BANKING ON PROBLEM SOLVING

by **ROBERT W. WIRTZ**

Pupil Activity Pages

BANK I

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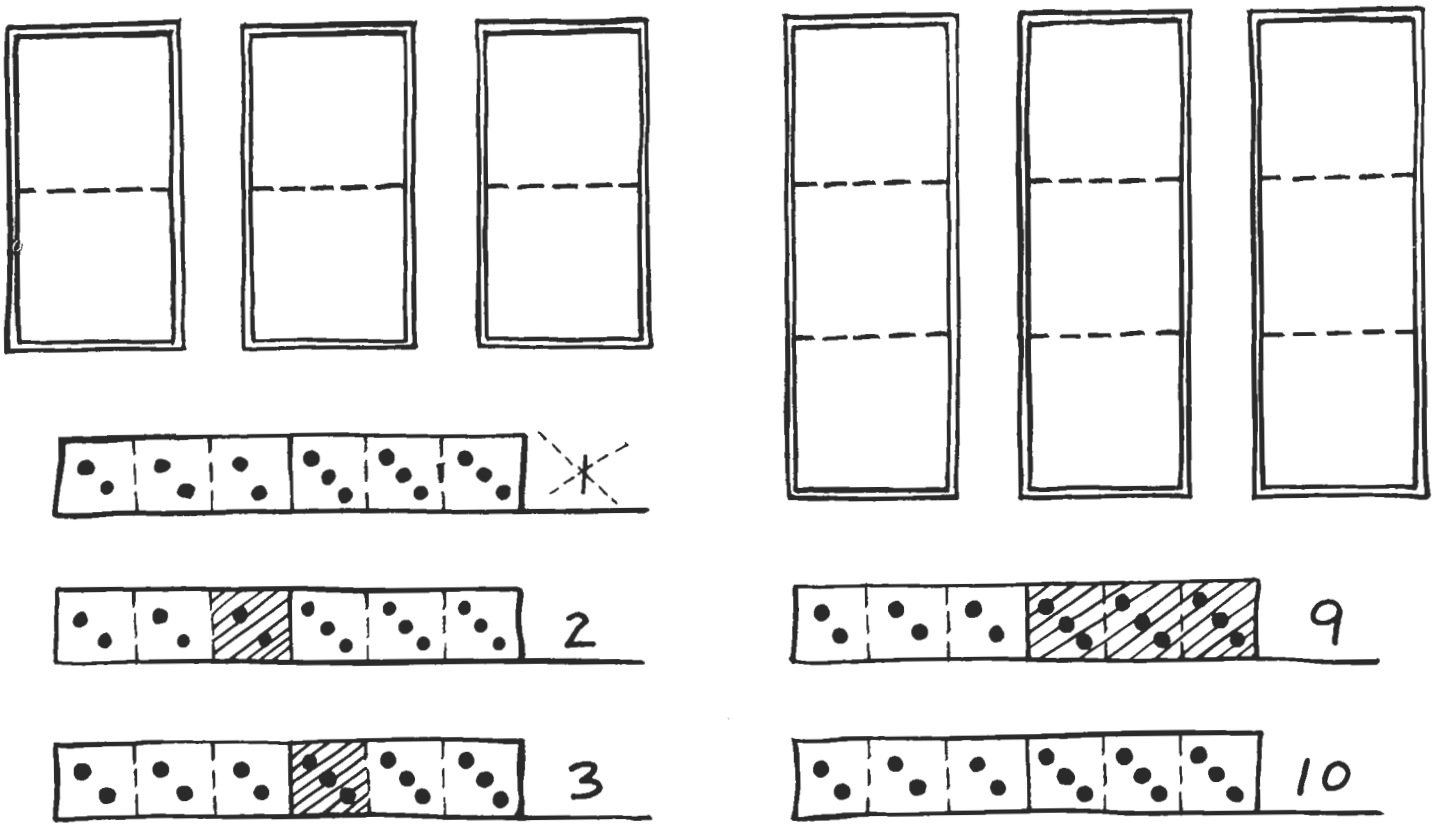

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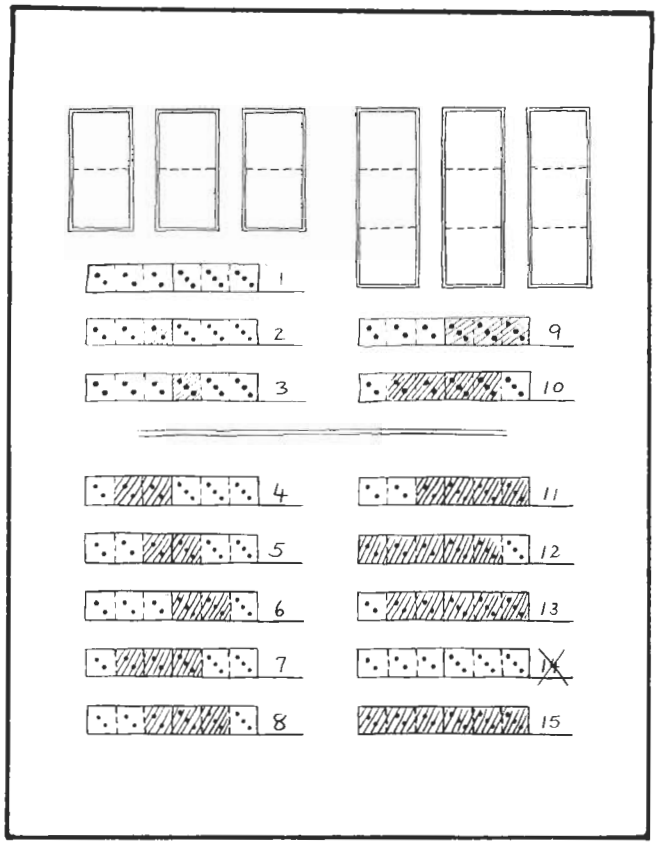
(Some of the appropriate responses are shown here.)



General Comments about "Windows and Panes"

This page begins a series of nine "Windows and Panes" activities, and it is helpful to make some general comments at the outset. A "window" is a group of squares framed by a double line. A "pane" is a small square inside the "window". There are some simple rules for placing counters on the windows pictured at the top of the page:

(1) If you put a counter on one pane in a window, you must put a counter on every pane in that window.
(2) You may not put more than one counter on any one pane.
(3) You may use as many windows as you like.

You may want to introduce these activities with a much more elementary version of "windows and panes". On this page, children are asked to place counters on windows with two or three panes, and there are only three of each kind of window available. A teacher, aide or volunteer could use the same problem in a different context with a group of beginners. They might gather around a table, and each person might get a group of counters.

"Can you put them in groups of three each? Depending on the number of counters, they may or may not be able to do so.

"Can you put them in groups with two or three in each group?" Yes, if we allow an unlimited number of groups of two or three each.

"How many groups of two and how many groups of three did you use?"

"Now let's all start with eight counters. Can you put them in groups of three each?" No, it cannot be done.

"Can you put them in groups with two or three in each group?" Yes, it can be done.

"How did you do it?" Some children may have four groups of two, and others one group of two and two groups of three.

"Who has the fewest groups?" (Those who have two groups of three and one group of two have only three groups.)

"Let's take another counter so everyone has nine. Please put them into groups of two or three—and make the fewest groups you can." There are two alternatives to make nine: three groups of two and one group of three (four groups), or three groups of three. The second alternative, of course, has fewest groups."

"Now use ten counters in all, and put them in groups of two or three. Use the fewest groups! How many groups do you have?" Some children will have five groups of two. Others might have two groups of two and two groups of three; just four groups.

"Try 11 counters. Use groups of two or three, and try for the fewest groups." Some may use four groups of two and one group of three; others may have three groups of three and one group of two. No one has fewer than four groups.

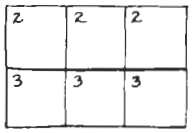
"Let's add another counter, to make 12 in all. How many different ways can we arrange them in groups of two or three?" Surprise! There are three possible outcomes:

1) six groups of two
2) three groups of two and two groups of three
3) four groups of three

At some point along the way it might be appropriate

to keep some kind of record to indicate these combinations.

A slightly different setting for this activity can introduce a new constraint for the problem and make it closer to the type of activity on this and the following pages. We can limit the number of piles of each kind. For example, a sheet of paper can show places for just three groups of each kind.



The new limitations are directly felt when one puts 12 counters on the page. Space is not provided for six groups of two or four groups of three—but three groups of two and two groups of three can be shown. Under the new constraint, this is the only solution.

Now, how many counters can be placed on the page in two *different* ways? Six counters can be shown as two groups of three or three groups of two. Nine counters can be shown as three groups of three or as three groups of two and one group of three.

The numbers of panes in the windows and the numbers of windows available can be varied in this activity, until children have a good sense of how to proceed.

Introducing the Problem

When children are ready to use the activity page, some time must be spent talking about how to record results. The number at the right of each strip of small boxes tells how many counters must be placed in the windows. After the counters have been placed, children should record their solution by coloring in the boxes with the dots that correspond to the windows they used. Each box with two dots represents a window with two panes; each box with three dots represents a window with three panes. If a number cannot be done, it should be crossed out.

In the example above the double line that calls for nine counters, there are two possible solutions. Children should be on the lookout for others that might have two solutions as well.

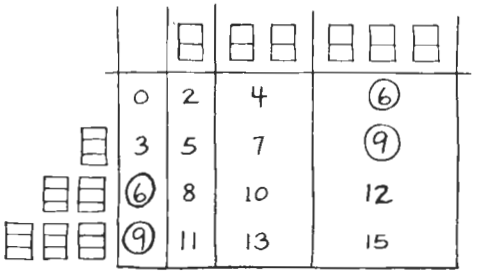
Talking about the Problem

(1) Which numbers of counters couldn't be placed in the windows according to the rules? (One counter, fourteen counters, and any number of counters larger than fifteen couldn't be placed.)
(2) Which numbers of counters could be placed in two different ways? (Six counters and nine counters.)

Extensions and Surprise Endings

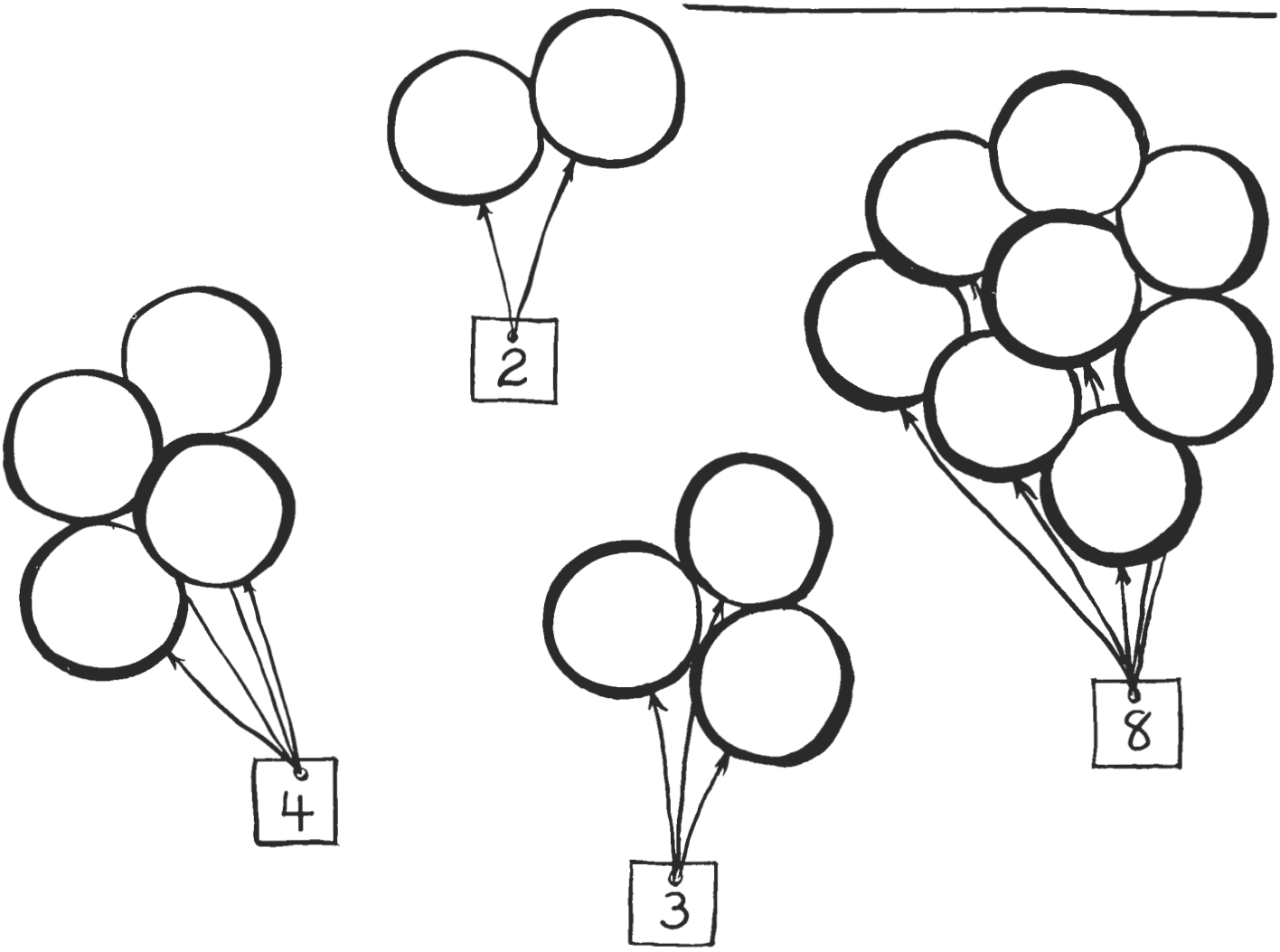
There is another way to approach the problem. How many panes are there all together? There are fifteen. To put on fourteen counters, or one less than fifteen, we would have to leave just one pane without a counter. That can't be done under the rules of the game. To place thirteen counters, which is two less than fifteen, we have to leave two panes out. That's easy — just leave out one of the windows with two panes, and put counters on all the rest. The same strategy can be used for 12, 11, 10, and so on.

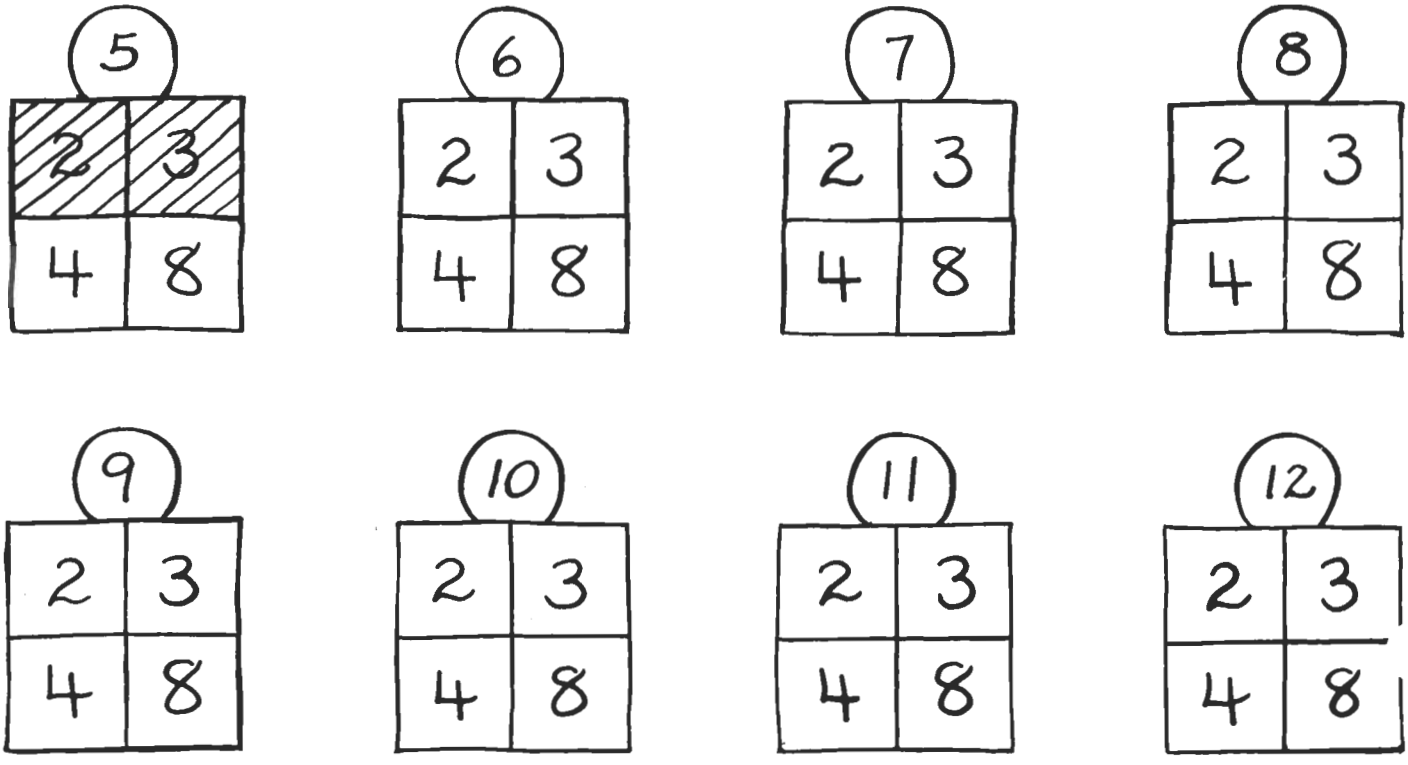
One could also make a chart of all the options for placing counters in this arrangement of windows and panes.



Clearly, all numbers occur in the chart except one and fourteen. Also, six and nine are the only numbers that occur in the chart twice, and they are the only numbers of counters that could be arranged in two different ways.

The activity on the reverse side of this page is taken from *Drill and Practice at the Problem Solving Level* (Curriculum Development Associates, 1974)





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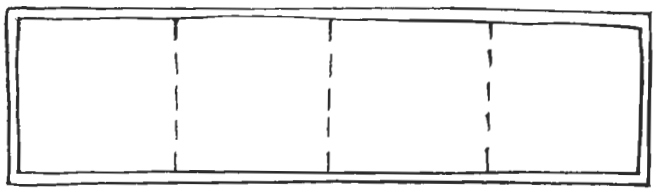
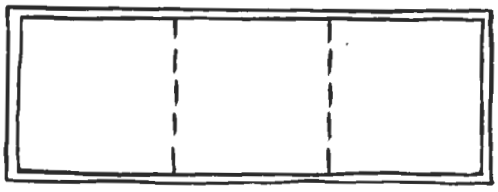
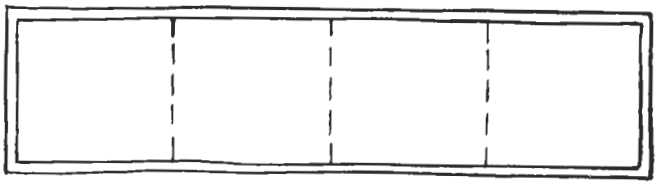
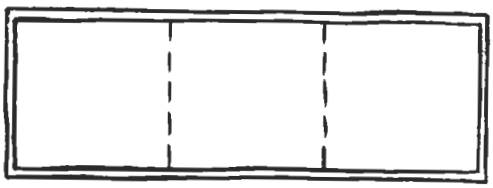
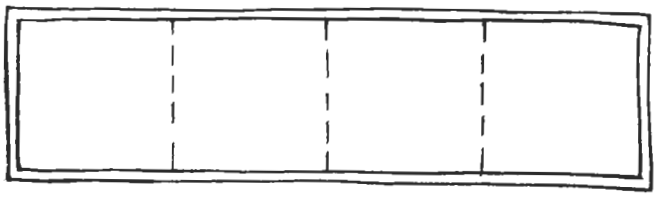
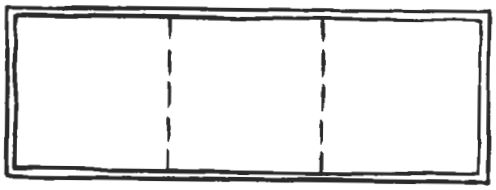
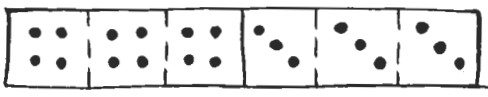
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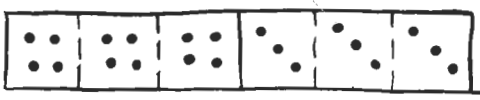
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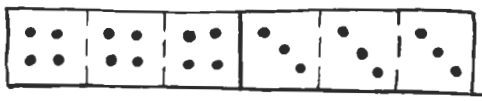
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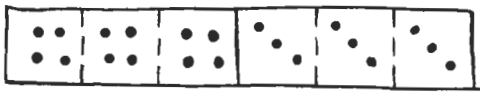
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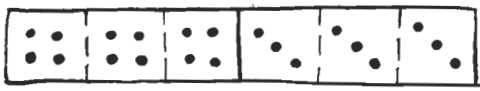
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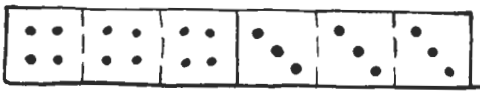
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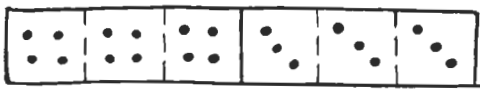
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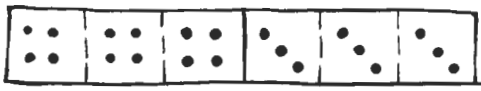
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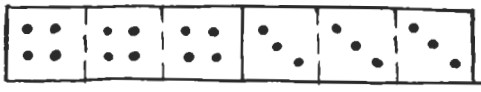
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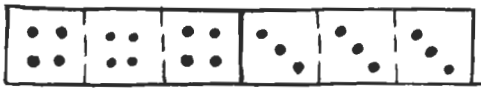
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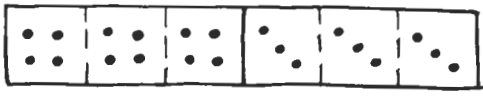
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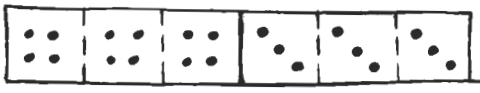
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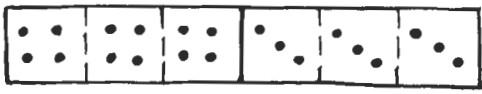
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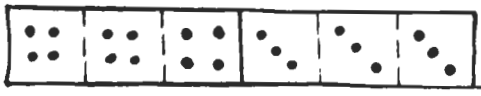
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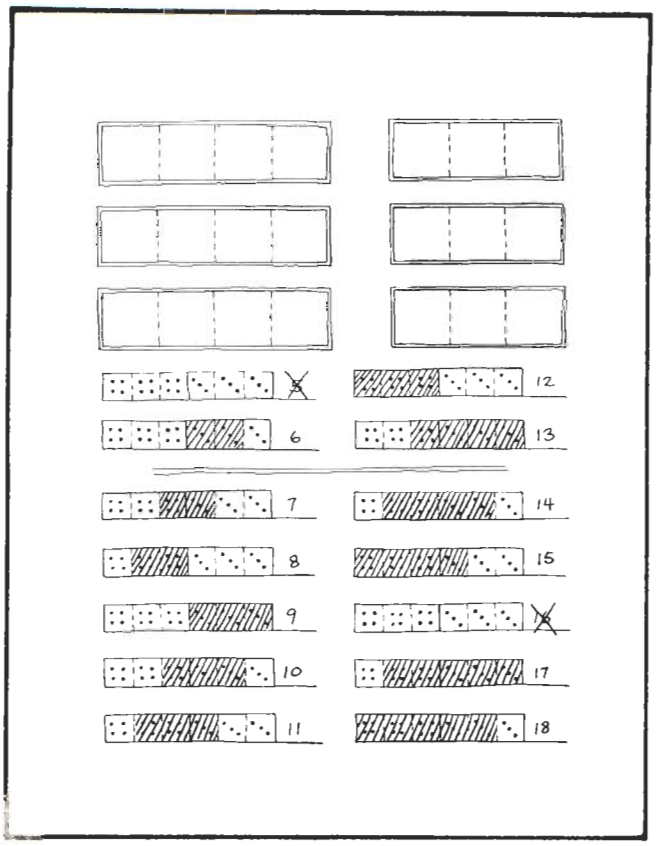


Introducing the Problem

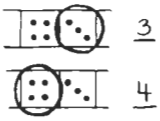
Rules for placing counters are the same as for the previous windows and panes problems. In this activity, the available windows have three panes each or four panes each.

The number at the right of each strip of small boxes tells how many counters must be placed in the windows. After the counters have been placed, children should record their solution by coloring in the boxes with the dots that correspond to the windows they used. Each box with four dots represents a window with four panes; each box with three dots represents a window with three panes. If a number cannot be done, it should be crossed out.

Talking about the Problem

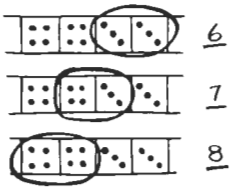
Which numbers of counters could not be placed? It is impossible to place five or 16 counters with this array of windows and panes.

While there is no rule for recording the combinations of windows used, it would be interesting to discuss one recording strategy when talking about the problem. A rather interesting pattern develops when we make it a point to first use those boxes on either side of the center line which separates the boxes of three from the boxes of four. For example, to show 3, use the first box of three to the *right* of the line; to show 4, use the first box of four to the *left* of the line.

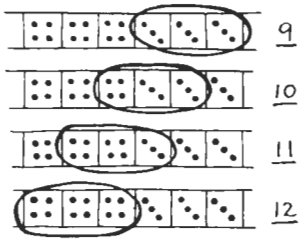


These choices suggest a movement from right to left, one box at a time.

To show 6, 7, and 8, we start with the two boxes of three to the right of the line and "move" one "step" to the left each time.

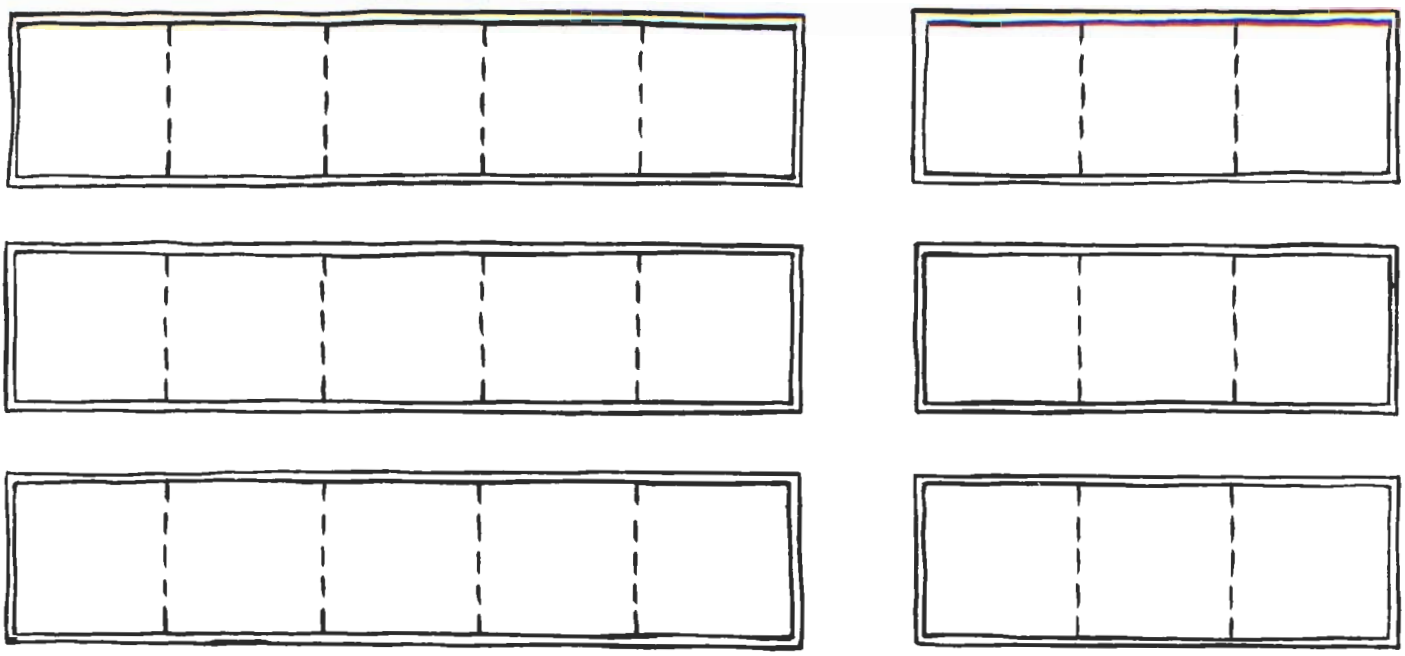


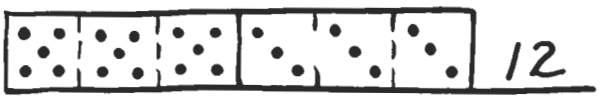
Combinations for 9, 10, 11 and 12 done in this way suggest a similar movement.

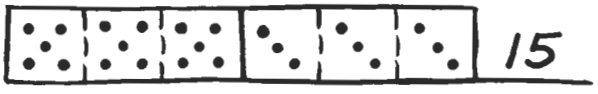


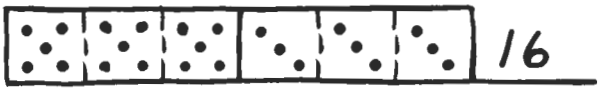
Combinations for 13, 14, and 15 "move" four boxes at a time, and combinations for 17 and 18 move five boxes at a time.

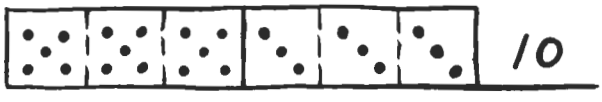
What is behind this movement? Quite simply, with each "step" to the left, three dots are dropped off, and four are gained. The net effect is the addition of one more dot.

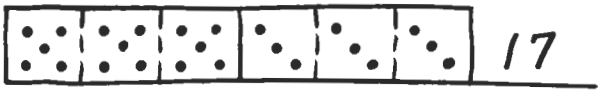


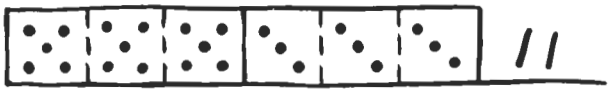


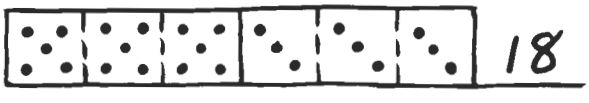












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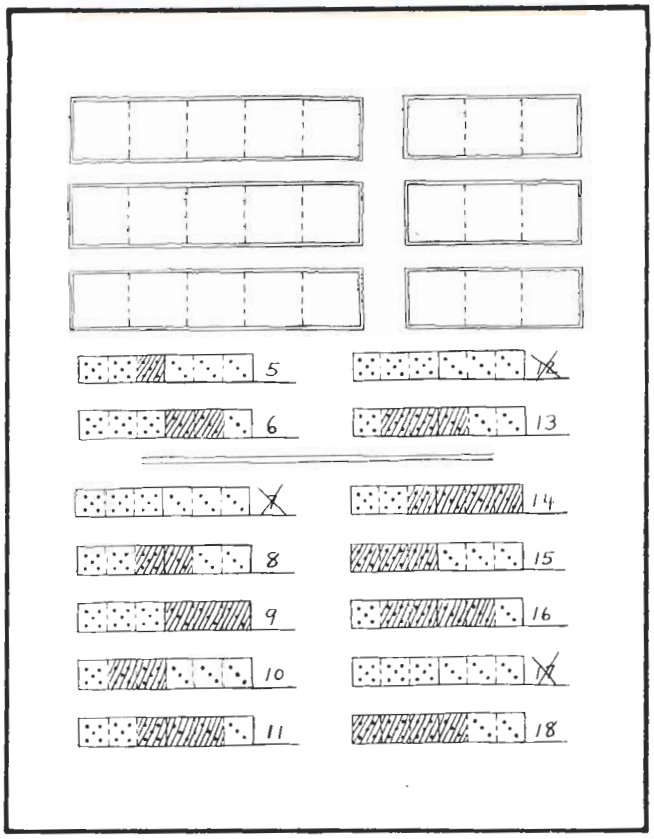


Introducing the Problem

Rules are the same as for previous windows and panes activities. Here we are using windows with five or three panes. Following the rules, we can place five counters by using just one window with five panes. Six can be done with two windows of three panes each. There is no way to do 12, but 13 can be done with two five-pane windows and one three-pane window. The fact that there is no way to do 12 suggests that others on the page may also be impossible under the rules.

Talking about the Problem

Which numbers couldn't you do? There is no way to place 7, 12 or 17 counters. (Nor is there any way to place 20, 22 or 23 counters.)

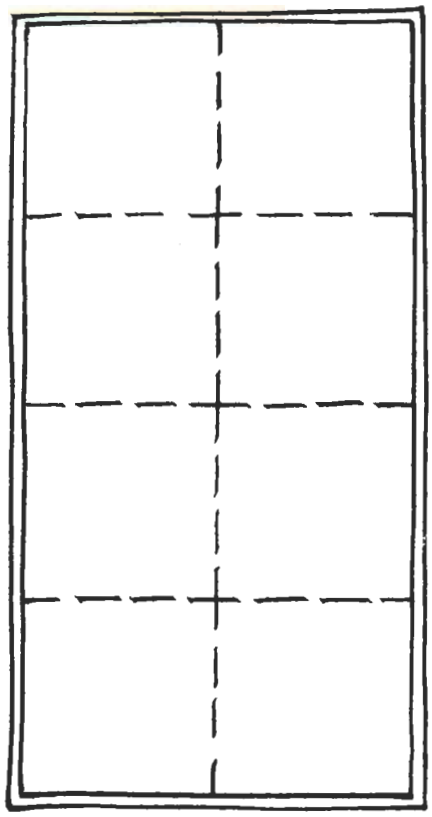
Are there any that can be done with different combinations? Of course, five counters could be placed in any one of the three five-pane windows, but we mean *essentially* different ways. The answer is no; no number can be shown in more than one way. (See chart below.)

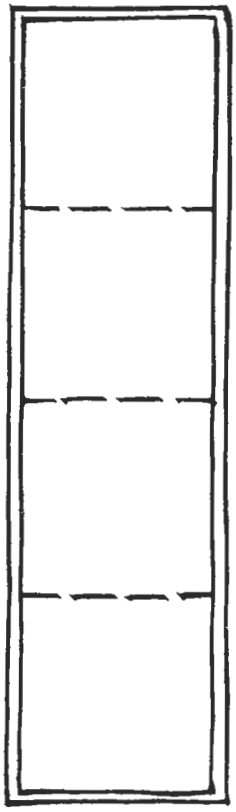
As in the previous activity, a chart can be made which shows all possible combinations for this set of windows.

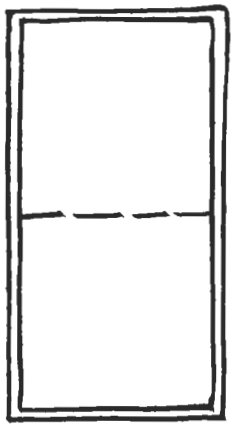
	0	3	6	9
	5	8	11	14
	10	13	16	19
	15	18	21	24

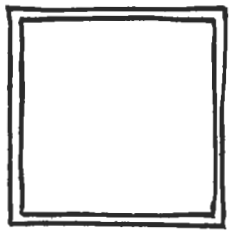
Again we see that it is impossible to show 7, 12, 17, 20, 22, or 23. They do not occur in the chart of all possible combinations.

There is another way to reason that some of these numbers cannot be done. If we put one counter on every pane in all the windows, how many counters would we use? We would need 24 counters. Now, how many would we have to remove in order to have 17 left? We would have to get rid of 7. But that is impossible under the rules. Reasoning in this way can save us a lot of work, especially when considering the larger numbers in the list.









8 4 2 (1) 1

8 (4) (2) 1 6

8 4 2 1 11

8 4 2 1 2

8 4 2 1 7

8 4 2 1 12

8 4 2 1 3

8 4 2 1 8

8 4 2 1 13

8 4 2 1 4

8 4 2 1 9

8 4 2 1 14

8 4 2 1 5

8 4 2 1 10

8 4 2 1 15

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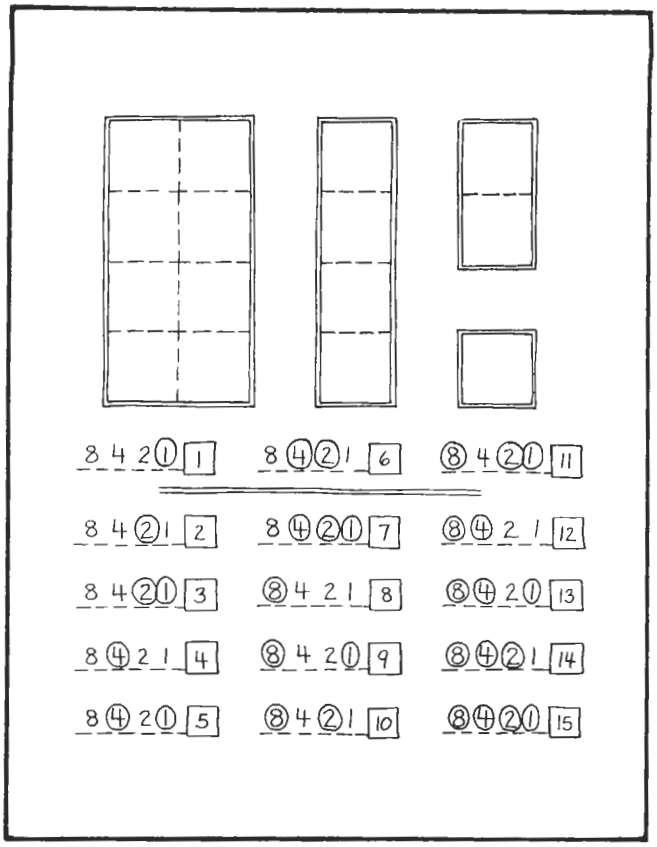


Introducing the Problem

Rules for this activity are the same as for previous windows and panes problems. In this variation, each window has a different number of panes. The numbers in the small squares indicate the numbers of counters that must be placed in the panes. For each problem, children should place the appropriate number of counters and then loop the numbers that correspond to the numbers of panes in the windows they used.

The three examples above the double line indicate that:

(a) it is permissible to loop a single number as was done for one counter,
(b) more than one number can be looped as for six and eleven counters, and
(c) the numbers looped do not have to be neighbors in the list.

Talking about the Problem

(1) Which examples did you think could not be done? (Actually, there is a solution for all examples, but this is not obvious at first.)

(2) Did you find two different ways of placing the counters for any of the numbers? (None of the numbers have two possible ways.)

Extensions and Surprise Endings

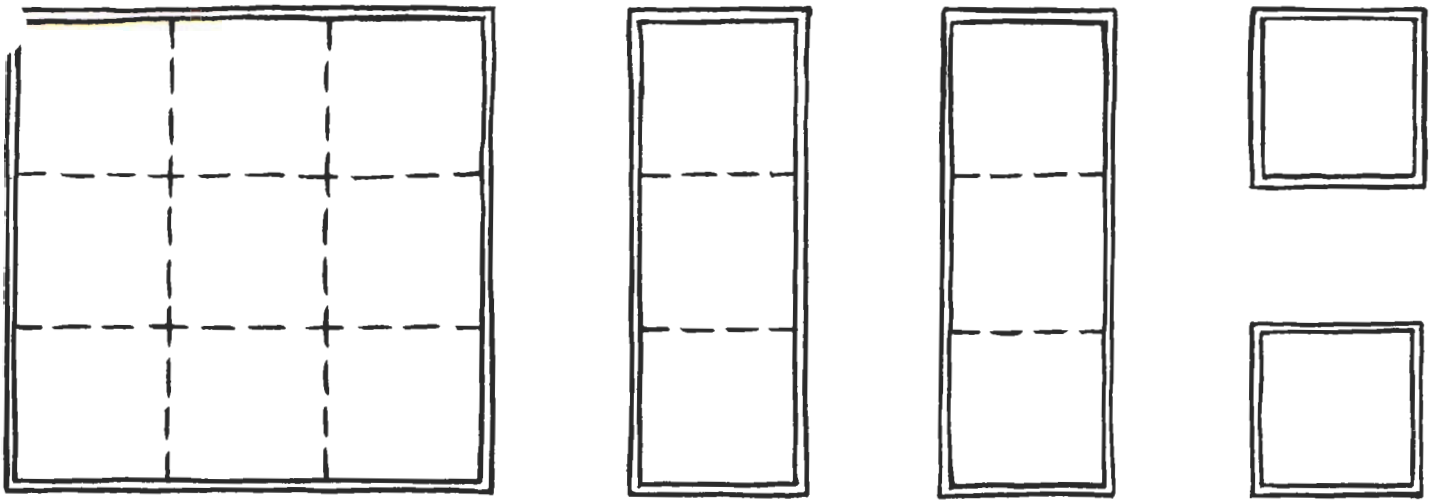
There are fifteen panes all together in this collection of windows. Every number of counters, 0 through 15, can be placed on these windows in one and only one way. The numbers 1, 2, 4, and 8 are a very special group of four numbers. No other selection of four numbers would provide unique combinations for all numbers, 0 through the sum of the four numbers. If we added a window with 16 panes, there would be a unique combination for all numbers, 0 through 31.

The Game of "One More and One Less"

This page can be used as a playing board for two children for the game of "One More and One Less". Rules for placing counters remain the same. Player A puts one counter on the board. "A" must do this by placing a counter in the window with one pane. Then player B must put "one more", or two counters on the board. To do this, "B" must move the counter played by "A" to the window with two panes, and add one more counter to fill up the window. Next, player A must add one more counter, and so on.

There is a surprising result. Every time player A is to add one more counter, the window with one pane is empty and ready for one more counter. But every time it is player B's turn, the counters on the board must be moved around to make a place for one more. All player A needs to do each time is place a counter in the empty window with one pane; player B has to do all the work in rearranging counters.

When all the panes are full, it is time to play "One Less" and player B goes first. "B" removes the counter on the 1-pane window. Then "A" must remove "one more". That requires removing a counter from the 2-pane window and moving the other counter to the 1-pane window. "B" simply removes that counter, leaving the 4-and 8-pane windows full. To remove "one more", player A takes one off the 4-pane window and moves the remaining 3 to the 2-pane and 1-pane windows. So the roles have been reversed; player A is doing all the work.



9, 3, 3, 1, (1) 1

9, (3, 3, 1), 1 7

(9, 3), 3, (1), 1 13

9, 3, 3, 1, 1 2

9, 3, 3, 1, 1 8

9, 3, 3, 1, 1 14

9, 3, 3, 1, 1 3

9, 3, 3, 1, 1 9

9, 3, 3, 1, 1 15

9, 3, 3, 1, 1 4

9, 3, 3, 1, 1 10

9, 3, 3, 1, 1 16

9, 3, 3, 1, 1 5

9, 3, 3, 1, 1 11

9, 3, 3, 1, 1 17

9, 3, 3, 1, 1 6

9, 3, 3, 1, 1 12

9, 3, 3, 1, 1 18

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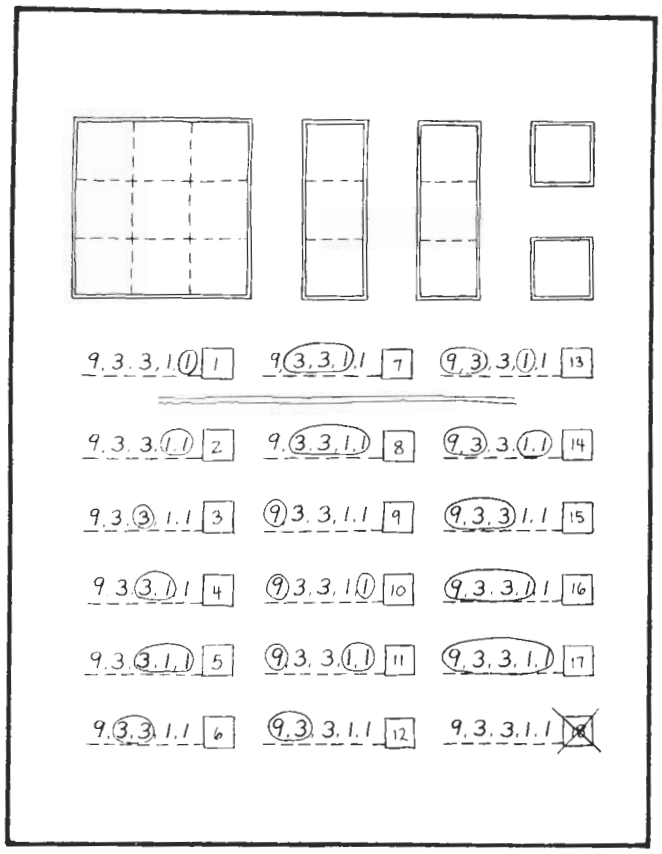


Introducing the Problem

Rules for placing counters are the same as for previous problems with windows and panes. The number in the small square indicates the number of counters to be placed in the windows. For each one, children should place the appropriate number of counters on the windows, and then loop the numbers in the list that correspond to the numbers of panes in the windows they used. For example, the record

$$9(3\,3\,1)1 = 7$$

indicates that two windows with three panes each and one window with one pane were used in placing seven counters.

The three problems used to introduce this activity show that sometimes a single number will be looped, sometimes "neighboring" numbers are looped, and sometimes numbers that are not "neighbors" will be looped in the record.

Talking about the Problem

(1) Can any numbers be done in more than one way? (No, assuming that we acknowledge that

$$(9\,3)3(1) = 13 \text{ and } (9)3(3\,1) = 13$$

represent essentially the same combination of windows: one with nine panes, one with three, and one with one.)

(2) How many panes are there all together? (There are 17 panes.)

(3) Which problems could not be done? (18 counters could not be placed.)

Extensions and Surprise Endings

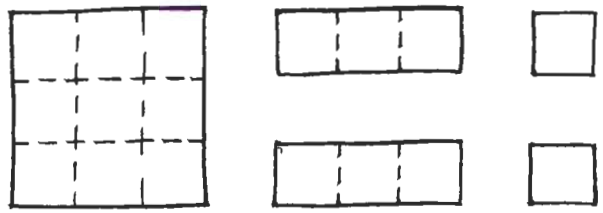
The "Human Computer" Game

A fun game can be played with a set of three large cards bearing the numbers "1", "3" and "9". Strings should be attached to the cards so the players can hang them around their necks. A team of four players is organized. Three players wear the cards, and one serves as the team coach.

A player can indicate the number on his card by raising one hand. He can also indicate twice the number on his card by raising two hands. Numbers are called out, and with the assistance of their coach, the team raises their hands to show the specified number. The group pictured below is responding to "7". The player wearing the three raises both hands, and the player wearing the one raises one hand (3 + 3 + 1 = 7).

Several groups of four players can be organized into teams and demonstrate their skill with the help of a coach.

With more able groups (adults included) a card which bears the number "27" can be added to the set. Now there will be a combination for all numbers through 80. This activity was developed by Wallace Manning of Idaho Falls for a workshop of the Idaho Council of Teachers of Mathematics.



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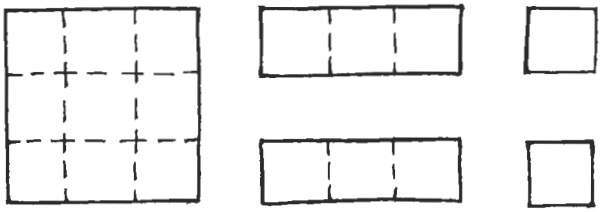

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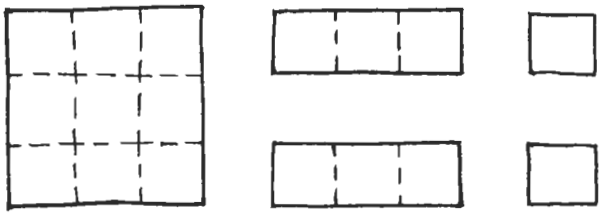

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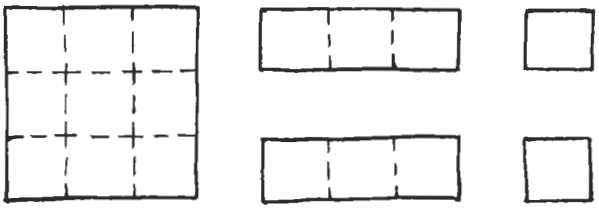

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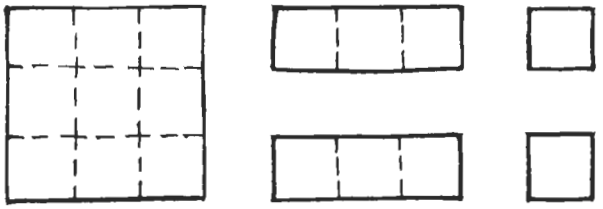

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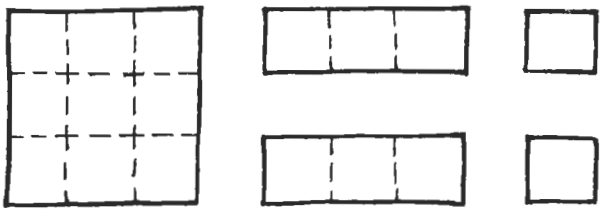

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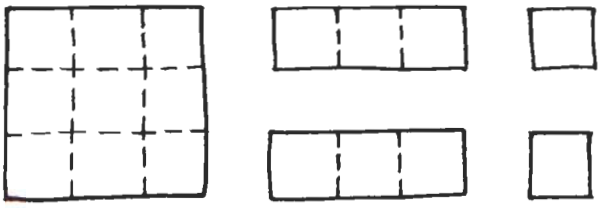

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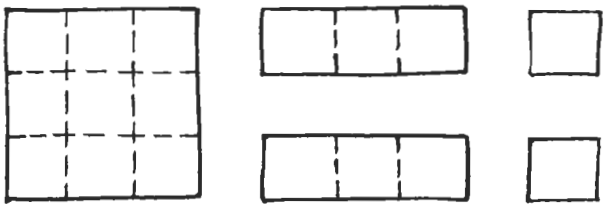

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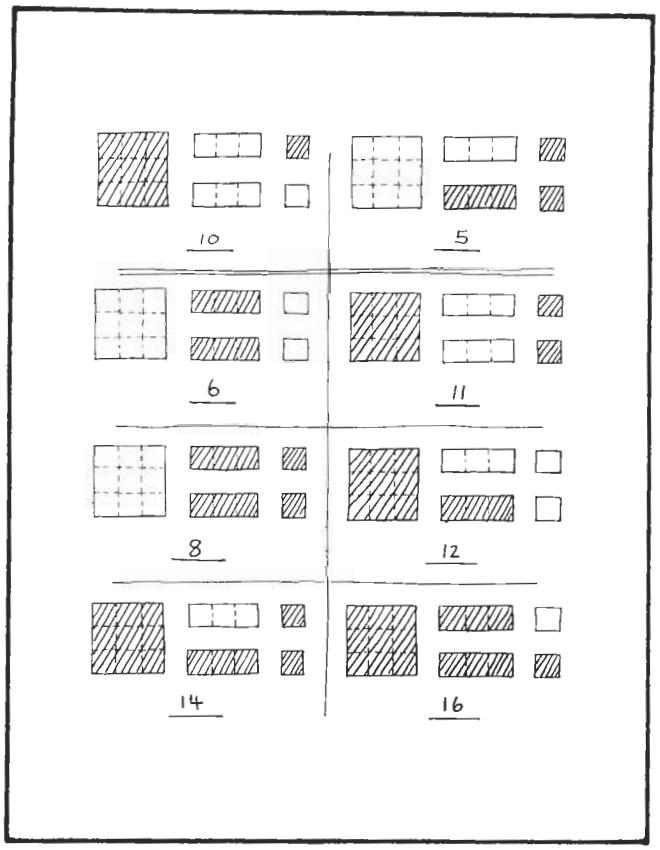
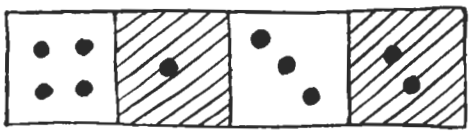


Introducing the Problem

Rules are the same as for previous windows and panes activities, except that this is a coloring activity. Instead of placing counters in the window panes, you must color the panes. If you color one pane in a window, you must color the whole window.

Activites that use movable objects have the unique quality of permitting trial and error without a record of mistakes. Coloring introduces a certain hazard—if you start out wrong, you must cope with the error in some way. Thus, if some children have difficulty with the activity, they might use small counters, arranging them on the windows until they have a solution. Then they can remove the counters and color in the windows they used. You might also provide children with plastic folders and grease pencils for this type of activity. Erasures can be easily made and there are no traces of the error.

Further ideas for using this page are given in the comments for previous windows and panes activities.



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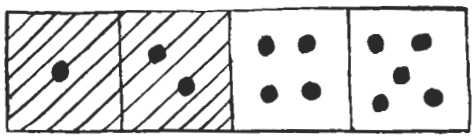

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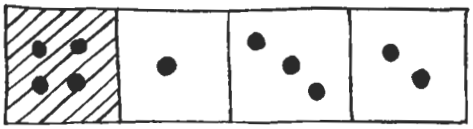

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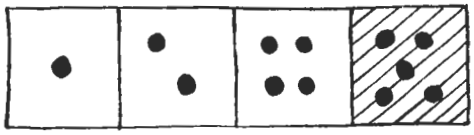

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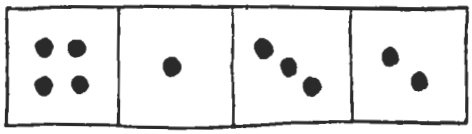

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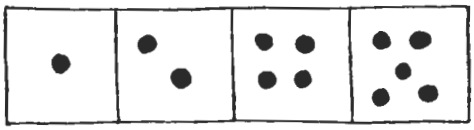

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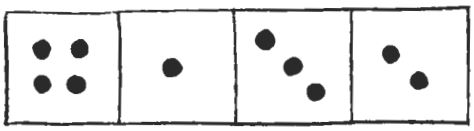

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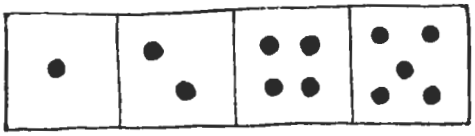

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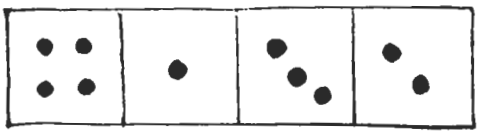

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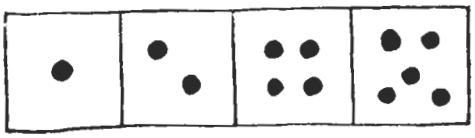

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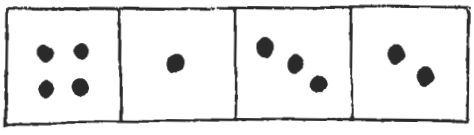

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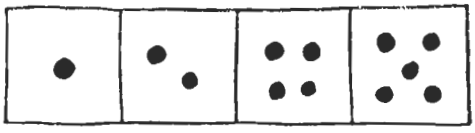

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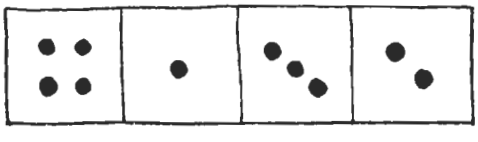

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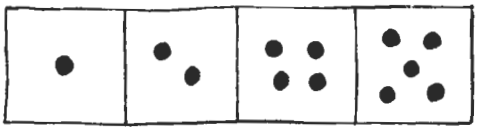

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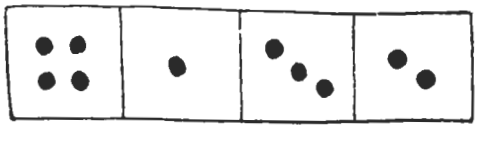

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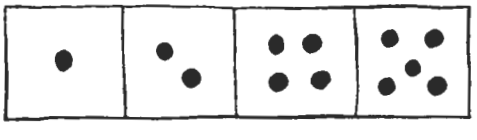

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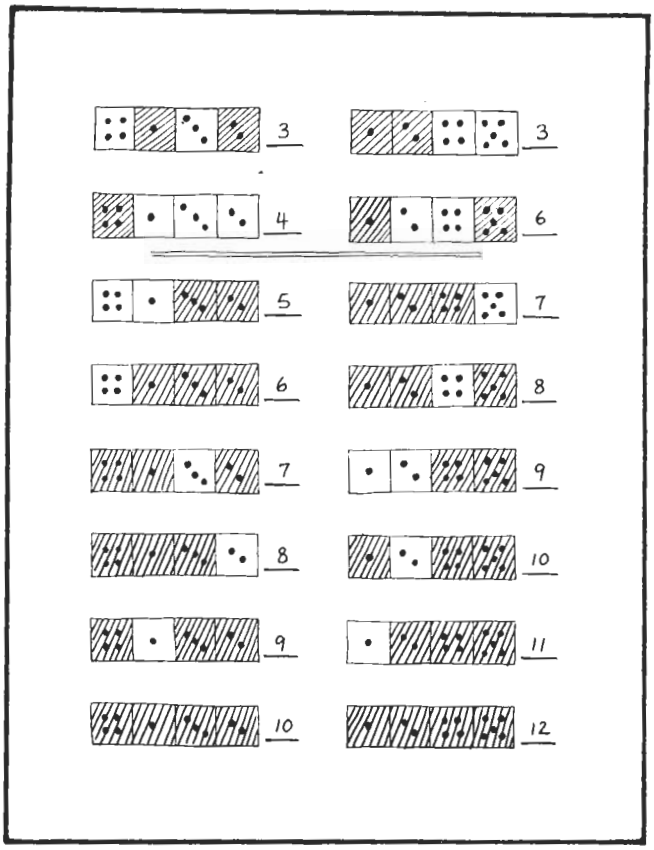
(Some of the appropriate responses are shown here.)



Introducing the Problem

Each strip of blocks in the left hand column on this page contains blocks with four, one, three, and two dots. Those in the right-hand column contain blocks with one, two, four and five dots. The number at the right of each strip indicates the number of dots to be colored in. Children should color in blocks containing dots that will sum to the specified number. Solutions for three of the problems above the double line are given. The fourth solution has been started, and could be completed by the group.

Talking about the Problem

(1) Was it possible to find a combination for each number in the left-hand column? (Yes.) Was it possible in the right-hand column also? (Yes.)

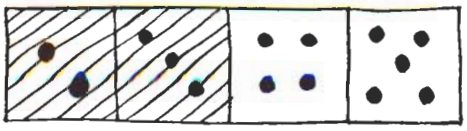
(2) How many dots are there in each diagram on the left side of the page? (Ten.) How many in each diagram on the right-hand side of this page? (Twelve.)

(3) How many numbers on the left-hand side could we show in more than one way? (3, 4, 5, 6, and 7.) Which numbers on the right-hand side could be shown in more than one way? (six and seven.)

Extensions and Surprise Endings

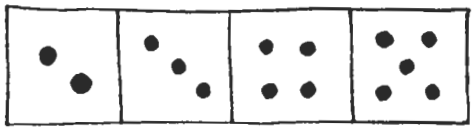
Suppose you wanted to choose four numbers so that you could find combinations for the most numbers, starting with one. Which four numbers would you choose? The combination 1, 2, 3, and 4 allows all numbers, 1 through 10, to be done. The combination 1, 2, 4, and 5 allows all numbers, 1 through 12.

Is there a combination that allows us to go further than 12? We need 1 to get 1, and 2 to get 2. Once we have 1 and 2, we can make 3, so we don't need 3. To get 4, we need 4. With 1, 2, and 4, we can go all the way to 7. If we then used 8 as the fourth number, we could show all numbers 0 through 15. Thus, the ideal combination is 1, 2, 4, and 8.

 5

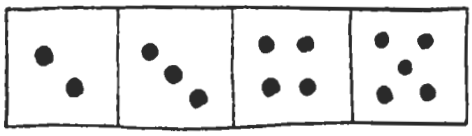
1	3	5	7

4

 6

1	3	5	7

6

 7

1	3	5	7

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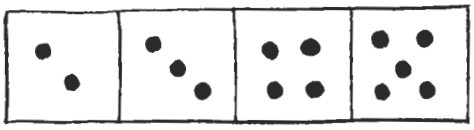 8

1	3	5	7

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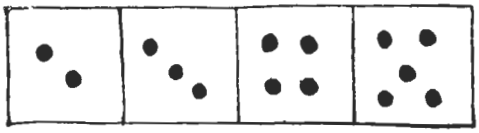 9

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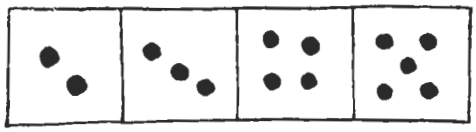
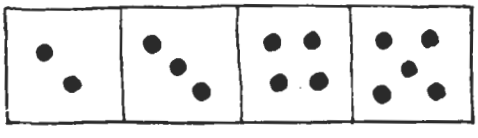
 10

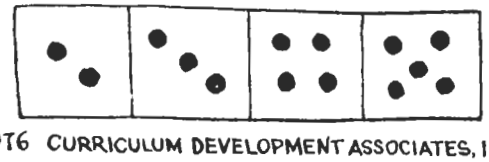
1	3	5	7

11

 11

1	3	5	7

12

 12

1	3	5	7

14

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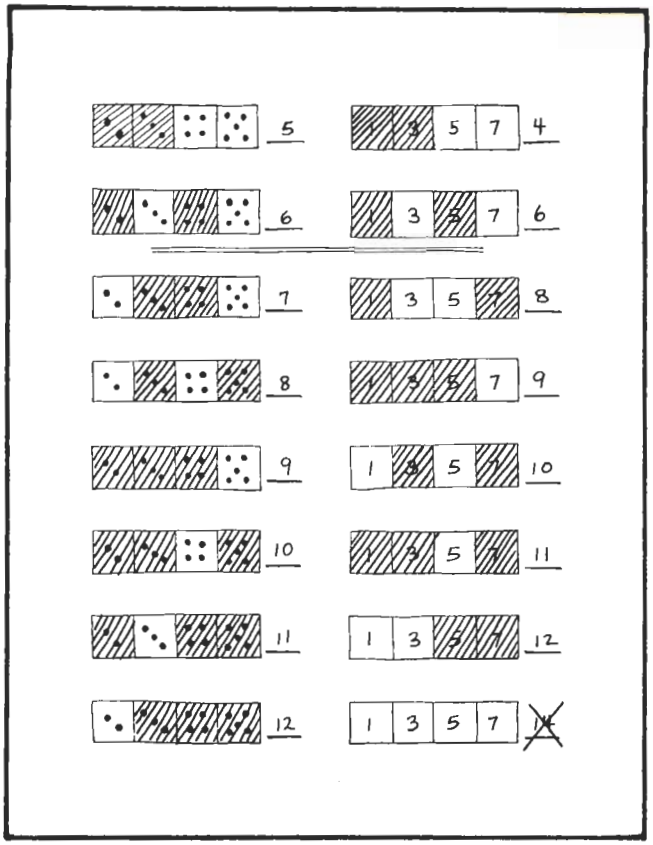
(Some of the appropriate responses are shown here.)



Talking about the Problem

(1) Were there any examples that couldn't be completed? (Yes; in the right-hand column, 14 cannot be done.)

(2) In the examples in the left-hand column, which of the numbers could be indicated in more than one way? (Seven and nine could be done in more than one way.)

(3) In the examples on the right-hand side, how many numbers could be indicated in more than one way? (None.)

(4) How many dots are there in each strip of blocks in the left-hand column? (There are 14.) Suppose you tried to show all numbers, 1 through 14. Which ones could not be shown? (One and thirteen.)

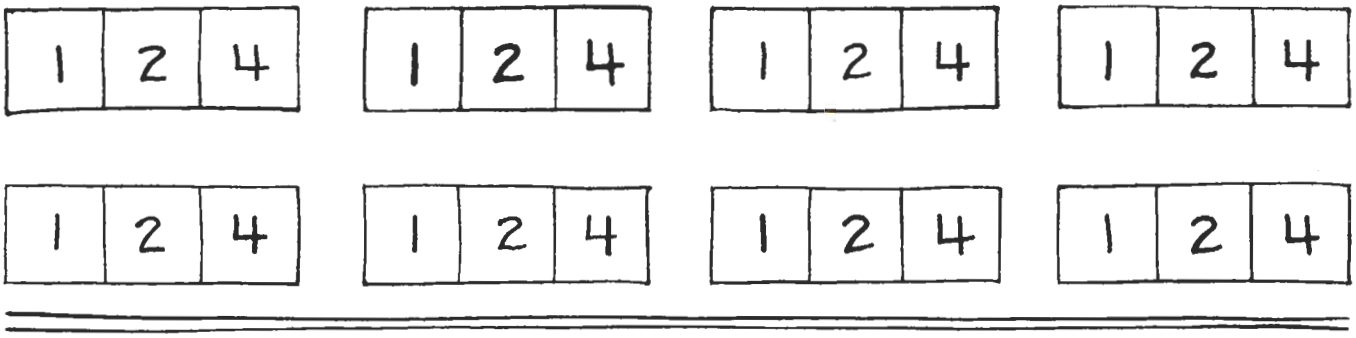
(5) What is the sum of the numbers in each strip on the right-hand side? (The sum is 16.) Suppose you tried to show all numbers, 1 through 16. Which ones could not be done? (Two and fourteen.)

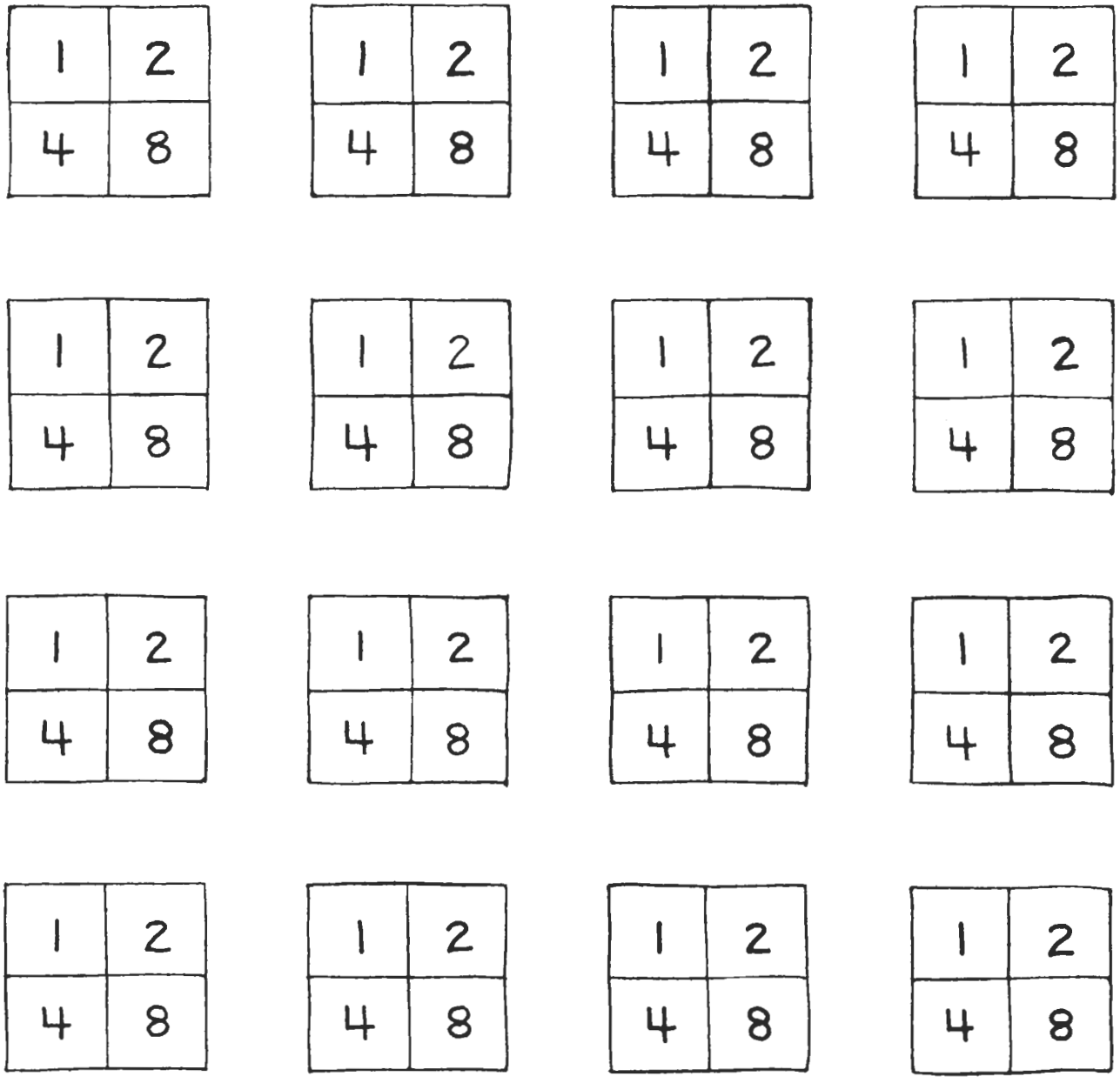
Introducing the Problem

Each strip of blocks on the left hand side of the page has a block with two dots, a block with three dots, a block with four, and one with five dots. The strip of blocks on the right-hand side uses only the abstract numerals "1", "3", "5" and "7". Children should color in blocks which indicate the sum given at the right of each strip.

Extensions and Surprise Endings

Once you know that each strip on the left has a total of fourteen dots, it might be easier to do the sums 8 through 12 by asking how many dots you should leave out. For example, to get 8, you could leave out 6; to get 9, you could leave out 5; to get 10, you could leave out 4. A similar procedure can be used for the examples in the right hand column.





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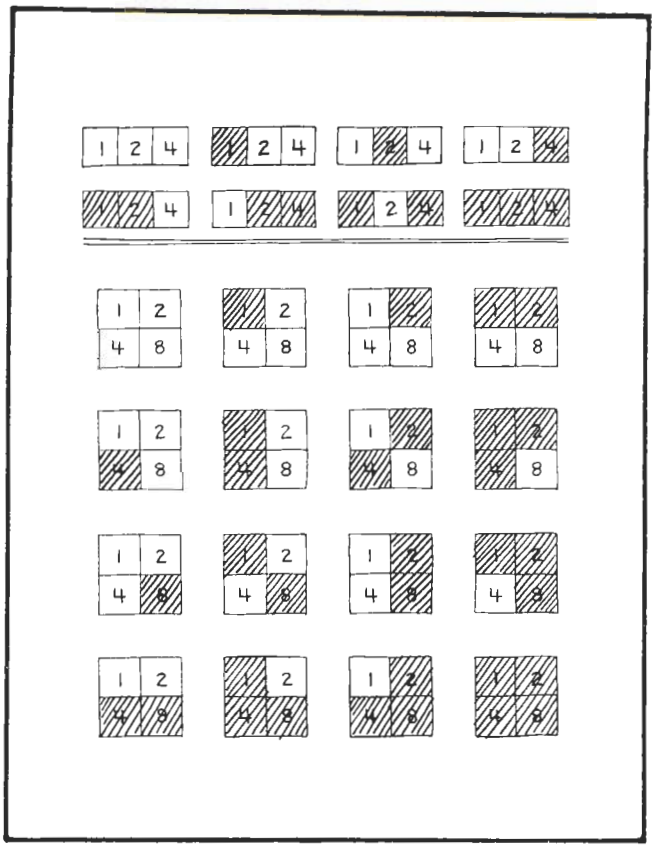


Introducing the Problem

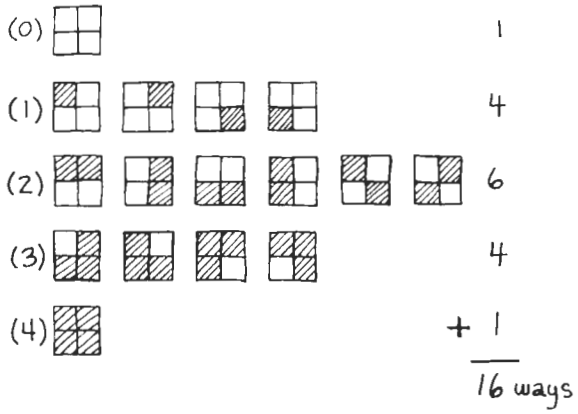
Arrays of blocks are shown which contain numbers. The problem is to color in patterns of blocks in as many different ways as possible. Before children color the sketches above the double line, they might place movable objects such as beans or buttons on the squares to indicate different combinations. After eight different combinations have been found (including one without any beans or buttons), they can be recorded by coloring in, crossing out, looping or any other method the children want to use.

The problem is the same below the double line, but there are now four blocks in each array.

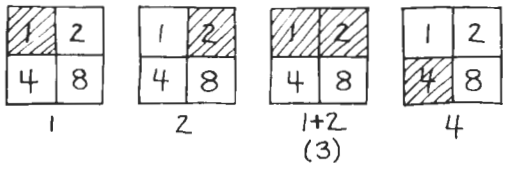
Talking about the Problem

Did you find 16 different ways to color in the 16 examples below the double line? Yes, if we include one in which no squares are colored. Are there more than 16 different combinations? No, unless someone changes the rules.

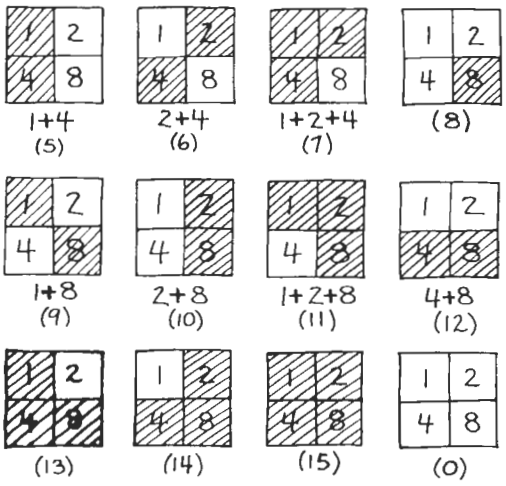
How many started with a plan? Did the plan hold up all the way? Someone who had a plan might give the first two or three examples and let the other children guess what combination came next. A typical plan starts by coloring no squares, then one at a time, two at a time, three at a time, and finally four at a time. A diagram of these results might look like this:



This plan pays no attention to the labels 1, 2, 4 and 8. Why do you think those labels were selected? Why not 1, 2, 3 and 4 or a, b, c and d? Does this kind of a beginning suggest a reason and a plan to follow?

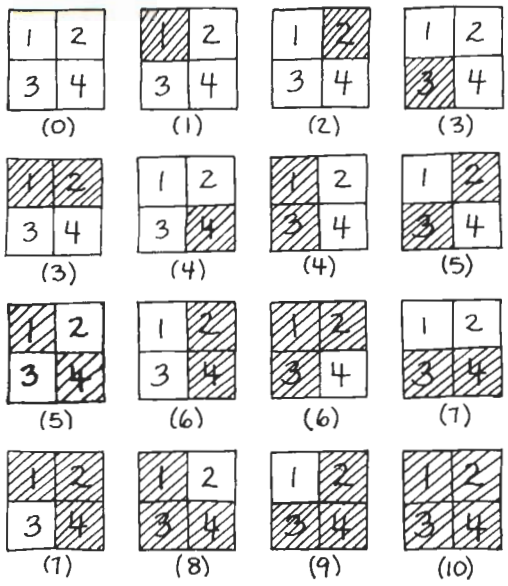


This plan might proceed;



The labels 1, 2, 3 and 4 could have been used, but

then we would have to require that all combinations for each number, 0 through 10, be shown:



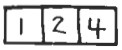
Now suppose we had other coloring problems that involved smaller numbers of little squares:

no. of squares colored	0	1	2	
different ways to color	1	2	1	4 total

no. of squares colored	0	1	
different ways to color	1	1	2 total

Extensions and Surprise Endings

Let's take a further look at the approach in which we colored one at a time, two at a time, and so on. Let's first consider the problem above the double line on this activity page which has three squares in the array. We will record the number of ways to color no squares at a time, one at a time, two at a time, and so on.

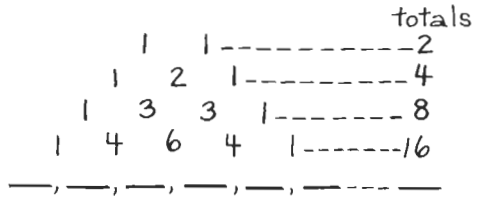


no. of squares colored	0	1	2	3	
different ways to color	1	3	3	1	8 total

The results from the problem below the double line can be summarized in a similar way.

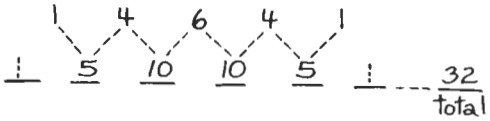
no. of squares colored	0	1	2	3	4	
different ways to color	1	4	6	4	1	16 total

Consider the second line in each chart above, that tells the number of different ways to color various numbers of squares. These lines of numbers can be arranged in an interesting way:



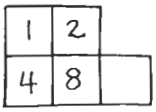
Except for the "1's" on the end of each line, each number is the sum of the two numbers "diagonally" above it:

If the pattern continues, the next row would be:

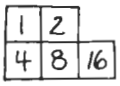


This pattern of numbers can be extended indefinitely. It arises in many different contexts in mathematics and has been named "Pascal's Triangle" after the French mathematician who studied it extensively. The patterns that can be discovered in Pascal's Triangle seem to be almost endless. Amateur and professional mathematicians continue to study its inner relationships.

Does that arrangement give you a hint of what might happen if there were five little squares to color in?



What would you suggest as a label for the fifth or additional box? '16' would be a good choice.



no. of squares colored	0	1	2	3	4	5	
different ways to color	1	5	10	10	5	1	32 total

The patterns we have noticed as a result of studying this little coloring problem can be generated in a very different way:

$$\begin{array}{cccc} 11 & 11 & 121 & 1331 \\ \times 1 & \times 11 & \times 11 & \times 11 \\ \hline 11 & 121 & 1{,}331 & 14{,}641 \end{array}$$

However, this method can be continued only with difficulty:

$$\begin{array}{r} 14641 \\ \times \quad 11 \\ \hline 15(10)(10)51 \end{array}$$

or

161,051

Clearly, the series 1, 2, 4, 8 and 16 in which each successive number is twice the previous number can be continued indefinitely. Can it be extended in the other direction? Yes, it can, if we use common fractions.

$$\ldots\ldots \tfrac{1}{16}, \tfrac{1}{8}, \tfrac{1}{4}, \tfrac{1}{2}, 1, 2, 4, 8 \ldots\ldots$$

This series of fractions suggests the way in which an inch is divided on some rulers.

The members of this series of numbers are often referred to as "powers of two" and are written in shorthand in this way:

$$\begin{aligned} 1 &= \square = 2^0 \\ 2 &= \square = 2^1 \\ 4 &= 2 \times 2 = 2^2 \\ 8 &= 2 \times 2 \times 2 = 2^3 \\ 16 &= 2 \times 2 \times 2 \times 2 = 2^4 \\ 32 &= 2 \times 2 \times 2 \times 2 \times 2 = 2^5 \end{aligned}$$

The "exponents" or little numbers written to the right and slightly above the '2' are obviously successive integers. The pattern has been maintained by an agreement that 1 will be written as '2^0' and 2 will be written as '2^1'. We can catch a glimpse of a powerful shortcut that can be explored later:

$1 = 2$	$4 \times 8 = 32$
$2 = 2^1$	$2^2 \times 2^3 = 2^5$
$4 = 2^2$	
$8 = 2^3$	$16 \times 8 =$ ____
$16 = 2^4$	$2^4 \times 2^3 = 2^7$
$32 = 2^5$	
$64 = 2^6$	$64 \div 16 =$ ____
$128 = 2^7$	$2^6 \div 2^4 = 2^2$
etc.	

We had extended the series using common fractions, and it turns out that we can also extend the "exponents" or "powers" by using a '-' Another glimpse of a shortcut appears:

$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
2^{-3}	2^{-2}	2^{-1}	2^0	2^1	2^2	2^3

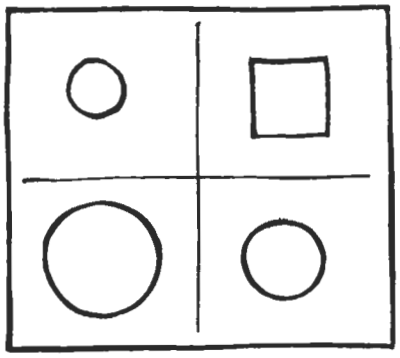
$\frac{1}{2} \times 8 =$ ___ and $4 \div 8 =$ ___

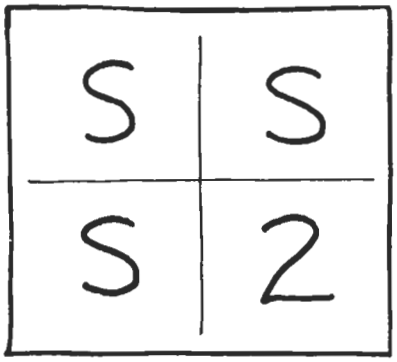
$2^{-1} \times 2^3 = 2^2$ $\qquad$ $2^2 \div 2^3 = 2^{-1}$

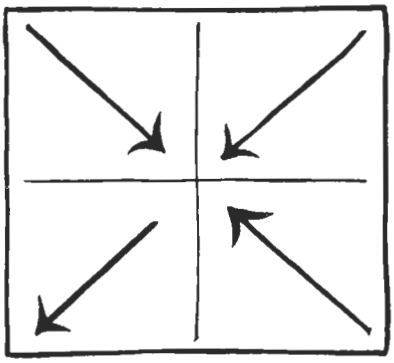
While such extensions may be interesting to only a small group of students, teachers and parents, they do introduce powerful notions that anticipate what lies ahead in mathematics in a context where the numbers are small and non-threatening.

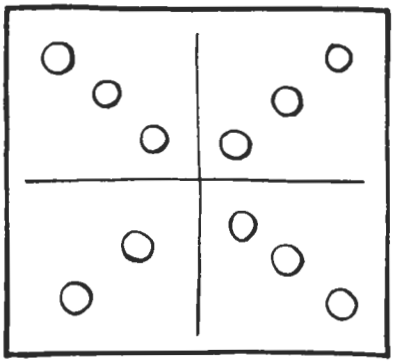
Another long, but very different exploration of this sequence of numbers (1, 2, 4, 8 and so on) can be found in the introductory book which accompanies this bank of problem solving activities.

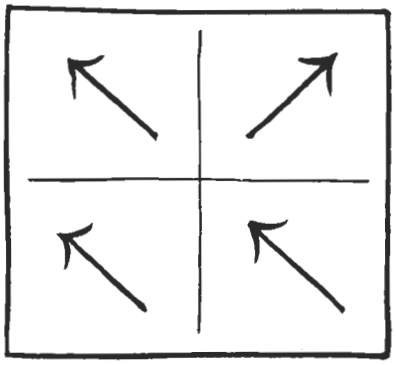
¿Cual es diferente de los otros tres? Which one isn't like the other three?

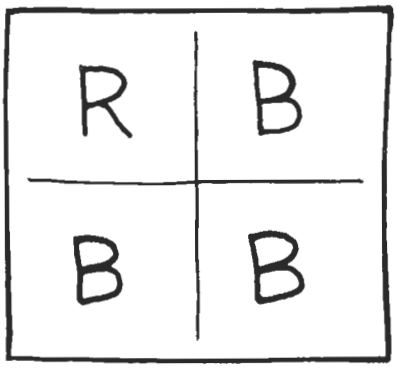
A 

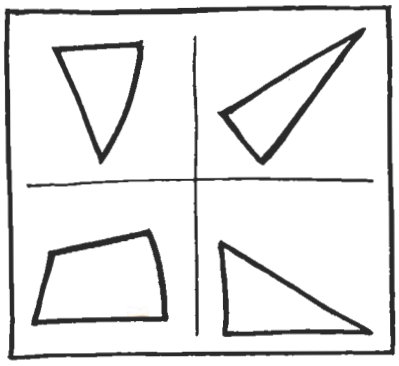
B 

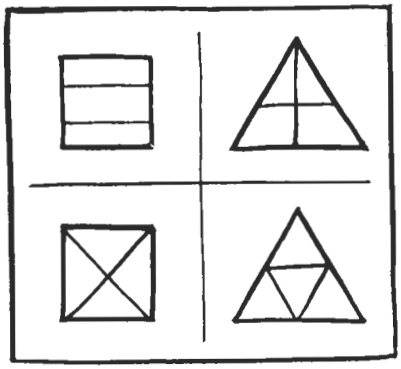
C 

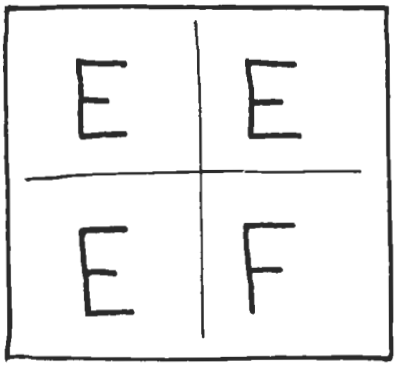
D 

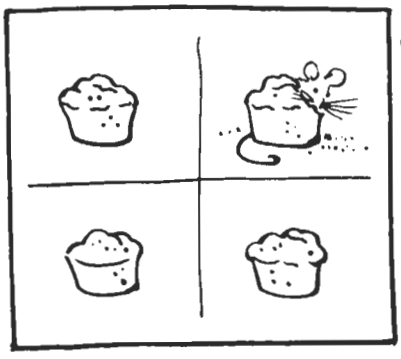
E 

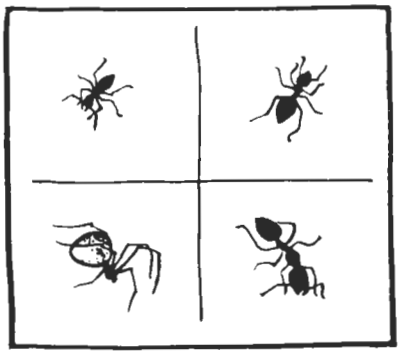
F 

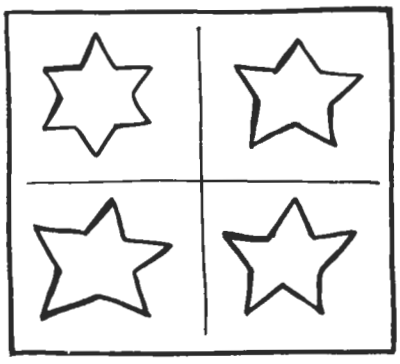
G 

H 

I 

J 

K 

L 

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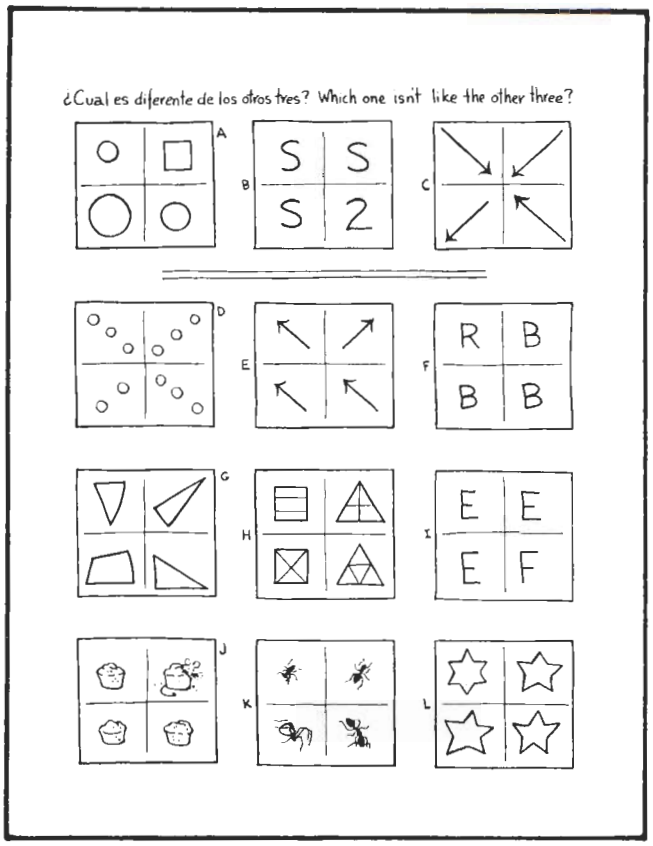


Introducing the Problem

Different pictures are shown in each set of four small boxes. Children consider the four sketches and decide which one does not belong; which one is not like the others.

