentence then we are forced to use the first analysis. But, as already noted, according to the first analysis the sentence (1′) says, not that a certain thing has a certain property, but that a certain property holds of each member of a certain set. The attempt to analyze (1′) as a subject-predicate sentence whose subject is 'each number' fails because 'each number' is not a noun. Attempts to make it a noun result in assertions that there are "general numbers" as well as "particular" numbers. That such attempts are doomed to failure is evidenced by the fact that if 'each number' were a noun then the two sentences:

each number is equal to $^+3$ or different from $^+3$

and:

each number is equal to $^+3$ or each number is different from $^+3$

would be equivalent [although they obviously aren't]; just as the sentences:

$^+2$ is equal to $^+3$ or different from $^+3$

and:

$^+2$ is equal to $^+3$ or $^+2$ is different from $^+3$

are equivalent.

[There is a third analysis of (0) suggested by:

$$^{+}2 \mid \leq \mid {}^{+}3,$$

according to which (0) says that $^{+}2$ has the relation of being less than or equal to $^{+}3$. This way of analyzing (0) will not concern us here, but it should be noted that the sentence still conveys the same information when analyzed in this way as it does when analyzed according to (a) or (b).]

We shall distinguish between the two analyses (a) and (b) by describing (0) either as

a subject-predicate sentence,

or as

a predicate-subject sentence.

Correspondingly, we shall think of (0) as being obtained either by filling the blank in:

(1) $\ldots \leq {}^{+}3$

by the numeral '$^{+}2$', or by filling the blank in:

(2) $^{+}2 \leq \text{---}$

by the numeral '$^{+}3$'. The expressions (1) and (2) are called predicates and, in each case, the numeral used to complete the sentence is called the subject of the sentence.

[According to the third way of analyzing (0) it is

a first subject-predicate-second subject sentence

whose predicate is '$\ldots \leq \text{---}$' and whose subjects are '$^{+}2$' and '$^{+}3$'.]

There are other ways of filling the blanks in the predicates (1) and (2) to obtain true-or-false sentences. For example, the sentence:

(1′) each number $\leq {}^{+}3$

is a false sentence, and:

(2′) $^{+}2 \leq$ some number

is a true sentence. As in the case of (0), we can analyze each of these sentences in two ways.

The following discussion is included in the COMMENTARY as background material for teachers who want to know how pronumerals get into generalizations. It is not expected that you will present these matters to your class. Also, you will want to read this discussion again and again as questions occur to you throughout the course.

On generalizations. --Consider the sentence:

(0) $$^{+}2 \leq {}^{+}3$$

We can analyze this sentence in several ways, two of which are suggested by the diagrams:

(a) $^{+}2 \mid \leq {}^{+}3$ (b) $^{+}2 \leq \mid {}^{+}3$

According to the analysis (a), the sentence (0) says that

$^{+}2$ has the property of being less than or equal to $^{+}3$.

According to the analysis (b), the sentence (0) says that

$^{+}3$ has the property of being greater than or equal to $^{+}2$.

Notice that whether we analyze (0) according to (a) or according to (b) it conveys the same information, but in the two cases it does so by referring to different properties and to different numbers.

Comparable analyses can be made of nonmathematical sentences. Consider:

John is in love with Mary.

Analysis (a) tells us that this sentence says that

John has the property of being in love with Mary.

Analysis (b) tells us that the sentence says that

Mary has the property of being loved by John.

According to analysis (a), the sentence refers to John and to the property of being in love with Mary. What does the sentence refer to according to analysis (b)?

In each case, he should agree that the two sentences have the same meaning. [If he doesn't, ask how he decides whether a given open sentence states a generalization.]

Now, point out that he should agree that (1) and:

(4′) either, for each x, $x < 2$, or, for each x, $x \not< 2$

say the same thing. So, (1′) and (4′) must have the same meaning. But, as the student should discover, while (1′) is true, (4′) is false. You may also point out that, while, for example, the statement '$3 \not< 2$' is the denial of the statement '$3 < 2$', and '$x \not< 2$' is the denial of '$x < 2$', statement (3′) is <u>not</u> the denial of statement (2′). The denial of (2′) is:

it is not the case that, for each x, $x < 2$

and this is to say that

there is an x such that $x \not< 2$,

a much weaker assertion than that made by (3′).

See the article "Variable Paradox" by Gertrude Hendrix in the June 1959 issue of <u>School Science and Mathematics</u>.

*

Answers for Part B [on page 2-27].

1. For each x, $3x + 7x = 10x$.
2. For each x, $(x + 2)7 = x7 + 2 \cdot 7$.
3. For each x, $[5(x + 5)4 - 100]\frac{1}{20} = x$.
4. For each x, $(x + 5)(x + 7) = xx + 12x + 35$.

*

After the students have had some practice in stating generalizations by writing 'For each x', etc., you may want to introduce the abbreviation '$\forall_x$'. Thus the commutative principle for addition may be written:

$$\forall_x \forall_y \; x + y = y + x.$$

[It is likely that the '$\forall$' was suggested by the word 'all'. Nevertheless, it is better to read '$\forall$' as 'for each' rather than as 'for all'.] We introduce this abbreviation formally on page 2-61.

*

10. (a) For each first real number, for each second real number,

first number - second number

equals

first number + the opposite of second number.

(b) For each x, for each y, $x - y = x + -y$.

*

Despite your previous efforts throughout the unit, you may still find students who will reply to Exercise 1 of Part A saying:

$$xy = yx.$$

Such a student may have either of two things in mind, and in either case is in trouble. He may really be thinking of '$xy = yx$' as providing a pattern for obtaining instances of the commutative principle (for multiplication). In this case, when he says that '$xy = yx$' is a statement of the commutative principle, he probably thinks of it as an abbreviation of:

> For each substitution of numerals for 'x' and 'y' in '$xy = yx$', the resulting sentence is true.

But, Al would object [see bottom of page 2-24] that this cannot be a statement of the commutative principle.

On the other hand, the student may think that it is proper to claim that '$xy = yx$' is a statement of the commutative principle because he takes '$xy = yx$' as saying the same thing as:

For each x, for each y, $xy = yx$.

Such a student may be asked whether he also takes the open sentence:

(1) either $x < 2$, or $x \nless 2$

to mean the same as:

(1′) for each x, either $x < 2$, or $x \nless 2$.

Presumably he will assent to this. Now, ask a similar question about the sentences:

(2) $x < 2$,

and: (2′) for each x, $x < 2$,

and about the sentences:

(3) $x \nless 2$,

and: (3′) for each x, $x \nless 2$.

*

In Exercises 3 and 4, we use pronumerals other than 'x' and 'y' to emphasize the point made by Al in his last comment [see bottom of page 2-26]. Your students may suggest using different pronumerals in other exercises also. Students should be introduced to subscripts as a device for enlarging the alphabet. In this connection see page 2-121.

*

5. (a) For each first real number, for each second real number, for each third real number,

   (first number + second number) × third number

   equals

   first number × third number + second number × third number.

   (b) For each x, for each y, for each z, $(x + y)z = xz + yz$.

6. (a) For each real number,

   that number + 0 equals that number.

   (b) For each x, $x + 0 = x$.

7. (a) For each real number,

   that number × 1 equals that number.

   (b) For each x, $x \cdot 1 = x$.

8. (a) For each real number,

   that number + the opposite of that number equals 0.

   (b) For each x, $x + -x = 0$.

9. (a) For each real number,

   that number × 0 equals 0.

   (b) For each x, $x \cdot 0 = 0$.

It is important that the students do the exercises at the top of page 2-27 immediately upon concluding the dialogue. The written statements using letter pronumerals should be recorded on a sheet of paper and inserted in the text at this point. Be sure that the leading phrases [quantifying phrases] are included. Students may use the expressions such as 'for each real number x' as the quantifying phrases if they wish. We don't because it is redundant. This is due to the fact that the domain of such variables as 'x' is understood [in this unit] to be the set of real numbers.

Answers for Part A.

1. (a) For each first real number, for each second real number,

   first number × second number equals second number × first number.

   (b) For each x, for each y, $xy = yx$.

2. (a) For each first real number, for each second real number,

   first number + second number equals second number + first number.

   (b) For each x, for each y, $x + y = y + x$.

3. (a) For each first real number, for each second real number, for each third real number,

   (first number × second number) × third number

   equals

   first number × (second number × third number).

   (b) For each a, for each b, for each c, $abc = a(bc)$.

   [Recall the convention that 'abc' is an abbreviation for '(ab)c'.]

4. (a) For each first real number, for each second real number, for each third real number,

   (first number + second number) + third number

   equals

   first number + (second number + third number).

   (b) For each x_1, for each x_2, for each x_3,

   $$x_1 + x_2 + x_3 = x_1 + (x_2 + x_3).$$

Say aloud each principle for the real numbers, first by using pronouns like 'first number', 'second number', etc., and then write it in concise form by using letter pronumerals.

1. Commutative principle for multiplication
2. Commutative principle for addition
3. Associative principle for multiplication
4. Associative principle for addition
5. Distributive principle for multiplication over addition
6. Principle for adding 0.
7. Principle for multiplying by 1.
8. Principle of opposites
9. Principle for multiplying by 0
10. Principle for subtraction

B. Each of the following is a general statement about numbers. Translate it into a sentence beginning with 'For each x, ...'.

Sample. Whatever real number you pick, if you multiply it by 2, and then multiply 3 by the product, the result is the product of 6 by the number chosen.

Solution. For each x, $3(x2) = 6x$.

1. Whatever real number you pick, if you multiply 3 by it, then multiply 7 by it, and then add the second product to the first, the result is the product of 10 by the chosen number.

2. For each real number you pick, if you add 2 to it and multiply the sum by 7, you get the same result as you would by adding the product of 2 by 7 to the product of the number you picked by 7.

3. Pick any real number. Add 5 to it. Multiply 5 by this sum. Multiply by 4. Subtract 100. Multiply by $\frac{1}{20}$. The result is the number you started with.

4. Pick a real number. Add 5 to it, and 7 to it. Multiply the sums. The result is the product of the chosen number by itself, plus the product of 12 by the chosen number, plus 35.

* * *

In Unit 1 you learned procedures for adding and multiplying real numbers. That is, you learned ways of solving problems like:

$^{+}5 + {}^{-}6 = ?$ $\qquad\qquad$ $^{-}3 \times {}^{-}5 = ?$

But, you were not asked to state the rules you followed. Let's look into the problem of stating a rule for, say, adding two negative numbers. Of course, you already know how to add such numbers. Can you state a rule which describes exactly what you do with the numbers to get the sum? Here is one such careful description.

> For each first real number, for each second real number,
> if the first real number is negative and
> the second real number is negative
> then the sum of the real numbers is the negative number which corresponds with the sum of the number of arithmetic which corresponds with the first real number and the number of arithmetic which corresponds with the second real number.

The description can be made briefer by using letter pronumerals. Here is a first attempt.

> For each x, for each y,
> if x is negative and y is negative
> then x + y is the negative number which corresponds with the sum of the number of arithmetic which corresponds with x and the number of arithmetic which corresponds with y.

Can we make further improvements? The phrase

'the number of arithmetic which corresponds with x'

can be abbreviated to '$|x|$' since, as you recall from Unit 1, the absolute value of a real number is the number of arithmetic which corresponds

The discussion on pages 2-28 and 2-29 should be read aloud in class. Here, at last, are descriptions of the procedures for adding and multiplying real numbers. These are procedures which the student has mastered by this time, but this is the first opportunity for him to consider the complexity of describing the procedure. Students should not be required to memorize any of these descriptions. The purpose of the exercises in Part C and Part D is to demonstrate the ease with which complicated rules can be stated when letter pronumerals are available.

*

Answers for Part C [on page 2-29].

1. $\forall_x \forall_y$ if x is positive and y is positive then $x + y = {}^{+}(|x| + |y|)$.
2. $\forall_x \forall_y$ if x is negative and y is positive then $xy = {}^{-}(|x| \cdot |y|)$.
3. $\forall_x \forall_y$ if x is positive and y is negative then $xy = {}^{-}(|x| \cdot |y|)$.
4. $\forall_x \forall_y$ if x is positive and y is positive then $xy = {}^{+}(|x| \cdot |y|)$.
5. $\forall_x \forall_y$ if x is negative and y is negative then $xy = {}^{+}(|x| \cdot |y|)$.

*

Answers for Part D [on pages 2-29 and 2-30].

1. This is the rule for adding a negative number to a positive number. [Notice that (a) could be written:

   $$\text{if } |x| > |y| \text{ then } \ldots,$$

   in which case (b) would be written:

   $$\text{if } |x| \leq |y| \text{ then } \ldots .$$

   If the condition of equality were not mentioned in either (a) or (b), we would need a third part:

   $$\text{(c)} \quad \text{if } |x| = |y| \text{ then } x + y = 0.$$

   When one generates instances from the rule as it is stated in Exercise 1, he might obtain an expression equivalent to '${}^{+}0$'. As mentioned on page 1-109, '${}^{+}0$' is <u>not</u> a name for a positive number. '${}^{+}0$' stands for the real number 0.]

2. $\forall_x \forall_y$ if x is negative and y is positive then

   (a) if $|x| \geq |y|$ then $x + y = {}^{-}(|x| - |y|)$

   and (b) if $|x| < |y|$ then $x + y = {}^{+}(|y| - |x|)$.

3. (a) $\forall_x\ x + 0 = x$. Principle for adding 0.

   (b) $\forall_x\ x \cdot 0 = 0$. Principle for multiplying by 0.

4. The commutative principle for addition.

5. The commutative principle for multiplication.

*

For the sake of completeness, we should have a description of the procedure for finding the opposite of a real number. Students should state this description and include it in their textbooks.

$\forall_x$ if x is nonnegative then $-x = {}^{-}|x|$, and

if x is negative then $-x = {}^{+}|x|$.

This description will be useful in giving justifications [if any are needed] for Part B on page 2-63. [Be sure that students read '$-x$' as 'the opposite of x' and not as 'negative x'.]

with it. So, a shorter description is:

For each x, for each y,
if x is negative and y is negative
then $x + y$ is the negative number
which corresponds with $|x| + |y|$.

We can make a final improvement by recalling that in order to form a name for a negative real number we just put a raised minus sign in front of a name of the corresponding number of arithmetic. So, a concise rule for adding negative numbers is:

For each x, for each y,
if x is negative and y is negative
then $x + y = {}^{-}(|x| + |y|)$.

* * *

C. State in a concise way the rule for

1. adding positive numbers.
2. multiplying a negative number by a positive number.
3. multiplying a positive number by a negative number.
4. multiplying a positive number by a positive number.
5. multiplying a negative number by a negative number.

D. 1. What rule is the following?

For each x, for each y,
if x is positive and y is negative then

(a) if $|x| \geq |y|$ then $x + y = {}^{+}(|x| - |y|)$

and (b) if $|x| < |y|$ then $x + y = {}^{-}(|y| - |x|)$.

2. State the rule for adding a positive number to a negative number.

3. State rules for adding the real number 0 and multiplying by the real number 0. [Do you know these rules by some other names?]

4. What principle of real numbers makes it unnecessary for you to remember the rule of Exercise 2 of Part D if you remember the rule of Exercise 1 of Part D?

5. What principle of real numbers makes it unnecessary for you to remember the rule of Exercise 3 of Part C if you remember the rule of Exercise 2 of Part C?

2.03 Generalizations.--Here is a generalization statement about numbers:

$$\text{For each } x, \quad 1 + 2x = 3x.$$

Translated into ordinary English, this statement tells you that no matter what real number you pick, if you multiply 2 by this number and add the product to 1, the result is the product of 3 by the chosen number. Do you believe what this generalization statement tells you? Whatever your answer is, you should be able to give evidence for your belief. If you believe that the generalization is true, you might try to justify it on the basis of the principles for real numbers. If you believe that it is false, you might try to find a counter-example, that is, a number such that when you multiply 2 by this number and add the product to 1, the result is different from the product of 3 by this number.

Actually, the generalization is false. A counter-example is 7.

$$1 + 2 \cdot 7 = 15, \qquad 3 \cdot 7 = 21, \qquad \text{and} \qquad 15 \neq 21.$$

[Notice that the generalization is false, despite the fact that $1 + 2 \cdot 1 = 3 \cdot 1$. A generalization is false even if it has only one false instance, no matter how many true instances it has.]

If you had first guessed that the generalization is true, this may have been because you thought that it is a consequence of the distributive principle for multiplication over addition. For example, you might have thought that:

$$(*) \qquad 1 + 2 \cdot 7 = (1 + 2)7$$

is an instance of the distributive principle. Why isn't it? [Or, you may have thought that (*) is an instance of one of the associative principles. Why isn't it?]

For your information, and to prepare you for questions which may arise, we amplify here the discussion on page 2-30 concerning generalizations and counter-examples.

As remarked earlier in the COMMENTARY, there are two kinds of generalization sentences--universal generalization sentences [beginning with 'for each', or, later in the text, with '$\forall$'] and existential generalization sentences [beginning with 'there exists a', or '$\exists$']. Because we have little to do with the latter kind of generalization sentence we have, in the text, used the word 'generalization' to refer to universal generalization sentences. And we shall continue this usage here.

A generalization is obtained by writing a quantifying phrase such as 'for each x, ' in front of a sentence. For example, consider the two sentences:

$x + 3 = 5$ and: for each y, $xy = yx$.

Both of these are open sentences, the first is an open equation and the second is an open generalization sentence. [Recall that open sentences are not statements. They are neither true nor false.] We can form generalization statements from these sentences by writing a 'for each x, ' in front of each:

(1) for each x, $x + 3 = 5$,

(2) for each x, for each y, $xy = yx$.

The first of these generalizations is false, and the second is true.

An instance of a generalization is, in the strict sense of 'instance', a sentence obtained by dropping the generalization's [left-most] quantifying phrase and substituting a numeral or a pronumeral expression for the corresponding pronumeral in the resulting sentence. Thus, each of the following is an instance of (1).

$2 + 3 = 5$; $7 + 3 = 5$; $^{-}4.76 + 3 = 5$; $(2y + 7) + 3 = 5$; $x + 3 = 5$.

[Notice that an instance of a generalization statement may be either a statement or an open sentence.]

The following are instances of (2).

for each y, $2y = y2$; for each y, $(z + 4)y = y(z + 4)$.

[And, the following are instances of the open generalization sentence 'for each y, $xy = yx$'.

$x3 = 3x$, $x(9 + 4) = (9 + 4)x$, $x(y + 1) = (y + 1)x$.]

However, we have used the word 'instance' in a slightly different sense, to describe sentences obtained from a generalization by dropping <u>all</u> initial quantifying phrases and then substituting for all of the corresponding pronumerals. In this sense, instances of (2) are such sentences as:

$$2 \cdot 3 = 3 \cdot 2, \quad 2(x - 5) = (x - 5)2,$$

$$(z + 4)y = y(z + 4), \quad (^{-}3 + 4)6 = 6(^{-}3 + 4).$$

[Such sentences would be more properly described as being <u>instances of instances</u> of (2).]

The preceding discussion has some application to the notion of counter-examples. It should be clear from the text that, for example, 7 is a counter-example to the generalization (1). For '$7 + 3 = 5$' is a false instance of (1). And, since (1) has a counter-example, (1) is, itself, false.

Consider, now, a more complicated false generalization, say:

(3) for each x, for each y, $8x + 3y = 11xy$.

Strictly speaking, 2 is a counter-example to (3), since the corresponding instance of (3):

(4) for each y, $8 \cdot 2 + 3y = 11 \cdot 2y$

is false. [Statement (4) is false because <u>it</u> has a counter-example. One such is 5; the statement '$8 \cdot 2 + 3 \cdot 5 = 11 \cdot 2 \cdot 5$' is false.]

Using 'instance' in the broader sense one might say that, since '$8 \cdot 2 + 3 \cdot 5 = 11 \cdot 2 \cdot 5$' is false, the ordered pair (2, 5) is, in a broader sense, a counter-example to (3). Although in the text we have used 'instance' in the broader sense, we have not wanted to introduce a corresponding broader meaning for 'counter-example'. And, to avoid raising the issues discussed above, we use the term 'counter-example' only in discussing generalizations having a single initial quantifying phrase.

Students can still explain why statement (3) is false by pointing out that it has the false sentence '$8 \cdot 2 + 3 \cdot 5 = 11 \cdot 2 \cdot 5$' among its consequences, thus appealing to the fact that consequences of true statements must be true. [In the broader sense, '$8 \cdot 2 + 3 \cdot 5 = 11 \cdot 2 \cdot 5$' is, as indicated above, an instance of the generalization. So, you could use 'instances' instead of 'consequences' in the explanation of why (3) is false.]

the difficulty which students encounter in conventional courses may be traced to the fact that in such courses one accumulates a multitude of generalizations [rules] without noticing the logical connections among them. Being unrelated and multitudinous, they are difficult to remember, let alone to understand. So, students in conventional courses often come to regard algebra as a dull game with innumerable and unrelated rules which the teacher divulges at critical points of the play.

*

Here is a quiz to review the inequality relations.

For each exercise, insert a '>', a '<' or an '=' so that the resulting sentence is true.

1.	−0.3	−0.2	2.	7.3	4.8
3.	1/6	1/7	4.	−8	$^{-}8$
5.	7	−9	6.	148	−14
7.	1/373	1/377	8.	−5.375	4.25
9.	−342	342	10.	−1/6	1/11

*

Answers for Quiz.

1. <	2. >	3. >	4. =	5. >
6. >	7. >	8. <	9. <	10. <

*

The last sentence at the bottom of page 2-32 is very important. The purpose of proof at this stage is not so much to convince oneself or others of the "correctness" of a generalization, but rather to exhibit the logical connections among generalizations. In proving the generalization:

$$\text{For each } x,\ 3(x2) = 6x,$$

the student is not trying to convince himself that this generalization is true. [Actually, he is as sure of this generalization as he is of the apm, the cpm, and the fact that $3 \cdot 2 = 6$.] What he is trying to do is to show that the generalization in question can be predicted from, or is a consequence of, the other generalizations and the computing fact.

Once students have become accustomed to proving generalizations, some will raise the question of why we don't try to prove one of the basic principles. This is an excellent question, and can be answered as follows: By 'proving a generalization' we mean deriving it from basic principles. Now, as it happens, it is impossible, for example, to derive the cpm from the other generalizations we have taken as basic principles [postulates]. We might be able to choose another set of generalizations which could serve as basic principles but would not include, for example, the cpm. We would then be able to derive the cpm from these new principles, or, as we would then say, we would be able to prove the cpm. Students have already noticed in Unit 1 that the ℓdpma is a consequence of the dpma and the cpm. So, the ℓdpma need not be included in a set of basic principles which includes the cpm and the dpma. Similarly, if we accept the cpm and the ℓdpma as basic principles, we need not include the dpma among the basic principles. Clearly, there is considerable freedom in choosing basic principles for the real numbers. In doing so, one tries to avoid including among the basic principles any generalization which can readily be derived from the others. Right now, students probably consider the principle for multiplying by 0 as a basic principle. Later in the unit, they will discover how to derive it from the others.

When students come upon a generalization [such as any of the true ones in Part A on pages 2-34 through 2-36] which they are inclined to accept, they should ask themselves whether their acceptance need be signalized by adjoining it to the set of basic principles, or whether it is derivable from their present set of basic principles. Such questioning is characteristic of much of what is known as "doing mathematics". Much of

Recall the role of the substitution rule [see TC[1-56]] in binding together the statements displayed toward the bottom of page 2-31. Here is an expanded form of the argument.

$$\dfrac{\dfrac{3 \cdot 2 = 6}{} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall_x \forall_y \forall_z (xy)z = x(yz)}{\forall_y \forall_z (3y)z = 3(yz)}}{\forall_z (3 \cdot 2)z = 3(2z)}}{(3 \cdot 2)5 = 3(2 \cdot 5)} \quad \dfrac{\forall_x x = x}{(3 \cdot 2)5 = (3 \cdot 2)5}}{3(2 \cdot 5) = (3 \cdot 2)5} \qquad \dfrac{\dfrac{\dfrac{\forall_x \forall_y xy = yx}{\forall_y 5y = y5}}{5 \cdot 2 = 2 \cdot 5} \quad \dfrac{\forall_x x = x}{3(5 \cdot 2) = 3(5 \cdot 2)}}{3(5 \cdot 2) = 3(2 \cdot 5)}}{3(5 \cdot 2) = (3 \cdot 2)5}}{3(5 \cdot 2) = 6 \cdot 5}$$

Note that in addition to inferences based on the substitution rule previously discussed, we make use of inferences from a universal generalization to its instances, such as:

$$\dfrac{\forall_x \forall_y xy = yx}{\forall_y 5y = y5}, \qquad \text{and:} \qquad \dfrac{\forall_y 5y = y5}{5 \cdot 2 = 2 \cdot 5}.$$

Just as inferences of the earlier mentioned sort [applications of the substitution rule] were acceptable to us because of the meaning we had adopted for '=', so inferences of this new kind are acceptable because of the meaning we have adopted for '$\forall$'.

From the expanded argument above, we can obtain a corresponding expanded form for the testing pattern by replacing each '5' by a '$\square$' [or by a copy of any other single pronumeral (except, in this case, 'y')].

Needless to say, we do not ask you to teach this form of presenting testing patterns to your students. But, at some point, you may find it helpful to point out, as you may already have done in Unit 1, that '$3(5 \cdot 2) = 3(2 \cdot 5)$' is a consequence of the cpm by virtue of the substitution rule and the principle that a thing is itself.

$$\dfrac{\dfrac{\text{cpm}}{5 \cdot 2 = 2 \cdot 5} \quad \dfrac{\forall_x x = x}{3(5 \cdot 2) = 3(5 \cdot 2)}}{3(5 \cdot 2) = 3(2 \cdot 5)}$$
[substitution of '$2 \cdot 5$' for second '$5 \cdot 2$'].

Also, the same rule and principle are used in establishing '$(3 \cdot 2)5 = 6 \cdot 5$'.

$$\dfrac{3 \cdot 2 = 6 \quad \dfrac{\forall_x x = x}{(3 \cdot 2)5 = (3 \cdot 2)5}}{(3 \cdot 2)5 = 6 \cdot 5}$$
[substitution of '6' for second '$3 \cdot 2$'].

The use of a test-pattern as a proof of a universal generalization statement is of basic importance for much that follows. Later in this unit students will use this procedure to obtain proofs for the generalizations concerning subtraction and division which justify their computing practices. And students will be called upon throughout their work in mathematics to give similar proofs. The trick of discovering such a test-pattern, by looking for a uniform way of verifying one or two chosen instances of the generalization, is worth emphasizing. Even after some practice, students may have difficulty in writing out a test-pattern without previously seeing how to verify a chosen instance. [You may have helped students in conventional classes to discover how to simplify an expression by suggesting that they "try it with numbers in place of letters".] The use of frames [as at the top of page 2-32] to isolate the substitutions used in obtaining the chosen instance makes it very easy to transform the verification of an instance into a proof of the generalization.

*

It is good practice to repeat in class the discussion on pages 2-31 and 2-32, but with a different generalization.

*

Note carefully that we do not claim that a testing pattern can be used in verifying every instance of the generalization in question. Life is too short! However, the testing pattern can be used to verify any instance that one might suggest. So, if one accepts the principles used in justifying the steps of the testing pattern, it is unreasonable for him to doubt any instance. And, he is in a strong position with regard to anyone who denies the generalization. For such a person is obligated to exhibit a counter-example [put up or shut up!], and one who has a testing pattern ready can afford to be complacent.

*

Although we take statements of computing facts [such as '$3 \cdot 2 = 6$'] for granted, they are, as indicated on TC[1-60, 61]c, consequences of our basic principles and definitions such as '$2 = 1 + 1$', '$3 = 2 + 1$', etc.

*

Now, consider another generalization about numbers:

$$\text{For each } x,\quad 3(x2) = 6x.$$

To make sure we understand what this says, let's look at a few instances. To write an instance, we substitute a numeral for 'x' in the open sentence which follows the 'For each x,'.

(1) $3(5 \cdot 2) = 6 \cdot 5$

(2) $3(8 \cdot 2) = 6 \cdot 8$

(3) $3(^{-}7 \cdot 2) = 6 \cdot {^{-}7}$

Statement (1) is true because $3(5 \cdot 2) = 3 \cdot 10 = 30$, and $6 \cdot 5 = 30$.

Statement (2) is true because $3(8 \cdot 2) = 3 \cdot 16 = 48$, and $6 \cdot 8 = 48$.

Statement (3) is true because $3(^{-}7 \cdot 2) = 3 \cdot {^{-}14} = {^{-}42}$, and $6 \cdot {^{-}7} = {^{-}42}$.

So, we have verified each of the three instances.

Do you think you could find a substitution for 'x' which would generate a false statement? If you think you couldn't, how can you be sure you're right? You certainly couldn't test and verify each instance as you did (1), (2), and (3). What you need is a method for testing any instance which you can be sure will verify each instance that you test. The method used in testing instances (1), (2), and (3) was to multiply the test number by 2, multiply 3 by this product, and compare this result with the product of 6 by the test number. But, it is not immediately clear that this computing method will result in a verification of each instance tested.

Consider another method of testing instance (1). Let's start with the expression on the left of '=' and try to transform it into the expression on the right of '='.

$3(5 \cdot 2) = 3(2 \cdot 5)$ [cpm]

$3(2 \cdot 5) = (3 \cdot 2)5$ [apm]

$(3 \cdot 2)5 = 6 \cdot 5.$ [$3 \cdot 2 = 6$]

Hence, $3(5 \cdot 2) = 6 \cdot 5$.

So, we say that '$3(5 \cdot 2) = 6 \cdot 5$' is a consequence of the commutative and associative principles for multiplication, and the computing fact that $3 \cdot 2 = 6$.

Let's use this method to test instance (2), this time putting a frame around the numeral for the test number.

$$3(\boxed{8} \cdot 2) = 3(2 \cdot \boxed{8}) \quad [\text{cpm}]$$

$$3(2 \cdot \boxed{8}) = (3 \cdot 2)\boxed{8} \quad [\text{apm}]$$

$$(3 \cdot 2)\boxed{8} = 6 \cdot \boxed{8}. \quad [3 \cdot 2 = 6]$$

Hence, $3(\boxed{8} \cdot 2) = 6 \cdot \boxed{8}$.

So, instance (2) is a consequence of the commutative and associative principles for multiplication, and the computing fact that $3 \cdot 2 = 6$.

Do you see how to test instance (3)? Just erase the numerals in the frames and write a '$^-7$' in each frame. Do you see that when you erase the numerals in the frames getting:

$$3(\boxed{} \cdot 2) = 3(2 \cdot \boxed{}) \quad [\text{cpm}]$$

$$3(2 \cdot \boxed{}) = (3 \cdot 2)\boxed{} \quad [\text{apm}]$$

$$(3 \cdot 2)\boxed{} = 6 \cdot \boxed{}. \quad [3 \cdot 2 = 6]$$

Hence, $3(\boxed{} \cdot 2) = 6 \cdot \boxed{}$.

you have a <u>testing pattern</u> which can be used to test any instance, and that such a test will always lead to a verification of the instance tested? In fact, the test-pattern shows you that each instance is a consequence of the commutative and associative principles for multiplication, and the fact that $3 \cdot 2 = 6$. So, we know that the generalization:

For each x, $3(x2) = 6x$

is a consequence of the commutative and associative principles for multiplication, and the computing fact that $3 \cdot 2 = 6$. The test-pattern is a <u>derivation</u> of the generalization from the cpm, the apm, and '$3 \cdot 2 = 6$'. For short, it is a <u>proof</u> of the generalization. The proof shows that if we accept the <u>premisses</u>:

For each x, for each y, $xy = yx$,
For each x, for each y, for each z, $xyz = x(yz)$,
$3 \cdot 2 = 6$,

then we must accept the <u>conclusion</u>:

For each x, $3(x2) = 6x$.

You should have just as much faith in the truth of the conclusion as you have in the truth of the premisses.

It may be instructive to introduce the class to the generalization:

$$\text{For each } x, \quad x(2x + 3) = 2(xx) + 3x$$

by means of the following "game". Each student picks a number but does not tell it to anyone. The teacher then asks that each student follow these instructions with regard to his chosen number:

(a) Multiply 2 by the number. Next, add 3 to this product. Then, multiply the chosen number by this sum, and record the product, drawing a loop around the answer.

(b) Now, multiply the chosen number by itself, and multiply 2 by this product. Multiply 3 by the chosen number, and add this product to the result of the previous multiplication. Draw a loop around the answer.

The teacher is able to predict that, for each student, the "looped answers" are the same. The two sequences of instructions are sufficiently different to cause some surprise at the outcome. The students will be sure that a generalization is involved. Now, they can be asked to state it. The first step in doing so is to write the universally quantifying phrase:

For each x,

Next, the steps in the instructions are followed one at a time [an activity which was explored in Part B on page 2-27].

$$\begin{array}{ll}
\text{(a)} & x \longrightarrow 2x \longrightarrow 2x + 3 \longrightarrow x(2x + 3) \\
\text{(b)} & \left.\begin{array}{l} x \longrightarrow xx \longrightarrow 2(xx) \\ x \longrightarrow 3x \end{array}\right\} \longrightarrow 2(xx) + 3x
\end{array}$$

So, the generalization in question is:

$$\text{For each } x, \quad x(2x + 3) = 2(xx) + 3x.$$

Now, you can proceed with developing a test-pattern for this generalization. [Notice that during the game, each student verified an instance, but the method of verification was not enlightening.]

As students become more adept at setting up test-patterns, you can ask them to make up generalizations which they can introduce to the class by way of a game as described above. The first time someone uses a false generalization should prove to be a most instructive occasion.

Now, let's consider still another generalization:

$$\text{For each } x, \quad x(2x + 3) = 2(xx) + 3x.$$

This generalization tells us, for example, that

$$7(2 \cdot 7 + 3) = 2(7 \cdot 7) + 3 \cdot 7.$$

Do you believe this instance? You could test it by computing. But, if we test it in such a way as to reveal a test-pattern, then we shall have proven not only the instance but the generalization itself.

$$\boxed{7}(2 \cdot \boxed{7} + 3) = \boxed{7}(2 \cdot \boxed{7}) + \boxed{7} \cdot 3 \quad [\ell\text{dpma}]$$

$$\boxed{7}(2 \cdot \boxed{7}) + \boxed{7} \cdot 3 = (2 \cdot \boxed{7})\boxed{7} + \boxed{7} \cdot 3 \quad [\text{cpm}]$$

$$(2 \cdot \boxed{7})\boxed{7} + \boxed{7} \cdot 3 = 2(\boxed{7} \cdot \boxed{7}) + \boxed{7} \cdot 3 \quad [\text{apm}]$$

$$2(\boxed{7} \cdot \boxed{7}) + \boxed{7} \cdot 3 = 2(\boxed{7} \cdot \boxed{7}) + 3 \cdot \boxed{7}. \quad [\text{cpm}]$$

Hence, $\boxed{7}(2 \cdot \boxed{7} + 3) = 2(\boxed{7} \cdot \boxed{7}) + 3 \cdot \boxed{7}.$

The test-pattern is easy to see if we erase the numerals from the frames, and even easier to see if we then replace the frames by a letter.

$$k(2k + 3) = k(2k) + k3 \quad [\ell\text{dpma}]$$

$$k(2k) + k3 = (2k)k + k3 \quad [\text{cpm}]$$

$$(2k)k + k3 = 2(kk) + k3 \quad [\text{apm}]$$

$$2(kk) + k3 = 2(kk) + 3k. \quad [\text{cpm}]$$

Hence, $k(2k + 3) = 2(kk) + 3k.$

This test-pattern can be used to verify any instance of the generalization:

$$\text{For each } x, \quad x(2x + 3) = 2(xx) + 3x.$$

So, it is a proof of this generalization. It shows that the generalization is a consequence of three principles for real numbers. What are they? [Notice that even when you have a test-pattern, you are still not able to test <u>every</u> instance of the generalization [Why not?]. However, the test-pattern gives you a sure-fire method of refuting anyone who claims to have a counter-example.]

EXERCISES

A. Each of the following is a generalization about real numbers. Some are true and some are false. Your job is to decide which, and in each case to give either a proof or a counter-example.

Sample 1. For each y, $7 + (3 + y) = y + 10$.

Solution. You may suspect that this is true, and be able to start writing a proof immediately. However, if you have difficulty in seeing how to construct a test-pattern, get to work on an instance. For example, try using the principles to verify:

$$7 + (3 + \boxed{5}) = \boxed{5} + 10.$$

We notice that the '$\boxed{5}$' is not in parentheses on the right side. This suggests transforming the left side '$7 + (3 + \boxed{5})$' by the associative principle for addition.

$$7 + (3 + \boxed{5}) = (7 + 3) + \boxed{5}.$$

Since $7 + 3$ is 10,

$$(7 + 3) + \boxed{5} = 10 + \boxed{5}.$$

Then next we can use the commutative principle for addition.

$$10 + \boxed{5} = \boxed{5} + 10.$$

We can now write a test-pattern.

$$7 + (3 + m) = (7 + 3) + m \quad [apa]$$

$$(7 + 3) + m = 10 + m \quad [7 + 3 = 10]$$

$$10 + m = m + 10. \quad [cpa]$$

Hence, $7 + (3 + m) = m + 10.$

1. For each t, $3(5t) = 15t$.

2. For each q, $3 + 6q = 9q$.

3. For each r, $3 + 6r = 3(1 + 2r)$.

4. For each x, $x \cdot 1 + x = 2x$.

5. No matter what number you pick, if you add it to 9 and add this sum to 1, you get 10 plus the chosen number.

Answers for Part A [on pages 2-34, 2-35, and 2-36].

1. $3(5t) = (3 \cdot 5)t$ [apm]

$(3 \cdot 5)t = 15t.$ [$3 \cdot 5 = 15$]

Hence, $3(5t) = 15t$.

So, the generalization 'For each t, $3(5t) = 15t$' is a consequence of the apm and the statement '$3 \cdot 5 = 15$'.

[Students should frequently be required to state, as we have done in the last sentence, what their test-patterns show. Also, to guard against the likelihood that students come to write test-patterns mechanically and forget their significance, students should be asked to use their test-patterns to verify instances of their generalizations. An appropriate request in connection with the answer for Exercise 1 is:

Use your test-pattern to show that $3(5 \cdot 863) = 15 \cdot 863$.

Of course, logically, the existence of a test-pattern relieves one, once and for all, of the necessity for testing instances of the corresponding generalization. But, pedagogically, the fact that it does so is perhaps best brought out by occasionally using, all the way through the course, test-patterns as patterns for testing instances.]

2. 4 is a counter-example. [$3 + 6 \cdot 4 = 27$, $9 \cdot 4 = 36$, and $27 \neq 36$.]

[Of course, each real number other than 1 is a counter-example. But, don't press for this.]

*

If students notice that $3 + 6 \cdot 1 = 9 \cdot 1$, don't let them conclude from this that the generalization 'For each q, $3 + 6q = 9q$' is sometimes true. A statement is either true or false, not sometimes one and sometimes the other. Similarly, it is nonsense to say of a generalization that it is always true.

*

Suppose a student claims to have a test-pattern for the generalization in Exercise 2:

$3 + 6q = (3 + 6)q$ [apa]

$(3 + 6)q = 9q.$ [$3 + 6 = 9$]

Hence, $3 + 6q = 9q$.

This is a common misapplication of the apa which we have tried to guard against through the exercises in Parts D and E on pages 2-20

and 2-21 and through the teaching suggestions on TC[1-44, 45]b, c and TC[1-50]. To handle this error, first ask for a statement of the apa. Then, take the pattern sentence from it:

$$(x + y) + z = x + (y + z)$$

and ask for the substitutions which the student has made to get:

$$3 + 6q = (3 + 6)q.$$

[It might even help to unabbreviate the sentence in question, thus obtaining:

$$3 + (6 \times q) = (3 + 6) \times q.$$

Frequently, the conventional omission of the times sign leads to this kind of misapplication.] Compelling the student to compare the pattern sentence of the basic principle with the sentence in his test-pattern will help him see the error. This is one of the advantages of being able to verbalize generalizations. [But, to keep the verbalizations from being mere parrot-like repetitions, we insist on non-verbal awareness first, as in Unit 1.]

*

3.

$$
\begin{aligned}
3 + 6r &= 3 + (3 \cdot 2)r && [6 = 3 \cdot 2]\\
3 + (3 \cdot 2)r &= 3 + 3(2r) && [\text{apm}]\\
3 + 3(2r) &= 3 \cdot 1 + 3(2r) && [\text{pm1}]\\
3 \cdot 1 + 3(2r) &= 3(1 + 2r). && [\ell\text{dpma}]
\end{aligned}
$$

Hence, $3 + 6r = 3(1 + 2r)$.

So, the generalization 'For each r, $3 + 6r = 3(1 + 2r)$' is a consequence of the statement '$6 = 3 \cdot 2$', the apm, the pm1, and the ℓdpma.

[Here is a place where students may ask if they can prove the generalization by transforming the right side into the left side. Of course they can, and they should compare the two test-patterns. The test-pattern for going from '$3(1 + 2r)$' to '$3 + 6r$' is just the complete reverse of the test-pattern given above.

$$
\begin{aligned}
3(1 + 2r) &= 3 \cdot 1 + 3(2r) && [\ell\text{dpma}]\\
3 \cdot 1 + 3(2r) &= 3 + 3(2r) && [\text{pm1}]\\
3 + 3(2r) &= 3 + (3 \cdot 2)r && [\text{apm}]\\
3 + (3 \cdot 2)r &= 3 + 6r. && [3 \cdot 2 = 6]
\end{aligned}
$$

Hence, $3(1 + 2r) = 3 + 6r$.

So, $3 + 6r = 3(1 + 2r)$.

Notice the additional final sentence in the test-pattern. It follows from the preceding one by virtue of a principle of logic, the symmetry of equality principle [See T C[1-56]b.]. We do not cite principles of logic as justifications of steps in a test-pattern. [For example, we have never cited the substitution rule in a test-pattern.] However, the last line of this test-pattern is necessary if the test-pattern is to be a proof of the generalization in Exercise 3.]

4.

$$x \cdot 1 + x = x \cdot 1 + x \cdot 1 \qquad [\text{pm1}]$$
$$x \cdot 1 + x \cdot 1 = x(1 + 1) \qquad [\ell\text{dpma}]$$
$$x(1 + 1) = x2 \qquad [1 + 1 = 2]$$
$$x2 = 2x. \qquad [\text{cpm}]$$

Hence, $x \cdot 1 + x = 2x.$

So, the generalization in question is a consequence of the pm1, the ℓdpma, the cpm, and the fact that $1 + 1 = 2$.

[At this point you should ask students to prove that

for each x, $x + x = 2x$.

They can do this by writing a test-pattern similar to the one for Exercise 4, or [preferably] they can give the following test-pattern:

$$x + x = x \cdot 1 + x \qquad [\text{pm1}]$$
$$x \cdot 1 + x = 2x. \qquad [\text{For each } x,\ x \cdot 1 + x = 2x.]$$

Hence, $x + x = 2x.$

This is an excellent opportunity for students to use a generalization already proven in establishing another one. [The generalization just established is one which the students will be asked to state, and prove, in Exercise 19 on page 2-36.] This opportunity will be repeated in Exercise 3 of Part C on page 2-37. [See the discussion for Exercise 3 on T C[2-37]b.]]

5. Students should state this generalization in concise form before writing a test-pattern.

$$1 + (9 + x) = (1 + 9) + x \qquad [\text{apa}]$$
$$(1 + 9) + x = 10 + x. \qquad [1 + 9 = 10]$$

Hence, $1 + (9 + x) = 10 + x.$

So, the generalization 'For each x, $1 + (9 + x) = 10 + x$' is a consequence of the apa and the statement '$1 + 9 = 10$'.

is introduced in Sample 2 on page 2-35. To deepen this appreciation you might assign generalizations like:

For each x, $2 + 5x + 7 + 4x = 9(1 + x)$,

and: For each y, $3(1 + 2y) + 6(5 + 3y) = 33 + 24y$,

to be proved before you get to Sample 2.

*

In connection with Sample 2, if no one suggests it, propose this step in the test-pattern:

$$3k + (9k - 2) = (3k + 9k) - 2 \quad [\text{apa}].$$

Again, recourse to an explicit comparison between the pattern sentence of the apa and the sentence in question will show that this is a misapplication. However, students may counter by asserting that they believe the generalization:

For each x, for each y, for each z,
$$x + (y - z) = x + y - z.$$

And, indeed, this is a theorem which they will prove later in Unit 2. They are prepared to prove it now. Let them do so if they wish.

$$x + (y - z) = x + (y + -z) \quad [\text{ps}]$$

$$x + (y + -z) = (x + y) + -z \quad [\text{apa}]$$

$$(x + y) + -z = x + y - z. \quad [\text{ps}]$$

Hence, $x + (y - z) = x + y - z.$

Now that it is proven, they can cite this generalization in support of the step:

$$3k + (9k - 2) = (3k + 9k) - 2.$$

This bit of pedagogy will help students see how generalizations are sometimes guessed at, then proved, and used.

6. 0 is a counter-example [and so is each real number other than 1]. [$8 \cdot 0 + 7 = 7$, $7 \cdot 0 + 8 = 8$, and $7 \neq 8$.] [If students misapply one of the commutative principles, be sure to require an explicit statement of the substitutions to be made in the pattern sentence of the principle. See the discussion in the COMMENTARY for Exercise 2.]

7. 1 is a counter-example [and so is each real number other than 0, $\sqrt{3}$, and $-\sqrt{3}$]. [$1 \cdot 1 \cdot 1 = 1$, $3 \cdot 1 = 3$ and $1 \neq 3$.] [Transforming 'xxx' to '3x' is a common error which probably arises from a mechanical approach to manipulating symbols and from the conventional omission of the times sign. A student is complimented for transforming '$x + x + x$' to '3x' and tries to generalize to transforming 'xxx' to '3x', also. He views the '3' as if it were an adjective instead of a noun. He should not explain the equivalence of '$x + x + x$' and '3x' as a case of having "3 of the same thing", but instead should see it as a consequence of the pm1, the ℓdpma, the cpm, and the facts that $1 + 1 = 2$ and $2 + 1 = 3$.]

8.
$$3x \cdot 4 = 4(3x) \quad [\text{cpm}]$$
$$4(3x) = 4 \cdot 3x \quad [\text{apm}]$$
$$4 \cdot 3x = 12x. \quad [4 \cdot 3 = 12]$$
Hence, $3x \cdot 4 = 12x$.

9.
$$9 + 7x + 3 = 7x + 9 + 3 \quad [\text{cpa}]$$
$$7x + 9 + 3 = 7x + (9 + 3) \quad [\text{apa}]$$
$$7x + (9 + 3) = 7x + 12. \quad [9 + 3 = 12]$$
Hence, $9 + 7x + 3 = 7x + 12$.

*

Note that the exercises in Part D of the Supplementary Exercises furnish students with test-patterns for which they are to supply the "reasons". [You may prefer to ask the abler students in advance to find test-patterns for the generalizations in question.] These exercises provide students with models of "sustained" proofs. They alert students to the need to use caution in skipping steps. They also help to build an appreciation for the abbreviated testing pattern which

6. For each r, $8r + 7 = 7r + 8$.

7. For each x, $xxx = 3x$.

8. For each x, $3x \cdot 4 = 12x$.

9. For each x, $9 + 7x + 3 = 7x + 12$.

[More exercises are in <u>Part D</u>, <u>Supplementary Exercises</u>.]

<u>Sample 2</u>. For each k, $3k + (9k - 2) = 12k - 2$.

<u>Solution</u>. Here is a test-pattern.

$$\begin{array}{rcll}
3k + (9k - 2) &=& 3k + (9k + -2) & [\text{ps}] \\
3k + (9k + -2) &=& 3k + 9k + -2 & [\text{apa}] \\
3k + 9k + -2 &=& (3 + 9)k + -2 & [\text{dpma}] \\
(3 + 9)k + -2 &=& 12k + -2 & [3 + 9 = 12] \\
12k + -2 &=& 12k - 2. & [\text{ps}]
\end{array}$$

Hence, $3k + (9k - 2) = 12k - 2$.

We can save almost half the writing if we abbreviate the test-pattern as follows.

$$\begin{array}{rcll}
3k + (9k - 2) &=& 3k + (9k + -2) & [\text{ps}] \\
&=& 3k + 9k + -2 & [\text{apa}] \\
&=& (3 + 9)k + -2 & [\text{dpma}] \\
&=& 12k + -2 & [3 + 9 = 12] \\
&=& 12k - 2. & [\text{ps}]
\end{array}$$

To save space horizontally, you may arrange your work as follows.

$$\begin{array}{ll}
3k + (9k - 2) & \\
& \} \ \text{ps} \\
= 3k + (9k + -2) & \\
& \} \ \text{apa} \\
= 3k + 9k + -2 & \\
& \} \ \text{dpma} \\
= (3 + 9)k + -2 & \\
& \} \ 3 + 9 = 12 \\
= 12k + -2 & \\
& \} \ \text{ps} \\
= 12k - 2. &
\end{array}$$

10. For each m, $3m - 7 + 5m = 8m - 7$.

11. For each q, $(2q \cdot 5)q = 10(qq)$.

12. For each x, $6x(3x) = 18(xx)$.

13. For each r, $3r - r = 3$.

14. For each m, $2m(3 + 5m) = 10(mm) + 6m$.

15. For each x, $x - 1 = 1 - x$.

16. For each y, $3y + 7 + 5y - 3 = 8y + 4$.

17. For each x, $3x + 4(x + 7) = 7(x + 4)$.

18. For each n, $4n(3n) = 7nn$.

19. For each number you pick, if you add it to itself, you get the product of 2 by the chosen number.

☆20. For each A, $A(A + 2) + A(A + 3) = 2\,AA + 5A$.

[More exercises are in <u>Part E</u>, <u>Supplementary Exercises</u>.]

* * *

Consider the following generalization:

(*) For each x, $(x + 3)(x + 7) = (x + 7)(x + 3)$.

Do you think it's true? Let's look at one of its instances:

$$(5 + 3)(5 + 7) = (5 + 7)(5 + 3).$$

Since '(5 + 3)' and '(5 + 7)' are numerals for real numbers, do you see that from the open sentence:

$$xy = yx$$

we could generate this instance? And, do you see that each instance of (*) can be generated from '$xy = yx$'? Hence, each instance of (*) follows from the commutative principle for multiplication:

For each x, for each y, $xy = yx$.

Therefore, each instance of (*) is true. So, (*) is true. In fact it is a consequence of the commutative principle for multiplication.

* * *

10.
$$
\begin{aligned}
& 3m - 7 + 5m \\
&= (3m + -7) + 5m && \text{ps} \\
&= 5m + (3m + -7) && \text{cpa} \\
&= (5m + 3m) + -7 && \text{apa} \\
&= (5 + 3)m + -7 && \text{dpma} \\
&= 8m + -7 && 5 + 3 = 8 \\
&= 8m - 7. && \text{ps}
\end{aligned}
$$

Hence, $3m - 7 + 5m = 8m - 7$.

So, the generalization 'For each m, $3m - 7 + 5m = 8m - 7$' is a consequence of the ps, the cpa, the apa, the dpma, and '$5 + 3 = 8$'.

11.
$$
\begin{aligned}
& (2q \cdot 5)q \\
&= [5(2q)]q && \text{cpm} \\
&= [5 \cdot 2q]q && \text{apm} \\
&= 10qq && 5 \cdot 2 = 10 \\
&= 10(qq). && \text{apm}
\end{aligned}
$$

Hence, $(2q \cdot 5)q = 10(qq)$.

[Note that the parentheses and the raised dot in the first line are necessary only because they appear in the given exercise. And the brackets in the second and third lines are inserted only to simplify reading. Our conventions would allow us to omit them.]

12.
$$
\begin{aligned}
& 6x(3x) \\
&= 6x3x && \text{apm} \\
&= 3(6x)x && \text{cpm} \\
&= 3 \cdot 6xx && \text{apm} \\
&= 18xx && 3 \cdot 6 = 18 \\
&= 18(xx). && \text{apm}
\end{aligned}
$$

Hence, $6x(3x) = 18(xx)$.

$$
\begin{aligned}
& (6x)(3x) \\
&= [(6x)3]x \\
&= [3(6x)]x \\
&= [(3 \cdot 6)x]x \\
&= [18x]x \\
&= 18(xx).
\end{aligned}
$$

Hence, $(6x)(3x) = 18(xx)$.

[Note the test-pattern on the right. It shows the grouping symbols which have been omitted [by convention] from the test-pattern on the left. In general, when explaining applications of the commutative and associative principles, you will find it helpful to introduce omitted grouping symbols. Notice that, as illustrated in the test-pattern on the right, you do not need to interpolate additional steps when introducing omitted grouping symbols.

Your attitude should be that the grouping symbols which have been omitted from the test-pattern on the left are really there, but happen to be invisible! In particular, <u>don't</u> say: $6x(3x) = (6x)(3x)$ because we can associate the '6' and the 'x'. They are already associated in '$6x(3x)$'.]

13. 0 is a counter-example [and so is each real number other than 3/2]. [$3 \cdot 0 - 0 = 0$, $3 = 3$, and $0 \neq 3$.]

Here it is instructive to ask for modifications in the generalization which would result in new generalizations which are derivable. You should get answers such as:

For each r, $3r - r = 2r$,

and: For each r, $3 + r - r = 3$.

[Here are test-patterns for these generalizations.

$$\begin{array}{ll}
3r - r & \\
= (2 + 1)r - r & \} \; 3 = 2 + 1 \\
= 2r + 1r - r & \} \; \text{dpma} \\
= 2r + r1 - r & \} \; \text{cpm} \\
= 2r + r - r & \} \; \text{pm1} \\
= 2r + r + -r & \} \; \text{ps} \\
= 2r + (r + -r) & \} \; \text{apa} \\
= 2r + 0 & \} \; \text{po} \\
= 2r. & \} \; \text{pa0}
\end{array}$$

Hence, $3r - r = 2r$.

$$\begin{array}{ll}
3 + r - r & \\
= 3 + r + -r & \} \; \text{ps} \\
= 3 + (r + -r) & \} \; \text{apa} \\
= 3 + 0 & \} \; \text{po} \\
= 3. & \} \; \text{pa0}
\end{array}$$

Hence, $3 + r - r = 3$.

Although students are not asked to think in terms of simplifications at this time, we are preparing the groundwork. Thus, when they notice that '$3r - r$' does not simplify to '3', it is natural for them to wonder about what it does simplify to.]

14.
$$\begin{aligned}
& 2m(3 + 5m) \\
&= 2m(5m + 3) && \text{cpa} \\
&= 2m(5m) + 2m3 && \ell\text{dpma} \\
&= 2[m(5m)] + 2(m3) && \text{apm} \\
&= 2[(5m)m] + 2(3m) && \text{cpm} \\
&= 2[5(mm)] + 2(3m) && \text{apm} \\
&= 2 \cdot 5(mm) + 2 \cdot 3m && \text{apm} \\
&= 10(mm) + 6m. && 2 \cdot 5 = 10,\ 2 \cdot 3 = 6
\end{aligned}$$

Hence, $2m(3 + 5m) = 10(mm) + 6m$.

[The third expression is transformed into the fourth by two applications of the apm.

$$\begin{aligned}
& 2m(5m) + 2m3 \\
&= 2[m(5m)] + 2m3 \\
&= 2[m(5m)] + 2(m3).
\end{aligned}$$

It is customary to combine these two steps into one, and to cite the principle just once. Similar elisions occur in transforming the fourth expression into the fifth, and the sixth into the seventh. In going from the seventh expression to the eighth, we again show only one step but we cite the two computing facts used. When students become skilled in giving proofs, they may even make more sweeping elisions, and justify the steps by citing several principles. [See TC[2-52]b, c.] This is permissible in those cases in which the student is sure of himself and is not just using a phrase such as 'cpm, apm' as a catch-all.]

[You may have wondered why we have '10(mm) + 6m' instead of '10mm + 6m' in Exercise 14, '10(qq)' instead of '10qq' in Exercise 11, and '18(xx)' instead of '18xx' in Exercise 12. When we introduce the exponent symbol '^{2}' in Unit 3, we shall there point out that, for example, 'q^2' is an abbreviation for '(qq)'. Hence, to go from '10qq' to '$10q^2$' requires the application of the apm. So, in Exercises 11, 12, and 14, we are trying to ready the students for the fact that it takes a basic principle to get from '10qq' to '$10q^2$'.]

15. Each real number other than 1 is a counter-example.

16.
$$
\begin{array}{ll}
3y + 7 + 5y - 3 & \\
 & \text{apa} \\
= 3y + (7 + 5y) - 3 & \\
 & \text{cpa} \\
= 3y + (5y + 7) - 3 & \\
 & \text{apa} \\
= 3y + 5y + 7 - 3 & \\
 & \text{dpma} \\
= (3 + 5)y + 7 - 3 & \\
 & 3 + 5 = 8 \\
= 8y + 7 - 3 & \\
 & \text{ps} \\
= 8y + 7 + -3 & \\
 & \text{apa} \\
= 8y + [7 + -3] & \\
 & 7 = 4 + 3 \\
= 8y + [4 + 3 + -3] & \\
 & \text{apa} \\
= 8y + [4 + (3 + -3)] & \\
 & \text{po} \\
= 8y + [4 + 0] & \\
 & \text{pa0} \\
= 8y + 4. &
\end{array}
$$

[Or, we could save some steps by citing the computing fact that $7 + -3 = 4$.]

Hence, $3y + 7 + 5y - 3 = 8y + 4$.

[In connection with this exercise students may suggest the short cut that "the things be rearranged so that the 'y's are together and the numerals are together". Of course, this is the short cut we hope students will become aware of. Some may even think that this is nothing but an application of the apa. They are not far off, but you should insist upon a precise verbalization. It might be stated as:

$$\forall_a \forall_b \forall_c \forall_d \; a + b + c + d = (a + c) + (b + d).$$

Then, they should prove it.

$$
\begin{array}{ll}
[(a + b) + c] + d & \\
 & \text{apa} \\
= [a + (b + c)] + d & \\
 & \text{cpa} \\
= [a + (c + b)] + d & \\
 & \text{apa} \\
= [(a + c) + b] + d & \\
 & \text{apa} \\
= (a + c) + (b + d). &
\end{array}
$$

Hence, $a + b + c + d = (a + c) + (b + d)$.

Compare with Exercise 4 on page 2-61.]

17.
$$\begin{aligned}
&3x + 4(x + 7) && \\
&= 3x + [4x + 4 \cdot 7] && \ell\text{dpma} \\
&= 3x + 4x + 4 \cdot 7 && \text{apa} \\
&= (3 + 4)x + 4 \cdot 7 && \text{dpma} \\
&= 7x + 4 \cdot 7 && 3 + 4 = 7 \\
&= 7x + 7 \cdot 4 && \text{cpm} \\
&= 7(x + 4). && \ell\text{dpma}
\end{aligned}$$

Hence, $3x + 4(x + 7) = 7(x + 4)$.

18. Each real number other than 0 is a counter-example. [Ask for a corrected statement of the generalization.]

19.
$$\begin{aligned}
&x + x && \\
&= x \cdot 1 + x \cdot 1 && \text{pm1} \\
&= x(1 + 1) && \ell\text{dpma} \\
&= x2 && 1 + 1 = 2 \\
&= 2x. && \text{cpm}
\end{aligned}$$

Hence, $x + x = 2x$.

[See the discussion of Exercise 4 on page TC[2-34]d.]

20.
$$\begin{aligned}
&A(A + 2) + A(A + 3) && \\
&= AA + A2 + [AA + A3] && \ell\text{dpma} \\
&= AA + A2 + AA + A3 && \text{apa} \\
&= AA + (AA + A2) + A3 && \text{cpa} \\
&= AA + AA + A2 + A3 && \text{apa} \\
&= AA1 + AA1 + A2 + A3 && \text{pm1} \\
&= AA1 + AA1 + (A2 + A3) && \text{apa} \\
&= AA(1 + 1) + A(2 + 3) && \ell\text{dpma} \\
&= AA2 + A5 && 1 + 1 = 2,\ 2 + 3 = 5 \\
&= 2(AA) + 5A && \text{cpm} \\
&= 2AA + 5A. && \text{apm}
\end{aligned}$$

The sequence: apa, cpa, apa, parallels exactly the sequence: apm, cpm, apm, in the answer for Exercise 12.

Hence, $A(A + 2) + A(A + 3) = 2AA + 5A$.

4.
$$\begin{aligned} & 3z(9z) \\ &= 3z9z && \text{apm} \\ &= 9(3z)z && \text{cpm} \\ &= 9 \cdot 3zz && \text{apm} \\ &= 27zz. && 9 \cdot 3 = 27 \end{aligned}$$

Hence, $3z(9z) = 27zz$.

5.
$$\begin{aligned} & 3(x + 4) + 3x \\ &= 3x + 3 \cdot 4 + 3x && \ell\text{dpma} \\ &= 3x + (3x + 3 \cdot 4) && \text{cpa} \\ &= 3x + 3x + 3 \cdot 4 && \text{apa} \\ &= (3 + 3)x + 3 \cdot 4 && \text{dpma} \\ &= 6x + 12. && 3 + 3 = 6,\ 3 \cdot 4 = 12 \end{aligned}$$

Hence, $3(x + 4) + 3x = 6x + 12$.

*

Quiz.

Complete each generalization [from the choices given] so that the resulting statement is true.

1. For each a, $5a + 2a =$ _______.
   (a) $(5 + a)\,2$ (b) $a + (5 \times 2)$ (c) $a + a + (5 + 2)$ (d) $a7$

2. For each n, for each p, $n + p + 2n + p =$ _______.
   (a) $2n + 2p$ (b) $5np$ (c) $(p + n)2 + n$ (d) $3(n + p)$

3. For each x, for each y, $4x \cdot 2y =$ _______.
   (a) $6xy$ (b) $xy \cdot 8$ (c) $2y + 4x$ (d) $8 + xy$

4. For each r, for each s, $(2r + r)s =$ _______.
   (a) $2sr + s$ (b) $(2r + s)r$ (c) $2sr + r$ (d) $2rs + rs$

*

Answers for Quiz.

[We give the letter which identifies the correct choice.]

1. d 2. c 3. b 4. d

3.

$$\begin{aligned}
& x + 4 + x - 2 & & \\
&= x + (x + 4) - 2 & & \text{cpa} \\
&= x + x + 4 - 2 & & \text{apa} \\
&= 2x + 4 - 2 & & \text{For each x, } x + x = 2x. \\
&= 2x + 4 + -2 & & \text{ps} \\
&= 2x + (4 + -2) & & \text{apa} \\
&= 2x + (2 + 2 + -2) & & 4 = 2 + 2 \\
&= 2x + [2 + (2 + -2)] & & \text{apa} \\
&= 2x + [2 + 0] & & \text{po} \\
&= 2x + 2 & & \text{pa0} \\
&= 2x + 2 \cdot 1 & & \text{pm1} \\
&= 2(x + 1). & & \ell\text{dpma}
\end{aligned}$$

Hence, $x + y + x - 2 = 2(x + 1)$.

[This is another good opportunity to point out that one can use previously proven generalizations as reasons for steps in a test-pattern. To make this quite clear, write on the board the first three lines of the test-pattern just given. Then follow these with:

$$\begin{aligned}
&= x \cdot 1 + x \cdot 1 + 4 - 2 & & \text{pm1} \\
&= x(1 + 1) + 4 - 2 & & \ell\text{dpma} \\
&= x2 + 4 - 2 & & 1 + 1 = 2 \\
&= 2x + 4 - 2. & & \text{cpm}
\end{aligned}$$

Now, call attention to the fact that you have just redone Exercise 19 of Part A on page 2-36, and that <u>re</u>doing it is a waste of time. Then, erase what you have just written and continue as in the answer given above for Exercise 3.]

The exercises in Part B are those for which the practice on pages 2-20 and 2-21 will be helpful to the students. Consider Exercise 2. The pattern sentence for that generalization is:

$$(y + 2)(y + 3) = (y + 2)y + (y + 2)3.$$

This sentence has the same pattern as:

$$x(y + 3) = xy + x3,$$

which has the same pattern as:

$$x(y + z) = xy + xz.$$

So, the given generalization is a consequence of the *ℓ*dpma.

*

Answers for Part B.

1. apm 2. *ℓ*dpma 3. cpa 4. *ℓ*dpma 5. *ℓ*dpma

6. apa 7. pm0

*

The purpose of Part C is to determine whether students have developed awarenesses of some of the simplification short cuts. The fact that they must choose among alternatives only one of which is "correct" will help to catch incipient errors. If necessary, instruct students that if they are uncertain, they can search for a counter-example. Be sure that you do not insist that students prepare <u>written</u> proofs for all thirteen exercises in one assignment.

*

Answers for Part C [on pages 2-37 and 2-38].

1. 5x + 8 [(c)] 2. 5y + 2 [(d)] 3. 2(x + 1) [(a)]

4. 27zz [(c)] 5. 6x + 12 [(b)]

<u>Proofs</u>.

1.
$$\begin{aligned} &3 + 5x + 5 \\ &= 5x + 3 + 5 \\ &= 5x + (3 + 5) \\ &= 5x + 8. \end{aligned} \quad \begin{matrix} \} \text{ cpa} \\ \} \text{ apa} \\ \} 3 + 5 = 8 \end{matrix}$$

Hence, $3 + 5x + 5 = 5x + 8$.

2.
$$\begin{aligned} &2y + 3y + 2 \\ &= (2 + 3)y + 2 \\ &= 5y + 2. \end{aligned} \quad \begin{matrix} \} \text{ dpma} \\ \} 2 + 3 = 5 \end{matrix}$$

Hence, $2y + 3y + 2 = 5y + 2$.

B. Each of the following generalizations is a consequence of one of the principles of real numbers. Tell which principle.

1. For each x, $(x + 4)(x + 3)(x + 5) = (x + 4)[(x + 3)(x + 5)]$.

2. For each y, $(y + 2)(y + 3) = (y + 2)y + (y + 2)3$.

3. For each k, $3k + 5 = 5 + 3k$.

4. For each A, $(2A + 7)[(8A + 1) + (5 + 2A)]$
$= (2A + 7)(8A + 1) + (2A + 7)(5 + 2A)$.

5. For each t, $(6t + 1)(7t) + (6t + 1)9 = (6t + 1)(7t + 9)$.

6. For each s, $3(s - 7) + 9 + 8(s - 15) = 3(s - 7) + [9 + 8(s - 15)]$.

7. For each x, $(xxx + xx + x + 1) \cdot 0 = 0$.

[More exercises are in Part F, Supplementary Exercises.]

C. Each of the following is the first part of a generalization. Your job is to complete the generalization [from the choices given] in such a way that you can prove it. [You should be prepared to give the proof.]

1. For each x, $3 + 5x + 5 =$ ____________.

(a) $8x + 5$ (b) $3 + 10x$ (c) $5x + 8$ (d) $13x$

2. For each y, $2y + 3y + 2 =$ ____________.

(a) $6y + 2$ (b) $2y + 5y$ (c) $(5 + 2)y$ (d) $5y + 2$

3. For each x, $x + 4 + x - 2 =$ ____________.

(a) $2(x + 1)$ (b) $2x + 6$ (c) $4x - 2$ (d) $5x - 2$

4. For each z, $3z(9z) =$ ____________.

(a) $27z$ (b) $12zz$ (c) $27zz$ (d) $12(zz)$

5. For each x, $3(x + 4) + 3x =$ ____________.

(a) $3(xx + 4)$ (b) $6x + 12$ (c) $6x + 4$ (d) $10x$

(continued on next page)

6. For each k, $2(k + 5) + 7(k + 4) =$ __________.

(a) $9(k + 9)$ (b) $9(kk + 9)$ (c) $9k + 9$ (d) $9k + 38$

7. For each m, $(3m)(5m)(4m) =$ __________.

(a) $60(mmm)$ (b) $180m$ (c) $12mmm$ (d) $60m$

8. For each p, $(p + 3) + (p + 5) + (p + 6) =$ __________.

(a) $4p + 14$ (b) $3p + 14$ (c) $ppp + 14$ (d) $p + 14$

9. For each k, $5k(2 + 3k) =$ __________.

(a) $25kk$ (b) $10k + 15kk$ (c) $25kkk$ (d) $10k + 3k$

10. For each r, $(3r + 7) - (3r + 7) =$ __________.

(a) $6r$ (b) $-6r + 14$ (c) 0 (d) $\cdot 6r - 7$

11. For each s, $5(3s + 1) \cdot [7(3s + 2)] =$ __________.

(a) $35(3s + 1)(3s + 2)$ (b) $12(6s + 3)$

(c) $35(6s + 3)$ (d) $(15s + 5)(3s + 2)$

12. For each x, $(x + 3)(x + 2) =$ __________.

(a) $xx + 6$ (b) $(x + 3)x + (x + 3)2$

(c) $2x + 5$ (d) $[(x + 3) + x] + [(x + 3) + 2]$

☆13. For each N, $(N + \frac{1}{2})(N + \frac{1}{2}) =$ __________.

(a) $2N + 1$ (b) $N + \frac{1}{2}N + \frac{1}{4}$

(c) $N(N + 1) + \frac{1}{4}$ (d) $NN + \frac{1}{4}$

6.	$9k + 38$	[(d)]	7.	$60(mmm)$	[(a)]
8.	$3p + 14$	[(b)]	9.	$10k + 15kk$	[(b)]
10.	0	[(c)]	11.	$35(3s + 1)(3s + 2)$	[(a)]
12.	$(x + 3)x + (x + 3)2$	[(b)]	13.	$N(N + 1) + \frac{1}{4}$	[(c)]

<u>Proofs</u>.

