GEOMETRY AND VISUALIZATION

Mathematics Resource Project

Preliminary Edition

Creative Publications

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

GEOMETRY
AND
VISUALIZATION

MATHEMATICS
RESOURCE
PROJECT

It is the intent of Congress that the resources developed by the Mathematics Resource Project will be made available within the school district using such material for inspection by parents or guardians of children engaged in educational programs or projects of that school district.

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ACKNOWLEDGMENTS

FOREWORD

THE SITUATION

I'LL TRY TO HAVE YOUR TESTS BACK BY TOMORROW.

YOUR HOMEWORK WAS NOT HANDED IN, ANN. THAT'S WHY YOU DIDN'T GET IT BACK!

YES, MARY, I KNOW YOU FINISHED EARLY— I'LL TRY TO HAVE SOME ACTIVITY CARDS FOR YOU TOMORROW.

TOM, YOU CAN GO OVER THAT WITH ME AFTER SCHOOL.

"YOU SHOULD KEEP THE GOOD PARTS OF THE "OLD" AND "NEW" MATH, USE THE LAB APPROACH, TEACH PROBLEM SOLVING AND . . . ."
— MATHEMATICS EDUCATOR

"YOU HAVE TO TAKE 6 CREDIT HOURS OF COURSE WORK EVERY 2 YEARS TO GET YOUR SALARY INCREASE."
— SUPERINTENDENT

"HAVE YOU WRITTEN YOUR OBJECTIVES FOR THIS LESSON?"
— PRINCIPAL

"I'M A PERSON, TOO! IT'S IMPORTANT FOR ME TO SUCCEED!"
— STUDENT

"WHY DIDN'T MY JOHNNY PASS THE XXX TEST FOR MINIMUM COMPETENCY?"
— PARENT

The demands on teachers are heavy. The fifth or sixth grade teacher with 25 to 30 students is often responsible for covering many subjects besides mathematics. The seventh or eighth grade teacher may be teaching only mathematics but be working with 125 to 150 students each day. Within this assignment the teacher must find time for correcting homework, writing and grading tests, discussions with individual students, parent conferences, teacher meetings and lesson preparations. In addition, the teacher may be asked to sponsor a student group, be present at athletic events or open houses, or coach an athletic team.

Demands are made on the teacher from other sources. Students, parents and educators ask that the teacher be aware of students' feelings, self-images and rights. School districts ask teachers to enlarge their backgrounds in mathematical or educational areas. The state may impose a list of student objectives and require teachers to use these to evaluate each student. There are pressures from parents for students to perform well on standardized tests. Mathematicians and mathematics educators are asking teachers to retain the good parts of modern mathematics, use the laboratory approach, teach problem solving as well as to increase their knowledge of learning theories, teaching strategies, and diagnosis and evaluation.

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WHERE TO START?

?

ISOLATIONVILLE
U.S.A.

RESOURCE CENTER
U.S.A.
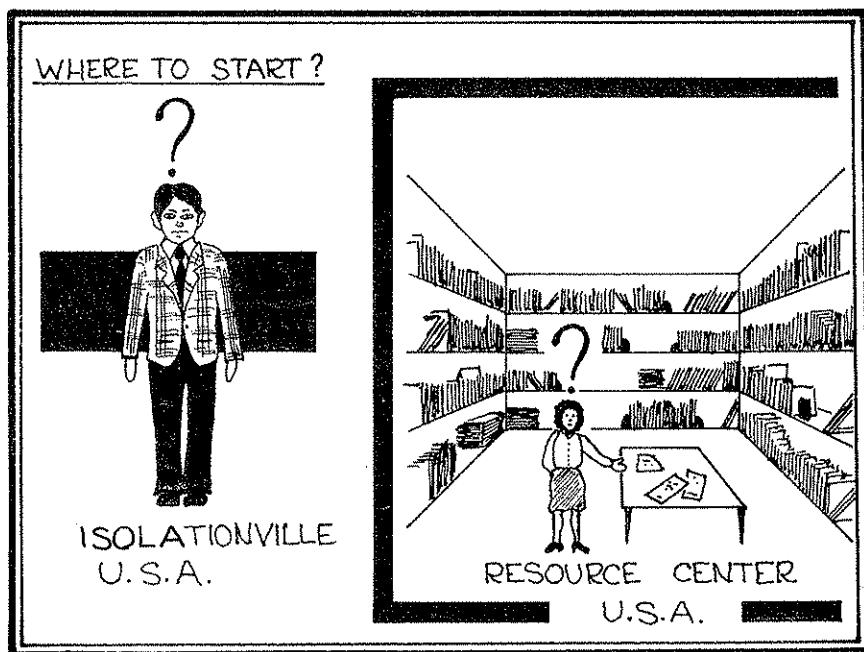
There is a proliferation of textbooks and supplementary material available. Much of this is related to the demands on teachers discussed above. The teacher in small outlying areas has little chance to see much of this material, while the teacher close to workshop and resource centers often finds the amount of available material unorganized and overwhelming.

The Mathematics Resource Project was conceived to help with these concerns. The goal of this project is to draw from the vast amounts of material available to produce topical resources for teachers. These resources are intended to help teachers provide a more effective learning environment for their students. From the resources, teachers can select classroom materials emphasizing interesting drill and practice, concept-building, problem solving, laboratory approach, and so forth. When completed the resources will include readings in content, learning theories, diagnosis and evaluation as well as references to other sources. A list of the resources is given below. A resource devoted to measurement and another devoted to problem solving have been proposed.

NUMBER SENSE AND ARITHMETIC SKILLS (preliminary edition, 1977)

RATIO, PROPORTION AND SCALING (preliminary edition, 1977)

GEOMETRY AND VISUALIZATION (preliminary edition, 1977)

MATHEMATICS IN SCIENCE AND SOCIETY (preliminary edition, 1977)

STATISTICS AND INFORMATION ORGANIZATION (preliminary edition, 1977)

INTRODUCTION

This is a preliminary edition of GEOMETRY AND VISUALIZATION. The resource is intended to help middle school and junior high school teachers by providing background materials and classroom ideas for use with students.

WHAT IS IN THIS RESOURCE?

The resource consists of the following components:

> Didactics
> Teaching Emphases
> Secondary Emphases
> Classroom Materials
> Teacher Commentaries
> Geoglossary
> Annotated Bibliography
> Selected Answers

The *Didactics* papers give information on:

Learning Theories

Teaching Techniques

Diagnosis and Evaluation

Goals and Objectives

The titles of the didactics papers in this resource are:

Planning Instruction in Geometry

The Teaching of Concepts

Goals through Discovery Lessons

Questioning

Teacher Self-Evaluation

A list of the *Didactics* papers for all of the resources is given on page 12.

The *Teaching Emphases* are a collection of processes, approaches and aids which are emphasized throughout the resource. These are:

Visual Perception

Graphic Representation

Calculators

Applications

Problem Solving

Estimation and Approximation

Laboratory Approaches

The *Secondary Emphases* discuss several mathematical threads which extend throughout this resource. These include:

> Straightedge and Compass Constructions
>
> Topics in Topology
>
> Symmetry and Motions
>
> Coordinate Geometry
>
> Maximum and Minimum Questions
>
> Limiting Processes
>
> Invariance

The *Classroom Materials* include:

Worksheets (can be duplicated for whole class use)

Activity cards (may be laminated and used as a nonconsumable card or used with other pages in a nonconsumable booklet)

Transparency masters

Games (individual, small group and whole class)

Bulletin board suggestions

Teacher-directed activities

Teacher ideas (ideas which teachers can develop into activities)

Teacher pages (mathematical background for teachers for specific student pages)

The *Teacher Commentaries* which appear before the sections and subsec-

tions of the classroom materials intend to:

Provide new mathematical information (historical, etc.)

