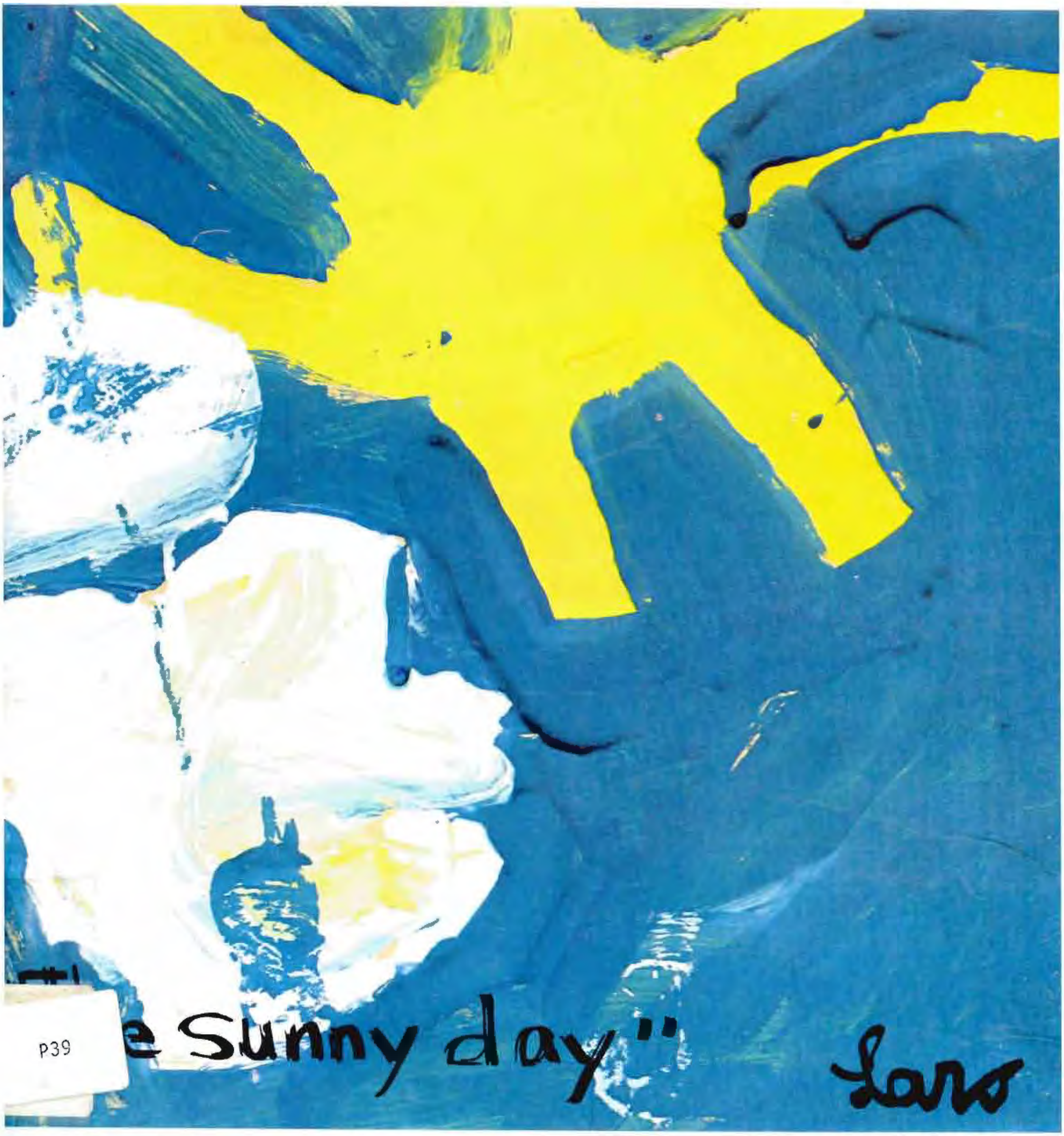
CDA math
ROBERT W. WIRTZ

MATHEMATICS FOR EVERYONE



P39

A Guide to CDA Math

Mathematics for Everyone						
Individualized Computation	a_1	b_1	c_1	d_1	e_1	f_1
	a_2	b_2	c_2	d_2	e_2	f_2
Patterns and Problems	a	b	c	d	e	f
Objectives and Criterion Referenced tests	My Progress a,b,and c			My Progress d,e,and f		
Drill and practice at the Problem solving level	An Alternative	pupil activity pages	tape cassettes			
Banking on Problem Solving	comments for teachers	pupil activity pages				

Cover Art

The Kay Goines Kindergarten class at River School.

The Kindergarten class of Pat Spencer at La Mesa School.

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Robert W. Wirtz

CDA math

A Friendly Math Associates Project

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TABLE OF CONTENTS

To the memory of Max Beberman

FOREWORD

The table of contents for *Mathematics for Everyone* can be thought of as a map, each chapter serving as a springboard for examining various aspects of the teaching of elementary school mathematics. As with any map, there is no one right way to travel, rather possibilities are presented. The "traveler" is best able to determine where he wants to go. Such determinations must be made as they are responsive to the needs of the communities of parents, teachers and administrators working together to develop a more humanistic mathematics program for their children.

In a very real sense, *Mathematics for Everyone* can be used to support all efforts to reinforce elementary school mathematics. Much of what is contained in *Mathematics for Everyone* can best be understood by those who are familiar with CDA Math. References to specific pages of instructional materials are not included: rather they can be found in annotated editions available to parents and teachers.

The chapter order I chose, as indicated by the Table of Contents, reflects my particular bias or sense of progression—the direction I chose to follow. Those in your community may very well decide to travel in different directions. For instance, you might choose to skip several sections and first consider "learning theory." Others may decide to begin with a look at the general outline of CDA Math materials and then see how those materials might fit into their current staff development program in which they are currently involved. There is no right order in which to read *Mathematics for Everyone*. The object of *Mathematics for Everyone* is satisfied if you who read it find a direction and content that will assist members of your community in working together to formulate and implement specific solutions to your problems, and find possibilities for a more rewarding educational experience in mathematics in your community.

Robert W. Wirtz

PART ONE

a. COMMUNITY PARTICIPATION

An Overview

The "Community Participation" we are talking about is not a well-defined program or a plan of action. It follows no set form or pattern. It is an opportunity that each community can respond to in its own way.

Two examples from my own personal experience may help get at the notion of how communities can work together: one had a "mathematical setting" and the other was far more general and probably more germane.

Brownsville was selected by the Texas State Department of Education to pilot an experiment in looking at elementary school arithmetic from a "problem solving" point of view.

One episode centered around finding ways to arrange four 4's so they indicate different numbers. For example:

$44 + 44 = 88$	$4+4+4-4=8$
$44 + 4 - 4 = 44$	$(4+4)\times(4+4)=64$
$4+4+4+4=16$	$(4-4)\times(4+4)=0$

This activity leads to a classical problem called "four 4's." **Can expressions using four 4's be found for all numbers 0 through 100?**

"Four 4's" is much too large a problem for one individual to undertake, but the problem offers an interesting opportunity for each member of a group to make a personal contribution. As an expression using four 4's is found for any whole number, 0 through 100, it is contributed to a group list.

Here are a few other random examples:

$(4\times4\times4)-4=60$	$(4\times4)+(4+4)=24$
$(44\div4)+4=15$	$(4\div4)\times44=44$
$(4\times4)+(4\times4)=32$	$(4+4)\times(4\div4)=8$

In the beginning, the list grows rapidly, and before long there are entries for 25 or 35 different numbers. But the flow soon slows down. Calls for help go out. Brothers, sisters, parents, friends are asked to join the search.

Someone is sure to suggest using the fact that

$$\sqrt{4}=2$$

"the square root of 4 equals 2." Beginners are not familiar with this symbol and it must be explained that "$\sqrt{4}$" is a special name for a number which when multiplied by itself equals 4, and $\sqrt{4}$ - 2 because $2 \times 2 = 4$.

This can lead to such expressions as:

$4+44+\sqrt{4}=50$	$\sqrt{4}+\sqrt{4}+\sqrt{4}+\sqrt{4}=8$
$(44\div4)+\sqrt{4}=13$	$(44\times\sqrt{4})-4=84$

This is just one of the expressions the more mathematically experienced members of the community may offer. Another is 4! (four factorial).

$$4!=4\times3\times2\times1=24$$
$$(3!=3\times2\times1 \text{ and } 5!=5\times4\times3\times2\times1,\text{etc.})$$
$$4\times4!=96$$
$$4\times4!+(4\div4)=97$$
$$4\times4!-(4\div4)=95$$

This community-wide search was successfully carried out in Brownsville. The episode that most often comes to mind happened when I went to the Board of Education office to meet with the Curriculum Director.

I gave the receptionist my name and explained I had an appointment. She looked at me for a moment and said: "Oh, you're from the math group and you can't see the Director until you tell me how to write 13 with four 4's."

At that time, I knew of only one way to get past the receptionist, and it involved decimals.

On a piece of paper I wrote:

$$4-.4=3.6$$
$$3.6\div.4=9$$
$$\text{so, } (4-.4)\div.4=9$$

"Is that enough?" I asked.

She looked at the paper. "That's right" was followed by a long pause.

Then came the explosion. She jumped to her feet and almost shouted. ". . . and 9 : 4 - 13; I've got it."

She opened the door with a flourish: "Please follow me; he's expecting you."

The other episode has to do with a Thanksgiving Dinner.

Several women were talking over coffee about the extent to which commercialism has tended to distort the original reasons for holidays.

Thanksgiving was a few weeks ahead — a long weekend and an opportunity to over eat in a traditional way.

"Let's do something different. Perhaps we can recapture some of the spirit of the first Thanksgiving."

The original plan was to organize a potluck Thanksgiving dinner in a church basement — everyone invited to attend and share with others. No admission, no speeches, no entertainment — just get together, eat together and talk together.

An announcement in the form of a leaflet was prepared and distributed through friends. Soldiers from a nearby army post were invited; notices were put on bulletin boards in homes for senior citizens. Transportation was arranged for those who needed it.

Even though there was almost no coverage by the press, the response was such that the site of the dinner had to be moved to the dining hall of the community's fair grounds.

So much food was promised that plans had to be made in advance to distribute the surplus to needy families in the community. Someone contributed a tank of helium gas so there would be lighter than air balloons for the children. Tables were decorated with branches and fruit and nuts.

The hall was filled to capacity.

While there was a steady flow of turkeys from the kitchen, the tables were filled with foods representing all the many ethnic groups in the community.

"What a nice idea" was heard over and over again.

The originators of "let's do something different" were tired but pleased.

Since then, the Community Thanksgiving Dinner has become a tradition in this community. The only way to accommodate the growing number of people who want to participate will be for others in the area to arrange local Community Thanksgiving Dinners.

There is a nation-wide groundswell beginning to surface — people feel a growing need to do things with others. They are becoming uncomfortable with the extent to which their lives are being manipulated by "outside forces" . . . such as commercialism's distortion of our holidays. They feel cheated that the old fashioned notion of neighbors living together and helping each other and having interests in common has somehow been replaced by impersonal "people who live on the block."

"Community participation," as we use it here, is an opportunity to work together in areas where there is general concern . . . not leaving it up to the experts who seem unable to meet the needs, not complaining that someone else is lagging; rather, finding a way, however difficult, to join in with others to make a difference.

Mathematics for Everyone speaks of a goal that's been missing for a long time . . . to replace the image of mathematics as too dull and too hard with a growing understanding that mathematics can be a source of enjoyment for all, thereby recouping a loss of our heritage.

The experts have failed in their efforts to help children develop the computation skills measured by standardized achievement tests. Untold millions of dollars of public funds have been expended without any general improvement. Successes that would tend to brighten the generally dismal picture are usually traceable to outstanding efforts by an individual or group of dedicated teachers whose achievements could spread if the parents were more involved in these efforts.

Mathematics for Everyone and all of the CDA Math materials are designed as resources to help broaden the opportunity for community participation in the search for more effective ways to improve elementary school mathematics and deepen everyone's appreciation of mathematics.

A Vision

Most people, old and young, are being robbed of their birthright to find enjoyment in mathematics. They do not realize they are already fully prepared to find fun and accomplishment in one of civilization's oldest and most fascinating pursuits.

The two main unwitting perpetrators of this robbery are School Mathematics and professional mathematicians. School Mathematics is dull, confusing and a source of continual frustration and defeat for most children. Professional mathematicians live comfortably in their own sparsely populated world and have lost much of their ability to communicate with the rest of us; their attempts—as in "new math"—are certainly well intentioned, but unsuccessful.

The resulting image of mathematics is that it is both dull and difficult. We and our children will never come to know a fascinating area of our rightful cultural heritage until we change that general image of mathematics—until we help children and adults alike to see the "real thing" for the first time.

"Arithmetic" is a general term that suggests the computational skills used by the supermarket shopper, engineers, accountants, carpenters, marketing experts in large corporations, and by the Internal Revenue Service.

I'm sure all of those mentioned, other than the shopper, would use "mathematics" to describe his manipulation of numbers and formulas according to rules he has learned.

While granting them license to use that term as descriptive of what they do, I would like to reserve it in this discussion for a very different activity—for the "**search for relationships** that aren't obvious."

"**Mathematics**", as I am using it here, **is what happens when people look at a situation as others do, but wonder if there isn't something more than meets the eye**—something worth telling others about if you find it. It's the excitement of the search, the disappointment of false tries, and the "sometime thing" of discovery and its thrill.

Unfortunately, it's necessary to offer at this point a "de-frightening pill" for most readers. The discussion is apt to seem headed toward the esoteric. Such is far from the fact.

In a recent Sesame Street episode, an old nursery rhyme is introduced. One moppet questions another, "As I was going to St. Ives, I met a man with 7 wives; each wife had seven sacks, each sack had seven cats, each cat had seven kits — kits, cats, sacks and wives — how many were going to St. Ives?"

And the victim seems to be taken in as he begins the laborious computation: 7 wives and 49 sacks, 7 x 49 or 343 cats; and 7 x 343 or 2,401 kits: 1 man, 7 wives, 49 sacks, 343 cats, 2,401 kits — 1 + 7 + 49 + 343 + 2,401 or 2801 . . . "Yes," said the second moppet, "you met 2,801 **coming from** St. Ives, but, of course, only you were **going to** St. Ives."

Now that's a switch — from dupe to show-off.

Here is a group of numbers and a very simple task: loop three in the list whose sum is **22.**

9	9	9
7	7	7
3	3	3
1	1	1

Try it. If you know too much, there is no problem. If you don't it will present some difficulty.

Of course, three 7's is close — 21, but not 22. Two 7's and a 9 is 23 — but not 22.

All such close combinations simply heighten the anxiety and intensify the sense of accomplishment when you finally realize that the sum of any three odd numbers has got to be an odd number. Twenty-two is even — the search is impossible.

Everyone is familiar with the multiplication tables —products of all possible pairs of single digit whole numbers.

One very simple example:

$$\begin{array}{r} 3 \\ \underline{\times 4} \\ 12 \end{array} \quad \text{and} \quad \begin{array}{r} 4 \\ \underline{\times 3} \\ 12 \end{array} \quad (1,2,3,4)$$

This example has a certain unique look if one considers the digits used — 1, 2, 3, 4 — all of which can be put in sequence and no digit is repeated.

To be sure the question is clear, we'll look at a more difficult example:

$$\begin{array}{r} 8 \\ \underline{\times 7} \\ 56 \end{array} \quad \text{and} \quad \begin{array}{r} 7 \\ \underline{\times 8} \\ 56 \end{array} \quad (5,6,7,8)$$

Here, we've used digits 5, 6, 7 and 8, once and only once.

How many more of the 100 "basic multiplication facts" have this same unique look?

The investigation can be broadened to look, also, for addition facts that use consecutive digits once, and only once, such as:

$$\begin{array}{r} 1 \\ \underline{+2} \\ 3 \end{array} \quad \text{and} \quad \begin{array}{r} 2 \\ \underline{+1} \\ 3 \end{array} \quad (1,2 \text{ and } 3)$$

Certainly, this search is all conducted on ground that is familiar to everyone. If there is any new element at all, it is that this simple question hasn't been raised before.*

I have a vision that elementary school teachers and the parents of the children in their classrooms can overcome a barrier that has grown up over a very long period of time—by changing the image of mathematics through making exciting searches such as we have just done. In a few years, we can accomplish a breakthrough to **mathematical literacy for everyone.**

* We feel a bit guilty in using those three examples to explain the "consecutive digits" search because you might have found them on your own—and we have been unable to find any others.

But, if you do feel cheated, you might open the boundaries to include such examples as:

$$\begin{array}{r} 13 \\ \underline{\times 4} \\ 52 \end{array} \quad (1,2,3,4,5) \quad \text{and} \quad \begin{array}{r} 789 \\ \underline{+264} \\ 1053 \end{array} \quad (\text{all 10 digits})$$

which qualify as using "consecutive digits".

Opportunities for small groups to "make a difference" seem to have almost disappeared as our society has grown ponderous, more complex, and technologically committed. So when such an opportunity does develop, it is exciting.

Many readers may wish this vision were realistic but, weighed down by their own past experiences with mathematics, feel they cannot participate; many will feel they are "not mathematically inclined." Others may be overwhelmed by what they might call the "magnitude of the task."

Please withhold any final judgment until we have considered new directions that are open to us.

The public image of mathematics as both dull and difficult is a gross misconception. This image has prevailed only because most people have never encountered mathematics that is both interesting and easy–not so easy it is trivial, but just elusive enough to require a little effort.

It takes more than words to break through the mistaken notions about mathematics, so let's do a little more together and then take a look at it.

Suppose a family gathers around a table to talk about different ways to make change for a quarter—with dimes, nickels and pennies. So the little ones can take part, members of the family can pool their resources to gather at least 2 dimes, 5 nickels and 25 pennies.

"How many different ways are there to make change for a quarter using these coins?"

Each member of the group can have a turn.

"We need someone to make a list of the combinations as we find them."

"(1) 2 dimes and 1 nickel
(2) 5 nickels
(3) 1 dime, 1 nickel and 10 pennies"*

* Hopefully, some readers will put down this book now and act out a scenario that begins with real live people. There is no satisfactory substitute for "doing it" yourself. If that isn't practical or if it doesn't seem reasonable, then please close the book and with paper and pencil find as many different ways as you can to make change for a quarter.

I'm going ahead to outline one path to the solution, so if you want to solve it with a group or in your own way, please don't read on until you are ready to compare results.

Personally, I'd like to find a way to record combinations without using so many words. And I would like to have some kind of simple plan as I complete my record — a kind of road that will lead to my destination. Here is my own little chart and a few entries that suggest my plan:

dimes	2	2	1	1	1	
nickels	1	0	3	2	1	
pennies	0	5	0	5	10	

Can you anticipate what combination comes next in my plan?

My first step was to find all combinations that included 2 dimes. The next step was to find all combinations that include 1 dime; so I started with 1 dime, 3 nickels and 0 pennies, and then I trade 1 nickel in for 5 more pennies for each new combination. I've completed this second step with the next combination in the list above — 1 dime, 0 nickels and 15 pennies.

All the combinations that are still unlisted will have 0 dimes — only nickels or pennies or both. If I start with 5 nickels as the next combination, new combinations arise as I trade those nickels in, one at a time, for 5 more pennies . . . which brings me to the end of the search.

dimes	2	2	1	1	1	1	0	0	0	0	0	0
nickels	1	0	3	2	1	0	5	4	3	2	1	0
pennies	0	5	0	5	10	15	0	5	10	15	20	25

There are 12 combinations, and I feel quite sure none has eluded my plan. The search has been neither long nor hard . . . but we now know something very few people have ever wondered about and bothered to work out.

I had done this little bit of mathematics many times with children and teachers and parents before I wondered about a related question: How many coins are used in each combination?

Since the answer to that question is simple arithmetic, I'll show the totals:

dimes	2	2	1	1	1	1	0	0	0	0	0	0
nickels	1	0	3	2	1	0	5	4	3	2	1	0
pennies	0	5	0	5	10	15	0	5	10	15	20	25
coins	3	7	4	8	12	16	5	9	13	17	21	25

Those totals, at first, may seem to be random. You might notice, however, that between the heavy vertical lines, each combination uses 4 more coins than the neighbor on its left. This is not surprising, because this happened as I traded 1 nickel for 5 pennies — 4 more coins.

But there is something that is far more interesting yet so simple you probably wouldn't think it was worth noting.

Those totals of the numbers of coins used are **all different.**

That means, that if you have a combination of dimes, nickels and pennies which is change for a quarter and tell me the "number of coins" — I can tell you exactly what those coins are!

Try it: think of a combination of coins that is change for a quarter; count the coins; look up that number in the bottom row of the chart we just completed. Above that number you will find the combination you are thinking of.

And now you have found out something about making change for a quarter that very few people have known before. It can hardly be considered important knowledge, but it certainly was not obvious at the outset.

MATHEMATICS — and this may surprise you — IS SIMPLY A SEARCH FOR RELATIONSHIPS THAT ARE NOT OBVIOUS!

If elementary school teachers and the parents of their children understood this and found that it was possible to see the whole subject in this light, then the vision of a mathematical breakthrough I dared to suggest is not as fanciful as it seemed at first.

As I have worked with children and teachers and parents over the years I've heard again and again the expression, "I sure wish I could have learned mathematics this way." It comes across in a kind of dreamlike way — "it would have been nice but now it's too late."

To catch a glimpse of what "might have been" and then to stop short of deciding to do something about it is too often the end of the matter. **Mathematics for Everyone** takes a look at what can be and suggests ways parents and teachers and children and everyone they influence can "make a difference."

Strategies

Visions become realities only when practical and effective strategies are developed and then conscientiously implemented. Goals are easy to state in attractive terms, but they must be pursued relentlessly as a way of life.

The pursuit of goals involves:

a clear assessment of the forces already available for a dependable foundation;

an identification of the several groups that must function together to accomplish productive community participation in the improvement of elementary mathematics; and

a sensible assessment of the readiness of each of these community groups to perceive the need for change, to accept responsibility for initiating and implementing it, and to function as a team.

PARENTS

Parents have been told by "new math" proponents that, unfortunately, they are not prepared to help their children in mathematics because they are not familiar with "sets, associative principles, inequalities, expanded notation, closed curves, missing addends, etc." . . . and these "concepts" are an important aspect of first grade mathematics.

And these same concerned parents are well aware that "test scores" in elementary mathematics have been eroding for a full decade. We can understand when some become angry and openly critical.

The growing tendency to avoid "homework" in mathematics, as well as other subjects, is understood by some parents as a strategy of the schools to discourage "meddling" in the educational process.

Because of this, some parents may be suspicious when they are urged to participate in a community effort to improve school mathematics. It will take time and patient efforts to build in parents a trust that "community participation" is anything more than a passing slogan.

Only a few can be expected, at first, to accept invitations to become part of open discussions about ways to work with children — to raise test scores and to help children learn more mathematics while developing a friendliness for numbers.

SCHOOL BOARDS

The most active "test watchers" are some school board members. They are charged with the responsibility to use public monies for plans and programs that pursue the goals of the community. They realize that "standardized achievement tests" have evolved as statements of society's minimum goals. Some school board members have hopes for education that are far above and beyond those minimums, but they realize that unsatisfactory test scores will scuttle their fondest hopes.

For years, these public officials have listened to the proponents of "new math" who asked for time: "test scores have suffered, but it takes time to build understanding; once we have improved the content, and teachers have been re-educated, the results will be different." School Boards were persuaded.

As scores kept slipping and as proponents of "new math" began complaining that "the tests don't test what we are teaching," elected representatives interpreted that as asserting that "society has set wrong goals." The only fair response a school board could make would be to charge the innovators with the responsibility of "re-educating the public." — And that's that.

The most disbelieving of our elected public officials began asking the schools to become "accountable." Their basic position was unassailable: "Our financial support will depend on demonstrations by school personnel that they can achieve specific goals." And this was, of course, just the language the "behaviorists" are tuned to hear. They offered their services to help state goals in terms of "observable" behavior.

School boards are fully aware that even when almost unlimited federal funds are made available to Title I projects, and goals are clearly defined in terms of society's minimum requirements, the educational community involved still is unable to show results.

SCHOOL ADMINISTRATORS

Charged with finding ways to "improve math scores," many administrators have turned to "management systems." They call on teachers to state their "objectives" and then to show by "criterion referenced tests" that their students, at the end of a given period, can demonstrate some specified behavior.

Administrators have, in the area of elementary school mathematics, welcomed almost every scheme to bring about desired change through some organizational system. But each scheme they try to implement flounders because they have failed to become concerned with the simple question: "Why do all these efforts fail?" Pursuing this question, they would find that children do not learn more mathematics because they have no reason to learn more mathematics.

TEACHERS

Teachers are where the action is . . . where learning takes place or it doesn't. They are first concerned with children — not with organizational schemes. And teachers are painfully aware that the results of their best efforts in helping children fall short of their expectations.

They have joined professional organizations, attended workshop after workshop, read the literature, taken college courses, searching for ways to help children learn mathematics. They ask for more "staff development" support, confident that they can find effective answers to the tough problems they face.

Hope lies in helping teachers with their search for better ways to help children learn. Of all the forces needed for community participation toward improving elementary school mathematics, the group which has shown the greatest willingness to face up to the real problems are classroom teachers.

Consequently, **Mathematics for Everyone,** while addressed to all concerned forces in the community, recognizes that change depends on supporting this powerful movement that classroom teachers have initiated and are pressing.

In making this assessment, I am aware that there are parents and school board members and administrators who are equally concerned. But, too many of these groups have found ways to "shift the buck"; only teachers know "the buck" stops in their classrooms.

COMMUNITY PARTICIPATION

Our "vision" depends on all these groups finally joining in a common effort to improve elementary school mathematics.

As a beginning, suppose a group of teachers came to the administration with a proposal: "We want your help and support in finding ways to improve scores on standardized achievement tests. Based on our experience, we would like to make this beginning. Obviously we are experimenting. Help us develop ways to evaluate our results over short periods of time, every 3 or 4 months, so we can modify our efforts in the light of results. Help us with 'pre-tests and post-tests' that help measure our effectiveness to meet the goal of improved computational proficiency."

In what more definitive way could a group of teachers announce that they want to be "accountable"?

What administration could say, "No"?

But there is a price tag.

Next, the Superintendent meets with his Board. He passes on the teachers' suggestion with a request that the Board give support. And the Superintendent explains that the teachers welcome and need all the community participation that can be generated.

The request for "accountability" which typically is initiated at the "summit" has now come from the classroom teachers, and no one feels he has been "pressured." "We all want something: let's get together and find out what we can do." And, all at once, "community participation" becomes a sensible approach to a problem everyone has been trying to push on someone else.

I want, at this point, to make two observations:

(1) I feel that teachers are deeply committed to finding ways to become more effective and want to help establish "professional development" as a basic part of the on-going educational program;

(2) While improving "standardized achievement test scores" is a unifying "first goal," achieving that

"minimum goal" permits us to set far more challenging and humanistic goals, such as developing a friendliness for numbers and shapes.

Mathematics for Everyone is addressed primarily to classroom teachers — but must also be interpreted as a plea to all parents, administrators, and school board members to find a way to change a "staff development effort" into a "community participation program."

Mathematics for Everyone is written with the hope that it will help classroom teachers play a more constructive role as initiators of an effort that can eventually develop into a full blown "community participation" thrust—teachers, administrators, parents and school board members working together on a problem that is common to all.

Some Activities: School or Home

In **Mathematics for Everyone,** I use the word "problem" in a specific way. To qualify as an appropriate "problem," a situation must meet the following criteria:

(1) The solver must understand the question;

(2) The solver must already have the skills required for the problem in his memory bank;

(3) And the problem must be concerned with relationships that are not obvious at the outset.

PROBLEM I

Consider the letters A, E, M, S, T: can you rearrange their order to spell out any recognizable 5-letter English words? If you respond by forming two words, such as

STEAM and MATES

that is evidence to me that we have satisfied the first two elements of what I consider to be an appropriate problem: you have clearly understood the question and already have sufficient information to answer it. So you are now ready for the problem.

How many such words are there? (which meets our third criterion, i.e. you don't know the answer.)

You have probably added "TAMES and MEATS and TEAMS" to this list — 5 words so far. And that's all I have been able to find. I am satisfied that no other combination would spell an English word with which I am familiar. However, if some dire consequence would follow in case a sixth 5-letter word I would recognize does exist, I would be compelled to make a list of all possible combinations.

How many combinations would I need to list so I could be sure that I have considered all possible combinations?

One approach would be to see this question as one of a series of questions that move from obvious to less than obvious: consider how many combinations can be made by rearranging 1 letter, 2 different letters, 3 different letters . . . 5 different letters.

This problem, initially cast in the setting of language arts, takes on an entirely new dimension as we look for a systematic method for discovering how many combinations I would need to list to be sure that I had missed none:

(1) We can see all the possible combinations for the two letters A and E by listing them,

$$A\boxed{E}\ ,\ E\boxed{A}$$

$$2 \times (1)$$

giving us two combinations.

(2) Then, we could build on this knowledge by listing the following 3-letter combinations,

$$M\boxed{AE} \quad A\boxed{ME} \quad E\boxed{MA}$$

$$M\boxed{EA} \quad A\boxed{EM} \quad E\boxed{AM}$$

$$3 \times (2 \times 1)$$

(3) Thus far, we have not taken any very surprising moves, but now we are ready to do so: (4 letter words) A,E,M,S:

A	E	M	S
EMS	AMS	EAS	EAM
ESM			
MES			
MSE	6	6	6
SEM			
SME			

$$4 \times (3 \times 2 \times 1) = 24$$

We were able to take that shortcut because we had established that three letter words combine to form 6 different combinations and we had four groups of 3 letter words to consider.

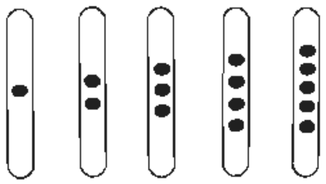
(4) Because of our new found shortcut (or relationship), the next step, representing the answer to our question of "how many" is easy: (5 letter words) A,E,M,S,T:

A	E	M	S	T
EMST 24	AMST 24	AEST 24	AEMT 24	AEMS 24

5 x (4 x 3 x 2 x 1)

PROBLEM 2

For our next example of a problem, we begin by taking a look at five beanstícks — a 1-stick, a 2-stick, a 3-stick, a 4-stick and a 5-stick — one of each:



(1) (2) (3) (4) (5)

You will notice that there are three different ways to pick up exactly 5 beans. (And we will use numbers to indicate the sticks rather than sketches of sticks themselves.)

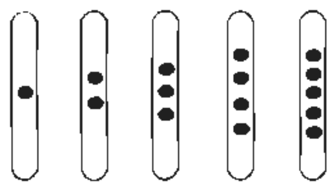
(5) (1 and 4) (2 and 3)

How many different ways to pick up 6 beans? The same stick may be used in different combinations. There are two ways that are obvious and one a little less than obvious.

(1 and 5) (2 and 4) ( )

We have already agreed that a stick may be used in more than one combination — so the three sticks with the fewest beans make a third combination — $1 + 2 + 3 = 6$.

Let's go extend the investigation — different ways to pick up 7 beans, 8 beans, 9 beans, etc. Remember we have only 5 sticks:



1 2 3 4 5

We might keep records of combinations in this form:

1			✓	✓		✓							
2		✓			✓	✓							
3		✓				✓							
4			✓		✓								
5	✓			✓									
	5	5	5	6	6	6	7						

If you want to continue this investigation without any intervention from me, all you need is pencil and paper.