6.

$$\begin{aligned}
& 2(k + 5) + 7(k + 4) && \\
&&& \ell\text{dpma} \\
= {} & 2k + 2\cdot 5 + [7k + 7\cdot 4] && \\
&&& 2\cdot 5 = 10,\ 7\cdot 4 = 28 \\
= {} & 2k + 10 + [7k + 28] && \\
&&& \text{apa} \\
= {} & 2k + 10 + 7k + 28 && \\
&&& \text{cpa} \\
= {} & 7k + (2k + 10) + 28 && \\
&&& \text{apa} \\
= {} & 7k + 2k + 10 + 28 && \\
&&& \text{dpma} \\
= {} & (7 + 2)k + 10 + 28 && \\
&&& \text{apa} \\
= {} & (7 + 2)k + (10 + 28) && \\
&&& 7 + 2 = 9,\ 10 + 28 = 38 \\
= {} & 9k + 38. &&
\end{aligned}$$

Hence, $2(k + 5) + 7(k + 4) = 9k + 38$.

7.

$$\begin{aligned}
& (3m)(5m)(4m) && \\
&&& \text{apm} \\
= {} & [3m5m](4m) && \\
&&& \text{cpm} \\
= {} & [5(3m)m](4m) && \\
&&& \text{apm} \\
= {} & [5\cdot 3mm](4m) && \\
&&& 5\cdot 3 = 15 \\
= {} & [15mm](4m) && \\
&&& \text{apm} \\
= {} & [15(mm)](4m) && \\
&&& \text{apm} \\
= {} & [15(mm)4]m && \\
&&& \text{cpm} \\
= {} & 4[15(mm)]m && \\
&&& \text{apm} \\
= {} & 4\cdot 15(mm)m && \\
&&& 4\cdot 15 = 60 \\
= {} & 60(mm)m && \\
&&& \text{apm} \\
= {} & 60(mmm). &&
\end{aligned}$$

Hence, $(3m)(5m)(4m) = 60(mmm)$.

[We spell out the first step:

$$(3m)(5m)(4m) \underset{\substack{\uparrow \\ \text{convention}}}{=} [(3m)(5m)](4m) \underset{\substack{\uparrow \\ \text{apm}}}{=} [((3m)5)m](4m) \underset{\substack{\uparrow \\ \text{convention}}}{=} [3m5m](4m).\ ]$$

[Some student might suggest proving a generalization like that in Exercise 3 on page 2-61 in order to apply it to this problem.]

8.
$$\begin{array}{lll}
 & (p + 3) + (p + 5) + (p + 6) & \\
= & [(p + 3) + p + 5] + (p + 6) & \text{apa} \\
= & [p + (p + 3) + 5] + (p + 6) & \text{cpa} \\
= & [p + p + 3 + 5] + (p + 6) & \text{apa} \\
= & [2p + 3 + 5] + (p + 6) & \text{For each x, } x + x = 2x. \\
= & [2p + (3 + 5)] + (p + 6) & \text{apa} \\
= & (2p + 8) + (p + 6) & 3 + 5 = 8 \\
= & [2p + 8 + p] + 6 & \text{apa} \\
= & [p + (2p + 8)] + 6 & \text{cpa} \\
= & [p + 2p + 8] + 6 & \text{apa} \\
= & p \cdot 1 + 2p + 8 + 6 & \text{pm1} \\
= & 1p + 2p + 8 + 6 & \text{cpm} \\
= & (1 + 2)p + 8 + 6 & \text{dpma} \\
= & 3p + 8 + 6 & 2 + 1 = 3 \\
= & 3p + (8 + 6) & \text{apa} \\
= & 3p + 14. & 8 + 6 = 14
\end{array}$$

[Students might suggest here the theorem '$\forall_x\ 1 \cdot x = x$' to save a step. [See below.]]

Hence, $(p + 3) + (p + 5) + (p + 6) = 3p + 14$.

[Furnishing a proof of the theorem:

For each x, $1 \cdot x = x$

is Exercise 1 on page 2-61, but your students can prove it now should they wish to, and they <u>should</u> prove it if they wish to use it. Here is a testing pattern.

$$\begin{array}{lll}
 & 1 \cdot x & \\
= & x \cdot 1 & \text{cpm} \\
= & x. & \text{pm1}
\end{array}$$

Hence, $1 \cdot x = x$.

[Notice that this theorem states an important property of the real

number 1 with respect to the operation of multiplication. Texts which assert that one can, for example, replace 'x' by '$1 \cdot x$' because "the coefficient of 1 is always understood" suggest to their readers that 'x' is an abbreviation for '$1 \cdot x$'. This is just plain wrong.]]

9.
$$\begin{aligned} & 5k(2 + 3k) \\ &= (5k)2 + (5k)(3k) && \ell\text{dpma} \\ &= (5k)2 + [(5k)3]k && \text{apm} \\ &= 2(5k) + [3(5k)]k && \text{cpm} \\ &= 2 \cdot 5k + 3 \cdot 5kk && \text{apm} \\ &= 10k + 15kk. && 2 \cdot 5 = 10,\ 3 \cdot 5 = 15 \end{aligned}$$

Hence, $5k(2 + 3k) = 10k + 15kk$.

10.
$$\begin{aligned} & (3r + 7) - (3r + 7) \\ &= (3r + 7) + -(3r + 7) && \text{ps} \\ &= 0. && \text{po} \end{aligned}$$

Hence, $(3r + 7) - (3r + 7) = 0$.

11.
$$\begin{aligned} & 5(3s + 1) \cdot [7(3s + 2)] \\ &= [5(3s + 1)7](3s + 2) && \text{apm} \\ &= 7[5(3s + 1)](3s + 2) && \text{cpm} \\ &= 7 \cdot 5(3s + 1)(3s + 2) && \text{apm} \\ &= 35(3s + 1)(3s + 2). && 7 \cdot 5 = 35 \end{aligned}$$

Hence, $5(3s + 1) \cdot [7(3s + 2)] = 35(3s + 1)(3s + 2)$.

12.
$$\begin{aligned} & (x + 3)(x + 2) \\ &= (x + 3)x + (x + 3)2. && \ell\text{dpma} \end{aligned}$$

Hence, $(x + 3)(x + 2) = (x + 3)x + (x + 3)2$.

13.

$$
\begin{array}{ll}
(N + \frac{1}{2})(N + \frac{1}{2}) & \\
= (N + \frac{1}{2})N + (N + \frac{1}{2})\frac{1}{2} & \} \; \ell\text{dpma} \\
= NN + \frac{1}{2}N + [N \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}] & \} \; \text{dpma} \\
= [NN + \frac{1}{2}N + N \cdot \frac{1}{2}] \quad + \frac{1}{2} \cdot \frac{1}{2} & \} \; \text{apa} \\
= NN + [\frac{1}{2}N + N \cdot \frac{1}{2}] \quad + \frac{1}{2} \cdot \frac{1}{2} & \} \; \text{apa} \\
= NN + [N \cdot \frac{1}{2} + N \cdot \frac{1}{2}] + \frac{1}{2} \cdot \frac{1}{2} & \} \; \text{cpm} \\
= NN + \quad N(\frac{1}{2} + \frac{1}{2}) \quad + \frac{1}{2} \cdot \frac{1}{2} & \} \; \ell\text{dpma} \\
= NN + N \cdot 1 + \frac{1}{4} & \} \; \frac{1}{2} + \frac{1}{2} = 1, \; \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \\
= N(N + 1) \quad + \frac{1}{4}\,. & \} \; \ell\text{dpma}
\end{array}
$$

Hence, $(N + \frac{1}{2})(N + \frac{1}{2}) = N(N + 1) + \frac{1}{4}$.

[This is the generalization underlying Exercise 10 on page 2-10. Do not fail to point out the usefulness of this generalization in computation.

$$8\frac{1}{2} \times 8\frac{1}{2} = 8 \times 9 + \frac{1}{4} = 72\frac{1}{4}$$

$$3.5 \times 3.5 = 3 \times 4 + 0.25 = 12.25$$

$$
\begin{aligned}
95 \times 95 = [9.5 \times 9.5] \times 100 &= [9 \times 10 + 0.25] \times 100 \\
&= (9 \times 10) \times 100 + 25 = 9025
\end{aligned}
$$

Similar interesting computational devices are based on these generalizations:

$$\forall_x \quad (x + \frac{1}{3})(x + \frac{2}{3}) = x(x + 1) + \frac{2}{9},$$

$$\forall_x \quad (x + \frac{1}{4})(x + \frac{3}{4}) = x(x + 1) + \frac{3}{16}.$$

These, of course, and the one in Exercise 13 are consequences of:

$$\forall_x \forall_y \; [x + y][x + (1 - y)] = x(x + 1) + y(1 - y). \qquad]$$

*

Here are some test items which you might like to include in a quiz.

Each of the following sentences is taken from a test-pattern. Your job is to write a name for the principle which justifies the sentence.

1. $2(5x + 7) = 2(5x) + 2 \cdot 7$ __________
2. $10 + [7k + 19] = 10 + 7k + 19$ __________
3. $2m + 5m + 9 = (2 + 5)m + 9$ __________
4. $(3x)(5y) = 3x5y$ __________
5. $10xx = 10(xx)$ __________
6. $p + 3 + p + 5 = p + (3 + p) + 5$ __________
7. $2x + x = 2x + x1$ __________
8. $5k(2 + 3k) = (2 + 3k)(5k)$ __________
9. $(x + 5)(x + 5) = x(x + 5) + 5(x + 5)$ __________
10. $(x + 5)(x + 5) = (x + 5)x + (x + 5)5$ __________

Answers.

1.	ℓdpma	2.	apa	3.	dpma	4.	apm	5.	apm
6.	apa	7.	pml	8.	cpm	9.	dpma	10.	ℓdpma

Notice that we do not use expressions such as 'equal sides' and 'equal angles'. Instead we say, for example, that an isosceles triangle has two congruent sides and two congruent angles, or that it has two sides of the same measure, or that the measures of two of its angles are equal. [See Unit 6, and Chapter 8 of the 24th Yearbook of the National Council of Teachers of Mathematics.]

*

Answers for Part A [on pages 2-39, 2-40, 2-41, and 2-42].

1. square, 20
2. rectangle, 26
3. parallelogram, 32
4. trapezoid, 33
5. triangle, 22
6. isosceles triangle, 18
7. equilateral triangle, 18
8. isosceles triangle, 24
9. rhombus, 24
10. quadrilateral, 25
11. circle, 8π [Answer to bracketed question: circumference]
12. isosceles trapezoid, 31
13. quadrilateral, 30
14. quadrilateral, 32 [In Exercises 13 and 14, students may give the name 'kite'.]
15. hexagon, 31
16. equilateral hexagon, 36
17. pentagon, 21
18. equilateral pentagon, 20
19. octagon, 39
20. equilateral octagon, 24
21. concave hexagon, 35
22. concave quadrilateral, 28

*

Answers for Part C [on page 2-44].

1. 18
2. 16
3. 18
4. 20
5. 42
6. 99
7. 66
8. 19
9. 27π
10. 30

In order to motivate the need for simplifying expressions, we introduce the problem of finding simple formulas for computing perimeters of geometric figures. Students who have had a good treatment of geometry in grades 7 and 8 will find the Exploration Exercises an easy review. If you are teaching this unit to 8th graders or to 7th graders, the work may be relatively new to the students. In particular, you may find a need for teaching students what angles are.

We do not regard an angle as an amount of rotation nor do we regard it as a wedge-shaped region. Instead, an angle consists of the points of two noncollinear rays which have the same vertex. [The vertex of the rays is the vertex of the angle; the half-lines obtained from the rays by deleting the vertex are the sides of the angle.]

We adopt the convention that the perimeter of a polygon is the sum of the measures of its sides and, so, is a number of arithmetic. When one wishes to call attention to the unit with respect to which the sides are measured, one may speak of the inch-perimeter, the foot-perimeter, etc. For example, the inch-measure of a side of a square, the length of each of whose sides is 2 ft., is 24. The inch-perimeter of such a square is 96, and its foot-perimeter is 8.

*

In Exercises 1 and 2, we think the students will identify these figures as a square and a rectangle, respectively, even though the angles are not indicated to be right angles. You may want to ask whether the pictures give all the information needed to insure that a square and a rectangle are being shown.

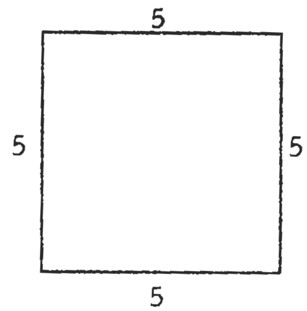
Similarly, when Exercises 16, 18, and 20 are being considered you may want to have a discussion as to how these figures differ from those pictured in Exercises 15, 17, and 19, respectively. The students will no doubt mention that in Exercises 16, 18, and 20 the figures have sides with equal measure; they may even say that the figures are equilateral. Familiarize them with the word 'regular' as applied to polygons by saying that if, for each figure, all of the angles have the same measure and all of the sides have the same measure, the figure is regular. Ask whether a polygon can be equilateral and not be equiangular. Have students make pictures to illustrate in the case of the equilateral quadrilateral and the equilateral hexagon.

*

EXPLORATION EXERCISES

A. Here are some geometric figures which you probably studied in earlier grades. See how many of these figures you remember. Name each figure and compute the distance around it [that is, compute the perimeter].

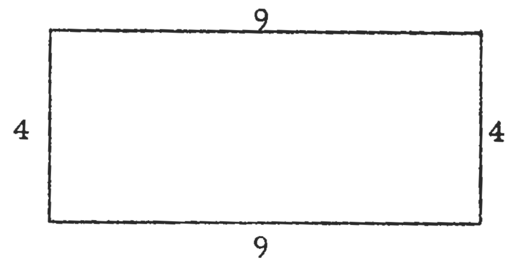
1.



Name _______________

Perimeter __________

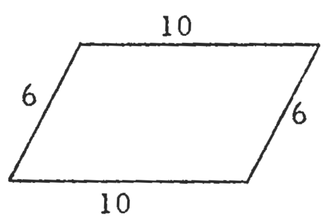
2.



Name _______________

Perimeter __________

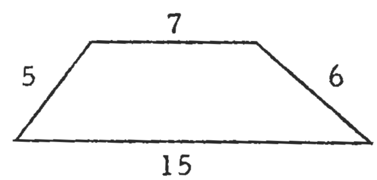
3.



Name _______________

Perimeter __________

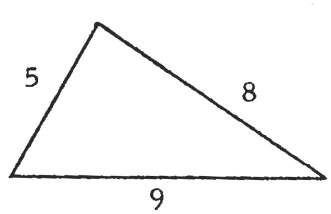
4.



Name _______________

Perimeter __________

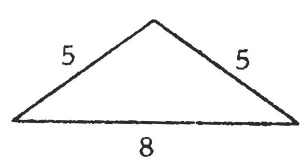
5.



Name _______________

Perimeter __________

6.

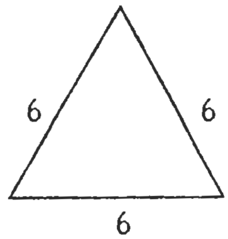


Name _______________

Perimeter __________

(continued on next page)

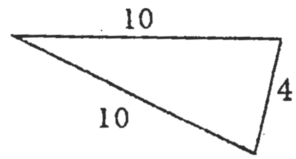
7.



Name ____________

Perimeter ________

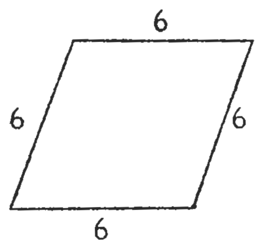
8.



Name ____________

Perimeter ________

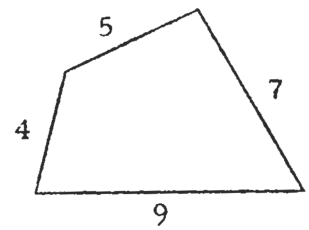
9.



Name ____________

Perimeter ________

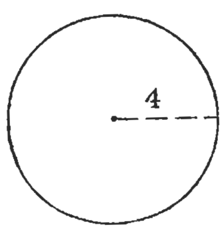
10.



Name ____________

Perimeter ________

11.

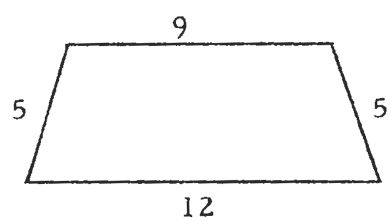


Name ____________

Perimeter ________

[Is there another word you use in Exercise 11 instead of 'perimeter'?]

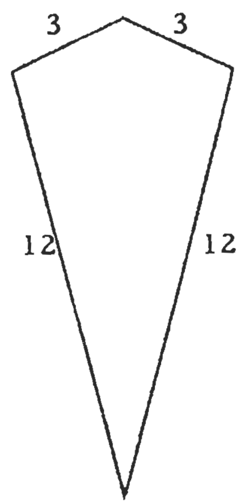
12.



Name ____________

Perimeter ________

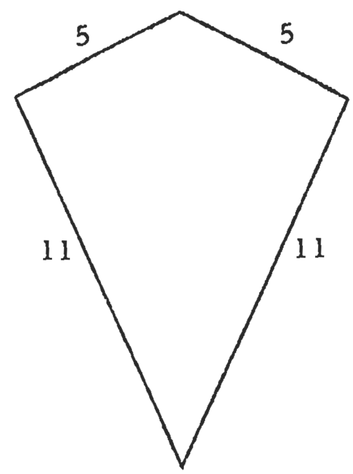
13.



Name ____________

Perimeter ____________

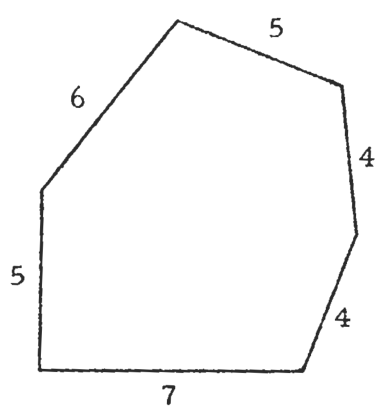
14.



Name ____________

Perimeter ____________

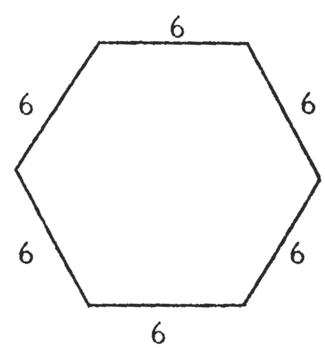
15.



Name ____________

Perimeter ____________

16.

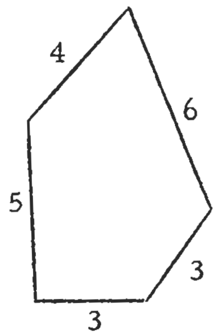


Name ____________

Perimeter ____________

(continued on next page)

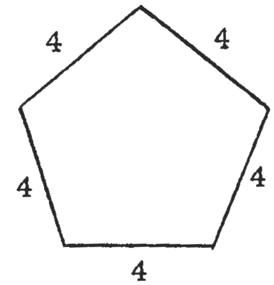
17.



Name ____________________

Perimeter _______________

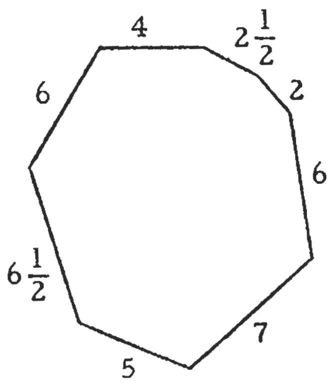
18.



Name ____________________

Perimeter _______________

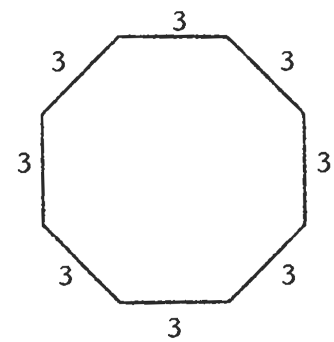
19.



Name ____________________

Perimeter _______________

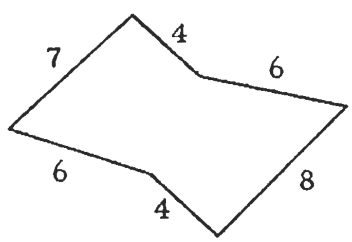
20.



Name ____________________

Perimeter _______________

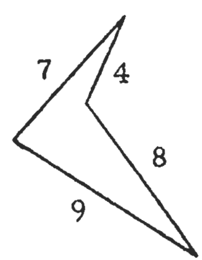
21.



Name ____________________

Perimeter _______________

22.



Name ____________________

Perimeter _______________

B. Use a ruler and compasses to make careful drawings of the figures described below. [Choose a convenient unit.]

1. A triangle with sides measuring 3, 5, and 7.
2. An equilateral triangle with one side measuring 2.
3. A rectangle with one side 2 units long and another side 5 units long.
4. A triangle whose perimeter is 12, and two of whose sides measure 3 and 4. [If you put two such triangles together with their longest sides matching, what kind of figure do you get?]
5. A kite with one side measuring 2 and another side measuring 5.
6. A square with one side 3 units long.
7. A circle with radius 2 units long.
8. A regular hexagon with side 2 units long. [Hint: Use the circle you drew for Exercise 7.]
9. An isosceles triangle with one side 3 units long and another side 1 unit long.
10. An isosceles triangle two of whose sides measure 3 and 4.
11. A circle whose circumference is 10π.
12. A rhombus [not a square], one of whose sides is 3 units long.
13. A parallelogram two of whose sides measure 3 and 5.
14. A square whose diagonal is 4 units long.
15. A parallelogram whose diagonals are 6 and 8 units long.
16. A square whose perimeter is 12.
17. A rectangle one of whose sides measures 3 more than twice the other, and whose perimeter is 18.

(continued on next page)

18. A kite with perimeter 20 and a side 4 units long.

19. An isosceles triangle whose perimeter is 30 and one of whose sides is twice as long as the other.

20. An equilateral triangle whose perimeter is 17.

C. Find the perimeter of each figure described below.

1. A rectangle with one side 4 units long and another side 7 units less than 3 times the first.

2. A square one of whose sides has the same length as a side of an equilateral triangle of perimeter 12.

3. A parallelogram whose sides have the same lengths as the sides of the rectangle in Exercise 1.

4. A quadrilateral whose sides are such that the average of their measures is 5.

5. An octagon whose sides are such that the average of their measures is $5\frac{1}{4}$.

6. A hexagon the measures of whose sides are consecutive whole numbers and whose longest side measures 19.

7. A parallelogram whose shorter side is 9 units long and the measure of whose longer side is twice the sum of the measure of its shorter side and 3.

8. An isosceles triangle whose base [the side whose length differs from that of the others] is 5 units long, and 9 units less than twice one of the other sides.

9. A circle whose diameter is 3 units shorter than twice the side of a square whose perimeter is 60.

10. An isosceles trapezoid the longer of whose parallel sides measures 12, the shorter of whose parallel sides measures 15 less than twice the measure of the longer, and the length of each of whose nonparallel sides is half the length of the shorter parallel side.

As mentioned in the COMMENTARY for pages 2-39 through 2-44, we have introduced the discussion of perimeters, at this point, to motivate simplification of pronumeral expressions. In order to accomplish this you will need to get agreement from your students that, by definition, the perimeter of a polygon is obtained by adding the measures of its sides taken consecutively in order "around" the figure.

*

The demonstration at the bottom of page 2-45 suggests that the formula '$P = 2(\ell + w)$' is a simpler formula for computing the perimeter of a rectangle than the formula '$P = \ell + w + \ell + w$'. Most of your students will know the formula '$P = 2(\ell + w)$', so this demonstration will deal with matters which are familiar to them.

*

The answers to the five 'why?'s are, respectively:

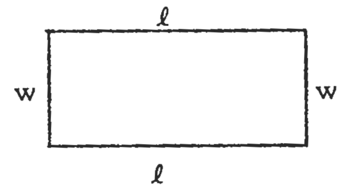
apa, pm1, ℓdpma, $1 + 1 = 2$, cpm.

*

The fact, brought out at the bottom of page 2-46, that simplifying a pronumeral expression amounts to proving a generalization, is important. We make more of this on page 2-60.

2.04 Simplification of expressions.--One of the uses of pronumerals and pronumeral expressions is to write formulas which can be used in solving problems. For example, in solving problems about the perimeters of rectangles, you might use a formula like:

(*) $$P = \ell + w + \ell + w.$$



Then, to compute the perimeter of a rectangle whose sides measure 4 and 5, you find the value of the expression '$\ell + w + \ell + w$' for the values 4 and 5 of 'w' and 'ℓ'. The value of '$\ell + w + \ell + w$' in this case is $5 + 4 + 5 + 4$. So, the perimeter is $5 + 4 + 5 + 4$. We simplify '$5 + 4 + 5 + 4$' to '18'.

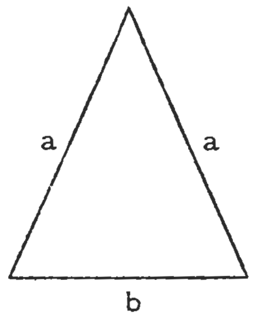
Now, every rectangle-perimeter problem which uses the formula (*) would involve a simplification like the one we went through in simplifying '$5 + 4 + 5 + 4$' to '18'. For example, another problem might require you to simplify '$87.6 + 49.3 + 87.6 + 49.3$'.

Let's look at a way of simplifying such numerical expressions, a way which can be generalized to all such expressions.

$$\begin{aligned}
& [(87.6 + 49.3) + 87.6] + 49.3 && \\
&\quad && \text{[why?]}______ \\
= \ & (87.6 + 49.3) + (87.6 + 49.3) && \\
&\quad && \text{[why?]}______ \\
= \ & (87.6 + 49.3)\cdot 1 + (87.6 + 49.3)\cdot 1 && \\
&\quad && \text{[why?]}______ \\
= \ & (87.6 + 49.3)(1 + 1) && \\
&\quad && \text{[why?]}______ \\
= \ & (87.6 + 49.3)\cdot 2 && \\
&\quad && \text{[why?]}______ \\
= \ & 2(87.6 + 49.3).
\end{aligned}$$

Does this suggest to you a simpler formula than (*)?
[Note that although measures of pieces of straight lines like the measures of the sides of a rectangle are numbers of arithmetic, in simplifying expressions for perimeters we can act as though these measures were nonnegative real numbers. Thus, we can use the principles for real numbers to justify our simplifications.]

Consider a formula for the perimeter of an isosceles triangle. From what we mean by 'perimeter', one such formula is:



$$P = a + b + a.$$

We can get a simpler formula by simplifying the expression '$a + b + a$'. Can you guess what the simpler expression will be? Here is how we use the principles of real numbers to simplify this expression.

$$\begin{array}{ll} (a + b) + a & \\ = a + (a + b) & \text{cpa} \\ = (a + a) + b & \text{apa} \\ = (a \cdot 1 + a \cdot 1) + b & \text{pm1} \\ = a(1 + 1) + b & \ell\text{dpma} \\ = a2 + b & 1 + 1 = 2 \\ = 2a + b. & \text{cpm} \end{array}$$

So, a simple formula is:

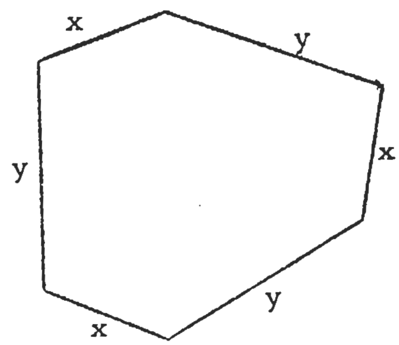
$$P = 2a + b.$$

Note that the simplification procedure from '$(a + b) + a$' to '$2a + b$' shows you that the generalization:

For each a, for each b, $a + b + a = 2a + b$

is a consequence of the principles [and the computing fact] listed on the right. This being so, you can use this generalization as you would any of the principles in justifying a step in another simplification problem.

For example, suppose you wanted a formula for finding the perimeter of a hexagon which looks like this.



From the meaning of 'perimeter', a formula is:

$$P = y + x + y + x + y + x.$$

By using the associative principle several times we can simplify:

$$y + x + y + x + y + x$$

to:

$$[y + x + y] + [x + y + x].$$

The generalization we obtained on page 2-46 justifies simplifying this to:

$$[2y + x] + [2x + y].$$

The commutative and associative principles for addition allow us to rephrase the expression as:

$$[2y + y] + [2x + x].$$

We should be able to simplify this expression to '3y + 3x'. To justify this, we can use the generalization:

For each k, $2k + k = 3k$.

[Give a proof of this last generalization.] Finally, the left distributive principle justifies going from:

$$3y + 3x$$

to:

$$3(y + x).$$

So, as you must have guessed from the beginning, a simpler formula is:

$$P = 3(y + x).$$

EQUIVALENT EXPRESSIONS

You have seen that the principles of real numbers can be used to simplify pronumeral expressions in just the same way as they are used in simplifying numerical expressions. In the work with perimeter formulas we simplified

'$\ell + w + \ell + w$' to '$2(\ell + w)$',

'$a + b + a$' to '$2a + b$',

and '$y + x + y + x + y + x$' to '$3(y + x)$'.

Since much of your work in mathematics will require skill in simplifying expressions, you will need to know not only how the principles are used in such simplifications, but also how to carry out the simplifications quickly and mechanically. You will learn to do this through practice, for by doing lots of simplifications you will discover short cuts. The important thing to remember about short cuts is that they are not magic, but that they are consequences of the principles, together with computing facts.

When you have a numerical expression like:

$$3 \times 7 + 5 \times -6$$

and simplify it to:

$$-9,$$

both the expression you start with and the final expression are numerals for the same number. And, we can state this fact by writing an equality sign between the two numerals, getting the true sentence:

$$3 \times 7 + 5 \times -6 = -9,$$

or, by saying that the numerals '$3 \times 7 + 5 \times -6$' and '-9' are <u>equivalent</u> <u>numerical</u> <u>expressions</u>.

[Note: We would not say that '$3 \times 7 + 5 \times -6$' equals '-9' because 'equal' has the same meaning as 'is the same as', and, clearly, the expressions '$3 \times 7 + 5 \times -6$' and '-9' are different. Although they are different, they stand for the same number, and this is what we mean when we say they are equivalent numerical expressions.]

A common way of discussing equivalent expressions is to say, for example, that '5x + 7' is a simpler expression for '3x + 7 + 2x'. This is correct if by 'for' you mean the same as 'to replace'. But, confusion arises if the hearer interprets 'for' to mean the same as 'to name'. So, it is wise to avoid the word 'for' in such contexts. [Of course, it is just plain wrong to say that '5x + 7' is a simpler name for '3x + 7 + 2x'.]

*

The bracketed sentences at the foot of page 2-48, and those at the foot of page 2-49, make an important point. Here is an amplification, should you want additional ammunition.

The expressions '4 + 3' and '5 + 2' are equivalent numerals [because they name the same number], but '4 + 3' and '5 + 2' are different expressions. Here is another way of saying what has been said above:

The sentence:

$$4 + 3 = 5 + 2$$

is true, but the sentence:

$$\text{'}4 + 3\text{'} = \text{'}5 + 2\text{'}$$

is false.

When we say that 4 + 3 and 5 + 2 are equal numbers, this is a conventional [but very sloppy] way of saying that 4 + 3 is the same number as 5 + 2, or, more briefly, of saying that 4 + 3 is 5 + 2.

Pronumeral expressions are equivalent if and only if "corresponding" substitutions convert the expressions into equivalent numerals. We rarely have occasion to say that expressions are equal. This is because we have only one way of naming an expression [by using single quotes]. So, the only true sentences of this sort we could write would be of the following kind:

$$\text{'}3 + 5\text{'} = \text{'}3 + 5\text{'}$$

$$\text{'}6x + 2y\text{'} = \text{'}6x + 2y\text{'}$$

etc.

Contrast this with the fact that not only are such sentences as:

$$4 + 3 = 4 + 3$$

and:

$$6 - 2 = 6 - 2$$

true, but so are sentences such as:

$$4 + 3 = 5 + 2$$

$$6 - 2 = 3 + 1.$$

*

If your class is a strong one, you may want them to write proofs which justify equivalence in the appropriate exercises in Part A on pages 2-50 and 2-51. Or, you may want to assign certain problems to individual students. If your class is weak, you may want them just to identify pairs of equivalent expressions, and give counter-examples in the case of pairs of nonequivalent expressions.

*

Quiz [to be used after discussing page 2-51].

Each exercise contains a pair of expressions. Tell which pairs are equivalent and which pairs are not.

1. $a(b + 2b)$
   $2ab + ab$
2. $2x + 2y$
   $2 + (x \times y)$
3. $2a(^{-}6b)(3c)$
   $(^{-}2ab)(9c)(2a)$
4. $8aa - 3a$
   $5a$
5. $aa + 2a$
   $a(a + 2)$
6. $8m + {}^{-}4n + 11n + 3m$
   $5m + (^{-}4n + 11n) + 6m$
7. $5m + (^{-}4n + 11n) + 6m$
   $11m + {}^{-}7n$
8. $8aa - 3a$
   $5aa$
9. $8m + {}^{-}4n + 11n + 3m$
   $11m + {}^{-}15n$
10. $\frac{1}{2}(2a + {}^{-}6b) + \frac{1}{4}(8a + {}^{-}4b)$
    $3a + {}^{-}4b$

*

Answers for Quiz.

Exercises 1, 5, 6, and 10 contain pairs of equivalent expressions; exercises 2, 3, 4, 7, 8, and 9 do not.

Now, suppose we have a <u>pronumeral</u> expression like:

$$7x + 3x + 5,$$

and simplify it by means of our principles and computing facts to:

$$10x + 5.$$

If we write an equality sign between the two pronumeral expressions:

$$7x + 3x + 5 = 10x + 5,$$

we get an <u>open sentence</u>. Can you generate a false sentence from this open sentence? The answer is 'no', and we can express this fact by stating the <u>generalization</u>:

For each x, $7x + 3x + 5 = 10x + 5$.

Another way of saying that you can't generate a false sentence from the open sentence:

$$7x + 3x + 5 = 10x + 5$$

is to say that '$7x + 3x + 5$' and '$10x + 5$' are <u>equivalent pronumeral expressions</u>. If you pick a value of 'x' and substitute a numeral for this value of 'x' in both '$7x + 3x + 5$' and '$10x + 5$', you get a pair of equivalent numerical expressions.

> Equivalent numerical expressions are numerals for the same number.
>
> Equivalent pronumeral expressions are expressions such that for each substitution both expressions have the same value.

[Note: We would not say that '$7x + 3x + 5$' equals '$10x + 5$', because they are different pronumeral expressions. Instead we say that '$7x + 3x + 5$' is equivalent to '$10x + 5$' meaning that the expressions have the same value for each value of 'x'.]

EXERCISES

A. Each of the following exercises contains a pair of expressions. In some cases, the expressions are equivalent. In the others, they are not. Tell which pairs are equivalent and which pairs are not. If you claim that two expressions are not equivalent, you should be able to support your claim by giving values of the pronumerals in the expressions which lead to different values for the two expressions. If you claim that two expressions are equivalent, you should be able to show how you can transform one expression into the other by using principles and computing facts.

Sample 1. $5x + y + 3x + 4y$

$6xy + 7xy$

Solution. I claim that these are not equivalent. So, I just write:

Not equivalent.

If I am asked to justify this answer, I should be able to give a substitution for 'x' and a substitution for 'y' for which the two expressions have different values. Suppose I try '2' for 'x' and '3' for 'y'. The corresponding value of '$5x + y + 3x + 4y$' is $5 \cdot 2 + 3 + 3 \cdot 2 + 4 \cdot 3$, that is, is 31. The corresponding value of '$6xy + 7xy$' is $6 \cdot 2 \cdot 3 + 7 \cdot 2 \cdot 3$, or 78.

Sample 2. $7x + 3y + 2x + 9y + 15$

$3(3x + 4y + 5)$

Solution. I notice the second expression can be transformed into '$9x + 12y + 15$' by the left distributive principle and some computing facts. I can transform the first into '$9x + 12y + 15$' by the use of the associative, commutative, and distributive principles along

Here is the sorting for the expressions given on TC[2-51]d.

(1) $3x + 2y + 7x + 5y$	(2) $3x(1 + y) + 2y(1 + x)$
(3) $4x + 5y + 2(3x + y)$	(4) $2(x + y) + x + 5xy$
(7) $3x + 7(x + y)$	(6) $y(5x + 2) + 3x$
(11) $2(2x + 3y) + 6x + y$	(12) $3x + 3xy + 2y + 2xy$
(13) $10x + 7y$	(24) $x(5y + 3) + 2y$
(21) $10(x + y) + {}^{-}3y$	(27) $2y + 5xy + 3x$

(5) $5xy + 12xy$	(8) $5(2xy + 1)$
(15) $17xy$	(26) $5 + 3y(\frac{10}{3}x)$
(16) $xy + 16xy$	

(9) $2(5x + 3y)$	(10) $5xy + 5y$
(23) $10x + 6y$	(17) $5(xy + y)$
(25) $9x + 6y + x$	(22) $5y(x + 1)$

(14) $10xy$	(19) $10x + y$
(18) $(2x)(5y)$	(20) $3x + 7x + y$

Here is a quiz which should help reveal the extent to which students have already developed techniques for simplifying expressions.

Sort these expressions into as few categories as possible with each category containing equivalent expressions only.

1. $3x + 2y + 7x + 5y$
2. $3x(1 + y) + 2y(1 + x)$
3. $4x + 5y + 2(3x + y)$
4. $2(x + y) + x + 5xy$
5. $5xy + 12xy$
6. $y(5x + 2) + 3x$
7. $3x + 7(x + y)$
8. $5(2xy + 1)$
9. $2(5x + 3y)$
10. $5xy + 5y$
11. $2(2x + 3y) + 6x + y$
12. $3x + 3xy + 2y + 2xy$
13. $10x + 7y$
14. $10xy$
15. $17xy$
16. $xy + 16xy$
17. $5(xy + y)$
18. $(2x)(5y)$
19. $10x + y$
20. $3x + 7x + y$
21. $10(x + y) + {}^{-}3y$
22. $5y(x + 1)$
23. $10x + 6y$
24. $x(5y + 3) + 2y$
25. $9x + 6y + x$
26. $5 + 3y(\frac{10}{3}x)$
27. $2y + 5xy + 3x$

[The sorting is on TC[2-51]e.]

7.

$$\begin{aligned}
& 6x(2 + 3x) + 5xx \\
&= 6x2 + 6x(3x) + 5xx && \} \ \ell\text{dpma} \\
&= 6x2 + 6x3x + 5xx && \} \ \text{apm} \\
&= 2(6x) + 3(6x)x + 5xx && \} \ \text{cpm} \\
&= 2 \cdot 6x + 3 \cdot 6xx + 5xx && \} \ \text{apm} \\
&= 12x + 18xx + 5xx && \} \ 2 \cdot 6 = 12,\ 3 \cdot 6 = 18 \\
&= 12x + [18xx + 5xx] && \} \ \text{apa} \\
&= 12x + [(18x + 5x)x] && \} \ \text{dpma} \\
&= 12x + (18 + 5)xx && \} \ \text{dpma} \\
&= 12x + 23xx. && \} \ 18 + 5 = 23
\end{aligned}$$

The expressions are equivalent.

8. Substitute '0' for 'y'. The corresponding value of '7 + 4y' is 7, and the corresponding value of '11y' is 0. Hence, the expressions are not equivalent.

9. Substitute '0' for 'x' and '1' for 'y'. The corresponding value of '3(x + 7y)' is 21, and the corresponding value of '3x + 7y' is 7. Hence, the expressions are not equivalent.

*

4. $$\begin{aligned} & 5p + 6 + 3p + 2 \\ &= [3p + (5p + 6)] + 2 && \text{cpa} \\ &= [(3p + 5p) + 6] + 2 && \text{apa} \\ &= (3p + 5p) + (6 + 2) && \text{apa} \\ &= (3 + 5)p + (6 + 2) && \text{dpma} \\ &= 8p + 8 && 3 + 5 = 8,\ 6 + 2 = 8 \\ &= 8p + 8 \cdot 1 && \text{pm1} \\ &= 8(p + 1). && \ell\text{dpma} \end{aligned}$$

The expressions are equivalent.

5. $$\begin{aligned} & 11x + 2y \\ &= (6 + 5)x + 2y && 11 = 6 + 5 \\ &= 6x + 5x + 2y && \text{dpma} \\ &= 6x + (5x + 2y) && \text{apa} \\ &= 6x + (2y + 5x) && \text{cpa} \\ &= 6x + 2y + 5x. && \text{apa} \end{aligned}$$

The expressions are equivalent.

6. Substitute '1' for 'a' and '0' for 'b'. The corresponding value of 'a + 3b + 5ab' is 1, and the corresponding value of '8ab' is 0. Hence, the expressions are not equivalent.

The reasons which should be supplied for the Solution to the Sample are, in order from the top to bottom:

apa; cpa; apa; apa; dpma; $7 + 2 = 9$, $3 + 9 = 12$;

$9 = 3 \cdot 3$, $12 = 3 \cdot 4$, $15 = 3 \cdot 5$; apm; ℓdpma; ℓdpma

*

Answers for Part A [which begins on page 2-50; see TC[2-48, 49, 50]b].

[For your convenience, for the exercises in which the two expressions <u>are</u> equivalent, we give the sequence of steps which shows the transformation of one expression into the other. For the exercises in which the two expressions <u>are not</u> equivalent, we give values for the pronumerals which lead to different values for the two expressions. Your students may suggest other values.]

1.
$$\begin{aligned} & 3a + 2b + 7a \\ &= 7a + (3a + 2b) && \text{cpa} \\ &= 7a + 3a + 2b && \text{apa} \\ &= (7 + 3)a + 2b && \text{dpma} \\ &= 10a + 2b. && 7 + 3 = 10 \end{aligned}$$

The expressions are equivalent.

2. Substitute '2' for 'x' and '1' for 'y'. The corresponding value of '$5x + 4 + 2y$' is 16, and the corresponding value of '$9x + 2y$' is 20. Hence, the expressions are not equivalent.

3.
$$\begin{aligned} & 8a(2b) + 3ab \\ &= (8a \cdot 2)b + 3ab && \text{apm} \\ &= [2(8a)]b + 3ab && \text{cpm} \\ &= 2 \cdot 8ab + 3ab && \text{apm} \\ &= 16ab + 3ab && 2 \cdot 8 = 16 \\ &= 16(ab) + 3(ab) && \text{apm} \\ &= (16 + 3)(ab) && \text{dpma} \\ &= 19(ab) && 16 + 3 = 19 \\ &= 19ab. && \text{apm} \end{aligned}$$

The expressions are equivalent.

with some computing facts. So, I know that the expressions are equivalent. Therefore, I just write:

Equivalent.

If someone disputes me, I can ask him to give me values of 'x' and 'y' for which the expressions have different values. If someone asks me how I <u>know</u> that they are equivalent, I would prove:

For each x, for each y,

$$7x + 3y + 2x + 9y + 15 = 3(3x + 4y + 5).$$

[You supply the reasons.]

$$\begin{aligned}
&\{[(7x + 3y) + 2x] + 9y\} + 15 \\
&= \{[7x + (3y + 2x)] + 9y\} + 15 \quad \underline{\qquad\qquad} \\
&= \{[7x + (2x + 3y)] + 9y\} + 15 \quad \underline{\qquad\qquad} \\
&= \{[(7x + 2x) + 3y] + 9y\} + 15 \quad \underline{\qquad\qquad} \\
&= \{(7x + 2x) + (3y + 9y)\} + 15 \quad \underline{\qquad\qquad} \\
&= \{(7 + 2)x + (3 + 9)y\} + 15 \quad \underline{\qquad\qquad} \\
&= \{9x + 12y\} + 15 \quad \underline{\qquad\qquad} \\
&= \{(3 \cdot 3)x + (3 \cdot 4)y\} + 3 \cdot 5 \quad \underline{\qquad\qquad} \\
&= \{3(3x) + 3(4y)\} + 3 \cdot 5 \quad \underline{\qquad\qquad} \\
&= 3\{3x + 4y\} + 3 \cdot 5 \quad \underline{\qquad\qquad} \\
&= 3(3x + 4y + 5). \quad \underline{\qquad\qquad}
\end{aligned}$$

1. $3a + 2b + 7a$

   $10a + 2b$

2. $5x + 4 + 2y$

   $9x + 2y$

3. $8a(2b) + 3ab$

   $19ab$

4. $5p + 6 + 3p + 2$

   $8(p + 1)$

5. $11x + 2y$

   $6x + 2y + 5x$

6. $a + 3b + 5ab$

   $8ab$

7. $6x(2 + 3x) + 5xx$

   $12x + 23xx$

8. $7 + 4y$

   $11y$

9. $3(x + 7y)$

   $3x + 7y$

* * *

We have been talking about pairs of equivalent expressions. What does this have to do with the problem of simplifying an expression?

To simplify an expression is to transform it into an equivalent one which is simpler. What do we mean by 'simpler'? In the perimeter problems, we said that '$2(\ell + w)$' was simpler than '$\ell + w + \ell + w$'. Do you agree? Why?

Often, when you are trying to decide which of two equivalent pronumeral expressions is the simpler, you think about which one would be easier to find values of. [Usually, this is the one with fewer marks in it.] When we ask you to simplify a pronumeral expression, you can use this "evaluation test". [But, in some later exercises you will transform an expression into an equivalent one which is not simpler to evaluate but which is simpler for some other purpose.]

* * *

B. Simplify.

Sample 1. $3x + 5 + 4x + 3$

Solution. $3x + 5 + 4x + 3$

$= (3x + 4x) + (5 + 3)$

$= 7x + 8.$

[As answer, we write: $7x + 8$. And, this means that we are claiming that, for each x, $3x + 5 + 4x + 3 = 7x + 8$.]

1. $a + 3 + 4a$	2. $6b + 5 + 2b$	3. $b + 6 + b$
4. $9 + 2c + 7$	5. $11a + 7 + a$	6. $4x + 5 + {}^{-}2x$
7. $r + 2r + 3r$	8. $6 + 7p + 2$	9. $8 + {}^{-}3k - 5$

Sample 2. $2x + 5y + 3 + 7x + 2y + 7$

Solution. $2x + 5y + 3 + 7x + 2y + 7$

$= (2x + 7x) + (5y + 2y) + (3 + 7)$

$= 9x + 7y + 10.$

Students should begin to develop mechanical short cuts for simplifying the expressions in Part B. Do not ask for justifications, unless disputes arise. In particular, if a student simplifies an expression correctly, do not ask him what he did. For example, if he simplifies

'3x + 7 + 2x' to '5x + 7'

and you ask him to tell what he did, he should answer that he added 2 to 3, got 5, wrote '5', wrote an 'x' after the '5', and then wrote a '+ 7' after that. This is a description of the mechanical short cut he has discovered. If you do ask any question at all, it should be: How do you know that '3x + 7 + 2x' and '5x + 7' are equivalent? To answer this question, he must refer to the principles. Suppose a student simplifies '3x + 7 + 2x' to '12x'. There are two ways to handle this. One is to supply a counter-example. This convinces the student that the two expressions are not equivalent. But, it may be more instructive for him to attempt to give a justification and then discover that where he thought he could use a principle he can't, or that he has misunderstood an abbreviation. For example, a student who simplifies '3x + 5y' to '8xy' may think he is using the distributive principle. A student who simplifies '7 + 2x' to '9x' may have misapplied the convention concerning the omission of grouping symbols.

*

In discussing these exercises, there will be times when you will want to talk about a sentence such as (Exercise 1):

$$a + 3 + 4a = 5a + 3.$$

There may be a tendency to want to say that this sentence is true. Of course, this sentence is an open sentence and is neither true nor false. What you should do is to prefix a quantifying phrase to the sentence, obtaining:

For each a, $a + 3 + 4a = 5a + 3$.

Then, you can talk about this generalization as being true.

*

In discussing the Samples it is good practice to do them in detail on the board and then point out to the students steps which they should be able to omit with safety [but which they should be able to supply if asked to]. The point here is to show students how the short cuts they develop can be justified, and to suggest a method of checking doubtful short cuts. For example, consider Sample 1.

$$\begin{aligned} & 3x + 5 + 4x + 3 \\ &= 3x + (5 + 4x) + 3 && \text{apa} \\ &= 3x + (4x + 5) + 3 && \text{cpa} \\ &= 3x + 4x + 5 + 3 && \text{apa} \\ &= (3x + 4x) + (5 + 3) && \text{apa} \\ &= (3 + 4)x + (5 + 3) && \text{dpma} \\ &= 7x + 8. && 3 + 4 = 7,\ 5 + 3 = 8 \end{aligned}$$

As previously remarked in the COMMENTARY, students who have difficulty using the apa can be helped by inserting all missing grouping symbols.

Students should by now be able to see, by merely looking at '$3x + 5 + 4x + 3$', that this expression is equivalent to '$(3x + 4x) + (5 + 3)$', and should be fairly confident that they can show that this is the case by referring to the apa and the cpa. So, if asked to justify their simplification of '$3x + 5 + 4x + 3$' to '$7x + 8$' by giving a testing pattern, such a student might reasonably write:

$$\begin{aligned} & 3x + 5 + 4x + 3 \\ &= (3x + 4x) + (5 + 3) && \text{apa, cpa} \\ &= (3 + 4)x + (5 + 3) && \text{dpma} \\ &= 7x + 8. && 3 + 4 = 7,\ 5 + 3 = 8 \end{aligned}$$

[Of course, in answering the exercises of Part B, students are <u>not</u> expected to give testing patterns. All that should be required as an answer is the simpler expression, in this case, '$7x + 8$'. Testing patterns come into the picture only when, and if, a student is required to show that his answer is correct.] However, a student who presents the abbreviated testing pattern should realize that the citation 'apa, cpa' is of the nature of a promissory note which he may be called upon to make good.

With practice, students will undoubtedly develop the ability to think their way through to the answer in the way suggested by:

$$\begin{aligned} &3x + 5 + 4x + 3 \\ &= (3 + 4)x + (5 + 3) && \text{apa, cpa, dpma} \\ &= 7x + 8. && 3 + 4 = 7,\ 5 + 3 = 8 \end{aligned}$$

*

Answers for Part B [on pages 2-52, 2-53, and 2-54].

1. $5a + 3$	2. $8b + 5$	3. $2b + 6$
4. $2c + 16$	5. $12a + 7$	6. $2x + 5$
7. $6r$	8. $7p + 8$	9. $^{-}3k + 3$

*

Testing pattern for the generalization corresponding to Sample 2.

$$\begin{aligned} &2x + 5y + 3 + 7x + 2y + 7 \\ &= 2x + 5y + (3 + 7x) + 2y + 7 && \text{apa} \\ &= 2x + 5y + (7x + 3) + 2y + 7 && \text{cpa} \\ &= 2x + 5y + 7x + 3 + 2y + 7 && \text{apa} \\ &= 2x + (5y + 7x) + 3 + 2y + 7 && \text{apa} \\ &= 2x + (7x + 5y) + 3 + 2y + 7 && \text{cpa} \\ &= 2x + 7x + 5y + 3 + 2y + 7 && \text{apa} \\ &= 2x + 7x + 5y + (3 + 2y) + 7 && \text{apa} \\ &= 2x + 7x + 5y + (2y + 3) + 7 && \text{cpa} \\ &= 2x + 7x + 5y + 2y + 3 + 7 && \text{apa} \\ &= (2x + 7x) + (5y + 2y) + 3 + 7 && \text{apa} \\ &= (2x + 7x) + (5y + 2y) + (3 + 7) && \text{apa} \\ &= (2 + 7)x + (5 + 2)y + (3 + 7) && \text{dpma} \\ &= 9x + 7y + 10. && 2 + 7 = 9,\ 5 + 2 = 7,\ 3 + 7 = 10 \end{aligned}$$

In Sample 3, when we give 'ab + 12c' as the simplest expression, we are anticipating the "1 times theorem" which is mentioned for the first time in the student text at the top of page 2-60. Your class may have proved this theorem in connection with Exercise 8 on page 2-38 [see TC[2-38]b]; if not, they should prove it now.

*

Answers for Part B, continued.

10. $10a + 2b$	11. $15x + 5y + 7$	12. $8c + 8d + 5$
13. $3x + 4y + 17$	14. $10m + 5n + 2$	15. $p + 10q + 14$
16. $14s + 6t$	17. $^{-}10x + ^{-}15y + ^{-}5$	18. $x + 13/8$
19. $(9/20)x + (11/15)y$	20. $4.8k + 3.8m$	21. $3.9p + 1.6$
22. $9ab + 7c$	23. $rs + t$	24. $9xx + 12x + 4$
25. $9mm + 7m + 12$	26. $9pq + 15rs$	27. $yy + ^{-}1y$
28. $20a + 39b$	29. $16a + 5b$	30. $24x + 21y$
31. $33x + 12$	32. $10xx + 17x + 4$	33. $28yy + 10y$
34. $3x + 13/3$	35. $4y + 9x + 1$	36. $15xx + 9x$
37. $3x + (24/7)y$	38. $36mnp$	39. $30mmn$

40. $21pppqqq$	41. $^{-}36xxxxyyy$	42. $^{-}8xxxyy$	43. $90xxxy$
44. $6aaabbb$	45. $24xxyyzz$	46. $^{-}1abbccd$	47. $(21/64)xyz$
48. $(1/15)mmm$	49. $(1/5)kmn$	50. $^{-}(1/6)abc$	51. $^{-}(5/28)rst$
52. $(3/22)efg$	53. $57.456xyz$	54. $0.006xyy$	55. $^{-}118.218ab$
56. $41xy$	57. $24ab$	58. $^{-}24pq$	59. $48mnp$

*

Answers for Part C [on pages 2-54 and 2-55].

1. $5x + 7$	2. $7m + 6$	3. $14k$
4. $9r + 9$	5. $8t + 3$	6. $15a + 3b + 7$
7. $16j + 14k + 2$	8. $9d + 5c$	9. $21x + 5y + 7$
10. $8m + 2k + 1$	11. $10d + 10r$	12. $(-3)x + 5y$
13. $(-1)k$	14. $10rs + 2t + 5u$	15. $21ab + 3c + 7d$
16. $6xyz + 7x + 3yz$	17. $8mn + 6n$	18. $21x + 8$
19. $25x + 24$		

10. $3a + 2b + 7a$

11. $6x + 2y + 7 + 9x + 3y$

12. $6c + 5 + 2c + 8d$

13. $x + 3y + 8 + 2x + y + 9$

14. $7m + 2 + 3m + 5n$

15. $12 + 6p + 3q + {}^{-}5p + 7q + 2$

16. $7s + 2s + 5s + 6t$

17. ${}^{-}3x + {}^{-}9y + {}^{-}7x + {}^{-}5 + {}^{-}6y$

18. $\frac{2}{3}x + \frac{3}{4} + \frac{1}{3}x + \frac{7}{8}$

19. $\frac{2}{5}y + \frac{1}{4}x + \frac{1}{3}y + \frac{1}{5}x$

20. $2.1k + 3.8m + 2.7k$

21. $0.8p + 7.9 + 3.1p + {}^{-}6.3$

<u>Sample 3.</u> $7ab + 3c + {}^{-}6ab + 9c$

<u>Solution.</u> $7ab + 3c + {}^{-}6ab + 9c$

$= (7ab + {}^{-}6ab) + (3c + 9c)$

$= ab + 12c.$

<u>Sample 4.</u> $3xx + 2x + 5xx + 5 + 9x$

<u>Solution.</u> $3xx + 2x + 5xx + 5 + 9x$

$= (3xx + 5xx) + (2x + 9x) + 5$

$= 8xx + 11x + 5$

$= x(8x + 11) + 5.$

[Either '8xx + 11x + 5' or 'x(8x + 11) + 5' is acceptable as an answer, although the latter is simpler according to our evaluation test. However, there are many places in mathematics where '8xx + 11x + 5' is more useful.]

22. $6ab + 5c + 3ab + 2c$

23. ${}^{-}10rs + 3t + 5rs + 6rs + {}^{-}2t$

24. $3x + 2xx + 9x + 7xx + 4$

25. $3mm + 2m + 5 + 6mm + 5m + 7$

26. $7pq + 3rs + 2pq + 12rs$

27. $6y + 2yy + 9 + {}^{-}1yy + {}^{-}7y + {}^{-}9$

28. $7(2a + 3b) + 6(a + 3b)$

29. $2(5a + b) + 3(b + 2a)$

30. $6(2x + y) + 3(4x + 5y)$

31. $3(5x + 1) + 9(1 + 2x)$

32. $4(1 + 3x) + 5(x + 2xx)$

33. $7(y + 3yy) + y(3 + 7y)$

34. $(1/2)(8 + 4x) + (1/3)(3x + 1)$

35. $(2/5)(10y + 15x) + (1/7)(21x + 7)$

36. $(1/3)(3xx + 6x) + 7(x + 2xx)$

37. $(3/7)(5x + 2y) + (2/7)(3x + 9y)$

(continued on next page)

<u>Sample 5</u>. $(3x)(2y)z$

<u>Solution</u>. $(3x)(2y)z$

$= (3 \cdot 2)xyz$

$= 6xyz.$

<u>Sample 6</u>. $(2ac)(3ab)(5abc)$

<u>Solution</u>. $(2ac)(3ab)(5abc)$

$= (2 \cdot 3 \cdot 5)aaabbcc$

$= 30aaabbcc.$

38. $(6m)(2n)(3p)$
39. $(2m)(3n)(5m)$
40. $p(3qp)(7pqq)$
41. $(6xx)(2xy)(^-3xyy)$
42. $(^-2x)(^-4xy)(^-1yx)$
43. $5(3x)(^-2x)(^-3xy)$
44. $(3ab)(^-2ab)(^-1ab)$
45. $(4xy)(2yz)(3xz)$
46. $(^-1ab)(^-1bc)(^-1cd)$
47. $(\frac{1}{2}x)(\frac{3}{4}y)(\frac{7}{8}z)$
48. $(\frac{2}{3}m)(\frac{1}{5}m)(\frac{1}{2}m)$
49. $(\frac{3}{7}k)(\frac{7}{9}m)(\frac{3}{5}n)$
50. $(\frac{^-1}{3}a)(\frac{^-2}{3}b)(\frac{^-3}{4}c)$
51. $(\frac{^-6}{7}t)(\frac{1}{3}r)(\frac{5}{8}s)$
52. $\frac{2}{5}(\frac{3}{4}e)(\frac{10}{11}f)(\frac{1}{2}g)$
53. $(7.2x)(3.8y)(2.1z)$
54. $(0.1x)(0.2y)(0.3y)$
55. $^-6.1(^-3.8a)(^-5.1b)$
56. $(3x)(2y) + (7x)(5y)$
57. $(9a)(2b) + (^-3a)(^-2b)$
58. $(4p)(^-3q) + 3(4p)(^-1q)$
59. $(7m)(2n)(^-3p) + (^-5m)(^-6n)(3p)$

<u>C</u>. Simplify.

1. $3x + 7 + 2x$
2. $5m + 2m + 6$
3. $8k + k + 5k$
4. $7r + 5 + 2r + 4$
5. $3t + 9 + 5t - 6$
6. $10a + 3b + 5a + 7$
7. $12j + 5k + 2 + 9k + 4j$
8. $2d + 3c + 7d + 2c$
9. $9x + 3y + 7 + 2y + 12x$
10. $5m + 2k + 3m + 1$
11. $7d + 5r + 3d + 5r$
12. $4x + 3y + (-7)x + 2y$
13. $(-1)k + (-5)k + 7k + (-2)k$
14. $7rs + 2t + 3rs + 5u$
15. $9ab + 3c + 7d + 12ab$
16. $6xyz + 2x + 3yz + 5x$
17. $3mn + 2n + 5nm + 4n$
18. $3(2x + 1) + 5(1 + 3x)$
19. $2(5x + 7) + 5(3x + 2)$

20. 36 + 45x
21. 732x + 864
22. 60y + 19
23. 31 + 81y
24. 38a + 57b + 19
25. 27x + 69y + 96z
26. 4x + 1
*27. (−21)y - 11
28. 36abc
29. 84xyz
30. 35abc
31. 30xxxx
32. 24xxyyy
33. 48xy
34. 16xxxyy
35. 3aaabb
36. (−10)xxyy
37. 44aab + 113abb
38. 49xxy + 50xyy
39. 12aa + 28ab + 27ac + 10bb + 22bc + 14cc
40. 40x + 19xy + 61y
41. 24xxx + 20xx + 30x
42. 20a + 4b + 7x + 11c
43. 4x + 3y + (−8)z
44. 24x + 8
45. 24x + 24
46. 92y - 11
47. 40xx + 53xy + 65yy
48. 40aa + 88ab + 48bb
49. 144aa + 96ab + 16bb
50. xx + 14x + 49 [or: (x + 7)(x + 7)]
51. aa + 9a + 14 [or: (a + 7)(a + 2)]
52. $\frac{3}{2}x + \frac{5}{4}xy + \frac{15}{4}y$
53. $\frac{3}{4}xx + \frac{23}{20}xy + yy + \frac{3}{4}x + \frac{3}{5}y$
54. $\frac{159}{700}ab$
55. $\frac{1}{6}xxy + \frac{2}{15}xyy$

*

Answers for Part D [on page 2-56].

1. 16x + 8
2. 7x + 16y
3. 28xx
4. 6x + 6y + 9z
5. 7A + 7B
6. 33k + 20
7. 56p + 77
8. 3x + 5y
9. (7k + 4j)(6k + 16j)
10. (3/4)x + (8/15)y

---

*If students write '−21y - 11', they are just applying the convention that '−21y' is an abbreviation for '(−21)y'. This convention is mentioned on TC[1-82]a [lines 8 and 9], and is more formally treated on page 2-69 in Exercise 4.

20. $6(5 + 4x) + 3(2 + 7x)$ 21. $4(61x + 72) + 8(61x + 72)$

22. $4(3 + 8y) + 7(4y + 1)$ 23. $12(2 + 5y) + 7(1 + 3y)$

24. $5(1 + 2a + 3b) + 7(2 + 4a + 6b)$

25. $3(3x + 2y + 5z) + 9(2x + 7y + 9z)$

26. $7x + 2 + (-1)(3x + 1)$ 27. $(-2)(4 + 3y) + (-3)(1 + 5y)$

28. $3a(2b)(6c)$ 29. $4x(3y)(7z)$ 30. $a(5b)(7c)$

31. $2x(3xx)(5x)$ 32. $2y(4xy)(3xy)$

33. $(-6)x(-8)y$ 34. $[(-2)xy][(-8)xyx]$

35. $[(-1)ab][(-3)aab]$ 36. $[(-2)xy][5yx]$

37. $2ab(a + 3b) + 5ab(2a + 7b) + 8ab(4a + 9b)$

38. $5xy(2x + 5y) + 7xy(5x + 3y) + 4xy(x + y)$

39. $4a(3a + 2b + 5c) + 5b(4a + 2b + 3c) + 7c(a + b + 2c)$

40. $7(2x + 3yx + 5y) + 4(8x + 2xy + 7y) + (-2)(3x + 5xy + y)$

41. $3x(5xx + 3x + 5) + 2x(4xx + 5x + 7) + x(xx + x + 1)$

42. $8a + 2b + 3x + 5c + 7x + 2b + 5a + 6c + 7a + (-3)x$

43. $4x + 2y + (-7)z + 6x + 3z + (-2)y + (-6)x + 3y + (-4)z$

44. $2[(7x + 3) + (5x + 1)]$ 45. $2[(3 + 5x) + (9 + 7x)]$

46. $5[(2y + 1) + (1 + y)] + 7[(8y - 1) + (3y - 2)]$

47. $4x[(3y + 2x) + (8x + 4y)] + 5y[(2x + 8y) + (3x + 5y)]$

48. $5a[(7a + b) + (7b + a)] + 6b[(2a + 3b) + (5b + 6a)]$

49. $12a[(8a + 3b) + (4a + b)] + 4b[(8a + 3b) + (4a + b)]$

50. $x(x + 7) + 7(x + 7)$ 51. $a(a + 2) + 7(a + 2)$

52. $\frac{1}{2}x(3 + y) + \frac{3}{4}y(x + 5)$ 53. $\frac{3}{4}x(x + y + 1) + \frac{1}{5}y(5y + 2x + 3)$

54. $(\frac{1}{5}a)(\frac{3}{5}b) + (\frac{2}{7}b)(\frac{3}{8}a)$ 55. $\frac{2}{3}(\frac{3}{4}xy)(\frac{1}{3}x) + \frac{1}{5}(\frac{5}{7}xy)(\frac{14}{15}y)$

[More exercises are in Part G, Supplementary Exercises.]

D. Complete each of the following into a true sentence by writing the simplest expression you can in the blank.

Sample. For each x, the sum of $(6x + 1)$ and $(3x + 4)$ is ____________.

Solution. The expression '$(6x + 1) + (3x + 4)$' would make the sentence true. But, an expression simpler than '$(6x + 1) + (3x + 4)$' and equivalent to it is '$9x + 5$'. So, write '$9x + 5$' in the blank.

1. For each x, the sum of $(9x + 3)$ and $(5 + 7x)$ is ____________.

2. For each x, for each y, the sum of $(2x + 7y)$ and $(5x + 9y)$ is ____________.

3. For each x, the product of $(7x)$ by $(3x + x)$ is ____________.

4. For each x, for each y, for each z, the sum of $(x + 3y + z)$, $(3x + 2y + 5z)$, and $(2x + y + 3z)$ is ____________.

5. For each A, for each B, the sum of $(3A)$ and $(4A + 7B)$ is ____________.

6. For each k, the sum of $(3k + 2)$ and the product of 6 by $(3 + 5k)$ is ____________.

7. For each p, the product of the sum of $(3p + 5)$ and $(5p + 6)$ by 7 is ____________.

8. For each x, for each y, the sum of $(3x)$ and $(5y)$ is ____________.

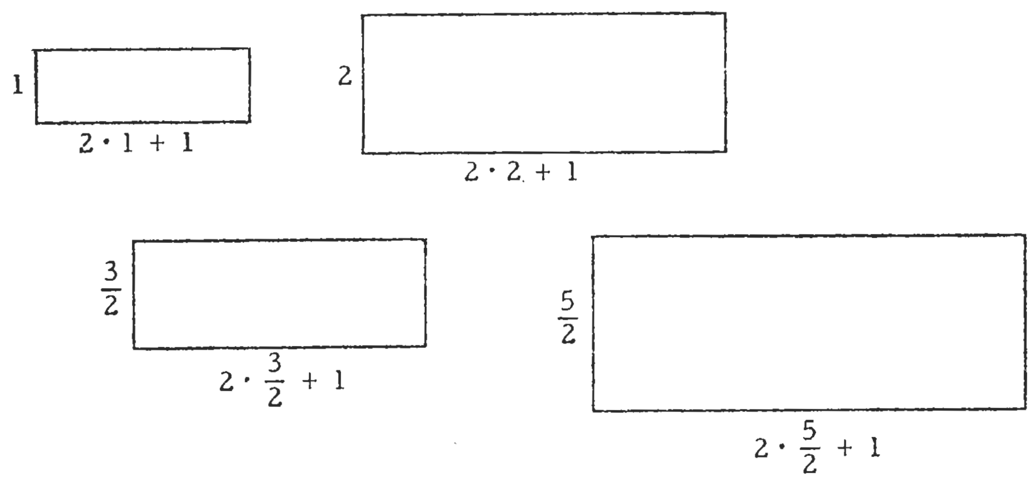
9. For each k, for each j, the product of the sum of $(2k + j)$ and $(3j + 5k)$ by the sum of $(5k + 7j)$ and $(k + 9j)$ is ____________.

10. For each x, for each y, the sum of $(\frac{1}{2}x + \frac{1}{3}y)$ and $(\frac{1}{4}x + \frac{1}{5}y)$ is ____________.

E. Write the simplest formula you can for the perimeter of each figure described.

Sample 1. A rectangle whose length measures 1 more than twice the measure of the width.

Solution. How many rectangles fit this description? Here are pictures of just a few of them.



Our job is to find a formula for computing the perimeters of all such rectangles. For the rectangles shown above, the perimeters are given by the expressions:

$$2[(2 \cdot 1 + 1) + 1],$$

$$2[(2 \cdot \frac{3}{2} + 1) + \frac{3}{2}],$$

$$2[(2 \cdot 2 + 1) + 2],$$

and: $$2[(2 \cdot \frac{5}{2} + 1) + \frac{5}{2}].$$

Do you see a pattern for these numerical expressions? Each of them can be obtained by substituting for 'x' in the pronumeral expression:

$$2[(2x + 1) + x].$$

So, a formula for the perimeter of such rectangles is the sentence:

$$P = 2[(2x + 1) + x].$$

We can simplify the expression on the right of the equality sign.

$$\begin{aligned} & 2[(2x + 1) + x] \\ = \; & 2[x + (2x + 1)] \\ = \; & 2[(x + 2x) + 1] \\ = \; & 2[3x + 1] \\ = \; & 6x + 2. \end{aligned}$$

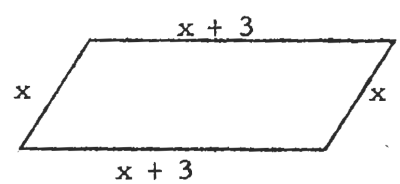
Hence, a simple formula is:

$$P = 6x + 2.$$

Note that the values of 'x' in this formula are the measures of the widths of rectangles which fit the given description. Let's check the formula with an example. Suppose we take the largest of the rectangles pictured on page 2-57, the one whose width measures $\frac{5}{2}$. According to the formula, the perimeter is $6 \cdot \frac{5}{2} + 2$, that is, the perimeter is 17. According to the description, the length measures 6; so, the perimeter is $2(6 + \frac{5}{2})$, which is 12 + 5, or 17.

Sample 2. A parallelogram whose longer side measures 3 more than the shorter side.