Give a rationale for teaching a topic

Suggest alternate ways to introduce or develop topics

Suggest ways to involve students

Highlight the classroom pages

Give more ideas on the teaching emphases

The *Geoglossary* gives informal definitions for the geometric terms used in this resource.

The *Annotated Bibliography* lists the sources which were used to develop this resource. These sources contain many additional ideas which can be of help to teachers.

HOW ARE THE PARTS OF THE RESOURCE RELATED?

The classroom materials are keyed to each other, to the teaching emphases, to the commentaries and to the didactics papers with symbols and teacher talk as shown on page 11.

The commentaries refer to specific classroom pages (cited in italics) and often a classroom page is shown reduced in size next to the discussion of the page. The commentaries relate the various teaching emphases to the mathematical topic of that subsection. When discussing methodology and teaching strategies, the commentaries refer to the appropriate didactics papers (cited in italics).

Each teaching emphasis includes a rationale and examples from the classroom materials. References are also made to the didactics papers (cited in italics).

Each secondary emphasis includes a list of classroom pages involving that emphasis.

The didactics papers refer to specific classroom pages (cited in italics) and occasionally a classroom page illustrating the topic of the paper is shown reduced in size.

HOW CAN THE RESOURCE BE USED?

The resource can be used by teachers to provide a more successful, varied and flexible mathematics curriculum and to obtain information about mathematics and didactics (teaching strategies, diagnosis and evaluation, learning theories and practices). The resource could also be used in inservice classes or workshops to emphasize problem solving, laboratory approaches, visual perception, etc.

More specifically the resource can be used:

- *As a source of ready-to-use activities to supplement and vary the curriculum.*

  Worksheets can be duplicated to provide a copy for each student; activity cards can be xeroxed and laminated for repeated use; transparencies can be made from a page to form the basis of a teacher-led discussion; or a page can be the focal point of a bulletin board.

- *To build basic skills in geometry.*

  For example, skills with drawing instruments, vocabulary skills and skills in computing area and volume can be improved by

using activities in the classroom materials (a classroom page is marked ⬚ when it involves skill building). Background for teaching some of these skills can be found in the commentaries and the teaching emphases section.

● *As a source of new or different ways to teach a topic.*

A variety of teaching approaches can be found in the classroom materials. The commentaries discuss additional options, and the five didactics papers in this resource give background in the teaching of geometry.

● *To help students understand ideas in geometry.*

Concrete models are used in many of the classroom activities. Attention is given to concept building in the classroom materials and in the didactics paper *The Teaching of Concepts*. Background for teachers is also provided through the teacher pages in the classroom materials, in the commentaries and in a geometry content section (not included in this experimental edition).

● *To gain some insight into difficulties in learning geometry.*

For example, some students have difficulties interpreting drawings of three-dimensional objects. This problem and related problems are discussed in the *Visual Perception* teaching emphasis, in the commentaries and in the didactics paper *Planning Instruction in Geometry*.

● *To improve attitudes.*

Many activities can be used to help students have a feeling of success and accomplishment. Some students who do not succeed in arithmetic activities may be very successful in activities involving geometric designs. Self-checking pages can help a student know when the activity is done correctly. Open-ended

activities help students have confidence in their own methods. The many visual and hands-on materials can be used to capture student interest.

● *To provide individual students or groups of students with material suitable for their needs and interests.*

For example, classroom materials are included for students who are interested in art, students who want to be challenged and students who need much experience with concrete examples in geometry.

● *To increase problem-solving abilities.*

Many classroom pages can be used to give problem-solving experiences. Each of these pages has a ❓ in the upper left-hand corner. Teacher hints and background for problem solving can be found in the *Problem Solving* teaching emphasis. The didactics paper *Goals through Discovery Lessons* and the main commentaries preceding each section give additional ideas on problem solving.

● *As a springboard for developing activities, units or curriculum.*

The classroom pages can be used as models for teacher-developed pages. An activity might have to be adapted to suit a specific teacher or class. For example, a page with too much reading might need to be rewritten as two pages. Activities can be developed from the pages of teacher ideas or from suggestions given in the commentaries or didactics papers. Units could be organized around a geometry topic, like polygons; a teaching emphasis, like problem solving; or a secondary emphasis, like topology.

● *To obtain information about curriculum trends and research in mathematics education.*

Many of the current trends in middle school mathematics are discussed in the teaching emphases section and keyed on the classroom pages. The didactics papers are an easy-to-read

synopsis of some of the research related to teaching mathematics.

* *To gain access to the many available sources for classroom ideas and teacher background.*

  The annotated bibliography can be used for selecting additional resources. Sources are also cited in the classroom pages, the commentaries, teaching emphases and didactics papers.

Teachers will decide which material is appropriate for their students. Since the pages do not list the prerequisites needed for an activity, teachers need to carefully examine each page. In general, each subsection is arranged with the easier, introductory pages first. However, since there are several topics within each subsection, it is possible to find introductory pages throughout a subsection.

It is not expected that all of the information in this resource will be read or used by teachers in one year. Several stages of use are possible. One stage might be to use some of the classroom pages to supplement the curriculum Another stage might be to organize a unit around a mathematical topic or teaching emphasis of the resource. A third stage could be to try a new approach to teaching (laboratory approach, problem solving, discovery lessons) as explained in the teaching emphases section and the various didactics papers. A fourth stage could be to use some of the sources cited throughout the resource and listed in the annotated bibliography. At any of these stages, teachers can add their own ideas and materials to the resource to personalize it and to keep it current.

FEATURES OF CLASSROOM PAGES

When a ditto master is made using the thermofax process, the material in blue will not reproduce. Thus, the student's copy will contain only the material printed in black. The corners are designed to describe the content on each page.

The symbols below identify the teaching emphases in this resource. Each of these is discussed in the section *Teaching Emphases*.

These are the topics of the page. The sub-section and section headings are useful for locating and refiling pages.

Enrichment (investigations or extensions)

Skill-building (drill and practice)

Introduction (concepts and meanings)

Visual Perception

Graphic Representation

Calculators

Applications

Problem Solving

Estimation and Approximation

Laboratory Approaches

AN EQUI=MEETING

1) DRAW A TRIANGLE. LABEL IT AS SHOWN.

2) ON EACH SIDE CONSTRUCT AN EQUILATERAL TRIANGLE.

   LABEL THE NEW VERTICES X, Y, Z AS SHOWN IN THE FIGURE.

3) JOIN Z TO B, A TO Y AND X TO C.

4) WHAT DO YOU NOTICE ABOUT $\overline{ZB}$, $\overline{AY}$ AND $\overline{CX}$?

IDEA FROM *Mathex, Junior Geometry, Teacher's Resource Book No. 9*
Permission to use granted by Encyclopaedia Britannica Publications, Ltd.
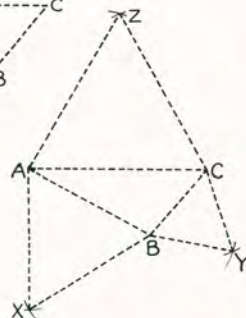
Any other blue material on the page is *teacher talk* or *answers*.