My results are these:

7 —	(2,5)	(3,4)	(1,2,4)
8 —	(3,5)	(1,2,5)	(1,3,4)
9 —	(4,5)	(1,3,5)	(2,3,4)
10 —	(1,4,5)	(2,3,5)	(1,2,3,4)

Did you find others? If so, please let me know, because those are all the combinations I've been able to find.

But there are 15 beans altogether, so let's proceed. How many ways to pick up 11 beans?

11 —

The going gets rougher. Two combinations are found without too much trouble (2,4,5) and (1,2,3,5). The third presents some problems.

After trying diligently you may come to the realization that there is no third combination. Good! But be introspective for a minute. "Trying" means considering combinations and rejecting them. And all the computations of this kind — finally rejected or useful — is the kind of practice children need to fix simple combinations in their memories — available on automatic recall.

As the going gets tougher, it's usually wise to see if there isn't a simpler way to look at the problem.

"Picking beansticks up" has another side to it — "leaving some down" . . . and it's time for us to look at this other side.

There are 15 beans in all; so "picking 11 up" is "leaving 4 down."

11 – (1, 2, 3, ~~4~~, 5) (~~1~~, 2, ~~3~~, 4, 5)

Clearly there are only two ways to do that.

Now the investigation of 12, 13, 14 and 15 can be completed with a few strokes of the pencil:

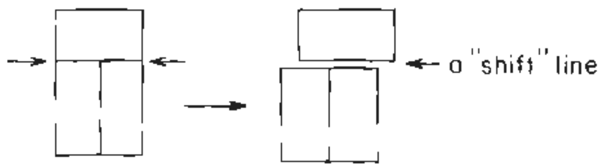
12 - (1, 2, ~~3~~, 4, 5)
~~(1, 2~~, 3, 4, 5)
13 - (1, 2, 3, 4, 5)
14 - (1, 2, 3, 4, 5)
15 - (1, 2, 3, 4, 5)

You may have the pleasure of crossing out the "2" to leave 13 and crossing out the "1" to leave 14, and crossing out none for 15. And that's the end of the problem.

PROBLEM 3

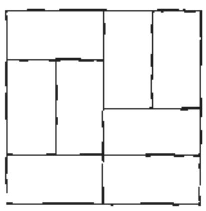
You will need some dominoes for this next problem — unless you want to sit back and let me do all the work (in which case you will miss much of the fun).

Our task is to build "earthquake-proof" walls with dominoes. Such a wall has no unbroken straight lines of mortar along which sections of the wall could slide if disturbed by a quake. If a straight line extends from one side to another the walls are not "earthquake-proof."



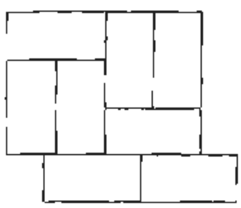
The top of the wall could shift from side to side.

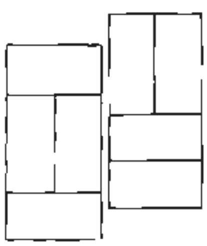
What is your judgment about this wall? Is it quake-proof?



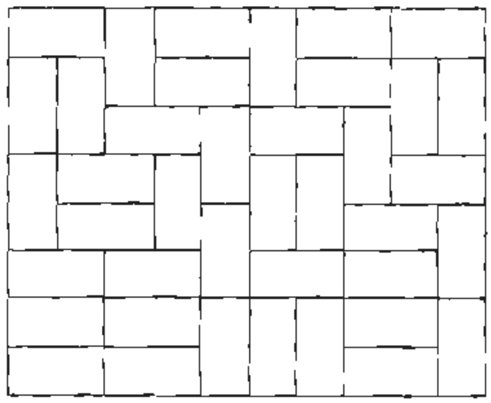
8 dominoes

Along how many lines could it shift?



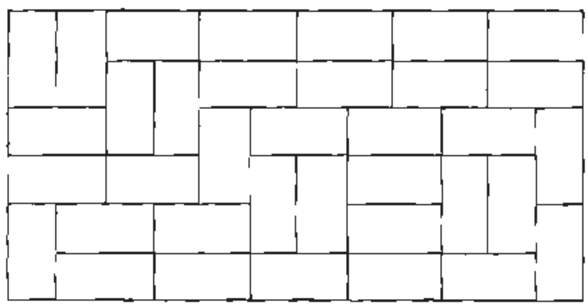


Can you find one or more shift lines in this wall?



40 dominoes (2-shift lines)

Study this wall as an armchair engineer in earthquake proofing. What do you think of it? Along how many lines could it shift? How many straight lines of mortar reach from one side to the other or from top to bottom?



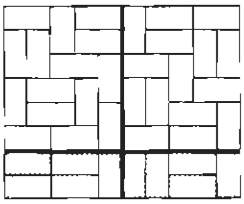
36 dominoes

As the engineer who built it, I am modestly proud of its ability to withstand earthquakes. I would feel better about it if it hadn't taken 36 dominoes to build.

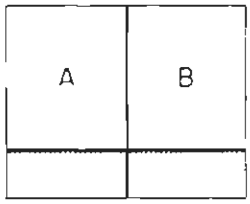
The height of quake-proof engineering power is to build a rectangular wall section using the **fewest dominoes** . . . and it can be done with less than half that many — with less than 18.

If you want to go back to building domino walls to meet this challenge please don't read on. When you have done enough building on your own, check back in.

I have already given you, in disguised form two of the best answers I know — using but 15 dominoes each. If you will draw in the two shift lines in the 40 domino diagram above, those lines divide it into 4 sections — two large and two small:



40 dominos (2-shift lines)



The two large sections are two different 15-domino earthquake-proof sections . . . the smallest number of sections I have ever been able to use to build an "earthquake-proof" section.

One might find the following sign in a store (in a state that allows limiting the number of items that can be bought):

SALE

Pencils 4¢ Erosers 3¢

LIMIT – No more than 2 of each to a customer

Lest some confusion arise, we begin with a few comments. If by "purchase" one does not admit a "purchase of 0¢," then the smallest amount of any purchase that can be made is 3¢. The largest purchase would be 2 pencils at 4¢ each (8¢) and 2 erasers at 3¢ each (6¢) or a total of 14¢.

I thought of this sign after reading a "story problem" in a third grade arithmetic text book. It went like this:

> "Mary bought a pencil for 4¢ and an eraser for 3¢, how much did Mary spend?"

It occurred to me that if the student knows that 4 + 3 = 7 then that's a demeaning question: and if the student doesn't know, then it is inappropriate for him.

I thought of substituting the sign above and asking other questions.

"Billy spent 11¢ . . . What did he buy?"

"Mary spent 10¢ . . . What did she buy?"

Neither question is too easy even for adults and neither is too difficult. The questions call for more than acts of automatic recall — each has a bit of a "problem" about it.

We agreed that 3¢ was the smallest and 14¢ the largest purchases that could be made. Are there any amounts between 3¢ and 14¢ that cannot be made under the terms of the sale?

4¢, 5¢, 6¢, 7¢, 8¢, 9¢, 10¢, 11¢, 12¢, 13¢?

How many such amounts are there?

How many different purchases are possible?

Is there any amount of money that can be spent in more than one way?

Questions in most text books are usually about single purchases — they are "customer" arithmetic. But there is also the arithmetic the clerk must know — "clerk" arithmetic. The clerk must handle many different purchases. Clerks might make up tables of all possible purchases so they don't have to keep doing the same arithmetic over and over again.

The clerk in the pencil and eraser store might make such a table as this:

		Erosers 3¢		
		0	1	2
Pencils 4¢	0	X	3¢	6¢
	1	4¢	7¢	10¢
	2	8¢	11¢	14¢

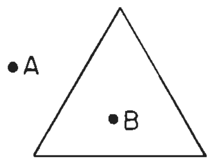
Now at a glance you can answer questions we asked above, quickly and without any doubts.

Between 3¢ and 14¢ the amounts that are not shown are 5¢, 9¢, 12¢ and 13¢.

If you want to know what Billy bought for 11¢ and Mary bought for 10¢, you can look them up in the chart.

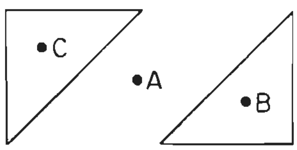
There are eight different purchases (excluding 0¢). Because no amount is repeated in the chart, there is no way to spend any amount in more than one way.

Suppose we think of triangles as small fences in a field. One triangle divides the field so two animals can be kept apart — we'll call these animals A and B.

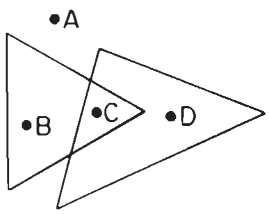


Now, can another fence be built without making a new enclosure? No, unless one fence is on top of the other.

Can you plan two triangular fences that creates just one more place — to separate animals A, B and C? That's easy



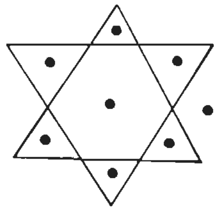
Now let's try to provide for 4 animals, using only 2 fences. The fences will need to cross each other in a way something like this.



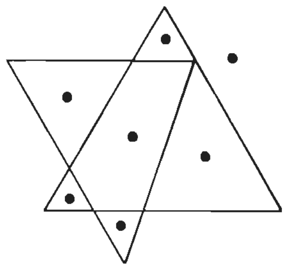
Can you arrange the two triangles to keep 5 animals apart? . . . 6 animals . . . 7 animals . . . 8 animals?

If you want to doodle away to find your own way to do it, please do — then come back and compare your results with those I am going to give.

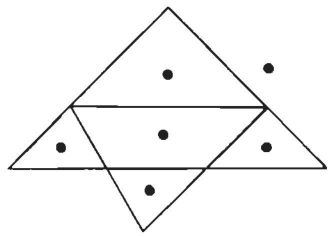
Perhaps showing an arrangement that separates 8 animals will give you a better feeling for different possibilities. The familiar Star of David arrangement meets these requirements.



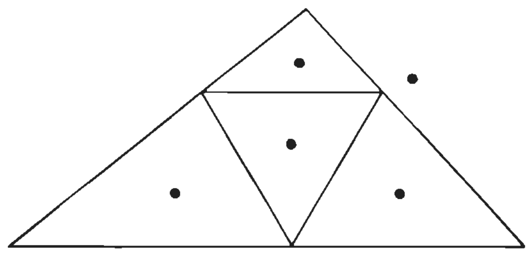
Now we can draw in one of the points and close up one fence . . . leaving 7.



We can repeat that on the other side, closing up another fence . . . so we can accommodate only 6 animals.



And yet again . . . to accommodate only 5.

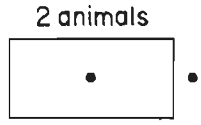


This completes the problem: arrangements for 2, 3, 4, 5, 6, 7 and 8 animals.

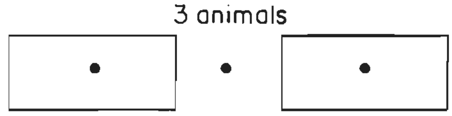
But this process introduces several related problems.

A. Suppose the fences had been rectangles instead of triangles.

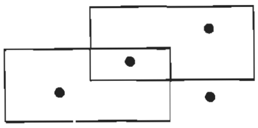
2 animals



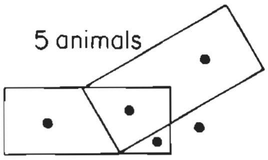
3 animals



4 animals



5 animals



. . . and continue the doodling until you have accommodated 6, 7, 8, 9 and 10 animals.

B: Suppose the fences are a triangle and a circle.

C: Suppose the fences are 2 circles.

D: Suppose the fences are a circle and a square.

None of these relationships was obvious when we started — they were there all the time, but they require some effort to see them, regardless of anyone's previous education.

The problems we have just considered were selected because they indicate the great variety of mathematics that is available to elementary school children — and to the entire community.

Mathematics in a Pluralistic Society

America is slowly coming to understand that our unique multi-cultural character is a source of inspiration and strength. The "melting pot," an unfortunate notion from another time, has denied to many the richness of their language and tradition. Fortunately, those who tenaciously resisted assimilation by retaining their identities and languages have reflected a spirit which is the essence of America. Education in the schools must begin to respond in new ways to the multi-cultural character of our children and of our society. The linguistic resources of the nation must be conserved; bilingualism must be welcomed as a blessing to be cultivated, not a problem to be eradicated.

Mathematics has a unique contribution to make to this more hopeful look toward the future. In mathematics, as in art and music, we can easily transcend the language barriers. As we do, we find that the content of mathematics is almost totally culture free — more culture free than either art or music.

Twelve can be evenly divided by 1, 2, 3, 4, 6 and 12 in all cultures.

That the square of the hypotenuse of a right triangle equals the sum of the squares of the other sides is universally recognized.

The sum of two odd numbers such as 7 + 9 is always an even number, 16 in this case.

That the circumference of a circle is about 3 1/7 times its diameter is common knowledge in all cultures.

Maria's family moved from Mexico to California when she was 7 years old. She knew no English, and, consequently, was assigned to first grade.

Subsequent tests suggested that Maria be placed in a special class for retarded children. Additional tests were administered by a person who had some facility with the Spanish language. The results were the same.

When her teacher learned that Maria was to be reassigned to a program for the retarded, there was a hasty conference in the principal's office because Maria had already finished half of level A in CDA math — a self-pacing, non-verbal introduction to arithmetic. Just before Christmas holiday, Maria had completed Level A and was well started into Level B. Her teacher expressed surprise: "But she's still counting to herself in Spanish" . . . (but writing numbers in English?)

Maria is actually gifted. She eventually overcame the language barrier and found her own way to be comfortable in a second culture . . . with children her own age.

Arithmetic, at first, was a haven for Maria; it was an oasis of familiarity in otherwise strange surroundings. From the first day, she was able to move ahead with the other children — doing the same arithmetic they were doing . . . and, in fact, demonstrated her real maturity to a sensitive teacher. Because Maria could count, manipulate objects, and write numbers, she succeeded at arithmetic, which from the beginning deals with counting, experimenting with things in different arrangements, and recording results. When Maria enrolled in her new school, she already understood the cross-cultural message of "+" and "—" signs and soon learned when the signs "x" and "÷" were appropriate.

Language is a common, useful, efficient means of communication between adults. It is a less reliable means of communication between adults and young children. Often universal, non-verbal means — such as sketches, moving objects around, and ordering numbers in different ways — lead more directly to understanding than do words. Such non-verbal activities provide precisely those experiences that later become the referent that gives meaning to words.

Children can do more and understand more than they can verbalize in the beginning. Children have much more information in their memory banks than they can express; they see many relationships they cannot yet put into words.

Consequently, non-verbal techniques are valuable — not only when there is a "language barrier" in the sense that participants think in different languages

— they also help all children operate comfortably on levels beyond the limits of their growing command of language.

I was asked once to serve as consultant to the schools in Los Angeles County that had undertaken the job of providing special facilities and specialized personnel for the deaf children of elementary school age.

"How can the curriculum be adjusted to meet the special needs of deaf children?" My answer was that it would depend entirely on what children can do.

In my 45-minute session with a group of six-year old deaf children, they grasped the idea of all four basic number operations, +, —, x and ÷, by moving blocks around and reporting their results. No oral or written language was used other than reports such as:

$$3 + 8 = 11 \qquad 3 \times 4 = 12$$

$$12 - 6 = 6 \qquad 11 \div 3 = 3\tfrac{2}{3}$$

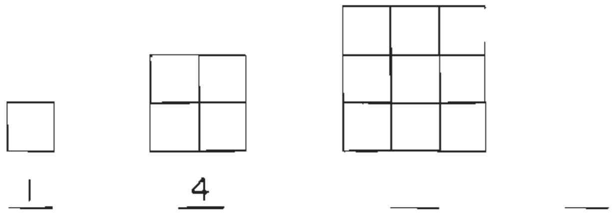
I had never been able to move so fast and so far with any 6-year-olds — and never have since except with other deaf children. I think the variable is that deaf children have a very highly developed capacity to "see" what's going on . . . their life depends on what they see. They are free from the static of sounds that hearing children experience — noise and words that are often confusing and distracting.

These experiences with the deaf led to my intensive search for a greater variety of non-verbal means of communication . . . not for deaf children alone, but as more reliable ways of communicating with all young children — particularly in situations where new ideas are introduced.

I had agreed to work with some primary children in Urbana, Illinois, and the teacher explained that one boy had just come from Japan, and that he did not speak or understand English. That was, of course, the signal for a "non-verbal" session.

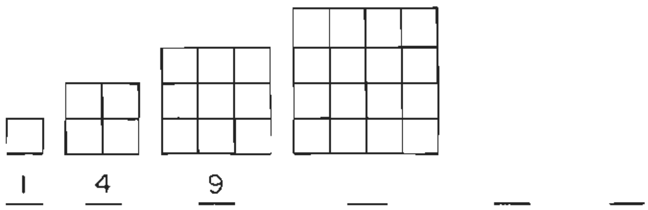
Nobu sat slumped in his seat with his head down.

I started writing on the blackboard.



Before filling in the third blank, I held out the chalk toward the group. Several hands shot out. I gave the chalk to a child who promptly wrote 9 under the array of 9 blocks.

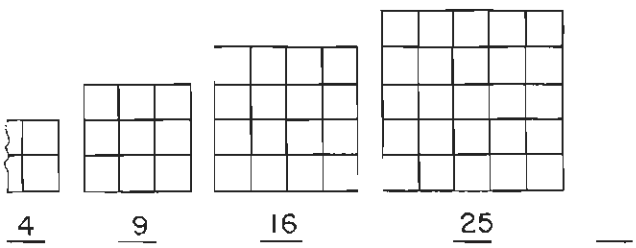
Then, I pointed to the space over the next blank and started to draw a line; but I stopped and held out the chalk, and a volunteer made a sketch,



but did not fill in the blank below. Nobu's hand was up! I gave him the chalk. He wrote "16" under the sketch. He looked at me. I smiled. He put the chalk on the board, indicating he wanted to do something more.

I smiled and nodded encouragement.

He drew a 5 x 5 array and wrote 25 on the line below the sketch.



Then Nobu put his chalk to the board, ready to draw another sketch; as I smiled and held out my hand for the chalk, a wide warm smile spread across his face. He gave me the chalk and sat down.

Nobu was ready to volunteer every time I held out the chalk, and he never faltered when it was his turn. Once a method had been devised to remove "the language barrier," Nobu was at home, eager, and successful.

(During such "non-verbal" sessions, the children remain silent, showing their displeasure with anyone who so much as drops a ruler even though there has been no request for silence.)

9 · 4 · 13

This means the same thing to everyone in our polyethnic, polycultural nation.

CDA Math is a first effort to begin building on the notion that mathematics has almost no cultural boundaries, and that non-verbal strategies in teaching early mathematics are more appropriate than oral or written language.

CDA Math is designed to value bilingualism. In all its pilot versions, the materials and teaching methods in CDA Primary Math were essentially non-verbal because of our commitment to communicate as freely and effectively as possible with young children. Words and other abstractions can interfere with understanding in the early years.

When CDA was approached to produce a Spanish edition, it appeared immediately that, in the existing primary materials, many pages contained no words and others contained so few that they could be written both in Spanish and English without omitting any of the mathematics. Consequently, we are able to present CDA Math at the primary levels as a bilingual program. All pupil materials, **Individualized Computations** (Levels A, B, and C) and **Patterns and Problems** (Levels A, B, and C), are printed in both Spanish and English on the same page.

At the intermediate level it was possible to continue a bilingual version in one of the strands of the CDA Math program. Levels D_1, E_1, and F_1 of the **Individual Computation** strand have instructions in both Spanish and English on the same page. However, at this intermediate level, the need for written language increases in mathematics and makes it harder to produce a program which is simultaneously bilingual. The number of words required to express an idea presents special design problems. We simply haven't yet devised sufficiently effective ways to present this level of mathematics in a non-verbal way on the printed page. Other media become more suited to such communication — animation, for example.

Because of this relatively heavy need for language described above, the **Patterns and Problems** strand, Levels D, E, and F are printed only in English.

BILINGUAL					
A	B	C	D	E	F
A_1	B_1	C_1	D_1	E_1	F_1
A_2	B_2	C_2	D_2	E_2	F_2

It is CDA's hope to encourage educators concerned with bilingual education to create school-based curriculum. We hope that from such effort a variety of bilingual curricula will emerge. CDA is eager to hear from interested schools to enable us to create and support co-development projects, especially those focusing on intermediate mathematics. CDA Math is designed to value bilingualism. We are concerned with the phenomenon that many children grow up in communities in which they seldom, if ever, think about the fact that the same idea can be, and indeed is, expressed in a language other than English. CDA Math hopes to help overcome such isolation, and the negative cultural evaluation of linguistic differences that so often exists in such situations. Page headings and explanations are provided in two different languages; these few words introduce a full page or more of mathematics that everyone does together regardless of his language or background.

We hope to make a small contribution toward (1) removing the language barrier for some children; (2) dispelling the misconception that language differences are more fundamental than they seem; and (3) presenting mathematics as a common aspect of all cultures.

As we appreciate more fully the unique universality of mathematics, its freedom from cultural differences — the vision of **Mathematics for Everyone,** assumes a new dimension. Mathematics can sweep aside cultural boundaries without threatening the identity or the integrity of any ethnic group.

CDA Math has done little more than suggest that mathematics can transcend cultural differences among all learners — adults and children. It has begun to provide all children with an opportunity to cultivate a bilingual background in our pluralistic society.

b. PEER GROUP STAFF DEVELOPMENT

Introduction

There is little doubt that teachers want help in teaching elementary mathematics. Teachers are painfully aware that their repertoire of effective activities in this field is limited. They want help!

Historically, the help has been provided by calling in visiting experts, who may have little real knowledge of the particular needs of individual teachers, and by dependence on Teacher's Editions that accompany children's texts. Teacher's Editions, to be saleable, must be universally applicable to all varieties of teaching situations. Those of the 70's include an exact copy of each of the pupil pages — with answers and directions to the teacher for using the pages. The most recent editions have a new feature — a "behavioral objective" for each pupil page: "The student will demonstrate that he can . . ." These "helps" suggest that classroom teachers and children will perform on the written commands in the margins or on the facing page.

What do you suppose might be the opinion of some future archeologist as he studies such an artifact? Would he wonder why a professional person needed to be told how to perform his job in such minute detail? Would he wonder why such a book was needed in an age that professed to individualize instruction —to meet the individual needs of children? Would not such a book, if used, dictate the curriculum and dictate the behavior of classroom teachers? Will history show that it did?

We hope that history will find that teachers eventually rejected the typical Teacher's Edition concept and became deeply involved in what we will call "peer-group staff development."

Teachers are sensitive to children, aware of their needs and often feel that most materials of instruction are not appropriate for use in their classrooms. They know, from experience, the kinds of activities that will involve children and the kinds that won't. They have few that are useful. They are searching for more.

We envision teachers undertaking the task of revising their mathematics curriculum, beginning with the most appropriate materials available, trying them in their classrooms, testing strategies, sharing results in an organized way with other teachers and parents, evaluating their progress in terms of observable results — in skill development and positive changes in attitude. At CDA, we call this process the "peer group approach to staff development" and it has worked successfully in schools throughout the U.S.

As such experimentation brings fresh air back into the classroom, teachers can begin taking stock of the new situation. What are the needs? What are the goals? What basic classroom strategies can we develop? What questions about content do we need to raise? What kind of evaluation is useful? How can the administration contribute most successfully? What does accountability mean? What about the current preoccupation with behavioral objectives?

Obviously, answers to these questions are not easy to work out. And no answer will, for long, be considered adequate. The effort has no ending — rather it becomes a way of life that changes continually; it leads toward an "unfinished curriculum."

If a staff development effort can show "better results" on standardized achievement tests, it will receive support and be guaranteed freedom to work toward other goals that are more exciting and more challenging.

In most school districts, a "track record" has been established in terms of some nationally normed instrument. The outcome of any change or innovation will be judged in terms of that district's own past results — its expectancy based on past experience.

If such results are non-existent or too fragmentary, some standardized instrument can be used as a "pre-test" and again as a "post-test" . . . to generate the "hard evidence" school boards want . . . and have a right to expect.

Such tests can be used periodically on a limited basis to provide an interim reading for teachers on progress toward minimal goals.

It becomes clear, at this point, that the administration has an important function because such "baby research" can not usually be organized and carried out by classroom teachers.

TEACHERS LEARNING FROM TEACHERS

One of the largest and most successful organized programs focusing on professional growth in mathematics was the Miller Mathematics Improvement program funded by a $500,000 a year appropriation from the California State Legislature. After several years, the annual appropriation failed to remain in the final budget. The Center for the Improvement of Mathematics Education (CIME) was established and is successfully carrying on the program without State funds.

Each summer, from 750 to 1,000 elementary teachers participate in workshops in several centers around the state. For two weeks they are involved in intensive workshop activities. Credit is granted by the University of California Extension, several state colleges or by private colleges in the vicinity of each workshop.

The staff of 40 leaders for these workshops, with few exceptions, are **elementary classroom teachers!** And these are not "teachers teaching teachers" as a current slogan goes: rather, the staff has as its goal the fullest involvement of every participant in searching for more effective teaching strategies.

In a summer workshop in San Jose (1972) "telling time" in the primary grades was being discussed. "Some get it and some don't" . . . "It's too hard for 6 and 7 year olds" . . . "Most 8 year olds pick it up whether it's taught or not."

A first grade teacher reported her experience.

"I got a digital electric clock and a standard electric clock with hands, and synchronized them. I made it plain to all the children I would under no condition talk about the clocks.

"That worked like a charm. Children would cluster around the clocks and talk about them. Before the year was over, all the children who wanted to tell time — and that was more than half — learned.

"If each teacher had such a pair of clocks, think of the time and anguish we could save ourselves . . . a 'teaching machine' of the most effective kind."

This is not teachers teaching teachers — it's teachers learning from teachers.

"Professional growth" will become a reality as classroom teachers and their principals and other curriculum leaders join with parents to focus on what can be done to help children learn more mathematics. It will not happen by importing expertise once in awhile in the currently popular fashion. Even the best "outside assistance" has a half-life of a week unless the people who work with children keep talking about what's going on in their own laboratories — where children are learning — in the classrooms and at home.

A Central Focus

(This section is from **Drill and Practice at the Problem Solving Level — An Alternative.**)

Curriculum Development Associates has a fundamental commitment to emphasize the relationship between staff development and curriculum development. Curriculum materials alone are not sufficient to achieve educational change. Instructional materials must support and encourage teaching that is responsive to both the cognitive and affective human needs of learners. Teachers and parents who understand the design and purpose of the curriculum and who are in agreement with the theory of learning which underlies it are in control of the personalization of the learner's experience.

Recognizing the important role of the teacher, administrators and parents in the whole process of educational change, CDA has made staff development an integral part of the total process of curriculum development and dissemination.

In the area of mathematics, more than in any other area of the curriculum, "outside experts" are called in to "give In-Service Training." And the "expertise" brought to the teachers and supervisors is "more background in mathematics." The assumption is, that to teach mathematics more effectively to young children, all you need is to know more mathematics: a patently absurd idea . . . and, of course, parents are completely ignored.

Very few curriculum generalists feel competent to provide leadership in the area of mathematics. They were frightened away by "Modern Math" and have not found a way to get back to urgent needs for innovation in the mathematics curriculum.

The "expertise" we need is insight into ways young children learn — not "learn mathematics" — simply the way young children learn! And this expertise is available in every school, in every district and in every community. The initial task is to direct this know-how into helping children learn the mathematics all teachers and parents know but can't communicate. And, if this becomes manageable, the mathematical horizons will broaden.

ACTION WORKSHOPS IN CALIFORNIA

During that 1973-74 academic year "action workshops" were held on the west coast focusing on **Drill and Practice at the Problem Solving Level** as a useful tool for initiating efforts to encourage skill development and positive attitudes toward mathematics on the part of elementary school children.

More than 2,000 teachers, principals and other administrators, school board members and parents participated . In most workshops opportunity for earning college credit was provided for those who wanted it.

Often the work-shops had the character of the beginnings of an "ongoing" peer group staff development. A first session began with exploring new activities and evaluating them in terms of learning-teaching situations. In this first session, participants finally selected several activities that seemed most appropriate for the learners who would test them.

In the two or three weeks between sessions, the participants would involve children in the activities selected and keep notes of interesting results – successes, failures, modifications that grew out of experiences, and anecdotes that suggested the learner's attitudes toward the activities.

The second session began with sharing experiences, which often had some flavor of "show and tell." Difficulties reported prompted suggestions from others and often led to discussion of generally troublesome problems; such as how to get past the barrier some children raise when they are faced with mathematics of any kind.

The latter part of the second session was similar to the latter part of the first; plans to try new activities or to try modifications of those already introduced.

Where local leadership was involved in these workshops, they became simply the first two episodes in an ongoing peer group effort.

PRINCIPALS ASSUME LEADERSHIP IN OREGON

In the summer of 1973, the N.S.F. supported Oregon Math Education Council joined with the elementary school principals' organization in that state to hold two one-week workshops, one at La Grande in Eastern Oregon and the other at Eugene.

Each workshop spent two days involved in learning more about **Drill and Practice at the Problem Solving Level** and familiarizing themselves with the tapes prepared for teachers by Dr. J. Richard Suchman.

The 150 participating principals began to see the problems of "elementary school mathematics" from a new standpoint. They began to see more deeply into some of the obstacles their teachers had been trying to overcome; and, at the same time, they began to see they had precisely the understanding and organizational expertise needed to generate "peer group" staff dvelopment in their own buildings (for which teachers could earn college credit, if they wanted it). The notion that lack of success in mathematics education was a "content" problem requiring "additional training in mathematics", was replaced by an understanding that good old-fashioned arithmetic needed to be presented in ways that showed deep respect for what teachers and parents already know about ways in which children learn. They began to realize that there were simple, inexpensive learning materials already available to help teachers and parents develop more effective teaching-learning strategies.

The principals recognized that the need for ongoing evaluation of results can only be met by close participation of supervisors and principals with their professional staffs. They faced up to their responsibility to provide support and encouragement and counsel for their classroom teachers and participating parents who are continually involved in the process of developing, testing and refining new strategies.

This group of principals developed a deeper understanding that a successful peer-group effort of any kind depends on the day-to-day involvement of the natural leadership in that local group — the principal — and not on outside "experts" and "advisers."

Upon returning to their districts, many of the principals involved in the workshops proceeded to help initiate "peer group" programs for professional growth.

A DISTRICT-WIDE PROGRAM

In the fall of 1973, Jacksonville, Florida recognized the urgency of improving basic mathematics skills on the part of elementary school children. They felt a real need for staff development, and created an implementation model for CDA Math which would utilize local resources and meet their particular goals.

I participated in an initial two-day session with the support of several staff development consultants from CDA. Key leadership people at the district level were trained in CDA Math, and they began to create their plan for implementation.

Shortly thereafter, a course based on **Drill and Practice at the Problem Solving Level** was initiated at Jacksonville University. Sixty elementary and secondary teachers and "resource teachers" participated. Three semester hours credit were available for this course. After they were trained, those 60 teachers became the in-service leaders for the school system. They in turn conducted staff development workshops for approximately 400 teachers. These workshops met for two hours, once weekly, for fourteen weeks, and teachers who participated gained "master plan" in-service credits which could be applied to certificate renewal requirements. Many of the teachers who enrolled in the in-service course also implemented the CDA primary mathematics program in their classrooms. According to reports from the leadership group, feedback from teachers using the materials has been overwhelmingly favorable.