This activity is used widely by the TV program "Sesame Street", and many children will already be familiar with the procedures. Children might first play the game using real objects. A table top that has been divided into four parts with masking tape serves nicely as a "game board". Children could work in two groups. The first group might choose four objects, arrange them on the table, and challenge the second group to find out which one doesn't belong, according to their plan. Then the second group could choose four objects and ask the first group to guess what scheme they used. Several experiences of this type with real objects will prepare them to play the game on the activity sheet.

In example A, most children will feel that the square doesn't belong. It is a different shape. But often a child will argue that the smallest circle or the largest circle doesn't belong because of its size. There is no "right" answer. A more appropriate concern is whether the responses are reasonable. Do they make sense? In example C, where four arrows are pictured, strong differences of opinion can arise.

"The bottom right arrow doesn't belong because it's the only one going toward the top."

"No, the bottom left arrow doesn't belong because it's the only one going away from the center."

"No, the top left one is different. It's the only one that goes the way we read, from left to right and from top to bottom."

Talking about the Problem

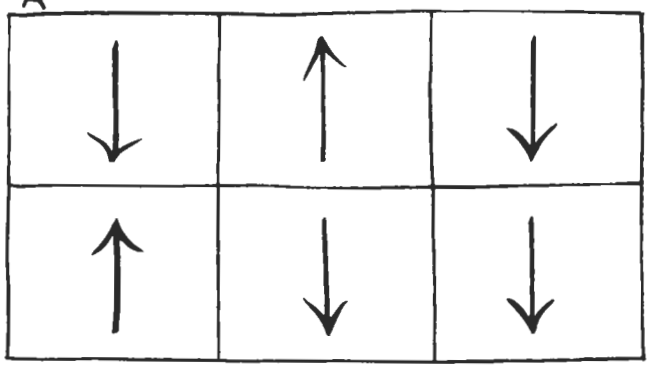
The discussion will probably reveal differences of opinion, particularly in problems E and H. In problem E, only one arrow (bottom right) points toward the center. Only one arrow (bottom left) points to the letter E. Only one (upper right) points in a different direction than the others. Only one (bottom right) would hit another arrow if it alone moved in the direction it is pointing. Only one arrow (bottom left) does not point toward one of the four corners of the large frame.

In problem H, only the upper left diagram has no triangles inside. It is also divided into three parts, while the others are divided into four. The upper right-hand shape is divided into different kinds of shapes. In the bottom right diagram, the sides of all the shapes are the same length, and there are no square corners in any of the shapes.

Skill in grouping by attributes can be encouraged by asking children to classify or sort objects according to size, shape, weight, uses, color, length, volume, texture, smell, and so on. Those involved in early childhood education must realize the importance of activities such as these. They are of far greater value in preparing children for success in school mathematics than the early introduction of arithmetic involving symbol manipulation with pencil and paper.

Pinta los dos que son diferentes. Please color in the two that are different.

A



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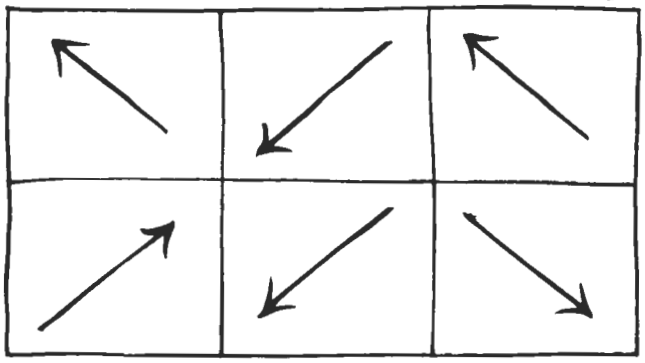

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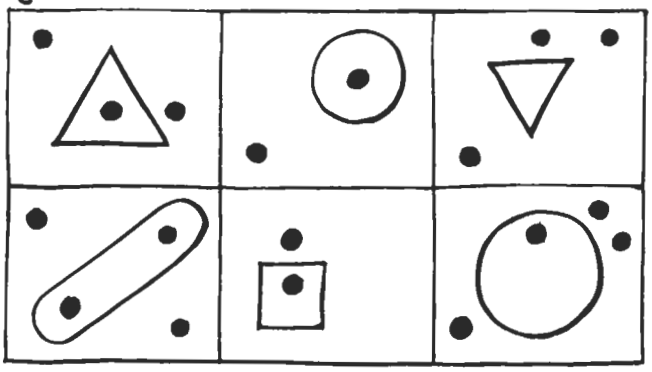

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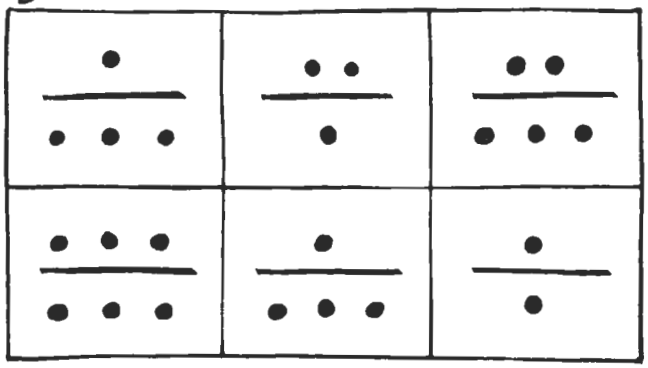
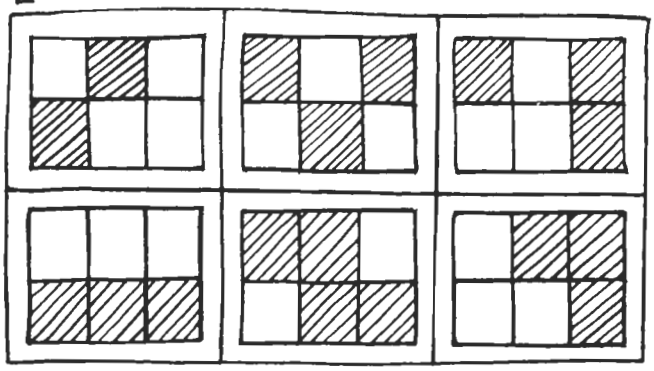


E



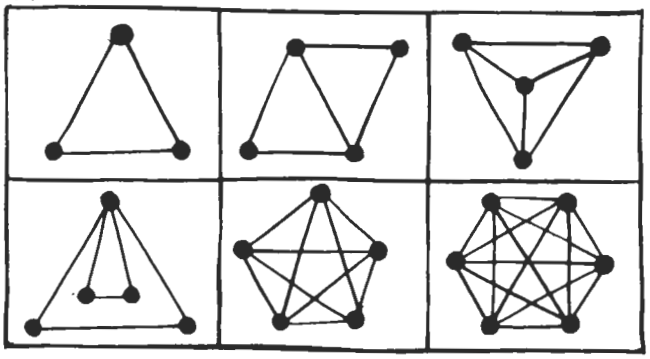
F

3	4	9	7	2	3
2	5	6	8	6	5
10	8	6	7	1	2
7	9	5	9	4	3

G

3 / 2	5 / 1	2 / 4
3 \ 3	6 \ 0	3 \ 4

H



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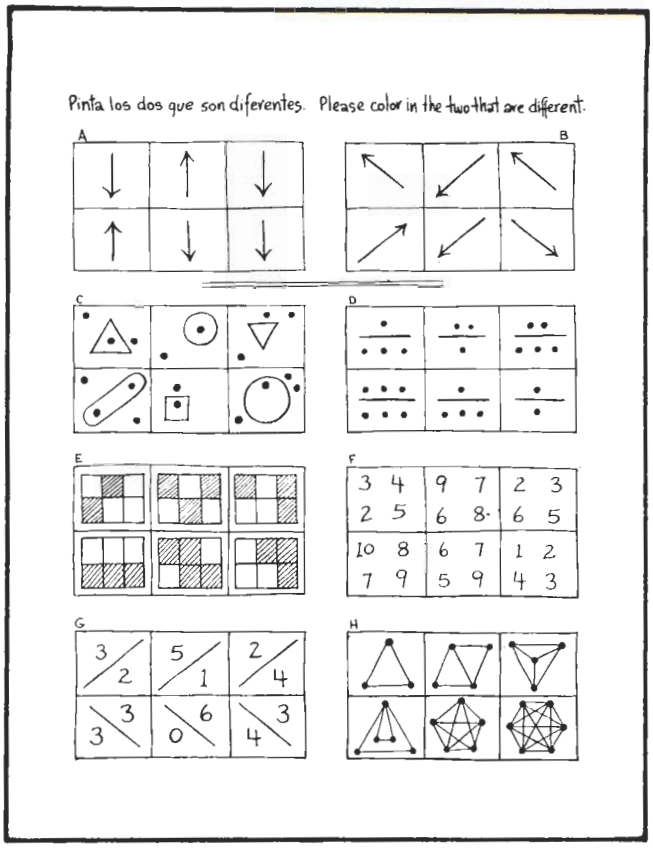
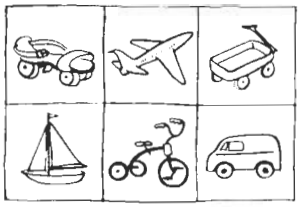


Introducing the Problem

Different pictures are shown in each of the six small boxes in each problem. Children should consider the six sketches and decide which two don't belong; which two aren't like the others. Hopefully, children will first be given lots of opportunities to play the game using real things. Perhaps a table could be sectioned off with masking tape, and various objects could be placed in the sections. Children might search for the *one* that doesn't belong, or for the *two* that don't belong, and so on.

This is a social activity which allows wide latitude for individuals to express their own ideas and preferences. A collection containing objects such as four oranges, an apple and a banana seems to call for a given response. But in these problems, the response usually depends on how you look at things.

Suppose we have the following objects arranged on a table.



Which two aren't like the others? Some answers will provoke heated discussions.

"The boat and airplane don't belong because they don't have wheels."

"But airplanes have wheels."

"Some seaplanes don't have wheels."

"Well, it's the boat and airplane because they don't have wheels on the ground all the time."

Another possibility:

"I think it's the boat and the tricycle because they don't have four wheels."

"But big jets have more than four wheels."

"O.K., the boat and the tricycle don't have four or more wheels."

Still another line of argument:

"It's the truck and the airplane because the others don't carry lots of luggage."

"Don't sailboats carry luggage?"

"Not usually."

After discussions and explanations about which two don't belong, the advocate is asked to remove the two that don't belong, to find two objects somewhere in the room that do belong, and to put them in the empty sections of the table. When these replacements have been made, the question is reopened: "Now which two don't belong?" After discussion, those two are taken off and two objects that do belong must be found. The game can go on for as many rounds as the participants want to play.

All the while the players are developing increasing familiarity with important mathematical ideas by going through the processes of classifying and sorting.

When the children begin to work on this page, they will still come up with different reasons for choosing the two that don't belong. In problem A, for example,

"All arrows are falling except two."

"Only two arrows have sharp ends pointing at each other."

"Only two have their tails together."

In example B,

"There are two that are going to hit each other while the rest will escape."

"All are pointing to the left except two."

"Only two are pointing at corners of the big frame."

"Which ones are those?"

"The one at the top on the left side and the one at the bottom on the right."

"But what about the middle arrow on the top? It would get to a corner sooner or later."

We need only be concerned with the reasonableness of the responses. Do they make sense? Can the reasons be put into words or otherwise clearly expressed?

Talking about the Problem

After children have made their own decisions about examples C through H, find out how many had similar explanations in each example. Discussion will again emphasize that the choices depend on how you look at things. Here are some of the many reasonable comments.

Problem C

"Four of the shapes have a single dot inside."

"Four of them have more than one dot outside."

"Only the bottom corner boxes have four dots."

"Only two of the shapes are circles."

"Only two of the shapes are triangles."

Problem D

"All have three dots below the line except the middle top and bottom right."

"Only two have exactly two dots above the line."

"Only two have the same number of dots above as below the line."

Problem E

"All except two have three shaded parts."

"Four have a group of three blank areas with sides that are together."

"Only two have white squares with sides that don't touch each other."

"Four have a single shaded part in the bottom row."

Problem F

"In four of the boxes, the numbers could be arranged in sequence: 1, 2, 3, 4; 2, 3, 4, 5; 6, 7, 8, 9; and 7, 8, 9, 10. The other two sets of numbers can't be put in sequence."

"Only two have 8's."

"Only two have 4's."

"Only two have the two smallest numbers on top."

"Only two have both a six and a seven."

Problem G

"Four have at least one even number; the other two have only odd numbers."

"Four examples have an even number on the bottom."

Problem H

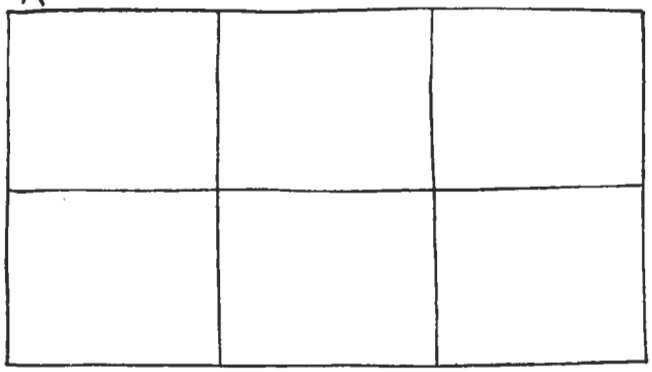
"In all examples but two, each dot is connected to every other dot."

"In only two cases is there a dot inside."

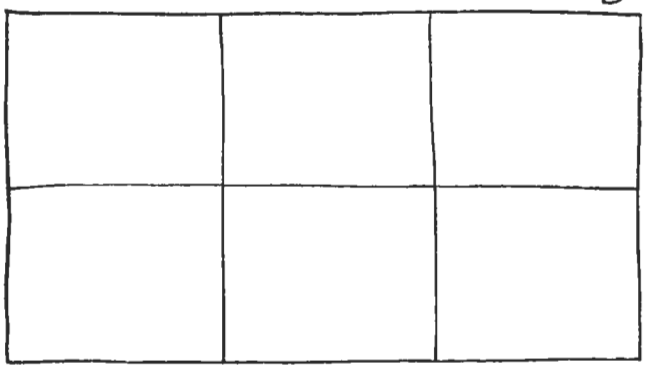
"There is a pair of diagrams with four dots and another pair with five dots, but the others don't belong to a pair."

Pinta los dos que son diferentes. Please color in the two that are different

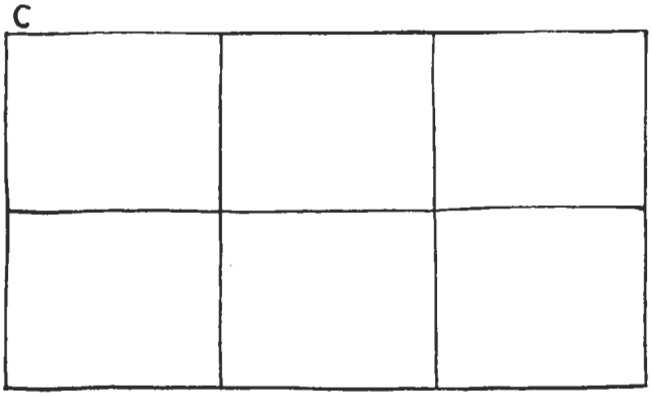
A



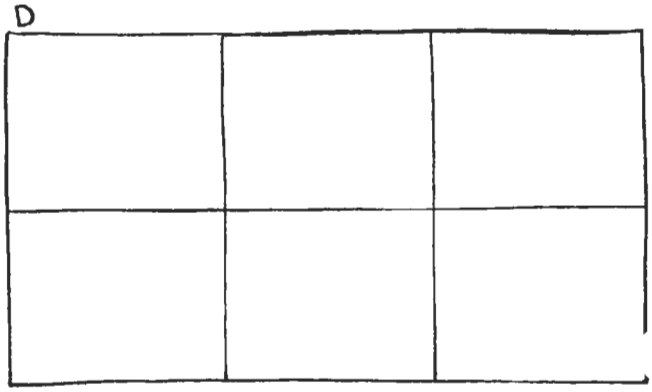
B



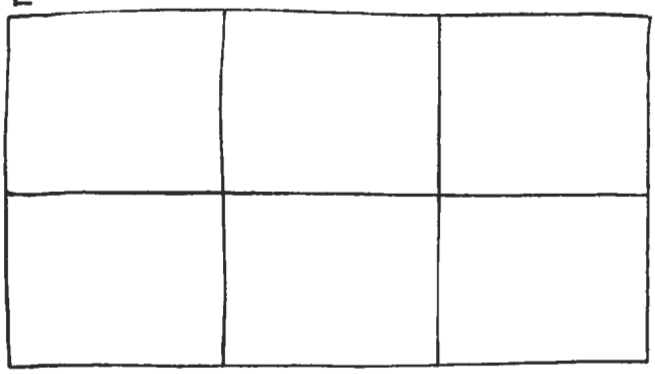
C



D



E



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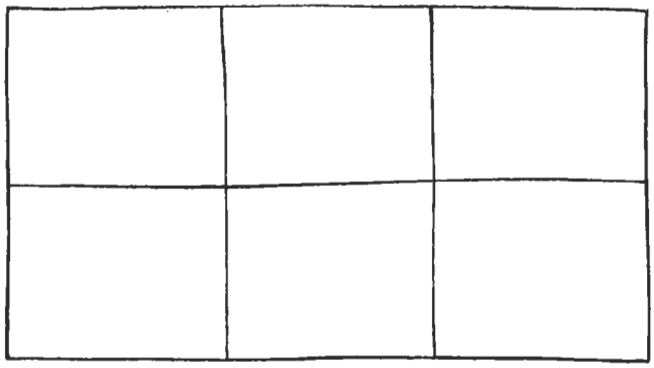
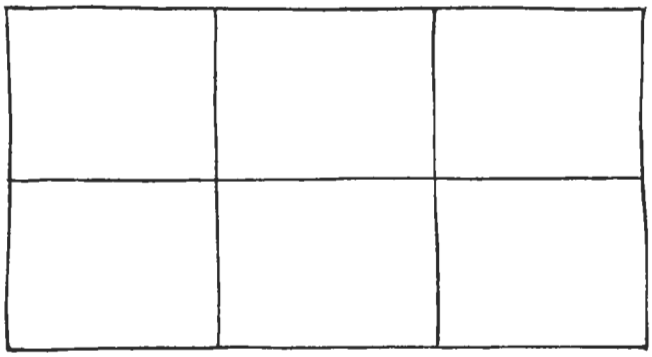
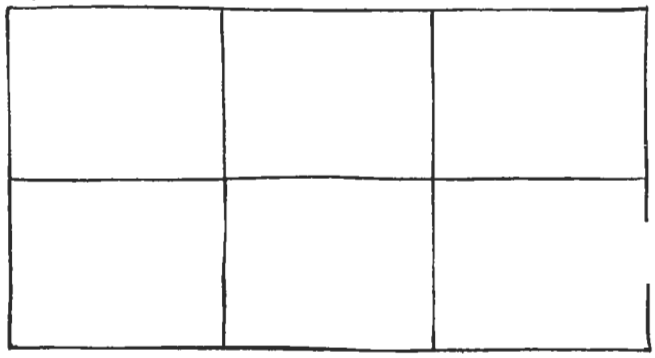


G



H



F11

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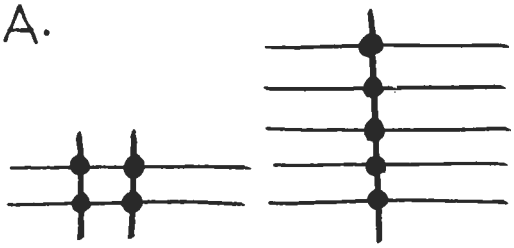
5 or less vertical lines on each side of the sketch.

5 o menos líneas verticales a cada del dibujo.

A.



4 + =

B.

+ = 8

C.

+ = 13

D.

+ = 9

E.

+ = 17

F.

+ = 16

G.

+ = 8

H.

+ = 6

I.

+ = 10

J.

+ = 13

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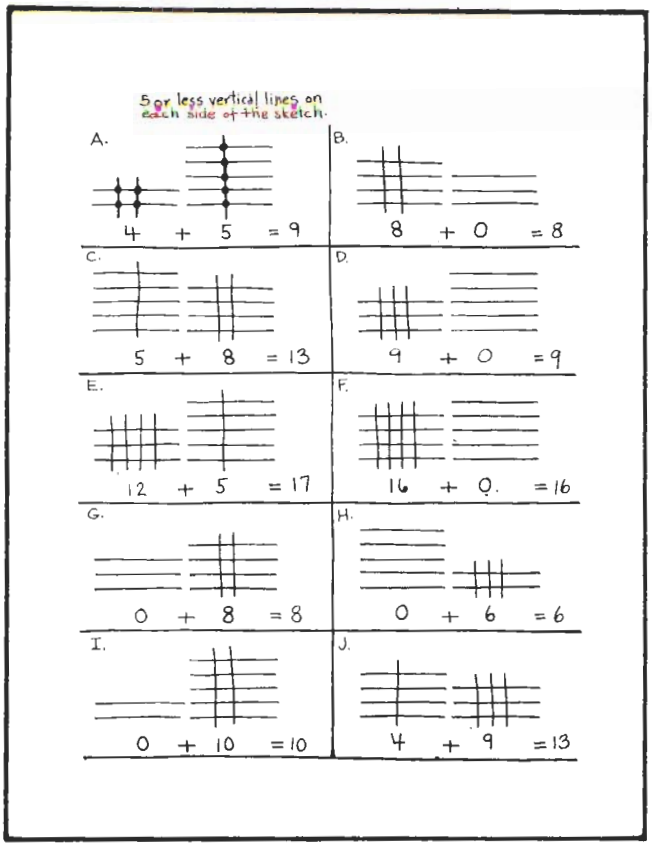
(Some of the appropriate responses are shown here.)



Introducing the Problem

Rules for this activity are simple. You may draw five or fewer vertical lines on either or both sides of the sketch. All vertical lines drawn on a side must cross all the horizontal lines on that side. A record is made to show the number of crossing points created on each side and the total number of crossing points created.

Sketch (A) is complete, but the report is not. We must record that five crossing points were created on the right-hand side of the sketch and that there is a total of nine crossing points. Sketch (B) is typical of all the problems that follow. The total number of crossing points is given and vertical lines must be drawn to create that number of points. In (B), there is only one way to draw an appropriate sketch — two vertical lines on the left side, and none on the right side. The report will read "8 + 0 = 8".

If beginners have not previously been introduced to this activity, please let them use sticks at first (half toothpicks will do nicely). They can lay the sticks on the sketch until they find a combination that "works" and then they can draw in the lines. This "toothpick arithmetic" eliminates any need for erasures. (See *Patterns and Problems*, Level C, of CDA Math for more activities with "toothpick arithmetic".)

It might be useful in the beginning to work out several examples which do not specify total numbers of crossing points. You might let a volunteer draw five or fewer lines (as many as the child chooses) on either or both sides and draw in the crossing points. The report is then written as a simple description of the result, and no "problem" is involved. This strategy allows the children to become familiar with the activity before they have to contend with a given sum; before the "problem" enters and brings with it a jump in complexity.

It is wise to suggest early on in these activities that actually drawing a dot on each crossing point is unnecessary. The dots need only be shown when difficulty or confusion occurs.

Talking about the Problem

Did you find any examples you couldn't do? Although there is a solution for each problem on this page, some children probably will not be able to do some of them, and these need to be talked about.

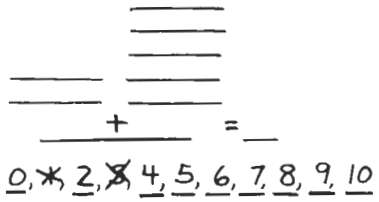
Using sketches like those on this page, what is the smallest total of crossing points that can be created? Zero; 0 + 0 = 0. What is the largest number that could be drawn in any sketch on this page? Five lines on each side of the sketch in (C) or (F) would lead to 25 + 20, or 45 crossing points. This would be the largest number

Extensions and Surprise Endings

How many of the examples can be completed with two different sketches — and two different reports? On this page, only (I) can be done in two ways:

10 + 0 = 10, or 0 + 10 = 10.

Let's remove the "five or fewer" limit and look at the different sketches with another question in mind. In each case, is there a largest number of crossing points that cannot be shown? For example, in (A), there are two horizontal lines on the left and five on the right. A record of those numbers of crossing points that can and cannot be done would look like this:

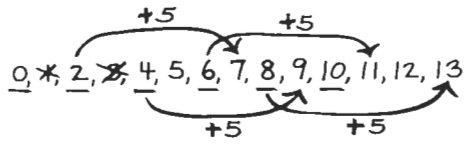


There is no way to show either one or three crossing points. Will there be a way to show all numbers larger than three? It seems likely, but can you be sure?

Well, we know we can show every even number using the two lines on the left side.

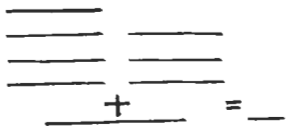
0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

All the odd numbers larger than six can be shown as an even number plus five:



Since we can easily show even numbers, and since we have five horizontal lines on the right side, we can do any even number plus five, for as large a number as we like.

What about the situation in problem B?



0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11

Is 5 the largest number of crossing points that can't be shown? Yes, but why?

The largest numbers that can't be shown in the various sketches are:

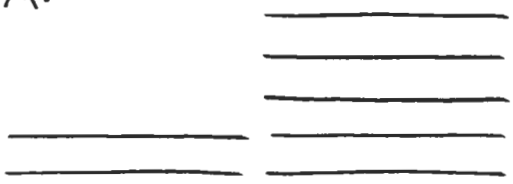
2's and 5's — 3
3's and 4's — 5
4's and 5's — 11
3's and 5's — 7

If you would like a bit of an extension to pursue, notice this:

$(2 \times 5) - (2 + 5) = 10 - 7 = 3$
$(3 \times 4) - (3 + 4) = 12 - 7 = 5$
$(4 \times 5) - (4 + 5) = 20 - 9 = 11$
$(3 \times 5) - (3 + 5) = 15 - 8 = 7$

It looks as if the largest number that can't be shown in any of the sketches is an odd number. This will in fact always be the case. Can you think of any reasons why this will be true?

A.



+ =

B.

+ =

C.

+ =

D.

+ =

E.

+ =

F.

+ =

G.

+ =

H.

+ =

I.

+ =

J.

+ =

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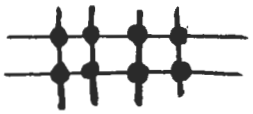
5 or less vertical lines on each side of the sketch.

5 o menos líneas verticales a cada del dibujo.

A.



8 + = 18

B.

+ = 24

C.

+ = 15

D.

+ = 26

E.

+ = 28

F.

+ = 29

G.

+ = 26

H.

+ = 30

I.

+ = 27

J.

+ = 31

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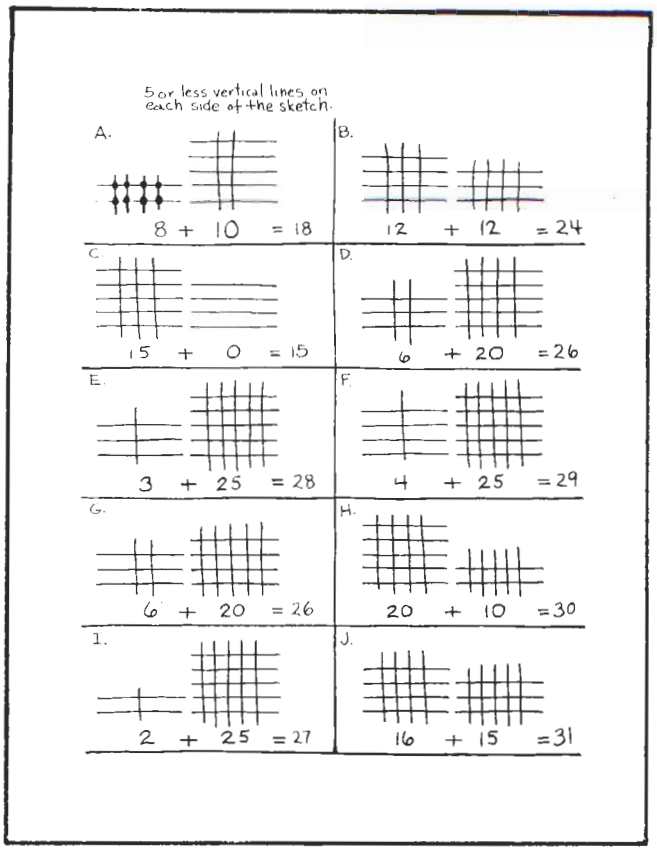


Introducing the Problem

If the children have not been involved recently with the previous activity, you may want to have them consider several examples from that page in order to review the rules for lines and crossing points.

In problem A on this page, the sketch shows eight of the required 18 crossing points. Ten more crossing points are needed. These can be created by drawing two vertical lines on the right-hand side of the sketch. The report can then be completed to show that 8 + 10 = 18.

Problem B simply specifies that 24 crossing points are to be shown. Some children may want to use six vertical lines on the left-hand side (6 × 4 = 24), or eight vertical lines on the right-hand side (8 × 3 = 24). But both of these solutions violate the rule that says only five or fewer vertical lines can be used on each side. The only valid solution to problem B shows 12 crossing points on each side: three vertical lines on the left and four on the right. The report would read 12 + 12 = 24.

Talking about the Problem

Were any problems given for which no appropriate sketch could be drawn? No, there is a solution for every problem on the page. Is there any example in which there could be two different sketches and two different reports? No, not if the rule is followed that limits the number of lines on each side to five. But suppose that you could draw as many lines as you like on each side. Now how many problems could have more than one solution? One could find se other solutions.

A.

8 + 10 = 18
(also 18 + 0 = 18)

B: 24 + 0 = 24
0 + 24 = 24
D: 21 + 5 = 26
E: 18 + 10 = 28
F: 24 + 5 = 29
G: 18 + 8 = 26

H: 30 + 0 = 30
10 + 20 = 30
0 + 30 = 30
I. 12 + 15 = 27
22 + 5 = 27
J. 28 + 3 = 31
4 + 27 = 31

Extensions and Surprise Endings

There is an activity called "loop arithmetic" in which numbers in a list are looped to show a given sum at the bottom of the column. For example,

2	4	5	3	3	4
2	4	5	3	3	4
2	4	5	3	3	4
2	4	5	3	3	4
2	4	5	3	3	4
5	3	4	5	5	5
5	3	4	5	5	5
5	3	4	5	5	5
5	3	4	5	5	5
5	3	4	5	5	5
18	24	15	26	28	29

Can you complete the other examples of "loop arithmetic" shown above? All of them can be done. Are these examples in any way related to the examples in the activity page we have just completed? Yes, they are simply another way of asking the same question posed in problems A through F. Essentially five of each of two numbers are available

to create a given sum. But in this activity we no longer have crossing points to count if we need them. We must rely entirely on our memory for addition facts.

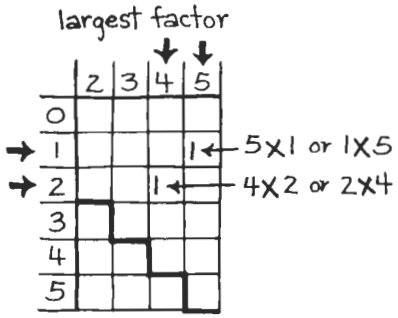
The basic questions asked in both "lines and crossing points" and "loop arithmetic" can be put in still another setting: "Please fill in the missing factors with numbers 0 through 5."

(×2)+(×5)=11
(×4)+(×3)=24
(×5)+(×4)=15
(×3)+(×5)=26
(×3)+(×5)=28
(×4)+(×5)=29

Suppose we wanted to find out which multiplication facts occurred on this and the previous activity page. We might be most interested in those with 2, 3, 4 and 5 as the largest factor. The solution to problem A on this activity page involves two facts:

4×2=8 or 2×4=8
and
5×2=10 or 2×5=10

We can build a chart and tally each multiplication fact that arises. When '4 × 2 = 8' occurs, it can also be considered as '2 × 4 = 8', and we don't want to show it in the frequency chart twice. We will agree to place our tally in the column headed by the larger factor. Here we show a tally for the facts '4 × 2 = 8' and '5 × 1 = 5'.



What would such a "frequency chart" look like for this activity page?

	2	3	4	5
0			|	
1	|	|	|	
2		||	|	||
3			||	||
4			|	|||
5				|||

What would it look like if the facts from the previous activity page were also shown? In the chart shown below, facts encountered on this page are indicated by a tally mark; those from the previous page are recorded with a dot.

	2	3	4	5
0	·	··	|	···
1	|	|	|·	···
2	·	||·	||·	||·
3		··	||·	||
4			|·	|||
5				|||

In summary:

	2	3	4	5
0	1	2	1	3
1	1	1	2	3
2	1	3	4	3
3		2	3	2
4			2	3
5				3

The more difficult facts in general occur more often than the easier ones, which need less practice. But the frequency is reasonably balanced. Activities such as "lines and crossing points", though they may look "unconventional", do in fact provide substantial practice in basic computational skill. It might be interesting to have the children themselves create such "frequency charts" for various activities.

A.

+ =

B.

+ =

C.

+ =

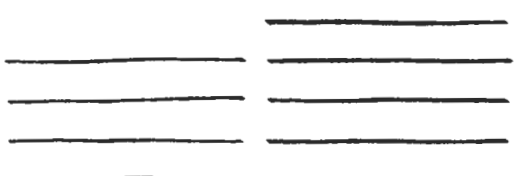
D.

+ =

E.

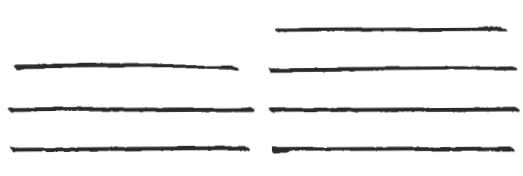
+ =

F.



+ =

G.

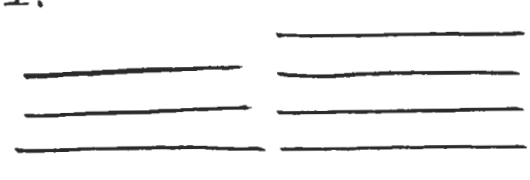


+ =

H.

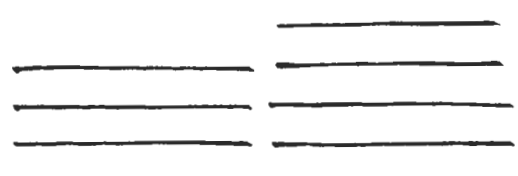
+ =

I.



+ =

J.



+ =

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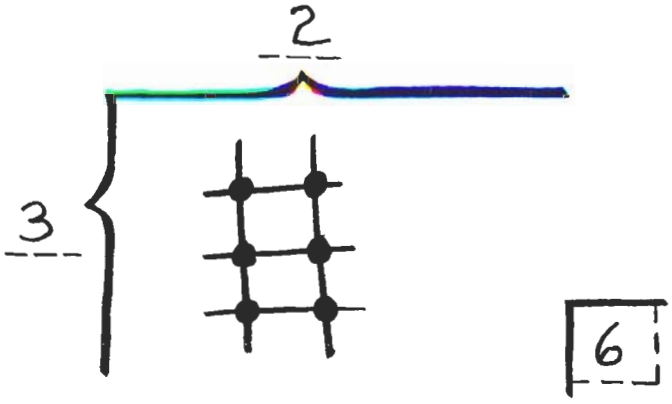

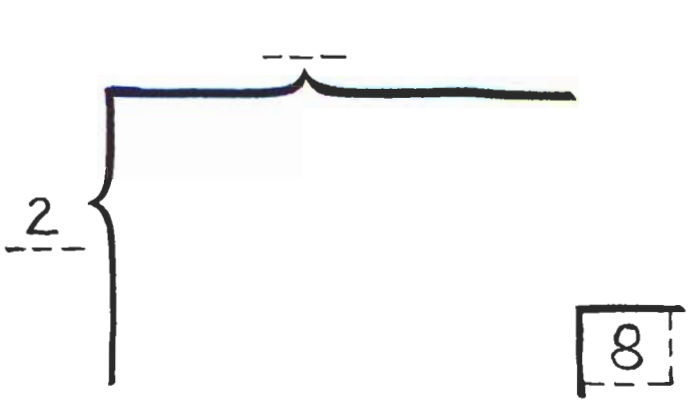

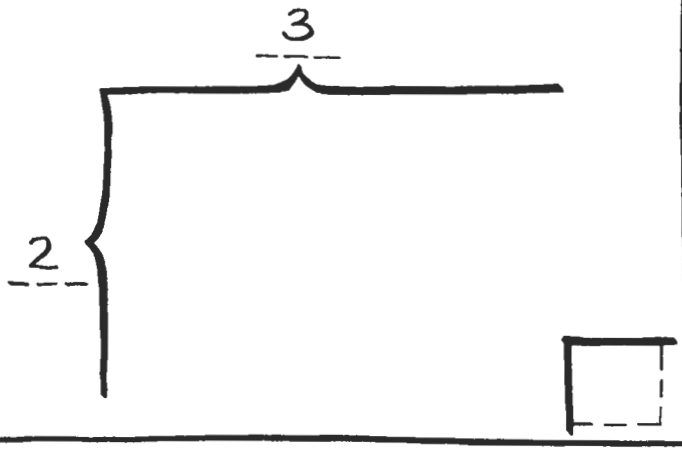

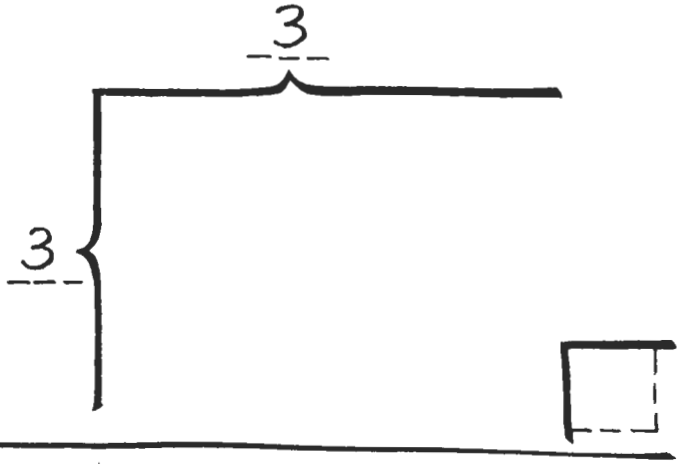

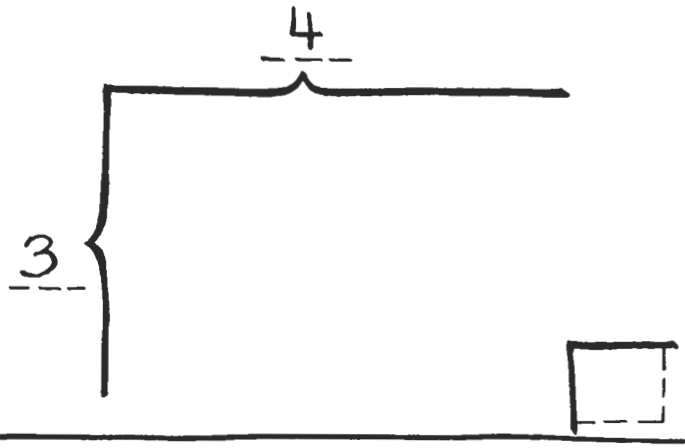

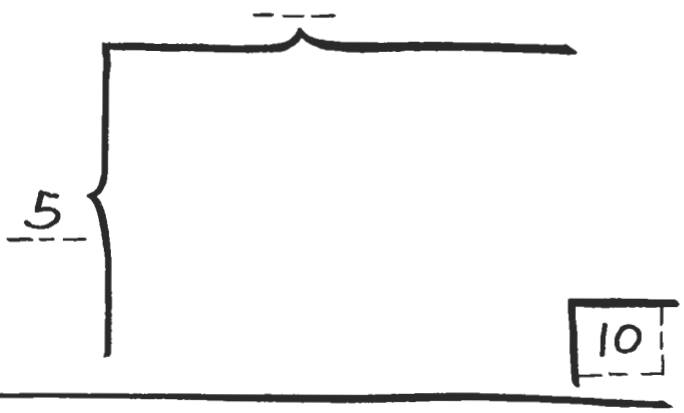

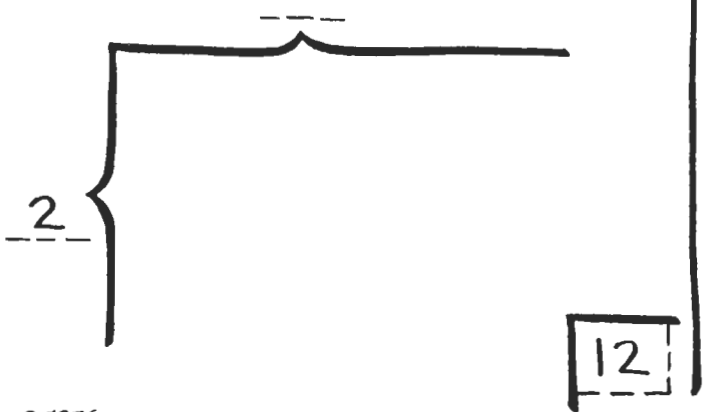

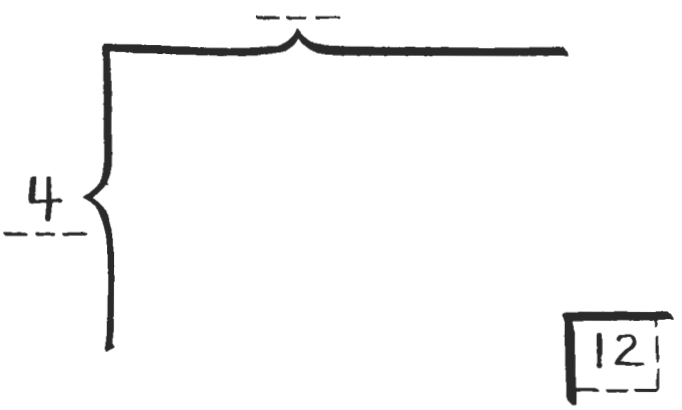

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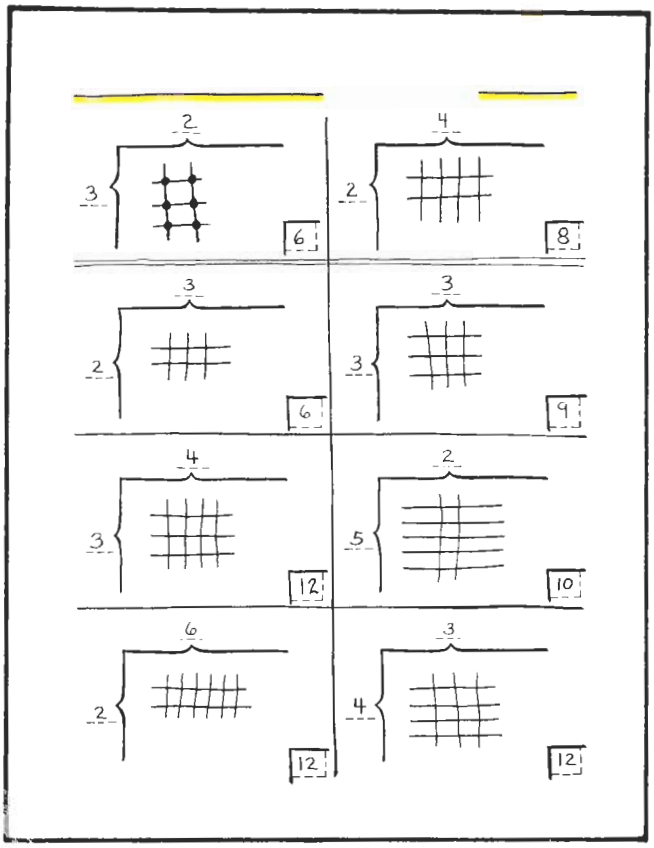
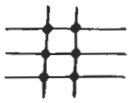
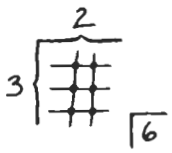


ntroducing the Problem

Suppose we draw a sketch on the overhead or chalkboard, or construct an arrangement on a flannel board that looks like this:

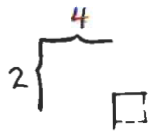


We want to talk about the "lines" and the "crossing points". There are two lines "going up and down" (vertical lines). There are three lines "going across" (horizontal lines). There are six "crossing points". We would like to record all of this information in a simple, direct way — a kind of shorthand. One way we might do this is:



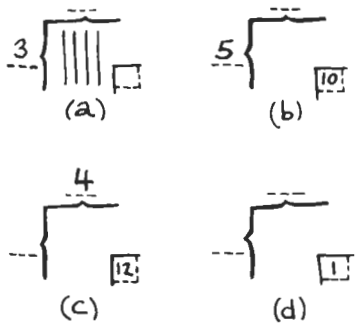
There are many other ways to make a record, but this is the method we will use on this activity page.

The following notation can be interpreted as a set of directions:



(a) Draw four vertical lines.
(b) Draw two horizontal lines, each of which crosses all the vertical lines.
(c) Find the number of crossing points created.
(d) Record that number in the box in the lower right-hand corner.

The notation can be varied to indicate other directions. Before work begins on the page, the variations shown below should be discussed with the children.



In (a), there are four vertical lines (and we assume that no more vertical lines are to be drawn.) We are asked to draw three horizontal lines and record in the small box the number of crossing points. In (b), we are asked to draw five horizontal lines, and then draw enough vertical lines to produce exactly ten crossing points. Two vertical lines will achieve this goal. In (c), we are asked to draw four vertical lines and then enough horizontal lines to produce exactly 12 crossing points. In (d), we are asked to produce a sketch with only one crossing point. There is only one possibility — one vertical line crossing one horizontal line.

Talking about the Problem

Did everyone draw the same kind of sketch in each example? (There is only one possibility for each problem on this page.) What would happen if the directions said to draw 0 vertical lines?

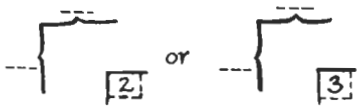
How could you complete the sketch? (Draw as many horizontal lines as you like.) How many crossing points would there be? (None, no matter how many lines you draw.)

Look at these directions:

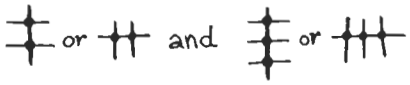
We are asked to use no vertical lines, and to get four crossing points. It can't be done!

Extensions and Surprise Endings

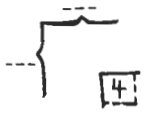
How many different sketches can you make for each of these two problems?



In each case, there are two appropriate sketches:



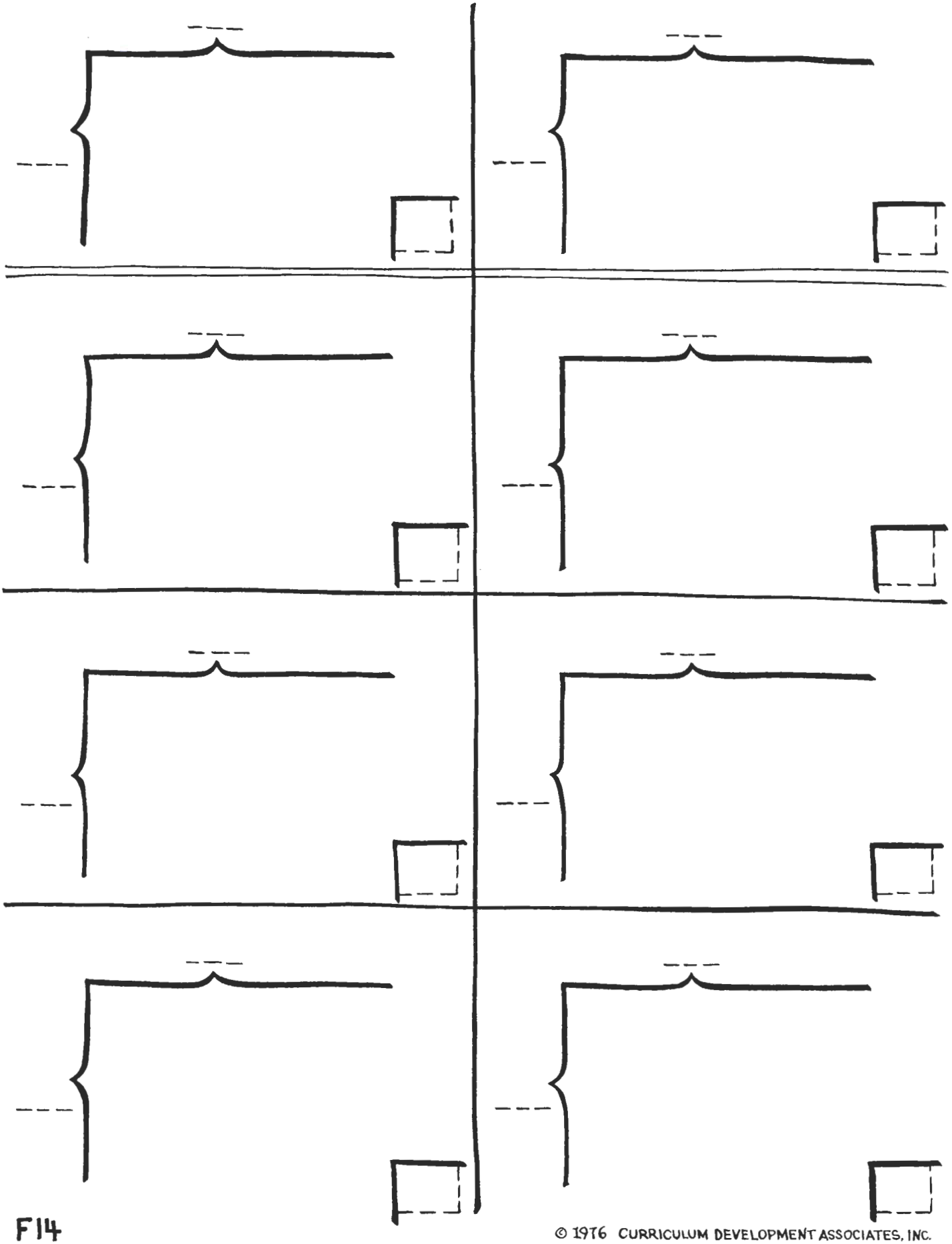
How many different sketches can you make that follow these directions?



There are three different sketches.

Can anyone find a number of crossing points that can be made with more than three different sketches? The answer is yes. The smallest number of crossing points for which more than three different sketches are possible is six. There are four possibilities.

Is there any number of crossing points that can be made with more than four different sketches? Yes; 12 can be shown in six different ways and 16 in five different ways. Such an investigation can be pursued as far as the children want to take it.



F14

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A. 6 4

2

6 4

2 8

B. 2

3

C.

D. 5

2

E.

2

6

F. 6

5

G. 0

3

H.

3

12

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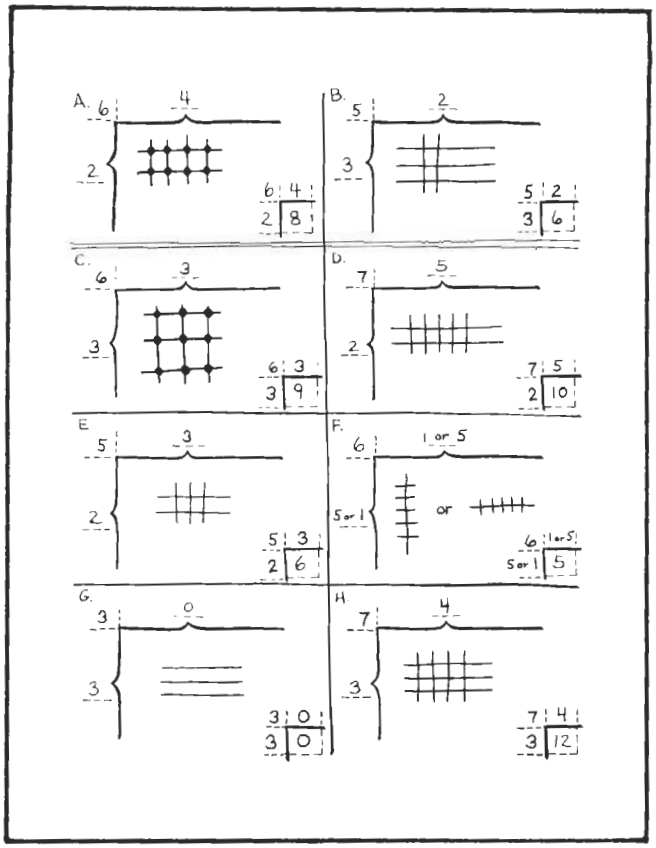
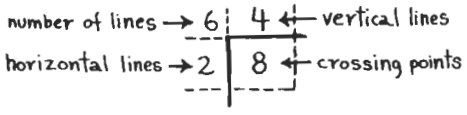
(Some of the appropriate responses are shown here.)



Introducing the Problem

If the children are not familiar with the previous activity page, please read the comments there about introducing "lines and crossing points". On this page another piece of information about the sketches is introduced. In the upper left-hand corner in these problems we record the total number of lines used in the sketch. Also, a record keeping device is introduced in the lower right-hand corner which provides a summary of information about the sketch.

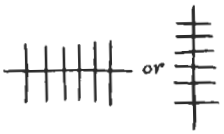


Our purpose in selecting this particular format for recording information becomes more apparent, if we eliminate two bits of information and the broken lines. It becomes a most familiar bit of notation.

Talking about the Problem

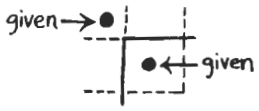
Did everyone draw the same sketch in each case? (There may be different sketches for (F).) Did everyone have the same summary of results? (There might be different summaries for (F).)

Did you notice any example that was different from the others? Problem (C) is the only one below the double line that has a completed sketch. Problem (G) is the only one in which 0 occurs. Is there an example in which there is no information about either the number of horizontal or the number of vertical lines? Yes, in (F), we are only given the total number of lines and the total number of crossing points. As a result, there is more than one sketch that is appropriate:



Extensions and Surprise Endings

Let's talk about problem (F). Will there always be two different sketches for problems in which you know the total number of lines and the total number of crossing points?

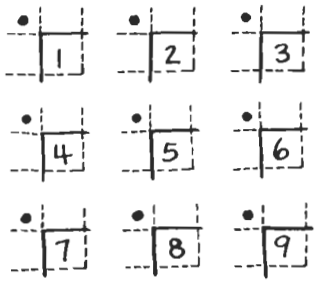


The answer is yes. If you have trouble convincing yourself, try working out several such problems.

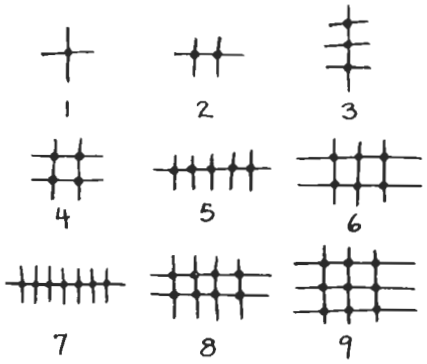
Will there ever be *more* than two different sketches possible when the number of lines and the number of crossing points are given? No — and that may be a bit surprising. (Try some examples if you're in doubt!)

In (F), the total number of lines used is greater than the number of crossing points. Can you think of other examples in which the total number of lines will be larger than the number of crossing points? This will be the case for any example with only one horizontal or one vertical line.

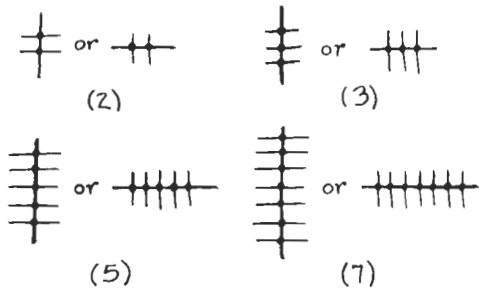
Suppose you only knew the number of crossing points.



In which cases can you draw a sketch that uses the same number or fewer lines than there are crossing points?



Four crossing points can be created with just four lines. Six, eight and nine crossing points can be created with five, six and six lines respectively. All the other numbers of crossing points (except for one crossing point) can be drawn in just two different ways. In each case both of them require more lines than there are crossing points.

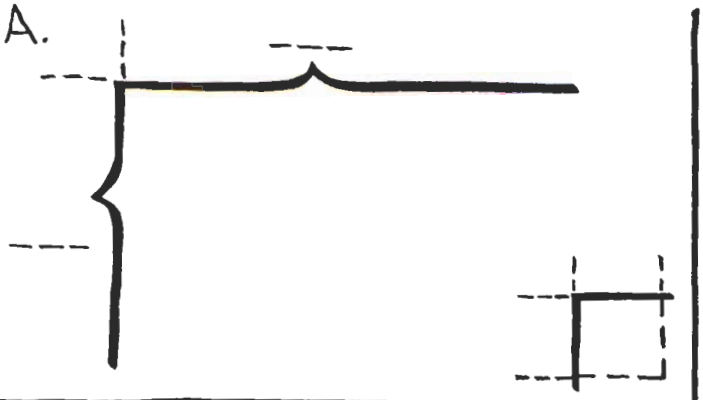


Notice that the numbers for which this is true are the prime numbers.

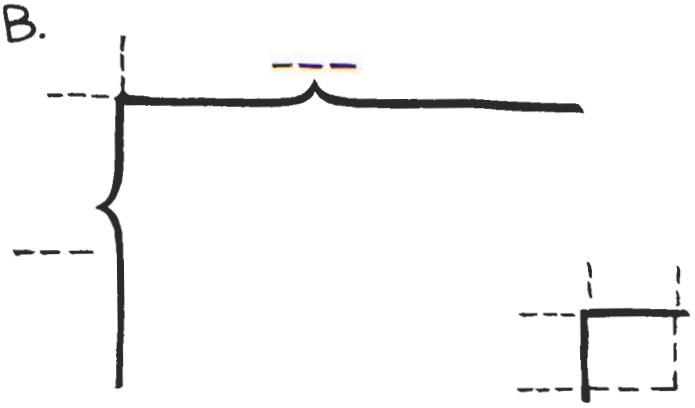
Which numbers of crossing points can be shown in three and only three different ways? After some investigation, we would begin to see a pattern: 4, 9, 25, 49, 121, etc. These are the prime numbers multiplied by themselves, or the "prime numbers squared".

Questions such as these, usually appropriate only for some intermediate grade children will help give them a new insight into "prime" and "composite" numbers. The next activity provides a format that can be used to keep track of such investigations.

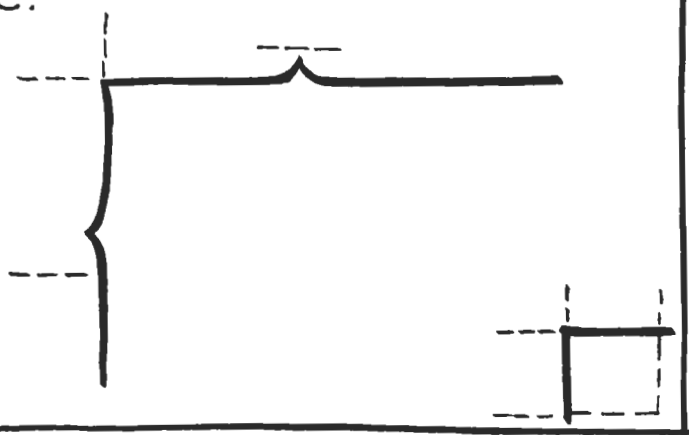
A.



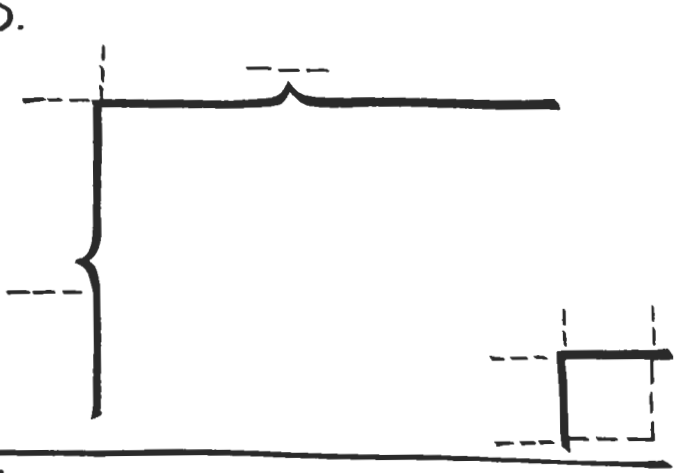
B.



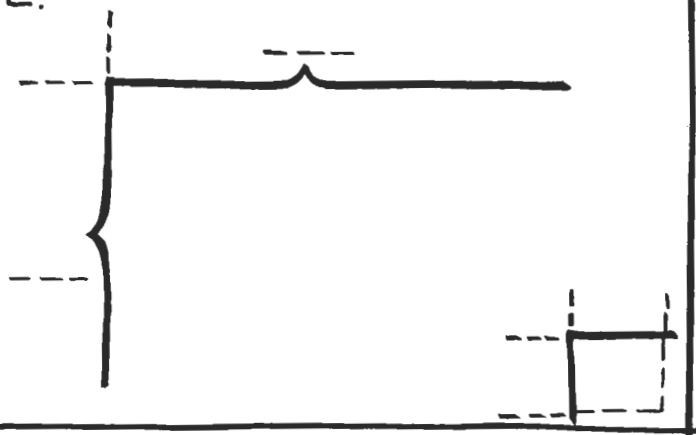
C.



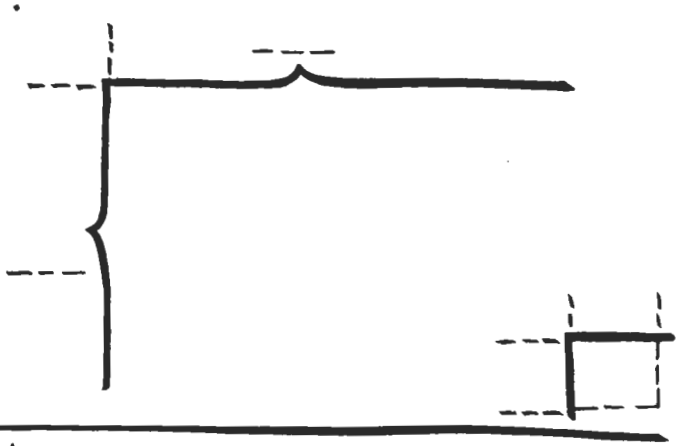
D.



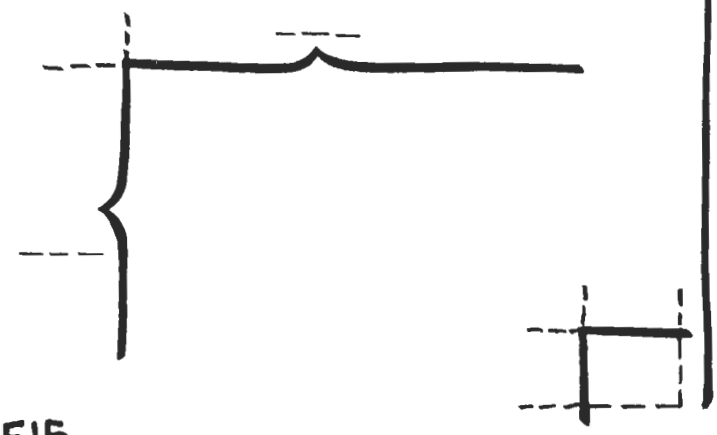
E.



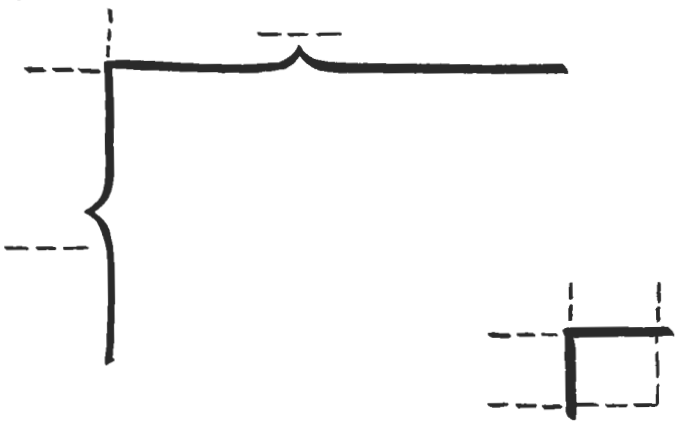
F.



G.



H.



F15

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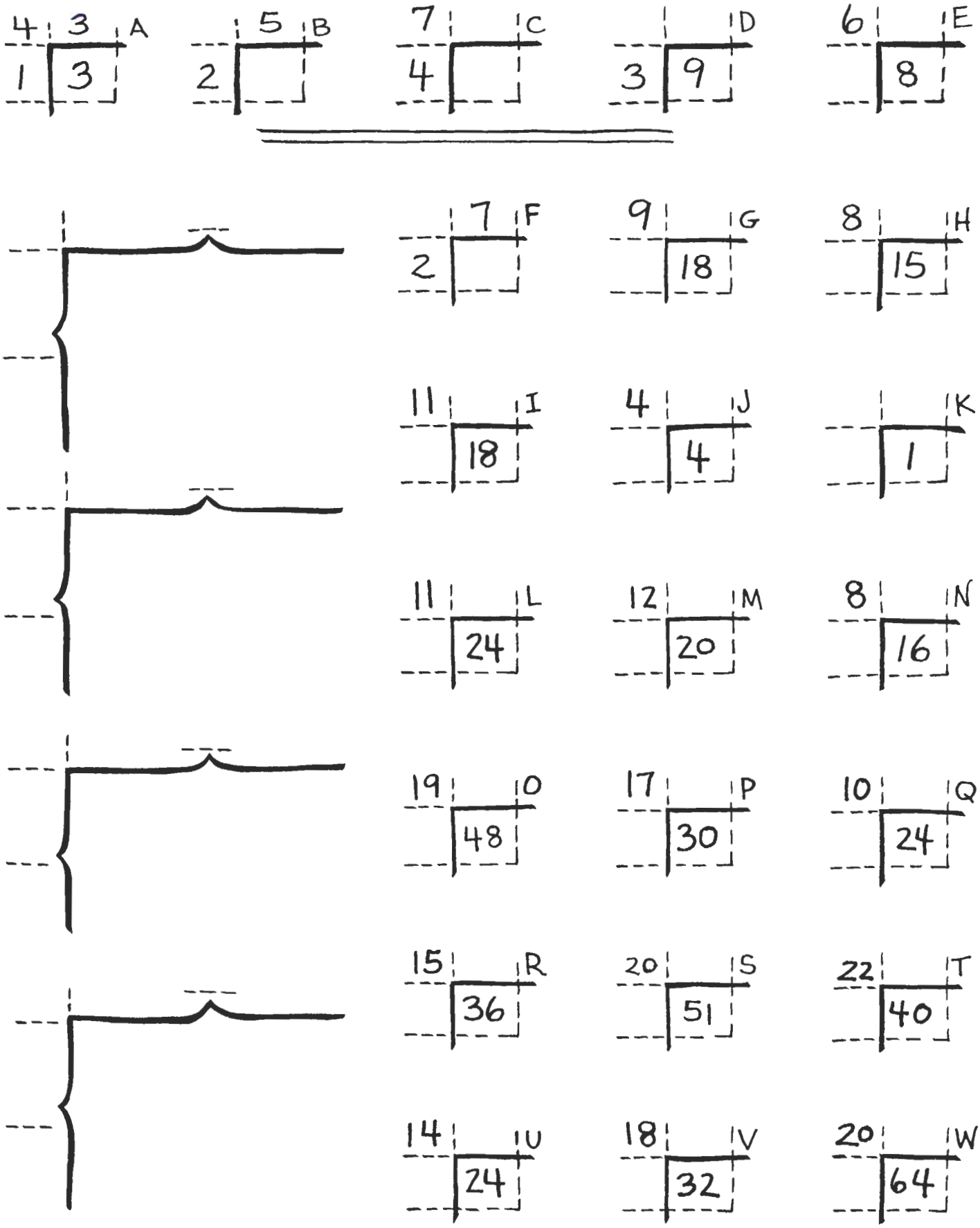

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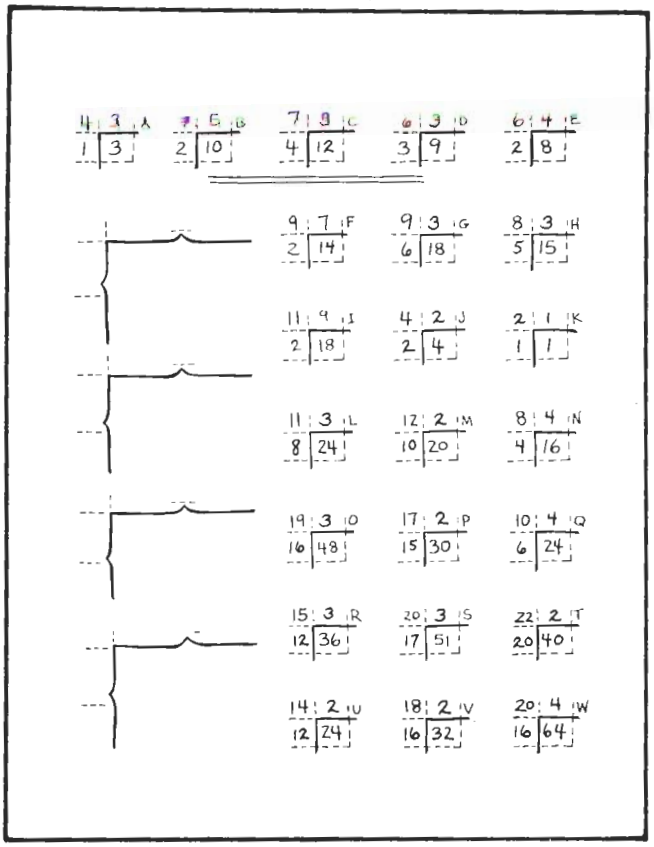
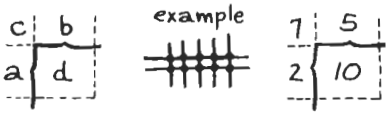
(Some of the appropriate responses are shown here.)



Introducing the Activity

If children have not been involved in either of the two previous activities, then they need to understand that each example on this page is a description of a sketch of "lines and crossing points". A completed description includes four pieces of information, located in the diagram below by a, b, c and d.

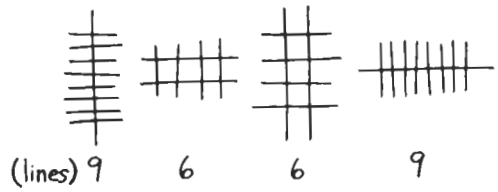


1) In position 'a', we record the number of horizontal lines (there are two in this example)
2) In 'b', we record the number of vertical lines (there are five in this example)
3) In 'c', we record the total number of lines in the sketch (there are seven in this example)
4) In 'd' we record the number of crossing points created (10 in the example)

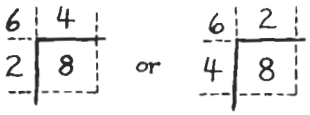
Examples A-D above the double line on this activity page are really simple pairs of directions for drawing a sketch. For example, problem (C) indicates that four horizontal lines and a total of seven lines should be drawn. Three vertical lines are needed to comply with this condition. Problem (D) calls for three horizontal lines and enough vertical lines to produce nine crossing points. Three vertical lines are needed.

Problem (E) is different. It is like all the examples below the double line. The total number of lines and the total number of crossing points is all the information that is given.

There are four different sketches that will produce the required eight crossing points:



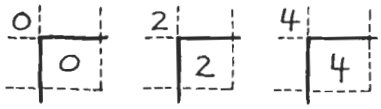
But since the sketch must have six lines, we can only use either (b) or (c) above. The completed example would be



As was the case here, there will often be two appropriate reports. Children should know that each one is equally acceptable. Both are "right answers".

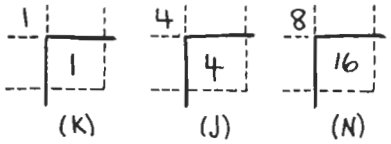
Talking about the Problem

Are there sketches that meet the requirements in each example? Yes. Could you imagine directions you couldn't follow? Consider these directions:

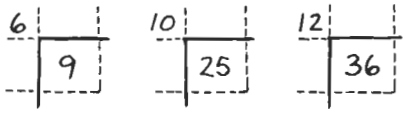


The first set of directions suggests a sketch that is already "drawn" before we put pencil to paper. It requires no horizontal and no vertical lines! The third set of directions is the same as problem (J) on this activity page, and requires two horizontal and two vertical lines. But the second set asks for the impossible. There is no way to use just two lines to create two crossing points.

Were there two different sketches for each example on the activity page? No. Which ones could be drawn in only one way?



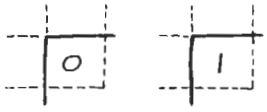
Can you think of other examples that give the total number of lines and the number of crossing points for which there is but one sketch?



Each such sketch we can think of will be a square arrangement, and has the same number of horizontal and vertical lines. The numbers of crossing points are 1, 4, 8, 16, 25, 36, etc. These numbers are called the "square numbers".

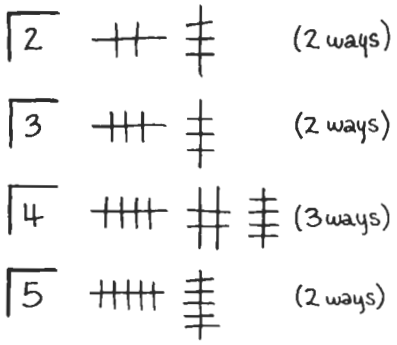
Extensions and Surprise Endings

Suppose we consider sketches in which all we know is the number of crossing points. Are there any situations in which there would be only one way to draw the sketch? Yes.

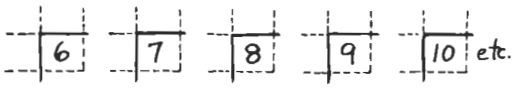


What about these directions?

How many different sketches are possible for each?



What other numbers of crossing points can be shown in more than one way under the rules? How about these?



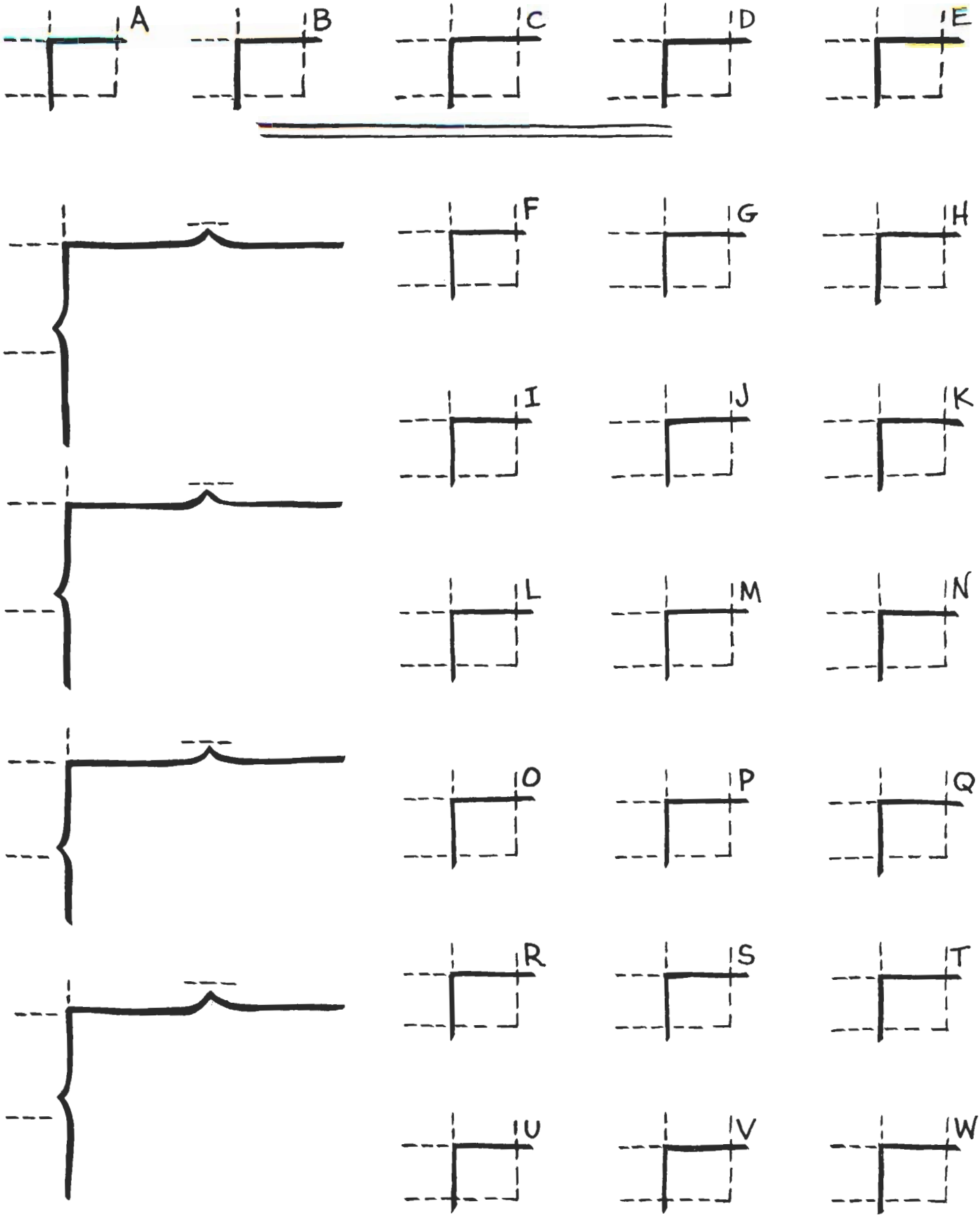
Of these, seven crossing points is the only number that cannot be shown in more than two ways. What about the numbers 11 through 20? How many can be shown in only two different sketches? 11, 13, 17, and 19 can be shown in only two different ways.

Take a good look at these numbers that have two and only two possible sketches: 2, 3, 5, 7, 11, 13, 17, 19, etc. They are called "prime numbers", and will come up time and again in mathematical investigations. Sometimes they are called the "unruly numbers" because they seem to have no regularity.

People often say that a prime number is a number with exactly two *different* factors — the number itself and one. This is another way of saying that if we consider the number as a number of crossing points, it will have exactly two possible sketches. This definition is arbitrary, and was chosen to exclude '1' from the family of primes. The whole number factors for '1' are not *different*.

Starting with the number four, all numbers that are *not* prime are called "composite". They have more than two factors.

$4 = 1 \times 2 \times 2$ or $4 = 1 \times 4$
$6 = 1 \times 2 \times 3$ or $6 = 1 \times 6$
$8 = 1 \times 2 \times 2 \times 2$ or $8 = 1 \times 8$
$9 = 1 \times 3 \times 3$ or $9 = 1 \times 9$
$10 = 1 \times 2 \times 5$ or $10 = 1 \times 10$



F16

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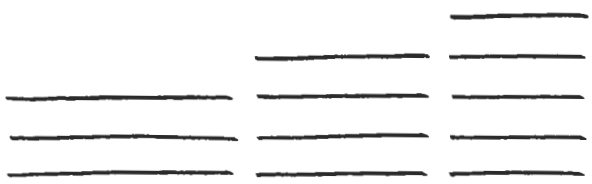
Diferentes maneras

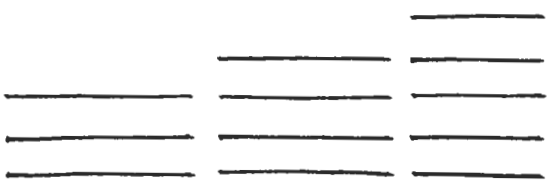
Different ways

A.



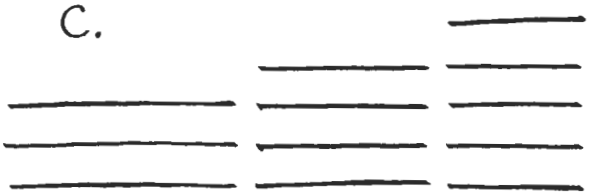
+ + = 14

B.



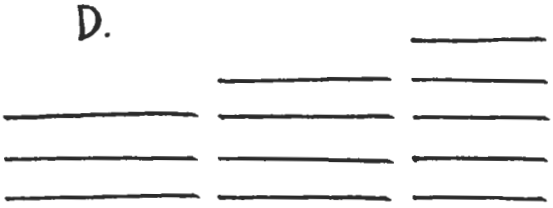
+ + = 14

C.



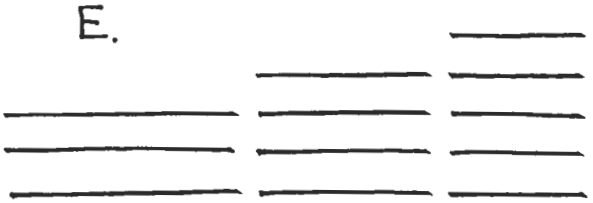
+ + = 14

D.



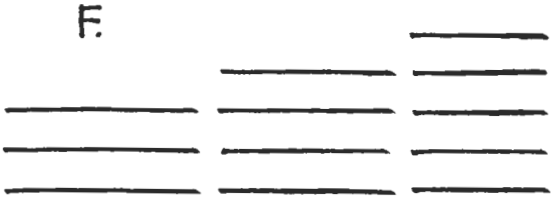
+ + = 23

E.



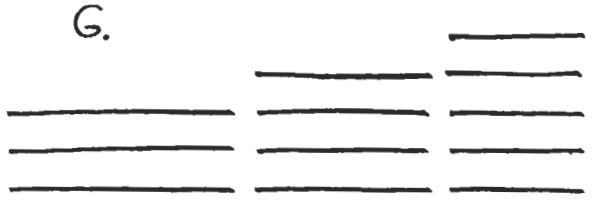
+ + = 23

F.



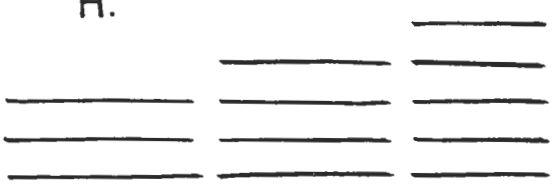
+ + = 23

G.



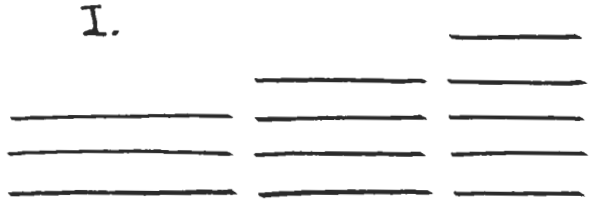
+ + = 23

H.



+ + = 23

I.



+ + = 23

J.

+ + = 23

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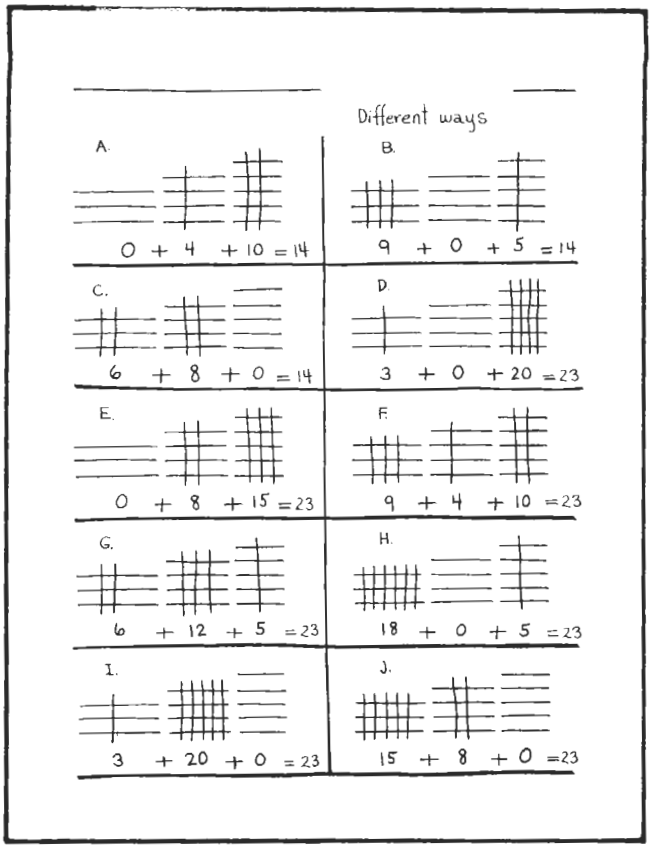
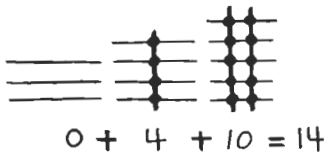
(Some of the appropriate responses are shown here.)



Introducing the Problem

If the children are familiar with either of the two previous activities, this activity will need little introduction. Once again it involves drawing vertical lines to create crossing points in each section of horizontal lines. In these sketches, a vertical line can produce three, four or five crossing points, depending on the section in which it is drawn. The number of crossing points created in each section is recorded below that section, and the total number of crossing points in the sketch is given at the right. In all examples on this page, the total number of crossing points is given. The problem is to draw appropriate sketches. However, here we lift the restriction that you must draw no more than five vertical lines in a section. You may draw as many as you like. Problems A, B and C each ask for a total of 14 crossing points. The others call for 23 crossing points. When two or more problems call for the same number of crossing points, the sketches must be different, and consequently the final reports must be different. One solution to problem A is



Once children understand the notion of "crossing points", they need not draw dots to indicate the points.

Can we find a solution to problem A that doesn't use any lines with five crossing points? Yes, two vertical lines in each of the other sections will give us a total of 14 crossing points.