If a page is referred to by a didactics paper, one of these symbols is used.

Learning Theories

Teaching Techniques

Diagnosis and Evaluation

Goals and Objectives

Here is the *type* of activity. This refers to the suggested use of the page.

Credit is given here to the *source* if the page is a direct copy. *Ideas from* other sources are also noted.

LIST OF PAPERS ON THE LEARNING THEORY
AND THE PLEASURABLE PRACTICE OF TEACHING

NUMBER SENSE AND ARITHMETIC SKILLS

Student Self-Concept

The Teaching of Skills

Diagnosis and Remediation

Goals through Games

RATIO, PROPORTION AND SCALING

Piaget and Proportions

Reading in Mathematics

Broad Goals and Daily Objectives

Evaluation and Instruction

GEOMETRY AND VISUALIZATION

Planning Instruction in Geometry

The Teaching of Concepts

Goals through Discovery Lessons

Questioning

Teacher Self-Evaluation

MATHEMATICS IN SCIENCE AND SOCIETY

Teaching for Transfer

Teaching via Problem Solving

Teaching via Lab Approaches

Middle School Students

STATISTICS AND INFORMATION ORGANIZATION

Components of Instruction--an Overview

Classroom Management

Statistics and Probability Learning

NOTE: A complete collection of all the papers from each resource
is available as a separate publication.

GENERAL CONTENTS

DIDACTICS

- Planning Instruction in Geometry
- The Teaching of Concepts
- Goals through Discovery Lessons
- Questioning
- Teacher Self-Evaluation

TEACHING EMPHASES

- Visual Perception
- Graphic Representation
- Calculators
- Applications
- Problem Solving
- Estimation and Approximation
- Laboratory Approaches
- Secondary Emphases

CLASSROOM MATERIALS

LINES, PLANES & ANGLES

- Commentary to LINES, PLANES & ANGLES
  - Lines
  - Planes
  - Angles
  - Symmetry and Motion

GEOGLOSSARY

ANNOTATED BIBLIOGRAPHY

SELECTED ANSWERS

INFORMAL GEOMETRY FOR ELEMENTARY AND JUNIOR HIGH SCHOOLS

> *Mathematics has been the subject of much curriculum reform in recent years. Because of this interest in the mathematics curriculum, especially geometry, The National Council of Teachers of Mathematics devoted one of their yearbooks to the teaching of geometry. The following draws from two articles: "Informal Geometry in Grades K-6" by Paul R. Trafton and John F. LeBlanc, and "Informal Geometry in Grades 7-14" by John C. Peterson in Geometry in the Mathematics Curriculum, Thirty-sixth Yearbook of The National Council of Teachers of Mathematics, 1973.*

WHY INCLUDE GEOMETRY IN THE ELEMENTARY OR JUNIOR HIGH CURRICULUM?

There are several justifications given for teaching geometry in elementary and junior high school. Among these are:

a. Students have a <u>natural curiosity and interest in geometrical ideas</u>, and they often can understand many geometrical relationships (when presented informally) that are difficult to prove formally.

b. Geometry is closely <u>related to the students' world</u> since it is a study of the space and shapes around them.

c. Geometrical ideas are applied in many areas of <u>science and technology</u>. Some of these ideas are taught in industrial arts, science and mechanical drawing classes, but they can also be introduced in mathematics classes.

d. Geometry is a unifying <u>theme of mathematics</u>. With a background in geometry, students have an alternate visual way of viewing topics in mathematics. An example is the number line which gives a represenational model of number. Another example is the geometric figures we use to help students understand fractions.

e. Geometry can involve students in <u>active inquiry, imaginative thought, discovery of relationships, the testing of conjectures, and critical analytical reasoning</u>. These processes are important in mathematics as well as many other fields.

f. The geometry of the upper grades and junior high gives informal experience with geometrical ideas. This forms a <u>background for a development of formal geometry</u> in the high school.

SOME GENERAL RECOMMENDATIONS

Although there has been little agreement on specific content, grade placement or approach for geometry, there have been some general recommendations for geometry curriculum in the elementary and junior high schools.

a. The study of geometry should be <u>related to the real world</u>. Students should be encouraged to explore the spatial relations in their surroundings and they should seek out examples of geometrical relations in the physical world.

b.   The geometry taught should fit the thinking and learning patterns of the students.

c.   The geometry should be informal, not formal.* Initially, students should handle geometric objects and discuss their properties in everyday language. Gradually, they will translate their experiences into more precise language, perhaps including definitions and symbols. From this informal approach, there would be much opportunity for active discovery, inductive reasoning, making and testing inferences or conjectures, and developing visual perception and imagination.

d.   The geometry taught should be mathematically cohesive. It does not need to be as structured as arithmetic, but it should be more than disjointed pieces and more than "fun and games." New learning should relate to previous learning and teachers should know why a particular idea is being taught and how this idea fits into a geometry unit.

A curriculum which fully satisfies these recommendations has not yet been written; however, the geometry curriculum is a current topic of interest in mathematics education. Research is attempting to determine how children think and learn about geometry, e.g., how middle schoolers handle the ideas of motion geometry.

SO . . . WHAT CAN THIS RESOURCE CONTRIBUTE?

Geometry Related to the Real World

Throughout the resource an attempt is made to relate geometrical ideas to actual objects. The commentaries include examples which can be used to discuss the occurrence of various shapes in the students' surroundings. The classroom pages use hands-on experiences to build the abstract ideas of angle, circle, polygon, ... The world symbol 🌐 on a page keys many of these ideas.

Thinking and Learning Patterns of Students

Concrete experiences are important in the learning of concepts, so many of the activities in this resource introduce concepts and relationships at a concrete level. Paper-folding, solid models and manipulatives are used to provide hands-on experiences in each of the sections. For those students who' are able to learn on a more abstract level, there are classroom pages which can be used as they are or adapted to suit the students. Many of the pages make use of a discovery approach, which gives students exposure to problem-solving processes. The *Problem Solving* teaching emphasis, *Planning Instruction in Geometry* and *Goals Through Discovery Lessons* discuss problem solving, some background from theoretical positions and the case for discovery learning.

*This does not refer to the formal geometry often taught to an advanced 9th-grade class.

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Informal Geometry

This resource was compiled because informal geometry is important as well as interesting for middle school students. Many classroom pages are designed to build, diagnose or utilize visual perception (marked with an 👁). Many of the activities encourage inductive reasoning or making and testing conjectures. The resource emphasizes active involvement and guided discovery, rather than solely visual or paper and pencil approaches. For example, the drawings of the conic sections below are helpful, but are not as concrete as cutting a cone of clay with wire and observing the resulting shapes.
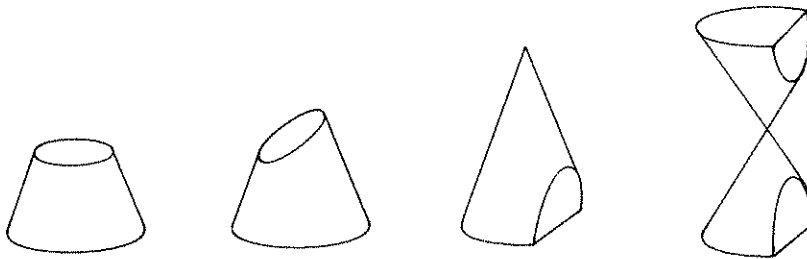
Mathematical Cohesiveness

Although this resource is not a curriculum for geometry and therefore does not provide geometry units as such, it has several features which can help provide mathematical cohesion to a geometry program. The classroom materials are organized into five sections: LINES, PLANES & ANGLES; POLYGONS & POLYHEDRA; CURVES & CURVED SURFACES; SIMILAR FIGURES; and AREA & VOLUME. Included in each section are commentaries discussing the content and teaching of the section topic. A teacher might, for example, design a unit on polyhedra, selecting activities from this resource, a textbook and other sources. This unit on polyhedra might begin with ideas about tetrahedra and cubes and culminate with generalities about regular polyhedra.