Solution. Instead of drawing several examples of such parallelograms, we draw only one, and label its sides with pronumeral expressions instead of numerals. If 'x' is a pronumeral whose values are the measures of the shorter sides of such parallelograms then, for such values of 'x', the values of 'x + 3' are the measures of the longer sides. Expressions for the perimeters of such parallelograms can be generated from the pronumeral expression:

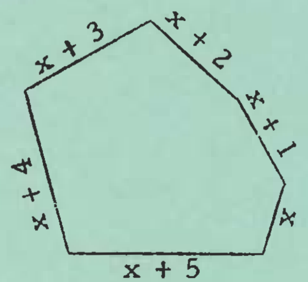


$$x + (x + 3) + x + (x + 3).$$

Hence, a formula for the perimeter of such parallelograms is '$P = x + (x + 3) + x + (x + 3)$'. A simpler formula is:

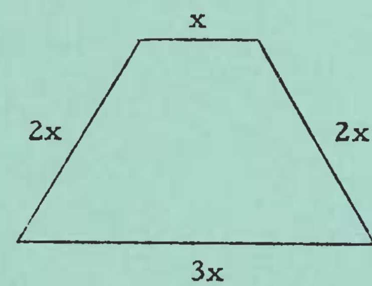
$$P = 4x + 6.$$

4.



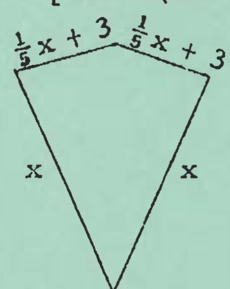
$P = 6x + 15$
[or: $3(2x + 5)$]

5.



$P = 8x$

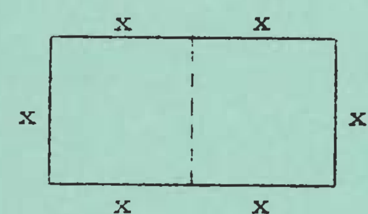
6.



$P = 2\frac{2}{5}x + 6$

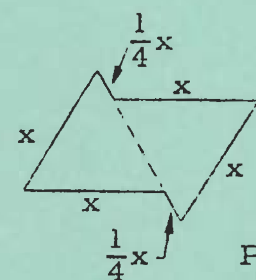
[Each of the shorter sides could be labeled with an 'x'; if so, each of the longer sides should be labeled with a '5x - 15'. If this is done, a formula for the perimeter would be '$P = 12x - 30$', and the values of 'x' are greater than 2.5.]

7.



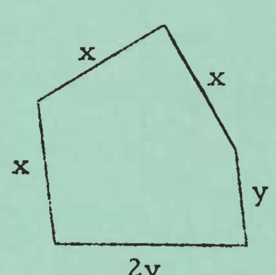
$P = 6x$

8.



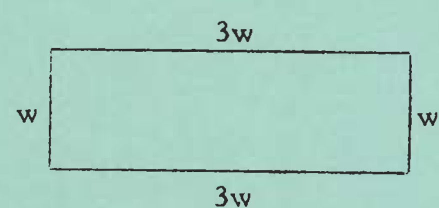
$P = 4\frac{1}{2}x$

9.



$P = 3x + 3y$

10.



$P = 8w$

Students should be instructed to include a labeled diagram for each of the exercises of Part E. The various parts of the diagram should be labeled with pronumeral expressions as shown in Sample 2 on page 2-58.

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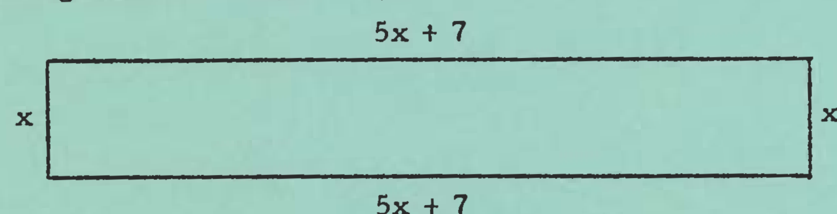
Answers for Part E [on pages 2-57, 2-58, and 2-59].

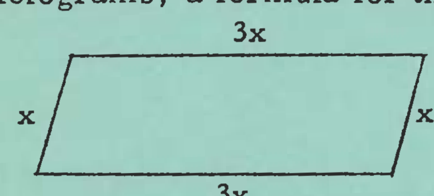
[There is more than one way to label the parts of these figures, and accordingly, more than one correct formula for the perimeter. In Exercises 1 and 6 we indicate an alternative. In other exercises you will doubtless want the class to consider other possibilities.]

1. Using 'x' as a pronumeral whose values are the measures of the shorter sides of such rectangles, a formula for the perimeter of such rectangles is '$P = 2(6x + 7)$'.



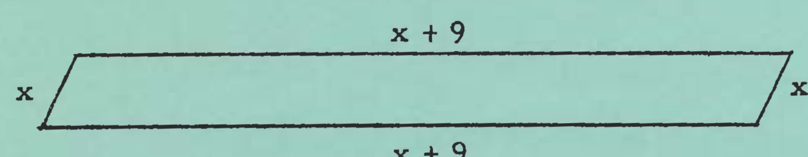
[Each of the longer sides could be labeled with an 'x'; if so, each of the shorter sides should be labeled with a '$\frac{x - 7}{5}$'. Then, the perimeter formula is '$P = 2(x + \frac{x - 7}{5})$', and the values of 'x' are greater than 7.]

2. Using 'x' as a pronumeral whose values are the measures of the shorter sides of such parallelograms, a formula for the perimeter is '$P = 8x$'.



For the remaining exercises, we give just a diagram and a formula.

3.



$$P = 2(2x + 9)$$

1. A rectangle whose longer side measures 7 more than 5 times the measure of the shorter side.

2. A parallelogram whose shorter side is $\frac{1}{3}$ as long as the longer side.

3. A parallelogram whose longer and shorter sides differ in measure by 9.

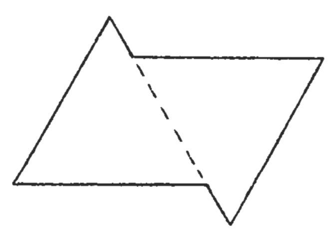
4. A hexagon the measures of whose sides are consecutive whole numbers. [Example: A hexagon with sides 2", 3", 4", 5", 6", and 7" long.]

5. An isosceles trapezoid for which the longer of its parallel sides is three times as long as the shorter of its parallel sides, and the sum of the measures of its nonparallel sides is the sum of the measures of its parallel sides.

6. A kite whose shorter side measures 3 more than $\frac{1}{5}$ the measure of its longer side.

7. A rectangle made by putting two squares next to each other.

8. A hexagon made by putting two equilateral triangles with equal perimeters next to each other so that the length of the overlap is $\frac{3}{4}$ the length of a side.



9. A pentagon, three of whose sides have the same measure, and, of the remaining two sides, the shorter side is $\frac{1}{2}$ as long as the longer.

☆10. A rectangle for which the average of the measures of its four sides is twice the measure of its shorter side.

[More exercises are in Part H, Supplementary Exercises.]

2.05 <u>Theorems</u> <u>and</u> <u>basic</u> <u>principles</u>.--In proving generalizations you probably noticed that you could have shortened your work if you had had principles like:

$$\text{For each } x, \quad 1 \cdot x = x,$$

and:

$$\text{For each } x, \quad \text{for each } y, \quad \text{for each } u, \quad \text{for each } v,$$

$$(xy)(uv) = (xu)(yv).$$

In fact, you may have already stated such principles and derived them from the ones mentioned in Unit 1.

Statements which can be derived from the <u>basic</u> principles are called <u>theorems</u>. For example, when you simplify the expression

'$3x + 2y + 7x + 1$' to '$10x + 2y + 1$',

you are claiming that the generalization:

$$\text{For each } x, \quad \text{for each } y, \quad 3x + 2y + 7x + 1 = 10x + 2y + 1$$

is a theorem. And, when you give a test-pattern for the generalization, you are showing that it is a theorem. So, you have already proven many theorems.

A theorem can be used in justifying a step in a proof in the same way that you have used the basic principles. Those that are most useful for this purpose are worth remembering and even giving names to. An example of such a theorem is the <u>left distributive principle for multiplication over addition</u>. Although this is one of the principles mentioned in Unit 1, it can be derived from the basic ones. Let's prove that for each x, for each y, for each z, $x(y + z) = xy + xz$.

$$x(y + z) = (y + z)x \quad [\text{cpm}]$$

$$(y + z)x = yx + zx \quad [\text{dpma}]$$

$$yx + zx = xy + xz. \quad [\text{cpm}]$$

$$\text{Hence,} \qquad x(y + z) = xy + xz.$$

[Notice that the commutative principle for multiplication was applied twice in the last step.] So, the left distributive principle for multiplication over addition is [as was mentioned in Unit 1] a consequence of the commutative principle for multiplication and the distributive principle for multiplication over addition.

In this section we collect basic principles for real numbers from which all other principles concerning the operations of addition, multiplication, opposition, subtraction, and division can be derived. [We might, of course, have selected another set of basic principles; there are many possible choices.] In Unit 1, students have become acquainted with the arithmetic of the real numbers and have become convinced of the truth of the basic principles displayed on page 2-61. In Unit 2, they have, so far, learned how to state these principles and have begun to learn how to derive other principles from them. In the remainder of this unit they will derive such principles, use them to justify computational short cuts, and gain skill, through practice, in applying these short cuts.

The procedure of taking a known subject matter [here, the arithmetic of the real numbers] and organizing it deductively, by choosing some true statements from it as basic principles [or postulates] and deriving others [theorems] from them, is a common one in mathematics, and is even more common in applications of mathematics.

There is a further step, peculiar to mathematics, in which one forgets entirely the "known subject matter" and, considering the postulates and theorems merely as sentences in an uninterpreted language, concentrates his attention on the logical connections among these sentences. [One might, for example, refrain from specifying the domain of the variables 'x', 'y', 'z', etc., and consider '0', '1', '+', '•', '−', '-', and '÷' as "undefined terms".] In doing this, one is paying strict attention to the "structure" exhibited by the original subject matter. This procedure has great value in that it enables one to perceive important similarities among different subject matters. For example, if, in this way, one empties of content the apa, the pa0, and the po, and considers the sentences which can be derived from these, he obtains an abstract deductive theory corresponding to the kind of structures known as <u>groups</u>. [If one adds the cpa, one obtains the theory of <u>commutative</u> groups.] If one, now, interprets these sentences by choosing some set as the domain of the variables, a member of this set to be named by '0', and operations on the members of this set to be named by '+' and '−', all this in such a way that the sentences of the abstract theory become true statements, one has described a <u>model</u> of the abstract theory. In the case under discussion, such a model is called <u>a group</u> [or, with the cpa, <u>a commutative group</u>]. It turns out that groups occur in many situations. So, it is very convenient to have one abstract theory which formalizes properties common to all groups.

You will probably be interested, in this connection, in the article "To teach modern algebra", by Carl H. Denbow, in The Mathematics Teacher, March, 1959. [Our distinction between organizing a known subject matter deductively and developing an abstract deductive theory corresponds with Professor Denbow's distinction between pragmatic postulational systems and abstract postulational systems.]

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The fact that the basic principles are true of the real number system means, in technical language, that the real number system is a field. [The rational number system and the complex number system are also fields.] There are many other sets of postulates for field theory. Our choice has been motivated largely by pedagogical considerations. We might, for example, have replaced the pa0 and the po by the statement:

For each x, for each y, there exists a z such that $x + z = y$,

and omitted the ps. We could then have proved that there is just one z such that, for each x, $x + z = x$, and introduced '0' as an abbreviation for 'the z such that, for each x, $x + z = x$' and '$-x$' to abbreviate 'the z such that $x + z = 0$'. And, we could have defined '$x - y$' as an abbreviation for '$x + -y$'. Obviously, such a procedure, while it has certain advantages from the point of view of a mathematician, would not be pedagogically sound. [There is the further difficulty that when, for example, we define '$x - y$' to be an abbreviation for '$x + -y$', this is really only a specimen definition. It does not tell us how '$3 - 2$' or '$y - 4$', or '$x - z$', for example, are to be unabbreviated; all it mentions is '$x - y$'. Although you can probably guess that '$3 - 2$' is an abbreviation for '$3 + -2$' from being told that '$x - y$' is an abbreviation for '$x + -y$', a definition which leaves one guessing is not a good one. Just because a correctly stated rule for unabbreviating expressions which contain '-' requires rather sophisticated language, we have taken the alternative of adopting the ps as a basic principle.]

Also, we might have treated division in a manner parallel to our treatment of subtraction. This would involve replacing the principle of quotients by a principle of reciprocals [like the po] and a principle for division [like the ps]. The difficulty of this approach is that, by the time students arrive at the ninth grade their notion of reciprocal is inextricably bound up with the notion of dividing into 1. Consequently, a definition of division in terms of reciprocation would certainly appear to be circular. It is as though opposition were first defined in terms of subtracting from 0, <u>previous to</u> a definition of subtraction in terms of opposition. If this block were not present, one could introduce a name, say '/', for the operation reciprocating, and take as basic principles:

$$\forall_{x \neq 0}\ x \cdot /x = 1,$$

and:

$$\forall_x \forall_{y \neq 0}\ x \div y = x \cdot /y.$$

[Compare these with the principle of opposites and the principle of quotients.] [One would, of course, very soon prove the theorem:

$$\forall_{x \neq 0}\ /x = 1 \div x.$$

And, just as a modified '−' is used as a name for subtraction, a modified '/' would be introduced as a name for division.]

69. $\forall_x \forall_y \forall_{z \neq 0}\ \frac{x}{z} - \frac{y}{z} = \frac{x - y}{z}$ [Part B, Ex. 1, p. 2-100]

[Th. 69 is the distributive theorem for division over subtraction.]

70. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{yz} - \frac{u}{vz} = \frac{xv - uy}{yvz}$ [Part B, Ex. 2, p. 2-100]

71. $\forall_x \forall_y \forall_{z \neq 0}\ x + \frac{y}{z} = \frac{xz + y}{z}$ [Part B, Ex. 3, p. 2-100]

[Th. 71 is the "mixed number" theorem.]

72. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ x \div \frac{y}{z} = x \cdot \frac{z}{y}$ [p. 2-101]

[Th. 72 is the "dividing by a fraction" theorem.]

73. $\forall_x \forall_{y \neq 0} \forall_{u \neq 0} \forall_{v \neq 0}\ \frac{x}{y} \div \frac{u}{v} = \frac{xv}{yu}$ [Part A, Ex. 1, p. 2-101]

[Th. 73 is the "dividing a fraction by a fraction" theorem.]

74. $\forall_{x \neq 0} \forall_{y \neq 0}\ \frac{1}{x/y} = \frac{y}{x}$ [Part A, Ex. 2, p. 2-101]

[Th. 74 is the "reciprocal of a fraction" theorem.]

75. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ \frac{x}{y} \div z = \frac{x}{yz}$ [Part A, Ex. 3, p. 2-101]

[Th. 75 is the "dividing a fraction" theorem.]

76. $\forall_x \forall_{y \neq 0}\ -\frac{x}{y} = \frac{-x}{y}$ [p. 2-103]

77. $\forall_x \forall_{y \neq 0}\ -\frac{x}{y} = \frac{x}{-y}$ [p. 2-103]

78. $\forall_x \forall_{y \neq 0}\ \frac{-x}{-y} = \frac{x}{y}$ [Part A, p. 2-103]

59. $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} \cdot \frac{u}{v} = \frac{xu}{yv}$ [p. 2-93]

[Th. 59 is the "multiplication of fractions" theorem.]

60. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ \frac{xz}{yz} = \frac{x}{y}$ [p. 2-94]

[Th. 60 is the "reducing of fractions by multiplication" theorem.]

61. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ \frac{x}{y} = \frac{x \div z}{y \div z}$ [p. 2-95]

[Th. 61 is the "reducing of fractions by division" theorem.]

62. $\forall_x \forall_y \forall_{z \neq 0}\ \frac{xy}{z} = \frac{x}{z}y$ [p. 2-96]

[Th. 62 is the "multiplying a fraction" theorem.]

63. $\forall_x \forall_{y \neq 0}\ \frac{x}{y} = x \cdot \frac{1}{y}$ [p. 2-97]

[Th. 63 is the "dividing is the same as multiplying by the reciprocal" theorem.]

64. $\forall_x \forall_{y \neq 0}\ \frac{xy}{y} = x$ [p. 2-97]

[Th. 64 is the "dividing is the inverse of multiplying" theorem.]

65. $\forall_{x \neq 0} \forall_y \forall_z\ \frac{xy + xz}{x} = y + z$ [p. 2-97]

66. $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0} \forall_{z \neq 0}\ \frac{xu}{yv} = \frac{(x \div z)u}{(y \div z)v}$ [p. 2-98]

67. $\forall_x \forall_y \forall_{z \neq 0}\ \frac{x}{z} + \frac{y}{z} = \frac{x + y}{z}$ [p. 2-99]

[Th. 67 is the distributive theorem for division over addition.]

68. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{yz} + \frac{u}{vz} = \frac{xv + uy}{yvz}$ [p. 2-99]

48. $\forall_x \forall_y \forall_{z \neq 0}$ if $xz = yz$ then $x = y$ [Sample, p. 2-90]

[Th. 48 is the cancellation principle for multiplication.]

49. $\forall_x \forall_{y \neq 0} \forall_z$ if $zy = x$ then $z = x/y$ [Ex. 1, p. 2-91]

[Th. 49 is the division theorem.]

50. $\forall_x\ x/1 = x$ [Ex. 2, p. 2-91]

[Th. 50 is the "dividing by 1" theorem.]

51. $\forall_{x \neq 0}\ x/x = 1$ [Ex. 3, p. 2-91]

[Th. 51 is the "dividing a nonzero number by itself" theorem.]

52. $\forall_x\ x/-1 = -x$ [Ex. 4, p. 2-91]

[Th. 52 is the "dividing by −1" theorem.]

53. $\forall_{x \neq 0}\ 0/x = 0$ [Ex. 5, p. 2-91]

54. $\forall_x \forall_{y \neq 0}$ if $x/y = 0$ then $x = 0$ [Ex. 6, p. 2-91]

[Th. 54 is the 0-quotient theorem.]

55. $\forall_x \forall_y$ if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$ [p. 2-91]

56. $\forall_x \forall_y$ if $xy = 0$ then $x = 0$ or $y = 0$ [p. 2-91]

[Th. 56 is the 0-product theorem.]

57. $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv}$ [p. 2-92]

[Th. 57 is the "addition of fractions" theorem.]

58. $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} - \frac{u}{v} = \frac{xv - uy}{yv}$ [p. 2-92]

[Th. 58 is the "subtraction of fractions" theorem.]

35. $\forall_x\forall_y\forall_z\ x - (y + z) = x - y - z$ [Part A, Ex. 4, p. 2-73]

36. $\forall_x\forall_y\forall_z\ x - (y - z) = x - y + z$ [Part A, Ex. 5, p. 2-73]

37. $\forall_x\forall_y\forall_z\ x + (y - z) = x - z + y$ [Part A, Ex. 6, p. 2-73]

38. $\forall_x\forall_y\forall_z\ x(y - z) = xy - xz$ [Part A, Ex. 7, p. 2-74]

[Th. 38 is the left distributive theorem for multiplication over subtraction (ℓdtms).]

39. $\forall_x\forall_y\forall_z\ (x - y)z = xz - yz$ [Part A, Ex. 8, p. 2-74]

[Th. 39 is the distributive theorem for multiplication over subtraction (dtms).]

40. $\forall_x\forall_y\forall_z\ x - (-y - z) = x + y + z$ [Part A, Ex. 9, p. 2-74]

41. $\forall_x\forall_y\forall_z\forall_u\ x - (y - z - u) = x - y + z + u$ [Part A, Ex. 10, p. 2-74]

42. $\forall_x\ 0 - x = -x$ [Part B, Ex. 1, p. 2-75]

[Th. 42 is the "subtracting from 0 is oppositing" theorem.]

43. $\forall_x\ x - 0 = x$ [Part B, Ex. 2, p. 2-75]

[Th. 43 is the "subtracting 0" theorem.]

44. $\forall_x\forall_y\forall_z\ x + z - (y + z) = x - y$ [Part B, Ex. 3, p. 2-75]

45. $\forall_x\forall_y\forall_z\ x - z - (y - z) = x - y$ [Part B, Ex. 4, p. 2-75]

46. $\forall_a\forall_b\forall_c\forall_d\ (a - b) + (c - d) = (a + c) - (b + d)$ [Part B, Ex. 5, p. 2-75]

47. $\forall_x\forall_y\forall_z$ if $z + y = x$ then $z = x - y$ [p. 2-89]

[Th. 47 is the subtraction theorem.]

24. $\forall_x \forall_y\ -xy = x(-y)$ [Part B, Ex. 1, p. 2-70]

25. $\forall_x \forall_y \forall_z\ -x(y + z) = -(xy) + -(xz)$ [Part B, Ex. 2, p. 2-70]

26. $\forall_x \forall_y \forall_z\ -x(-y + -z) = xy + xz$ [Part B, Ex. 3, p. 2-70]

27. $\forall_x\ x \cdot -1 = -x$ [Part B, Ex. 4, p. 2-70]

[Th. 27 is the theorem for multiplying by −1.]

28. $\forall_x\ -x = -1 \cdot x$ [Part C, p. 2-70]

[Th. 28 is the −1 times theorem.]

29. $\forall_x \forall_y\ (x + y) + -y = x$ [p. 2-71]

[Th. 29 is the "adding the opposite is the inverse of adding" theorem.]

30. $\forall_x \forall_y\ (x + y) - y = x$ [p. 2-71]

[Th. 30 is the "subtraction is the inverse of addition" theorem.]

31. $\forall_x \forall_y \forall_z\ x - yz = x + -yz$ [Part A, Sample, p. 2-71]

32. $\forall_x \forall_y\ x - y + y = x$ [Part A, Ex. 1, p. 2-72]

[Th. 32 is the "addition is the inverse of subtraction" theorem.]

33. $\forall_x \forall_y\ -(x - y) = y - x$ [Part A, Ex. 2, p. 2-72]

[Th. 33 is the distributive theorem for opposition over subtraction.]

34. $\forall_x \forall_y \forall_z\ x + (y - z) = x + y - z$ [Part A, Ex. 3, p. 2-73]

11. $\forall_x \forall_y \forall_z$ if $x = y$ then $xz = yz$ [Ex. 4, p. 2-66]

[Th. 11 is the uniqueness principle for multiplication.]

12. $\forall_x \forall_y \forall_z$ if $x = y$ then $zx = zy$ [Ex. 5, p. 2-66]

[Th. 12 is the left uniqueness principle for multiplication.]

13. $\forall_u \forall_v \forall_x \forall_y$ if $u = v$ and $x = y$ then $u + x = v + y$ [Ex. 6, p. 2-66]

14. $\forall_u \forall_v \forall_x \forall_y$ if $u = v$ and $u + x = v + y$ then $x = y$ [Ex. 7, p. 2-66]

15. $\forall_x$ $x \cdot 0 = 0$ [Ex. 8, p. 2-66]

[Th. 15 is the pm0.]

16. $\forall_x \forall_y$ if $x + y = 0$ then $-x = y$ [p. 2-68]

[Th. 16 is the 0-sum theorem.]

17. $\forall_x$ $- -x = x$ [Part A, Sample, p. 2-69]

18. $\forall_x \forall_y$ $-(x + y) = -x + -y$ [Part A, Ex. 1, p. 2-69]

[Th. 18 is the distributive theorem for opposition over addition.]

19. $\forall_x \forall_y$ $-(x + -y) = y + -x$ [Part A, Ex. 2, p. 2-69]

20. $\forall_x \forall_y$ $-(xy) = x \cdot -y$ [Part A, Ex. 3, p. 2-69]

21. $\forall_x \forall_y$ $-(xy) = -xy$ [Part A, Ex. 4, p. 2-69]

[Note that '$-xy$' is an abbreviation for '$(-x)y$' and for '$-x \cdot y$'.]

22. $\forall_x \forall_y$ if $x = -y$ then $-x = y$ [Part A, Ex. 5, p. 2-69]

23. $\forall_x \forall_y$ $-x \cdot -y = xy$ [Part B, Sample 2, p. 2-70]

1. $\forall_x \forall_y \forall_z\ x(y + z) = xy + xz$ [p. 2-60]

   [Th. 1 is the left distributive principle for multipication over addition.]

2. $\forall_x\ 1 \cdot x = x$ [Ex. 1, p. 2-61]

   [Th. 2 is the 1 times theorem.]

3. $\forall_x \forall_a \forall_b \forall_c\ ax + bx + cx = (a + b + c)x$ [Ex. 2, p. 2-61]

   [Th. 3 is the extended distributive theorem.]

4. $\forall_x \forall_y \forall_a \forall_b\ (ax)(by) = (ab)(xy)$ [Ex. 3, p. 2-61]

   [Th. 4 is the product rearrangement theorem.]

5. $\forall_x \forall_y \forall_a \forall_b\ (a + x) + (b + y) = (a + b) + (x + y)$ [Ex. 4, p. 2-61]

   [Th. 5 is the sum rearrangement theorem.]

6. $\forall_x \forall_y \forall_z$ if $x = y$ then $x + z = y + z$ [p. 2-64]

   [Th. 6 is the uniqueness principle for addition.]

7. $\forall_x \forall_y \forall_z$ if $x + z = y + z$ then $x = y$ [p. 2-65]

   [Th. 7 is the cancellation principle for addition.]

8. $\forall_x \forall_y \forall_z$ if $x = y$ then $z + x = z + y$ [Ex. 1, p. 2-66]

   [Th. 8 is the left uniqueness principle for addition.]

9. $\forall_x \forall_y \forall_z$ if $z + x = z + y$ then $x = y$ [Ex. 2, p. 2-66]

   [Th. 9 is the left cancellation principle for addition.]

10. $\forall_x \forall_y$ if $x = y$ then $-x = -y$ [Ex. 3, p. 2-66]

    [Th. 10 is the uniqueness principle for opposition.]

[The theorems in Exercises 2, 3, and 4 are not too important in themselves. They serve merely as samples of the kind of "sweeping" theorem a student uses as a basis for short cuts in simplification. It is probably better to have students justify a short cut by saying 'commutative, associative, and distributive principles' instead of by referring to special theorems like those in Exercises 2, 3, and 4. However, you should occasionally challenge such answers to make sure they are not being used as substitutes for 'I don't know'.]

*

For your convenience [in using the COMMENTARY] we have prepared a list of the more important theorems proved in Unit 2. This list contains theorems already proved and theorems which will be proved later in the unit. [We do not expect that each student will have proved each theorem!] The theorems are numbered consecutively in the list, and [in the COMMENTARY] we shall refer to them by number. You may want your students to compile a list like this, but it would be silly and wasteful of time if they were required to memorize the theorems [especially by number]. It will probably be more helpful if they assign names to the theorems [as we have done in many cases], and either refer to the theorems by these names, or write them out in full when they use them in proofs.

Answers for Exercises on page 2-61.

1. Prove: $\forall_x \; 1 \cdot x = x$. [Theorem 2. See TC[2-61]c.]

$$\begin{aligned} & 1 \cdot x \\ &= x \cdot 1 && \text{cpm} \\ &= x. && \text{pm1} \end{aligned}$$

Hence, $1 \cdot x = x$.

2. Prove: $\forall_x \forall_a \forall_b \forall_c \; ax + bx + cx = (a + b + c)x$. [Theorem 3]

$$\begin{aligned} & ax + bx + cx \\ &= (a + b)x + cx && \text{dpma} \\ &= (a + b + c)x. && \text{dpma} \end{aligned}$$

Hence, $ax + bx + cx = (a + b + c)x$.

3. Prove: $\forall_x \forall_y \forall_a \forall_b \; (ax)(by) = (ab)(xy)$. [Theorem 4]

$$\begin{aligned} & (ax)(by) \\ &= axby && \text{apm} \\ &= a(xb)y && \text{apm} \\ &= a(bx)y && \text{cpm} \\ &= abxy && \text{apm} \\ &= (ab)(xy). && \text{apm} \end{aligned}$$

Hence, $(ax)(by) = (ab)(xy)$.

4. Prove: $\forall_x \forall_y \forall_a \forall_b \; (a + x) + (b + y) = (a + b) + (x + y)$. [Theorem 5]

$$\begin{aligned} & (a + x) + (b + y) \\ &= a + x + b + y && \text{apa} \\ &= a + (x + b) + y && \text{apa} \\ &= a + (b + x) + y && \text{cpa} \\ &= a + b + x + y && \text{apa} \\ &= (a + b) + (x + y). && \text{apa} \end{aligned}$$

Hence, $(a + x) + (b + y) = (a + b) + (x + y)$.

For definiteness and ease of reference we state <u>the basic principles</u> which we shall use in this unit. [The last basic principle will be discussed later, but we state it here for completeness.] We also introduce an abbreviation for 'for each'. It is '$\forall$'.

<u>Commutative principles</u>

$\forall_x \forall_y \quad x + y = y + x.$

$\forall_x \forall_y \quad xy = yx.$

<u>Associative principles</u>

$\forall_x \forall_y \forall_z \quad x + y + z = x + (y + z).$

$\forall_x \forall_y \forall_z \quad xyz = x(yz).$

<u>Distributive principle</u>

$\forall_x \forall_y \forall_z \quad (x + y)z = xz + yz.$

<u>Principles for 0 and 1</u>

$\forall_x \quad x + 0 = x.$

$\forall_x \quad x \cdot 1 = x.$

<u>Principle of Opposites</u>

$\forall_x \quad x + -x = 0.$

<u>Principle for Subtraction</u>

$\forall_x \forall_y \quad x - y = x + -y.$

<u>Principle of Quotients</u>

$\forall_x \forall_{y \neq 0} \quad (x \div y) \cdot y = x.$

[Notice that we have not included the principle for multiplying by 0. This is because we shall later derive it from the principles just stated.]

EXERCISES

Prove the following theorems.

1. $\forall_x \quad 1 \cdot x = x.$ ["The 1 times theorem"]

2. $\forall_x \forall_a \forall_b \forall_c \quad ax + bx + cx = (a + b + c)x.$ ["Extended distributive theorem

3. $\forall_x \forall_y \forall_a \forall_b \quad (ax)(by) = (ab)(xy).$ ["Product rearrangement theorem"]

4. $\forall_x \forall_y \forall_a \forall_b \quad (a + x) + (b + y) = (a + b) + (x + y).$ ["Sum rearrangement theorem"]

Up to now in Unit 2 you have learned about pronumerals and how they can be used in stating generalizations about real numbers. You have seen how to derive theorems from the basic principles, and how to use the basic principles and the theorems in finding short cuts for simplifying expressions. These short cuts apply to problems involving addition and multiplication. Now we want to work toward short cuts dealing with opposition, subtraction, and division. We shall prove theorems like:

$$\forall_x \forall_y \ x - y = -(y - x),$$

and use them to simplify expressions like:

$$3b - (a - b).$$

So, your two main purposes for the rest of this unit are to prove theorems which will help you develop more short cuts in simplifying expressions, and to gain skill in applying these short cuts. The skills you acquire in simplifying expressions will be used throughout the rest of your work in mathematics.

Before beginning this section you may find it helpful to reread the COMMENTARY for page 1-80, for you are now beginning the "more adequate treatment" mentioned there.

*

Answers for Part A.

1. $-{}^{+}3$ [or: ${}^{-}3$]	2. $-{}^{-}7$ [or: ${}^{+}7$]	3. $--4$ [or: 4]
4. $-+8.2$ [or: -8.2]	5. $---3$ [or: ${}^{-}3$]	6. -0 [or: 0]

*

In doing the exercises of Part B, it is imperative that students read the expression '$-x$' as 'the opposite of x' rather than as 'negative x'. [The description of the procedure for finding the opposite of a real number given on TC[2-29, 30] may be helpful in giving justifications if any are needed.]

*

Answers for Part B.

1. F	2. F	3. T	4. T
5. T	6. T	7. T	8. T
9. T			

*

Answers for Part C.

In each of Exercises 1, 3, 4, 5, 6, 7, and 8 the blank should be filled with an oppositing sign.

2.06 Oppositing and subtracting.--We saw in Unit 1 that in order to define the operation subtraction it was convenient to have the operation oppositing [and we introduced the minus sign as a name for this operation]. Oppositing is such that

$$(*) \qquad \text{For each } x, \quad x + -x = 0.$$

We call (*) the principle of opposites. [Read '$x + -x$' as 'x plus the opposite of x' and not as 'x plus negative x'.]

EXERCISES

A. Give the opposite of each number listed.

1. $^+3$ 2. $^-7$ 3. -4 4. $+8.2$ 5. $- -3$ 6. 0

B. True or false?

1. For each x, $+x$ is a positive number.
2. For each x, $-x$ is a negative number.
3. For each x, if x is negative then $-x$ is positive.
4. For each x, if x is positive then $-x$ is negative.
5. For each x, if x is 0 then $-x$ is 0.
6. For each x, if $-x$ is positive then x is negative.
7. For each x, if $-x$ is negative then x is positive.
8. For each x, if $+x$ is positive then x is positive.
9. For each x, if $+x$ is negative then x is negative.

C. Fill the blank with an oppositing sign to make the sentence true unless the sentence is already true.

1. $^+534 + \underline{\qquad} {}^+534 = 0$
2. $- {}^+534 + \underline{\qquad} {}^+534 = 0$
3. $-{}^-721 + \underline{\qquad} -{}^-721 = 0$
4. $^-721 + \underline{\qquad} {}^-721 = 0$
5. $^+5 \cdot -{}^+2 = \underline{\qquad}({}^+5 \cdot {}^+2)$
6. $^+5 \cdot -{}^-2 = \underline{\qquad}({}^+5 \cdot {}^-2)$
7. $-{}^+2 \cdot {}^+7 = \underline{\qquad}({}^+2 \cdot {}^+7)$
8. $-{}^-2 \cdot {}^+7 = \underline{\qquad}({}^-2 \cdot {}^+7)$
9. $-{}^+3 \cdot -{}^+4 = \underline{\qquad}({}^+3 \cdot {}^+4)$
10. $-{}^-3 \cdot -{}^+4 = \underline{\qquad}({}^-3 \cdot {}^+4)$
11. $-{}^+3 \cdot -0 = \underline{\qquad}({}^+3 \cdot 0)$
12. $-{}^-3 \cdot 0 = \underline{\qquad}({}^-3 \cdot 0)$

ADDITION PRINCIPLES

Suppose Rita and Rhoda each picks a real number. Then, Aaron picks a real number and tells both Rita and Rhoda to add it to hers. If Rita and Rhoda pick the same number [and both add correctly], do they get the same sum?

Suppose Rita and Rhoda each picks a real number. Then, Aaron picks a real number and tells both Rita and Rhoda to add it to hers. If they both get the same sum [and both add correctly], did they pick the same number?

These situations illustrate two important properties of addition. Let's state the principle which expresses the property illustrated in the first situation. For each number Rita picks, for each number Rhoda picks, and for each number Aaron picks, if Rita's number is the same as Rhoda's number then Rita's number plus Aaron's number is the same as Rhoda's number plus Aaron's number. For short:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x + z = y + z.$$

For example, this tells us that, since $8 = 4 \times 2$, $8 + 7 = 4 \times 2 + 7$. We accept this reasoning because $8 + 7 = 8 + 7$ and, supposing that $8 = 4 \times 2$, '4×2' is another name for 8. So, substituting '4×2' for the second '8', we see that $8 + 7 = 4 \times 2 + 7$.

Here is a test-pattern for this generalization.

Suppose that $x = y$.

Since $x + z = x + z$, [$\forall_a\ a = a$.]

it follows that $x + z = y + z$.

Hence, if $x = y$ then $x + z = y + z$.

The generalization:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x + z = y + z$$

is called <u>the uniqueness principle for addition</u>. ["If you add a number to a number you get a <u>unique</u> sum".]

The uniqueness principle for addition [Theorem 6] and the cancellation principle for addition [Theorem 7] make up what, in Unit 3, we call the addition transformation principle. They could be used [at an appropriate time; not now] together with the po and the pa0 to justify the transposition-short cut used in solving equations. [The uniqueness and cancellation principles might just as well be called uniqueness and cancellation theorems.]

Students who have difficulty distinguishing between the two principles may be helped by asking them to consider pairs of sentences such as:

> If John is well then he is in school today.
> If John is in school today then he is well.

and:

> If two angles are right angles then they have the same number of degrees.
> If two angles have the same number of degrees then they are right angles.

*

As pointed out at the top of page 2-65, the uniqueness principle for addition is a logical [as opposed to a mathematical] principle in the sense that it can be proved without making use of any of our basic principles for real numbers. The proof given in the text makes use only of rules and principles of logic. In case you have found the "tree-form" of proof used in the COMMENTARY for pages 1-56, 2-31, and 2-32 helpful, here is such a proof of the uniqueness principle.

$$\dfrac{\dfrac{\overset{*}{x = y} \qquad \dfrac{\forall_x\, x = x}{x + z = x + z}}{x + z = y + z}}{\text{if } x = y \text{ then } x + z = y + z}\,{}^{*}$$

The first inference is of a kind mentioned in the COMMENTARY for pages 2-31 and 2-32--an example of inferring an instance of a universal generalization from this generalization. The second inference is an example of the substitution rule. The final inference is of a new kind, called conditionalizing--from any sentence ['x + z = y + z'] one

can infer any conditional sentence which has the former as its consequent. [For example, from 'the moon is made of green cheese' one may infer 'if the moon is a satellite then the moon is made of green cheese'.] The asterisks indicate the applicability of another rule of logic according to which if [as is here the case] the antecedent of the conditional ['$x = y$'] is one of the premisses of the argument, then the conditional is a consequence of the remaining premisses [in this case, of the premiss '$\forall_x\ x = x$']. So, the proof shows that the sentence 'if $x = y$ then $x + z = y + z$' is a consequence of the logical principle '$\forall_x\ x = x$'. Hence, the same is true of the statement:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x + z = y + z.$$

You may wish at this time to introduce your students to the substitution rule of logic [See TC[1-56]a, b.]. If so, the following remarks may prove helpful.

Suppose we have the premisses

'$8 = 5 + 3$' and '$9 + 8 > 12$'.

Then we infer from these premisses the conclusion

'$9 + (5 + 3) > 12$'.

The inference we have just made [''substitution''] is acceptable because we have decided to use '=' to refer to identity. So, in assuming the premiss '$8 = 5 + 3$', we are assuming that '8' and '$5 + 3$' are names for the same number, and in view of this, the conclusion says the same thing as the second premiss. We can show the pattern of this inference [that is, the derivation of the conclusion from the two premisses] by using pronumerals:

$$\frac{a = b \quad c + a > d}{c + b > d}$$

We shall say that '$c + b > d$' is a consequence of the premisses '$a = b$' and '$c + a > d$' because statements generated from the open sentences are related in this premisses-conclusion way. [We cannot justify the inference of the open sentence conclusion from the open sentence premisses as we did in the case of the statements. Since 'a' and 'b' are pronumerals, it makes no sense to say that they are names for the same number.] Note that while the first premiss must be an equation, the second premiss [and the conclusion] need not be.

Thus, in summary:

> the first inference, (1), is an example of inferring from a universal generalization [in this case, the uniqueness principle for addition] one of its instances;
>
> the second inference is [as mentioned above] an example of modus ponens;
>
> the third inference is of the same kind as (1) [in this case, the premiss being the apa];
>
> the fourth inference is an example of the substitution rule;
>
> the fifth, seventh, ninth, and eleventh inferences are like (1) [the universal generalizations which are the premisses of these inferences are, respectively, the apa, the po, the pa0, and the pa0];
>
> the sixth, eighth, tenth, and twelfth inferences are examples of the substitution rule;
>
> the thirteenth [and final] inference is an example of conditionalizing.

Hence, we see that the cancellation principle for addition is a consequence of the logical uniqueness of addition principle [which is itself a consequence of '$\forall_x\ x = x$'], and of certain mathematical principles, the apa, the po, and the pa0. Since we do not usually cite logical principles in proofs, we may say that the cancellation principle for addition is a consequence of the apa, the po, and the pa0.

The proof of the cancellation principle for addition is similar in structure to that of the uniqueness principle. Lack of space precludes exhibiting a tree-form proof, but a slightly abbreviated one can be obtained by tacking together the following pieces.

$$(2)\ \dfrac{x + z \overset{*}{=} y + z \qquad (1)\ \dfrac{\text{uniqueness principle for addition}}{\text{if } x + z = y + z \text{ then } x + z + -z = y + z + -z}}{x + z + -z = y + z + -z}$$

$$\begin{matrix}(3)\\(4)\end{matrix}\ \dfrac{\dfrac{\text{apa}}{x + z + -z = x + (z + -z)} \qquad x + z + -z = y + z + -z}{x + (z + -z) = y + z + -z}$$

$$\begin{matrix}(5)\\(6)\end{matrix}\ \dfrac{\dfrac{\text{apa}}{y + z + -z = y + (z + -z)} \qquad x + (z + -z) = y + z + -z}{x + (z + -z) = y + (z + -z)}$$

$$\begin{matrix}(7)\\(8)\end{matrix}\ \dfrac{\dfrac{\text{po}}{z + -z = 0} \qquad x + (z + -z) = y + (z + -z)}{x + 0 = y + 0}$$

$$(13)\ \dfrac{\begin{matrix}(11)\\(12)\end{matrix}\ \dfrac{\dfrac{\text{pa0}}{y + 0 = y} \qquad \begin{matrix}(9)\\(10)\end{matrix}\ \dfrac{\dfrac{\text{pa0}}{x + 0 = x} \qquad x + 0 = y + 0}{x = y + 0}}{x = y}}{\text{if } x + z = y + z \text{ then } x = y}\ *$$

The kinds of inference are those used before, with the exception of the last inference in the first block. This is an example of *modus ponens*--from a conditional sentence together with its antecedent, one may infer the consequent of the conditional.

After students have read Rita's explanation, you should have them consider examples like this.

> Suppose Aaron had chosen 6 as his number, and Rita told him that the sum, when she added Aaron's number to hers, was 13. How could Aaron use addition to find out what number Rita had picked?

Take several examples of this kind, and then have the class pursue the following.

> Suppose Rita picked 73 and Aaron picked 59; without doing any computing, tell what expression represents Rita's sum.
>
> Write '73 + 59' on the board, and ask what you should add to 73 + 59 to get back 73, the number Rita picked. The students will tell you [we think] that you should add the opposite of 59. So, write a '+ −59' to the right of '73 + 59'.
>
> Then you should go through this sequence.

$$
\begin{array}{ll}
\ 73 + 59 + -59 & \\
 & \Big\}\ \text{apa} \\
= 73 + (59 + -59) & \\
 & \Big\}\ \text{po} \\
= 73 + \qquad 0 & \\
 & \Big\}\ \text{pa0} \\
= 73. &
\end{array}
$$

If necessary, take a few more examples like this. The students should then be ready for the proof on page 2-65.

[Notice that in the proof of the uniqueness principle we did not use any of the basic principles or theorems. We did cite a <u>principle of logic</u>--"a thing is equal to itself". So, the uniqueness principle for addition is itself a theorem of logic.]

Let's turn now to the second situation. Rita, Rhoda, and Aaron each picked a number. Rita and Rhoda added Aaron's number to their numbers and got the same sum. When they told Aaron this, he said, "You both chose the same number." How did he know?

Rita figured it out this way:

> "When I told him <u>my</u> sum, he knew that he could get my number from it by adding the opposite of his number. And, he knew he could get your number, Rhoda, by adding the opposite of his number to <u>your</u> sum. And, since we had the same sum, he knew he would come out with the same number both times [uniqueness principle]."

Here is a statement of the principle Aaron used:

$$\forall_x \forall_y \forall_z \text{ if } x + z = y + z \text{ then } x = y.$$

It is called <u>the cancellation principle for addition</u>.

Rita's explanation can be translated into a proof.

Suppose that	$x + z = y + z$.	
Then	$x + z + -z = y + z + -z$,	[uniqueness principle]
	$x + (z + -z) = y + (z + -z)$,	[apa, apa]
	$x + 0 = y + 0$,	[po, po]
and	$x = y$.	[pa0, pa0]
Hence,	if $x + z = y + z$ then $x = y$.	

Principles and theorems of logic are not usually cited in proofs. We cited the uniqueness principle in this last proof for the sake of clarity. Thus, the cancellation principle is a consequence just of the apa, the po, and the pa0.

Notice in both these proofs that the last sentence is a <u>conditional sentence</u>, that is, a sentence of the form: if... then___. Notice also that the first line in each proof is a supposition of the "if-part" and the next-to-last line is the "then-part". So, the proof consists in using the if-part together with principles or theorems to derive the then-part.

EXERCISES

Prove these theorems.

1. $\forall_x \forall_y \forall_z$ if $x = y$ then $z + x = z + y$. ["Left uniqueness principle for addition"]

2. $\forall_x \forall_y \forall_z$ if $z + x = z + y$ then $x = y$. ["Left cancellation principle for addition"]

3. $\forall_x \forall_y$ if $x = y$ then $-x = -y$. ["Uniqueness principle for opposition"]

4. $\forall_x \forall_y \forall_z$ if $x = y$ then $xz = yz$. ["Uniqueness principle for multiplication"]

5. $\forall_x \forall_y \forall_z$ if $x = y$ then $zx = zy$. ["Left uniqueness principle for multiplication"]

☆6. $\forall_u \forall_v \forall_x \forall_y$ if $u = v$ and $x = y$ then $u + x = v + y$.

☆7. $\forall_u \forall_v \forall_x \forall_y$ if $u = v$ and $u + x = v + y$ then $x = y$.

*

Let us now prove the principle for multiplying by 0. What do our basic principles tell us about 0?

$x + 0 = x$ [pa0]

Can we get an expression containing an '$x \cdot 0$' out of this?

$x(x + 0) = xx$ [left uniqueness principle for multiplication]

$xx + x \cdot 0 = xx$ [ℓdpma]

Now we have an '$x \cdot 0$' on one side. Can we get a '0' on the other?

$xx + x \cdot 0 = xx + 0$ [pa0]

$x \cdot 0 = 0$ [left cancellation principle for addition]

So, the principle for multiplying by 0 is a consequence of the principle for adding 0, the left distributive principle for multiplication over addition, and the left cancellation principle for addition. Earlier, we claimed that we could derive the principle for multiplying by 0 from the basic principles stated on page 2-61. This can be done because the ℓdpma and the left cancellation principle can be derived from our basic principles.

*

☆8. Derive the pm0 directly from the basic principles.

Answers for Exercises.

1. Prove: $\forall_x \forall_y \forall_z$ if $x = y$ then $z + x = z + y$. [Theorem 8]

Suppose that	$x = y.$
Since	$z + x = z + x,$
it follows that	$z + x = z + y.$

Hence, if $x = y$ then $z + x = z + y$.

2. Prove: $\forall_x \forall_y \forall_z$ if $z + x = z + y$ then $x = y$. [Theorem 9]

Suppose that	$z + x = z + y.$	
Then	$z + x + -z = z + y + -z,$	[uniqueness principle]
	$x + z + -z = y + z + -z,$	[cpa]
	$x + (z + -z) = y + (z + -z),$	[apa]
	$x + 0 = y + 0,$	[po]
and	$x = y.$	[pa0]

Hence, if $z + x = z + y$ then $x = y$.

[Notice that again, for the sake of clarity, we cite a uniqueness principle.] [There is another proof using the cpa and the cancellation principle proved on page 2-65.]

3. Prove: $\forall_x \forall_y$ if $x = y$ then $-x = -y$. [Theorem 10]

Suppose that	$x = y.$
Since	$-x = -x,$
it follows that	$-x = -y.$

Hence, if $x = y$ then $-x = -y$.

4. Prove: $\forall_x \forall_y \forall_z$ if $x = y$ then $xz = yz$. [Theorem 11]

Suppose that	$x = y.$
Since	$xz = xz,$
it follows that	$xz = yz.$

Hence, if $x = y$ then $xz = yz$.

[The cancellation principle for multiplication is proved in the Sample of the Exercises on page 2-90. The proof uses the principle of quotients.]

5. [Similar to Exercise 4.] [Theorem 12]

☆6. Prove: $\forall_u \forall_v \forall_x \forall_y$ if $u = v$ and $x = y$ then $u + x = v + y$. [Theorem 13]

Suppose that	$u = v$ and $x = y$.
Since	$u + x = u + x$,
it follows that	$u + x = v + y$.

Hence, if $u = v$ and $x = y$ then $u + x = v + y$.

[The "double substitution" carried out here can be justified by the earlier substitution rule together with a new kind of inference for dealing with conjunctions. Here is a tree-form proof for the theorem of Exercise 6.

$$\dfrac{\dfrac{\dfrac{\overset{*}{u = v \text{ and } x = y}}{x = y} \qquad \dfrac{\dfrac{\overset{*}{u = v \text{ and } x = y}}{u = v} \qquad \dfrac{\forall_x \; x = x}{u + x = u + x}}{u + x = v + x}}{u + x = v + y}}{\text{if } u = v \text{ and } x = y \text{ then } u + x = v + y}\,{}^{*}$$
]

☆7. Prove: $\forall_u \forall_v \forall_x \forall_y$ if $u = v$ and $u + x = v + y$ then $x = y$. [Theorem 14]

Suppose that $u = v$ and $u + x = v + y$.

Then	$u + x + -u = v + y + -u$,	
and	$-u = -v$.	
So,	$u + x + -u = v + y + -v$,	
	$x + u + -u = y + v + -v$,	[cpa]
	$x + (u + -u) = y + (v + -v)$,	[apa]
	$x + 0 = y + 0$,	[po]
and	$x = y$.	[pa0]

Hence, if $u = v$ and $u + x = v + y$ then $x = y$.

[The theorems in Exercises 6 and 7 are those which justify the "addition method" of solving systems of equations.]

☆8. Prove: $\forall_x\ x0 = 0$. [Theorem 15]

$$
\begin{array}{rcll}
x + 0 &=& x & [pa0] \\
x(x + 0) &=& xx & [uniqueness] \\
(x + 0)x &=& xx & [cpm] \\
xx + 0x &=& xx & [dpma] \\
xx + x0 &=& xx & [cpm] \\
xx + x0 &=& xx + 0 & [pa0] \\
xx + x0 + -(xx) &=& xx + 0 + -(xx) & [uniqueness] \\
x0 + xx + -(xx) &=& 0 + xx + -(xx) & [cpa] \\
x0 + [xx + -(xx)] &=& 0 + [xx + -(xx)] & [apa] \\
x0 + 0 &=& 0 + 0 & [po] \\
x0 &=& 0 & [pa0]
\end{array}
$$

So, the principle for multiplying by 0 is a consequence of the pa0, the dpma, the cpm, the cpa, the apa, and the po. Note that we do not include the two uniqueness principles cited in the proof in this list because they are logical principles. If we wanted to include <u>all</u> principles needed in establishing the principle for multiplying by $\overline{0}$ it would, as the proofs of the uniqueness principles show, be sufficient to add '$\forall_x\ x = x$'.

The above proof for the pm0 shows how such a proof as that given on page 2-66 can be expanded to show directly that its conclusion follows from basic principles. Just replace each theorem used in the proof by a derivation of that theorem from basic principles. When one does this, it is often possible to shorten the resulting proof. For example, in the expanded proof above, the second line can be deleted if in the third line one replaces '[cpm]' by '[uniqueness]'. Also, the sixth line may be deleted if one also deletes the '+ 0' from the seventh line, and a '0 +' from each of the succeeding lines.

The shorter proof given on page 2-66 has the advantage that it shows that the full strength of the po is not needed in order to derive the pm0. So, for example, a similar proof would show that the pm0 holds for numbers of arithmetic although there is for these no po.

Another way, to clarify what the 0-sum theorem tells one, is to do the following. Ask the class to suppose that two students have each picked a number. Suppose [point to students as you say this] that Mary adds her number to John's and finds that the sum is 0. What conclusion follows? The class should conclude that the opposite of John's number is Mary's number. And, this conclusion is justified by the 0-sum theorem.

Suppose the sum is not 0. What can we conclude? The conclusion is that the opposite of John's number is not Mary's number. But, this conclusion is not justified by the 0-sum theorem ['if $x + y \neq 0$ then $-x \neq y$' is the converse of the contrapositive of 'if $x + y = 0$ then $-x = y$'.] It is justified by the principle of opposites. The principle of opposites could have been stated:

(1) $\forall_x \forall_y$ if $-x = y$ then $x + y = 0$.

And, this is logically equivalent to:

(2) $\forall_x \forall_y$ if $x + y \neq 0$ then $-x \neq y$.

In case students have trouble seeing the logical equivalence of (1) and (2), you might find the following device helpful. Consider all possible pairs of real numbers. The principle of opposites tells us that for each pair for which the opposite of the first number is the second, the first number plus the second is 0. Now, suppose we locate a pair for which the first number plus the second is not 0. This can't be a pair for which the opposite of the first number is the second [because, by (1), if it were, the sum would be 0]. But, this is precisely what (2) tells us.

[On the other hand, the 0-sum theorem tells us that if we locate a pair whose sum is 0 then this is a pair for which the opposite of the first number is the second.]

[You can also contrast converses with contrapositives by using the example of the set of pairs of all angles. For those pairs which are pairs of right angles, it is the case that the angles have the same number of degrees. Contrapositively, for those pairs for which the angles do not have the same number of degrees, you can conclude that they are not pairs of right angles. But, conversely, if we locate a pair for which the angles have the same number of degrees, it is not necessarily the case that this is a pair of right angles.]

*

The reference in the next-to-bottom line on page 2-68 should, strictly speaking, be to the 0-sum theorem, rather than to (*). [And, although your students will probably not raise this point, it was (*), rather than the 0-sum theorem, which we used in Unit 1.]

*

It is important that students continue to be aware that proofs such as the ones just discussed <u>are</u> proofs <u>because</u> they furnish patterns for testing instances of generalizations. When taken literally, such a testing pattern as the one for the 0-sum theorem is meaningless, in the same sense as an open sentence is meaningless. However, if one substitutes numerals for 'x' and 'y', he obtains a paragraph which should convince a reader that the corresponding instance of the 0-sum theorem is a consequence of the principle of opposites and the left cancellation principle for addition. For example, consider the following paragraph.

> Suppose that $^+5 + {}^-5 = 0$. Now, [by the principle of opposites] $^+5 + -{}^+5 = 0$. So, $^+5 + -{}^+5 = {}^+5 + {}^-5$. Therefore, [by the left cancellation principle for addition] $-{}^+5 = {}^-5$. Hence, if $^+5 + {}^-5 = 0$ then $-{}^+5 = {}^-5$.

*

Before moving to the theorems of Part A on page 2-69, be sure students understand what the 0-sum theorem tells them. As noted above, it tells them that if $^+5 + {}^-5 = 0$ then $-{}^+5 = {}^-5$. [So, because $^+5 + {}^-5 = 0$, $-{}^+5 = {}^-5$]. You might ask at this time how to prove that the opposite of the opposite of $^+5$ is $^+5$, that is, how to prove that $- -{}^+5 = {}^+5$. The 0-sum theorem does this very neatly. To show that the opposite of $-{}^+5$ is $^+5$, just show that $-{}^+5 + {}^+5 = 0$. By the cpa, $-{}^+5 + {}^+5 = {}^+5 + -{}^+5$. And, by the po, $^+5 + -{}^+5 = 0$. Thus, $-{}^+5 + {}^+5 = 0$. So, by the 0-sum theorem, $- -{}^+5 = {}^+5$. This should prepare them for the proof of the Sample on page 2-69 [Theorem 17].

Another simple consequence of the 0-sum theorem, which you may want to ask students for, is '$-0 = 0$'. The proof is very simple:

$$\left.\begin{aligned} &0 + 0 \\ &= 0. \end{aligned}\right\} \text{pa0}$$

Hence, $-0 = 0$. [0-sum theorem]

[This proof is also given in the answer for Exercise 2 on TC[2-75]a.]

Throughout the subsection beginning on page 2-67 be especially careful to read '$-$' as 'the opposite of'. Of course, you won't read it as 'negative'; but if you read it as 'minus', some students translate this to 'negative'!

*

After studying the proofs on pages 2-64 and 2-65, proving the theorems on page 2-66, and reading the proof on page 2-67, students should have no difficulty in fulfilling the request at the foot of page 2-67. Of course, all they have to do is copy the test-pattern given on this page, replacing each '$^-485$' by a '9832'. The proof of (*) at the top of page 2-68 is equally simple. Here it is.

Suppose that	$x + y = 0.$	
Now,	$x + -x = 0.$	[po]
So,	$x + y = x + -x.$	
Therefore,	$y = -x.$	[left cancellation]

Hence, if $x + y = 0$ then $y = -x$.

The 0-sum theorem [Theorem 16 which is displayed near the middle of page 2-68] is proved in the same way. Just replace the last three lines of the proof given above by:

So,	$x + -x = x + y.$	
Therefore,	$-x = y.$	[left cancellation]

Hence, if $x + y = 0$ then $-x = y$.

*

In both proofs we have made implicit use of the symmetry of equality. In the first proof, in order to apply the substitution rule to obtain the third line we need '$0 = x + -x$', rather than '$x + -x = 0$'; and in the second proof we need '$0 = x + y$', rather than '$x + y = 0$'. In the former, the additional step would be:

$$\frac{x + -x = 0 \qquad \dfrac{\forall_x\, x = x}{x + -x = x + -x}}{0 = x + -x}$$

In the latter proof, the additional step is similar to this, but with '$x + y$' in place of '$x + -x$'. [See TC[1-56].] You are not likely to have students who will see any point in such a step, and we advise against straining to develop an interest in such subtleties at this time.

THE PRINCIPLE OF OPPOSITES

The principle of opposites:

$$\forall_x \; x + -x = 0$$

tells us that for each real number there is a real number [the opposite of the first] which when added to the first gives the sum 0. In Unit I we mentioned that we would be able to prove that there couldn't be <u>two</u> numbers such that the sum of each with the given number is 0. For example, take the number $^-485$. The principle of opposites tells us that $^-485 + -^-485 = 0$. We are now able to predict that there isn't another number [that is, a number different from the opposite of $^-485$] which when added to $^-485$ gives the sum 0. In other words, we can now show that no matter what number you pick, if you add it to $^-485$ and get 0, that number must be the opposite of $^-485$. Let's state this generalization:

$$\forall_y \text{ if } {^-485} + y = 0 \text{ then } y = -{^-485}.$$

and build a test-pattern.

Suppose that	$^-485 + y = 0$.	
Now,	$^-485 + -^-485 = 0$.	[po]
So,	$^-485 \div y = {^-485} + -^-485$.	
Therefore,	$y = -^-485$.	[left cancellation prin.]
Hence,	if $^-485 + y = 0$ then $y = -^-485$.	

Could you write a test-pattern for the generalization:

$$\forall_y \text{ if } 9832 + y = 0 \text{ then } y = -9832?$$

Do so right here.

Now, consider the generalization:

$$(*) \qquad \forall_x \forall_y \text{ if } x + y = 0 \text{ then } y = -x.$$

We leave to you the job of writing a proof of (*).

Do you see how generalization (*) provides a way of showing that a second number is the opposite of a first number? Suppose a first boy picks a number, and a second boy picks a number. How can the boys use (*) to find out if the second number is the opposite of the first? Just find out if the first number plus the second number is 0. If it is 0 then the second number is the opposite of the first.

In Unit 1 you used this idea when you wanted to show, for example, that the opposite of $({}^{+}4 + {}^{-}7)$ is ${}^{-}4 + {}^{+}7$. What we used there was the theorem:

$$\forall_x \forall_y \text{ if } x + y = 0 \text{ then } -x = y.$$

We shall call this <u>the 0-sum theorem</u>. [Do you see that it is equivalent to (*)?] According to the 0-sum theorem, in order to show that

$$-\underbrace{({}^{+}4 + {}^{-}7)}_{x} = \underbrace{{}^{-}4 + {}^{+}7}_{y},$$

it is sufficient to show that

$$\underbrace{({}^{+}4 + {}^{-}7)}_{x} + \underbrace{({}^{-}4 + {}^{+}7)}_{y} = 0.$$

Let's do so.

$$\begin{aligned} &({}^{+}4 + {}^{-}7) + ({}^{-}4 + {}^{+}7) \\ &= ({}^{+}4 + {}^{-}4) + ({}^{+}7 + {}^{-}7) && \text{[Why?]} \\ &= \quad 0 \quad + \quad 0 && \text{[Why?]} \\ &= \quad 0. && \text{[Why?]} \end{aligned}$$

So, since $({}^{+}4 + {}^{-}7) + ({}^{-}4 + {}^{+}7) = 0$, it follows from (*) that

$$-({}^{+}4 + {}^{-}7) = {}^{-}4 + {}^{+}7.$$

Now, let's replace each occurrence of 'a' by a '$^{-}5$' and each occurrence of 'b' by a '$^{+}6$'.

$$
\begin{aligned}
& (^{-}5 + {}^{+}6) + (-{}^{-}5 + -{}^{+}6) \\
&= (^{-}5 + {}^{+}6) + (-{}^{+}6 + -{}^{-}5) && \text{cpa} \\
&= {}^{-}5 + {}^{+}6 + -{}^{+}6 + -{}^{-}5 && \text{apa} \\
&= {}^{-}5 + (^{+}6 + -{}^{+}6) + -{}^{-}5 && \text{apa} \\
&= {}^{-}5 + 0 + -{}^{-}5 && \text{po} \\
&= {}^{-}5 + -{}^{-}5 && \text{pa0} \\
&= 0. && \text{po}
\end{aligned}
$$

Hence, $(^{-}5 + {}^{+}6) + (-{}^{-}5 + -{}^{+}6) = 0$.

So, $-(^{-}5 + {}^{+}6) = -{}^{-}5 + -{}^{+}6$. [0-sum theorem]

The principles cited as justifications refer to the numbers $^{-}5$ and $^{+}6$ and to the operations addition and opposition. When one cites these principles in writing the testing pattern, he should always have in mind [perhaps not consciously] the verification of instances.

Now, since the testing pattern can be used to verify any sentence obtained by substituting numerals for the pronumerals in:

$$-(a + b) = -a + -b,$$

we conclude that the generalization:

$$\forall_a \forall_b \; -(a + b) = -a + -b$$

is a theorem.

juggling attitude may be found in the student's invention of colloquialisms such as:

> commute the 'x' and the 'y',
> associate the 'y' with the 'z',
> distribute the 'z' over the '(x + y)'.

The use of such colloquialisms would lead one to think that the student believes that the basic principles apply to numerals or to pronumerals rather than to numbers. Now, colloquialisms facilitate communication and it would be idiotic to forbid their use in the classroom. But it is imperative that the student understand that such colloquialisms cannot be taken literally. This understanding and the concomitant understanding of what one is doing when he proves a generalization can be achieved by reviewing occasionally the idea about patterns for testing instances which were discussed in Section 2.03 [pages 2-31ff.]. For example, after the students have completed Part A, you might open a discussion on what is really accomplished in the proof of the theorem in Exercise 1. Here is a sketch of such a discussion.

To convince someone that the generalization:

$$\forall_a \forall_b \; -(a + b) = -a + -b$$

is a theorem, we show him a pattern by means of which he could verify any sentence obtained by substituting numerals for 'a' and 'b' in the open sentence:

$$-(a + b) = -a + -b.$$

For example, suppose he wants to verify the sentence:

$$-({}^{-}5 + {}^{+}6) = -{}^{-}5 + -{}^{+}6$$

One testing pattern which we presented as a proof of the generalization is the following.

$$\begin{aligned}
&(a + b) + (-a + -b) \\
&= (a + b) + (-b + -a) && \text{cpa} \\
&= a + b + -b + -a && \text{apa} \\
&= a + (b + -b) + -a && \text{apa} \\
&= a + 0 + -a && \text{po} \\
&= a + -a && \text{pa0} \\
&= 0. && \text{po}
\end{aligned}$$

Hence, $a + b + (-a + -b) = 0$.

So, $-(a + b) = -a + -b$. [0-sum theorem]

5. Prove: $\forall_x \forall_y$ if $x = -y$ then $-x = y$. [Theorem 22]

Suppose that	$x = -y$.	
Since	$y + -y = 0$,	[po]
it follows that	$y + x = 0$.	
Since	$x + y = y + x$,	[cpa]
it follows that	$x + y = 0$.	
So,	$-x = y$.	[0-sum theorem]

Hence, if $x = -y$ then $-x = y$.

[We can abbreviate <u>this</u> proof by omitting the steps which indicate more or less explicitly the use of the substitution rule. Thus:

Suppose that	$x = -y$.	
Then	$y + x = 0$.	[po]
So,	$x + y = 0$,	[cpa]
and	$-x = y$.	[0-sum theorem]

Hence, if $x = -y$ then $-x = y$.]

An alternative proof:

Suppose that	$x = -y$.	
Then	$-x = --y$.	[uniqueness of opposition [Th. 10]]
So,	$-x = y$.	[$\forall_x\ --x = x$ [Th. 17]]

Hence, if $x = -y$ then $-x = y$.

This theorem and the 0-sum theorem justify the useful result:

If the sum of any first number and any second number is 0 then each is the opposite of the other.

*

This is a good time to inject a note of caution concerning the attitude students may be developing toward proofs. Some students develop a mechanical proficiency in giving proofs which rivals the proficiency customarily developed in conventional courses with respect to manipulating algebraic expressions. In other words, they become proficient in juggling symbols and may tend to lose sight of what it is they are accomplishing when they prove a generalization. Some evidence of this

4. Prove: $\forall_x \forall_y\ -(xy) = -xy$. [Theorem 21]

$$\begin{aligned} & xy + -xy \\ &= (x + -x)y && \text{dpma} \\ &= 0y && \text{po} \\ &= y0 && \text{cpm} \\ &= 0. && \text{pm0} \end{aligned}$$

Hence, $xy + -xy = 0$.

So, $-(xy) = -xy$. [0-sum theorem]

[The theorem in Exercise 4 can be regarded as a corollary of the one in Exercise 3.

$$\begin{aligned} & -(xy) \\ &= -(yx) && \text{cpm} \\ &= y \cdot -x && \forall_x \forall_y\ -(xy) = x \cdot -y \quad [\text{Th. } 20] \\ &= -xy. && \text{cpm} \end{aligned}$$

Hence, $-(xy) = -xy$.]

*

Note that the theorems of Exercises 3 and 4 relate the operations opposition and multiplication, just as those of Exercises 1 and 2 relate the operations opposition and addition, and as the dpma relates multiplication and addition. The theorems of Parts A and B are *not* rules for adding or multiplying positive and negative numbers. [*Such* rules have been discussed on pages 2-28 and 2-29.] For this reason, we have previously stressed the importance of reading, say '$-x$' as 'the opposite of x', and have warned that confusion results from reading it as 'negative x'.

To drive this point home, have students give several instances of, say, the theorem of Exercise 3, and see that they come up with some like '$-({}^-3 \cdot {}^+2) = {}^-3 \cdot -{}^+2$' and '$-({}^+3 \cdot {}^-2) = {}^+3 \cdot -{}^-2$'.

*

That $-{}^{-}3 = {}^{+}3$ and $-{}^{-}2 = {}^{+}2$ follows from the rules for oppositing given on TC[2-29, 30].]

*

2. Prove: $\forall_c \forall_d\ -(c + -d) = d + -c$. [Theorem 19]

$$\begin{aligned} & c + -d + (d + -c) \\ &= -d + c + (-c + d) && \text{cpa} \\ &= -d + c + -c + d && \text{apa} \\ &= -d + (c + -c) + d && \text{apa} \\ &= -d + 0 + d && \text{po} \\ &= -d + d && \text{pa0} \\ &= d + -d && \text{cpa} \\ &= 0. && \text{po} \end{aligned}$$

[As illustrated in the preceding treatment of Exercise 1, there are many ways of deriving the sentence '$c + -d + (d + -c) = 0$' from the basic principles. You might ask your brighter students to find several such derivations.]

Hence, $c + -d + (d + -c) = 0$.

So, $-(c + -d) = d + -c$. [0-sum theorem]

[A very short proof of the theorem in Exercise 2, but one which does not use the 0-sum theorem, is the following:

$$\begin{aligned} & -(c + -d) \\ &= -c + --d && \forall_x \forall_y\ -(x + y) = -x + -y \ \text{[Th. 18]} \\ &= -c + d && \forall_x\ --x = x \ \text{[Th. 17]} \\ &= d + -c. && \text{cpa} \end{aligned}$$

Hence, $-(c + -d) = d + -c$.

A theorem such as this one which follows quickly from another theorem is known as a corollary of the second theorem. So, the theorem in Exercise 2 is a corollary of the theorem [Th. 18] in Exercise 1.]

3. Prove: $\forall_p \forall_q\ -(pq) = p \cdot -q$. [Theorem 20]

$$\begin{aligned} & pq + p \cdot -q \\ &= p(q + -q) && \ell\text{dpma} \\ &= p0 && \text{po} \\ &= 0. && \text{pm0} \end{aligned}$$

Hence, $pq + p \cdot -q = 0$.

So, $-(pq) = p \cdot -q$. [0-sum theorem]

There are many ways of developing a testing pattern for Exercise 1 [and for each of the other exercises]. We give two of these here. [The last two lines are the same for both.]

Prove: $\forall_a \forall_b\ -(a + b) = -a + -b$. [Theorem 18]

$$\begin{aligned}
&a + b + (-a + -b) \\
&= a + b + (-b + -a) && \text{cpa} \\
&= a + b + -b + -a && \text{apa} \\
&= a + (b + -b) + -a && \text{apa} \\
&= a + 0 + -a && \text{po} \\
&= a + -a && \text{pa0} \\
&= 0. && \text{po}
\end{aligned}
\qquad\Bigg\|\qquad
\begin{aligned}
&a + b + (-a + -b) \\
&= a + b + -a + -b && \text{apa} \\
&= a + (b + -a) + -b && \text{apa} \\
&= a + (-a + b) + -b && \text{cpa} \\
&= a + -a + b + -b && \text{apa} \\
&= (a + -a) + (b + -b) && \text{apa} \\
&= 0 + 0 && \text{po} \\
&= 0. && \text{pa0}
\end{aligned}$$

Hence, $a + b + (-a + -b) = 0$.