The final report would be '6 + 8 + 0 = 14'. Still another solution would have the record '9 + 0 + 5 = 14'.

Talking about the Problem

There were three problems that required 14 crossing points. But there are seven problems asking for 23 crossing points. That means we must find seven *different* ways to make 23 crossing points with this array of lines. Did anyone have a plan in the beginning or develop one along the way? One strategy is to first figure out how many ways a sketch can be drawn using no vertical lines in the section with five horizontal lines; then to figure how many ways a sketch can be drawn using just one line in the "5's section".

3 + 20 + 0 = 23
15 + 8 + 0 = 23

and

6 + 12 + 5 = 23
18 + 0 + 5 = 23

A continuation of this strategy yields three more solutions, bringing the total to seven.

9 + 4 + 10 = 23
0 + 8 + 15 = 23
3 + 0 + 20 = 23

Extensions and Surprise Endings

Suppose we begin to summarize these results in a chart. We might record the number of vertical lines

drawn in each of the three sections for a given solution.

3's	4's	5's	
1	0	4	23
0	2	3	23
3	1	2	23
2	3	1	23
6	0	1	23

Notice that in the fourth entry in the table, (2, 3, 1), there are three 4's (12 crossing points) indicated. These three 4's can be interchanged with four 3's (which also represent 12 crossing points). If we make this switch, the result is described by the fifth entry in the table (6, 0, 1). So, when we have a combination that involves at least three 4's, such as the first one shown below, we know that we can "exchange" three 4's for four 3's to get a second different combination.

3's	4's	5's	
1	5	0	23
5	2	0	23

If we make a record of the numbers of different ways to sketch certain numbers of crossing points using the array of lines on this page, a very interesting pattern develops for numbers beyond 16 crossing points.

no. of points	16	17	18	19	20	21	22	23	24	25	26	27	etc.
no. of sketches	4	4	4	5	6	6	6	7	8	8	8	9	

In verifying these results or in extending the investigation to even larger numbers of crossing points, the notion of "exchanging" three 4's for four 3's is a most useful shortcut. For example, one solution for creating 48 crossing points uses one 3, ten 4's and one 5. Three more combinations can be written down without much computation. In each of these new combinations, a group of three 4's is replaced by a group of four 3's.

3's	4's	5's	
1	10	1	48
5	7	1	48
9	4	1	48
13	1	1	48

Another solution for 48 uses just twelve 4's.

Again, four more combinations can found by successive exchanges of a group of three 4's for a group of four 3's.

3's	4's	5's	
0	12	0	48
4	9	0	48
8	6	0	48
12	3	0	48
16	0	0	48

(Answers for the two following pages are shown below.)

All factors should be 9 or less.

A.

$(2 \times 0)+(9 \times 1) = 0+9 = 9$
$(2 \times 6)+(9 \times 0) = 12+0 = 12$
$(2 \times 3)+(9 \times 2) = 6+18 = 24$
$(2 \times 5)+(9 \times 3) = 10+27 = 37$
$(2 \times 1)+(9 \times 5) = 2+45 = 47$
$(2 \times 8)+(9 \times 4) = 16+36 = 52$
$(2 \times 4)+(9 \times 6) = 8+54 = 62$
$(2 \times 7)+(9 \times 7) = 14+63 = 77$
$(2 \times 2)+(9 \times 9) = 4+81 = 85$
$(2 \times 9)+(9 \times 8) = 18+72 = 90$

B.

$(8 \times 1)+(3 \times 1) = 8+3 = 11$
$(8 \times 2)+(3 \times 0) = 16+0 = 16$
$(8 \times 0)+(3 \times 7) = 0+21 = 21$
$(8 \times 3)+(3 \times 5) = 24+15 = 39$
$(8 \times 4)+(3 \times 3) = 32+9 = 41$
$(8 \times 5)+(3 \times 6) = 40+18 = 58$
$(8 \times 6)+(3 \times 4) = 48+12 = 60$
$(8 \times 8)+(3 \times 2) = 64+6 = 70$
$(8 \times 7)+(3 \times 9) = 56+27 = 83$
$(8 \times 9)+(3 \times 8) = 72+24 = 96$

C.

$(2 \times 5)+(1 \times 6) = 10+6 = 16$
$(1 \times 5)+(2 \times 6) = 5+12 = 17$
$(0 \times 5)+(4 \times 6) = 0+24 = 24$
$(5 \times 5)+(0 \times 6) = 25+0 = 25$
$(3 \times 5)+(3 \times 6) = 15+18 = 33$
$(6 \times 5)+(6 \times 6) = 30+36 = 66$
$(4 \times 5)+(8 \times 6) = 20+48 = 68$
$(9 \times 5)+(5 \times 6) = 45+30 = 75$
$(8 \times 5)+(7 \times 6) = 40+42 = 82$
$(7 \times 5)+(9 \times 6) = 35+54 = 89$

D.

$(0 \times 4)+(1 \times 7) = 0+7 = 7$
$(5 \times 4)+(0 \times 7) = 20+0 = 20$
$(2 \times 4)+(2 \times 7) = 8+14 = 22$
$(1 \times 4)+(3 \times 7) = 4+21 = 25$
$(4 \times 4)+(5 \times 7) = 16+35 = 51$
$(6 \times 4)+(4 \times 7) = 24+28 = 52$
$(7 \times 4)+(6 \times 7) = 28+42 = 70$
$(3 \times 4)+(9 \times 7) = 12+63 = 75$
$(9 \times 4)+(7 \times 7) = 36+49 = 85$
$(8 \times 4)+(8 \times 7) = 32+56 = 88$

18

All factors should be 9 or less.

E.

$(2 \times 4)+(3 \times 0) = 8+0 = 8$
$(2 \times 1)+(3 \times 2) = 2+6 = 8$
$(2 \times 0)+(3 \times 7) = 0+21 = 21$
$(2 \times 3)+(3 \times 5) = 6+15 = 21$
$(2 \times 6)+(3 \times 3) = 12+9 = 21$
$(2 \times 9)+(3 \times 1) = 18+3 = 21$
$(2 \times 2)+(3 \times 8) = 4+24 = 28$
$(2 \times 5)+(3 \times 6) = 10+18 = 28$
$(2 \times 8)+(3 \times 4) = 16+12 = 28$
$(2 \times 7)+(3 \times 9) = 14+27 = 41$

F.

$(3 \times 5)+(1 \times 8) = 15+8 = 23$
$(1 \times 5)+(3 \times 8) = 5+24 = 29$
$(0 \times 5)+(4 \times 8) = 0+32 = 32$
$(7 \times 5)+(0 \times 8) = 35+0 = 35$
$(6 \times 5)+(2 \times 8) = 30+16 = 46$
$(5 \times 5)+(6 \times 8) = 25+48 = 73$
$(2 \times 5)+(8 \times 8) = 10+64 = 74$
$(4 \times 5)+(7 \times 8) = 20+56 = 76$
$(8 \times 5)+(5 \times 8) = 40+40 = 80$
$(9 \times 5)+(9 \times 8) = 45+72 = 117$

G.

$(0 \times 4)+(2 \times 9) = 0+18 = 18$
$(5 \times 4)+(0 \times 9) = 20+0 = 20$
$(6 \times 4)+(1 \times 9) = 24+9 = 33$
$(4 \times 4)+(4 \times 9) = 16+36 = 52$
$(8 \times 4)+(3 \times 9) = 32+27 = 59$
$(7 \times 4)+(5 \times 9) = 28+45 = 73$
$(3 \times 4)+(7 \times 9) = 12+63 = 75$
$(1 \times 4)+(8 \times 9) = 4+72 = 76$
$(2 \times 4)+(9 \times 9) = 8+81 = 89$
$(9 \times 4)+(6 \times 9) = 36+54 = 90$

H.

$(6 \times 0)+(7 \times 1) = 0+7 = 7$
$(6 \times 2)+(7 \times 0) = 12+0 = 12$
$(6 \times 1)+(7 \times 2) = 6+14 = 20$
$(6 \times 6)+(7 \times 3) = 36+21 = 57$
$(6 \times 7)+(7 \times 4) = 42+28 = 70$
$(6 \times 5)+(7 \times 6) = 30+42 = 72$
$(6 \times 4)+(7 \times 8) = 24+56 = 80$
$(6 \times 3)+(7 \times 9) = 18+63 = 81$
$(6 \times 8)+(7 \times 5) = 48+35 = 83$
$(6 \times 9)+(7 \times 7) = 54+49 = 103$

19

Todos los factores deben ser 9 o menos.

All factors should be 9 or less.

A.

(2x)+(9x)= + =9

(2x)+(9x)= + =12

(2x)+(9x)= + =24

(2x)+(9x)= + =37

(2x)+(9x)= + =47

(2x)+(9x)= + =52

(2x)+(9x)= + =62

(2x)+(9x)= + =77

(2x)+(9x)= + =85

(2x)+(9x)= + =90

B.

(8x)+(3x)= + =11

(8x)+(3x)= + =16

(8x)+(3x)= + =21

(8x)+(3x)= + =39

(8x)+(3x)= + =41

(8x)+(3x)= + =58

(8x)+(3x)= + =60

(8x)+(3x)= + =70

(8x)+(3x)= + =83

(8x)+(3x)= + =96

C.

(x5)+(x6)= + =16

(x5)+(x6)= + =17

(x5)+(x6)= + =24

(x5)+(x6)= + =25

(x5)+(x6)= + =33

(x5)+(x6)= + =66

(x5)+(x6)= + =68

(x5)+(x6)= + =75

(x5)+(x6)= + =82

(x5)+(x6)= + =89

D.

(x4)+(x7)= + =7

(x4)+(x7)= + =20

(x4)+(x7)= + =22

(x4)+(x7)= + =25

(x4)+(x7)= + =51

(x4)+(x7)= + =52

(x4)+(x7)= + =70

(x4)+(x7)= + =75

(x4)+(x7)= + =85

(x4)+(x7)= + =88

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All factors should be 9 or less.

Todos los factores deben ser 9 o menos.

E.

(2x)+(3x)= + =8

(2x)+(3x)= + =8

(2x)+(3x)= + =21

(2x)+(3x)= + =21

(2x)+(3x)= + =21

(2x)+(3x)= + =21

(2x)+(3x)= + =28

(2x)+(3x)= + =28

(2x)+(3x)= + =28

(2x)+(3x)= + =41

F.

(x5)+(x8)= + =23

(x5)+(x8)= + =29

(x5)+(x8)= + =32

(x5)+(x8)= + =35

(x5)+(x8)= + =46

(x5)+(x8)= + =73

(x5)+(x8)= + =74

(x5)+(x8)= + =76

(x5)+(x8)= + =80

(x5)+(x8)= + =117

G.

(x4)+(x9)= + =18

(x4)+(x9)= + =20

(x4)+(x9)= + =33

(x4)+(x9)= + =52

(x4)+(x9)= + =59

(x4)+(x9)= + =73

(x4)+(x9)= + =75

(x4)+(x9)= + =76

(x4)+(x9)= + =89

(x4)+(x9)= + =90

H.

(6x)+(7x)= + =7

(6x)+(7x)= + =12

(6x)+(7x)= + =20

(6x)+(7x)= + =57

(6x)+(7x)= + =70

(6x)+(7x)= + =72

(6x)+(7x)= + =80

(6x)+(7x)= + =81

(6x)+(7x)= + =83

(6x)+(7x)= + =103

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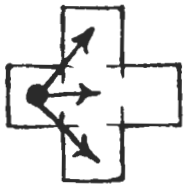
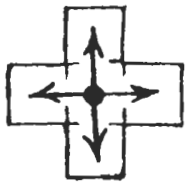
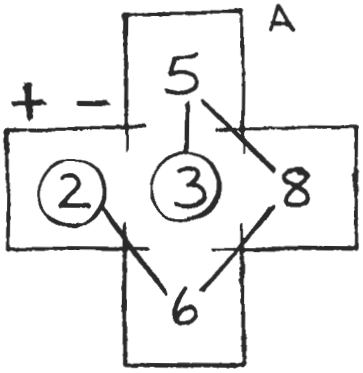
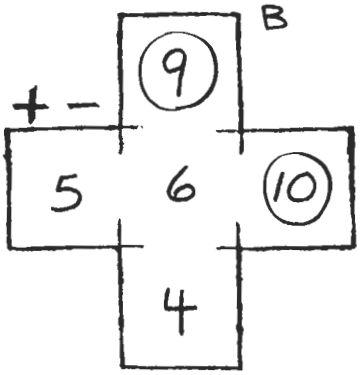
ENCONTRANDO EL CAMINITO
adición y sustracción

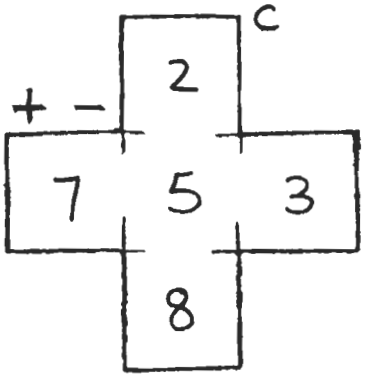
HIDDEN PATHWAYS
addition and subtraction



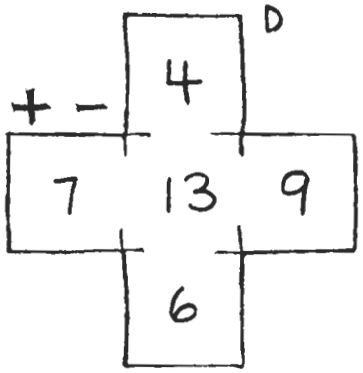


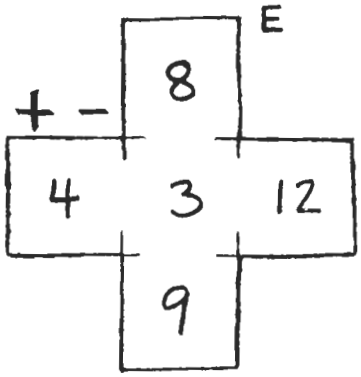


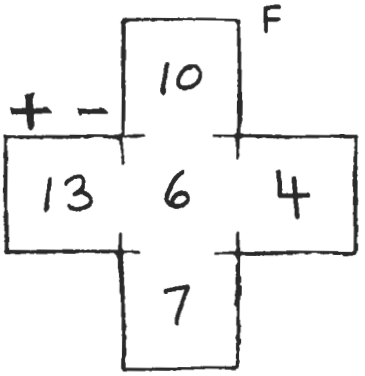


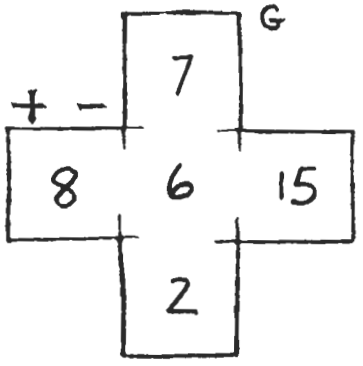


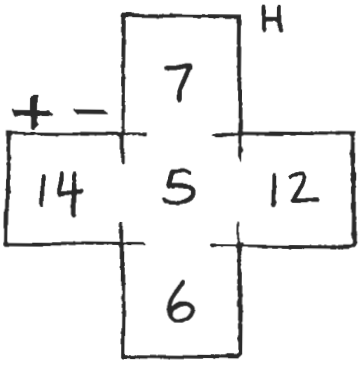
¿Puedes hacerlos todos? Can you do them all?

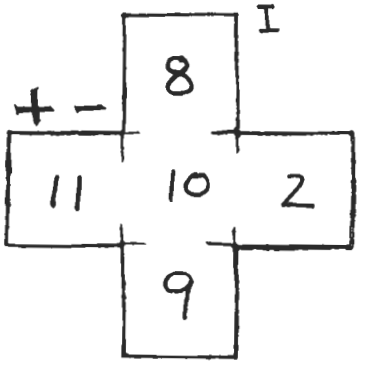












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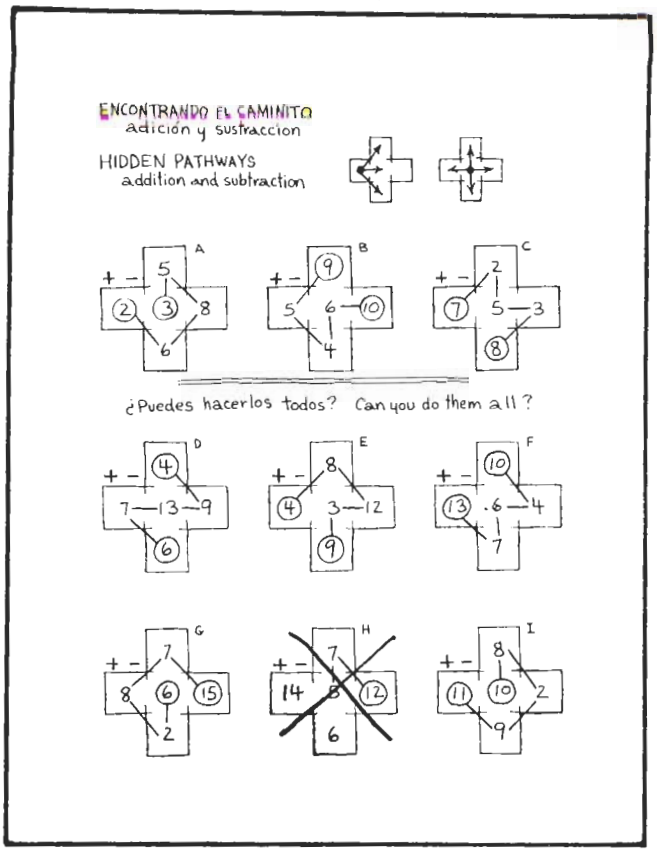
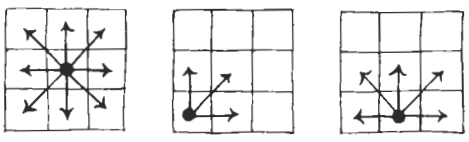
(Some of the appropriate responses are shown here.)



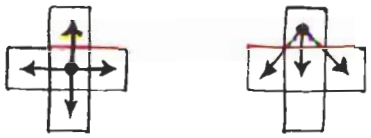
Introducing the Problem

"Hidden Pathways" is an activity that can involve the use of multiplication and division or addition and subtraction facts. This page begins a series of seven pages of variations on the "Hidden Pathways" problem. The comments given here cover all of those pages.

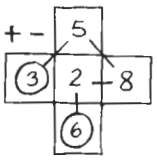
The most challenging versions of "Hidden Pathways" allow the use of all four operations. In variations on this page addition and subtraction facts are used. The object of the game is to find a "pathway" through all of the numbers in the array using addition and subtraction, such that each number is included once and only once on the pathway. Moves from one number to another can be one space horizontally, vertically, or diagonally in any direction, much as the king can move in chess.



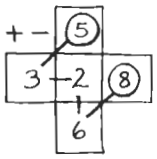
In the arrays on this page, there are four possible moves from the center square, and three possible moves from each of the other squares.



Every pathway has two endpoints. It turns out that the pathways are reversible and that there may be more than one pathway through the numbers. For example,

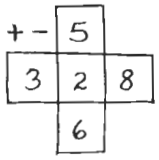


3 + 5 = 8 and 8 − 2 = 6
and in reverse
6 + 2 = 8 and 8 − 5 = 3



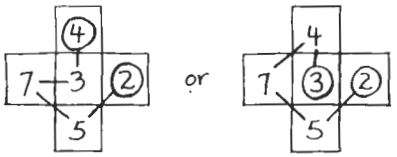
8 − 6 = 2 and 2 + 3 = 5
and in reverse
5 − 3 = 2 and 2 + 6 = 8

In problems A and B on this page, the starting and ending points are indicated by circles. But in the rest of the problems, there are no clues. Several possible starts can lead to dead ends, as shown below:



Here are some false starts
8 − 2 = 6 8 − 5 = 3
5 − 2 = 3 2 + 3 = 5
3 + 2 = 5

Occasionally, there are two or more basic pathways for the same array:



Before setting to work on the problems below the double line, children should know that for at least one of them, no one has found a pathway that includes all numbers.

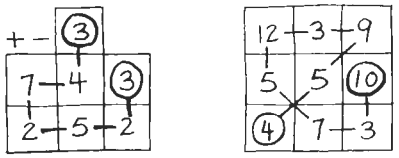
Talking about the Problem

Were there any problems for which you couldn't find a pathway? We have not yet found a pathway for problem H. Most children will have recognized this as the one with no solution. Some may even be able to make an argument based on trial and error as to why it cannot be done. No matter where you start, you always run into trouble somewhere along each route.

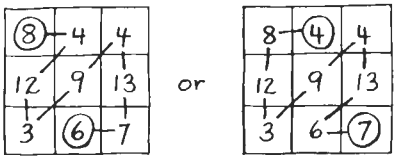
Extensions and Surprise Endings

Using Larger Arrays

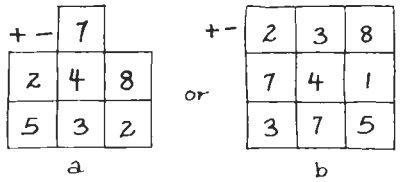
The problem can be made more complex by increasing the number of entries in the array. As the number of entries increases, the starting points are more difficult to locate, as in the following examples:



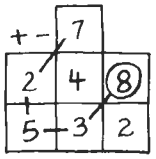
In a given 3 by 3 arrangement, there is usually just one basic path, although there may be several different places along that path that could be starting points. It is difficult to construct an example that will support two basically different pathways, but it can be done.



Of course, there are arrangements for which no hidden pathway has been found. We make it a point to include at least one of these in every group of problems, and tell children that one such problem is there. If they think that they have found one that can't be done, they learn to persevere a bit longer, for it may be just more elusive than the others. Some children even find ways to argue for impossibility. Consider the problems below:

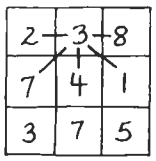


If there is only one way into or out of a number, it has to be an endpoint. In example (a) above, the number 8 must be an endpoint, since there is only one way for it to be on a pathway that involves only addition and subtraction. But if we follow that pathway, we come to a "dead end".



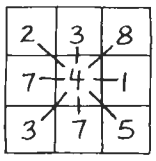
There is no way to connect 7, 4, and 2. So 8 cannot be on a pathway, and since it can't be included, we can't follow the rule that all numbers must be included once and only once.

In the second example we were considering, one might reason that no pathway can include the number two. None of the following trios can be used in true addition or subtraction facts:



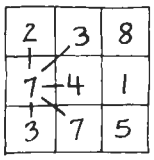
2, 3, 8
2, 3, 1
2, 3, 4
2, 3, 7

or



2, 4, 3 2, 4, 5
2, 4, 8 2, 4, 7
2, 4, 1

or

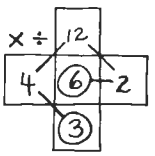


2, 7, 3
2, 7, 4
2, 7, 7

Since 2 cannot be a part of any pathway, no pathway can include all numbers. Such investigations and arguments are forerunners of mathematical proofs.

Using Multiplication and Division

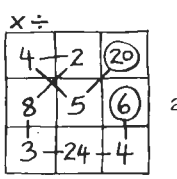
Another variation on this activity uses multiplication and division instead of addition and subtraction. The procedures are basically the same.



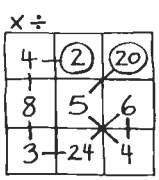
$6 \times 2 = 12$ and $12 \div 4 = 3$

or

$3 \times 4 = 12$ and $12 \div 2 = 6$



and



While there may be different pathways, as in the problem above, in all cases either endpoint can be a starting point.

$20 \div 5 = 4,\ 4 \times 2 = 8,\ 8 \times 3 = 24,\ 24 \div 4 = 6$
and $6 \times 4 = 24,\ 24 \div 3 = 8,\ 8 \div 2 = 4,\ 4 \times 5 = 20$
or
$2 \times 4 = 8,\ 8 \times 3 = 24,\ 24 \div 6 = 4,\ 4 \times 5 = 20$
and $20 \div 5 = 4,\ 4 \times 6 = 24,\ 24 \div 3 = 8,\ 8 \div 4 = 2$

Some problems have been made for which no one has found a complete pathway.

15	2	8
5	4	7
3	12	21

a

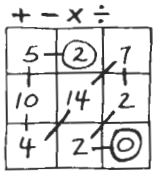
4	28	20
24	5	7
6	20	4

b

In (a), we might notice that the only numbers given in the array that are factors of 21 are 3 and 7. So only 3 and 7 could be along a pathway to or from 21. But they are not positioned so they can be connected. In problem (b), the number 20 occurs twice, so there would have to be two pairs of factors whose product is 20. The only factors of 20 in the array are two 4's and one 5. One of the 20's could be connected to both 5 and 4, but since there is only one 5, the other 20 cannot be on the pathway.

"No Operations Barred"

The most lively hunts for hidden pathways develop when "no operations are barred". Addition, subtraction, multiplication, and division are all permitted in any order. For example,



$2 \times 5 = 10,\ 10 + 4 = 14,\ 14 \div 7 = 2,\ 2 - 2 = 0$
(and of course, in reverse)
$0 + 2 = 2,\ 2 \times 7 = 14,\ 14 - 4 = 10,\ 10 \div 5 = 2$

As the pathways become longer, trial and error is almost indispensable in determining where to begin. Let's look at this problem:

			+ − × ÷
12	2	4	
3	6	11	10
4	7	8	2

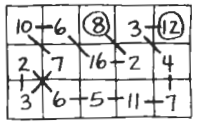
Since 7 and 11 have no factors other than themselves and 1, they must be on sections of pathways that involve addition and subtraction. There are two possible parts of the path that might involve 7: $3 + 4 = 7$, or $4 + 7 = 11$. But 11 can be included in just one section: $11 - 7 = 4$ (or $4 + 7 = 11$). Thus, 11 must be an endpoint.

With that beginning, the pathway unfolds:

$11 - 7 = 4;\ 4 \times 3 = 12;\ 12 \div 2 = 6;\ 6 + 4 = 10;\ 10 - 8 = 2$
(and the reverse path)

When no operations are barred, it becomes increasingly difficult to find the examples that apparently cannot be completed, for there are so many different possibilities. A pathway can be found for at least one of the following examples:

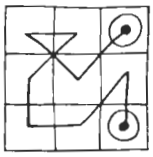
10	12	2	14	3
2	7	4	3	6
3	5	6	11	9



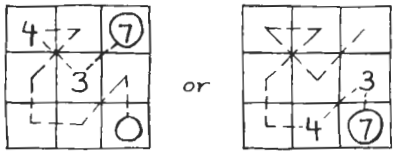
Are pathways possible for both?

Constructing "Hidden Pathway" Problems

Until you become more familiar with constructing hidden pathway problems, you may begin by designing a pathway to be followed:

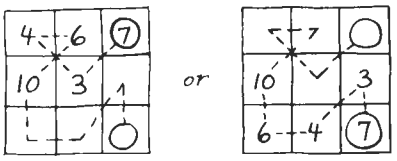


Then start with any number and decide to add or subtract or multiply or divide by another number. Now you can fill in three cells of the array, using either end of your planned pathway as a starting point. For example, using $7 - 3 = 4$, we would have

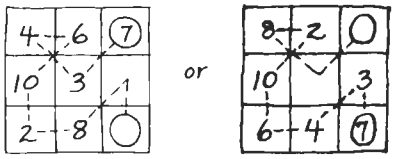


Going on from 4, add or subtract another number, and you have the next two steps along the pathway.

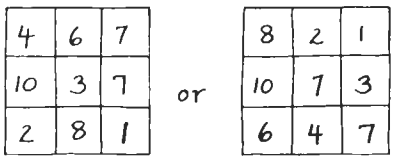
Suppose we choose 4 + 6 = 10:



Now start with 10 and add or subtract anything you like. If we choose 10 - 2 = 8, we have the next two cells:



Finally add or subtract some number from 8 to complete the example. After the pathway plans are removed, the problem would look like this.



If you look away for a few minutes, you will probably forget which numbers are at the ends of your pathway—and you, too, will have to use some trial and error to find a starting point that works!

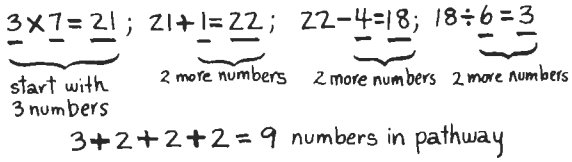
It is usually surprising to both adults and children that examples are so easy to construct and yet often are difficult to solve. This leads to challenges: "Here's one I made up; do you think you can find the pathway?"

Circular Pathways

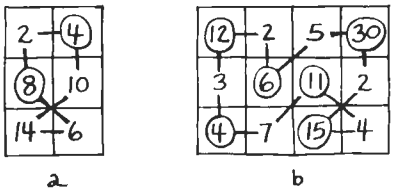
Thus far we have considered examples of pathways connecting 5, 7, 9, 11, or 15 numbers. Why did we always use an odd number of numbers? Three numbers are required to begin a pathway. For example,

$$3 \times 7 = 21$$

Each additional statement adds two new numbers. So all together there will be an odd number of numbers, for we start with three numbers and then keep adding numbers in pairs:

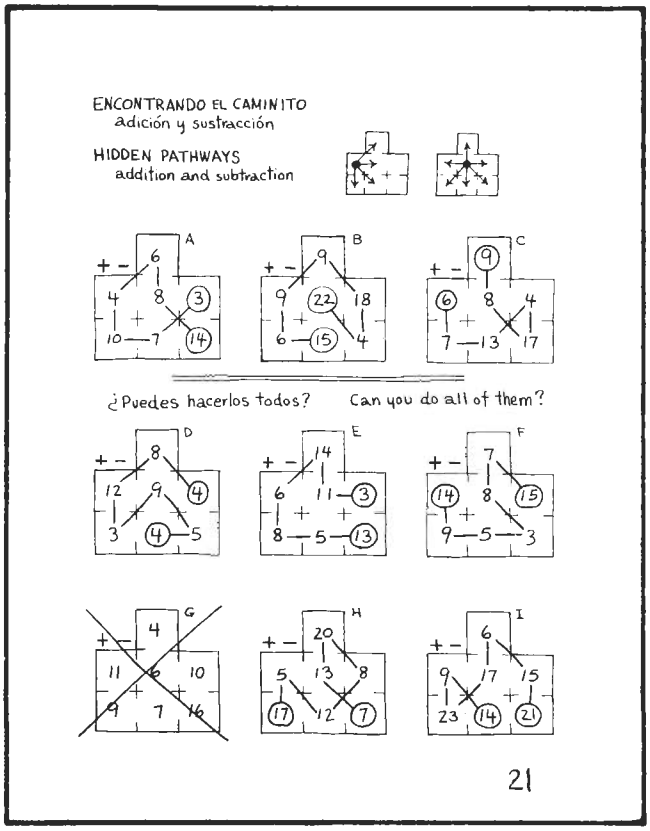


Some children might like to make up "circular" pathways, in which ends are joined together. Here are two examples:

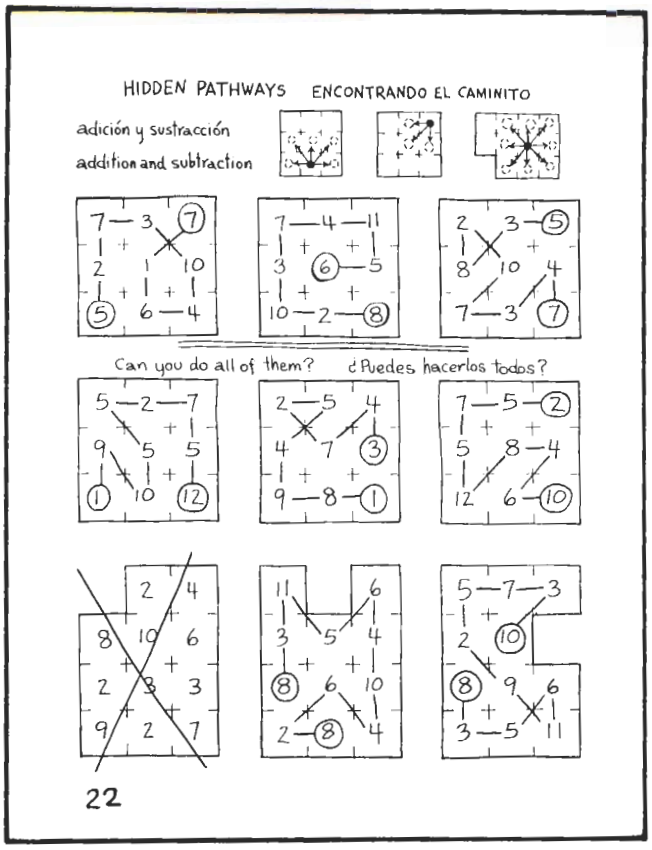


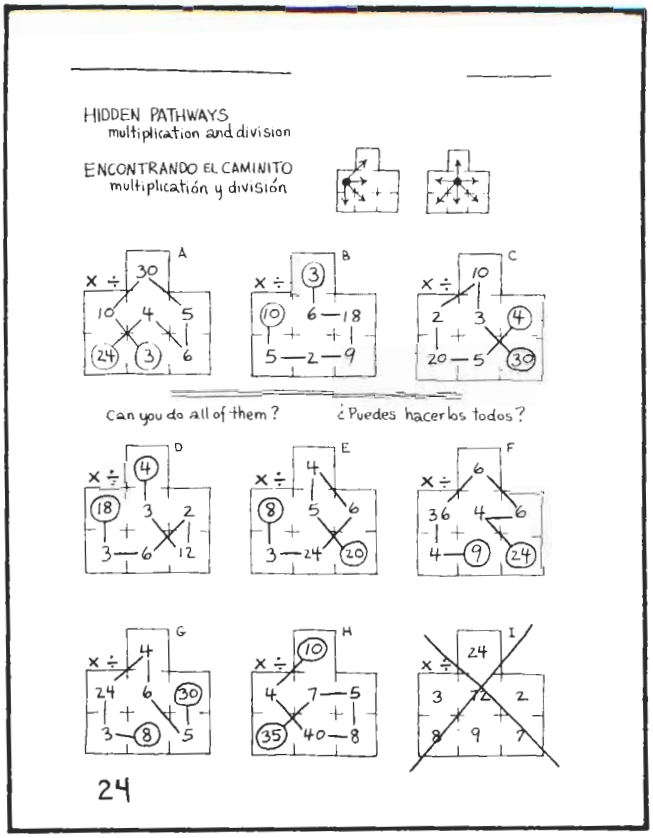
These circular paths can be entered at the points indicated by circles. Such pathways require an even number of cells.

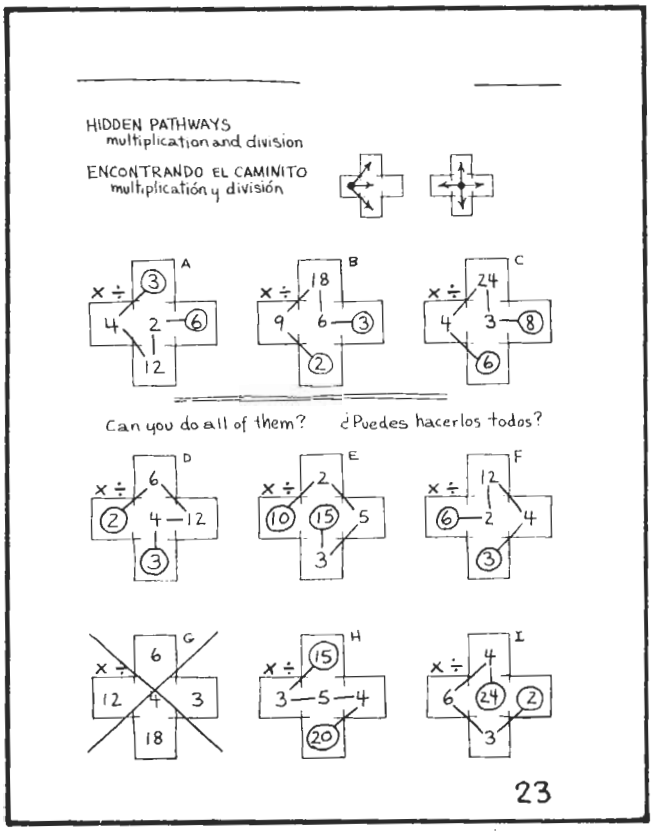
(Some of the appropriate responses for the activities on the following pages are shown here.)

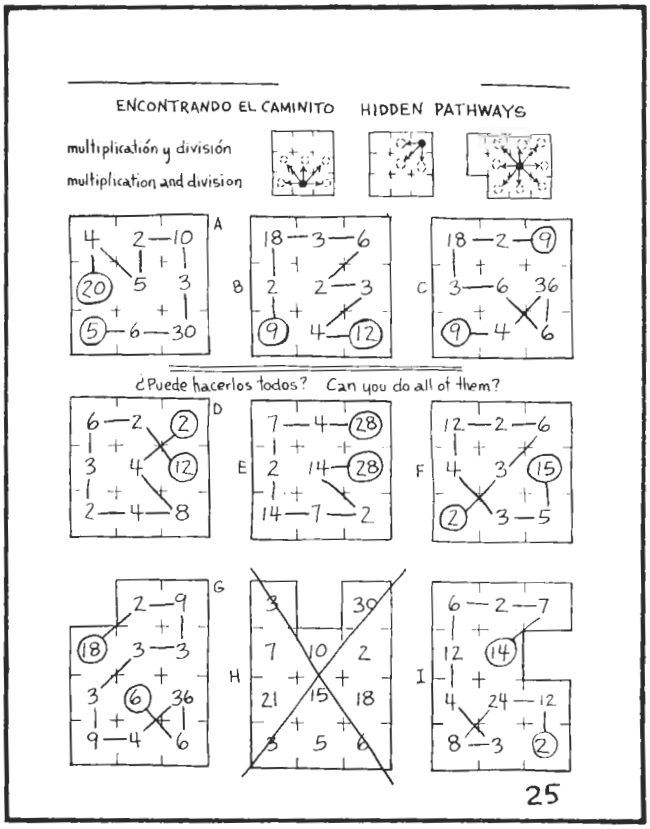


(Some of the appropriate responses are shown here.)









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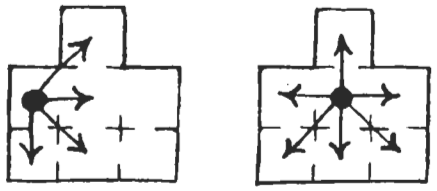
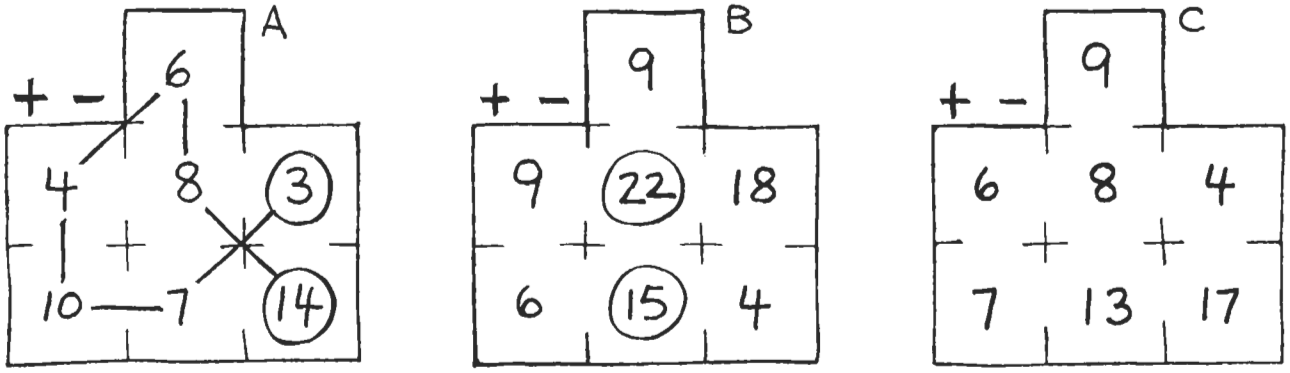
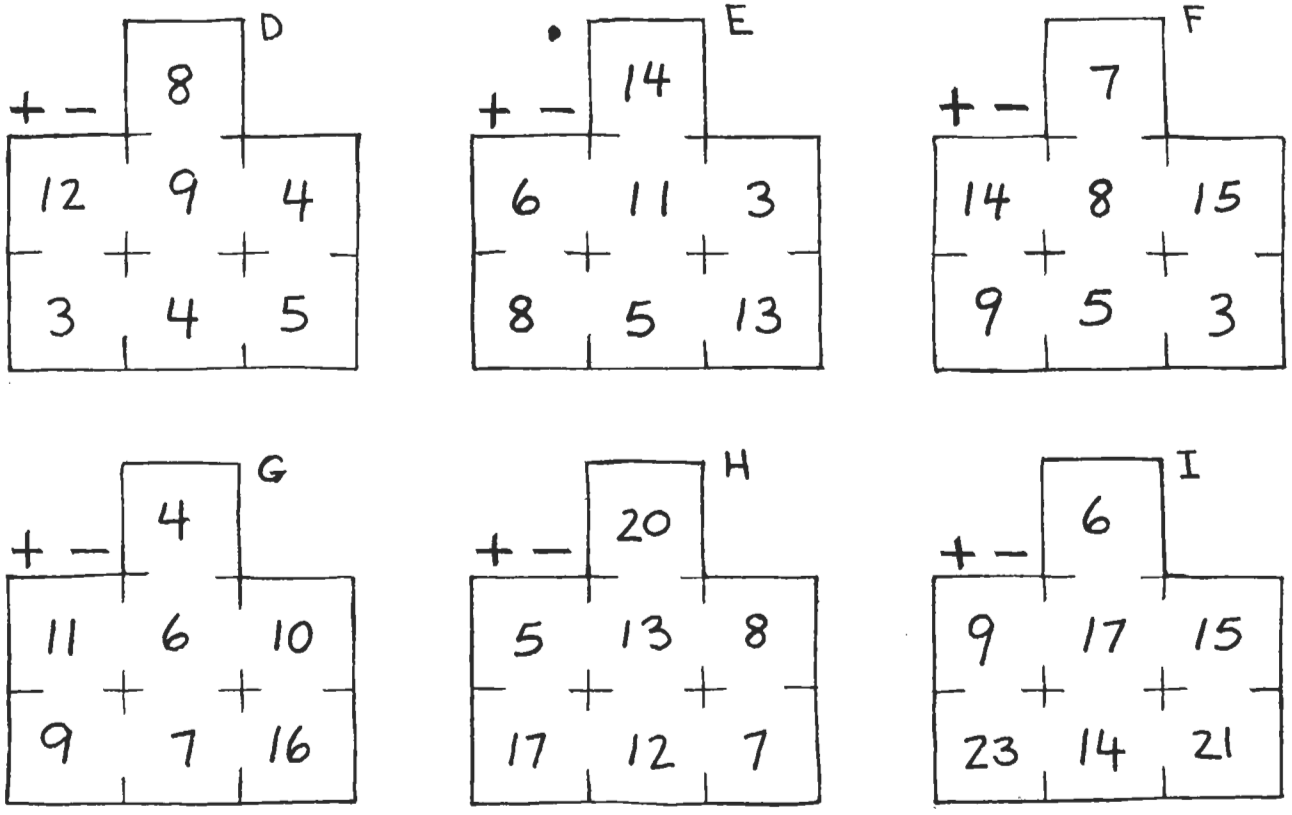
adición y sustracción

HIDDEN PATHWAYS
addition and subtraction





¿Puedes hacerlos todos? Can you do all of them?



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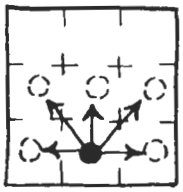
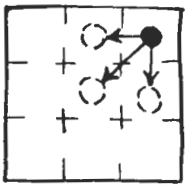
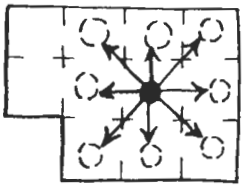
HIDDEN PATHWAYS ENCONTRANDO EL CAMINITO

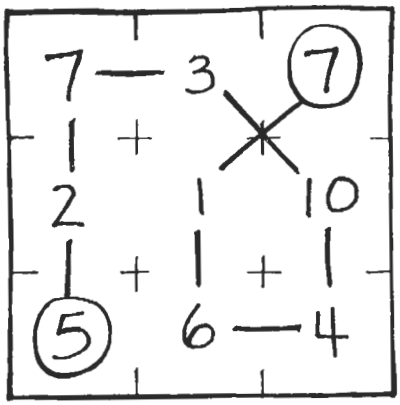
adición y sustracción

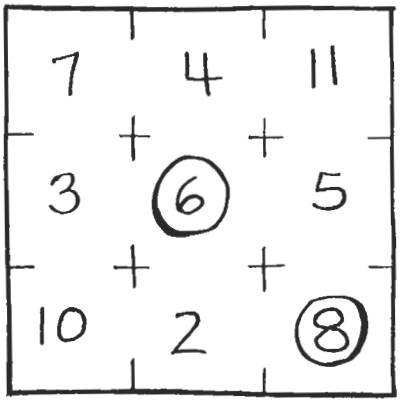
addition and subtraction

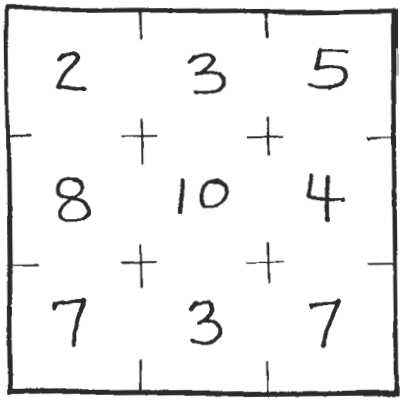




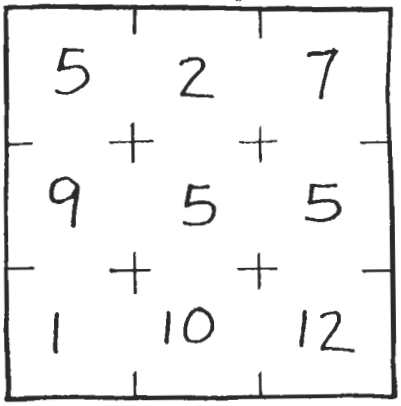


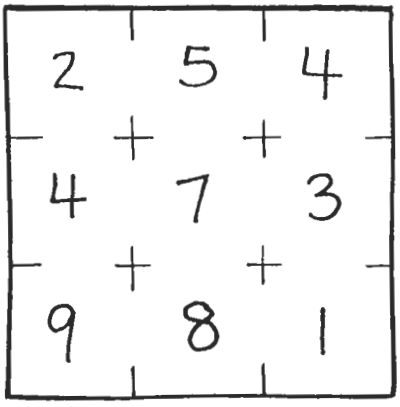


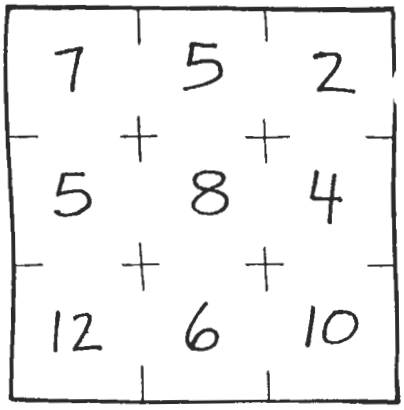


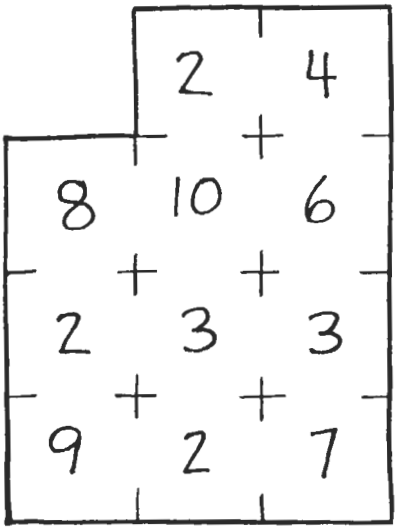


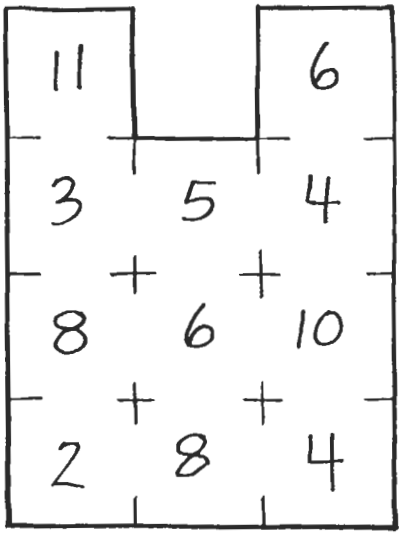
Can you do all of them? ¿Puedes hacerlos todos?

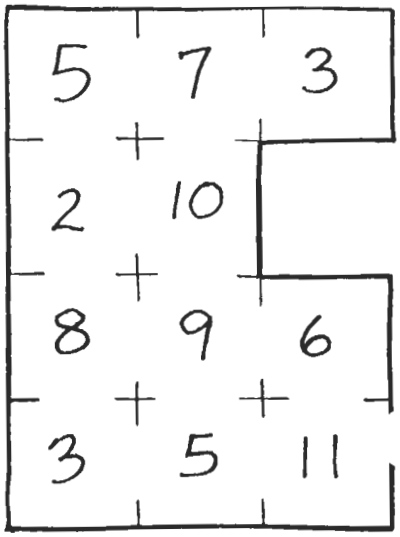












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HIDDEN PATHWAYS
multiplication and division

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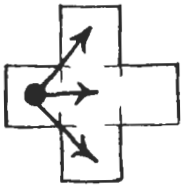
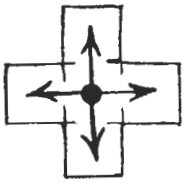
multiplicatión y división





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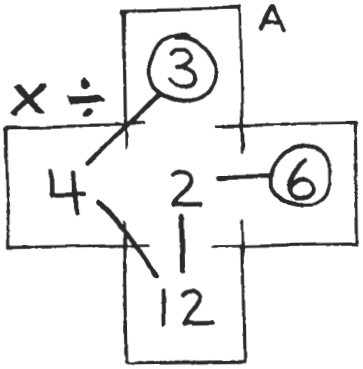

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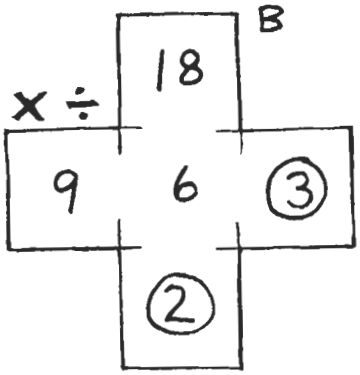

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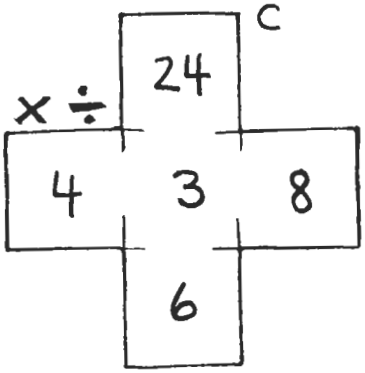
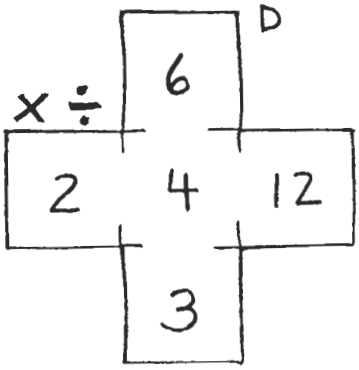


Can you do all of them? ¿Puedes hacerlos todos?

D



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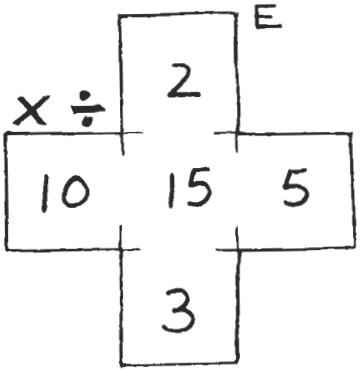

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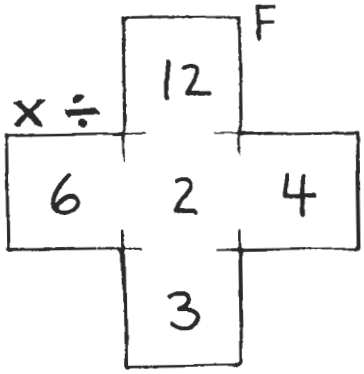

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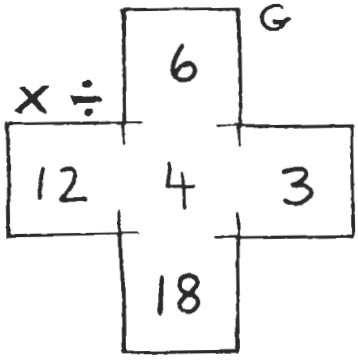

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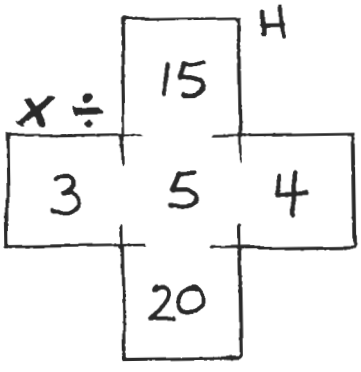

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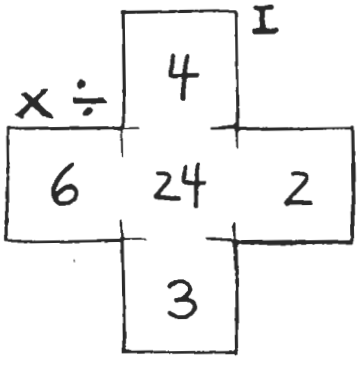

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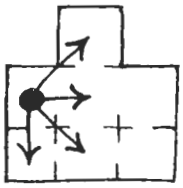
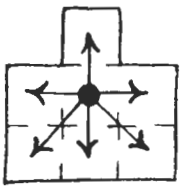
HIDDEN PATHWAYS
multiplication and division

ENCONTRANDO EL CAMINITO
multiplicación y división





A

× ÷	30	
10	4	5
24	3	6

B

× ÷	3	
10	6	18
5	2	9

C

× ÷	10	
2	3	4
20	5	30

Can you do all of them? ¿Puedes hacerlos todos?

D

× ÷	4	
18	3	2
3	6	12

E

× ÷	4	
8	5	6
3	24	20

F

× ÷	6	
36	4	6
4	9	24

G

× ÷	4	
24	6	30
3	8	5

H

× ÷	10	
4	7	5
35	40	8

I

× ÷	24	
3	72	2
8	9	7

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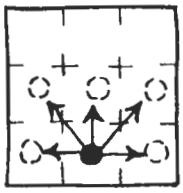
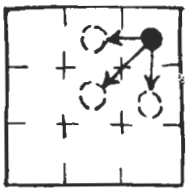
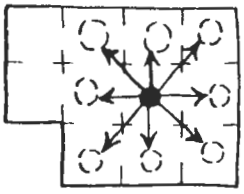
ENCONTRANDO EL CAMINITO HIDDEN PATHWAYS

multiplicatión y división

multiplication and division







A

4	2	10
20	5	3
5	6	30

B

18	3	6
2	2	3
9	4	12

C

18	2	9
3	6	36
9	4	6

¿Puede hacerlos todos? Can you do all of them?

D

6	2	2
3	4	12
2	4	8

E

7	4	28
2	14	28
14	7	2

F

12	2	6
4	3	15
2	3	5

G

	2	9
18	3	3
3	6	36
9	4	6

H

3		30
7	10	2
21	15	18
3	5	6

I

6	2	7
12	14	
4	24	12
8	3	2

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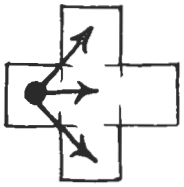
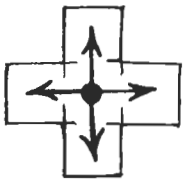
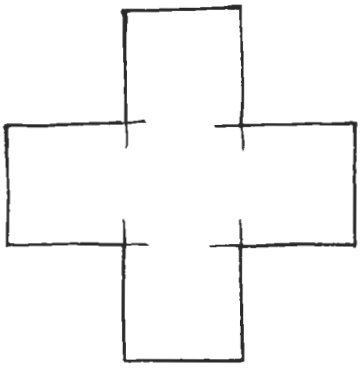
ENCONTRANDO EL CAMINITO

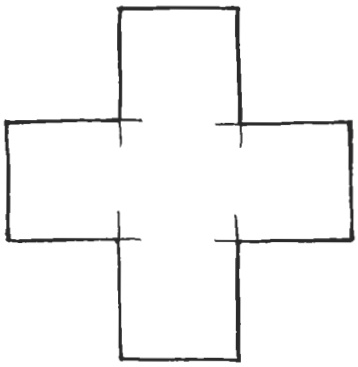
HIDDEN PATHWAYS

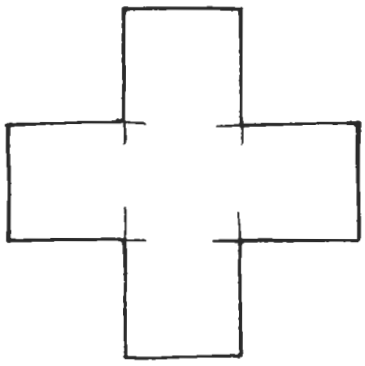
usando _____
using

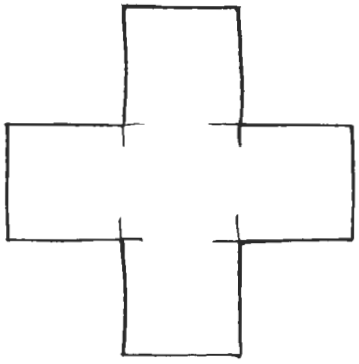


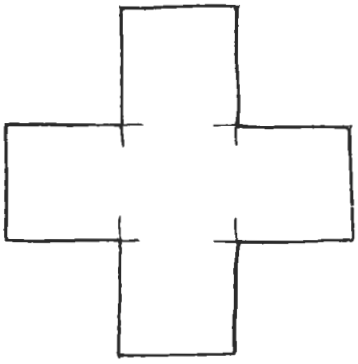


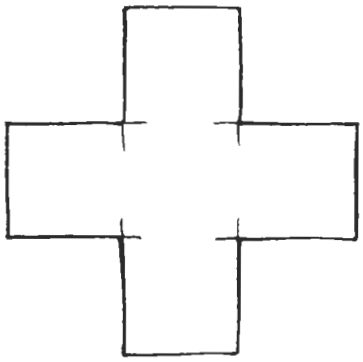


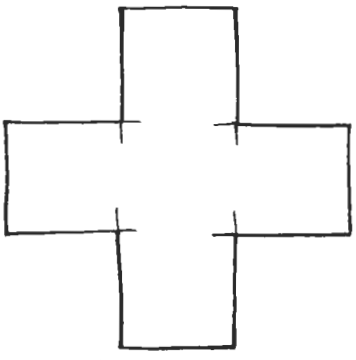


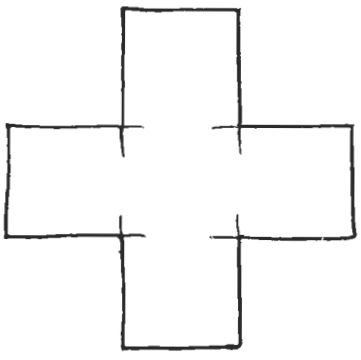


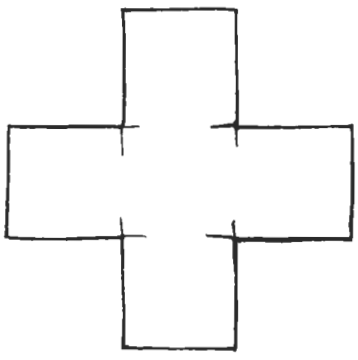












F20 and F23

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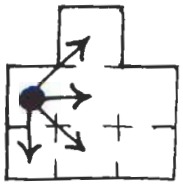
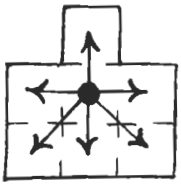
HIDDEN PATHWAYS

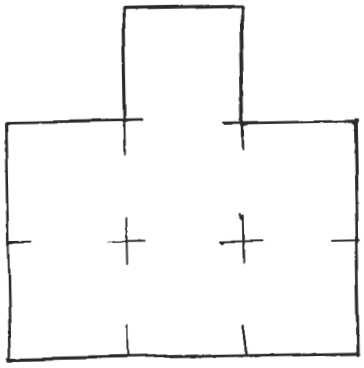
ENCONTRANDO EL CAMINITO

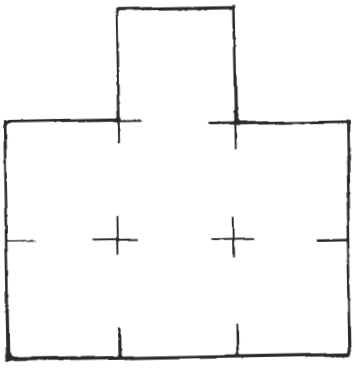
using _____

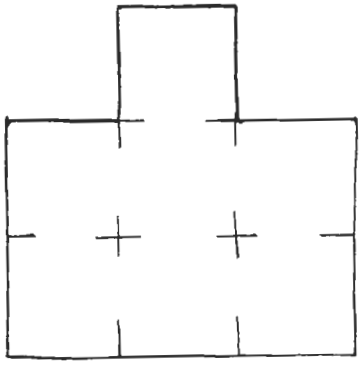
usando

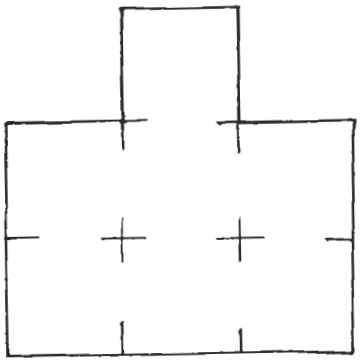


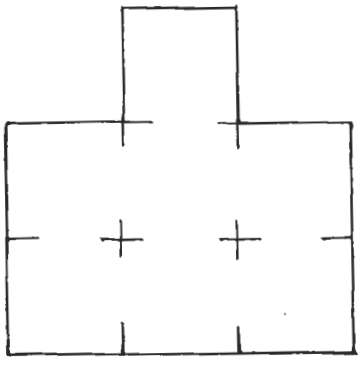


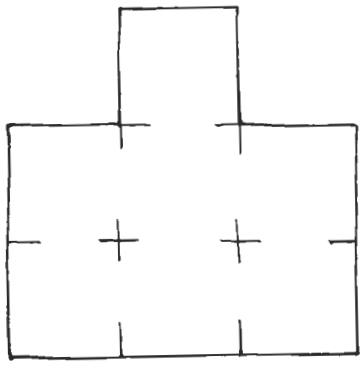


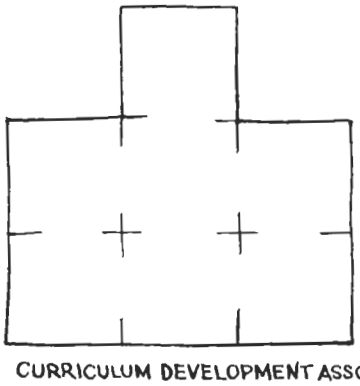


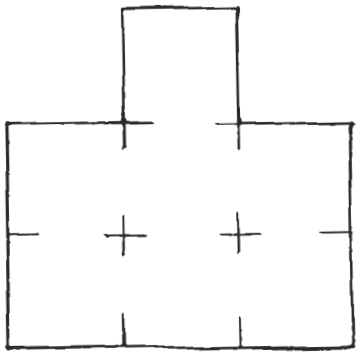


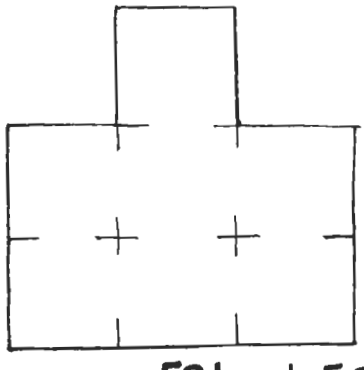












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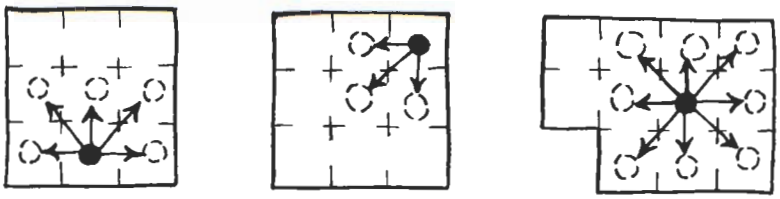
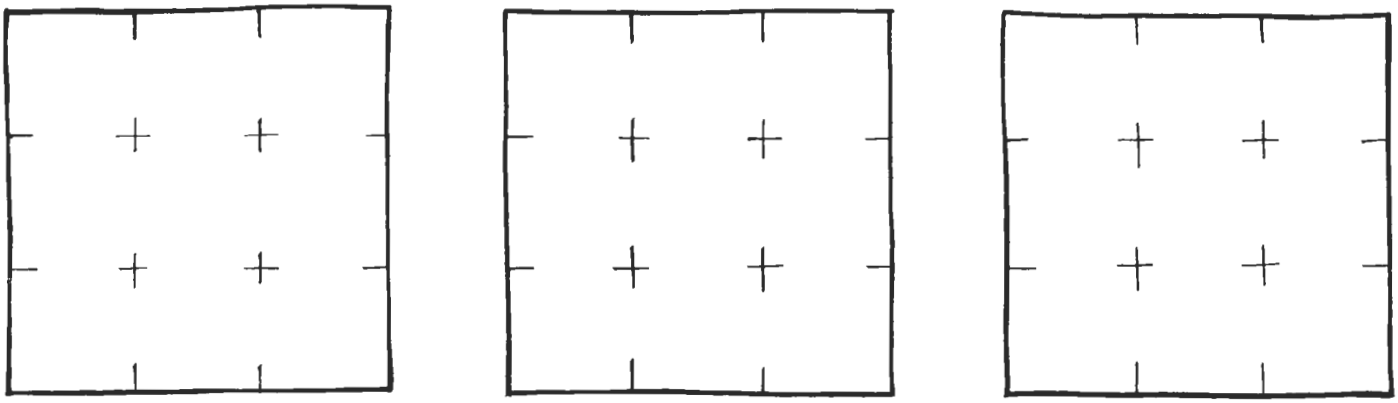
F21 and F24

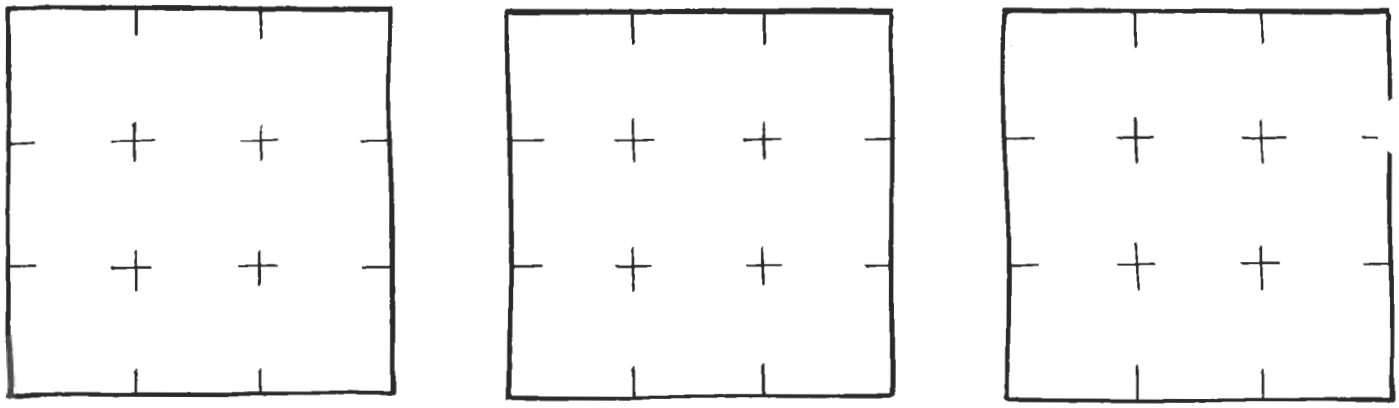
ENCONTRANDO EL CAMINITO

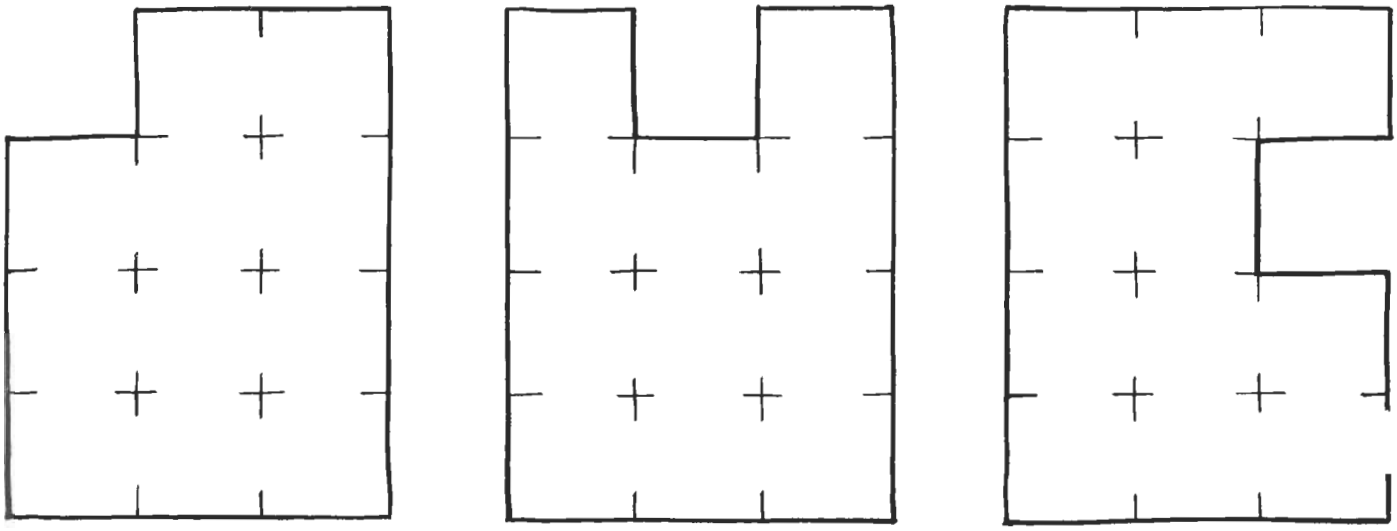
HIDDEN PATHWAYS

usando
using _____









F22 and F25

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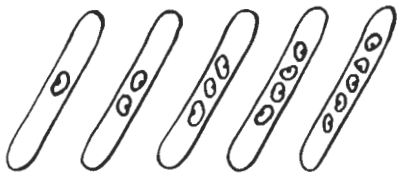
How many different combinations?
.... using a beanstick for 1, 2, 3, 4 and 5.

¿Cúantas combinaciones diferentes hay?
...usando un "patito de a diez" para 1, 2, 3, 4 y 5.



5 1 2 3 4 5 | 1 2 3 4 5 | 1 2 3 4 5

6 1 2 3 4 5 | 1 2 3 4 5 | 1 2 3 4 5

7 1 2 3 4 5 | 1 2 3 4 5 | 1 2 3 4 5

8 1 2 3 4 5 | 1 2 3 4 5 | 1 2 3 4 5

9 1 2 3 4 5 | 1 2 3 4 5 | 1 2 3 4 5

10 1 2 3 4 5 | 1 2 3 4 5 | 1 2 3 4 5

11 1 2 3 4 5 | 1 2 3 4 5 | 1 2 3 4 5

12 1 2 3 4 5 | 1 2 3 4 5 | 1 2 3 4 5

13 1 2 3 4 5 | 1 2 3 4 5 | 1 2 3 4 5

14 1 2 3 4 5 | 1 2 3 4 5 | 1 2 3 4 5

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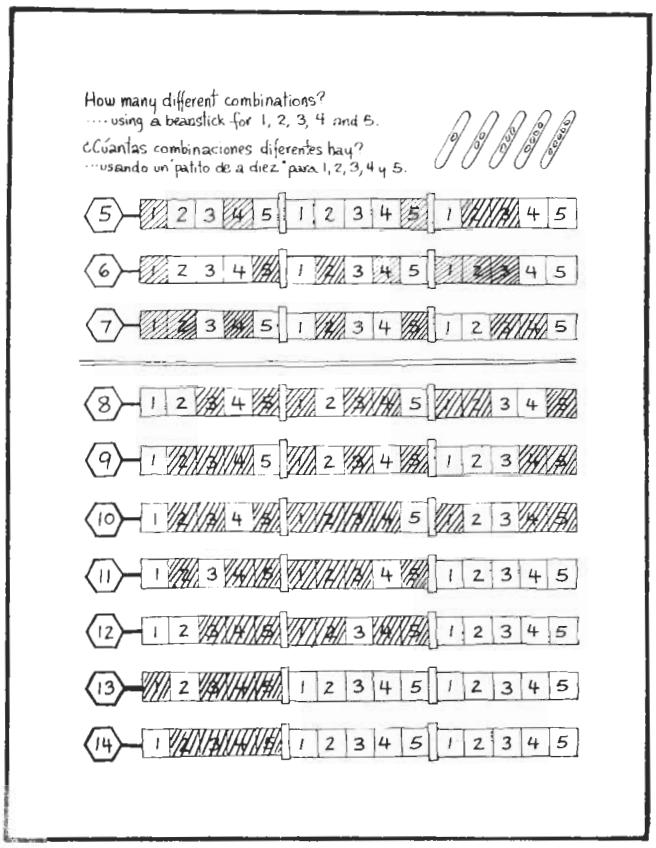


Introducing the Problem

Some children will prefer to use real beansticks for this activity. They will need a 1-stick, 2-stick, 3-stick, 4-stick and 5-stick. The number in the hexagon at the left of each row of numbers tells how many beans must be picked up. Any or all of the five beansticks can be used. Each block of numbers, 1 through 5, provides a place to record a combination of sticks the child used to pick up the given number of beans.

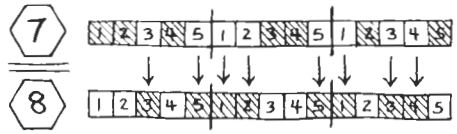
In the first example, we could get five beans by picking up a 1-stick and a 4-stick. Thus, we color in the "1" and the "4" in the first block of five numbers. But we could also get five beans by simply picking up the 5-stick. So, we color the "5" in the second block of five numbers. Is there another way to pick up five beans?

Talking about the Problem

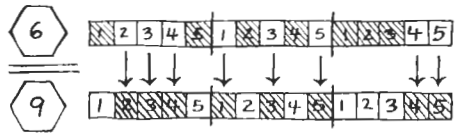
This problem is more difficult than it seems because it is rather easy to find *three* different combinations for the numbers, 5, 6, 7, 8, 9, and 10. But with 11, difficulty arises. Almost everyone expects to find *three* ways, but can't do it. Likewise with 12. With 13 and 14, only one way can be found in each case.

Extensions and Surprise Endings

Did anyone notice that the solutions for seven beans and eight beans can be seen as the same question?

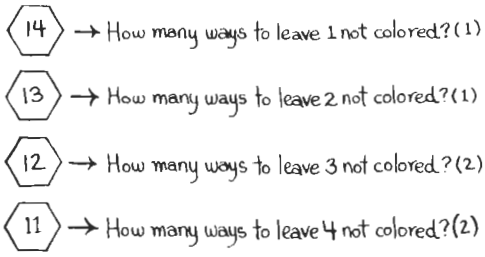


There is a total of fifteen beans. If you pick up seven, then you leave eight down. So, to show combinations for eight, color in what you didn't color in for seven. For nine, all you need to do is reverse the coloring for six.



For ten, simply reverse the coloring scheme for five.

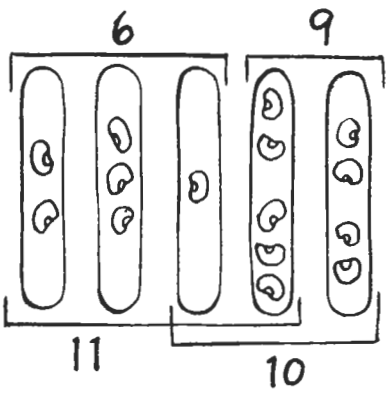
Really, the easiest approach to this problem would be to start from the bottom of the page and work up.



How would the problem change if we started out with just four beansticks: the 1-, 2-, 4-, and 8-sticks? Amazing! The results would reveal that:

(a) there is a stick or a combination of sticks for all numbers one through fifteen, and
(b) there is only one stick or combination for any number one through fifteen.

There is one and only one way to pick up one through fifteen beans. There is no other combination of four sticks for which statements (a) and (b) are true.



2 3 1 5 4 _ 6 ~~7~~ ~~8~~ 9 10 11 ~~12~~ 13 ~~14~~ 15

_ 6 7 8 9 10 11 12 13 14 15

1 3 5 2 4 _ 6 7 8 9 10 11 12 13 14 15

3 4 2 1 5 _ 6 7 8 9 10 11 12 13 14 15

_ 6 7 8 9 10 11 12 13 14 15

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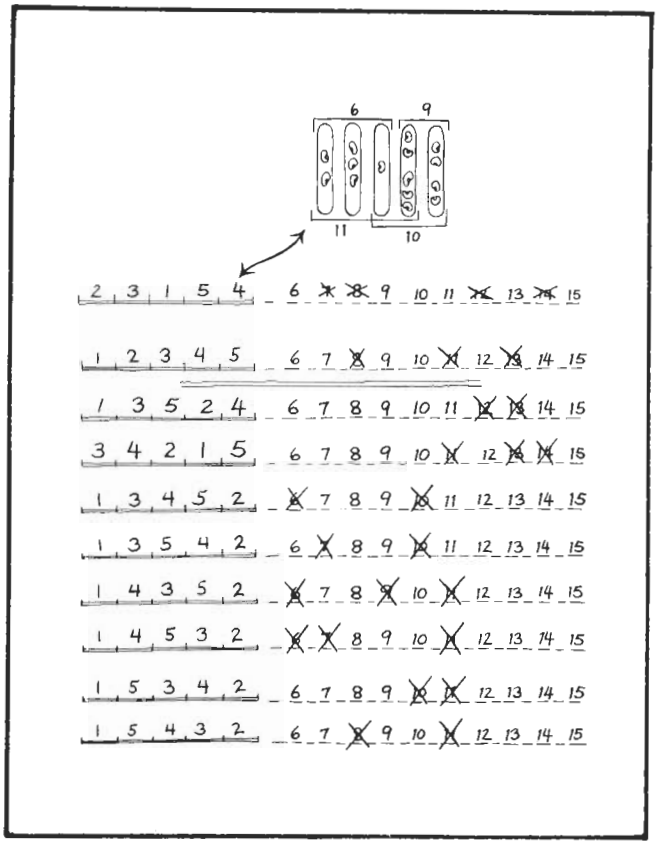
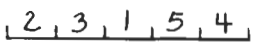


Introducing the Problem

Children may feel most comfortable in this activity if they have their own set of beansticks. They will need a 1-stick, 2-stick, 3-stick, 4-stick, and 5-stick. Beansticks should be arranged side by side, as shown in the diagram on the activity page. The arrangement should be recorded as shown in the example.

The object of the game is to find two or more beansticks whose beans add up to 6; then two or more whose beans add up to 7, then to 8, to 9, and so on for each number through 15. However, the beansticks chosen *must be neighbors;* that is, they must be next to one another. Children vary the arrangements of sticks and for each arrangement, they try to find a combination for each number, 6 through 15. If a number cannot be done, that number should be circled or crossed out in the record. With the arrangement shown in the diagram at the top of the activity page,



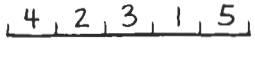
there is no way to show *neighbors* for 7, 8, 12, or 14.

As they work at the problem, children should be on the lookout for an arrangement of the beansticks that will allow them to do all the numbers, 6 through 15.

Talking About the Problem

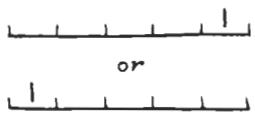
(1) Did anyone find an arrangement that allowed all numbers to be done? An arrangement that allowed all but one? An arrangement that allowed all but two? There is no arrangement such that all numbers can be done. Nor is there an arrangement that allows all numbers but one. Several arrangements can be found that allow all but two numbers to be done.

(2) For what arrangement did you have the most numbers crossed out? The arrangement

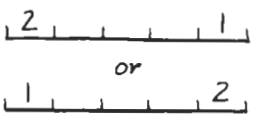


has no combination of neighbors for five numbers — 7, 8, 12, 13, and 14.

(3) Suppose we wanted to arrange the sticks so that we could get combinations for as many numbers, 6 through 15, as possible. How could we do this? We might start by figuring out how the sticks should be arranged to allow 15, then to allow 14, to allow 13, and so on. Since there are 15 beans all together on the five sticks, any arrangement will allow you to do 15. Fourteen is a bit more difficult. To do 14, we have to be able to leave just one bean out. Therefore, the 1-stick has to be at one of the ends of the arrangement. So the plan begins to take shape:

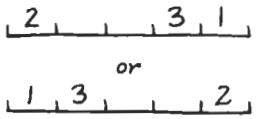


To get 13, we have to be able to leave 2 beans out. Thus, the 2-stick must be on the other end of the arrangement. So we have

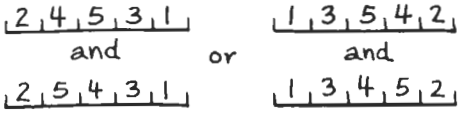


With the 1-stick and the 2-stick on the ends, the three remaining sticks (the 3-, 4-, and 5-sticks) will be neighbors and will add to 12, no matter how they are arranged (3 + 4 + 5 = 12).

So, we go on to look at 11. To show 11, we must leave 4 beans out. Thus, we have to put the 3-stick next to the 1-stick:



Now we have the 4-stick and the 5-stick left, and these can be placed in two different ways, leading to four possible arrangements:



None of these combinations can show 10 beans as neighbors. It is also impossible to show seven beans with either of the top two arrangements, or six beans with either of the bottom two. There will always be at least two numbers that can't be shown.