There are other ways to organize geometry than by geometric objects. A unit could be organized around a geometric relation such as congruence or similarity. In addition, there are various threads or themes such as symmetry, constructions or limits which can provide a basis for organization. A brief discussion of some of these threads and a list of related student pages are given in the SECONDARY EMPHASES section. A teacher could use one of these emphases as the focus for a unit and choose from ideas throughout all five sections of the resource. The introduction gives more details about using this resource.

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

PLANNING INSTRUCTION IN GEOMETRY

As the many types of classroom activities in this resource indicate, geometry in the middle school can involve much more than vocabulary and measurement. Solid figures, for example, often receive little attention in this country. Yet, solid figures offer opportunities to relate mathematics to everyday living, to develop aspects of space perception and to practice problem solving techniques through discovery lessons. In general, European schools do more work with geometry and do it earlier. What can their experiences teach us about instruction in geometry?

LEVELS OF THINKING IN GEOMETRY

Two European teachers, the van Hieles, have described five levels of thinking in geometry. [1959]

Level 0 At this level students identify a figure by its appearance as a whole. That is, when a student calls something a square, he is reacting to the total figure, not to interrelationships among the sides and among the angles. Indeed, such interrelationships may not even be noticed. The student does not yet consider a square to be a special rectangle or a rhombus to be a special parallelogram. He can, however, produce or copy a specified figure on a geoboard.

Level 1 At this level the student is alert to the parts of figures and how they are related. For example, that a figure is a square indicates that all the sides are congruent, each angle is a right angle, and perhaps that the diagonals are perpendicular to each other and bisect each other and the angles of the square. If a figure is described as having both pairs of opposite sides parallel, it will be acknowledged to be a parallelogram. Even so, the properties may not be so well organized that, for example, a rhombus is regarded as a parallelogram.

Level 2 Here properties and classifications assume an organization. That one property follows from another can be partially digested (no full understanding of a deductive proof can be assumed, however). The figure and the interrelations of its parts are linked. Hence, definitions can be called for (or given) and the student is comfortable with calling a square a special rectangle or a special rhombus, for example.

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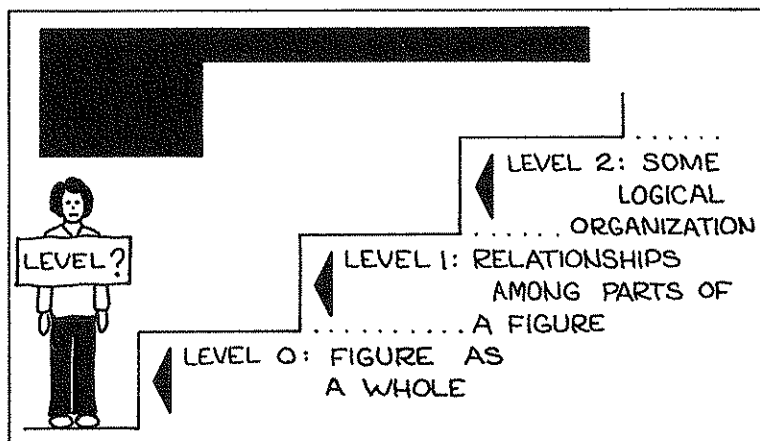
<u>Levels 3 and 4</u> The highest levels deal with the nature and variations of a deduc-
tive organization of geometry (Level 3) and finally with geometry as an abstract
system in which, for instance, "point" can be interpreted in any way consistent with
the axioms of the system (Level 4). Clearly most middle school students are think-
ing at Levels 0, 1 or 2.

The van Hieles are adamant on one
excellent point: *"Two people who are
reasoning on two different levels can-
not understand each other."* [1959,
p. 201] *We've got to address the stu-
dent on his level.* (Have <u>you</u> ever been
a student at Level n when the instructor
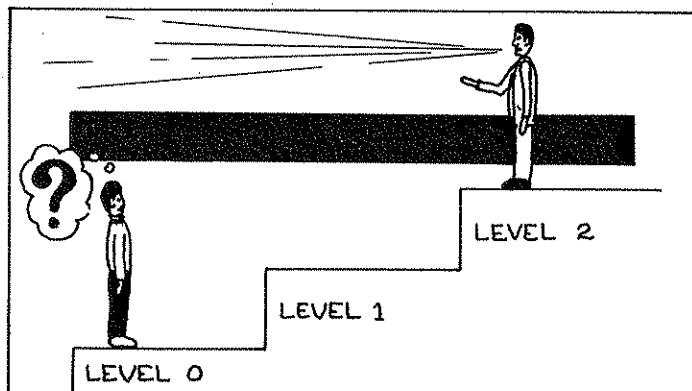
seemed to be at Level n+2?) For example, a textbook may start the treatment of geo-
metry by bringing in definitions, a Level 2 activity, whereas many students may still
be operating at Level 0, reacting to geometric figures as wholes. Van Hiele feels
that it may be possible to skip levels by learning things <u>rotely</u>, but only with the
risk of subsequent rapid forgetting (in Freudenthal, 1973, pp. 125-126).

LINKING THE LEVELS

Instructional activity for a
student should be based on his level
of thinking. Besides enriching his
thinking at that level, the lessons
should be moving him toward the next
level. Since even an occasional
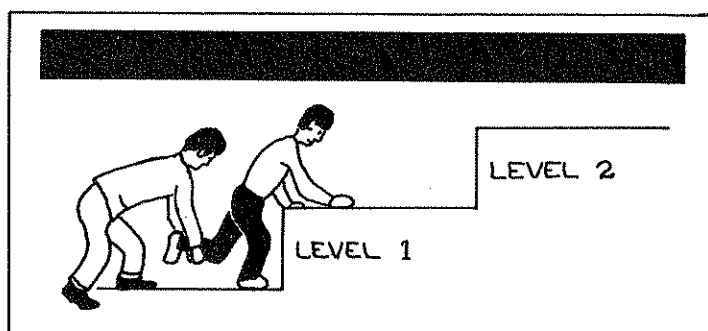
college student may not know rudimentary geometry vocabulary, we should check our
middle school students for knowledge of the basic shapes: square, rectangle, tri-
angle, circle. The concepts of parallelogram, trapezoid, rhombus and other more
specialized ideas will need development or at least review. Geoboard work--showing
a figure given its name, or naming and copying a given figure--would strengthen
vocabulary and give the students chances to notice some of the interrelationships
among parts (e.g., that the opposite sides of a parallelogram have the same length).