Prior to implementing the CDA Math program, the Jacksonville school system had defined a comprehensive set of performance objectives for students in elementary mathematics. Those who were trained in the CDA approach keyed the math materials to the performance objectives and are using the CDA curriculum and methodology as the means to achieve those objectives. Thus the mathematics program is being centered on the CDA curriculum and the unique staff development program designed for Jacksonville teachers by the leadership group in cooperation with CDA.

A PERSONAL EXPERIENCE

Several years ago, I participated in weekly seminars of classroom teachers and interested parents who expressed a genuine desire to know more about the "New Math."

The participants listened patiently through the first session as they were introduced to "new terminology" and "new topics." But I knew we were already bogged down.

What to do?

That night I was doodling on a napkin as the gloom kept closing in. I had been thinking a bit about magic squares, and wrote out two classic examples —one of the order 3 (a 3 by 3 arrangement) and one of the order 4 (a 4 by 4 arrangement).

Then, for no reason other than compulsive curiosity, I wondered why I had never considered magic squares of the order 2 (a 2 by 2 arrangement). So I drew such a grid — and the napkin looked like this:

?

8	1	6
3	5	7
4	9	2

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

The sums of all rows, columns, and diagonals must be the same number.

Try it for 2 by 2 arrangements:

Do you run into trouble?

Well, it is rather easy to satisfy the condition that the sum for all rows and columns be the same number.

But the diagonals are stubborn. Start with the diagonals having the same sums:

1	2	3
3	4	7
4	6	

7	9	16
1	3	4
8	12	

0	10	10
3	13	16
3	23	

Rather conclusive evidence that the only magic square of the order 2 is one in which all numbers are the same, such as:

5	5
5	5

or:

0	0
0	0

etc.

For no particular reason — standard doodler style — I put an extra box on each example, so they looked like this:

1	2	3
2	1	3
3	3	

1	2	3
3	4	7
4	6	

7	9	16
1	3	4
8	12	

There's no question about what ought to be written in those boxes — 6, 10 and 20, respectively.

All the doodles "worked" the same way. So, try to invent one that won't "work." Try big numbers; try fractions; call for help from negative and positive integers. Failure!

This is how "Cross Number Puzzles" were born.

The next session with the teachers and parents began with a few examples I had thought up that "worked"; the entry in the extra box was in no doubt. They all "work".

7	3	
4	6	

11	9	
1	4	

$\frac{1}{2}$	$\frac{1}{3}$	
$\frac{1}{4}$	$\frac{2}{3}$	

"All right. Now, take out pencil and paper and see if you can find others that work."

In a short time, volunteers began to report successes. Each was put up on the chalkboard until it was full.

Disarmingly easy. But the participants were no longer listening politely. They were buzzing; the "noise level" was rising.

"Well, since this is so easy, maybe the investigation ought to be widened, to find examples that won't work."

The noise level settled down. Quiet prevailed.

Then the buzzing started. Finally someone had the courage to assert, "It's impossible!"

As soon as this claim was made, there was an argument. "No, it isn't!" And the counter-example was given; it included a computational error! Laughter.

Big numbers were tried. Common fractions, decimal fractions, positive and negative integers were tried. But each example "worked" if the arithmetic was right.

The teachers in the workshop were urged to present this little surprising experience to their classes; and it was agreed that the next session would begin with reports about what happened.

The entire format of this series of seminars was revamped that night. Thereafter, each meeting would begin with reports about what happened in the classroom, and it would end with a discussion of an idea to explore during the intervening week.

The enthusiasm generated was infectious. Soon my role had changed to that of a "moderator." And the children in the classes of these teachers received a new look at arithmetic.

The Cross Number Puzzle grew up as new facets were explored in many classrooms. Wednesday night reporting sessions were exciting.

A question arose: Does the puzzle "work" only for addition? No one knew, but everyone wanted to find out. This investigation led to the surprising result that, with a few rough spots to avoid (or plow through), it "worked" in addition, subtraction, multiplication and division. For example:

+

2	1	3
5	7	12
7	8	15

−

9	4	5
5	3	2
4	1	3

×

2	5	10
1	7	7
2	35	70

÷

24	6	4
4	2	2
6	3	2

A "rough spot" occurs in subtraction. Either you must plan ahead or be ready to employ positive and negative integers and zero. Another happens in division. Either you must plan ahead or be ready to divide by a fraction.

Teachers reported that the children were anxious to fill sheets of papers with examples they made up themselves—testing, testing, testing—searching for an example that wouldn't "work." They soon gave up hope of finding a counter-example—involving small numbers. But would big numbers behave in the same way? Would common fractions and decimal fractions and "mixed numbers" follow the same rules? Pupils pushed their study out to the edges of their computational abilities and began stretching, wanting to move into new areas of computation. These investigations were interesting to children and involved the very practice in computation that leads to proficiency.

The seminar participants began to see mathematics from a new view point—as a fascinating search for relationships that aren't obvious at first.

CURRICULUM MATERIALS, STAFF DEVELOPMENT AND DISSEMINATION

Traditionally, dissemination has been in the hands of salesmen, advertising agencies, and profit-motivated publishers. Staff development has been a concern of educators who generally shy away from published materials of instruction. For the most part, "in-service training" has become synonymous with more "methods courses." Recently, "action workshops" have become popular, but they are often no more than sessions on "how to make your own material" for a single high-interest activity. "How to teach" discussions maintain a professional aloofness from specific curriculum materials.

Yet it is hard to disagree with the notion that the most effective staff development must be vitally concerned with the materials that teachers and parents are using with their children. Theory gets carried into practice only when it has something to say about more effective use of materials that are in the hands of the learners. "Action workshops" would be more helpful if they were addressed to working with basic curriculum materials rather than to interesting side trips.

If staff development must be centrally concerned with materials of instruction, then professional educators must become less "publisher shy." At the same time, they must be careful to select curriculum materials that are the most appropriate for children.

This is a tough new assgnment, and the problem is made even more difficult by the fact that most basic materials of instruction are not appropriate for children, especially in mathematics.

Curriculum Development Associates decided to hire no salesmen and use no advertising agencies. It chose to look to the education professional—to those whose main interest is professional growth and staff development—for support in the dissemination of the curricula they publish. To be successful in this effort, CDA knew it would have to publish only materials that met the highest educational standards.

Thus far, CDA has published **Man: A Course of Study**, a curriculum created by the Education Development Center under the leadership of Jerome Bruner, and the CDA Elementary Mathematics program, created by the Associates themselves. The successful dissemination of **Man: A Course of Study** is a matter of record. Central to the dissemination effort has been the International Faculty for **Man: A Course of Study,** a core group of professional educators in school systems and colleges and universities throughout the world who have a commitment to this curriculum, and who organize and conduct workshops and seminars to introduce this outstanding curriculum to other educators. The International Faculty supports the staff development requisite that CDA has made an integral part of every implementaton of **Man: A Course of Study.**

A dissemination model patterned, in part, after this first successful model is emerging for the CDA Mathematics program. We are beginning to build an International Faculty for CDA Mathematics from a core of dedicated professionals who are committed to the program and its philosophy. These faculty members will support the staff development requirement that CDA has once again made an integral part of implementation of the mathematics program.

Through the International Faculty, CDA can provide professional services directed toward helping teachers and school districts break through to better elementary school mathematics. CDA has complete confidence that the education profession will welcome this pioneering effort and recognize curriculum-based staff development as a valid professional dissemination model requiring the commitment of educators to bring about constructive educational change.

Strategies

Feature articles in the **Wall Street Journal** (May 31, 1973) and in **Newsweek** (June 25, 1973) reported that "math scores are declining" on a nationwide scale. Both suggested that "new math" is responsible. Morris Kline, in his recent book, **Why Johnny Can't Add,** places responsibility on "the new-math people."

That "math scores are declining" is indisputable evidence that we, as a profession, are not meeting the minimum requirements society has set for us. But finding "the culprit" is dangerously simplistic and, in this case, leads away from finding basic causes. The problem is much more complex—and "new math" is only one of several factors that has contributed to the serious problem we face today—one of the least significant factors.

A somewhat wider perspective takes into account other and more significant causes of our growing failure to meet society's expectations. Consider some of those causes.

FACTOR ONE

Over the past two decades, there has been a dramatic shift in the everyday image of arithmetic. Today, we live in a world in which arithmetic is done for us. A trip to the supermarket provides clear evidence of this. With the advent of "unit pricing" the transformation is almost complete. And the final act at the checkout counter—determining the amount of change due the customer—is done by a machine. The little store around the corner—mecca for children in an earlier day—is long gone . . . and with it the powerful motivation for young children to learn arithmetic. Everywhere, computations are done for us by machines so reliable there is no point in checking the answers they print out.

Adding machines and other calculators have become common household appliances. Electronic pocket-sized computers have broken the $50.00 price barrier and plans are underway to push the price below $25.00. What sophisticated cash registers and unit pricing have done inside the supermarket, pocket calculators are doing on the outside. Not long ago, doing everyday arithmetic was everybody's responsibility. Today, machines are doing more and more of our arithmetic for us. Thus a powerful social motivation to develop arithmetic skills has, to a significant degree, faded into the past.

FACTOR TWO

Over the same two decades there has been a shift in elementary schools from a policy of "promotions" based on achievement to a policy called "social promotions" based on age and without regard for achievement. Many of us learned our addition facts and multiplication tables because we had to. The threat of being "held back" or "failed" is no longer available as a prod for reluctant learners. Children today have an option not to learn. Thus, another motivation teachers used with varying degrees of effectiveness in the past has been taken from them.

The motivational pattern has changed in these two respects—machines do our everyday arithmetic and children are promoted regardless of their progress toward stated goals. And both of these changes are irreversible. Electronic equipment will invade our lives more deeply, and few children will ever again be stigmatized as failures by being held back in school. Any program to improve math scores then, must find new motivations—new answers to the question: "Why should I learn to add?"

FACTOR THREE

Now let "new math" make its entrance. It was not conceived to replace motivations that had paled. Rather, it was born with Sputnik when the Russians beat us into space.

Tremendous amounts of money were made available by the government through the National Science Foundation for developing new mathematics curriculum and training specialists. Listening to mathematicians and turning a deaf ear to the judgements of classroom teachers, the SMSG (School Mathematics Study Group) led the "new math" revolution. They set about to "clean up" mathematics; first in the high school curriculum, then the junior high, the intermediate, and finally the primary and kindergarten curricula.

At no point along the way could they show that their "good mathematics" in elementary school textbooks was contributing toward improved scores on standardized achievement tests. In fact, evidence indicated quite the opposite. Elementary teachers were overwhelmed. They tried to find out why kindergartners should talk about "equivalent sets" and why first graders should be confronted by "expanded notation" and "the associative principle." As long as SMSG had laundered the content, it was the thing to teach. Supporters of SMSG controlled the leadership and the policy of the National Council of Teachers of Mathematics.

By administrative decision, one district after another "went into the new math." SMSG enthusiasts from the colleges came down to tell teachers how to handle this new curriculum and to provide classroom teachers with "more adequate mathematical backgrounds." Local curriculum leaders, especially principals, did not feel qualified to become involved and turned toward administrative arrangements in hope for change. Almost everyone was confused: teachers, administrators, parents and children, and few dared to admit it.

FACTOR FOUR

Gradually, after the usual 4 to 10 years textbook publishers need to reflect popular innovations, materials of instruction have cemented "new math" into the elementary school curriculum. Many teachers saw no way out but to shelve their "old fashioned" ways and let the textbook take over.

As a consequence, when given the task of subtracting 25 from 68, first grade children are taught to do the following:

$$\begin{array}{rl} 68 = & \boxed{60} + \boxed{8} \\ -25 = & -\boxed{20} - \boxed{5} \longleftarrow (25 = 20 - 5?) \\ \hline & \boxed{40} + \boxed{3} = \boxed{43} \end{array}$$

Please try to imagine what must be done to persuade young children to conceptualize such an abstraction.

Some authors introduce parentheses to minimize the difficulty:

$$\begin{array}{rl} 68 = & \boxed{60} + \boxed{8} \\ -25 = & -\left(\boxed{20} + \boxed{5}\right) \\ \hline & \boxed{40} + \boxed{3} = \boxed{43} \end{array}$$

And please try again to imagine what cajoling it must take to get children to write what looks like "8 + 5 = 3."

If you could somehow overcome these serious notational difficulties and reach 43 at the end of the route, it is still absurd to expect a six year old child to believe that "68 — 25 = 43" because each of the steps along the way followed logical principles.

FACTOR FIVE

While old motivations were fading into history and no effort was made to replace them, while "new math" was brought in by administrative decree, and while textbooks were gearing up to reflect the fruits of the revolution in content, IPI (Individually Prescribed Instruction) made its appearance.

Most classroom teachers have discovered that the "management system" (IPI, etc.) imposed on them are institutionalizing basic approaches to teaching mathematics that have already failed — math scores have declined steadily for 15 years. Teacher complaints are met by accusations that teachers don't want to be "accountable." But it is manifestly wrong to ask teachers to be accountable and then prescribe an approach which they know has failed in the past.

This is not to say that teachers know of approaches that will meet the goals of skill development that have been set and reverse the downward tendency in test scores. The experience of many teachers has convinced them that simply imposing a "management system" on a textbook-oriented curriculum will not produce the desired results.

The **Wall Street Journal** and **Newsweek** seem to suggest that maybe we got a little off the track with "new math" and that it's probably time to jump back on. They simply haven't done their homework. "New math" is only one of the contributing factors.

SUGGESTED GUIDELINES

Let's reject any "new-math-is-**the**-culprit" judgement and face up to the problems of reversing a trend that has moved us away from what society expects of its elementary schools.

The purpose of this chapter is to suggest some guidelines that reject the traditional notion of a "fixed curriculum" and look toward its opposite — the "unfinished curriculum." These suggestions are made with a full realization that we must take into account the historical developments that have led to a deplorable situation. No "bandaid approach" is realistic; rather, we must perform major surgery.

Let's start with an outline of broad proposals and then fill in some specifics behind each one, remembering that the goal of CDA Math is to find ways of returning mathematics to the entire community.

I. We can begin showing new deep-seated respect for classroom teachers' judgements about what is appropriate for young children—respect for the tactics teachers have developed and tested and refined in the past while working with children in their own classrooms.

II. We can help build a theory of learning that uses terms teachers already understand—a theory that is based on what teachers already know about the ways in which children learn.

III. We can create expanding resources of people, materials and activites from which teachers can draw to meet ther needs.

IV. We can redefine content and base it squarely on the mathematics which teachers and parents already know and use.

V. We can find a solution to the dilemma facing teachers in the area of skill development. They know that they do not provide enough drill and practice and that the drill and practice they do provide turns children away from mathematcs. They are confronted with the situation that "not enough is already too much," and they need to find a way out.

VI. We can create conditions under which school principals and others concerned with instruction and curriculum development can with confidence once again fulfill their responsibilities in the area of elementary school mathematics, including ongoing staff development. They can help groups of teachers plan and carry out periodic checks to demonstrate children's progress, at least toward that achievement which standardized tests are designed to measure.

VII. We can encourage colleges to drop their attitude of academic detachment and search for ways to participate more fully in an effort to meet society's minimum requirements of skill development and to move on toward more demanding goals.

PROPOSAL I

The "unfinished curriculum" starts with the ideas and materials and activities teachers have devised over the years to help them communicate mathematics as they understood it. When "modern math" shattered whatever confidence teachers had, these devices were shelved, but not forgotten. These homespun tactics had been developed, tested in the classroom, and revised in response to children's needs. The aforementioned article in the **Wall Street Journal** reports that Lee Mahon, curriculum director for San Francisco City Schools, said: "Most San Francisco teachers seem to feel the new-math textbooks should be put in the closet and left there." She commented that most teachers have replaced the texts with mimeographed math lessons they have written themselves. The newspaper also reports that classroom teachers throughout the nation are in rebellion against standard textbooks.

If classroom teachers are once again urged to trust their own professional judgement and use strategies and tactics they feel are appropriate for their children, they will revive the best out of their past experience and welcome new ideas that are useful in helping to meet society's minimum requirements. As teachers uncover both old and new resources and try them in their classrooms, children's reactions will be a determining factor in decisions about what is effective, what needs reshaping, and what needs to be discarded. Teachers are continually growing and children have various and changing needs. Consequently, the curriculum will never be "finished" and no two curricula will be precisely alike.

PROPOSAL II

In creating and building their own curriculum, teachers and their local support leadership need a rationale far more explicit than hunches and guesses. They need a theory of learning that is simple, widely inclusive and illuminating.

Such a theory will be discussed later in sections under "learning theory."

PROPOSAL III

Everyone is encouraged to learn that essentially all elementary arithmetic can be summarized in very simple terms:

- arithmetic starts with counting and recording the results of counting.
- arithmetic is especially concerned with four basic experiments: addition, subtraction, multiplication and division.
- arithmetic is a continual search for reliable ways to avoid one-by-one counting.

Teachers and parents will welcome the notion that "pencil and paper arithmetic" can be introduced as a convenient way to keep track of experiments with real things. Here are some formats that might be used in performing these basic experiments and for keeping records of both the experiments and their outcomes:

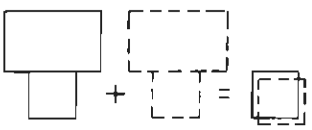


Fig. 1

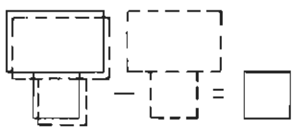


Fig. 2

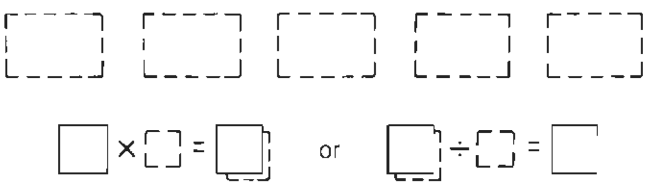


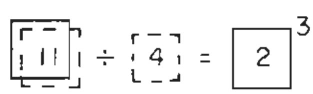
Fig. 3

The addition experiment (see fig. 1) and record keeping include these steps: 1) Put some objects in the large solid-line frame—the one on the left. 2) In the small box below the frame, write the number of objects in the frame. 3) Put some other objects in the frame on the right and record the number of objects in the small box below the frame. 4) Find the total number of objects used and record that number in the little box on the right—the double line box.

The subtraction experiment (see fig. 2) and record keeping include these steps: 1) Put some objects in the double-line frame—the frame on the left. 2) Record the number in the double-line box below. 3) Move as many counters as you want to the broken-line frame—the one on the right—and record that number. 4) In the box on the right—a solid line box—show the number of counters left in the frame on the left.

The multiplication experiment (see fig. 3) and record keeping include these steps: 1) Put as many counters as you want (including none) in one of the broken-line frames. 2) Report that number in the solid-line box. 3) Put the same number of counters as there are in the first frame in as many broken-line frames as you want. 4) Note the number of broken-line frames used in the small broken-line box after the "x" sign. 5) In the double-line box on the right of the "—." sign, show the total number of counters used.

The division experiment (see fig. 3) and record keeping include these steps: 1) Start with as many counters as you like. 2) Record the number in the double-line box to the left of the "—" sign. 3) Place check marks above as many large broken-line frames as you want to use. 4) Record the number in the small broken-line box. 5) Distribute as many of the counters as possible in the large frames already checked so there are the same number in each frame. 6) In the small solid-line box to the right of the "—." sign, report the number of counters in each large frame and the number left over, if any. For example, if eleven counters were chosen at the beginning and four large frames were checked, the experiment would be reported thus:



As soon as a child can record these procedures, he can make up his own examples to satisfy his own style and inclinations. He understands the concepts of addition, subtraction, multiplication and division of whole numbers.

As teachers use this and similar formats and procedures for getting children involved in the four basic operations, they begin to see "pencil and paper arithmetic" in a new light. It is a means for recording what the learner has done with things — a way of describing an experiment and its outcome.

It is significant to note that teachers are helping children learn their arithmetic in the world of things and there is no separation between "computations" and "applications." It is apparent to them that no separation will develop as long as what happens with things remains the final arbiter of what's "right" and "wrong" — of what's reasonable and unreasonable in arithmetic. With this approach, "mysteries" cannot develop, and common sense is enshrined as arithmetic's Solomon.

PROPOSAL IV

As teachers are given the opportunity and urged to build their own "unfinished curriculum", they welcome all resources that provide materials and suggest activities for them to try with children. And, if they are equipped with some explicit theory of learning, they can make discriminating choices.

Teachers are painfully aware that standard textbooks are devoid of appropriate ideas and activities for young children. The self-confidence of some teachers has been so completely shattered that they cling to the published "scope and sequence" in a text. Given time, and the example of their peers who are already building their own curricula, they too will begin to rely on their own ability to decide what meets the needs of their children.

It is understandable that publishers know that their primary goals and purposes are determined by the law of the marketplace. It would be an economic mistake to depart very far from "series" that have been selling. Each cautious revision is submitted to the sales force and marketing division for final decisions. Any alteration in format must be cost accounted carefully, so the profit factor can be maintained.

But it must be possible for some group, large or small, to demonstrate that it is economically feasible to produce materials tailored to the needs of teachers and children. At the risk of appearing to personalize this issue—but with the strong conviction that one's "remembered experience" is a strong and legitimate way to illustrate a point—let me share with you an experience I had with two distinguished colleagues several years ago. **Developing Insights into Elementary Mathematics** was designed by its authors, Wirtz, Beberman and Botel, to test this hypothesis. **Developing Insights** is a set of resources from which teachers can choose those which are most appropriate for use in the classroom. Teachers are given rights to reproduce the materials from the masters provided. The authors prepared page-by-page comments for teachers. Both volumes were submitted to several leading publishers in the form of camera-ready art—all set for the final step in printing.

One by one, the publishers reached the same conclusion—it's not in the best interest of book publishers to grant reproduction rights to teachers. That would mean the sale of only one book per classroom—and no need ever to replace it!

So the authors printed it themselves and submitted it for adoption in California. It was adopted by the state and furnished without charge to every teacher, first through eighth grade.

This encouraged the authors to create Curriculum Development Associates, Inc., an organization dedicated to publishing and disseminating those materials that big book companies shy away from, even though those materials are needed by teachers.

In the area of mathematics, CDA has gone on to develop and publish a second set of resource books for teachers and **CDA Primary Math** — which is an example of an unfinished curriculum.

In the area of social studies, CDA publishes the NSF-sponsored **Man: A Course of Study** developed by Education Development Center with leadership from Jerome Bruner. This powerful curriculum had been rejected by all leading publishers because it includes a serious commitment to staff development. It is economically feasible to produce materials teachers need in a format that provides a wide variety of choices.

Such curricula have clear and identifiable designs, yet teachers understand them well enough to have a sense of control over the materials. The nature of the materials invites them to adapt, modify, and go beyond the original curriculum structure. This is the paradox of the unfinished curriculum: because it has a finite clarity, it creates conditions for infinite modification.

Testing

Very early on, someone, be he skeptic, realist or antagonist, will remind us that "test scores" — standardized achievement test scores — are stacked against any effort that departs from tradition. Standardized achievement tests are institutionalized tradition and will stand as a barrier to change.

This is a strange mixture that must be sorted out.

Standardized achievement tests are, in a very real sense, society's way of setting minimum standards for progress in what it calls "mathematics" or "arithmetic."

We accept that as fact. The people who pay the bills have a right to set standards for those they employ to help children "measure up."

What do "test scores" show over the past decade? Everyone knows — for the past fifteen years those scores have been declining. (See **Wall Street Journal,** May 31, 1973; and **Newsweek,** June 25, 1973.)

So, traditional approaches have failed to produce expected test results.

Now, let's get back to our vision. Suppose we attribute this failure to show satisfactory standardized achievement scores to the fact that the mathematics offered children is both dull and too hard, and risk the assertion that mathematics made interesting and easy would lead to better standardized achievement test scores — what can the skeptics and realists and antagonists say?

"Show us" . . . is their logical challenge.

So, our vision can move toward reality only as we can help children perform at a level beyond the expectancies on standardized achievement tests.

We accept this condition — "Go your own way, but your accomplishments will be judged in our terms." This is not too unreasonable for us to handle.

Each level of CDA Math — A through F — has two distinct components — (1) **Individualized Computation** and (2) **Patterns and Problems** . These organized materials are easily recognized as a "basic program." In fact these materials can be used by those who still feel that a successful mathematics program requires a textbook.

The title of our resource book, **Drill and Practice at the Problem Solving Level,** which is an integral part of CDA Math further illustrates that attention is given both to cognitive and affective goals. The first part of that title is an unabashed call for "drill and practice" followed by the reservation "at the problem solving level" . . . or, in other words, no return to pages of unmotivated, repetitious, unrelated examples.

The overall CDA strategy is to focus central attention on objectives that lie squarely in the **affective domain.**

If this decision seems inconsistent with the tactical approach suggested above, consider for a moment that it is quite impossible to work successfully toward building positive attitudes toward mathematics and oneself as a learner without development of skills and acquisition of knowledge.

There may be some useful purpose in considering the cognitive and affective aspects of human behavior as separable — for discussion, at least. But it is very hard to imagine a situation in real life in which skills and attitudes are not inextricably tangled. Negative attitudes impair skill development. Lack of adequate skills leads to feelings of failure.

Considering positive attitudes and skill development are, in effect, two ways of looking at the same phenomena — human learning.

Currently popular "instructional management systems" are concerned only with the observable and measurable cognitive development. They have demonstrated that it is clearly possible to develop curriculum that both fosters negative attitudes toward mathematics and shatters children's self-esteem. Young learners who dislike mathematics populate the lower quartile when test scores are in. (Only a few children with unfriendly feelings about numbers find their way to the top quartile — those who have been persuaded to bite the bitter bullet.)

But there are no exceptions in the converse situation: all children who enjoy mathematics can be found in the upper reaches on any measurement of skill development and knowledge acquisition.

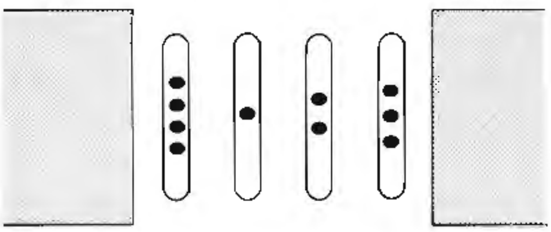
If a child has been regularly exposed to mathematics for several years and is still on a friendly basis with numbers, he must have acquired more skills and knowledge than standardized achievement test norms require. (These norms have been shamefully depressed over the years by the performance of large numbers of children who have learned to dislike mathematics.)

We wholeheartedly support those pioneers in curriculum development who do not require regular exposure to school mathematics from the outset and are willing to wait for those times when mathematical situations arise as part of a less contrived curriculum.

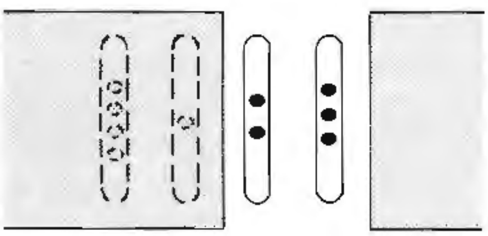
But again, we return to reality. Such a departure from tradition might reveal that Mary hadn't learned the mathematics by her seventh year that test makers think she ought to have learned. And remember that legislators, school board members and parents want "better scores" on thoses tests. They show little confidence that Mary has developed powerful methods to learn what she really wants to learn and feels she needs to learn.

Thus, our compromise in CDA Math: we provide organized materials for systematic exposure to a humanized elementary mathematics program. Let's look at an example of this kind of exposure:

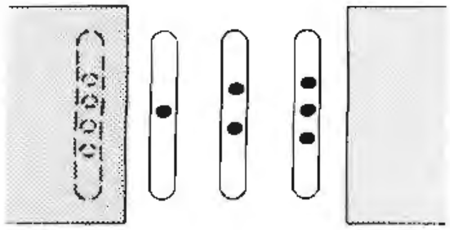
We have 4 beansticks with 1, 2, 3 and 4 beans glued on them . . . or any similar collection of materials. Put the beansticks in front of you in this way:



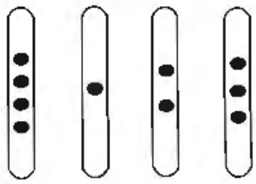
You need two masks that can be moved in from either side — your hands, perhaps. Can you move in masks so you see only 5 beans?



Can you mask off all except 6 beans?

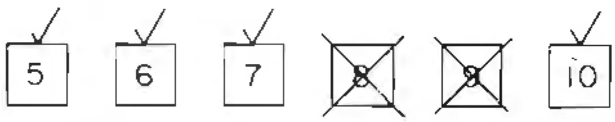


Yes! But, what about masking off 7, 8, 9 and 10 beans? . . . there are a total of 10 beans.



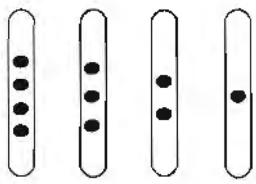
Here is a way to keep track of what you can and can't do. Check the numbers you can mask off — and cross out those you can't.

Our efforts lead to these records:

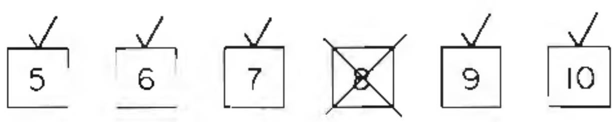


Two crossed out add up to four yeses (√) and two noes (x).

Now, move the beansticks around, perhaps in this way:

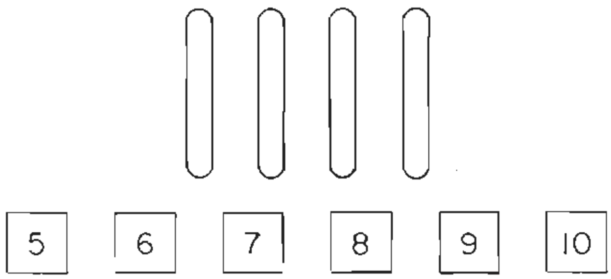


Check and cross out again:



This time only one was crossed out. That's better.

Now it's your turn to arrange the sticks — 1, 2, 3 and 4 —

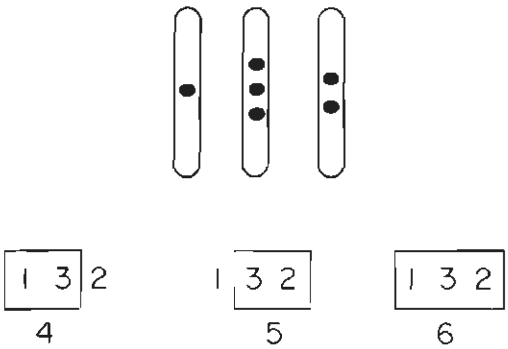


Hopefully, you can arrange them so that there are neighbors for **all** numbers 5 through 10.

With pencil and paper you can work at this problem — remembering all the time that beginners have real-life beansticks to move around.

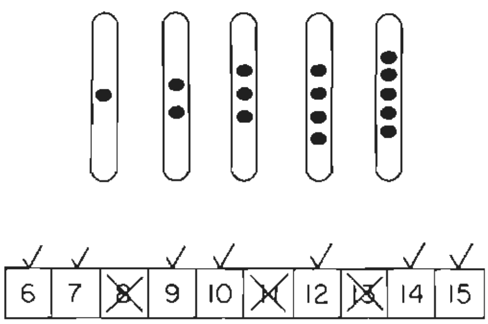
Perhaps I ought to admit to you that I have never found an arrangement that didn't have at least one cross in its report (but don't let that stop you from looking).

But, there is an arrangement of 3 sticks — 1-stick, 2-stick and 3-stick — which has neighbors for 4, 5 and 6 beans.