(4) What would happen if you could use any five beansticks and they didn't have to be all different? The arrangement:

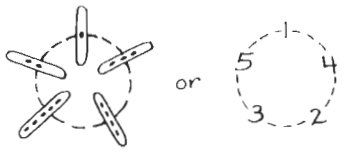
1, 1, 7, 3, 3

can show all numbers of beans, 6 through 15:

6 = 3 + 3	11 = 1 + 7 + 3
7 = 7	12 = 1 + 1 + 7 + 3
8 = 1 + 7	13 = 7 + 3 + 3
9 = 1 + 1 + 7	14 = 1 + 7 + 3 + 3
10 = 7 + 3	15 = 1 + 1 + 7 + 3 + 3

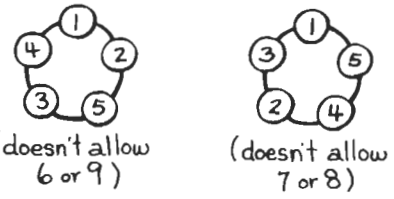
Extensions and Surprise Endings

Suppose the sticks were arranged in a circle:

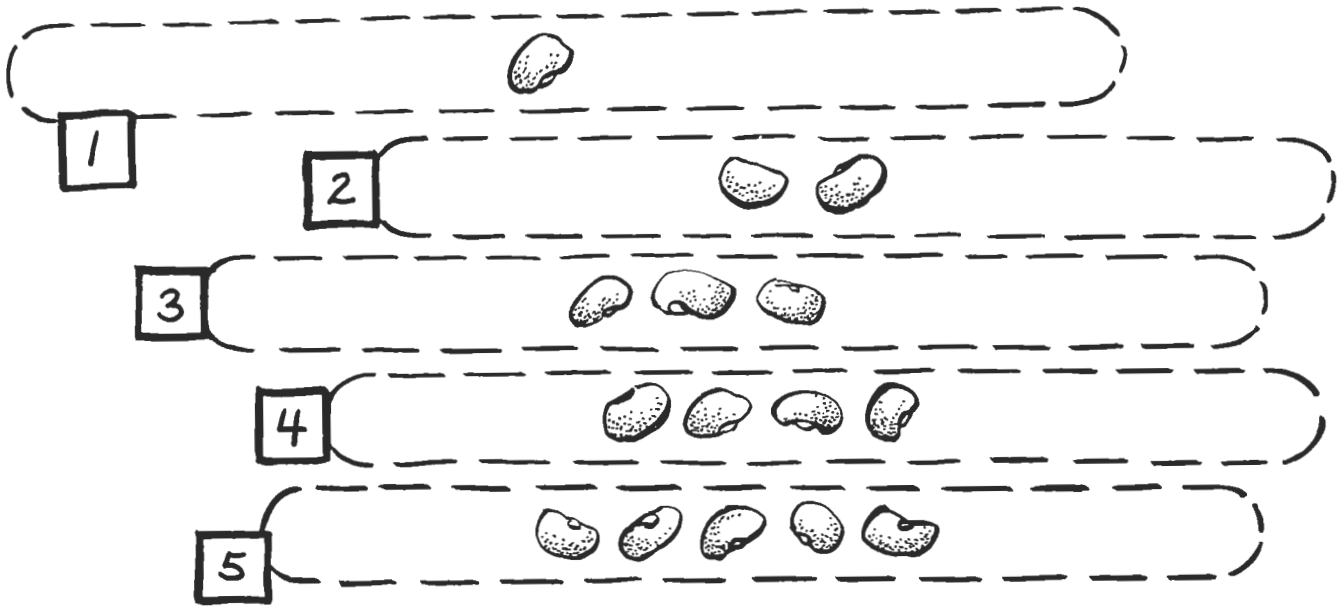


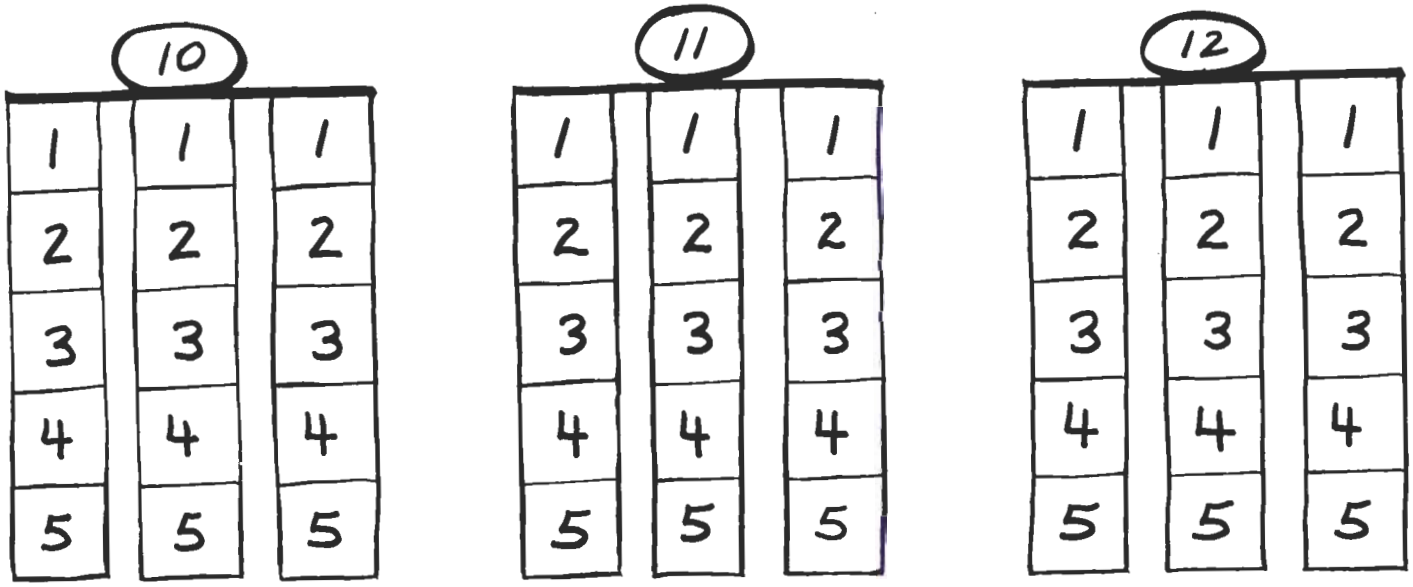
Most circular arrangements allow you to show all numbers, 1 through 15, as individual sticks or neighbors around the circle. There are just twelve basically different circular arrangements, if we consider arrangements such as these to be the same:

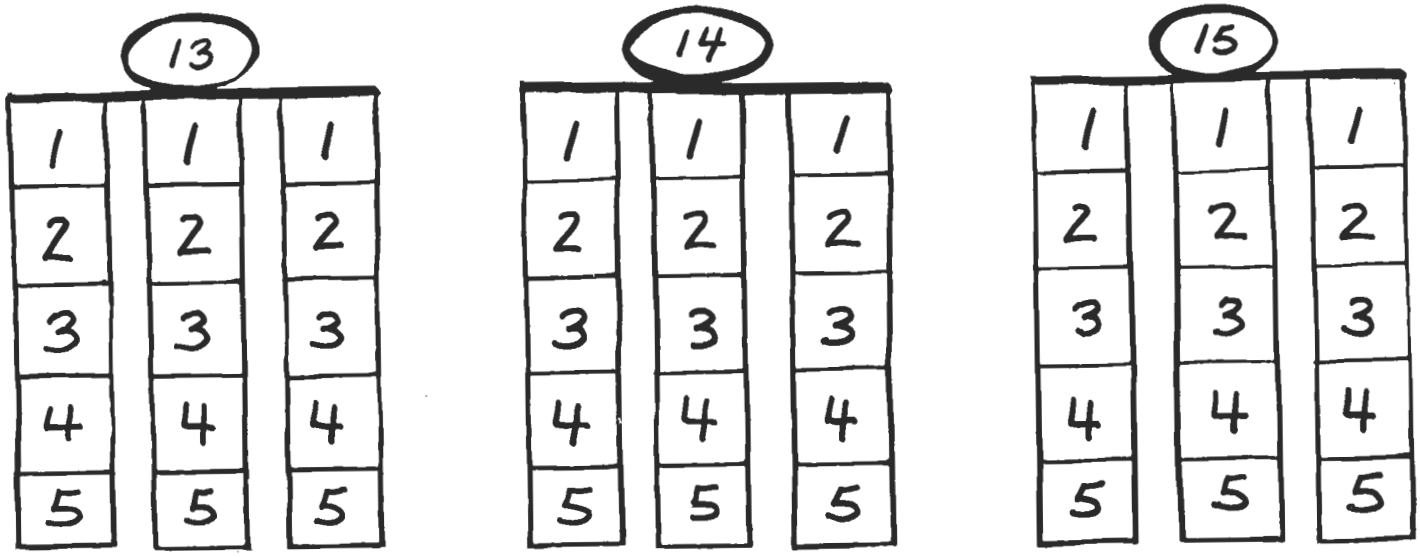
There are just two basic arrangements which do not allow all numbers to be done:



The activity on the reverse side of this page is taken from *Drill and Practice at the Problem Solving Level* (Curriculum Development Associates, 1974)







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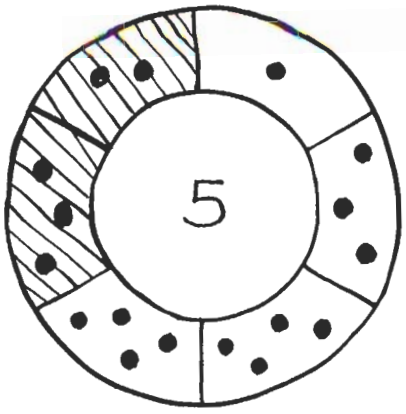

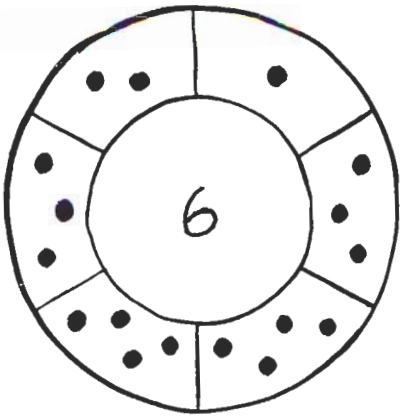

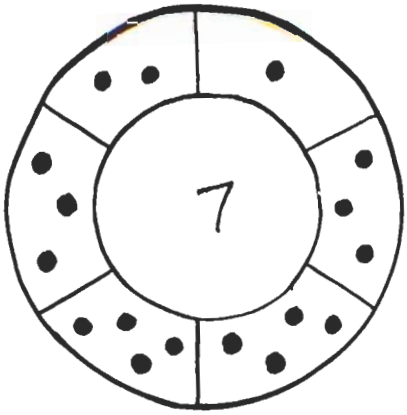

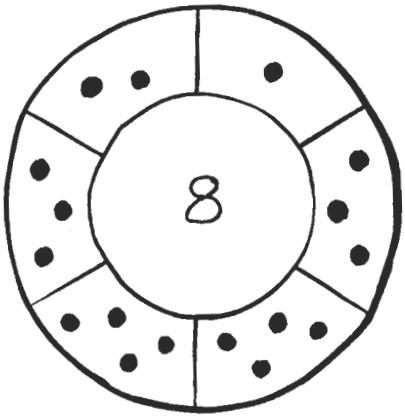

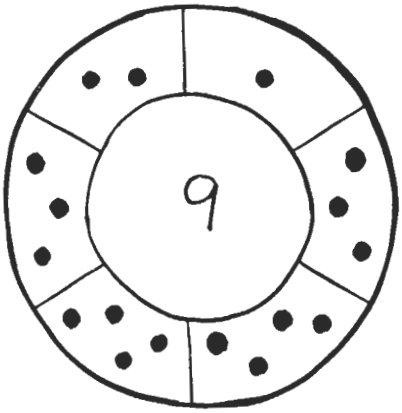

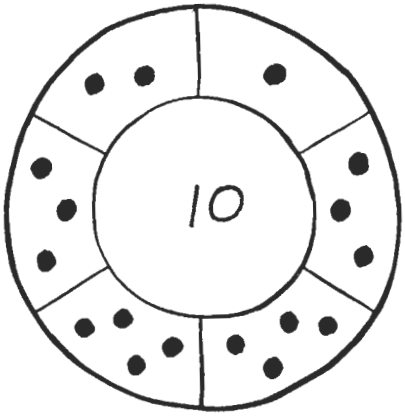

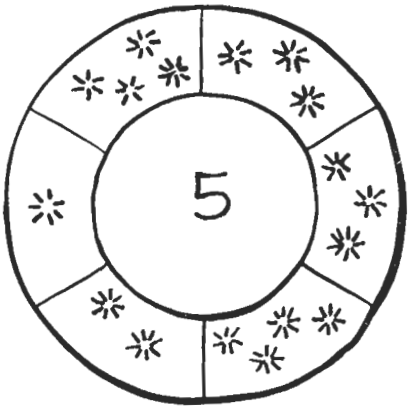

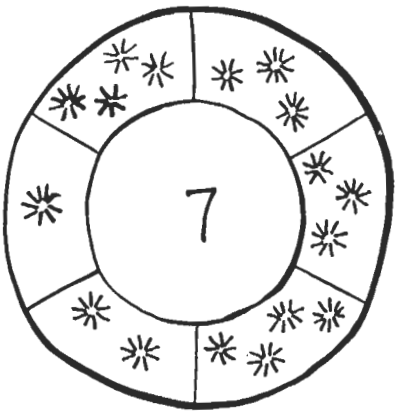

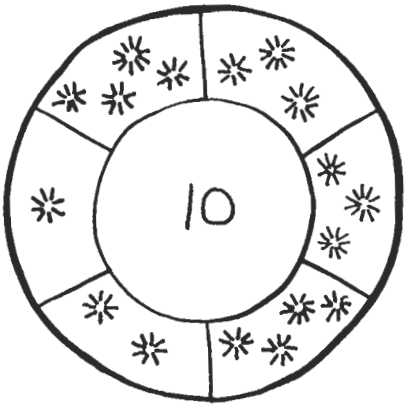

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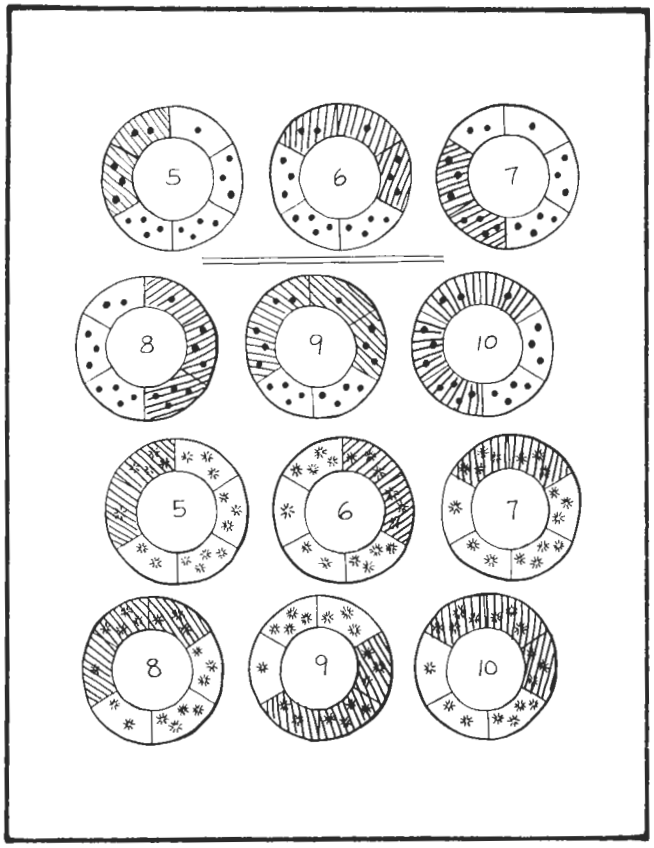
(Some of the appropriate responses are shown here.)



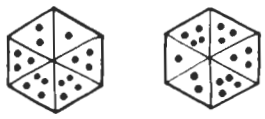
Introducing the Problem

On this page circles are pictured containing sections that have various numbers of dots. Neighboring sections (sections that share a side) must be colored which have a total of dots equal to the number given in the center of the circle. This activity is similar to a beanstick activity in which a 1-stick, a 2-stick, two 3-sticks and two 4-sticks are arranged in a circle as shown below:

Five beans can be shown with the 3-stick and the 2-stick that are neighbors. No two neighbors have a total of six beans, but the neighbors with two, one, and three beans have a total of six beans. The pair of neighbors with three and four beans will have a total of seven. The two sticks with four beans on each will have a total of eight beans, as will the combination of neighbors with four, three and one beans. There are two combinations with nine beans; one of them uses three neighbors and the other uses four neighbors. There are two combinations with ten beans; one with three and the other with four neighbors.

This same problem can be presented in still a different way, using a set of six cardboard triangles. The triangles should have sides that are all the same length. Counters can be glued onto the triangles.

These loose cards can be put into the arrangement suggested by the problems that are given on this activity page.



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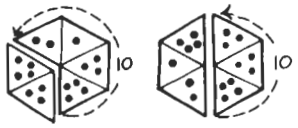
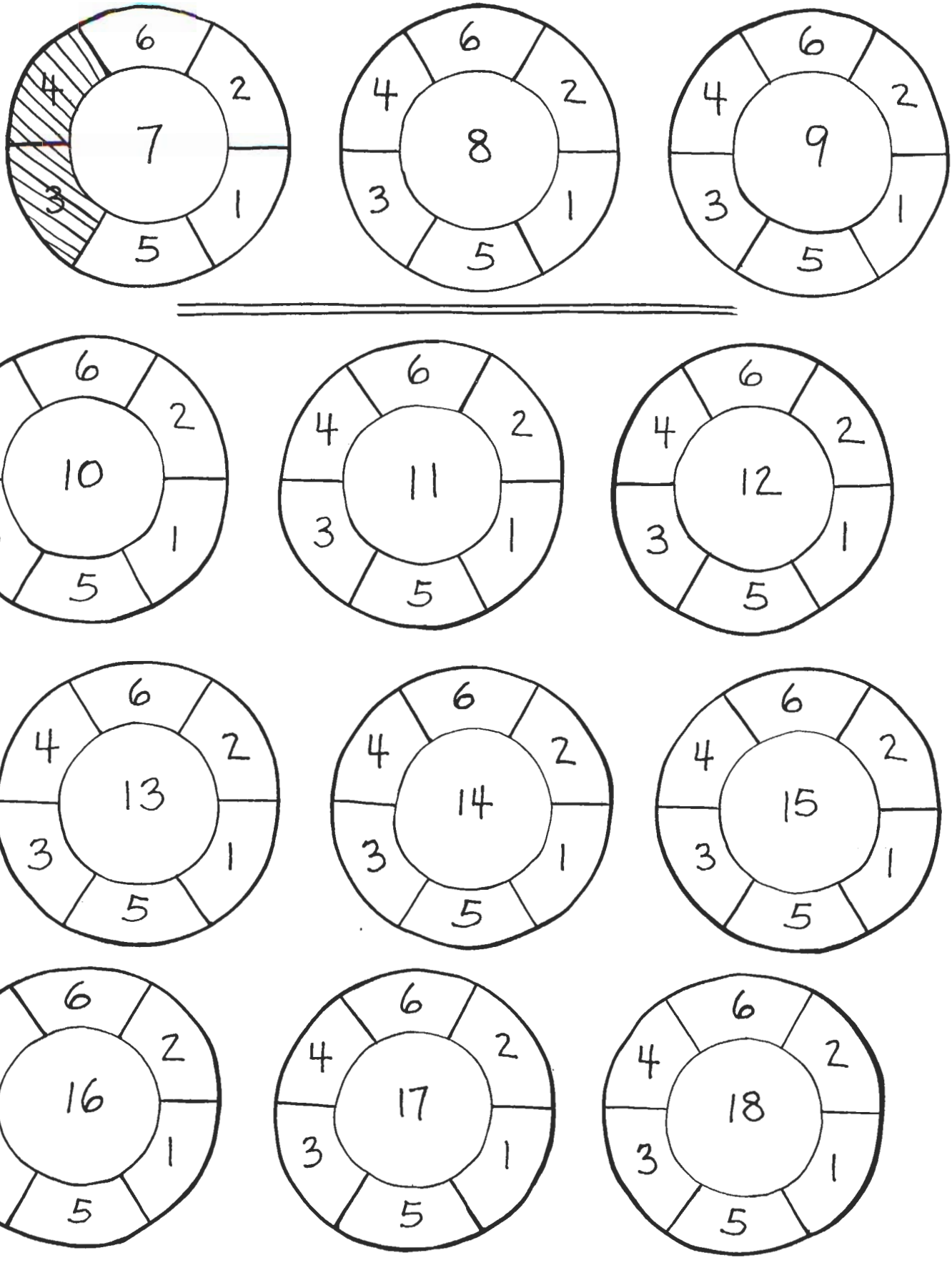
Using the cards, groups of neighbors can be pulled out of the configuration and considered more easily. For example, to get ten counters,



Talking about the Problem

If you consider the arrangement shown in the first six examples on the page, how many different sums can be shown using just two neighbors? We can show five sums: 3, 4, 5, 7 and 8. Which sums can be shown with just two neighbors in the other arrangement shown on this page? We can show four sums: 3, 5, 6 and 7.

How many dots are there in each circle in the first two rows on the page? There are 17. How many in each circle in the last two rows on the page? There are also 17. The same numbers of dots are included on all of the circles. They are simply arranged differently.



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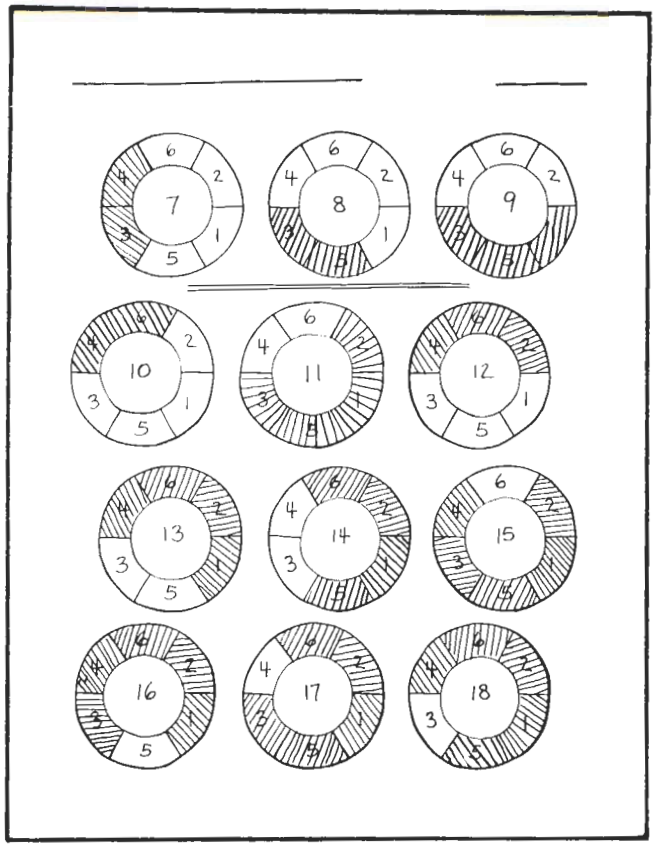
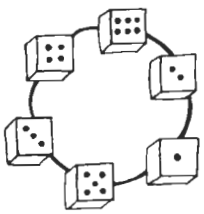
(Some of the appropriate responses are shown here.)



Introducing the Problem

If children have worked with the activity on the previous page, this one will need little introduction. Once again, the problem is to color in neighboring sections in the circles which have numbers that sum to the number in the center of the circle.

You may want to introduce the activity for some children in another setting. Six dice can be arranged in a circle so that 1, 2, 3, 4, 5 and 6 are represented in the same order as they are shown on the wheels on this page.



Can we find combinations of neighboring dice to represent all numbers, 7 through 18? If there are two possible ways to make a number, list both of them.

Talking about the Problem

What numbers can be shown with more than one combination? The numbers 8, 9, 12 and 13 can be shown in more than one way. Are there combinations for numbers that are larger than 18? Yes, combinations can be found for 19, 20 and 21.

Extensions and Surprise Endings

If we were allowed to use just one section of the circle at a time, how many numbers could we make? We could make just six: 1, 2, 3, 4, 5 and 6. If we could use two sections at a time, how many numbers could we make? We could make five: 3, 6, 7, 8 and 10. Let's compile this information in a chart:

number of neighbors

0 ---- 0
1 ---- 1, 2, 3, 4, 5, 6
2 ---- 3, 6, 7, 8, 10
3 ---- 8, 9, 12, 13, 14, 15, 18
4 ---- 11, 12, 13, 14, 15, 18
5 ---- 16, 17, 18, 19, 20
6 ---- 21

Now suppose that we rearrange this information and look for interesting patterns.

number of neighbors

0 --- 0] 0+21=21
6 --- 21]

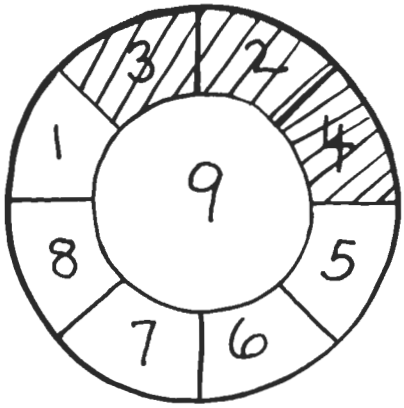
1 --- 1, 2, 3, 4, 5, 6] 6+15=21
5 --- 20, 19, 18, 17, 16, 15]

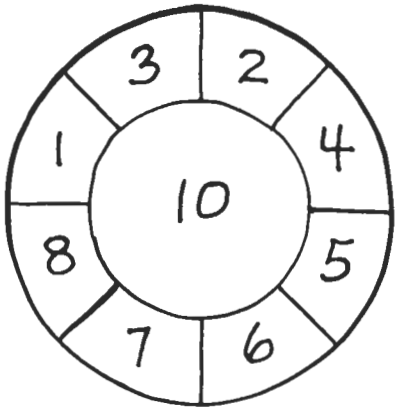
2 --- 3, 6, 7, 8, 10] 10+11=21
4 --- 18, 15, 14, 13, 11]

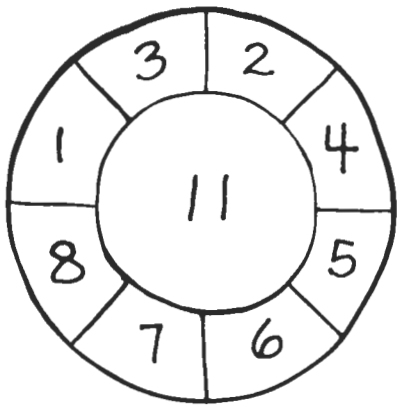
(3 --- 8, 9, 12, 13] 13+8=21
3 --- 13, 12, 9, 8]

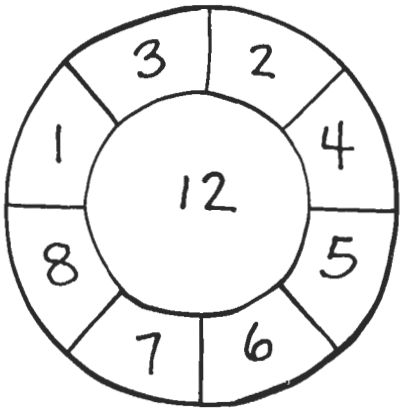
This result should not be too surprising. The sum of all the numbers in the circle is 21.

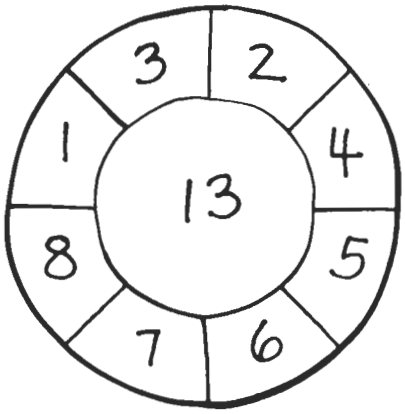
When one section or a combination of neigboring sections is selected to represent some number, then the neighbors that are left out must add up to 21 minus that number. For example, if a combination of neighbors is used to show 9, the remaining neighbors must add up to 12, since 21 - 9 = 12. If one section is colored, then five neighbors are not colored; if two are colored, then four are not. This is why the patterns emerged in the table we just created.

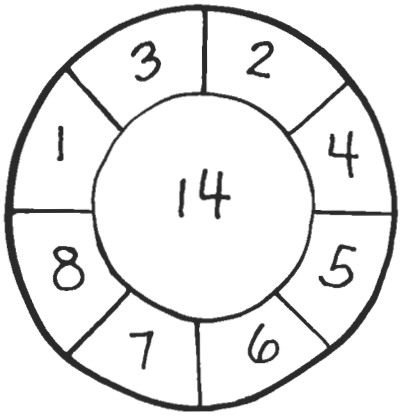


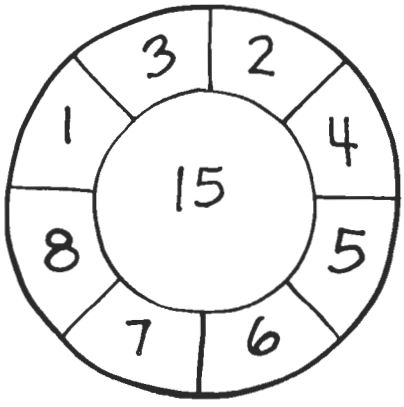


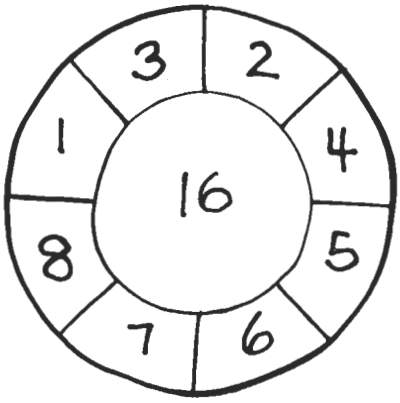


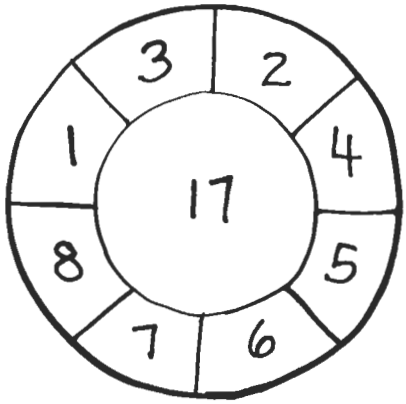


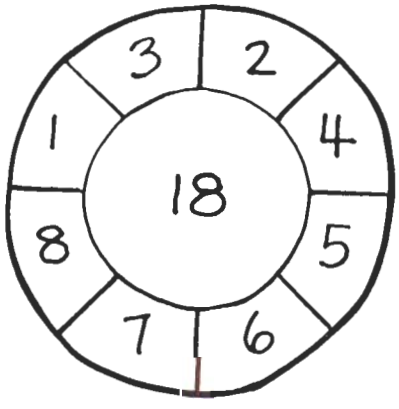


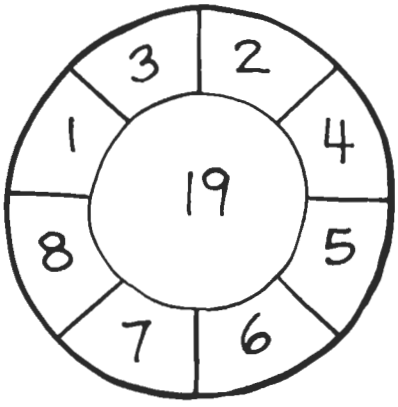


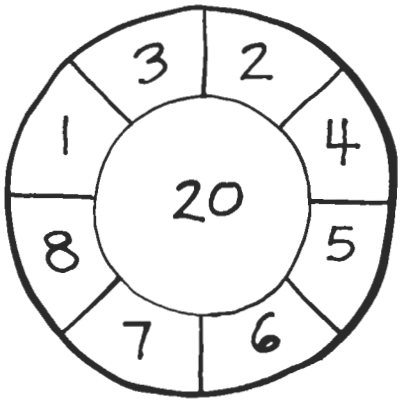












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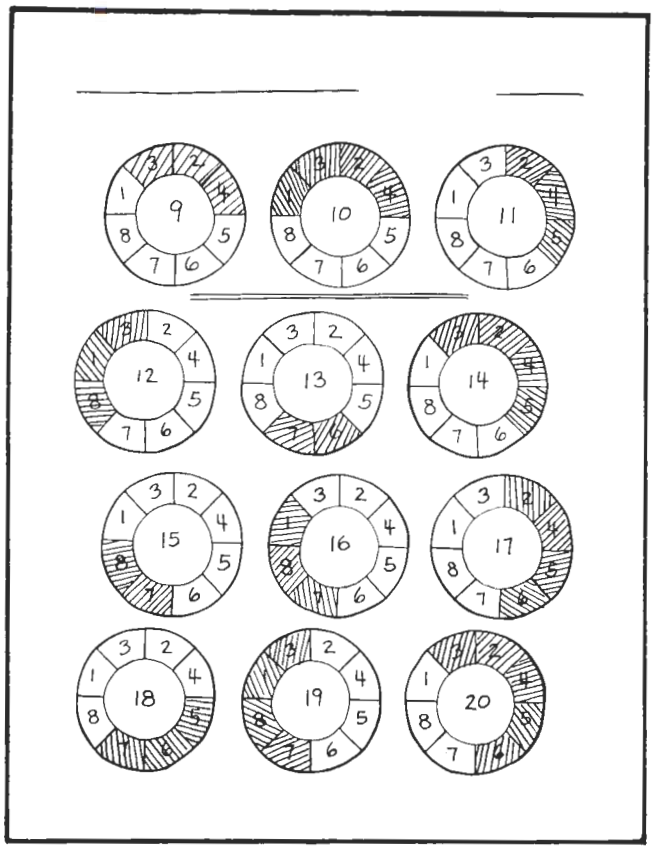
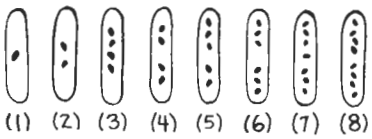
(Some of the appropriate responses are shown here.)



Introducing the Problem

This page is clearly a simple extension of the previous activities. We now have eight numbers around the circle, instead of six. All of these "Sums of Neighbors" activities grew out of "Beanstick Neighbors" activities, and you might want to use beansticks to introduce this activity to some of the children. Suppose we had eight beanstick "neighbors" living in a row in order from one to eight.



Can neighbors be found that have nine beans? Ten beans? Eleven beans? What is the smallest number of beans that cannot be shown with neighbors? Sixteen is the smallest. What other numbers up to and including 20 cannot be shown as neighbors? 19 is the only one that can't be shown. Of the numbers larger than twenty, 23, 29, 31, 32 and 34 cannot be shown. Of course, since there is a total of 36 beans, any number larger than 36 cannot be shown.

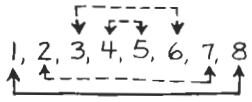
The activity on this page can be looked at as a record of a similar activity which would use eight beansticks in a circular arrangement. Notice that in this arrangement the 2-stick and the 3-stick have changed places. All the other numbers are in their natural order as you go clockwise around the circle. Neighbors which add to the number in the center of the circle are to be colored in.

Talking about the Problem

Could you find a combination for every number? Yes, there is one for each. Why do you think that the two and the three were switched around? It turns out that if the numbers were in their natural order, there would be no combination for 17. There would also be no way to do 19. Since the total of all the numbers is 36, if 17 cannot be done, neither can 19, for 36 - 17 = 19.

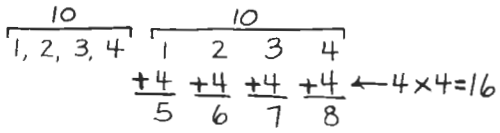
Extensions and Surprise Endings

Let's consider the numbers 1, 2, 3, 4, 5, 6, 7 and 8. Can you see an easy way to find the sum of all those numbers?



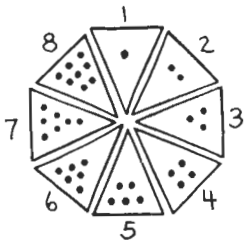
The sum of the first and last is 9. Moving in from both ends, we find three more sums of nine: 7 + 2 = 9, 6 + 3 = 9, and 4 + 5 = 9. Thus, we have four nines, and 4 × 9 = 36.

A person who bowls might recognize the 1, 2, 3 and 4 as suggesting the arrangement of the ten bowling pins. He might look at the problem in this way:

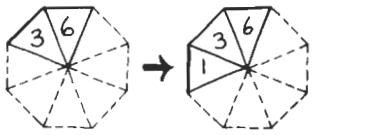


Then, 10 + 10 + 16 = 36.

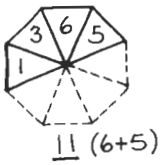
There is a game suggested by this activity that can be played with eight cardboard triangles. The triangles should be pieces of an octagon, and should have dots on them corresponding to the numbers one through eight.



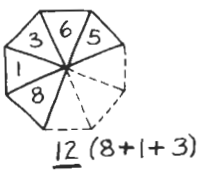
The problem is to build an octagon, piece by piece, so that combinations of neighbors can be found for all numbers, 9 through 36. As each new piece is added to the array, the next number or numbers must be able to be made. For example, suppose we start with the triangles that have three and six dots in order to make nine. Next we can add the triangle with one dot, so that there is a combination for ten. In the sketches below, we will use numerals to indicate the number of dots.



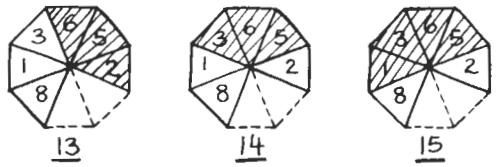
Now we need to provide a combination for 11. One way to accomplish that is to place the triangle with five dots next to the one with six.



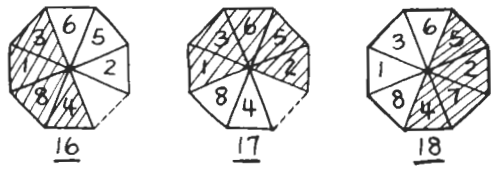
Next we need 12. If we place the triangle with eight dots next to the one with one, we can get 12 by using 8, 1 and 3.



Placing the triangle with two dots next to the one with five makes possible a combination for 13 (2 + 5 + 6 = 13). Without adding any more triangles at this point, we can also do 14 (5 + 6 + 3 = 14) and 15 (5 + 6 + 3 + 1 = 15).



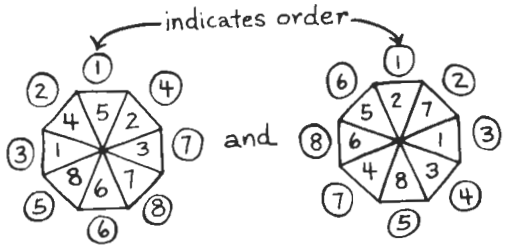
If we now place the triangle with four dots next to the one with eight, we will have combinations for 16 and 17. Completing the octagon with the triangle with seven dots makes possible a combination for 18.



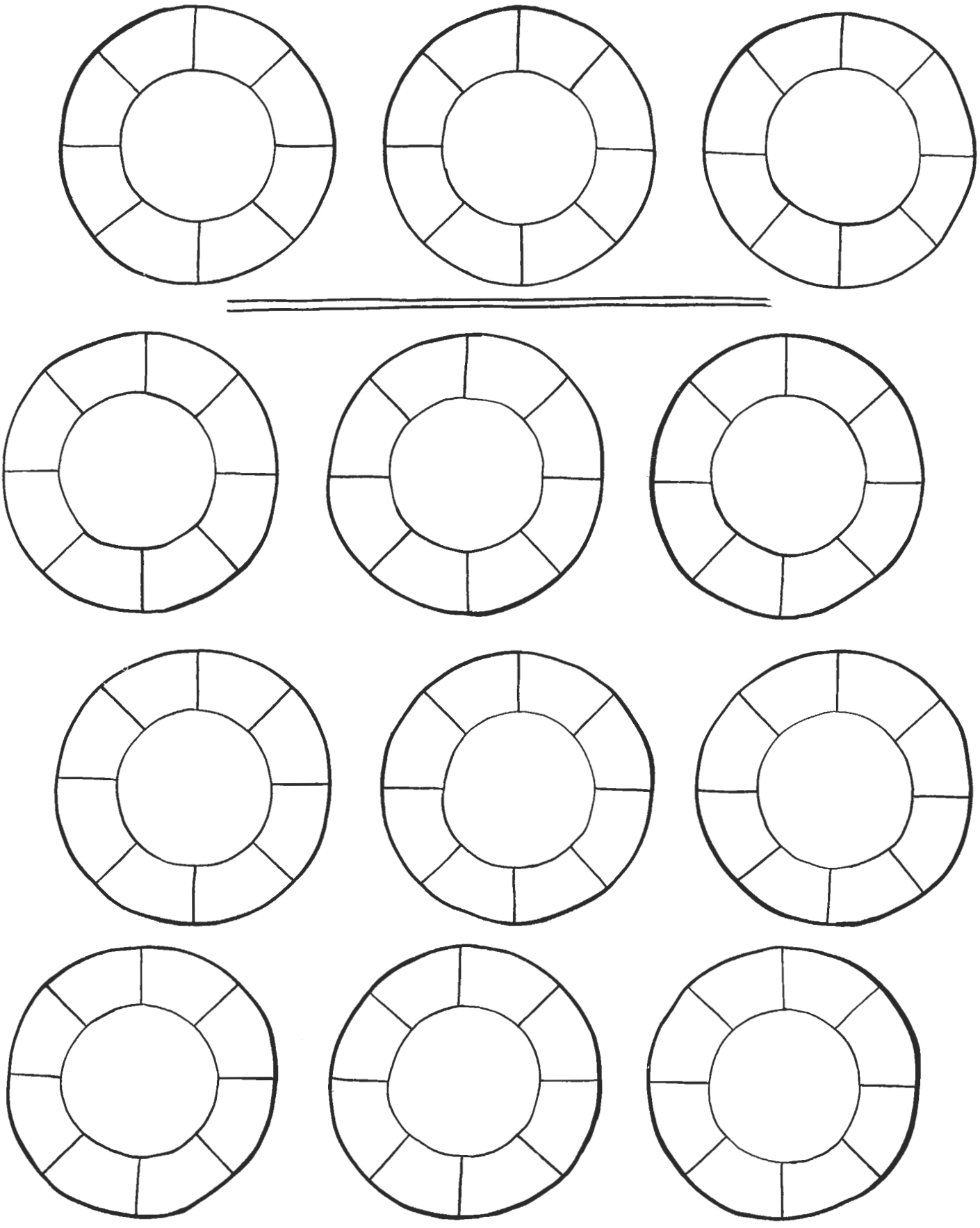
Now, is there a combination for all numbers, 19 through 36? Of course there is! To get 19, just use those triangles not used for 17 (since 36 - 17 = 19). To get 20, use those not included in the combination for 16, and so on.

Are there other solutions to this problem? There are several.

Here are two more. The numbers in the circles outside the octagon indicate the order in which the triangles were placed.



Variations on this problem can be easily worked out with six triangles that build into a hexagon. The numbers can be changed, or some numbers might even be repeated.



F30

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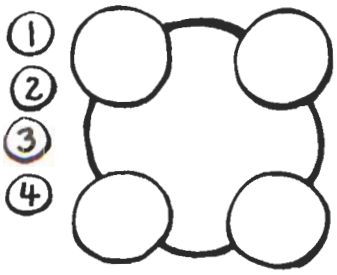
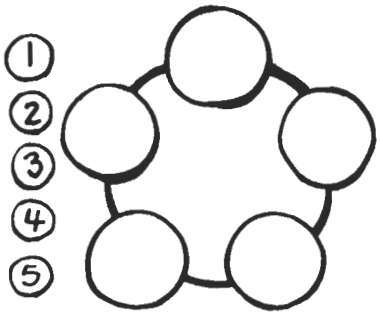
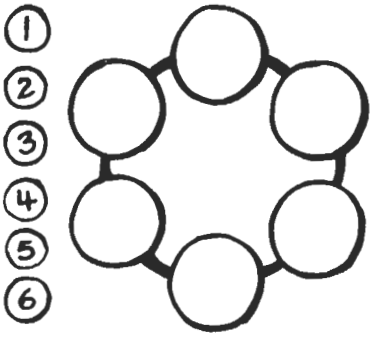


5=_____



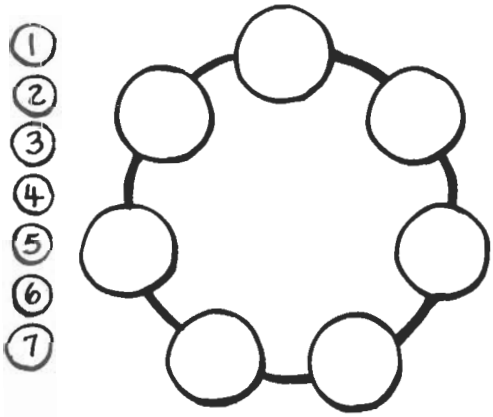
6=_____

7=_____



7=_____ 9=_____

8=_____ 10=_____

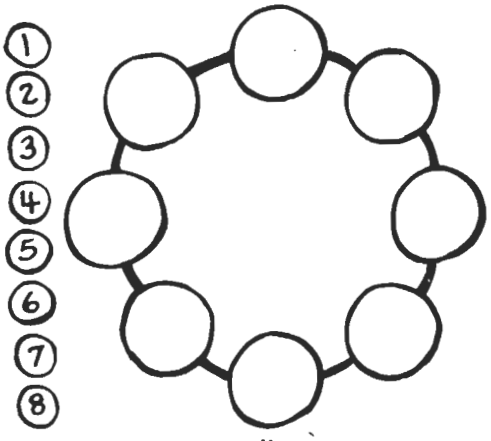


8=_____ 11=_____

9=_____ 12=_____

10=_____ 13=_____

14=_____



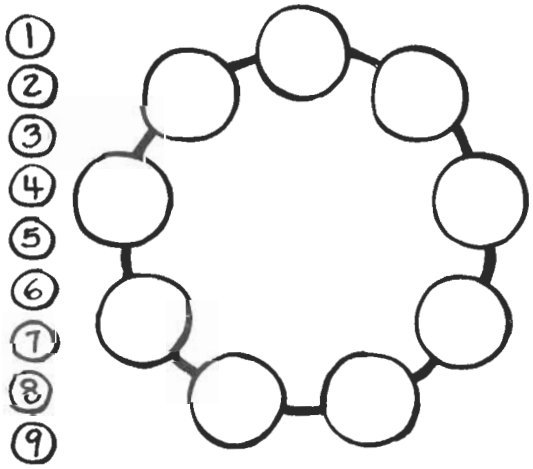
9=_____ 14=_____

10=_____ 15=_____

11=_____ 16=_____

12=_____ 17=_____

13=_____ 18=_____



10=_____ 16=_____

11=_____ 17=_____

12=_____ 18=_____

13=_____ 19=_____

14=_____ 20=_____

15=_____ 21=_____

22=_____

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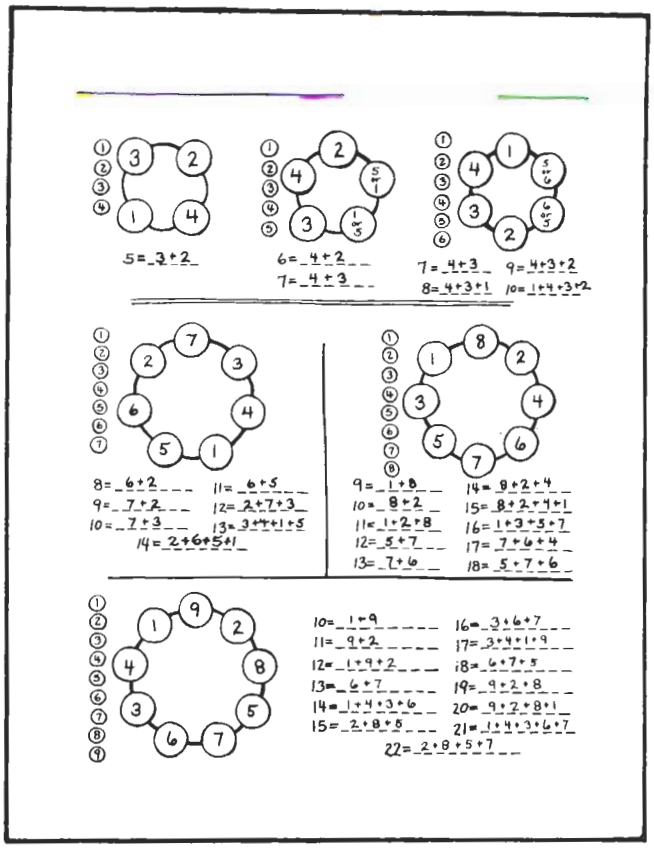
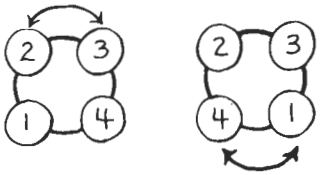
(Some of the appropriate responses are shown here.)



Introducing the Problem

Pat Washington, a teacher we know, designed this activity and made the following claim: "If you can take the numbers from the small circles and arrange them in the large circles in such a way that there are neighbors for each of the sums listed under the array, your arrangement will have neighbors for all numbers, one through the sum of all the numbers you used.

The problem is to find out whether Pat's claim is sound. Let's look at the first problem above the double line. Here are two basically different arrangements that would give us neighbors for five:



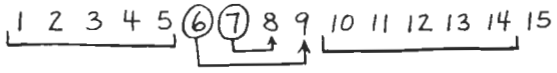
Now the sum of the numbers around the circle is 10. Can you show neighbors for all numbers 1 through 10? Yes; one, two, three and four can be easily shown by using one circle at a time. We started by arranging the circles to show five, so we know that five can be done. By leaving one of the four circles out at a time, you can show combinations for six, seven, eight and nine. The sum of all four circles is ten.

Many children will have more success if they use small circles of cardboard with numbers written on them. This encourages trial and error and eliminates the need for erasures if an unsuccessful start is made.

Talking about the Problem

Is it possible to arrange the numbers to show all of Pat's sums in each example? Yes, it can be done. What do you think of Pat's claim? It certainly seems to be true.

Consider the second example above the double line that uses the numbers one through five. The sum of these numbers is 15. We can look at the problem in this way:

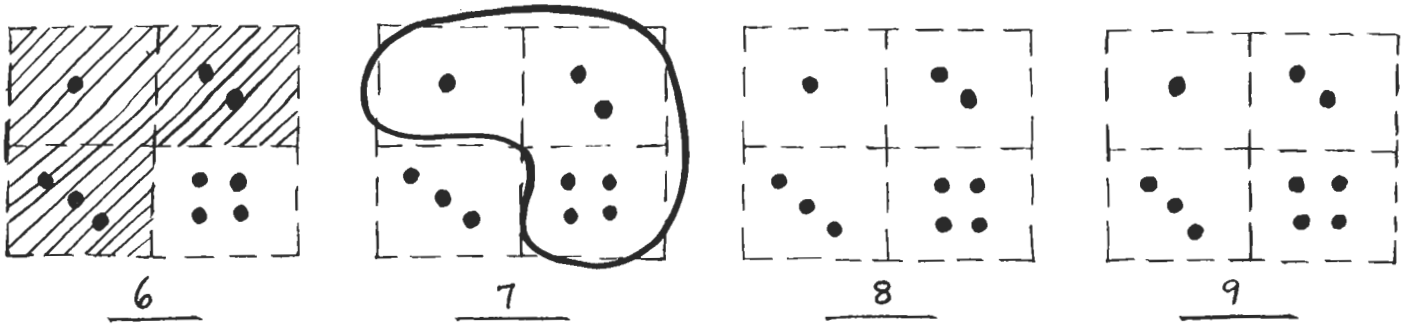


The numbers one through five can be shown by using one circle at a time. The numbers 10 through 14 can be shown by using all *but* one circle at a time: 15 - 5 = 10, 15 - 4 = 11, 15 - 3 = 12, 15 - 2 = 13, and 15 - 1 = 14. Fifteen can be shown by using all five circles. This cuts the problem down to considering 6, 7, 8 and 9. If there is a combination for 6, then those not included in that combination will be a combination for 9, since 15 - 6 = 9. If there is a combination for 7, then those not included will sum to 8, because 15 - 7 = 8. So really, we only need to insure that we can do 6 and 7 in order to conclude that we will be able to do all numbers, 1 through 15.

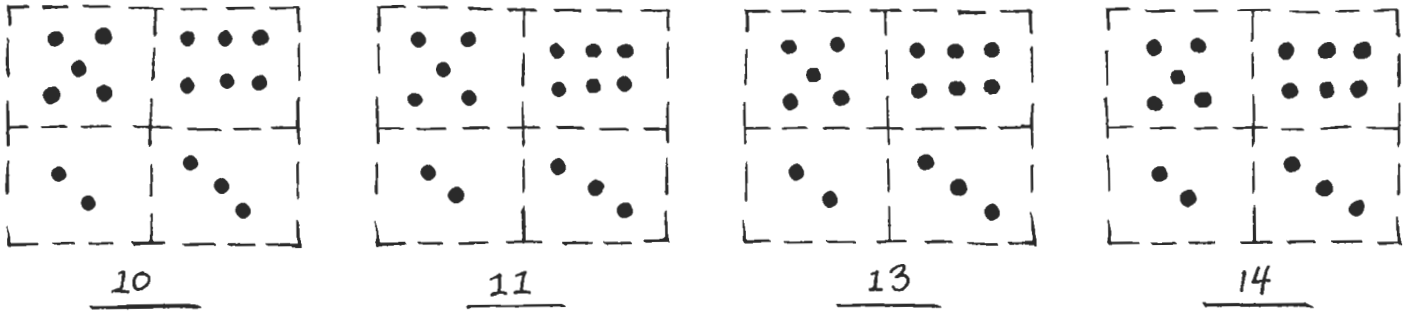
This is an example of a powerful mathematical strategy we might call "cutting the problem down to manageable size".

Encierra los tres cuadritos en una rueda o píntalos de un solo color.

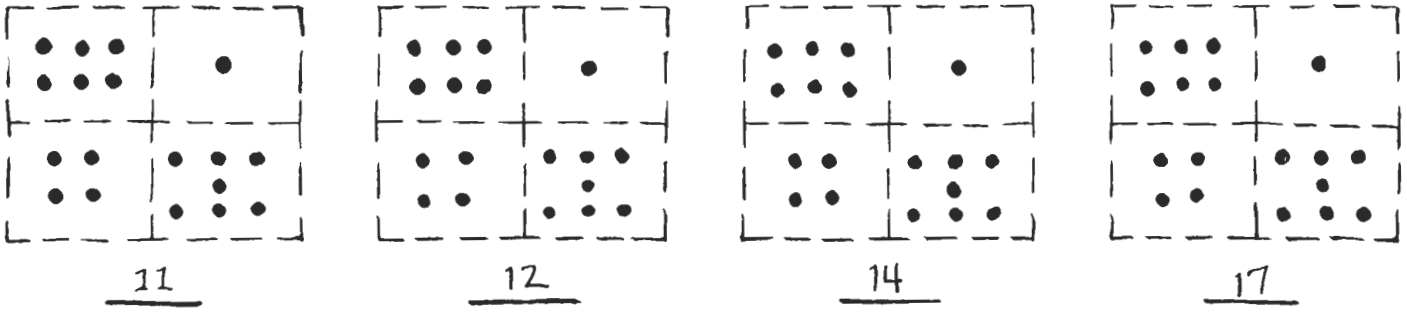
Please color-in or loop 3 blocks.



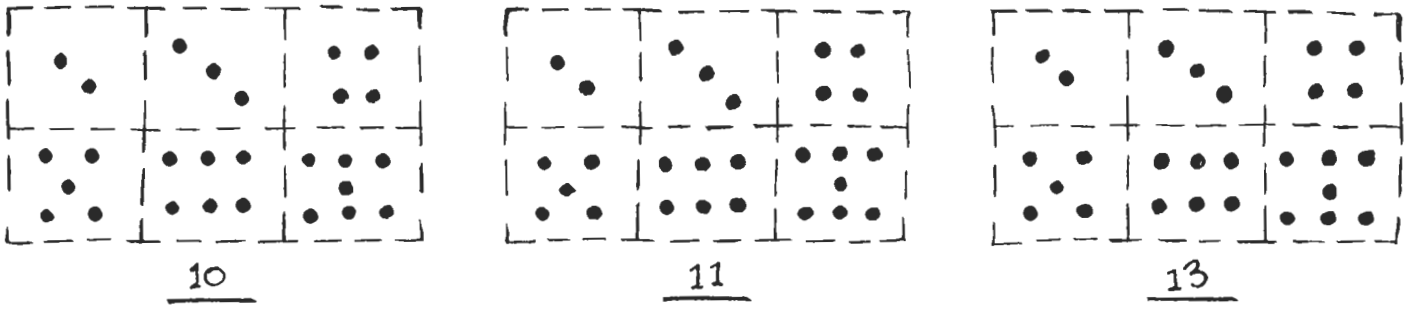
6 7 8 9



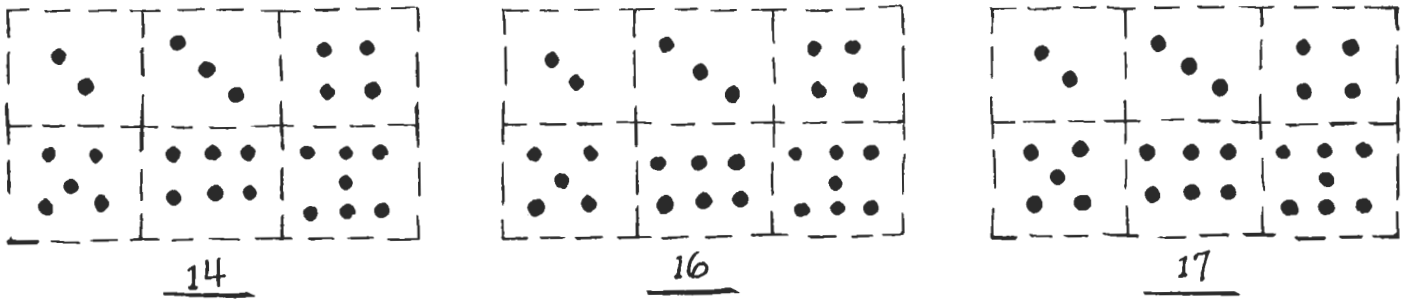
10 11 13 14



11 12 14 17



10 11 13



14 16 17

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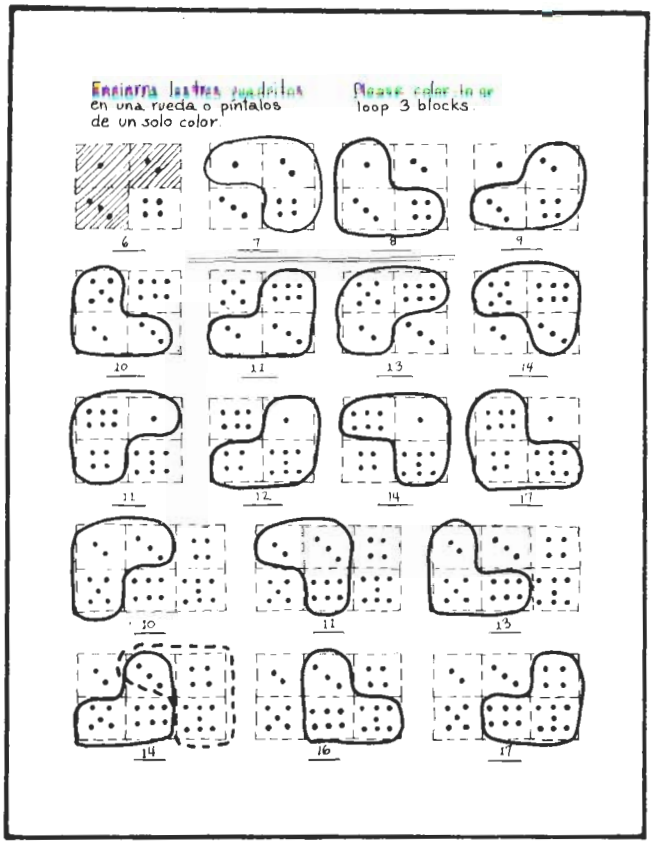
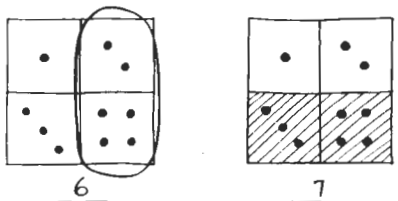


Introducing the Problem

Several blocks containing dots are shown in each diagram. Children are asked to loop three of the blocks so that the number of dots looped is the number indicated under the diagram. In the first two examples, other combinations of blocks are possible to make the number indicated, but they do not meet the requirement that *three* blocks be used.



Talking about the Problem

In some of the problems in the fourth and fifth rows, children may have found different ways to color three blocks, and these should all be explored. Other questions can be pursued as well.

(1) Looking back over the examples on this page, in which examples could you color in or loop just two blocks with the number of dots indicated?

Row 1 – 6, 7
Row 2 – 11
Row 3 – 11
Row 4 – 11 (2 ways), 13
Row 5 – None

(2) In which examples could you color in or loop four blocks with the number of dots indicated?

(This is only possible in the fifth row for 14 and 16.)

Extensions and Surprise Endings

It could be helpful to notice that there is an easy way to solve the problems in the first three rows. In the problems above the double line, there are ten dots all together in each diagram. There are 16 dots all together in each diagram in the second row, and 18 dots in each one in the third row. Coloring or looping exactly three blocks is the same as leaving out one of the four blocks. Since there is a total of ten dots in each diagram in the top row of problems, to loop six dots, we need to leave out four. To loop seven, we must leave out three, and so on.

10 – 4 = 6	1, 2, 3, ✱	(1 + 2 + 3 = 6)
10 – 3 = 7	1, 2, ✱, 4	(1 + 2 + 4 = 7)
10 – 2 = 8	1, ✱, 3, 4	(1 + 3 + 4 = 8)
10 – 1 = 9	✱, 2, 3, 4	(2 + 3 + 4 = 9)

In the second row, there is a total of sixteen dots, so:

16 – 6 = 10	2, 3, 5, ✱
16 – 5 = 11	2, 3, ✱, 6
16 – 3 = 13	2, ✱, 5, 6
16 – 2 = 14	✱, 3, 5, 6

In the third row, each problem shows eighteen dots. Thus:

18 – 7 = 11	1, 4, 6, ✱
18 – 6 = 12	1, 4, ✱, 7
18 – 4 = 14	1, ✱, 6, 7
18 – 1 = 17	✱, 4, 6, 7

It is clear that there are only four ways to leave out one block in these problems. So there are just four different ways to color in or loop three blocks at a time.

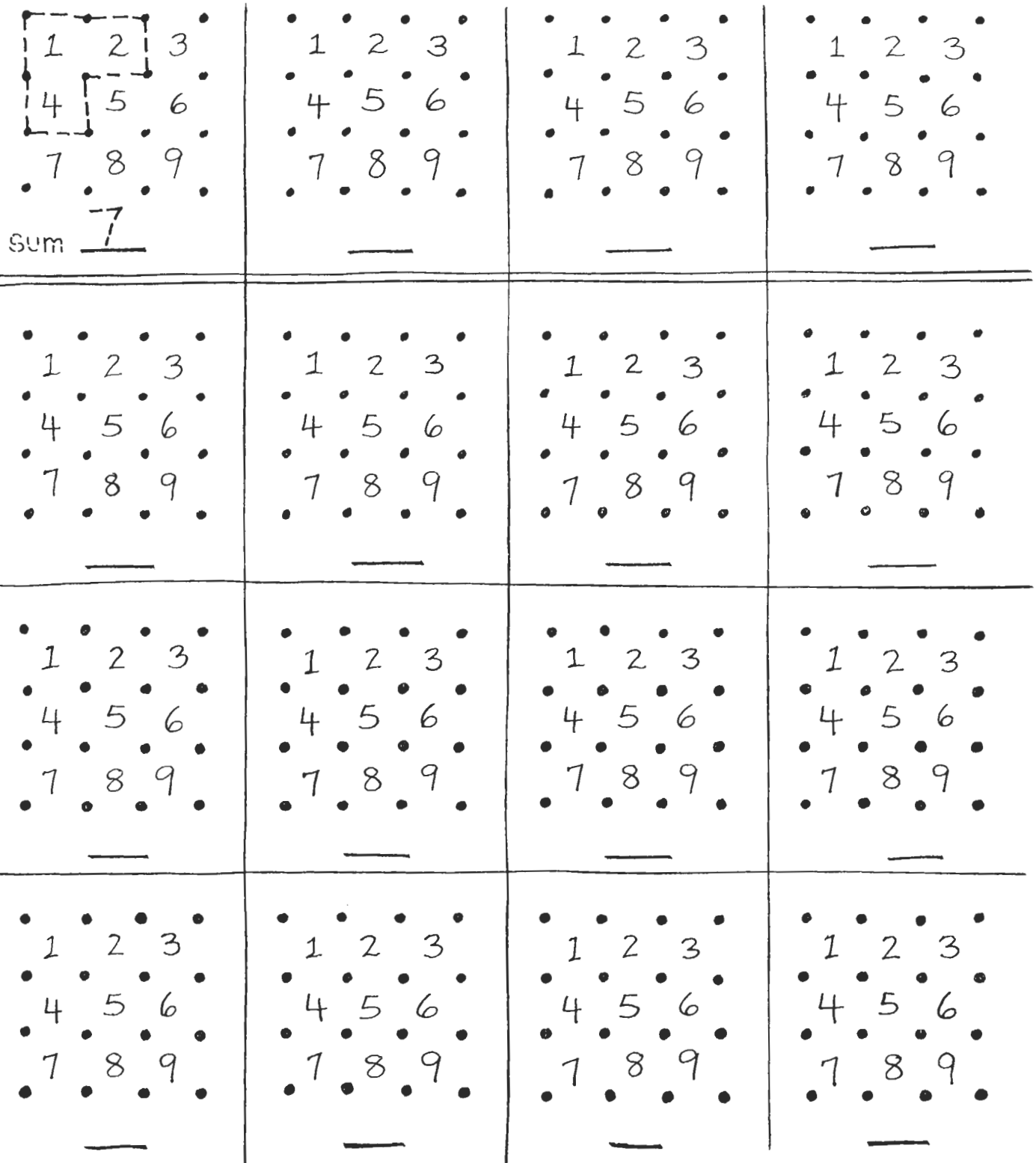
Cardboard squares can be made with various numbers of dots on each one. The squares can be arranged according to one of the diagrams on the activity page, and children can pick up squares to make the number of dots designated. Or, paper diagrams like those on the page might be made to fit a geoboard. Children could use rubber bands to loop the three blocks that would have the number of dots required.

FENCING 3 NEIGHBORS

Can you make all 16 sums on this page different?

ACORRALANDO 3 VECINOS

¿Puedes sacar sumas diferentes en los 16 ejemplos de esta pagina?



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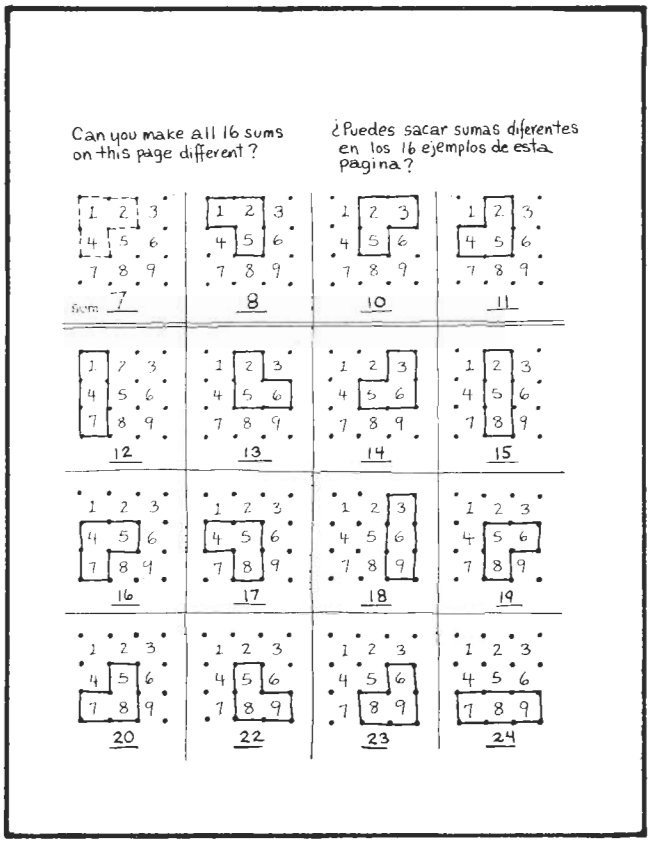
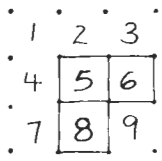
(Some of the appropriate responses are shown here.)

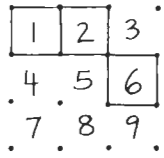


Introducing the Problem

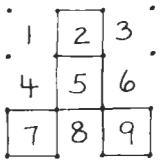
There are 16 fencing arrays on the activity page. Each one contains the numbers 1 through 9. "Neighbors" in the array are blocks which have a side in common. For example, 5, 6, and 8 are neighbors:



The blocks containing 1, 2 and 6 are not neighbors. While 1 and 2 are neighbors, neither one is a neighbor to 6, for the block containing 6 only shares a point with 2, not a side:

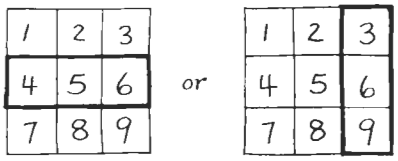


Likewise, while 2 and 5 are neighbors, neither one is a neighbor to 7 or 9.

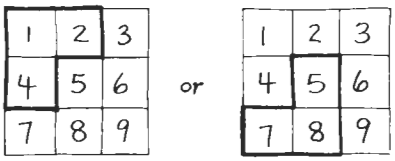


In this activity we will be looking for groups of three neighbors. There are just two basic arrangements of three neighbors:

(a) three in a row



(b) a "dog leg" arrangement



The problem asks for 16 sets of three neighbors which have different sums. As each set of neighbors is fenced, its sum should be recorded on the line below the array.

What is the smallest number you can fence with three neighbors? (6 is the smallest; 1 + 2 + 3 = 6). What is the largest number you can fence with three neighbors? (24 is the largest; 7 + 8 + 9 = 24).

If we can fence three neighbors for 6, for 7, for 8, 9, 10, and so on up through 24, (in other words, every sum from the smallest to the largest) there would be 19 different sums to show. This activity page only has space to show 16 different sums. So, three sums will be left out. Children can work at the page and later compare results.

Talking About the Problem

The class might make a tally sheet for the numbers 6 through 24 and check off those sums that were done. If different ways were found to make a particular sum, they might also be recorded.

(1) Were there any numbers that nobody did? There are two sums that are impossible with this arrangement of numbers: 7 and 21.

(2) Suppose we arranged the numbers 1 through 9 in a different order:

1	2	3
6	5	4
7	8	9

Now which numbers can't be shown as the sum of a trio of neighbors? There are still two such numbers: 7 and 23.

Extensions and Surprise Endings

(1) There is a very special way to arrange the numbers 1 through 9 as a "magic square". In this arrangement, the sum of each row, column and diagonal is 15.

8	1	6
3	5	7
4	9	2

Now which numbers, 6 through 24, cannot be shown as the sum of a trio of neighbors? The smallest sum that *can* be shown is 9; (1 + 3 + 5 = 9). Therefore, 6, 7, and 8 cannot be shown. The largest sum that *can* be shown is 21; (9 + 5 + 7 = 21). Thus 22, 23, and 24 cannot be shown. It turns out that 10, 11, 19, and 20 cannot be done either. Notice the pattern in these numbers that can't be done

these cannot be done
6 – 7 – 8 – 9 – 10 – 11 – 12 – 13 – 14 – 15
24 – 23 – 22 – 21 – 20 – 19 – 18 – 17 – 16
these cannot be done

(2) Can you arrange the numbers 1 through 9 in a 3 by 3 grid so that all numbers, 6 through 24, can be shown as the sum of a trio of neighbors? Frankly, the author does not know yet. The closest he has come are arrangements with just two sums that can't be done:

1	2	3
5	4	6
7	9	8

(14? or 22?)

1	2	3
5	7	4
9	8	6

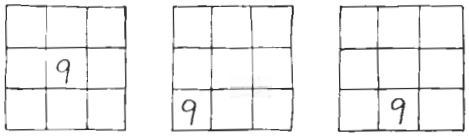
(7? or 11?)

These were interesting because the exceptions in one of them, 14 and 22, were exactly the doubles of the exceptions in the other: 7 and 11. Another arrangement was found with just two impossible sums:

1	2	3
5	4	6
9	8	7

(20? or 23?)

Someone may want to explore a line of reasoning that begins like this. There are just three basic positions for the 9. It must be in the center, in a corner, or in a side square:



Also, to be able to get 24, 23, and 22, there are three trios required that involve the 9:

9, 8 and 7; (9 + 8 + 7 = 24)
9, 8 and 6; (9 + 8 + 6 = 23)
and 9, 8 and 5; (9 + 8 + 5 = 22)
[or 9, 7 and 6; (9 + 7 + 6 = 22)]

Thus, we know that 9, 8 and 7 must be neighbors; 9, 8 and 6 must be neighbors, and so on.

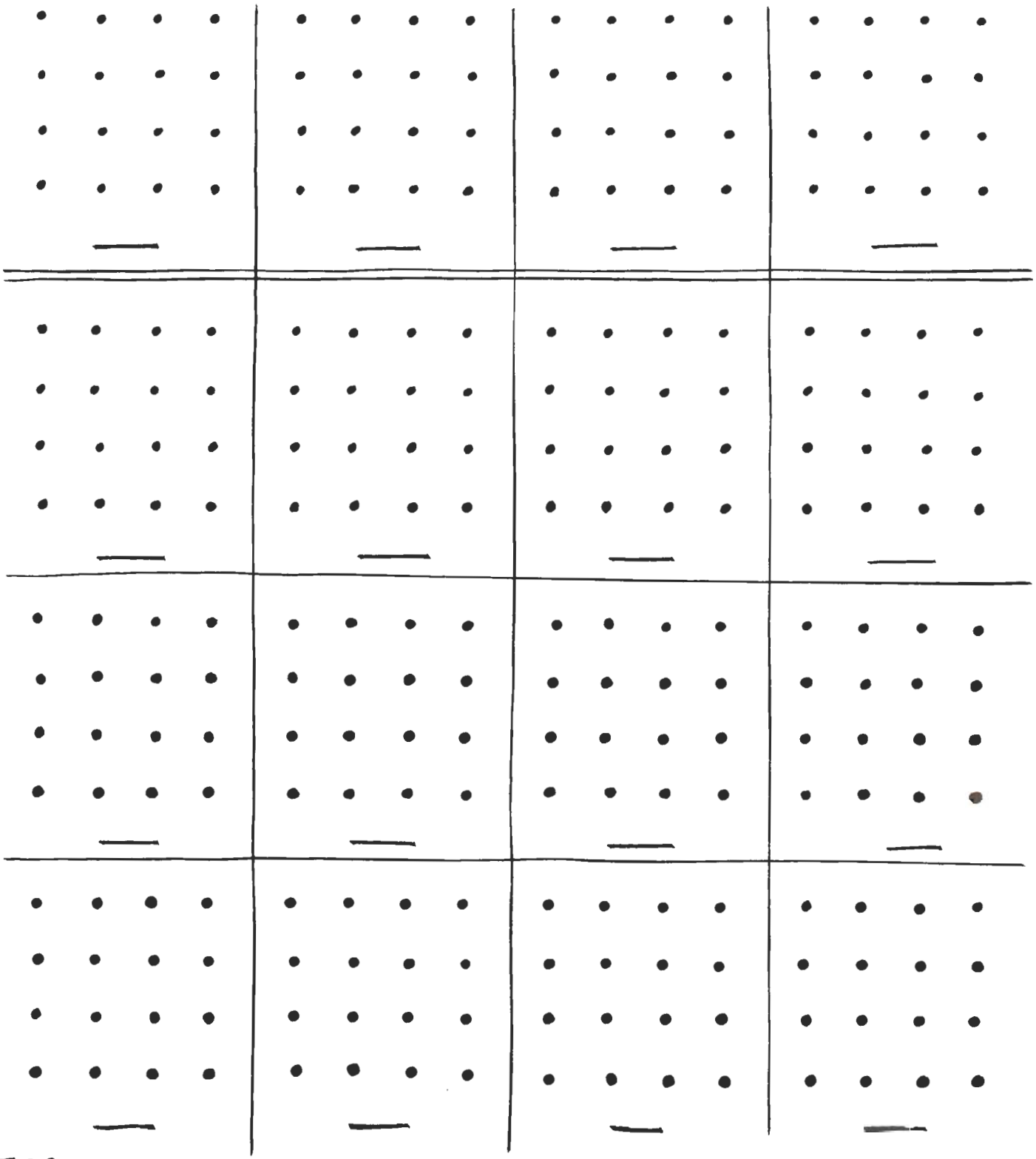
If any one finds an arrangement that allows all sums, 6 through 24, or proves conclusively that it can't be done, please write to the author!

FENCING NEIGHBORS

Can you make all 16 sums on this page different?

ACORRALANDO VECINOS

¿Puedes sacar sumas diferentes en los 16 ejemplos de esta pagina?



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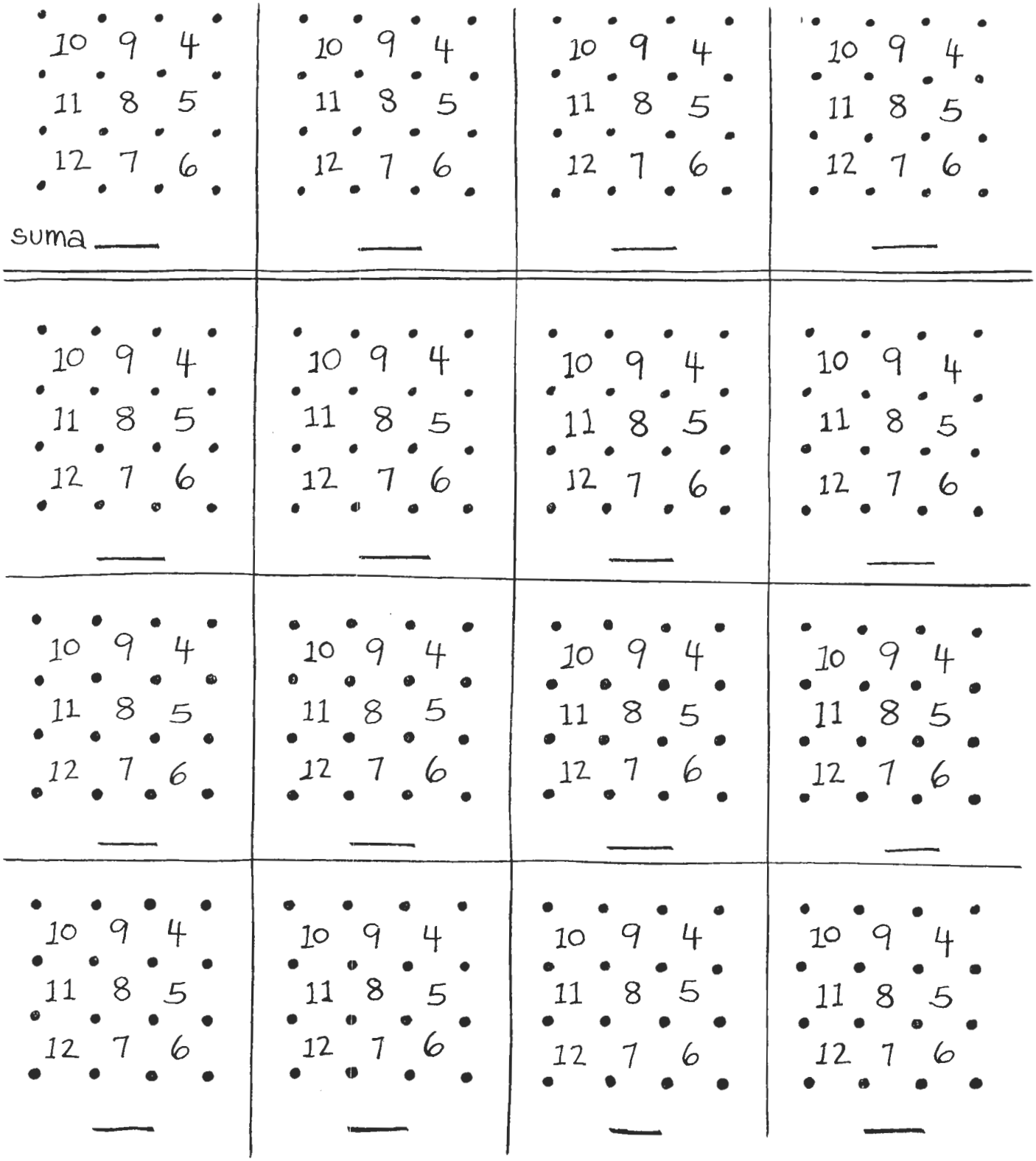
ACORRALANDO 3 VECINOS

¿Puedes sacar sumas diferentes en los 16 ejemplos de esta pagina?

FENCING 3 NEIGHBORS

Can you make all 16 sums on this page different?



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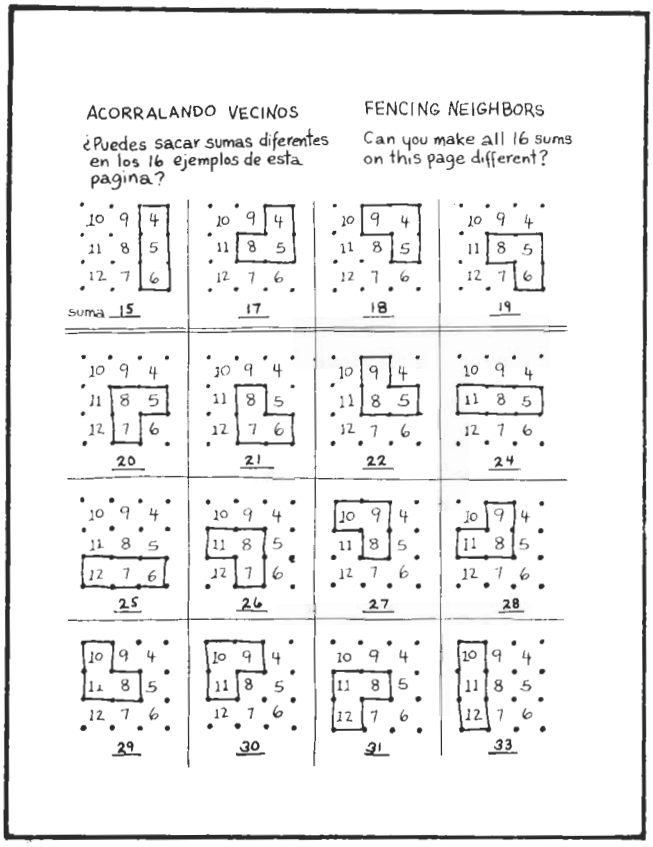
(Some of the appropriate responses are shown here.)



Introducing the Problem

Rules for fencing three neighbors are the same as for previous fencing activities. In this variation, each array uses the numbers 4 through 12. Again, the problem is to find 16 sets of three neighbors that have different sums.

The smallest sum that can be shown is 15 (4 + 5 + 6 = 15). The largest sum is 33 (10 + 11 + 12 = 33). If all numbers 15 through 33 could be shown as sums, we would have 19 different sums: 15, 16, 17, . . . 33. Of course, since there are only 16 arrays on this page for recording, the class will have to compare results to see if all 19 can be done.

Talking About the Problem

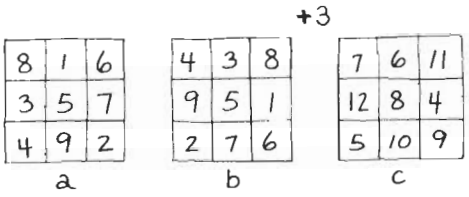
Again, the class might make a tally sheet and indicate those numbers that could and couldn't be done. It turns out that there are two numbers that cannot be shown as trios: 16 and 32. The only three numbers in the array that add to 16 are 4, 5, and 7. The 4 and 5 are neighbors, but the 7 is not a neighbor to either one. The only three numbers in the array that add to 32 are 12, 11, and 9. Again, these three are not neighbors.

Extensions and Surprise Endings

Children might be encouraged to explore the question: "Can you find a different arrangement for the numbers 4 through 12 so that there *is* a trio for all numbers, 15 through 33?" This problem is closely related to the one posed in the previous activity.

On the previous activity page, we noted that there were two numbers that could not be shown as a sum of three neighbors: 7 and 23. Each of the numbers in the array on this page can be thought of as three more than some number in the previous array. Thus, the sum of any *three* numbers in this array would be increased by 3 + 3 + 3, or 9. We might guess then, that 7 + 9, or 16, and 23 + 9, or 32 could not be shown as sums of trios. This was exactly the case. So what we learned in the previous activity has value here if we notice the relationship between the two arrays.

Recall our investigation of a "magic square" arrangement for the numbers 1-9. This arrangement is shown in (a) below. If it is rotated one quarter turn clockwise, it becomes the array shown in (b). Then, if we again add three to each number in the array, we would have the arrangement shown in (c).



Recall that with the arrangement in (a), we could not find trios for 6, 7, 8, 10, 11, 19, 20, 22, 23, and 24. It shouldn't be much of a surprise then, that in figure (c) we will not be able to find combinations of three neighbors for:

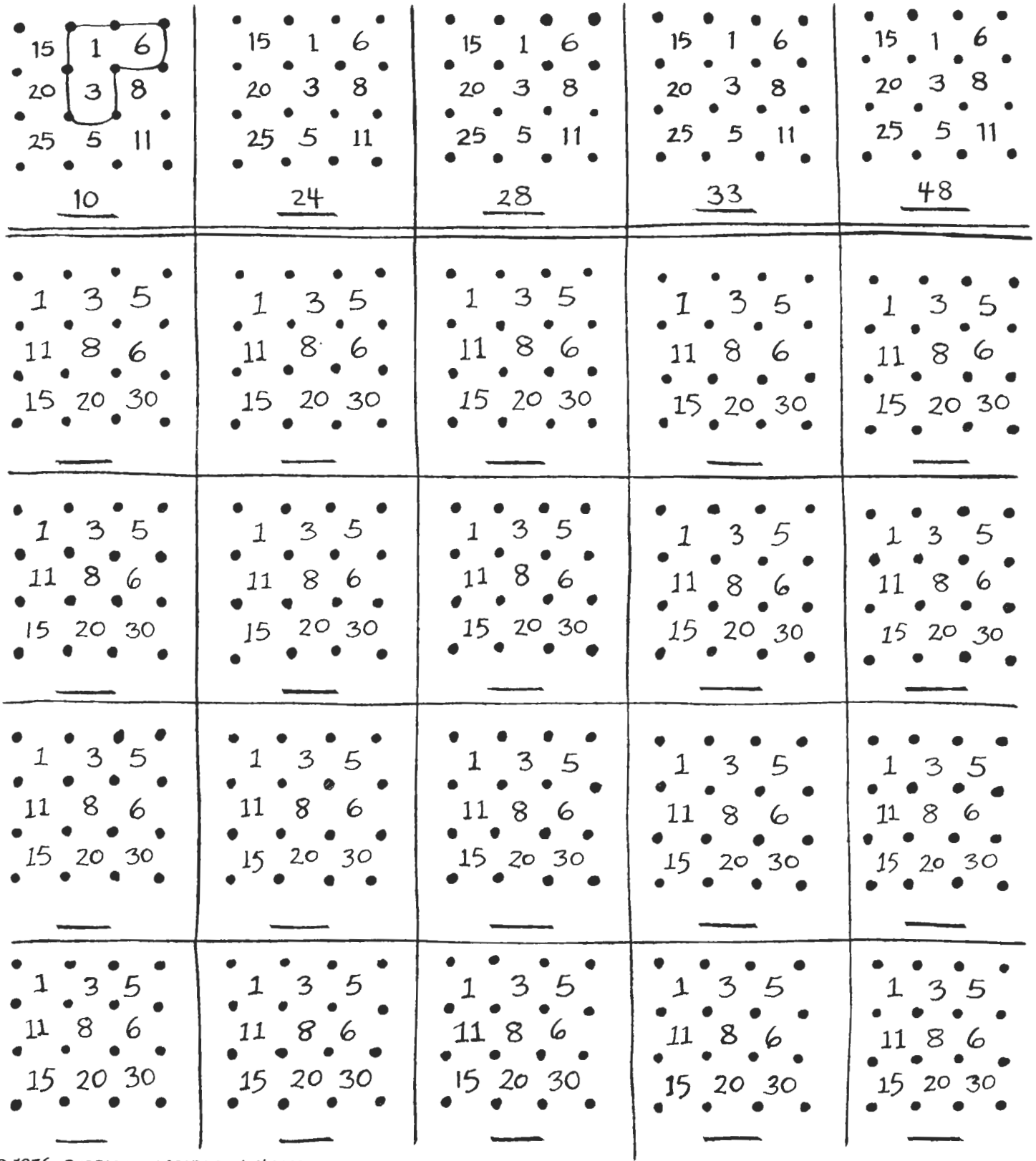
6 + 9, or 15
7 + 9, or 16
8 + 9, or 17
10 + 9, or 19
etc.

ACORRALANDO 3 VECINOS

¿Puedes sacar sumas diferentes en los 20 ejemplos que están debajo de la línea doble?

FENCING 3 NEIGHBORS

Can you make all sums different in the 20 examples below the double line?



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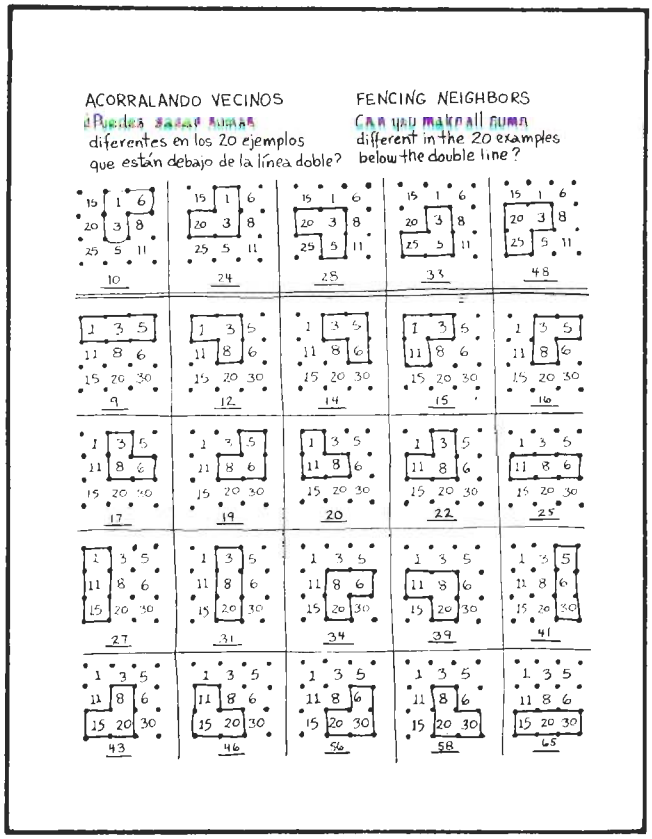


Introducing the Problem

Rules for fencing three neighbors are the same as for previous fencing activities. Again, the problem is to find 20 sums that are all different. The problem will give in to trial and error, but the going can get rough. This would be a good time to encourage children to follow some kind of a plan in solving the problem.

Talking About the Problem

Children might be asked to share their various plans. One plan that is useful looks at all possible ways to fence trios. We can begin by remembering that there are essentially two ways of fencing three neighbors:

(a) in a row (or column)
(b) in a "dog leg" arrangement.