For Level 0 students, our aim is to give them experiences oriented toward
Level 1, the awareness of the relationships among the parts of a figure. In saying,

". . . it is the manipulation of figures which causes a structure to be born. This nourishes the thinking at Level 1," [1959, p. 205] van Hiele joins hands with Piaget. Inhelder, a Piaget co-worker, believes that ". . . children with richer possibilities of manipulative and visual tactile explorations have better spatial reasoning" (in Sherman, 1967, p. 296). The hands-on activities in *Symmetry* or *Reflection Methods* (from Symmetry and Motions) or *As Easy as 1, 2, 3* (in POLYGONS & POLYHEDRA: Polygons) and *Surveying Solid Shapes* (in POLYGONS & POLYHEDRA: Polyhedra) could be used to help the students move toward Level 1.

Since these activities can be used to invite the study of the <u>parts</u> of a figure and how they are related, they pave the way to Level 1 thinking. Many more relationships for Level 1 can be established through activities like those suggested in *Tessellations with Triangles* and *Tessellations* from POLYGONS AND POLYHEDRA: Polygons. Once the proper Level 1 background is set up, occasional forays toward a Level 2 type of thinking can be tried--*Interior Angles of a Polygon 1* or *Are You in Shape?* (from the same subsection), for example.

WHERE TO START?

Perhaps because of the influence of the logical development of high school courses, many middle school text series start their developments of geometry by dealing with points, segments, rays, lines and planes, then move to plane figures and end with solid figures. (This sequence is so common that it is intentionally reflected in the organization of the classroom materials in this resource.) Such an order is <u>not</u> essential. For middle school geometry, there are neither logical nor psychological necessities for starting with points, segments, etc., and ending with solids.

For example, you may choose to reorganize your students' work with geometry to begin with cubes, rectangular solids, pyramids, cones, . . . and to develop and review the points, segments, angles, squares, triangles, etc., terminology and ideas with these solid figures. One could, for example, use as a basis the following ideas from POLYGONS & POLYHEDRA: Polyhedra. For the students at Level 0, one could collect solids, empty boxes, cans, pieces of scrap from the shop, etc., to introduce vocabulary and encourage an alertness to solid figures. Making shell and skeletal models (e.g., *Constructing Polyhedra Models,* *Constructing Nets of Irregular Tetrahedra*) would give a no-threat activity and provide lots of chances to introduce vocabulary dealing with lower dimensions (segment, point, angle, parallelogram, rectangle, . . .). Exploration of the figures (*Surveying Solid Shapes,* or the more technical *Vertices, Faces and Edges*) could naturally lead to some Level 1 thinking,

as could some drawing practice with *Two-D or Not Two-D* and *Two Dimensional Representations*. (One would keep in mind the difficulty of "reading" drawings of solid figures. For example, the pages *These Are, These Are Not.* and *You Decide..* give opportunities for moving toward Level 1 thinking, but use the more abstract pictorial mode. The reading of drawings can be enhanced by making drawings. See *Graphic Representations* in the TEACHING EMPHASES section.) By branching off from the basework with solid figures, one could certainly intermingle activities with polygons, angles, segments or measurement from earlier parts of the resource or from later sections (e.g., AREA & VOLUME).

Many Europeans favor starting geometry work with three-dimensional objects not only because of their emphasis on physical experiences with (three-dimensional) concrete materials, but also because of the belief that too much work with only plane figures can "deaden" one's spatial perception (see Freudenthal, 1973, pp. 408-409). We do live in a 3-D world.

SPATIAL PERCEPTION

Perception Deception in the Lines subsection illustrates how we can be deceived by drawings and brings up the matter of visual perception. Optical illusions aside, problems with perception can arise when we represent a three-dimensional figure (a cube, say) on a two-dimensional surface (e.g., the chalkboard). We may make a good sketch of a cube, but some students may not "see" the cube at all. Drawings of three-dimensional objects may not be "concrete" enough for many students. (See *Visual Perception* in the TEACHING EMPHASES section for

A cube? . . . Or a hexagon with segments inside?

more discussion.) Let us consider here a few experimental results dealing with spatial perception which have implications for geometry instruction.
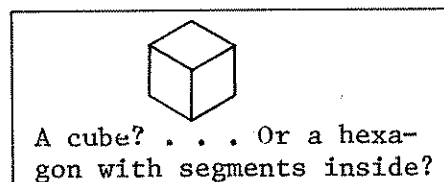
Individual Differences

You may notice unusual differences in individuals during the work in geometry, particularly during work presented by drawings. Students vary so greatly in their abilities at visual perception that the usual "top" students may not do so well as usual, or some of the usually weaker students may do very well. One interesting aspect of these individual differences is that girls in general do not seem to do so well as boys on tasks involving judging or manipulating spatial figures. [Tyler, 1965, p. 245] There are, of course, many girls who excel at such tasks, just as there are many boys who do not do well. However, this on-the-average difference is apparent

by the middle grades [Davis, 1973; Karnovsky, 1974], is more pronounced with compli-
cated tasks, and seems to increase through the teens [Sherman, 1967]. Plausible
explanations lie in (a) the different amounts of spatial-processing practice involved
in the activities usually associated with culturally-imposed sex roles (boys are
expected to be more physically active with mechanical things such as model building,
blocks, sports), and (b) the greater verbal facility usually noted with girls, which
enables them to do many things without the physical action required by the less-verbal
boys (for additional explanations see Maccoby and Jacklin, 1974).

Whatever the reason for the differences, there is some evidence (e.g., Wolfe,
1970) that the spatial visualization ability of junior high boys and girls can be
improved by instruction. Rather than relying entirely on picture-presented material,
you may wish to use several of the student pages involving some aspect of visual per-
ception to attempt to enable the weaker students to improve their spatial ability.

Piagetian Tasks

The (admittedly approximate) age guidelines for mastery of some of Piaget's geo-
metry tasks would lead one to expect that most middle school students can handle them.
Many can, but research suggests that many (and even some adults) cannot. Two examples
follow, to illustrate that (a) some geometry tasks may best serve as diagnostics and
(b) some students may profit from preliminary or supplementary work with hands-on
experiences.

The first example has to do with predicting the shape of an object's shadow.
[Piaget and Inhelder, 1956, ch. 7]
Shadow Stumpers in SIMILAR
FIGURES would give some information on
a student's ability to visualize the
shape of a shadow; students who are
successful could be asked about the
shape of the shadow when the sun is
shining at an angle. The ages indicat-
ed by Piaget and Inhelder suggest that
middle school students should be able to predict the shapes. Failure to do so may
indicate a slower rate of cognitive development or perhaps a lack of prior experience.
It may be that the student could profit from taking a flashlight into a darkened cor-
ner, shining it on objects and noting the shadows as the light is moved. Using
specific Piagetian tasks to develop ability with those tasks does not always result in
improved performance [Flavell, 1963, p. 377]; knowing this may help us avoid exasper-
ation with students who just can't seem to see the shadows.