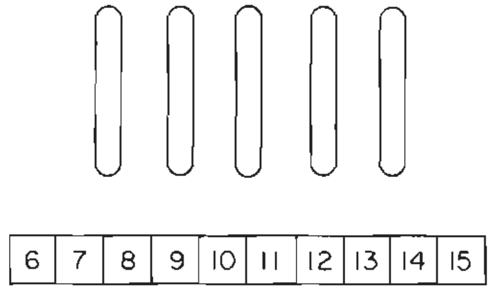


How well can you do with 5 sticks — 1-stick, 2-stick, 3-stick, 4-stick and 5-stick?

Here is a sort of natural order:



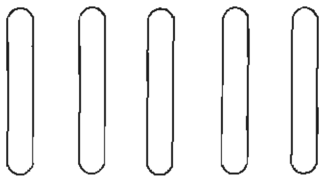
Try again — this time making your own arrangement of that particular group of 5 beansticks.



Can you find an arrangement so you can mask out all numbers of beans 6 through 15?

That is a problem that children and adults can work at together.

If together you can't find such a combination, then struggle with the problem of finding an arrangement so only one of the numbers in the report shows "can't do."



My private guess is that adults thinking and working at this problem have seldom if ever felt as close to their children as "problem solvers."

"Problem solving" in mathematics brings children and their parents and their teachers into the same arena.

If a few parents and teachers are threatened by this, that is their problem; most parents and teachers will revel in such intimacy!

Think backwards now. Is working at these problems — problems not all all demeaning to adults — preparing children for standardized achievement tests?

Had you ever thought before that adults and children could grope ahead toward a solution with such minimal difference — such closeness? . . : encouraging lots and lots of simple computation done willingly because they serve an immediate purpose?

How do you feel at this point about the vision of a breakthrough? What reservations do you still have?

It must be clear now why we invite the use of standardized achievement tests or CRT's (criterion referenced tests) to check us for "accountability." Any regular exposure to a humanistically designed mathematics program that places top priority on developing a positive attitude toward mathematics and a growing self esteem on the part of the learner will have equipped him to score high on any reasonable test of computational facility. (He would also, of course, more fully reveal his growth on instruments designed to assess growth in his ability to solve real mathematical problems — such as the criterion referenced "Problem Solving Activities" designed and published by CDA.)

At this point, we repeate the basic objectives of CDA Math:

1. Our tactics call for careful attention to both the cognitive and affective needs of children.
2. Our strategy focuses on the helping to develop in a learner a positive attitude toward mathematics and a growing self-esteem.
3. We invite outside evaluation with any instrument that represents society's notions of progress.

A mathematics curriculum that places the highest priority on developing positive attitudes provides rich opportunities for growth in human terms even though the results be evaluated with sterile SAT's or currently fashionable "criterion referenced tests for mastery of basic skills."

If staff development effort can show "better results" on standardized achievement tests it will receive support and guaranteed freedom to work toward other goals that are more exciting and more challenging.

In most school districts, a "track record" has been established in terms of some nationally-normed instrument. The outcome of any change or innovation will be judged in terms of that district's own past results —its expectancy based on past experience.

If such results are non-existent or too fragmentary, some standardized instrument can be used as a "pre-test" and again as a "post-test" . . . to generate the "hard evidence" school boards want . . . and have a right to expect.

Such tests can be used periodically, on a limited basis, to provide an interim reading for teachers on progress toward minimal goals.

It becomes clear, at this point, that the administration has an important function because such "baby research" usually can not be organized and carried out by classroom teachers.

Instruments to evaluate development of attitudes toward mathematics are difficult to design though significant progress has been made in this direction. (See the bank of "criterion referenced" tests by CDA entitled "Problem Solving Activities".) But such instruments are not as crucial as one might think. Since progress in the cognitive and affective domain are inseparable, we can for the time being use SAT's to meet two different needs:

a. to demonstrate accountability in the area of skill development (growth in the cognitive domain), and . . .
b. to provide indicators indirectly though quite reliably — of the attitudes of the learners (growth in the affective domain).

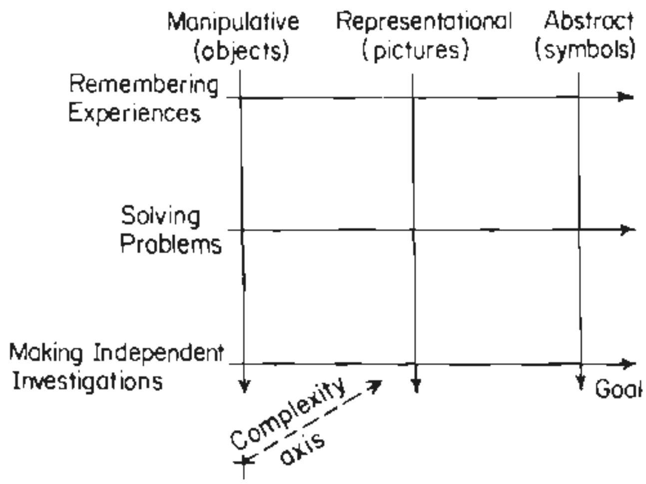
c. LEARNING THEORY

A Statement of a Theory

Any serious effort to develop an appropriate curriculum or provide for appropriate staff development should reflect a basic and consistent learning theory — organized beliefs about the ways in which children learn.

At the outset, I will be as direct and explicit as I can about the theoretical basis that is the determining factor in the positions I take. This theoretical basis is simple, widely inclusive, and for me, illuminating.

It recognizes two developmental hierarchies suggested by the following diagram:



1. All ideas a child fully internalizes must be met first in the world of three dimensional things— the **manipulative** world. Later, the same ideas can be encountered and understood in sketches and pictures and diagrams — the **representational** world. Eventually, the child can work with these same ideas expressed in symbols— the **abstract** world.
2. All learning begins as the senses bring a myriad of data to the mind. Some data is retained, and the child's memory bank grows — he is **remembering experiences.** Soon, a child can use these experiences to determine relationships that are not obvious at the outset — he is **solving problems.** Eventually, the child himself is able to pose problems and work toward solving them — he is **making independent investigations.**

In terms of this theory, the goal of education is to help children develop a taste for carrying out independent investigations at the abstract level (with increasing degrees of complexity).

Movement from left to right in the matrix is suggested by some of the research of Jean Piaget. Vertical movement, from top to bottom, is suggested by some of the research of Jerome Bruner.

Both movements are hierarchical, beginning with the child's confrontation with the physical world around him and moving toward handling abstraction, making independent investigations — (and moving to higher levels of difficulty). But movement in those two directions takes place in different ways, with different tempos, with varying possibilities for productive adult intervention.

MANIPULATIVE TO REPRESENTATIONAL TO ABSTRACT

Movement from manipulating objects to manipulating symbols is generally very slow.

The meaning of any abstraction is precisely the sum total of the experiences that are its referent. Everyone is familiar with this fact in language development. A child may learn to say the word "dog" and spell it correctly — and yet it may have no meaning for him. Its meaning depends entirely on his experience with dogs, pictures of dogs, stories about dogs, etc.

I had occasion to fly with my family to Chicago and drive through the country-side to my home town. David saw his first "silo."

"What's that?"

"A silo."

"Is that another silo over there?" and he began finding silos in every direction.

"What are silos for?"

From my personal experiences with silos as a child, I filled David in as best I could.

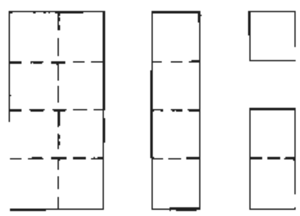
The peculiar sweet-sour smell of a silo full of feed corn came back to me . . . but I didn't try to describe it; perhaps someday we could visit a silo.

Confrontations with the real world of things is an inundation of input coming in through all the senses. Pictures are sterile by comparison unless they call up earlier experiences.

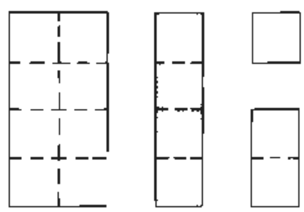
There is another fundamental difference between manipulative and representational situations.

Two episodes some six or seven years apart helped me see the differences and its significance (repeated here from **Drill and Practice — an alternative).**

I introduced Joanne at age 5 to the game of "coloring panes and windows." There are windows with 1, 2, 4 and 8 panes. The rule for coloring is that if you color any pane, you must color all the panes in that window.



In successive sketches she colored in 1 pane, 2 panes . . . and 7 panes. She wanted to color 8 panes in the sketch and began by coloring the 4-pane window.



(after all, this start had worked for 4, 5, 6 and 7 panes.)

"That's wrong! . . . what can I do?"

"Just cross out that window and try again."

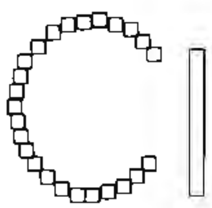
She complied reluctantly. As she finished coloring in the 8-pane window, her comment was: "Daddy, I don't want to play that game any more."

At that moment, I thought back very many years to Clarence, a 6-year-old boy who was having trouble getting started in school. He was in a group of "immature children" who were being considered as kindergartners.

Clarence was a big boy, usually accounting for more than his share of commotion. He often played for long periods of time building roads and castles with Cuisenaire Rods.

One day his interest in moving the blocks around became intense. He worked with a purpose. He had isolated the little 1-centimeter cubes and was moving them around, making adjustment in the position of one cube after another. The following sketch suggests certain stages of his work.

"See! . . . that's the first letter in my name!" And with but a moment's hesitation, he picked up the longest rod (10 cm) and placed it next to his "C"—



"and that's the second part of my name."

The teacher asked if he would like to draw the letters with a pencil. He sure would! . . . and did. Before he went home that night he could write his name.

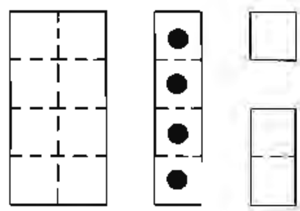
It was years later when windows and panes were the problem that this message from Clarence finally came through loud and clear to me. "Let me first meet an activity at the manipulative level where I can set a goal, work toward it, getting closer — and finally announce 'I've got it' . . . **with no history of the trials and errors that preceded the achievement."**

Joanne had been drawn into an activity at the "representational level" . . . and the characteristic of that level is that there is a history of less than successful efforts, because the experiment was recorded in print.

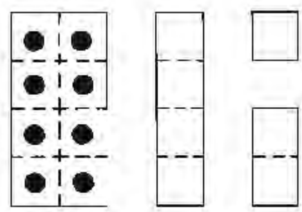
Had Clarence been asked to "copy the dotted line" his results might look something like this:

— a discouraging history of his failures to "stay on the line."

And if Joanne could have met "windows and panes" in a "manipulative activity" she would not have been discouraged. Suppose she had been asked to place 8 counters on a sketch with the rule that if you put a counter on one pane of a window you must place a counter on each pane in the window — and she had started with



She wouldn't have asked what to do to correct her false start. She would have moved the 4 counters over and added 4 more:

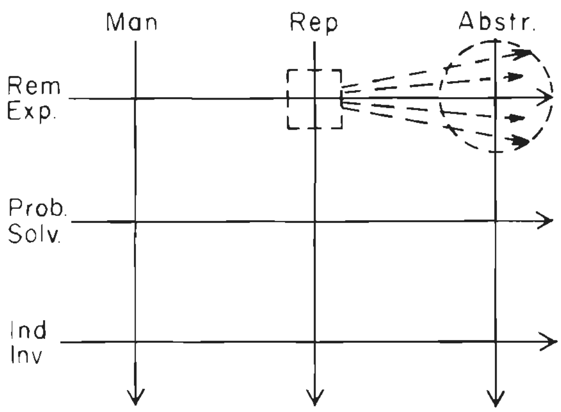


"I got it . . . and now I'm going to try 9 counters."

Children will internalize fully only those notions they meet first in a real world context, with input coming in through all the senses and with complete freedom to engage in trial and error—with no record of the failures that lead to ultimate accomplishment.

Clarence and Joanne are eloquent in their pleas that we concern ourselves more deeply with the ways children learn.

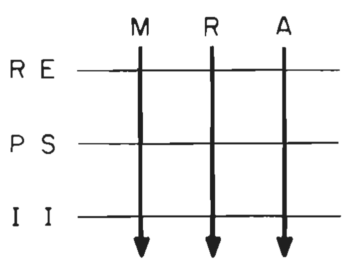
We can now see why traditional materials of instruction have contributed to dismal results: they start with a few pictures and race out to manipulating abstractions — abstractions that have little meaning because they have such inadequate referents. Positioned in our 3 x 3 matrix, they look like this.



As we proceed to design alternative materials, we need to search long and experiment tirelessly to find appropriate activities and situations at the manipulative levels when any idea is first introduced.

REMEMBERING EXPERIENCE — PROBLEM SOLVING AND INDEPENDENT INVESTIGATION

Movement from top to bottom in our two-dimensional matrix suggests a very different aspect of learning.



As the child experiences the world around him his mind is literally bombarded by sensory data. In some fashion he filters this information and retains a portion, adding it to his memory bank . . . the act of "remembering experiences."

This kind of learning is common to all animals—remembering experiences. A dog remembers where his dinner bowl is placed, etc.

The human mind is capable of a very different kind of learning which we refer to as "problem solving."

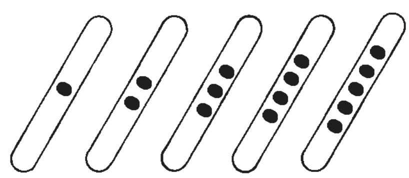
We need to define what "problem solving" means in this context. It does not mean those 25 exercises on page 42: we refer to them as "examples." They are a kind of test: Can you perform on command the tricks you have been taught and supposed to remember?

By "problem solving," we mean working at a situation that is new, bringing remembered experience to bear on that "problem" in an effort to uncover something that is not obvious.

An "appropriate" problem is one the solver understands and has sufficient experience to work productively toward a solution.

There is another anecdote described in **Drill and Practice — an alternative,** that warrants repeating here.

Michael was 6 years old. He was introduced to this set of beansticks:



"Can you pick up 1 bean?" . . . "Sure."

"What about 2 beans? . . . 3 beans? . . . 4 beans?"

When trying for 4 beans he first picked up the 5-stick — counted it, and picked up the 4-stick. He had no trouble with "5 beans."

"How about 6 beans, Michael?"

"Naw — there's no stick with that many beans on it."

"But you can pick up more than 1 stick at a time."

"Oh . . . I got it."

"What about 7 beans, Michael?"

He tried several combinations to find one with just 7 beans. "I got it . . . and now I'm going to try for 8."

"I got it . . . and now I'm going to try for 9."

He considered each number until he ran out of sticks. He held up all 5 sticks in one hand answering "Fifteen: I got it, and that's all there is to it!"

And Michael is in the "mentally retarded" group!

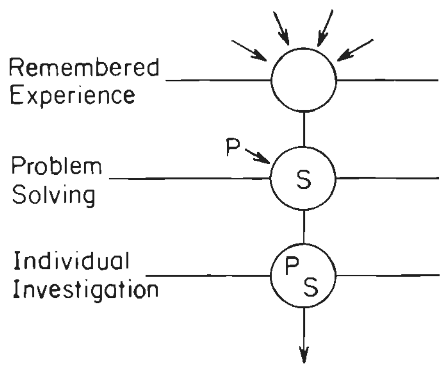
He had been introduced to 5 beansticks he could move around. He could touch the beans and count them.

The request for 6 beans and then 7 beans confronted him with problems. At "8 beans" Michael took over on his own and completed an "independent investigation."

He demonstrated convincingly that beginners, even though they may have less than usual mental development showing, can display the full range of human thinking when they find themselves in an appropriate situation and urged on beyond simply remembering experiences. And he was proud of what he had done.

Michael may develop very slowly toward operating comfortably at the abstract level, but that need not prevent him from experiencing the thrill that awaits all learners who develop a taste for making independent investigations.

With the help of our 3 x 3 matrix, we might summarize this movement this way:



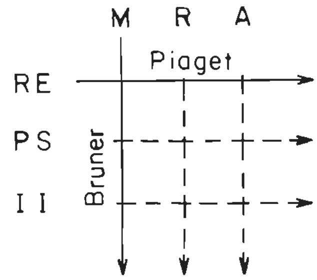
At the "remembering experiences" level, all input comes from outside. The mind is much like a sponge, soaking up what it can or wants to.

At the "problem solving" level, the problem is posed from outside, but the learner takes over and solves it in his own way.

At the level of "independent investigation," the learner poses the problem for himself and works toward a solution in his own way. At this most mature level, learning is a fully internalized process independent of any outside intervention.

As children move to the level of independent investigation they take over responsibility for their own continuing education.

As already mentioned, left to right movement in this 3 x 3 matrix is suggested by the findings of Piaget and vertical movement by the work of Bruner.



It is instructive to realize why the reports of these two seminal thinkers in learning theory seem so different.

Piaget is an observer interested in the "natural" development of learning — reporting the ways children learn. He sits outside the learning process exercising great care not to "contaminate" the phenomena he is analyzing.

Bruner, on the other hand, wants to contaminate the learning process — intervene in and influence it as effectively as possible finding ways to encourage learners to make independent investigation a way of life.

Piaget has become firmly convinced that, in the overall view, children mature slowly, developing according to a kind of natural time table as they move from sensory awareness of the world about them to manipulating abstract symbols and finally developing abstract systems.

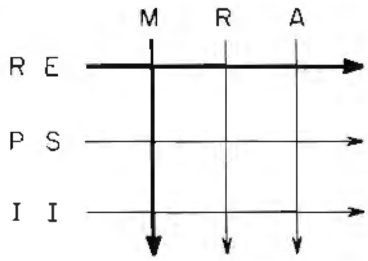
Bruner looks rather at cultivating the uniqueness of human thinking — the ability to solve problems and carry out independent study. And, he is keenly aware that individual learners can, in certain situations, traverse the matrix from top to bottom with amazing speed.

From Theory to Practice

Now, we need to demonstrate that what may seem neat at the theoretical level can be translated into practice:

1. You will see the tactics CDA Math uses to introduce all basic operations to beginners as a means of keeping records of arranging and rearranging counters.
2. You will see a representative activity that begins with problem solving using counters, moves to problem solving based on sketches and then on to problem solving at the "abstract level." Lastly, you will see how CDA Math approaches problem solving in "story problems" or applications.

We have discussed in the previous sections two basic aspects of learning suggested by horizontal and vertical movement in a 3 x 3 matrix:



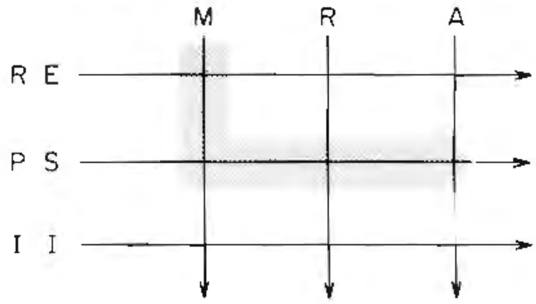
This extension of the discussion will deal with a basic strategy for moving through this matrix and consider examples of tactics that might be used successfully.

The traditional strategy, as mentioned before, is to introduce ideas with a few pictures and sketches, and then jump to abstractions — all at the level of "remembered experience."

Consequently, arithmetic becomes an exercise in remembering as many facts and processes as possible . . . preparing to "perform on command" at the abstract level.

This is understandable. Such performance at the abstract level is what standardized achievement tests are all about. It is the goal of school board administrations and parents. If you are a third grade teacher, you know the fourth grade teacher is worried about the same kind of performance. That's the way it has been. Of course, all the evidence shows that this strategy of teaching for command performances has failed.

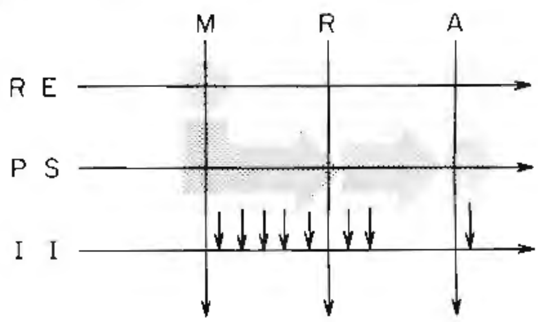
CDA Math has a very different basic strategy. We can suggest it in over-simplified form in this way:



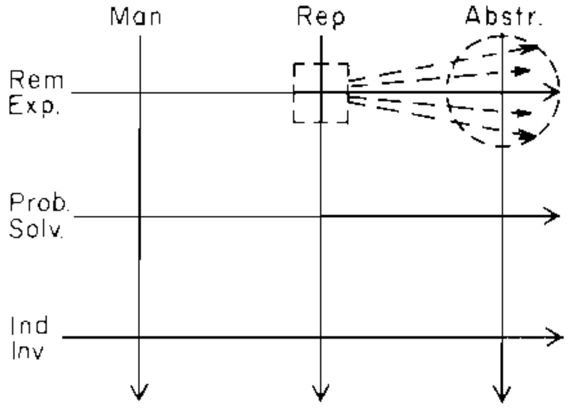
In words: introduce the basic ideas of arithmetic at the manipulative level; move as soon as possible to the problem solving level; and then move across at this level so eventually children are "solving problems at the abstract level."

Please note at the outset that our goals are not the same as those presumed by the standardized achievement tests . . . but those test goals are more than satisfactorily achieved in reaching our ultimate goal. In other words, a child who can do "problem solving" at the abstract level obviously must be able to "compute" — he is simply computing with a purpose — to solve problems. And along the way he has probably often responded to the opportunity to solve problems he made up himself.

Perhaps the following diagram is more suggestive of the ground covered by a child who enters the CDA Math series:



Traditional textbooks for beginners present some pictures of objects as referent for moving on quickly to the manipulation of symbols — to abstraction. We have suggested this program with the following sketch:



Teachers are thus given the full responsibility to develop appropriate manipulative activities that will provide experiences upon which the meaning of later abstractions depend. Textbook writers seem to feel that, *if they talk about "manipulative" in the teachers* manuals and include a picture on the pupil page, they have fulfilled their responsibilities . . . after all, that's all that can be done on two-dimensional pages.

CDA Math has designed materials that ask beginners to use a two-dimensional page as a place to arrange and rearrange objects and for keeping records of what happens.

A student will be ready for this "record keeping" only after he has internalized the concept of the four operations. This undoubtedly happened before the child came to school. What child does not understand about getting more to go with what he has? . . . about losing or giving away some? . . . about sharing what he has, equally?

What the child needs to be encouraged to do is verbalize the operations — to formulate the ideas in his own words before learning "how adults say it." Early experiences with any idea, at any level, should include periods of:

— "Talking Math" as a group,

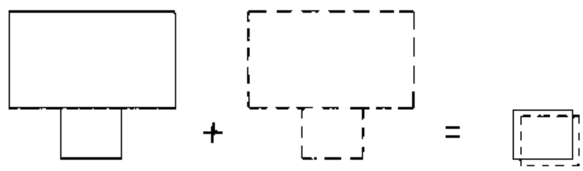
— a chance to talk about his activities with an interested peer or adult,

— a sheet of blank paper on which to record in his own way what he has done.

The formats of **Individualized Computation, Levels A_1 and A_2** are essentially game-boards. The playing pieces are counters of any kind. A record form is given in which to record outcomes. Numbers are written to report the results of counting. The signs :, —, > and - indicate four different sets of directions.

This skill is all children need to learn to demonstrate the fact that they understand the basic operations of arithmetic. They are ready for **Individualized Computations, Levels A_1 and A_2**.

Playing pieces are counters of any kind. A record of each episode is kept using numerals (or numbers, whichever sounds better to you).



"Please put some counters in one of the frames and show how many you used (writing in the small box below the frame). Put some counters in the other frame and show how many you used. In the double-line frame on the right, show how many counters you used altogether."

The symbol for this game is " · " and adults complete the record:

$$\boxed{4} + \boxed{3} = \boxed{7}$$

Adults read it "four plus three equals seven" but children ought to be encouraged, at first, *to use their own language to verbalize what they have done.*

And now, the children are encouraged to play the game and record what they do using as many counters as they like.

They are truly "in the driver's seat." Each child determines how many counters he or she will use. Some children will stay with less than 5 in each frame; others will crowd the frames and find themselves asking "how do you write seventeen?"

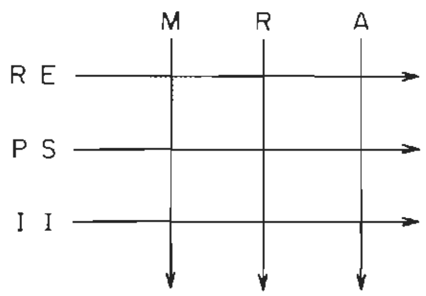
Some will develop plans so that one example is related to another, such as leaving 5 in one frame and increasing the number in the other frame as they carry out a series of experiments.

Usually, these plans have a general purpose — not having to start over from scratch each time; and they sometimes result in the elimination of one-by-one counting, thus anticipating what lies ahead of them in elementary school arithmetic.

They will encounter no surprises because they know all there is to know about what we call "addition of whole numbers." The help they need is generally limited to writing larger numbers to report what they have done. And the activity is completely "individualized" even though some children may have the same "game board."

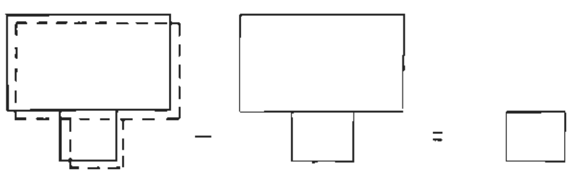
This is a totally new concept of "pencil and paper" arithmetic. The only "abstraction" that arises is using a code to show what you have done — describing what you have done and what you see in front of you.

Such introduction to arithmetic can be suggested with this sketch:



. . . In terms of the learning theory we are proposing, this is a most appropriate beginning.

The second activity uses a very similar gameboard format. It's designated sign is "—".



"Please put some counters in the frame on the left, and show the number of counters you are using; move some of those counters (including all or none) to the other frame, and report this number moved; finally, in the box on the right, report the number left in the starting frame."

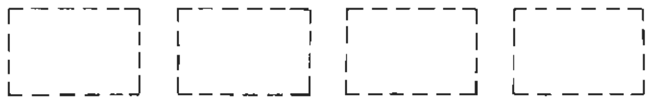
Adults read a final report such as

$$\boxed{7} - \boxed{3} = \boxed{4}$$

as "seven minus three equals four" but before we use adult language, the child ought to be encouraged to describe the experience in his own way.

And now that the learner knows all there is to know about subtraction of whole numbers, he is in the driver's seat in complete command of where he goes.

A third game board format is quite different:

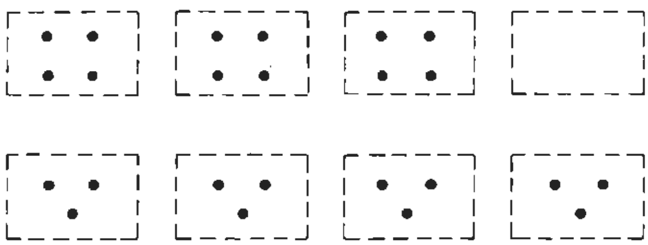


$$\square \times \square = \square$$

"Please decide how many counters are going to be placed in each of how many frames; and note these plans in the report forms," such as

$$\boxed{4} \times \boxed{3}$$

This can be read in two different ways: 4 counters in each of 3 frames or 4 frames with 3 counters in each. When carried out, these possible results will occur:



Any ambiguity is resolved by the fact that the final report of how many counters are involved is "12" in both cases:

$$\boxed{3} \times \boxed{4} = \boxed{12}$$

$$\boxed{4} \times \boxed{3} = \boxed{12}$$

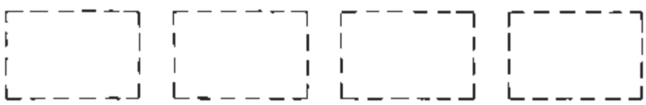
Children should be encouraged to describe what they have done in their own words.

Stop a moment to think about this question: at what time in a child's development is this activity appropriate?

If we look it up in the standard scope and sequence chart, it is reserved for those few children who reach the last chapter in the second grade text; but most children encounter it in the middle or later part of the third grade text . . . and in both cases they are asked to understand what's going on with only a few pictures and lots of words.

If "appropriateness" is the central criteria for determining "scope and sequence" in curriculum design, "multiplication" is an activity for beginners . . . as soon as they begin recording the results of what they do with things. Later the activity can move to the representational levels, considering sketches, diagrams, and pictures. When the learner eventually uses symbols alone, they are generalizations for a common aspect noticed in many and varied previous experiences.

The final kind of experiment in the sequence we are outlining uses a game board very much like that last one and the symbol for it is "÷".

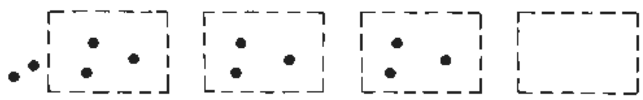


$$\boxed{} \div \boxed{} = \boxed{}$$

"Please start with as many counters as you like, such as 11, and keep a note of that by writing '11' in the small box on the left; decide how many frames to use, and report that choice in the small box after the '÷', such as:

$$\boxed{11} \div \boxed{3}$$

"Please distribute the 11 counters in the three frames, ending with the same number of counters in each frame:"



. . . if one tries to put 4 in each frame he is 1 short; if he puts 3 in each frame, 2 are left over. This outcome can be reported:

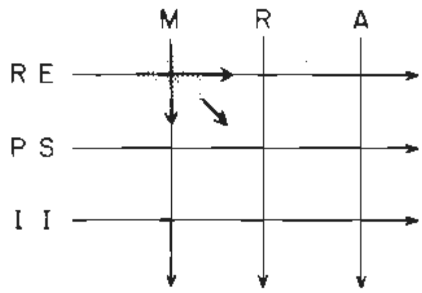
$$\boxed{11} \div \boxed{3} = \boxed{3}^{\underline{2}}$$

— "three in each frame and two left over."

Again, the learner steps into the driver's seat by making up his own experiments, carrying them out and reporting the results. He knows all there is to know about the operation of dividing whole numbers.

The main content of **Individualized Computation, A¹ and A²** is performing these four basic experiments with objects and noting results. It recognizes that children already understand the four basic "concepts" involved, but need help and experience in keeping records that accurately report those experiments.

This general tactic can be placed in our learning theory matrix in this way:



The arrows suggest that very early some children move out on their own — toward "abstraction" and toward "independent investigation."

"Do I have to use counters?" . . . of course not, as long as you can get right answers in some other way.

Also, we've found that many children want to use tallymarks instead of counters ("representations" of counters) often devising ingenious ways to draw many examples in a very limited space. Others prefer sketches that suggest real objects.

So, the question is raised: "How soon can the children work without objects?" The answer: "Whenever they feel confident to make statements before carrying out actual experiments." A few children will prematurely want to put the counters away — willing to guess; but errors are clear evidence they are not ready. "Let's do those experiments with counters — so you can know that what you say is reasonable."

A few children may be more timid than they need be — arranging objects even though they know in advance what the outcome will be. They can be prodded gently: "Do you know what would happen if you carried out this experiment?"

$$\boxed{10} - \boxed{2} = \boxed{}$$

"Can you find the answer without counting beans?"