There are three rows and three columns. The six sums that can be gotten from these are

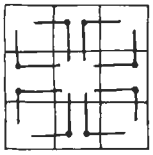
1	3	5
11	8	6
15	20	30

$1+3+5=9$ $1+11+15=27$
$11+8+6=25$ $3+8+20=31$
$15+20+30=65$ $5+6+30=41$

There are eight ways to use a corner cell or the center cell as the "point" of a "dog leg" arrangement. From these, we get the following sums:

$1+3+11=15$ $8+3+11=22$
$5+3+6=14$ $8+3+6=17$
$30+6+20=56$ $8+6+20=34$
$15+11+20=46$ $8+20+11=39$

There are two ways to use the middle cell on each of the four sides as the "point" of the dog leg. So, we have eight more sums:



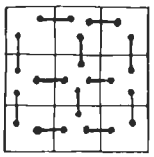
$1+3+8=12$ $30+20+8=58$
$5+3+8=16$ $15+20+8=43$
$5+6+8=19$ $15+11+8=34$
$30+6+8=44$ $1+11+8=20$

Thus, we have shown a total of 22 different trios. If we glance back over the sums, we see that 34 occurs as a sum for two different combinations:

$$8+6+20=34$$
and
$$15+11+8=34$$

So we really have just 21 different sums. Twenty of these can be recorded on the activity sheet.

How many different sums can be indicated if we just use two neighbors rather than three? There will 12 sums:



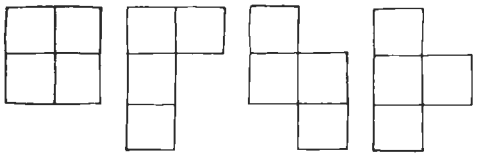
12 pairs:

$1+3=4$ $1+11=12$
$3+5=8$ $11+15=26$
$11+8=19$ $3+8=11$
$8+6=14$ $8+20=28$
$15+20=35$ $5+6=11$
$20+30=50$ $6+30=36$

1	3	5
11	8	6
15	20	30

Since 11 occurs twice, there are really just eleven *different* sums.

Someone might like to begin exploring the sums from groups of four neighbors. There are four basic arrangements to consider:



(4 ways) (16 ways) (8 ways) (8 ways)

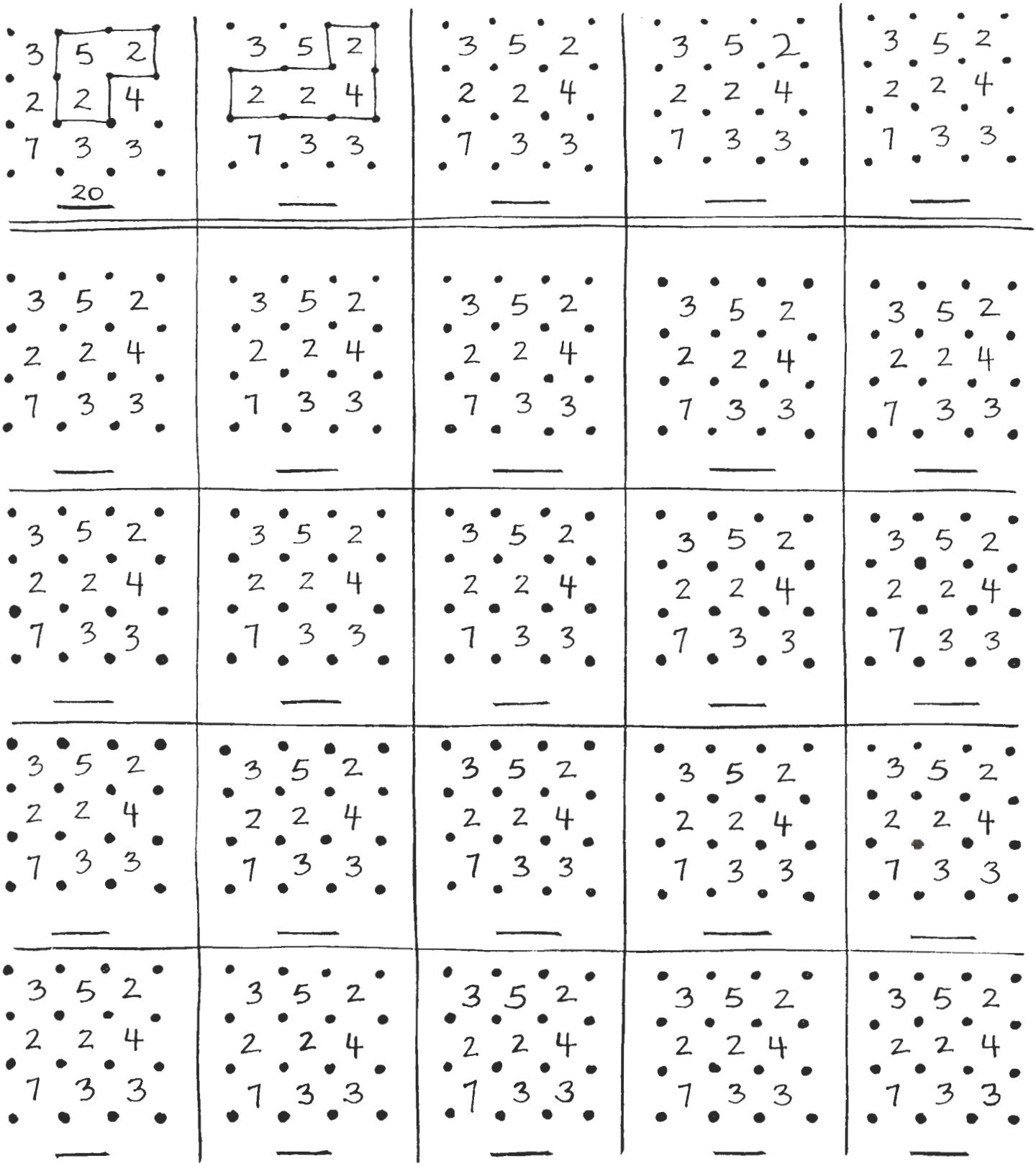
All together there will be 36 ways to combine four neighbors. How many *different* sums are there?

FENCING NEIGHBORS

Can all the 25 products be different ?

ACORRALANDO VECINOS

¿ Puedes ser diferentes los 25 resultados ?



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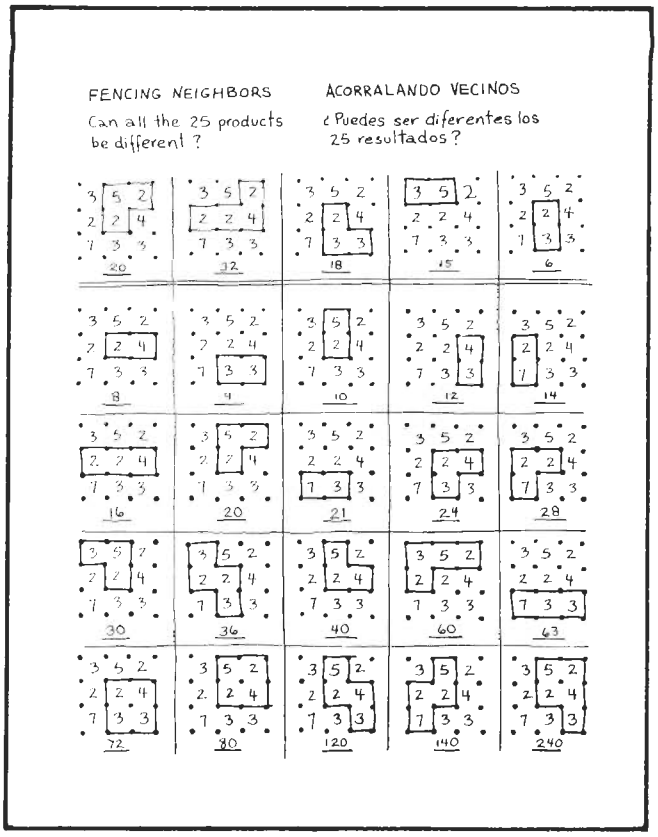


Introducing the Problem

"Neighbors" are still considered to be blocks that share a common side. In this variation, children are asked to fence neighbors and indicate their *PRODUCT* on the line below the array. They are not restricted to using three neighbors. Any number of neighbors can be used. The problem asks for 25 different products. Once again, children should be encouraged to develop some kind of plan for their search.

Talking About the Problem

In sharing plans and results, the children may want to keep a tally of all the products they found.

One plan that we found useful and efficient may not have been used by any of the children, but there would be value in looking at it as a strategy for tackling the problem. Instead of choosing neighbors and finding their products, we chose specific products and looked for neighbors. Since 1 does not appear in the array, we know right off that we will not be able to find neighbors whose product is a "prime" number; that is, a number whose only factors are itself and 1. So we only need to consider the "composite", or non-prime numbers. The first composite number to consider is 4.

4 = 2×2	13 is prime
5 is prime	14 = 2×7
6 = 2×3	15 = 3×5
7 is prime	16 = 2×2×4
8 = 2×4	17 is prime
9 = 3×3	18 = 2×3×3
10 = 2×5	19 is prime
11 is prime	20 = 2×2×5
12 = 3×4	21 = 3×7

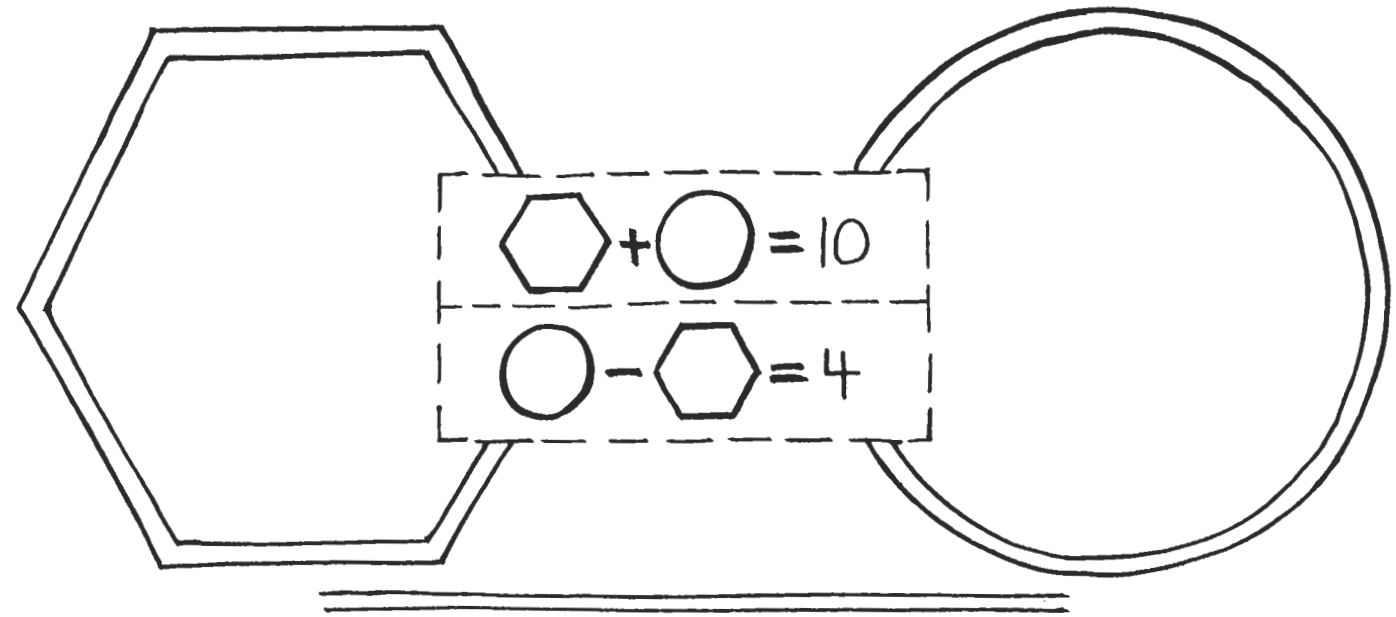
The first composite we *cannot* show is 22, because 11 is not in the array. But we have already found 12 different products. The problem called for 25 different products, so we must find 13 more. Continuing with our composites,

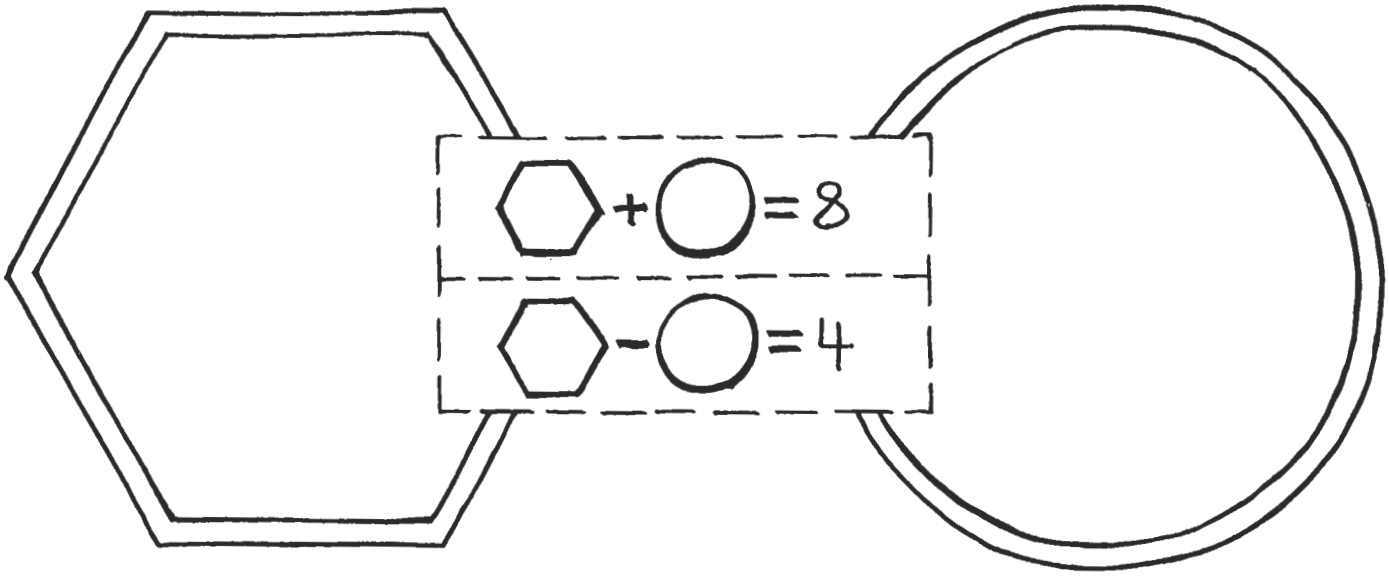
24 = 2×3×4	30 = 2×3×5
25 can't be done	31 is prime
26 can't be done	32 = 2×2×2×4
27 can't be done	33 can't be done
28 = 2×2×7	34 can't be done
29 is prime	35 can't be done

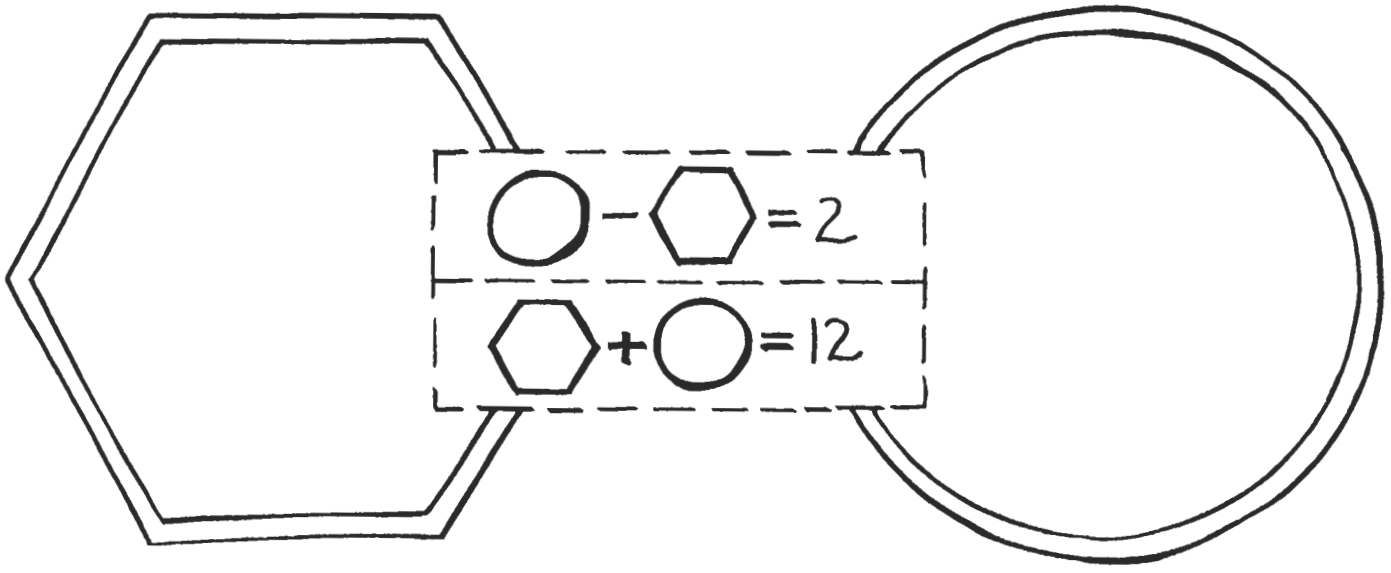
We have four more products, so we are up to 16 so far. From 35 on, the numbers that can be shown as products of neighbors thin out, but we can still find as many as we need. And all our products can be less than 100.

36 = 3×3×4	72 = 2×4×3×3
40 = 2×4×5	80 = 2×2×4×5
42 = 2×3×7	84 = 2×2×3×7
48 = 2×2×3×4	90 = 2×3×3×5
60 = 2×2×3×5	96 = 2×2×2×4×3
63 = 3×3×7	

Thus far we have found a total of 27 different products — two more than are needed. There are more that can be found.







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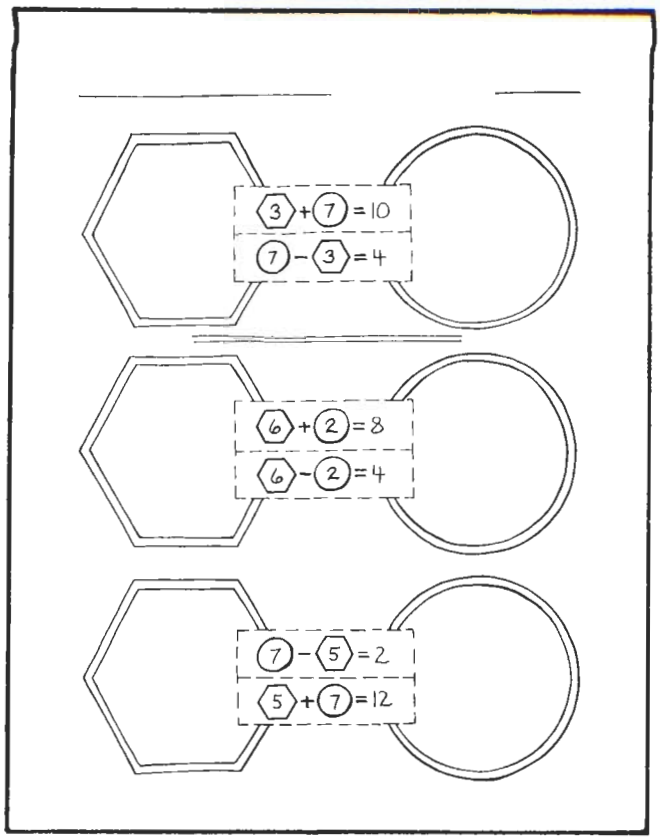
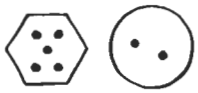


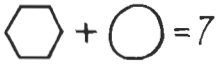
Introducing the Problem

This activity and the five that follow can be used in different ways to span the distance between children just beginning and children approaching algebra. Beginners can be introduced to this activity as a simple way to record what they see as they consider two groups of counters. Suppose a few counters are placed in two large frames, one hexagonal and the other circular.



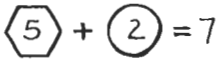
How many counters are in the hexagon? How many are there in the circle? There are five in the hexagon and two in the circle. These answers can be recorded by putting the numbers inside the appropriate shape:

How many counters are there all together in both shapes? There are seven, and we could indicate that in this way:



This notation can be read: "The number of counters in the hexagon plus the number of counters in the circle is (or equals) seven." Any other language that communicates the same information is also appropriate.

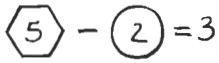
The entire situation can be written as



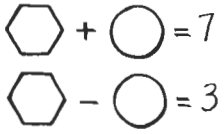
"There are five counters in the hexagon, two counters in the circle, and seven counters all together."

We might notice that there are more counters in the hexagon than in the circle. In fact, there are three more in the hexagon.

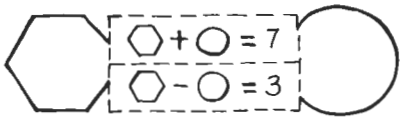
We could write this observation:



Both of the facts described above can be recorded in this way:



On this activity page, such information is written in the boxes connecting the two frames.



If we put counters in the frames as before, there would be five in the hexagon and two in the circle. Are there other arrangements of counters which

would have a total of seven counters in the two shapes? Yes, in fact there are eight different ways.

hexagon		circle
0	and	7
1	and	6
2	and	5
3	and	4
4	and	3
5	and	2
6	and	1
7	and	0

Are there other arrangements of counters with three more in the hexagon than in the circle? Yes.

3 and 0
4 and 1
5 and 2
6 and 3
etc.

This list could go on forever!

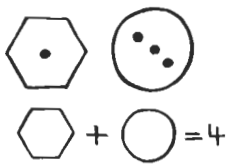
Is there any arrangement other than five in the hexagon and two in the circle for which there are

(a) seven counters all together *and*

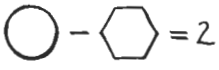
(b) three more in the hexagon than in the circle?

If there is any question, try moving counters around in different ways, looking for another arrangement that fits. The search will lead to the conclusion that we have found the only combination that meets both requirements.

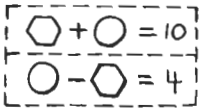
Here is an arrangement with one counter in the hexagon and three in the circle. All together there are four counters.



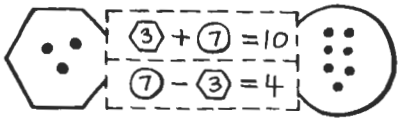
There are more in the circle than in the hexagon. We can show that in this way:



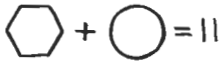
Now we are ready to talk about the first example on the activity page.



What do these two descriptions mean? The first tells us that there are ten counters all together in both shapes. The second tells us that there are four more counters in the circle than in the hexagon. How many counters will we need to place in the shapes so that both statements are true? We know that there are ten counters all together, so that's the number of counters we need. Can you arrange them so there are four more in the circle than in the hexagon? The only arrangement that meets both requirements — ten in all, and four more in the circle than in the hexagon — is three in the hexagon and seven in the circle.



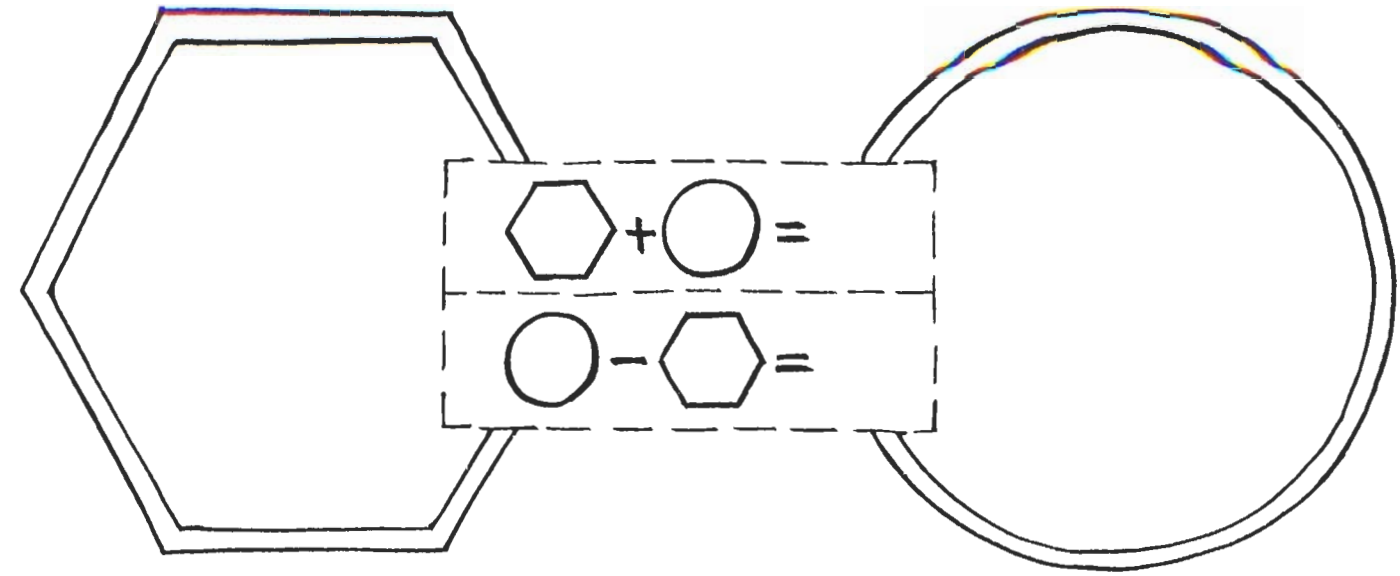
Children (and many adults) will feel the need to move counters around as they look for arrangements that meet the requirements for each problem on the page. The more they do this, the more convinced they will be that such strange looking statements as:

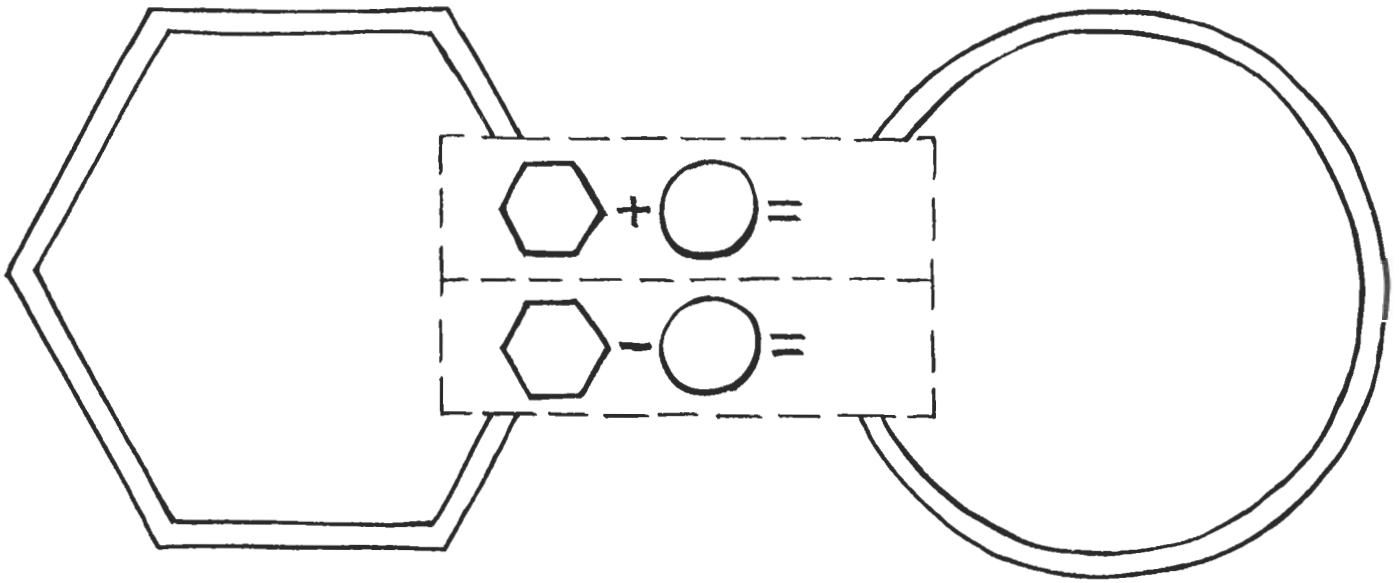


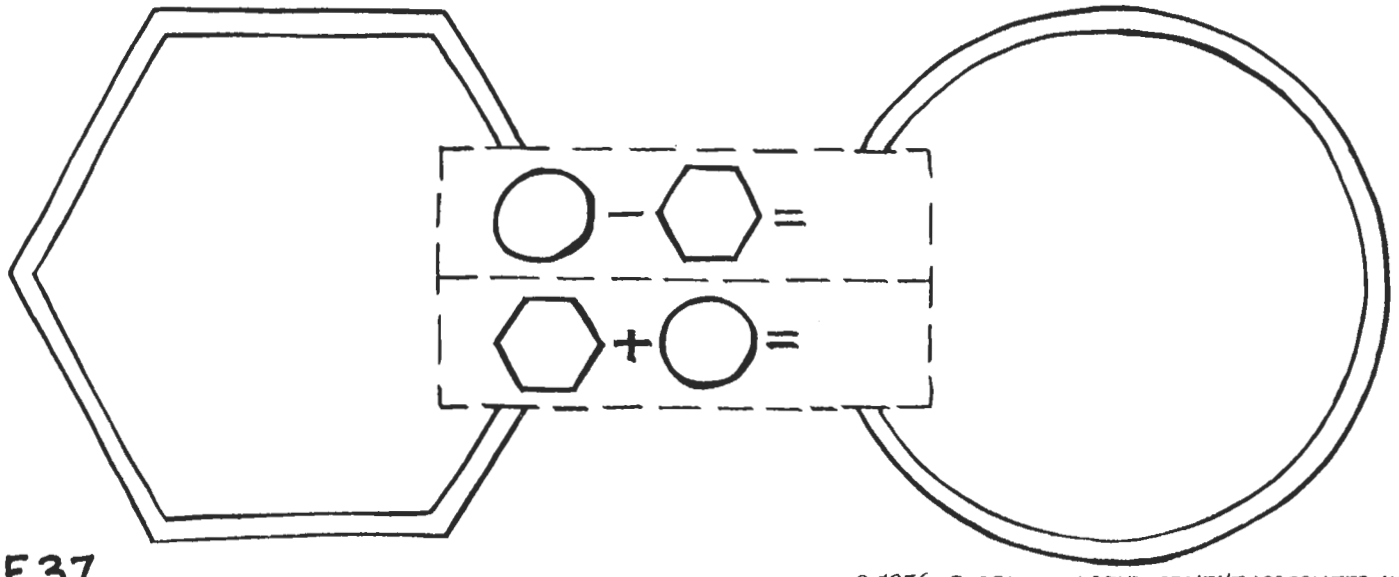
can be interpreted as saying something reasonable about counters. They may even be prepared along the way to handle such expressions as 'x + y = 11' with no feeling of mystery and confusion.

Talking about the Problem

Did you find a solution for each problem on the page? There is a unique solution for each one. What strategies or plans did you use to solve the problems? Time might be taken to allow the children to talk about the ways in which they solved the problems.







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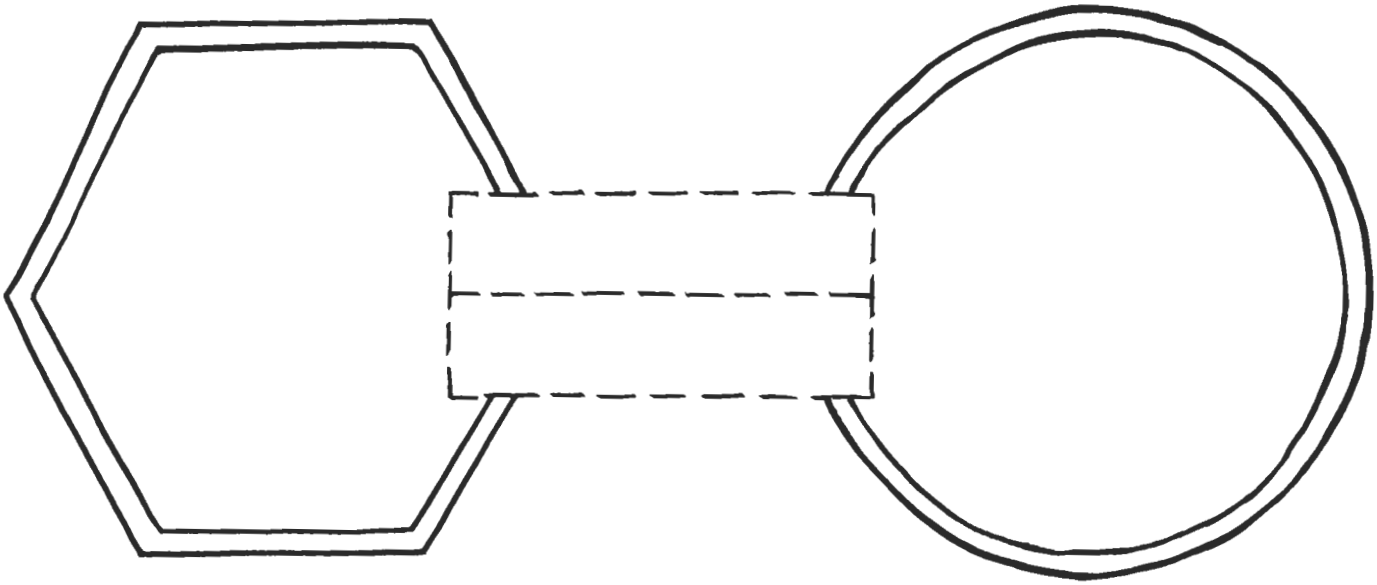

⬡ + ○ = 7	○ − ⬡ = 3
⬡ − ○ = 3	○ + ⬡ = 11
⬡ + ○ = 5	○ − ⬡ = 3
⬡ − ○ = 3	○ + ⬡ = 9
⬡ − ○ = 6	○ + ⬡ = 8
⬡ + ○ = 10	○ − ⬡ = 2
⬡ + ○ = 12	○ + ⬡ = 10
⬡ − ○ = 2	○ − ⬡ = 0

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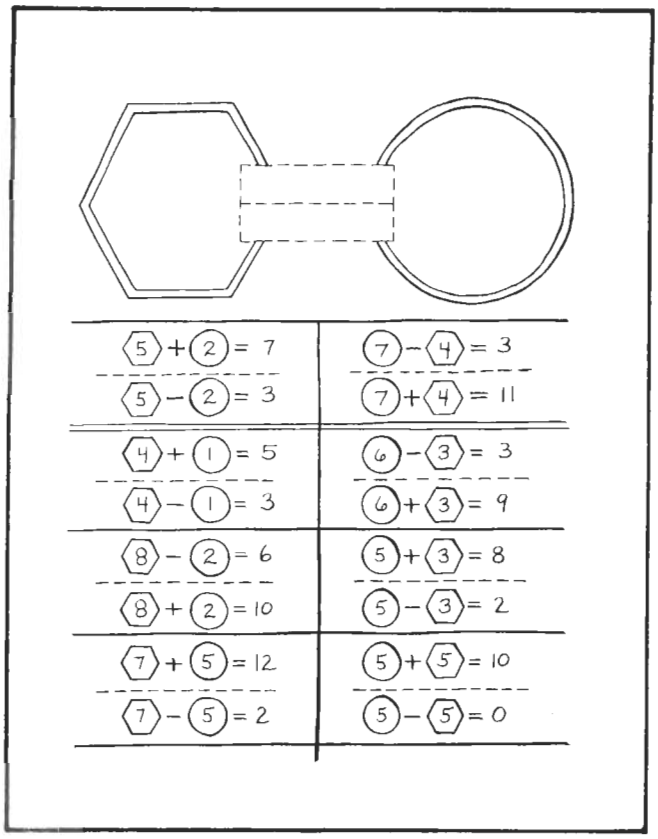


Introducing the Problem

If children are familiar with the preceding activity, this one will need only minimal introduction. The statements about sums and differences are not given in the rectangular boxes connecting the two frames. Rather, eight pairs of statements are given below the frames, each referring to a different arrangement of counters.

The example on the left above the double lines has two pieces of information: (1) all together there are seven counters in the two shapes and (2) there are three more counters in the hexagon than in the circle. If we put '5' into the hexagon and '2' into the circle, both statements will be true. So, the boxes can be filled in to read:

$$\langle 5 \rangle + (2) = 7$$
$$\langle 5 \rangle - (2) = 3$$

In the other example above the double line, the first statement refers to the difference and the second to the sum of counters to be placed in the two frames. In which frame are there more counters? There are more in the circle. How many more are there in the circle? There are three more in the circle. How many counters are there all together? There are eleven. The appropriate arrangement is four in the hexagon and seven in the circle. In the six problems below the double line, a maximum of 12 counters is used, so the necessary arithmetic is minimal.

Talking about the Problem

Let's look back over the problems for any clues that might be useful. Look at each pair of sums and differences. Do they have anything in common? Some sums are odd numbers and others are even numbers. The same is true of differences. But, if the sum in a pair is even, the difference is also even. If the sum in a pair is odd, the difference is also odd. In each pair, both the sum and the difference are odd, or both even. Why does that happen?

If a sum of two numbers is odd, then one of the addends must be odd.

2 + 3 = 5	7 + 4 = 11
5 + 4 = 9	3 + 4 = 7
_ + 7 = 9	_ + 2 = 5

If one number is even and the other is odd, the difference is also odd.

5 − 2 = 3	11 − 4 = 7
9 − 4 = 5	8 − 3 = 5
11 − 8 = _	10 − 9 = _

So, if one number of counters is an even number and the other is an odd number, then both the sums and differences will be odd.

odd + even = odd	odd − even = odd
even + odd = odd	even − odd = odd

But what if both numbers of counters are odd or both are even?

6 + 2 = 8	3 + 7 = 10
6 − 2 = 4	7 − 3 = 4

In either case, both sums and differences are even.

even + even = even odd + odd = even
even − even = even odd − odd = even

Consider the first example on the page. The final report is

(5) + (2) = 7
(5) − (2) = 3

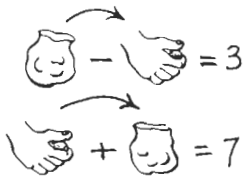
Is there any significance to the fact that the arrangement of counters is five in the hexagon and two in the circle and

7 + 3 = 10 and 10 ÷ 2 = 5
7 − 3 = 4 and 4 ÷ 2 = 2

Consider the other example above the double line. There are seven counters in the circle and four counters in the hexagon. Also,

3 + 11 = 14 and 14 ÷ 2 = 7
11 − 3 = 8 and 8 ÷ 2 = 4

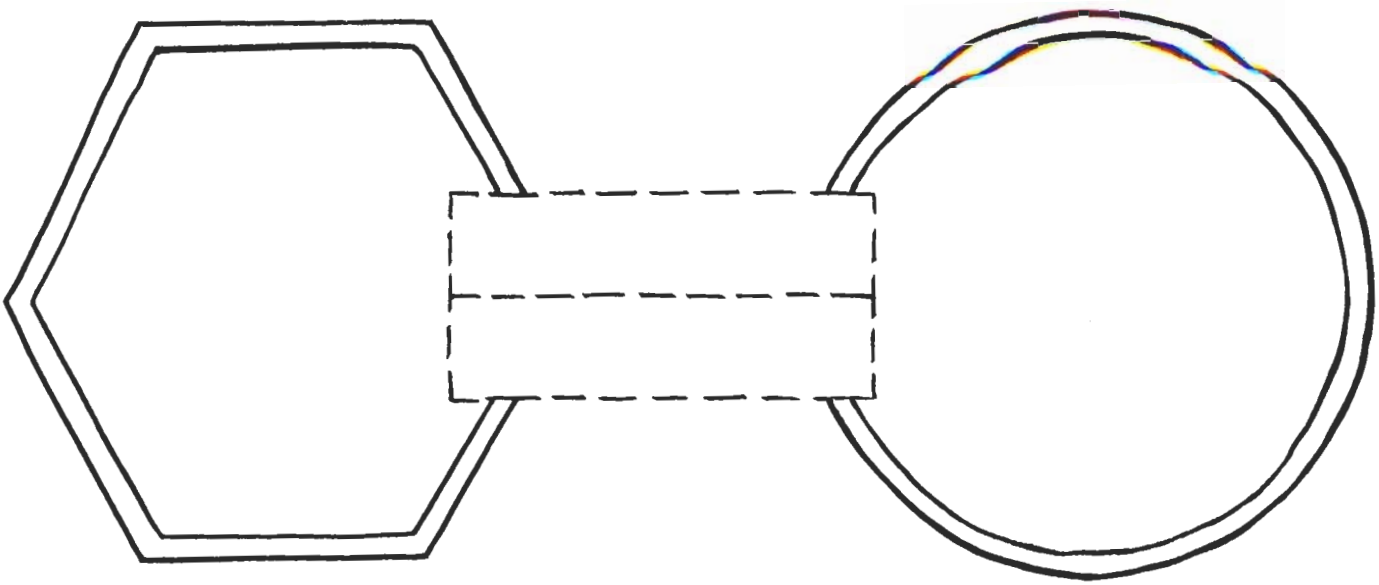
Why does this happen? Suppose that you have two bags of stones. You know that the bags have the same number of stones in each, but you don't know exactly how many there are in either one. You take a handful of stones out of one bag and put it into the other bag.



Now you count the stones in each bag. There are three stones left in the first bag, and the other bag has seven stones in it. So you would know that the bags must have had ten stones in all at the start, and since they had the same number, each had five stones. The arithmetic we used here is simple: 3 + 7 = 10, and 10 ÷ 2 = 5. It is the same simple arithmetic we looked at above that was so surprising. A similar explanation using bags of stones to illustrate can be given for "7 − 3 = 4, and 4 ÷ 2 = 2."

If learners test this "shortcut" arithmetic in all the examples on the page, the results hold up. Few will understand why it happens. Some will not be able to follow or believe the explanation that uses bags of stones to illustrate, and you may find other ways to explain. But everyone will appreciate that we are trying to demonstrate to ourselves why we can predict the results.

We are searching for ways to understand why the results of moving "things" around reveal patterns we can then use to predict outcomes. The results are observable. The patterns we see in the records of outcomes may not be clear, but there is surely a connection between the results and the patterns. The focus is on seeing the relationship between the way things behave and the way symbols can be manipulated. If this connection between things and symbols can be maintained, algebra will make good sense, and will mystify no one.



⬡ + ○ =	○ − ⬡ =
⬡ − ○ =	○ + ⬡ =
⬡ + ○ =	○ − ⬡ =
⬡ − ○ =	○ + ⬡ =
⬡ − ○ =	○ + ⬡ =
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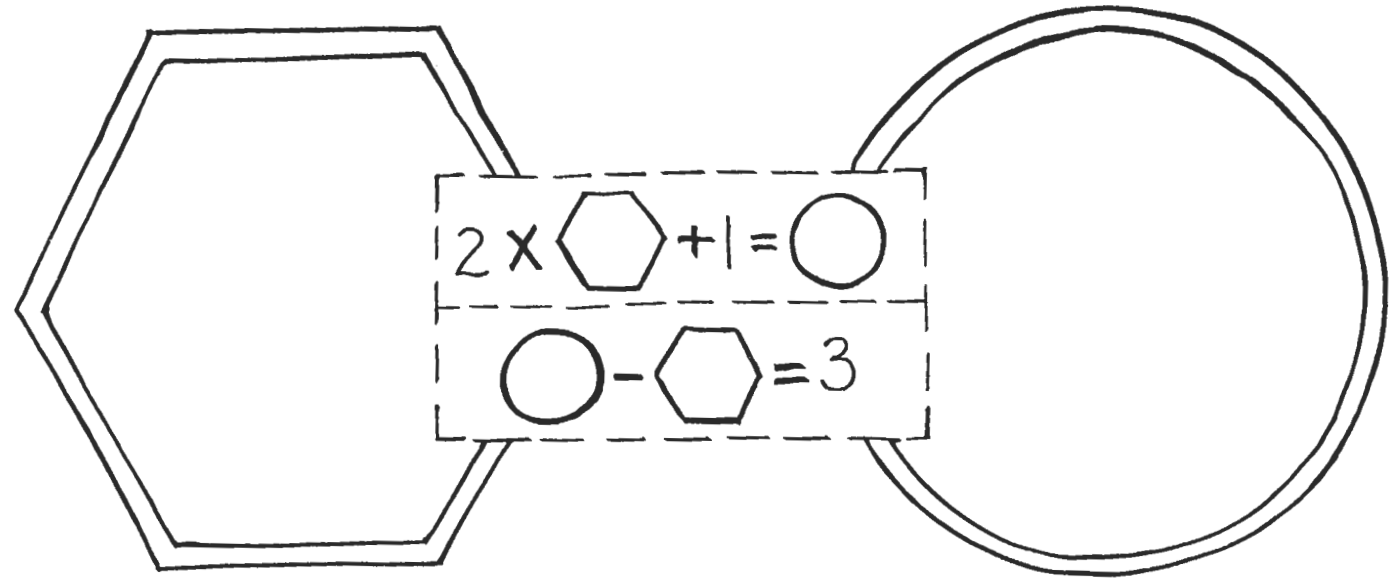

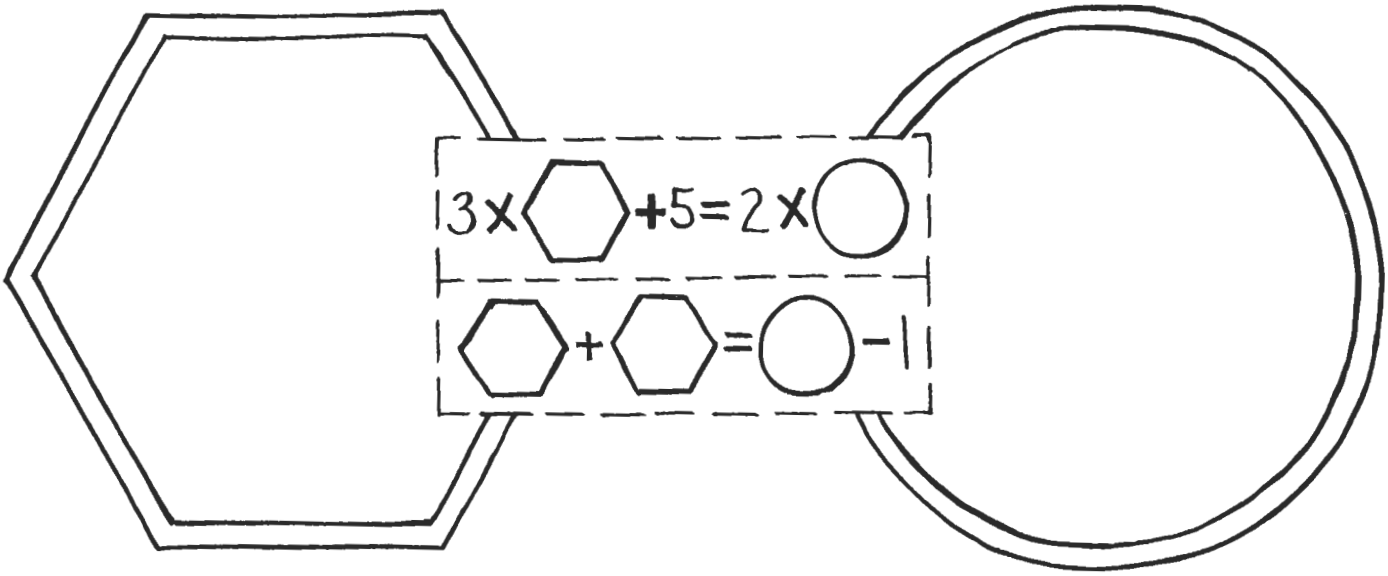

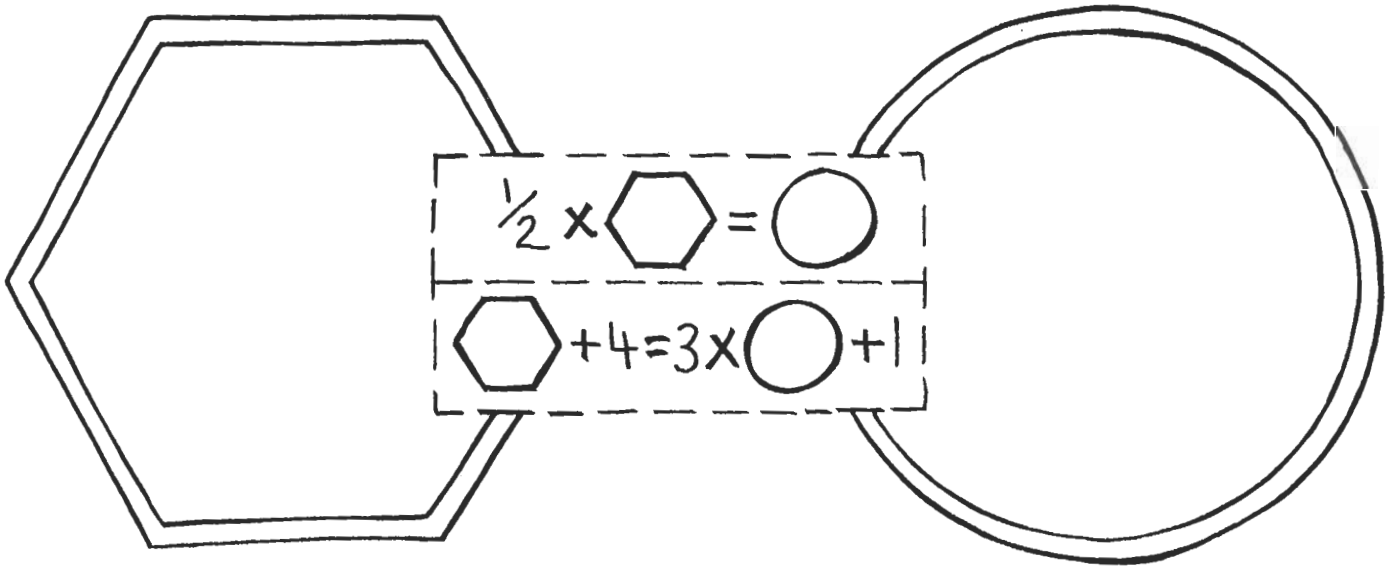

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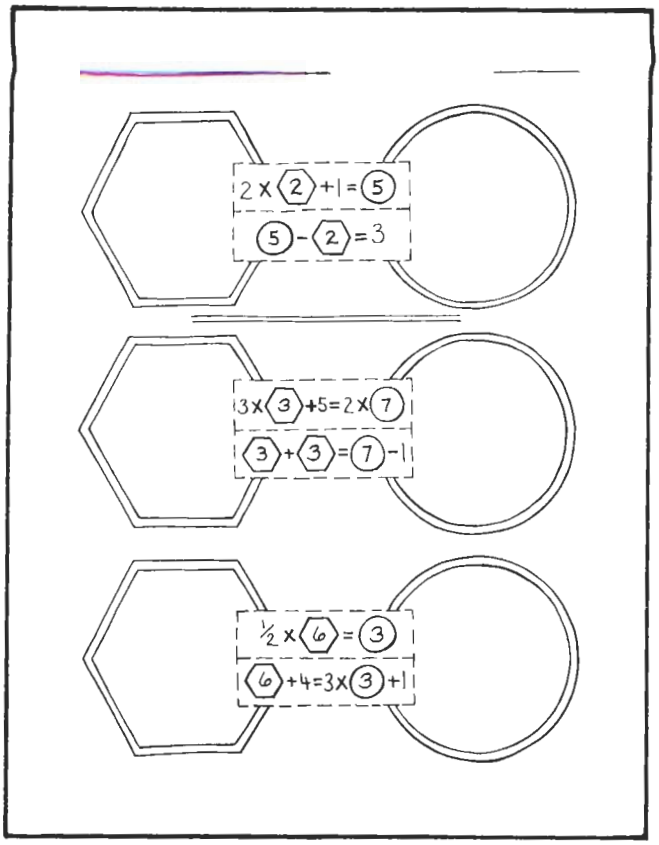
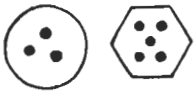


Introducing the Problem

The two preceding activities are essential prerequisites for this activity. They have been concerned with observing and recording the differences and sums of numbers of objects arranged in two frames. Now we are concerned with finding ways to indicate other relationships between the numbers of objects arranged in two frames.

Suppose we have three counters in the circle and five in the hexagon.



We have already considered sums and differences and could say:

⬡ + ⬭ = 8

and

⬡ − ⬭ = 3

But we could also point to other relationships. For example,

⬡ + 1 = 2 x ⬭

This notation could be read: "One more than the number of counters in the hexagon is twice the number of counters in the circle." Or,

3 x ⬡ = 5 x ⬭

which says: "Three times the number of counters in the hexagon is the same as five times the number of counters in the circle." There are 15 counters in both cases. Another relationship can be expressed:

⬡ + 2 = ⬭

It says: "If you add two to the number of counters in the hexagon, it will be the same number as the number of counters in the circle." Still another:

2 ⬭ = ⬡ + 1

This says: "Twice the number of counters in the circle is one more than the number of counters in the hexagon." The central concern here is with helping children transform the relationships they see into an unfamiliar notation.

If you have five cows in a field shaped like a hexagon and three cows in another circular field, you can write down some relationships you notice between the numbers of cows.

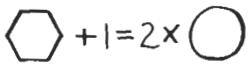
⬭ + 2 = ⬡

This notation could be read: "If there were two more cows in the circular field, there would be as many as there are in the hexagonal field." Another statement:

⬡ + ⬭ = 2 x ⬭ + 2

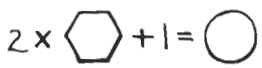
This one says: "If the cows in the two fields got together, there would be two more than twice the

number of cows in the circular field." Still another:



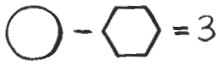
This one can be read: "If there was another cow in the hexagonal field, there would be twice as many as in the circular field."

As children become familiar with this way of expressing relationships, we can consider this activity page. The example above the double line presents two expressions. The first is:

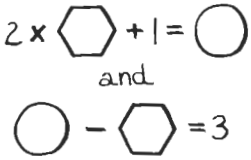


One way to read that notation is: "One more than twice the number of counters in the hexagon is the number of counters in the circle." Or, "Double the number of counters in the hexagon and add one and you will have the number of counters in the circle."

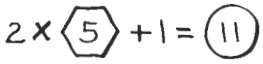
The second statement is a familiar one:



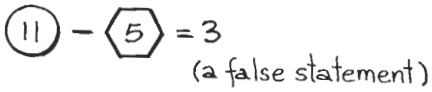
It says: "There are three more counters in the circle than in the hexagon." Or, "If you added three counters to the hexagon, it would have as many as the circle." Now, given those two statements, can you arrange the counters in the frames so that both statements are true?



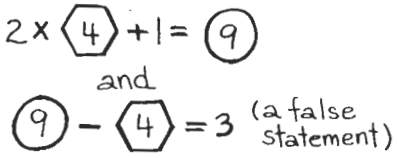
Let's try five in the hexagon.



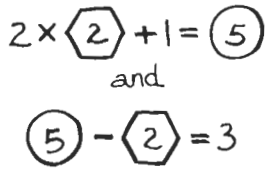
That would mean



Try again. If we put four in the hexagon,



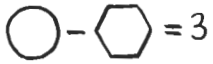
Try again. If we put two counters in the hexagon,



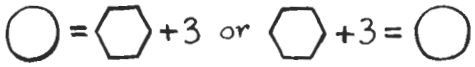
Finally we have at least one solution!

Talking about the Problem

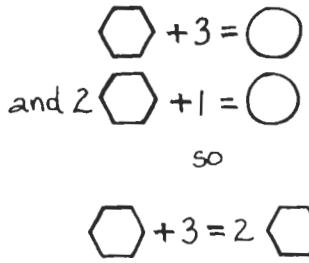
Did you find a solution for each problem? There is a solution for every one. Are you satisfied with the one solution, or did you wonder if there were other solutions? Let's go back to the problem above the double line. The expression



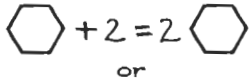
can be put differently as

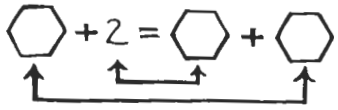


Then we can say:

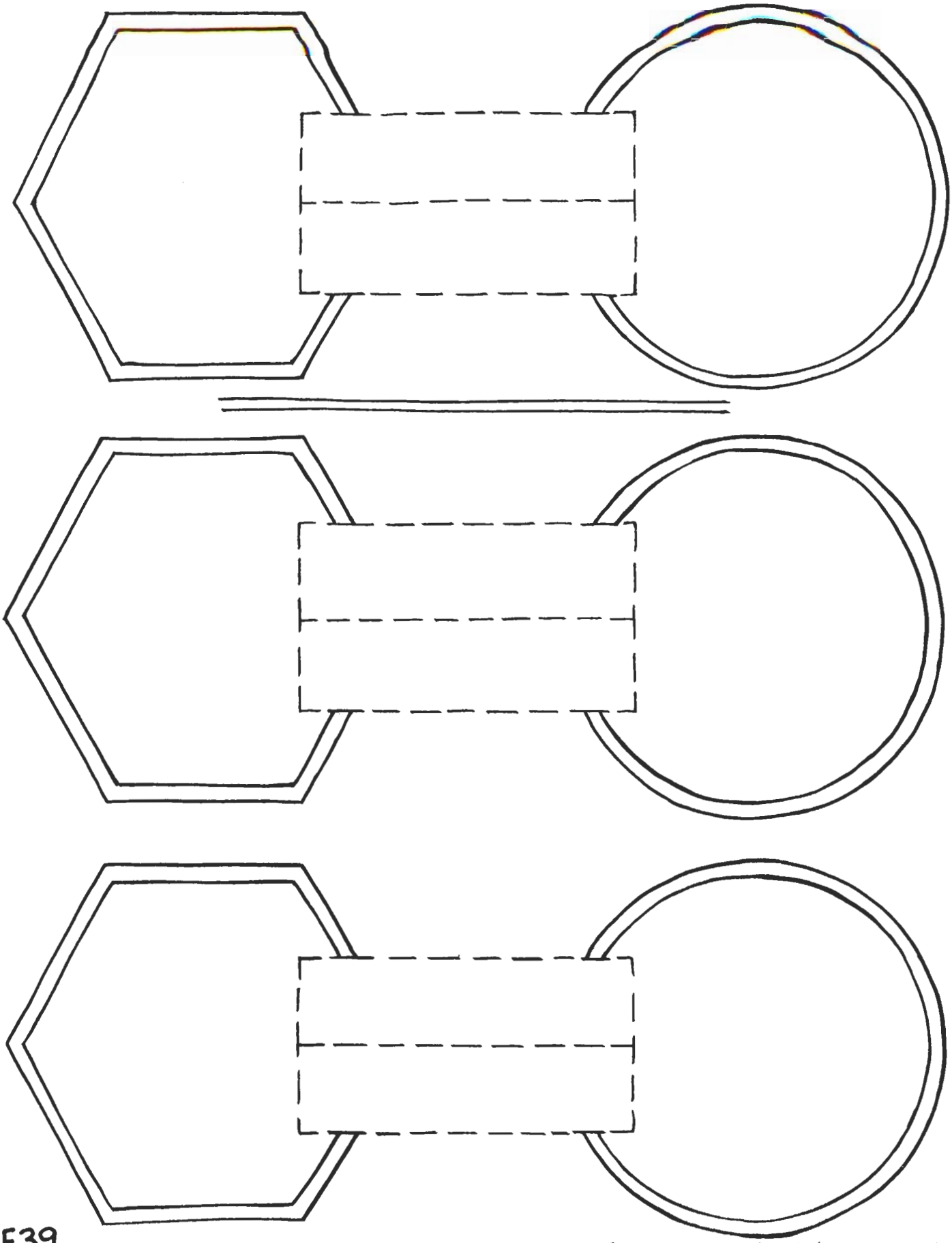


Suppose we subtract one from each of those two statements. We would have:





So, there have to be two counters in the hexagon. There is no other solution.



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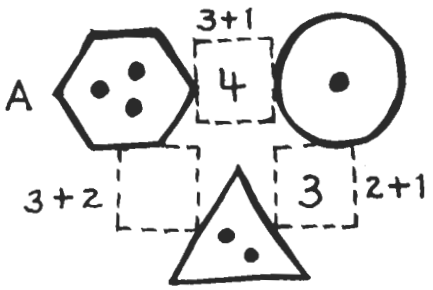

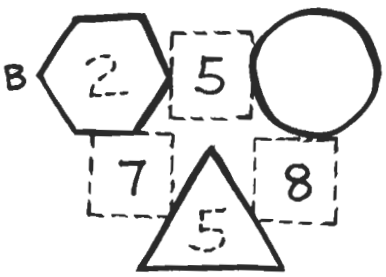

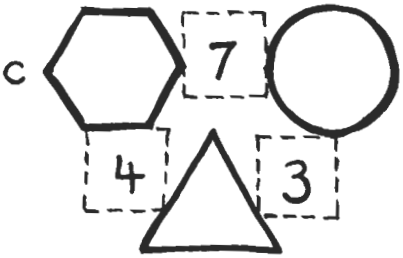

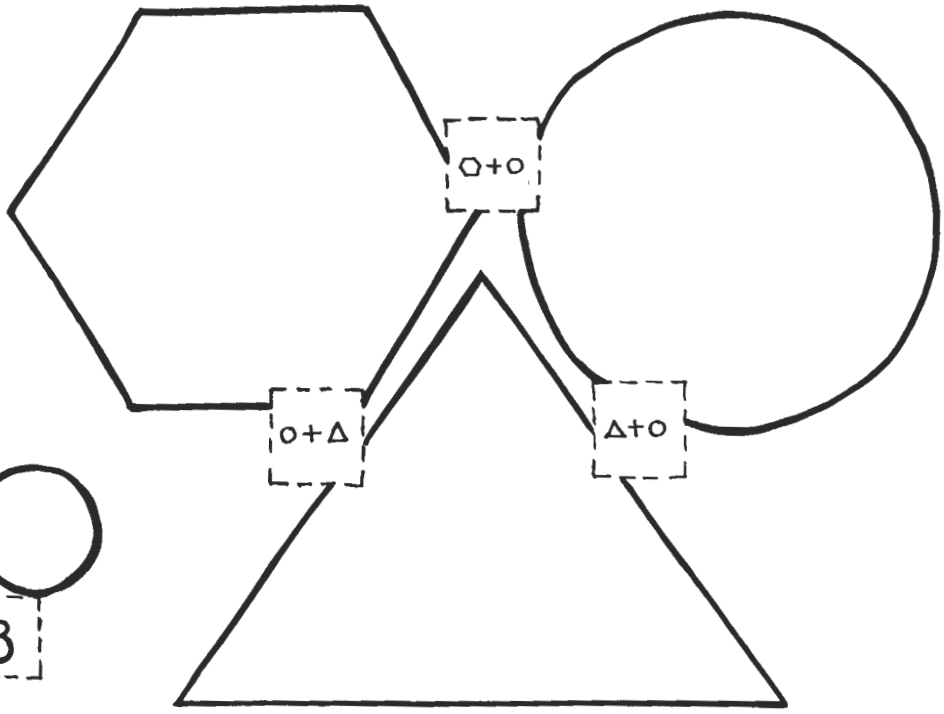

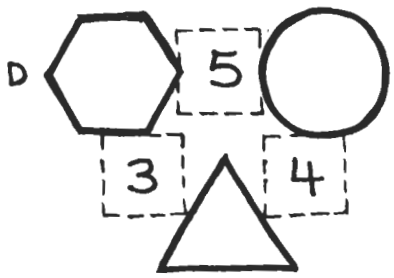

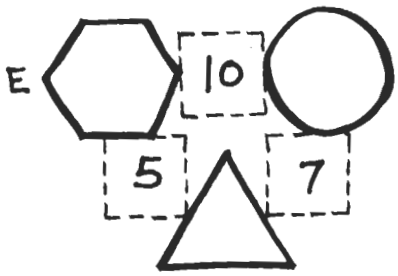

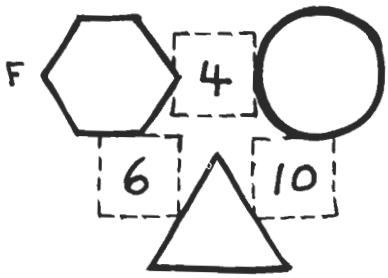

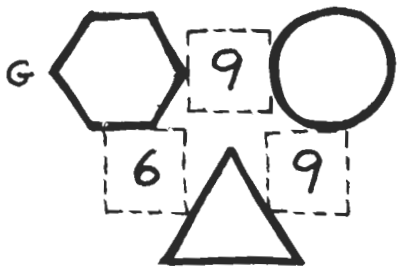

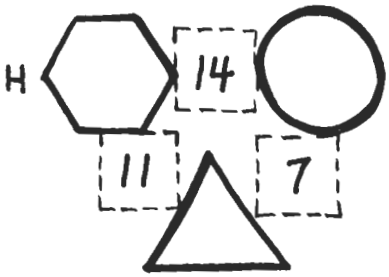

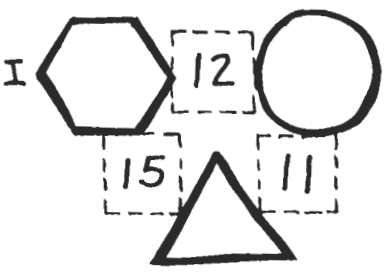

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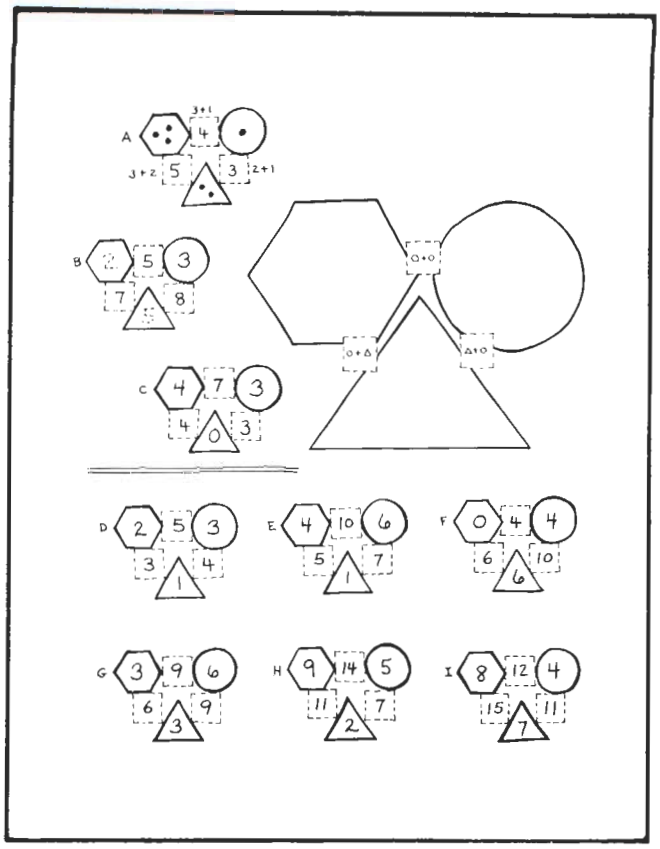
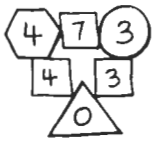


Introducing the Problem

Three shapes are pictured in each diagram with small boxes connecting each pair of shapes. Suppose we were to put some counters in each shape, and record in each small box the sum of the counters in the two shapes it touches. If we then removed all the counters, we would have a record like that pictured in problem C.

In problem A, we can see the counters in the shapes. Sums have been recorded in two small boxes. The third sum, which is 5, must be recorded in the third box. In problem B, all three sums are shown, and the numbers of counters for the hexagon and triangle are given. It is easy to find the number that goes in the circle. The sum of counters in the hexagon and circle is 5; there are 2 counters in the hexagon; therefore there must be 3 counters in the circle.

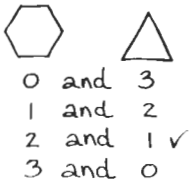
In problem C, only the sums are given. This is the case for problems D through I as well. Essentially, each problem is a record of the sums that were produced when the counters were placed. The question is : How were the counters arranged? Even adults will find it helpful to use counters in the large frames at the top of the page when they are searching for a solution. In problem C, there is only one arrangement of counters that will work: four in the hexagon, three in the circle, and none in the triangle.



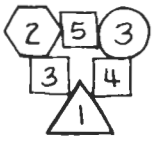
Talking About the Problem

(1) Did everyone get the same solution for each problem? Is it possible to get another solution for any one of the problems? As a matter of fact, there can be only one solution if all three sums are given.

(2) Did anyone find a way to shortcut the work? Did anyone use a plan? Some children may have noticed that the sum of the numbers in the small boxes is exactly twice the number of counters needed. Some may have noticed that trials and errors can be reduced by considering the smallest sum first. For example, in D, the possible sums for 3 are:



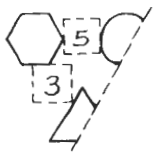
The third possibility which places two counters in the hexagon and one counter in the triangle leads to the solution. The circle must have three counters.



Extensions and Surprise Endings

There is an approach which eliminates the need for trial and error and leads directly to the solution.

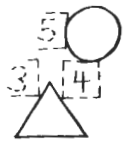
Consider the information given below:



The sum of counters in the hexagon and circle is 5; the sum in the hexagon and triangle is 3. We can use a different notation to say this same thing:

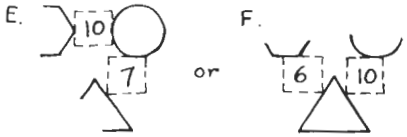
$$\hexagon + \bigcirc = 5$$
$$\hexagon + \triangle = 3$$

The number of counters in the hexagon — whatever it is — remains the same. Thus, there must be two more counters in the circle than there are in the triangle. We know that there are four counters all together in the circle and the triangle:



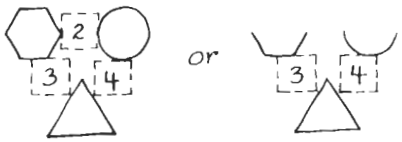
We also know that there are two more counters in the circle than in the triangle. So the real question is: what two numbers have a difference of 2 and a sum of 4? Three and one fill the bill, so there must be three counters in the circle and one in the triangle. Once we know that, it is a simple matter to figure that there must be two counters in the hexagon.

The strategy just described will work in any of the problems on the activity page. For example:

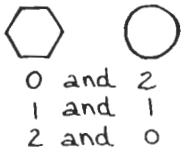


In E, there will be three more in the hexagon than in the triangle. In F there will be four more in the circle than in the hexagon.

(1) Consider this problem:



Since the number of counters in the triangle remains the same, there must be one more counter in the circle than in the hexagon. But since the sum of those two must be 2, there are only three options:

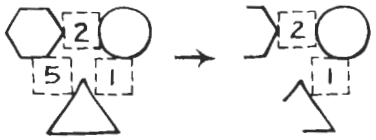


In no case is there just one more in the circle than in the hexagon.

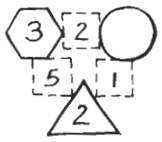
This situation leads us to consider "fractions" of beans. If we could put 1/2 bean in the hexagon and 1-1/2 beans in the circle, the problem would be solved, for

$$1\tfrac{1}{2} - \tfrac{1}{2} = 1$$
$$1\tfrac{1}{2} + \tfrac{1}{2} = 2$$

(2) Consider this problem:



There is one more counter in the hexagon than in the triangle. Since the sum of those numbers is 5, there must be three counters in the hexagon and two in the triangle.

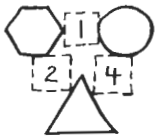


But now to get the other sums to work, we must invent "anticounters".

When a counter and an "anti-counter" meet, they both disappear into thin air. If three counters and one anti-counter meet, the vanishing act takes place and two counters are left. Anti-counters might also be called "negative counters" and we could write about them in this way:

$$3 + {}^{-}1 = 2$$

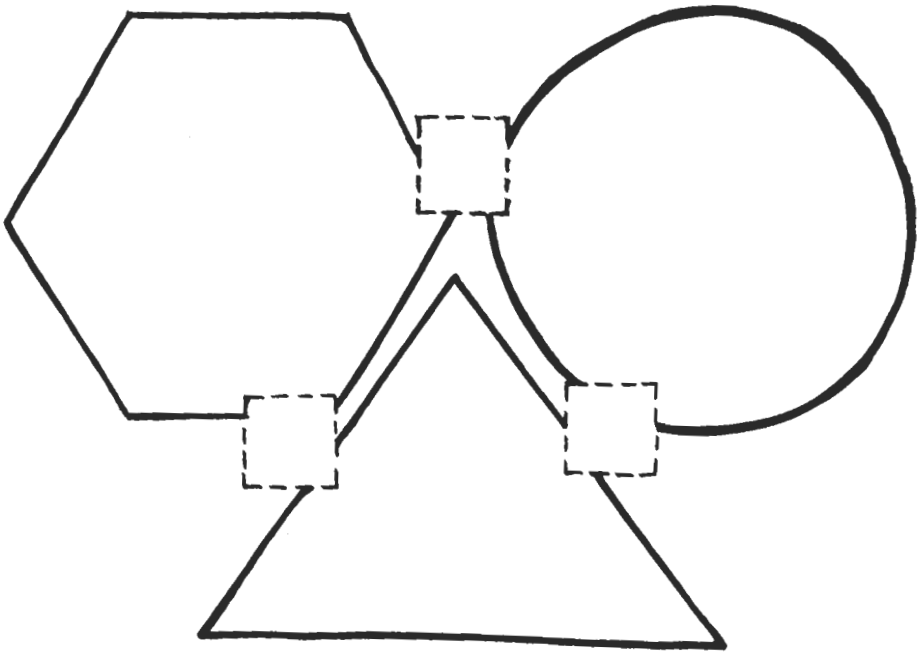
In examples such as:



fractions of counters and fractions of anti-counters will come up:

In the triangle — 2½ counters
In the circle —— 1½ counters
In the hexagon — ½ "anti-counter"!

These suggested extensions suggest a novel and powerful approach to elementary algebra, particularly if they are considered in combination with the activities on the following pages.



4 + 9 = 13

5 + 4 = 9

9 + 5 = 14

1 + 2 = 3

1 + △ = 4

⬡ + △ = 5

⬡ + △ = 6

○ + ⬡ = 9

△ + ○ = 11

⬡ + △ = 5

○ + △ = 10

○ + ⬡ = 7

⬡ + △ = 11

⬡ + ○ = 9

△ + ○ = 14

△ + ○ = 7

⬡ + △ = 10

○ + ⬡ = 11

○ + ⬡ = 8

△ + ⬡ = 12

△ + ○ = 14

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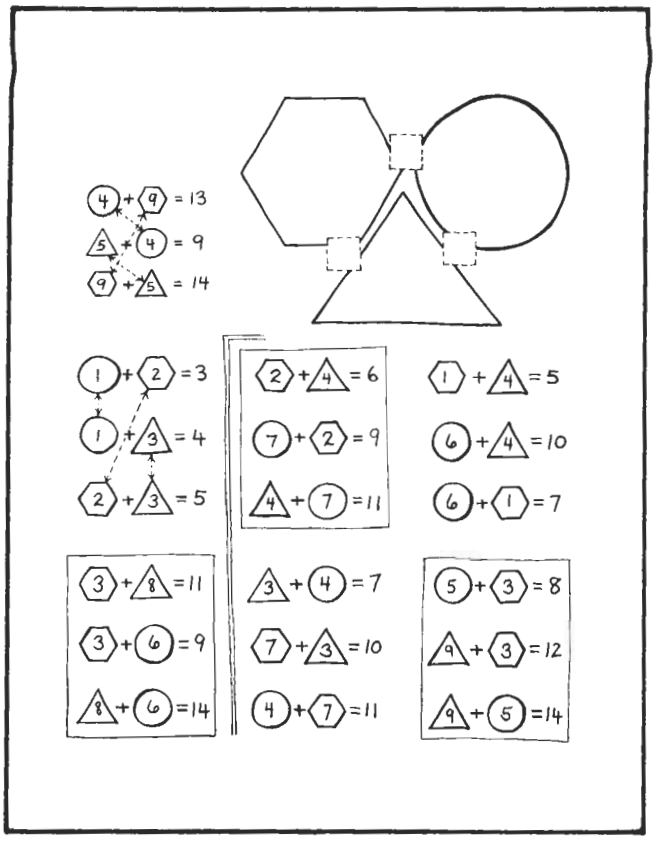


Introducing the Problem

This activity is a natural extension of the previous activity. There are groups of three statements which give information about how counters were arranged in the shapes in the large diagram. The problem is to figure how many counters would have been in each shape to make the statements true. If the frames in the statements have the same shape, the same number must be written in them. If the frames have different shapes, the numbers written in them may be different or may be the same. It may be helpful to use counters in the large frames to figure solutions, as was done in the previous activity.

The first problem to the left of the double line has been completed. The second problem has been partially solved. Once we write a '2' in the hexagon in the third statement, we can easily see that '3' belongs in both triangles.

In the third problem to the left of the double line, the number statement

$$\hexagon + \triangle = 11$$

says: "The number of counters in the hexagon plus the number of counters in the triangle equals 11." Likewise, the statement

$$\hexagon + \bigcirc = 9$$

says: "The number of counters in the hexagon plus the number of counters in the circle equals 9."

And the statement

$$\triangle + \bigcirc = 14$$

says: "The number of counters in the triangle plus the number of counters in the circle equals 14."

The solution is:

$$\hexagon = 3,\ \bigcirc = 6 \text{ and } \triangle = 8$$

There should be 3 counters in the hexagon, 6 in the circle, and 8 in the triangle.

Talking About the Problem

After discussing any strategies the children used, it might be enlightening to explore this strategy, if no one has used it. We can look at pairs of statements such as those in the problem beyond the double line.

$$\hexagon + \triangle = 6 \text{ and } \bigcirc + \hexagon = 9$$

Now

$$\text{If } \bigcirc + \hexagon = 9, \text{ then } \hexagon + \bigcirc = 9$$

We can then write the first statement together with the modified second statement, and we can indicate the sums with sketches of counters.

$$\hexagon + \triangle = \text{::}$$
$$\hexagon + \bigcirc = \text{:::}$$

These statements say: "The number of counters in the hexagon plus the number of counters in the triangle is 6. The number in the hexagon plus the number in the circle is 9." Of course, the number in the hexagon stays the same. Thus, since the first sum is 6 and the second sum is larger than 6, namely 9, there must be three more counters in the circle than in the triangle. We also know that

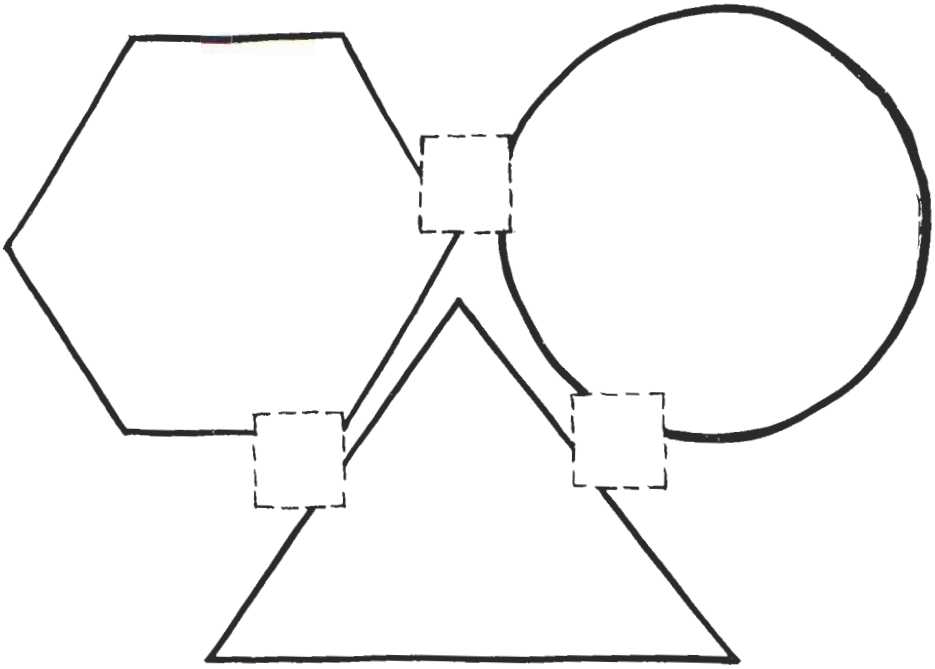
$$\triangle + \bigcirc = 11$$

Thus, we need to find two numbers whose difference is 3 and whose sum is 11; 4 in the triangle and 7 in the circle would do the job. Then we can easily figure that there must be 2 in the hexagon.

This same activity will arise later in high school algebra in a chapter titled: "Solving Families of Simultaneous Equations". Unfortunately, learners will probably have no counters, even in the beginning. The problem we just discussed would appear as:

Solve for x, y and z.

$$x + y = 6$$
$$z + x = 9$$
$$y + z = 11$$



1 + 5 = 6

+ 1 = 8

+ = 12

+ = 7

+ = 8

+ = 9

+ = 13

+ = 17

+ = 14

+ = 13

+ = 8

+ = 9

+ = 10

+ = 15

+ = 13

+ = 10

+ = 16

+ = 12

+ = 11

+ = 15

+ = 12

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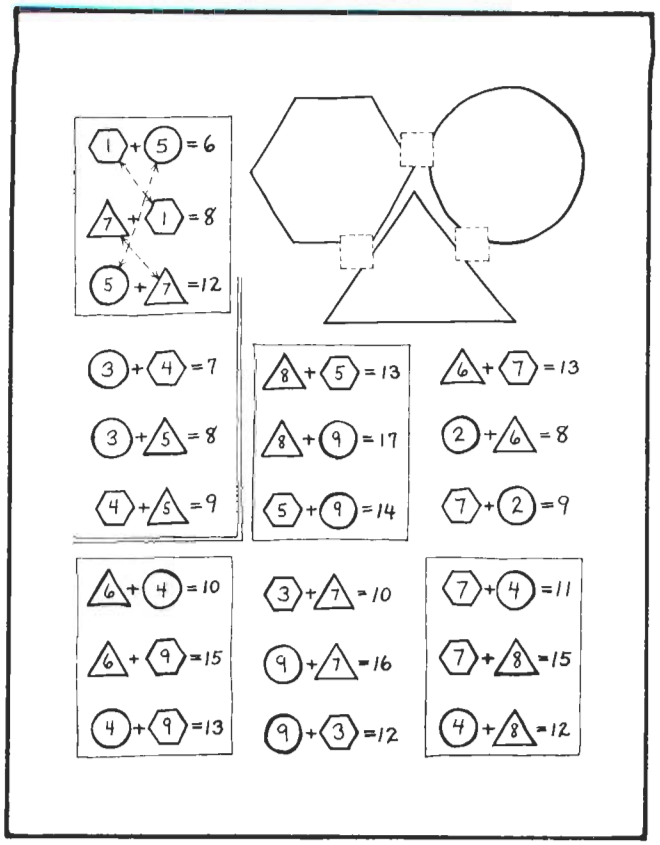


Introducing the Problem

The procedures for doing this activity are similar to those given for the two previous activities. The problem is once again to find numbers to complete each group of three statements. When working on this page the focus of the activity might be shifted slightly, and children might be encouraged to find out more about the three sums in each example. Do groups of three sums have anything in common? Is there anything special about the numbers that are sums?

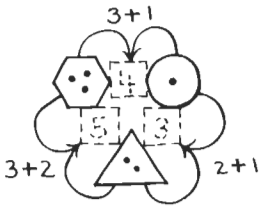
Talking About the Problem

(1) Could we select any three numbers as sums and find arrangements of counters to fit? No, not unless we allow fractions of counters and invent "anti-counters" (discussed in comments for a previous activity).

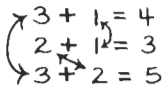
(2) There are solutions using counting numbers *for all examples on this activity page.* What similarities do you notice in the groups of sums? Someone may notice that in every example there are either

(a) three even sums, or
(b) two odd sums and one even sum.

Because (a) and (b) are true, the "sum of the sums" in each example is an *even* number. Why does this happen? Consider the following situation:



The number of counters in each shape is included in *two* sums:



So, the total of the three sums must be *twice* the number of counters used; hence it must be an even number. In this case,

total of sums $4 + 3 + 5 = 12$

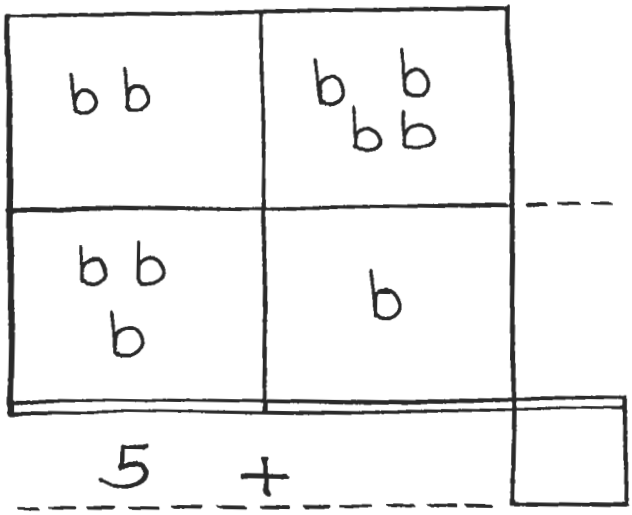
and

total of counters $3 + 1 + 2 = 6$

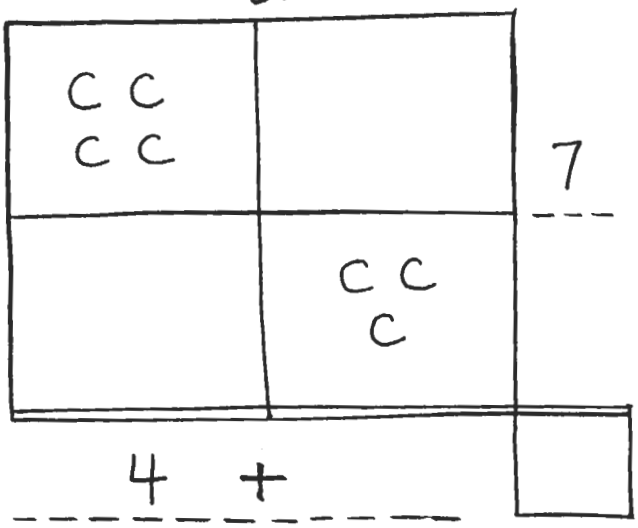
Thus, if there is one odd sum in an example, there must be another odd sum so that the total of the two odd sums is an even number. The third sum must then be even to insure that the total of all three sums will be even. Of course, if all three of the sums are even to begin with then the total will also be even.

This line of reasoning will also lead us to the conclusion that the total of the three sums will be exactly *twice* the number of counters used. So it is possible to figure out right away how many counters are needed for all three shapes.