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The second example deals with predicting the out-
line of a piece obtained from a straight cut through a
simple solid. Boe [1968] found that American youngsters
in grades 8, 10 and 12 were not as proficient at pre-
dicting such shapes as she thought Piaget said they
would be. Davis [1973] repeated the experiment with
grades 6, 8 and 10 but provided a practice session during which the students actually
cut some irregular styrofoam solids. Results were slightly higher than Boe's, but
the student population was different and, even with some practice, average student
performances in grades 6 and 8 were only 66% and slightly under 80%, respectively.
Sections from oblique cuts were the hardest ones for students to predict. Work with
actual cutting of solids (clay, styrofoam, oranges, bread, . . .) with specific
attention to the outlines obtained might supplement the student page, *Cross Section*,
in POLYGONS AND POLYHEDRA: Polyhedra. Once again, the activity might best be re-
garded as diagnostic, giving some information about the student's spatial perception.
(More information on Piaget is in *Piaget and Proportions* in Ratio, Proportion and
Scaling, Mathematics Resource Project.)
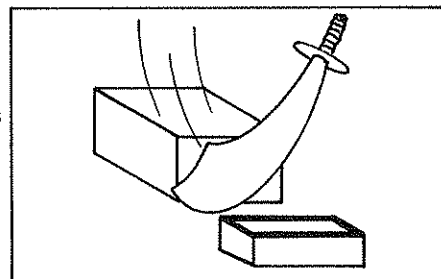
SUMMARY

In planning instruction in geometry, there may be benefits in considering the
van Hieles' first levels of thinking: Level 0, in which a figure is regarded *in
toto*; Level 1, in which interrelationships among parts are used; and Level 2, in
which some degree of logical organization has taken place. It is more important
that we work at the student's level, giving instruction that leads to the next level.
Recognizing the models available in real-world objects, you may wish to build your
geometry work on 3-D figures. You may also wish to include experiences with spatial
perception, keeping in mind that some students may not give their usual performance.
Finally, work with Piaget-type tasks may give data on students' cognitive development.

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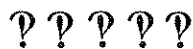
1. Examine one of the textbooks you use to see at what level (*a là* van Hiele) geo-
    metric topics are introduced.

2. Choose some geometric figure (parallelogram, rhombus, cube, . . .) and outline
    the Level 0 and Level 1 work you would use for the concept.

3. The topic of visual perception has many aspects (see *Visual Perception* in the
    TEACHING EMPHASES section; McKim, 1972):
        --finding figures embedded in a complicated drawing
        --finding the identical match to a given figure

--completing a partially-given figure or pattern

--remembering a design

--visualizing a figure after a rotation or reflection

--visualizing cross sections

--visualizing an object from different viewpoints

--mentally manipulating an unfolded pattern

--mentally putting pieces together to form a given figure

Which aspects do these student pages seem directed toward? (a is in the Symmetry and Motions subsection, b in POLYGONS & POLYHEDRA: Polygons, and c-f in POLYGONS & POLYHEDRA: Polyhedra)

    a.   *Which Way Will the Arrow Point?*

    b.   *The Tantalizing Tangram*

    c.   *Fold-Ups*

    d.   *Several Views of Cubes*

    e.   *Cross Sections*

    f.   *Seeing It Like It Is*

4. Using the aspects in number 3, evaluate these activities as possible exercises in visual perception:

    a. Have students copy a geoboard figure; have them copy it, imagining that the geoboard has been turned 90° clockwise.

    b. Project a sketch of a cube. Have students hold cubes so that the sketch shows how their cubes would look to them. Have them hold the cube so that the sketch shows how the cube would look to their neighbor or you. (See the commentary for POLYGONS & POLYHEDRA.)

5. Here is another Piagetian experiment you might try. [Piaget and Inhelder, 1956, ch. 13] Pass out slips of paper like those in the drawing. Partly fill a transparent jar with colored water and show it to the class, keeping the jar in an upright position. Then put a piece of paper around the jar, tilt it as in the drawing and ask the students to show how the water would look then. Predict beforehand how you think your students will do.

6. Davis [1973] noted sex differences with his sectioning-of-solids tasks (boys performed better than girls).

    a. You might want to ask your students to sketch the outlines as you hold a "knife" to suggest cuts of different sorts (perpendicular to an axis of symmetry, along an axis of symmetry, oblique to an axis of symmetry) through a rectangular prism, a cube, a circular cylinder, and a circular cone. If your results show a girl-boy difference, try to account for this difference by thinking of specific everyday-life experiences that might give one sex better background for such tasks.

    b. You might also want to compare performance on some of the student pages dealing with visual perception--e.g., *Which Way Will the Arrow Point?* in Symmetry and Motions.

7. Plan a unit in geometry for one of your classes, starting the work with solid figures.

    a. Include work to get to Level 0 and to give Level 1 background.

    b. Include work with manipulatives.

8. Here are some Piagetian-type tasks dealing with area. You may wish to use similar tasks as informal diagnostics. (Some other ideas are in Wagman [1975]. See also the commentaries in AREA & VOLUME.)

   a. <u>Conservation of area</u> In each pair the figure to the left is cut (or pieces are moved) to get the figure to the right. You might like to ask students over a range of ages whether the areas remain the same. (You can make up your own tasks with tangram pieces—see *Are Squares Larger?* in AREA & VOLUME: Area.)
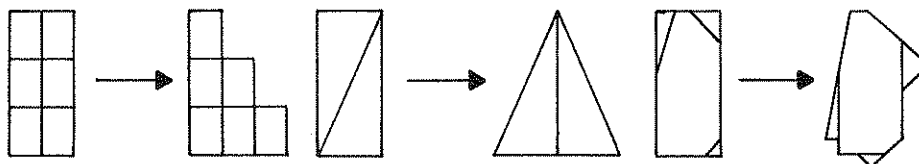
   b. <u>Conservation of comple- mentary ("left-over") area</u> Surprisingly, some students who conserve area in tasks like those in part (a) do not realize that if a constant area within a region is transformed,

   THE WIDE OPEN SPACES

   1) Pete Piajay has six pictures, each with dimensions of 10 cm x 20 cm, that he wants to arrange in a 60 cm x 75 cm poster. Pete thinks arrangement (b) leaves the most area uncovered. Is he correct? _____ Explain. _____

   a)   b)   c)

   the "left-over" part still has the same area. One version is given to the right (from *The Wide Open Spaces* in the AREA & VOLUME section). You can design other versions by placing figures like those in part (a) within regions.

9. Tasks similar to those in number 8 can be devised for conservation of volume. (See *Lake & Island Board* in AREA & VOLUME: Volume.) Of the conservation abilities studied, conservation of <u>displaced volume</u> is the latest to develop. If you have some cubes that do not float on water, you might try this task with some of your students. Build a structure with the cubes in a transparent container and add enough water to cover the cubes. Mark the water level. Ask what will happen to the water level if you take the structure apart and spread the cubes out. [Piaget, Inhelder & Szeminska, 1964, ch. XIV]
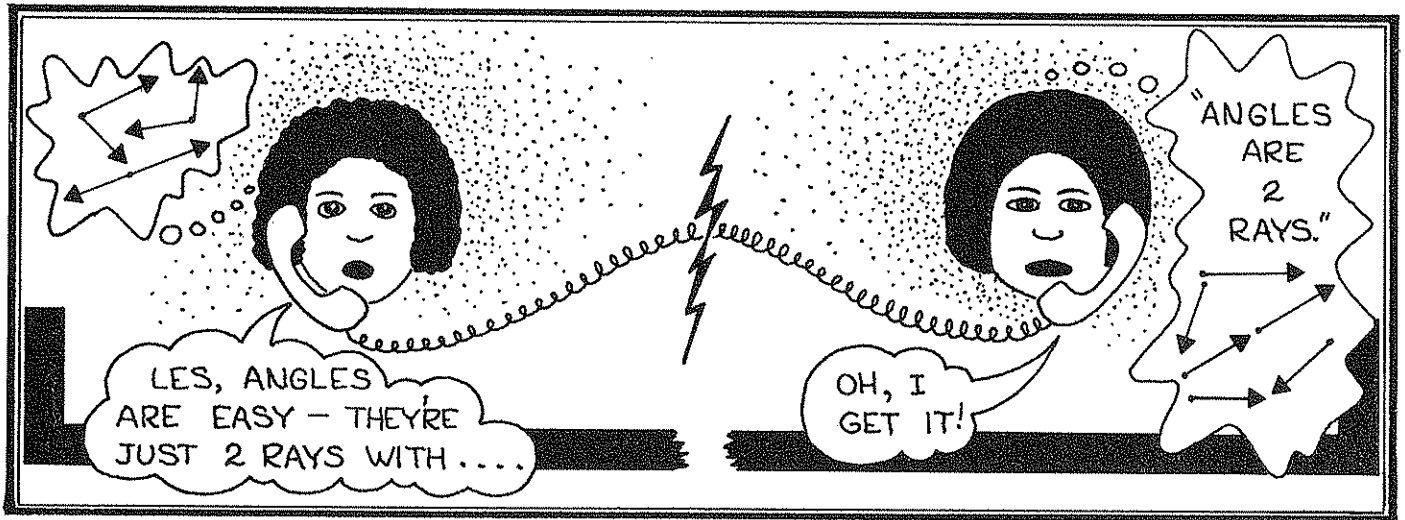
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GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