Prodding, yes, but pushing, no! Some beginners are so fearful of being "wrong" they are reluctant to risk any new venture. If they are pushed by adults, they feel threatened and their fears tend to immobilize them. Their usual defense in such situations is to make wild guesses . . . perhaps with a smile. *There is no stigma attached to an answer that is* clearly a "guess," but if you **try** and the answer is *wrong — that's failure!*

*There is an arrow in the 3 x 3 matrix above suggesting movement toward making "independent in*vestigations" because, for instance, some beginners wonder how many different experiments can be carried out and reported by completing this "open sentence."

$$\boxed{} + \boxed{} = \boxed{5}$$

Some may pursue this idea to find out that there are 6 ways — 1 more than 5: and predict that there will be 10 different ways to complete:

$$\boxed{} + \boxed{} = \boxed{9}$$

and 101 ways to complete:

$$\_\_\_ + \_\_\_ = 100$$

There is a surprise for the child who asks the same question about this situation:

$$\boxed{} - \boxed{} = \boxed{5}$$

"There are more than you could ever write." And

$$105 - 100 = 5$$

is enough to make that clear. One child announced that there are "affinity" ways to do it.

The introduction of arithmetic for beginners just outlined must be accompanied from the outset with appropriate problem-solving activities at the manipulative level. And as the elementary program unfolds, moving children slowly from manipulation to experiences with representations of objects and finally to work at the abstract levels (manipulating symbols), problem solving activities must play a central role.

Before we go on to some representative activities found in CDA Math, we should first consider a stubborn dilemma everyone knows about but seems to turn away from with regrets and without resolution.

We all realize that practice is a requirement for committing facts and procedures to memory. This is commonly expressed as "kids don't get enough drill and practice."

It is also generally understood that "it's the 'drill *and practice' that turns kids off" to arithmetic.*

Simply stated the dilemma is: "Not enough is already too much."

A child with a healthy, inquiring mind will rebel against large doses of unmotivated repetitions — drill and practice.

But that same child, with a healthy, inquiring mind, will eagerly practice what he should eventually remember if he encounters it as requirements in solving a problem he wants to solve. The desire to find a solution to a problem provides the motivation that has been missing.

In **Drill and Practice at the Problem Solving Level,** there are about 300 pages of activities placed all along the axis from "manipulative to abstract" including varying degrees of complexity.

One strand of CDA Math is developed in a series of six levels called **Patterns and Problems**. And, in **Individualized Computation** drill and practice are consistently set in a problem solving context.

Now, let's select a rather simple idea and look at it as it might come up at several stops along the continuum from "manipulation to abstraction." To illustrate the fact that interesting problems often do not require large numbers, each variation will involve no numbers larger than 21.

We can start out in kindergarten with children who are learning to count. There is a supply of counters.

"Please count out 11 beans." After each child has his counters and has checked to be sure: "Can you arrange them in piles of 3's and 4's?"

Do you think it can be done?

Kindergartners have no trouble at all. They almost always start making piles of 3 and reach this point:

with 2 not used. Since piles of 4 are also allowed, one extra is added to each of the 2 piles. . one pile of 3 and two piles of 4.

If it seemed appropriate, some method could be used to show such combinations in this form of:

11 ⟶ 1(•••) and 2(••••)

or

11 ⟶ 1(3) and 2(4)

"Did everyone do it with one 3 and two 4's?"

"Do you think you can do it in a different way? Try 11 counters again." Again everyone ends up with a 3 and two 4's.

"Please start over. but this time we'll begin with 12 counters. Can you arrange 12 counters in piles of 3's and 4's?"

Again, the children have no trouble, usually making them all in piles of 3's.

12 ⟶ 4(3) and 0(4)

"Is there another way to arrange 12 counters in 3's or 4's?"

It turns out, of course, that there is another arrangement — three piles of 4 —

12 4 (3) and 0 (4)

and

12 0 (3) and 3 (4)

Which numbers of counters up to 12 can be arranged in 3's and 4's in two different ways?

Starting with 3 counters and adding one more each time would produce these results:

3 → 1 (3) and 0 (4)

4 → 0 (3) and 1 (4)

5 → (can't be done)

6 → 2 (3) and 0 (4)

7 → 1 (3) and 1 (4)

8 → 0 (3) and 2 (4)

9 → 3 (3) and 0 (4)

10 → 2 (3) and 1 (4)

11 → 1 (3) and 2 (4)

12 → 0 (3) and 3 (4)

or

4 (3) and 0 (4)

Perhaps a few would like to go on. How about you?

Would you care to predict what's going to happen with 13, 14 and 15 counters? Can each of those numbers of counters be arranged in more than one way as piles of 3's and 4's?

Spotted through this little investigation we've looked at in a kindergarten situation, were questions you probably had to stop and think about . . . and you were engaged in "drill and practice" in a problem solving context.

A favorite game for beginners is "Balloons and Bunches." The board has bunches with 2, 3, 4 and 8 balloons. Any small movable objects can be used for playing pieces:

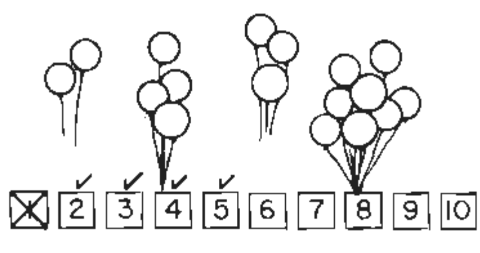
There are three simple rules:

(1) No more than 1 counter on any one balloon.

(2) If you put a counter on one balloon you **must put a counter on each balloon in that bunch.**

(3) You may use as many bunches as you like (e.g. to make 5, you can use a bunch of 3 balloons and a bunch of 2 balloons.)

Here is one possible game board:



If you can put a given number of counters on a bunch according to the rules, that number is checked (√) in the row of boxes; if you cannot, then cross out that number. The first 5 examples have already been worked.

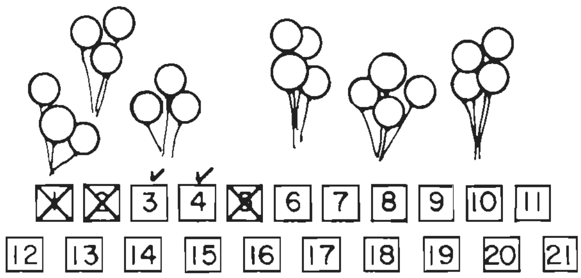
Can you put exactly 6 counters on the board? . . . 7 counters? . . . 8 counters? . . . 9 counters? . . . 10 counters?

When I have played this game with groups of adults, they reply "yes" in "chorus" for 6, 7 and 8 . . . but for 9 there is usually a moment's hesitation and then a mixture of "noes" and "yeses" in about equal proportions.

At this point we have an operational definition for a problem that is appropriate for beginners; one that's not too hard for them but hard enough so adults disagree about the answer until they have thought about it for awhile.

Once the rules of "Balloons and Bunches" are clear, many variations are easily available and children need no further introduction.

Here is another game board and report form:



There is no way to put exactly 1 or 2 counters on the board, and no difficulty with 3 and 4. There is no way to use just 5 counters. What about 6? . . . 7? . . . 8?, etc.

Notice there are 21 balloons altogether.

Please stop and consider combinations for all other whole numbers less than and including 21. After you've done this, we'll compare results.

How many times did you have to stop and think for at least a moment? The amount of pleasure in each case was probably directly proportional to the effort required (and the success achieved).

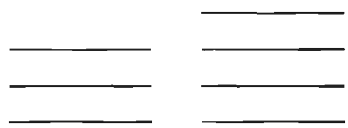
Were you surprised that while it took some time to realize there was no way to use 16 counters, you could easily use 17 and 18?

And using 19 or 20 is as difficult as using 1 or 2; in fact, using 19 or 20 would be like putting a counter on all 21 balloons and then picking up just 1 or 2 without breaking the rules. Using 16 counters is comparable to leaving just 5 balloons empty — clearly impossible.

Did you feel demeaned in any way by this little investigation . . . even though it was designed in part to provide large quantities of drill and practice for beginners?

Next, consider an activity at the "representational level" we call "lines and crossing points." It can be introduced without any complexity and then quickly be changed to "problem-solving."

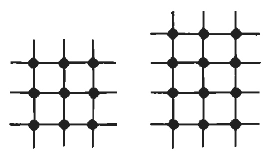
A diagram has the following arrangement of lines:



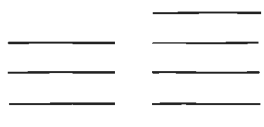
Three parallel horizontal lines on one side and four on the other.

The activity calls for adding vertical lines and reporting the number of crossing points on each side as well as the total. To stay within given limits we will permit no more than 3 vertical lines on each side.

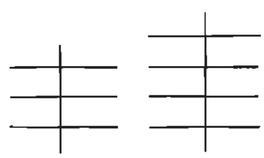
Here are some examples one might do:



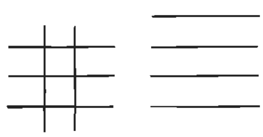
9 + 12 = 21



0 + 0 = 0

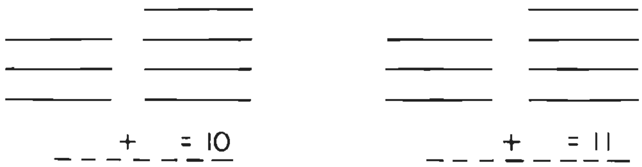


3 + 4 = 7



6 + 0 = 6

But there is no motivation to keep on completing such sketches. So, as soon as the learners are familiar with this activity they are confronted with problems:



These sketches ask you to draw vertical lines crossing all the lines on one side or another to produce 10 crossing points and 11 crossing points, etc.

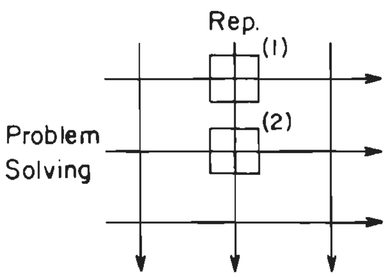
Remembering that there is a restriction of 3 or less lines on each side, how many different sketches can be drawn — sketches that lead to different reports?

In the sketches above, there are 6 different reports given:

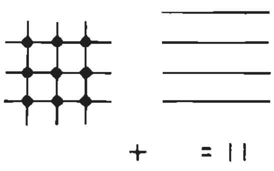
9 + 12 = 21	0 + 0 = 0
3 + 4 = 7	6 + 0 = 6
6 + 4 = 10	3 + 8 = 11

How many more can arise in this activity? (You might stop at this point and try your hand at this problem.)

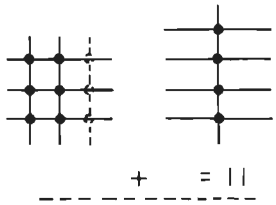
In terms of the matrix we are using to suggest a theory of learning, "lines and crossing points" was described as a "representational" activity not yet involving problem-solving and then described at the "problem-solving" level.



Some children have faced "lines and crossing points" problems before they were adequately prepared. Asked to make 11 crossing points, they may begin in this way:



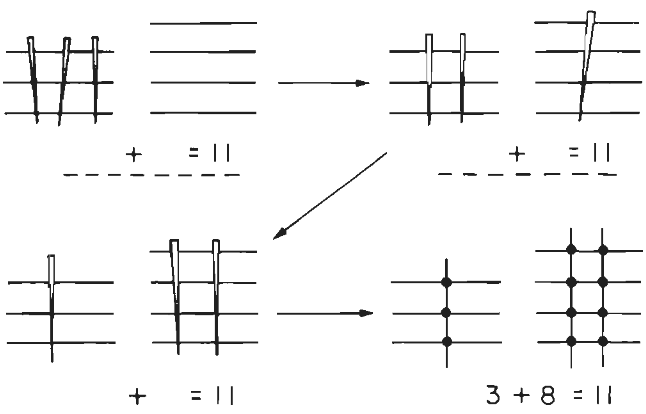
What to do? As in most all "representational" versions of activities, to make a move is to write or draw something that requires erasures to change. After one erasure, the problem still isn't solved.



But, 10 is the number represented. It would take another erasure, but few beginners handle that much frustration. The problem is, therefore, not appropriate in its present form. Retreat is in order.

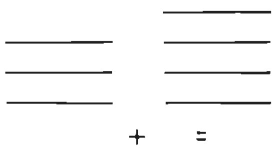
A first grade teacher suggested a way out — changing the activity into a manipulative counterpart children have named " toothpick 'arithmetic'. "

The same sketch is used, but vertical lines are not drawn until tooth picks are arranged to reveal the solution.



When frustration develops beyond normal levels, it is a signal to move back in the matrix — in this case, to remove the "problem" aspect, or to retain the problem but recast the activity in a manipulative context.

This side trip left unresolved a question that arose earlier. How many different reports can we generate using 3 or less lines on each side of the sketch?



One way to tackle this problem is to consider each number of crossing points 21 or less — can they or can they not be produced?

✓			✓	✓		✓	✓	✓	✓	✓	✓
0	~~1~~	~~2~~	3	4	~~5~~	6	7	8	9	10	11

✓	✓	✓	✓		✓	✓			✓
12	13	14	15	~~16~~	17	18	~~19~~	~~20~~	21

That is 16. But there may be some numbers that can arise from two different sketches.

For example:

0 + 12 = 12   12 + 0 = 12

— two different sketches with the same number of crossing points, but we've violated the condition that the maximum of vertical lines on either side is 3 . . . and one of the sketches uses 4 lines on the left side.

Are there some numbers of crossing points that can arise from 2 different sketches?

No amount of hard work will lead to any such situations and we are left with 16 different sketches . . . after quantities of drill and practice.

One reason children enjoy "loop arithmetic" is that they don't have to write any numbers at all — they simply loop a number or series of numbers in a list so the sum of those numbers is the number given at the bottom. Here is an example:

3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
3	4	5	6	7	8	9	10	11

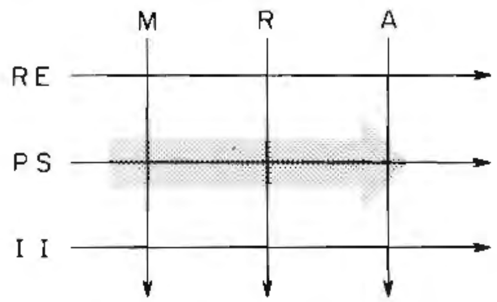
There's no way to indicate 5. How about 8, 9, 10 and 11?

Since the sum of all numbers in the list is 21 — consider sums from 11 through 21.

Much earlier in this discussion, we looked at a problem involving "pencils @ 4¢ and erasers at 3¢" . . . with a limit of 2 each to a customer. If the limit were raised to "3 to a customer" the sale sign would read:

SALE	
Pencils	*Erasers*
4¢	*3¢*
Limit: 3 of each to a customer.	

You already know a lot about this situation — three 3's and three 4's — whether it's piles of 3's and 4's, balloons in bunches of 3's and 4's, looping entries in a list of 3's and 4's, or 4¢ pencils and 3¢ erasers with a limit of 3 each to a customer . . . the same arithmetic arises time and time again, but it is always incidental to finding answers to less than obvious questions. This is the basic strategy we propose as a means of resolving the dilemma of "not enough is already too much" — the strategy of providing "drill and practice at the problem solving level." The investigation of 3's and 4's moved from manipulating objects to working with diagrams and finally working with abstractions.



In working with children, this movement is very slow — a kind of natural development from immature to more mature, a growing process.

And, whereas for a special reason we focused on a single basic situation, 3's and 4's, the problems can have unlimited variety.

INDEPENDENT INVESTIGATION

Making independent investigation is realizing the fullest potential of the human mind . . . the goal of education. It's at this point that the learner takes the initiative to pose his own problems and work toward solutions; learning becomes a way of life.

Learners cannot be pushed into making investigations because that "push" would come from outside and not from the learner himself, but often are drawn in by honest involvement of an adult in the problem.

An urge to discover more about something often grows out of problem-solving activities. Problems typically arise within clearly defined limits — such as 3 or less pencils at 4¢ and 3 or less erasers at 3¢. Uncovering relationships that exist within those tight boundary conditions leads to wondering what happens if some of those conditions were relaxed.

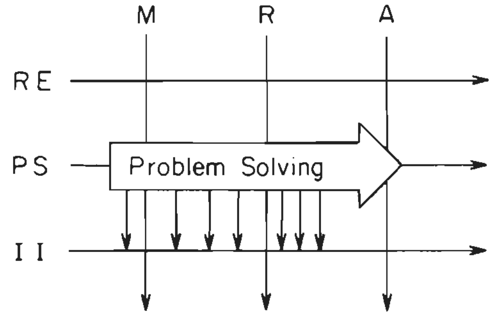
How does the situation change if the price of pencils were changed to 5¢ . . .to 6¢? Or, since you can't spend 16¢ under the stated terms of the sale, what is the result of adding a third item — notebooks at 16¢ each but no more than 3 to a customer?

Earlier, we found there were 12 ways to make change for a quarter with pennies, nickels and dimes. How does the situation change if we wonder how many different combinations there are that add up to 28¢ . . . or 35¢?

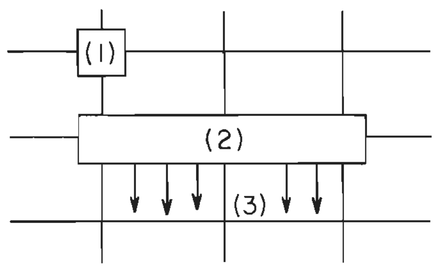
Problem-solving is almost a precondition for undertaking self-initiated searches for new information.

Consequently, the strategy of **Drill and Practice at the Problem Solving Level** creates a climate most conducive to original work. With the learner in the driver's seat, solving problems in his own way, there is always the temptation to celebrate success by choosing his own destination.

The arrows in the diagram below suggest these opportunities:



The three basic elements in our strategy might be suggested in our matrix in this way:



(1) To introduce the basic operations of arithmetic at the manipulative level,

(2) To provide abundant "drill and practice at the problem solving level," and

(3) To encourage learners who are inclined to make independent investigations.

Our ultimate goal is: learners who involve themselves in independent investigations at the abstract level — solving problems of increasing complexity.

Too Abstract Too Soon

Countless teachers of beginners in mathematics have voiced this complaint in one form or another. After a few pages of sketches, textbooks rush on to manipulation of symbols — for which no adequate referent has been built in the student's experience.

Perhaps the most glaring example of this is the rush to "place value." This is most damaging because place value is such a productive source of shortcuts to avoid one-by-one counting. Such a fundamentally important notion ought to be introduced with meticulous care — with full attention to the ways children learn. The language of place value is replete with abstractions — "tens" and "ones," the "tens place" and the "ones place." Such terms should be postponed until they enter as convenient ways to talk about ideas that the learners have encountered in many ways: 10-sticks and loose beans, dimes and pennies, blocks stacked 10-high and loose blocks, etc.

Beginners do not talk nonsense about "10-sticks" and "loose beans," but they will make strange statements about "tens" and "ones."

Standard textbooks are so anxious to get to the abstractions of "place value" they plunge in before children have encountered addition facts with sums as large as 10. Consider the "scope and sequence" suggested by the chapter headings of two widely-used 1st Grade textbooks.

SERIES A—FIRST GRADE

1—Sets and Numbers to 6 .... page 1
2—Sets and Numbers to 10 .... page 31
3—Joining and separating sets .... page 57
4—Combinations to 5 .... page 89
5—*Tens and ones* .... page 121
6—Combinations to 7 .... page 183
7—Combinations to 10 .... page 215
8—Hundreds, tens and ones .... page 277
etc.

SERIES B—FIRST GRADE

1—Geometric shapes .... page 1
2—Numbers and numerals .... page 21
3—Numbers and numerals .... page 31
4—*Place Value* .... page 81
5—Basic addition and subtraction
Combinations through 10 .... page 131

In Series A, Chapter 4 is concerned with "combinations to 5" only. But Chapter 5 deals with "tens and ones" in which children are asked $18 + 2 =$ ____ (on page 134) and 3 tens and 2 ones $=$ ____ (on page 149). Then after reviewing "combinations to 5," a chapter is devoted to combinations to 6 and 7.

In Series B, Chapter 3, while labeled "numbers and numerals," actually presents addition and subtraction through 9. Chapter 4 introduces "place value" and children are asked $40 + 9 =$ ____ before they have considered $1 + 9$ ____.

This rather surprising unevenness is the result of a compulsion to get to "expanded notation" . . . perhaps the most inappropriate and damaging "new math" innovation in first grade. The purpose of "expanded notation" is to justify asking this question:

$$\begin{array}{r} 68 \\ \underline{-25} \end{array}$$

Text A directs the learners to approach that question in this way:

$$\begin{array}{rcl} 68 & = & 60+8 \\ \underline{-25} & = & \underline{-20-5} \end{array}$$

How much brain washing has it taken to get children to rewrite 25 as $20 - 5$?

This kind of thing disturbed the authors of text B. They used parentheses a la high school algebra to lead the children to do the following:

$$\begin{array}{rcl} 68 & = & (60+8) \\ \underline{-25} & \underline{=} & \underline{-(20+5)} \\ & & 40+ \end{array}$$

To rewrite 25 as $20 + 5$ is eminently more reasonable. So we proceed, and now children are confronted with a situation in which they would normally write

$$\begin{array}{r} 8 \\ +5 \\ \hline 13 \end{array}$$

but they are instructed to write "3" because

$$\begin{array}{r} (+8) \\ -(+5) \\ \hline (+3) \end{array}$$

And that becomes reasonable only after one has learned about operations with positive and negative integers.

But suppose we somehow get through this long and tedious process and write "43" over on the right

$$\begin{array}{rl} 68 = & (+) \\ -25 = & -(+) \\ \hline & (+) \qquad = 43 \end{array}$$

The first grade child is supposed to believe he has found a resonable answer to the original question (68 — 25 = ____) because he has followed a long sequence of logically defensible manipulations.

The wages of such sin run high.

During the first week of school, this question:

$$\begin{array}{r} 19 \\ +3 \\ \hline \end{array}$$

was written on the chalk board of 4 second grade classrooms (in a high socio-economic community). Most children were reluctant to risk answering. All who did, wrote:

$$\begin{array}{r} 19 \\ +3 \\ \hline 112 \end{array}$$

except one child who wrote:

$$\begin{array}{r} 19 \\ +3 \\ \hline 1|12 \end{array}$$

A ray of hope!

"How do you read that number?"

"A hundred and twelve" . . . and the hope was crushed.

To the whole group, "Let's read what's on the board" . . .

"Nineteen plus three equals one hundred and twelve" . . . with no sign of discomfort.

"Do you mean if you had nineteen cookies and I gave you three more you would have one hundred twelve cookies?"

They laughed: "Of course not!" . . . and some said they would have 22 cookies.

A comment by W. W. Sawyer is germane.

> "The depressing thing about arithmetic badly taught is that it destroys a child's intellect, and to some extent, his integrity. Before they are taught arithmetic, children will not give their assent to utter nonsense; afterwards they will. Instead of looking at things and thinking about them, they will make wild guesses in the hope of pleasing a teacher or an examiner."

A somewhat similar anecdote from the middle grades involved one of the daughters of the late Max Beberman, who had, with her own money, purchased a book of drill in arithmetic at the supermarket. She asked her father to correct her work which he agreed to do — ("only because she had spent her own money for it," Max explained).

The page was filled with subtraction problems all of which required double regrouping. Here is the first example and her answer:

$$\begin{array}{r} 152 \\ -83 \\ \hline 131 \end{array}$$

. . . because, we assume she argued that, 3 — 2 = 1 and 8 — 5 = 3. All her calculations were without error . . . but all the conclusions were equally unreasonable.

Max asked his daughter to get out the paper clips — count out 152 paper clips — give him 83 — and count those she had left. Max kept track of the results along side the completed example:

$$\begin{array}{r} 152 \\ -83 \\ \hline 131 \end{array} \qquad \begin{array}{r} 152 \\ -83 \\ \hline 69 \end{array}$$

"What do you think of that?" Max asked.

"What you mean, Daddy?"

"Which one is right?"

"They are both right" . . . and pointing to her original work she said, "That's the way you do it with pencil and paper." Then pointing to the other record she added. "And that's the way you do it with paper clips."

That the world of "pencil and paper arithmetic" had anything to do with the world of "things" was a foreign notion.

These are all too familiar consequences of children who may be able to parrot back place-value language but do not grasp the idea of place-value in any useful way.

PLACE VALUE AS A CODING DEVICE

In the first place, there is no need to talk about place-value since the child has no more trouble with the idea that 9 is followed by 10 in the counting sequence, than that I is followed by J in the alphabet. He has seen 10, 11, and 12 on the clock; 10, 11, 12 and 13 on the TV Channel selector — and numbers up to 31 on the calendar.

The child who mastered the basic structure of his spoken language learns to count as far as he likes as soon as he wants to. As a coding system, the rules for saying and writing the "next number" are disarmingly simple. Children appreciate in a nonverbal way the beautiful simplicity of a code that uses only 10 symbols and a very simple set of rules to write a never-ending sequence of numbers (or numerals).

I once saw two girls on hands and knees in the hall writing numbers with a felt tipped pen.

"What are you doing today?"

"We're going to find out how far we can count with numbers on this roll of adding machine paper."

I took my son to the printers some hundred miles away — to help him know what "Daddy does for a living".

"Let's count trucks."

"What is a truck?"

"Anything that's got 2 tires on a back wheel."

So, some mobile homes became trucks. We were driving near a train track and met a train.

"I'm going to count that as a truck because it has so many wheels."

We counted to 497.

On the way back, he began where he had left off — "498, 499, 500".

"Let's do something else, Daddy." And you can be sure I was grateful.

On another family excursion, he turned again to counting. "I'm going to count to 100." Fine.

There was nothing at the time that interested him more, so he counted by ones to 100.

"Do you want me to do it again?"

"Not really," I admitted.

"I didn't think you would."

This first significant idea about our system of numeration is that it is an easily decipherable code — and that's an important fact for most 6-year old beginners.

There is a temptation to write a "terminal behavior objective" for "place-value in first grade." It would go something like this: "Given a felt pen and a long strip of adding machine paper, and a request to write the numbers 0, 1, 2, 3, etc., the child would perform with 85% accuracy." The trouble with that objective is that it would call for "reteaching" if the child counted by 3's after minimal compliance — 0, 1, 2, 3, 6, 9, 12, 15, 18, etc. — or showed disrespect in some other way.

BEYOND CODING

But there is more to "place value" than a uniquely simple and completely adequate "code" — it is the resource we tap for powerful strategies to avoid one-by-one counting.

Traditional materials of instruction show a pervading anxiety to tap that resource as soon and as fully as possible. But they also display a disregard for what we know about the way young children learn . . . and they plunge ahead into expanded notation.

$$\begin{array}{r} 68 = \text{etc.} \\ \underline{-25} = \underline{\text{etc.}} \\ \text{etc.} = \text{etc.} \end{array}$$

They begin manipulating "tens" and "ones" not realizing that, from the learner's viewpoint, those are abstractions without referents and therefore meaningless.

Few six-year-olds are prepared to handle the abstractions implicit in talk of "tens and ones." They are asked to understand that in "22," the "2" on the left is worth "ten times" the "2" on the right.

This is too much to ask of the child who most reluctantly comes to understand that a dime is worth more than a nickel even though it is smaller. It took the reality of the candy counter to convince him.

To introduce the abacus as a concrete model assumes an agreement that a bead on one wire is worth 10 beads on another wire. It asks the child to accept the same abstraction inherent in place value.

"Tens" and "ones" introduced too soon are pure abstractions without referents and therefore without meaning. The learner has two options — ignore what's happening (a healthy response) or learn to parrot back language that has no meaning (and that is the road to confusion).

Beansticks, rods, Unifix cubes, or other manipulatives offer a sensible alternative to tens and ones.

The manipulatives can easily be structured to demonstrate the usefulness of "place-value" . . . by considering only 10-sticks and loose beans or 10-rods and unit cubes, etc.

Suppose we put away all beansticks except those with exactly 10 beans on them so we have only 10-sticks and loose beans.

First we check all the 10-sticks so everyone is satisfied there are 10 beans on each.

"What's the quickest way you can think of to pick up 10 beans?" . . . a 10-stick, of course.

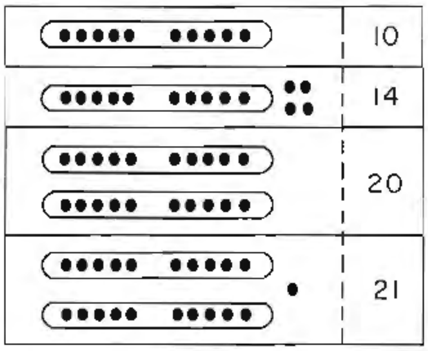
"Let's count out 14 loose beans."

"Is there a quicker way to get 14 beans?" . . . a 10-stick and 4 loose beans . . . so the 14 loose beans are returned to the pile and the more efficient selection is made.

"How about 20 beans?" . . . This question may lead to some discussion. Start with a 10-stick and add loose beans until you have 20. Use two 10-sticks. That is quicker, but will that be 20? The truth of that can be verified by counting beans on two sticks.

"How about 21 beans?" . . . one more than 20 — two 10-sticks and one loose bean.

Next to each selection the number of beans is indicated.



In the first example, what do you see? "One 10-stick and no loose beans."

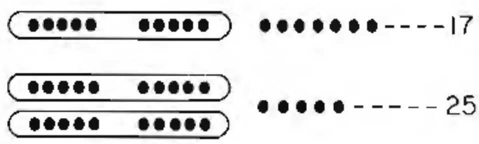
"How do we write 10?" . . . a 1 on the left and a 0 on the right.

"In the next example, how did we get 14 beans? . . . and how do we write 14?"

After studying each of the 4 examples, language can be introduced that emphasizes the relationship between the physical arrangement of counters and the number that answers the question "how many?"

"the 10-stick place"	"the loose-bean place"
1	0
1	4
2	0
2	1

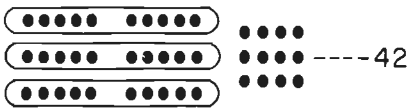
"Does that pattern hold for 17 and 25?"



"Let's put those 17 beans and those 25 beans together: What would we have?"

By counting, there are "three 10-sticks and twelve loose beans."

"But how many beans altogether?" By counting on from 30, we find there are 42 — and note the total.



"What's happened to our 10-stick place and loose bean place pattern?"

10-sticks → ← loose beans
4 2

At this point there are two very different ways to resolve the problem and it would depend entirely on the teacher and the learners.

One option is to consider the following:

10-sticks → ← loose beans
3 12

It is a logical extension of the pattern — numbers of 10-sticks on the left and loose beans on the right. But it looks very much like 312 . . . and we don't have that many beans.

This confusion can be overcome. What about

3, 12 or 3(12), etc.

We use precisely this same strategy in recording 2 feet and 11 inches (2' 11")

So we might write

10-sticks	loose beans
1,	7
+2,	5
3,	12

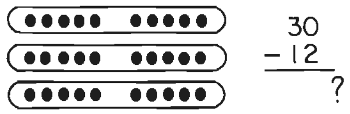
The other option (instead of, or following the first) is to raise the possibility of "trading" ten loose beans for one 10-stick whenever possible before reporting the total — using a record keeping method that will come up later:

$$\begin{array}{r} 1 \\ 17 \\ +25 \\ \hline 42 \end{array}$$

a 10-stick
and
two loose beans

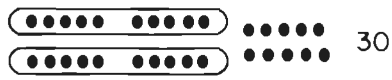
"Place value" is no longer abstract — the position indicates whether you are referring to 10-sticks or loose beans — to "things" that are being moved around.

Very soon, the following question would lead to a reasonable solution: "If I have three 10-sticks, what must you do to give me 12 beans?" or:

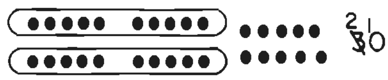


As the matter appears at first, the direction could be practically carried out by removing a 10-stick . . . but that's not enough.