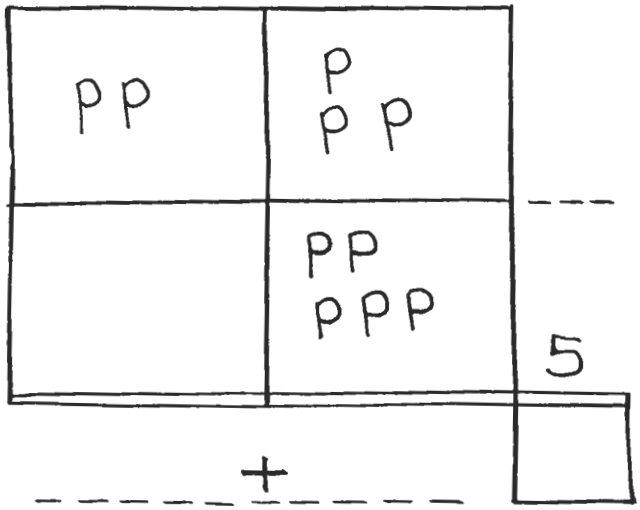
A.



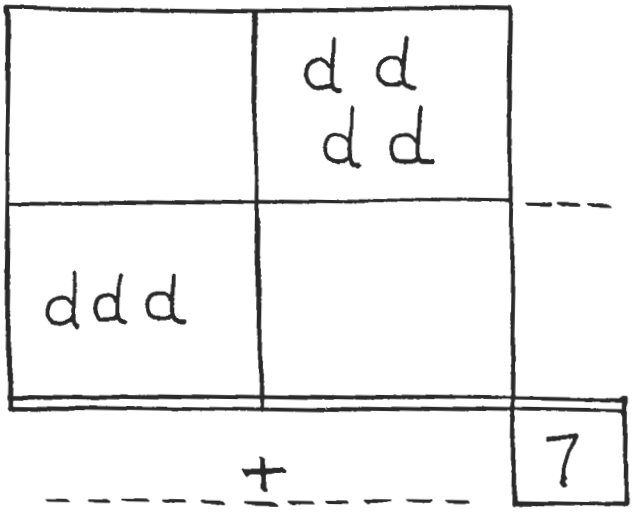
B.



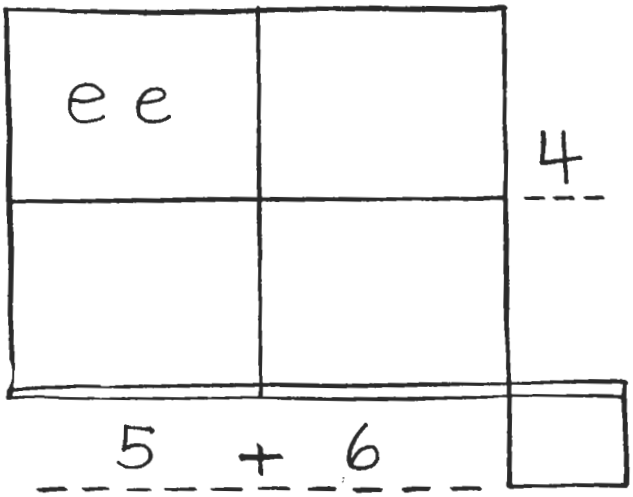
C.



D.



E.



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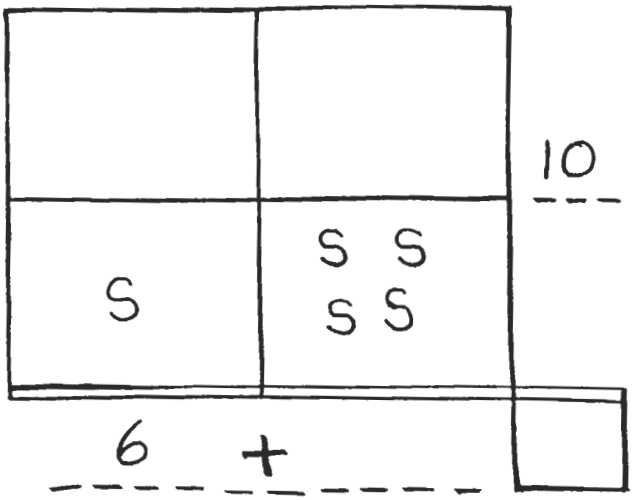

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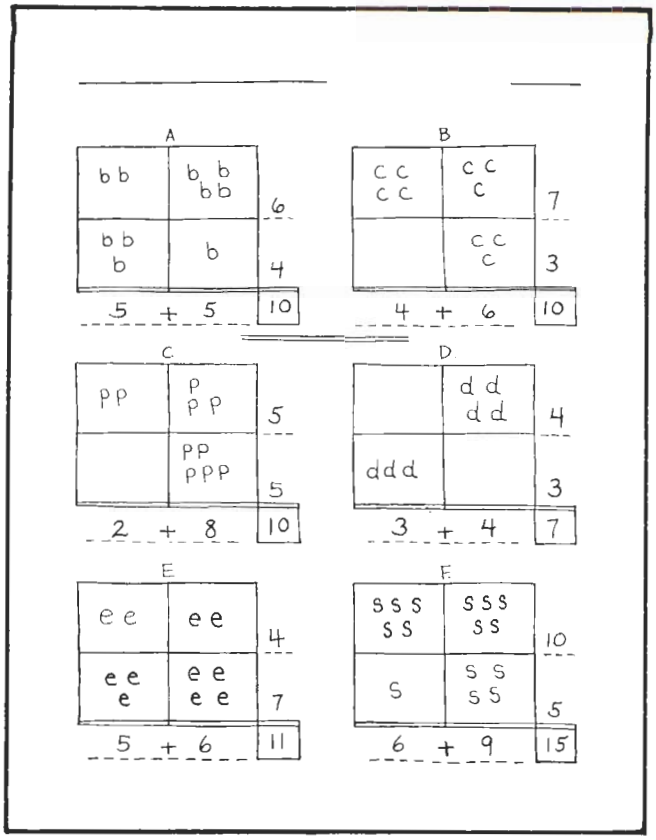


Introducing the Problem

Many years ago we named this activity the "cross number problem". In a cross number problem, there are four numbers (or numbers of objects) in a two-by-two array. Six sums must be found and recorded

(1) the sum of the numbers in the two top boxes

(2) the sum of the numbers in the two bottom boxes

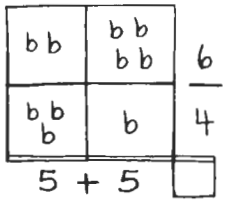
(3) the sum of the numbers in the two left-hand boxes

(4) the sum of the numbers in the two right-hand boxes

(5) the sum of (1) and (2) above

(6) the sum of (3) and (4) above.

For example, in Problem A, there is a total of six letters in the top two small boxes. There is a total of four letters in the bottom two; there is a total of five letters in the two left-hand boxes and five in the right-hand boxes. If we record these results to the right and at the bottom of the figure we have:



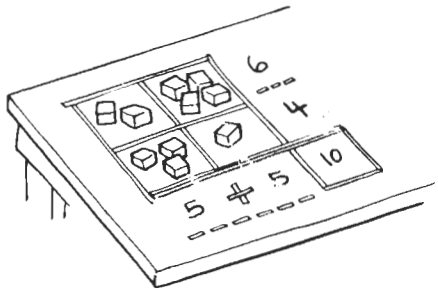
Now 5 + 5 = 10, and 6 + 4 = 10, so we record '10' in the small corner box at the lower right of the figure. Whenever it happens that the number we write in that corner box is the correct sum for both of the addition facts, we say that the problem "works".

Problem B introduces a new twist. The problem indicates that there are to be seven c's in the two top sections; but there are only four c's shown. There is a simple rule to follow when drawing additional letters. You can draw in letters only if the section in which you draw them was originally empty. So, we must draw three c's in the top right-hand box.

There is to be a total of four c's in the two left-hand boxes. There are already that many in the top left section. Consequently, we do not need to draw in any c's. Now there is a total of three c's in the bottom two sections, and six c's in the two right-hand sections. So we have, in effect, a situation which works if we write '10' in the corner box.

$$\begin{array}{r} 7 \\ +\ 3 \\ \hline 4 + 6 = \boxed{10} \end{array}$$

If there is any confusion, a table can be divided into four sections with masking tape. Then objects can be used instead of letters, and numerals written on chunks of tagboard can be used to indicate sums. In such a context, problem A would look like this:



As children understand the procedure more fully, they may be able to move on to the activity on this page. Some may need to act out such cross number problems on a table until their confidence grows, and save the problems on the printed page for later.

Talking about the Problem

Even though you may not be at all surprised that these problems "work", this outcome often fascinates children. In fact, if cross number problems are introduced as in the activity pages which follow, most adults are surprised and often fascinated when they do work.

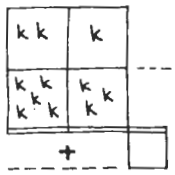
Do all the examples on this page "work"? It turns out that they do. If some children are puzzled, let them talk to each other — argue, disagree, wonder about this or that. The only source of real difficulty might be with the rule that you can draw letters in sections *only if* they were originally empty. Letters must be written in all examples except D. While there are two empty sections there, they must remain empty to make the problem work.

Please consider being non-committal about any connection between the total number of letters and the number in the box in the lower right hand corner. However, if some children notice the relationship, they might be encouraged to talk to others about it.

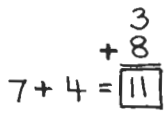
Extensions and Surprise Endings

If children have not noticed the relationship between the total number of letters and the number written in the lower right-hand corner, time might be taken to consider the matter. How many b's are drawn in problem A? There are ten. What number did we write in the little box there? We wrote '10'. Is it an accident that they are the same number? What happens in the other examples? It turns out that in every example, the number in the box is the total number of letters in the array.

Let's see if we can make up an example that works. Different volunteers can suggest the number of letters to write in each of the four sections. This may lead to something like:



Does it work? Let's see.



It does work! Other examples put together without any planning will also work. You might then suggest that the search begin for examples that *don't* work. This is much more difficult. In fact, if any don't work, there is a mistake or a misunderstanding somewhere. Why?

Suppose that the letters in problem A were cows, and that the lines were fences. If Farmer Brown drove the cows in the northern fields to the east, there would be six cows; if he drove those in the southern fields to the east, there would be four cows. Then if he drove them all south to market, he would have ten cows to sell. But suppose all gates are on the south side of the fields. He must drive the cows south: five from the western fields, and five from the eastern fields. Then he could drive all ten to market. It would certainly be surprising if he didn't end up with the same number of cows regardless of the direction in which he drove them!

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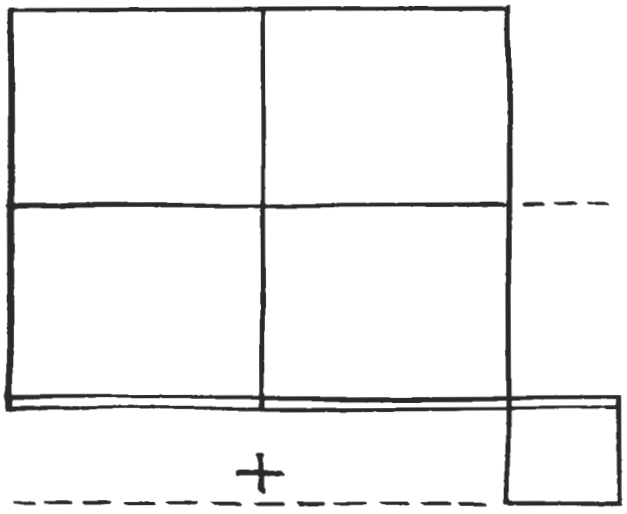

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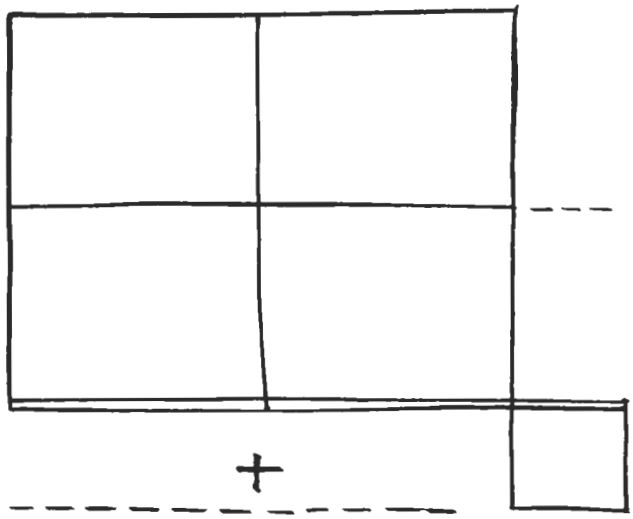

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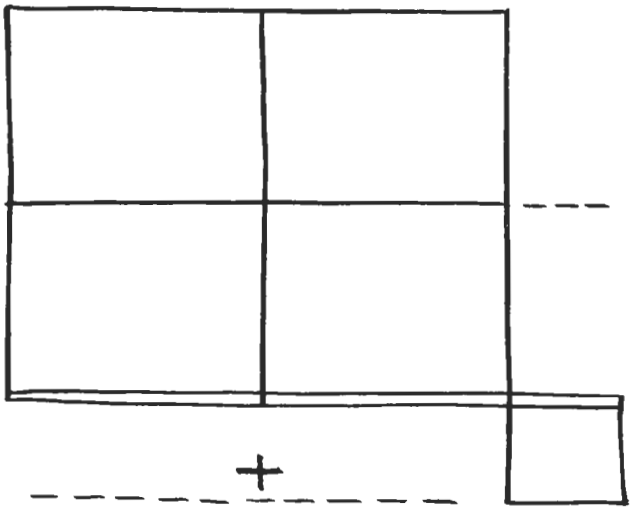

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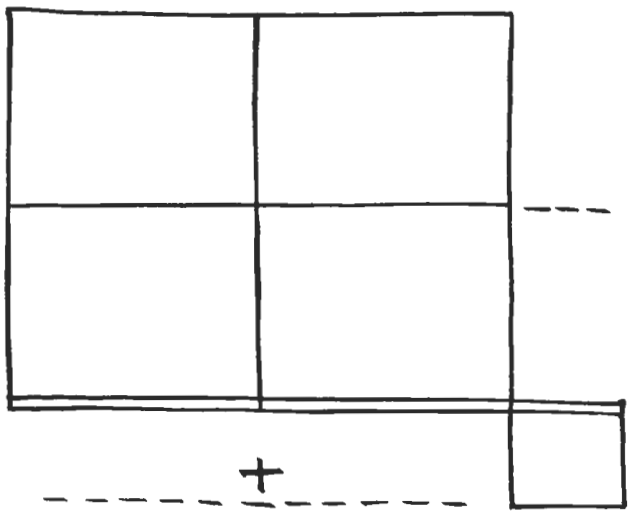

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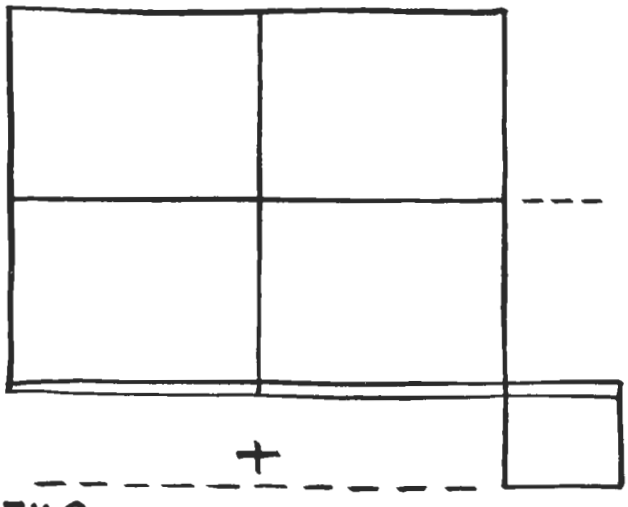

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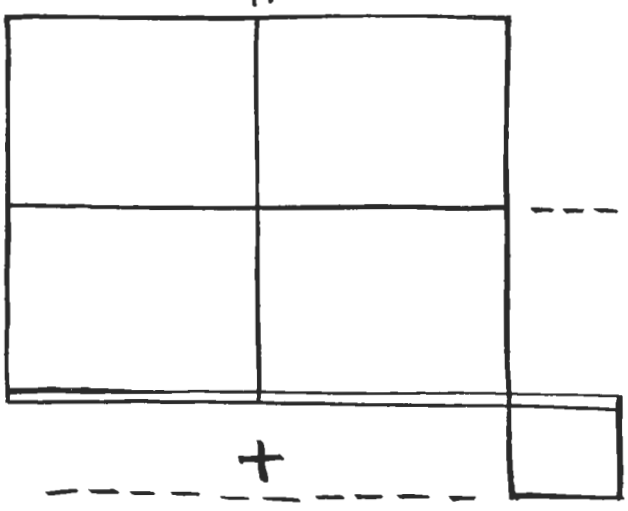

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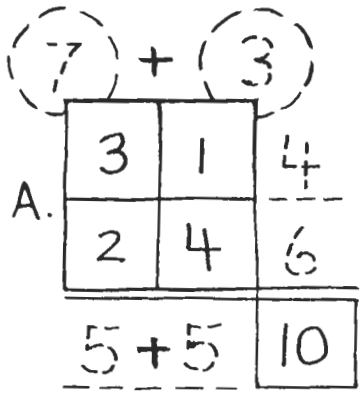

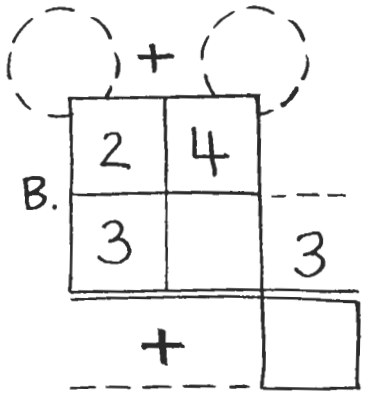

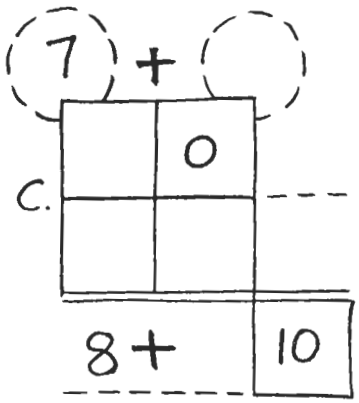

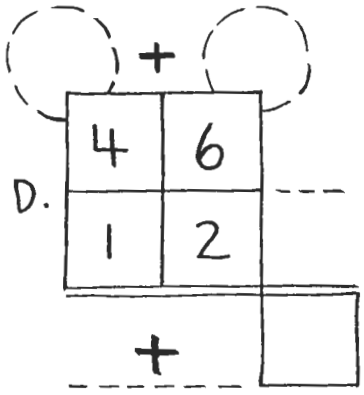

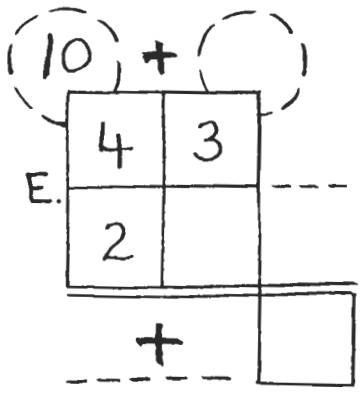

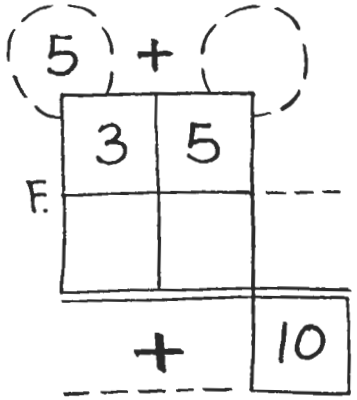

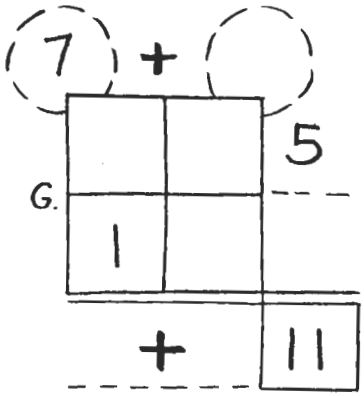

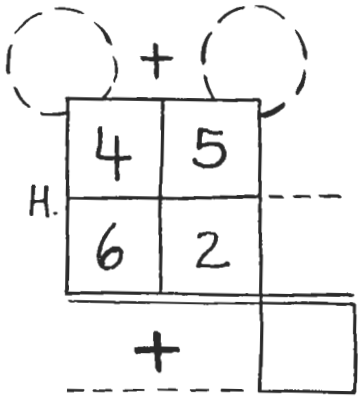

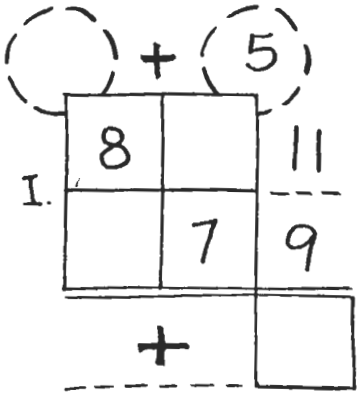

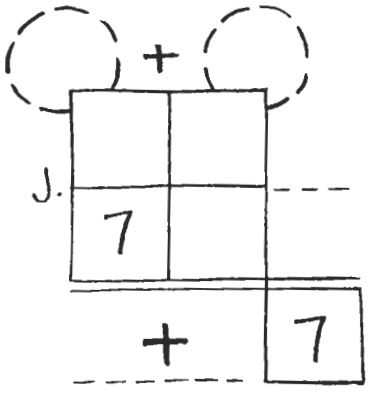

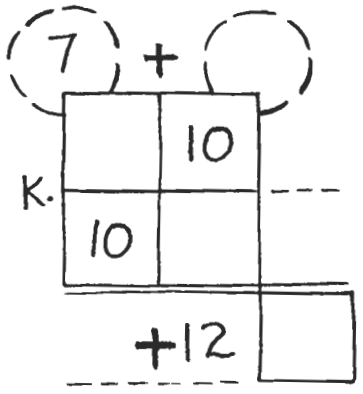

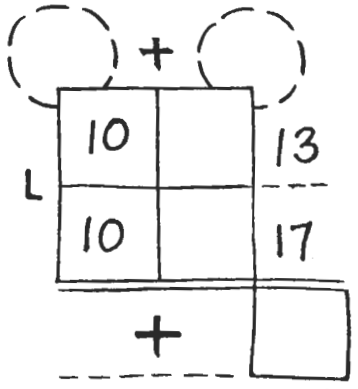

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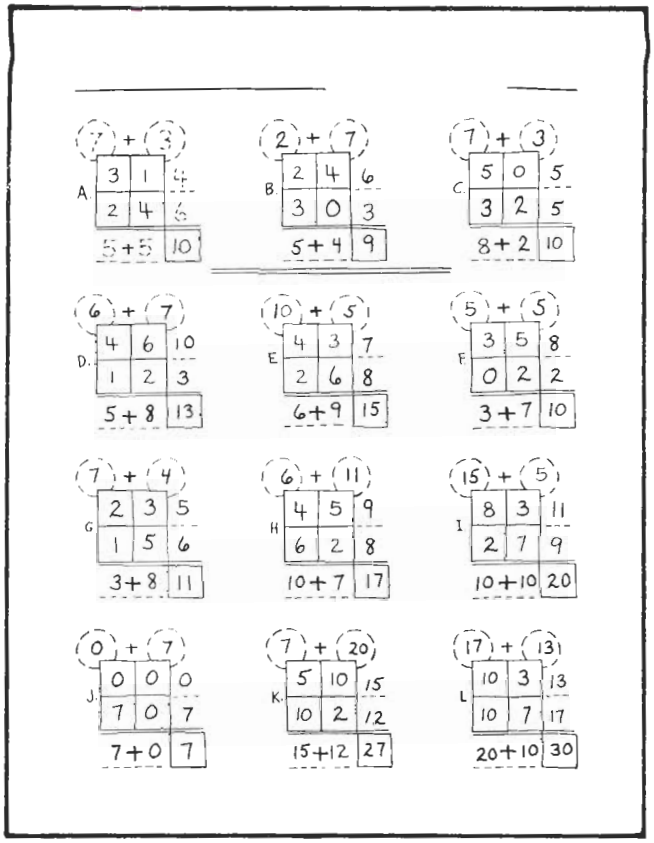


Introducing the Problem

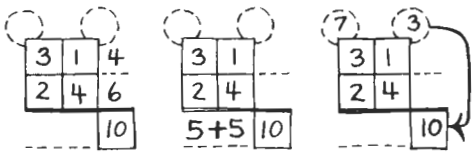
In these "cross number problems" we would like to find four numbers to write in a two-by-two arrangement in such a way that there are three ways to get the same sum:

(1) by getting the sum of the two top boxes, the sum of the two bottom boxes, and then the sum of those two sums;

(2) by getting the sum of the two left-hand boxes, the sum of the two right-hand boxes, and then the sum of those two sums; and

(3) by getting the sums along each diagonal and then taking the sum of those two sums.

For example, in problem A,



When it happens that all three sums are the same, we say that the problem "works". Each example on this page is to be completed so that it "works".

In problem B, one of the numbers in the two-by-two array is missing. But we do know that there must be a total of three in the bottom two boxes. Thus, it is easy to figure that the missing number must be 0, since 3 + 0 = 3. If we complete the arithmetic, we find that the problem works if we put '9' in the small box in the bottom right-hand corner.

In problem C, there is very little information given. There is a '0' in the upper right-hand box. Where do we start? We know that the sum along one diagonal is 7, that the sum of the numbers in the two left-hand boxes is 8, and that the number in the small box at the lower right is 10. We could start any place and use trial and error. This strategy would eventually lead to numbers that would work, but there is another way to start. Notice that the bottom line, "8 + __ = 10", can be made into a true statement:

$$8 + \underline{2} = 10$$

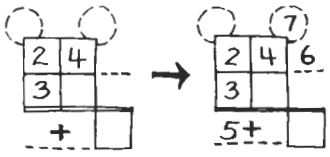
That helps, for now we know that the sum of the numbers in the two right-hand boxes is two and that one of them is zero. So we know that '2' must be in the bottom right-hand box. That helps with the diagonal, for we know that $2 + \underline{5} = 7$. The rest is simple arithmetic and in the end, all three sums are equal to 10.

Talking about the Problem

Did you find numbers so that all of the examples work? They can all be completed so that they work. Did everyone complete each example in the same way? Could any examples be made to work in more than one way? No, each one has a unique solution.

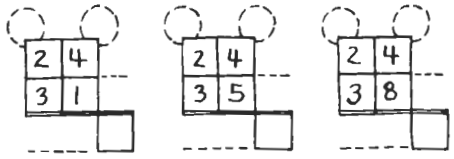
Extensions and Surprise Endings

What would happen if we removed one of the pieces of information given in any example? Suppose that in problem B, for instance, we remove the '3' given as the sum of the two lower boxes. That would give us



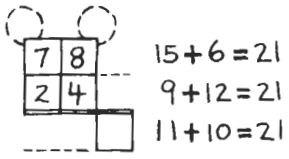
Now where do we go? Suppose we try each of the numbers 1, 5 and 8 in the fourth box in the

arrangement. We would have three different examples:



If we complete the arithmetic properly, they all work with triple sums of 10, 14 and 17 respectively. There seems to be no special charm about 1, 3 or 8. Could we put any number in that fourth box in this problem? How about 0 or 10? Both of those will work as well. Could we find a number to place in that empty cell so that the example won't work? Try 100. Many children think that big numbers behave in their own ways that are unlike those of small numbers. But it will turn out that 100 will work as well! Someone might hazard the opinion that fractions will spoil the works. Let the children who can handle such arithmetic test that opinion.

Suppose we remove two pieces of information in those examples that start out with four numbers given. What happens to the solution? That's a question that children can explore. Suppose we throw all the information away and start from scratch. Will *any* four numbers work? Try creating an array with any four numbers that come into your head — like the last four digits of your telephone number:



It works! It begins to look as if our search for examples that don't work will be unsuccessful. If there are any doubts, the story about Farmer Brown given at the end of the comments on the previous activity page may help.

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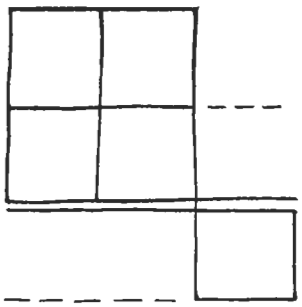

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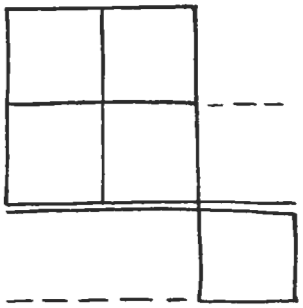

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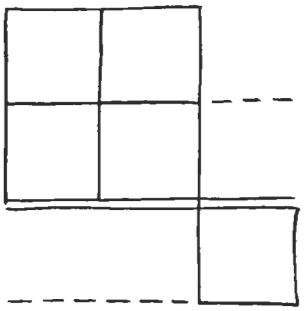

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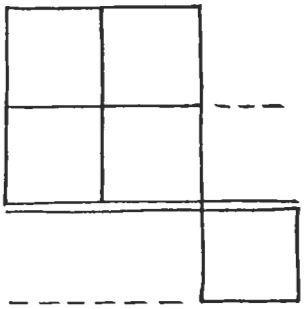

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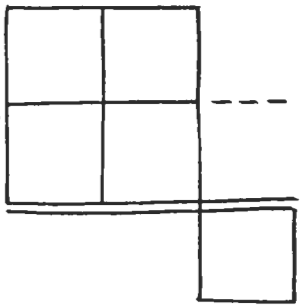

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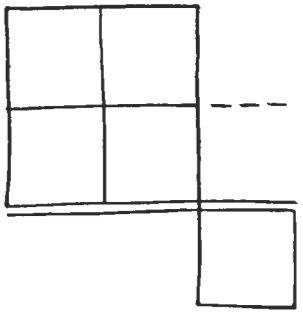

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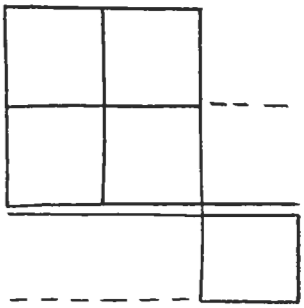

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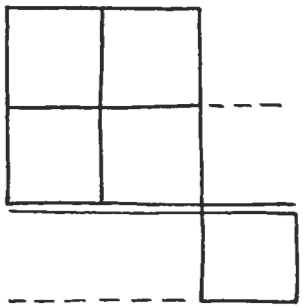

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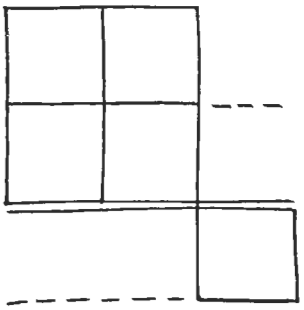

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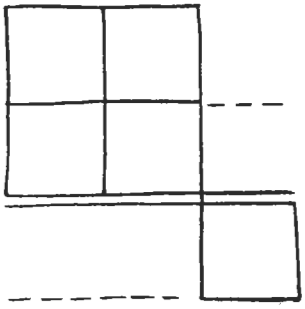

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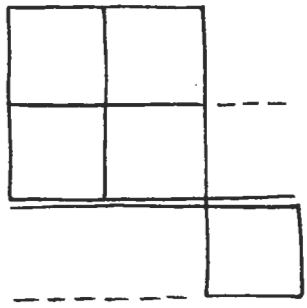

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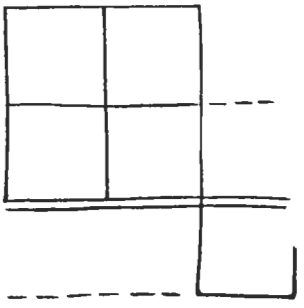

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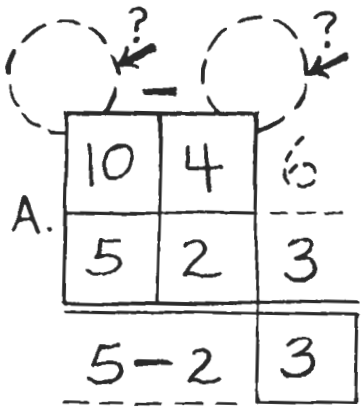

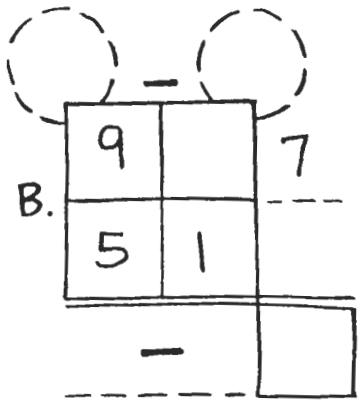

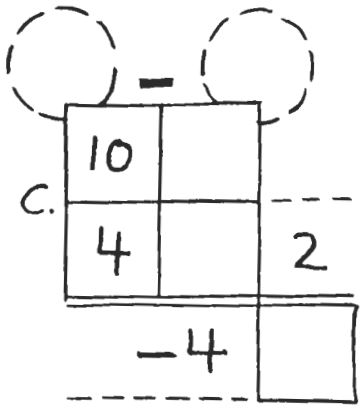

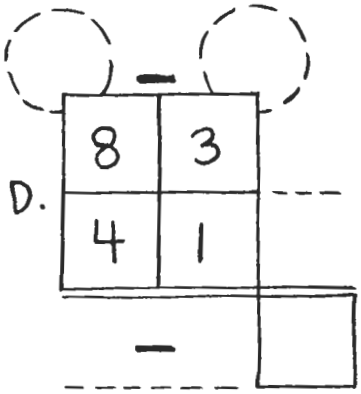

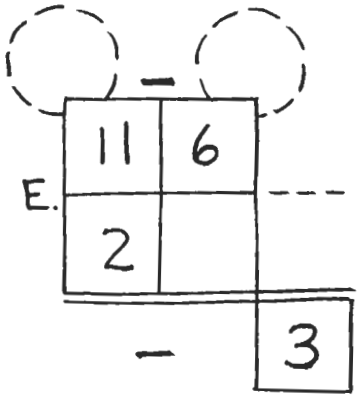

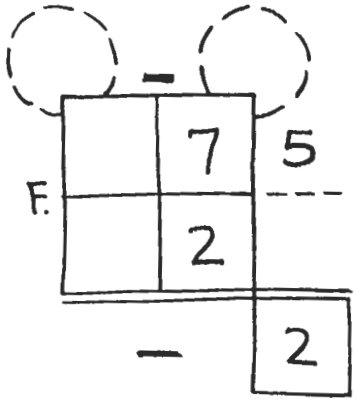

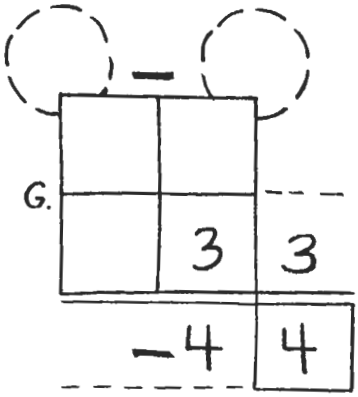

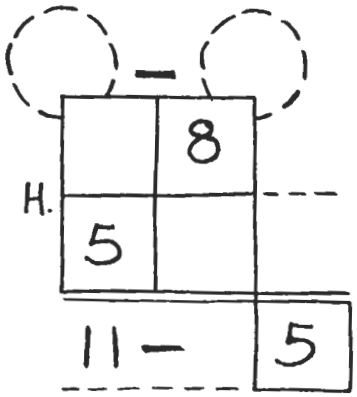

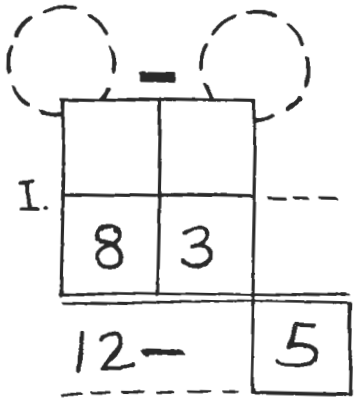

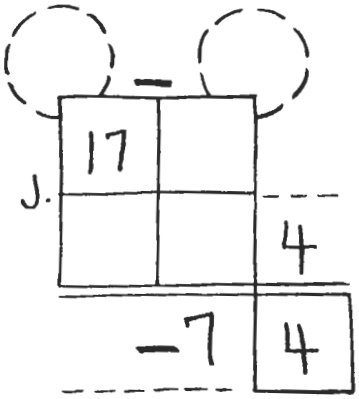

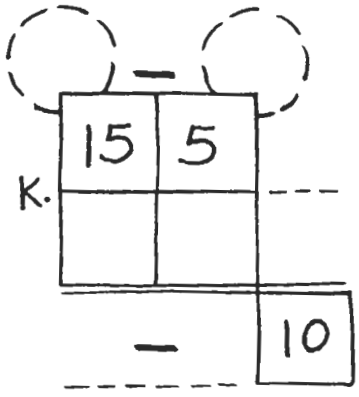

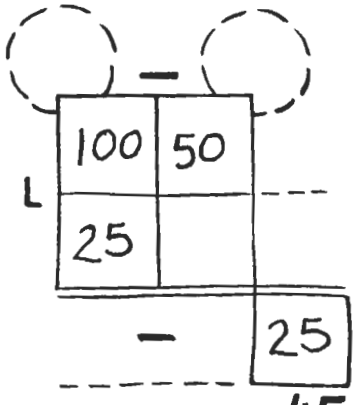

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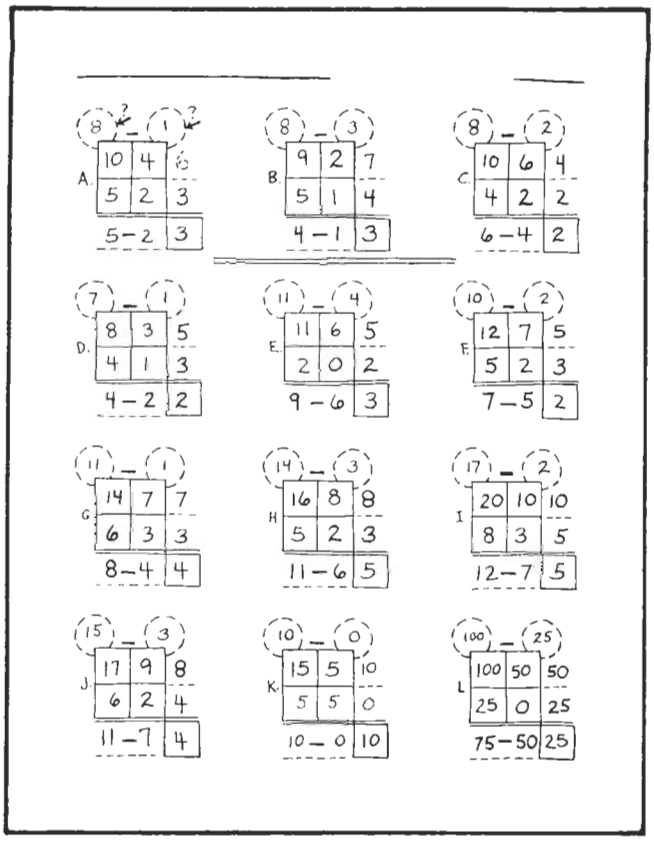
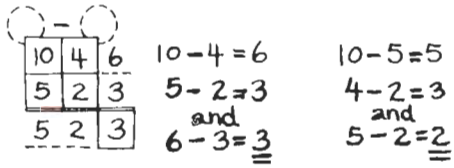
(Some of the appropriate responses are shown here.)



Introducing the Problem

We assume that the children are familiar with the previous cross number problems which involved addition. In this variation we will be subtracting instead of adding, as we did with the previous activity. But trouble develop s right at the outset — serious "ear trouble". Everything goes well initially:



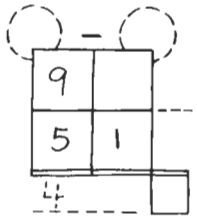
But we get nowhere when we look at the diagonals and subtract: 10 - 2 = 8 (or 2 - 10 = -8), and 5 - 4 = 1 (or 4 - 5 = -1). We can't find a way for the ears to give us a difference of three.

Let's try problem B. The empty cell in the two-by-two arrangement must be '2', so that the top line would indicate '9 - 2 = 7'. Everything begins to work toward a '3' in the corner: 9 - 2 = 7; 5 - 1 = 4; and 7 - 4 = 3. Also, 9 - 5 = 4; 2 - 1 = 1; and 4 - 1 = 3. But there is more "ear trouble": 9 - 1 = 8 (or 1 - 9 = -8) and 5 - 2 = 3 (or 2 - 5 = -3). That creates a real impasse. So we plod on to problem C. Again everything works — except the ears. Don't commit yourself to anything about the situation except that we can expect a lot of "ear trouble" as we move ahead.

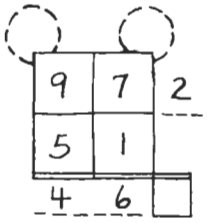
Talking about the Problem

Did you continue to have "ear trouble"? Lots of it! Did you find a way to complete each problem so that it "worked" except for the diagonals? There is at least one solution that works for each one. How many pieces of information are given in each problem? There are four, except in problem K, which has only three. Do you have any ideas about why that is a special case?

Suppose we remove one of the four bits of information from problem B. If we eliminate the '7', for example, we have:

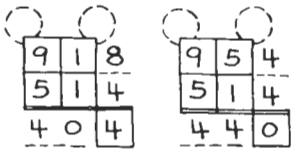


We have to write '4' as the difference of 9 and 5, and '4' as the difference of 5 and 1. Then we might try '7' in the open cell. That leads to:



And now we've got more than "ear trouble".

Try numbers other than seven in that fourth cell. You will discover that all numbers, 1 through 5, will work, if we don't always consider the ears.

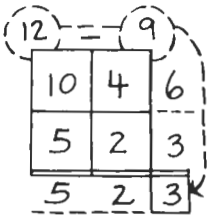


It might be worth noting that the first example above suggests that ear trouble might not *always* develop: 9 - 1 = 8, 5 - 1 = 4, and 8 - 4 = 4, which is the number in the lower corner box. Perhaps the repeated "1's" make this a special case.

It turns out that in this subtraction version of the cross number problem, we need to carefully plan ahead to make it work horizontally, vertically and diagonally.

Extensions and Surprise Endings

I need to introduce a special note. A third grade child once told me that she could clear up all that "ear trouble". I urged her to check her work, since I didn't want her to be embarrassed in front of the class. She persisted, "You see," she said, "when you come to the diagonals, you simply add into the ears and then subtract. See?"



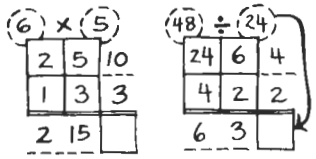
"It works!"

"Yes," I cautioned, "it works in that example, but what about the other examples?"

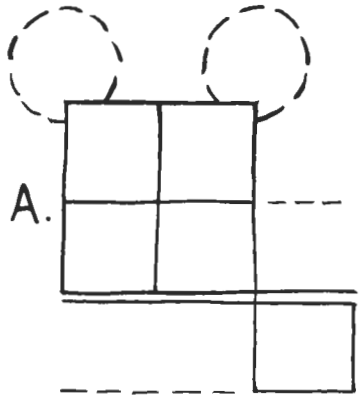
"Well," she put it to me, "why don't you find out for yourself? That's what I had to do!"

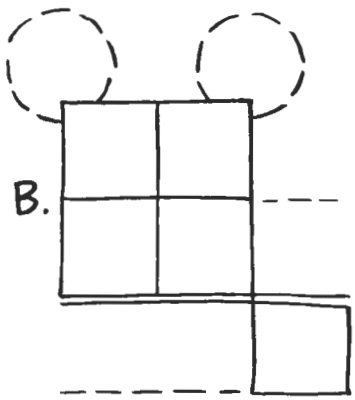
That was my comeuppance. Was she right?

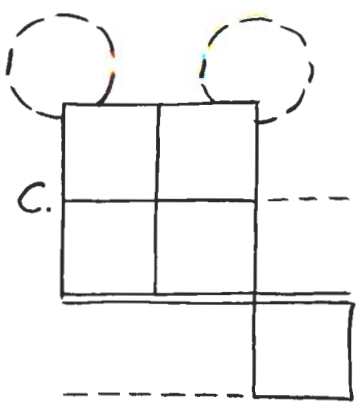
There is a sequel. A few days later I told this story to a fifth grade group. Billy was sitting in the corner and looked at me with a blank stare. Slowly he raised his hand. As I called on him, he closed his eyes. "I'll bet the cross number problems will work fine in multiplication, but in division you'll have to plan ahead or hit fractions. Also, you will have ear trouble, but if you multiply on the diagonals, you will get out of trouble and it will all work fine."

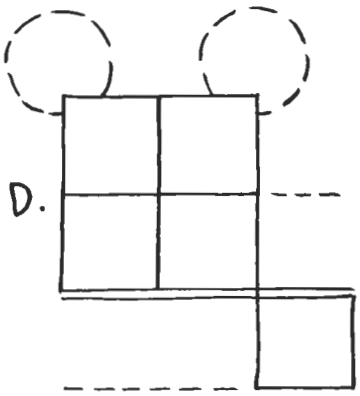


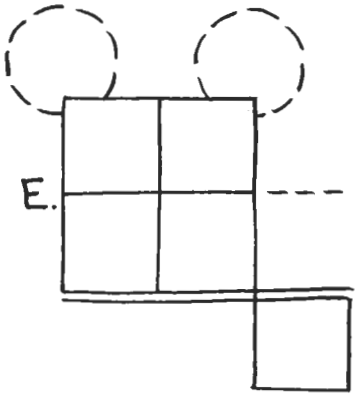
This time I didn't have to ask if this was a special case. Billy had put to use a fact I knew well but hadn't utilized. There is a very close parallel between the way addition and subtraction are related and the way multiplication and division are related. Billy's intutition had outdistanced my more extensive knowledge. Those are the kinds of classroom experiences that are both humbling and thrilling.

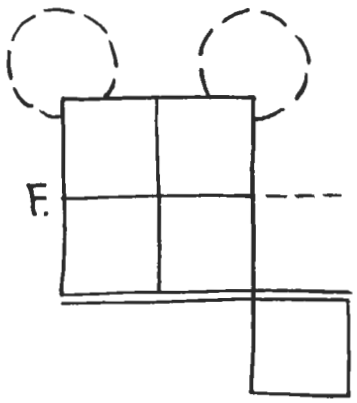


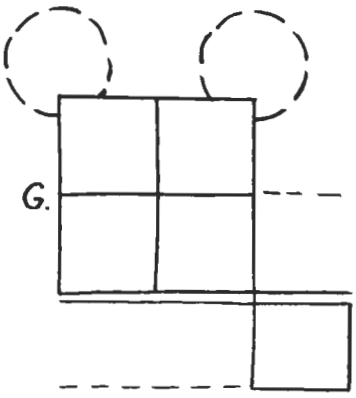


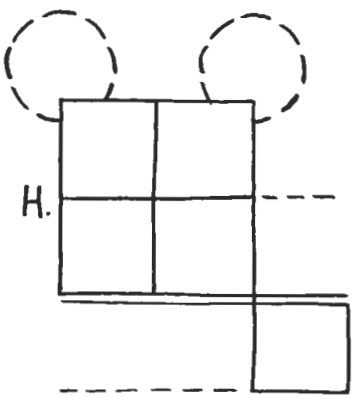


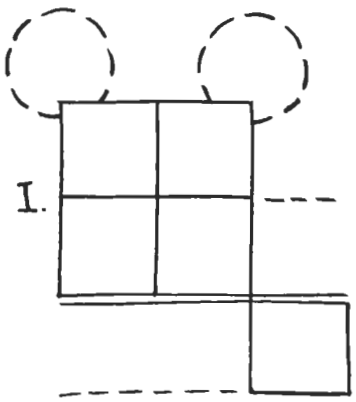


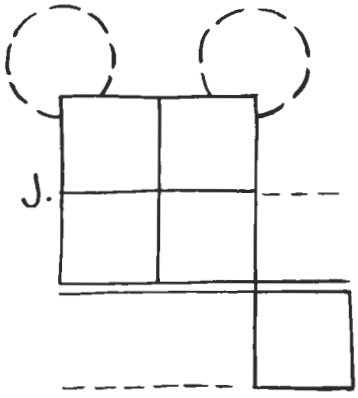


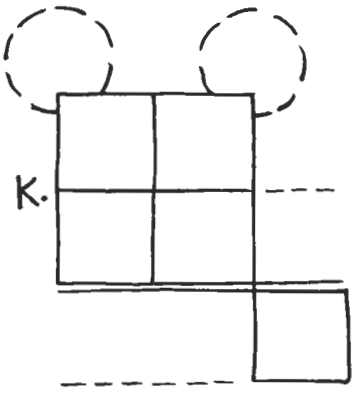


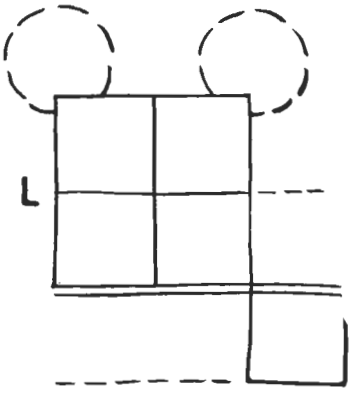












F45 and F44

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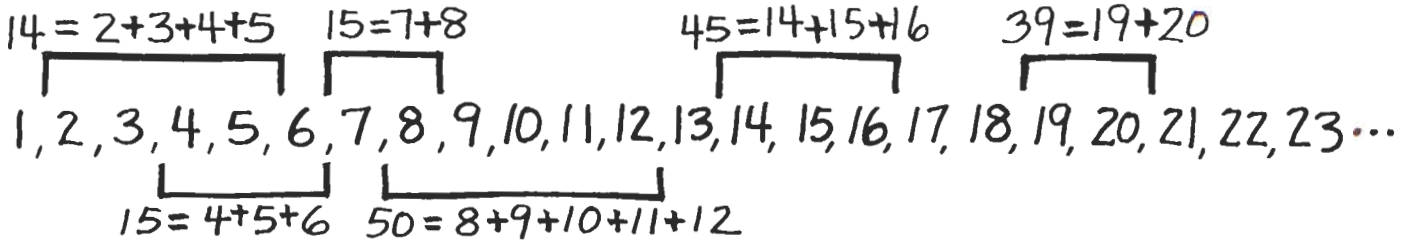

1 = ~~X~~	19 = 9+10	37 =
2 = ~~X~~	20 =	38 =
3 = 1+2	21 =	39 =
4 = ~~X~~	22 =	40 =
5 =	23 =	41 =
6 =	24 =	42 =
7 =	25 =	43 =
8 =	26 =	44 =
9 =	27 =	45 =
10 =	28 =	46 =
11 =	29 =	47 =
12 =	30 =	48 =
13 =	31 =	49 =
14 =	32 =	50 =
15 =	33 =	51 =
16 =	34 =	52 =
17 =	35 =	53 =
18 =	36 =	54 =

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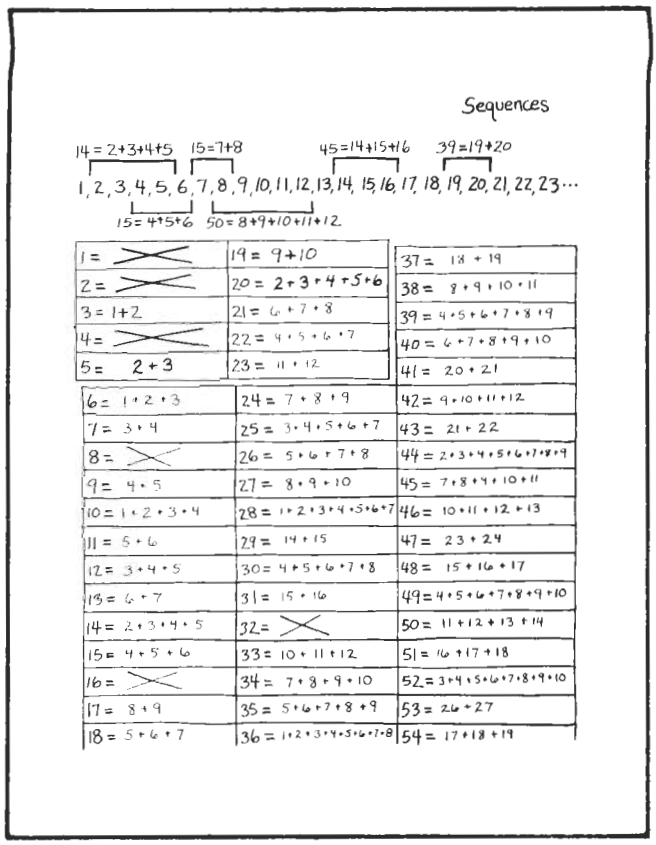
(Some of the appropriate responses are shown here.)



Introducing the Problem

In this activity, we are given a list of positive whole numbers (1, 2, 3, 4, . . .). The list can go on as far as you want it to — and then still further. A "sequence" of numbers can be thought of as a group of "neighbors" in the list. The problem is to find out which numbers, 1 through 54, can be indicated as the sum of a sequence of two or more numbers from the list. The requirement that we use a sequence of two or more numbers rules out solutions such as:

$$1 = 1 \text{ and } 2 = 2 \text{ and } 4 = 4$$

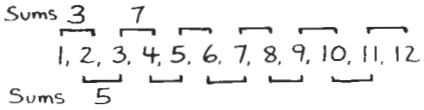
The smallest number than can be made as the sum of two or more neighbors is 3:

$$3 = 1 + 2$$

A sequence may contain as many numbers as needed. For example, a sequence of four numbers is required to show 10:

$$10 = 1 + 2 + 3 + 4$$

Since sums must be shown for 54 numbers, it is probably wise to look for some shortcuts to reduce the work. Before individual work begins, the question could be asked: "Are there any patterns that might help?" Let's look at pairs in the list.



A pattern is apparent and may lead to the reasonable conclusion that all odd numbers, except 1, can be taken care of quite easily.

However, the even numbers may present some difficulty. In fact, neither 2 nor 4 can be shown as the sum of a sequence. If the sum of a sequence is to be even, it must have at least three members. This is because the sum of any pair of neighbors is an odd number. For example, to get 6 as a sum, we must use 1, 2, and 3.

$$6 = 1 + 2 + 3$$

Now suppose that we add 1 to each of the three numbers in the sequence used for 6:

$$2 + 3 + 4 = 9$$

The sum of the three numbers increased by 3, and the result was 9, an odd number. But if we add 1 again to each number in that sequence, we have

$$3 + 4 + 5 = 12$$

The sum increased by three again, and we got another even number.

$$6 = 1 + 2 + 3$$
$$9 = 2 + 3 + 4$$
$$12 = 3 + 4 + 5$$

This pattern can be extended by adding two to each member of the sequence.

$$18 = 5 + 6 + 7$$
$$24 = 7 + 8 + 9$$
$$30 = 9 + 10 + 11$$

Look carefully at the sums of the sequences above. It appears that in each case, the sum equals three times the middle term in the sequence.

$6 = 1+2+3$ or 3×2
$12 = 3+4+5$ or 3×4
$18 = 5+6+7$ or 3×6 etc.

The even sums that have been generated are also multiples of three. Such numbers are also multiples of six. Thus, it will be easy to find sums for:

6, 12, 18, 24, 30, 36, 42, 48 and 54

At this point, you may wish to let children go on with the investigation on their own. The following could be discussed either before or after independent work.

Suppose we consider groups of four neighbors. We might start with 1, 2, 3, 4; go on to 2, 3, 4, 5; then 3, 4, 5, 6, and so on.

$10 = 1+2+3+4$
$14 = 2+3+4+5$
$18 = 3+4+5+6$

Let's look at the first sequence. The sum of the middle terms, 2 and 3, is 5. The sum of the end terms, 4 and 1, is also 5. The total sum, 10, is 2 times 5.

Sum 5
$\underbrace{1+2+3+4}_{5}$ or $2 \times 5 = 10$

This pattern continues for each sequence of four numbers:

$\overset{7}{2+3+4+5}$	$\overset{9}{3+4+5+6}$	$\overset{11}{4+5+6+7}$
$\underbrace{\quad\quad}_{7}$	$\underbrace{\quad\quad}_{9}$	$\underbrace{\quad\quad}_{11}$
$2 \times 7 = 14$	$2 \times 9 = 18$	$2 \times 11 = 22$

We can see that these sequences of four numbers are generating sums that are the doubles of the odd numbers, starting with 5. So, we have an easy way to find more numbers in our list, namely:

10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50 and 54

Earlier we showed a way to write sequences for all odd numbers and for all multiples of 6. So the only numbers left are:

8, 16, 20, 28, 32, 40, 44 and 52

Talking About the Problem

Sequences with sums of 8 and 16 are baffling. But all the others — 20, 28, 40, 44, and 52 will give in to hard work. Each of these numbers can be seen as a product that has at least one odd factor:

$20 = 4 \times 5$ $\quad$ $44 = 4 \times 11$
$28 = 4 \times 7$ $\quad$ $52 = 4 \times 13$
$40 = 8 \times 5$

Now, it should be clear that all sequences have either an odd number or an even number of terms. We showed earlier that if a sequence has three terms, its sum equals three times the middle term.

$6 = 1+\underline{2}+3 = 3 \times \underline{2}$
$9 = 2+\underline{3}+4 = 3 \times \underline{3}$
etc.

We could go on to show that if a sequence has five terms, its sum equals five times the middle term; if it has seven terms, its sum equals seven times the middle term. (Try it!)

— — — (3 times middle term)
— — — — — (5 times middle term)
— — — — — — — (7 times middle term)

So all sums of an odd number of terms have at least one factor that is odd; namely, 3, 5, 7, 9, etc.

If the sequence has an even number of terms it will have a pair in the middle whose sum is odd, for the sum of any two consecutive numbers is odd. It turns out that the sum of the numbers on either side of these two middle digits equals this same sum of the middle digits. (See above examples of sequences of four numbers.) Likewise, the sum of the next lower and higher numbers equals the sum of the middle numbers. So every sequence with an even number of digits must have a sum that equals: (the sum of the two middle digits multiplied by the number of pairs in the sequence.) Since the sum of the middle digits is odd, the sum of the sequence has an odd factor.

middle term
— — (2 × an odd number)
— — — — (3 × an odd number)
— — — — — — (4 × an odd number)

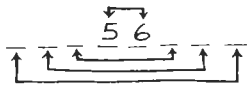
So prospects look bright for finding a sum for our leftover numbers, for they each have at least one odd factor. It turns out that each one can be written as a sum of a sequence in one of the patterns shown above:

20 or 4×5 — _ _ 4 _ _

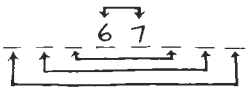
28 or 4×7 — _ _ _ 4 _ _ _

40 or 8×5 — _ _ 8 _ _

44 or 4×11 — _ _ _ 5 6 _ _ _



52 or 4×13 — _ _ _ 6 7 _ _ _



So we have:

$20 = 2+3+\underline{4}+5+6$

$28 = 1+2+3+\underline{4}+5+6+7$

$40 = 6+7+\underline{8}+9+10$

$44 = 2+3+4+\underline{5+6}+7+8+9$

$52 = 3+4+5+\underline{6+7}+8+9+10$

We still haven't found a sequence for 2, 4, 8, 16, or 32. But we just showed that the sum of every sequence of whole numbers must have at least one odd factor. What numbers are there that have no odd factor other than 1?

$2 = 2$ $\quad$ $16 = 2 \times 2 \times 2 \times 2$

$4 = 2 \times 2$ $\quad$ $32 = 2 \times 2 \times 2 \times 2 \times 2$

$8 = 2 \times 2 \times 2$ $\quad$ etc.

The reasoning we have done is a nice proof that no number from this group can be written as the sum of a sequence of whole numbers. This is a special group of numbers called the "powers of two". They are often written

$2 = 2^1$ $\quad$ $16 = 2^4$

$4 = 2^2$ $\quad$ $32 = 2^5$

$8 = 2^3$ $\quad$ $64 = 2^6$ etc.

The proof that has just been shown is not obvious in the beginning, but some elementary school children can follow the argument and be convinced by the conclusion.

The author's experience with "mathematical proof" should not happen to anyone. In sixth grade, he came across the "proof" given in the *Book of Knowledge* that "2 = 1". Later in high school, his geometry teacher "proved" that "all triangles are isosceles". It took many years for him to trust mathematical proofs. How much better for children to encounter valid proofs they can understand, believe, and feel excited about.

This activity and the two that follow were developed by Dr. Lindsay Simmons, professor of education at University of California, Humboldt.

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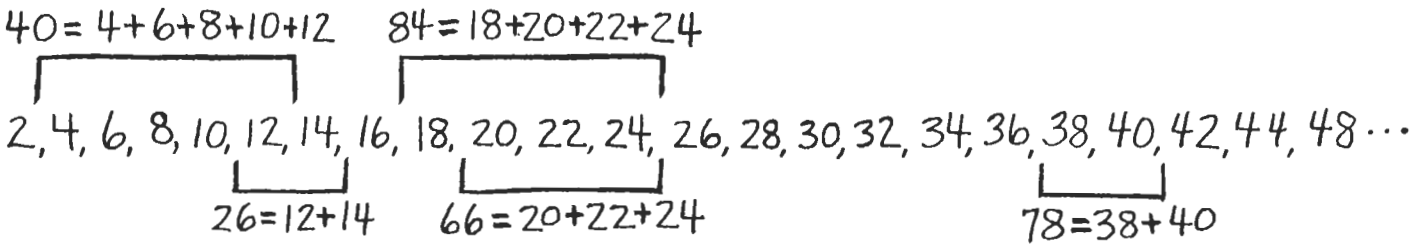

2 = ~~X~~	36 =	70 =
4 = ~~X~~	38 =	72 =
6 = 2+4	40 =	74 =
8 =	42 =	76 =
10 =	44 =	78 =
12 =	46 =	80 =
14 =	48 =	82 =
16 =	50 =	84 =
18 =	52 =	86 =
20 =	54 =	88 =
22 =	56 =	90 =
24 =	58 =	92 =
26 =	60 =	94 =
28 =	62 =	96 =
30 =	64 =	98 =
32 =	66 =	100 =
34 =	68 =	102 =

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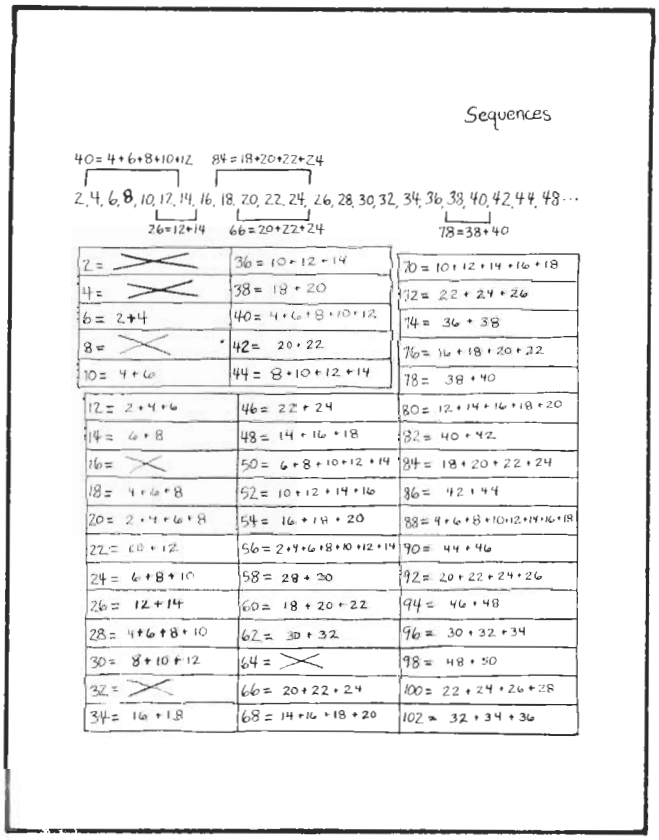


Sequences

40 = 4+6+8+10+12 84 = 18+20+22+24

2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 48 ···

26 = 12+14 66 = 20+22+24 78 = 38+40

2 = ~~X~~	36 = 10+12+14	70 = 10+12+14+16+18
4 = ~~X~~	38 = 18+20	72 = 22+24+26
6 = 2+4	40 = 4+6+8+10+12	74 = 36+38
8 = ~~X~~	42 = 20+22	76 = 16+18+20+22
10 = 4+6	44 = 8+10+12+14	78 = 38+40
12 = 2+4+6	46 = 22+24	80 = 12+14+16+18+20
14 = 6+8	48 = 14+16+18	82 = 40+42
16 = ~~X~~	50 = 6+8+10+12+14	84 = 18+20+22+24
18 = 4+6+8	52 = 10+12+14+16	86 = 42+44
20 = 2+4+6+8	54 = 16+18+20	88 = 4+6+8+10+12+14+16+18
22 = 10+12	56 = 2+4+6+8+10+12+14	90 = 44+46
24 = 6+8+10	58 = 28+30	92 = 20+22+24+26
26 = 12+14	60 = 18+20+22	94 = 46+48
28 = 4+6+8+10	62 = 30+32	96 = 30+32+34
30 = 8+10+12	64 = ~~X~~	98 = 48+50
32 = ~~X~~	66 = 20+22+24	100 = 22+24+26+28
34 = 16+18	68 = 14+16+18+20	102 = 32+34+36

In roducing the Problem

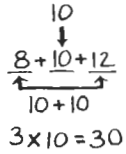
This is a thinly disguised variation of the preceding activity. Nonetheless, it can be considered quite independently, and could precede or follow the investigation of sums of consecutive whole numbers. In this variation, the given list from which "neighbors" must be taken contains just the even numbers. The problem is to find sums of sequences for the even numbers, 2 through 102.

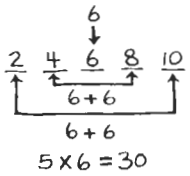
It might be helpful to again look at the two possible kinds of sequences: (1) those which have an odd number of terms, and (2) those which have an even number of terms. Let's look at some sequences with odd numbers of terms.

3 terms	5 terms
2+4+6 = 12	2+4+6+8+10 = 30
8+10+12 = 30	6+8+10+12+14 = 50
10+12+14 = 36	8+10+12+14+16 = 60

Notice the middle term in each one. Then notice that the numbers immediately to the right and to the left of the middle term add up to *twice* the middle term.

Also, if there are more than three terms, the next number to the right and the next number to the left also add up to twice the center term.





Thus, the sum of a sequence with an odd number of terms is always equal to some odd number times the middle term. In other words, any number that can be shown as a sum of a sequence of an odd number of terms *must* have an odd factor.

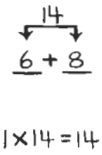
Now suppose that the sequence has an even number of terms. Must its sum have an odd factor as well? Let's consider some sums that have an even number of terms.

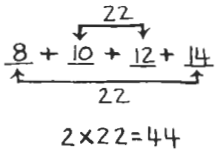
2 terms	4 terms
2+4=6	2+4+6+8=20
6+8=14	8+10+12+14=44
20+22=42	20+22+24+26=92

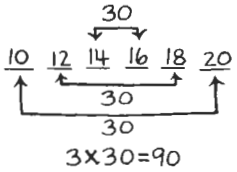
6 terms

2+4+6+8+10+12=42
6+8+10+12+14+16=66
10+12+14+16+18+20=90

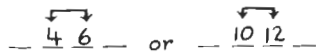
What relationship is there between the sum of the terms and the terms themselves? Notice that if we start with the center pair of terms and then pair the numbers off by taking each next larger with each next smaller term, the sums of these pairs are all the same.







Thus, to get the sum of the sequence, find the sum of the center pair and multiply by the number of pairs in the sequence. Certainly if there is an *odd* number of pairs, the sum will have an odd factor. Let's look at two possible pairs of middle terms.



The "average" of 4 and 6 is 5. The "average" of 10 and 12 is 11. Both "averages" are odd numbers. Is the average of two even numbers always odd? Another way to ask the same question is : "When you divide the sum of two consecutive even numbers by two, will you always get an odd number?" If so, we would know that the sum of two even numbers must have an odd factor.

There is a formal way to reason. Any even number can be thought of as (2n), where n stands for a whole number (0, 1, 2, 3, 4, 5, . . .). The next even number to (2n) could be written as (2n + 2). If we add these two together, we would have

$$(2n) + (2n+2) \text{ or } 4n+2$$

Half of (4n + 2) is (2n + 1), which is the form for an odd number.

If this formal argument is a bit heavy, we can get evidence to support the same conclusion by looking at several pairs of even numbers in our sequence. Can their sum be written as the product of 2 and some odd number?

$2+4 = 2\times 3$ $\quad$ $12+14 = 2\times 13$
$8+10 = 2\times 9$ $\quad$ $20+22 = 2\times 21$
$14+16 = 2\times 15$ $\quad$ $24+26 = 2\times 25$

The evidence is convincing. It turns out that in any sequence with an even number of terms, the sum of the two middle terms has an odd factor. So there will always be an odd factor in any multiple of the sum of the two middle terms.

Thus, whether the sequence has an odd or an even number of terms, the sum will have an odd factor. Consequently, we will be unable to find a sequence of two or more even numbers to make sums for numbers that have no odd factors. That means that we can't find sequences for 2, 4, 8, 16, 32, 64, and so on — the powers of 2 again.

The activity on the reverse side of this page is taken from *Drill and Practice at the Problem Solving Level* (Curriculum Development Associates, 1974)

8 + 4 + 2 + 1 = ___	8 4 2 1 = 9
8 + 4 − 2 + 1 = ___	8 4 2 1 = 7
8 + 4 + 2 1 = 13	8 4 2 1 = 5
8 − 4 − 2 − 1 = ___	8 4 2 1 = 3

10 + 4 + 3 + 2 = ___	12 6 3 1 = 22
10 + 4 + 3 − 2 = ___	12 6 3 1 = 20
10 + 4 − 3 2 = 13	12 6 3 1 = 16
10 4 3 2 = ___	12 6 3 1 = 14
10 4 − 3 − 2 = 9	12 6 3 1 = 10
10 4 3 2 = 7	12 6 3 1 = 8
10 4 3 2 = 5	12 6 3 1 = 4
10 4 3 2 = 1	12 6 3 1 = 2

FAVOR DE HACER DIFFERENTES TODOS LOS RESULTADOS.

PLEASE MAKE ALL RESULTS DIFFERENT

9 + 7 + 5 + 3 = 24	12 5 4 2 = ___
9 7 5 3 = 18	12 5 4 2 = ___
9 7 5 3 = 14	12 5 4 2 = ___
9 7 5 3 = 10	12 5 4 2 = ___
9 7 5 3 = 8	12 5 4 2 = ___
7 5 9 3 = 6	12 5 4 2 = ___
9 7 5 3 = 4	12 5 4 2 = ___
7 5 9 3 = 0	12 5 4 2 = ___

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SUMAS DE SERIES

SUMS OF SEQUENCES

$27 = 6+9+12$ $90 = 18+21+24+27$

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57 ...

$150 = 24+27+30+33+36$ $93 = 45+48$

3=	54=	105=
6=	57=	108=
9= 3+6	60=	111=
12=	63=	114=
15=	66=	117=
18=	69=	120=
21=	72=	123=
24=	75=	126=
27=	78=	129=
30=	81=	132=
33=	84=	135=
36=	87=	138=
39=	90=	141=
42=	93=	144=
45=	96=	147=
48=	99=	150=
51=	102=	153=

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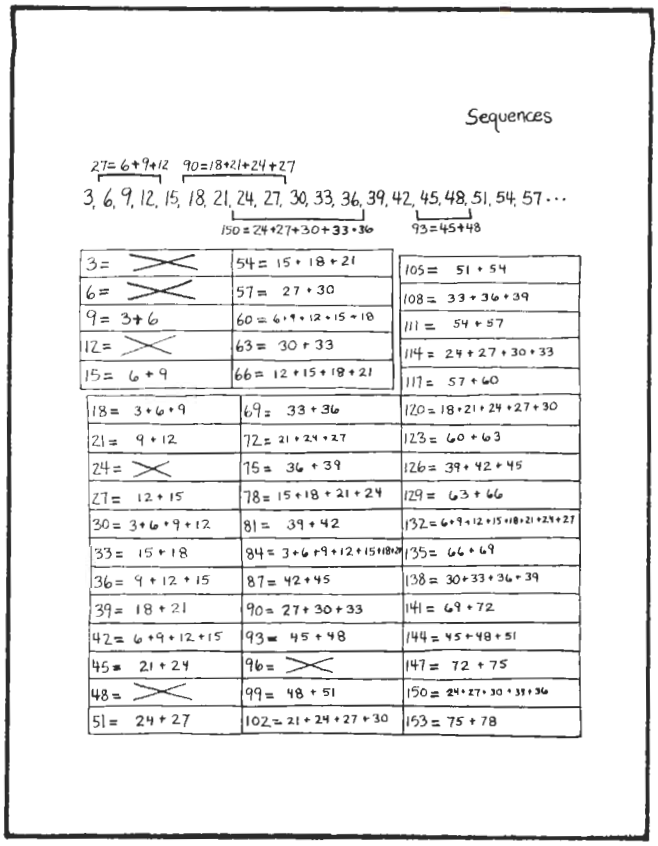
(Some of the appropriate responses are shown here.)



Sequences

27 = 6+9+12 90 = 18+21+24+27

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57 ...

150 = 24+27+30+33+36 93 = 45+48

3= X	54 = 15 + 18 + 21	105 = 51 + 54
6= X	57 = 27 + 30	108 = 33 + 36 + 39
9 = 3+6	60 = 6+9+12+15+18	111 = 54 + 57
12 = X	63 = 30 + 33	114 = 24 + 27 + 30 + 33
15 = 6 + 9	66 = 12 + 15 + 18 + 21	117 = 57 + 60
18 = 3+6+9	69 = 33 + 36	120 = 18+21+24+27+30
21 = 9 + 12	72 = 21 + 24 + 27	123 = 60 + 63
24 = X	75 = 36 + 39	126 = 39 + 42 + 45
27 = 12 + 15	78 = 15 + 18 + 21 + 24	129 = 63 + 66
30 = 3+6+9+12	81 = 39 + 42	132 = 6+9+12+15+18+21+24+27
33 = 15 + 18	84 = 3+6+9+12+15+18+21	135 = 66 + 69
36 = 9 + 12 + 15	87 = 42 + 45	138 = 30+33+36+39
39 = 18 + 21	90 = 27 + 30 + 33	141 = 69 + 72
42 = 6+9+12+15	93 = 45 + 48	144 = 45+48+51
45 = 21 + 24	96 = X	147 = 72 + 75
48 = X	99 = 48 + 51	150 = 24+27+30+33+36
51 = 24 + 27	102 = 21 + 24 + 27 + 30	153 = 75 + 78

Introducing the Problem

This activity is another variation on a previous activity in this series. Here we are given a list of the multiples of three that are whole numbers. The problem is to find sums of sequences from this list for the numbers given, which are also multiples of three. The search will probably turn up nothing that is essentially new. However, some leading questions might produce a different approach to the problem.

Talking About the Problem

At least some of the following discussion should take place before children set to work independently on the problem.

Let's look at our list of numbers, beginning with 6, in terms of their prime factors.

$6 = 3 \times 2$ (can't be done)
$9 = 3 \times 3$
$12 = 3 \times 2 \times 2$ (can't be done)
$15 = 3 \times 5$
$18 = 3 \times 3 \times 2$
$21 = 3 \times 7$
$24 = 3 \times 2 \times 2 \times 2$ (can't be done)
$27 = 3 \times 3 \times 3$
$30 = 3 \times 5 \times 2$
$42 = 3 \times 7 \times 2$
$45 = 3 \times 3 \times 5$
$48 = 3 \times 2 \times 2 \times 2 \times 2$ (can't be done)

What is true about the factors of 6, 12, 24 and 48 that is not true of factors of the other multiples of 3? They have 3 as their *only* odd prime factor. All the other numbers have at least two odd prime factors. We might wonder whether the sum of any sequence from our list must also have at least two odd prime factors. If it does, then we won't be able to find sums for the numbers that have just one odd prime factor, namely 6, 12, 24, 48, and so on.

Our investigation might begin again with the observation that a sequence has to have an odd number of terms or an even number of terms. Look at a sequence that has an odd number of terms, such as the one shown below. The sum of the sequence will be equal to the number of terms in the sequence times the middle term (which of course, is a multiple of three).

$9 + 12 + 15 = 3 \times (3 \times 2 \times 2)$ (no. of terms; 12+12)

$15 + 18 + 21 + 24 + 27 = 5 \times (3 \times 7)$ (21+21; 21+21)

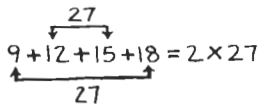
The sums of all such sequences must have at least two prime factors that are odd: the 3 which is a factor of the middle term, and some other odd prime that is a factor of the number of terms in the sequence. A few examples should make this a bit clearer:

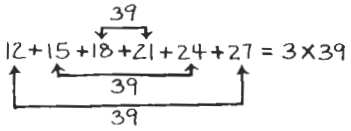
$6 + 9 + 12 = 27 = 3 \times (3 \times 3)$ (number of terms; 9+9; factors of 9)

$9 + 12 + 15 + 18 + 21 = 75 = 5 \times (3 \times 5)$ (number of terms; factors of 15)

Now, what if the sequence has an even number of terms? Its sum will certainly have 3 as one of its factors, since all its terms are multiples of 3. But must it have a second prime factor that is odd? The

sum of an even number of terms will be a multiple of the sum of the two midterms, as in the following:





Now, the sum of every pair of middle terms must have 3 as a factor (since both will be multiples of three).

$3+6=9=3+(3+3)$
$6+9=15=(3+3)+(3+3+3)$
$9+12=21=(3+3+3)+(3+3+3+3)$

If one of the pairs of middle terms is the sum of an odd number of threes, the second of the pair is the sum of an even number of threes (for it contains one more or one less three). Together, the sum of the pair must contain an odd number of threes, for an even number added to an odd number is always an odd number. Thus, the sum of the middle pair must have factors of 3 and at least one other odd prime number. Since the sum of the whole sequence is a multiple of this middle pair, that sum must also have factors of 3 and at least one other odd prime number.

$9+12=(3\times3)+(4\times3)=7\times3$
$12+15=(4\times3)+(5\times3)=9\times3$
$15+18=(5\times3)+(6\times3)=11\times3$

To summarize:

(a) If a sequence has an odd number of terms, its sum must have at least two prime factors that are odd (namely 3, and one or more others).

(b) If the sequence has an even number of terms, its sum will be the product of three and another odd number.

This is another way of saying that we can only find sequences for numbers that have at least *two* odd prime factors. So if a number has three as the lone prime factor, it can't be shown as the sum of an odd number of terms or as the sum of an even number of terms — it can't be done! In our list, the numbers that have 3 as their lone prime factor are

$6=3\times2$
$12=3\times2\times2$
$24=3\times2\times2\times2$
$48=3\times2\times2\times2\times2$
$96=3\times2\times2\times2\times2\times2$
etc.

We have put together a nice little mathematical proof here that no matter how far we extend our record, there will always be "holes" — or numbers that can't be done. They will be those numbers that have three as their *only* prime factor. We have not yet *proved* that those are the *only* "holes" that will occur, but it does turn out that there are no others.

Extensions and Surprise Endings

A closer look at the numbers that we could not do will reveal a relationship between them and the numbers we could not do in the previous activities.

$6=3\times2$
$12=3\times4$
$24=3\times8$
$48=3\times16$
etc.

The powers of two rise again!

The activity on the reverse side of this page is taken from *Drill and Practice at the Problem Solving Level* (Curriculum Development Associates, 1974)

FAVOR DE HACER DIFFERENTES TODOS LOS RESULTADOS.

PLEASE MAKE ALL RESULTS DIFFERENT.

11 − 6 + 3 + 2 = ___	11 6 3 2 = ___
11 + 6 + 3 2 = 18	11 6 3 2 = ___
11 − 6 − 3 2 = 0	11 6 3 2 = ___
11 6 3 2 = ___	11 6 3 2 = ___

12 7 4 1 = ___	13 6 4 2 = ___
12 7 4 1 = ___	13 6 4 2 = ___
12 7 4 1 = ___	13 6 4 2 = ___
12 7 4 1 = ___	13 6 4 2 = ___
12 7 4 1 = ___	13 6 4 2 = ___
12 7 4 1 = ___	13 6 4 2 = ___
12 7 4 1 = ___	13 6 4 2 = ___
12 7 4 1 = ___	13 6 4 2 = ___

14 8 5 1 = ___	15 6 5 3 = ___
14 8 5 1 = ___	15 6 5 3 = ___
14 8 5 1 = ___	15 6 5 3 = ___
14 8 5 1 = ___	15 6 5 3 = ___
14 8 5 1 = ___	15 6 5 3 = ___
14 8 5 1 = ___	15 6 5 3 = ___
14 8 5 1 = ___	15 6 5 3 = ___
14 8 5 1 = ___	15 6 5 3 = ___

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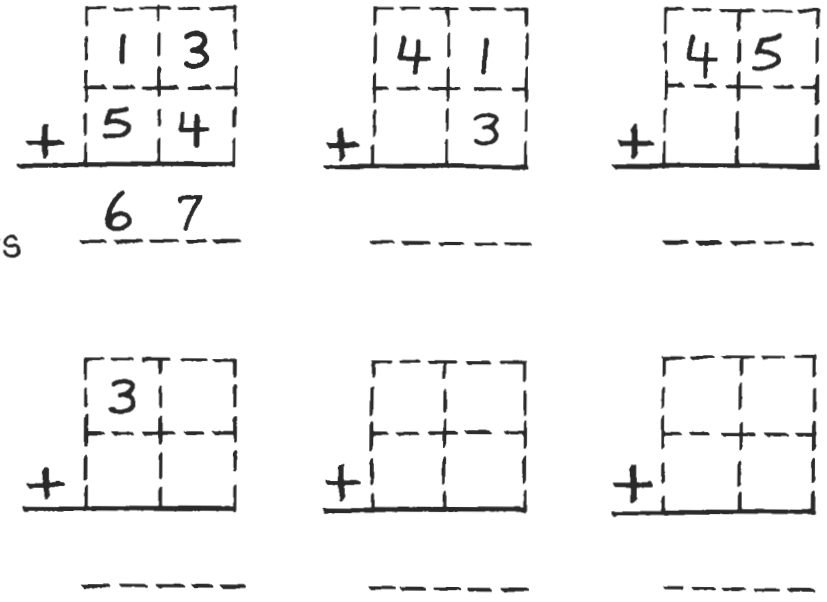
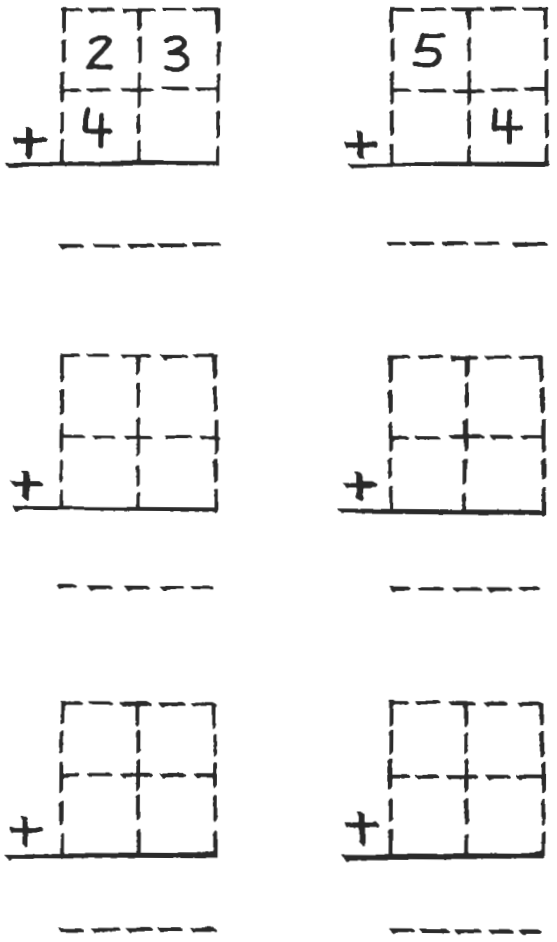
1	3	4	5

Please use all four digits in each example. All answers should be different.

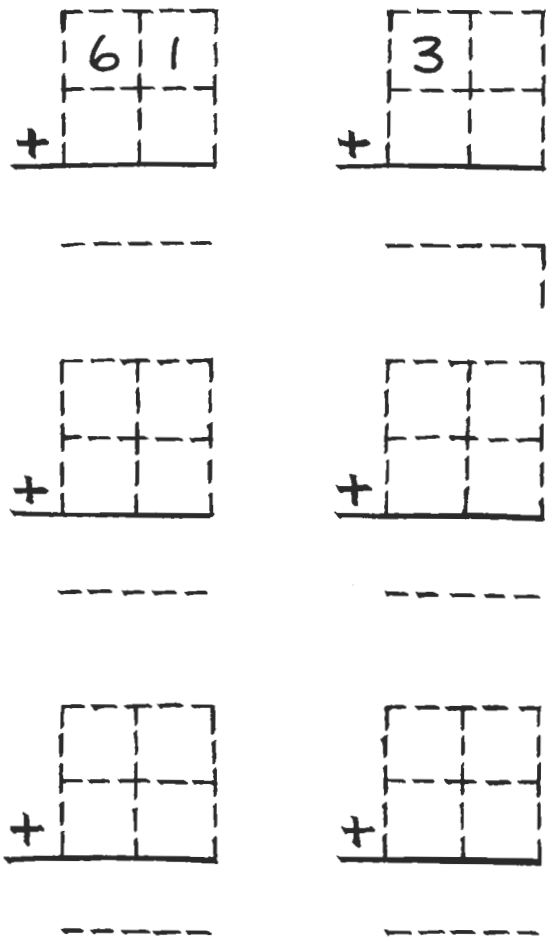
Usa los cuatros digitos en cada ejemplo. Todos los resultados deben ser diferentes.



2	3	4	5



1	2	3	6



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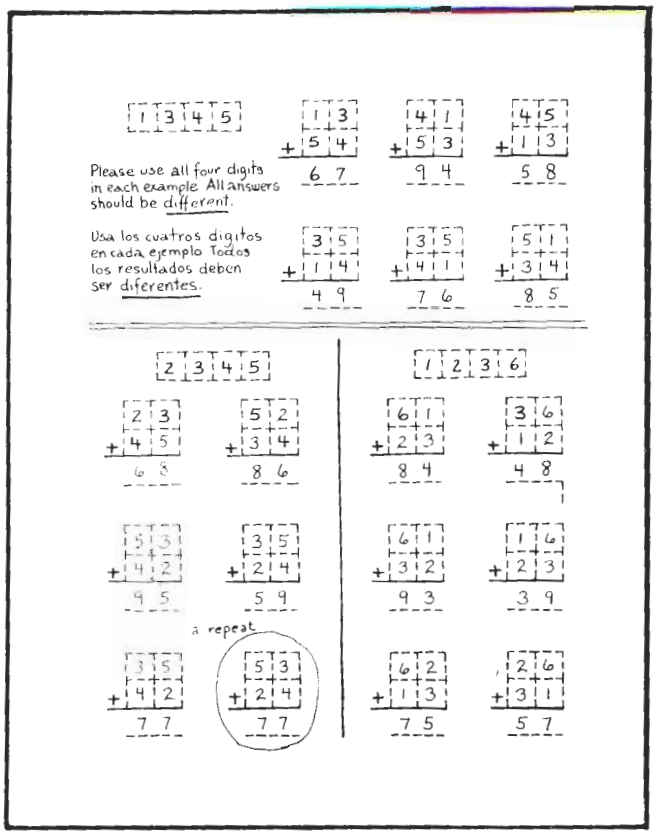


Introducing the Problem

Four digits are given in each problem. The four digits are used to make up addition examples. Each number must be used once and only once in each example, and all six sums in the group must be different.