THE TEACHING OF CONCEPTS

> LES, ANGLES ARE EASY — THEY'RE JUST 2 RAYS WITH....

> OH, I GET IT!

> "ANGLES ARE 2 RAYS."

HOW CAN WE TELL?

What is adequate evidence that a student has indeed "got it," when "it" is a new concept? Certainly we would expect the student to be able to <u>choose and recognize all examples</u> of the concept ("mark all the trapezoids") and to <u>produce examples</u> ("sketch a trapezoid"). Since he will need to communicate about the concepts, at some stage he should be able to <u>interpret and produce the word</u> (or symbol) for the concept.

Should he also be able to give a definition of the concept? Although definitions phrased at the proper level can help in concept formation, there is a danger of over-relying on verbalization. The important thing is the idea, and a rote recitation of a definition does <u>not</u> assure that the idea is present. "Learning can become ununverbalized, which means that the concepts learned are highly inadequate in their references to actual situations. The learner, one may note, 'does not really know the meaning of the word,' even though he can use it correctly in a sentence." [Gagné, 1970, p. 187] (There is, incidentally, agreement that a student <u>can</u> possess an idea <u>without</u> verbalization—see number 16 on page 9.) Similarly, we teachers must guard against relying too much on <u>our</u> words and incorrectly assuming that the students are understanding them. Fortunately, in teaching geometry there are more trustworthy ways of communicating with unsophisticated learners than by relying on statements like "$\forall p, q \in \Pi, p \parallel q \longleftrightarrow p \cap q = \emptyset$" or even by using accurate but awkward definitions (try writing a definition of alternate interior angles). The importance of communicating on the student's level was touched on in *Planning Instruction in Geometry* in the first section.

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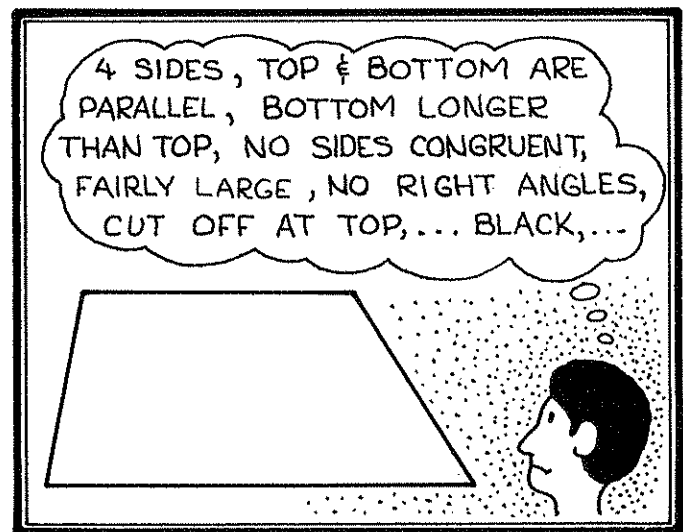## GUIDELINES TO CONSIDER WHEN TEACHING CONCEPTS

The above suggests that throughout our teaching of concept, we'll want to use every opportunity to have the student give examples and choose examples of the concept--that is, <u>have the student do something which gives us feedback about his grasp of the concept</u>. Secondary evidence might be sought from verbalizations. What are some other points to keep in mind when teaching a concept? The following guidelines have support from learning theory and research findings.

Use an Adequate Number of Examples

Exactly how many "adequate" means is not clear, but it certainly means more than one. To expect a student to grasp the concept of a regular polygon with only one example is probably unrealistic, even if a definition is given. It is safe to say that a concept which involves greater abstraction or more relationships and ideas will require more examples. The concept of a regular polygon should be illustrated with more examples than the concept of a triangle. The moral is, have extra examples <u>ready</u> if feedback from the class indicates that they are not getting the idea.

Vary Irrelevant Properties in Examples

It is not just the number of examples that is important, it is also their quality. The (hypothetical) student in the figure is forming several impressions from the example of a trapezoid, not all of which are pertinent to the concept. Some of these irrelevant properties, *if always present* in examples, may become part of the student's trapezoid concept. A variety of carefully planned examples can help to cancel out attention to these wrong impressions. Our student should drop the following impressions as being part of his trapezoid concept if he also sees the accompanying examples.

> 4 SIDES, TOP & BOTTOM ARE PARALLEL, BOTTOM LONGER THAN TOP, NO SIDES CONGRUENT, FAIRLY LARGE, NO RIGHT ANGLES, CUT OFF AT TOP,... BLACK,...

Irrelevant property   Trapezoids

"top and bottom are parallel"   (examples show that
                                        parallels need not
                                        be at top and bottom)

"bottom longer than top"   (example shows error
                                        of this idea)

"no sides congruent"   (examples show that two
                                        or even three sides can
                                        be congruent)

"no right angles"   (example shows there may
                                        be two right angles)

"balanced on bottom"

This sharpening of a student's concept by varying irrelevant properties in examples will also give some protection against underfgeneralizing, that is, failing to recognize some figures as examples. Students who perhaps subconsciously feel that trapezoids must have their parallel sides horizontal may not recognize trapezoids which don't have such an orientation. Or, as is likely with the *Angles, Not Angels* exercise to the right, some students will see only three angles, having adopted the irrelevant criterion "doesn't have anything inside it" and thus rejecting an angle like ∠BAD.

Name four different angles.
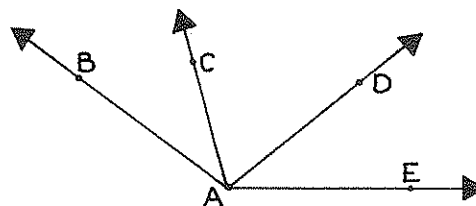
Use Some Nonexamples as Well as Examples

Doing this enables the student to focus on concept-relevant properties by noting their absences. It helps to see not only examples of a concept but also things which resemble, but are not, examples of that concept (e.g., as in *These Are. These Are Not* in the Polygons subsection or *You Decide..* in Polyhedra). Seeing nonexamples enables students to recognize the insufficiency of an incomplete concept. "All sides congruent" may have worked fine as a criterion for a regular polygon with both the examples in the figure, but its inadequacy becomes evident when nonexamples b and d

regular polygons   not regular polygons
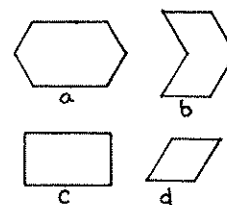
are shown. Hence, nonexamples should help students avoid <u>overgeneralizing</u>, that is,
incorrectly identifying some nonexamples as being examples. Under the faulty "all
sides congruent" criterion for regular polygons, a student would have overgeneral-
ized by identifying <u>b</u> and <u>d</u> as regular polygons.