If we made a trade — a beanstick for 10 loose beans, the situation is as follows:

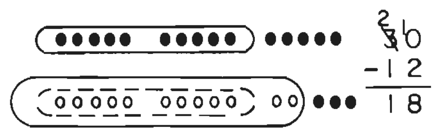


But, the following report fits the situation more accurately; the symbols show what we have done.



We have two 10-sticks and 10 loose beans.

Now we can remove one 10-stick and 2 loose beans, leaving one 10-stick and 8 loose beans, reporting the whole transaction in this way:



Each mark that is made helps build a record of what was required to complete the removal of just 12 beans.

Beginners can be introduced to arithmetic as a way of recording what they do with things. And as complexities arise they can be considered at that same level — with abstraction carefully avoided.

WHAT ABOUT "TENS AND ONES?"

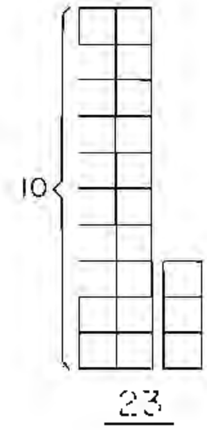
Teachers and parents sometimes become impatient — when will the children get to "tens" and "ones" so they don't have to be manipulating things all the time?

One answer might be "what's the hurry — children are already doing arithmetic that formerly was postponed until much later." And standardized achievement tests will show they are performing "beyond expectancy."

Another answer might be "whenever they want to work without moving things around" — and some take this step very early.

But for some time to come, CDA Math does not require all learners to "work without beans."

With beansticks and beans available, the structure of place-value will be considered in pictures or sketches of 10-sticks and loose beans along with sketches of blocks — piles of blocks 10-high and loose blocks.



"Two 10-high stacks and three more blocks."

The significant difference between sketches and the real thing is that sketches cannot be moved around; there is little opportunity for trial and error. And, of course, this development is appropriate only after learners have had sufficient introduction to the ideas and problems at the full manipulative level.

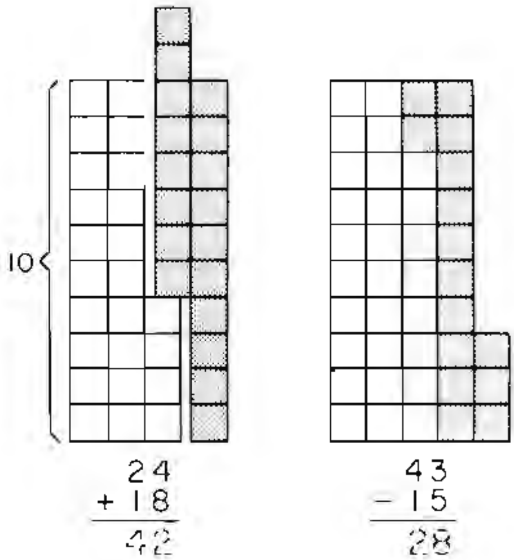
A basic resource in CDA Math is a 2-volume set — **Developing Insights into Elementary Mathematics — Operations with Whole Numbers** — 177 pages, black and white, for reproduction and use in the classroom (and a volume describing possible uses of these materials).

Developing Insights, after introducing the four basic operations in manipulative contexts, is an exercise in providing sketches and pictures illustrating in two dimensions development of counting and all the basic algorisms or processes — through long division with 2-digit divisors.

When this arithmetic arises $\begin{array}{r} 2{,}174 \\ -1{,}025 \\ \hline \end{array}$

there is a sketch of 2,174 blocks arranged in 1000's, 100's, 10's and 1's in two different ways.

Learners are not required to use these sketches but they are available to those who need them.

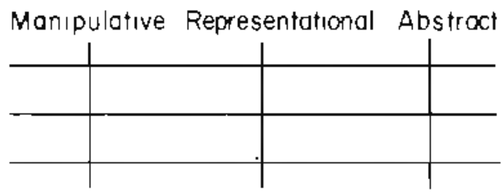


Now anytime someone wants to talk about "tens" and "ones" the children understand the notion referred to.

If such abstract language is delayed long enough, it can be introduced with no discussion at all. The referent has been solidly established and words are names for what the child already knows — that "prose" is what he has been writing all the time.

PLACE VALUE AT THE ABSTRACT LEVEL

Recalling the matrix that suggests our theory of learning on the left to right axis,



we are confronted with the problem, once again, of deciding when a child is ready to move on from the representational to the abstract level — when a child is ready to make the transition from using pictures as referents to using the shortcuts involved in mental calculations. The clearest insightful experience I can recall is found in a passage in "Mathematics in Primary Schools," (Curriculum Bulletin No. 1, The Schools Council, Her Majesty's Stationery Office, London, 1969, pp. 26-27):

"In a class of fifty seven-year-olds one boy was outstanding in his quick response when asked to add and subtract numbers less than 100. For example, when he was asked to add 9 and 7 he gave the answer, 16, immediately; for 19 and 7, however, there was a pause before the answer, 26, was given. The boy explained, when questioned, that he had counted-on in ones from 19 to find the answer. Fixed horizontally to the walls of the crowded classroom was the piece of structural apparatus to which reference was made in the previous chapter. This was a number strip, 100 inches long, and strips 1 in. to 10 in. in length, made by the teacher herself. The boy was asked to use the strips to verify his answers to 9 + 7 and 19 + 7. This done, he turned away from the strip and was asked to add 29 and 7. Again there was a pause, suggesting that he was counting-on in ones, before the answer, 36, was given. Once more the boy was asked to check his answer using the strips. The questioning continued, step by step: 39 + 7, 49 + 7 and so on. Each time the boy was asked to give his answer before verifying this by using the strips. Each time he admitted that he counted on from the larger number to find the answer, until he reached 89 + 7. At this, his face lit up and he said: 'I don't need to count anymore; I know. It's 96.' For the first time this boy had seen the pattern of his responses: 16, 26, 36, 46 . . . Because he was very intelligent, this boy was able, at this stage, to make an abstraction, for when asked to add 8 and 3 he replied: '11, and now I can tell you 18 + 3, and 28 + 3, and 58 + 3 without counting-on.'

"When children can give an immediate answer to questions like 68 + 6 they need further experience with pairs of two-figure numbers, such as 68 + 26, 38 + 46, and so on until these, too, can be performed mentally by an efficient method (that is, without counting-on in ones at any stage). For example, to add 68 and 26, some may add the tens first, 80, the units next, 14, and give the answer: 94. Others may add the units first and tens second. Some add 68 and 20, 88, and then add 6: 94. Now if at the last step they count-on in ones from 88, this is not an efficient method and it usually means that the child is not yet ready to benefit from practice in two-figure addition (tens and units). **This then is a crucial test of readiness for practice in written computation with tens and units: the ability to add 2-figure numbers mentally by an efficient method. To achieve this, children need a secure knowledge of a place value since they have been able to discriminate between the tens and units in the numbers added."**

d. LEARNING MATERIALS

Introduction

One day, I was pushed to characterize CDA Math, Levels A through F, in "a few words." After pulling back from the request momentarily, I answered that the materials are a half-way house for teachers with textbook-addiction!

And at that moment, I had a flashback to an experience at the University of Illinois.

The late Dr. Max Beberman had invited Edith Biggs to work with the UICSM project at the University. Because Ms. Biggs was considered an initiator and leader of the British Infant School movement in England, I kept asking her to explain how we could effectively initiate a similar movement in the United States. No matter how I pushed, she would answer, "You just get it started, and it will spread." She demonstrated what she meant about the first part of her statement —"get it started"— by rearranging a kindergarten classroom with "learning centers" before the children arrived, and showed us a replica of a British Infant School day.

It was thrilling to see a master teacher at work with young children. But there was no convincing evidence that it would "spread."

During her visit, she was joined by a colleague of hers, Ms. Maryanne Perry.

At the first opportunity — at dinner with Ms. Biggs, Ms. Perry and Max Beberman — I asked Maryanne, "How did the British Infant School movement get started?"

"Oh, don't you know?"

"No."

"Well, Hitler bombed the cities. We had to take the children out into the countryside without anything . . . no apparatus and no textbooks."

Ms. Biggs supported her emphatically: "That's right: that's the way it happened."

If Hitler bombed the American cities, the teachers and administrators would move our children out into the safety of the countryside and "make do" . . . and American education would never be quite the same again.

Our teachers know that textbooks in mathematics are not helping them achieve their goals; and perceptive parents, when they look at those texts, are dismayed, confused, frightened and discouraged.

But, unless the physical safety of the children is challenged, teachers and parents will continue to feel a need for some organized materials of instruction that will not precipitously shift the total responsibility for the mathematical education of children on them — they simply are not confident they could meet the challenge. They are deeply concerned about giving every child a chance to grow intellectually; yet, if the bombs were to fall, they would not hesitate for a moment.

CDA Math Levels A through F has been designed as an alternative to traditional textbooks, moving away from rigidity and prescription, step by step toward a humanization that can still be fully accountable in terms of standardized achievement test scores.

In other words, CDA Math, Levels A through F, finds no reason to quarrel with those who cannot see beyond skill development and acquisition of knowledge, beyond scores — it simply charts an alternate route. It respects children as human beings, it introduces children to the excitement of mathematics, it satisfies standardized test requirements, and, without bombs, leads out into the countryside.

A New Approach to Content

Please look away, for a moment, from familiar scope and sequence charts in elementary school mathematics and ask yourself some new questions.

What do children know when they enter school at 4 or 5 or 6 years of age? . . . How much have they learned? . . . How have they learned it?

Of course, children's most surprising accomplishment is learning to speak a language on their own. Already, they have a firm grasp of one of the most complex structures they will ever encounter. How they accomplish this amazing feat is one of the more challenging secrets learning theorists try to find out.

One thing is clear — these preschoolers learn this complex structure on their own. No one explained language to them or provided organized drill and practice.

It's rather sobering to have to agree with William Hull: "If we taught children to speak, they'd never learn." (Quoted by John Holt in **How Children Fail**).

At this time, we can only marvel at very young children's ability to learn on their own. They have demonstrated they are in control of a powerful learning system. They are in the driver's seat and getting where they want to go.

Appreciation of this accomplishment gives considerable credence to the assertion that children can learn anything they want to learn. Others go further, claiming that children learn **only** what they want to learn.

Perhaps less spectacular, but still worth noting, children have developed a power to classify in many different ways — most of which are so complex they cannot be verbalized.

An infant can locate one of his parents in a sea of hundreds of faces, most of which are quite similar.

Children know the difference between dogs and cats . . . yet, an adult would be hard pressed to verbalize what characteristics cats or dogs have in common that makes dogs and cats distinguishable.

Children have learned the "body language" of other children and of adults. They are adept at decoding every facial expression in their parents' repertoire. They quickly decode the individual body languages of each teacher they come to know.

They have developed tactical strategies to cope with many different situations — desire, denial, acceptance, rejection, inclusion, exclusion, disappointment, anger, fear, disinterest, anxiety, competition — each individual child in his own way. And, when we say a child has not yet learned to "cope with this or that," we are actually expressing a hope that, with help, he can learn to handle situations in a different way.

What have children learned they will use later in mathematics?

One highly developed skill is that of "classifying" — of noticing a relationship that enables one to say this belongs but that does not belong. The idea of a determining relationship is what mathematicians call "a function."

Sesame Street children play versions of this game as someone sings a song: "Which one of these isn't like the other, which one of these doesn't belong? . . ."

Sometimes four objects are displayed such as a tennis racket, a baseball, a basketball and a football. "Which one of these isn't like the other?" The children will point to the tennis racket. "Why doesn't it belong?" . . . and their answers are vivid proof that children invent language when they need to.

Suppose the four objects were 3 rectangular boxes and a round hat-box. That's a simple choice until you notice they all have lids except one of the rectangular boxes.

Now "it all depends on how you look at things."

There is no particular charm in limiting the choices to four things — why not try five or six objects and ask, "Which two don't belong?"

But really, the nicest version requires a box with a great variety of toys and other objects.

It starts with a selection of 6 objects. Then — "Which two don't belong?" . . . "That one and that one don't" . . . "Please put the ones that don't belong back in the big box of things . . . and find two replacements that do belong!"

After watching these episodes on Sesame Street, even preschoolers make up their own version on the floor singing, "Which one of these isn't like the others?" . . . challenging other children and any adults who will join in.

The notion of a determining relationship is perhaps the most pervasive, unifying idea of all mathematics.* Which numbers belong to this list . . . 0, 3, 6, 9, 12 . . . and which don't? Which of the triangles are right triangles? Which numbers have a remainder of 3 when divided by 7? Which whole numbers are factors of 24? Which pairs of lines are perpendicular? Are these triangles similar?

Beginners already have a sharp eye for noticing determining relationships . . . a profound mathematical idea.

Gertrude Hendriks, at the University of Illinois, made a large set of dominoes with randomly arranged pictures of objects instead of the usual "domino arrangement" of spots.

In the beginning, no domino had more than four pictures on either of its halves. All pictures were of different objects.

Rules were communicated simply by the initiator saying "Yes, that fits" or "No, that doesn't fit." If a player had no piece that "fits", he would be asked to pick from the "bone pile" until he finds one that can fit.

Four-to five-year-olds would play this game for long periods of time and protest if time ran out. None of them was ever observed "counting" the pictures.

When dominoes with five objects were introduced, most children lost interest almost as soon as they encountered a group of 5. Some would play on, but began to count whenever a 5-object domino came up to be compared with a 4 or 5-object piece; only after counting both would they be willing to decide that it did or did not fit. But the interest lagged very soon.

Very young children have much more number sense than many think. The problem is that they cannot communicate what they know. Watching what children do gives far more reliable insight into what children have learned than any kind of verbal communication.

A common example of this fact is the child who says, "She readed me the book." The first reaction of many is the child is "wrong" — he hasn't learned the language very well.

But what has the child done? He has discovered the general rule for forming the past tense of a verb, and he has applied it in a new situation. He has never heard people say "readed." He's invented a word that demonstrates an understanding of the structure of the langauge. Sometime later, he will learn that the structure is full of inconsistencies and the past tense of the verb "to read" is one of them. To learn that the present and past tense are spelled the same way, but pronounced differently must be disconcerting.

Paul is a kindergarten construction worker, erecting buildings, sometimes with grand proportions, out of all available materials. Each has a theme and a style. Sometimes, they sprawl as if he were making a 3-dimensional floor plan of a shopping complex. Other times he tries for height.

One day, he built an airport with parking areas, restaurants, baggage areas — and he would eagerly tell about each.

The structure was symmetrical — for every piece on the right side, there was an identical piece on the other. He would never, in his conversational rambling, ever refer to his plan.

I handed him an odd triangularly-shaped block. There was no other block like it around.

Paul looked at his structure, then at the other blocks not already in place. Finally, he placed the block on the line of symmetry he had established and went on about his building.

Children learn at free play. Most of their amazing store of knowledge, most of their skills, have been acquired through play. At play, they are squarely in the driver's seat, in full momentum making

*To use the mathematician's label "function" for the notion won't contribute a thing to our discussion at this point.

choices, reacting to the consequences of their choices, making new decisions, fully involved in trial and error — with "feedback" loops in full operation: they act, evaluate the results and modify what they do based on what happens.

As children mature, they develop another style of learning we might call "goal-oriented," for want of a better formulation. Paul had set a goal he wanted to reach and stuck to it until he was satisfied with his results. My intervention with an odd-shaped block was momentarily a puzzler, but he found a satisfactory solution and went on about his own business.

The fashionable strategy in "early childhood education" is to begin introducing children as early as possible to adult-selected goals. The desire is to hasten the child's "cognitive development."

What used to be considered first grade arithmetic (as well as content in other areas) is slowed down for introduction in kindergarten. *This has got to be one of the most mindless, misguided and most counterproductive trends of the day.*

There are two destructive aspects to this recent distortion:

1. The "first grade arithmetic" of textbooks was most inappropriate for six-year-olds and is totally inappropriate for five-year-olds.
2. It shows utter disregard for the learning systems and styles children have already developed and used to achieve truly fantastic results.

The practitioners of these "cognitive pushers" will point to their achievements as proof of their innovations: "See, the children can do it."

But, the results they point to are dangerously deceptive, to say the least. Any animal, at almost any age, can be taught to perform on command if the reward is attractive enough or the consequences of not performing are dire enough.

But, such training dims the child's demonstrated ability to learn what he wants to learn. It yanks him out of the driver's seat and prescribes activities in which there is no intrinsic reward for the child — no intrinsic reason for him to learn.

There is a movement among the dispensers of funds for early childhood education (ECE) to make allocations dependent on pushing "first grade work down to kindergarten" and describing the desired "performance" in "behavioral terms."

In this way, the expertly structured "management system" is stretched to include ECE, and teachers who know and respect children are held "accountable" to treat them as less than human.

However, the misguided movers behind this distorted view of early childhood education will not be moved, at all, by our criticism of their programs. Their convictions are honest and firm.

There is no defense that will protect children from these managers except a vigorous offense.

In planning such an offense, we need not be hurried into hastily designed alternatives, because we know those "cognitive pushers" can show only the thinnest and most short-lived of results.

With time, we can develop strategies and tactics that follow from two related goals:

1. To find content that is appropriate for young children, and
2. To find strategies that show deep respect for the ability of children to learn what they want to learn.

Beyond this, we need a bit of faith that children, treated as learners, will learn more than trainers can "teach" . . . and will handle any "criterion referenced test" more successfully than those who were merely trained for the test.

Focusing on what the child as a learner is doing is much more revealing than depending on verbal communication.

Almost all children will demonstrate understanding of the four basic operations of addition, subtraction, multiplication and division — including division with remainders — before they come to kindergarten.

Perhaps the best example is "subtraction." Can you imagine the child who doesn't know about "take-away" before he is 1 year old?

"Take-away " to a two-year-old, must fill up a rather substantial part of his world. After being the victim of the activity, he has become mobile and makes it more of a sometimes-you-win, sometimes-you-lose game.

There is nothing about the "concept of subtraction" he doesn't know at a visceral level.

Of course, children are familiar with putting groups of things together and having more — taking some away and having less. Given a group of objects up to 4, very young children can make other piles like it; and they can pass out cookies so everyone has his share, coping with "left-overs" in several different ways.

Preschool children already understand the "basic concepts" of addition, subtraction, multiplication and division — with remainders.

What is it, precisely, that he still needs to learn?

In the beginning, he needs to learn ways to communicate what he knows — socially acceptable ways to express what he already knows.

Then, after learning one-by-one counting and ways to report his results, he will begin the long search for reliable ways and means to avoid one-by-one counting (including the shortest short-cut of all — memory — which is also the least reliable).

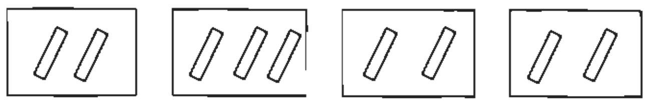
In this view, the child comes to school having demonstrated that he has a highly developed learning system; he understands the unifying notion "which ones belong and which don't"; he has internalized the concepts of the four basic operations of arithmetic.

At this point, it is our task to develop a curriculum that is appropriate for these children.

This is the goal towards which CDA Math is oriented.

In more specific terms, we fully appreciate the fact that beginners already have internalized the concepts of numbers and of the four basic operations with objects, but beginners do not know socially acceptable ways to communicate what they know. "Concept development" has already taken place, and we are left with what is essentially a "coding problem" . . . children can play the games, but they don't know how to record what they do.

Imagine a "which one isn't like the others, which one doesn't belong" episode with 2 trucks in each of 3 shallow boxes and 3 trucks in the fourth box:



If a child points to the box with 3 trucks, you can be sure he understands the concept of "twoness," even though he may not have a name for it yet. So we introduce the name — "two in this box, two in this box, two in this box — 'not two' in the other box."

Pointing to the box with 3 trucks: "Can you make this one so it is like the others?" . . . Now they all have two.

"Let's count the trucks in each box — one, two — one, two — one, two — one, two — now they all have 2."

And writing "2" on the board or on a piece of paper — "and here's the sign for 'two'."

"Can we put the trucks back the way they were?"

"Now there are 3 in this box. Can we make the other boxes like this one?" . . . and after another truck is put in each of the other boxes, "Let's count them — one, two, three — one, two, three," etc.

"This is the sign for three — '3'."

After children demonstrate, beyond any doubt, that they have the concept of 2 and of 3, they are given the verbal names for those ideas. After repeating the names several times, they are given the symbols "2" and "3." This is an elementary "coding" activity. The next step is to help the children make symbols that are recognizable as "2" and "3," etc.

The first section of **Patterns and Problems, Level A** is designed to help develop a child's ability to draw symbols others can read.

Major Objectives of CDA

CDA strategy is simply to focus attention on objectives that lie squarely in the affective domain . . . attitudes toward mathematics and the learner's self-esteem.

But we must hasten to add that we feel justified in making this "switch" only because we are confident that this plan is the most effective strategy to meet society's minimum requirements. We welcome those who "pay the bill" to hold us fully accountable in terms of standardized achievement test results or scores.

We cannot imagine a mathematics program designed to build a learner's feeling of competence and a friendly attitude toward numbers that in some way could ignore skill development. Focusing on the affective domain cannot avoid attention to cognitive development.

The converse is not true; it is easy to imagine programs that mobilize strategies and tactics to promote cognitive development and ignore the attitudes of the learner. We see them all around us. Their "behaviorist" designers turn away from the "learner's" feelings because those feelings are beyond the behaviorist's expertise in "measurement."

We are prepared to consider objectives that are very hard to measure, even hard to state — "objectives" we can only put down in "subjective terms."

Here are some formulations that suggest CDA Math objectives. The list is neither definitive nor all-inclusive; ideas overlap, certain words are used in ways we need to talk more about later, but our basic goals will begin to emerge.

SOME CDA MATH OBJECTIVES

1. Encouraging perseverance
2. Putting the learner in the driver's seat
3. Solving substantial problems
4. Building a pride in the learner of what he already knows and what he finds out
5. Helping the learner record that knowledge
6. Providing opportunities and recognition for independent investigation
7. Developing a friendliness for numbers and shapes. (Please notice that specific content crept in only in this final formulation.)

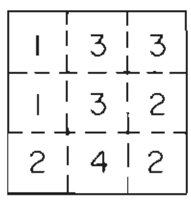
Rather than dealing with the goals in detail, let's try a little "definition by example." No behaviorist would be comfortable with this approach. We are probing far beyond the narrow "cognitive domain," knowing that a learner's attitudes about mathematics and about himself as a participant are decisive.

In each example below, our focus is on the feelings of the child, but inescapably, that concern must come to grips with mathematical ideas. And in each example below, the learner considers content far more complex than any traditional scope and sequence would dare include.

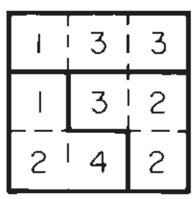
EXAMPLE I

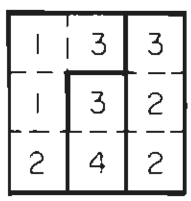
"Fencing" has become a favorite activity of many children over a wide spectrum of ages.

In the example to follow, students are asked to "fence in" series of numbers whose sum in this case is exactly 7. Within each fence, adjoining cells must have a common border and no number is permitted inside more than a single fence. Whenever possible, all numbers given should lie within a fence. Here is an example:



There are at least two possible solutions.

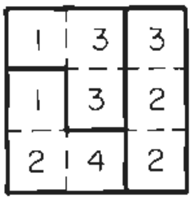




Each is as good as the other . . . but neither is already in the learner's memory bank.

Are there other solutions?

Yes, at least one other.



Are there others?

What is the "objective" of this activity?

CDA's primary objective is to present situations that require problem-solving, finding solutions not already in the memory. If the learner is satisfied with a single solution, he has already experienced the feeling of having solved a problem. He can find other solutions, each with its own rewarding outcome. He may wonder if he has found all the solutions possible, thereby encouraging him to make his own independent investigation.

Learners look forward to "fencing" as an opportunity to use basic number facts as a problem-solving tool.

If this activity were found in traditional materials of instruction (which is highly improbable) the "objective" would probably be spelled in some such way as: "The child will practice finding sums with 2, 3 and 4 addends."

EXAMPLE II

As another example, let's start with a standard behavioral objective such as this: "Objective: given a simple subtraction problem, the student will find the difference by finding the missing addend."

To the uninitiated, this might sound a bit intriguing, "finding the missing addend" . . . maybe it's a hunt or a search game. But consider the example to illustrate the search:

"$11 - 6 = a$
"Since $5 + 6 = 11$
"Then $11 - 6 = 5$"

This same arithmetic could arise in a very different context. The following is an activity that developed as we looked for ways to humanize the subject of mathematics, and to initiate genuine "searches" as motivation for young learners.

Consider a list of randomly selected numbers. Can you find pairs of numbers in the list below that can be used to complete the following statements?

3 , 6 , 7 , 9 , 11 , 15

3 + 7 = 10	___ − ___ = 5
___ + ___ = 13	___ − ___ = 8
___ + ___ = 14	___ − ___ = 9

(This list was not really selected at random because we wanted to include the "$11 - 6 = a$.")

Can any of the examples be completed in more than one way with numbers "From the List" such as:

$11 - 3 = 8$ and
$15 - 7 = 8$?

"From the List" has an unlimited number of variations. Searches of this kind can be tailored to focus on any area of skill development that seems appropriate.

The central "objectives" of CDA Math are clearly in the "affective domain" yet provide activities designed to provide the drill and practice that "skill development" exercises of the traditional kind fail, in fact, to achieve when there is no intrinsic reward. Pages of examples destroy any interest the learner might have. These pages come through to him as a continual check-up on how well he can perform on command . . . with no other purpose at all.

EXAMPLE III

This same "From the List" activity can indicate another aspect of CDA objectives. We will add a few numbers to the same list and ask for statements that are true. But this time, all numbers in the sentences must come only from the list. Numbers from the list may be repeated as often as you like. Can you find examples to complete the blanks with different and true statements?

2 , 3 , 5 , 6 , 7 , 9 , 10 , 11 , 14 , 15 , 17

2 + 3 = 5	17 − 10 = 7	2 x 5 = 10
___ + ___ = ___	___ − ___ = ___	___ x ___ = ___
___ + ___ = ___	___ − ___ = ___	___ x ___ = ___

Can you make up other statements in any form using as many numbers from the list as you want such as:

$6 + 7 - 3 = 10$ $\qquad$ $7 \times 9 - 3 = 6 \times 10$

________ $\qquad$ __________ etc

No specific number of examples is required. "Just find as many as you can in 10 or 15 minutes".

This is clearly an open-ended activity. There is no one way to complete it in any particular length of time.

The activity is taken from a bank of evaluative activities (Problem Solving Activities, published by CDA). Our goal is to develop ways to measure changes in the affective domain.

Professor Art Costa, California State University at Sacramento, described this as an activity designed to find a reading on a "perseverance scale." And as a matter of fact, our experimental work has shown that any attempt to evaluate different responses has provided no more useful information than the raw score of the number of responses each child made. (Discounting for errors did not change the results in any significant way.)

In "Webster's" words: Perseverance is the "act or quality of persevering: persistence; steadfast pursuit of an undertaking or aim," . . . and, in "theology — continuance in a state of grace until it is succeeded by a state of glory."

"Perseverance" is a positive factor in almost any problem–solving effort. Children who have always been told what to do next, display resistance when they are confronted with a problem-solving situation. Learners whose diet is rich in looking for solutions to real problems develop growing perseverance.

CDA does not shy away from such a "subjective" goal; rather we are at work trying to develop activities that nurture "perseverance" and finding ways to measure growth in this area.

EXAMPLE IV

Children come to school already knowing considerable mathematics for which they are seldom given any credit at all.

Recently, I put our two youngest to bed, going through the usual ritual of a story, a song or two, tuck-ins and goodnight kisses. But that's not the end. The last ditch effort to delay the inevitable is the request "Will you sit for 5 minutes?" Of course that is much too long, so we negotiate.

After we had finally reached a settlement on 3 minutes, Mark (nursery school) said: "Then you sit on my bed for one minute and then a half a minute more; and one minute on Paul's bed and then a half a minute more."

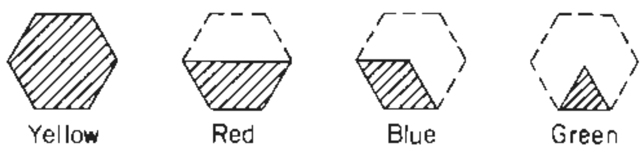
Paul (kindergarten) had another proposal: "Why don't you sit on my bed for one minute and on Mark's bed for one minute and one minute in the chair?"

If we could find ways to elicit such mathematical ideas from children, help them to communicate those ideas to others and eventually help them to record them in a socially acceptable way, their grasp of the subject of mathematics would shock most adults.

Instead, we ignore what they already know by entering them in the deadly scope and sequence set down in some publishing house.

EXAMPLE V

Shortly after the bedtime story just recounted, I was working with a group of second grade children. Each small group had a supply of "pattern blocks." There were several pieces of each of 4 basic shapes, which we referred to by their colors.



"Hey, two red ones fit on top of the yellow one." "So do 3 blues . . . and 6 greens."

Shortly, we began writing down the relationship they noticed: $2R = 1Y$, $3B = 1Y$, $6G - 1Y$, etc., much as some teachers do with cuisenaire rods and other structured materials.

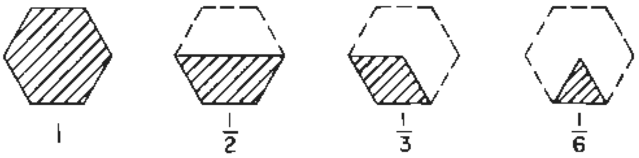
Some of the statements were more complex:

$R + B + G \quad Y \quad R + B \quad 5G$

I explained that adults sometimes talk about these same ideas but use a different kind of language.

Adults may decide that they could call the big yellow hexagon "1." Then because it takes 2 reds to make a 1, they would call it "½" . . . and so on with "1/3" and "1/6."

So, we began using adult labels:



Then, we wrote down relationships the children noticed using these new labels:

½ + ½ = 1

and shortly a child said:

"1/3 + 1/6 = ½"

which was very clear to everyone. But, an observing teacher could not be contained, blurting out, "They can't do that because they don't know about common denominators."

But, it was quite clear they truly had "done it," and the investigation proceeded.

A major objective of CDA Math is to elicit from children the mathematics they already know to help them uncover new relationships, and then to learn to write down what they know in socially acceptable mathematical shorthand.

CONCLUSION

Contrived "behavioral objectives" for each page or activity in textbooks are manifestations of crude "management systems" designed to produce results in skill development and knowledge acquisition . . . systems that have been tried and proved to be ineffective at best, and often damaging to a learner's healthy growth.

CDA Math will not misguide teachers and parents by indicating any low level goals for any page or activity. We will rather design materials that support the teacher and parent efforts to develop positive attitudes toward mathematics and a growing self-esteem in each learner.

We will use simple annotations on the page of the teachers' editions, along with answers, to help clarify the activity . . . but no "behavioral objectives" and no "prescriptions."

We assume that each teacher and parent will find unique strategies to make these materials most useful with learners in organized staff development or at home.

CDA Math - A Basic Program

A map of these materials is provided on the inside of the front cover of this book.

— **Individualized Computation,** A_1 thru F_2

— **Patterns and Problems,** A thru F

— **My Progress** — objectives and criterion referenced tests

CDA Math has two quite separate developments — **Individualized Computation**, Levels A through F and **Patterns and Problems,** Levels A through F.

This division reflects two different aspects of mathematics. One is computation — addition, subtraction, multiplication and division — with increasing degrees of complexity. The other is patterns and problems that can be explored without involving difficult computations.