No regrouping is required in any of the problems, since the sum of any two of the four given digits is never larger than 9. Because sums must be different, only one of a "family" of sums such as the following can be included:

$$\begin{array}{cccc} 14 & 15 & 34 & 35 \\ +35 & +34 & +15 & +14 \\ \hline 49 & 49 & 49 & 49 \end{array}$$

Talking About the Problem

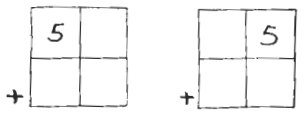
(1) Were there any examples for which you could not find six different sums? In the problem using 2, 3, 4 and 5, only five different sums can be found. This limitation arises because there is one pair of digits whose sum is the same as the other two.

$$2+5=7 \text{ and } 3+4=7$$

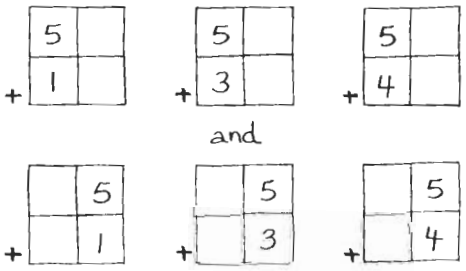
As a result, reversing the digits in the tens and units place does not affect the sum.

$$\begin{array}{r} 23 \\ +54 \\ \hline 77 \end{array} \text{ and } \begin{array}{r} 32 \\ +45 \\ \hline 77 \end{array}$$

(2) Were there any examples for which you found more than six different sums? It turns out that with four digits, no more than six different sums are possible. One way to see this is to look at what happens when one of the four given digits is placed first in one column, and then in the other. For example, using 1, 3, 4 and 5 as our digits, we might choose 5. It could be placed in the tens column or the units column:



Now, there are three other digits — 1, 3, and 4 — that could be placed in the same column as the 5:



In each case shown above, the other two digits would be placed in the empty spaces. It makes no difference which one is on top, for the sum will be the same. Thus, using four digits, there are only six different sums possible.

Extensions and Surprise Endings

Look at the sums in each group of six problems. Did anyone notice that the sum of the digits in each sum within a group is the same number? In the problems above the double line, every sum has two digits that add to 13. In the problems to the left of the heavy vertical line, the sum of the digits in each sum is 14. In the third group of problems, every sum has digits that add to 12.

a. 49, 94, 58, 85, 67 and 76 → 13
b. 59, 94, 68, 86 and 77 → 14
c. 39, 93, 48, 84, 75 and 57 → 12

This should not really be so surprising; for it is also true that the sum of the four given digits is the same number.

a. 1 + 3 + 4 + 5 = 13
b. 2 + 3 + 4 + 5 = 14
c. 1 + 2 + 3 + 6 = 12

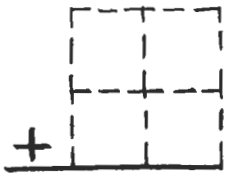
Would this same thing occur if we used digits which would necessitate regrouping? Consider the four digits 2, 5, 7, and 8. The six different sums would be:

$$\begin{array}{rrr} 25 & 27 & 27 \\ +78 & +58 & +85 \\ \hline 103 & 85 & 112 \end{array}$$

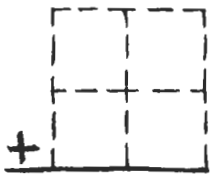
$$\begin{array}{rrr} 52 & 72 & 72 \\ +87 & +85 & +58 \\ \hline 139 & 157 & 130 \end{array}$$

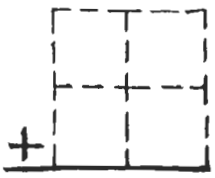
The sum of the digits in each total is either 4 or 13. But notice that the sum of the digits in 13 is 4. This result is related to the procedure of "casting out nines", an old device accountants used in checking their computations.

Please use all four digits in each example. All answers should be different.

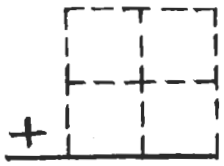


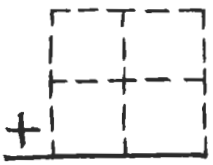
6 7

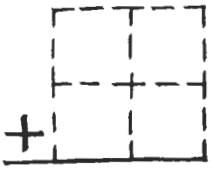


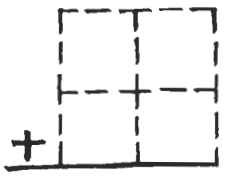


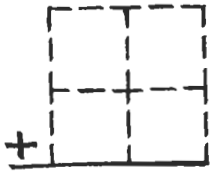
Usa los cuatros digitos en cada ejemplo. Todos los resultados deben ser diferentes.

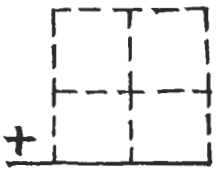


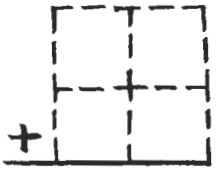


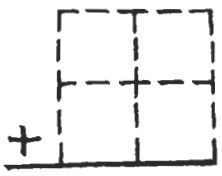


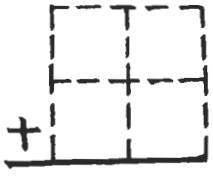


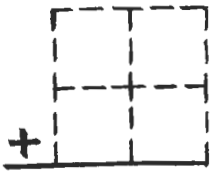


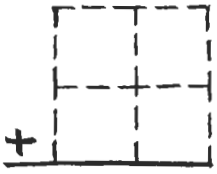


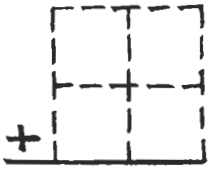


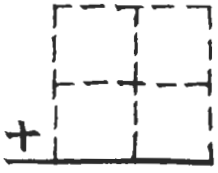


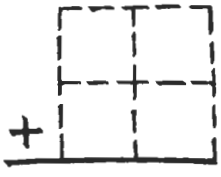


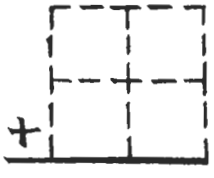












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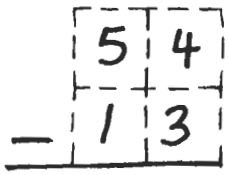
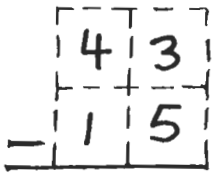
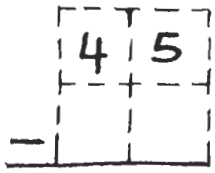
1	3	4	5

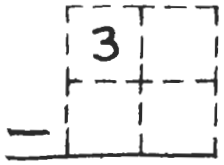
Please use all four digits in each example. All answers should be <u>different</u>.

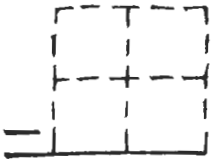


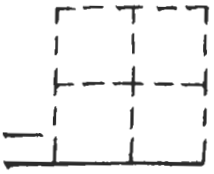




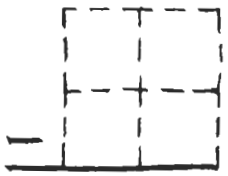
Usa los cuatro digitos en cada ejemplo. Todos los resultandos deben ser <u>diferentes</u>.

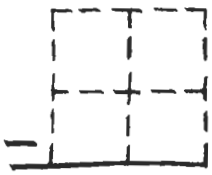


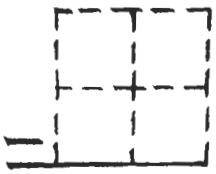


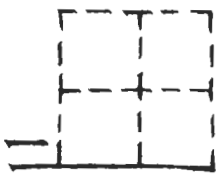


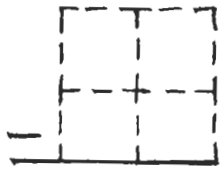
2	3	5	9

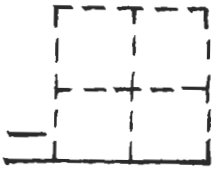


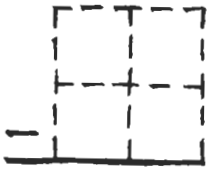


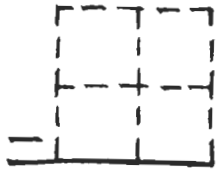


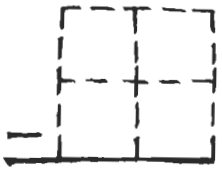


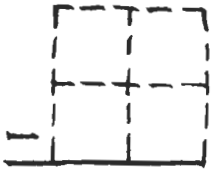


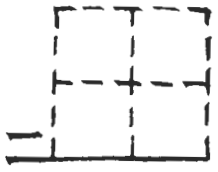












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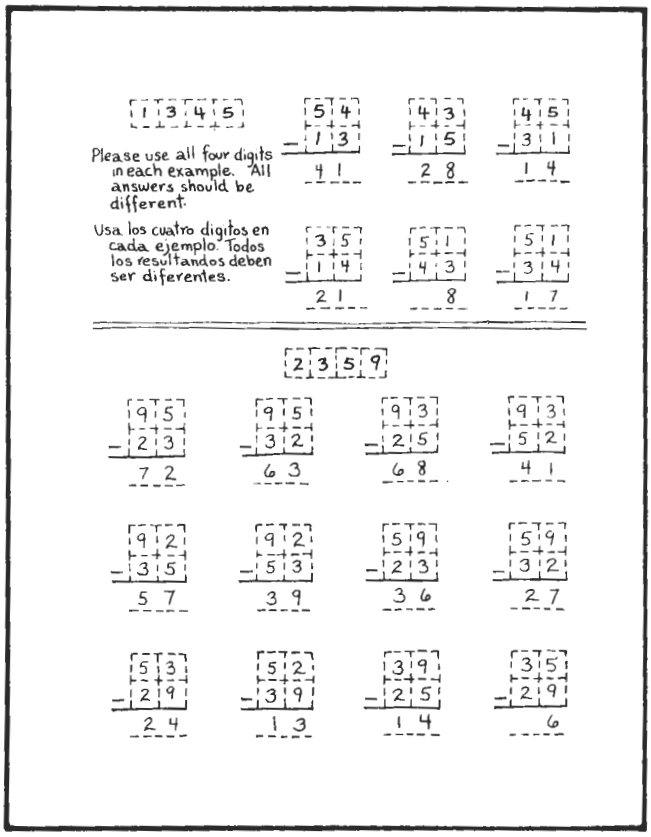


Introducing the Problem

Work on this activity focuses attention on the notion of place value as it arises in the subtraction algorism. Four digits are given for each group of problems. Using these four digits, children make up subtraction examples that fit the given format. The differences obtained in each set of examples should be positive whole numbers and should all be different. Each of the four numbers must be used once and only once in each example within the group. Working with the four numbers 1, 3, 4 and 5, it quickly becomes clear that the digits can be arranged to yield a negative number for a difference. For example,

$$\begin{array}{r} 13 \\ -\,54 \\ \hline \end{array}$$

has no solution that is a positive whole number.

It may be helpful to do some group exploration with the problem before setting children to work independently. We might talk about what happens when we arrange the digits in certain ways. How can the digits 1, 3, 4 and 5 be arranged in an example to yield the *largest* difference? One approach is to select the smallest digit and the largest digit for the tens place. Then, arrange the other two digits so that the larger is on top:

$$\begin{array}{r} 5_ \\ -1_ \\ \hline \end{array} \rightarrow \begin{array}{r} 54 \\ -13 \\ \hline 41 \end{array}$$

What arrangement will lead to the *smallest* difference? We should choose digits for the tens column that will have the smallest difference. Using the digits 1, 3, 4 and 5, we have two options:

$$\begin{array}{r} 5_ \\ -4_ \\ \hline \end{array} \text{ and } \begin{array}{r} 4_ \\ -3_ \\ \hline \end{array}$$

Each of these options has two possible completions:

$$\begin{array}{r} 5_ \\ -4_ \\ \hline \end{array} \rightarrow \begin{array}{r} 53 \\ -41 \\ \hline \end{array} \text{ and } \begin{array}{r} 51 \\ -43 \\ \hline \end{array}$$

$$\begin{array}{r} 4_ \\ -3_ \\ \hline \end{array} \rightarrow \begin{array}{r} 45 \\ -31 \\ \hline \end{array} \text{ and } \begin{array}{r} 41 \\ -35 \\ \hline \end{array}$$

The arrangement

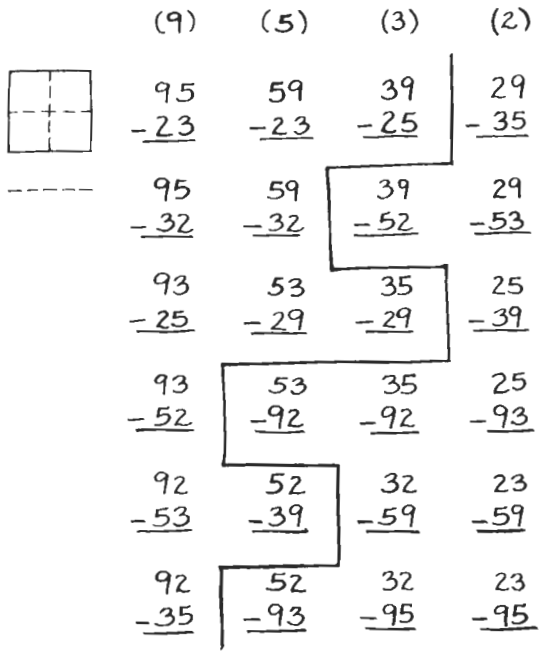
$$\begin{array}{r} 41 \\ -35 \\ \hline 6 \end{array}$$

will yield the smallest difference.

Talking About the Problem

How many "different differences" did you find using the numbers 2, 3, 5 and 9? Did anyone use a plan or strategy? One possible plan is to consider all possible combinations. We can do this by first placing the 9 in the tens place in the top number, and generating all possible examples. Then we can replace the 9 with the 5 and find all possible examples. The list can be completed by placing the 3

and then the 2 in that same position. There are the 24 possible combinations:



It turns out that exactly half of them have differences that are positive whole numbers. Thus there are 12 "different differences" that can be found.

Extensions and Surprise Endings

Let's go back to the problem at the top of the page. We have six examples already. How many more can we find with different answers? Will there be twelve different answers, as in the other problem?

We can look at the more general case. Let's consider any four different digits. Let 'a' be the largest and 'd' be the smallest so that

$$a > b > c > d \quad \text{(such as 7, 5, 3 and 2)}$$

Since we have said that the differences have to be positive numbers, we only have to consider what happens in the tens place. Suppose we put 'a' in the tens place for the top number. Then any one of the other numbers could be placed in the tens column for the lower number, and the difference would be positive, since 'a' is larger than the other three. Once we have selected a digit to go below 'a', there are two ways to arrange the remaining two digits:

$$\underline{a},c \quad \underline{a},d \qquad \underline{a},b \quad \underline{a},d \qquad \underline{a},b \quad \underline{a},c$$
$$-b,d \quad -b,c \qquad -c,d \quad -c,b \qquad -d,c \quad -d,b$$

If 'b' is in the tens place for the top number, only 'c' or 'd' could be in the tens place for the bottom number. If 'a' were there, the difference would not be positive, since a>b. So the only possible examples with 'b' in that place are:

$$\underline{b},a \quad \underline{b},d \qquad \underline{b},a \quad \underline{b},c$$
$$-c,d \quad -c,a \qquad -d,c \quad -d,a$$

If 'c' is in the tens place for the top number, there are only two possible arrangements that will yield positive differences:

$$c,a \quad c,b$$
$$-d,b \quad -d,a$$

So, we can see that all together there will be twelve possible positive differences when four different digits are used.

Will all twelve differences be different no matter what four digits are used? No. Consider 2, 4, 6 and 8.

$$\begin{array}{r} 86 \\ -42 \\ \hline 44 \end{array} \qquad \begin{array}{r} 68 \\ -24 \\ \hline 44 \end{array}$$

What happens if the digits are not all different, and we used digits such as 2, 3, 3 and 8? It is not enough to consider only the tens place, for the units place could be the source of trouble and lead to a difference that was not positive.

$$\begin{array}{r} 38 \\ -32 \end{array} \quad \text{and} \quad \begin{array}{r} 32 \\ -38 \end{array}$$

Variations on the Activity

Suppose that we choose six digits and use them to make up examples that use three digit numbers. Children can be asked to make up examples that meet certain conditions, such as:

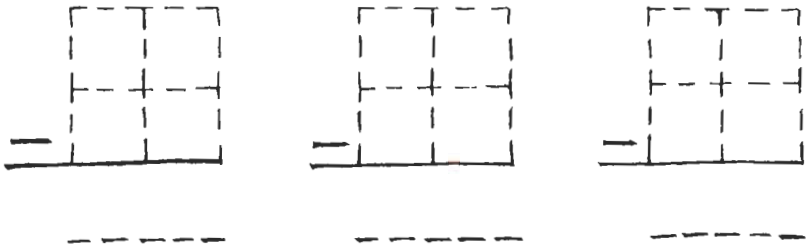
(a) no regrouping is required
(b) regrouping is required in the units column only
(c) regrouping is required in the tens column only
(d) regrouping is required in both the tens and the units columns.

(a)	(b)	(c)	(d)
654	653	625	624
-123	-124	-134	-135

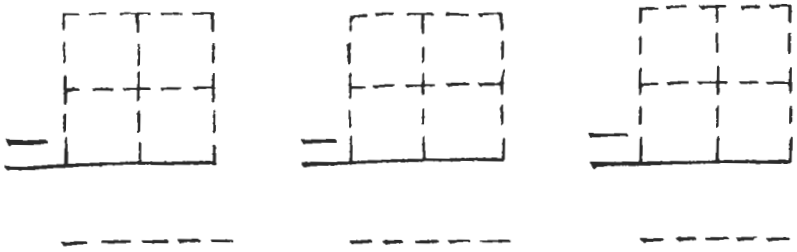
NAME

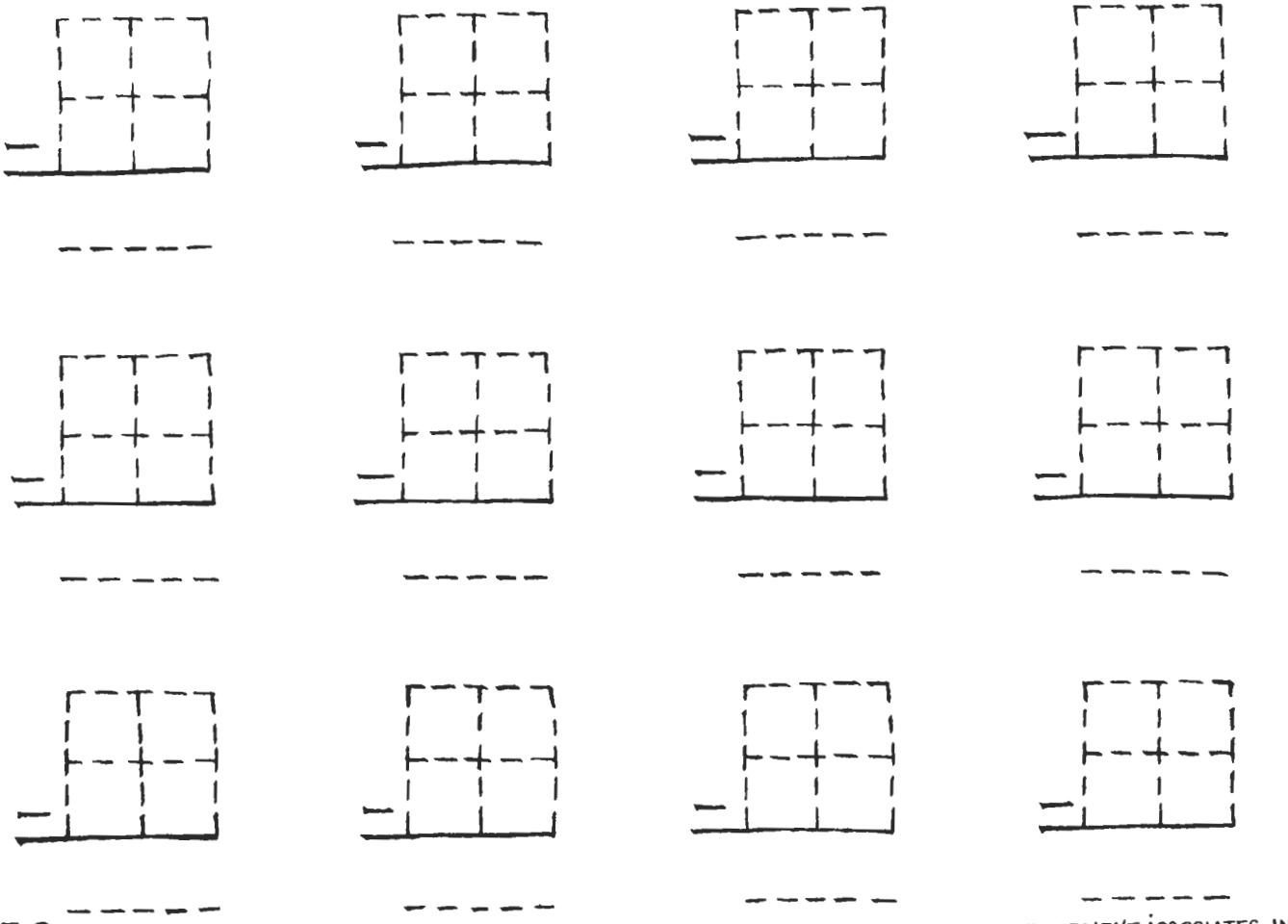
DATE

Please use all four digits in each example. All answers should be different.



Usa los cuatro digitos en cada ejemplo. Todos los resultandos deben ser diferentes.





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1	4	6	8

Please use all four digits in each example. Make all products different....

Usa los cuatro dígitos de cada ejemplo. Haz todos los productos diferentes....

.... less than 1,000.	.... greater than 1,000.
6 8 x 1 4	1 8 x 6 4
x	x

1	3	4	5

.... menos de 1,000.	.... más de 1,000.
5 4 x 1 3	3 1 x 4 5
x	x
x	x
x	x

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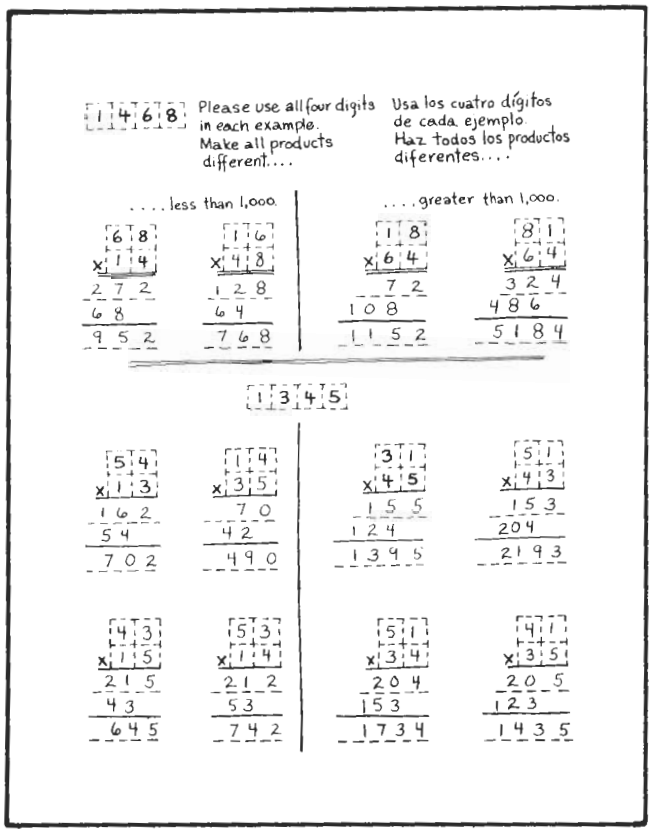

Introducing the Problem

As in the two previous activities, four digits are given. Children make up multiplication examples to fit the given format using each digit once and only once in each example. The object is to make up examples that have different products. Products are called for that are less than 1,000 and greater than 1,000.

While talking about the problem above the double line, we might want to think about how the digits could be arranged to get the *smallest* product. The two smallest of the four digits should be in the tens columns. That arrangement will lead to two different products.

$$\begin{array}{r} 16 \\ \times 48 \\ \hline 768 \end{array} \quad \text{and} \quad \begin{array}{r} 18 \\ \times 46 \\ \hline 828 \end{array}$$

Generally we would have to do the calculations to determine which would be smaller.

What arrangement would lead to the *largest* product? It certainly seems that the two largest digits should go in the tens column. Again, there are two possible products:

$$\begin{array}{r} 81 \\ \times 64 \\ \hline 5184 \end{array} \quad \text{or} \quad \begin{array}{r} 84 \\ \times 61 \\ \hline 5124 \end{array}$$

Calculation will reveal the larger of the two products.

Talking About the Problem

The class might like to make a composite record of all the different products that were found using the numbers 1, 3, 4 and 5. These are the possibilities: (The number in parentheses is the "sum of the digits in the sum," which we will tend to later.)

$$\begin{array}{r} 31 \\ \times 45 \\ \hline 1395 \end{array} \quad \begin{array}{r} 31 \\ \times 54 \\ \hline 1674 \end{array} \quad \begin{array}{r} 13 \\ \times 45 \\ \hline 585 \end{array} \quad \begin{array}{r} 13 \\ \times 54 \\ \hline 702 \end{array}$$

(18) → (18) → (18) → (9)

$$\begin{array}{r} 41 \\ \times 35 \\ \hline 1435 \end{array} \quad \begin{array}{r} 41 \\ \times 53 \\ \hline 2173 \end{array} \quad \begin{array}{r} 14 \\ \times 35 \\ \hline 490 \end{array} \quad \begin{array}{r} 14 \\ \times 53 \\ \hline 2173 \end{array}$$

(13) → (13) → (13) → (13)

$$\begin{array}{r} 51 \\ \times 34 \\ \hline 1734 \end{array} \quad \begin{array}{r} 51 \\ \times 43 \\ \hline 2193 \end{array} \quad \begin{array}{r} 15 \\ \times 34 \\ \hline 510 \end{array} \quad \begin{array}{r} 15 \\ \times 43 \\ \hline 645 \end{array}$$

(15) → (15) → (6) → (15)

Are there any other arrangements that will lead to different products? No. All possible arrangements are included. We started by selecting the 3 and the 1 as the digits for the top number, and wrote all four possible examples. In a similar way, we wrote all arrangements that used 4 and 1, or 5 and 1 as the top two digits. There are five products that are less than 1,000 and seven that are greater than 1,000.

Look at the sum of the digits in each product, given in the parentheses under each example.

18 — 18 — 18 — 9
13 — 13 — 13 — 13
15 — 15 — 6 — 15

A pattern seems to emerge for each group of four examples, but two flaws occur: the 9 in the top row and the 6 in the bottom row. But suppose that for all the two digit sums, we note the sum of *their* digits. Then we would have

9 — 9 — 9 — 9
4 — 4 — 4 — 4
6 — 6 — 6 — 6

And that's a pattern without flaws. This result is the basis for a procedure called "casting out nines", used by accountants to check their computations before the days of the computer. Some children may want to pursue a further investigation of casting out nines.[1] Others will simply have an added respect for number relationships that are not obvious at the outset — and that's the name of the game in mathematics.

[1] For more discussion, see *Drill and Practice at the Problem Solving Level: Activity Pages*, published by Curriculum Development Associates, pages 166-168.

Change for a quarter.

Cambio de a 25 centavos.

	D	N
1	2	1
2	1	
3	0	

	D	N	P
4	2	0	

	D	N	P
5	1	2	
6	1	1	
7	1	0	

	D	N	P
8	0	4	
9	0	3	
10	0		
11	0		
12	0		

How many ways can you find to make change for $1.^{00}$ without pennies?

¿De cuántas maneras puedes cambiar un dólar sin usar monedas de centavos?

	H	Q	D	N
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				

	H	Q	D	N
15				
16				
17				
18				
19				
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25				
26				
27				
28				

	H	Q	D	N
29				
30				
31				
32				
33				
31				
35				
36				
37				
38				
39				
40				
41				
42				

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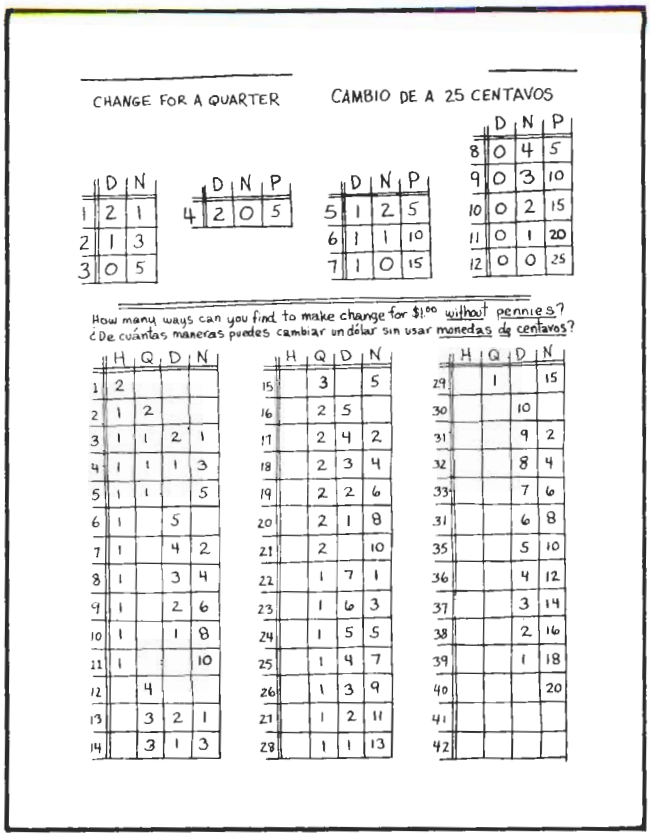

CHANGE FOR A QUARTER

	D	N
1	2	1
2	1	3
3	0	5

	D	N	P
4	2	0	5

CAMBIO DE A 25 CENTAVOS

	D	N	P
5	1	2	5
6	1	1	10
7	1	0	15

	D	N	P
8	0	4	5
9	0	3	10
10	0	2	15
11	0	1	20
12	0	0	25

How many ways can you find to make change for $1.00 without pennies?
¿De cuántas maneras puedes cambiar un dólar sin usar monedas de centavos?

	H	Q	D	N
1	2			
2	1	2		
3	1	1	2	1
4	1	1	1	3
5	1	1		5
6	1		5	
7	1		4	2
8	1		3	4
9	1		2	6
10	1		1	8
11	1			10
12		4		
13		3	2	1
14		3	1	3

	H	Q	D	N
15		3		5
16		2	5	
17		2	4	2
18		2	3	4
19		2	2	6
20		2	1	8
21		2		10
22		1	7	1
23		1	6	3
24		1	5	5
25		1	4	7
26		1	3	9
27		1	2	11
28		1	1	13

	H	Q	D	N
29		1		15
30			10	
31			9	2
32			8	4
33			7	6
34			6	8
35			5	10
36			4	12
37			3	14
38			2	16
39			1	18
40				20
41				
42				

Introducing the Problem

This activity is the first of a series of problems which investigates various coin combinations. The problem above the double line on this page asks about ways to make change for a quarter, and helps children see the importance and usefulness of some kind of a plan or strategy for solving such a problem. While random trial and error might eventually lead to a solution, it can also breed confusion and frustration. The problem above the double line can be used to introduce children to the notion of developing a plan for solving a problem. The question is: "How many ways are there to make change for a quarter, using nickels, dimes and pennies?"

The plan suggested by the charts simply considers four kinds of combinations:

(1) combinations without pennies

(2) combinations with pennies:

(a) using two dimes

(b) using one dime

(c) using no dimes

It turns out that there are just 12 combinations.

The main problem below the double line on this activity page asks for combinations of coins worth exactly $1.00, which do not include any pennies. Children are free to select their own plan to get a solution. Two popular plans have the following beginnings:

H	Q	D	N
2			
1	2		
1	1	2	1
1	1	1	3
1	1	0	5

H	Q	D	N
			20
		1	18
		2	16
		3	14
		4	12

One plan seems to be designed to get combinations using the big coins out of the way first. The other is concerned first with finding all combinations which do not use quarters or halves.

Talking about the Problem

What different plans did children develop? Time should be taken to discuss each one. Some of the discussion might take place in the context of an "educated guessing game". A child shows the first few items in his or her record, and the other children try to guess the plan used.

Consider the following plan:

H	Q	D	N
2			
1	2		
1	1	(2)	1
1	1	1	3
1	1	0	5
1	0	(5)	0
1	0	4	2
1	0	3	4
1	0	2	6
1	0	1	8
1	0	0	10

There is a noteworthy feature of this plan. Once the third combination (one half dollar, one quarter, two dimes and one nickel) has been established, the next two combinations follow directly. The two dimes can be exchanged, one at a time, for two nickels. When five dimes come up in the sixth combination listed, the next five combinations can be easily found by exchanging one dime at a time for two nickels. At this point, all combinations that use a half dollar are exhausted. So, we can go on to combinations that

use quarters — first four quarters; then three quarters, two dimes and one nickel. Then, we can resume the strategy of "turning in" one dime for two nickels.

H	Q	D	N
	4		
	3	②	1
	3	1	3
	3	0	5
	2	⑤	0
	2	4	2
	2	3	4
	2	2	6
	2	1	8
	2	0	10
	1	⑦	1
	1	6	3

etc.

When we exhaust combinations using quarters, we start on dimes. The combination of ten dimes and no nickels yields the last ten combinations, if the strategy of exchanging one dime for two nickels is continued.

H	Q	D	N
0	0	10	0
		9	2

etc.

All together, there are 40 possible combinations of coins that make $1.00, none of which use pennies.

Extensions and Surprise Endings

Now let's look back to the problem above the double line which considers change for a quarter. We could have approached it in another way. There were three combinations that did not use pennies. One nickel was used in the first of these three combinations, three in the second, and five in the third. If we consider each of these combinations in turn and employ a "trading in" strategy for each, there will be nine new combinations. In the first combination, the one nickel can be traded for five pennies. In the second combination, each of the three nickels can be traded in turn for five pennies, to yield three new combinations. The third combination which used five nickels will yield five new combinations.

D	N	P
2	①	0
	0	5

D	N	P
1	③	0
	2	5
	1	10
		15

D	N	P
0	⑤	0
	4	5
	3	10
	2	15
	1	20
		25

Thus nine new combinations (1 + 3 + 5 = 9) could be derived from the three combinations that did not use pennies, and all together there will be 12 different combinations. This line of reasoning provides a short way to figure the possible combinations for a quarter.

We could have noticed that the original three combinations without pennies have a total of nine nickels.

D	N
2	1
1	3
0	5

9

Each nickel can be turned in, one at a time, for pennies. Thus there will be nine combinations, and 3 + 9 = 12

Now suppose that we allow pennies to be used in our problem about change for $1.00. How many new combinations can we find? If you start out looking for them one at a time, you will probably break down, bog down, or get a different result each time. There is a much shorter and surer method. All new combinations can be seen as a result of starting with a combination that included nickels, and then trading those nickels, one at a time, for pennies. The 40 combinations without pennies use a total of 252 nickels, so:

40 without pennies
252 with pennies
292 all together

The activity on the reverse side of this page is taken from *Drill and Practice at the Problem Solving Level* (Curriculum Development Associates, 1974)

RULES: USE ALL DIGITS GIVEN ONCE AND ONLY ONCE IN EACH SUM; AND NO NUMBERS LARGER THAN 100.

REGLAS: USE TODAS LAS CIFRAS DADAS UNA VEZ Y SÓLO UNA VEZ EN CADA SUMA; Y USE NÚMEROS MÁS GRANDE QUE 100.

0+1+2 = 3	=
12+0 = 12	=
=	=
=	=

PLEASE LIST NUMBERS FROM SMALLEST TO LARGEST.

FAVOR DE ALISTAR LAS SUMAS DESDE LA MÁS PEQUEÑA HASTA LA MAS GRANDE.

3 12 ___ ___ ___ ___

FAVOR DE NOTAR LAS RESTAS : 9
PLEASE NOTE DIFFERENCES : ___ ___ ___ ___ ___

=	=
=	=
=	=
=	=

SUMAS:
SUMS: ___ ___ ___ ___ ___ ___ ___

RESTAS:
DIFFERENCES: ___ ___ ___ ___ ___ ___

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CHANGE FOR A DOLLAR

using more than 10 coins

	50¢ H	25¢ Q	10¢ D	5¢ N	1¢ P	no. coins
1	0	1	5	5	0	11
2	0	0	9	2	0	
3	0	3	0	0		
4						20

CAMBIO DE UN DÓLAR

usando más de 10 monedas

	H	Q	D	N	P	no. coins
5						12
6						
7						
8						

How many ways using 10 or fewer coins?

	H	Q	D	N	P	no. coins
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						
11						
12						
13						

¿De cuántas maneras usando 10 monedas o menos?

	H	Q	D	N	P	no. coins
14						
15						
16						
17						
18						
19						
20						
21						
22						
23						
24						
25						
26						

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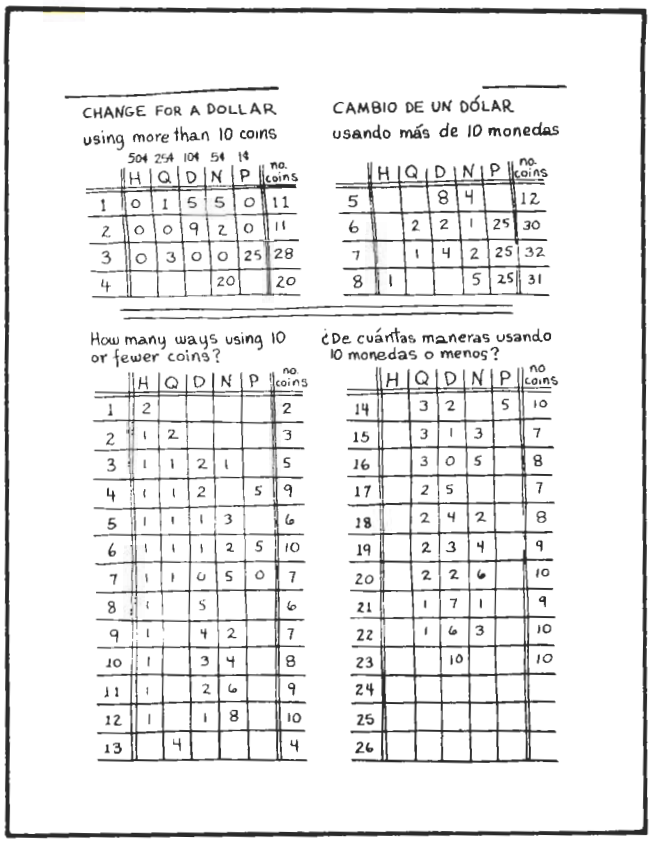

CHANGE FOR A DOLLAR
using more than 10 coins

	H (50¢)	Q (25¢)	D (10¢)	N (5¢)	P (1¢)	no. coins
1	0	1	5	5	0	11
2	0	0	9	2	0	11
3	0	3	0	0	25	28
4				20		20

CAMBIO DE UN DÓLAR
usando más de 10 monedas

	H	Q	D	N	P	no. coins
5			8	4		12
6		2	2	1	25	30
7		1	4	2	25	32
8	1			5	25	31

How many ways using 10 or fewer coins?

	H	Q	D	N	P	no. coins
1	2					2
2	1	2				3
3	1	1	2	1		5
4	1	1	2		5	9
5	1	1	1	3		6
6	1	1	1	2	5	10
7	1	1	0	5	0	7
8	1		5			6
9	1		4	2		7
10	1		3	4		8
11	1		2	6		9
12	1		1	8		10
13		4				4

¿De cuántas maneras usando 10 monedas o menos?

	H	Q	D	N	P	no. coins
14		3	2		5	10
15		3	1	3		7
16		3	0	5		8
17		2	5			7
18		2	4	2		8
19		2	3	4		9
20		2	2	6		10
21		1	7	1		9
22		1	6	3		10
23			10			10
24						
25						
26						

Introducing the Problem

In this activity, children are asked to find combinations of coins that make $1.00, and to note the numbers of coins that they used. In the problem above the double line, more than ten coins must be used to make $1.00. The first combination is given and uses 11 coins. The second combination is also given and uses 11 coins as well. The third entry in the table is not complete. We must decide what other coins are needed to make $1.00, and then record our result. Since three quarters are used, but no dimes or nickels are used, there must be 25 pennies in this combination, a total of 28 coins all together. The fourth line in the table simply specifies that 20 coins are to be used. An easy solution comes to mind quickly: 20 nickels will make $1.00. Are there any other combinations of 20 coins that would make $1.00? This is not an easy question. It requires attention to three variables: (1) the number of coins, (2) the total value of the coins, and (3) the denominations of the coins involved. A full discussion of this investigation is given under "Extensions and Surprise Endings".

In the table at the right above the double line, the first line requires a combination of 12 coins, but there are no restrictions on the other three combinations.

The problem below the double line asks for 26 different ways to make change for $1.00, using no more than ten coins each time. Once again, children should be encouraged to develop some sort of plan for finding the combinations. It would probably be useful to review the "trading in" strategies discussed in the comments for the previous activity.

Talking about the Problem

How many different ways did you find to make change for $1.00 using no more than ten coins? It turns out that there are just 23 different ways. What plans or strategies were used? Time should be taken to discuss the various plans that children worked out.

Extensions and Surprise Endings

How many ways are there to make change for $1.00, using exactly 20 coins? The question is simplified once we realize that every combination must have either 15, 10, 5 or 0 pennies. Thus, we can break the problem up into four smaller problems:

(1) If 15 pennies are used, can 85¢ be made with five other coins?

(2) If ten pennies are used, can 90¢ be made with ten other coins?

(3) If five pennies are used, can 95¢ be made with 15 other coins?

(4) If no pennies are used, can $1.00 be made with 20 coins?

Let's look at the first case in which we use 15 pennies. We need to make the remaining 85¢ with five coins. Suppose that we use one half dollar. We then must make 35¢ with four coins. If we use a quarter as one of the four coins, the remaining 10¢ must be made with three coins. This is impossible. So there can be no quarters in the combination to make 35¢. If there is a combination that uses four coins to make 35¢, it must be limited to dimes and nickels. Three dimes and one nickel will do the trick.

H	Q	D	N	
1		3	1	85¢ in 5 coins

Now suppose we use no half dollars at all. We might look at the other possibilities to make 85¢ with five coins. If we use quarters, we can only use three, two or one. If we use three quarters (75¢), we must make 10¢ with two coins. If we use two quarters, we must make 35¢ with three coins, and so on.

Q	P
3	15
2	15
1	15
0	15

	D	N	
10¢	0	2	2 coins
35¢	✓		3 coins
60¢	✓		4 coins
85¢	✓		5 coins

Three quarters and two nickels is the only combination for 85¢ that includes quarters and uses exactly five coins. So, thus far we have two different combinations for $1.00 that use 15 pennies:

H	Q	D	N	P
1	0	3	1	15
0	3	0	2	15

Before we go on to consider using ten, five or no pennies in our search for combinations of exactly 20 coins to make a dollar, we can note that no other combination can use a half dollar.

H	Q	D	N	P	
1				10	9 coins more (40¢)
1				5	14 coins more (45¢)
1				0	19 coins more (50¢)

If we used a half dollar with ten pennies, (a total of 60¢ with 11 coins), we would have to make up the remaining 40¢ with 9 coins. But nine nickels is 45¢, more than 40¢. If we used one half dollar with five pennies (a total of 55¢ with 6 coins), we would have to get 45¢ with 14 coins. If we used one half dollar with no pennies, we would need 50¢ with 19 coins. None of these combinations can be made. So we don't have to worry about half dollars any more. This makes our work a bit simpler as we go on to consider other possible combinations of 20 coins. This kind of "cutting the problem down in size" eventually leads to the following results:

	H	Q	D	N	P
(1)	1	0	3	1	15
(2)	0	3	0	2	15
(3)	0	2	0	8	10
(4)	0	1	4	5	10
(5)	0	0	8	2	10
(6)	0	1	0	14	5
(7)	0	0	4	11	5
(8)	0	0	20	0	0

Looking back over these results, we might notice some useful relationships. First, 30¢ can be made with two different combinations that use the same numbers of coins: one quarter and five pennies, or six nickels. Once we know this, we can consider making a "trade" whenever we see that we have at least one quarter and five pennies in a combination. For example, the second combination in the chart below shows two quarters, eight nickels, and ten pennies.

H	Q	D	N	P
0	3	0	2	15
	2	0	8	10
	1	0	14	5
	0	0	20	0

We can make a "trade" of one quarter and five pennies for six nickels, which would give us the third combination shown. Another similar trade yields the fourth combination shown.

Secondly, 40¢ can be made with four dimes or with one quarter and three nickels. The same initial combination just considered (two quarters, eight nickels and ten pennies) gives in to two different trades when we employ this exchange.

H	Q	D	N	P
0	2	0	8	10
0	1	4	5	10
0	0	8	2	10

CHANGE FOR A DOLLAR
using more than 10 coins

	50¢ H	25¢ Q	10¢ D	5¢ N	1¢ P	no. coins
1	0	1	5	5	0	11
2	0	0	9	2	0	
3	0	3	0	0		
4						20

CAMBIO DE UN DÓLAR
usando más de 10 monedas

	H	Q	D	N	P	no. coins
5						12
6						
7						
8						

	H	Q	D	N	P	no. coins
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						
11						
12						
13						

	H	Q	D	N	P	no. coins
14						
15						
16						
17						
18						
19						
20						
21						
22						
23						
24						
25						
26						

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Making $1.00 with the same number of coins – in two different ways.

no. of coins	50¢	25¢	10¢	5¢	1¢
6	1	1	1	3	0
		3	2	1	
7	0	2	5	0	0

no. of coins	50¢	25¢	10¢	5¢	1¢
8		3			
	1				
9			7		
10					
		2		6	
11			9		
12		3			
13					
14					
15					
16					
17					
18					

Cambiando $1.00 con el mismo número de moneda, de dos maneras diferentes.

no. de monedas	50¢	25¢	10¢	5¢	1¢
19					15
				12	
20	0	0	8	2	

no. de monedas	50¢	25¢	10¢	5¢	1¢
21					
22					
23					
24					
25					
26					
27					
28					
29					
30					
31					

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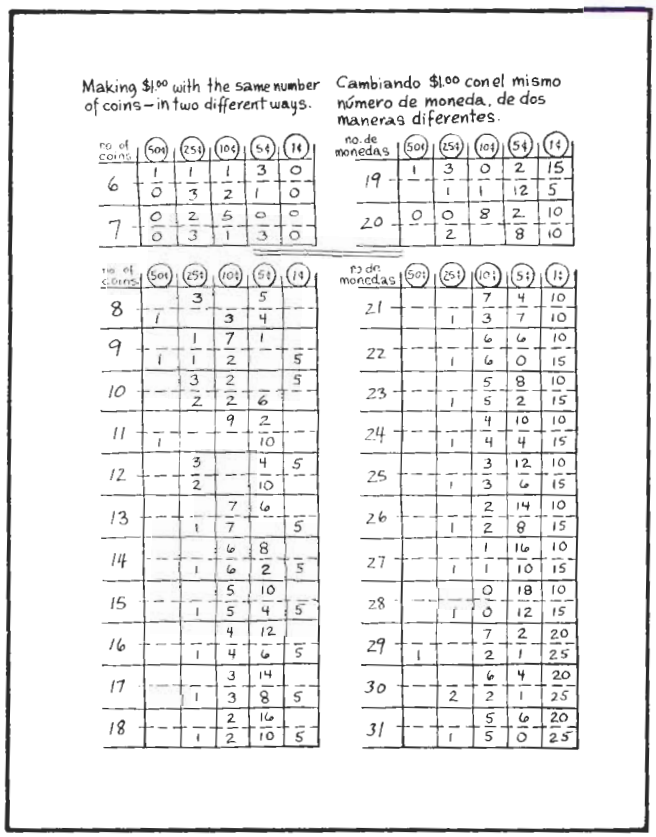


Making $1.00 with the same number of coins—in two different ways.

no. of coins	(50¢)	(25¢)	(10¢)	(5¢)	(1¢)
6	1	1	1	3	0
	0	3	2	1	0
7	0	2	5	0	0
	0	3	1	3	0
8	1	3		5	
			3	4	
9	1	1	7	1	
	1	1	2		5
10		3	2		5
		2	2	6	
11	1		9	2	
				10	
12		3		4	5
		2		10	
13			7	6	
		1	7		5
14			6	8	
		1	6	2	5
15			5	10	
		1	5	4	5
16			4	12	
		1	4	6	5
17			3	14	
		1	3	8	5
18			2	16	
		1	2	10	5

Cambiando $1.00 con el mismo número de moneda, de dos maneras diferentes.

no. de monedas	(50¢)	(25¢)	(10¢)	(5¢)	(1¢)
19	1	3	0	2	15
		1	1	12	5
20	0	0	8	2	10
		2		8	10
21			7	4	10
		1	3	7	10
22			6	6	10
		1	6	0	15
23			5	8	10
		1	5	2	15
24			4	10	10
		1	4	4	15
25			3	12	10
		1	3	6	15
26			2	14	10
		1	2	8	15
27			1	16	10
		1	1	10	15
28			0	18	10
		1	0	12	15
29	1		7	2	20
			2	1	25
30			6	4	20
		2	2	1	25
31			5	6	20
		1	5	0	25

Introducing the Problem

This activity asks for combinations of specific numbers of coins that make $1.00. Once a number of coins is specified, we must look for two *different* combinations that use that number of coins. Since there are 22 numbers of coins specified in the problems below the double line, and since each calls for two different combinations, there are really 44 problems ahead! One would be well advised to look for shortcuts.

The problems above the double line give an opportunity to look at an approach that might be useful. In the chart at the left, two combinations that use six coins are specified. One combination is given for seven coins, and the second must be found. Of course, a seven-coin combination uses one more coin than a six-coin combination, and though obvious, that is a useful bit of information. For we might notice that if, in any combination, one dime is exchanged for two nickels, the total *value* of the combination will remain the same, but there will be one more coin. Thus, when we notice that a combination for six coins uses a dime, that dime could be "traded" for two nickels, and we would have a seven-coin combination.

two nickels, and we would have a seven-coin combination.

	H	Q	D	N	P
6	0	3	2	1	0
7	0	3	1	3	0

Now this seven-coin combination uses one dime. That dime could be traded for two nickels, providing a combination that uses eight coins:

	H	Q	D	N	P
6	0	3	2	1	0
7	0	3	1	3	0
8	0	3			0

?

Three quarters, no dimes and five nickels is the combination we would get that uses eight coins.

Such discussion hopefully will alert children to a most helpful tactic for adding one more coin to any combination that uses dimes.

	H	Q	D	N	P
8	0	2	4	2	0
9	0	2	3	4	0
10	0	2	2	6	0
11	0	2	1	8	0
12	0	2	0	10	0

Of course, the trading can also go the other way. Two nickels can be traded for one dime to get a combination that uses one less coin.

There are other relationships that are helpful for finding different combinations of coins that have the same value and use the same number of coins. Here are two useful ones:

(1) 40¢ can be made with four coins in two ways:

(a) four dimes

(b) one quarter and three nickels

(2) 30¢ can be made with six coins in two ways:

(a) six nickels

(b) one quarter and five pennies

Talking about the Problem

Attention should be given to the "trading" strategy discussed above. Some children have probably used it successfully in their work. We might consider the following situation to reinforce the point. Given a single combination of 22 coins, namely, six dimes, six nickels, and ten pennies, a whole sequence of combinations for the chart on this page can be found. First, a second combination of 22 coins can be found by changing six nickels for one quarter and five pennies.

	H	Q	D	N	P
22	0	0	6	6	10
	0	1	6	0	15

Then, two combinations that use one more coin (23 coins) can be found by changing one dime for two nickels in each of the two combinations for 22 coins.

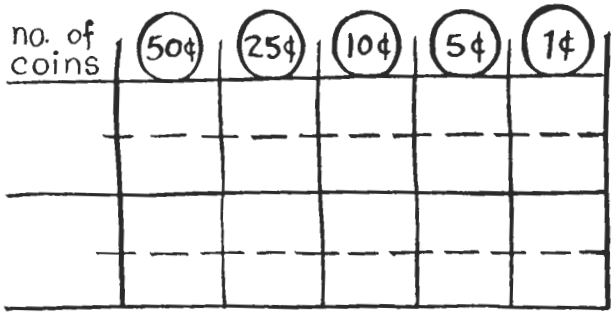
	H	Q	D	N	P
22	0	0	6	6	10
	0	1	6	0	15
23	0	0	5	8	10
	0	1	5	2	15

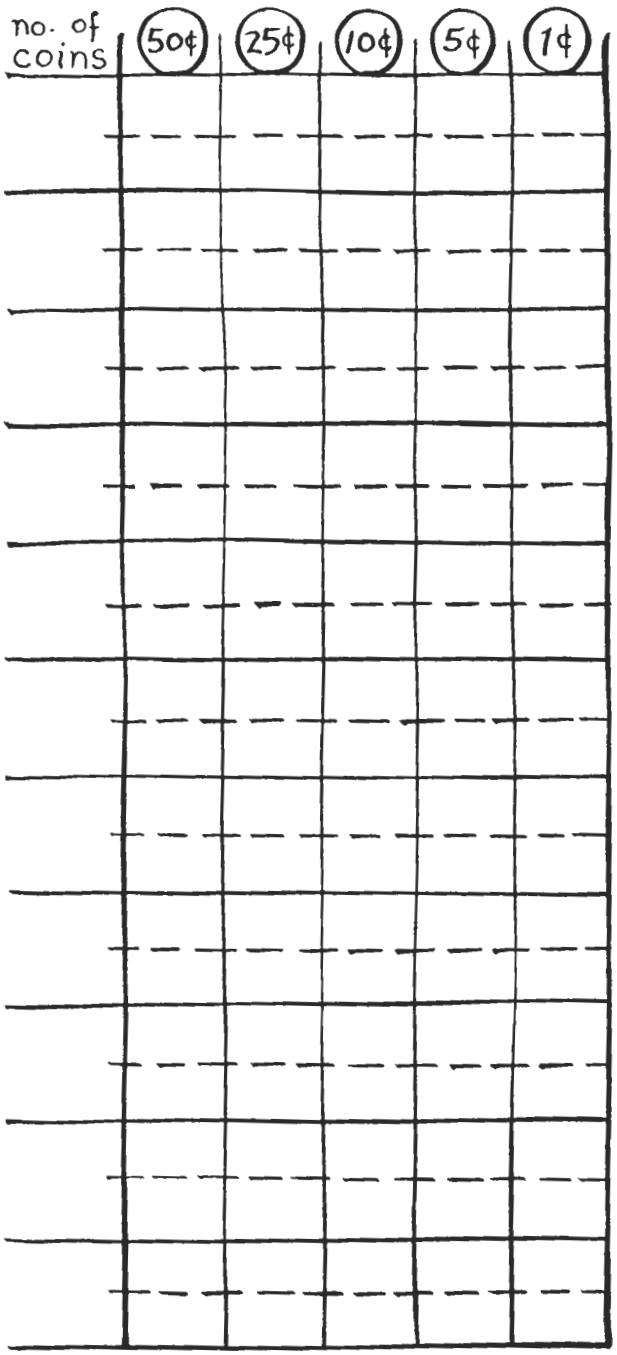
Now, once we have two combinations for 23 coins, we can repeat the process of trading one dime for two nickels to get two new combinations for 24 coins, and so on until the dimes run out.

	H	Q	D	N	P
22			6	6	10
		1	6	0	15
23			5	8	10
		1	5	2	15
24			4	10	10
		1	4	4	15
25			3	12	10
		1	3	6	15
26			2	14	10
		1	2	8	15
27			1	16	10
		1	1	10	15
28			0	18	10
		1	0	12	15

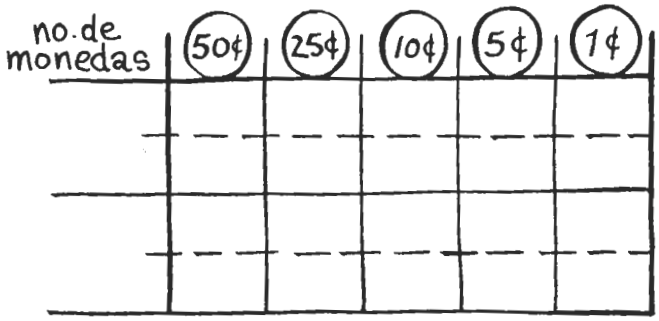
By using this type of strategy, the 44 combinations required can be generated from a very few key combinations.

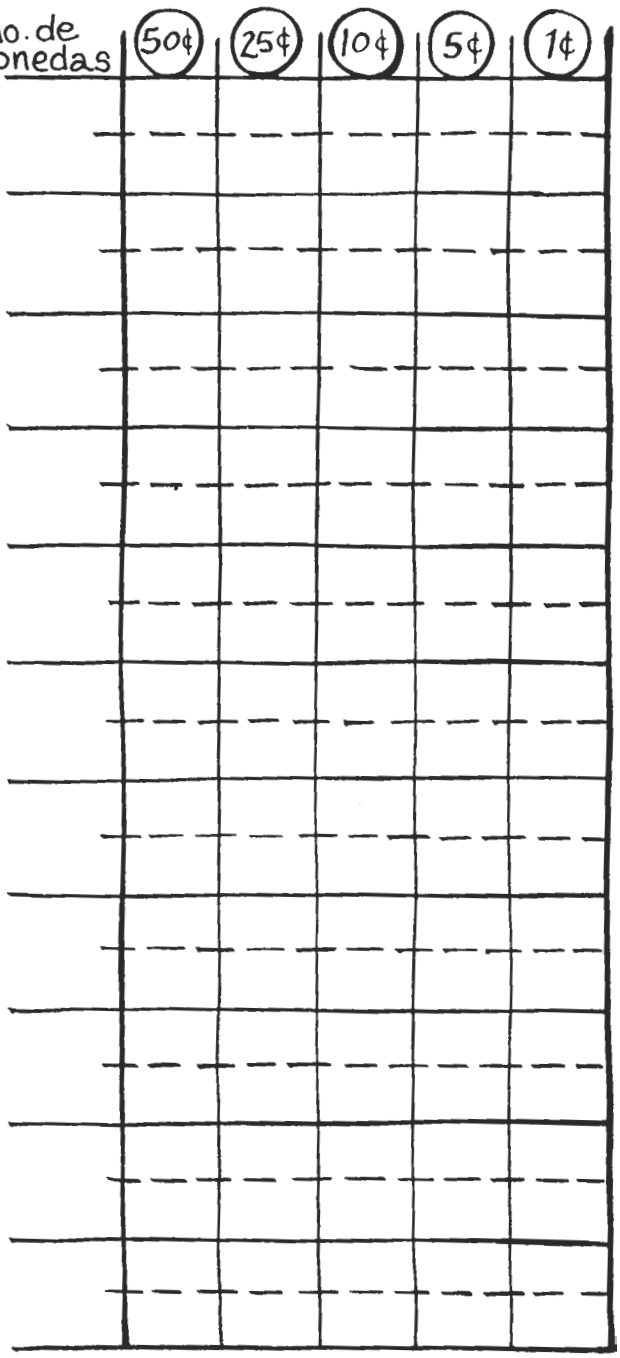
Making $1.00 with the same number of coins – in two different ways.





Cambiando $1.00 con el mismo número de moneda, de dos maneras diferentes.





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Making $1.00 with the same number of coins – in two different ways.

no. of coins	50¢	25¢	10¢	5¢	1¢
6	1	0	5	0	0
	0	3	2	1	0
8	0	3	0	5	0

no. of coins	50¢	25¢	10¢	5¢	1¢
10					
12					
14					
16					
18					
20					
22					
24					
26					
28					
30					

Cambiando $1.00 con el mismo número de moneda, de dos maneras diferentes.

no. de monedas	50¢	25¢	10¢	5¢	1¢
33	0	0	7	1	25
36					

no. de monedas	50¢	25¢	10¢	5¢	1¢
39					
42					
45					
48					
51					
54					
57					
60					
63					
66					
69					

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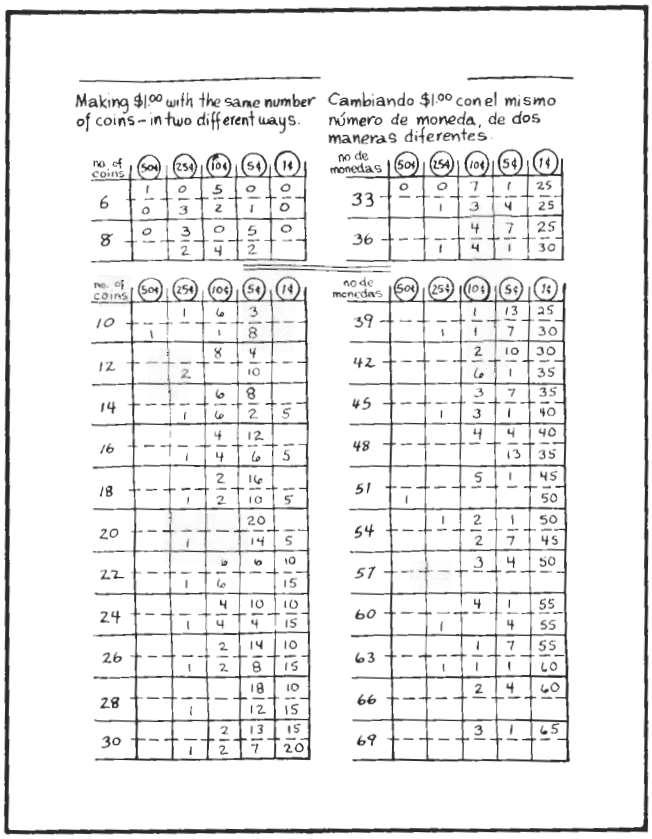
(Some of the appropriate responses are shown here.)



Making $1.00 with the same number of coins – in two different ways.

no. of coins	50¢	25¢	10¢	5¢	1¢
6	1 / 0	0 / 3	5 / 2	0 / 1	0 / 0
8	0 / –	3 / 2	0 / 4	5 / 2	0
10	– / 1	1 / –	6 / 1	3 / 8	– / –
12	– / –	2 / –	8 / –	4 / 10	– / –
14	– / 1	1 / –	6 / 6	8 / 2	5 / –
16	– / –	1 / –	4 / 4	12 / 6	5 / –
18	– / –	1 / –	2 / 2	16 / 10	5 / –
20	– / –	1 / –	– / –	20 / 14	5 / –
22	– / –	1 / –	6 / 6	6 / –	10 / 15
24	– / –	1 / –	4 / 4	10 / 4	10 / 15
26	– / –	1 / –	2 / 2	14 / 8	10 / 15
28	– / –	1 / –	– / –	18 / 12	10 / 15
30	– / –	1 / –	2 / 2	13 / 7	15 / 20

Cambiando $1.00 con el mismo número de moneda, de dos maneras diferentes.

no de monedas	50¢	25¢	10¢	5¢	1¢
33	0 / –	0 / 1	7 / 3	1 / 4	25 / 25
36	– / –	– / 1	4 / 4	7 / 1	25 / 30
39	– / –	– / 1	1 / 1	13 / 7	25 / 30
42	– / –	– / –	2 / 6	10 / 1	30 / 35
45	– / –	– / 1	3 / 3	7 / 1	35 / 40
48	– / –	– / –	4 / –	4 / 13	40 / 35
51	– / 1	– / –	5 / –	1 / –	45 / 50
54	– / –	1 / –	2 / 2	1 / 7	50 / 45
57	– / –	– / –	3 / –	4 / –	50 / –
60	– / –	– / 1	4 / –	1 / 4	55 / 55
63	– / –	– / 1	1 / 1	7 / 1	55 / 60
66	– / –	– / –	2 / –	4 / –	60 / –
69	– / –	– / –	3 / –	1 / –	65 / –

Introducing the Problem

While this activity may look similar to that on the preceding page, it is structured to enable further insight into the problem solving strategies discussed there. The left-hand column on this page considers combinations of even numbers of coins, 6 through 30. Each pair of combinations requires two more coins than the previous one. The right-hand column considers numbers of coins that differ by three: 33, 36, 39, and so on up to 69.

When working in the left-hand column, it is helpful to notice that there are at least two "trades" that result in the addition of two more coins to the combination:

(1) one quarter for two dimes and a nickel (one coin for three)

(2) two dimes for four nickels (two coins for four)

If we start with the combination of three quarters, two dimes and one nickel (six coins all together), we can trade the quarters, one at a time for two dimes and a nickel, and get combinations of 8, 10 and 12 coins.

(1Q → 2D, 1N)

	H	Q	D	N	P
6		3	2	1	
8		2	4	2	
10		1	6	3	
12			8	4	

Then, trading two dimes at a time for four nickels until we run out of dimes, we get combinations of 14, 16, 18 and 20 coins.

(2D → 4N)

	H	Q	D	N	P
12			8	4	
14			6	8	
16			4	12	
18			2	16	
20			0	20	

Now, the problem asks for two different combinations for each number of coins. We can get some help in finding a second combination for each of these numbers of coins if we make use of two trades that do not alter the number of coins in a combination or their total value:

(a) 40¢ can be made with four dimes or one quarter and three nickels.

(b) 30¢ can be made with six nickels or one quarter and five pennies.

(a) 40¢ .... 4D and 1Q, 3N
(b) 30¢ .... 6N and 1Q, 5P

	H	Q	D	N	P	
12	0	0	8	4	0	(a)
	0	1	4	7	0	
14	0	0	6	8	0	(a)
	0	1	2	11	0	
16	0	0	4	12	0	(a)
	0	1	0	15	1	
18	0	0	2	16	0	(b)
	0	1	2	10	5	
20	0	0	0	20	0	(b)
	0	1	0	14	5	

Some of these "trading" relationships are useful when working in the right-hand column, but we also need some relationships that will help us jump from

one coin combination to a combination with *three* more or three fewer coins. Some that are helpful are listed below.

(a) 30¢ can be made with three dimes and six nickels

(b) 30¢ can be made with three dimes or one quarter and five pennies

(c) 15¢ can be made with three nickels or one dime and five pennies

(d) 50¢ can be made with one half dollar or one quarter, two dimes and one nickel.

(a) 30¢ ···· 3D and 6N
(b) 30¢ ···· 3D and 1Q, 5P
(c) 15¢ ···· 3N and 1D, 5P
(d) 50¢ ···· 1H and 1Q, 2D, 1N

For example, using a trade described by (d) above, and one trade that doesn't change the number of coins, (one quarter and five pennies for six nickels) we can get:

	H	Q	D	N	P	
51	1	0	0	0	50	trade "d"
54	0	1	2	1	50	a "same no." trade
	0	0	2	7	45	

These trading strategies can be used to generate large sections of the right-hand column. In the diagram below, the letters at the right refer to the type of trade that was made.

	H	Q	D	N	P	
33			7	1	25	(a)
36			4	7	25	(a)
39			1	13	25	(b)
42			2	10	30	(b)
45			3	7	35	(b)
48			4	4	40	(b)
51			5	1	45	

It turns out that we cannot find a second combination for 57, 66 or 69 coins. Let's try 57 coins. We will look at combinations using 55, 50 and 45 pennies.

(a) 55 P leaves — 45¢ in 2 coins
(b) 50 P leaves — 50¢ in 7 coins (3D, 4N)
(c) 45 P leaves — 55¢ in 12 coins

(a) 45¢ cannot be made with two coins, so we cannot use 55 pennies.

(b) 50¢ can be made with seven coins in only one way: three dimes and four nickels. You might be tempted to use a quarter, but notice that if you do, the problem becomes one of making 25¢ with six coins, which is impossible.

(c) 55¢ cannot be made with 12 coins since even 12 nickels (the smallest denomination after a penny) would be more than that.

No combination is possible that uses fewer than 45 pennies, for then too many coins are needed to make up the balance of the dollar. For example, if we used 40 pennies, we would have to make 60¢ with 17 coins. That is clearly impossible.

Suppose we consider 66 coins using 65, 60 or 55 pennies.

(a) 65 P leaves — 35¢ in 1 coin
(b) 60 P leaves — 40¢ in 6 coins (2D, 4N)
(c) 55 P leaves — 45¢ in 11 coins (but 11N = 55¢)

The one combination that works is the only one that can be found. A second combination does not exist.

Suppose we consider 69 coins using 65 or 60 pennies.

(a) 65 P leaves — 35¢ in 4 coins (3D, 1N)
(b) 60 P leaves — 40¢ in 9 coins (but 9N = 45¢)

Once again, the combination that works is the only one possible.

So we must finally conclude that there is just one possible combination for 57, 66 and 69 coins.

The activity on the reverse side of this page is taken from *Drill and Practice at the Problem Solving Level* (Curriculum Development Associates, 1974)

SUMAS:

SUMS:

RESTAS:

DIFFERENCES:

SUMAS:

SUMS:

RESTAS:

DIFFERENCES:

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$5.00 usando 50 monedas, sin usar monedas de a diez.

$5.00 using 50 coins without dimes.

3	Dollars...	$3.00	0 D....	$0.00	1 D.... $.
1	Halves...	.50	6 H....	.	___ H.... .
4	Quarters..	1.00	___ Q....	.	___ Q.... .
2	Nickels...	.10	___ N....	.	___ N.... .
___	Pennies..	.	30 P....	.	40 P.... .
50		$5.00	50	$5.00	50 $5.00

___	Dollars...	$.	___ D....	$.	___ D.... $.
___	Halves...	.	___ H....	.	___ H.... .
___	Quarters..	.	___ Q....	.	___ Q.... .
___	Nickels...	.	___ N....	.	___ N.... .
___	Pennies...	.	___ P....	.	___ P.... .
50		$5.00	50	$5.00	50 $5.00

___	Dollars...	$.	___ D....	$.	___ D.... $.
___	Halves...	.	___ H....	.	___ H.... .
___	Quarters..	.	___ Q....	.	___ Q.... .
___	Nickels...	.	___ N....	.	___ N.... .
___	Pennies...	.	___ P....	.	___ P.... .
50		$5.00	50	$5.00	50 $5.00

___	Dollars...	$.	___ D....	$.	___ D.... $.
___	Halves...	.	___ H....	.	___ H.... .
___	Quarters..	.	___ Q....	.	___ Q.... .
___	Nickels...	.	___ N....	.	___ N.... .
___	Pennies..	.	___ P....	.	___ P.... .
50		$5.00	50	$5.00	50 $5.00

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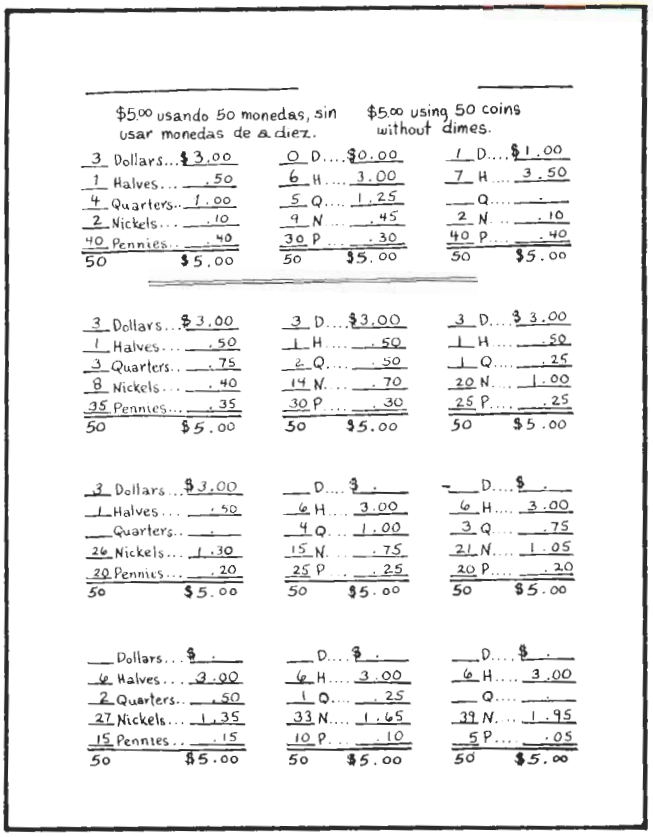
(Some of the appropriate responses are shown here.)



$5.00 usando 50 monedas, sin usar monedas de a diez.

3 Dollars	$3.00	
1 Halves	.50	
4 Quarters	1.00	
2 Nickels	.10	
40 Pennies	.40	
50	$5.00	

$5.00 using 50 coins without dimes.

0 D	$0.00	
6 H	3.00	
5 Q	1.25	
9 N	.45	
30 P	.30	
50	$5.00	

1 D	$1.00	
7 H	3.50	
Q	.	
2 N	.10	
40 P	.40	
50	$5.00	

3 Dollars	$3.00	
1 Halves	.50	
3 Quarters	.75	
8 Nickels	.40	
35 Pennies	.35	
50	$5.00	

3 D	$3.00	
1 H	.50	
2 Q	.50	
14 N	.70	
30 P	.30	
50	$5.00	

3 D	$3.00	
1 H	.50	
1 Q	.25	
20 N	1.00	
25 P	.25	
50	$5.00	

3 Dollars	$3.00	
1 Halves	.50	
Quarters	.	
26 Nickels	1.30	
20 Pennies	.20	
50	$5.00	

D	$.	
6 H	3.00	
4 Q	1.00	
15 N	.75	
25 P	.25	
50	$5.00	

D	$.	
6 H	3.00	
3 Q	.75	
21 N	1.05	
20 P	.20	
50	$5.00	

Dollars	$.	
6 Halves	3.00	
2 Quarters	.50	
27 Nickels	1.35	
15 Pennies	.15	
50	$5.00	

D	$.	
6 H	3.00	
1 Q	.25	
33 N	1.65	
10 P	.10	
50	$5.00	

D	$.	
6 H	3.00	
Q	.	
39 N	1.95	
5 P	.05	
50	$5.00	

Introducing the Problem

This coin problem asks for combinations of exactly fifty coins that have a value of $5.00. It would have been easy to show $5.00 as 50 dimes, but since dimes are excluded from use, we have to do more work. Clues are given for the three combinations above the double line. The first one specifies ten coins in the combination which have a total value of $4.60. So we only need to find 40 coins that have a value of 40¢. Of course, 40 pennies will fill the bill. In the second combination, 36 coins are already indicated which have a total of $3.30. So we need a combination of 14 coins (quarters and nickels) with a value of $1.70: five quarters ($1.25) and nine nickels ($.45). The third combination specifies 41 coins with a combined value of $1.40. Thus, $3.60 must be made up with nine coins. This is not difficult; seven half dollars ($3.50) and two nickels ($.10) will work.

Talking about the Problem

Below the double line, there is space to record nine more combinations, and no clues at all are given. Since the three combinations above the double line have either 30 or 40 pennies in each, we might begin by trying 35 pennies. That would leave $4.65 to be made with 15 coins. Suppose we start by using four dollars. We would than have 39 coins worth $4.35 and would need eleven coins worth 65¢ to complete the combination.

39 coins worth $4.35 ···· 11 coins for 65¢

That can be done easily with nine nickels (45¢) and two dimes (20¢). But dimes are not allowed! Without dimes, we can't find an appropriate combination of eleven coins. So we can start over and try using *three* dollars with our 35 pennies. We could also try two dollars. We would have:

38 coins worth $3.35 ···· 12 coins for $1.65
37 coins worth $2.35 ···· 13 coins for $2.65

We will have real trouble making $2.65 with 13 coins, but we can get $1.65 with 12 coins. So, another combination of 50 coins worth $5.00 is:

3	D	$3.00
1	H	.50
3	Q	.75
8	N	.40
35	P	.35
50		$5.00

Now compare this combination with the first one above the double line:

3	D	$3.00
1	H	.50
4	Q	1.00
2	N	.10
40	P	.40
50		$5.00

(a)

3	D	$3.00
1	H	.50
3	Q	.75
8	N	.40
35	P	.35
50		

(b)

Combination (b) has one less quarter and five fewer pennies than (a), but (b) has six more nickels than (a). This is evidence that one combination of six coins worth 30¢ has been exchanged for another combination of six coins worth 30¢. Since combination (b) still has quarters and pennies, we could keep on making that trade to get other different combinations until we ran out of the coins needed to make the trade.

D	H	Q	N	P
3	1	4	2	40
3	1	3	8	35
3	1	2	14	30
3	1	1	20	25
3	1	0	26	20

If we use this same trading scheme with the second example above the double line, it is even more productive. The exchange can be made in both directions, since we start out with more than six nickels. Notice that as the number of quarters decreases by one and the number of pennies decreases by five in each successive combination, the number of nickels increases by six. This number pattern reflects the trade.

D	H	Q	N	P
0	6	6	3	35
0	6	5	9	30
0	6	4	15	25
0	6	3	21	20
0	6	2	27	15
0	6	1	33	10
0	6	0	39	5

This bonanza might suggest that we look for combinations with a large number of quarters and pennies in order to make lots of trades. Here are some examples:

	D	H	Q	N	P
(11)	0	2	14	4	30
(12)	0	2	13	10	25
(13)	0	2	12	16	20
(14)	0	2	11	22	15
(15)	0	2	10	28	10
(16)	0	2	9	34	5
(17)	0	2	8	40	0

	D	H	Q	N	P
(18)	2	0	10	3	35
(19)	2	0	9	9	30
(20)	2	0	8	15	25
(21)	2	0	7	21	20
(22)	2	0	6	27	15
(23)	2	0	5	33	10
(24)	2	0	4	39	5
(25)	2	0	3	45	0

Sometimes a given combination will give in to trades "in both directions", as in the following example:

D	H	Q	N	P	
1	3	8	3	35	
1	3	7	9	30	↑
1	3	6	15	25	
1	3	5	21	20	↓
1	3	4	27	15	
1	3	3	33	10	
1	3	2	39	5	
1	3	1	45	0	

Extensions and Surprise Endings

A note from Robert Wirtz:

By starting with the following combinations and turning in one quarter and five pennies for six nickels, I have found that an additional 39 combinations can be generated. If you find more, please let me know!

D	H	Q	N	P	combinations
3	1	4	2	40	5
2	4	2	2	40	3
2	0	10	3	35	8
1	7	0	2	40	7
1	3	8	3	35	8
0	6	6	3	35	7
0	2	14	4	30	7
					39

The activity on the reverse side of this page is taken from *Drill and Practice at the Problem Solving Level* (Curriculum Development Associates, 1974)

DIGITS USED
CIFRAS USADAS

SUMAS:

SUMS:

RESTAS:

DIFFERENCES:

DIGITS USED
CIFRAS USADAS

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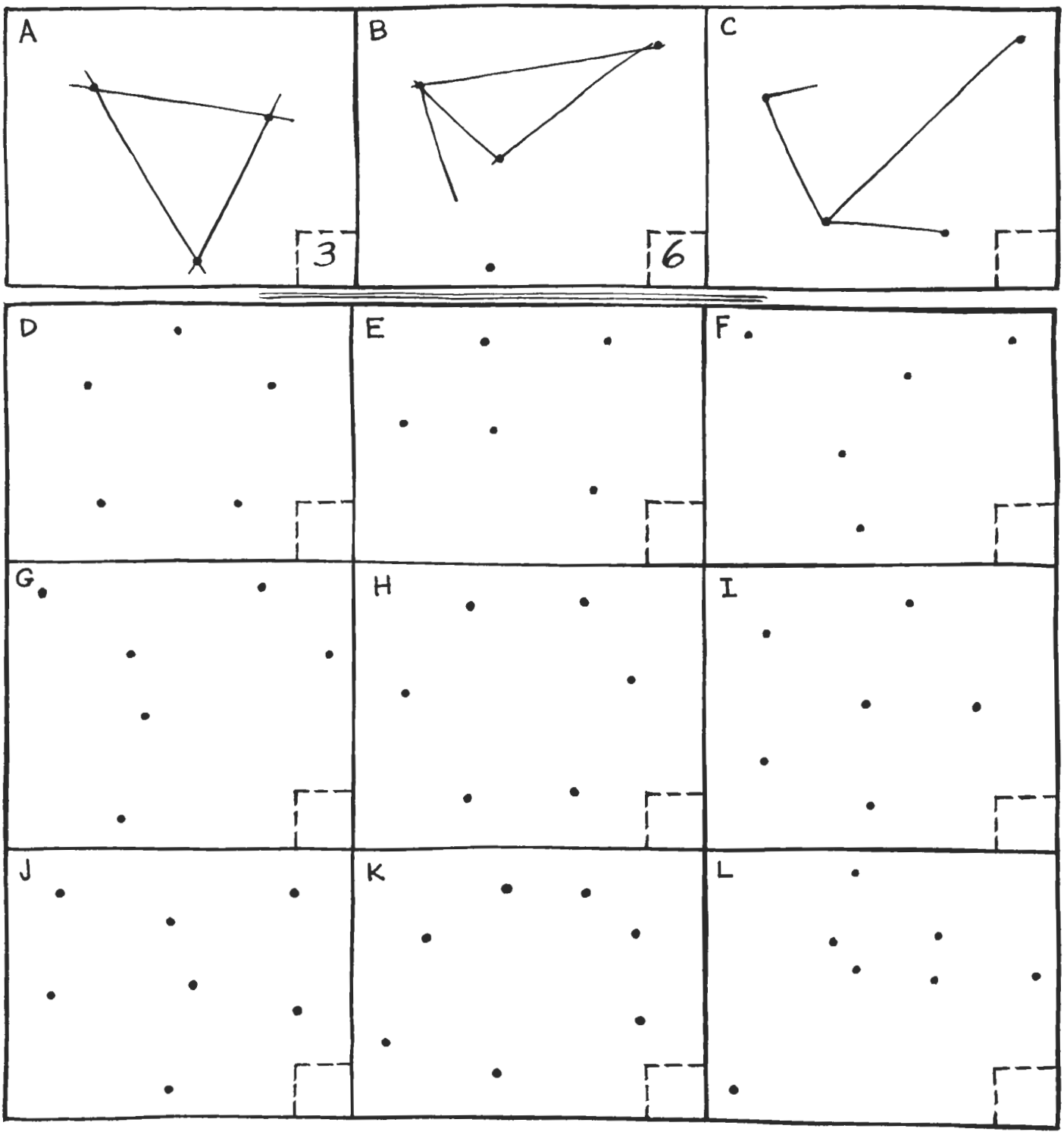
¿Cuantos lineas se necesitan para unir todos los puntos de cada diagrama?

How many lines are needed to connect all points in each diagram?



What do you notice? ¿Qué notas?

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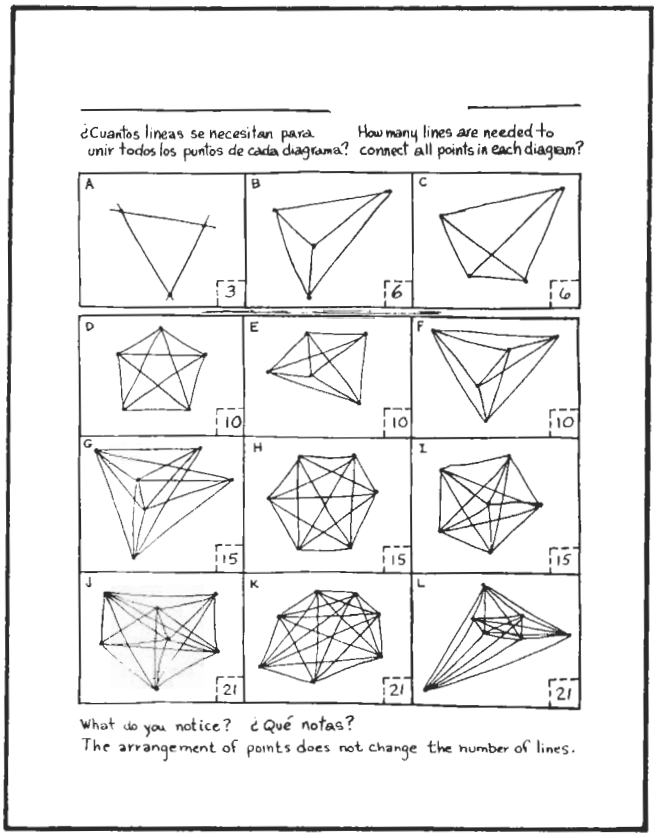


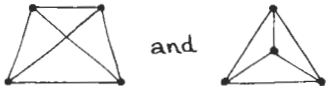
Introducing the Problem

Each diagram shows a configuration of points. The problem is to connect all possible pairs of points in the configuration with straight lines. The number of lines needed to do this is recorded in the small square in the bottom right-hand corner of each frame. When three points are shown, as in problem A, all pairs can be connected in basically one way. A triangle will be formed by the lines.

When four points are considered, as in problems B and C, there are two seemingly different shapes formed by the connecting lines:

(a) a four-sided figure, with opposite corners connected by diagonal lines

(b) a triangle with a point "inside" which is connected to each vertex of the triangle:



In both cases, six lines are required to connect all pairs.

Problems D, E and F on this page have five points in the configuration; problems G, H and I have six points; and problems J, K and L have seven points. It might be useful to have children suggest some ways to keep track of the lines they draw, since the diagrams become complicated and it is difficult to count the lines. While using a straightedge generally leads to diagrams that are easy to read, the lines need not be straight. They simply must connect two points and not touch any other points.

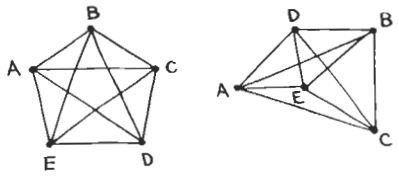
Talking about the Problem

Did anything interesting happen as you solved the problems on this page? Some children will probably notice that the number of lines required to connect all pairs of a given number of dots is the same, regardless of the arrangement of the dots.

Consider examples B and C, each of which has four dots. Both figures can be seen as sketches of a three-dimensional tetrahedron (a four-sided solid with four triangular faces). In fact when any four points are designated (no three in a straight line) and all pairs are connected, the drawing can be seen as a tetrahedron.

Think of the dot in each sketch above at the "top" of the three-dimensional shape. In (a) you see only two of the sides; in (b), you are looking down on the top, seeing just three sides but not the "bottom". In (c), you see only one side.

Another way to approach these problems is to label the points with letters and then list the connecting lines. The labels for the points need follow no particular pattern.



In each case, dot A will be connected to all other dots:

$\overline{AB}$ $\overline{AC}$ $\overline{AD}$ $\overline{AE}$ (4)

Dot B is already connected to A, but it can be connected to all other dots:

$\overline{BC}$ $\overline{BD}$ $\overline{BE}$ (3)

Dot C is already connected to all dots except D and E:

$\overline{CD}$ $\overline{CE}$ (2)

Dot D is already connected to all dots except E:

$\overline{DE}$ (1)

Finally, dot E is already connected to all dots. So all together there are ten lines (4 + 3 + 2 + 1 = 10) needed to connect every pair of dots.