So, $-(a + b) = -a + -b$. [0-sum theorem]

[This testing pattern shows that the generalization '$\forall_a \forall_b\ -(a + b) = -a + -b$' is a consequence of the apa, the cpa, the po, the pa0, and the 0-sum theorem. Hence, the generalization is a theorem. Some of your students may like to check the proof of the 0-sum theorem to discover that it is a consequence of the po and the left cancellation principle, and then check the proof of the latter to see that it is a consequence of the apa, the cpa, the po, and the pa0. They will then see that Theorem 18 is actually a consequence of these same basic principles.]

*

You will notice that one uses the distributive theorem for opposition over addition in conventional courses when one talks about how to remove parentheses which are preceded by a minus sign. It might be profitable to follow the discussion of Exercise 1 with a few applications of the theorem. For example, students can be asked to transform each of the following expressions into equivalent expressions which do not contain grouping symbols:

$$-({}^-3 + {}^+2), \qquad -({}^-5 + {}^-8), \qquad -(3 + 8).$$

Answers:

$${}^+3 + {}^-2, \qquad {}^+5 + {}^+8, \qquad -3 + -8.$$

[Note that what Theorem 18 tells us is that $-({}^-3 + {}^+2) = -{}^-3 + -{}^+2$.

You may feel that Part A on page 2-69 should be replaced by Part C on page 2-70. The reason for using the 0-sum theorem in Part A rather than the -1 times theorem of Part C is that it is desirable that students be fully aware that finding the opposite of a number means finding a second number which when added to the first gives 0. [The establishing of this awareness was also the main point of Part C on page 1-84.] If they are aware of this, they will have less trouble with division. For, in order to discover [and organize for themselves] theorems about quotients they need to be aware of the analogous fact that finding the quotient of a pair of numbers means finding a number whose product by the divisor is the dividend [cf. the division theorem, page 2-86].

Although the 0-sum theorem is of fundamental importance and should, for this and the reason given in the preceding paragraph, be stressed, the -1 times theorem is, of course, the basis for numerous short cuts, and, for this reason, should not be neglected. We hope that the treatment of these two theorems in the text will provide the proper emphasis for each.

*

Answers for Part A.

1. Although the theorem to be proved is stated in the exercise, there is some advantage to be gained by considering, first, how such a theorem might be discovered. Starting at this point, we are looking for a theorem which will answer questions like: What is the opposite of $^-5 + ^+6$? The 0-sum theorem tells us that we shall have the answer to this question if we find a number which, when added to $^-5 + ^+6$, gives the sum 0. Now, if we add, first $-^+6$ and then $-^-5$, it is clear that the resulting sum will be 0. So, one answer is '$-^+6 + -^-5$'. This suggests the theorem '$\forall_x \forall_y\ -(x + y) = -y + -x$'. An equally satisfactory answer is '$-^-5 + -^+6$'. This suggests '$\forall_x \forall_y\ -(x + y) = -x + -y$'. And this may be more appealing than the former theorem because we see in it a kind of distributivity. In fact, we can call it <u>the distributive theorem for opposition over addition.</u>

   The point to be stressed in the preceding discussion is the power of the 0-sum theorem to suggest other theorems dealing with opposition.

EXERCISES

A. Let's use the 0-sum theorem to prove some other theorems about opposites.

Sample. The opposite of the opposite of a number is that number.

Solution. $\forall_a \; --a = a.$

[The 0-sum theorem tells us that to prove this, we should prove that, for each a,

$$\underbrace{-a}_{\substack{\uparrow \\ x}} + \underbrace{a}_{\substack{\uparrow \\ y}} = 0.$$

Then, the 0-sum theorem would enable us to conclude that

$$-\underbrace{-a}_{\substack{\uparrow \\ x}} = \underbrace{a}_{\substack{\uparrow \\ y}}.$$

Let's write a test-pattern .]

$$\begin{aligned} &-a + a \\ &= a + -a \\ &= 0. \end{aligned} \quad \begin{aligned} &\} \text{ cpa} \\ &\} \text{ po} \end{aligned}$$

Hence, $-a + a = 0.$

So, $--a = a.$ [0-sum theorem]

This shows that the generalization '$\forall_a \; --a = a$' is a consequence of the commutative principle for addition, the principle of opposites, and the 0-sum theorem.

1. $\forall_a \forall_b \; -(a + b) = -a + -b.$ ["Distributive theorem for opposition over addition"]

2. $\forall_c \forall_d \; -(c + -d) = d + -c.$

3. $\forall_p \forall_q \; -(pq) = p \cdot -q.$

4. $\forall_x \forall_y \; -(xy) = -xy.$ [Note that '$-xy$' is an abbreviation for '$(-x)y$' and for '$-x \cdot y$'.]

5. $\forall_x \forall_y$ if $x = -y$ then $-x = y.$

B. Prove these theorems.

Sample 1. $\forall_m \; - - -m = -m.$

Solution.
$$\left.\begin{aligned} & - - -m \\ &= -m. \end{aligned}\right\} \forall_a \; - -a = a.$$

Sample 2. $\forall_u \forall_v \; -u \cdot -v = uv.$

Solution.
$$\begin{aligned} & -u \cdot -v && \\ &= -(-u \cdot v) && \Big\} \; \forall_p \forall_q \; -(pq) = p \cdot -q. \\ &= -[-(uv)] && \Big\} \; \forall_x \forall_y \; -(xy) = -xy. \\ &= uv. && \Big\} \; \forall_a \; - -a = a. \end{aligned}$$

1. $\forall_x \forall_y \; -xy = x(-y).$

2. $\forall_x \forall_y \forall_z \; -x(y + z) = -(xy) + -(xz).$

3. $\forall_x \forall_y \forall_z \; -x(-y + -z) = xy + xz.$

4. $\forall_x \; x \cdot -1 = -x.$

C. Prove the "−1 times theorem":

$$\forall_x \; -x = -1 \cdot x.$$

Then, use it to prove as many of the theorems in Parts A and B as you can. For example, here is a proof of the theorem of Exercise 1 of Part A.

$$\begin{aligned} & -(a + b) && \\ &= -1 \cdot (a + b) && \Big\} \; -1 \text{ times theorem} \\ &= -1 \cdot a + -1 \cdot b && \Big\} \; \ell\text{dpma} \\ &= -a + -b. && \Big\} \; -1 \text{ times theorem} \end{aligned}$$

Sample 1 of Part B provides another illustration of the notion of a corollary of a theorem. It is a corollary of the theorem in the Sample of Part A. In fact the theorem in the Sample of Part A is the basis of the short cut that "pairs of opposition signs cancel out". To drive this point home, ask students [after Sample 1 has been discussed] to prove:

$$\forall_x \; - - - - - - - - x = -x.$$

*

To make sure students do not regard the theorem in Sample 2 as one which tells them that "the product of two negative numbers is a positive number", ask them to apply the theorem in transforming '$-{}^-2 \cdot -{}^+3$' to '${}^-2 \cdot {}^+3$'.

*

Answers for Part B.

1. Prove: $\forall_x \forall_y \; -xy = x(-y)$. [Theorem 24]

$$\left.\begin{array}{l} -xy \\ = -(xy) \\ = x \cdot -y. \end{array}\right\} \begin{array}{ll} \forall_x \forall_y \; -(xy) = -xy & \text{[Th. 21]} \\ \forall_x \forall_y \; -(xy) = x \cdot -y & \text{[Th. 20]} \end{array}$$

Hence, $-xy = x(-y)$.

[A few students may have difficulty in seeing how the transformation of '$-xy$' into '$-(xy)$' is justified by Theorem 21. The pattern sentence of Theorem 21 is '$-(xy) = -xy$'. But, because of the symmetry of equality, this pattern sentence is equivalent to '$-xy = -(xy)$', which is the pattern used in the proof.]

2. Prove: $\forall_x \forall_y \forall_z \; -x(y + z) = -(xy) + -(xz)$. [Theorem 25]

$$\left.\begin{array}{l} -x(y + z) \\ = -xy + -xz \\ = -(xy) + -(xz). \end{array}\right\} \begin{array}{ll} \ell\text{dpma} & \\ \forall_x \forall_y \; -(xy) = -xy & \text{[Th. 21]} \end{array}$$

Hence, $-x(y + z) = -(xy) + -(xz)$.

[An alternative proof is on TC[2-70]b.]

Alternative proof.

$$\left.\begin{aligned} & -x(y+z) \\ &= -[x(y+z)] \\ &= -(xy+xz) \\ &= -(xy) + -(xz). \end{aligned}\right\} \begin{aligned} &\forall_x\forall_y\ -(xy) = -xy \quad [\text{Th. } 21] \\ &\ell\text{dpma} \\ &\forall_x\forall_y\ -(x+y) = -x + -y \quad [\text{Th. } 18] \end{aligned}$$

Hence, $-x(y + z) = -(xy) + -(xz)$.

3. Prove: $\forall_x\forall_y\forall_z\ -x(-y + -z) = xy + xz$. [Theorem 26]

$$\left.\begin{aligned} & -x(-y+-z) \\ &= -(x\cdot -y) + -(x\cdot -z) \\ &= --(xy) + --(xz) \\ &= xy + xz\,. \end{aligned}\right\} \begin{aligned} &\forall_x\forall_y\forall_z\ -x(y+z) = -(xy) + -(xz) \quad [\text{Th. } 25] \\ &\forall_x\forall_y\ -(xy) = x\cdot -y \quad [\text{Th. } 20] \\ &\forall_x\ --x = x \quad [\text{Th. } 17] \end{aligned}$$

Hence, $-x(-y + -z) = xy + xz$.

Alternative proof.

$$\left.\begin{aligned} & -x(-y+-z) \\ &= -x\cdot -y + -x\cdot -z \\ &= xy + xz. \end{aligned}\right\} \begin{aligned} &\ell\text{dpma} \\ &\forall_x\forall_y\ -x\cdot -y = xy \quad [\text{Th. } 23] \end{aligned}$$

Hence, $-x(-y + -z) = xy + xz$.

4. Prove: $\forall_x\ x\cdot -1 = -x$. [Theorem 27]

$$\left.\begin{aligned} & x\cdot -1 \\ &= -(x\cdot 1) \\ &= -x. \end{aligned}\right\} \begin{aligned} &\forall_x\forall_y\ -(xy) = x\cdot -y \ [\text{Th. } 20] \\ &\text{pm1} \end{aligned}$$

Hence, $x\cdot -1 = -x$.

*

Answers for Part C.

Prove: $\forall_x \; -x = -1 \cdot x$. [Theorem 28]

$$\begin{aligned} & -x \\ &= x \cdot -1 \\ &= -1 \cdot x. \end{aligned} \quad \begin{aligned} &\left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \forall_x \; x \cdot -1 = -x \text{ [Th. 27]} \\ &\left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \text{cpm} \end{aligned}$$

Hence, $-x = -1 \cdot x$.

*

In using the -1 times theorem in proving the theorems in Parts A and B, students may [and some should] raise the question of "circularity". Perhaps, in proving the -1 times theorem, we have used, at least implicitly, some of the theorems which we now propose to use the -1 times theorem to prove! As a matter of fact, this is the case. In the proof given above for the -1 times theorem we cited the theorem of Exercise 4 of Part B, and in the proof given for this theorem we cited Exercise 3 of Part A. So, it would, with this development, be circular to use the -1 times theorem in proving the theorem of Exercise 3 of Part A. However, this state of affairs can be corrected by giving an alternative proof for the -1 times theorem. The following is one such suitable proof.

$$\begin{aligned} & x + -1 \cdot x \\ &= 1 \cdot x + -1 \cdot x \\ &= (1 + -1) \cdot x \\ &= 0 \cdot x \\ &= x \cdot 0 \\ &= 0. \end{aligned} \quad \begin{aligned} &\left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \text{the 1 times theorem [Th. 2]} \\ &\left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \text{dpma} \\ &\left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \text{po} \\ &\left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \text{cpm} \\ &\left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \text{pm0} \end{aligned}$$

Hence, $x + -1 \cdot x = 0$.

So, $-x = -1 \cdot x$. [0-sum theorem]

We can now, without danger of circularity, use the -1 times theorem in proving some of the theorems of Parts A and B. Such a proof for the theorem of Exercise 1 of Part A is given in the text. And, such proofs for the theorems of Exercises 3 and 4 of Part A and Exercises 1, 2, and 4 of Part B are readily constructed. As examples, here are

proofs for the theorems of Exercise 4 of Part A and Exercise 2 of Part B.

$$\begin{array}{ll}
-(xy) & \\
 & \} \ -1 \text{ times th.} \\
= -1(xy) & \\
 & \} \ \text{apm} \\
= -1xy & \\
 & \} \ -1 \text{ times th.} \\
= -xy. &
\end{array}$$

Hence, $-(xy) = -xy$.

$$\begin{array}{ll}
-x(y + z) & \\
 & \} \ -1 \text{ times th.} \\
= -1x(y + z) & \\
 & \} \ \ell\text{dpma} \\
= -1xy + -1xz & \\
 & \} \ \text{apm} \\
= -1(xy) + -1(xz) & \\
 & \} \ -1 \text{ times th.} \\
= -(xy) + -(xz). &
\end{array}$$

Hence, $-x(y + z) = -(xy) + -(xz)$.

But, if one attempts to use the -1 times theorem in proving the theorem of Exercise 2 of Part A, he hits a snag. Let's see where.

$$\begin{array}{ll}
-(c + -d) & \\
 & \} \ -1 \text{ times theorem} \\
= -1(c + -1d) & \\
 & \} \ \ell\text{dpma} \\
= -1c + -1(-1d) & \\
= -1c + \ ? &
\end{array}$$

What are we to do with '$-1(-1d)$'? By the -1 times theorem, this is equivalent to '$-1(-d)$' and, again to '$- -d$'. So, we could use the theorem of the Sample. But, we have not yet used the -1 times theorem to prove this. Another possibility is to use the theorem of Exercise 4 of Part A to show that '$-1(-1d)$' is equivalent to '$- -1d$'. But, again, we need the Sample if we are to replace '$- -1$' by '1'. A third alternative is to cite the apm to justify reducing '$-1(-1d)$' to '$-1 \cdot -1d$'. But, we have not yet reproven the theorem of Sample 2 of Part B which we would need to cite if we were to claim that $-1 \cdot -1 = 1$. Evidently, if we are to carry out the program in Part C, we need, in addition to the -1 times theorem, either the Sample of Part A, or Sample 2 of Part B, or at least one of the simpler theorems '$- -1 = 1$' and '$-1 \cdot -1 = 1$'. And, so, we must make an additional use of the 0-sum theorem. The point of all this is that the -1 times theorem alone is not enough.

Probably, if we are to follow up the idea which suggests using the -1 times theorem, the most satisfactory way out of the difficulty we have

come upon is to prove that $-1 \cdot -1 = 1$. Here is a way of doing so.

$$
\begin{array}{ll}
-1 \cdot -1 & \\
 & \}\ -1 \text{ times theorem} \\
= - -1. &
\end{array}
$$

But,

$$
\begin{array}{ll}
-1 + 1 & \\
 & \}\ \text{cpm} \\
= 1 + -1 & \\
 & \}\ \text{po} \\
= 0. &
\end{array}
$$

So, $- -1 = 1$. [0-sum theorem]

Hence, $-1 \cdot -1 = 1$.

We can now complete the proof we began for the theorem of Exercise 2 of Part A.

$$
\begin{array}{ll}
 & \}\ \text{apm} \\
= -1c + -1 \cdot -1d & \\
 & \}\ -1 \text{ times theorem} \\
= -c + -1 \cdot -1d & \\
 & \}\ -1 \cdot -1 = 1 \\
= -c + 1d & \\
 & \}\ 1 \text{ times theorem} \\
= -c + d & \\
 & \}\ \text{cpa} \\
= d + -c. &
\end{array}
$$

Hence, $-(c + -d) = d + -c$.

Proofs based on the -1 times theorem together with the theorem '$-1 \cdot -1 = 1$' can now be constructed easily enough for the Sample of Part A, Exercise 5 of Part A, and Samples 1 and 2 and Exercise 3 of Part B.

*

It may have occurred to you that '$-1 \cdot -1 = 1$' states a computing fact. What you probably have in mind is that $^{-}1 \cdot {}^{-}1 = 1$ [and this is a computing fact] and that $-1 = {}^{-}1$ [and this is a consequence of the computing fact that $1 + {}^{-}1 = 0$ together with the 0-sum theorem]. So, however you look at it, the 0-sum theorem comes into the picture. Hence, it is more satisfactory to base the proof of '$-1 \cdot -1 = 1$' on the 0-sum theorem, the -1 times theorem, and basic principles, as we have done, and to avoid unnecessary citations of computing facts.

add 5 (3, 8)

Inverse of add 5 (8, 3)

Here are test patterns for the two theorems which precede the exercises.

$\forall_x \forall_y \; (x + y) + -y = x$ [Theorem 29]

$$\begin{array}{ll} & (x + y) + -y \\ & \quad \} \text{ apa} \\ = & x + (y + -y) \\ & \quad \} \text{ po} \\ = & x + 0 \\ & \quad \} \text{ pa0} \\ = & x. \end{array}$$

Hence, $(x + y) + -y = x$.

$\forall_x \forall_y \; (x + y) - y = x$ [Theorem 30]

$$\begin{array}{ll} & (x + y) - y \\ & \quad \} \text{ ps} \\ = & (x + y) + -y \\ & \quad \} \text{ apa} \\ = & x + (y + -y) \\ & \quad \} \text{ po} \\ = & x + 0 \\ & \quad \} \text{ pa0} \\ = & x. \end{array}$$

Hence, $(x + y) - y = x$.

[The first theorem states that adding the opposite of a real number is the operation inverse to adding the real number. The second theorem states that subtracting a real number [also] is the inverse of adding the real number.]

*

In class discussion you may find it helpful to refer back to the subsection "Does Absolute Valuing Have An Inverse", pages 1-107 and 1-108.

SUBTRACTION

In Unit 1 you learned that the inverse of adding a real number is adding the opposite of that number. We can express this idea by the generalization:

$$\forall_x \forall_y \ (x + y) + -y = x.$$

Write a test-pattern for this theorem.

The principle for subtraction:

$$\forall_x \forall_y \ x - y = x + -y$$

tells us that adding the opposite of a real number is the same operation as subtracting the number. So, we see that the inverse of adding a real number is subtracting the real number. That is, that

$$\forall_x \forall_y \ (x + y) - y = x.$$

Prove this last theorem by writing a test-pattern.

EXERCISES

A. Each of the following exercises contains several sentences-to-be-completed. After you complete the sentences in an exercise, state the theorem which is illustrated by the completed sentences and be prepared to prove it.

Sample.

(a) $7 - 4 \cdot 5 = 7 + \_\_\_\_ \cdot 5$

(b) $^-9 - 8 \cdot 13 = {}^-9 + \_\_\_\_ \cdot 13$

(c) $5 - {}^-2 \cdot 5 = 5 + \_\_\_\_ \cdot 5$

(d) $-8 - -3 \cdot 7 = -8 + \_\_\_\_ \cdot 7$

Theorem: ______________________________

(continued on next page)

Solution.

(a) $7 - 4 \cdot 5 = 7 + \underline{-4} \cdot 5$

(b) $^{-}9 - 8 \cdot 13 = {}^{-}9 + \underline{-8} \cdot 13$

(c) $5 - {}^{-}2 \cdot 5 = 5 + \underline{-^{-}2} \cdot 5$

(d) $^{-}8 - -3 \cdot 7 = {}^{-}8 + \underline{--3} \cdot 7$

Theorem: $\underline{\forall_x \forall_y \forall_z \quad x - yz = x + -yz}$

[The sentences illustrate the theorem that subtracting the product of a first number by a second number is the same as adding the product of the opposite of the first number by the second number.]

Here is a proof of this theorem.

$$\begin{aligned} & x - yz \\ &= x + -(yz) \quad \text{ps} \\ &= x + -yz. \quad \forall_x \forall_y \; -(xy) = -xy. \end{aligned}$$

1. (a) $^{+}9 - {}^{-}5 + \_\_\_\_\_ = {}^{+}9$

(b) $^{-}3 - {}^{+}7 + \_\_\_\_\_ = {}^{-}3$

(c) $^{-}8 - {}^{+}2 + {}^{+}2 = \_\_\_\_\_$

(d) $\_\_\_\_\_ - {}^{-}7 + {}^{-}7 = {}^{-}43$

Theorem: ______________________________

2. (a) $-(70 - \_\_\_\_\_) = 90 - 70$

(b) $-(81 - \_\_\_\_\_) = 35 - 81$

(c) $-(10 - \_\_\_\_\_) = -3 - 10$

(d) $-(\_\_\_\_\_ - {}^{-}19) = {}^{-}19 - {}^{-}3$

Theorem: ______________________________

You should require all students to fill the blanks in Part A and to write the relevant generalizations. It is not necessary to require each student to write proofs of all of them!

*

Answers for Part A [on pages 2-72, 2-73, and 2-74].

1. (a) $^{-}5$ (b) $^{+}7$ (c) $^{-}8$ (d) $^{-}43$

<u>Theorem</u>: $\forall_x \forall_y\ x - y + y = x$ [Theorem 32]

<u>Proof</u>:

$$\begin{aligned} & x - y + y \\ &= x + -y + y && \text{ps} \\ &= x + (-y + y) && \text{apa} \\ &= x + (y + -y) && \text{cpa} \\ &= x + 0 && \text{po} \\ &= x. && \text{pa0} \end{aligned}$$

Hence, $x - y + y = x$.

[When a student writes a proof of a theorem on the board, it will be easier for other students to follow his development if the theorem is written first, with a proof below it.]

*

In the answers for the remainder of the exercises, to save time and space, <u>we</u> shall omit the words 'Theorem' and 'Proof', and just give the theorem with a proof below it. <u>We</u> shall also omit the last line of the proof: Hence, ... = ---- .

*

2. (a) 90 (b) 35 (c) -3 (d) $^{-}3$

$\forall_x \forall_y\ -(x - y) = y - x$ [Theorem 33]

$$\begin{array}{ll}
\ -(x - y) & \\
 & \text{ps} \\
= -(x + -y) & \\
 & \forall_x \forall_y\ -(x + -y) = y + -x \ \ [\text{Th. } 19] \\
= y + -x & \\
 & \text{ps} \\
= y - x. &
\end{array}$$

[As usual there are alternative proofs. One may, for example, use the -1 times theorem.

$$\begin{array}{ll}
\ -(x - y) & \\
 & -1 \text{ times theorem} \\
= -1(x - y) & \\
 & \text{ps} \\
= -1(x + -y) & \\
 & \ell\text{dpma} \\
= -1x + -1 \cdot -y & \\
 & -1 \text{ times theorem} \\
= -x + --y & \\
 & \forall_x\ --x = x \ \ [\text{Th. } 17] \\
= -x + y & \\
 & \text{cpa} \\
= y + -x & \\
 & \text{ps} \\
= y - x. &
\end{array}$$

If, with a view to consistency of method, we had modified the proof above to use '$-1 \cdot -1 = 1$' in place of Theorem 17, the proof would have contained eleven lines rather than the present eight. In either case, the contrast with the previous 4-line proof is a striking indication of the value of using previous theorems <u>other</u> than the -1 times theorem.]

Alternative proof.

$$
\begin{aligned}
& x - (y - z) \\
&= x + -(y - z) && \text{ps} \\
&= x + (z - y) && \forall_x \forall_y\ -(x - y) = y - x \ [\text{Th. } 33] \\
&= x + (z + -y) && \text{ps} \\
&= x + (-y + z) && \text{cpa} \\
&= (x + -y) + z && \text{apa} \\
&= x - y + z. && \text{ps}
\end{aligned}
$$

6. (a) 9 (b) 58 (c) $^{-}7$ (d) 3

$\forall_x \forall_y \forall_z\ x + (y - z) = x - z + y$ [Theorem 37]

$$
\begin{aligned}
& x + (y - z) \\
&= x + (y + -z) && \text{ps} \\
&= x + (-z + y) && \text{cpa} \\
&= x + -z + y && \text{apa} \\
&= x - z + y. && \text{ps}
\end{aligned}
$$

7. (a) 3 (b) 18 (c) 17 (d) 6

$\forall_x \forall_y \forall_z\ x(y - z) = xy - xz$ [Theorem 38]

$$
\begin{aligned}
& x(y - z) \\
&= x(y + -z) && \text{ps} \\
&= xy + x \cdot -z && \ell\text{dpma} \\
&= xy + -(xz) && \forall_x \forall_y\ -(xy) = x \cdot -y \ [\text{Th. } 20] \\
&= xy - xz. && \text{ps}
\end{aligned}
$$

3. (a) 60 (b) 84 (c) $^{-}3$ (d) $^{-}31$

$\forall_x \forall_y \forall_z \; x + (y - z) = x + y - z$ [Theorem 34]

$$\begin{aligned}
& x + (y - z) \\
&= x + (y + -z) && \text{ps} \\
&= x + y + -z && \text{apa} \\
&= x + y - z. && \text{ps}
\end{aligned}$$

4. (a) 2 (b) 9 (c) $^{-}6$ (d) 5

$\forall_x \forall_y \forall_z \; x - (y + z) = x - y - z$ [Theorem 35]

$$\begin{aligned}
& x - (y + z) \\
&= x + -(y + z) && \text{ps} \\
&= x + (-y + -z) && \forall_x \forall_y \; -(x + y) = -x + -y \text{ [Th. 18]} \\
&= (x + -y) + -z && \text{apa} \\
&= x - y - z. && \text{ps}
\end{aligned}$$

5. (a) 5 (b) 7 (c) 12 (d) $^{-}5$

$\forall_x \forall_y \forall_z \; x - (y - z) = x - y + z$ [Theorem 36]

$$\begin{aligned}
& x - (y - z) \\
&= x - (y + -z) && \text{ps} \\
&= x - y - -z && \forall_x \forall_y \forall_z \; x - (y + z) = x - y - z \text{ [Th. 35]} \\
&= x - y + --z && \text{ps} \\
&= x - y + z. && \forall_x \; --x = x \text{ [Th. 17]}
\end{aligned}$$

[An alternative proof is on TC[2-73, 74]b.]

3. (a) $30 + (90 - 60) = 30 + 90 -$ _____

   (b) $72 + (53 - 84) = 72 + 53 -$ _____

   (c) $^-9 + (^-3 - {}^+7) = {}^-9 +$ _____ $- {}^+7$

   (d) $15 + (29 - {}^-31) = 15 + 29 -$ _____

   Theorem: ____

4. (a) $10 - (5 +$ _____ $) = 10 - 5 - 2$

   (b) $7 - ($ _____ $+ 4) = 7 - 9 - 4$

   (c) $^-5 - ($ _____ $+ {}^-3) = {}^-5 - {}^-6 - {}^-3$

   (d) $5 - (5 +$ _____ $) = 5 - 5 - 5$

   Theorem: ____

5. (a) $9 - (3 - 5) = 9 - 3 +$ _____

   (b) $7 - (10 - 7) = 7 - 10 +$ _____

   (c) $11 - ($ _____ $- 5) = 11 - 12 + 5$

   (d) $^-3 - (^-9 - {}^-5) = {}^-3 - {}^-9 +$ _____

   Theorem: ____

6. (a) $5 + (12 - 9) = 5 -$ _____ $+ 12$

   (b) $61 + (37 - 58) = 61 -$ _____ $+ 37$

   (c) $^-2 + (^-7 - 5) = {}^-2 - 5 +$ _____

   (d) $3 + (^-4 - {}^-6) =$ _____ $- {}^-6 + {}^-4$

   Theorem: ____

(continued on next page)

7. (a) $5(6 - 3) = 5 \cdot 6 - 5 \cdot \underline{\qquad}$

(b) ${}^-5(7 - 18) = {}^-5 \cdot 7 - {}^-5 \cdot \underline{\qquad}$

(c) $4 \cdot 3 - 4 \cdot 17 = 4(3 - \underline{\qquad})$

(d) $6(9 - {}^-2) = 6 \cdot 9 - \underline{\qquad} \cdot {}^-2$

Theorem: $\underline{\hspace{12cm}}$

8. (a) $(9 - 3)8 = 9 \cdot 8 - 3 \cdot \underline{\qquad}$

(b) $(4 - 14)51 = 4 \cdot 51 - \underline{\qquad} \cdot 51$

(c) $6 \cdot 19 - 8 \cdot 19 = (6 - \underline{\qquad})19$

(d) $-3 \cdot 48 - 3 \cdot 48 = (\underline{\qquad} - 3)48$

Theorem: $\underline{\hspace{12cm}}$

9. (a) $5 - (-90 - \underline{\qquad}) = 5 + 90 + 60$

(b) $7 - (\underline{\qquad} - 13) = 7 + 81 + 13$

(c) $6 - (\underline{\qquad} - {}^-1) = 6 + {}^-3 + {}^-1$

(d) $4 - (-15 - {}^-9) = 4 + 15 + \underline{\qquad}$

Theorem: $\underline{\hspace{12cm}}$

10. (a) $15 - (17 - 13 - 19) = 15 - 17 + \underline{\qquad} + 19$

(b) $21 - (-8 - 10 - -3) = 21 - \underline{\qquad} + 10 + -3$

(c) $-5 - (-7 - 18 - 25) = \underline{\qquad} - -7 + 18 + 25$

(d) $60 - (60 - 30 - 30) = 60 - 60 + 30 + \underline{\qquad}$

Theorem: $\underline{\hspace{12cm}}$

8. (a) 8 (b) 14 (c) 8 (d) $^{-}3$

$\forall_x \forall_y \forall_z \; (x - y)z = xz - yz$ [Theorem 39]

$$\begin{aligned}
&(x - y)z \\
&= z(x - y) && \text{cpm} \\
&= zx - zy && \ell\text{dtms [Th. 38]} \\
&= xz - yz. && \text{cpm}
\end{aligned}$$

Alternative proof.

$$\begin{aligned}
&(x - y)z \\
&= (x + -y)z && \text{ps} \\
&= xz + -yz && \text{dpma} \\
&= xz - yz. && \forall_x \forall_y \forall_z \; x - yz = x + -yz \text{ [Th. 31]}
\end{aligned}$$

9. (a) 60 (b) -81 (c) $-^{-}3$ (d) $^{-}9$

$\forall_x \forall_y \forall_z \; x - (-y - z) = x + y + z$ [Theorem 40]

$$\begin{aligned}
&x - (-y - z) \\
&= x + -(-y + -z) && \text{ps} \\
&= x + --(y + z) && \forall_x \forall_y \; -(x + y) = -x + -y \text{ [Th. 18]} \\
&= x + (y + z) && \forall_x \; --x = x \text{ [Th. 17]} \\
&= x + y + z. && \text{apa}
\end{aligned}$$

10. (a) 13 (b) -8 (c) -5 (d) 30

$\forall_x \forall_y \forall_z \forall_u \; x - (y - z - u) = x - y + z + u$ [Theorem 41]

$$\begin{aligned}
&x - (y - z - u) \\
&= x - (y - z) + u && \forall_x \forall_y \forall_z \; x - (y - z) = x - y + z \text{ [Th. 36]} \\
&= x - y + z + u. && \text{[Th. 36]}
\end{aligned}$$

3. $\forall_x \forall_y \forall_z\ (x + z) - (y + z) = x - y$ [Theorem 44]

$$
\begin{array}{ll}
(x + z) - (y + z) & \\
 & \text{cpa} \\
= (x + z) - (z + y) & \\
 & \forall_x \forall_y \forall_z\ x - (y + z) = x - y - z \ \text{[Th. 35]} \\
= (x + z) - z - y & \\
 & \forall_x \forall_y\ (x + y) - y = x \ \text{[Th. 30]} \\
= x - y. &
\end{array}
$$

4. $\forall_x \forall_y \forall_z\ (x - z) - (y - z) = x - y$ [Theorem 45]

$$
\begin{array}{ll}
(x - z) - (y - z) & \\
 & \text{ps} \\
= (x + -z) - (y + -z) & \\
 & \forall_x \forall_y \forall_z\ x + z - (y + z) = x - y \ \text{[Th. 44]} \\
= x - y. &
\end{array}
$$

5. $\forall_a \forall_b \forall_c \forall_d\ (a - b) + (c - d) = (a + c) - (b + d)$ [Theorem 46]

$$
\begin{array}{ll}
(a - b) + (c - d) & \\
 & \forall_x \forall_y \forall_z\ x + (y - z) = x + y - z \ \text{[Th. 34]} \\
= (a - b) + c - d & \\
 & \forall_x \forall_y \forall_z\ x + (y - z) = x - z + y \ \text{[Th. 37]} \\
= a + (c - b) - d & \\
 & \text{[Th. 34]} \\
= (a + c) - b - d & \\
 & \forall_x \forall_y \forall_z\ x - (y + z) = x - y - z \ \text{[Th. 35]} \\
= (a + c) - (b + d). &
\end{array}
$$

Answers for Part B.

1. $\forall_x \; 0 - x = -x$ [Theorem 42]

$$\begin{aligned} & 0 - x \\ &= 0 + -x && \text{ps} \\ &= -x + 0 && \text{cpa} \\ &= -x. && \text{pa0} \end{aligned}$$

2. $\forall_x \; x - 0 = x$ [Theorem 43]

$$\begin{aligned} & x - 0 \\ &= x + -0 && \text{ps} \\ &= x + 0 && -0 = 0 \leftarrow \left[\underline{\text{Proof}}\text{: } 0 + 0 = 0. \text{ So, by the 0-sum theorem, } -0 = 0.\right] \\ &= x. && \text{pa0} \end{aligned}$$

The following interesting alternative proof for Theorem 43 was submitted by a student.

$$\begin{aligned} & x - 0 \\ &= -(0 - x) && \forall_x \forall_y \; -(x - y) = y - x \text{ [Th. 33]} \\ &= -(0 + -x) && \text{ps} \\ &= -(-x + 0) && \text{cpa} \\ &= - -x && \text{pa0} \\ &= x. && \forall_x \; - -x = x \text{ [Th. 17]} \end{aligned}$$

B. State the theorems referred to in the following exercises, and be prepared to prove them.

1. If I subtract a number from 0, I get the opposite of the number.

   Theorem: ______________________________

2. If I subtract 0 from a number, I get the number.

   Theorem: ______________________________

3. Rita and Rhoda each picks a real number. Aaron picks a number and tells each girl to add his number to hers. Then Rhoda subtracts her sum from Rita's sum, and finds that she gets the same answer as she would have if she had subtracted her original number from Rita's original number.

   Theorem: ______________________________

4. Just like Exercise 3 except that the girls subtract Aaron's number from each of theirs.

   Theorem: ______________________________

5. Albert and Beulah and Charles and Dora each picks a real number. Beulah subtracts hers from Albert's and Dora subtracts hers from Charles'. They add the differences. Then, Charles adds his number to Albert's and Dora adds her number to Beulah's. The girls subtract their sum from the boys' sum, and get the same result as when they added the differences.

   Theorem: ______________________________

C. Complete each of the following from the choices given to make a true generalization. [There is only one correct choice.]

1. $\forall_x \forall_y \forall_z$ $x - (-y - z) =$ __________.

(a) $x - y - z$ (b) $x + y - z$ (c) $x - y + z$ (d) $x + y + z$

2. $\forall_x \forall_y \forall_z$ $-(x - y - z) =$ __________.

(a) $x + y + z$ (b) $-x + y - z$ (c) $-x + y + z$ (d) $x - y + z$

3. $\forall_x \forall_y \forall_z$ $x - yz =$ __________.

(a) $x - y - z$ (b) $x + y - z$ (c) $x + -yz$ (d) $x + -y \cdot -z$

4. $\forall_a \forall_b \forall_c \forall_d$ $-(a - b)(c - d) =$ __________.

(a) $(a + b)(c + d)$ (b) $(b - a)(d - c)$

(c) $(b - a)(c + d)$ (d) $(b - a)(c - d)$

5. $\forall_a \forall_b \forall_c \forall_d$ $-(a - b) - (c - d) =$ __________.

(a) $-a - b - c - d$ (b) $b - a + d - c$

(c) $-a + b - c - d$ (d) $b + a + d - c$

6. $\forall_u \forall_v \forall_x \forall_y$ $-u(x + y) - v(x - y) =$ __________.

(a) $-ux + uy - vx - vy$ (b) $-ux - uy - vx + vy$

(c) $-ux - uy + vx - vy$ (d) $-ux + -uy + -vx + -vy$

7. $\forall_x \forall_y$ $8x - 2y - 5x + 7y =$ __________.

(a) $6xy - 2xy$ (b) $3x - 9y$ (c) $8x$ (d) $3x + 5y$

8. $\forall_a \forall_b \forall_c$ $3a - 2(a - b + c) =$ __________.

(a) $a - b + c$ (b) $a + 2b - 2c$

(c) $3a - 2 - a - b + c$ (d) $3a - 2 - a + b - c$

9. $\forall_x \forall_y \forall_z$ $3x - x - x - 2y - y - z =$ __________.

(a) $x - 3y - z$ (b) $3x - 2 - z$

(c) $-5x - 3y - z$ (d) $3 - x - 2 - z$

10. $\forall_a \forall_b$ $-(3a - 5b)(4a - 2b)(5a - 7b) =$ __________.

(a) $(5b - 3a)(2b - 4a)(7b - 5a)$ (b) $(3a - 5b)(2b - 4a)(7b - 5a)$

(c) $-(5b - 3a)(2b - 4a)(7b - 5a)$ (d) $(5b + 3a)(4a + 2b)(5a + 7b)$

The purpose of Part C is to give the student an opportunity to continue to develop and to use short cuts based on the theorems he has seen in Parts A and B. There is the same opportunity here to correct misconceptions as was afforded by Part C on pages 2-37 and 2-38. Students will begin using short cuts in earnest in Part D on pages 2-77 through 2-80.

*

Answers for Part C.

1.	$x + y + z$	[(d)]	2.	$-x + y + z$	[(c)]
3.	$x + -yz$	[(c)]	4.	$(b - a)(c - d)$	[(d)]
5.	$b - a + d - c$	[(b)]	6.	$-ux - uy - vx + vy$	[(b)]
7.	$3x + 5y$	[(d)]	8.	$a + 2b - 2c$	[(b)]
9.	$x - 3y - z$	[(a)]	10.	$(5b - 3a)(2b - 4a)(7b - 5a)$	[(a)]

*

Here is a quiz which will get at some of the ideas developed in Part A. [This is a <u>long</u> quiz and perhaps should be given for homework.]

Sort the following expressions into as few categories as possible with each category containing only equivalent expressions.

1.	$a - (b - c)$	2.	$-a + c - b$	3.	$a(b - c)$
4.	$c + a + b$	5.	$c - a - b$	6.	$-ab + ac$
7.	$a + c - b$	8.	$c - a + b$	9.	$-a(b + c)$
10.	$-a(-b - c)$	11.	$c + a - b$	12.	$c - b + a$
13.	$ab - ac$	14.	$-(b - c) + a$	15.	$-c + a - b$
16.	$-[c - (a - b)]$	17.	$-ab - ac$	18.	$-(b - a - c)$
19.	$-a(b - c)$	20.	$-(c - a - b)$	21.	$-a + c + b$
22.	$-(ab - ac)$	23.	$-(ac + ab)$	24.	$-(-c + b - a)$
25.	$-(ab + ac)$	26.	$-a + (c - b)$	27.	$ac - ab$
28.	$-(-ab - ac)$	29.	$c - (-a - b)$	30.	$a - b + c$
31.	$a - b - c$	32.	$(a - b) - c$	33.	$-(c + a) - b$
34.	$-(ac - ab)$	35.	$-(b - a) - c$	36.	$(a - b) + c$
37.	$ab + ac$	38.	$b - (a + c)$	39.	$c - (-b - a)$
40.	$a(b + c)$				

*

Here is the sorting for the quiz.

<table>
<tr>
<td rowspan="2">(1) $a - (b - c)$
(7) $a + c - b$
(11) $c + a - b$
(12) $c - b + a$
(14) $-(b - c) + a$
(18) $-(b - a - c)$
(24) $-(-c + b - a)$
(30) $a - b + c$
(36) $(a - b) + c$</td>
<td>(2) $-a + c - b$
(5) $c - a - b$
(26) $-a + (c - b)$</td>
<td>(3) $a(b - c)$
(13) $ab - ac$
(34) $-(ac - ab)$</td>
</tr>
<tr>
<td>(4) $c + a + b$
(29) $c - (-a - b)$
(39) $c - (-b - a)$</td>
<td>(6) $-ab + ac$
(19) $-a(b - c)$
(22) $-(ab - ac)$
(27) $ac - ab$</td>
</tr>
<tr>
<td>(8) $c - a + b$
(21) $-a + c + b$</td>
<td>(9) $-a(b + c)$
(17) $-ab - ac$
(23) $-(ac + ab)$
(25) $-(ab + ac)$</td>
<td>(10) $-a(-b - c)$
(28) $-(-ab - ac)$
(37) $ab + ac$
(40) $a(b + c)$</td>
</tr>
<tr>
<td rowspan="2">(15) $-c + a - b$
(16) $-[c - (a - b)]$
(31) $a - b - c$
(32) $(a - b) - c$
(35) $-(b - a) - c$</td>
<td>(20) $-(c - a - b)$</td>
<td>(33) $-(c + a) - b$</td>
</tr>
<tr>
<td>(38) $b - (a + c)$</td>
<td></td>
</tr>
</table>

For maximum benefit, students should be required to write the expressions [not just the corresponding numerals] in columns. This will compel them to compare equivalent expressions.

Answers for Part D [on pages 2-77, 2-78, 2-79, and 2-80].

1. $-3a + 2b$
2. $8x - 8y$ [or: $8(x - y)$]
3. $3k - 2r$
4. $-2q + 7$
5. $7x + 8y$
6. $-4a$
7. $6t - s$
8. $4x + y$
9. $2x + 5y$
10. $3m - 9n$
11. $-5xx + 3x$
12. $5aa - 3a + 13b$
13. $12a - 3b - 6c$
14. $3x - 10y$
15. $3r - 2k - 7rk$
16. $16t - 3st - 6s$
17. $(7 + \pi)x - 9y$
18. $9.7a - 2.7b$
19. $(a + b + c)x$
20. $a + 4cc + 4ac$
21. $b + 4d + bd$
22. $(x - y)(u + v)$
23. $24a - 45b$ [or: $3(8a - 15b)$]
24. $14m - 30n$ [or: $2(7m - 15n)$]
25. $10x - 29y$
26. $-6x - 22y - z$
27. $24a - 9b$
28. $-7M + 19N + 21$
29. $-26x - 20y$ [or: $-2(13x + 10y)$]
30. $-13a - 38b$
31. $-4aa + 15ab + 3bb$
32. $36xx + 31xy$ [or: $x(36x + 31y)$]
33. $4x + 3y$
34. $8a + 6b$ [or: $2(4a + 3b)$]
35. $-m + n$ [or: $n - m$]
36. $k - j$
37. $13x - 5y - 3$
38. $10r + 5s + 7$
39. $-8a - 3b$ [or: $-(8a + 3b)$]
40. $-3x - 6y$ [or: $-3(x + 2y)$]

D. Simplify.

Sample 1. $3x - 2y - 7x$

Solution. $3x - 2y - 7x$

$$= 3x + -2y + -7x$$

$$= 3x + -7x + -2y$$

$$= 3x - 7x - 2y$$

$$= (3 - 7)x - 2y$$

$$= -4x - 2y.$$

[Read '$-4x - 2y$' as 'the opposite of 4, times x, minus, 2 times y'.]

[Note: In doing simplification exercises you do not need to write all the steps. As in earlier simplification exercises, you should look for short cuts, and be able to justify them by referring to principles or to theorems which you can prove from the principles.]

1. $5a + 2b - 8a$
2. $3x - 7y + 5x - y$
3. $9k - 3r - 6k + r$
4. $7p - 2q - 6p - p + 7$
5. $8x - y - x + 9y$
6. $5a - a - a - 7a$
7. $2s + 5t - 3s + t$
8. $5x + 3y + -x + -2y$
9. $x + 3y - -x - -2y$
10. $8m - 3n - 6n + -5m$
11. $2xx + 3x - 7xx$
12. $5aa + 6b - 3a + 7b$
13. $7a - 3b - 6c + 5a$
14. $-x - 3y + 4x - 7y$
15. $-2k + 3r - 7rk$
16. $9t - 3st + 7t - 6s$
17. $\pi x + 7x - 9y$
18. $3.4a - 2.7b + 6.3a$
19. $(a + b)x + cx$
20. $a + 2c(a + 2c) + 2ac$
21. $b + 2d + (b + 2)d$
22. $(x - y)u + (x - y)v$

[More exercises are in Part I, Supplementary Exercises.]

Sample 2. $5(3x - 4y) + 6x$

Solution. $5(3x - 4y) + 6x$

$$\begin{aligned}
&= 5(3x) - 5(4y) + 6x \\
&= (5 \cdot 3)x - (5 \cdot 4)y + 6x \\
&= 15x - 20y + 6x \\
&= 15x + -20y + 6x \\
&= 15x + 6x - 20y \\
&= 21x - 20y.
\end{aligned}$$

23. $9(2a - 5b) + 6a$
24. $7(2m - 3n) - 9n$
25. $8(x - 3y) + 2x - 5y$
26. $3(-2x - 7y) - y - z$
27. $2(4a - 5b) + 3(7a - 3b) + 5(2b - a)$
28. $9(2M - 3N) + 11(N - M) + 7(3 - 2M + 5N)$
29. $-6(5x + 3y) + 4x - 2y$
30. $-3(2a + 13b) - 7a + b$
31. $-4a(a - 3b) + 3b(a + b)$
32. $-7x(-5x - 4y) + xx + 3yx$

[More exercises are in Part I, Supplementary Exercises.]

Sample 3. $3r + 5s - (2r - 7s)$

Solution. $3r + 5s - (2r - 7s)$

$$\begin{aligned}
&= 3r + 5s - 2r + 7s \\
&= r + 12s.
\end{aligned}$$

33. $5x + 2y - (x - y)$
34. $7a + 5b - (-a - b)$
35. $3m - 2n - (4m - 3n)$
36. $8k - 3j - (7k - 2j)$
37. $11x - (3 - 2x + 5y)$
38. $15r - (5r - 2s - 7) + 3s$
39. $a - (3b + 7a) - 2a$
40. $x - (6x - 3y) - (9y - 2x)$

[More exercises are in Part I, Supplementary Exercises.]

Answers for Part D [on pages 2-79 and 2-80].

41. $38y - 14x$ [or: $2(19y - 7x)$]

42. $38m + 18n$ [or: $2(19m + 9n)$]

43. $-3k + 6j$ [or: $3(2j - k)$]

44. $36r - 45s$ [or: $9(4r - 5s)$]

45. $28x + 13y$

46. $20x + 45y$ [or: $5(4x + 9y)$]

47. $62y$

48. $x + xz + 5yz + 15 + 20y - z$

49. $7A + 26B - 24$

50. $-26xz + 14yz$ [or: $2z(7y - 13x)$]

51. $-12aa - ab + 15bb$

52. $4x - 12y$ [or: $4(x - 3y)$]

53. $-x + 22y + 6$

54. $-14 + 23a - 38b$

55. $-45 + 21x + 21y$ [or: $3(7x + 7y - 15)$]

56. $-13a - 131b$

57. $-70xyz$

58. $60abc$

59. $66mpq$

60. $-42ABB$

61. $-24XYZ$

62. $-XYZ$

63. $30aaabbb$

64. $42xxyy$

65. $-3ppqr$

66. $6kkkmmm$

67. $-13aab + 27abb - abc$ [or: $-ab(13a - 27b + c)$]

68. $4x + 22xy - 36xx$ [or: $2x(2 + 11y - 18x)$]

69. $18bb + 48ab - 15aa$ [or: $3(6bb + 16ab - 5aa)$]

70. $12x - 24xx + 26xy + 125y - 75yy$

71. $9x + 1.5y$

72. $-14gg + 5gh + 2gy - 10g$

73. $-11\triangle\triangle + 2\triangle\square$

74. $-33rst + 35stt$

75. $-11pq - 4p + 8q$

76. $21ccd - 2cdd + 24cd$ [or: $cd(21c - 2d + 24)$]

Sample 4. $5x - 6(x - 3y)$

Solution. $5x - 6(x - 3y)$

$= 5x + -6(x + -3y)$

$= 5x + [-6x + -6 \cdot -3y]$

$= 5x + [-6x + (-6 \cdot -3)y]$

$= 5x + [-6x + (6 \cdot 3)y]$

$= 5x + [-6x + 18y]$

$= 5x + -6x + 18y$

$= 5x - 6x + 18y$

$= (5 - 6)x + 18y$

$= -1x + 18y$

$= -x + 18y.$

41. $3y - 7(2x - 5y)$
42. $8m - 6(-3n - 5m)$
43. $7k - 2(5k - 3j)$
44. $18r - 9(5s - 2r)$
45. $4x - 3y - 8(-3x - 2y)$
46. $5(2x + 7y) - 10(-x - y)$
47. $9x - (x - y) - 3(5x - 11y) - 7(-2x - 4y) - 7x$
48. $-4x - z(x - 2y) - 5(-x - 3 - 4y) - z(1 - 2x - 3y)$
49. $6A - 3(A - B + 1) - 7(2A + 3 - 5B) - 6(2B - 3A)$
50. $-z(11x - 2y) - 3z(5x - 4y)$
51. $-2a(6a - 2b) - 5b(a - 3b)$
52. $-x - 3y - 4(-x + 2y) - y + x$
53. $5x + 4y - 6(x - 3y - 1)$
54. $-5(2 - 3a + 5b) - 7(1 - a + 2b) - (-a - b - 3)$
55. $-6(5 - x - 3y) - 3[5 - 3x - 4y - (2x - 3y)]$
56. $-2[3(a - b) - 7(2a - 3b)] - 5[-6(-a - 3b) - (-a - b)]$

[More exercises are in Part I, Supplementary Exercises.]

<u>Sample</u> <u>5</u>. $3a(-2b)(-4c)(-d)$

<u>Solution</u>. $3a(-2b)(-4c)(-d)$

$= -[3a(2b)(4c)d]$

$= -[(3 \cdot 2 \cdot 4)\ abcd]$

$= -24abcd.$

<u>Sample</u> <u>6</u>. $-5x(-2y)(-3z)(-2x)$

<u>Solution</u>. $-5x(-2y)(-3z)(-2x)$

$= (5 \cdot 2 \cdot 3 \cdot 2)xxyz$

$= 60xxyz.$

57. $7x(2y)(-5z)$

58. $6a(-2b)(-5c)$

59. $-2m(-3p)(11q)$

60. $2A \cdot 7B \cdot -3B$

61. $-3X \cdot -2Y \cdot -4Z$

62. $-X \cdot -Y \cdot -Z$

63. $3ab \cdot 2ab \cdot 5ab$

64. $-2x \cdot -3y \cdot -xy \cdot -7$

65. $-3q \cdot -p \cdot -rp$

66. $-2mk \cdot -3km \cdot -m \cdot -k$

67. $3ab(-2a + 7b) - 7aab + 5abb - acb + bba$

68. $-2x(3 - 2y + 7x) - 5x(4x - 3y - 2) - 2xx + 3xy$

69. $-3a(5a - 2b) - 6b(-7a - 3b) - 3b \cdot 2a - 6a \cdot -b$

70. $4x[3 - 2(3x - 7y)] - 5y[-4 - 3(7 - 2x - 5y)]$

71. $1.8(-2.5y) + 6(3x + 2y) - 3(3x + 2y)$

72. $-5g(3g - 2h) + \frac{1}{2}g(2g + 4y) - 5g \cdot h + 10 \cdot -g$

73. $-\triangle(3\triangle - 4\square) - 10\triangle(-\triangle - \square) - 3\triangle(4\square) + 3\triangle(-6\triangle)$

74. $-\frac{3}{5}s(4t)(-1\frac{2}{3}r) + 3s(-2r)(5t) - 7st(r - 5t)$

75. $6p(-3q) - 2(3p - 5q) + 2(p - q) - 7q(-p)$

76. $5cd(9c - 4d) - 4c(9d) + 5d \cdot 12c + 6cd(3d - 4c)$

[More exercises are in <u>Part</u> <u>I</u>, <u>Supplementary</u> <u>Exercises</u>.]

2.07 Division.--In Unit 1 you learned that dividing by a number is the inverse of multiplying by that number. For example,

$$({}^{+}3 \times {}^{-}4) \div {}^{-}4 = {}^{+}3$$

$$\text{and} \quad ({}^{-}12 \times {}^{+}5) \div {}^{+}5 = {}^{-}12.$$

If you wish to do a division problem, such as:

$${}^{+}16 \div {}^{-}2 = ?,$$

you can do it by solving the problem:

$$? \times {}^{-}2 = {}^{+}16.$$

You also learned that there are several numerals which name the quotient of ${}^{+}16$ by ${}^{-}2$. Probably the simplest looking is '${}^{-}8$'. Others are '${}^{+}16 \div {}^{-}2$' itself, and the fractions '$\frac{{}^{+}16}{{}^{-}2}$' and '${}^{+}16/{}^{-}2$'.

DOES MULTIPLYING BY 0 HAVE AN INVERSE?

Since division is the inverse of multiplication, and subtraction is the inverse of addition, division and subtraction have much in common. But, they differ in one important respect.

It is always possible to subtract a second real number from a first real number, but you can divide a first real number by a second real number only if the second is not 0.

Let's look into this problem of "dividing by 0". Do you recall from Unit 1 the description of an operation and an inverse operation as sets of pairs of numbers? We can construct a list of pairs which belong, for example, to dividing by 2, by first constructing a list of pairs which belong to multiplying by 2. You get a pair which belongs to dividing by 2 by interchanging the components of a pair which belongs to multiplying by 2.

multiplying by 2		
(5, 10)	(17, 34)	(−2, −4)
(0, 0)	(9, 18)	(1, 2)
(−1, −2)	$(\frac{3}{5}, \frac{6}{5})$	$(\frac{3}{2}, 3)$
(10, 20)	(18, 36)	(2.5, 5)
(4.5, 9)	$(\frac{1}{2}, 1)$	(−4, −8)
	. . .	

dividing by 2		
(10, 5)	(34, 17)	(−4, −2)
(0, 0)	(18, 9)	(2, 1)
(−2, −1)	$(\frac{6}{5}, \frac{3}{5})$	$(3, \frac{3}{2})$
(20, 10)	(36, 18)	(5, 2.5)
(9, 4.5)	$(1, \frac{1}{2})$	(−8, −4)
	. . .	

To do a problem which involves dividing by 2, for example:

$$18 \div 2 = ?$$

we look for a pair which belongs to dividing by 2 and which has first number 18. This is the pair (18, 9). So, the solution of the problem '18 ÷ 2 = ?' is 9.

Now, let's get a list of pairs which belong to "dividing by 0". This set of pairs ought to be the inverse of multiplying by 0.

multiplying by 0			
(4, 0)	(−3, 0)	(0, 0)	$(\frac{3}{5}, 0)$
(−2, 0)	(10, 0)	(17, 0)	$(\pi, 0)$
(93, 0)	. . .	(378, 0)	

So, some of the pairs in "dividing by 0" would be

"dividing by 0"			
(0, 4)	(0, −3)	(0, 0)	$(0, \frac{3}{5})$
(0, −2)	(0, 10)	(0, 17)	$(0, \pi)$
(0, 93)	. . .	(0, 378)	

Now, let's try to use this list to solve a "division by 0" problem:

$$93 \div 0 = ?.$$

We want to find a pair belonging to "division by 0" whose first number is 93. A glance at the list shows us no such pair. Perhaps we need a bigger list! Do you think a bigger list would help? The principle for multiplying by 0 tells us that there can be <u>no</u> pair in the multiplying by 0 list with second number 93. So, there can be no pair in the "dividing by 0" list with first number 93. This means that there is no solution to the problem:

$$93 \div 0 = ?.$$

So, this means that marks such as '$93 \div 0$' and '$\frac{93}{0}$' and '93/0' are not numerals. <u>There is no number whose name is '$93 \div 0$' or '$\frac{93}{0}$' or '93/0'</u>.

What about the problem:

$$0 \div 0 = ?.$$

Once again, we go to the list for "dividing by 0" to seek a pair, this time one with first number 0. The principle for multiplying by 0 tells us that every pair in the "dividing by 0" list has 0 for first number [Explain.]. So, if '$0 \div 0$' <u>were</u> a numeral, it would have to be a numeral for each number! If this were the case, we would have, for example,

$$0 \div 0 = 17 \text{ and } 0 \div 0 = 10.$$

And, this would mean that

$$17 = 10,$$

which is certainly <u>not</u> the case. The only way out of this unpleasant situation is to decide, as in the case of '$93 \div 0$', that '$0 \div 0$' is not a numeral.

We conclude, then, that "dividing by 0" is not an operation; multiplying by 0 does not have an inverse. A short way of saying this is:

YOU CAN'T DIVIDE BY ZERO!

[NOBODY CAN!]

EXERCISES

A. You have seen that you can't divide by 0. Here are some exercises which ask you to divide 0 by numbers other than 0. Fill in the blanks to make true sentences.

1. $0 \div 8 =$ ________ because ________ $\cdot 8 = 0$.

2. $0 \div -3 =$ ________ because ________ $\cdot -3 = 0$.

3. $\frac{0}{5} =$ ________ because ________ $\cdot 5 = 0$.

4. For each $x \neq 0$, $\frac{0}{x} =$ ______ because, for each x, ______ $\cdot x = 0$.

B. Fill in the blanks to make true sentences.

1. For each x, $x \div 1 =$ ______ because, for each x, ______ $\cdot 1 = x$.

2. For each x, $x \div -1 =$ ______ because, for each x, ______ $\cdot -1 = x$.

3. For each $x \neq 0$, $x/x =$ ______ because, for each x, ______ $\cdot x = x$.

[Give names to the first generalization in each of Exercises 1 and 2].

The exercises in Part A are placed immediately after the rather strongly stated injunction because there is a tendency for students to feel that as far as division is concerned, 0 must be left out of consideration. Having learned that you can't divide by 0, students also conclude [incorrectly] that 0 can't be divided by a number.

*

Answers for Part A.

1. 0, 0 2. 0, 0 3. 0, 0 4. 0, 0

*

Answers for Part B.

1. x, x 2. $-x$, $-x$ 3. 1, 1

[The theorems of Part B are Theorem 50, Theorem 52, and Theorem 51 respectively. Their proofs are assigned as exercises on page 2-91.]

*

Exercise 4 of Part A and Exercise 3 of Part B contain the first examples of a restricted quantifying phrase: For each $x \neq 0$. [The next occurrence is in (*) on page 2-85--'$\forall_{y \neq 0}$'.] The phrase 'For each $x \neq 0$' may be read as 'For each x unequal to 0', or 'For each x other than 0', or 'For each x different from 0', or 'For each nonzero x'.

We have already accepted the convention that [in the absence of restrictions] the set of values of our pronumerals is the set of real numbers. A restriction such as '$y \neq -1$' tells you that the set of values of the pronumeral 'y' is the set of real numbers different from -1. And this means that in working with expressions which contain 'y', the only admissible values of 'y' are real numbers different from -1. Consider the generalization:

$$(\Phi) \qquad \forall_{x \neq 0}\ \frac{0}{x} = 0.$$

Since the set of admissible values of 'x' does not contain 0, this generalization does not have as instances meaningless expressions like:

$$`\frac{0}{0} = 0\text{' and '}\frac{0}{2 \cdot 5 - 10} = 0\text{'.}$$

[Note that since '0/0' and '0/(2 • 5 - 10)' are neither numerals nor pronumeral expressions, the expressions displayed above are <u>not</u> sentences.] So, with this convention as to the meaning of restricted quantifying phrases, each expression which results from substituting a numeral for an <u>admissible</u> value of 'x' in the pattern sentence of (⚓) is a sentence [instance], and is a consequence of generalization (⚓).

Note that it is not correct to paraphrase (⚓) by:

$$(⚓⚓) \qquad \forall_x \text{ if } x \neq 0 \text{ then } \frac{0}{x} = 0.$$

This expression is not a sentence because, if it were, it would have:

$$\text{if } 0 \neq 0 \text{ then } \frac{0}{0} = 0$$

as a consequence. But, this last expression is not a sentence [since '0/0' is not a numeral] and, so, has no meaning. Since we do not want a sentence to have meaningless consequences, we do not regard (⚓⚓) as a sentence. [Sometimes it is necessary to substitute a pronumeral expression, rather than a numeral, for a restricted pronumeral [as might occur when writing a test-pattern]. The procedure to be followed in such cases is best illustrated in context. For this, see COMMENTARY for Sample 3 on page 2-89.]

In (1) - (6) read '?' as 'what'.

*

The purpose of the subsection on page 2-85 is to lead students, through the recollection of some computation facts, to re-examine their belief that multiplication by a nonzero number has an inverse, and to recognize (*) as a statement of this belief.

*

The principle of quotients, referred to in the next-to-last sentence on page 2-85, has already been given on page 2-61, and appears again at the top of page 2-86.

*

As already remarked, on TC[2-60]c, our treatment of division is <u>not</u> analogous to our treatment of subtraction. You may find it helpful to reread <u>that</u> COMMENTARY now.

*

The division theorem [see page 2-86] plays the role with respect to division which the 0-sum theorem plays with respect to opposition. Just as the latter suggests other theorems concerning opposites [see TC[2-69]a; in particular, the discussion of Exercise 1 of Part A], and plays an important part in the proofs of such theorems, so the division theorem suggests other theorems concerning quotients, and is a powerful tool for use in proving them. [Some illustrations to this effect are given in the last paragraph on page 2-86 and another occurs in the first paragraph on page 2-87.]

*

The division theorem bears a stronger formal resemblance to (*) on page 2-68 than it does to the equivalent 0-sum theorem. In some cases it would be slightly more convenient to have a more exact analogue of the latter. On this point, see the COMMENTARY for Exercise 2 on page 2-91.

DOES MULTIPLYING BY A NONZERO NUMBER HAVE AN INVERSE?

Tell how many solutions there are for the problem:

$$(1) \quad ? \cdot 0 = 93.$$

Tell how many solutions there are for the problem:

$$(2) \quad ? \cdot 0 = 0.$$

We have seen that the first problem has no solutions, and that the second problem has many. It was these answers which showed us that multiplying by 0 does not have an inverse.

Now, tell how many solutions there are for the problem:

$$(3) \quad ? \cdot {}^{-}2 = {}^{+}16.$$

Clearly, $^{-}8$ is one solution. Is there another solution? How about these problems:

$$(4) \quad ? \cdot {}^{-}2 = {}^{+}12 \qquad (5) \quad ? \cdot {}^{-}2 = {}^{-}41 \qquad (6) \quad ? \cdot {}^{-}2 = 0.$$

How many solutions does each of these have?

When we say that multiplying by $^{-}2$ has an inverse operation, we mean precisely that for each first number, you can find just one number whose product by $^{-}2$ is that first number. That is, we mean that each problem like (3), (4), (5), and (6) has exactly one solution. In other words,

for each x, there is just one z such that $z \cdot {}^{-}2 = x$.

What we have said about multiplying by $^{-}2$ could be said about multiplying by any nonzero number. So, multiplying by each nonzero real number has an inverse just if

$$(*) \qquad \forall_x \forall_{y \neq 0} \text{ there is just one } z \text{ such that } z \cdot y = x.$$

Since we believe it to be the case that multiplication by each nonzero real number does have an inverse, we want (*) to be a theorem. One way to make sure that it is a theorem would be to take (*) itself as one of our basic principles. But, as you will see, it is sufficient to take the principle of quotients as a basic principle. Using the principle of quotients and the other basic principles we can derive (*).

QUOTIENTS

The principle of quotients:

$$\forall_x \forall_{y \neq 0}\ (x \div y) \cdot y = x$$

tells us, for example, that, since $5 \neq 0$, $6 \div 5$ is a real number whose product by 5 is 6. In fact, it tells us that, for each x, and for each nonzero y,

there is at least one z [the quotient of x by y] such that $z \cdot y = x$.

Hence, in order to establish (*), it is sufficient to show that there is no number other than $x \div y$ whose product by y is x--that is, it is sufficient to prove:

$$\forall_x \forall_{y \neq 0} \forall_z \text{ if } z \cdot y = x \text{ then } z = x \div y.$$

We shall call this generalization the division theorem, and you will prove it later in an exercise.

Since '$\frac{x}{y}$' is an abbreviation for '$(x \div y)$', the principle of quotients can be written:

$$\boxed{\forall_x \forall_{y \neq 0}\ \frac{x}{y} \cdot y = x.}$$

The principle of quotients tells you, for example, that

$$(6 \div 2) \cdot 2 = 6, \quad \frac{36}{4} \cdot 4 = 36, \quad \frac{1}{9} \times 9 = 1, \quad \frac{3 \cdot 7}{5 \cdot 7}(5 \cdot 7) = 3 \cdot 7.$$

Similarly, the division theorem can be written:

$$\forall_x \forall_{y \neq 0} \forall_z \text{ if } z \cdot y = x \text{ then } z = \frac{x}{y}.$$

It tells us, for example, that

$$\text{if } 3 \cdot 2 = 6 \text{ then } 3 = \frac{6}{2}.$$

So, since $3 \cdot 2 = 6$, we know from the division theorem that $3 = \frac{6}{2}$;

$$\frac{6}{2} = 3 \ \underline{\text{because}}\ \ 3 \cdot 2 = 6.$$

Similarly,

$$\frac{36}{4} = 9 \ \underline{\text{because}}\ \ 9 \cdot 4 = 36,$$

$$(5 \times 6) \div 6 = 5 \ \underline{\text{because}}\ \ 5 \cdot 6 = 5 \times 6,$$

$$\frac{3 \cdot 7}{5 \cdot 7} = \frac{3}{5} \ \underline{\text{because}}\ \ \frac{3}{5}(5 \cdot 7) = (\frac{3}{5} \cdot 5)7 = 3 \cdot 7.$$

[In the last example we also used the principle of quotients [Where?]. What other principle did we use?]

Point out casually to students that the division theorem is equivalent to:

$$\forall_x \forall_{y \neq 0} \forall_z \text{ if } z \cdot y = x \text{ then } x \div y = z.$$

*

The proof of the division theorem is Exercise 1 on page 2-91. [Students have, on page 2-89, a proof of an analogous "subtraction theorem" which should serve them as a model.]

*

Notice that each of the four "<u>because</u>-sentences" at the bottom of page 2-86 is of the form:

$$\frac{x}{y} = z \text{ \underline{because} } z \cdot y = x.$$

Such a sentence records a derivation of the form:

$$\dfrac{\dfrac{z \cdot y = x \qquad \dfrac{\text{the division theorem}}{\text{if } z \cdot y = x \text{ then } z = x \div y}}{z = x \div y}}{\dfrac{x}{y} = z}$$

*

The fourth of the "<u>because</u>-sentences" anticipates the proof of Theorem 60, which is required of students on page 2-94. The discussion of Sample 1 of Part B on page 2-88 furnishes another illustration of how the division theorem can be used to suggest other theorems concerning quotients, and how the principle of quotients and the division theorem cooperate in the proofs of such theorems.

*

Answers to questions in brackets at bottom of page 2-86: In justifying the second equality; the apm.

*

It may be helpful to use the following as a 'pattern sentence" for the division theorem:

if quotient · divisor = dividend

then quotient = dividend ÷ divisor.

Your students may find the proof at the top of page 2-87 easier to follow if they have seen the pattern sentence referred to on TC[2-86]. [Also, see TC[2-91, 92]a.]

*

Answers for Part A.

1. $\frac{7}{3} + \frac{5}{3}$

2. $6 \times \frac{4}{3}$

3. $\frac{6 \times 4}{3}$

4. $\frac{9 \times 5}{2 \times 4}$

5. $\frac{9 \times 5}{2} \times 4$

6. $\frac{6}{2} \times \frac{9}{3}$

7. $\dfrac{\frac{6}{2} \times 9}{3}$

8. $\dfrac{\frac{6}{2 \times 9}}{3}$

9. $\dfrac{\frac{10}{5}}{\frac{4}{3}}$

[The convention that, in a complex fraction, the "principal" fraction-bar is longer than the others is easier to grasp from an oral explanation accompanied by chalk board illustrations than from a written discussion. So, we have left this explanation to you.]

Another use of the division theorem is to prove such generalizations as:

$$\forall_x \ \frac{5x - 5}{5} = x - 1.$$

Here is a test-pattern.

$$\begin{aligned} (x - 1)5 &= x5 - 1 \cdot 5 && [\text{dtms}] \\ &= 5x - 5 \cdot 1 && [\text{cpm}] \\ &= 5x - 5. && [\text{pm1}] \end{aligned}$$

Hence, $\quad x - 1 = \dfrac{5x - 5}{5}.$ $\quad$ [division theorem]

Notice that while the division sign '÷', like '+', '×', and the subtraction sign '-', has associated with it a pair of grouping symbols [which, according to the conventions you learned in Unit 1, can often be omitted], the fraction-bar '——' does not require grouping symbols. For example, although

'$3 \times (5 \div 6)$' <u>may</u> <u>not</u> be abbreviated to '$3 \times 5 \div 6$'

[since the latter is an abbreviation for '$(3 \times 5) \div 6$'],

'$3 \times (5 \div 6)$' <u>may</u> be abbreviated to '$3 \times \frac{5}{6}$'.

Furthermore, the use of the fraction-bar instead of '÷', often permits one to omit other grouping symbols. For example,

'$3 \times [(5 + 4) \div (6 - 3)]$' may be abbreviated to '$3 \times \frac{5 + 4}{6 - 3}$',

thus getting rid of two pairs of parentheses, as well as the pair of brackets. In general, the outermost grouping symbols in both the numerator and the denominator of a fraction may be omitted.

EXERCISES

<u>A</u>. Abbreviate each of the following expressions by using fraction-bars instead of division signs and as few grouping symbols as necessary.

1. $(7 \div 3) + (5 \div 3)$
2. $6 \times (4 \div 3)$
3. $(6 \times 4) \div 3$
4. $9 \times 5 \div (2 \times 4)$
5. $9 \times 5 \div 2 \times 4$
6. $6 \div 2 \times (9 \div 3)$
7. $6 \div 2 \times 9 \div 3$
8. $6 \div (2 \times 9) \div 3$
9. $10 \div 5 \div (4 \div 3)$

D. You can use the division *theorem* [and the basic principles] to simplify expressions containing fractions. Simplify each of the following, and show how the division theorem justifies the simplification.