Individualized Computation, as the title implies, provides the opportunity for each child to move ahead at his own rate, in his own way. It is self-correcting and requires a minimum of adult intervention for most pupils. It is completely multi-graded. Whenever a student finishes one book, he moves to the next. Children in any class may be working at widely separated levels of difficulty.

Patterns and Problems is designed to encourage maximum interaction within heterogeneous groups. Difficult computations are unnecessary and avoided. Regardless of the wide separations in "individualized computation," the group can come together to investigate patterns and problems. A child's lack of ability to handle computational difficulties is dealt with elsewhere; it need not interfere with looking for numerical and spatial relationships that are not obvious. Most of these patterns can be uncovered with minimal computations.

This basic division allows for maximum interaction and individualization.

INDIVIDUALIZED COMPUTATION

Level A_1 assumes the learner can count to 10 and write corresponding numerals. The first section of **Patterns and Problems,** Level A, is designed to help develop this facility, and progress will indicate readiness to work in **Individualized Computation,** Level A_1.

The "content" has a simple, unifying idea — that the four basic processes of addition, subtraction, multiplication and division can be understood in terms of manipulating things and reporting results in the special shorthand of arithmetic.

Levels A_1 and A_2 ask children to move counters on the page as they carry out the familiar experiments signaled by $+$, $-$, $\times$ and $\div$, and keep track of their "experiments." We hope to develop an unshakable notion on the part of the learner that the truth or falsity of a statement in arithmetic depends on what would happen if an appropriate experiment were carried out with things.

A_1 and A_2 are devoted almost exclusively to arranging and rearranging "things" on the page with an occasional suggestion that we will soon move on to working with diagrams — (the 'representational level') — and eventually to manipulating symbols (the abstract level).

A_1 and A_2 are *completely non-verbal.* Any "individualized" program in first grade that depends on written language requires continual explanations by an adult and in large measure defeats an important purpose — to release teachers and aides for constructive work with those learners who need it most.

Individualized Computation, A_1 through F_2 is essentially non-verbal. Language is a dangerous means of communication between adults (including authors) and children.

In an airport in Pittsburgh, I overheard this exchange:

Two boys, dressed alike, were watching the activity on the landing field. A gentleman behind them asked: "Are you boys twins?"

One boy turned around to answer, "Yeah, we're twins."

A few moments later the gentleman asked, "How old are you?"

"I'm six and he's five."

"Oh, then you aren't twins — you're just dressed the same."

The other boy turned around, explaining, "We're just watching the same airplanes."

While spoken language is such an unreliable means of communication, written language has additional hazards.

For these reasons, CDA Math has pioneered the use of all kinds of non-verbal techniques. (As an un-expected result, few changes were required to provide language in both Spanish and English.)

Levels B_1 and B_2 depend more and more on diagrams and sketches — with movable counters for those needing them. And the opportunities grow for children to make statements they believe using symbols only.

"Pencil and paper arithmetic," as many teachers refer to manipulation of symbols, comes in, particularly at the intermediate level, but sketches and diagrams never disappear because they are so much more effective as a means of communication than words. Presentation of new ideas is never attempted without sketches and diagrams — often sketches that suggest objects such as beansticks that are familiar — and available if necessary.

Individualized Computation, A through F, is sequenced to help develop necessary computational skills through simple long division and operations with both common fractions and decimal fractions. It is straighforward and uncluttered. It assumes that most applications of computation will come up in **Patterns and Problems.**

Some answers are provided on each page. These serve a dual purpose. They are useful in answering the perennial question, "What am I supposed to do here?" An instructive answer would be to ask the learner if the answers given would help; and the usual call for help can be extinguished over a period of time.

The second purpose of providing answers on the page is that it can serve as a device to let the child know immediately whether he is right or wrong — providing "immediate reinforcement."

Several methods are used to make answers available on the page. In the beginning, the answers are given at the side, but as close to the example as possible. About half of the answers are provided interspersed with "on your own" examples.

Later, answers are moved to the bottom of the page, but not in order, thereby discouraging "answer-copying."

Still further, particularly in Levels D, E and F, almost all the answers are given in a "multiple choice" format . . . each is randomly entered in a list of from three to five possibilities.

In Levels A, B and C (both 1 and 2) there is an incorrect answer in each sequence of pages between test papers. These mistakes are pointed out in the **Answers and Annotations** and can be used in any way that seems most effective.

(A middle grade student who had been having more than usual difficulty suddenly took off with enthusiasm. Her mother explained something the teacher hadn't been aware of: "It all happened when Marjorie found a mistake in the answers in her book. If she was smart enough to correct an author's mistake, maybe she was pretty smart after all.")

Whenever answers are provided, it is a temptation for some children to "answer-copy," and this is a problem that must be dealt with. It may be necessary to put the book away for awhile, using similar material from the "other resources" that are described in other sections.

But a word of caution — what appears as "cheating" may be something else. An investigation of a group of eight-year-old children produced a surprising result. They were using a programmed reading series with answers on the side. Almost every child observed looked at the answer given before writing in the place provided.

But test results were favorable.

This led to a further look at "cheating." Actually, the children were playing the game according to the rules — answering the questions before looking at the answers given. They were so reluctant to be wrong they checked first before writing in their responses.

In Levels A, B, and C, there is a single sequence — A_1, A_2, B_1, B_2, C_1 and C_2.

However, Levels D, E and F have a different sequencing.

Levels D_1, E_1 and F_1 are designed for pupils who have had difficulty and do not have the preparation usually expected of them.

Levels D_2, E_2 and F_2 assume better preparation.

There are three basic routes through these six books and many variations of those routes.

Levels D_1, E_1 and F_1, while less demanding, prepare a child to perform at or above grade level expectancy at the end of the intermediate grades . . . and may be as far as some want to go.

Levels D_2, E_2 and F_2 may be enough for some students — they may not need any other help.

Still others may find it helpful to work through all six books — D_1, D_2, E_1, E_2, F_1 and F_2, in that order.

In diagrammatic form these routes might look like this:

$D_1 - E_1 - F_1 \rightarrow$	$D_1 \quad E_1 \quad F_1$	$D_1 \quad E_1 \quad F_1$
$D_2 \quad E_2 \quad F_2$	$D_2 - E_2 - F_2 \rightarrow$	$D_2 \quad E_2 \quad F_2$

As teachers become more familiar with the materials and their students, they may find other routes more appropriate for individual children.

In Levels D, E and F (1 and 2) a sequencing was generally developed to facilitate the introduction of each book to a child or a group of children. The formats that recur in the book are all introduced in the first section — before the first test. The teacher has the opportunity of providing special help in the beginning, knowing that the formats will reappear.

Throughout the whole **Individualized Computation** series of twelve books, curriculum-imbedded tests are included at the end of each sequence of from six to twelve pages. There are no answers given on these pages — children are "on their own."

If they demonstrate an ability to perform adequately on the test, they color in an appropriate block on the "progress chart" located on the inside of the back cover.

On the backs of each test in Levels A, B and C (1 and 2), there is a letter to parents explaining what the child has been doing in "computation" and what lies ahead. Suggestions are included of ways the parents could help strengthen the child's computational and general mathematical facility.

MY PROGRESS

My Progress — Primary (Levels A, B and C) and **My Progress — Intermediate** (Levels D, E and F) are a compilation of these tests. Inserted in the teacher's copy, which includes answers, are "performance objectives" for each of the "criterion referenced" tests.

Many teachers have found this a most helpful "management system"— complying with requirements in a way that is not at all burdensome.

Historically, CDA Math was developed as a total program and then reshaped in the light of classroom experiences. When a form emerged that produced results — including better standardized achievement test scores — tests were designed to help check on the child's individual progress. Finally, "performance objectives" were written that are consistent with the tests.

We know that this procedure is the reverse of what certain "planners" are trying to push on education. They would begin with "performance objectives," then produce "criterion referenced tests," and then produce materials to prepare children for the tests. This procedure makes consideration for the child an afterthought. Test results show such efforts are not productive.

If the classical "management system" concept has some positive contribution to make, its disciples have been unable, thus far, in elementary mathematics education, to prove it by obtaining positive results.

Individualized Computation, Levels B_1 and B_2 through F_1 and F_2 is designed to help children with the on-going search for ways to avoid one-by-one counting. The two basic strategies in this extension are (1) to commit the "basic facts" to automatic recall, and (2) to recognize the "place value" plan of our system of numeration as the source of powerful shortcuts that make one-by-one counting unnecessary.

Children develop computational skills at greatly varying rates of progress. So, **Individualized Compu-**

tation Levels A_1, A_2 through F_1, F_2 are self-pacing and self-correcting with most children requiring a minimum amount of outside intervention.

All language in Levels A_1, A_2, B_1, B_2, C_1, C_2 is given both in English and in Spanish. Levels D_1, E_1, and F_1 are also bilingual.

We have been careful to design materials providing the fullest opportunity for each child to feel he is firmly in the driver's seat. We hope learners will keep their own learning machines going "full speed ahead."

PATTERNS AND PROBLEMS

This is the creative side of mathematics — a face of mathematics very few people have met.

Computation is a matter of skill development . . . a matter of training — being able to perform on command or when the need arises. It is an indispensable tool for advancement in education; it is sometimes helpful in the work-a-day world we live in (when machines aren't readily available); it's a must to show higher standardized achievement test scores. And all of these requirements are very real — though they tend to be dull at best.

But, one more compelling reason to develop computational facility is that finding patterns and solving problems usually requires some computation — not a lot, but some. And computational difficulties, unless minor, will intrude to dampen enjoyment.

For example: it is quite possible to find out almost all one needs to know about operations with "fractions" and never introduce any number larger than 12 (12 eggs in a dozen). Further, it would be quite possible to find out anything one needs to know about operations with "positive and negative integers and zero," and rule out all numbers larger than 10.

Patterns and Problems, beyond the first part of Level A, is almost completely "episodic." Levels D, E and F are subtitled "Jumping Around in Mathematics," inviting learners to develop an "exploring" attitude, opening opportunities for them to make independent investigations.

TALK SESSIONS

Patterns and Problems was designed for use with heterogeneous groups of any size.

An activity can be introduced with some kind of class discussion in order to promote a maximum of interaction. "Talk sessions" are important to a child's development in mathematics because "communication of ideas" is an essential part of the subject. It is so frustrating to "have it in your head" and to be unable to share with others.

We hope the adults will not usurp "talk time." There is more to be learned about children through listening than by talking. Children usually learn more from talking and listening to other children than they do when adults assume an "I'll tell you" stance.

Another good reason to let children do most of the talking is that they often come up with more interesting ideas and questions than the teacher or the book had in mind.

I was once observing a model lesson in Toronto about "finite and infinite sets." The teacher was well prepared. He had a table full of tools and elicited from the group that it was a "set of tools" . . . and that it was a "finite set."

"Can you give me an example of an infinite set?"

"Sure," a student called out, "the stars."

"Now really," the teacher admonished, "stars don't go on forever like the set of whole numbers."

"Yes, they do," came a quick rejoinder.

Another student entered the argument. "The stars aren't infinite — you would just need a very long time to count them."

"Right," the teacher agreed.

"I still think the stars are infinite because while you are counting them more are being born."

That was enough for the teacher. This kind of argument was not part of his lesson plan scenario. So he brought the discussion "back to earth."

The activities in **Patterns and Problems** are simply thought-starters. If they impel the group out into the cosmos, then they have served their purpose.

We are fully aware that goals which prepared materials are intended to reach are trivial in comparison with adventures out into the unknown which children are willing to undertake. Following the development as it unfolds in "the book" must always be a course that can be followed until something happens that can involve the learners more fully, releasing their imaginations, their creativity, their desire to know something they want to know.

"Talk sessions" may start as looking at a particular problem. The goal is not "that problem"— the goal is "the most engrossing problem" that arises.

"Solving problems and finding patterns" is the underlying theme — and, in an alive group, "talk sessions" will very often go far afield and far beyond the rather prosaic problems and patterns that initiated the discussion.

Teachers and parents who can't find time for "talk sessions" have a questionable sense of priorities.

If you look back over the week and find that "talk times" were overlooked, that ought to be a signal that the next week's plans would include "double time" for talk about arithmetic.

MEASUREMENT

Some of the activities have been selected because they require extensive though modest computations. Teachers may find it useful to bring these developments in to meet a need for more practice.

Other activities are off the beaten path, thereby providing more variety than learners have had available before. Many have a very light touch.

Many opportunities are provided for making many different kinds of measurement investigations. These are, in large part, inspired by John Holt through his book **What Do I Do Monday** (E. P. Dutton & Co., Inc., New York, 1970. Also available in paperback as a Delta Book.) Chapters 14, 15, 16, 17 and 18 are a fascinating source of measuring activities with fascinating ideas for children competing with themselves to demonstrate their ability to reach a goal they set for themselves.

When John Holt was asked recently what he would add if he were rewriting these chapters, he said he would begin by asking, "What is it children see adults measuring?" He pointed out that in much a child does on his own he is imitating adults — playing "hospital," "school," "store"— taking on adult roles. Later, children imitate characters they meet on television — adults, again. In this sense, he said, "Children are trying to escape from childhood."

Some parents are weight-watchers — and children are ready to play the same game, keeping records of results. A bathroom scale is a most versatile piece of equipment in the classroom.

Imagine a situation in which a child weighing himself can simultaneously pull up on a fixed object, such as a bar that can be set at any height. After finding his weight, how much higher can he make the scale go by pulling upon that fixture? . . . only the difference is noteworthy. Does the increase differ if checked in the morning and again shortly before the end of the school day? Can he increase the difference by practicing each day?

If the bar is set higher, how much difference can be accomplished by pushing up — with the bar at different heights?

Doctors are "pulse counters" and measurers of blood pressure. Carpenters and other builders are always measuring, often using fascinating collapsible rules and automatically rewinding metal tapes . . . clearly such devices are more adult than classroom foot rulers marked off in quarter inches.

A stop watch is an adult measuring tool used in many different "grown-up" athletic events. Some schools might find a source of supermarket scales that are being replaced by more modern versions — "surplus" scales headed for the junk pile.

Perhaps a school could collect all kinds of measuring devices to be catalogued and then "checked out" as from a library — and returned. In some factories, such collections of tools are called "tool cribs"; workers check out the tools they need, and return them when finished — signing them "out" and "in."

Measurement is a recurring theme in **Patterns and Problems,** Levels A thru F.

Patterns and Problems is primarily designed as a resource of opportunities for children to keep their natural learning systems in high gear in a mathematical environment. One of its goals is to offer a way out of a dilemma that is widely known but seldom discussed in public—"Not enough is already too much." This is an ominous cloud that hangs over elementary arithmetic at all grade levels. And it will not go away if you ignore it.

Patterns and Problems offers this solution: if "drill and practice" is raised to the "problem-solving level," then, and only then, will children put up with "enough."

Before coming to school children have been solving problems and making their own investigations . . . *when they wanted to.* Certainly "learning language" required continuous problem-solving and investigating. They have already demonstrated amazing accomplishments.

Not all of what children learn has "utilitarian value" in the sense that learning to speak has. They learn to play many games that are just fun. They enjoy working at puzzles — if they are not too hard. Any form of "hide and seek" seems to have a special appeal both for the searchers and for the hiders.

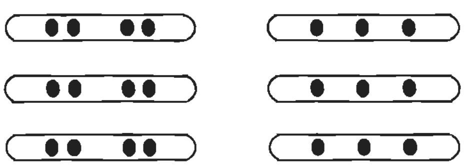
"Can you find a large book with a picture of the moon on its cover?" . . . the search is on; and if there is such a book, it will be found with squeals and other expressions of satisfaction from everyone involved.

Dump a can of beansticks on a table and ask if everyone will help find all the 4-sticks (popsicle sticks with 4 beans glued on them). Children will pick through the pile for a long time; each discovery of a 4-stick is a happy event and impetus to renew the search.

If cans or other receptacles are provided, one for each kind of stick (those with the same number of beans on them), the whole table top covered with sticks will begin soon to be cleared and the sticks properly sorted. The element of "search" fascinates most children. There is uncertainty involved, frustration, success, resolution.

The element of "search" is characteristic of most appropriate mathematical problems. A typical problem begins with obvious examples and "search" enters the activity as soon as outcomes are less than obvious.

Consider a group of three 4-sticks and three 3-sticks.



"Can you pick up just 1 bean?" . . . No. "Two beans?" . . . No. "Three beans?" . . . Yes, "Four?" . . . yes. "Five?" . . . after some hesitation there is general agreement that it can't be done.

From this point on, uncertainty enters almost every step of the way, because any larger numbers of beans will require combinations of sticks.

If you count all the beans, there is a total of 21.

Suppose we keep track of results with a simple system using checks (√) to mean "I can do it" and crosses (x) to mean "I can't find a combination with that number."

The results already considered are marked in this system:

√ √ √
3 4 ~~5~~ 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

What about 6, 7, 8, 9, 10, 11, etc.?

Please join the search for combinations for each of these numbers of beans. Remember, all you have is three 3-sticks and three 4-sticks.

Beginners will find the going rough with 6, 7, 8 and 9. You, the reader, ought to begin slowing down at 10, 11 and 12. Beginners might be asked to go no further than 12. You ought to persevere all the way to the full count of 21.

For beginners and adults alike, as soon as we pass beyond obvious examples (which are of little or no interest), difficulty enters. Most adults find 10 and 11 or both rather elusive. But eventually you hit on a combination — urged on by the feeling of accomplishment.

Any good problem for beginners is one that requires even adults to push beyond simple recall, to think in a characteristically human way.

As you look back over this little problem, notice that it was not demeaning even to adults. (How long did you spend looking for a combination of 16 beans before reaching the conclusion it can't be done?)

How much "drill and practice" did you do even though you didn't need it? . . . using small multiples of 3 and 4 . . . 3, 6, 9 and 4, 8, 12 . . . and adding to make combinations? The same computations as a typical page of unrelated examples would turn most people off — children and adults — because there is no reason to do them.

Notice, also, that the problem arises not because of the difficulty of the computation. Most "patterns and problems" can be explored without computational hazards.

Thus **Patterns and Problems** can be used with heterogeneous groups of any size providing a maximum of interaction between learners.

EVALUATION

The vogue in education is to limit evaluation to the "cognitive domain"— skill development and acquisition of knowledge.

A growing appreciation for patterns and developing a taste for problem-solving is squarely in the "affective domain"— attitudes, emotions and feelings. Since these human characteristics are difficult to measure, they are treated as "unmeasurables."

A creative approach to mathematics is discouraged by the fact that there is no standardized achievement test or "criterion referenced test" that can evaluate progress in the affective domain. We wish the "behaviorists" would broaden their expertise to find answers for these harder questions.

CDA Math has made a pioneering attempt to find answers to the harder questions — design "criterion referenced" tests that can measure learners' appreciation of patterns and ability to solve mathematical problems.

The initial effort is a collection of Problem Solving Activities designed to measure progress in the area of mathematics that concerns us in **Patterns and Problems.**

Our purpose is not to perfect a definite instrument. We are simply unwilling to equate "it's difficult" with "it can't be done." A slogan comes to mind: "If it's difficult, we'll do it tomorrow; if it's impossible, it will take us a little longer."

This "first attempt" to evaluate growth in the "affective domain" is essentially a request for the educational community to join in a concerted effort to find a solution to problems the behaviorists call "impossible."

Comments on our efforts will be deeply appreciated; but diverse pioneering efforts by many others are required because we have only begun to probe an area where "angels fear to tread."

EPILOGUE

An Experience with a Mathematician

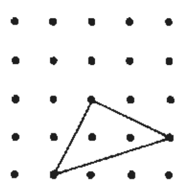
Several years ago, I found I had agreed to be two places at once and asked a colleague at the University of Illinois. Dr. Hyman Gabai, to talk to a P.T.A. meeting.

He protested, arguing he couldn't think of anything to say that they would want to hear, but finally agreed to try.

Hyman is a research mathematician with a fine sensitivity to other people and is conscientious to a fault. I was confident he would find a way to meet the challenge.

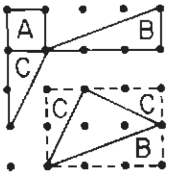
A few days before the P.T.A. meeting Hyman asked to talk about his plans. We sat down and he tried his ideas with me.

"You know, you can make lots of different shapes on a geoboard with a rubber band." He had such a board with 5 rows of pins spaced equally in both directions. He put a rubber band on the board in the following shape:



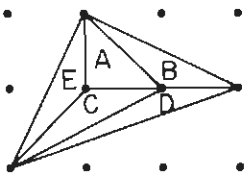
"Now you probably know that you can find the area of this kind of triangle by thinking of it as part of the area of a rectangle."

With pencil and a sheet of paper on which there was a 5 by 5 array of dots, he continued his explanation. He knew I was familiar with most of what he said, but he was really practicing for the P.T.A.

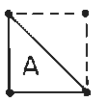


"Let's agree that the smallest square is 1 unit of area ... A in the sketch. B is half of 3 units of area, or $1\frac{1}{2}$ units; C is half of 2 units of area. The rectangle around the triangle is 2 x 3 units of area ... 6 units. So the area of the triangle is $6-(C+C+B)$ or $6-(1+1+1\frac{1}{2})$ and $6-3\frac{1}{2}=2\frac{1}{2}$. The area of the triangle is $2\frac{1}{2}$ units.

"We might call that the outside approach. Now there's another way to find that area—I'll call it the inside approach: I break up the inside into as many small triangles as possible—something like this:



"We are faced with the problem of finding the area of each of those triangles labeled A, B, C, D and E and we fall back on the outside approach with each.



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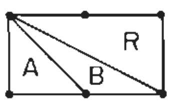

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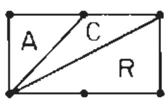

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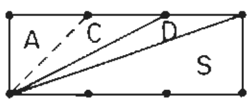

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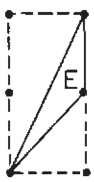

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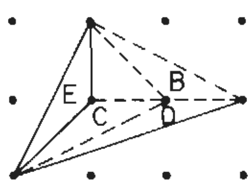


(6)

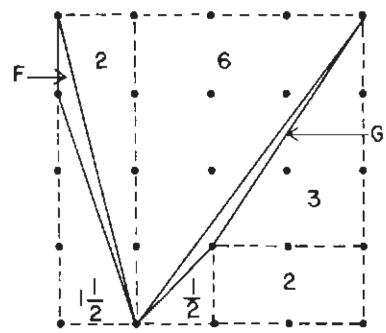
"(1) Triangle A is obviously half of a 1 unit square. (2) Triangle R is half of 2 units — or 1 unit. Triangle A and triangle B together are the same size as R or 1 unit. Since A is $\frac{1}{2}$ unit, so is B. (3) If you look at sketches 2 and 3, you will notice that B and C are the same size — each $\frac{1}{2}$ unit. (4) The main diagonal cuts this 3 unit area in half — so S must be $\frac{1}{2}$ of 3 or $1\frac{1}{2}$ units. Since A is $\frac{1}{2}$ unit and C is $\frac{1}{2}$ unit, then D must also be $\frac{1}{2}$ unit. (5) And E turns out to be $\frac{1}{2}$ unit also. (6) This sketch shows that B, C and E are all the same shape, just twisted around or flipped over.

"So, all 5 of those little triangles in our original area regardless of their particular shapes have the same area — $\frac{1}{2}$ unit."

That came as quite a surprise to me and I took a jump ahead.

"Will all 3-peggers on the geoboard have an area of $\frac{1}{2}$ unit?"

"Here's the geoboard, you can try some others. But remember — no pegs inside and no pegs touching any of the 3 sides."

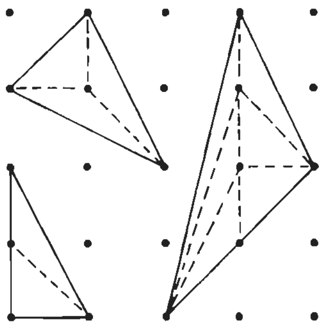


Triangle F is a slice through a 4 unit rectangle with a 2 unit triangle on one side and $1\frac{1}{2}$ unit triangle on the other — leaving $\frac{1}{2}$ unit for F.

Triangle G is a slice through a 3 x 4 unit area — or 12 units. On one side is a 6 unit triangle. On the other is a 3 unit triangle, a 2 unit rectangle, and a $\frac{1}{2}$ unit triangle: $6+3+2+\frac{1}{2}=11\frac{1}{2}$ and $12-11\frac{1}{2}=\frac{1}{2}$.

I tried on a larger geoboard to find a 3-pegger that had some area other than $\frac{1}{2}$ unit but I found no exception.

Then I thought back to the larger triangle we began with and the question arose, "Can the inside of any triangle be broken up into 3-peggers?"

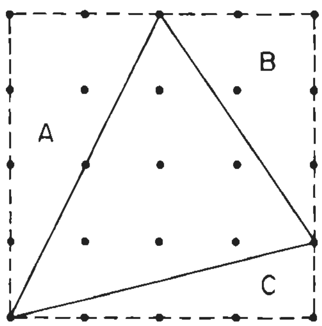


"Of course we haven't really proved anything," Hyman cautioned, "but you probably have strong feelings that all triangles you can make on a geoboard can be broken down into 3-peggers that all have an area of ½ unit; and that all you need to do is count the 3-peggers, divide by 2 and the result is the number of units in the whole area.

"Do you think the P.T.A. would find that little result surprising and interesting?"

I assured him it was sure worth a try.

"But that's not all," Hyman continued. "You might find that breaking up into little 3-peggers and counting the number of them becomes more and more tedious as the starting triangles get larger." He stretched a rubber band on the geoboard to make his point.

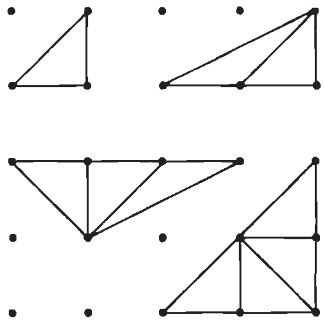


"In fact, the outside method would probably be much more efficient. The whole board has an area of 16 units, A is ½ of 8 or 4 units, B is ½ of 6 or 3 units, and C is ½ of 4 or 2 units; 9 units of area." $(4 + 3 + 2 = 9.)$ $16 - 9 = 7$ units of area.

I admitted it was probably more efficient but not nearly as much fun.

"Well," mathematician Gabai went on, "maybe we can improve on the efficiency of the inside method.

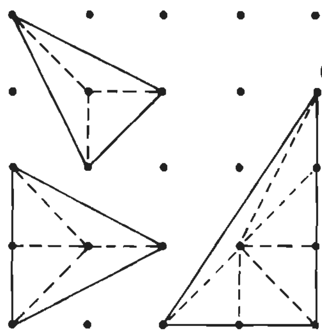
"You will notice that after the first 3 pegs, each additional peg on the outside adds a new 3-pegger inside:



Pegs	3-Peggers
3	1
4	2
5	3
6	4

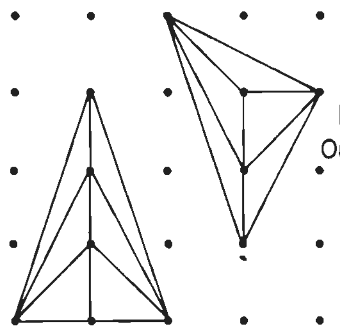
"So, if there are no pegs inside, just count the pegs on the outside, subtract 2 and that's the number of 3-peggers you will have.

"Now, what is the result of having a peg inside?"



Pegs Outside	Pegs Inside	No. of 3-Peggers
3	1	3
4	1	4
5	. . .	. . .
6	1	6

"It seems that a peg inside adds two 3-peggers. Now, if that holds for more pegs inside, a 3-pegger with 2 inside would break up into $1 + 2 + 2$ or 5 three-peggers.



Pegs Outside	Pegs Inside	No. of 3-Peggers
3	2	5
4	2	6

"These examples support our guess.

"So the 'inside approach' becomes a kind of 'pegs-on-the-outside, pegs-on-the-inside' approach.

"If there are no pegs on the inside, the triangle can be broken up into 2 less 3-peggers than pegs. So, outside pegs minus 2 indicates the number of

3-peggers with ½ unit of area each; and each inside peg adds 2 three-peggers with 1 unit each.

"Let E stand for outside pegs (Edges) and i for inside pegs, and we can say: $(E - 2) + (2i) =$ number ½ units 3-peggers.

$$\frac{(E-2)+2i}{2} = \text{Area}$$

"So the number of units in the whole area must be half the number of 3-peggers.

"Now we can clean that up a little but say the same thing. If there are no pegs inside, the area is 1 less than half the number of pegs—

$$\frac{E}{2} - 1$$

"And each peg inside adds a unit of area, so we can put that all together and say:

$$\frac{E}{2} - 1 + i = \text{Area} \qquad \frac{E}{2} + i - 1 = \text{Area}$$

"or as a set of directions: find the number of outside pegs, divide by 2; add the number of pegs inside; finally subtract 1 and that's the number of units in the area.

"Now you have both your fun and some real efficiency.

"Do you think the P.T.A. members will like that?"

"Not only will they like it, so would children. They could have some fun with their friends."

"Here's a geoboard: make any old triangle you want to; don't let me see it; how many pegs does the rubber band touch? and how many inside? . . . Then the area of your triangle is _ __ _ units. It would seem almost like magic."

"Oh, but that's not all," Hyman added.

"What do you mean?"

"Well, I've been talking about triangles you can make on a geoboard with a rubber band. Wouldn't you like to know something about squares and rectangles and shapes with as many sides as you want?"

"Now wait, Hyman," I warned, "I do, and I'll work on it, but I think you've gone far enough for one evening at the P.T.A. meeting. Maybe if you don't over do it they'll ask you for a return engagement."

"But I only have one more statement to make."

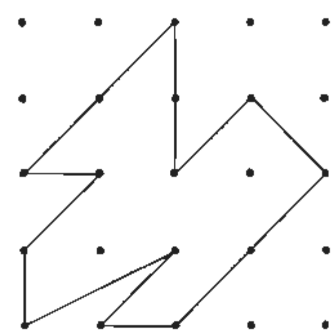
"What's that?"

"Simply that the formula without any changes works just as well for all shapes as it does for triangles. Can't I tell that to the P.T.A.?"

"You win," was all I could say.

And I tried one example on the geoboard:

$$\frac{14}{2} + 2 - 1 = 8$$



You can verify it for yourself — that that shape has an area of 8 units.

Hyman had a last remark: "Mathematicians know this as Pick's Theorem, named after its originator who lived in the late 1800's."

Not only did the P.T.A. hear about Pick's Theorem, but a growing number of children and parents from coast to coast now have found out about it.

Hyman Gabai let the rest of us in on a delightful bit of mathematical lore that's been lying around for the better part of a century.

There must be many other Hyman Gabai's in the fraternity of mathematicians who will ferret out of their past experience some notes that are appropriate for a talk to a P.T.A.

At some time or another, even some of us may chance upon interesting mathematical surprises that occur when working with numbers and shapes. The presentation of those can provide a stimulating springboard for discussion during staff development meetings, P.T.A. meetings and with children in the classroom.

PART TWO

A MEMO
from the author

TABLE OF CONTENTS

A Memo is Part II of *Mathematics for Everyone.* Fc the convenience of adults working with children i the classroom, the following material has been printe separately and inserted in Teacher's Editions of / B, C, D, E, and F.

'OREWORD

With Bob Beck as Editor and General Manager, "Friendly Math Associates" was created in Carmel, California, to undertake the task of adding an intermediate component to the already existing CDA Math Primary program. This seemed to be a modest task at first, because I had already outlined the extension into the middle grades. Lieselotte Esser, with whom I have worked closely for 15 years, came from Pennsylvania to act as "art and production manager."