If we summarize the results of our investigation on this page in a table, other patterns begin to emerge. The differences between successive numbers in the second line in the table seem to increase by one each time.

dots to connect	2	3	4	5	6	7	8	9
lines required		3	6	10	15	21	28	
differences:		3	4	5	6	7		

If we extend this pattern of numbers on both ends, we have:

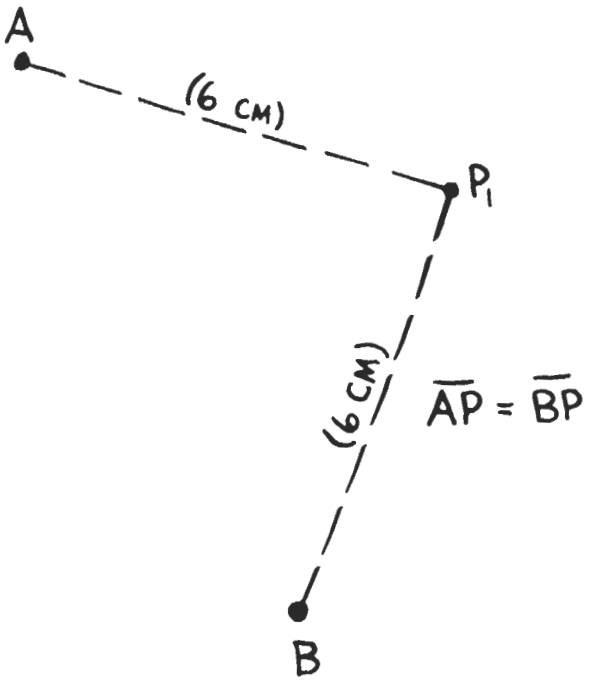
dots to connect	1	2	3	4	5	6	7	8	9
lines required	0	1	3	6	10	15	21	28	36

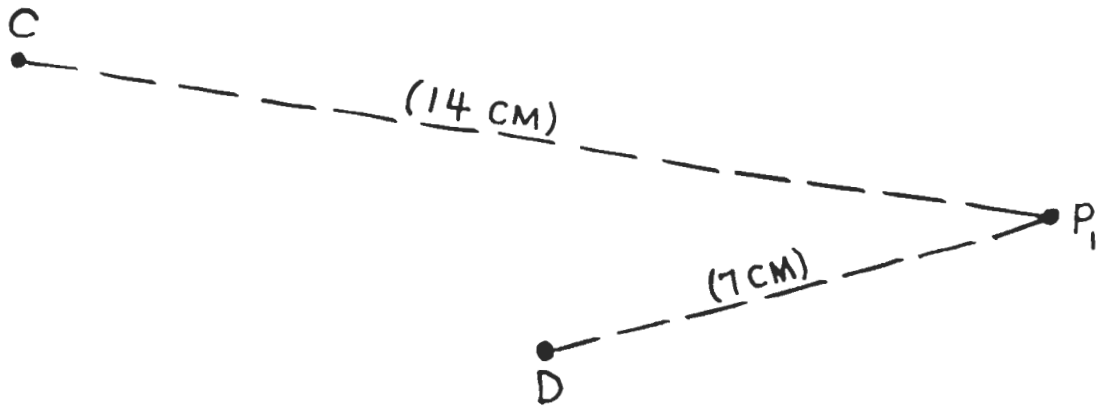
We are led to say that if there are two dots, one line will be required to connect them. If there is a single dot, no lines will be required because there is no pair. At the other end of the chart, 28 lines will be needed to connect eight dots in pairs; 36 lines to connect nine dots in pairs, and so on.

Extensions and Surprise Endings

There is a famous problem about a party situation in which each guest shakes hands with each other guest. No two people shake hands with each other more than once. How many handshakes take place? Suppose there were nine people at the party. How many handshakes would there be? We might start by working up toward nine. With two people, there would be one handshake; with three people, three handshakes; with four people, six handshakes, and so on. Or, we might consult our chart above, letting dots be guests and lines be handshakes. Nine guests would perform 36 handshakes.

NOTES





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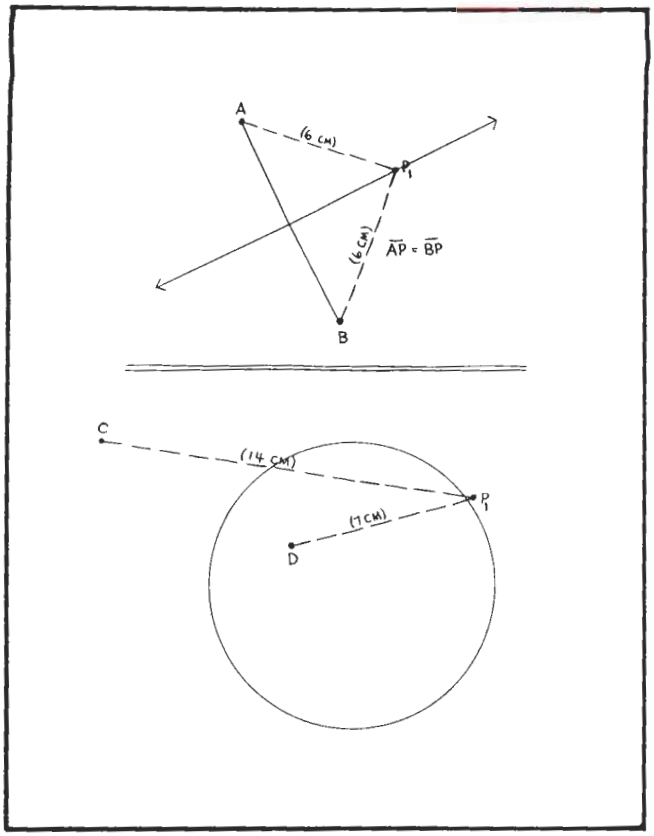


Introducing the Problem

This activity is concerned with finding sets of points that satisfy a certain condition. In the sketch above the double line, the point labelled 'P_1' is the same distance from point A as it is from point B. In fact, it is six centimeters from each. Can you find other points that have the same distance from A as they do from B?

You may use a ruler or any other instrument. You might pick a point "by eye" and check its distance from each of points A and B by using a string or marks on the edge of a paper. All methods are "legal". One easy way to find another point that has the same distance to A as it does to B is to draw a straight line from A to B, measure the distance, and mark the midpoint of the line.

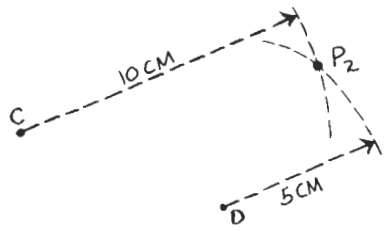
Can you find a point that is eight centimeters from both A and B? Nine centimeters from both? As points are found by a variety of methods and marked on the paper, they all appear to lie along a straight line. Furthermore, the line seems to be perpendicular to line AB and to go through its midpoint.

The problem below the double line is quite *different. Points must be found which are twice as far* away from point C as they are from point D. In the diagram, P_1 is such a point. The line from P_1 to C to 14 centimeters long. The line from P_1 to D is only 7 centimeters long. Thus, P_1 is twice as far from C as it is from D.

Don't expect the points to fall in a straight line. Since quite a few points must be found before any familiar line or shape is suggested, it might be useful to have children work in pairs or small groups and share their results with each other.

Talking about the Problem

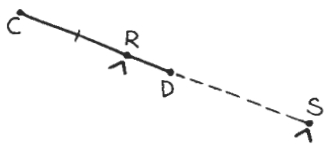
Did you find many points that are twice as far from C as from D? What methods did you use to find them? Did anyone use a compass? If you set a compass with its points 10 centimeters apart and draw an arc with the center at C, and then set the points five centimeters apart and draw an arc with the center at D, you would have this result:



What can you say about the point P_2 where the arcs cross? It is five centimeters from D and 10 centimeters from C, so it is twice as far from C as it is from D.

If we draw the line CD and measure it, how can we find another point that satisfies our condition? We can take a point that is two thirds of the way from C to D. Call it point 'R'. It will be twice as far from C as it is from D.

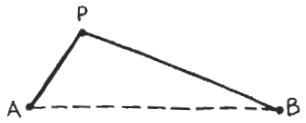
If we extend the line CD beyond D, how can we find another point twice as far from C as from D? Simply pick a point that is the same distance from D as D is from C. Call this point 'S'.



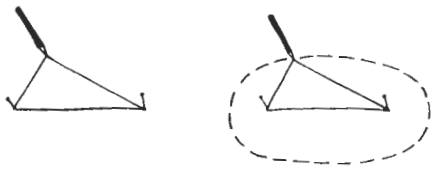
As more and more points are found, they will all seem to lie on a circle that has the line RS as a diameter.

Extensions and Surprise Endings

Consider another variation. Draw a triangle and label its three vertices A, P and B. Measure the distance from A to P and from P to B. Then add those distances. Suppose the sum of the distances is 21 centimeters.



Now the problem is to find other points whose combined distance from points A and B is also 21 centimeters. An interesting way to find these points is to place a pin at point A and one at point B. Then place the point of a pencil at P. Now make a loop of string that will "fit" exactly when stretched around the pins and the pencil point. (In other words, a loop of string that is the length of the perimeter of the triangle.) Now move the pencil, keeping the string stretched reasonably taut. The resulting figure will be an oval.

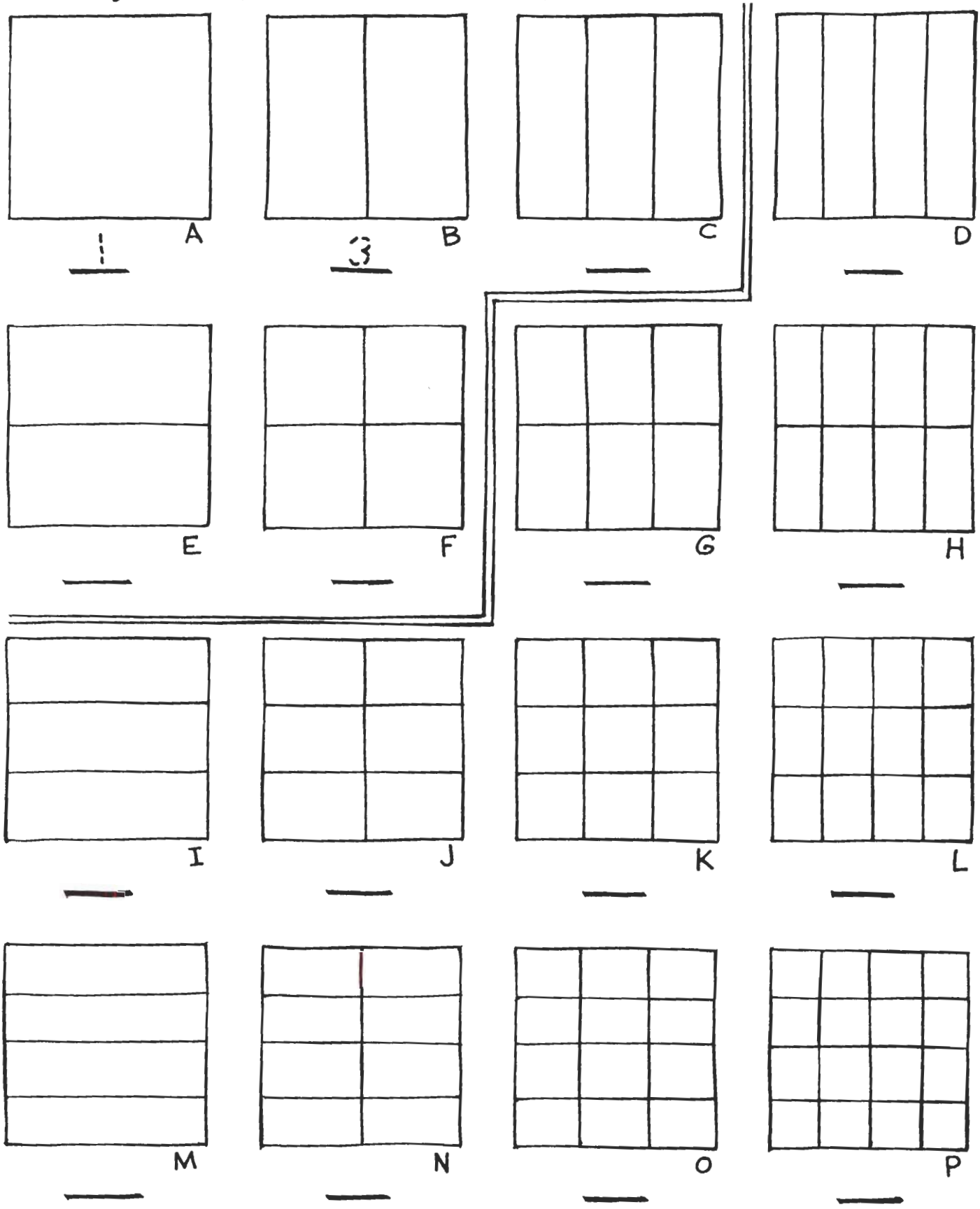


Every point on the oval will have the same combined distance from points A and B, since the length of string that stretched from A to P to B stayed the same as you were drawing the oval.

NOTES

How many squares and rectangles can you see?

¿Cuántos cuadritos y paralelogramos encuentras?



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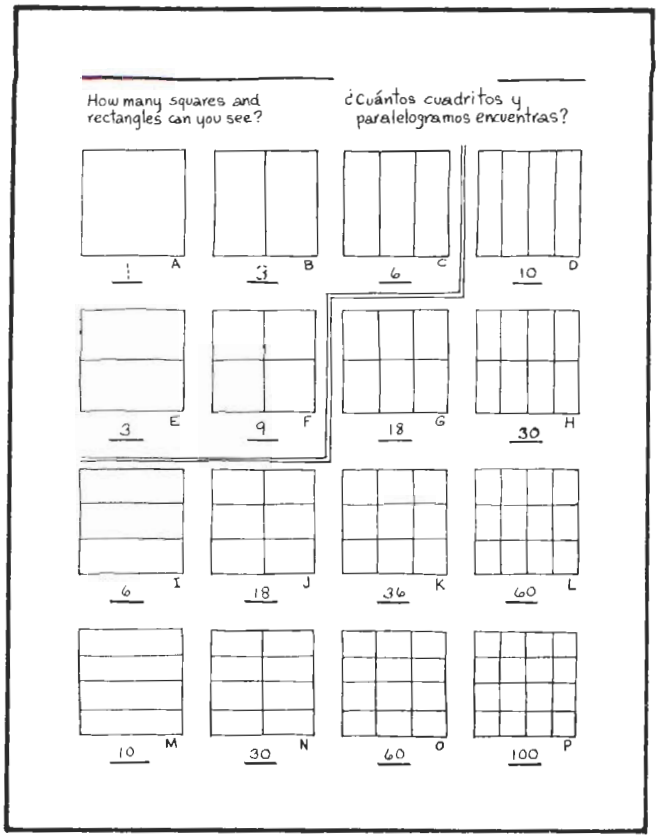
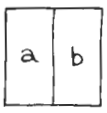


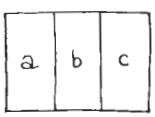
Introducing the Problem

Each diagram shows a large square containing squares or rectangles. The problem is to find as many squares or rectangles as you can in each diagram. Beginning with problem B, it is helpful to assign a letter to each small area so that the squares and rectangles can be identified by using one or more of the letters. For example, if the areas in problem (B) are designated (a) and (b) as shown in the diagram below, we can find three squares or rectangles. There are two rectangles which we record as "a" and "b", and there is the large square which we will record as "ab" to indicate that it is made up of the two small rectangles, a and b.



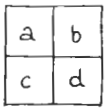
a, b and ab
(3)

The letters become increasingly useful as the number of small areas increases. In problem C, six squares or rectangles can be found.



a, b, c
ab, bc, abc (6)

And in problem F, there are nine that can be identified:

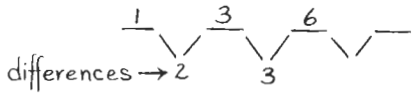


a, b, c, d
ab, cd, ac, bd (9)
abcd

Anyone about to work on this problem should know ahead of time that a very large number of squares and rectangles can be found in problem P. There are more than 75! This suggests that one ought to look for some pattern or regularity along the way that could be used as a shortcut.

Talking About the Problem

It might be useful to ask the group to report any interesting patterns they notice as the work proceeds. Someone might look at the first three results in the top row; namely, 1, 3 and 6. Writing down the differences between neighbors, we have



One might predict that the next difference would be 4. If that were true, the fourth number in the top row would be 10 (i.e., 6 + 4). This prediction is a good one; ten squares or rectangles can be found in problem (D).

It should be clear that the examples in the leftmost vertical column are exactly the examples in the top row, given a quarter turn. Problems (B) and (E) are essentially the same, as are (C) and (I), and (D) and (M). Thus, the diagrams A, E, I and M should have the same numbers of squares and rectangles as their counterparts: 1, 3, 6, and 10 respectively.

Extension and Surprise Endings

One interesting relationship can be noticed by studying three of the examples above the double line:

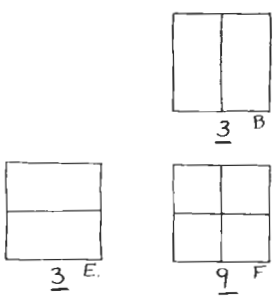
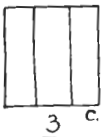
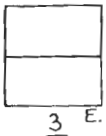


Diagram (F), which has nine squares or rectangles, in a sense is a combination of the other two. Diagram (B) has one vertical line; diagram (E) has one horizontal line; and diagram (F) has one vertical and one horizontal line. Three squares or rectangles were found in (B) and three were found in (E). Could it be significant that there were nine in (F), and that 3 × 3 = 9?

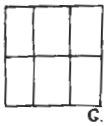
Let's see if that relationship holds on a similar group of three examples:



3 C.

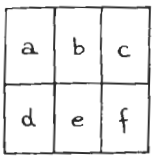


3 E.



— C.

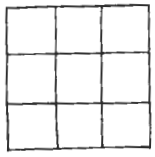
Does (G) have 3 × 6, or 18 squares or rectangles? We should label each small area, and then look at the possibilities:



1 at a time: a, b, c, d, e, f – (6)
2 at a time: ab, bc, de, ef, ad, be, cf – – (7)
3 at a time: abc, def – – – (2)
4 at a time: abde, bcef – (2)
5 at a time: (none)
6 at a time: abcdef – – – (1)
18

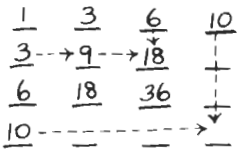
The prediction was borne out.

If there is still some doubt, we might check out one more problem. Consider problem (K). We know that examples (C) and (I) showed six squares or rectangles. Our new theory would predict that there would be 6 × 6, or 36 squares or rectangles in (K).



1 at a time: – – – – 9
2 at a time: – – – – 12
3 at a time: – – – – 6
4 at a time: – – – – 4
5 at a time: – – – – 0
6 at a time: – – – – 4
7 and 8 at a time: – – 0
9 at a time: – – – – 1
36

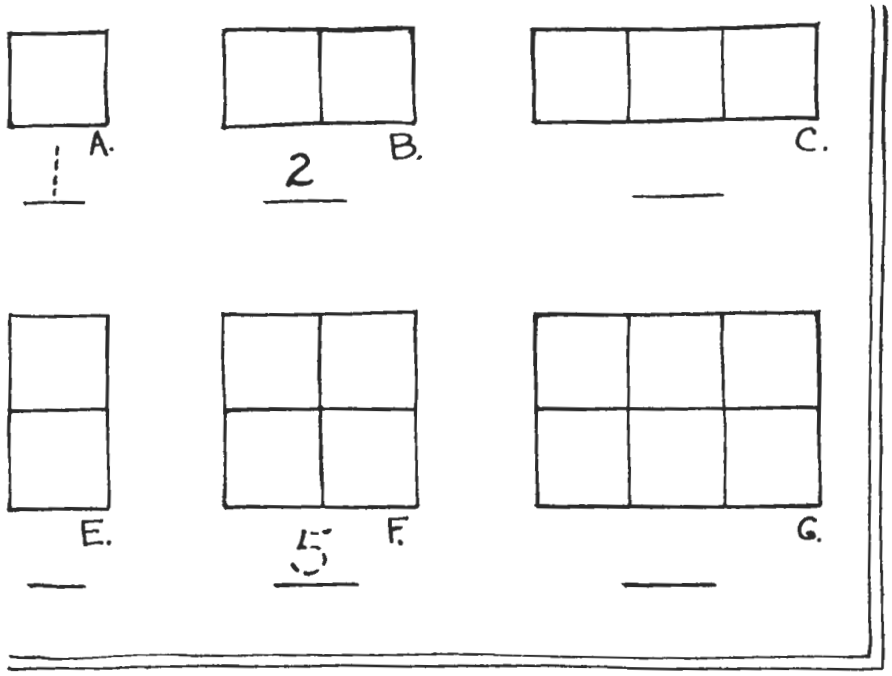
While these examples do not give a formal proof, they give us strong evidence to support a suspicion that there will be 10 × 10, or 100 squares or rectangles in the last diagram in the bottom row. We can summarize the examples we have just worked out (and their quarter turn versions) with this array:

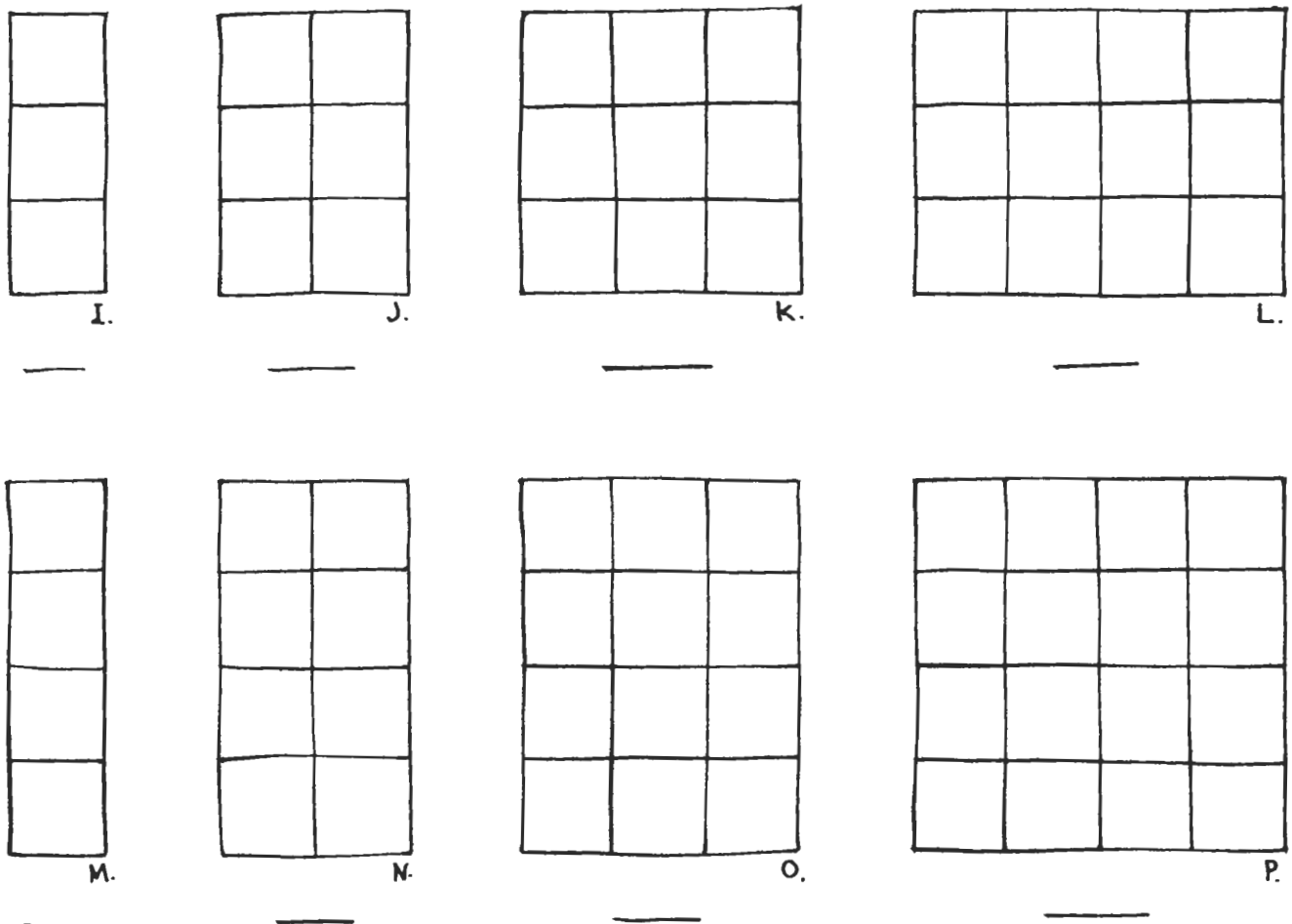


If there is still any doubt that the right-hand column and the bottom row will read: 10, 30, 60 and 100, we have no other recourse at this point than to "work them out" the hard way. Our predictions will be supported if the investigations are careful and complete!

NOTES

How many squares can you see? ¿Cuántos cuadritos encuentras?





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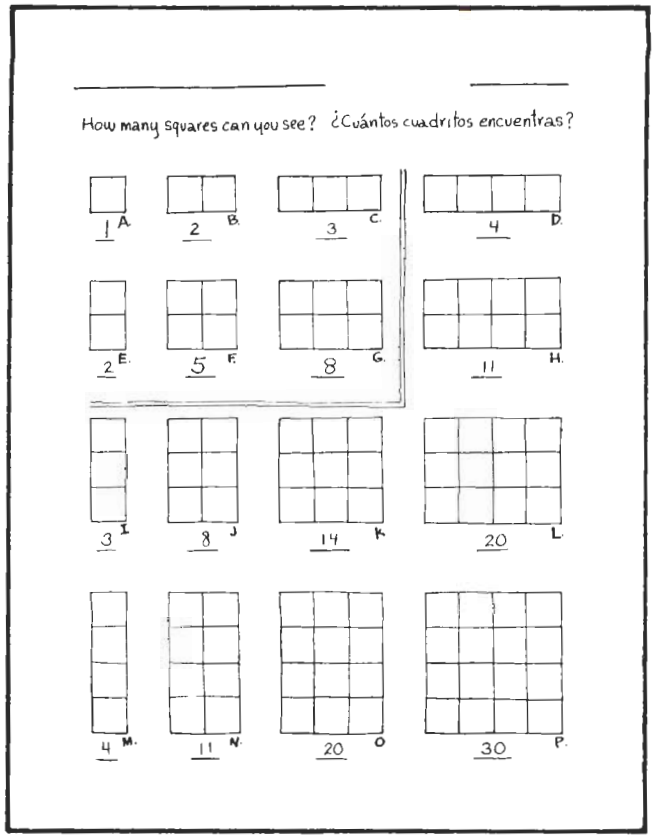


Introducing the Problem

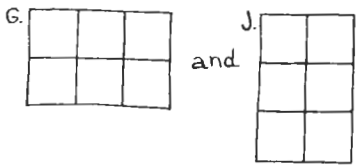
Each diagram shows an array of small squares. The problem is to find as many squares as possible in each diagram. How many squares are there in the diagrams in the top row and in the left-hand column? The answer is obvious. There are 1, 2, 3 and 4 squares respectively as you move along the top row and down the left column.

Now consider the remaining problems above the double line. Some are not so obvious. In (F), there are four small squares. But one also needs to notice that the outside of the diagram is also a square, so there are five squares all together. In problem G, there are six small squares, and two larger squares, but the outside of the whole figure is a rectangle, not a square. So there are 6 + 2, or eight squares all together. Thus, for the problems above double line we have the results:

number of squares

1	2	3	---
2	5	8	---
---	---	---	---
---	---	---	---

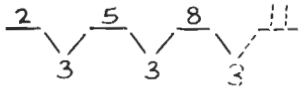
We can also be sure of four other results. Those along the top row and left column are obvious and were considered at the start. Example J can be seen as the same array as example G above the double line, in which we found eight squares.



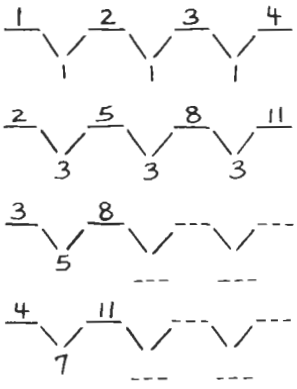
Talking about the Problem

Notice that a pattern has begun in the numbers of squares in the second row and in the second column from the left. It seems that three squares are added with each successive diagram.

Would you expect that figures H and N will continue this pattern?

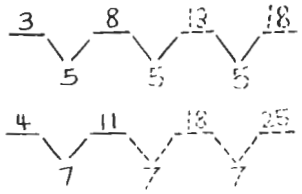


They do, for adding two little squares to an end of a two-by-three arrangement will provide two new small squares and one new large square, or a total of three more squares. Now one might be tempted to make another prediction by noticing this situation:

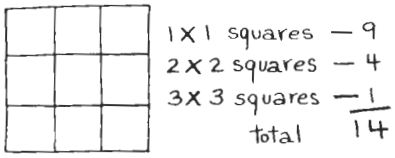


Row 1 shows differences of 1 between successive numbers of squares. Row 2 shows differences of 3; row 3 begins with a difference of 5; row 4 begins with a difference of 7. Those differences are successive odd numbers; 1, 3, 5 and 7. If the pattern keeps up,

we would complete those two bottom rows like so:

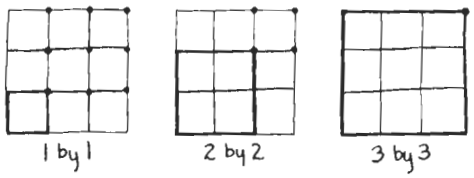


This prediction leads us to say that we will probably find 13 squares in problem K. Let's have a look at it.



Prediction: thirteen. Result: fourteen! Let this be a powerful warning that while patterns and regularities can often lead to amazing shortcuts, they sometimes run into unpredicted flaws or interruptions.

Perhaps we ought to check our search for squares in problem K to be sure that we haven't made a mistake. Let's look at the situation this way:

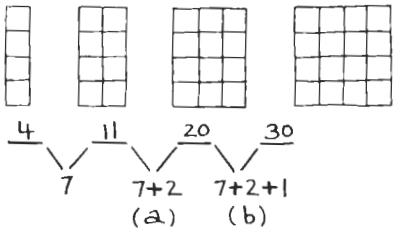


In each of the figures above, one square is outlined with a heavy line and a dot is shown in its upper right hand corner. This square could be moved around into different positions in the array so that the dot in its corner is placed on top of each of the other dots shown within the array. In the first figure, the one-by-one square could be moved into eight other positions. In the second figure, the two-by-two square could be moved into three other positions. In the third figure, there is only one position for the three-by-three square. If we total up all such possibilities, we have 14 (9 + 4 + 1) positions for squares of these sizes. Our prediction that the results in the third row would continue in the pattern did not account for the fact that a new square came into the picture with the square three-by-three array.

As we moved from I to J (or C to G), we added three little squares and two larger squares. As we moved to K, we added three little squares and two larger squares. But a new three-by-three square was also added. Thus six new squares were introduced, rather than the expected five.

Movement from H to L (or N to O) will add four new small squares, three new two-by-two squares, and two three-by-three squares. That is a total of nine (4 + 3 + 2 = 9) new squares, instead of the seven that we might have expected.

Another surprise awaits us when we move to problem P, because a new four-by-four square appears along with more three-by-three, two-by-two, and one-by-one squares. Thus we have:



But now look at the arrays along the diagonal on this page, namely the sketches in problems A, F, K and P. A nice pattern emerges — this time without flaws. In A, a one-by-one array, there was one (1 × 1 = 1) square. In F, a two-by-two array, there was one (1 × 1 = 1) large square and four (2 × 2 = 4) small squares. In K, there was one (1 × 1 = 1) large square, four (2 × 2 = 4) medium or two-by-two squares, and 9 (3 × 3 = 9) small squares. In P, there was one (1 × 1 = 1) large square, four (2 × 2 = 4) squares that were three-by-three, nine (3 × 3 = 9) squares that were two-by-two, and 16 (4 × 4 = 16) small squares. These observations can be summarized:

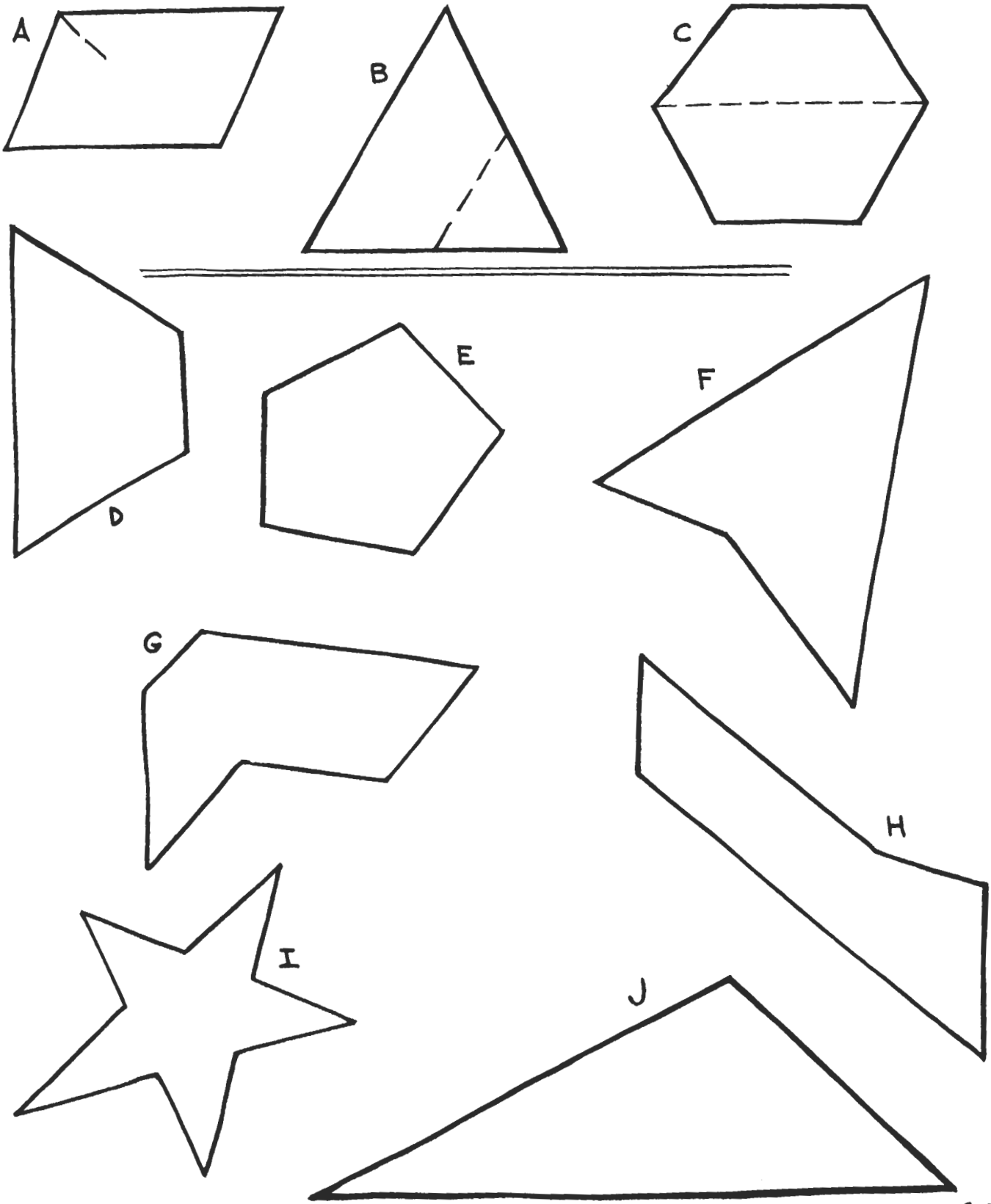
size of array	number of squares
1×1	1×1=1
2×2	(1×1)+(2×2)=5
3×3	(1×1)+(2×2)+(3×3)=14
4×4	(1×1)+(2×2)+(3×3)+(4×4)=30

On the basis of this emerging pattern, we would predict that in a five-by-five array, there would be 55 squares. That is in fact the case.

NOTES

Divide each shape so all pieces are acute triangles.

Divide todas las formas, de modo que todas las piezas resulten triangulos agudos.



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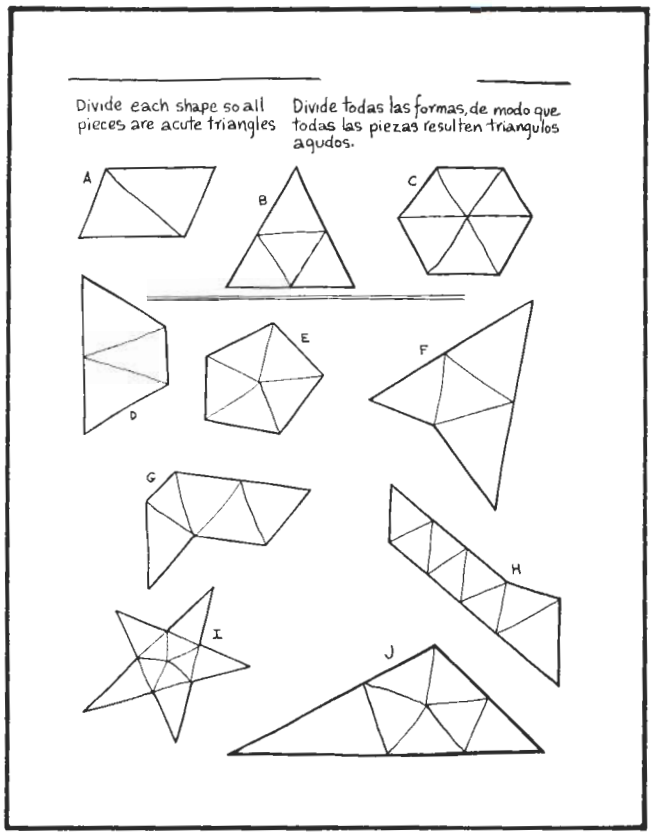
(Some of the appropriate responses are shown here.)

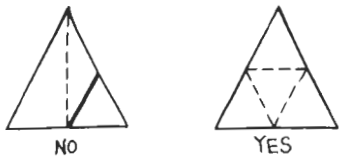


Introducing the Problem

All triangles can be classified as acute, right or obtuse. An *acute* triangle is one in which all three angles are "sharper than a square corner"; that is, they measure less then 90 degrees. A triangle is called *right* if one angle is a square corner. In an *obtuse* triangle, one of the angles is "larger than a square corner"; it measures more than 90 degrees.

The problem on this activity page is to use straight lines to break up the interior of each shape so that all the smaller shapes formed by drawing the lines are *acute* triangles. Another way of saying this is that if the shape were cut along all of the lines drawn, every resulting piece should be an acute triangle. There is no limit to the number of lines that can be drawn, except that it must be a finite number of lines.

A clue is given for the solution to problem A — simply complete the diagonal to form two acute triangles. The triangle in problem B was an acute triangle to begin with. The line that has been drawn creates one acute triangle and a four-sided shape. But a diagonal in that four-sided shape would not help as it did in problem A, for it would create one obtuse and one right triangle. However, drawing two other lines to create four small triangles as shown below finishes the job.



A solution to problem C is obvious. Drawing the other two main diagonals will create six acute triangles. A handy way to tell whether an angle is acute, right or obtuse is to use the square corner of a sheet of paper. The corner should be placed at the vertex of the angle to be checked, and one edge of the paper should be lined up along one side of the angle. If the paper covers the other side of the angle, it is acute. If it fits exactly, it is right. If the other side of the angle "shows", it is obtuse.

Some children may get bogged down in drawing too many lines in each figure to the point where the triangles get so small that it is impossible to tell whether they are acute, right or obtuse. Sometimes it is helpful to suggest using the "fewest number" of lines, to actually specify a certain number of lines to be used, or to simply give a limit to the number of lines that can be used.

Talking about the Problem

Most of the examples will give in to the combined efforts of a group, but problem J stumps even most adults. It has a simple but elusive solution. Some children might like to save that problem to work at on their own time, or perhaps take it home for some help.

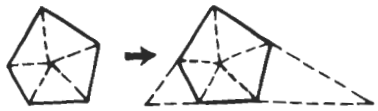
Extensions and Surprise Endings

A bit of clear reasoning would help in problem J. Clearly the obtuse angle at the "top" of the triangle must be divided. Extending the line which divides that angle all the way to the opposite side would either create two right angles, or one acute and one obtuse angle. So nothing would be accomplished by drawing such a line. But there is no requirement that the line dividing the obtuse angle be extended all the way to the other side. If we stop the line before it gets to the opposite side and then draw *two* lines

from that point to the opposite side, we can form an acute triangle and two quadrilaterals similar to those in problem D.

This strategy is a useful one in mathematics. It reduces a problem we don't know how to solve to one(s) that are familiar.

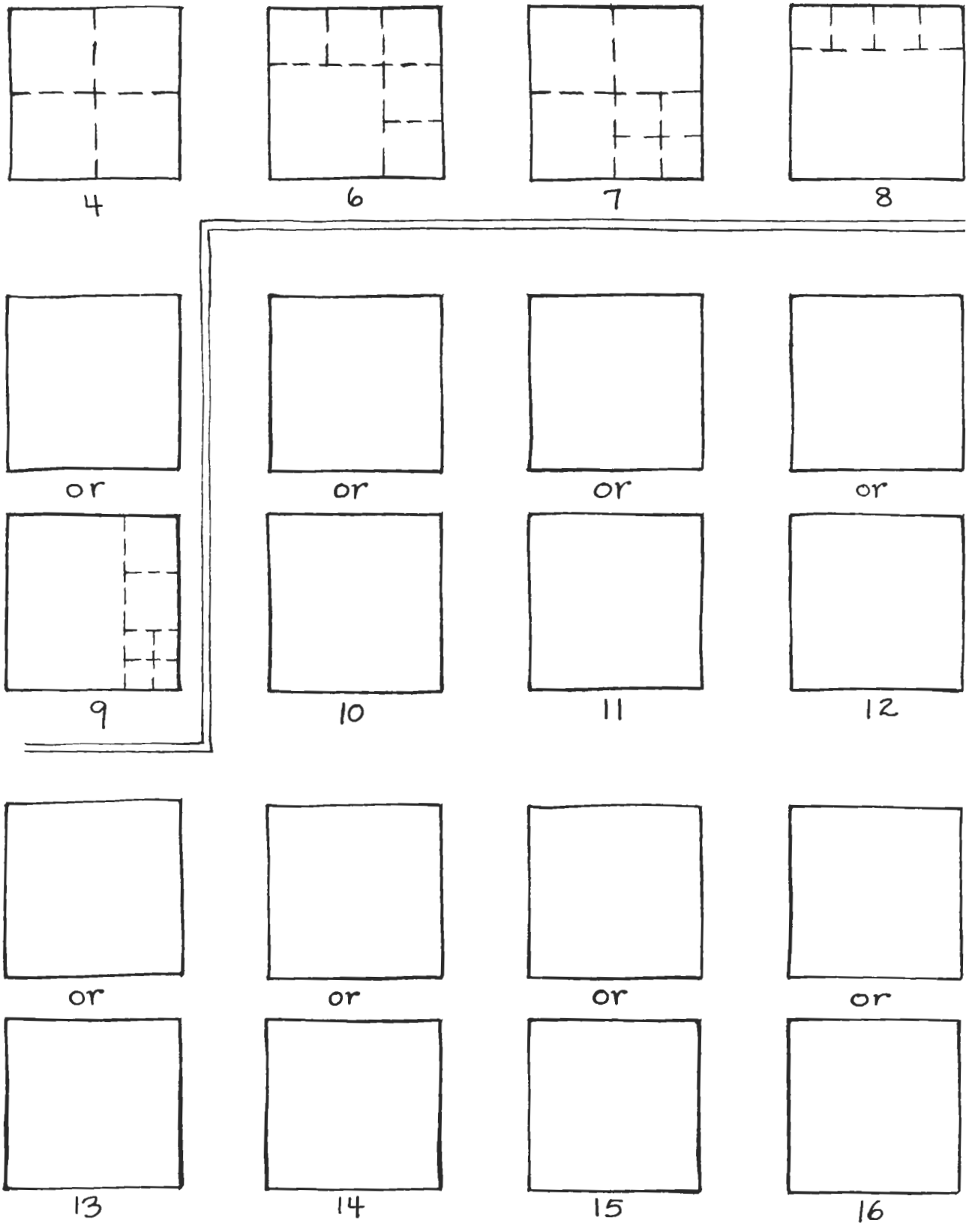
Problems E and I were included along the way as clues for problem J. Each of them involved a pentagon, and it is just a small step from a pentagon to a triangle such as that in problem J.



In our experience, elementary school children find the solution to problem J as often and as quickly as teachers, parents or other adults. Mathematical problems often narrow the generation gap!

NOTES

Haciendo más cuadritos. Making more squares.



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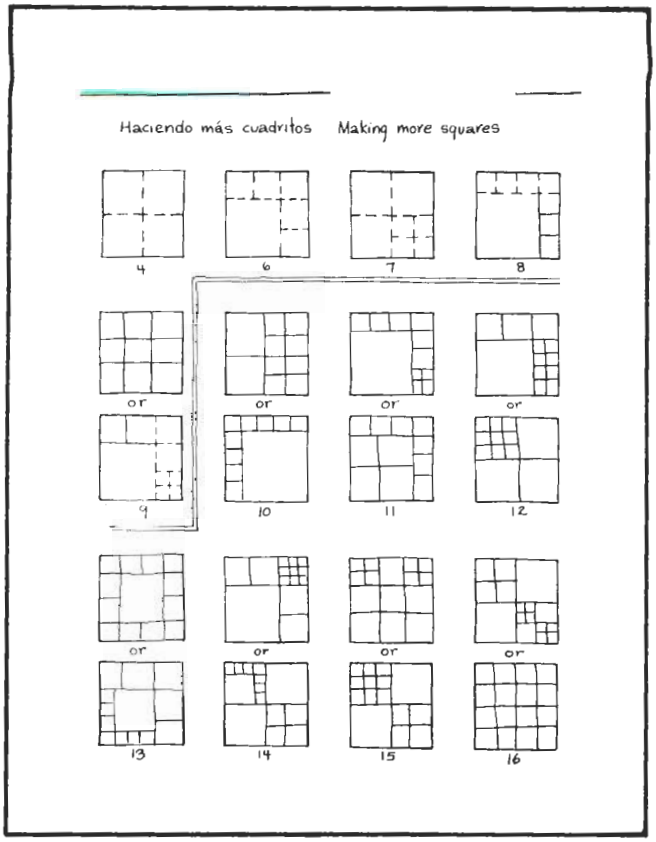
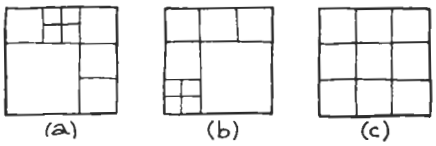
(Some of the appropriate responses are shown here.)



Introducing the Problem

The problem on this page is to divide the interior of each diagram into exactly the number of squares indicated below the sketch. The squares that are created may be the same or different sizes. The first three problems are completed. In the fourth, we must draw three more small squares on either the left or the right-hand side.

Each problem after the first four asks for *two* basically different solutions. In the first pair asking for nine squares, we might find these solutions.



Diagrams (a) and (b) above are not essentially different. Both have one large, four medium-sized, and four small squares. But diagram (c) is very different from the other two.

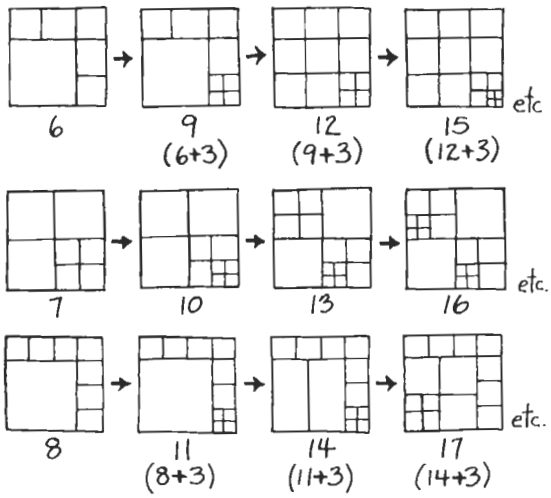
Talking about the Problem

Could you find two basically different solutions for each problem? There are such solutions for each, although problems calling for 10 and 11 squares are quite difficult.

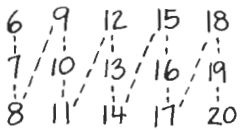
Did you discover any shortcuts for adding squares? To add three more squares to any given array, one simply divides one of the squares into four small squares. The net gain is three more squares. In which problems can the squares all be of the same size? Those that call for 9 and 16 squares allow squares of the same size. You can expect that the same will be true for 25, 36, 49, and so on.

Extensions and Surprise Endings

You may have noticed that no problem called for five squares. A square cannot be divided into exactly five squares. Can a square be divided into any number of squares greater than five, no matter how large the number is? We noticed that one square can be divided into four squares. When this is done, the resulting number of squares is three more than the original number; we start with one square and end up with four.



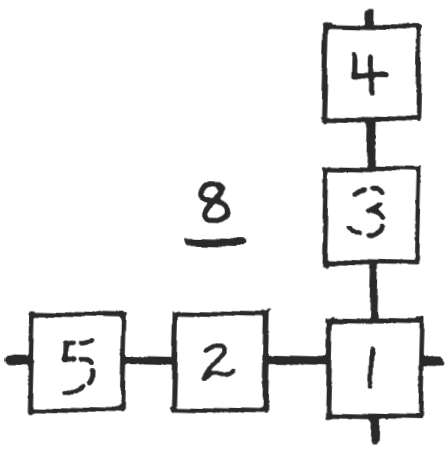
If these three series of squares are extended, they will eventually include all numbers greater than five, no matter how large the numbers.

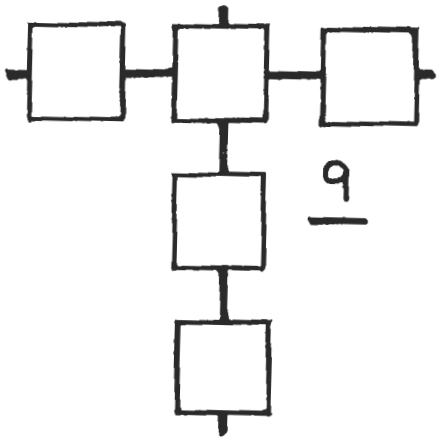


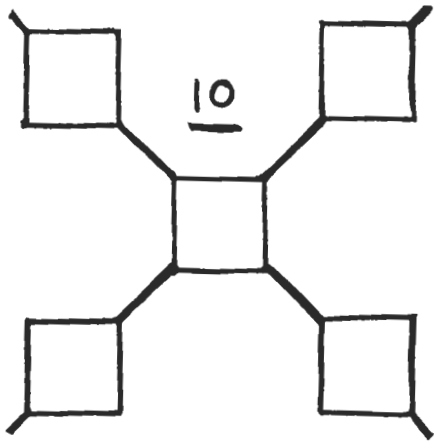
SUMAS "MÁGICAS"

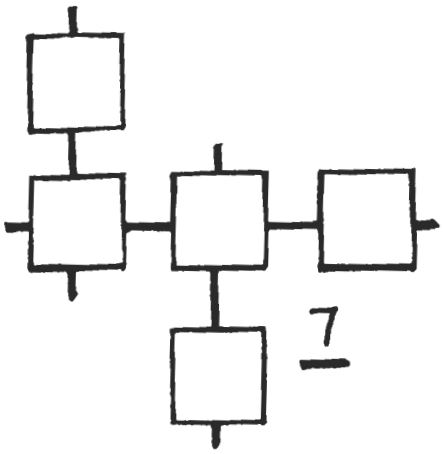
MAGIC SUMS

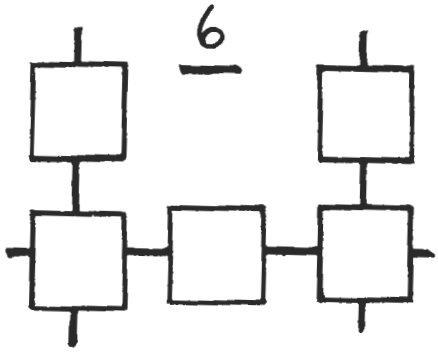
usa 1, 2, 3, 4 y 5
use 1, 2, 3, 4 and 5

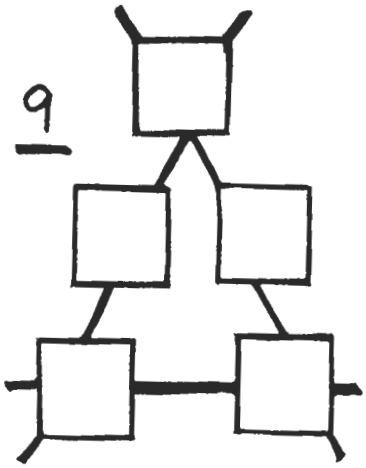




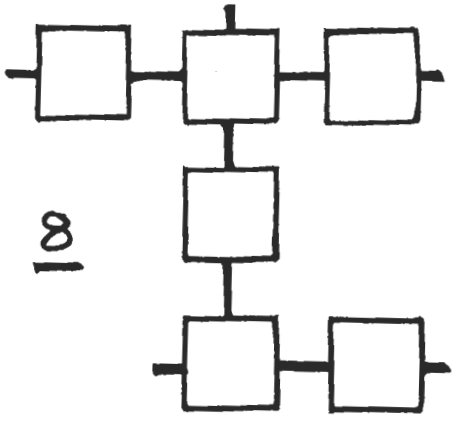


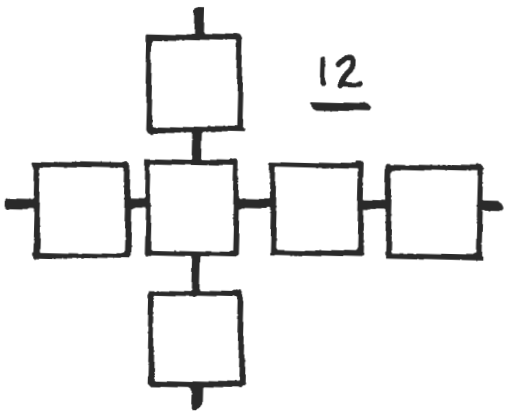


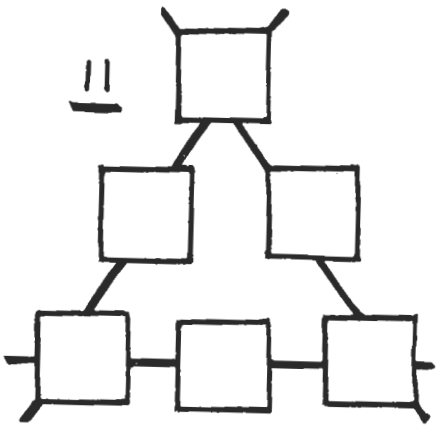




usa 1, 2, 3, 4 5 y 6
use 1, 2, 3, 4, 5 and 6







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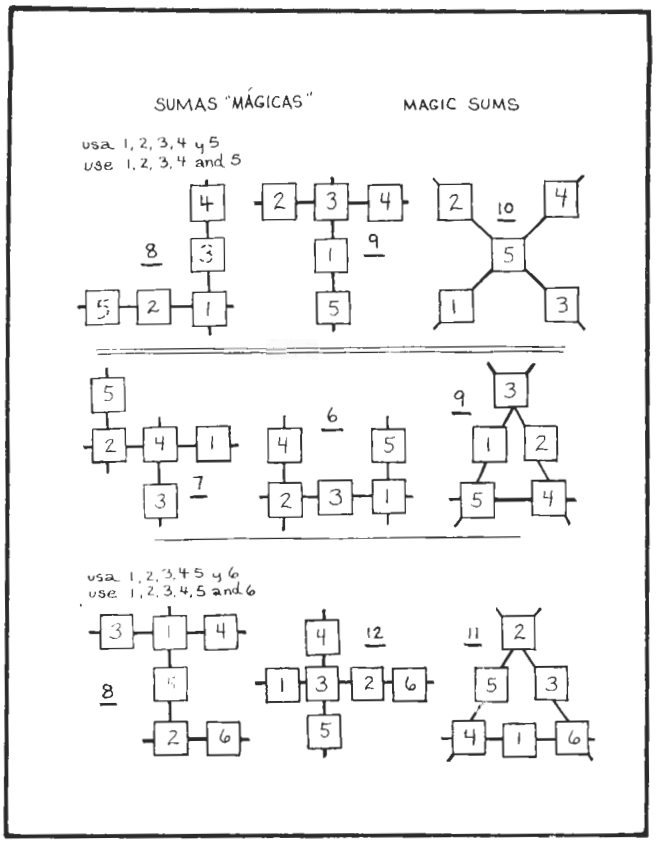
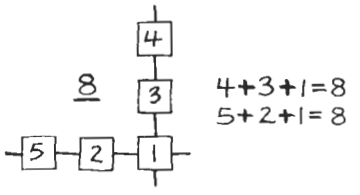
(Some of the appropriate responses are shown here.)



Introducing the Problem

Various arrays of small boxes connected by heavy lines are shown on this page. Numbers are given which are to be written in the boxes in such a way that the numbers along each heavy line add up to the same "magic sum". For example, in the first problem, 8 is the "magic sum", and the numbers on each heavy line add up to 8.



Children will probably find these problems easier to solve if they use small pieces of cardboard marked with the given numbers and move them around on the array. The first two rows of problems must use the numbers 1, 2, 3, 4, and 5. In the third row, the numbers 1 through 6 must be used.

Talking About the Problem

After solutions have been worked out using trial and error, it might be interesting and instructive to point out that *every problem can be solved directly* by "arguments" or "reasoning".

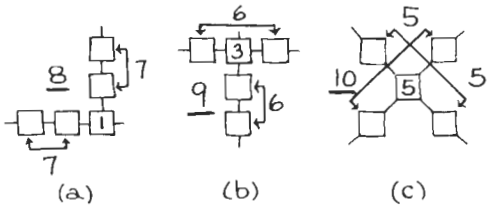
The three problems in the first row have a common characteristic. In each one, we are looking for two groups of three numbers whose sums are both equal to the magic number. Further, in all three problems, one of the five numbers must appear in both groups. Consider the possibilities for the given magic sums: 8, 9, and 10. Remember that we are using the numbers 1, 2, 3, 4 and 5.

$8 = 1+2+5 \quad 8 = 1+3+4$ (7)

$9 = 1+3+5 \quad 9 = 2+3+4$ (6)

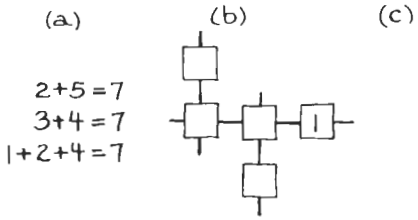
$10 = 1+4+5 \quad 10 = 2+3+5$ (5)

In each case, there are only two combinations of three numbers for the given sum. In each case, one number appears in both combinations. This is the number that must go in the box where the lines cross.



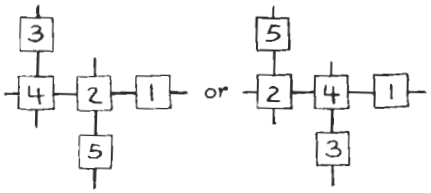
Once this common number has been placed, we must then find numbers for the other boxes. In (a), we need two pairs, each of whose sum is 7: 3+4, and 2+5. In (b), we need two pairs, each of whose sum is 6: 1+5 and 2+4. In (c), we need two pairs with 5 as each sum: 1+4 and 2+3.

The first problem in the second row requires sums of 7. We might begin by writing down all ways to make 7 using 2 or 3 numbers from the list. There are not too many possibilities:



Clearly the row with three addends must use 1, 2, and 4. Since 1 does not occur in the other sums for 7,

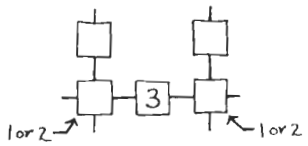
it must be placed in the box at the right in that row, which is clear of the two vertical heavy lines. The numbers 2 and 4 will go in the other two boxes. It makes no difference which goes where. Clearly the 5 then goes in the same column as the 2, and the 3 goes with the 4.



The next problem is essentially the same situation. There are two sums with two addends and one with three addends. The magic sum is 6. The only possibilities for sums of 6 are:

$$6 = 1 + 2 + 3$$
$$6 = 2 + 4$$
$$6 = 1 + 5$$

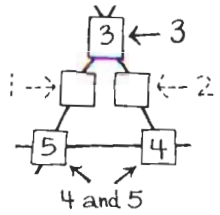
In the first sum, 1+2+3, 3 is the only number that doesn't appear in either of the other sums. Thus, it must occupy the center square. The numbers 1 and 2 can then go in the other squares in that row in any order. The array can be completed with the sums 2+4 and 1+5.



The rightmost problem in the middle row is different from the others in several respects. Again, let's write down the possible ways to make 9 as a sum using 1, 2, 3, 4 and 5.

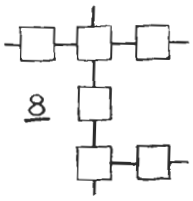
$$9 = 4 + 5$$
$$9 = 1 + \boxed{3} + 5$$
$$9 = 2 + \boxed{3} + 4$$

Notice that there is only one sum with two addends. Thus, these numbers must go in the row with two cells. The number 3 occurs in both of the other expressions. So it must be in the cell common to the two rows of three numbers:



Now it is simple to fill in the remaining squares with the 1 and the 2.

In the row of problems at the bottom of the page, six numbers must be arranged. In the first problem, we need two combinations of three numbers whose sum is 8, and which have a common term.

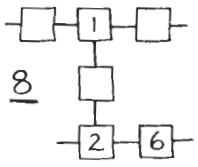


Possible combinations are:

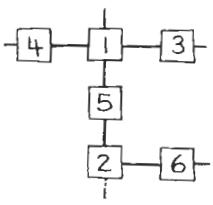
$$8 = 1 + 2 + 5$$
$$8 = \underline{1} + 3 + 4$$

↖ common term

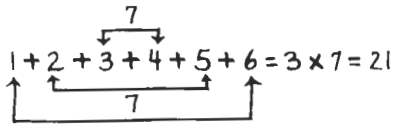
The number 1 is common to both, so it must go in the square at the center of the top row. The 6 was not used in either sum, so it must go in the row with two squares, in the right hand square: Then 2 must be placed in the bottom row with 6.



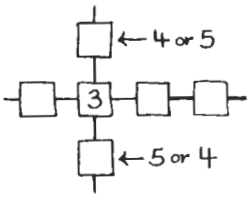
The number 5 must go in the column between 1 and 2. Finally, 4 and 3 can be placed in the top row.



In the next problem, we need two sums of 12 using the numbers 1, 2, 3, 4, 5 and 6. Let's find the sum of the numbers we are using.

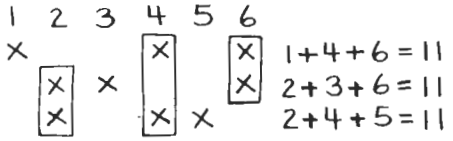


Two sums of 12 would total 24, which is 3 more than the total of the available numbers. So, somehow the three must get counted twice. That means we must put the 3 in the box that occurs in both rows.

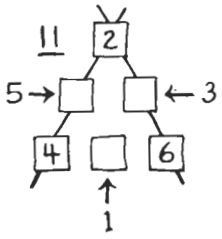


Now we need a pair of numbers which will combine with 3 to make 12. The pairs 4 and 5 or 3 and 6 would suffice. But 3 has already been used. Thus, 4 and 5 must go in the column of three squares. The remaining numbers, 6, 2, and 1, whose sum is 9, can be used to fill the empty boxes in the horizontal row.

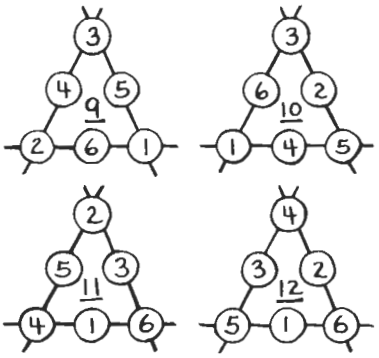
The final problem might seem to be much more difficult. But let's look at a simple table indicating all combinations of three numbers, 1 through 6, whose sum is the specified number 11.



Three of the six numbers occur in two of the sums, and this is the requirement in the problem. Those three numbers must be placed in the corner squares where the lines cross. The rest is simple arithmetic.



The magic sum for this array could be changed to any number, 9 through 12. We would have:



While all the problems in this activity will give way sooner or later to random trial and error, children ought to have an opportunity to consider a quite different approach. In each instance the scope of the problem is quite limited. The number of sums is specified, the amount of the sum is given, and only the numbers 1 through 5 or 1 through 6 are permitted.

While we must show great respect for trial and error as a legitimate problem solving strategy, there are other problem solving strategies that may be more direct. One of these is reasoning from what we know to what we do not know. Children should be encouraged to move on to these more sophisticated problem solving strategies as soon as they show a willingness to do so.

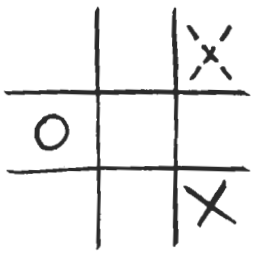
TIC-TAC-TOE

You are "X"
It's your play.

EL "GATO"

Tu eres "X"
A ti te toca.

A



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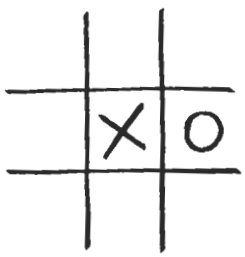

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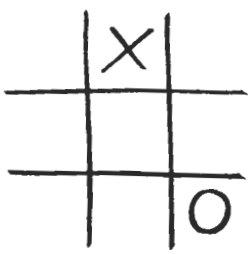

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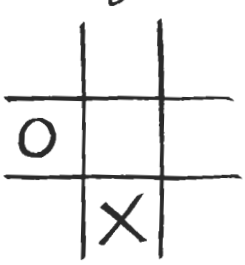

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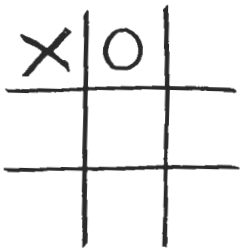

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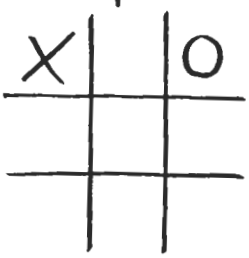

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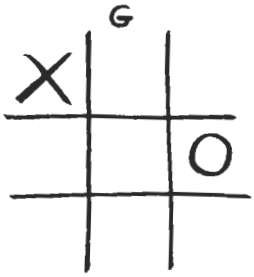

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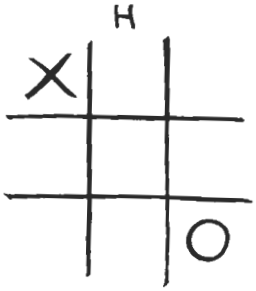

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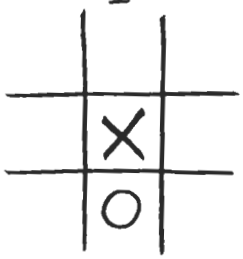

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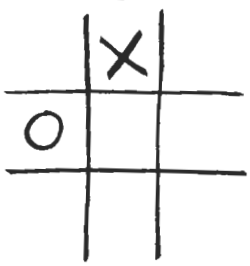

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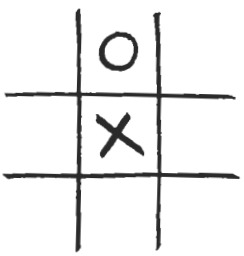

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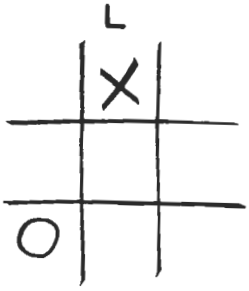

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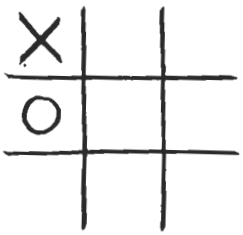

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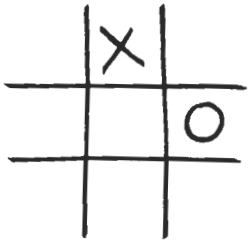

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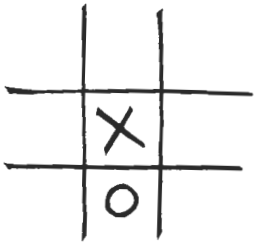

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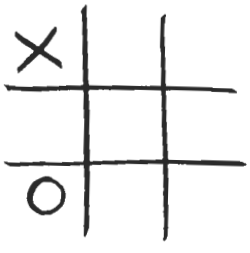

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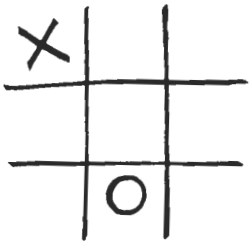

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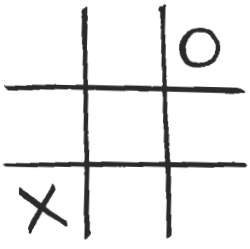

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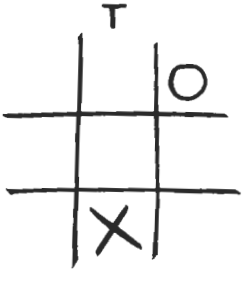

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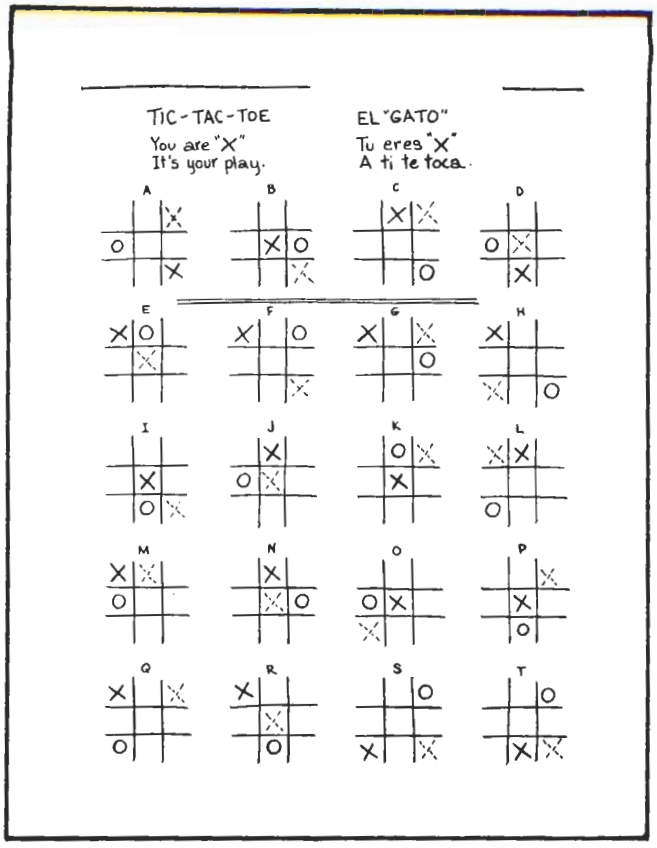
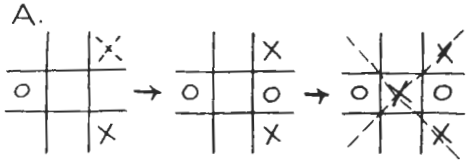
(Some of the appropriate responses are shown here.)



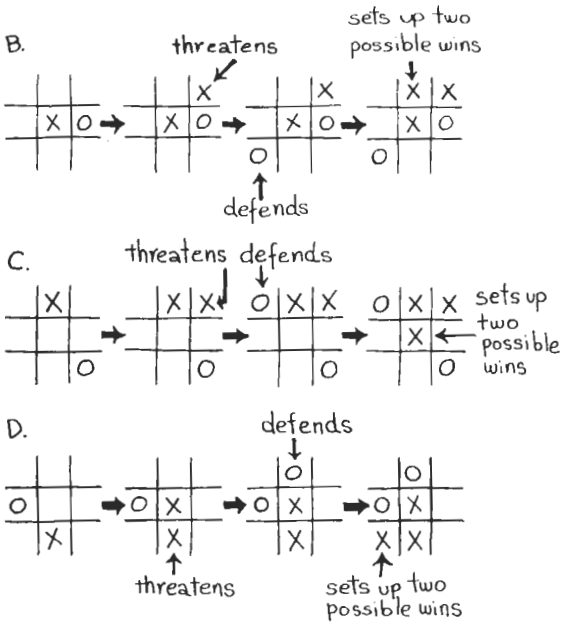
Introducing the Problem

Please don't pass by this activity simply because it does not use numbers or geometrical shapes. Mathematics is involved as we probe problems of defensive and winning strategies in games like tic-tac-toe.

In each array on this page, the two first moves of the tic-tac-toe game are given. The problem is to indicate a second play for player "X" which will insure that "X" will win. For example, in problem A, the second play indicated by the dotted "X" forces player "O" to play between the two X's. Then, if "X" plays in the center, he is set up for two possible diagonal wins, and "O" cannot stop them both.

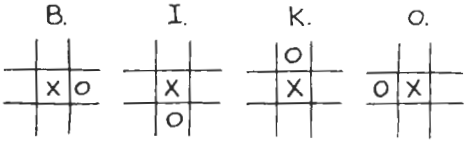


For each of the 20 starts pictured on this page, it is possible for "X" to select a second move that can lead to a win regardless of how well player "O" defends. The general strategy is for "X" to find a *second move that forces "O" to prevent an* immediate win by "X". Player "X" then makes a move that sets up two possible wins. This might be called the "threaten-defend-win" strategy.

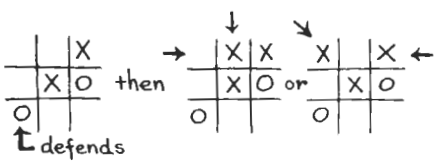


Talking About the Problem

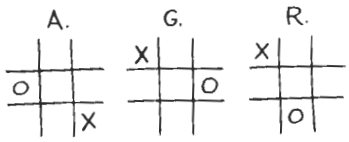
Did anyone notice that there are not 20 *different* situations on this page? Consider these examples:



In each case, the first player (X) took the center, and the second player (O) chose a side cell. Any one side is as good or as bad as another. To set up a sure win, the first player simply needs to play on either side of the "O". This move threatens, and forces "O" to stop a diagonal win. "X" then has two choices for setting up two wins.

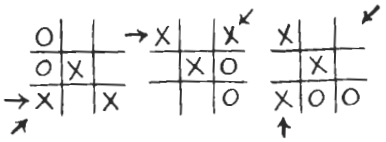


The situation in any given array does not basically change with a turn of the playing area or a "flip" over a diagonal. For example, in problems A, G, and R, the situation is essentially the same. Player "X" first chose a corner, and player "O" chose a side cell that was not next to "X".



A second play by "X" in the center threatens in the same way in all cases, and forces "O" to defend by preventing a diagonal win.

This defensive play forces "X" to take a corner to prevent an immediate win by "O" and at the same time sets up two possible wins for "X".



How many different situations are actually represented in the problems on this page? It turns out that there are only seven. Problems that are essentially the same require the same strategy for a sure win.

The next activity page looks at the problem of selecting a first move for the second player. The seemingly great variety of situations there can again be reduced to manageably few unique conditions.

NOTES

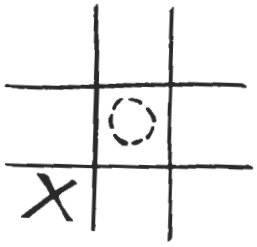
TIC-TAC-TOE

You are "O"
It's your play.

EL "GATO"

Tu eres "O"
A ti te toca.

A



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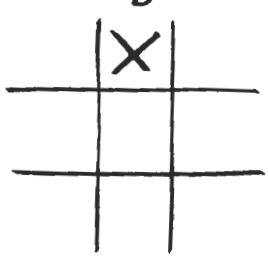

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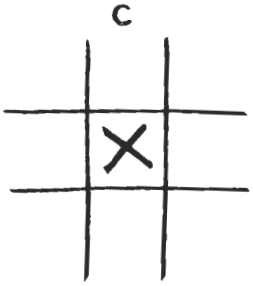

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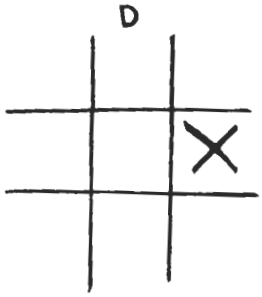

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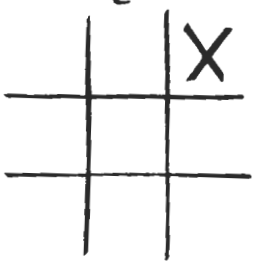

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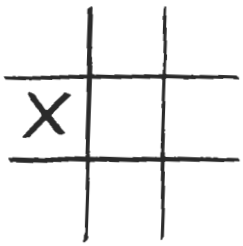

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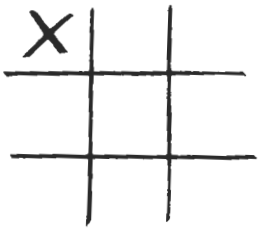

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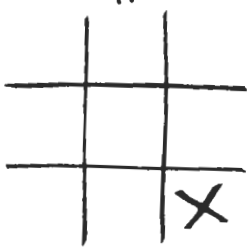

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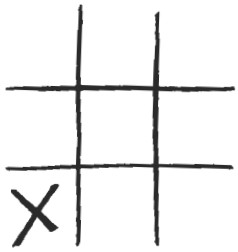

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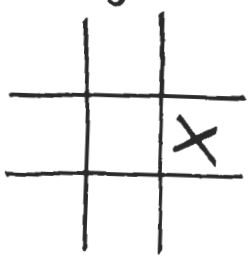

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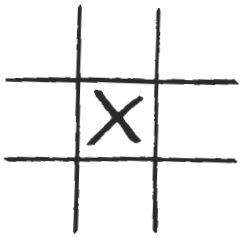

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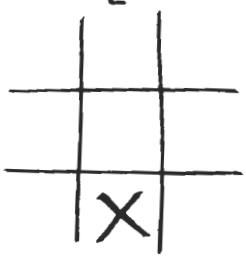

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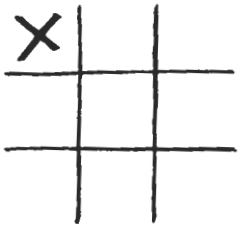

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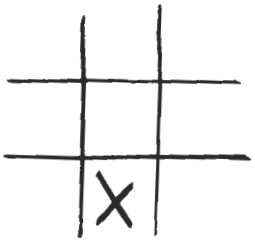

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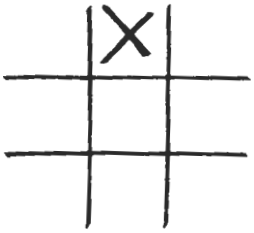

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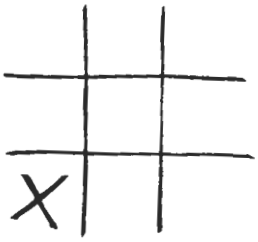

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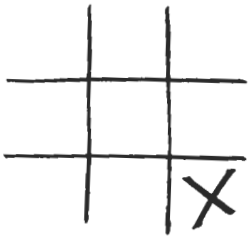

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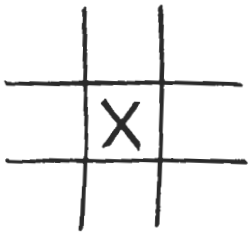

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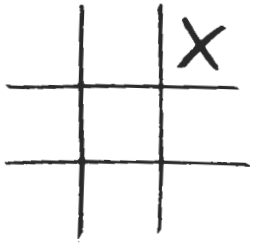

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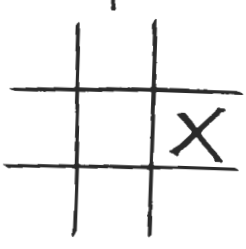

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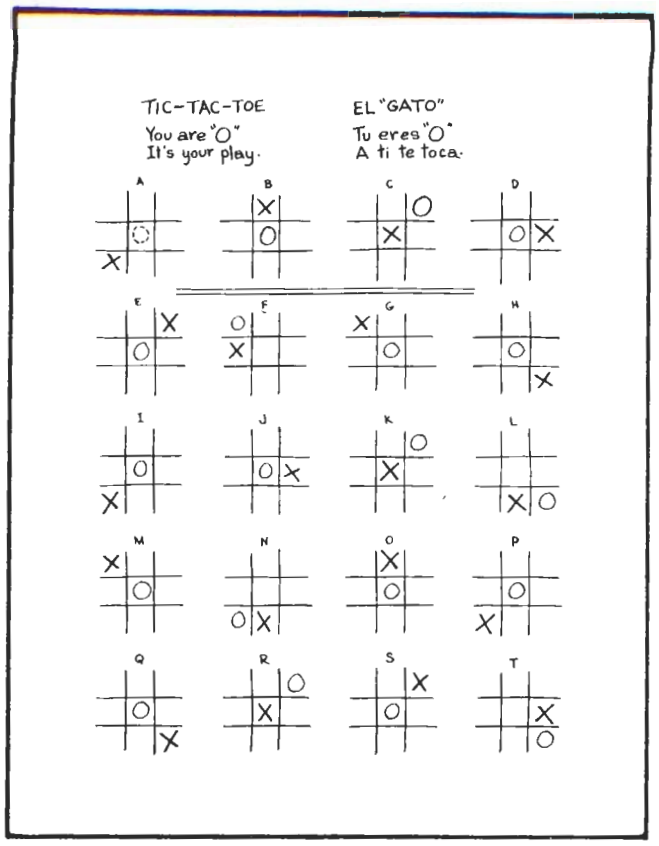
(Some of the appropriate responses are shown here.)



Introducing the Problem

This activity can stand on its own or can follow the previous investigation. First plays are indicated for "X" in each tic-tac-toe array, and the problem is to select a first move for the second player which is a "sure defense". (Sure, that is, assuming that the defender is not caught napping!)

It might seem that the second player must be ready to respond to each of nine possible opening moves. If there were nine possible openings and eight possible responses to each, then there would be 72 situations (8 × 9) to consider. But such need not be the case. You might want to encourage the children to look at the situations very carefully to see if some of the problems are really the same.

Talking About the Problem

After children have shared their results, the class as a whole might take a deeper look at the problem. It turns out that there are basically only three first plays that can be made:

(1) in a corner,

(2) in the center, or

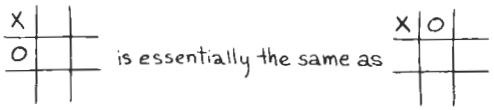
(3) in a side cell.

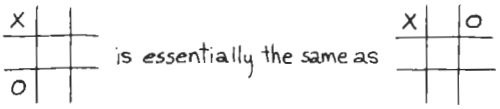
All corner opening plays are essentially alike; all side plays are also alike; and there is only one center play. Let's look at what would happen for each possible opening move.

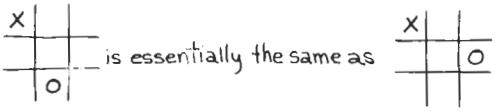
(1) An Opening Corner Play

If the first player selects a corner, the only *different* options for the second player's response are indicated by dots in the diagram below:

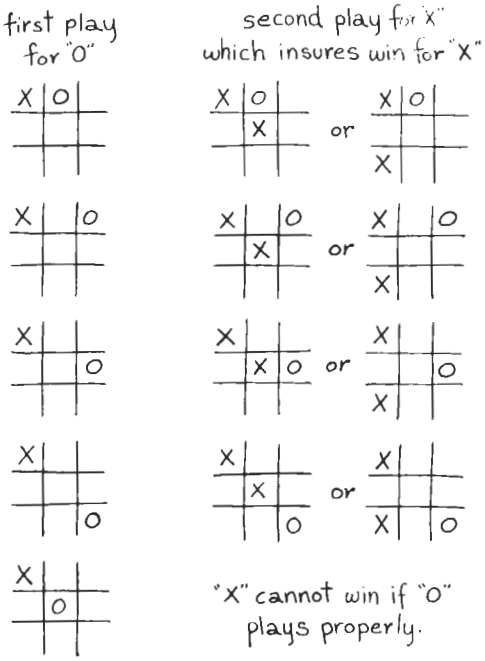
The other options are essentially no different than those pictured above:



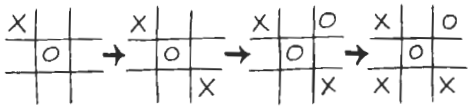




Let's look at what would happen as a result of each of these choices:



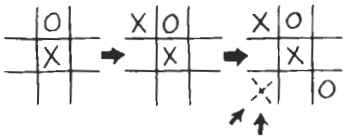
The only safe response to a corner "X" is the center square. But even then, "0" has to be careful. If "X" responds by taking the opposite corner, and "O" takes either remaining corner, "X" will surely win.



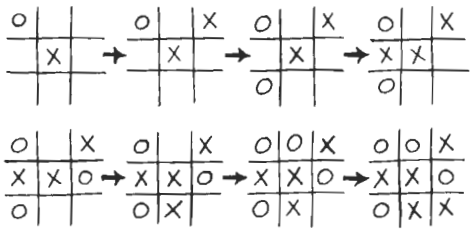
Any other second play by "O" would be safe.

(2) An Opening Play in the Center

The second possible opening for "X" is the center square. This forces "O" to take a corner or a side. Choosing any side is fatal; the play would go like this:

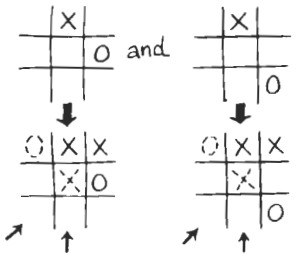


However, if "O" responds with a corner play, that will force an "old cat's game".



(3) An Opening Play in a Side

An opening play for "X" in a side square is the weakest possible beginning move. But there are still two pitfalls that "O" must avoid.



Let's look at the possible first plays in terms of their possible wins:

(a) A play in the center has four possible wins;
(b) A play in a corner has three possible wins; and
(c) A play in the side has two possible wins:

So the entire story of a sure defense is simple. If you can play in the center, do so; if you can't, then play in a corner. . . and never lose! A good offense is a play in the corner or the center. If the second player doesn't play a corner, the first player can always find a second play that is a winner.

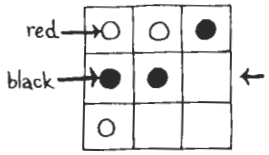
That's really all there is to the game of tic-tac-toe. There are really very few essentially different options. When children realize that they can master such a seemingly complex game with this kind of simple analysis, they may begin to appreciate the power of reasoning or thinking about a situation.

Extensions and Surprise Endings

There are variations of the tic-tac-toe game which some children might like to explore. They can be played with checkers on a tic-tac-toe board.

"Three Checkers Each"

Players take turns putting on three checkers each. A starting situation might look like this:



Red must play next by moving any one of his checkers already on the board to any empty square. Black than does the same. For example, in the diagram above, it would be unwise for red to move the checker from the bottom left hand corner, for black will then take that position and win on the diagonal. The game goes on until red or black occupies all three positions on a row, column or diagonal.

NOTES