Sample 1. $\frac{2}{3} + \frac{5}{7}$

Solution. The division theorem tells us that, for each x, and for each nonzero y,

$$\text{if } (\frac{2}{3} + \frac{5}{7})y = x \quad \text{then } \frac{2}{3} + \frac{5}{7} = \frac{x}{y}.$$

So, to simplify '$\frac{2}{3} + \frac{5}{7}$' requires finding simple numerals for a number x and a nonzero number y such that $(\frac{2}{3} + \frac{5}{7})y = x$. Now, for each y, $(\frac{2}{3} + \frac{5}{7})y = \frac{2}{3}y + \frac{5}{7}y$. If we use the value $3 \cdot 7$ for 'y', the corresponding value of '$\frac{2}{3}y + \frac{5}{7}y$' is $\frac{2}{3}(3 \cdot 7) + \frac{5}{7}(3 \cdot 7)$. The principle of quotients together with the associative and commutative principles for multiplication tell us that this value is $2 \cdot 7 + 5 \cdot 3$. So, we know that

$$(\frac{2}{3} + \frac{5}{7})(3 \cdot 7) = 2 \cdot 7 + 5 \cdot 3,$$

and the division theorem tells us that $\frac{2}{3} + \frac{5}{7} = \frac{2 \cdot 7 + 5 \cdot 3}{3 \cdot 7}$. So, '$\frac{2}{3} + \frac{5}{7}$' simplifies to '$\frac{29}{21}$'.

[Note: Do you see that this justifies, in this instance, the rule for "adding fractions by finding a common denominator"?]

Sample 2. $\frac{a}{10} - \frac{b}{5}$

Solution. $(\frac{a}{10} - \frac{b}{5})10 = \frac{a}{10} \cdot 10 - \frac{b}{5} \cdot 10 = a - 2b$.

So, by the division theorem, '$\frac{a}{10} - \frac{b}{5}$' simplifies to '$\frac{a - 2b}{10}$'.

[We say that '$\frac{a - 2b}{10}$' is simpler because it is a single fraction, whereas '$\frac{a}{10} - \frac{b}{5}$' contains two fractions.]

1. $\frac{7}{8} + \frac{2}{3}$
2. $\frac{3}{5} - \frac{2}{15}$
3. $\frac{8}{3} + 0.7$
4. $\frac{x}{3} + \frac{x}{5}$
5. $\frac{y}{6} - \frac{y}{2}$
6. $\frac{3x}{2} - \frac{5y}{7}$

Even though your students know a rule for "adding fractions", they should solve the exercises of Part B by the method illustrated in the samples. The main point of these exercises is to show students how solutions to problems involving fractions can be discovered by paying attention to the division theorem, and how such solutions can be verified by the use of this theorem and the principle of quotients. Also, these exercises prepare students for proving Theorem 57 and Theorem 58 on page 2-92.

Students will have opportunity to practice the short cuts suggested by these theorems while doing the exercises on pages 2-100ff.

*

You may find it helpful to write the solution for Sample 1 as follows.

$$
\begin{aligned}
&\left(\frac{2}{3} + \frac{5}{7}\right)(3 \cdot 7) \\
&= \frac{2}{3}(3 \cdot 7) + \frac{5}{7}(3 \cdot 7) && \text{dpma} \\
&= \frac{2}{3}(3 \cdot 7) + \frac{5}{7}(7 \cdot 3) && \text{cpm} \\
&= \left(\frac{2}{3} \cdot 3\right)7 + \left(\frac{5}{7} \cdot 7\right)3 && \text{apm} \\
&= 2 \cdot 7 + 5 \cdot 3 && \text{pq} \\
&= 29. && 2 \cdot 7 = 14,\ 5 \cdot 3 = 15,\ 14 + 15 = 29
\end{aligned}
$$

Hence, $\left(\frac{2}{3} + \frac{5}{7}\right)21 = 29$. $[3 \cdot 7 = 21]$

So, $\frac{2}{3} + \frac{5}{7} = \frac{29}{21}$. [division theorem]

[In applying the principle of quotients, we also use the fact that $3 \neq 0$ and that $7 \neq 0$. We shall, in the future, call attention to such additional hypotheses, but it is not a serious fault if students take such numerical facts for granted and so fail to cite them.]

*

Here is an expanded form of the Solution for Sample 2.

$$\begin{aligned}
&\left(\frac{a}{10} - \frac{b}{5}\right)10 \\
&= \frac{a}{10}\cdot 10 - \frac{b}{5}\cdot 10 && \text{dtms} \\
&= \frac{a}{10}\cdot 10 - \frac{b}{5}(5\cdot 2) && 10 = 5\cdot 2 \\
&= \frac{a}{10}\cdot 10 - \left(\frac{b}{5}\cdot 5\right)2 && \text{apm} \\
&= a - b2 && \text{pq } [10 \neq 0,\ 5 \neq 0] \\
&= a - 2b. && \text{cpm}
\end{aligned}$$

Hence, $\left(\frac{a}{10} - \frac{b}{5}\right)10 = a - 2b$.

So, by the division theorem $[10 \neq 0]$, $\frac{a}{10} - \frac{b}{5} = \frac{a - 2b}{10}$.

*

Answers for Part B [on pages 2-88 and 2-89].

1. $\frac{37}{24}$
2. $\frac{7}{15}$
3. $\frac{10.1}{3}$ [or: $\frac{101}{30}$]
4. $\frac{8x}{15}$
5. $\frac{-2y}{6}$ [or: $\frac{-y}{3}$]
6. $\frac{21x - 10y}{14}$

*

If your students have difficulty in seeing the purpose of Part B, it may be because it appears to them that we are trying to teach them something they already know. For example, it is the case that they know from grade school how to transform

$$`\frac{7}{8} + \frac{2}{3}' \quad \text{into} \quad `\frac{37}{24}'.$$

What we are concerned with here is proving that the sum of the quotient of 7 by 8 and the quotient of 2 by 3 is the quotient of 37 by 24. To show that $7 \div 8 + 2 \div 3$ is the quotient of 37 by 24, we show that $(7 \div 8 + 2 \div 3)24$ is 37 and use the division theorem.

The purpose of Part B, then, is to show that the grade school rule for "adding fractions" is consistent with our basic principles. For someone who didn't know the grade school rule, Part B would be a means for his discovering it.

*

Notice that we are using a colloquialism when we say "adding fractions", since fractions are numerals, not numbers. But, the colloquialism is well-established.

*

Quiz.

Simplify.

1. $17m - 7n + 8p - 5n - 10m$	2. $2.5s + 1.3r + 9.4s - 10.6r$
3. $\frac{4}{9}c - \frac{1}{3} - \frac{5}{9}c + 3$	4. $\frac{1}{5}x - \frac{1}{4}y + \frac{4}{5}x - \frac{2}{7}y$
5. $5(3 - 2y) + 7y - 10$	6. $aa + 2ab - 3a(a + 2b)$
7. $n - (5n - 2r) - (7r - 3n)$	8. $4(-c - d) - 2(-2c - 2d)$
9. $-3e \cdot -4f \cdot -5fg$	10. $0.5x(-3y)(-4z)$

*

Answers for Quiz.

1. $7m - 12n + 8p$	2. $11.9s - 9.3r$
3. $-\frac{1}{9}c + \frac{8}{3}$	4. $x - \frac{15}{28}y$
5. $5 - 3y$	6. $-2aa - 4ab$
7. $-n - 5r$	8. 0
9. $-60effg$	10. $6xyz$

9. $\frac{32m + 21k}{12km}$, $[k \neq 0,\ m \neq 0]$

10. $\frac{15y + 14x}{35xy}$, $[x \neq 0,\ y \neq 0]$

11. $\frac{10y + 21x}{35xy}$, $[x \neq 0,\ y \neq 0]$

12. $\frac{ad + cb}{bd}$, $[b \neq 0,\ d \neq 0]$

*

In motivating proofs of the cancellation principle for multiplication [given on page 2-90] and the division theorem we make use of students' familiarity with subtraction by showing how the theory of subtraction could have been developed in a manner parallel to our development of the theory of division. Make sure that students are aware of the formal similarity between (1) [on page 2-89] and the principle of quotients, and between (2) and the division theorem.

Here is an expanded form of the Solution for Sample 3.

$$\left.\begin{aligned}&\left(\frac{3}{x} + \frac{5}{y}\right)(xy)\\ &= \frac{3}{x}(xy) + \frac{5}{y}(xy)\end{aligned}\right\} \text{dpma}$$

$$\left.\begin{aligned}&= \frac{3}{x}(xy) + \frac{5}{y}(xy)\\ &= \frac{3}{x}(xy) + \frac{5}{y}(yx)\end{aligned}\right\} \text{cpm}$$

$$\left.\begin{aligned}&= \frac{3}{x}(xy) + \frac{5}{y}(yx)\\ &= \left(\frac{3}{x}\cdot x\right)y + \left(\frac{5}{y}\cdot y\right)x\end{aligned}\right\} \text{apm}$$

$$\left.\begin{aligned}&= \left(\frac{3}{x}\cdot x\right)y + \left(\frac{5}{y}\cdot y\right)x\\ &= 3y + 5x.\end{aligned}\right\} \text{pq; } x \neq 0,\ y \neq 0$$

Hence, [for $x \neq 0$ and $y \neq 0$,] $\left(\frac{3}{x} + \frac{5}{y}\right)(xy) = 3y + 5x$.

So, $\frac{3}{x} + \frac{5}{y} = \frac{3y + 5x}{xy}$. [division theorem; $xy \neq 0$]

This testing pattern is, because of the restrictions required when citing the principle of quotients, a proof of the theorem:

$$\forall_{x \neq 0} \forall_{y \neq 0}\ \frac{3}{x} + \frac{5}{y} = \frac{3y + 5x}{xy}.$$

In contrast with the premisses '$3 \neq 0$' and '$7 \neq 0$' mentioned in the COMMENTARY for Sample 1, page 2-88, it is essential that students <u>do</u> cite such restrictions as '$x \neq 0$' and '$y \neq 0$'. For, recognition of these is necessary if one is to formulate the generalization proved by the testing pattern, or if one is to state properly the answer to the corresponding simplification problem.

*

Answers for Part B, continued.

7. $\frac{5b + 2a}{ab}$, [$a \neq 0$, $b \neq 0$]

8. $\frac{21y - 10x}{14xy}$, [$x \neq 0$, $y \neq 0$]

[Actually, in solving Exercise 8, one comes out with the restrictions '$2x \neq 0$', and '$7y \neq 0$'. The reduction of these to '$x \neq 0$' and '$y \neq 0$', respectively, should cause students no concern at this time. That '$2x \neq 0$' is a consequence of '$x \neq 0$' and '$2 \neq 0$' follows from the 0-product theorem [See page 2-91.].]

<u>Sample 3.</u> $\frac{3}{x} + \frac{5}{y}$

<u>Solution.</u> $(\frac{3}{x} + \frac{5}{y})(xy)$

$= \frac{3}{x}(xy) + \frac{5}{y}(xy)$
$= 3y + 5x.$ [providing that neither 'x' nor 'y' has the value 0]

So, by the division theorem, $\forall_{x \neq 0} \forall_{y \neq 0}\ \frac{3}{x} + \frac{5}{y} = \frac{3y + 5x}{xy}$.

We write our answer as: $\frac{3y + 5x}{xy}$, $[x \neq 0,\ y \neq 0]$.

7. $\frac{5}{a} + \frac{2}{b}$ 8. $\frac{3}{2x} - \frac{5}{7y}$ 9. $\frac{8}{3k} + \frac{7}{4m}$

10. $\frac{3}{7x} + \frac{2}{5y}$ 11. $\frac{2}{7x} + \frac{3}{5y}$ 12. $\frac{a}{b} + \frac{c}{d}$

PROVING THE DIVISION THEOREM

You have seen how the division theorem can be used in simplifying expressions. It is now time to show that the division theorem is actually a theorem. That is, to derive it from the principle of quotients and the other basic principles. As preparation for doing so, let's consider an analogous situation concerning subtraction.

For subtraction, the analogue of the principle of quotients is:

$$(1) \qquad \forall_x \forall_y\ (x - y) + y = x,$$

and the analogue of the division theorem is:

$$(2) \qquad \forall_x \forall_y \forall_z \text{ if } z + y = x \text{ then } z = x - y.$$

Now, we can use (1) to derive (2) as follows.

Suppose that $z + y = x$.
Then $z + y = (x - y) + y$, [(1)]
and $z = x - y$. [cancellation principle]

Hence, if $z + y = x$ then $z = x - y$.

However, besides (1) we had to use the cancellation principle for addition. Now, it turns out that we can use (1) to derive the cancellation principle [without using either the principle of opposites or the principle for subtraction]. Turn the page to see how.

Suppose that $x + z = y + z$.

Then $x + z + (0 - z) = y + z + (0 - z)$,

$$x + [z + (0 - z)] = y + [z + (0 - z)], \quad [apa]$$

$$x + [(0 - z) + z] = y + [(0 - z) + z], \quad [cpa]$$

$$x + 0 = y + 0, \quad [(1)]$$

and $x = y$. [pa0]

Hence, if $x + z = y + z$ then $x = y$.

So, we <u>could</u> use (1) as a basic principle in place of the principle of opposites and the principle for subtraction. Analogously, you can use the principle of quotients [together with our other basic principles] to prove <u>the cancellation principle for multiplication</u>:

$$\forall_x \forall_y \forall_{z \neq 0} \text{ if } xz = yz \text{ then } x = y.$$

Then, use this and the principle of quotients to prove the division theorem:

$$\forall_x \forall_{y \neq 0} \forall_z \text{ if } z \cdot y = x \text{ then } z = x \div y.$$

[The principle of quotients and the division theorem together tell us that multiplication by each nonzero real number has an inverse. That is, (*) on page 2-85 is a consequence of the principle of quotients and our other basic principles.]

EXERCISES

Prove each of the following theorems.

<u>Sample</u>. "Cancellation principle for multiplication"

$$\forall_x \forall_y \forall_{z \neq 0} \text{ if } xz = yz \text{ then } x = y.$$

<u>Solution</u>. Suppose that $xz = yz$.

Then $xz(1 \div z) = yz(1 \div z)$,

$$x[z(1 \div z)] = y[z(1 \div z)], \quad [apm]$$

$$x[(1 \div z)z] = y[(1 \div z)z], \quad [cpm]$$

$$x \cdot 1 = y \cdot 1, \quad [pq;\ z \neq 0]$$

and $x = y$. [pm1]

Hence, [for $z \neq 0$,] if $xz = yz$ then $x = y$.

[Note that in applying the principle of quotients we had to

Students ought to be able to state the cancellation principle for multiplication before they see it on page 2-90. One way of handling this is to write the uniqueness principle for addition and the cancellation principle for addition, one under the other, on the board. Then, to one side, write the uniqueness principle for multiplication, and ask for a statement of the cancellation principle for multiplication. It is likely that you will get:

$$\forall_x \forall_y \forall_z \text{ if } xz = yz \text{ then } x = y.$$

This, of course, is false. Let the students discover the error by going through the following routine. Ask the class to suppose that Mary has given you a number and that John has given you a number, and that you multiply Mary's number by 3 and you also multiply John's number by 3. It turns out that the products are the same. What can the class conclude about Mary's number and John's number? Now, ask the class to suppose that Jack has given you a number, that Jill has given you a number, and that you multiply each number by 0. Again, it turns out that the products are the same. What conclusion follows? Clearly, one cannot correctly conclude that Jack and Jill picked the same number. So, the correct statement of the cancellation principle for multiplication is:

$$\forall_x \forall_y \forall_{z \neq 0} \text{ if } xz = yz \text{ then } x = y.$$

*

Compare the proof given at the top of page 2-90 for the cancellation principle for addition with the proof given on page 2-65.

If we were to develop the theory of subtraction using (1) on page 2-89 as a basic principle [in place of the po and the ps], we would need an additional basic principle, '$\forall_x \; -x = 0 - x$' to enable us to prove theorems concerning opposition. Similarly, if we now wanted to introduce the operation reciprocation [see TC[2-60]c], we would adopt '$\forall_{x \neq 0} \; /x = 1 \div x$' as an additional basic principle.

*

Some students may profit from discovering a new proof, like that at the top of page 2-90, for the <u>left</u> cancellation principle for addition.

*

The bracketed sentence just before 'EXERCISES' is merely a repetition of what was said in the first paragraph on page 2-86.

*

Note the parallelism between the proofs on page 2-90 of the two cancellation principles.

*

Some students may wish to state the left cancellation principle for multiplication. It can be proved in an analogous manner, or derived from the cancellation principle for multiplication and the cpm.

*

Quiz.

Abbreviate each of the following expressions by using fraction-bars instead of division signs and as few grouping symbols as necessary.

1. $15 \div (5 \times 6) \div 2$
2. $12 \div 3 \times 5 \div 4$
3. $10 \div 2 \times (14 \div 7)$
4. $18 \div 9 \div (5 \div 3)$
5. $7 \times 8 \div (13 \times 4)$
6. $(13 \div 2) + (9 \div 2)$
7. $7 \times 8 \div 13 \times 4$
8. $(21 \times 4) \div 7$
9. $12 \times (5 \div 2)$

*

Simplify.

10. $\frac{5}{8} - \frac{7}{16}$
11. $0.9 + \frac{7}{2}$
12. $\frac{b}{5} - \frac{b}{30}$
13. $\frac{8}{x} + \frac{12}{y}$
14. $\frac{7c}{3} - \frac{5c}{4}$
15. $\frac{r}{s} + \frac{t}{u}$

*

Answers for Quiz.

1. $\dfrac{\frac{15}{5 \times 6}}{2}$
2. $\dfrac{\frac{12}{3} \times 5}{4}$
3. $\frac{10}{2} \times \frac{14}{7}$
4. $\dfrac{\frac{18}{9}}{\frac{5}{3}}$
5. $\frac{7 \times 8}{13 \times 4}$
6. $\frac{13}{2} + \frac{9}{2}$
7. $\frac{7 \times 8}{13} \times 4$
8. $\frac{21 \times 4}{7}$
9. $12 \times \frac{5}{2}$
10. $\frac{3}{16}$
11. $\frac{22}{5}$ [or: 4.4]
12. $\frac{b}{6}$
13. $\frac{8y + 12x}{xy}$, [$xy \neq 0$]
14. $\frac{13c}{12}$
15. $\frac{ru + ts}{su}$, [$su \neq 0$]

Notice that '$\forall_x \forall_y$ if $x = 0$ or $y = 0$ then $xy = 0$' is a rather immediate consequence of the pm0, and that an equivalent statement is:

$$\forall_x \forall_y \text{ if } xy \neq 0 \text{ then } x \neq 0 \text{ and } y \neq 0.$$

Together with the 0-product theorem these tell us that

$$\forall_x \forall_y \; xy = 0 \text{ if and only if } x = 0 \text{ or } y = 0$$

and that

$$\forall_x \forall_y \; xy \neq 0 \text{ if and only if } x \neq 0 \text{ and } y \neq 0.$$

*

The answers to the 'Why?'s in the testing pattern on page 2-92 are, from top to bottom: dpma, cpm, apm.

The test-pattern for the "subtraction of fractions" theorem [Theorem 58] is obtained from that for the "addition of fractions" theorem [Theorem 57] given on page 2-92 by replacing each '+' by a '-' and changing the citation 'dpma' to 'dtms'. [The dtms is Theorem 39.]

*

In your class discussion of the subsection which begins on page 2-92 you should be sure, whenever a theorem has been proved, to ask your students to state some of its instances.

The 0-product theorem [which is (2′) on page 2-91 and listed as Theorem 56 on TC[2-61]g] is, as you will recognize, of fundamental significance in equation-solving. Our immediate concern with it will be to use it [more conveniently in the equivalent form (2)] to show that, for example, '$2x \neq 0$' is implied by '$2 \neq 0$' and '$x \neq 0$' [See COMMENTARY for Exercise 8 of Part B, page 2-89.].

Students should have no difficulty in proving (1), but they may have some difficulty in recognizing that (1) and (1′) are logically equivalent and that (2) and (2′) are logically equivalent. Let us consider, first, (1) and (1′). To "see" that these say the same thing, begin by considering pairs of sentences like:

$\left\{\begin{array}{l}\text{If John is well then he is in school today.}\\ \text{If John is not in school today then he is not well.}\end{array}\right.$

and:

$\left\{\begin{array}{l}\text{If today is a legal holiday then the bank is closed today.}\\ \text{If the bank is not closed today then today is not a legal holiday.}\end{array}\right.$

Once it has been established that the two sentences of such pairs are equivalent it should be sufficient to notice that one can pair up <u>instances</u> of (1) and (1′) in this way. So, (1) and (1′) are equivalent generalizations.

The equivalence of (2) and (2′) poses an additional problem. It should by now be clear that (2) is equivalent to:

(*) $\forall_x \forall_y$ if $xy = 0$ then it is not the case that $x \neq 0$ <u>and</u> $y \neq 0$.

To recognize that (*) is equivalent to (2′) it is necessary to see that 'it is not the case that $x \neq 0$ and $y \neq 0$' is equivalent to '$x = 0$ or $y = 0$'. One way of looking at this is to realize that if one cannot do both of two things then one <u>must</u> decide not to do one or the other of them [and, conversely]. [For a discussion of conditionals and their contrapositives, see TC[2-67, 68]c.]

*

5. $\forall_{x \neq 0}\ \frac{0}{x} = 0$ [Theorem 53]

$$\left.\begin{array}{l} 0 \cdot x \\ = x \cdot 0 \end{array}\right\} cpm \qquad \left.\begin{array}{l} = x \cdot 0 \\ = 0. \end{array}\right\} pm0$$

So, $0 \cdot x = 0$.

Hence, $\frac{0}{x} = 0$. [division theorem; $x \neq 0$]

6. $\forall_x \forall_{y \neq 0}$ if $\frac{x}{y} = 0$ then $x = 0$ [Theorem 54]

Suppose that $\frac{x}{y} = 0$.

Then $\frac{x}{y} \cdot y = 0 \cdot y$,

$x = 0 \cdot y$, [pq; $y \neq 0$]

$x = y \cdot 0$, [cpm]

and $x = 0$. [pm0]

Hence, [for $y \neq 0$,] if $\frac{x}{y} = 0$ then $x = 0$.

*

Notice how Theorem 53 and Theorem 54 complement one another. Theorem 53 is equivalent to '$\forall_x \forall_{y \neq 0}$ if $x = 0$ then $\frac{x}{y} = 0$'. Hence, these two theorems could be combined into one:

$$\forall_x \forall_{y \neq 0}\ [\frac{x}{y} = 0 \text{ if and only if } x = 0]$$

Such "if and only if generalizations" will be introduced in Unit 3.

*

2. $\forall_x \frac{x}{1} = x$ [Theorem 50]

$x \cdot 1 = x.$ [pm1]

Hence, $x = \frac{x}{1}.$ [division theorem; $1 \neq 0$]

So, $\frac{x}{1} = x.$

[In the future, we shall [as your students probably do] take the symmetry of equality for granted and, so, replace '$x = \frac{x}{1}$' in the second line of the proof above by '$\frac{x}{1} = x$', and omit the third line.]

3. $\forall_{x \neq 0} \frac{x}{x} = 1$ [Theorem 51]

$1 \cdot x = x.$ [1 times theorem]

So, $\frac{x}{x} = 1.$ [division theorem; $x \neq 0$]

4. $\forall_x \frac{x}{-1} = -x$ [Theorem 52]

$$\left.\begin{array}{l} -x \cdot -1 \\ = x \cdot 1 \end{array}\right\} \forall_x \forall_y \; -x \cdot -y = xy \; \text{[Th. 23]}$$
$$\left.\begin{array}{l} = x \cdot 1 \\ = x. \end{array}\right\} \text{pm1}$$

So, $-x \cdot -1 = x.$

Hence, $x/-1 = -x.$ [division theorem; $-1 \neq 0$]

Alternatively, the first three lines of the proof might be:

$$\left.\begin{array}{l} -x \cdot -1 \\ = --x \end{array}\right\} \text{multiplying by } -1 \text{ theorem [Th. 27]}$$
$$\left.\begin{array}{l} = --x \\ = x. \end{array}\right\} \forall_x \; --x = x \; \text{[Th. 17]}$$

Answers for Exercises.

1. $\forall_x \forall_{y \neq 0} \forall_z$ if $zy = x$ then $z = \frac{x}{y}$ [Theorem 49]

   Suppose that $zy = x$.

   Then $zy = \frac{x}{y} \cdot y$, [pq; $y \neq 0$]

   and $z = \frac{x}{y}$. [cancellation principle; $y \neq 0$]

   Hence, [for $y \neq 0$,] if $zy = x$ then $z = \frac{x}{y}$.

[Note the parallelism between this proof and that given on page 2-89 for statement (2).]

*

To get started on the remaining proofs, it may help the students if you review the "pattern sentence" for the division theorem which was suggested on TC[2-86]:

if quotient $\cdot$ divisor = dividend

then quotient = dividend $\div$ divisor.

For example, to prove the theorem in Exercise 4, we must derive the sentence:

$$\frac{\overset{\text{dividend}}{x}}{\underset{\text{divisor}}{-1}} = \overset{\text{quotient}}{-x}.$$

So, first we derive the sentence:

$$\overset{\text{quotient}}{-x} \cdot \overset{\text{divisor}}{-1} = \underset{\text{dividend}}{x}.$$

And then we use this sentence together with the division theorem [$-1 \neq 0$] to derive:

$$-x = \frac{x}{-1}.$$

By symmetry of equality, this is equivalent to the sought-for sentence.

introduce the restriction '$z \neq 0$'. So, the test-pattern proves:

$\forall_x \forall_y \forall_{z \neq 0}$ if $xz = yz$ then $x = y$, not: $\forall_x \forall_y \forall_z$ if $xz = yz$ then $x = y$.]

1. $\forall_x \forall_{y \neq 0} \forall_z$ if $zy = x$ then $z = \frac{x}{y}$. ["Division theorem"]

2. $\forall_x \ \frac{x}{1} = x$. 3. $\forall_{x \neq 0} \ \frac{x}{x} = 1$. 4. $\forall_x \ \frac{x}{-1} = -x$.

5. $\forall_{x \neq 0} \ \frac{0}{x} = 0$. 6. $\forall_x \forall_{y \neq 0}$ if $\frac{x}{y} = 0$ then $x = 0$.

THE 0-PRODUCT THEOREM

It is easy to derive the generalization:

$$\forall_x \forall_y \text{ if } x = 0 \text{ then } xy = 0$$

from the commutative principle for multiplication and the principle for multiplying by 0 [Do so.], So, this generalization is a theorem. Do you think that you could prove:

$$\forall_x \forall_y \text{ if } xy = 0 \text{ then } x = 0?$$

Since $1 \cdot 0 = 0$ but $1 \neq 0$, this second generalization is not a theorem. How can you "correct" this generalization to get a theorem?

The generalization:

(1) $\qquad \forall_x \forall_{y \neq 0}$ if $xy = 0$ then $x = 0$

is a theorem. Here is a proof.

Suppose that $xy = 0$.

Then $xy = 0y$. [pm0; cpm]

So, $x = 0$. [cancellation principle for multiplication; $y \neq 0$]

Hence [for $y \neq 0$], if $xy = 0$ then $x = 0$.

Notice that (1) is logically equivalent to:

(1') $\qquad \forall_x \forall_{y \neq 0}$ if $x \neq 0$ then $xy \neq 0$.

And, (1') is logically equivalent to:

(2) $\qquad \forall_x \forall_y$ if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

Also, (2) is logically equivalent to:

(2') $\qquad \forall_x \forall_y$ if $xy = 0$ then $x = 0$ or $y = 0$.

We shall call (2') <u>the 0-product theorem</u>. Sometimes when we refer to this theorem, you will want to think of the equivalent generalization (2).

SIMPLIFYING EXPRESSIONS CONTAINING FRACTIONS

In doing Exercise 12 on page 2-89 you actually proved the theorem:

$$\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv}.$$

This theorem justifies the rule you learned in an earlier grade for "adding fractions". So, the rule you learned follows from the basic principles. This is the case for all the rules you learned about fractions. You will use these same rules in simplifying pronumeral expressions such as:

$$\frac{15xy}{3x} + 4x(\frac{3y}{2x} + \frac{7y}{4x}) + \frac{6ab}{2xy} \times \frac{5xy}{4ab} \div \frac{3x}{2}.$$

But, before you do so, you will want to show that these rules also follow from the basic principles. This amounts to stating generalizations which justify the rules and showing that these sentences are theorems.

Adding or subtracting fractions

For this you have two rules based on the generalizations:

$$\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv},$$

and:

$$\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} - \frac{u}{v} = \frac{xv - uy}{yv}.$$

Here is a test-pattern for the first generalization.

$$\begin{aligned} &(\frac{x}{y} + \frac{u}{v})(yv) && \text{Why?} \\ &= \frac{x}{y}(yv) + \frac{u}{v}(yv) && \text{Why?} \\ &= \frac{x}{y}(yv) + \frac{u}{v}(vy) && \text{Why?} \\ &= \frac{x}{y}yv + \frac{u}{v}vy && \text{pq; } [y \neq 0,\ v \neq 0] \\ &= xv + uy. \end{aligned}$$

So, $(\frac{x}{y} + \frac{u}{v})(yv) = xv + uy.$

Hence, $\frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv}.$ [division theorem; $yv \neq 0$]

Since, by the 0-product theorem, the restrictions '$y \neq 0$' and '$v \neq 0$' imply the restriction '$yv \neq 0$', we may disregard the latter. So, the above test-pattern is a proof of the first generalization.

Now, write a test-pattern for the theorem which justifies the rule for subtracting fractions.

Proof of Theorem 59, the "multiplication of fractions" theorem.

$$\left.\begin{array}{l}(\frac{x}{y}\cdot\frac{u}{v})(yv) \\ = (\frac{x}{y}\cdot y)(\frac{u}{v}\cdot v)\end{array}\right\}\text{ product rearrangement theorem [Th. 4]}$$

$$\left.\begin{array}{l}= (\frac{x}{y}\cdot y)(\frac{u}{v}\cdot v) \\ = xu.\end{array}\right\}\text{ pq; } y \neq 0,\ v \neq 0$$

So, $(\frac{x}{y}\cdot\frac{u}{v})(yv) = xu$. Hence, $\frac{x}{y}\cdot\frac{u}{v} = \frac{xu}{yv}$. [division theorem; $yv \neq 0$]

So, [for $y \neq 0$, $v \neq 0$,] $\frac{x}{y}\cdot\frac{u}{v} = \frac{xu}{yv}$. [0-product theorem]

[The citation '0-product theorem' justifies omitting '$yv \neq 0$' from the bracket following 'So,'.] The first step in this proof may be expanded by paralleling the proof given for Theorem 4 [see answer for Exercise 3 on TC[2-61]a], or the citation of Theorem 4 may be replaced by 'apm, cpm'.

*

If the proof of Theorem 59 is assigned as homework, some student should then present it at the board. In the ensuing discussion you may find it helpful to suggest a motivation for the proof by taking up the problem of simplifying, say, '$\frac{2}{3}\cdot\frac{5}{7}$', and treating this in the manner of the Solution for Sample 1 on page 2-88.

If, on the other hand, you wish to lead students to discover the proof of Theorem 59 in class, you may find it helpful to refer them to the Solution of Sample 1 on page 2-88 and suggest that they begin by trying to parallel the discussion there with '+' replaced by '$\cdot$'.

*

Answers for Exercises [on pages 2-93 and 2-94].

[As in earlier simplification exercises [see TC[2-52]a], students should not be asked for justifications unless disputes arise. This remark applies also to later simplification exercises.]

1. $\frac{6kx}{35my}$, [$m \neq 0$, $y \neq 0$]
2. $\frac{99rr}{20ss}$, [$s \neq 0$]
3. $\frac{acd}{5bb}$, [$b \neq 0$]
4. $\frac{20pm}{21jk}$, [$j \neq 0$, $k \neq 0$]
5. $\frac{6}{xyz}$, [$x \neq 0$, $y \neq 0$, $z \neq 0$]
6. $\frac{63ekr}{80}$

* * *

Multiplying fractions

$$\frac{3}{5} \times \frac{7}{8} = ?$$

Since $(\frac{3}{5} \times \frac{7}{8}) \times (5 \times 8) = 3 \times 7$, it follows from the division theorem that

$$\frac{3}{5} \times \frac{7}{8} = \frac{3 \times 7}{5 \times 8}.$$

The rule about multiplying numerator-numbers and multiplying denominator-numbers is justified by this generalization:

$$\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0} \ \frac{x}{y} \cdot \frac{u}{v} = \frac{xu}{yv}.$$

Prove this theorem.

EXERCISES

Simplify.

Sample 1. $\frac{2a}{3b} \cdot \frac{5c}{7d}$

Solution.
$$\left.\begin{aligned} &\frac{2a}{3b} \cdot \frac{5c}{7d} \\ &= \frac{(2a)(5c)}{(3b)(7d)} \end{aligned}\right\} [3b \neq 0;\ \ 7d \neq 0]$$
$$= \frac{10ac}{21bd}.$$

Answer. $\frac{10ac}{21bd}$, $[b \neq 0,\ \ d \neq 0]$

[Notice that, since $3 \neq 0$, the 0-product theorem tells us we can replace the restriction '$3b \neq 0$' by '$b \neq 0$'.]

1. $\frac{3x}{7y} \cdot \frac{2k}{5m}$
2. $\frac{9r}{2s} \cdot \frac{11r}{10s}$
3. $\frac{a}{b} \cdot \frac{cd}{5b}$
4. $\frac{-5p}{7j} \cdot \frac{-4m}{3k}$
5. $\frac{1}{x} \cdot \frac{2}{y} \cdot \frac{3}{z}$
6. $\frac{7k}{2} \cdot \frac{3r}{-5} \cdot \frac{3e}{-8}$

(continued on next page)

<u>Sample 2.</u> $\frac{w}{z}(\frac{x}{y} + \frac{u}{v})$

<u>Solution.</u>
$$\begin{aligned} &\frac{w}{z}(\frac{x}{y} + \frac{u}{v}) \\ &= \frac{w}{z}(\frac{xv + uy}{yv}) \\ &= \frac{w(xv + uy)}{z(yv)}. \end{aligned} \quad \begin{aligned} &\} [y \neq 0,\ v \neq 0] \\ &\} [z \neq 0,\ yv \neq 0] \end{aligned}$$

<u>Answer.</u> $\frac{w(xv + uy)}{zyv}$, $[y \neq 0,\ v \neq 0,\ z \neq 0]$

7. $\frac{3}{a}(\frac{2}{b} + \frac{5}{c})$ 8. $\frac{r}{6}(\frac{t}{s} - \frac{u}{2})$ 9. $\frac{k}{m}(\frac{2x}{y} + \frac{3z}{u})$

* * *

<u>Reducing fractions</u>

$$\frac{6}{10} = \frac{3 \times 2}{5 \times 2} = \frac{3}{5}. \qquad \frac{6}{18} = \frac{1 \times 6}{3 \times 6} = \frac{1}{3}.$$

$$\frac{2.5}{15} = \frac{2.5 \times 10}{15 \times 10} = \frac{25}{150} = \frac{1 \times 25}{6 \times 25} = \frac{1}{6}.$$

The generalization which justifies these illustrations is:

$$\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ \frac{xz}{yz} = \frac{x}{y}.$$

Here is the beginning of a test-pattern for this generalization; you should complete it.

$$\begin{aligned} &\frac{x}{y}(yz) \\ &= \\ &\vdots \end{aligned} \quad \} \text{ aprn}$$

Here is another test-pattern for the same generalization.

$$\begin{aligned} &\frac{xz}{yz} \\ &= \frac{x}{y} \cdot \frac{z}{z} \\ &= \frac{x}{y} \cdot 1 \\ &= \frac{x}{y}. \end{aligned} \quad \begin{aligned} &\} \text{ multiplication theorem for fractions; } [y \neq 0,\ z \neq 0] \\ &\} \text{ Why?} \\ &\} \text{ Why?} \end{aligned}$$

An alternative solution for Sample 2 begins with an application of the ℓdpma. This leads to the answer:

$$\frac{wx(zv) + wu(zy)}{zy(zv)}, \; [y \neq 0, \; v \neq 0, \; z \neq 0].$$

If students suggest this procedure, point out that the alternative answer must be equivalent to the one given in the text [since both fractions are equivalent to '$\frac{w}{z}(\frac{x}{y} + \frac{u}{v})$'], and use this as a motivation for the material on reducing fractions which follows the exercises.

Some students may also suggest using a least common denominator. This technique is taken up on page 2-99.

Students will discover when doing exercises on pages 2-106 and 2-107 that, in problems like Sample 2, it is sometimes simpler to begin by using the ℓdpma.

*

Answers for Exercises, continued.

7. $\frac{3(2c + 5b)}{abc}$, $[a \neq 0, b \neq 0, c \neq 0]$

8. $\frac{r(2t - su)}{12s}$, $[s \neq 0]$

9. $\frac{k(2xu + 3yz)}{myu}$, $[m \neq 0, y \neq 0, u \neq 0]$

*

Completion of first test-pattern for Theorem 60.

$$\left.\begin{aligned} &= (\tfrac{x}{y} \cdot y)z \\ &= xz. \end{aligned}\right\} pq; \; y \neq 0$$

So, $\frac{x}{y}(yz) = xz$. Hence, $\frac{xz}{yz} = \frac{x}{y}$. [division theorem; $yz \neq 0$]

So, [for $y \neq 0$, $z \neq 0$,] $\frac{xz}{yz} = \frac{x}{y}$. [0-product theorem]

Answers for 'Why?'s in second test-pattern:

$\forall_{x \neq 0}\; x/x = 1$, $[z \neq 0]$, and: pm1.

A second proof.

$$\left.\begin{array}{l}\dfrac{xy}{z}\\[2ex] = \dfrac{xy}{z\cdot 1}\end{array}\right\}\text{pm1}$$

$$\left.\begin{array}{l}= \dfrac{xy}{z\cdot 1}\\[2ex] = \dfrac{x}{z}\cdot\dfrac{y}{1}\end{array}\right\}\forall_x\forall_{y\neq 0}\forall_{u\neq 0}\forall_{v\neq 0}\ \frac{x}{y}\cdot\frac{u}{v} = \frac{xu}{yv},\ [z\neq 0,\ 1\neq 0][\text{Th. } 59]$$

$$\left.\begin{array}{l}= \dfrac{x}{z}\cdot\dfrac{y}{1}\\[2ex] = \dfrac{x}{z}y.\end{array}\right\}\forall_x\ x/1 = x\ [\text{Th. } 50]$$

Hence, [for $z \neq 0$,] $\frac{xy}{z} = \frac{x}{z}y$.

*

6. $\frac{13x}{17y}$, $[y \neq 0]$

7. $\frac{5}{2}$, $[x \neq 0]$

8. $\frac{7}{5}$, $[a \neq 0,\ b \neq 0,\ c \neq 0]$

9. $\frac{25z}{6}$, $[x \neq 0,\ y \neq 0]$

10. $\frac{11x}{13y}$, $[y \neq 0,\ a + b \neq 0]$

Answers for 'Why?'s in the proof of Theorem 61 on page 2-95:

$$\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ \frac{xz}{yz} = \frac{x}{y},\ [y \neq 0,\ z \neq 0],\ \text{and: pq; } z \neq 0.$$

*

When discussing instances of Theorem 61, make sure that students see that this theorem justifies such simplifications as

$$\text{`}\frac{\frac{3}{2}}{\frac{7}{2}}\text{'} \quad \text{to} \quad \text{`}\frac{3}{7}\text{'}, \quad \text{and} \quad \text{`}\frac{\frac{6x+5}{x(x+1)}}{\frac{x-4}{x(x+1)}}\text{'} \quad \text{to} \quad \text{`}\frac{6x+5}{x-4}\text{'}.$$

*

Answers for Part A [on page 2-96].

1. $\frac{3}{7}$ 2. $\frac{7}{11}$ 3. $\frac{2}{9}$ 4. $\frac{1}{4}$ 5. $\frac{1}{5}$

*

Students may be helped in seeing the connection between the theorem to be proved and the problem of transforming '(12x) ÷ 3' to '(12 ÷ 3)x' if they first state the theorem as:

$$\forall_x \forall_y \forall_{z \neq 0}\ (xy) \div z = (x \div z)y.$$

A proof of '$\forall_x \forall_y \forall_{z \neq 0}\ \frac{xy}{z} = \frac{x}{z}y$'. [Theorem 62]

$$\begin{aligned} &(\tfrac{x}{z}y)z \\ &= \tfrac{x}{z}(yz) && \text{apm} \\ &= \tfrac{x}{z}(zy) && \text{cpm} \\ &= (\tfrac{x}{z}z)y && \text{apm} \\ &= xy. && \text{pq; } z \neq 0 \end{aligned}$$

So, $(\frac{x}{z}y)z = xy$. Hence, $\frac{xy}{z} = \frac{x}{z}y$. [division theorem; $z \neq 0$]

[Another proof of Theorem 62 is on TC[2-95, 96]b.]

Perhaps you are more familiar with the rule for reducing fractions in which you divide numerator-number and denominator-number by the same number. Here are two examples.

$$\frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}. \qquad \frac{6}{18} = \frac{6 \div 6}{18 \div 6} = \frac{1}{3}.$$

And, you probably shorten the work by using cancel marks like this.

$$\frac{\overset{3}{\cancel{6}}}{\underset{5}{\cancel{10}}} = \frac{3}{5} \qquad \frac{\overset{1}{\cancel{6}}}{\underset{3}{\cancel{18}}} = \frac{1}{3}$$

These procedures for reducing fractions are justified by

$$\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \ \frac{x}{y} = \frac{x \div z}{y \div z}.$$

This generalization can be proved by applying the theorem stated on page 2-94.

$$\left.\begin{aligned} &\frac{x \div z}{y \div z} \\ &= \frac{(x \div z)z}{(y \div z)z} \end{aligned}\right\} \text{Why?}$$

$$\left.\begin{aligned} &= \frac{(x \div z)z}{(y \div z)z} \\ &= \frac{x}{y}. \end{aligned}\right\} \text{Why?}$$

Sometimes the process of reducing fractions is used in a disguised form in carrying out a simplification. For example:

$$\frac{7}{8} \times \frac{6}{11} = ? \qquad \frac{7}{\underset{4}{\cancel{8}}} \times \frac{\overset{3}{\cancel{6}}}{11} = \frac{21}{44}.$$

But, this is really a short cut for the following procedure:

$$\frac{7}{8} \times \frac{6}{11} = \frac{7 \times 6}{8 \times 11} = \frac{7 \times 6}{11 \times 8} = \frac{7}{11} \times \frac{\overset{3}{\cancel{6}}}{\underset{4}{\cancel{8}}} = \frac{7 \times \overset{3}{\cancel{6}}}{11 \times \underset{4}{\cancel{8}}} = \frac{7 \times \overset{3}{\cancel{6}}}{\underset{4}{\cancel{8}} \times 11} = \frac{7}{\underset{4}{\cancel{8}}} \times \frac{\overset{3}{\cancel{6}}}{11}.$$

So, the rule for cancelling before multiplying is justified by the multiplication theorem for fractions, the theorem for reducing fractions by dividing, and the commutative principle for multiplication.

EXERCISES

A. Reduce these fractions.

1. $\frac{15}{35}$ 2. $\frac{28}{44}$ 3. $\frac{18}{81}$ 4. $\frac{16}{64}$ 5. $\frac{19}{95}$

Sample 1. $\frac{12x}{15y}$

Solution. In reducing this fraction we can apply the theorem for reducing fractions by dividing.

$$\frac{12x}{15y} = \frac{12x \div 3}{15y \div 3} = \frac{4x}{5y}, \quad [y \neq 0].$$

[Notice that in simplifying '$12x \div 3$' to '$4x$' we assume that '$(12x) \div 3$' and '$(12 \div 3)x$' are equivalent. In other words, we assume that the generalization:

$$\forall_x \forall_y \forall_{z \neq 0} \ \frac{xy}{z} = \frac{x}{z}y$$

is a theorem. Prove this theorem.]

Sample 2. $\frac{25a}{35b}$ Solution. $\frac{\overset{5}{\cancel{25}}a}{\underset{7}{\cancel{35}}b} = \frac{5a}{7b}, \quad [b \neq 0]$

6. $\frac{26x}{34y}$ 7. $\frac{45x}{18x}$ 8. $\frac{21abc}{15abc}$ 9. $\frac{50xyz}{12xy}$ 10. $\frac{44x(a+b)}{52y(a+b)}$

* * *

There are several theorems which follow easily from the theorem:

$$(*) \qquad \forall_x \forall_y \forall_{z \neq 0} \ \frac{xy}{z} = \frac{x}{z}y$$

which you were asked to prove in Sample 1. These theorems justify some of the short cuts you have learned. Of course, you don't need these theorems to justify the short cuts because you can use earlier theorems for this purpose. For example, consider the problem:

$$\frac{3}{5} \times 7 = ?$$

In grade school you may have learned to do this problem this way:

$$\frac{3}{5} \times 7 = \frac{3}{5} \times \frac{7}{1} = \frac{3 \times 7}{5 \times 1} = \frac{3 \times 7}{5}.$$

$\forall_{x \neq 0}\forall_y\forall_z\ \dfrac{xy + xz}{x} = y + z$ [Theorem 65]

$$\begin{aligned}
&\frac{xy + xz}{x} && \Big\} \ \ell\text{dpma} \\
&= \frac{x(y + z)}{x} && \Big\} \ \forall_x\forall_y\forall_{z \neq 0}\ \frac{xy}{z} = \frac{x}{z}y,\ [x \neq 0]\ [\text{Th. } 62] \\
&= \frac{x}{x}(y + z) && \Big\} \ \forall_{x \neq 0}\ x/x = 1,\ [x \neq 0]\ [\text{Th. } 51] \\
&= 1(y + z) && \Big\} \ 1 \text{ times theorem [Th. 2]} \\
&= y + z.
\end{aligned}$$

Hence, [for $x \neq 0$,] $\dfrac{xy + xz}{x} = y + z$.

A second proof.

$$\begin{aligned}
&(y + z)x && \Big\} \ \text{cpm} \\
&= x(y + z) && \Big\} \ \ell\text{dpma} \\
&= xy + xz.
\end{aligned}$$

So, $(y + z)x = xy + xz$.

Hence, $\dfrac{xy + xz}{x} = y + z$. [division theorem; $x \neq 0$]

A third proof.

$$
\begin{aligned}
&(x \cdot \frac{1}{y})y \\
&= x(\frac{1}{y} \cdot y) && \} \text{ apm} \\
&= x1 && \} \text{ pq; } y \neq 0 \\
&= x. && \} \text{ pm1}
\end{aligned}
$$

So, $(x \cdot \frac{1}{y})y = x$. Hence, $\frac{x}{y} = x \cdot \frac{1}{y}$. [division theorem; $y \neq 0$]

*

$\forall_x \forall_{y \neq 0}\ \frac{xy}{y} = x$ [Theorem 64]

$$
\begin{aligned}
&\frac{xy}{y} \\
&= \frac{x}{y} \cdot y && \} \ \forall_x \forall_y \forall_{z \neq 0}\ \frac{xy}{z} = \frac{x}{z}y, \text{ [} y \neq 0 \text{] [Th. 62]} \\
&= x. && \} \text{ pq; } y \neq 0
\end{aligned}
$$

Hence, [for $y \neq 0$,] $\frac{xy}{y} = x$.

A second proof.

$xy = xy$.

Hence, $\frac{xy}{y} = x$. [division theorem; $y \neq 0$]

*

We give three proofs for Theorem 63, and two proofs for each of Theorems 64 and 65. [These three theorems are given in the bottom half of page 2-97.]

$\forall_x \forall_{y \neq 0}\ \frac{x}{y} = x \cdot \frac{1}{y}$ [Theorem 63]

$$
\begin{aligned}
& \frac{x}{y} \\
& \qquad \text{pm1} \\
&= \frac{x1}{y} \\
& \qquad \text{cpm} \\
&= \frac{1x}{y} \\
& \qquad \forall_x \forall_y \forall_{z \neq 0}\ \frac{xy}{z} = \frac{x}{z}y,\ [y \neq 0]\ [\text{Th. } 62] \\
&= \frac{1}{y} \cdot x \\
& \qquad \text{cpm} \\
&= x \cdot \frac{1}{y}.
\end{aligned}
$$

Hence, [for $y \neq 0$,] $\frac{x}{y} = x \cdot \frac{1}{y}$.

A second proof.

$$
\begin{aligned}
& \frac{x}{y} \\
& \qquad \text{pm1} \\
&= \frac{x1}{y1} \\
& \qquad \text{cpm} \\
&= \frac{x1}{1y} \\
& \qquad \forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} \cdot \frac{u}{v} = \frac{xu}{yv},\ [1 \neq 0,\ y \neq 0]\ [\text{Th. } 59] \\
&= \frac{x}{1} \cdot \frac{1}{y} \\
& \qquad \forall_x\ x/1 = x\ \ [\text{Th. } 50] \\
&= x \cdot \frac{1}{y}.
\end{aligned}
$$

Hence, [for $y \neq 0$,] $\frac{x}{y} = x \cdot \frac{1}{y}$.

This method is correct and is justified by the theorem for dividing by 1 [Exercise 2 on page 2-91], the multiplication theorem for fractions, and the principle for multiplying by 1. But, a short cut is:

$$\frac{3}{5} \times 7 = \frac{3 \times 7}{5},$$

and this is justified by (*). [In fact, the long way of doing the problem may suggest a proof of (*).]

Consider the problem of simplifying the expression:

$$\frac{4x + 6y}{2}.$$

Here is one procedure for simplifying:

$$(1) \qquad \frac{4x + 6y}{2} = (4x + 6y)\frac{1}{2} = 2x + 3y.$$

Another way to do this is as follows:

$$(2) \qquad \frac{4x + 6y}{2} = \frac{(2x + 3y)2}{2} = 2x + 3y.$$

Still a third way is:

$$(3) \qquad \frac{4x + 6y}{2} = \frac{2(2x) + 2(3y)}{2} = 2x + 3y.$$

Each of these procedures is justified in part by a special theorem.

In (1) we used the generalization:

$$\forall_x \forall_{y \neq 0} \ \frac{x}{y} = x \cdot \frac{1}{y}.$$

[Dividing by a number is the same as multiplying by its reciprocal.]

In (2) we used the generalization:

$$\forall_x \forall_{y \neq 0} \ \frac{xy}{y} = x.$$

[The inverse of multiplying by a nonzero number is dividing by that number.]

In (3) we used the generalization:

$$\forall_{x \neq 0} \forall_y \forall_z \ \frac{xy + xz}{x} = y + z.$$

Each of these theorems is an easy consequence of (*), and an even easier consequence of the division theorem. <u>Prove these three theorems</u>.

Here is another easy consequence of (*):

$$(\dagger) \qquad \forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0} \forall_{z \neq 0} \; \frac{xu}{yv} = \frac{(x \div z)u}{(y \div z)v}.$$

Prove it.

* * *

B. Simplify.

1. $\frac{2}{7} \times \frac{6}{11}$
2. $\frac{3}{8} \times \frac{14}{9}$
3. $\frac{a}{5} \times \frac{b}{6}$
4. $a \times \frac{b}{c}$
5. $10z \cdot \frac{1}{z}$
6. $xy \cdot \frac{z}{y}$
7. $10a \times \frac{3b}{4c}$
8. $\frac{3x}{2(a+b)} \cdot 8x(a+b)$
9. $\frac{2m}{5n} \times 15np$
10. $(ab) \div a$
11. $(16axy) \div (4xy)$
12. $(30x) \div 5$
13. $(3x) \div 2$
14. $\frac{5y}{2}$
15. $\frac{64xyz}{4y}$
16. $\frac{8+10}{2}$
17. $\frac{3x+6y}{3}$
18. $\frac{(3x) \cdot (6y)}{3}$
19. $\frac{15ab + 25ac}{5a}$
20. $\frac{9x(y+z) - 3u(y+z)}{3(y+z)}$
21. $\frac{6xy - 8yz}{2y}$
22. $\frac{2x}{3} \cdot \frac{5y}{7}$
23. $\frac{y}{x} \cdot \frac{3x}{8}$
24. $\frac{2k}{5m} \cdot \frac{7n}{8k}$
25. $\frac{3st}{5rq} \cdot \frac{70r}{30s}$
26. $\frac{2x}{9(a+b)} \cdot \frac{12(a+b)}{16x}$
27. $\frac{1}{ab} \cdot \frac{1}{bc}$
28. $\frac{3}{a} \cdot \frac{5}{b}$
29. $\frac{3}{a} + \frac{5}{b}$
30. $\frac{3x}{2y} + \frac{7y}{5x}$
31. $\frac{1}{p} - \frac{1}{q}$
32. $\frac{12a}{3c} - \frac{5b}{15d}$
33. $\frac{a+x}{x} + \frac{b+y}{y}$

* * *

Here is a proof for (†) which is at the top of page 2-98.

$\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0} \forall_{z \neq 0}\ \frac{xu}{yv} = \frac{(x \div z)u}{(y \div z)v}$ [Theorem 66]

$$\left.\begin{array}{l} \frac{xu}{yv} \\ = \frac{(xu) \div z}{(yv) \div z} \end{array}\right\} \forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ \frac{x}{y} = \frac{x \div z}{y \div z},\ [yv \neq 0,\ z \neq 0]\ [\text{Th. } 61]$$

$$\left.\begin{array}{l} = \frac{(xu) \div z}{(yv) \div z} \\ = \frac{(x \div z)u}{(y \div z)v}. \end{array}\right\} \forall_x \forall_y \forall_{z \neq 0}\ \frac{xy}{z} = \frac{x}{z}y,\ [z \neq 0]\ [\text{Th. } 62]$$

Hence, [for $y \neq 0$, $v \neq 0$, $z \neq 0$,] $\frac{xu}{yv} = \frac{(x \div z)u}{(y \div z)v}$. [0-product theorem]

*

Answers for Part B.

1. $\frac{12}{77}$
2. $\frac{7}{12}$
3. $\frac{ab}{30}$
4. $\frac{ab}{c}$, $[c \neq 0]$
5. 10, $[z \neq 0]$
6. xz, $[y \neq 0]$
7. $\frac{15ab}{2c}$, $[c \neq 0]$
8. 12xx, $[b \neq -a]$
9. 6mp, $[n \neq 0]$
10. b, $[a \neq 0]$
11. 4a $[x \neq 0,\ y \neq 0]$
12. 6x
13. 1.5x
14. 2.5y
15. 16xz, $[y \neq 0]$
16. 9
17. x + 2y
18. 6xy
19. 3b + 5c, $[a \neq 0]$
20. 3x - u, $[z \neq -y]$
21. 3x - 4z, $[y \neq 0]$
22. $\frac{10xy}{21}$
23. $\frac{3y}{8}$, $[x \neq 0]$
24. $\frac{7n}{20m}$, $[m \neq 0,\ k \neq 0]$
25. $\frac{7t}{5q}$, $[r \neq 0,\ q \neq 0,\ s \neq 0]$
26. $\frac{1}{6}$, $[a + b \neq 0,\ x \neq 0]$
27. $\frac{1}{abbc}$, $[a \neq 0,\ b \neq 0,\ c \neq 0]$
28. $\frac{15}{ab}$, $[a \neq 0,\ b \neq 0]$
29. $\frac{3b + 5a}{ab}$, $[a \neq 0,\ b \neq 0]$
30. $\frac{15xx + 14yy}{10xy}$, $[x \neq 0,\ y \neq 0]$
31. $\frac{q - p}{pq}$, $[p \neq 0,\ q \neq 0]$
32. $\frac{12ad - bc}{3cd}$, $[c \neq 0,\ d \neq 0]$
33. $\frac{y(a + x) + x(b + y)}{xy}$, $[x \neq 0,\ y \neq 0]$

A second proof.

$$\begin{aligned}
&\frac{x}{yz} + \frac{u}{vz} \\
&= \frac{x}{yz}\cdot 1 + \frac{u}{vz}\cdot 1 && \Big\}\ \text{pm1} \\
&= \frac{x}{yz}\cdot\frac{v}{v} + \frac{u}{vz}\cdot\frac{y}{y} && \Big\}\ \forall_{x \neq 0}\ x/x = 1,\ [v \neq 0,\ y \neq 0]\ [\text{Th. } 51] \\
&= \frac{xv}{(yz)v} + \frac{uy}{(vz)y} && \Big\}\ \text{multiplication of fractions theorem; } yz \neq 0 \neq vz,\ v \neq 0 \neq y\ [\text{Th. } 59] \\
&= \qquad \vdots
\end{aligned}$$

[Notice that the second of the proofs given for Theorem 68 amounts to replacing the first step in the first proof by a proof of Theorem 60. Students are often taught to go through steps like those in the second proof in order to simplify an addition problem. They should be able to use the short cut suggested by Theorem 60 instead.]

*

Quiz.

Simplify.

1. $\frac{a}{b}\cdot ab$
2. $\frac{7c}{3d}\cdot 27de$
3. $42n \div 14$
4. $\frac{70cde}{5d}$
5. $\frac{(9r)\cdot(15s)}{3}$
6. $\frac{14kn - 35np}{7n}$
7. $\frac{12b(c + d) - 4a(c + d)}{4(c + d)}$
8. $\frac{3r}{15(x + y)}\cdot\frac{60(x + y)}{12r}$
9. $\frac{7ab}{12cd}\cdot\frac{36c}{21b}$
10. $\frac{4t(x + a)}{5t} - \frac{1.6(ra + rx)}{2r}$

*

Answers for Quiz.

1. aa, $[b \neq 0]$
2. $63ce$, $[d \neq 0]$
3. $3n$
4. $14ce$, $[d \neq 0]$
5. $45rs$
6. $2k - 5p$, $[n \neq 0]$
7. $3b - a$, $[c + d \neq 0]$
8. 1, $[-x \neq y,\ r \neq 0]$
9. $\frac{a}{d}$, $[bcd \neq 0]$
10. 0, $[t \neq 0 \neq r]$

Lowest common multiples are discussed in Unit 4.

*

Proof of the distributive theorem for division over addition [Theorem 67].

$$\left.\begin{aligned}&(\frac{x}{z} + \frac{y}{z})z \\ &= \frac{x}{z}z + \frac{y}{z}z\end{aligned}\right\} \text{dpma}$$

$$\left.\begin{aligned}&= \frac{x}{z}z + \frac{y}{z}z \\ &= x + y.\end{aligned}\right\} \text{pq; } z \neq 0$$

So, $(\frac{x}{z} + \frac{y}{z})z = x + y$. Hence, $\frac{x}{z} + \frac{y}{z} = \frac{x + y}{z}$. [division theorem; $z \neq 0$]

*

A proof of '$\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{yz} + \frac{u}{vz} = \frac{xv + uy}{yvz}$'. [Theorem 68]

$$\left.\begin{aligned}&\frac{x}{yz} + \frac{u}{vz} \\ &= \frac{xv}{(yz)v} + \frac{uy}{(vz)y}\end{aligned}\right\} \forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ \frac{xz}{yz} = \frac{x}{y},\ [v \neq 0,\ y \neq 0][\text{Th. } 60]$$

$$\left.\begin{aligned}&= \frac{xv}{(yz)v} + \frac{uy}{(vz)y} \\ &= \frac{xv}{yvz} + \frac{uy}{yvz}\end{aligned}\right\} \text{cpm, apm}$$

$$\left.\begin{aligned}&= \frac{xv}{yvz} + \frac{uy}{yvz} \\ &= \frac{xv + uy}{yvz}.\end{aligned}\right\} \forall_x \forall_y \forall_{z \neq 0}\ \frac{x}{z} + \frac{y}{z} = \frac{x + y}{z},\ [yvz \neq 0]\,[\text{Th. } 67]$$

Hence, [for $y \neq 0$, $v \neq 0$, $z \neq 0$,] $\frac{x}{yz} + \frac{u}{vz} = \frac{xv + uy}{yvz}$. [0-product theorem]

[Another proof of Theorem 68 is on TC[2-99]b.]

Least common denominator

Consider the problem:

$$\frac{3}{7} + \frac{2}{7} = ?$$

You wouldn't want to use the addition theorem for fractions:

$$\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0} \; \frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv}.$$

to solve this problem. You would probably solve it this way:

$$\frac{3}{7} + \frac{2}{7} = \frac{3 + 2}{7} = \frac{5}{7}.$$

That is, you would use the generalization:

$$\forall_x \forall_y \forall_{z \neq 0} \; \frac{x}{z} + \frac{y}{z} = \frac{x + y}{z}.$$

[Use the division theorem to give a very quick proof of this theorem. Do you see that this theorem tells you that division distributes over addition? So, we can call this the distributive theorem for division over addition. Do you think that there is a left distributive theorem for division over addition? If there were, it would follow that $\frac{7}{9} + \frac{7}{5} = \frac{7}{9 + 5}$!]

Similarly, you wouldn't want to use the addition theorem for fractions to solve the problem:

$$\frac{2}{15} + \frac{7}{20} = ?$$

Instead you might proceed as follows:

$$\frac{2}{15} + \frac{7}{20} = \frac{2}{3 \cdot 5} + \frac{7}{4 \cdot 5} = \frac{2 \cdot 4}{(3 \cdot 5)4} + \frac{7 \cdot 3}{(4 \cdot 5)3} = \frac{2 \cdot 4 + 7 \cdot 3}{3 \cdot 4 \cdot 5}.$$

So, $\frac{2}{15} + \frac{7}{20} = \frac{29}{60}$. [Notice the way in which the least common denominator, '$3 \cdot 4 \cdot 5$', was found.] Explain the step:

$$\frac{2}{3 \cdot 5} + \frac{7}{4 \cdot 5} = \frac{2 \cdot 4}{(3 \cdot 5)4} + \frac{7 \cdot 3}{(4 \cdot 5)3},$$

and then explain the last step.

This procedure is justified by the generalization:

$$\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \forall_u \forall_{v \neq 0} \; \frac{x}{yz} + \frac{u}{vz} = \frac{xv + uy}{yvz}.$$

Prove it.

EXERCISES

A. Simplify.

1. $\frac{7}{18} + \frac{3}{14}$ 2. $\frac{5}{6} - \frac{3}{34}$ 3. $\frac{11}{24} + \frac{17}{81}$ 4. $\frac{1}{2} + \frac{2}{3} - \frac{4}{5}$

5. $\frac{3}{xy} + \frac{5}{yz}$ 6. $\frac{5}{2x} - \frac{3}{2y}$ 7. $\frac{9}{5a} + \frac{3}{15b}$ 8. $\frac{9}{5a} \cdot \frac{3}{15b}$

9. $\frac{7}{2rs} + \frac{3}{8st}$ 10. $5 + \frac{3}{16}$ 11. $a + \frac{b}{c}$ 12. $6.5 + 5.02$

B. Prove these theorems.

1. $\forall_x \forall_y \forall_{z \neq 0}\ \frac{x}{z} - \frac{y}{z} = \frac{x - y}{z}$.

2. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{yz} - \frac{u}{vz} = \frac{xv - uy}{yvz}$.

3. $\forall_x \forall_y \forall_{z \neq 0}\ x + \frac{y}{z} = \frac{xz + y}{z}$.

* * *

Dividing fractions

Consider the problem:

$$3 \div \frac{5}{7} = ?$$

The grade school rule tells you to "invert" the '$\frac{5}{7}$' to get '$\frac{7}{5}$', and then multiply 3 by $\frac{7}{5}$. Let's justify this procedure.

To solve the problem '$3 \div \frac{5}{7} = ?$' is to find the number whose product by $\frac{5}{7}$ is 3. That is,

$$? \times \frac{5}{7} = 3.$$

But, $\frac{7}{5} \times \frac{5}{7} = 1$ [Why?], and $3 \times 1 = 3$. So,

$$3 \times (\frac{7}{5} \times \frac{5}{7}) = 3.$$

Hence,

$$(3 \times \frac{7}{5}) \times \frac{5}{7} = 3.$$

So,

$$3 \div \frac{5}{7} = 3 \times \frac{7}{5}.$$

Answers for Part A.

1. $\frac{38}{63}$ 2. $\frac{38}{51}$ 3. $\frac{433}{648}$ 4. $\frac{11}{30}$

5. $\frac{3z + 5x}{xyz}$, $[x \neq 0,\ y \neq 0,\ z \neq 0]$

6. $\frac{5y - 3x}{2xy}$, $[x \neq 0,\ y \neq 0]$

7. $\frac{9b + a}{5ab}$, $[a \neq 0,\ b \neq 0]$

8. $\frac{9}{25ab}$, $[a \neq 0,\ b \neq 0]$

9. $\frac{28t + 3r}{8rst}$, $[r \neq 0,\ s \neq 0,\ t \neq 0]$

10. $\frac{83}{16}$

11. $\frac{ac + b}{c}$, $[c \neq 0]$

12. 11.52 [or: $\frac{288}{25}$]

*

Answers for Part B.

1. [This is Theorem 69, the distributive theorem for division over subtraction. Its proof is like that of Theorem 67, given on TC[2-99].]

2. [This is Theorem 70. It can be proved as Theorem 68 was proved on TC[2-99].]

3. The "mixed number" theorem. [Theorem 71]

$$\begin{aligned} &(x + \tfrac{y}{z}) \cdot z \\ &= xz + \tfrac{y}{z} \cdot z \quad \Big\} \text{ dpma} \\ &= xz + y. \quad \Big\} \text{ pq; } z \neq 0 \end{aligned}$$

So, $(x + \frac{y}{z}) \cdot z = xz + y$.

Hence, $x + \frac{y}{z} = \frac{xz + y}{z}$. [division theorem; $z \neq 0$]

[Another proof of Theorem 71 is on TC[2-100]b.]

A second proof.

$$
\begin{aligned}
& x + \frac{y}{z} \\
&\quad \text{dividing is the inverse of multiplying theorem; } z \neq 0 \\
&= \frac{xz}{z} + \frac{y}{z} \\
&\quad \forall_x \forall_y \forall_{z \neq 0}\ \frac{x}{z} + \frac{y}{z} = \frac{x+y}{z},\ [z \neq 0]\ [\text{Th. } 67] \\
&= \frac{xz + y}{z}.
\end{aligned}
$$

Hence, [for $z \neq 0$,] $x + \frac{y}{z} = \frac{xz + y}{z}$.

A third proof.

$$
\begin{aligned}
& x + \frac{y}{z} \\
&\quad \forall_x\ x/1 = x\ [\text{Th. } 50] \\
&= \frac{x}{1} + \frac{y}{z} \\
&\quad \text{addition of fractions theorem; } 1 \neq 0,\ z \neq 0\ [\text{Th. } 57] \\
&= \frac{xz + y \cdot 1}{1 \cdot z} \\
&\quad \text{cpm} \\
&= \frac{xz + y \cdot 1}{z \cdot 1} \\
&\quad \text{pm1} \\
&= \frac{xz + y}{z}.
\end{aligned}
$$

Hence, [for $z \neq 0$,] $x + \frac{y}{z} = \frac{xz + y}{z}$.

[Note that the first two proofs are conceptually much simpler, as well as shorter.]

*

Notice how the division theorem motivates the "invert-and-multiply" rule. To answer the question '$3 \div \frac{5}{7} = ?$', it is sufficient, according to the division theorem, to find a number whose product by $\frac{5}{7}$ is 3. That $3 \times \frac{7}{5}$ is such a number is guaranteed by the apm, the "multiplication of fractions" theorem, the cpm, the "dividing a number by itself" theorem, and the pm1. [See proof at top of TC[2-101]a.]

Answers for Part B [on pages 2-101 and 2-102].

1. $\frac{6}{7}$ [Note: One neat way of arriving at the answer for Exercise 1 is to use Theorem 60 and Theorem 61 as follows. $(3/5) \div (7/10) = (6/10) \div (7/10) = 6/7$.]

2. $\frac{11}{3}$ 3. $\frac{1}{8}$ 4. 30 5. $\frac{5a}{6}$, $[b \neq 0]$

6. $\frac{15}{2y}$, $[a \neq 0,\ x \neq 0,\ y \neq 0]$ 7. $\frac{2y}{c}$, $[abcxz \neq 0]$

[Note that, for typographical reasons, we state the restriction for Exercise 7 as '$abcxz \neq 0$' rather than as '$a \neq 0,\ b \neq 0,\ c \neq 0,\ x \neq 0,\ z \neq 0$'. Students should be able to explain why these are equivalent.]

8. $\frac{2}{7}$ 9. $\frac{60}{13}$ 10. $\frac{3}{5}$, $[xy \neq 0]$ 11. $\frac{3}{2ac}$, $[abc \neq 0]$

12. $\frac{7}{10}$ 13. $\frac{85}{24}$ 14. $\frac{9}{2}$ 15. $\frac{76}{77}$

16. $\frac{2x + 1}{3x - 2}$, $[x \neq 0,\ x \neq \frac{2}{3}]$ 17. $\frac{b + a}{b - a}$, $[ab \neq 0,\ b \neq a]$

18. $\frac{11}{20}$, $[x \neq 0]$ 19. $\frac{a + b + 8}{a + b - 5}$, $[0 \neq a + b \neq 5]$

[Note that the restriction for Exercise 19 is an abbreviation for '$a + b \neq 0,\ a + b \neq 5$'.]