Friendly Math Associates grew from one desk in an office to a group of dedicated and talented people. The goals began changing. "Can't we start from scratch and create a whole new program to span the elementary grades?"

As soon as the question was posed, the answer was, "Yes, let's do exactly that" . . . and now it's been done — at least the 1974 Edition of CDA Elementary Math is completed and already activity is under way to establish a "Catalog" in the spirit of the "Whole Earth Catalog" — a periodical written and edited by classroom teachers and administrators and parents.

Friendly Math Associates, in helping to create and produce an elementary school math program that asserts the primacy of humanistic goals has drawn together this energetic and challenging group of people. They have helped create CDA Math not as an end, but rather as a solid beginning to help teachers find effective answers to hard problems; but there will always be hard problems. And there is a need for a group to face up squarely to the difficulties that confront classroom teachers every day, searching for ways to answer cries for "HELP."

The "tasks" Friendly Math Associates undertook at first have, under the sensitive guidance of Bob Beck, developed into a crusade. Every problem that arose was always resolved in terms of "what do classroom teachers need?" . . . and there is no end to the challenge of that question.

I would like to list those who have helped make CDA Elementary Math possible. Each has contributed an indispensable piece, but to specify the particular contributions of each would require a volume. To each of these people, I owe a full measure of gratitude.

Robert W. Wirtz

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Suggestions About "Placement"

How can a teacher determine the point of entry into CDA Math that is most appropriate for a particular child and for a particular group of children?

BEGINNERS

The real question with beginners is "When is it appropriate to introduce them to **any** organized materials?" After that time has come, then the individual beginner should start with *Individualized Computation*, A_1 and groups of beginners can start with *Patterns and Problems*, Level A.

When are such materials appropriate?

The answer must focus attention on attitudes, not on skill development. Children, whether individuals or groups, are ready to encounter new material when their successes motivate them to move ahead. The crucial criterion for an individual child to move into *Individualized Computation* is not "as soon as he can," but, rather, when he "wants to" and, after starting, continues to "want to." Similarly, the criterion of readiness for a group of children to enter into *Patterns and Problems* is not at the point when most of the children "are able," but, rather, "when they can move ahead successfully" . . . when the group activities develop a "friendly feeling for numbers" in all participants.

Elsewhere, I have spoken out as forcefully as I can against misguided efforts to "push first grade performance down to kindergarten" and to describe desired "performance" in "behavioral terms" with "cognitive goals." The "management systems" have strongly influenced Early Childhood Education programs to train children much as Skinner trains his pigeons. For these people, cognitive goals are central and are to be pursued without concern for the attitudes of children. These "managers" fail to learn from the fact that such "training" does not show lasting results. Once the pressures they depend on are relaxed, the behavior they want is replaced by behavior that is more consistent with the child's natural style.

When in doubt about whether or not children are ready, hold back. If there is an optimum time for entry into CDA Math, it is far better to begin later than to begin too soon. However, if the time for entry seems to be approaching, the teacher should initiate activities and observe the children's attitudes during their involvement in these new activities, rather than merely noting their ability to comply with directions.

There is no appropriate placement tactic that does not depend squarely on the teacher's professional judgment . . . and this is true for newcomers to CDA Math in later grades as well as for beginners. However, there are several factors a teacher can implement to help determine placement.

Small piles of activity pages from *Drill and Practice at the Problem Solving Level* might be made available. Those who show interest need only a word or two of explanation and they will be able to move ahead on their own. When children indicate a desire for more activities of that kind, they can try *Individualized Computation*, Level A.

Groups of children who enjoy writing numbers and talking about them are ready to try *Patterns and Problems*, Level A. But whenever children who had been introduced to these materials need to be pushed to move on, they would learn more by turning to games and by handling materials with which they are more familiar.

If local school district guidelines insist on "placement tests," those requirements are adequately met by *My Progress—Primary Level*. The children's ability to handle the first test for Level A, will indicate whether or not they can handle beginning activities in CDA Math, Level A. I earnestly hope, however, that if these tests are required, each teacher will be concerned primarily with the child's attitude toward the tested activity when deciding whether CDA Math, Level A, is appropriate at that time.

NEWCOMERS

By "newcomers," I mean children who have already been introduced to materials other than CDA Math.

Again, *My Progress—Primary* and *My Progress—Intermediate* can be used to find the point at which each child is "able" to enter CDA Math and move ahead on his own. But scores on those tests provide an indication only of previous skill development, unless the teacher takes on the responsibility of assessing the child's attitude toward the activities within the test. The child's feelings about what he is asked to do in arithmetic are better indicators of future development than the test scores. Negative attitudes such as inattention, reluctance, numerous erasures, little progress, grumbling and the like are frequently manifestations of fear and should serve as a warning signal to the teacher to reduce the stress on the child by presenting him with material that is more familiar . . . or simply waiting for a while.

Age, grade placement and test scores are dangerous criteria for placement in CDA Math unless they are considered as tentative indicators that will help a teacher make this important professional decision. Such a judgment must depend mainly on the teacher's familiarity with CDA Math and on a familiarity with the child and the class as a whole. The appropriate level of materials and timing of their introduction to a newcomer is largely determined by a child's previously established image of the subject of mathematics and of himself as a learner of mathematics. Adequate time is required to enable a teacher to become familiar with each child's need before making any decision on placement, despite the fact that many administrators would prefer an immediate decision in the interest of record keeping.

Many teachers have come to know their children and CDA Math materials by selecting activities from *Drill and Practice at the Problem Solving Level* . . . activities the teachers can introduce with confidence and enjoyment. In the beginning, these teachers make their selections without any special concern about "content area." Rather, their concern is focused on deciding whether the activity is something the children will enjoy and whether they will want to do more of the same, even though the level of difficulty might rise a little.

The activities in *Drill and Practice at the Problem Solving Level* are based on the same view of content and the same learning theory as the rest of CDA Math (Levels A through F). Children's reactions to these activities will signal, to teachers who are watching how children feel, appropriate points of entry into the body of these CDA Math materials. Some children have developed a pervading sense of defeat when confronted with mathematics. There is no other way than moving them back to the point where they can handle the activity with confidence, where their faith in their ability to learn mathematics can be re-established — where they can begin developing a "friendliness for numbers." If the signals sent out by some intermediate grade children suggest to the teacher that they should start with quite elementary activities, these alert teachers should feel comfortable acting on that information.

After the foregoing discussion, I hope that I can assume that no child will be introduced to *Individualized Computation*, Level D, simply because he is in the fourth grade or because he is nine years old.

"Counting-On Club"

Arithmetic begins with learning to count by ones. After that, it is a never-ending search for shortcuts to avoid one-by-one counting.

In the beginning, many children are reluctant to "count-on" — they seem compelled to begin again at "1". For example: suppose the children have two piles of counters, they have counted the number in each, and recorded the results . . .

o o   o o o
o o   o o o

4 + 6 = ____

. . . and they are asked "How many altogether?" Most beginners will start counting from "1". As they were introduced to arithmetic, there was no alternative when asked, "How many?" It may be the case that some children think that to start counting from "1" is one of the rules of the game.

Somewhere along the way, someone, on his own or in response to gentle urging, will "count-on from 4" . . . "five, six . . . ten" and announce that the total is "10".

That's the time to stop everything else and have "Ruth" explain what she had just done. An overhead projector would help show the whole group how to count-on.

Then Ruth's name goes up on the chalkboard as president of the "Counting-On Club". She can wear a badge to proclaim her status. Then anyone can join the club by demonstrating the ability to "count-on."

Later, "Dave" will see the economy of counting-on from the bigger number:

4 + 6 = ____

"Six, seven, eight, nine, ten."

Dave becomes President of the Select Group that "counts-on from the bigger number."

These are but indications of many ways we can dramatize the important notions of arithmetic. Children need to hear often that arithmetic is basically a search for shortcuts to avoid one-by-one counting and in the most dramatic ways we can invent.

Counting-on, or using information you already have to avoid one-by-one counting is a tactic some children have not yet learned by the time they reach the intermediate grades.

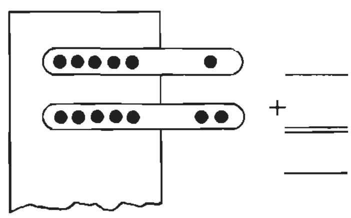
Bruce was in the fourth grade, already two years behind his peers. He had been through a rough experience — bounced from one relative to another, one foster home to another — one prospective adoption to another. In arithmetic, he was a "finger counter." Given sketches or diagrams, he had no confidence whatever unless he could count, one by one.

The principal of his new school felt he ought to be moved closer to his peers — a promotion.

Bruce was excited. For the first time he asked his new adoptive parents (who have now legally adopted him) to help him catch up in "arithmetic and reading."

How do you begin with a 12-year-old finger counter?

"Bruce, here is a page showing a special set of beansticks. The questions are: How many on each stick — and how many all together?" Here is the first example:



Bruce started counting.

"Wait. Let's talk about the beansticks. Are they just any old beansticks?"

(Every stick on the page had 5 on the left and 0, 1, 2, 3 or 4 on the right.)

"No, all the sticks have 5 beans on one side."

"Okay, how many beans are there on the first stick?" He started to point to each bean and count.

"But you said there were 5 beans on that side, so you don't have to count them."

"Oh — there are 6" . . . "So, write that down. And on the next stick?"

"Seven!" No counting, a big smile as he wrote it down.

"Bruce, how many on both sticks together?"

Bruce began to count.

"Wait a minute. Look at the stick. You told me there were 5 on the left side of all the sticks. How much is 5 and 5?"

"Ten." . . . "So how many on the left side of the pair of sticks?"

"Ten." . . . "and how many altogether on the two sticks?"

"Thirteen!" His face lost its usual tenseness and he grinned confidently. He wrote "13" and tore through the entire page — with the same verve or joie de vivre seen a few days later when his friend let him drive his "mini-bike" with complete abandon and in violation of all family rules and the laws of the state.

Bruce had finally qualified for the "Counting-on Club."

Adults may also become more confident as they come to understand that there is no mystery at all in arithmetic — simply shortcuts to avoid counting — (and that "memory" is the shortest shortcut of all.)

"Story Problems"

(an excerpt from Drill and Practice at the Problem Solving Level — An Alternative . . . pages 32 and 33)

When children have learned "2" as a symbol for twoness, they can be asked to tell a story that has the idea of "two" in it. Here is one kindergartner's response:

"My brother and I were hungry. We went out in the woods. He climbed one tree and I climbed another. He picked a piece of fruit and ate it. So did I. The trees said, 'Ouch, ouch!' "

Later children can be asked to tell stories that have the ideas of both two and three. "I watched channel 2 for three hours . . . I walked two blocks to the store and ate three hamburgers . . . I've only got two dolls and I wish I had three instead."

When children become quite sure that collections of two things and three things taken together are five things and that the idea is reported in the form "$2 + 3 = 5$," such a statement can be considered a "headline" or a title. What can you tell about that fits the headline?

"Two monsters were chasing three other monsters. I'm glad all five of them were running away from me."

"I have a two-wheeler and my little brother has a tricycle. We've got five wheels."

As a report of an experiment, a child wrote "$6 - 1 = 5$" on the board. The teacher asked, "Can you make up a story to fit that number sentence?" One boy came up to the front of the room. He walked over to the teacher and kissed her before explaining, "I had six kisses. I gave Mrs. Burns one of them. I have five left."

Making up stories or describing situations to fit number sentences can be an exciting highlight for children. It is an invitation to show individuality, to amuse, to get things out. It puts the learner in the driver's seat.

"My headline is '$50 + 50 = 100$'. A truck full of dynamite exploded and blew the truck driver into 100 pieces. Half of them landed on the beach and half of them fell in the ocean."

A fourth grade boy used "$2\frac{1}{2} + 4\frac{1}{2} = 7$" as a headline for his story: "On the way home from school I got my shoes muddy. Then I walked across the living room rug. My mother bawled me out for $2\frac{1}{2}$ hours. My father bawled me out for $4\frac{1}{2}$ hours. How long did it take my parents to say nothing?"

As thought-starters for headline-story sessions, one inventive primary teacher devoted a bulletin board to recording information children reported:

"It is 45 steps to the principal's office."

"Our room is 30 feet long."

"The dictionary weighs 4 pounds."

"There are 7 windows in our room."

"There are 5 days of school each week."

"Milk is 7¢ a carton."

"There are 28 children in our class," etc.

Each fact was on a separate piece of tagboard so the information could be changed easily. Children were encouraged to use that information as "thought-starters" suggesting situations for real or imagined happenings or applications.

In almost every teacher's guide and textbook for pre-service training there is buried somewhere the advice ". . . and sometimes let the children make up their own story problems." But the bulk of advice these guides and textbooks offer could hardly be calculated to more effectively separate arithmetic and its applications. Teachers are encouraged to have children look for "cue words" in story problems. "And" usually suggests addition; "less than" usually suggests subtraction; "of" usually suggests multiplication.

A girl in a British infant school developed her own set of rules which she found quite reliable: if both numbers are large, then use addition or subtraction; if both are small, multiply; if one is small and the other large, then divide!

"Story problems" are a source of frustration for most teachers and children. If a year's moratorium on all such problems were declared, children would probably perform at a higher level on the "applications" part of the standardized achievement tests at the end of the year. Many children would forget all the frustrations and the rules they were supposed to remember. They would read the problems and try to figure out what the question is, and then use common sense. The arithmetic is usually on the simple-minded side.

The most productive strategy would be to have children spend a year away from the problems that authors dream up for textbooks. Instead, they should make up and solve their own problems. Our experience is that most children will invent and solve problems far more difficult than textbook authors would dare include. Children write about what they understand and there are no semantic hurdles for them to overcome.

If "pencil and paper arithmetic" is separated from "applications" it can only develop as memorization without understanding, and applications are an entirely new encounter. If arithmetic is introduced as a means for keeping records of a child's experiments and their outcome, then applications are an integral part of the learning experience. As representations and symbols take the place of objects to be manipulated, the intimate connection between arithmetic and the real world is maintained as children talk about their own real and imaginary experiences.

Later, when the need develops in science, in social studies or any other content areas to use numbers to help describe an experiment or situation, this is no new idea. It has been the role of arithmetic from the very beginning to keep records of experiments with things and to report about goings-on in the real world.

Memorizing the Basic Facts and Processes

Unless, at some point, children have "automatic recall" of basic facts and processes, they fall far behind when computations are required. While the "new math" proponents often argued that "understanding is more important," they had no answer for a decade of declining test results.

How do you memorize your telephone number, your address, a poem, the spelling of an unusual name, the National Anthem, a recipe, a formula?

My feeling is that two teachers I know have suggested a breakthrough on this question that seems to have escaped decades of research. Marilyn Burns and Mary Lorton have worked with beginners with a keen sensitivity to their responses in a great variety of situations. Before outlining their conclusions, I would like to recall other approaches to this same troublesome problem.

In "Teachers College" in the early 30's, I was introduced to the psychological theories of Thorndike. He maintained that we do a disservice to children to have them memorize the "tables" . . . "0 plus 1 equals 1, 1 plus 1 equals 2, 2 plus 1 equals 3," etc. . . . and "0 times 3 equals 0, 1 times 3 equals 3, 2 times 3 equals 6, 3 times 3 equals 9," etc. The result of learning this litany is that when the child is confronted with the question, "How much is 8 times 3?" he must begin with "0 times 3" and go through the whole chart until he comes to the item he wants.

An example of this phenomenon is the difficulty one has with answering the question: "In the Pledge of Allegiance to the Flag of the United States of America, what word precedes the expression 'for which'?" To answer, most people must recite to themselves the entire pledge until they reach that phrase before they can answer that "republic" is the preceding word.

Thorndike's advice was to have the children practice the "basic facts" in "random order"—so each is fixed as a single item in the memory—not part of a lengthy list.

$$\begin{array}{r} 7 \\ \times 2 \\ \hline \end{array} \quad \begin{array}{r} 3 \\ \times 5 \\ \hline \end{array} \quad \begin{array}{r} 8 \\ \times 4 \\ \hline \end{array} \quad \text{etc.}$$

So "memorizing the tables," as I had done in school, became passé . . . it was replaced by "random drill."

Later, the behaviorists from Watson to Skinner have added "reinforcement" — immediate reward of some kind for each correct response — as a powerful motivation to remember. Using this technique, Skinner has trained pigeons to pick defective pills off pharmaceutical production-line belts with greater efficiency than human beings (which is a compliment to Skinner's skills and a greater compliment to human beings).

The logical projection of this tactic would be "pennies for pupils" . . . teachers supplied with, let us say, $500 per year of pennies — 50,000 pennies — ready to reward immediately each correct response a child makes. (And if our goals do not reach beyond "skill development", this would produce better test scores than most programs based on "performance-objectives-in-the-cognitive-domain" are producing today.)

Marilyn Burns and Mary Lorton searched deeply into their own experiences and observed children at work and play. "How do people memorize?" Their conclusion was that at first people generally remember "auditory patterns" — they remember what they say often enough to themselves or "out loud." And when called upon for an act of "memory," the reply is a remembered "auditory pattern."

When confronted with this question:

$$\begin{array}{r} 37 \\ \times 9 \\ \hline \end{array}$$

I know I say to myself, "9 times 7 is 63, put down the 3 and carry 6" and I write down:

$$\begin{array}{r} {\scriptstyle 6} \\ 37 \\ \times 9 \\ \hline 3 \end{array}$$

I am calling up auditory patterns I have committed to memory!

I selected that particular fact — $9 \times 7 = 63$ — because it has always been more troublesome than most other facts. While I went through the "auditory pattern" stage of recall with all the facts, most of them eventually blend into "visual patterns." Now when I see

$$\begin{array}{r} 3 \\ \times 2 \\ \hline \end{array} \quad \begin{array}{r} 6 \\ \times 5 \\ \hline \end{array} \quad \begin{array}{r} 4 \\ \times 4 \\ \hline \end{array} \quad \text{etc.}$$

my immediate response is "6, 30, 16" . . . each is an immediate response.

The conclusions of Marilyn Burns and Mary Lorton, as I interpret them, would suggest that beginners would benefit more from many opportunities to say out loud and to themselves the full expression suggested by

$$\begin{array}{r} 5 \\ +6 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{r} 3 \\ \times 4 \\ \hline \end{array} \quad \text{etc.}$$

"five plus six is eleven" or "five plus six equals eleven" if you prefer and "four times three equals twelve," etc.

"Choral Response" is a tactic that can use the insight of Burns/Lorton to help groups of children get on with memorization.

In this activity, a group of children is urged to tell, for example, the basic addition facts they find most difficult to remember. These are written on the board — answers included:

$$\begin{array}{r} 8 \\ +5 \\ \hline 13 \end{array} \quad \begin{array}{r} 9 \\ +7 \\ \hline 16 \end{array} \quad \begin{array}{r} 8 \\ +9 \\ \hline 17 \end{array} \quad \begin{array}{r} 7 \\ +4 \\ \hline 11 \end{array} \quad \text{etc.}$$

After the list is completed, the leader points to one of the examples and raises his hand signalling "think-time." When the leader brings his hand down, everyone is to whisper the full expression as loudly as he can — "eight plus nine is seventeen."

There is usually full group participation because the activity threatens no one; if a child's response is wrong, no one else knows it. If the response is right, there is "immediate reinforcement."

After considering the list once or twice in random order, the answers are erased, and the tempo quickened. Children take over the role of leader . . . and begin seeing how fast they can move the chorus through the examples.

The appropriate timespan of this activity is short. Interest is intense; children are keyed to a high pitch — it often becomes feverish.

In another session, the chorus is limited to whispering as loud as they can "only the answers." When the leader holds up his hand signalling "think time" a member can prepare by repeating an "auditory pattern" to himself or he may have developed confidence in a more visual pattern — saying to himself only the answer.

As the tempo increases this time, it may gently persuade everyone to shift from auditory to visual patterns whenever possible.

Choral response is a highly motivating and effective tactic whenever "memory" is the desired outcome, regardless of the content area.

My youngest children often ask for nursery rhymes at bedtime. While many of the well known rhymes have a rich heritage, I am convinced they are basically "jibberish" to the children. "Peter, Peter, Pumpkin Eater . . ."

Sometimes they ask me to say them as fast as I can, or they join in if the pace is slowed down.

One night, instead of a nursery rhyme, I began, "One plus one is two; two plus two is four; three plus three is six . . ." and on up to "ten plus ten is twenty".

When I finished they asked, "Will you do that one again, Daddy?"

I slowed it down to the "altogether pace" and we went up to "ten plus ten is twenty". While they had been uncertain along the way, they both knew that last line and said it with a flourish.

"Do it again, Daddy!"

So, I started with "ten plus ten is twenty" and worked down to "one plus one is two."

At least once a week they now ask to do "one plus one is two" . . . always at the slow pace so they can join in.

The producers of Sesame Street understand the importance of helping young children develop useful auditory patterns. Almost every child who listens to the program at age four comes to kindergarten knowing the auditory patterns of "counting to twenty" and "the alphabet."*

As teachers and parents appreciate more fully that most beginners first become familiar with the facts of arithmetic as auditory patterns, they will find many ways to stimulate children to use those patterns often — and parents and teachers will, themselves, use those patterns often with children. However, children who have memorized a visual pattern prefer to respond with crisp answers rather than longer expressions, and they should be encouraged to do so.

Marilyn Burns and Mary Lorton have made a most significant contribution to the on-going search for more efficient ways to help children memorize the basic facts and processes necessary for computational facility.

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* Some may not yet have learned to count a pile of 17 blocks reliably, but they know the sounds and need only to break them up . . . one — two — three — four . . . as they consider objects in some order and answer the question "How many?" with the sound associated with the last item.

Variety and CDA Math

CDA Math, in recognizing the importance of humanistic goals as integral to its instructional program, has seen the clear need for including a variety of presentations or formats. One way of attaining this variety is in the use of pictures, stories and graphic formats which deal with specific mathematical experiences. In terms of our learning theory grid, some of these different kinds of stories start at simple representational levels while others fall on a continuum, finally ending up with rather explicit mathematical proofs.

The covers used on CDA materials have been chosen from original works of art done by children. These lively and authentic covers remind children, parents and teachers that the materials included are for children. There is a clear recognition of and respect for the abilities of children in all areas of performance.

This notion extends to the ultimate use of any CDA Math page for purposes that are particular to an individual child–that go beyond the normal intent of the page. Friendliness with numbers is more likely to be achieved if the child considers the page a personal possession, a potential extension of one's own personality.

Specifically this means that in addition to computing, experimenting or recording, a child might want to decorate, color, doodle or dally on a page. All of this is acceptable. Neatness is not the sole virtue; nothing is sacred about a page because it duplicates the answer book. If a child feels free to react openly to a page of arithmetic, the revelations there may provide invaluable insights into the attitudes of that child about math, school, the child's own self and all those around.

Even a casual observation of the enthusiasm with which children welcome comic books has led CDA Math to incorporate that format as a means of reaching children in the affective domain. By the time many children reach the intermediate grades, they have developed a fear of mathematics textbooks or, at best, a boredom with mathematics. The lightness and humor of comic books which deal with mathematical situations, directly or indirectly, helps to reestablish a friendliness for numbers in a way that no amount of formal presentation or practice (computational drill) could accomplish.

In addition to comic books, CDA has included several partially illustrated mathematical stories, sometimes using comic book characters, to enhance and help explain mathematical ideas. Beside providing human and sometimes humorous situations, the illustrations help to clarify difficulties presented by the nature of the mathematical problems. At first profusely illustrated, the stories become less so as the complexity of the mathematics develops.

I think that the introduction of rigorous mathematical proofs (see *Patterns and Problems*, Level F) has never been attempted at the elementary school level in such a head-on fashion. CDA has included a few such proofs, written in vocabulary understandable to elementary school children. These proofs, carried out in careful, step-by-step progressions, are neither frivolous nor demeaning. Many children will not be ready to follow the arguments. However, such proofs provide the opportunity for those children who have a special interest in and aptitude for mathematics to face up to a challenging and exciting look into the thinking which goes into substantial mathematical discovery.

"Diagnosis" and "Prescription"

Preoccupation with skill development and lack of attention to the feeling of learners has spawned the "diagnosis-prescription-diagnosis" cycle — (language borrowed from the medical profession).

The idea is that diagnostic tests will pin-point the skill which children need, but have not acquired. Then materials are selected to remedy this particular weakness.

I have been asked to provide references in CDA Math materials so that if a child is having trouble with regrouping in subtraction, a teacher can locate materials to help overcome this weakness. Beyond cross references (in the annotations) to *Drill and Practice at the Problem Solving Level* and references to *Developing Insights into Elementary Mathematics*, there is no attention to isolated skills.

When a child is "having trouble with regrouping in subtraction" and others around him are not, he is probably bogged down by arithmetic in general, and he will have trouble with whatever skill the order of the day requires. His trouble is probably a symptom of a generally unfriendly attitude toward numbers and an appropriate response must go to the basic cause. (More on this in a moment.)

If on the other hand, the difficulty is not basic and is simply an isolated breakdown in communication, then it can best be overcome by the peers who have "got it." Children have a way of helping each other that adults do not understand. If the materials written by adults or the explanations made by teachers, aides or parents have not been clearly understood by some of the children, those children who did understand are the best resource available for those who did not.

However, very often a child's trouble lies far below the surface — in a fear of mathematics — an unfriendly feeling about the subject. No cross reference to more of the same will help overcome his problem. He needs a change — and finding a different activity that is responsive to his needs requires all the highest capabilities of a teacher or a parent.

Again, *Drill and Practice at the Problem Solving Level* is one source designed to supply a large variety of high-interest activities at all levels of difficulty. And while these activities sometimes seem like games to the casual observer, each is designed to move the learner from a completely non-threatening situation and help shift him from the role of passenger to that of driver. As such, he will explore what he thinks are the boundaries of his abilities and find he can push further than he thought he could.

Until an activity can be found in which the learner is not overcome by frustration, he will learn more by doing no "pencil and paper arithmetic."

At the risk of being redundant, I would like to suggest again that listening to children is more revealing than all the tests that have been devised. And by "listening" I mean trying very hard to hear what the child is trying to say.

A particularly effective listening technique is to paraphrase what you think the child is trying to say, asking if that is what he means. He will answer honestly, and if the response is negative, try another formulation, and another, if necessary, until he tells you that you finally got through to what he is thinking about. Very often, when you do get there, you find that what's really on his mind is far distant from where the discussion began . . . deep in the feelings of the child.

(Teachers and parents can learn much more about this powerful means of communication in "Parent Effectiveness Training" (PET) by Dr. Thomas Gordon.)

When techniques of evaluation are related foremost to goals in the "affective domain," we can learn to know children much better and are thus better prepared to help them grow as learners.

Answers Provided on Pupil Pages

Throughout *Individualized Computation, Levels A through F*, some "answers" to examples are given on all pages.

The technique varies. In Levels A_1, A_2, B_1, B_2 answers are shown on the left side of the page and are easily found as the child works. Slightly more than half of the answers are provided — the rest are "on your own."

Later, this technique is gradually shifted toward a multiple choice format. There is an answer box in the lower right hand corner of the page. Rows and columns are labeled with letters.

Adjacent to most examples, there is a letter — meaning that the answer can be found in the row or column indicated by that letter. If a child's answer appears in that row or column, he can be reasonably sure it is correct; if it doesn't appear, he should check his work. If he still feels he is correct, he should challenge the answer given — books are written by humans who also make mistakes.

To foster a healthy skepticism about other people's answers, some answers given are deliberately "wrong." These are pointed out in the annotations so teachers are alerted to them. And all children have an opportunity to "catch the author."

I am sure that other mistakes will escape even the best proof-readers . . . and there will be other mistakes not pointed out in the annotations.

Finally, there are situations in which the correctness of an answer depends on how one interprets the question. Regardless of the answer given, any answer is correct if it is an appropriate response to a reasonable interpretation of the question posed. If the author is ambiguous, and he sometimes is, there may be several right answers. The test is "reasonability." Every answer a child is ready to defend is worthy of careful consideration.

Human knowledge is expanded by people who search for new and more comprehensive answers.

"Jumping Around"

Patterns and Problems, Levels A through F, bear a subtitle: "Jumping Around in Mathematics."

I hope all teachers will find a more effective way to order the activities in *Patterns and Problems* — a sequence that meets the needs of their children more adequately. Perhaps different groups within a class may "jump around" in an order that the group, itself, helped to select.

In the intermediate grades, some individual students or small groups can select chapters or developments that are harder and more demanding, while others may skip those same sections.

Patterns and Problems, Levels, A, B and C, have a sequence some teachers may find useful. In the first two-thirds (64 pages) most of the basic notions have been developed. They are extended in the last third (32 pages) but the difficulty is not increased. Annotations indicate when the activity is repeated in the last third and the page numbers are listed.

Consequently, all learners can sometimes work successfully "in the back of the book" — not only those who are more mature and able. And we hope this organization will help dispel any notion on the teacher's part that the book needs to be "covered."

Notes to Parents

In *Individualized Computation, Levels A, B and C*, tests are spotted along the way based on the material covered. On the back of each test page, there is a letter addressed to parents that can be signed by the teacher.

These tests and letters help parents know what has been going on at school in mathematics and suggest what lies immediately ahead. Suggestions are included of activities parents can help initiate in the home, such as making bean-sticks to explore bean-stick problems.

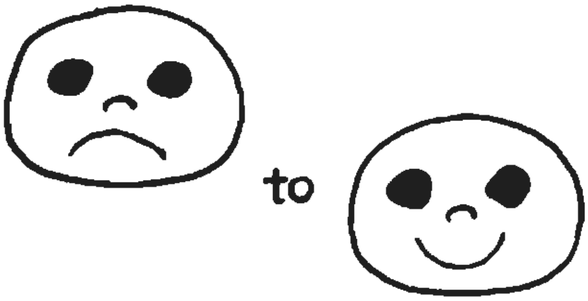
Parents, in the past, have been made to feel the schools would rather they didn't help their children in mathematics. The "notes to parents" leave no question — their help is needed, earnestly solicited and welcomed.

(Translations of these letters are made available in Spanish with rights to reproduce. As other ethnic communities want similar translations they will be provided.)

Space is provided on these combined test-and-letters-to-parent pages so the teacher can add whatever seems most helpful.

Perhaps these letters to parents can contribute toward the ultimate goal of community participation in the improvement of elementary school mathematics.

At the end of each test, there is a face with eyes and nose but no mouth. The child is asked to show "How do you feel?" by adding the mouth



and all variations in between.

The emphasis on the child's feeling or attitudes is a recurring theme throughout the parent letters, stressing the importance of helping children develop a real friendliness for numbers as the most effective preparation for continuing to succeed in mathematics.

Peer Group Teaching

Many classroom teachers feel they are beseiged with questions throughout the day that make it virtually impossible to get anything done. The call goes out for smaller classes, more preparation time and more aides. Any or all of these changes may help, but such changes are often impractical in terms of economy of time and money. But, more importantly, in all but the most extreme cases, such changes may be unnecessary.

Classroom teachers do need help, but many of them are overlooking their most available and effective resource — the very children they are teaching. In the past, the notion prevailed that a good classroom was one in which a principal could walk by and hear a pin drop. Obviously, given silence in the classroom as the gauge of excellence, children were not encouraged to work together or ask each other questions. However, times have changed and there is now emerging an understanding on the part of parents, teachers and administrators that a low buzz of activity and discussion is often an indication that learning *is* going on.

It is now possible for teachers to shift part of the responsibility of individual explanations to other chil-