3. The "dividing a fraction" theorem. [Theorem 75]

$$\left.\begin{aligned}&\frac{x}{yz}z\\ &= \frac{xz}{yz}\\ &= \frac{x}{y}.\end{aligned}\right\}\begin{aligned}&\forall_x\forall_y\forall_{z \neq 0}\ \frac{xy}{z} = \frac{x}{z}y,\ [yz \neq 0]\ [\text{Th. } 62]\\ &\forall_x\forall_{y \neq 0}\forall_{z \neq 0}\ \frac{xz}{yz} = \frac{x}{y},\ [y \neq 0,\ z \neq 0]\ [\text{Th. } 60]\end{aligned}$$

So, $\frac{x}{yz} \cdot z = \frac{x}{y}$. Hence, $\frac{x}{y} \div z = \frac{x}{yz}$. [division theorem; $z \neq 0$]

Hence, [for $y \neq 0$, $z \neq 0$,] $\frac{x}{y} \div z = \frac{x}{yz}$. [0-product theorem]

A second proof.

$$\left.\begin{aligned}&(\tfrac{x}{y} \div z)(yz)\\ &= (\tfrac{x}{y} \div z)(zy)\\ &= \tfrac{x}{y} \div z \cdot z \cdot y\\ &= \tfrac{x}{y} \cdot y\\ &= x.\end{aligned}\right\}\begin{aligned}&\text{cpm}\\ &\text{apm}\\ &\text{pq; } z \neq 0\\ &\text{pq; } y \neq 0\end{aligned}$$

So, $(\frac{x}{y} \div z)(yz) = x$. Hence, $\frac{x}{y} \div z = \frac{x}{yz}$. [division theorem; $yz \neq 0$]

Hence, [for $y \neq 0$, $z \neq 0$,] $\frac{x}{y} \div z = \frac{x}{yz}$. [0-product theorem]

A third proof.

$$\left.\begin{aligned}&\frac{x}{y} \div z\\ &= \frac{x}{y} \cdot \frac{1}{z}\\ &= \frac{x \cdot 1}{yz}\\ &= \frac{x}{yz}.\end{aligned}\right\}\begin{aligned}&\forall_x\forall_{y \neq 0}\ \frac{x}{y} = x \cdot \frac{1}{y},\ [z \neq 0]\ [\text{Th. } 63]\\ &\text{multiplication of fractions theorem; } y \neq 0 \neq z\ [\text{Th. } 59]\\ &\text{pm1}\end{aligned}$$

Hence, [for $y \neq 0$, $z \neq 0$,] $\frac{x}{y} \div z = \frac{x}{yz}$.

Proof of the "dividing by a fraction" theorem. [Theorem 72]

$$\begin{aligned}
&\left(x\cdot\frac{z}{y}\right)\cdot\frac{y}{z} \\
&= x\left(\frac{z}{y}\cdot\frac{y}{z}\right) && \text{apm} \\
&= x\cdot\frac{zy}{yz} && \text{multiplication of fractions theorem; } y\neq 0,\ z\neq 0\ \text{[Th. 59]} \\
&= x\cdot\frac{yz}{yz} && \text{cpm} \\
&= x\cdot 1 && \forall_{x\neq 0}\ x/x=1,\ [yz\neq 0]\ \text{[Th. 51]} \\
&= x. && \text{pm1}
\end{aligned}$$

So, $\left(x\cdot\frac{z}{y}\right)\cdot\frac{y}{z}=x$. Hence, $x\div\frac{y}{z}=x\cdot\frac{z}{y}$. [division theorem; $\frac{y}{z}\neq 0$]
Hence, [for $y\neq 0$, $z\neq 0$,] $x\div\frac{y}{z}=x\cdot\frac{z}{y}$. [0-product theorem; Th. 54]

*

Answers for Part A.

1. The "dividing a fraction by a fraction" theorem. [Theorem 73]

$$\begin{aligned}
&\frac{x}{y}\div\frac{u}{v} \\
&= \frac{x}{y}\cdot\frac{v}{u} && \text{Th. 72; } u\neq 0,\ v\neq 0 \\
&= \frac{xv}{yu}. && \forall_x\forall_{y\neq 0}\forall_u\forall_{v\neq 0}\ \frac{x}{y}\cdot\frac{u}{v}=\frac{xu}{yv},\ [y\neq 0,\ u\neq 0]\text{[Th. 59]}
\end{aligned}$$

2. The "reciprocal of a fraction" theorem. [Theorem 74]

$$\begin{aligned}
&\frac{1}{\frac{x}{y}} \\
&= 1\cdot\frac{y}{x} && \text{Th. 72; } x\neq 0,\ y\neq 0 \\
&= \frac{y}{x}. && \forall_x\ 1\cdot x=x\ \text{[Th. 2]}
\end{aligned}$$

Justify the invert-and-multiply rule in general by proving that

$$\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \; x \div \frac{y}{z} = x \cdot \frac{z}{y}.$$

EXERCISES

A. Prove these theorems.

1. $\forall_x \forall_{y \neq 0} \forall_{u \neq 0} \forall_{v \neq 0} \; \frac{x}{y} \div \frac{u}{v} = \frac{xv}{yu}.$

2. $\forall_{x \neq 0} \forall_{y \neq 0} \; \frac{1}{\frac{x}{y}} = \frac{y}{x}.$

3. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \; \frac{x}{y} \div z = \frac{x}{yz}.$

B. Simplify.

1. $\frac{3}{5} \div \frac{7}{10}$

2. $1 \div \frac{3}{11}$

3. $\frac{3}{4} \div 6$

4. $0.3 \div 0.01$

5. $\frac{a}{2b} \div \frac{3}{5b}$

6. $\frac{5a}{3xy} \div \frac{2a}{9x}$

7. $\frac{13xyz}{2abc} \div \frac{26xz}{8ab}$

8. $\frac{\frac{2}{9}}{\frac{7}{9}}$

9. $\frac{\frac{8}{13}}{\frac{2}{15}}$

10. $\frac{\frac{x}{y}}{\frac{5x}{3y}}$

11. $\frac{\frac{9}{8ab}}{\frac{3c}{4b}}$

Sample. $\dfrac{\frac{2}{5} + \frac{3}{4}}{\frac{1}{10} + \frac{1}{4}}$

Solution. We could simplify this by simplifying the numerator and then the denominator, and then use the theorem for dividing fractions. Perhaps an easier way is to "clear the numerator and denominator of fractions" by using the theorem about multiplying numerator-number and denominator-number by the same nonzero number.

$$\frac{\frac{2}{5} + \frac{3}{4}}{\frac{1}{10} + \frac{1}{4}} = \frac{(\frac{2}{5} + \frac{3}{4})20}{(\frac{1}{10} + \frac{1}{4})20} = \frac{8 + 15}{2 + 5} = \frac{23}{7}.$$

[Why did we multiply the numerator- and denominator-numbers by 20?]

12. $\dfrac{\frac{2}{3} + \frac{1}{2}}{\frac{3}{2} + \frac{1}{6}}$ 13. $\dfrac{\frac{2}{9} + \frac{1}{4}}{\frac{1}{3} - \frac{1}{5}}$ 14. $\dfrac{\frac{3}{5} + 0.3}{0.7 - \frac{1}{2}}$ 15. $\dfrac{5 + \frac{3}{7}}{3 + \frac{5}{2}}$

16. $\dfrac{2 + \frac{1}{x}}{3 - \frac{2}{x}}$ 17. $\dfrac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}}$ 18. $\dfrac{\frac{2}{5x} + \frac{1}{3x}}{\frac{1}{2x} + \frac{5}{6x}}$ 19. $\dfrac{1 + \frac{8}{a+b}}{1 - \frac{5}{a+b}}$

DIVISION AND OPPOSITION

In earlier sections you proved theorems about opposites of sums, products, and differences.

$$\forall_x \forall_y \quad -(x + y) = -x + -y.$$

$$\forall_x \forall_y \quad -(xy) = -xy = x \cdot -y.$$

$$\forall_x \forall_y \quad -(x - y) = -x - -y.$$

It is now natural to ask about the opposite of a quotient. From the theorems on the opposites of sums and differences, one might suspect that the opposite of a quotient is the quotient of opposites. For example, that

$$-\frac{18}{3} = \frac{-18}{-3} \text{ and } -\frac{-9}{-18} = \frac{9}{18}.$$

But, a bit of computing shows that these statements are false. Moreover, division is more closely related to multiplication than it is to addition or subtraction, so we should expect to get a better clue from the theorem on the opposite of a product. This suggests, for example, that

$$-\frac{18}{3} = \frac{-18}{3} \text{ and that } -\frac{18}{3} = \frac{18}{-3}.$$

Computation shows that these statements are true. So, we should investigate the generalization:

$$(*) \qquad \forall_x \forall_{y \neq 0} \quad -\frac{x}{y} = \frac{-x}{y} = \frac{x}{-y}.$$

A simpler version of the "same" proof:

$$\left.\begin{array}{l} \frac{-x}{-y} \\ = -\frac{-x}{y} \\ = --\frac{x}{y} \\ = \frac{x}{y}. \end{array}\right. \quad \begin{array}{l} \text{Th. 77; } y \neq 0 \\ \text{Th. 76; } y \neq 0 \\ \forall_x \; --x = x \;\; [\text{Th. } 17] \end{array}$$

An alternative proof:

$$\begin{array}{l} \frac{-x}{-y} \\ = \frac{x \cdot -1}{y \cdot -1} \\ = \frac{x}{y}. \end{array} \quad \begin{array}{l} \forall_x \; x \cdot -1 = -x \;\; [\text{Th. } 27] \\ \forall_x \forall_{y \neq 0} \forall_{z \neq 0} \; \frac{xz}{yz} = \frac{x}{y}, \; [y \neq 0 \neq -1] \; [\text{Th. } 60] \end{array}$$

*

Note that Theorem 76 might have been proved earlier and used in deriving subtraction theorems from addition theorems. For example, note the following proof of Theorem 58.

$$\begin{array}{l} \frac{x}{y} - \frac{u}{v} \\ = \frac{x}{y} + -\frac{u}{v} \\ = \frac{x}{y} + \frac{-u}{v} \\ = \frac{xv + -uy}{yv} \\ = \frac{xv - uy}{yv}. \end{array} \quad \begin{array}{l} \text{ps} \\ \forall_x \forall_{y \neq 0} \; -\frac{x}{y} = \frac{-x}{y}, \; [v \neq 0] \; [\text{Th. } 76] \\ \forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0} \; \frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv}, \; [y \neq 0, \; v \neq 0] \; [\text{Th. } 57] \\ \forall_x \forall_y \forall_z \; x - yz = x + -yz \;\; [\text{Th. } 31] \end{array}$$

This is one way to remind students that <u>subtracting</u> is <u>adding the opposite</u>. And, the more conscious they are of this, the better.

[Some students may suggest that (b) could be proved by using the division theorem as in the following. [As mentioned earlier in the COMMENTARY, there are many ways to prove most of the theorems; we usually give no more than two of them.]

$$\begin{aligned}
&\frac{x}{-y}\cdot y \\
&= \frac{1x}{-1y}\cdot y && \text{1 times theorem and } -1 \text{ times theorem} \\
&= \frac{1}{-1}\cdot\frac{x}{y}\cdot y && \forall_x\forall_{y\neq 0}\forall_u\forall_{v\neq 0}\ \frac{x}{y}\cdot\frac{u}{v} = \frac{xu}{yv},\ [-1\neq 0 = y][\text{Th. } 59] \\
&= -1\cdot\frac{x}{y}\cdot y && \text{dividing by } -1 \text{ theorem} \\
&= -1\left(\frac{x}{y}\cdot y\right) && \text{apm} \\
&= -1x && \text{pq; } y\neq 0 \\
&= -x. && -1 \text{ times theorem}
\end{aligned}$$

So, $\frac{x}{-y}\cdot y = -x$. Hence, $\frac{x}{-y} = \frac{-x}{y}$. [division theorem; $y\neq 0$]

And, $-\frac{x}{y} = \frac{-x}{y}$ [Th. 76; $y\neq 0$] So, $-\frac{x}{y} = \frac{x}{-y}$.]

*

Answer for Part A.

$\forall_x\forall_{y\neq 0}\ \frac{-x}{-y} = \frac{x}{y}$ [Theorem 78]

$$\begin{aligned}
&\frac{-x}{-y} \\
&= -\frac{x}{-y} && \forall_x\forall_{y\neq 0}\ -\frac{x}{y} = \frac{-x}{y},\ [-y\neq 0]\ [\text{Th. } 76] \\
&= --\frac{x}{y} && \forall_x\forall_{y\neq 0}\ -\frac{x}{y} = \frac{x}{-y},\ [y\neq 0]\ [\text{Th. } 77] \\
&= \frac{x}{y}. && \forall_x\ --x = x\ [\text{Th. } 17]
\end{aligned}$$

Hence, [for $y\neq 0$,] $\frac{-x}{-y} = \frac{x}{y}$. [$\forall_{y\neq 0}$ if $y\neq 0$ then $-y\neq 0$]

Using the division theorem:

$$\left.\begin{array}{l} -\frac{x}{y}\cdot -y \\ = \frac{x}{y}\cdot y \\ = x. \end{array}\right. \quad \begin{array}{l} \}\ \forall_x\forall_y\ -x\cdot -y = xy\ [\text{Th. } 23] \\ \}\ pq;\ y \neq 0 \end{array}$$

So, $-\frac{x}{y}\cdot -y = x$.

Hence, $-\frac{x}{y} = \frac{x}{-y}$. [division theorem; $-y \neq 0$]

Hence, [for $y \neq 0$,] $-\frac{x}{y} = \frac{x}{-y}$. [?]

The citation indicated by the '?' is needed to justify the assumption that, for each y, if $y \neq 0$ then $-y \neq 0$. One can derive a theorem to this effect from the 0-product theorem, the -1 times theorem and '$-1 \neq 0$'. Here is a proof.

$$\begin{array}{lrl} \text{Suppose that} & y \neq 0. & \multirow{2}{*}{$\Big\}\ \forall_x\forall_y$ if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$, [$-1 \neq 0$] [Th. 55]} \\ \text{Then} & -1y \neq 0, & \\ & & \Big\}\ \forall_x\ -x = -1x\ [\text{Th. } 28] \\ \text{and} & -y \neq 0. & \end{array}$$

Hence, if $y \neq 0$ then $-y \neq 0$.

Here is another proof [it involves contraposition].

Suppose that $-y = 0$.

Then $y + -y = y + 0,$ } po and pa0

and $0 = y.$

Hence, if $-y = 0$ then $y = 0$.

So, if $y \neq 0$ then $-y \neq 0$.

Proofs of (b) on page 2-103. [Theorem 77]

Because there is no left distributive theorem for division over (or under!) addition, the use of the 0-sum theorem is not as direct in this case as it is in proving Theorem 76. If one attempts to use it one may be led to the following proof.

$$
\begin{aligned}
&\frac{x}{y} + \frac{x}{-y} && \Big\} \ \forall_x \ x \cdot -1 = -x \ \text{[Th. 27]} \\
&= \frac{x}{y} + \frac{x}{y \cdot -1} && \Big\} \ \forall_x \forall_{y \neq 0} \forall_{z \neq 0} \ \frac{x}{y} \div z = \frac{x}{yz}, \ [y \neq 0 \neq -1]\text{[Th. 75]} \\
&= \frac{x}{y} + \frac{x}{y} \div -1 && \Big\} \ \forall_x \ x/-1 = -x \ \text{[Th. 52]} \\
&= \frac{x}{y} + -\frac{x}{y} \\
&= \quad \vdots
\end{aligned}
$$

But, one already sees a "direct" proof of Theorem 77:

$$
\begin{aligned}
&-\frac{x}{y} && \Big\} \ \forall_x \ x/-1 = -x \ \text{[Th. 52]} \\
&= \frac{x}{y} \div -1 && \Big\} \ \forall_x \forall_{y \neq 0} \forall_{z \neq 0} \ \frac{x}{y} \div z = \frac{x}{yz}, \ [y \neq 0 \neq -1]\text{[Th. 75]} \\
&= \frac{x}{y \cdot -1} && \Big\} \ \forall_x \ x \cdot -1 = -x \ \text{[Th. 27]} \\
&= \frac{x}{-y}.
\end{aligned}
$$

Hence, $-\frac{x}{y} = \frac{x}{-y}$.

[Another proof of Theorem 77 is on TC[2-103]c.]

Proofs of (a) on page 2-103. [Theorem 76]

Using the 0-sum theorem:

$$\begin{aligned}
&\frac{x}{y} + \frac{-x}{y} && \Big\} \forall_x \forall_y \forall_{z \neq 0}\ \frac{x}{z} + \frac{y}{z} = \frac{x+y}{z},\ [y \neq 0]\ [\text{Th. } 67]\\
&= \frac{x + -x}{y} && \Big\} \text{po}\\
&= \frac{0}{y} && \Big\} \forall_{x \neq 0}\ 0/x = 0,\ [y \neq 0]\ [\text{Th. } 53]\\
&= 0.
\end{aligned}$$

Hence, $-\frac{x}{y} = \frac{-x}{y}$. [0-sum theorem]

Using the division theorem:

$$\begin{aligned}
&(-\frac{x}{y})y && \Big\} \forall_x \forall_y\ -(xy) = -xy\ [\text{Th. } 21]\\
&= -(\frac{x}{y}y) && \Big\} \text{pq; } y \neq 0\\
&= -x.
\end{aligned}$$

So, $(-\frac{x}{y})y = -x$.

Hence, $-\frac{x}{y} = \frac{-x}{y}$. [division theorem; $y \neq 0$]

Using the -1 times theorem:

$$\begin{aligned}
&-\frac{x}{y} && \Big\} -1 \text{ times theorem}\\
&= -1 \cdot \frac{x}{y} && \Big\} \text{cpm}\\
&= \frac{x}{y} \cdot -1 && \Big\} \forall_x \forall_y \forall_{z \neq 0}\ \frac{xy}{z} = \frac{x}{z}y,\ [y \neq 0]\ [\text{Th. } 62]\\
&= \frac{x \cdot -1}{y} && \Big\} \forall_x\ x \cdot -1 = -x\ \ [\text{Th. } 27]\\
&= \frac{-x}{y}.
\end{aligned}$$

Hence, $-\frac{x}{y} = \frac{-x}{y}$.

Let's try to prove that

(a) $$\forall_x \forall_{y \neq 0} \; -\frac{x}{y} = \frac{-x}{y}.$$

One way to attack this would be through the 0-sum theorem. That is, we try to show that, for each x and each nonzero y,

$$\frac{x}{y} + \frac{-x}{y} = 0.$$

This is easy to do.

$$\frac{x}{y} + \frac{-x}{y} = \frac{x + -x}{y} = \frac{0}{y} = 0.$$

Another way to prove (a) is to use the division theorem. That is, we try to show that, for each x and each nonzero y,

$$(-\frac{x}{y})y = -x.$$

Carry out the proof yourself.

Still a third way of proving (a) is to use the -1 times theorem. Hint: $-\frac{x}{y} = -1 \cdot \frac{x}{y} = \ldots$. Finish this third proof of (a).

To complete the proof of (*), we need to prove:

(b) $$\forall_x \forall_{y \neq 0} \; -\frac{x}{y} = \frac{x}{-y}.$$

Prove this in at least two ways.

EXERCISES

A. The theorem about the product of opposites is:

$$\forall_x \forall_y \; (-x)(-y) = xy.$$

There is an analogous theorem for quotients. State it and prove it.

B. Simplify by finding equivalent expressions with fewer minus signs.

Sample 1. $-\frac{-3}{-7}$

Solution. $-\frac{-3}{-7} = -\frac{3}{7}.$

(continued on next page)

Sample 2. $-\frac{-x(a - b)}{-y(c - d)}$

Solution. $-\frac{-x(a - b)}{-y(c - d)}$

$$\left.\begin{aligned} &= -\frac{-[x(a - b)]}{-[y(c - d)]} \\ &= -\frac{x(a - b)}{y(c - d)} \end{aligned}\right\} [y(c - d) \neq 0]$$

$$= \frac{-[x(a - b)]}{y(c - d)}$$

$$= \frac{x \cdot -(a - b)}{y(c - d)}$$

$$= \frac{x(b - a)}{y(c - d)}.$$

Answer. $\frac{x(b - a)}{y(c - d)}$, $[y \neq 0,\ c \neq d]$

[An equally good answer is: $\frac{x(a - b)}{y(d - c)}$, $[y \neq 0,\ c \neq d]$.]

1. $-\frac{-5}{8}$ 2. $-\frac{9}{-7}$ 3. $-\frac{7 - x}{2}$ 4. $-\frac{8 + y}{8 - y}$

5. $-\frac{-3 - y}{5}$ 6. $\frac{-(x - 3)}{-(3 - y)}$ 7. $-\frac{-(x - 3)}{-(3 - y)}$ 8. $\frac{4 - x - y}{m - 2}$

9. $\frac{(a - b)(c - d)}{-(a + b)}$ 10. $-\frac{(x - 1)(2 - x)(3 - x)}{-(x + 4) \cdot -(x + 5)}$

*

[Part J of the Supplementary Exercises provides computational practice in simplifying fractions which do not contain pronumerals.]

*

C. Simplify.

1. $\frac{40x}{5}$ 2. $\frac{54y}{9}$ 3. $\frac{72a}{-8}$ 4. $\frac{-16a}{4}$

5. $\frac{-30p}{-6}$ 6. $\frac{18xy}{2}$ 7. $\frac{36abc}{-9}$ 8. $\frac{-17rs}{-1}$

9. $\frac{6xy}{1/2}$ 10. $\frac{-7x}{-2}$ 11. $\frac{-204xyz}{2}$ 12. $\frac{-27xy}{-3}$

13. $\frac{5(a + b)}{5}$ 14. $\frac{-6(x - y)}{6}$

Answers for Part B [which begins on page 2-103].

1. $\frac{5}{8}$ 2. $\frac{9}{7}$ 3. $\frac{x - 7}{2}$

4. $\frac{8 + y}{y - 8}$, $[y \neq 8]$ 5. $\frac{3 + y}{5}$ 6. $\frac{x - 3}{3 - y}$, $[y \neq 3]$

7. $\frac{x - 3}{y - 3}$ [or: $\frac{3 - x}{3 - y}$], $[y \neq 3]$ 8. $\frac{4 - (x + y)}{m - 2}$, $[m \neq 2]$

9. $\frac{(b - a)(c - d)}{a + b}$ [or: $\frac{(a - b)(d - c)}{a + b}$], $[a \neq -b]$

10. $\frac{(1 - x)(2 - x)(3 - x)}{(x + 4)(x + 5)}$, $[-5 \neq x \neq -4]$

*

Answers for Part C [on pages 2-104 through 2-108].

1.	$8x$	2.	$6y$	3.	$-9a$	4.	$-4a$
5.	$5p$	6.	$9xy$	7.	$-4abc$	8.	$17rs$
9.	$12xy$	10.	$\frac{7x}{2}$	11.	$-102xyz$	12.	$9xy$
13.	$a + b$			14.	$y - x$		

15. $5y$, $[x \neq 0]$

16. $5ab$, $[a \neq 0]$

17. $3x$, $[x \neq 0 \neq y]$

18. $\frac{3y}{u}$, $[x \neq 0 \neq u]$

19. 1, $[xyz \neq 0]$

20. $9aa$, $[b \neq 0]$

21. -17, $[a \neq 0]$

22. $\frac{4x}{y}$, $[xyz \neq 0]$

23. $\frac{-2}{y}$, $[xy \neq 0]$

24. -1, $[a \neq 0]$

25. $\frac{3n}{m}$, $[mn \neq 0]$

26. $\frac{6bc}{xy}$, $[axy \neq 0]$

27. $\frac{5ax}{14bb}$, $[b \neq 0]$

28. $\frac{15kk}{2p}$, $[mp \neq 0]$

29. $\frac{c}{z}$, $[abxyz \neq 0]$

30. $\frac{-18dr}{7cs}$, $[crs \neq 0]$

31. $\frac{4xy}{3}$, $[xy \neq 0]$

32. $\frac{3pps}{4t}$, $[rst \neq 0]$

33. $\frac{10ay}{9bx}$, $[bxy \neq 0]$

34. $\frac{27x}{8a}$, $[by \neq 0,\ a \neq 0]$

35. $\frac{14ccr}{3as}$, $[acprs \neq 0]$

36. 4, $[a \neq 5]$

37. $\frac{5}{18}$, $[x \neq 2,\ x \neq -7]$

38. $\frac{45}{2}$, $[x \neq -1,\ x \neq -5]$

39. $\frac{2(x + 5)(x + 2)}{3(x + 4)}$, $[x \neq -1, -3, -4]$

40. $\frac{5}{7(a - 4)}$, $[a \neq -3,\ 4]$

41. $\frac{3}{5}$, $[x \neq -9,\ 11]$

42. $9d + 3e$

43. $m - 3$

44. $3 - x$

45. $2x + 3$

46. $2(3c - d)$

47. $2(3c - d)$

48. $-p - 2r$

49. $3(3x - 5y)$

50. $4(1 - 20a)$

51. $\frac{u + v}{2}$

52. $-a(4b + c)$

53. $9(3f - 4g)$

54. $10xy - 14$

55. $2(2x - 3y)$

56. $\frac{a + 3b}{4}$

57. $2y + z$, $[x \neq 0]$

58. $3b - 7c$, $[a \neq 0]$

59. $2y + 3$, $[xyz \neq 0]$

60. $7q - 2$, $[ap \neq 0]$

61. $9xy + 1$, $[z \neq 0]$

62. $1 - 6bc$, $[a \neq 0]$

63. 3, $[x \neq -4]$

64. $x + 7$, $[x \neq 2]$

65. $-(y + 9)$, $[y \neq 3]$

66. $-m - 7$, $[m \neq 5]$

67. $7a$, $[a \neq 1]$

68. $2x$, $[x \neq -3]$

15. $\frac{15xy}{3x}$

16. $\frac{20aab}{4a}$

17. $\frac{9xxy}{3xy}$

18. $\frac{9xy}{3xu}$

19. $\frac{5xyz}{5xyz}$

20. $\frac{-18aab}{-2b}$

21. $\frac{17a}{-a}$

22. $\frac{24xxyz}{6xyyz}$

23. $\frac{-6xy}{3xyy}$

24. $\frac{9aa}{-9aa}$

25. $\frac{-15mnn}{-5mmn}$

26. $\frac{-12abc}{-2axy}$

27. $\frac{5a}{6b} \times \frac{3x}{7b}$

28. $\frac{5k}{3m} \times \frac{9mk}{2p}$

29. $\frac{2xy}{3ab} \times \frac{12abc}{8xyz}$

30. $\frac{-3cd}{2rs} \times \frac{12rr}{7cc}$

31. $\frac{3xx}{-2y} \times \frac{-8yy}{9x}$

32. $\frac{5tp}{-3sr} \cdot \frac{9ssrp}{-20tt}$

33. $\frac{4a}{3b} \div \frac{6x}{5y}$

34. $\frac{9ax}{4by} \div \frac{2aa}{3by}$

35. $\frac{7apc}{2rs} \div \frac{3aap}{4rrc}$

36. $\frac{a - 5}{2} \times \frac{8}{a - 5}$

37. $\frac{x + 7}{3(x - 2)} \times \frac{5(x - 2)}{6(x + 7)}$

38. $\frac{9(x + 1)}{3(x + 5)} \times \frac{15(x + 5)}{2(x + 1)}$

39. $\frac{4(x + 5)(x + 3)}{9(x + 1)} \cdot \frac{3(x + 1)(x + 2)}{2(x + 3)(x + 4)}$

40. $\frac{5(a + 3)}{6(a - 4)} \div \frac{7(a + 3)}{6}$

41. $\frac{2(9 + x)}{7(11 - x)} \div \frac{10(x + 9)}{21(11 - x)}$

42. $\frac{18d + 6e}{2}$

43. $\frac{5m - 15}{5}$

44. $\frac{3x - 9}{-3}$

45. $\frac{4.6x + 6.9}{2.3}$

46. $\frac{2(9c - 3d)}{3}$

47. $\frac{2}{3}(9c - 3d)$

48. $\frac{1}{7}(-7p - 14r)$

49. $\frac{3}{5}(15x - 25y)$

50. $(5 - 100a)\frac{4}{5}$

51. $\frac{2u + 2v}{4}$

52. $\frac{12ab + 3ac}{-3}$

53. $(9f - 12g) \div \frac{1}{3}$

54. $\frac{5xy - 7}{1/2}$

55. $\frac{18x - 27y}{9/2}$

56. $\frac{\frac{1}{2}a + \frac{3}{2}b}{2}$

57. $\frac{8xy + 4xz}{4x}$

58. $\frac{3ab - 7ac}{a}$

59. $\frac{8xyyz + 12xyz}{4xyz}$

60. $\frac{2ap - 7apq}{-ap}$

61. $\frac{27xyz + 3z}{3z}$

62. $\frac{18abc - 3a}{-3a}$

63. $\frac{3(x + 4)}{x + 4}$

64. $\frac{(x - 2)(x + 7)}{x - 2}$

65. $\frac{(y - 3)(y + 9)}{-(y - 3)}$

66. $\frac{(m - 5)(m + 7)}{5 - m}$

67. $(a - 1) \cdot \frac{7a}{(a - 1)}$

68. $2(x + 3) \cdot \frac{x}{x + 3}$

Sample 1. $28\left[\frac{x+18}{4} - \frac{3}{7}(x-3)\right]$

Solution. $28\left[\frac{x+18}{4} - \frac{3}{7}(x-3)\right]$

$= 28 \cdot \frac{x+18}{4} - 28\left[\frac{3}{7}(x-3)\right]$

$= 7(x+18) - 4 \cdot 3(x-3)$

$= 7x + 126 - 12(x-3)$

$= 7x + 126 - (12x - 36)$

$= 7x + 126 - 12x + 36 = 162 - 5x.$

69. $12\left(\frac{a}{3} + \frac{a}{4}\right)$ 70. $8\left(\frac{3n}{4} - \frac{5n}{2}\right)$ 71. $8\left(\frac{3}{4}n - \frac{5}{2}n\right)$

72. $18\left(\frac{5x}{6} + \frac{5x}{9}\right)$ 73. $24\left(\frac{2x}{3} - \frac{3y}{8}\right)$ 74. $14\left(\frac{x}{7} - \frac{y}{2}\right)$

75. $6\left(\frac{a+2}{3} + \frac{a+3}{2}\right)$ 76. $10\left(\frac{2k}{5} - \frac{3k}{2}\right)$ 77. $15\left(\frac{3x}{5} - \frac{5x}{3}\right)$

78. $8\left(\frac{a+5}{2} - \frac{a+1}{4}\right)$ 79. $12\left(\frac{4y+5}{2} - \frac{2y+9}{3} - \frac{5y-4}{4}\right)$

80. $6\left(\frac{x+11}{6} - \frac{10-x}{3}\right)$ 81. $24\left[\frac{y-7}{8} - \frac{9-y}{3} + \frac{y}{6}\right]$

82. $12\left[\frac{1}{3}(x+5) - 4 - \frac{x-8}{4} + \frac{1}{2}\right]$

Sample 2. $2(a-5)\left(\frac{a}{a-5} - \frac{3}{2}\right)$

Solution. $2(a-5)\left(\frac{a}{a-5} - \frac{3}{2}\right)$

$$\left.\begin{aligned} &= 2(a-5)\frac{a}{a-5} - 2(a-5)\frac{3}{2} \\ &= 2a - (a-5)3 \end{aligned}\right\}[a \neq 5]$$

$= 2a - 3a + 15$

$= -a + 15.\ \ [a \neq 5]$

Answer. $-a + 15$, $[a \neq 5]$, or: $15 - a$, $[a \neq 5]$

69. $7a$ 70. $-14n$ 71. $-14n$ 72. $25x$

73. $16x - 9y$ 74. $2x - 7y$ 75. $5a + 13$ 76. $-11k$

77. $-16x$ 78. $2a + 18$ 79. $y + 6$ 80. $3x - 9$

81. $15y - 93$ 82. $x + 2$ 83. $3 - k$, $[k \neq 0]$

84. $5 - 6a$, $[a \neq 0]$ 85. $7(a + 1)$, $[a \neq 4]$ 86. $10 - a$, $[a \neq 0]$

87. 33, $[x \neq 0]$ 88. $5s - 21$, $[s \neq 0]$ 89. $b - 5$, $[b \neq -5]$

90. $7(2 - n)$, $[n \neq 0]$ 91. $x + 6$, $[x \neq -2]$ 92. $5(b - 5)$, $[b \neq -3]$

93. $4(6 - y)$, $[y \neq -4, 0]$ 94. $3(xx - 7x + 21)$, $[x \neq 0, 3]$

95. $m - 3$, $[m \neq -3, 0]$ 96. $3(6rr - 3r + 4)$, $[r \neq 0, \frac{4}{3}]$

*

Students should see additional samples before going on with the exercises. [For one thing, their work on the previous exercises may otherwise lead them to make the common mistake of forgetting the denominator when simplifying a sum or difference of fractions.]

<u>Sample</u>. $\frac{4x}{3y} + \frac{5x}{6z}$

<u>Solution 1</u>.

$$\left.\begin{aligned} &\frac{4x}{3y} + \frac{5x}{6z} \\ &= \frac{4x(2z)}{3y(2z)} + \frac{5xy}{6zy} \end{aligned}\right\} [y \neq 0,\ z \neq 0]$$

$$= \frac{8xz + 5xy}{6yz}.$$

<u>Solution 2</u>.

$$\frac{4x}{3y} + \frac{5x}{6z}$$

$$\left.\begin{aligned} &= \frac{\left(\frac{4x}{3y} + \frac{5x}{6z}\right)(6yz)}{6yz} \\ &= \frac{8xz + 5xy}{6yz}. \end{aligned}\right\} [yz \neq 0]$$

*

97. $(-2x)/15$ 98. $(25y)/14$ 99. $(29x - 2)/6$

100. $(7a)/8$ 101. $(8b - 1)/10$ 102. $(23x - 15)/15$

103. $\frac{13}{2x}$, $[x \neq 0]$ 104. $\frac{26}{3y}$, $[y \neq 0]$

105. $\frac{44 + 15k}{5k}$, $[k \neq 0]$ 106. $\frac{168z - 5}{24z}$, $[z \neq 0]$

107. $\frac{21r - 46}{14r}$, $[r \neq 0]$ 108. $\frac{9t + 47}{20}$

109. $\frac{9t + 47}{20}$ 110. $\frac{3a + 14}{10}$ 111. $\frac{a - 3}{12}$

112. $\frac{r + 13}{6}$ 113. $\frac{4 - 3s}{10}$ 114. $\frac{z - 2}{10}$

115. $\frac{-c}{12}$ 116. $\frac{2(n + 5)}{9}$ 117. $\frac{105 - 7y}{18}$

118. $\frac{8a + 19}{(a + 2)(a + 3)}$, $[a \neq -3, -2]$ 119. $\frac{31 - 3x}{(x - 5)(x - 1)}$, $[x \neq 1, 5]$

120. $\frac{11x - 42}{(x - 3)(x - 4)}$, $[x \neq 3, 4]$ 121. $\frac{11 - y}{(y - 7)(y - 5)}$, $[y \neq 5, 7]$

☆122. $\frac{xx - 40}{x(x - 8)}$, $[x \neq 0, 8]$ ☆123. $\frac{5(y - 1)}{y(2y + 1)}$, $[y \neq -\frac{1}{2}, 0]$

☆124. $\frac{a - 11}{(a - 3)(a + 5)}$, $[a \neq -5, 3]$ ☆125. $\frac{4yy + 5y + 18}{12(y + 2)}$, $[y \neq -2]$

☆126. $\frac{8x - 9}{5(x - 8)}$, $[x \neq 8]$ ☆127. $\frac{-4x(x + 5)}{(x + 3)(x + 7)}$, $[-7 \neq x \neq -3]$

128. $\frac{9x - 3}{2 - x}$, $[x \neq 0, 2]$ 129. $\frac{3(6t - 1)}{2(21t + 1)}$, $[0 \neq t \neq -\frac{1}{21}]$

130. $\frac{rs}{s + r}$, $[rs \neq 0,\ r \neq -s]$

131. $\frac{4a(3ab + 1)}{3b(4ab + 1)}$, $[b \neq 0 \neq a,\ ab \neq -\frac{1}{4}]$

132. $\frac{xy(x - y)}{3x + 2y}$, $[x \neq 0 \neq y,\ 3x + 2y \neq 0]$

133. $\frac{bc}{a(b + c)}$, $[abc(b + c) \neq 0]$

134. -1 135. $7/17$

TC[2-107, 108]

83. $k\left(\frac{3}{k} - 1\right)$

84. $2a\left(\frac{5}{2a} - 3\right)$

85. $7(a - 4)\left(\frac{5}{a - 4} + 1\right)$

86. $2a\left(\frac{5}{a} - \frac{1}{2}\right)$

87. $6x\left(\frac{4}{x} + \frac{3}{2x}\right)$

88. $7s\left(\frac{5}{7} - \frac{3}{s}\right)$

89. $2(b + 5)\left(\frac{b}{b + 5} - \frac{1}{2}\right)$

90. $12n\left(\frac{2n + 1}{2n} - \frac{3n - 2}{3n} - \frac{7}{12}\right)$

91. $2(x + 2)\left(\frac{3}{2} - \frac{x}{x + 2}\right)$

92. $8(b + 3)\left(\frac{b - 2}{b + 3} - \frac{3}{8}\right)$

93. $5y(y + 4)\left(\frac{6}{5y} - \frac{2}{y + 4}\right)$

94. $3x(x - 3)\left(\frac{x}{x - 3} - \frac{7}{x}\right)$

95. $m(m + 3)\left(\frac{2}{m + 3} - \frac{1}{m}\right)$

96. $9r(3r - 4)\left(-\frac{1}{3r} + \frac{2r}{3r - 4}\right)$

*

97. $\frac{x}{5} - \frac{x}{3}$

98. $\frac{2y}{7} + \frac{3y}{2}$

99. $\frac{9x}{2} - \frac{1}{3} + \frac{x}{3}$

100. $\frac{3a}{4} + \frac{a}{8}$

101. $\frac{4b}{5} - \frac{1}{10}$

102. $\frac{x}{3} + \frac{x}{5} + x - 1$

103. $\frac{8}{x} - \frac{3}{2x}$

104. $\frac{7}{y} + \frac{5}{3y}$

105. $\frac{9}{k} - \frac{1}{5k} + 3$

106. $\frac{2}{3z} - \frac{1}{2z} + 7 - \frac{3}{8z}$

107. $1 - \frac{2}{7r} + \frac{1}{2} - \frac{3}{r}$

108. $\frac{7t + 1}{10} - \frac{t - 9}{4}$

109. $\frac{1}{10}(7t + 1) - \frac{1}{4}(t - 9)$

110. $\frac{a + 4}{2} - \frac{a + 3}{5}$

111. $\frac{4a + 1}{2} - \frac{2a + 3}{3} - \frac{5a - 1}{4}$

112. $\frac{r + 1}{2} - \frac{r - 5}{3}$

113. $\frac{6s - 3}{5} - \frac{3s - 2}{2}$

114. $\frac{z - 7}{5} + 2 - \frac{z + 8}{10}$

115. $\frac{c - 2}{6} - \frac{c - 4}{4} - \frac{2}{3}$

116. $3 + \frac{n - 1}{3} - \frac{n + 14}{9}$

117. $\frac{y + 9}{9} + \frac{1}{3} - \frac{y - 7}{2} + 1$

Sample 3. $\dfrac{5}{a - 2} - \dfrac{6}{a}$

Solution.
$$\left.\begin{aligned}&\frac{5}{a - 2} - \frac{6}{a}\\ &= \frac{5a - 6(a - 2)}{(a - 2)a}\end{aligned}\right\}[a \neq 2,\ a \neq 0]$$

$$= \frac{5a - 6a + 12}{(a - 2)a}$$

$$= \frac{-a + 12}{(a - 2)a}\ .$$

Answer. $\dfrac{-a + 12}{(a - 2)a}$, $[a \neq 2,\ a \neq 0]$

118. $\dfrac{3}{a + 2} + \dfrac{5}{a + 3}$

119. $\dfrac{4}{x - 5} - \dfrac{7}{x - 1}$

120. $\dfrac{9}{x - 3} + \dfrac{2}{x - 4}$

121. $\dfrac{2}{y - 7} - \dfrac{3}{y - 5}$

☆122. $\dfrac{5}{x} + \dfrac{3}{x - 8} + 1$

☆123. $\dfrac{11}{2y + 1} - \dfrac{5}{y} + \dfrac{4}{2y + 1}$

☆124. $\dfrac{6}{a - 3} + \dfrac{2}{a + 5} - \dfrac{7}{a - 3}$

☆125. $\dfrac{3}{4} - \dfrac{y}{y + 2} + \dfrac{y}{3}$

☆126. $\dfrac{x + 1}{x - 8} + \dfrac{3}{5} + \dfrac{2}{x - 8}$

☆127. $\dfrac{x}{x + 3} - \dfrac{2x}{x + 7} - \dfrac{3x}{x + 3}$

128. $\dfrac{\frac{3}{2} - \frac{1}{2x}}{\frac{1}{3x} - \frac{1}{6}}$

129. $\dfrac{3 - \frac{1}{2t}}{7 + \frac{1}{3t}}$

130. $\dfrac{1}{\frac{1}{r} + \frac{1}{s}}$

131. $\dfrac{a + \frac{1}{3b}}{b + \frac{1}{4a}}$

132. $\dfrac{x - y}{\frac{2}{x} + \frac{3}{y}}$

133. $\dfrac{\frac{1}{a}}{\frac{1}{b} + \frac{1}{c}}$

134. $\dfrac{1}{1 - \dfrac{1}{1 - \dfrac{1}{1 + 1}}}$

135. $\dfrac{1}{2 + \dfrac{1}{2 + \dfrac{1}{2 + 1}}}$

[More exercises are in Part K, Supplementary Exercises.]

Exercises 122 through 127 are optional, of course. However, we anticipate that your better students will do them. It would be instructive to ask them to justify their solutions by means of the basic principles or theorems previously studied.

Here is a solution for Exercise 122.

$$\begin{aligned}
&\frac{5}{x} + \frac{3}{x-8} + 1 \\
&\qquad \text{apa} \\
&= \frac{5}{x} + \left(\frac{3}{x-8} + 1\right) \\
&\qquad \text{cpa} \\
&= \frac{5}{x} + \left(1 + \frac{3}{x-8}\right) \\
&\qquad \text{mixed number theorem; } x \neq 8 \\
&= \frac{5}{x} + \frac{x - 8 + 3}{x - 8} \\
&\qquad \text{ps, apa, and } -8 + 3 = -5 \\
&= \frac{5}{x} + \frac{x - 5}{x - 8} \\
&\qquad \text{addition of fractions theorem; } 0 \neq x \neq 8 \\
&= \frac{5(x - 8) + (x - 5)x}{x(x - 8)} \\
&\qquad \ell\text{dtms, dtms, } 5 \cdot 8 = 40 \\
&= \frac{5x - 40 + (xx - 5x)}{x(x - 8)} \\
&\qquad \text{ps, apa, cpa, apa, po, pa0, ps} \\
&= \frac{xx - 40}{x(x - 8)}.
\end{aligned}$$

Hence, $\frac{5}{x} + \frac{3}{x-8} + 1 = \frac{xx - 40}{x(x - 8)}$.

Here is a solution for Exercise 124.

$$\frac{6}{a-3} + \frac{2}{a+5} - \frac{7}{a-3}$$

ps and $\forall_x \forall_{y \neq 0}\ -\frac{x}{y} = \frac{-x}{y}$, $[a \neq 3]$[Th. 76]

$$= \frac{6}{a-3} + \frac{2}{a+5} + \frac{-7}{a-3}$$

cpa

$$= \frac{-7}{a-3} + \left(\frac{6}{a-3} + \frac{2}{a+5}\right)$$

apa

$$= \frac{-7}{a-3} + \frac{6}{a-3} + \frac{2}{a+5}$$

$\forall_x \forall_y \forall_{z \neq 0}\ \frac{x}{z} + \frac{y}{z} = \frac{x+y}{z}$, $[a \neq 3]$[Th. 67]

$$= \frac{-7+6}{a-3} + \frac{2}{a+5}$$

$-7 + 6 = -1$; addition of fractions theorem; $-5 \neq a \neq 3$

$$= \frac{-1(a+5) + 2(a-3)}{(a-3)(a+5)}$$

ℓdpma, $\forall_x\ -1x = -x$, ℓdtms, $2 \cdot 3 = 6$

$$= \frac{-1a + -5 + (2a - 6)}{(a-3)(a+5)}$$

apa, cpa, apa, ps, apa

$$= \frac{2a + -1a + (-5 + -6)}{(a-3)(a+5)}$$

dpma, $2 + -1 = 1$, $\forall_x\ 1x = x$, $-5 + -6 = -11$, ps

$$= \frac{a - 11}{(a-3)(a+5)}$$

Hence, $\frac{6}{a-3} + \frac{2}{a+5} - \frac{7}{a-3} = \frac{a-11}{(a-3)(a+5)}$.

[The first four steps in this derivation are worth noting. If one followed the convention "perform additions and subtractions in order from left to right" the test-pattern would be much longer.]

There is a more extensive treatment of inequations in Unit 3. [See pages 3-100ff. and the accompanying COMMENTARY.]

*

Students who wish to prove the generalization in Sample 1 can proceed as follows.

$$\left.\begin{array}{l} x + 7 - x \\ = x + 7 + -x \\ = 7 + x + -x \\ = 7. \end{array}\right. \begin{array}{l} \}\ \text{ps} \\ \}\ \text{cpa} \\ \}\ \forall_x \forall_y\ (x + y) + -y = x\ \ [\text{Th. } 29] \end{array}$$

So, since 7 is a positive number, $x + 7 > x$.

*

Answers for Part A [on pages 2-109 and 2-110].

[For some exercises which deal with false generalizations we give below a related true generalization.]

1. $\forall_x\ x - 3 < x$. True.

2. $\forall_x\ 2x > x$. False. [$\forall_x$ if $x > 0$ then $2x > x$.]

3. $\forall_x\ x \div 2 < x$. False. [$\forall_x$ if $x > 0$ then $x \div 2 < x$.]

4. $\forall_x\ xx \geq x$. False. [$\forall_x$ if $x \geq 1$ or $x \leq 0$ then $xx \geq x$.]

5. $\forall_x \forall_y$ if $y - x > 0$ then $x - y < 0$. True.

6. $\forall_x \forall_y$ if $y - x \leq 0$ then $x - y > 0$. False.

7. $\forall_x \forall_y \forall_z$ if $x > y$ and $y > z$ then $x > z$. True.

8. $\forall_x \forall_y \forall_z$ if $x < y$ and $z < y$ then $x < z$. False.

9. $\forall_x \forall_y \forall_z$ if $y < z$ then $x \div y > x \div z$. False. [More properly, 'Nonsense'.]

 [$\forall_{x > 0} \forall_{y > 0} \forall_{z > 0}$ if $y < z$ then $x \div y > x \div z$.]

2.08 Comparing real numbers.--In Unit 1 you learned a procedure for deciding which of two real numbers is the larger. If you have a first number and a second number and if you can get the first number by adding a positive number to the second number, which is the larger number?

Suppose Rita picks a first number, Rhoda picks a second number, and Rhoda subtracts her number from Rita's. If the difference is a positive number, who picked the larger number? If the difference is not a positive number, who picked the larger?

We can summarize this subtraction procedure for telling which is the larger number as follows:

For each x, for each y,

(a) if $x - y$ is a positive number then $x > y$, and

(b) if $x - y$ is not a positive number then $x \ngtr y$.

EXERCISES

A. Each of the following exercises involves a generalization. In some cases the generalization is true, in others it is false. For each exercise, state the generalization in a concise way, and tell whether you think it is true or false.

Sample 1. If I add 7 to a first number, I get a second number which is larger than the first number.

Solution. If you need to, you can try a few examples to get the "feel" of the generalization.

$1 + 7 = 8$ and $8 > 1$

$^-5 + 7 = 2$ and $2 > {}^-5$

$^-483 + 7 = {}^-476$ and $^-476 > {}^-483$

The generalization is:

$$\forall_x \; x + 7 > x.$$

I think it's true.

Sample 2. If I add a second number to a first number, I get a third number which is larger than the first number.

The generalization is:

$$\forall_x \forall_y \; x + y > x.$$

This is false, and here's why. Add $^-5$ to 8; you get 3 which is smaller than the first number, 8.

1. If I subtract 3 from a first number, I get a second number which is smaller than the first number.
2. If I multiply a first number by 2, I get a second number which is larger than the first number.
3. If I divide a first number by 2, I get a second number which is smaller than the first number.
4. If I multiply a number by itself, I get the same or a larger number.
5. If I subtract a first number from a second number and get a positive difference, I would get a negative difference if I reversed the order of subtracting.
6. If I subtract a first number from a second number and get a nonpositive difference, I would get a positive difference if I reversed the order of subtracting.
7. If a first number is greater than a second number, and the second number is greater than the third number, then the first number is greater than the third number.
8. If a first number is less than a second number and a third number is less than this second number, then the first number is less than the third number.
9. I divide a first number by a second number, and I also divide the first number by a third number. If the second number is less than the third number, the first quotient is greater than the second quotient.

Answers for Part B.

1. T	2. T	3. T	4. T	5. T
6. F	7. T	8. F	9. T	10. T
11. T	12. F	13. F	14. F	15. T
16. F	17. F	18. T	19. T	20. T

*

We have given no basic principles from which students could derive theorems concerning absolute values. Using '$|\ldots|$' as an abbreviation for '$^{+}|\ldots|$' we could take for such a basic principle:

$$\forall_x \text{ (if } x \geq 0 \text{ then } |x| = x \text{ and if } x < 0 \text{ then } |x| = -x).$$

*

Quiz.

1. Which of the expressions given below is equivalent to '$|-9 - {}^{-}13|$'?

(a) $|9 - -13|$ (b) $|-9 + -13|$ (c) $|-9 - 13|$
(d) $|9 + 13|$ (e) $|9 + -13|$

2. Which of the following statements is true?

(a) $|6| - |-3| > |6| + |3|$ (b) $|6| - |-3| > |6| - |3|$
(c) $|6| - |-3| < |6 - -3|$ (d) $|6| - |-3| < |-6| - |3|$
(e) $|6| - |-3| > |6 - 3|$

3. $\forall_x$ if $x < 0$ then _______.

(a) $x + |x| > x - x$ (b) $|x| < -x$ (c) $|x| = x$
(d) $\frac{1}{x} < 0$ (e) $\frac{1}{x} < x$

*

Answers for Quiz.

[We give the letter which identifies the correct choice.]

1. (e) 2. (c) 3. (d)

B. True or false?

1. $\forall_x \ |x| \geq x$. [An instance of this generalization is: $|{}^-5| \geq {}^-5$. To make sense out of this statement, we must assume that '$|{}^-5|$' is being used as an abbreviation for '$^+|{}^-5|$'. See Unit 1, page 1-110.]

2. $\forall_x \forall_y \ xy \leq |x| \cdot |y|$.

3. $\forall_x \ |-x| = |x|$.

4. $\forall_x \ -|xx| = x \cdot -x$.

5. $\forall_x \ |x(-x)| = xx$.

6. $\forall_x \forall_y \ |x + y| = |x| + |y|$.

7. $\forall_x \forall_y \ |x + y| \leq |x| + |y|$.

8. $\forall_x \forall_y \ |x - y| \leq |x| - |y|$.

9. $\forall_x \forall_y \forall_u \forall_v$ if $x > y$ and $u > v$ then $x + u > y + v$.

10. $\forall_x \forall_y \forall_z$ if $x > y$ then $x + z > y + z$.

11. $\forall_x \forall_y \forall_z$ if $x + z > y + z$ then $x > y$.

12. $\forall_x \forall_y \forall_z$ if $x > y$ then $xz > yz$.

13. $\forall_x \forall_y \forall_z$ if $xz > yz$ then $x > y$.

14. $\forall_{x \neq 0} \forall_{y \neq 0}$ if $x > y$ then $\frac{1}{x} < \frac{1}{y}$.

15. $\forall_x \forall_y \forall_{z > 0}$ if $y < x$ then $yz < xz$.

16. $\forall_x \forall_y \forall_{z < 0}$ if $x > y$ then $xz > yz$.

17. $\forall_x \forall_{y > 0} \ |xy| \not> -xy$.

18. $\forall_{x < 0} \forall_{y > 0} \ |xy| \not< -xy$.

19. $\forall_x \forall_y$ if $x \geq y$ then $x - y \geq 0$.

20. $\forall_{x > 0}$ if $\frac{1}{x} > 2$ then $x < \frac{1}{2}$.

MISCELLANEOUS EXERCISES

A. For each open sentence, tell what value of the pronumeral can be used to generate a true sentence.

1. $4\,\square + 1 = 9$
2. $8\,\diamond - 2 = 38$
3. $2\,\bigtriangledown + \bigtriangledown = 12$
4. $9\,\triangle - 2\,\triangle = 28$
5. $3x - 17 = 1$
6. $2y + 12 = 18$
7. $5 - 3A = -7$
8. $20 + 3B = 50$
9. $2m + 5m = 35$
10. $p - 4p = 17$
11. $\frac{1}{2}y = 12$
12. $\frac{1}{4}x + 7 = 22$
13. $\frac{1}{3}z - 5 = 75$
14. $7 + \frac{1}{3}y = 15$
15. $\frac{x - 4}{3} = 12$
16. $\frac{5 - y}{9} = 2$
17. $\frac{2A + 1}{4} = 8$
18. $\frac{12 - 7k}{2} = 5\frac{1}{2}$
19. $x = x + 9$
20. $|x - 3| = 5$
21. $3(x + 2) = 51$
22. $5(x + 7) + 8 = 108$
23. $8x + 2 - 5x = 27$
24. $4y + 6 - 17y = 6$
25. $3(x + 2) + 2(x - 3) + 9x = 42$

B. For each of the following expressions, write <u>three</u> expressions which are equivalent to it.

1. $3x + 2y + 9$
2. $x + 5x - 7$
3. $5 + 9 - 6$
4. $18 - 2x + 25$
5. $6(r - 3s)$
6. $4xx + 2x$
7. $4x(2y - 3z)$
8. $3ab + 5ac$
9. $\frac{1}{2}x - \frac{2}{3}y$
10. $\frac{11 - 6x}{9}$
11. $27xy$
12. $33xy \div (3x)$

Answers for MISCELLANEOUS EXERCISES.

A.

1.	2	2.	5	3.	4	4.	4	5.	6
6.	3	7.	4	8.	10	9.	5	10.	$-\frac{17}{3}$
11.	24	12.	60	13.	240	14.	24	15.	40
16.	-13	17.	$15\frac{1}{2}$	18.	$\frac{1}{7}$	19.	none	20.	8, -2
21.	15	22.	13	23.	$\frac{25}{3}$	24.	0	25.	3

B. [For each exercise, we list three expressions which are equivalent to the given ones; these are not the only ones which could be written. Your students will doubtless suggest others. You may need to review the description of equivalent expressions which is found on page 2-49.]

1. $2y + 3x + 9$, $2y + 9 + 3x$, $9 + 3x + 2y$
2. $6x - 7$, $-7 + 6x$, $5x + x - 7$
3. $9 + 5 - 6$, $-6 + 5 + 9$, 8
4. $-2x + 18 + 25$, $25 + 18 - 2x$, $43 - 2x$
5. $6r - 18s$, $(r - 3s)6$, $-18s + 6r$
6. $2x + 4xx$, $2(x + 2xx)$, $2x(2x + 1)$
7. $4x \cdot 2y - 4x \cdot 3z$, $(2y - 3z)4x$, $8xy - 12xz$
8. $5ac + 3ab$, $5ca + 3ba$, $(3b + 5c)a$
9. $\frac{x}{2} - \frac{2y}{3}$, $-\frac{2}{3}y + \frac{1}{2}x$, $\frac{3x - 4y}{6}$
10. $\frac{11}{9} - \frac{6x}{9}$, $\frac{11}{9} - \frac{2}{3}x$, $\frac{-6x + 11}{9}$
11. $9 \cdot 3xy$, $\frac{54xy}{2}$, $xy \cdot 27$
12. $\frac{33xy}{3x}$, $[x \neq 0]$; $11y$, $[x \neq 0]$; $\frac{11xy}{x}$, $[x \neq 0]$

C. [The numerals given here name the numbers 9, 12, 14, 17, 18, 49, −3, −12, and −14. We use the simplest names as column headings, and under each of them give the other numerals which name the same number.]

14	17	−14
$5 \times 2 + 4$	$23 - 2 \cdot 3$	$7 \div -.5$
$\frac{2}{3} \cdot 21$	$\frac{40 + -6}{2}$	${}^-6 \cdot {}^-2 - {}^-2 \cdot {}^-13$
$11 + 3 \cdot 1$	$.5 \times \lvert 1 - 35 \rvert$	$7(3 - 5)$
$2 + 3 \times 4$	$8 \times 2 + 1$	$-8 + 3({}^-2)$
7×2	$\frac{30}{4} + \frac{19}{2}$	$-\frac{1}{3} \cdot (-42)(-1)$
50% of 28	$-\frac{13 \cdot 3 + 4 \cdot 3}{-3}$	$87\frac{1}{2}$% of −16
$-7(2)(-1)$	$7 \cdot 2 + 3$	$\frac{6 + 2(10 + 1)}{-2}$
$2(5 + 2)$	$2 + (6 - 1)(2 + 1)$	$5 \cdot -2 - 4$
$2 \cdot 5 + 2 \cdot 2$	$\frac{9 + 3(11 - 2\frac{2}{3})}{2}$	$\frac{-17}{2} - \frac{-11}{-2}$
$11 \cdot 2 - 16 \cdot \frac{1}{2}$	20% of 85	
$-7 \cdot -2$	2% of 850	
$(6 + 1)(9 - 7)$	.02% of 85000	
$-7 \cdot -(5 - 3)$		
700% of 2		

−3	9	−12
$9 \times 1 - 6 \times 2$	$\frac{8}{1 + 1} + 5$	$6(8 - 6) - 24$
$5(2 - 7) + 22$	$3 \times 1 + 3 \times 2$	$\frac{-18}{5} - \frac{-42}{-5}$
$-1 \cdot 5 + -2 \cdot -1$	$8(2 - \frac{7}{8})$	4% of −300
$\frac{-5}{8} + \frac{5}{-2} + \frac{-1}{-8}$	$(2 + 5) \div 3 + 7 - \frac{1}{3}$	

12	18	49
$-3 \cdot -4$	$6(2 + 1)$	$-8 \cdot -2 + 3 \cdot 11$

Answers for Part B, continued.

13. 2x + 2. This is not equivalent to the given expression because 1 is a counter-example. If I substitute '1' for 'x' in both expressions, the given expression has the value 3 and the expression I wrote has the value 4.

[The explanation given as the answer for Exercise 13 is the kind of argument a student should give in each case to establish the fact that the expression he has written is not equivalent to the given one. For the remainder of the answers, we merely give an expression which is not equivalent to the given one, and we give values of the pronumerals which will yield different values for the two expressions.]

14.	3y + z;	substitute '1' for 'y' and '0' for 'z'.
15.	4k;	substitute '0' for 'k'.
16.	10st;	substitute '2' for 's' and '2' for 't'.
17.	xx;	substitute '2' for 'x'.
18.	7x;	substitute '2' for 'x'.
19.	3y;	substitute '2' for 'x' and '1' for 'y'.
20.	7xxy;	substitute '2' for 'x' and '1' for 'y'.

*

For each of the following pronumeral expressions, write <u>one</u> expression which contains the same pronumerals but which is <u>not</u> equivalent to it, <u>and</u> <u>prove</u> that they are not equivalent.

13. $2x + 1$	14. $3yz$
15. $6k - 2$	16. $7s \times 3t$
17. $3xx - 2x$	18. $7 + 0x$
19. $\frac{3y + x}{x}$	20. $\frac{14xxyyy}{2xyy}$

C. Rearrange these numerals into columns with all numerals for the same number in the same column.

$5 \times 2 + 4$	$9 \times 1 - 6 \times 2$	$23 - 2 \cdot 3$	$7 \div -.5$	
$\frac{2}{3} \cdot 21$	$\frac{8}{1 + 1} + 5$	$-3 \cdot -4$	$11 + 3 \cdot 1$	$6(2 + 1)$
$\frac{40 + -6}{2}$	${}^-6 \cdot {}^-2 - {}^-2 \cdot {}^-13$	$2 + 3 \times 4$	7×2	$7(3 - 5)$
$.5 \times \lvert 1 - 35 \rvert$	$-8 + 3({}^-2)$	$8 \times 2 + 1$	50% of 28	
$\frac{30}{4} + \frac{19}{2}$	$-7(2)(-1)$	$2(5 + 2)$	$-\frac{1}{3} \cdot (-42)(-1)$	
$2 \cdot 5 + 2 \cdot 2$	$11 \cdot 2 - 16 \cdot \frac{1}{2}$	$-7 \cdot -2$	$(6 + 1)(9 - 7)$	
$87\frac{1}{2}\%$ of -16	$5(2 - 7) + 22$	$-\frac{13 \cdot 3 + 4 \cdot 3}{-3}$		
$\frac{6 + 2(10 + 1)}{-2}$	$3 \times 1 + 3 \times 2$	$-7 \cdot -(5 - 3)$	$5 \cdot -2 - 4$	
$7 \cdot 2 + 3$	$-8 \cdot -2 + 3 \cdot 11$	$\frac{-17}{2} - \frac{-11}{-2}$	$8(2 - \frac{7}{8})$	
$2 + (6 - 1)(2 + 1)$	$-1 \cdot 5 + -2 \cdot -1$	$6(8 - 6) - 24$		
$\frac{9 + 3(11 - 2\frac{2}{3})}{2}$	$\frac{-18}{5} - \frac{-42}{-5}$	$\frac{-5}{8} + \frac{5}{-2} + \frac{-1}{-8}$		
20% of 85	2% of 850	$.02\%$ of 85000		
$(2 + 5) \div 3 + 7 - \frac{1}{3}$	4% of -300	700% of 2		

D. State the generalization involved in each of the following descriptions and prove it.

Sample. If I multiply a first number by itself and add the first number to this product, I get the same result I would have gotten if I had multiplied the first number by a number which is 1 more than the first number.

Solution. [I try a special case first to get the "feel" of this generalization.

$$8 \times 8 + 8 = 8 \times (8 + 1).]$$

The generalization is:

$$\forall_x \; xx + x = x(x + 1).$$

Proof.

$$\begin{aligned} & xx + x \\ = \; & x \cdot x + x \cdot 1 \quad \text{pm1} \\ = \; & x(x + 1). \quad \ell\text{dpma} \end{aligned}$$

1. The square of the double of a number is 4 times the square of the number. [The square of 7 is 49, of 8 is 64, of −3 is 9; the double of 3 is 6, of 15 is 30, of −4 is −8.]

2. The product of the opposites of two numbers is the product of the two numbers.

3. If I subtract 8 times a first number from 13 times the first number, I get 5 times the first number.

4. Pick a number. Multiply it by 10. Add the number you started with. Divide the sum by 11. What number do you get?

5. Pick a number. Add 9 to it to get a second number. Subtract 9 from the first number to get a third number. Take the average of the second and third numbers. What number do you get?

<u>**D**</u>. 1. $\forall_x \; x2(x2) = 4(xx)$

$$\begin{aligned} & x2(x2) && \\ &= xx(2 \cdot 2) && \text{Th. 4} \\ &= (2 \cdot 2)(xx) && \text{cpm} \\ &= 4(xx). && 2 \cdot 2 = 4 \end{aligned}$$

2. $\forall_x \forall_y \; -x \cdot -y = xy$ [This is Theorem 23.]

$$\begin{aligned} & -x \cdot -y && \\ &= -(x \cdot -y) && \text{Th. 21} \\ &= - -(xy) && \text{Th. 20} \\ &= xy. && \text{Th. 17} \end{aligned}$$

3. $\forall_x \; 13x - 8x = 5x$

$$\begin{aligned} & 13x - 8x && \\ &= 13x + -(8x) && \text{ps} \\ &= 13x + -8x && \text{Th. 21} \\ &= (13 + -8)x && \text{dpma} \\ &= (5 + 8 + -8)x && 13 = 5 + 8 \\ &= [5 + (8 + -8)]x && \text{apa} \\ &= (5 + 0)x && \text{po} \\ &= 5x. && \text{pa0} \end{aligned}$$

4. $\forall_x \; (x10 + x) \div 11 = x$

$$\begin{aligned} & (x10 + x) \div 11 && \\ &= (x10 + x \cdot 1) \div 11 && \text{pm1} \\ &= x(10 + 1) \div 11 && \ell\text{dpma} \\ &= x \cdot 11 \div 11 && 10 + 1 = 11 \\ &= x. && \text{Th. 64; } 11 \neq 0 \end{aligned}$$

5. $\forall_x\ [x + 9 + (x - 9)] \div 2 = x$

$$\begin{aligned}
&[x + 9 + (x - 9)] \div 2 && \\
&= \{x + [9 + (x - 9)]\} \div 2 && \text{apa} \\
&= \{x + [x - 9 + 9]\} \div 2 && \text{cpa} \\
&= (x + x) \div 2 && \text{Th. } 32 \\
&= (x \cdot 1 + x \cdot 1) \div 2 && \text{pm1} \\
&= [x(1 + 1)] \div 2 && \ell\text{dpma} \\
&= (x \cdot 2) \div 2 && 1 + 1 = 2 \\
&= x. && \text{Th. } 64;\ 2 \neq 0
\end{aligned}$$

6. $\forall_x \forall_y \forall_z\ (y - z) - (x - z) = y - x$

$$\begin{aligned}
&(y - z) - (x - z) && \\
&= (y + -z) + -(x + -z) && \text{ps} \\
&= (y + -z) + (-x + - -z) && \text{Th. } 18 \\
&= (y + -x) + (-z + - -z) && \text{Th. } 5 \\
&= y + -x + 0 && \text{po} \\
&= y + -x && \text{pa0} \\
&= y - x. && \text{ps}
\end{aligned}$$

7. $\forall_x \forall_y \forall_z\ z(y - x) + z(x - y) = 0$

$$\begin{aligned}
&z(y - x) + z(x - y) && \\
&= z[(y - x) + (x - y)] && \ell\text{dpma} \\
&= z[(y - x) + -(y - x)] && \text{Th. } 33 \\
&= z \cdot 0 && \text{po} \\
&= 0. && \text{pm0}
\end{aligned}$$

11. Students should see that here [and in Exercise 12] they would be foolish to try to state the generalization. But thinking about these exercises may lead them to the solution suggested for Exercise 10. [You may tell students that when they study mathematical induction in a year or so they will discover easy ways of stating and proving generalizations such as these.] The sum of the numbers is the number of numbers multiplied by the average of the first and last, or it is half the number of numbers multiplied by the sum of the first and last.

<u>E</u>.

1.	$2x + 5$	2.	$15y$	3.	x	4.	x
5.	$x + 1$	6.	$-2a + 8b$	7.	$x - 5$	8.	$7x - 7$
9.	$x + 9$	10.	$8x - 2$			11.	$2x - 2y + 4z$
12.	$9xx + 6x$	13.	$x + 2y + z + 1$			14.	$x - 7y$
15.	$-8x$	16.	$a + b$			17.	$x - y$
18.	$a + 1$	19.	$8 \bigcirc + 15 \square + 3$			20.	$3mm + 2mn$
21.	$18 \triangledown - 44 \square + 3$					22.	$2r + 2 + \frac{r}{t}$
23.	$9 + 27a$	24.	$\frac{26}{7x}$			25.	$\frac{12x - 29}{(x - 2)(x - 3)}$
26.	$\frac{-11a - 6}{(2a - 3)(3a - 2)}$	27.	$\frac{y}{x}$			28.	$\frac{9x - 60}{15 - 5x}$

*

[Note that phrases like 'exceeds' and 'decreased by' already have connotations which refer to numbers of arithmetic. In order to conform to conventional terminology, we introduce these words in Exercises 4 - 10, and 14 in connection with <u>real numbers</u>. Students will have to extend the meanings of these words to the point where it makes sense to say that -6 exceeds -5 by -1.]

8. $\forall_x \; x(x - 3) + x(x + 3) = 2(xx)$

$$\begin{aligned}
& x(x - 3) + x(x + 3) && \} \ \ell\text{dpma} \\
&= x[(x - 3) + (x + 3)] && \} \ \text{cpa} \\
&= x[x - 3 + (3 + x)] && \} \ \text{apa} \\
&= x[x - 3 + 3 + x] && \} \ \text{Th. 32} \\
&= x[x + x] && \} \ \text{pm1} \\
&= x[x \cdot 1 + x \cdot 1] && \} \ \ell\text{dpma} \\
&= x[x(1 + 1)] && \} \ 1 + 1 = 2 \\
&= x(x \cdot 2) && \} \ \text{apm} \\
&= xx2 && \} \ \text{cpm} \\
&= 2(xx).
\end{aligned}$$

9. $\forall_{x \neq 0} \forall_{y \neq 0} \; \frac{1}{x} + \frac{1}{y} = \frac{x + y}{xy}$

$$\begin{aligned}
& \frac{1}{x} + \frac{1}{y} && \} \ \text{Th. 57; } x \neq 0,\ y \neq 0 \\
&= \frac{1 \cdot y + 1 \cdot x}{xy} && \} \ \ell\text{dpma} \\
&= \frac{1 \cdot (y + x)}{xy} && \} \ \text{Th. 2, cpa} \\
&= \frac{x + y}{xy}.
\end{aligned}$$

10. $\forall_x \; x + (x + 1) + (x + 1 + 1) + (x + 1 + 1 + 1) + (x + 1 + 1 + 1 + 1)$
$+ (x + 1 + 1 + 1 + 1 + 1) + (x + 1 + 1 + 1 + 1 + 1 + 1)$
$+ (x + 1 + 1 + 1 + 1 + 1 + 1 + 1) + (x + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1)$
$+ (x + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1)$
$= 5[x + (x + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1)].$

[This is a case where the instructions 'and prove it' should not be taken too seriously. It will be amply sufficient if some students notice that, for any number x, the sum of the first and tenth numbers is the same as the sum of the second and ninth, etc., and that this sum occurs 5 times.]

6. I decrease each of a first number and a second number by a third number to get a fourth number and a fifth number, respectively. The difference of the fourth number from the fifth number is the difference of the first number from the second number.

7. I subtract a first number from a second number, and then subtract the second number from the first number. Then I pick a third number and multiply it by each of the differences. The sum of these products is 0.

8. I multiply a first number by a second number which is 3 less than the first. I multiply the first number by a third number which is 3 more than the first. I add the products and get twice the square of the first.

9. The sum of the reciprocals of two numbers is the sum of the numbers divided by their product.

10. Pick a first number. Add 1 to it to get a second number. Add 1 to the second number to get a third number. Add 1 to the third number to get a fourth number. Keep this up until you get a tenth number. The sum of the ten numbers is 5 times the sum of the first and tenth numbers.

11. Continue the process of adding 1 in Exercise 10 until you get a thousandth number. The sum of the thousand numbers is 500 times the sum of the first and thousandth numbers.

☆12. Continue the process of adding 1 in Exercise 10 until you get bored. What is the sum of the numbers you get?

E. Complete each sentence to a true one by writing the simplest expression you can in the blank.

1. For each x, the sum of 2x and 5 is ____________________.

2. For each y, the product of 5 by 3y is ____________________.

3. For each x, the difference of 7 from x + 7 is ____________.

4. For each x, x + 7 exceeds 7 by ____________________.

5. For each x, 3x exceeds 2x - 1 by ____________________.

6. For each a, for each b, 3a + 2b exceeds 5a - 6b by ____________________.

7. For each x, x decreased by 5 is ____________________.

8. For each x, 7x decreased by 7 is ____________________.

9. For each x, x increased by 9 is ____________________.

10. For each x, 5x increased by 3x - 2 is ____________________.

11. For each x, for each y, for each z, the sum of 5x - 4y + 6z and -3x + 2y - 2z is ____________________.

12. For each x, the product of 3x by the sum of 4x + 1 and 1 - x is ____________________.

13. For each x, for each y, for each z, the difference of x - y + 1 from the sum of x + z + 1 and x + y + 1 is ____________________.

14. For each x, for each y, 3x - 2y exceeds 2x + 5y by ____________________.

15. For each x, the difference of the product of x + 9 by x from the product of x + 1 by x is ____________________.

16. For each a, for each b, the difference of ____________________ from 3a - 2b is 2a - 3b.

17. For each x, for each y, for each $z \neq 0$, the quotient of 3xz - 3yz by 3z is ____________________.

18. For each a, for each $b \neq 0$, the quotient of $ab + b$ by b is ________________________.

19. For each $\bigcirc$, for each $\square$, the sum of $3\bigcirc + 5\bigcirc$ and $9\square + 6\square + 3$ is ________________________________.

20. For each m, for each n, the product of $15m + 10n$ by $\frac{1}{5}m$ is ________________________.

21. For each $\square$, for each $\bigtriangledown$, the difference of $45\square - 9 - 15\bigtriangledown$ from $3\bigtriangledown + 12 + \square - 18$ is

________________________.

22. For each r, for each $t \neq 0$, the quotient of $6rt + 4t - 2rt + 2r$ by $2t$ is ____________________.

23. For each a, $13a$ increased by $9 + 14a$ is ________________.

24. For each $x \neq 0$, the sum of $\frac{3}{x}$ and $\frac{5}{7x}$ is ________________.

25. For each x other than 2 and 3, the sum of $\frac{5}{x - 2}$ and $\frac{7}{x - 3}$ is ________________________.

26. For each a other than $\frac{3}{2}$ and $\frac{2}{3}$, the difference of $\frac{9}{2a - 3}$ from $\frac{8}{3a - 2}$ is ____________________________________.

27. For each $x \neq 0$, for each $y \neq 0$, for each $z \neq 0$, the product of $\frac{-8xy}{3yz}$ by $\frac{-3yz}{8xx}$ is ______________________________.

28. For each $x \neq 0$ and $\neq 3$, the quotient of $\frac{3}{5} - \frac{4}{x}$ by $\frac{1}{x} - \frac{1}{3}$ is ________________________.

F. Rearrange these numerals into columns with all numerals for the same number in the same column.

$-\frac{3}{5}$	$\frac{3 - 8}{-3}$	$-\frac{3/4}{5/4}$	$\frac{3 \times 7}{5 \times 7}$
$\frac{8 - 3}{-3}$	$\frac{3 - 8}{1 - 4}$	$-\frac{-2 \times -3}{-2 \times -5}$	$\frac{3}{-5}$
$\frac{4 - 9}{10 - 7}$	$-\frac{-5}{-3}$	$\frac{3 \times -1}{5 \times -1}$	$\frac{6}{10}$
$\frac{+3}{+5}$	$\frac{6 - 16}{-6}$	$-\frac{+3}{+5}$	$\frac{3/4}{5/4}$
$-\frac{5}{-3}$	$\frac{9 - 4}{7 - 10}$	$\frac{-3}{5}$	$\frac{-2 \times -3}{2 \times 5}$
$-\frac{3 - 8}{-3}$	$-\frac{3 - 8}{4 - 1}$	$\frac{3 \times -7}{5 \times 7}$	$-\frac{3}{-5}$
$\frac{4 - 9}{7 - 10}$	$-\frac{8 - 3}{1 - 4}$	$\frac{2 \times 3}{2 \times 5}$	$\frac{3}{5}$
$\frac{-3}{-5}$	$-\frac{-5}{3}$	$-\frac{5}{3}$	$\frac{3 - 8}{3}$
$-\frac{-3}{5}$	$-\frac{3 - 8}{3}$	$\frac{8 - 3}{3}$	$\frac{2 \times -3}{5 \times -2}$
$-\frac{-3}{-5}$	$\frac{5}{3}$	$\frac{-5}{-3}$	$\frac{-(7 - 2)}{-(9 - 6)}$

F. [The numerals given here name the numbers $-\frac{3}{5}$, $\frac{5}{3}$, $\frac{3}{5}$, and $-\frac{5}{3}$. We use the simplest names as column headings, and under each of them give the other numerals which name the same number.]

$-\frac{3}{5}$	$\frac{5}{3}$	$\frac{3}{5}$	$-\frac{5}{3}$
$-\frac{3}{5}$	$\frac{3-8}{-3}$	$\frac{3 \times 7}{5 \times 7}$	$\frac{8-3}{-3}$
$-\frac{3/4}{5/4}$	$\frac{3-8}{1-4}$	$-\frac{-3}{5}$	$\frac{4-9}{10-7}$
$-\frac{-2 \times -3}{-2 \times -5}$	$-\frac{3-8}{3}$	$\frac{3 \times -1}{5 \times -1}$	$-\frac{-5}{-3}$
$\frac{3}{-5}$	$\frac{8-3}{3}$	$\frac{6}{10}$	$\frac{9-4}{7-10}$
$-\frac{+3}{+5}$	$\frac{6-16}{-6}$	$\frac{+3}{+5}$	$-\frac{3-8}{-3}$
$\frac{-3}{5}$	$-\frac{5}{-3}$	$\frac{3/4}{5/4}$	$-\frac{5}{3}$
$-\frac{-3}{-5}$	$-\frac{3-8}{4-1}$	$\frac{2 \times -3}{5 \times -2}$	$\frac{3-8}{3}$
$\frac{3 \times -7}{5 \times 7}$	$\frac{5}{3}$	$\frac{-2 \times -3}{2 \times 5}$	
	$\frac{4-9}{7-10}$	$-\frac{3}{-5}$	
	$-\frac{8-3}{1-4}$	$\frac{2 \times 3}{2 \times 5}$	
	$\frac{-5}{-3}$	$\frac{3}{5}$	
	$-\frac{-5}{3}$	$\frac{-3}{-5}$	
	$\frac{-(7-2)}{-(9-6)}$		

G. 1. a, b, c, d, e, g, i, k, ℓ, m, n

2. c, d, e, h, i, j, k, ℓ

3. a, b, d, f

4. a, b, c, e, h, i, j, k

5. (a) 0 (b) 1, −1

6. (a) T (b) F (c) T (d) T (e) T

H.

1. 1007	2. −830	3. $3\frac{511}{512}$
4. $1\frac{5}{99}$	5. 26π	6. 25π
7. 49π	8. 42	9. $158\frac{14}{17}$
10. 4850	11. 21	12. $\frac{63}{2}$
13. 900	14. 64π	15. 155.625π
16. 141	17. 99π	18. 18.5
19. 60.48	20. $\frac{8}{15}\pi$	21. 193.948
22. 64.6468	23. 4.03	24. $\frac{4}{3}$
25. 265	26. −90	27. 362.1375
28. 1029.56	29. 11421	30. $\frac{5}{24}$
31. $\frac{21}{10}$	32. $\frac{21}{10}$	33. $\frac{20}{89}$

G. 1. Which of the following numerals name the opposite of -6?

(a) $--6$ (b) $+6$ (c) $-(7 - 13)$

(d) -3×-2 (e) -1×-6 (f) $-1 \times -2 \times -3$

(g) $-1 \times 2 \times -3$ (h) $1 \times 2 \times -3$ (i) 2×3

(j) $\frac{6}{-1}$ (k) $\frac{-6}{-1}$ (l) $-\frac{6}{-1}$

(m) $\frac{-18}{-3}$ (n) $(5 - 3) \times (5 - 2)$

2. Which of the following numerals name the reciprocal of $-\frac{3}{8}$?

(a) $\frac{3}{8}$ (b) $\frac{8}{3}$ (c) $-\frac{8}{3}$ (d) $\frac{8}{-3}$

(e) $\frac{-8}{3}$ (f) $\frac{-\frac{3}{8}}{-1}$ (g) $-1 \times -\frac{3}{8}$ (h) $-1 \times \frac{8}{3}$

(i) $\frac{-16}{6}$ (j) $\frac{16}{-6}$ (k) $-\frac{16}{6}$ (l) $1 \div -\frac{3}{8}$

3. Which of the following numerals name the opposite of $(6 - 3 + 7)$?

(a) $-(6 - 3 + 7)$ (b) $-6 + 3 - 7$ (c) $-6 - 3 + 7$

(d) $-1 \times (6 - 3 + 7)$ (e) $6 + 3 - 7$ (f) $(6 - 3 + 7) \div -1$

4. Which of the following numerals name the opposite of $\frac{3 - 8}{-2 + 7}$?

(a) $-1 \times \frac{3 - 8}{-2 + 7}$ (b) $\frac{3 - 8}{-2 + 7} \div -1$ (c) $\frac{8 - 3}{-2 + 7}$

(d) $\frac{3 - 8}{7 - 2}$ (e) $\frac{3 - 8}{2 - 7}$ (f) $\frac{3 + 8}{2 + 7}$

(g) $\frac{-2 + 7}{3 - 8}$ (h) $\frac{-3 + 8}{-2 + 7}$ (i) $\frac{-1 \times (3 - 8)}{-2 + 7}$

(j) $\frac{3 - 8}{-1 \times (-2 + 7)}$ (k) $-1 \times \frac{-2 + 7}{3 - 8}$

5. (a) Which number is its own opposite?

(b) Which number is its own reciprocal?

(continued on next page)

6. True or false?

   (a) If you multiply a number by -1, the product is the opposite of the given number.

   (b) If you divide a number by -1, the quotient is the reciprocal of the given number.

   (c) If you divide a number by -1, the quotient is the opposite of the given number.

   (d) If you divide -1 by a nonzero number, the quotient is the opposite of the reciprocal of the given number.

   (e) If you divide -1 by a nonzero number, the quotient is the reciprocal of the opposite of the given number.

H. Evaluate each of the following pronumeral expressions using the given values of the pronumerals. Answers should be in simplest form.

Sample. $a + (n - 1)d$; '7' for 'a', '13' for 'n', '4' for 'd'

Solution.
$$\begin{aligned} & 7 + (13 - 1)4 \\ = {} & 7 + 12 \cdot 4 \\ = {} & 7 + 48 \\ = {} & 55. \end{aligned}$$

1. $\frac{n}{2}(a + \ell)$; '19' for 'n', '71' for 'a', '35' for 'ℓ'

2. $\frac{n}{2}[2a + (n - 1)d]$; '20' for 'n', '6' for 'a', '-5' for 'd'

3. $\frac{r\ell - a}{r - 1}$; '2' for 'a', '$\frac{1}{2}$' for 'r', '$\frac{1}{512}$' for 'ℓ'

4. $\frac{a}{1 - r}$; '1.04' for 'a', '.01' for 'r'

5. $2\pi r$; '13' for 'r'

6. $\frac{1}{2}rC$; '5' for 'r', '10π' for 'C'

7. πrr; '7' for 'r'

8. $\frac{1}{2}bh$; '14' for 'b', '6' for 'h'

9. $\frac{(n - 2)180}{n}$; '17' for 'n'

10. $\frac{n(n - 3)}{2}$; '100' for 'n'

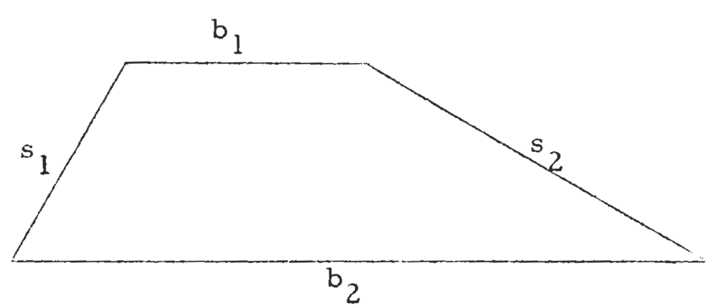
*

In some of the expressions which follow, you will see pronumerals like 'd_1' and 'd_2'. The small numerals written at the lower right of the letter are called <u>subscripts</u>, and they are used to indicate that 'd_1' and 'd_2' are different pronumerals. [Read 'd_1' and 'd_2' as 'dee sub one' and 'dee sub two'.]

Using subscripts enables you to manufacture an unlimited number of pronumerals from just one letter. They are especially helpful in writing and remembering formulas. For example, the formula:

$$P = b_1 + s_1 + b_2 + s_2$$

can be used to compute the perimeter of a trapezoid.



The values of 's_1' and 's_2' are the measures of the nonparallel <u>sides</u>, and the values of 'b_1' and 'b_2' are the measures of the <u>bases</u>.

*

11. $\frac{1}{2}d_1d_2$; '6' for 'd_1', '7' for 'd_2'

12. $\frac{h(b_1 + b_2)}{2}$; '3' for 'h', '6' for 'b_1', '15' for 'b_2'

13. s(s - a)(s - b)(s - c); '15' for 's', '5' for 'a', '12' for 'b', '13' for 'c'

14. $\pi r(\ell + r)$; '4' for 'r', '12' for 'ℓ'

15. $\frac{1}{3}\pi rrh$; '7.5' for 'r', '8.3' for 'h'

(continued on next page)

16. $\frac{1}{2}(P_1 + P_2)\ell$; '19' for '$P_1$', '28' for '$P_2$', '6' for '$\ell$'

17. $\pi\ell(r_1 + r_2)$; '9' for 'ℓ', '3' for 'r_1', '8' for 'r_2'

18. $aa + bb + cc$; '1.5' for 'a', '2.0' for 'b', '3.5' for 'c'

19. $2(ab + ac + bc)$; '2.1' for 'a', '3.4' for 'b', '4.2' for 'c'

20. $\frac{E}{720} \cdot 4\pi rr \cdot \frac{r}{3}$; '36' for 'E', '2' for 'r'

21. $\frac{h}{6}(B_1 + B_2 + 4M)$; '7.1' for 'h', '20.3' for 'B_1', '31.2' for 'B_2', '28.1' for 'M'

22. $aa + bb + 2abc$; '5.1' for 'a', '3.2' for 'b', '0.87' for 'c'

23. $\frac{kr}{t}$; '1.3' for 'k', '6.2' for 'r', '2' for 't'

24. $\frac{v_1 - v_0}{t}$; '60' for 'v_1', '40' for 'v_0', '15' for 't'

25. $v_0t + \frac{1}{2}att$; '3' for 'v_0', '20' for 'a', '5' for 't'

26. $r\left(\frac{k_2 - k_1}{t}\right)$; '9' for 'r', '50' for 'k_2', '80' for 'k_1', '3' for 't'

27. $G \cdot \frac{m_1m_2}{dd}$; '0.0000000666' for 'G', '250000' for 'm_1', '8700000' for 'm_2', '20' for 'd'

28. ma + Rv; '41' for 'm', '25' for 'a', '0.12' for 'R', '38' for 'v'

29. $m \cdot \frac{vv}{r}$; '94' for 'm', '27' for 'v', '6' for 'r'

30. $\frac{1}{p} + \frac{1}{q}$; '8' for 'p' and '12' for 'q'

31. $\dfrac{1}{\frac{1}{p} + \frac{1}{q}}$; '7' for 'p' and '3' for 'q'

32. $\frac{pq}{p + q}$; '7' for 'p' and '3' for 'q'

33. $\frac{r_1t}{m(r_1 + r_2) - t(m - 1)}$; '10' for 'r_1', '20' for 'r_2', '1.0' for 't', '1.5' for 'm'

I. Complete each sentence to a true one by writing the simplest expression you can in the blank.

1. (a) If eggs cost 60 cents a dozen, 3 dozen eggs cost ______ cents.

   (b) For each $x > 0$, if eggs cost 60 cents a dozen, x dozen eggs cost ______ cents. [Why 'For each $x > 0$,' instead of just 'For each x,'?]

   (c) For each $x > 0$, if eggs cost 60 cents a dozen, x eggs cost ______ cents.

   (d) For each $x > 0$, if eggs cost 60 cents a dozen, $(x + 3)$ dozen eggs cost ______ cents.

2. (a) For each $x > 0$, if one pencil costs 2 cents, x pencils cost ______ cents.

   (b) For each $x > 0$, if one pencil costs 3 cents, $x + 5$ pencils cost ______ cents.

   (c) For each $x > 0$, if a dozen pencils cost 30x cents then 2 pencils cost ______ cents.

   (d) For each $x > 0$, for each $y > 0$, if 3 pencils cost y cents then x pencils cost ______ cents.

3. (a) For each $x > 0$, the perimeter of a square with one side 2x units long is ______.

   (b) For each $y > 0$, the perimeter of a square with one side $(3y + 7)$ units long is ______.

   (c) For each $x > 0$, for each $y > 0$, the perimeter of a square with one side $\left(\frac{y}{8} + \frac{x}{2}\right)$ units long is ______.

4. (a) If there are 100 sheets of paper in a pile 1 inch thick, there are ______ sheets of paper in a pile 9 inches thick, and 575 sheets in a pile ______ inches thick.

   (b) For each $x > 0$, if there are 75 sheets of paper in a pile 1 inch thick then there are ______ sheets of paper in a pile 2x inches thick.

The quantifying phrases in the generalization sentences of Part I indicate that the generalizations refer to real numbers. Strictly speaking this is not correct. The generalizations are about numbers of eggs, numbers of pencils, distances, lengths of segments, etc., and these are [cardinal numbers and] numbers of arithmetic rather than real numbers. [See bracketed note at bottom of page 2-45.] So, for example, the proper quantifying phrase in Exercise 1(b) is:

For each number of arithmetic $x > 0$, .

['$x > 0$' instead of just 'x' because one wouldn't ordinarily concern himself with the price of 0 dozen eggs.] As it now stands, the generalization in Exercise 1(b) refers to the positive real numbers. An instance of the correctly completed generalization is:

if eggs cost 60 cents a dozen, $^+4$ dozen eggs cost $60 \cdot {}^+4$ cents.

To make sense out of this, one would translate this to:

if eggs cost 60 cents a dozen, 4 dozen eggs cost $60 \cdot 4$ cents.

You may want to make a brief mention of these matters in class. We make more of this in Unit 3.

<u>I.</u>

1. (a) 180 (b) $60x$ (c) $5x$ (d) $60(x + 3)$
2. (a) $2x$ (b) $3x + 15$ (c) $5x$ (d) $\frac{xy}{3}$
3. (a) $8x$ (b) $4(3y + 7)$ (c) $\frac{y}{2} + 2x$
4. (a) 900, $5\frac{3}{4}$ (b) $150x$ (c) 300aabbc (d) $\frac{125x}{y}$
5. (a) 21k (b) $3tt + 4t + 6$
6. (a) $10x + 8$ (b) $2x$ (c) $4x + 8$ (d) $\frac{a}{2} + 4b$ (e) 2
7. (a) 45 (b) $15x$ (c) $12x$ (d) 26
8. (a) $3x - 12$ (b) $6x - 9$ (c) $3x - 28$ (d) 4tr
9. (a) 144 (b) $2.4xy$ (c) $.9abc$ (d) $\frac{50abc}{zy}$ (e) $\frac{4x + 5y}{50}$
10. (a) $35x$ (b) $90x + 195$ (c) $y + 1$

(c) For each $a > 0$, for each $b > 0$, for each $c > 0$, if there are $20ab$ sheets of paper in a pile 1 inch thick, there are ______ sheets in a pile $15abc$ inches thick.

(d) For each $x > 0$, for each $y > 0$, if there are 125 sheets of paper in a pile y inches thick then there are ______ sheets of paper in a pile x inches thick.

5. (a) For each $k > 0$, the perimeter of an equilateral triangle with one side $7k$ units long is ______.

(b) For each $t > 0$, if the perimeter of an equilateral triangle is $9tt + 12t + 18$, one of its sides is ______ units long.

6. (a) For each $x > 0$, the perimeter of a rectangle whose dimensions are $3x$ units by $2x + 4$ units is ______.

(b) For each $x > 0$, if the perimeter of a rectangle is $18x + 4$, and if two of the sides are each $(7x + 2)$ units long, then each of the other two sides is ______ units long.

(c) For each $x > 0$, for each $y > 0$, if one dimension of a rectangle is $(2y + 4)$ units and the perimeter is $4[2(x + 3) + y]$, the other dimension is ______ units.

(d) For each $a > 0$, for each $b > 0$, the perimeter of a rectangle whose dimensions are $\frac{a}{4}$ units by $2b$ units is ______.

(e) For each $x > 0$, the perimeter of a rectangle whose dimensions are $\frac{x}{2}$ units by $4x$ units is ______ times the perimeter of one whose dimensions are $\frac{x}{4}$ units by $2x$ units.

7. (a) If a car travels 1 mile in $\frac{3}{2}$ minutes, it will travel 30 miles at this rate in ______ minutes.

(b) For each $x > 0$, if a car travels 1 mile in $\frac{3}{2}$ minutes, it will travel $10x$ miles at this rate in ______ minutes.

(c) For each $x > 0$, if a car travels 1 mile in $\frac{3}{2}$ minutes, it will travel ______ miles at this rate in $18x$ minutes.

(d) For each $x > 0$, if a car travels $13x$ miles in $\frac{x}{2}$ minutes, it will travel ______ miles at this rate in 1 minute!

(continued on next page)

8. (a) For each $x > 4$, if a side of an equilateral triangle is $(x - 4)$ units long, the perimeter is ______. [Why 'For each $x > 4$,' instead of 'For each $x > 0$,'?]

(b) For each $x > \frac{3}{2}$, if a side of an equilateral triangle is $(2x - 3)$ units long, the perimeter is ______.

(c) For each $x > \frac{28}{3}$, if a side of a square is $\left(\frac{3x}{4} - 7\right)$ units long, the perimeter is ______ units.

(d) For each $t > 0$, for each $r > 0$, if the perimeter of a square is $16tr$, a side of the square is ______ units long.

9. (a) If a man borrows \$1200 at an annual interest rate of 3%, the total simple interest due at the end of 4 years is ______ dollars.

(b) For each $x > 0$, for each $y > 0$, if a man borrows $30x$ dollars at an annual interest rate of 4%, the total simple interest due at the end of $2y$ years is ______ dollars.

(c) For each $a > 0$, for each $b > 0$, for each $c > 0$, if a man borrows $15a$ dollars at an annual interest rate of b%, the total simple interest due at the end of $6c$ years is ______ dollars.

(d) For each $y > 0$, for each $z > 0$, for each $a > 0$, for each $b > 0$, for each $c > 0$, if a man borrows ______ dollars at an annual interest rate of $4y$%, the total simple interest due at the end of $3z$ years is $6abc$ dollars.

(e) For each $x > 0$, for each $y > 0$, if a person borrows x dollars at 4% per annum and y dollars at 5% per annum, the total simple interest due on these two loans at the end of 2 years is ______ dollars.

10. (a) For each $x > 0$, a pile of coins consisting of x nickels and $3x$ dimes is worth ______ cents.

(b) For each $x > 0$, a pile of coins consisting of $2x$ nickels, $(3x + 2)$ dimes, and $(2x + 7)$ quarters is worth ______ cents.

(c) For each $y > 0$, a pile of coins consisting of ______ nickels, $(7y + 2)$ dimes, and $(5y + 3)$ quarters is worth $(200y + 100)$ cents.

J. 1. T 2. T 3. T 4. F 5. F

1. T	2. T	3. T	4. F	5. F
6. T	7. F	8. T	9. F	10. F
11. T	12. T	13. F	14. T	15. F

K. 1. \$24.3475 [or: \$24.35]

2. This problem gives insufficient data to make it possible to answer the question asked.

3. 42 miles per hour

4. 143 hits

5. \$172.05

6. \$81.59 [The wording of this problem makes it possible to interpret it in more than one way. A more precise statement of the problem we intended is 'If Mrs. Smith buys a sofa whose original marked price was \$99.50 at a discount of 18%, how much does she pay for the sofa?'.]

7. 25

8. \$7.00

9. $45 + 5.8\pi$ [If 3.14 is used as an approximation for π, the circumference of the outer edge of the sidewalk is approximately 63 feet.]

10. $133\frac{1}{3}\%$

J. True or false?

1. $5 \geq -5$
2. $-1.4 \leq -1.3$
3. $-9 \not> -7$
4. $7 \neq |-7|$
5. $-6 \geq |-6|$
6. $7 - 3 = |3 - 7|$
7. $\frac{2}{73} < \frac{3}{110}$
8. $-.062 = \frac{-31}{500}$
9. $0 \not< |2 - 100|$
10. $(3 - 5)(6 - 7) = (5 - 3)(6 - 7)$
11. $(859 - 384)(7842 - 9257) = (384 - 859)(9257 - 7842)$
12. $\frac{8 - 2}{5 - 6} = \frac{2 - 8}{6 - 5}$
13. $\frac{58 - 72}{69 - 93} = \frac{58 - 72}{93 - 69}$
14. $\frac{(583 - 729)(864 - 275)}{(421 - 593)(684 - 275)} = \frac{(275 - 864)(583 - 729)}{(421 - 593)(275 - 684)}$
15. $(785 - 359)(621 - 256)(16 - 34) = (58 - 97)(42 - 42)(83 - 75)$

K. Solve these problems.

1. A television set is advertised for \$97.39. If the set costs the store 75% of this price, what is the store's margin?

2. A recipe includes 2 cups of sugar, 3 tablespoons of citron, and a pinch of salt. If the recipe is to be increased to take care of 8 people, how many more cups of sugar will be needed?

3. A train travels from station A to station B in 2 hours and 20 minutes. If the distance between the two stations is 97 miles, what is the average rate of the train? [Round to the nearest mile per hour.]

4. A baseball player's batting average at the end of a season is .302. If he was "at bat" 473 times during the season, how many hits did he get?

5. Mr. Alexander has a \$15,000 life insurance policy. If the annual premium rate is \$11.47 per thousand dollars of insurance, what is his annual premium?

6. If Mrs. Smith buys a sofa selling at \$99.50 at a discount of 18%, how much does she pay for the sofa?

7. How many hours of baby sitting at 55 cents per hour will it take to accumulate \$13.75?

(continued on next page)

8. Mrs. Ashton buys a radio selling at $112. She gives $20 as a down payment, and agrees to pay $8.25 each month for a year. How much is she paying for the privilege of buying the radio on the installment plan?

9. A cement sidewalk is placed around a flower bed. If the flower bed has a circumference of 45 feet and the sidewalk is 2.9 feet wide, what is the circumference of the outer edge of the sidewalk?

10. If the wholesale price of an article is 75% of the retail price, what percent of the wholesale price is the retail price?

<u>L</u>. [Make a careful sketch for each problem.]

1. What is the perimeter of a rectangle whose smaller dimension is 3 units and whose larger dimension is 3 units more than 3 times the smaller dimension?

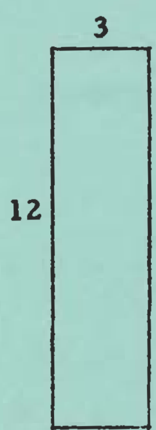
2. A <u>semicircle</u> is drawn on each side of the rectangle in Exercise 1 and the sides of the rectangle are erased leaving a figure with four bulges. What is the perimeter of this figure?

3. What is the perimeter of a parallelogram if each of its longer sides measures 2 more than either of its shorter sides, and if the measure of one of its shorter sides is 1 less than the average of the measures of the four sides?

4. The measure of one of the longer sides of a rectangle is 3.5 times the measure of one of the shorter sides. The sum of the measures of the four sides is 6 times the measure of a shorter side. What is the perimeter?

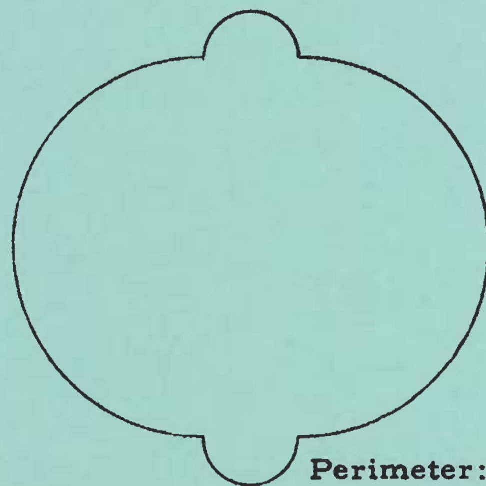
5. The midpoints of the four sides of a square are connected to form another square. If the perimeter of the smaller square is 8, estimate the perimeter of the larger square.

L. 1.



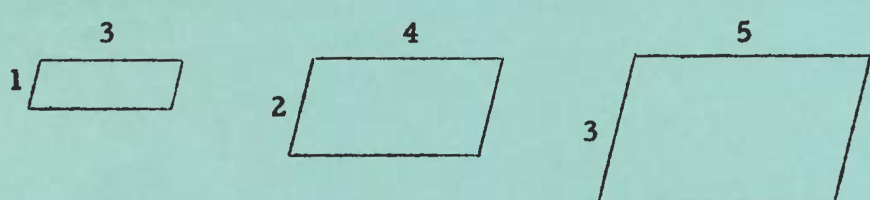
Perimeter: 30

2.



Perimeter: 15π

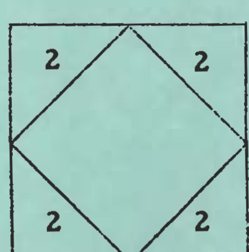
3. An infinite number of parallelograms would have sides whose measures agree with the conditions of the problem. Here are pictures of some of them.



So, it is impossible to determine the perimeter, since the information does not refer to a particular parallelogram.

4. The data given in this problem are inconsistent since, if the measure of each of the longer sides is 3.5 times the measure of one of the shorter sides, the sum of the measures of the four sides would be 9 [and not 6] times the measure of a shorter side. [But, even if one were given these consistent data, it would be impossible to compute the perimeter since they would not be sufficient.]

5.



Perimeter of larger square is approximately 11.3

Students will probably estimate that the perimeter is "between 10 and 12".

M.

1. 3△	2. 8m	3. 2△ + □	4. 2□
5. 11△	6. 7x	7. 0	8. 26x
9. 4c + 2d	10. 5x	11. 72a	12. 4g + 20
13. 11x - 5	14. 30□	15. 3t + 14	16. 8y + 3
17. 40○	18. 4u + 6y	19. 12n + 6	20. 10x
21. 15e + 3	22. 3h + 4j	23. 11△	24. 13a + 3
25. 3p + n	26. 16r	27. 14y - 4z	28. 19 - a
29. 0	30. 2	31. 3a + 9	32. 3d + c
33. 7 - r	34. 11x - 6	35. 5h	36. 18b + 9c
37. 7c + 8d	38. 7x - 8y	39. −40m	40. −32r + 2s
41. 2y	42. 7□	43. 6 - a	44. b - 2
45. 3	46. 9x	47. 9g - 5	48. 6g + 3
49. 8a	50. 0.7n + 0.7	51. 8w - 15	52. 3h + 3
53. 3d - 19	54. 2j + 2	55. 8 - 6t	56. 10p - 5
57. 7r - 7	58. 7s - 8	59. 8f - 4	60. 20 - 5k
61. 9 - 5a	62. 4e - 2	63. r + 5	64. 3x - 3
65. 12d - 21	66. 5g - 5	67. −2p + 2	68. 6z + 6
69. 2n + 8	70. 2m - 4	71. 4k + 5	72. 8t + 9
73. 2xx - xy + 3yy		74. 3aa + 2ab - 2bb	
75. 36aa + 6ab - 6bb		76. 9r + 9s	
77. 5xy		78. 7x + 10	
79. xxx - xxy - yyx - yyy - x + y		80. 8x - 2	
81. 2y + 3b - 6	82. 7x - 2	83. 5y + 11	84. y + 30
85. 2y + 64	86. 9y + 3	87. y - 10	88. 3a + 3b
89. 3a + 5b	90. a + 4b	91. 4n	92. −11ab
93. −147cd	94. n - 4r	95. 19y	96. 23st

M. Simplify.

1. $\triangle + 2\triangle$
2. $3m + 5m$
3. $\triangle + \triangle + \square$
4. $3\square - \square$
5. $5\triangle + 6\triangle$
6. $15x - x - 7x$
7. $5\hexagon - 3\hexagon - 2\hexagon$
8. $15x + 11x$
9. $3c + d + c + d$
10. $2x + 3x$
11. $60a + 10a + 2a$
12. $3g + 12 + g + 8$
13. $7x - 5 + 4x$
14. $45\square - 15\square$
15. $5t + 8 - 2t + 6$
16. $16 + 6y - 13 + 2y$
17. $56\hexagon - 16\hexagon$
18. $6u + 4y - 2u + 2y$
19. $5n + 6 + 7n$
20. $5x + 4x + x$
21. $9e + 6e + 3$
22. $2h + 3j + h + j$
23. $8\triangle + 2\triangle + \triangle$
24. $4a + (9a + 3)$
25. $p + (n + 2p)$
26. $9r + 10r - 3r$
27. $8y + (6y - 4z)$
28. $8 + [5 + (6 - a)]$
29. $b + 4b - 5b$
30. $\square - (\square - 2)$
31. $4a + (9 - a)$
32. $d + (c + 2d)$
33. $8r + (7 - 9r)$
34. $7x + (4x - 6)$
35. $4g - (-5h + 4g)$
36. $3(6b + 3c)$
37. $\frac{1}{2}(14c + 16d)$
38. $\frac{1}{3}(21x - 24y)$
39. $12(\frac{2}{3}m - 4m)$
40. $-8(4r - \frac{1}{4}s)$
41. $3(y)(2)(\frac{1}{3})$
42. $2\frac{1}{3}\square + 4\frac{2}{3}\square$
43. $3(a + 2) - 4a$
44. $2(b - 1) - b$
45. $3(c + 1) - 3c$
46. $4x + 7x - 2x$
47. $7g - 5 + 2g$
48. $8 + 6g - 5$

(continued on next page)

49. $7a - 2a + 3a$

50. $0.8n - 0.1n + 0.7$

51. $7w - 15 + w$

52. $5h + 3 - 2h$

53. $10d - 7d - 19$

54. $7j + 2 - 5j$

55. $8 - 9t + 3t$

56. $2p - 5 + 8p$

57. $11r - 7 - 4r$

58. $2s - 8 + 5s$

59. $7f - 4 + f$

60. $13 - 5k + 7$

61. $9 + a - 6a$

62. $2e - 2 + 2e$

63. $2r + (5 - r)$

64. $5x - (2x + 3)$

65. $13d - (d + 21)$

66. $3g - (5 - 2g)$

67. $5p + (2 - 7p)$

68. $9z - (3z - 6)$

69. $6n - (4n - 8)$

70. $3m + 4 - (m + 8)$

71. $5k + 7 - (k + 2)$

72. $7t - (-t - 9)$

73. $2x(x + y) - 3y(x - y)$

74. $a(3a + 3b) - b(a + 2b)$

75. $4a(9a - 3b) + 3b(6a - 2b)$

76. $2(5r - 5r) + 3(3r + 3s)$

77. $3x(3y - 2) - 2x(2y - 3)$

78. $5(2x + 2) + x(3x - 3) - 3xx$

79. $xx(x - y) - yy(x + y) - (x - y)$

80. $3(x - 4) + 5(x + 2)$

81. $2(y + 3) + 3(b - 4)$

82. $2(2x + 1) + (3x - 4)$

83. $3(y + 5) + 2(y - 2)$

84. $5(y + 2) - 4(y - 5)$

85. $6(y + 8) - 4(y - 4)$

86. $5(y + 3) + 4(y - 3)$

87. $3(y - 2) - 2(y + 2)$

88. $2(2a + b) - (a - b)$

89. $2(2a + 3b) - (a + b)$

90. $4(a + b) - 3a$

91. $\frac{56n}{14}$

92. $\frac{-132ab}{12}$

93. $\frac{21cd}{-\frac{1}{7}}$

94. $\frac{15n - 60r}{15}$

95. $\frac{38xy}{2x}$

96. $\frac{-46stt}{-2t}$

97. $\frac{-4n}{r}$	98. $7ab - 1$	99. $32uv - 48$
100. $2r - 12$	101. $\frac{2xx}{9bb}$	102. $-\frac{x}{yy}$
103. $\frac{8b}{3cc}$	104. $-\frac{3bbc}{xyy}$	105. $9a$
106. $2 - 38b$	107. $5 + 14y$	108. $23 - 15c$
109. 102	110. $73 - 15m$	111. $3n + 16$
112. $2tt + 28t - 8$	113. $\frac{5b}{24}$	114. $\frac{10r - 3}{20}$
115. $\frac{h + 25}{12}$	116. $\frac{336x - 7}{42x}$ [or: $\frac{48x - 1}{6x}$]	
117. $\frac{-3d - 16}{10}$	118. $\frac{k + 67}{18}$	119. $\frac{-3c + 19}{c - 2}$
120. $\frac{17a - 15}{a(2a - 3)}$	121. $\frac{10f + 27}{7(f - 5)}$	122. $\frac{-4gg - 26g}{(g + 6)(g + 7)}$
123. $\frac{5x + 30}{3x - 45}$	124. $\frac{y + 5}{y - 5}$	125. $\frac{3}{8}$
126. $\frac{26}{103}$	127. $\frac{rt}{3r - 5t}$	128. $\frac{ad - bc}{ad + bc}$

97. $\frac{52nnr}{-13nrr}$

98. $\frac{35abd - 5d}{5d}$

99. $\frac{8uv - 12}{\frac{1}{4}}$

100. $\frac{(r - 6)(r - 7)}{\frac{1}{2}(r - 7)}$

101. $\frac{5xy}{3ab} \times \frac{2xa}{15yb}$

102. $\frac{-3xxy}{-2yyz} \times \frac{6yz}{-9xyy}$

103. $\frac{4ab}{7bcc} \div \frac{3ac}{14bc}$

104. $\frac{5abbc}{3xyyz} \div \frac{-5axby}{9bzxy}$

105. $28\left(\frac{a}{4} + \frac{a}{14}\right)$

106. $63\left(\frac{2b + 1}{7} + \frac{b - 1}{9} - \frac{3b}{3}\right)$

107. $2y\left(\frac{5}{2y} + 7\right)$

108. $30\left(\frac{c + 5}{5} - \frac{2c - 1}{10} + \frac{3c + 2}{-6}\right)$

109. $12e\left(\frac{7}{e} + \frac{6}{4e}\right)$

110. $42\left[\frac{1}{7}(m + 4) - \frac{1}{2}(m - 2) + \frac{1}{6}\right]$

111. $5(n + 2)\left(\frac{8}{5} - \frac{n}{n + 2}\right)$

112. $16t\left(\frac{2t - 1}{2t} + \frac{3t + 2}{4} - \frac{5t - 2}{8}\right)$

113. $\frac{b}{3} - \frac{b}{8}$

114. $\frac{5r}{10} - \frac{3}{20}$

115. $\frac{h + 7}{4} - \frac{h - 2}{6}$

116. $\frac{4}{7x} - \frac{1}{2x} + 8 - \frac{5}{21x}$

117. $\frac{d - 3}{5} - \frac{d + 2}{2}$

118. $\frac{k + 9}{9} + \frac{1}{3} - \frac{k - 7}{18} + 2$

119. $\frac{c + 6}{c - 2} + \frac{5}{c - 2} - 4$

120. $\frac{13}{2a - 3} + \frac{5}{a} - \frac{6}{2a - 3}$

121. $\frac{f + 4}{f - 5} + \frac{3}{7} + \frac{2}{f - 5}$

122. $\frac{g}{g + 6} - \frac{2g}{g + 7} - \frac{3g}{g + 6}$

123. $\dfrac{\frac{1}{3} + \frac{2}{x}}{\frac{1}{5} - \frac{3}{x}}$

124. $\dfrac{1 + \frac{5}{y}}{1 - \frac{5}{y}}$

125. $\dfrac{\frac{2}{k} - \frac{3}{k}}{\frac{1}{3k} - \frac{3}{k}}$

126. $\dfrac{\frac{1}{5r} + \frac{2}{3r}}{\frac{7}{2r} - \frac{1}{15r}}$

127. $\dfrac{1}{\frac{3}{t} - \frac{5}{r}}$

128. $\dfrac{\frac{a}{b} - \frac{c}{d}}{\frac{a}{b} + \frac{c}{d}}$

TEST

I. In these exercises make the substitutions indicated and tell whether the resulting sentence is true or false.

'-1' for 'a' '4' for 'b'

'-3' for 'c' '$\frac{1}{-5}$' for 'r'

'2' for 'y''

1. $2a + 5 = 7a - 5$
2. $3c + 2c = 5c$
3. $y + 3y = 25$
4. $b + 8 = 6 - b$
5. $8r + 6 = 4 - 2r$

II. In these expressions make the substitutions listed below and simplify the resulting numeral.

'2' for 'x' '4' for 'z

'-3' for 'y' '0' for 'a'

1. $(3 - 2y)4x$ 2. $\dfrac{4y - 2ax}{3y}$ 3. $\dfrac{3x - 2yz}{3x + 4y}$

4. $\dfrac{\frac{1}{3}az}{-\frac{2}{5}xy}$ 5. $\dfrac{3xy + 4xz}{5ax - \frac{1}{2}yz}$

III. Use single quotes to punctuate the following paragraph so that it makes sense.

I wanted to have a dog as a pet because a dog is a nice animal for a pet. But I could not have a dog for a pet, because a dog is not and cannot be a pet. A dog is not even an animal. If I had a dog for a pet, I could have a lot of fun playing with it. But I could not even walk a dog!

Answers for TEST.

I. 1. F 2. T 3. F 4. F 5. T

II. 1. 72 2. $\frac{4}{3}$ 3. -5 4. 0 5. $\frac{7}{3}$

III. I wanted to have a dog as a pet because a dog is a nice animal for a pet. But I could not have a 'dog' for a pet, because a 'dog' is not and cannot be a pet. A 'dog' is not even an animal. If I had a dog for a pet, I could have a lot of fun playing with it. But I could not even walk a 'dog'!

IV. 1. $4y$ 2. $4x + 2$ 3. $19n - 28$

4. $-4a - 8$ 5. $\frac{5}{2}x - y$ 6. $-\frac{2}{5}xxyy$

7. $6x - 41y$ 8. $-9r - 11$ 9. $\frac{nnn}{27}$

10. $\frac{2}{15}cd$ 11. $\frac{-14e - 5f}{6}$ 12. $-\frac{9}{14}g - h$

13. $13b$, $[a \neq 0]$ 14. $22bc$, $[a \neq 0]$ 15. $\frac{-z}{2u}$, $[abxyu \neq 0]$

16. $20xz$, $[a \neq 0 \neq n]$ 17. $\frac{3b - a}{10}$

18. $\frac{5n - 4}{2n}$, $[n \neq 0]$ 19. $\frac{ab}{5dd}$, $[abcd \neq 0]$

20. $6c$, $[c \neq 0]$ 21. $2x - 7$ 22. $15x - 4$, $[aq \neq 0]$

23. $\frac{27d - 4dd - 6}{(d - 6)(d - 2)}$, $[d \neq 6, 2]$

24. $\frac{rs(r + s)}{4s - 5r}$, $[rs \neq 0, 4s - 5r \neq 0]$

IV. Simplify.

1. $y + y + y + y$

2. $3x + 2 + x$

3. $7n + 12n - 28$

4. $5a + 4 - 9a - 12$

5. $\frac{x}{2} + 2(x + y) - 3y$

6. $\frac{4}{5}x \cdot \frac{3}{2}xy \cdot -\frac{1}{3}y$

7. $-2(4x + 3y) + 7(2x - 5y)$

8. $12r - 3(7 + 2r) + 5(-3r + 2)$

9. $\left(\frac{1}{3}n\right)\left(\frac{1}{3}n\right)\left(\frac{1}{3}n\right)$

10. $\left(-\frac{1}{5}c\right)\left(-\frac{2}{3}d\right)$

11. $\frac{2}{3}e - \frac{5}{6}f - 3e$

12. $-\frac{1}{7}g - \frac{3}{8}h - \frac{1}{2}g - \frac{5}{8}h$

13. $\frac{-169ab}{-13a}$

14. $\frac{176abc}{8a}$

15. $\frac{-6xy}{5ab} \times \frac{5baz}{12xyu}$

16. $\frac{35nx}{3a} \times \frac{12az}{7n}$

17. $\frac{a + b}{2} - \frac{3a + b}{5}$

18. $\frac{3n - 2}{2n} + \frac{n - 1}{n}$

19. $\frac{-4c}{-5d} \div \frac{-20cd}{-5ab}$

20. $7c\left(\frac{6}{c} \div \frac{7}{c}\right)$

21. $\frac{10x - 35}{5}$

22. $\frac{4aq - 15\,aqx}{-aq}$

23. $\frac{3}{d - 6} - \frac{4d}{d - 2}$

24. $\frac{r + s}{\frac{4}{r} - \frac{5}{s}}$

V. In these exercises, write in each blank the principle or theorem [<u>not just its name</u>] which justifies the step.

1. $\forall_x \forall_y \forall_z \ (y + x)z = xz + yz.$

$(y + x)z$
$= z(y + x)$ ______
$= z(x + y)$ ______
$= zx + zy$ ______
$= xz + yz.$ ______

2. $\forall_x \forall_y \forall_z \ x + y - z = (y - z) + x.$

$x + y - z$
$= x + y + -z$ ______
$= x + [y + -z]$ ______
$= [y + -z] + x$ ______
$= (y - z) + x.$ ______

3. $\forall_a \forall_b \ -(a - b) = b - a.$

$-(a - b)$
$= -1(a - b)$ ______
$= -1(a + -b)$ ______
$= -1a + -1 \cdot -b$ ______
$= -1a + 1b$ ______
$= 1b + -1a$ ______
$= 1b + -a$ ______
$= 1b - a$ ______
$= b - a.$ ______

V. 1. $\forall_x \forall_y\ xy = yx$

$\forall_x \forall_y\ x + y = y + x$

$\forall_x \forall_y \forall_z\ x(y + z) = xy + xz$

$\forall_x \forall_y\ xy = yx$

2. $\forall_x \forall_y\ x - y = x + -y$

$\forall_a \forall_b \forall_c\ a + b + c = a + (b + c)$

$\forall_x \forall_y\ x + y = y + x$

$\forall_x \forall_y\ x - y = x + -y$

3. $\forall_x\ -1 \cdot x = -x$

$\forall_x \forall_y\ x - y = x + -y$

$\forall_x \forall_y \forall_z\ x(y + z) = xy + xz$

$\forall_x\ -1 \cdot x = -x$; $\forall_x\ - -x = x$ [or: $\forall_x \forall_y\ -x \cdot -y = xy$]

$\forall_x \forall_y\ x + y = y + x$

$\forall_x\ -1 \cdot x = -x$

$\forall_x \forall_y\ x - y = x + -y$

$\forall_x\ 1x = x$

VI.	(1) F	(2) F	(3) T	(4) F	(5) T
	(6) T	(7) F	(8) F	(9) F	(10) T

VII.	1. ✓	2. ✓	3. ____	4. ✓	5. ✓
	6. ____	7. ✓	8. ____	9. ____	10. ____

VI. In each blank at the right, indicate with a 'T' or an 'F' whether the statement is true or false.

1. For each x, $+x$ is a positive number. (1) ________

2. For each a, $a = {}^{+}|a|$. (2) ________

3. For each x, for each y, the product of x by y is xy. (3) ________

4. For each c, for each d, $c \cdot -d \leq 0$. (4) ________

5. For each $\square$, for each $\triangle$, if $\square + \triangle = 0$, then $\square = -\triangle$. (5) ________

6. For each m, for each n, $-m \cdot -n = mn$. (6) ________

7. For each x, for each y, $|x + y| \geq |x| + |y|$. (7) ________

8. For each r, for each s, $rs > 0$. (8) ________

9. For each m, $8m + m = 8mm$. (9) ________

10. For each s, for each t, $-(t - s) = s - t$. (10) ________

VII. Examine the pairs listed below, and put a '$\checkmark$' in the blank alongside each one whose first component is greater than or equal to its second component.

1. $(.5,\ .5 \times .5)$	________	2. $(-\lvert 42\rvert,\ -\lvert -42\rvert)$	________
3. $(-12,\ 10)$	________	4. $(-5,\ -8)$	________
5. $(-10,\ -26)$	________	6. $(\lvert -5\rvert,\ \lvert -6\rvert)$	________
7. $(-\frac{1}{4},\ -\frac{1}{2})$	________	8. $(\frac{-23}{7},\ \frac{-7}{23})$	________
9. $(-1000,\ -999)$	________	10. $(-200,\ 0)$	________

VIII. 1. List 3 pairs of real numbers which belong to the operation multiplying by -2.

2. List 3 pairs of real numbers which belong to the operation which is the inverse of adding $^{-}5$.

3. List 3 pairs of real numbers which belong to the operation multiplying by zero.

4. List 3 pairs of real numbers which belong to the operation which is the inverse of multiplying by $^{-}3$.

5. List 3 pairs of real numbers which belong to the operation dividing by $-\frac{1}{2}$.

IX. Complete with the simplest expression which makes the statement true.

1. For each n, the sum of 3n and 10 is _________.

2. For each r, the product of 8 by $-3r$ is __________.

3. For each c, the difference of 12 from $c + 12$ is _________.

4. For each y, y increased by 9 is __________.

5. For each a, for each b, 5a decreased by $2a + 7b$ is _________.

6. For each x, for each $y \neq 0$, the quotient of $3xy - 6y + 9xy - 3y$ by 3y is _____________.

7. For each $x > 0$, for each $y > 0$, the perimeter of a rectangle with short side 3x units long and long side 2y units long is _______________.

8. For each $a > 0$, if 3a objects are to be distributed equally among 5 persons, then each person will receive ___________ objects.

9. For each $m > 0$, a car traveling at a steady rate of 3m miles an hour will travel 150 miles in __________ hours.

VIII. 1. (0, 0), (1, −2), (2, −4), (−1, 2), (−3, 6), ...

2. (0, 5), (3, 8), (10, 15), (−1, 4), (−2, 3), ...

3. (0, 0), (4, 0), (7, 0), (−3, 0), (−5, 0), ...

4. (6, −2), (−6, 2), (3, −1), (12, −4), (−15, 5), ...

5. (0, 0), (2, −4), (5, −10), (−2, 4), (−3, 6), ...

IX.

1. $3n + 10$	2. $-24r$	3. c
4. $y + 9$	5. $3a - 7b$	6. $4x - 3$
7. $6x + 4y$	8. $\frac{3a}{5}$	9. $\frac{50}{m}$

X.

1. C	2. F	3. E	4. H	5. D
6. A	7. K	8. B	9. G	10. I
11. J				

X. Each of the following statements is a consequence of one of the principles for real numbers. Below them are the names of these principles, each being preceded by a letter. In the blank at the left of each statement, write the letter corresponding to the principle of which the statement is a consequence.

__________ 1. $4 + (5 + 8) = (4 + 5) + 8$

__________ 2. $(-3 + 0) + 7 = -3 + 7$

__________ 3. $-4 \cdot 12 + 7 \cdot 12 = (-4 + 7)12$

__________ 4. $^{-}13 \cdot 2 \cdot 1 = {}^{-}13 \cdot 2$

__________ 5. $(9 \cdot 8)5 + 10 = 9(8 \cdot 5) + 10$

__________ 6. $15 + (6 + 2) = (6 + 2) + 15$

__________ 7. $(-19 + 7) - 15 = (-19 + 7) + (-15)$

__________ 8. $-3[(5 + 2)6] = -3[6(5 + 2)]$

__________ 9. $9(8 \cdot 0) = 9 \cdot 0$

__________ 10. $578 + -578 = 0$

__________ 11. $63 + 2 = \frac{63 + 2}{-13} \cdot -13$

A. Commutative principle for addition

B. Commutative principle for multiplication

C. Associative principle for addition

D. Associative principle for multiplication

E. Distributive principle for multiplication over addition

F. Principle for adding 0

G. Principle for multiplying by 0

H. Principle for multiplying by 1

I. Principle of opposites

J. Principle of quotients

K. Principle for subtraction

SUPPLEMENTARY EXERCISES

A. Find the value of each of the following pronumeral expressions for the given values of the pronumerals.

Pronumeral	'x'	'y'	'u'	'v'
Value	5	3	2	−4

1. $x + y$
2. $u + v$
3. $u - v$
4. $2x$
5. $7y$
6. $8u$
7. $63 + u$
8. $6 \cdot 3 + y$
9. $6.3 - u$
10. $11 - x$
11. $x - 11$
12. $-3x$
13. $-4v$
14. $y - x$
15. $x - y$
16. $6y - 2$
17. $4u + 5$
18. $5 + 4u$
19. $3 + 7x$
20. $(3 + 7)x$
21. $3x + 7x$
22. $3 + 3u$
23. $3(1 + u)$
24. $3 + u$
25. $1 + 5y$
26. $6 + y$
27. $6y$
28. xx
29. $xx - 55$
30. vv
31. $1 + uu$
32. $(1 + u)u$
33. $u + uu$
34. $2x + y$
35. $2x + 3y$
36. $2(x + 3y)$
37. $2uxy$
38. $3uvx$
39. $-5xvy$
40. $x + 5y + u$
41. $7x - 2y - v$
42. $3u - 2v + 4x$
43. $6x + 2v - 7$
44. $19 - 3x - 3x$
45. $12 + 2u - 7$
46. $\frac{3x}{y}$
47. $\frac{7y}{3u}$
48. $\frac{9v}{8y}$
49. $\frac{-2v}{7u}$
50. $\frac{x - y}{5}$
51. $\frac{x + y}{y - x}$
52. $\frac{xx - 1}{yy + 2}$
53. $-\frac{u + v}{2v}$
54. $\frac{5x - 3u}{7v + y}$
55. $2(x + 3) + 5(x - 7)$
56. $3(u + 2) + 7(8 + u)$
57. $4(7 - y) - x + y$
58. $5(2x + 4) - 3(1 - 2y)$
59. $8(u - v + x) + 5(u - v + x)$
60. $7u - 3x(2 - 5y + v)$

Answers for SUPPLEMENTARY EXERCISES.

A.

1. 8	2. −2	3. 6
4. 10	5. 21	6. 16
7. 65	8. 21	9. 4.3
10. 6	11. −6	12. −15
13. 16	14. −2	15. 2
16. 16	17. 13	18. 13
19. 38	20. 50	21. 50
22. 9	23. 9	24. 5
25. 16	26. 9	27. 18
28. 25	29. −30	30. 16
31. 5	32. 6	33. 6
34. 13	35. 19	36. 28
37. 60	38. −120	39. 300
40. 22	41. 33	42. 34
43. 15	44. −11	45. 9
46. 5	47. $\frac{7}{2}$	48. $-\frac{3}{2}$
49. $\frac{4}{7}$	50. $\frac{2}{5}$	51. −4
52. $\frac{24}{11}$	53. $-\frac{1}{4}$	54. $-\frac{19}{25}$
55. 6	56. 82	57. 14
58. 85	59. 143	60. 269

B.	1.	9	2.	25	3.	21	4.	47	5.	-40
	6.	17	7.	-15	8.	$-\frac{3}{4}$	9.	49	10.	2
	11.	35	12.	$-389\frac{1}{4}$	13.	$-\frac{1}{7}$	14.	$-\frac{49}{6}$	15.	-3
	16.	49	17.	45	18.	20	19.	19	20.	28
	21.	-13	22.	32	23.	-120	24.	3	25.	$-\frac{3}{2}$

26. No answer; when the substitutions are made, the resulting expression is meaningless.

27.	$-\frac{2}{9}$	28.	$\frac{14}{5}$	29.	$\frac{5}{14}$	30.	-9	31.	$9\frac{1}{3}$
32.	-19	33.	$\frac{24}{7}$	34.	-36.5	35.	-30	36.	22
37.	$-\frac{7}{3}$	38.	$\frac{14}{3}$	39.	6	40.	-168		

B. In each of the following pronumeral expressions make the substitutions listed below and simplify the resulting numeral.

'7' for 'm',	'0' for 'y',
'2' for 'x',	'−3' for 'z',
'$\frac{3}{4}$' for 'v',	'−8' for 'u'.

1. $m + x - y$
2. $3m + 2x$
3. $6x - 3z + 59y$
4. $5x - 2u + 3m$
5. $6u + 8v \div x$
6. $-x - z - 2u$
7. $12v - 3y + 8z$
8. $-v + 3x + 2z$
9. $11m - 2x + 3u$
10. $3(x \div z) + 5$
11. $x(2m - x) + 11$
12. $5z(x - 3u) + v$
13. $\frac{x + z + y}{m}$
14. $\frac{m}{x} \cdot \frac{m}{z}$
15. $\frac{z}{m} \cdot m$
16. $(m + y)(m - y)$
17. $(m + x)(m - x)$
18. $2m + 3x + 4.2y$
19. $2(m + 2x) + z$
20. $mx + my + mx$
21. $z(m + x) + mx$
22. $4xv + 8mv + xu$
23. $-5zu + 3xuzy$
24. $3m - (x - 2u)$
25. $\frac{m + (z + x) + 4z}{2x}$
26. $\frac{m}{z} \div \frac{y}{m}$
27. $\frac{2m}{3x} \cdot \frac{x}{mz}$
28. $\frac{mx + yz}{m - x}$
29. $\frac{x - z}{2m + 3y}$
30. $\frac{xz + mz}{3x + z}$
31. $\frac{2}{3}x + \frac{4}{3}v + m$
32. $(\frac{1}{3}z)(12v) - 5x$
33. $(\frac{2}{7}xm)(\frac{4}{7}xv)$
34. $2y + 4z - 3.5m$
35. $8z + 5y - 3x$
36. $5mx - 4yz + 8xz$
37. $\frac{m}{z} + \frac{y}{x} \div \frac{z}{m}$
38. $\frac{4mz}{3xz}$
39. $\frac{25xz + 2my}{5z - 5x}$
40. $3x[4(z - 2x) - 117y]$

C. Use each of the following open sentences to generate a statement. Tell whether the statement is true or false. Try to generate some true statements and some false statements.

1. $4y - 6 = 30$
2. $10m - m + 1 = 10$
3. $4 = 4b - b + b$
4. $x - \frac{1}{2}x - 2 = 1$
5. $a - 3b = -3b + a$
6. $7.5r - 22.5 = (3 - r)7.5$
7. $c - \frac{1}{3}c - 7 = 1$
8. $18 + 3(y - 5) = 12$
9. $4ww - 8.5 = 7.5$
10. $(2m + 3)(2m - 3) = 4mm - 9$
11. $10(g - 5) + 30 = 20$
12. $1.005 + d < d$
13. $16 - (s - 10) > 0$
14. $5(n - 1) - 15 \leq 5$
15. $7d - (d - 10) \geq 0$
16. $p = p - 1$
17. $|x - 1| = 7$
18. $|x + 2| + 8 = 5$
19. $rr - 26r + 169 = (r - 13)(r - 13)$
20. $\frac{4}{9}kk - 64 = (\frac{2}{3}k - 8)(\frac{2}{3}k + 8)$

C. [For Exercises 1-4, 6-9, 11, and 17, we simply list the numbers which will give true statements.]

1. 9 2. 1 3. 1 4. 6

5. Each number will give a true statement.

6. 3 7. 12 8. 3 9. 2, −2

10. Each number will give a true statement.

11. 4

12. No number will give a true statement.

13. Each number smaller than 26 will give a true statement, and these are the only numbers which do.

14. Each number not greater than 5 will give a true statement, and these are the only numbers which do.

15. Each number not smaller than $-\frac{5}{3}$ will give a true statement, and these are the only numbers which do.

16. No number will give a true statement.

17. 8, −6

18. No number will give a true statement.

19. Each number will give a true statement.

20. Each number will give a true statement.

D.

1. (a) apm
   (b) apm
   (c) cpm
   (d) apm
   (e) $4 \cdot 2 = 8$
   (f) apm

2. (a) apa
   (b) cpa
   (c) apa
   (d) dpma
   (e) $2 + 3 = 5$
   (f) apa
   (g) $5 + 8 = 13$

3. (a) ℓdpma, ℓdpma
   (b) apm, apm
   (c) $2 \cdot 7 = 14$, $3 \cdot 2 = 6$, $3 \cdot 5 = 15$
   (d) pml
   (e) apa
   (f) apa
   (g) cpa
   (h) apa
   (i) dpma
   (j) apa
   (k) $14 + 6 = 20$, $2 + 15 = 17$

D. Here are test-patterns for generalizations. Your job is to give the reasons for the steps in the proof.

1. For each x, $4x(2x) = 8(xx)$.

$4x(2x) = 4x2x$ [__________] (a)

$4x2x = 4(x2)x$ [__________] (b)

$4(x2)x = 4(2x)x$ [__________] (c)

$4(2x)x = 4 \cdot 2xx$ [__________] (d)

$4 \cdot 2xx = 8xx$ [__________] (e)

$8xx = 8(xx)$. [__________] (f)

2. For each y, $2y + 5 + 3y + 8 = 5y + 13$.

$2y + 5 + 3y + 8 = 2y + (5 + 3y) + 8$ [__________] (a)

$2y + (5 + 3y) + 8 = 2y + (3y + 5) + 8$ [__________] (b)

$2y + (3y + 5) + 8 = (2y + 3y) + 5 + 8$ [__________] (c)

$(2y + 3y) + 5 + 8 = (2 + 3)y + 5 + 8$ [__________] (d)

$(2 + 3)y + 5 + 8 = 5y + 5 + 8$ [__________] (e)

$5y + 5 + 8 = 5y + (5 + 8)$ [__________] (f)

$5y + (5 + 8) = 5y + 13$. [__________] (g)

3. For each k, $2(7k + 1) + 3(2k + 5) = 20k + 17$.

$2(7k + 1) + 3(2k + 5) = 2(7k) + 2 \cdot 1 + [3(2k) + 3 \cdot 5]$ [_____, _____](a)

$2(7k) + 2 \cdot 1 + [3(2k) + 3 \cdot 5] = (2 \cdot 7)k + 2 \cdot 1 + [(3 \cdot 2)k + 3 \cdot 5]$ [_____, _____](b)

$(2 \cdot 7)k + 2 \cdot 1 + [(3 \cdot 2)k + 3 \cdot 5] = 14k + 2 \cdot 1 + [6k + 15]$ [____, ____, ____](c)

$14k + 2 \cdot 1 + [6k + 15] = 14k + 2 + [6k + 15]$ [__________](d)

$14k + 2 + [6k + 15] = 14k + 2 + 6k + 15$ [__________](e)

$14k + 2 + 6k + 15 = 14k + (2 + 6k) + 15$ [__________](f)

$14k + (2 + 6k) + 15 = 14k + (6k + 2) + 15$ [__________](g)

$14k + (6k + 2) + 15 = 14k + 6k + 2 + 15$ [__________](h)

$14k + 6k + 2 + 15 = (14 + 6)k + 2 + 15$ [__________](i)

$(14 + 6)k + 2 + 15 = (14 + 6)k + (2 + 15)$ [__________](j)

$(14 + 6)k + (2 + 15) = 20k + 17$. [_____, _____](k)

E. Each of the following is a generalization about real numbers. Some are true and some are false. Your job is to decide which, and in each case to give either a counter-example or a proof.

1. For each x, $1 + 3x + 5 = 3x + 6$.
2. No matter what number you pick, if you add 4 to it, and multiply 2 by this sum, the result is the product of 2 by the chosen number, plus 8.
3. For each t, $3t + 9t = 12t$.
4. For each k, $3(5k) + 6(2k) = 27k$.
5. For each m, $9 + m + 7 = 16 + m$.
6. For each y, $3 + 8y = 11y$.
7. For each r, $5 + 10r = 5(1 + 2r)$.
8. For each k, $3k + 5 + 9k = 17k$.
9. For each t, $7t + 6 + 8t = 15t + 6$.
10. For each q, $2q + 9 + 7q = 9(q + 1)$.
11. For each a, $3a(7a) = 21(aa)$.
12. For each r, $7r + 1 + 3r + 5 = 10r + 6$.
13. For each n, $n(n + 2) + 6n = nn + 8n$.

F. Each of the following generalizations is a consequence of one of the principles of real numbers. Tell which principle.

1. For each x, $3x(x + 5) = (x + 5)(3x)$.
2. For each y, $7 + (y + 3) + 5 = (y + 3) + 7 + 5$.
3. For each r, $(2r + 1)(3r + 7) = 2r(3r + 7) + 1(3r + 7)$.
4. For each k, $7k + 3 - (8k + 5) = 7k + 3 + -(8k + 5)$.
5. For each y, $3(y + 12) + 3(y + 11) = 3[(y + 12) + (y + 11)]$.
6. For each m, $(m + 4)(m + 3)(m + 1) = (m + 4)[(m + 3)(m + 1)]$.
7. For each x, $3(x - 5) + 6(x + 9) + 12(x - 2) = 3(x - 5) + [6(x + 9) + 12(x - 2)]$.

E. 1.
$$\begin{aligned} &1 + 3x + 5 \\ &= 3x + 1 + 5 && \text{cpa} \\ &= 3x + (1 + 5) && \text{apa} \\ &= 3x + 6. && 1 + 5 = 6 \end{aligned}$$

Hence, $1 + 3x + 5 = 3x + 6$.

2. [For each x, $2(x + 4) = 2x + 8$.]
$$\begin{aligned} &2(x + 4) \\ &= 2x + 2 \cdot 4 && \ell\text{dpma} \\ &= 2x + 8. && 2 \cdot 4 = 8 \end{aligned}$$

Hence, $2(x + 4) = 2x + 8$.

3.
$$\begin{aligned} &3t + 9t \\ &= (3 + 9)t && \text{dpma} \\ &= 12t. && 3 + 9 = 12 \end{aligned}$$

Hence, $3t + 9t = 12t$.

4.
$$\begin{aligned} &3(5k) + 6(2k) \\ &= (3 \cdot 5)k + (6 \cdot 2)k && \text{apm} \\ &= 15k + 12k && 3 \cdot 5 = 15,\ 6 \cdot 2 = 12 \\ &= (15 + 12)k && \text{dpma} \\ &= 27k. && 15 + 12 = 27 \end{aligned}$$

Hence, $3(5k) + 6(2k) = 27k$.

5.
$$\begin{aligned} &9 + m + 7 \\ &= 7 + (9 + m) && \text{cpa} \\ &= 7 + 9 + m && \text{apa} \\ &= 16 + m. && 7 + 9 = 16 \end{aligned}$$

Hence, $9 + m + 7 = 16 + m$.

6. False; each number other than 1 is a counter-example.

7.

$$\begin{aligned} & 5 + 10r \\ &= 5 + 5 \cdot 2r && 10 = 5 \cdot 2 \\ &= 5 + 5(2r) && \text{apm} \\ &= 5 \cdot 1 + 5(2r) && \text{pm1} \\ &= 5(1 + 2r). && \ell\text{dpma} \end{aligned}$$

Hence, $5 + 10r = 5(1 + 2r)$.

8. False; each number other than 1 is a counter-example.

9.

$$\begin{aligned} & 7t + 6 + 8t \\ &= 8t + (7t + 6) && \text{cpa} \\ &= 8t + 7t + 6 && \text{apa} \\ &= (8 + 7)t + 6 && \text{dpma} \\ &= 15t + 6. && 8 + 7 = 15 \end{aligned}$$

Hence, $7t + 6 + 8t = 15t + 6$.

10.

$$\begin{aligned} & 2q + 9 + 7q \\ &= 7q + (2q + 9) && \text{cpa} \\ &= 7q + 2q + 9 && \text{apa} \\ &= (7 + 2)q + 9 && \text{dpma} \\ &= 9q + 9 && 7 + 2 = 9 \\ &= 9q + 9 \cdot 1 && \text{pm1} \\ &= 9(q + 1). && \ell\text{dpma} \end{aligned}$$

Hence, $2q + 9 + 7q = 9(q + 1)$.

11.

$$\begin{aligned} & 3a(7a) \\ &= 3a7a && \text{apm} \\ &= 7(3a)a && \text{cpm} \\ &= 7 \cdot 3aa && \text{apm} \\ &= 21aa && 7 \cdot 3 = 21 \\ &= 21(aa). && \text{apm} \end{aligned}$$

Hence, $3a(7a) = 21(aa)$.

12.
$$\begin{array}{ll}
 & 7r + 1 + 3r + 5 \\
 & \qquad \text{cpa} \\
= & 3r + (7r + 1) + 5 \\
 & \qquad \text{apa} \\
= & 3r + 7r + 1 + 5 \\
 & \qquad \text{dpma} \\
= & (3 + 7)r + 1 + 5 \\
 & \qquad \text{apa} \\
= & (3 + 7)r + (1 + 5) \\
 & \qquad 3 + 7 = 10,\ 1 + 5 = 6 \\
= & 10r + 6.
\end{array}$$

Hence, $7r + 1 \div 3r + 5 = 10r + 6$.

13.
$$\begin{array}{ll}
 & n(n + 2) + 6n \\
 & \qquad \text{ldpma} \\
= & nn + n2 + 6n \\
 & \qquad \text{cpm} \\
= & nn + 2n + 6n \\
 & \qquad \text{apa} \\
= & nn + (2n + 6n) \\
 & \qquad \text{dpma} \\
= & nn + (2 + 6)n \\
 & \qquad 2 + 6 = 8 \\
= & nn + 8n.
\end{array}$$

Hence, $n(n + 2) + 6n = nn + 8n$.

<u>F</u>. 1. cpm 2. cpa 3. dpma 4. ps

5. ldpma 6. apm 7. apa

62. $\left(-\frac{1}{9}\right)c + 2\frac{2}{3}$ 63. $5\frac{2}{5}x + \left(-13\frac{2}{5}\right)y$ 64. 14n + 26y

65. (−3)r + s 66. (−3)z + 5w 67. 14c + 12d

68. 27m + 43n 69. (−20)y - 5 70. 22 + 31w

71. 50 + 29t 72. 14p + 36 73. (−7)a + (−7)b

74. 10w + 18u + 4v 75. 22bb + 36b 76. 35aa + 40a

77. 31xx + 48x 78. 46xxy + 72xy + 24xyy

79. 22a + 42b 80. 4c + 2d 81. $\frac{1}{2}$w + (−3)u

82. 7rtt 83. 2m + 5n

84. 2sst + 10tt + 4ss + (−3)t + (−3)s

85. 8aaa + ab + (−5)aab + (−4)bb + (−6)abb

86. $\frac{1}{8}$ttt 87. $\frac{1}{6}$gh 88. 39ee

89. 54rst 90. (−72)bcd 91. 105nnnnn

92. (−63)eeff 93. 4.5rs 94. 84aabc

95. (−30)ccd 96. (−72)△△□ 97. 34.5□○

98. (−540)ppqq 99. 108aab + 77abb 100. 4d + $6\frac{3}{4}$c + 5

H. 1. P = 16a 2. P = $3\frac{1}{2}$n 3. P = $3\frac{1}{2}$b

4. P = $9\frac{1}{2}$c 5. P = $2\frac{3}{5}$m 6. P = 5s

7. P = $3\frac{1}{3}$r 8. P = 4πx 9. P = 6y + 4

10. P = 8w 11. P = 3x + 9 12. P = 3a + 6

13. P = 4s 14. P = 12r + $2\frac{1}{2}$

15. P = 2m + $6\frac{1}{2}$n + 1 16. P = 7d + $3\frac{1}{2}$

<u>G</u>.

1. 8a 2. 19b 3. 11x 4. 25t

5. 50r 6. 4x 7. 17x 8. 22y

9. 14a + 3 10. 22a + 1 11. 10b + 115 12. 8k + 31

13. 10r + 2 14. 10p + 12 15. 9x + 7 16. 4x + 5

17. 6.2p 18. 12w + 4 19. 9r + 9 20. 26d + 31

21. 14a + 2b + 2c 22. 19m + 20n 23. 9x + 3y + 5

24. 12a + 2b + 11 25. 11t + 18s 26. 16m + 7

27. 23s + 4r 28. 14x + 13y + 5 29. $\frac{4}{5}a + b$

30. $\frac{11}{18}b + 5$ 31. (−3)k 32. M

*33. (−1)u + (−1)v 34. (−3)t - 8 35. 2x + 1

36. 3b + 8 37. 26k + 21 38. 50a + 5

39. 5j + 8 40. 4x 41. (−3)x

42. 15x + 54 43. 16f + 67 44. 8x + 10

45. 9x + 15 46. 7a + 25 47. 2aab + 3abb + 5

48. 5ab + b + 5a 49. (−21)t 50. (−3)c + (−29)d + 13

51. 7m + (−12)n + 8p 52. $x + \left(-\frac{15}{28}\right)y$ 53. 11.9s + (−4.1)r

54. 12j + 11k + (−14)h + 14 55. (−11)c

56. (−10)u + (−14)v + 12 57. 13d + (−6)e

58. $\frac{3}{8}y + \left(-\frac{7}{8}\right)c + \frac{2}{3}$ 59. $1\frac{1}{12}t + \left(-\frac{9}{10}\right)s$

60. 6.3g + (−.9)h 61. (−.6)e + (−8.6)f

---

*For Exercise 33, the answer '−u + −v' is acceptable. However, if a student gives this, you should ask him to state the generalization he is using [$\forall_x\ -x = -1 \cdot x$]. Tell him that this generalization will be proved later in the unit [See page 2-70.].

G. Simplify.

1. $3a + 5a$
2. $7b + 12b$
3. $2x + 9x$
4. $8t + 17t$
5. $9r + 41r$
6. $x + 3x$
7. $9x + 5x + 3x$
8. $4y + 7y + 11y$
9. $2a + 3 + 12a$
10. $7a + 1 + 15a$
11. $6b + 4b + 115$
12. $31 + k + 7k$
13. $7r + 2 + 3r$
14. $6p + p + 12 + 3p$
15. $6x + 3x + 7$
16. $x + 2x + 5 + x$
17. $5p + 0.5p + 0.7p$
18. $11w + w + 4$
19. $4r + 9 + 5r$
20. $12 + 7d + 19 + 9d + 10d$
21. $6a + 2b + 8a + 2c$
22. $4m + 9n + 15m + 11n$
23. $2x + 5 + 3y + 7x$
24. $11a + 2b + 5 + a + 6$
25. $4t + 13s + 7t + 5s$
26. $4m + 2m + 7 + 10m$
27. $14s + r + 3r + 9s$
28. $9x + 7y + 5 + 5x + 6y$
29. $\frac{1}{5}a + \frac{1}{4}b + \frac{3}{4}b + \frac{3}{5}a$
30. $\frac{1}{3}b + \frac{1}{6}b + 5 + \frac{1}{9}b$
31. $(-9)k + (-6)k + 12k$
32. $M + 6M + (-5)M + (-1)M$
33. $u + (-1)v + (-2)u$
34. $7t + 4 + (-10)t + -12$
35. $x + (x + 1)$
36. $2(b + 4) + b$
37. $3(7k + 7) + 5k$
38. $10(5a) + 5$
39. $2(j + 4) + 3j$
40. $3x + 4(.25x)$
41. $x + 2(-2x)$
42. $7x + (8x + 54)$
43. $9f + (7f + 67)$
44. $3x + 5(x + 2)$
45. $4x + 5(x + 3)$
46. $2(a + 5) + 5(a + 3)$
47. $2a(ab) + 3b(ab) + 5$
48. $b(2a + 1) + a(3b + 5)$
49. $(-1)t + (-5)t + (-9)t + (-6)t$
50. $15c + (-9)d + 13 + (-18)c + (-20)d$
51. $17m + (-7)n + 8p + (-5)n + (-10)m$

(continued on next page)

52. $\frac{1}{5}x + \left(-\frac{1}{4}\right)y + \frac{4}{5}x + \left(-\frac{2}{7}\right)y$

53. $2.5s + 7.5r + 9.4s + (-11.6)r$

54. $12j + -5 + 11k + (-7)h + 19 + (-7)h$

55. $(-1)c + (-4)c + (-9)c + (-3)c + 6c$

56. $9u + (-2)v + 12 + (-19)u + (-12)v$

57. $9d + (-7)e + 6d + (-2)d + e$

58. $\frac{3}{8}y + -\frac{1}{3} + \left(-\frac{7}{8}\right)c + 1$

59. $\frac{1}{3}t + \left(-\frac{2}{5}\right)s + \frac{3}{4}t + \left(-\frac{1}{2}\right)s$

60. $2.6g + 4.5h + 3.7g + (-5.4)h$

61. $4.8e + (-9.6)f + (-5.4)e + f$

62. $\frac{4}{9}c + -\frac{1}{3} + \left(-\frac{5}{9}\right)c + 3$

63. $7\frac{1}{5}x + \left(-14\frac{2}{5}\right)y + \left(-1\frac{4}{5}\right)x + y$

64. $3(3n + 2y) + 5(n + 4y)$

65. $4[r + (-5)s] + (-7)[r + (-3)s]$

66. $\left(-\frac{1}{5}\right)(25z + 15w) + \frac{2}{3}(3z + 12w)$

67. $(-10)(5c + 6d) + 8(8c + 9d)$

68. $4(3m + 2n) + 5(3m + 7n)$

69. $(-1)(10y + 3) + (-2)(5y + 1)$

70. $7(6 + 5w) + (-4)(5 + w)$

71. $8(5 + 3t) + 5(2 + t)$

72. $4(3p + 7) + 2(4 + p)$

73. $7(a + b) + (-14)(a + b)$

74. $6(w + 3u) + 4(w + v)$

75. $4b(b + 3) + 6b(3b + 4)$

76. $5a(a + 4) + 10a(3a + 2)$

77. $3x(x + 9) + 7x(3 + 4x)$

78. $2xy(5x + 36) + 4xy(6y + 9x)$

79. $6(2a + 4b) + 10a + 18b$

80. $\frac{1}{5}(10c + 5d) + \frac{1}{3}(3d + 6c)$

81. $\frac{1}{6}(3w + 6v) + \left(-\frac{1}{3}\right)(3v + 9u)$

82. $3rt(2r + 5t + 4) + (-2)rt(6 + 3r + 4t)$

83. $6(m + n) + (-3)(2m + n) + (-2)(n + m) + 4(m + n)$

84. $2(sst + tt) + 4(ss + 2tt) + (-3)(t + s)$

85. $2(3aaa + ab) + (-1)(5aab + ab) + (-6)(bb + abb) + 2(bb + aaa)$

86. $\left(\frac{1}{2}t\right)\left(\frac{1}{2}t\right)\left(\frac{1}{2}t\right)$

87. $\left[\left(-\frac{1}{4}\right)g\right]\left[\left(-\frac{2}{3}\right)h\right]$

88. $1.2(5e)(6.5e)$

89. $9r(3s)(2t)$

90. $c(8b)(-9)d$

91. $5n(7nn)(3nn)$

92. $(4.5ef)(2e)(-7)f$

93. $(-3)r(-1.5)s$

94. $7a(-3)b(-4)ac$

95. $3c(-2)d(5c)$

96. $4\triangle(3\square)(-6)\triangle$

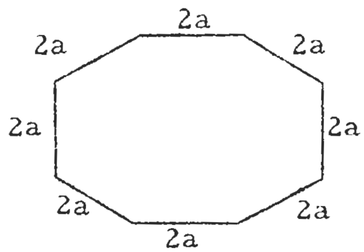
97. $-2.3(5\square)(-3)\bigcirc$

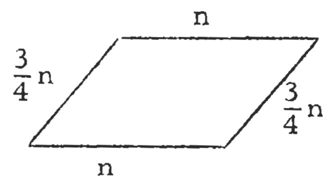
98. $[(6p)(-3)q][(-2)3p(-5)q]$

99. $7ab[(2a + 5b) + (2b + 6a)] + 4ab[(9a + 4b) + (3b + 4a)]$

100. $\frac{1}{6}[(12d + 18) + (9d + 30c)] + \frac{1}{8}[(4d + 2c) + (16 + 12c)]$

H. Write the simplest formula you can for the perimeter of each figure pictured below.

1.


2.


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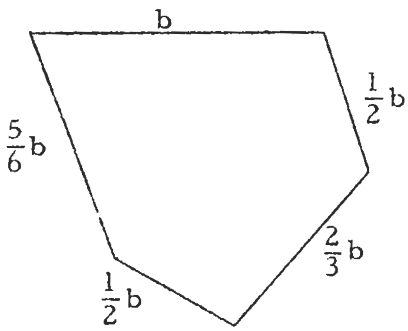
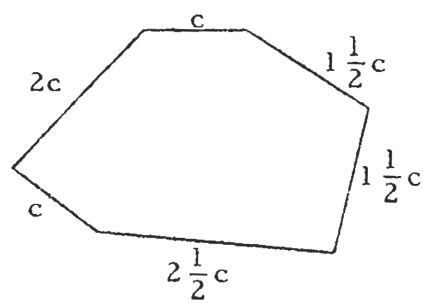
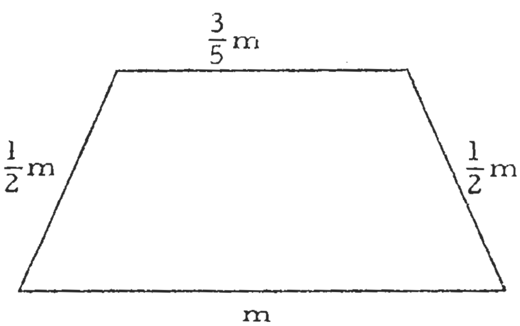


4.


5.


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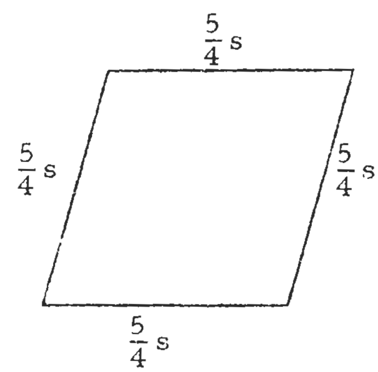
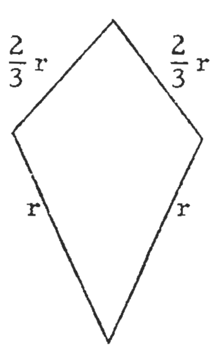
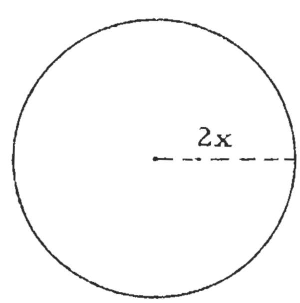


7.


8.


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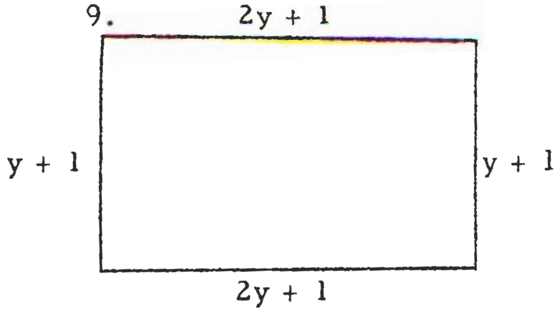
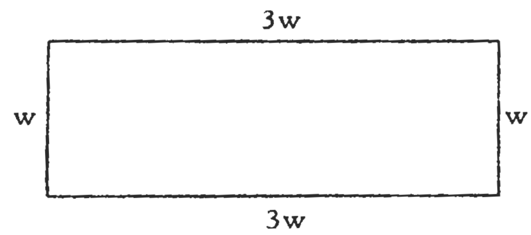
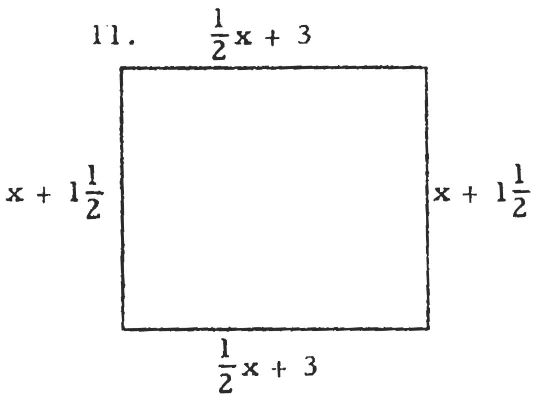
9.



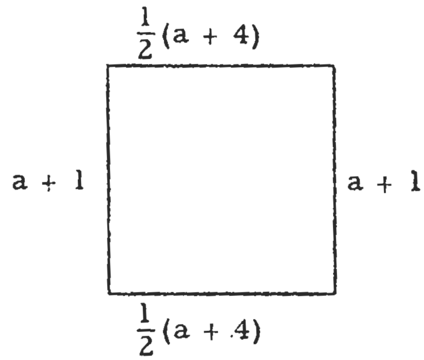
10.



11.



12.



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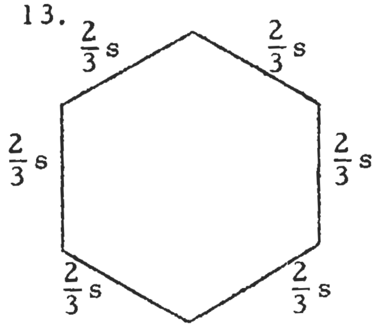

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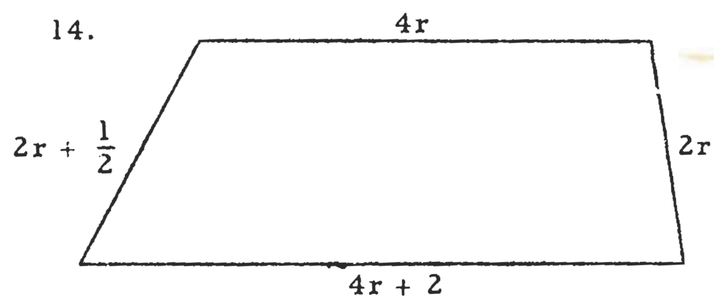

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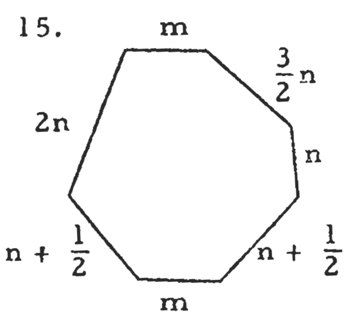

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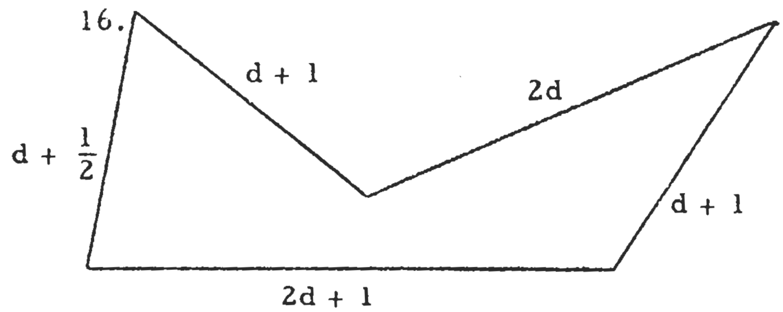

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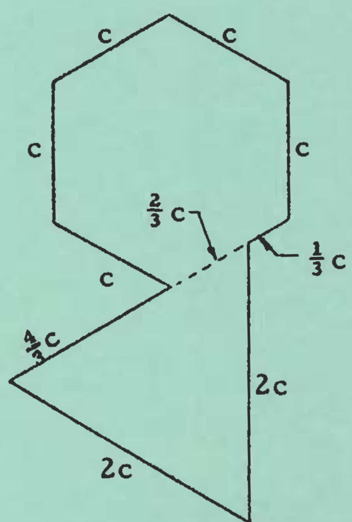
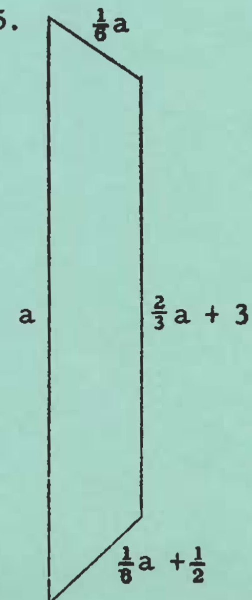
24.



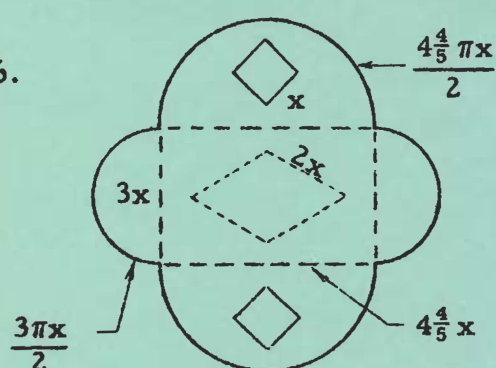
$P = 10\frac{2}{3}c$

25.



$P = 2a + 3\frac{1}{2}$

26.



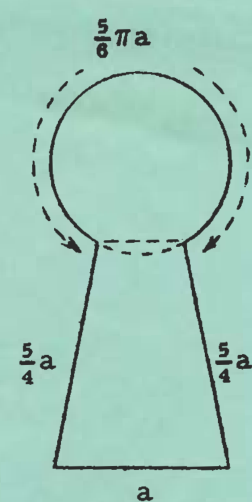
rectangle: $P = 15\frac{3}{5}x$

square: $P = 4x$

rhombus: $P = 8x$

figure formed by four semicircles: $P = \frac{39\pi x}{5}$

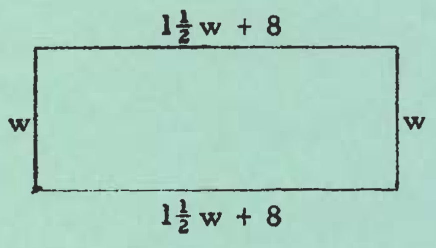
27.



$P = a(\frac{5\pi}{6} + 3\frac{1}{2})$

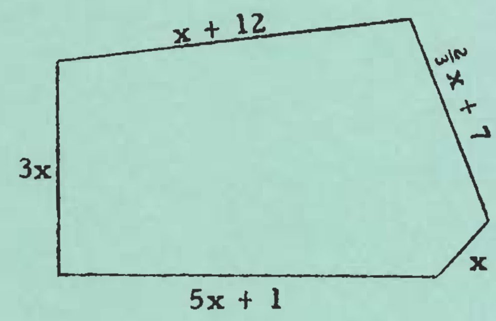
[or: $P = a(\frac{5\pi + 21}{6})$]

19.



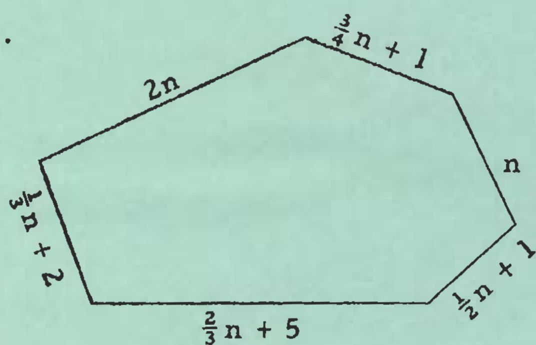
$P = 5w + 16$

20.



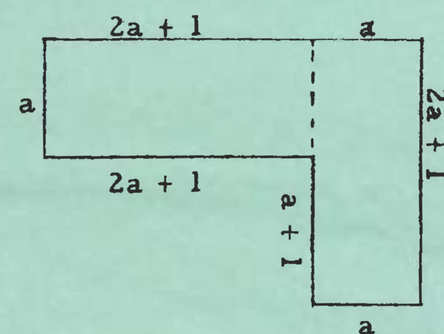
$P = 10\frac{2}{3}x + 20$

21.



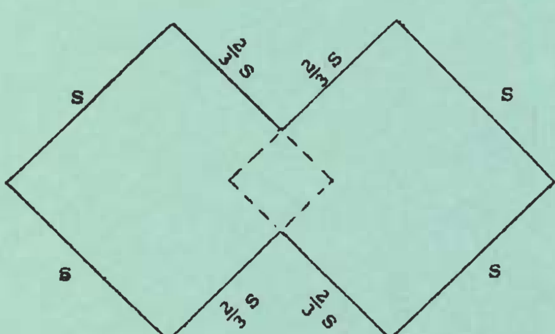
$P = 5\frac{1}{4}n + 9$

22.



$P = 10a + 4$

23.

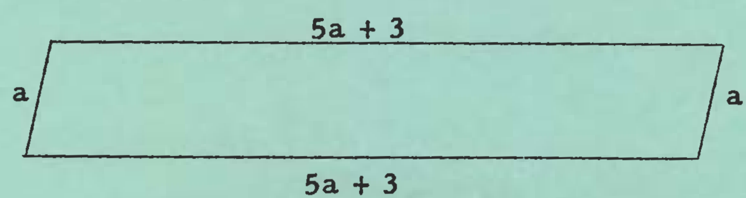


$P = 6\frac{2}{3}s$

[Each of the shorter sides could have been labeled with an 's'; if so, each of the longer sides should be labeled with a '$\frac{3}{2}s$'. If this is done, a formula for the perimeter would be 'P = 10s'.]

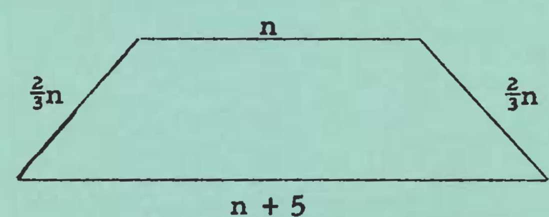
Students should be instructed to include a labeled diagram for each of the exercises from Exercise 17 through Exercise 27 inclusive. Of course, there is more than one way to label the parts of these figures, and accordingly, more than one correct formula for the perimeter. In Exercises 17 and 23 we indicate alternatives. In the other exercises you will doubtless want the class to consider other possibilities.

17. Using 'a' as a pronumeral whose values are the measures of the shorter sides of such parallelograms, a formula for the perimeter of such parallelograms is '$P = 12a + 6$'.



[Each of the longer sides could be labeled with an 'a'; if so, each of the shorter sides should be labeled with an '$\frac{a - 3}{5}$'. Then, the perimeter formula is '$P = 2(a + \frac{a - 3}{5})$', and the values of 'a' are greater than 3.]

18. Using 'n' as a pronumeral whose values are the measures of the shorter of the parallel sides of such trapezoids, a formula for the perimeter is '$P = 3\frac{1}{3}n + 5$'.



For the remaining exercises, we give just a diagram and a formula.

Write the simplest formula you can for the perimeter of each figure described below.

17. A parallelogram whose longer side measures 3 more than 5 times the measure of its shorter side.

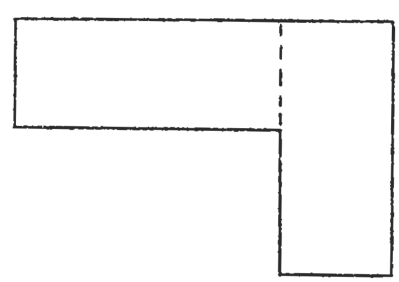
18. An isosceles trapezoid with its non-parallel sides measuring $\frac{2}{3}$ as long as the shorter of the parallel sides, and with the longer of its parallel sides measuring 5 more than the shorter one.

19. A rectangle whose length measures 8 more than $\frac{3}{4}$ the sum of the measures of its two shorter sides.

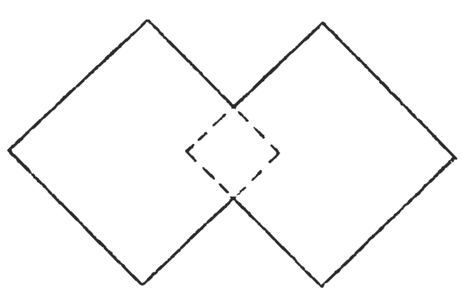
20. A pentagon whose longest and shortest sides differ in measure by 12; of the three remaining sides, one is three times as long as the shortest side of the pentagon; another is 7 units longer than $\frac{2}{3}$ the length of the shortest side; the last is 1 unit longer than 5 times the length of the shortest side.

21. A hexagon which has
    first side measuring 1 more than $\frac{1}{2}$ the fourth side;
    second side measuring 2 more than $\frac{1}{3}$ the fourth side;
    third side measuring 1 more than $\frac{3}{4}$ the fourth side;
    fifth side measuring twice the fourth side, and
    sixth side measuring 5 more than $\frac{2}{3}$ the fourth side.

22. A hexagon made by placing two rectangles together like this: The rectangles have the same perimeter, and the length of each rectangle is one unit more than twice its width.



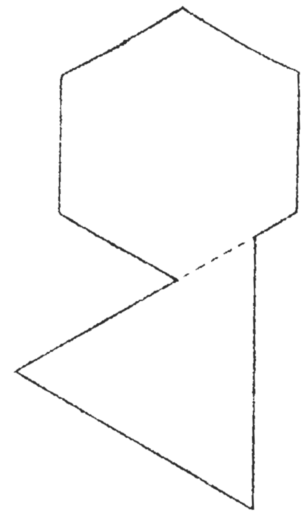
23.



An octagon formed by placing two squares with equal perimeters in this position. The intersecting sides divide their lengths in a 2 to 1 ratio.

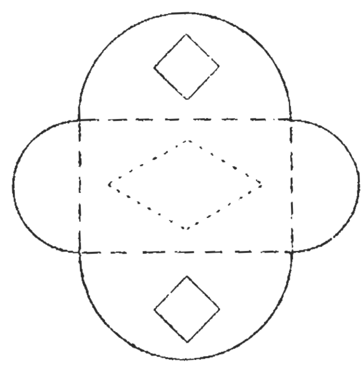
(continued on next page)

24. A nonagon made by placing an equilateral triangle next to a regular hexagon so that $\frac{1}{3}$ the length of a side of the triangle overlaps a side of the hexagon. A side of the triangle is twice as long as a side of the hexagon.

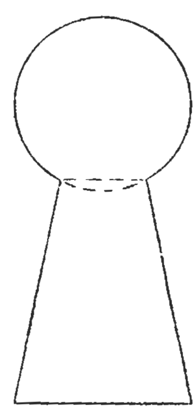


25. A trapezoid the shorter of whose parallel sides measures 3 more than $\frac{2}{3}$ the measure of the longer parallel side, one of whose non-parallel sides is $\frac{1}{6}$ as long as the longer parallel side, and whose other non-parallel side is $\frac{1}{2}$ unit more than $\frac{1}{6}$ the longer parallel side.

26. A formal flower garden is laid out as in this sketch. The rectangular part is $1\frac{3}{5}$ times as long as it is wide; each of the curved edges is a semicircle; the large diamond-shaped part is a rhombus whose side is $\frac{2}{3}$ of the width of the rectangular part; the small squares each have a side measuring $\frac{1}{3}$ the width of the rectangle. Find the formulas for the perimeter of each of the following: the rectangle, the large rhombus, each square, and the figure formed by the four semicircles.



27. A keyhole is of the shape pictured here. The lower part is trapezoidal in shape, with a base edge of the same measure as the diameter of the circular part, and with its non-parallel edges each $1\frac{1}{4}$ times as long as the base edge. The trapezoidal part intersects the circular part in such a way that $\frac{1}{6}$ of the circle is missing. Find a formula for the perimeter of the keyhole.



I. 1. $5x + 3y$ 2. $3y - 6z$

3. This cannot be simplified. 4. $5c - 2x$

5. $2x - 20y$ 6. $10r + 4t$ 7. $-\frac{1}{3}x + \frac{1}{2}y$

8. $-\frac{2}{15}a + \frac{2}{15}b$ 9. $ab + 12c$ 10. $-xy - 10zy$

11. $3\frac{4}{5}x - 20y$ 12. $-\frac{3}{5}x - \frac{2}{3}y$ 13. $2r + 7s - 2$

14. $5t - 21$ 15. $2m - 2mn$ 16. $-5xy - 5x - 6y$

17. $-1\frac{1}{2}x - \frac{3}{4}y$ 18. $-\frac{9}{20}a - \frac{2}{5}b$ 19. $3x$

20. $8rs + 3rr + 5$ 21. $-6j + 11$ 22. $-\frac{5}{8}x + 1$

23. $7a - 12k$ 24. $40r - 17s$ 25. $3j - 11m$

26. $-13r - 23t$ 27. $11X - 25Y$ 28. $5p + 2r - 25s + 7$

29. $-12 - 10k$ 30. $-17x - 7y$ 31. $2kk - 5km + 8$

32. $3yy + 22xy - x + xx$ 33. $8a + 12b$

34. $13r + 8s$ 35. $-4n + p$ 36. $2c - 3d$

37. $25y - 9 + 3z - 8x$ 38. $29g + 8h + 6$

39. $-6f - 6q$ 40. $-3s$ 41. $17x - 48$

42. $2x + 5$ 43. $9a - 6$ 44. $15x - 12$

45. $-7x + 15$ 46. $-20b - 216$ 47. $-8x$

48. $8x - 6$ 49. $7z + 10$ 50. $2x - 2$

51. $-8y - 3$ 52. $-3y + 8$ 53. $x - 3$

54. $2x + 9$ 55. $7b + 16$ 56. $10 + 5s$

57. $16 - 16q$ 58. $4x - 12$ 59. $30 - 8y$

60. $4r - 7$ 61. $10x - 5$ 62. $-20x - 216$

I. Simplify.

1. $7x + 3y - 2x$
2. $9y - 5z - 6y - z$
3. $6a + 5b + 1$
4. $7c + 3x - 2c - 5x$
5. $3x - 13y - 7y - x$
6. $8r + 2r - 5t + 9t$
7. $\frac{1}{3}x + \frac{1}{4}y - \frac{2}{3}x + \frac{1}{4}y$
8. $\frac{1}{5}a + \frac{1}{3}b - \frac{1}{3}a - \frac{1}{5}b$
9. $3ab + 7c - 2ab + 5c$
10. $8xy - 3zy - 7yz - 9xy$
11. $4x - 3y - 17y - \frac{1}{5}x$
12. $-x - x - \frac{2}{3}y + \frac{7}{5}x$
13. $2r + 1 + 7s - 3$
14. $8t - 15 - 6 - 3t$
15. $-3m - 2mn + 5m$
16. $-2xy - 3yx - 5x - 6y$
17. $\frac{1}{2}x - \frac{3}{4}y - 2x$
18. $-\frac{1}{5}a - \frac{3}{5}b - \frac{1}{4}a + \frac{1}{5}b$
19. $(3 + k)x - kx$
20. $5 + 3r(s + r) + 5sr$
21. $3 - 2j + 4(2 - j)$
22. $1 + \frac{1}{2}x - \frac{1}{4}x + \frac{1}{8}x - x$

23. $7(a - 3k) + 9k$
24. $5(8r - 2s) - 7s$
25. $2(j - 5m) + j - m$
26. $2(-5r - 8t) - 3r - 7t$
27. $9(X - 2Y) + 3(2X - Y) + 4(-X - Y)$
28. $2(6p - 7r) + 9(r - 2s) + 7(1 + r - s - p)$
29. $-5(1 + 3k) + 5k - 7$
30. $-2(5x + 3y) - y - 7x$
31. $2k(k - 5m) + 5mk + 8$
32. $-3y(-y - 8x) - x(2y + 1 - x)$

33. $9a + 11b - (a - b)$
34. $12r + 7s - (-r - s)$
35. $6n - 11p - (10n - 12p)$
36. $15c - 10d - (13c - 7d)$
37. $25y - (9 - 3z + 8x)$
38. $37g - (8g - 3h - 6) + 5h$
39. $f - (6q + 3f) - 4f$
40. $t - (9t - 7s) - (10s - 8t)$

41. $5x - 3(16 - 4x)$
42. $5(x + 1) - 3x$
43. $3a + 2(3a - 3)$
44. $7x - 4(3 - 2x)$

(continued on next page)

45. $2x + 3(5 - 3x)$

46. $7b - 9(3b + 24)$

47. $2x + 5(-2x)$

48. $3(2x - 2) + 2x$

49. $3z - 2(-5 - 2z)$

50. $4x - (2x + 2)$

51. $3(-1 - 2y) - 2y$

52. $2(y + 4) - 5y$

53. $7x - 3(2x + 1)$

54. $5x + 3(3 - x)$

55. $2(8 + 2b) + 3b$

56. $2(5 - s) + 7s$

57. $2(8 - 3q) - 10q$

58. $3(x - 4) + x$

59. $5(6 - 3y) + 7y$

60. $r - (7 - 3r)$

61. $8x - (5 - 2x)$

62. $7x - 9(3x + 24)$

63. $4a - 3(8 - a)$

64. $3x + 2(19 - 3x)$

65. $x - (11 - 2x)$

66. $4y - 2(3y - 4)$

67. $2(3y) - 5y$

68. $k - [-3k - (-2k - 1)]$

69. $a + b + 2(a - b)$

70. $2(2a + b) - 3(3a - 2b)$

71. $3(3x - 5) + 4(5x - 3)$

72. $4(4n - 1) - 5(n + 2)$

73. $2(5x - 3) - 3(3x + 11)$

74. $4(3s + 5) - 3(s - 3)$

75. $3xx - 2[x - (3 - 2x)]$

76. $2xx - [x - (1 - 2x)]$

77. $7 - [x + (3 - 2x)]$

78. $-(2a - 4) - 18 + a$

79. $5k - 4(k - 1)$

80. $4(z + 1) - 7z$

81. $5 - 3(x - 3)$

82. $4y - (5y - 1)$

83. $8n - 9(3p - 4n)$

84. $11r - 7(-2s - 6r)$

85. $12j - 3(4h - 5j)$

86. $21t - 8(9u - 2t)$

87. $5a - 2b - 7(-4a - 3b)$

88. $6(3c + 8d) - 12(-c - d)$

89. $1.5e - (.5e - 2.5f) - 4(4e - 10f) - 6(-3e - 5f) - 8e$

90. $-7p - q(p - 3n) - 6(-p - 2 - 5n) - q(2 - p - 4n)$

91. $7C - 2(C - D + 3) - 8(3C + 4 - 4D) - 7(2D - 2C)$

92. $-x(13z - 3u) - 3x(6z - 5u)$

93. $-3r(7r - 3s) - 9s(r - 2s)$

63. 7a - 24 64. −3x + 38 65. 3x - 11

66. −2y + 8 67. y 68. 2k - 1

69. 3a - b 70. −5a + 8b 71. 29x - 27

72. 11n - 14 73. x - 39 74. 9s + 29

75. 3xx - 6x + 6 76. 2xx - 3x + 1 77. 4 + x

78. −a - 14 79. k + 4 80. −3z + 4

81. 14 - 3x 82. −y + 1 83. 44n - 27p

84. 53r + 14s 85. 27j - 12h 86. 37t - 72u

87. 33a + 19b 88. 30c + 60d 89. −5e + 72.5f

90. −p + 7nq + 12 + 30n - 2q 91. −5C + 20D - 38

92. −31xz + 18ux 93. −21rr + 18ss 94. −38g + 7h

95. 4d + 45e + 26 96. −33 + 27c - 45b

97. −71 + 43v + 21u 98. 97k - 226n

99. 30abz 100. 8abc 101. −7rrr

102. 6rrr 103. −24ABC 104. 3XYZ

105. 48rrsstt 106. 6aaabbbccc 107. 400tt

108. 0 109. −31r + 43rs - 49rr - 4ss

110. 11jk + 10jj - 21kkj + 5jjk 111. −3uw + 7vw + 2uv - u + v + w

112. 12a + 48aa + 23ab + 5b + 2bb + 1

113. −40x - 15xx + 10 114. 15y - 16yy + 15

115. 13xx + 3x - 4 116. 4xx - 3x - 4

117. 10xyz 118. −7.66 + 49.04x

119. −8.7 + 1.6n 120. −6abc

94. $-14g - 8(-h + 3g) - h$

95. $17d + 19e - 13(d - 2e - 2)$

96. $-4(8 - 2c + 7b) - 6(1 - 3c + 3b) - (-c - b - 5)$

97. $-8(7 - v - 2u) - 5[3 - 4v - 3u - (3v - 2u)]$

98. $-5[4(k - n) - 9(3k - 4n)] - 2[-8(-k - 4n) - (-k - n)]$

99. $-2a(5b)(-3z)$

100. $4a(-b)(-2c)$

101. $6r(-\frac{1}{6}r)(7r)$

102. $7r(-\frac{2}{7}r)(-3r)$

103. $-A \cdot -3B \cdot -8C$

104. $-X \cdot --Y \cdot ---3Z$

105. $2rs \cdot -3st \cdot -8tr$

106. $-abc \cdot -3ab \cdot -2bc \cdot -ca$

107. $(5 \cdot -4 \cdot -t)(-5 \cdot 4 \cdot -t)$

108. $8ab \cdot 1 \cdot 7.53a \cdot 0 \cdot 4ab$

109. $-8r(2 - 5s + 6r) - 3r(5 - s - r) - 4(rr + ss)$

110. $-2j(3k - 5j) + 7kj(1 - 3k) - 5jk(-2 - j)$

111. $-3w(u - 2v) - v(-w - 2u) - (u - v - w)$

112. $8a[5 - 3(1 - 2a - b)] - [4a - 1 - b(5 - a + 2b)]$

113. $9(-3x) - 2(-4x - 5) - 3x[2 - 5(-1 - x)]$

114. $4y(3 - 2y) + 7(1 - yy) - y[-(3 - y)] + 8$

115. $3(xx - 2x - 3) - 2x(-5x - 3) - 3(-x - 2) - 1$

116. $-(x - 3)(-2x) - (-xx - x + 1) - (3 - 2x - xx)$

117. $\frac{1}{2}(-4x)(-8y)z - \frac{2}{9}(3y)(-2z)(-3x) - \frac{1}{7}(-14xy)(-z)$

118. $2.5(3 - 1.2x) - 7.8(4.1 - 6.3x) - 2.9(-x - 5.8)$

119. $\frac{2}{3}(12 - .6n) - \frac{1}{9}(18 - 2.7) - \frac{5}{6}(18 - 2.4n)$

120. $.3(2a)(-2b)c - .2(-3c)(-8b)(-a) - .4(-2b)(3a)(-4c)$

J. Simplify.

1. $\frac{3}{5} \times 5$

2. $\frac{{}^-3}{{}^+4} \times {}^+4$

3. $\frac{6}{7} \times (3 + 4)$

4. ${}^-5 \times \frac{4}{{}^-5}$

5. $81 \times \frac{17}{100 - 19}$

6. ${}^+5 \times \frac{4}{7 - 12}$

7. $\frac{1}{2} \times \frac{9}{1/2}$

8. $\frac{3}{4} \times \frac{5}{7}$

9. $\frac{2}{5} \times \frac{8}{3}$

10. $\frac{{}^-7}{{}^-5} \times \frac{{}^+2}{{}^-3}$

11. $-\frac{4}{9} \cdot -\frac{4}{9}$

12. $\frac{17}{9} \times \frac{101}{102}$

13. $4 \div \frac{3}{5}$

14. ${}^-7 \div \frac{5}{8}$

15. ${}^-2 \div \frac{{}^-3}{{}^-7}$

16. $\frac{4 \times 9}{9}$

17. $\frac{{}^-8 \times {}^-5}{{}^-5}$

18. $\frac{{}^+6 \times 2/5}{2/5}$

19. $\frac{7 \times 4}{9 \times 4}$

20. $\frac{38}{42}$

21. $\frac{{}^-160}{{}^-970}$

22. $7 \times \frac{5}{3}$

23. $4 \times \frac{1}{5}$

24. $7 \times \frac{1/7}{15}$

25. $\frac{4}{5} \div 3$

26. $\frac{{}^+5}{{}^+17} \div {}^-2$

27. $\frac{{}^-1}{{}^-3} \div 2$

28. $\frac{3}{4} \div \frac{2}{7}$

29. $\frac{{}^-8}{5} \div \frac{1}{{}^-3}$

30. $\frac{7}{{}^-3} \div \frac{4}{{}^-7}$

31. $-\frac{5}{4} \div \frac{31}{49}$

32. $\frac{2 - 7}{4} \div \frac{3}{4 + 7}$

33. $\frac{12}{5} \div \frac{61}{5}$

34. $\frac{13}{33} \div \frac{{}^-2}{33}$

35. $\frac{6 \cdot 4}{5 \cdot 7} \div \frac{11}{5 \cdot 7}$

36. $\frac{15}{5 + 3} \div \frac{17}{2 \cdot 4}$

37. $\dfrac{\frac{16}{3}}{\frac{15}{3}}$

38. $\dfrac{\frac{18}{7}}{\frac{{}^-43}{5 + 2}}$

39. $\dfrac{\frac{{}^-65}{12}}{\frac{{}^+65}{4 \cdot 3}}$

40. $\frac{4}{5} + \frac{3}{5}$

41. $\frac{17}{32} + \frac{28}{32}$

42. $\frac{14 - 3}{19} + \frac{14 + 3}{19}$

J.	1.	3	2.	$^{-}3$	3.	6	4.	4
	5.	17	6.	$^{-}4$	7.	9	8.	$\frac{15}{28}$
	9.	$\frac{16}{15}$	10.	$-\frac{14}{15}$	11.	$\frac{16}{81}$	12.	$\frac{101}{54}$
	13.	$\frac{20}{3}$	14.	$-\frac{56}{5}$	15.	$-\frac{14}{3}$	16.	4
	17.	-8	18.	6	19.	$\frac{7}{9}$	20.	$\frac{19}{21}$
	21.	$\frac{16}{97}$	22.	$\frac{35}{3}$	23.	$\frac{4}{5}$	24.	$\frac{1}{15}$
	25.	$\frac{4}{15}$	26.	$-\frac{5}{34}$	27.	$\frac{1}{6}$	28.	$\frac{21}{8}$
	29.	$\frac{24}{5}$	30.	$\frac{49}{12}$	31.	$-\frac{245}{124}$	32.	$-\frac{55}{12}$
	33.	$\frac{12}{61}$	34.	$-\frac{13}{2}$	35.	$\frac{24}{11}$	36.	$\frac{15}{17}$
	37.	$\frac{16}{15}$	38.	$-\frac{18}{43}$	39.	-1	40.	$\frac{7}{5}$
	41.	$\frac{45}{32}$	42.	$\frac{28}{19}$	43.	$\frac{33}{17}$	44.	$\frac{33}{17}$
	45.	2	46.	$\frac{29}{21}$	47.	$\frac{67}{52}$	48.	$\frac{31}{21}$
	49.	$\frac{11}{40}$	50.	$\frac{11}{40}$	51.	$-\frac{1}{6}$	52.	-5
	53.	-1	54.	-1	55.	-5	56.	$-\frac{1}{2}$
	57.	1	58.	5	59.	2	60.	$-\frac{1}{4}$
	61.	7	62.	11	63.	9379	64.	$\frac{12}{11}$
	65.	$\frac{1}{4}$	66.	1	67.	$\frac{56}{95}$	68.	$\frac{38}{11}$
	69.	$\frac{44}{145}$	70.	20	71.	$\frac{441}{20}$	72.	$-\frac{322}{27}$

43. $\frac{65}{17} + \frac{^-32}{17}$

44. $\frac{65}{17} - \frac{32}{17}$

45. $\frac{43}{15} - \frac{13}{15}$

46. $\frac{2}{3} + \frac{5}{7}$

47. $\frac{3}{4} + \frac{7}{13}$

48. $\frac{5}{6} + \frac{9}{14}$

49. $\frac{7}{8} + \frac{^-3}{5}$

50. $\frac{7}{8} - \frac{3}{5}$

51. $\frac{1}{3} - \frac{1}{2}$

52. $\frac{-10}{2}$

53. $\frac{-4}{3} \div \frac{4}{3}$

54. $\frac{5 \times -2}{5 \times 2}$

55. $\frac{10}{^-2}$

56. $\frac{3}{4} + \frac{5}{-4}$

57. $\frac{3}{8} - \frac{5}{^-8}$

58. $\frac{-10}{-2}$

59. $\frac{3}{4} + \frac{-5}{-4}$

60. $\frac{3}{8} - \frac{-5}{-8}$

61. $\frac{7 \cdot 5 + 7 \cdot 2}{7}$

62. $\frac{60 + 6}{6}$

63. $\frac{3 \cdot 9378 + 3}{3}$

64. $\frac{9 + 3}{9 + 2}$

65. $\frac{16}{64}$

66. $\frac{10 + 3}{5 \cdot 2 + 3}$

<u>Sample</u> $5\frac{1}{3} \div 3\frac{1}{5}$

<u>Solution.</u>

$$5\frac{1}{3} \div 3\frac{1}{5}$$

$$= \frac{16}{3} \div \frac{16}{5}$$

$$= \frac{16}{3} \times \frac{5}{16}$$

$$= \frac{5}{3}$$

67. $4\frac{1}{5} \div 7\frac{1}{8}$

68. $9\frac{1}{2} \div 2\frac{3}{4}$

69. $^-2\frac{1}{5} \div {}^-7\frac{1}{4}$

70. $2\frac{1}{7} \times 9\frac{1}{3}$

71. $5\frac{1}{4} \times 4\frac{1}{5}$

72. $^-2\frac{1}{3} \times {}^+5\frac{1}{9}$

(continued on next page)

73. 82% of 424
74. 62.5% of 848
75. 0.375% of 16
76. 15% of ? is 9
77. 27% of ? is 51
78. 6.3% of ? is 126
79. 35 is ? % of 140
80. 12 is ? % of 75
81. 94 is ? % of 23.5
82. $\frac{1}{5} + 0.6$
83. $9.35 + \frac{3}{2}$
84. $8.375 - \frac{5}{8}$
85. 4.87×0.02
86. 3.92×2.81
87. 5.31×0.00025
88. $28.8 \div 1.2$
89. $6250 \div 0.025$
90. $0.00441 \div 30$
91. $9.02 \times 7\frac{1}{4}$
92. $0.038 \times 2\frac{1}{9}$
93. $8\frac{1}{7} \times 7.077$
94. $8.3(2\frac{1}{3} + 5.04)$
95. $8.3 \div (2\frac{1}{3} + 5.04)$
96. $15\frac{1}{7} - 6.21$
97. $\dfrac{8 + \frac{1}{3}}{4 + \frac{1}{5}}$
98. $\dfrac{\frac{1}{2} + \frac{1}{7}}{2}$
99. $\dfrac{3 - \frac{1}{4}}{\frac{1}{2} - \frac{3}{8}}$
100. $\dfrac{\frac{2}{9} \div \frac{1}{4}}{\frac{8}{3} - \frac{1}{2}}$
101. $\dfrac{\frac{5}{12} - \frac{1}{6}}{\frac{2}{3} + \frac{1}{4}}$
102. $\dfrac{\frac{1}{2} - \frac{1}{3} \div \frac{5}{8} - \frac{7}{12}}{\frac{3}{8} - \frac{1}{6} + \frac{5}{12} - \frac{7}{2}}$
103. $\dfrac{-\frac{3}{4}}{8}$
104. $\dfrac{1}{\frac{3}{5}}$
105. $\dfrac{\frac{5}{6}}{3}$
106. $\frac{5}{6} \div \dfrac{-1}{\frac{2}{3}}$
107. $\frac{1}{2} \div \dfrac{4}{\frac{5}{12}}$
108. $\dfrac{\frac{2}{3}}{\frac{4}{9}} \div \dfrac{\frac{1}{5}}{\frac{3}{10}}$
109. $\dfrac{-\frac{2}{5} \times \frac{3}{4} \times \frac{2}{3}}{-\frac{1}{5} \times -\frac{2}{3} \times -\frac{6}{7}}$
110. $\dfrac{\frac{5}{9} \times -\frac{3}{10}}{\frac{5}{9} + -\frac{3}{10}}$
111. $\dfrac{\frac{1}{3} + \frac{1}{7}}{\frac{1}{3} \times \frac{1}{7}}$
112. $\dfrac{1}{\frac{1}{5} + \frac{1}{7}}$
113. $\dfrac{1}{\frac{2}{3} - \frac{6}{7}}$
114. $\dfrac{\frac{2}{9} - \frac{3}{5}}{\frac{2}{9} + \frac{3}{5}}$

An expression such as '82% of ...' should be translated into '... × 0.82'. [See page TC[1-59].]

73. 347.68	74. 530	75. 0.06

*

Exercise 76 can be translated into 'What number multiplied by 0.15 is 9?'. By the principle of quotients, one such number is $9 \div 0.15$. And, by the division theorem, this is the only such number. [An intuitive method of handling the problem is to note that since 15% of the number in question is 9, 5% of it is 3. So, 100% of it is 60.]

76. 60	77. $188\frac{8}{9}$	78. 2000

*

Translate Exercise 79 into '140 multiplied by what number times $\frac{1}{100}$ is 35?'. This number is $\frac{35}{140} \times 100$.

79. 25	80. 16	81. 400

*

82. 0.8	83. 10.85	84. 7.75	85. 0.0974
86. 11.0152	87. 0.0013275	88. 24	89. 250000
90. 0.000147	91. 65.395	92. $\frac{361}{4500}$	93. 57.627
94. $\frac{45899}{750}$	95. $\frac{1245}{1106}$	96. $\frac{6253}{700}$	97. $\frac{125}{63}$
98. $\frac{9}{28}$	99. 22	100. $\frac{17}{78}$	101. $\frac{3}{11}$
102. $-\frac{5}{69}$	103. $-\frac{3}{32}$	104. $\frac{5}{3}$	105. $\frac{5}{18}$
106. $-\frac{5}{9}$	107. $\frac{5}{96}$	108. $\frac{9}{4}$	109. $\frac{7}{4}$
110. $-\frac{15}{23}$	111. 10	112. $\frac{35}{12}$	113. $-\frac{21}{4}$
114. $-\frac{17}{37}$			

K.

1. $3a$ 2. $7n$ 3. $6c$ 4. $6d$
5. $17r$ 6. $11rs$ 7. $-8xyz$ 8. $41ab$
9. $27ef$ 10. $\frac{11}{3}r$ 11. $-14n$ 12. $41ab$
13. $1200d$ 14. $-11.1x$ 15. $4p$, $[n \neq 0]$
16. $8s$, $[r \neq 0]$ 17. $3a$, $[ad \neq 0]$ 18. $\frac{3d}{c}$, $[ac \neq 0]$
19. 1, $[rst \neq 0]$ 20. $13ee$, $[f \neq 0]$ 21. -21, $[w \neq 0]$
22. $\frac{4c}{d}$, $[cde \neq 0]$ 23. $\frac{-9}{u}$, $[xu \neq 0]$ 24. -1, $[n \neq 0]$
25. $\frac{16e}{d}$, $[de \neq 0]$ 26. $\frac{13dx}{uv}$, $[cuv \neq 0]$
27. $\frac{6xk}{yj}$, $[yj \neq 0]$ 28. $\frac{2aa}{3bd}$, $[bcd \neq 0]$
29. $\frac{12x}{5y}$, $[rsy \neq 0]$ 30. $\frac{y}{3z}$, $[abcxz \neq 0]$
31. $\frac{-5r}{3s}$, $[jks \neq 0]$ 32. ac, $[bcrs \neq 0]$
33. $\frac{2}{27}$, $[y \neq 0]$ 34. $\frac{c}{5b}$, $[abcx \neq 0]$
35. $\frac{-4cd}{bb}$, $[abcd \neq 0]$ 36. 4, $[x \neq 9]$
37. $\frac{3}{2}$, $[x \neq -2,\ y \neq 5]$ 38. 15, $[a \neq -4,\ -3]$
39. $\frac{3(y-1)}{2(y-5)}$, $[y \neq 2,\ 3,\ 5]$ 40. $\frac{x+2}{6(x-3)(x-3)}$, $[x \neq 3]$
41. -6, $[x \neq 3,\ -3]$

K. Simplify.

1. $\frac{39a}{13}$ 2. $\frac{63n}{9}$ 3. $\frac{42c}{7}$ 4. $\frac{-18d}{-3}$

5. $\frac{-34r}{-2}$ 6. $\frac{44rs}{4}$ 7. $\frac{96xyz}{-12}$ 8. $\frac{-41ab}{-1}$

9. $\frac{9ef}{\frac{1}{3}}$ 10. $\frac{-11r}{-3}$ 11. $\frac{-98n}{7}$ 12. $\frac{-205ab}{-5}$

13. $\frac{-960d}{-.8}$ 14. $\frac{-33.3x}{3}$

15. $\frac{16np}{4n}$ 16. $\frac{24rs}{3r}$ 17. $\frac{12aad}{4ad}$ 18. $\frac{15ad}{5ac}$

19. $\frac{8rst}{8rst}$ 20. $\frac{-52eef}{-4f}$ 21. $\frac{21w}{-w}$ 22. $\frac{28ccde}{7cdde}$

23. $\frac{-18xu}{2xuu}$ 24. $\frac{11nn}{-11nn}$ 25. $\frac{-48dee}{-3dde}$ 26. $\frac{-26cdx}{-2cuv}$

27. $\frac{8x}{3y} \times \frac{9k}{4j}$ 28. $\frac{3ab}{7cd} \times \frac{14ac}{9bb}$ 29. $\frac{-8r}{-4s} \times \frac{6xs}{5yr}$

30. $\frac{-5ab}{9xc} \times \frac{3cxy}{-5abz}$ 31. $\frac{-2k}{-3j} \times \frac{5rj}{-2sk}$ 32. $\frac{abcc}{-rs} \times \frac{-sr}{bc}$

33. $\frac{1}{3y} \div \frac{9}{2y}$ 34. $\frac{x}{2a} \div \frac{-5xb}{-2ac}$ 35. $\frac{-3a}{-2b} \div \frac{-3ab}{8cd}$

36. $\frac{x - 9}{3} \times \frac{12}{x - 9}$ 37. $\frac{x + 2}{3(y - 5)} \times \frac{9(y - 5)}{2(x + 2)}$

38. $\frac{18(a + 3)}{5(a + 4)} \times \frac{25(a + 4)}{6(a + 3)}$ 39. $\frac{2(y - 1)(y - 3)}{9(y - 2)} \times \frac{27(y - 2)}{4(y - 3)(y - 5)}$

40. $\frac{7(x + 2)}{9(x - 3)} \div \frac{14(x - 3)}{3}$ 41. $\frac{8(3 - x)}{5(3 + x)} \div \frac{4(x - 3)}{15(x + 3)}$

(continued on next page)

42. $\frac{6a - 18}{6}$

43. $\frac{5x - 5}{-5}$

44. $\frac{3.9b - 7.8}{1.3}$

45. $\frac{2b + 50c}{2}$

46. $\frac{3}{5}(20a - 5b)$

47. $\frac{1}{9}(-9n - 63t)$

48. $\frac{2}{7}(28c - 49d)$

49. $(8 - 32s)\frac{3}{8}$

50. $\frac{7gh - 11}{1/3}$

51. $\frac{6c + 6d + 6e}{-6}$

52. $\frac{14nr + 2ns}{-2}$

53. $(12g - 26k) \div \frac{1}{2}$

54. $\frac{8xy + 8}{8}$

55. $\frac{-5 - 5kj}{-5}$

56. $\frac{\frac{1}{3}a + \frac{2}{3}y}{3}$

57. $\frac{6rs - 11rt}{r}$

58. $\frac{5nq - 8nqr}{-nq}$

59. $\frac{9effg + 27efg}{3efg}$

60. $\frac{45abc + 5c}{5c}$

61. $\frac{21cde - 7c}{7c}$

62. $\frac{24g - 8e}{-4}$

63. $\frac{7(a + 5)}{a + 5}$

64. $\frac{(n - 3)(n + 5)}{n - 3}$

65. $\frac{(a - 4)(a + 7)}{-(a - 4)}$

66. $\frac{(c - 8)(c + 9)}{8 - c}$

67. $(x - 3) \cdot \frac{3x}{x - 3}$

68. $5(a + 7) \cdot \frac{a}{a + 7}$

69. $15\left(\frac{n}{3} + \frac{n}{5}\right)$

70. $14\left(\frac{3c}{7} - \frac{9c}{2}\right)$

71. $10\left(\frac{4}{5}a - \frac{7}{2}b\right)$

72. $21\left(\frac{5r}{3} + \frac{5r}{7}\right)$

73. $32\left(\frac{3x}{4} - \frac{5y}{2}\right)$

74. $18\left(\frac{n}{9} - \frac{n}{6}\right)$

75. $12\left(\frac{d + 1}{4} + \frac{d + 2}{6}\right)$

76. $16\left(\frac{e + 4}{2} - \frac{e + 1}{8}\right)$

77. $8\left(\frac{3n}{2} - \frac{5n}{4}\right)$

78. $24\left(\frac{3k + 2}{4} - \frac{4k + 1}{6} - \frac{2k - 3}{8}\right)$

79. $28\left(\frac{3u}{7} - \frac{5u}{2}\right)$

80. $15\left[\frac{1}{5}(d + 3) - 5 - \frac{d - 6}{3} + \frac{1}{5}\right]$

81. $9\left(\frac{d + 13}{9} - \frac{12 - d}{3}\right)$

82. $36\left[\frac{t - 5}{9} - \frac{8 - t}{4} + \frac{t}{12}\right]$

83. $r\left(\frac{5}{r} - 1\right)$

84. $3x\left(\frac{7}{3x} - 4\right)$

85. $5(s - 3)\left(\frac{4}{s - 3} + 2\right)$

86. $7x\left(\frac{3}{x} - \frac{1}{7}\right)$

87. $10d\left(\frac{5}{d} + \frac{4}{5d}\right)$

88. $9g\left(\frac{7}{9} - \frac{5}{g}\right)$

42. $a - 3$
43. $1 - x$
44. $3b - 6$
45. $b + 25c$
46. $12a - 3b$
47. $-n - 7t$
48. $8c - 14d$
49. $3 - 12s$
50. $21gh - 33$
51. $-c - d - e$
52. $-7nr - ns$
53. $24g - 52k$
54. $xy + 1$
55. $1 + kj$
56. $\frac{1}{9}a + \frac{2}{9}y$ [or: $\frac{a + 2y}{9}$]
57. $6s - 11t$, $[r \neq 0]$
58. $-5 + 8r$, $[nq \neq 0]$
59. $3f + 9$, $[efg \neq 0]$
60. $9ab + 1$, $[c \neq 0]$
61. $3de - 1$, $[c \neq 0]$
62. $-6g + 2e$
63. 7, $[a \neq -5]$
64. $n + 5$, $[n \neq 3]$
65. $-(a + 7)$, $[a \neq 4]$
66. $-c - 9$, $[c \neq 8]$
67. $3x$, $[x \neq 3]$
68. $5a$, $[a \neq -7]$
69. $8n$
70. $-57c$
71. $8a - 35b$
72. $50r$
73. $24x - 80y$
74. $-n$
75. $5d + 7$
76. $6e + 30$
77. $2n$
78. $-4k + 17$
79. $-58u$
80. $-2d - 33$
81. $4d - 23$
82. $16t - 92$
83. $5 - r$, $[r \neq 0]$
84. $7 - 12x$, $[x \neq 0]$
85. $10s - 10$, $[s \neq 3]$
86. $21 - x$, $[x \neq 0]$
87. 58, $[d \neq 0]$
88. $7g - 45$, $[g \neq 0]$

89. $3a - 7$, $[a \neq -7]$

90. $28 - 22r$, $[r \neq 0]$

91. $3d + 35$, $[d \neq -5]$

92. $4z - 19$, $[z \neq -2]$

93. $-11k + 35$, $[k \neq 0, -5]$

94. $7ee - 63e + 315$, $[e \neq 0, 5]$

95. $2b - 2$, $[b \neq 0, -2]$

96. $36vv - 6v + 15$, $[v \neq 0, \frac{5}{2}]$

97. $\frac{-3a}{28}$

98. $\frac{29x}{15}$

99. $\frac{25r - 3}{6}$

100. $\frac{11d}{6}$

101. $\frac{6h - 1}{12}$

102. $\frac{17m - 12}{12}$

103. $\frac{22}{3g}$, $[g \neq 0]$

104. $\frac{27}{2p}$, $[p \neq 0]$

105. $\frac{38 + 15s}{3s}$, $[s \neq 0]$

106. $\frac{4 + 270b}{30b}$, [or: $\frac{2 + 135b}{15b}$], $[b \neq 0]$

107. $\frac{20c - 69}{15c}$, $[c \neq 0]$

108. $\frac{16j + 68}{45}$

109. $\frac{16j + 68}{45}$

110. $\frac{4t + 30}{21}$

111. $\frac{-17n - 28}{60}$

112. $\frac{z + 25}{12}$

113. $\frac{6 - u}{12}$

114. $\frac{e + 7}{8}$

115. $\frac{-A - 15}{24}$

116. $\frac{B + 54}{10}$

117. $\frac{-3f + 38}{8}$

89. $4(a + 7)\left(\frac{a}{a + 7} - \frac{1}{4}\right)$

90. $30r\left(\frac{3r + 1}{3r} - \frac{5r - 3}{5r} - \frac{11}{15}\right)$

91. $4(d + 5)\left(\frac{7}{4} - \frac{d}{d + 5}\right)$

92. $9(z + 2)\left(\frac{z - 1}{z + 2} - \frac{5}{9}\right)$

93. $6k(k + 5)\left(\frac{7}{6k} - \frac{3}{k + 5}\right)$

94. $7e(e - 5)\left(\frac{e}{e - 5} - \frac{9}{e}\right)$

95. $b(b + 2)\left(\frac{3}{b + 2} - \frac{1}{b}\right)$

96. $12v(2v - 5)\left(-\frac{1}{4v} + \frac{3v}{2v - 5}\right)$

97. $\frac{a}{7} - \frac{a}{4}$

98. $\frac{3x}{5} + \frac{4x}{3}$

99. $\frac{11r}{3} - \frac{1}{2} + \frac{r}{2}$

100. $\frac{5d}{3} + \frac{d}{6}$

101. $\frac{3h}{6} - \frac{1}{12}$

102. $\frac{m}{4} + \frac{m}{6} + m - 1$

103. $\frac{9}{g} - \frac{5}{3g}$

104. $\frac{10}{p} + \frac{7}{2p}$

105. $\frac{13}{s} - \frac{1}{3s} + 5$

106. $\frac{3}{5b} - \frac{1}{3b} + 9 - \frac{4}{30b}$

107. $1 - \frac{3}{5c} + \frac{1}{3} - \frac{4}{c}$

108. $\frac{5j + 1}{9} - \frac{j - 7}{5}$

109. $\frac{1}{9}(5j + 1) - \frac{1}{5}(j - 7)$

110. $\frac{t + 6}{3} - \frac{t + 4}{7}$

111. $\frac{5n + 1}{3} - \frac{3n + 4}{4} - \frac{6n - 1}{5}$

112. $\frac{z + 1}{3} - \frac{z - 7}{4}$

113. $\frac{5u - 2}{4} - \frac{4u - 3}{3}$

114. $\frac{e - 5}{4} + 3 - \frac{e + 7}{8}$

115. $\frac{A - 3}{8} - \frac{A - 3}{6} - \frac{3}{4}$

116. $7 + \frac{B - 2}{5} - \frac{B + 12}{10}$

117. $\frac{f + 8}{8} + \frac{1}{4} - \frac{f - 5}{2} + 1$

(continued on next page)

118. $\frac{3}{n-4} - \frac{5}{n+2}$

119. $\frac{10}{a-3} - \frac{9}{a-5}$

120. $\frac{6}{c-3} + \frac{3}{c-6}$

121. $\frac{y+5}{y-3} + \frac{4}{y-3} - 5$

122. $\frac{7}{d} + \frac{5}{d-3}$

123. $\frac{12}{3y+2} - \frac{7}{y} + \frac{6}{3y+2}$

124. $\frac{10}{g-5} + \frac{4}{g+7}$

125. $\frac{5}{8} - \frac{e}{e+3} + \frac{e}{5}$

126. $\frac{r+2}{r-7} + \frac{4}{5} + \frac{3}{r-7}$

127. $\frac{t}{t+5} - \frac{3t}{t+6} - \frac{4t}{t+5}$

128. $\dfrac{\frac{5}{3} - \frac{1}{5x}}{\frac{1}{5x} + \frac{1}{6}}$

129. $\dfrac{\frac{1}{f} - \frac{1}{g}}{\frac{1}{f} - \frac{1}{g}}$

130. $\dfrac{5 - \frac{3}{4r}}{6 + \frac{1}{2r}}$

131. $\dfrac{2}{\frac{1}{k} + \frac{1}{m}}$

132. $\dfrac{x - \frac{1}{3y}}{y - \frac{1}{5x}}$

133. $\dfrac{a+b}{\frac{2}{a} - \frac{3}{b}}$

134. $\dfrac{\frac{2}{x}}{\frac{3}{y} + \frac{4}{z}}$

135. $\dfrac{1}{3 - \dfrac{1}{3 - \dfrac{1}{3-1}}}$

118. $\frac{-2n + 26}{(n - 4)(n + 2)}$, $[n \neq 4, -2]$

119. $\frac{a - 23}{(a - 3)(a - 5)}$, $[a \neq 3, 5]$

120. $\frac{9c - 45}{(c - 3)(c - 6)}$, $[c \neq 3, 6]$

121. $\frac{24 - 4y}{y - 3}$, $[y \neq 3]$

122. $\frac{12d - 21}{d(d - 3)}$, $[d \neq 0, 3]$

123. $\frac{-3y - 14}{y(3y + 2)}$, $[y \neq 0, -\frac{2}{3}]$

124. $\frac{14g + 50}{(g - 5)(g + 7)}$, $[g \neq 5, -7]$

125. $\frac{8ee + 9e + 75}{40(e + 3)}$, $[e \neq -3]$

126. $\frac{9r - 3}{5(r - 7)}$, $[r \neq 7]$

127. $\frac{-6tt - 33t}{(t + 5)(t + 6)}$, $[t \neq -5, -6]$

128. $\frac{50x - 6}{6 + 5x}$, $[x \neq 0, -\frac{6}{5}]$

129. 1, $[f \neq 0 \neq g \neq f]$

130. $\frac{20r - 3}{24r + 2}$, $[r \neq 0, -\frac{1}{12}]$

131. $\frac{2km}{k + m}$, $[k \neq 0 \neq m \neq -k]$

132. $\frac{15xxy - 5x}{15xyy - 3y}$, $[y \neq 0 \neq x,\ 15xyy - 3y \neq 0]$

133. $\frac{aab + abb}{2b - 3a}$, $[a \neq 0 \neq b,\ 2b - 3a \neq 0]$

134. $\frac{2yz}{3xz + 4yx}$, $[y \neq 0 \neq z,\ 3xz + 4yx \neq 0,\ x \neq 0]$

135. $\frac{5}{13}$