As another example, part of a
follow-up to *Don't Get Stuck on This*
(in Polygons) might include a question
such as, "Show a quadrilateral that is
<u>not</u> a parallelogram," or have students
decide whether the figures to the right
are parallelograms.
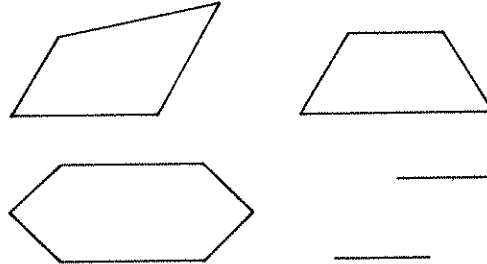
Experience indicates these three points about using nonexamples:

● Introduce nonexamples <u>after</u> the students have experienced some examples.

● Choose nonexamples which closely resemble examples. To use a circle or a triangle
as a nonexample of squares would not add much precision to a middle-school student's
concept of a square, whereas a nonsquare rhombus might.

● You may have to force attention to nonexamples ("Why isn't this an example?") since
students usually do not process the information in nonexamples very completely.

<u>Use a Variety of Representation Modes, Especially Concrete Ones</u>

The common categories of representation modes are <u>concrete</u>, <u>pictorial</u> (semi-
concrete), and <u>symbolic</u> (including verbal). A circle could be represented by a
hula hoop (concrete), by a drawing (pictorial), or by the word "circle" (symbolic).
While there is consensus that learning for <u>young</u> children should start with concrete
situations, some observers attach great importance to concrete representations for
<u>all</u> students: "The learner must . . . begin with concrete situations . . . additional
learnings will be useless as sheer verbalizations, unless the student knows the con-
cept by reference to a <u>class of concrete situations</u>." [Gagné, 1970, pp. 177, 179]
"Reactions to the world of concrete objects are the foundation stones from which the
structure of abstract ideas arises." [Van Engen, 1953, p. 80] An activity based on
Constructing Polyhedra Models (in the Polyhedra subsection), for example, surely
would enrich a student's concept of a polyhedron. In the same way a student's

concept of a polygon could be enhanced
by encountering polygons in non-
mathematical contexts like *Poly-Art*
(in the Polygons subsection) or in the
shapes of traffic signs.

Textbooks and chalkboards auto-
matically work with pictorial and
symbolic representations. Exactly how
literally the term "concrete" must be
interpreted is not clear. "Concrete"
seems to be a relative term, varying
in its meaning depending on a stu-
dent's background. To eighth grade
students, $3 + 2 = 2 + 3$ may be a "con-
crete" representation of $a + b = b + a$,
whereas 3 blocks and 2 blocks may be
required for "concrete" work with pri-
mary students. Since the meaning of
"concrete" will vary from student to student, our concern should be not to overlook
the fact that for many middle school students, concept development may require exper-
ience with physically concrete representations.

Use Definitions Judiciously

So far all the guidelines have been about examples. Definitions and verbal state-
ments, of course, do play an important role in the classroom. Let's review the
dangers:

1. Operating solely with words and symbols may "lose" a lot of students.

2. Presenting a definition does not mean it has communicated fully (or even
been heard or read, for that matter).

3. A student recital of a definition is not proof in itself that the concept
has been mastered (nor is failure to verbalize a definition evidence that the student
has no idea at all).

How can we use definitions? Good sense and research offer a few ideas:

1. Be certain that the students know the words you're using to define the new
concept. You may have to stop, examine each phrase, discuss its meaning, and

demonstrate or refer to an example. Just because "ray" was "covered" three weeks ago
does not mean "ray" is remembered today. Another hint: Saying, "Does everyone
remember what a ray is?" and seeing how many nods and raised hands there are may give
you much _less_ reliable information than, "Everyone draw a ray. When you finish, look
at your neighbor's," and a quick tour of the room.

2. Emphasize all the parts of a definition. If several conditions must be
satisfied ("both equilateral and equiangular"), emphasize the "and," and use non-
examples which fail on one or more of the conditions (see the "not regular polygons"
figures above). Underlining or circling words or even a simple thing like putting
each condition on a separate line may help. [Markle, 1975, p. 5] Occasionally you
will run across an "or" definition (e.g., a conic section is a circle _or_ an ellipse
_or_ a hyperbola _or_ a parabola) or one with an implied "or" (e.g., "an isosceles tri-
angle is a triangle with at least two congruent sides" means "an isosceles triangle
is a triangle with two congruent sides _or_ three congruent sides"). Give careful
attention to such statements since some students fail to interpret them correctly.

3. Use examples and nonexamples with the definition. The combination is better
than definitions or examples alone   [Klausmeier, et al., 1974]

Experiment with Your "Moves"

A considerable amount of research over the last several years has been devoted to
analyzing what the teacher does in the classroom. This analysis has identified
several types of teacher "moves" and has provoked studies to see what seems to be the
best sequences of "moves". (More complete explanations are given in Henderson [1967,
1970] and Cooney, Davis and Henderson [1975].)

Here is a simplified version of the moves involved in teaching a concept. One
type of move would be to give an example (E); giving a nonexample (not-E) would be
another type; and stating a definition (D) would be another. The research questions
with major classroom implications have fallen along these lines: When I teach a
concept, should I give the definition first and then several examples (D,E,E,E,E), or
reverse the order (E,E,E,E,D)? Where do nonexamples fit in best--is D,E,E,not-E,
not-E more effective than E,not-E,D,E,not-E? How many examples should be used? To
date, research findings do not strongly suggest a "best" way.

It is likely that the nature and difficulty of the concept involved, along with
learner characteristics, will be so important that a universal "maxim-for-moves" is
a long way off. Nevertheless, you may choose to keep in touch with the research on
moves (e.g., Rector and Henderson, 1970; Gaston and Kolb, 1973; Dossey and Henderson,
1974) and will likely wish to "experiment" with moves in your own classes.

An exciting--but at the same time dismaying--aspect of teaching is that we can--and must--try different approaches. Using the guidelines above will not <u>assure</u> successful learning of concepts. But they may help your planning, your imagination, and your skill in presenting concepts to your students.

? ? ? ? ?

1. Sketch examples which should remove each focus on an irrelevant property arising from the single example of a parallelogram
   a. "leans to the right"
   b. "wider than it is tall"
   c. "most of top over bottom"
   d. "leans"
   e. "top and bottom parallel," not noticing other parallels
   f. "top and bottom are horizontal"
   g. size

   (given example)

2. Sketch examples to remove these irrelevant foci for the concept of supplementary angles:
   a. "adjacent angles"
   b. "unequal"
   c. "above a horizontal line"
   d. "one ray pointing to the upper right"

3. a. Give some irrelevant properties which <u>might</u> mislead students who see this single example of an isosceles triangle.
   b. Sketch an example to squelch each of the irrelevant properties you noticed in part a.

4. Give specific examples, nonexamples and concrete representations you might use in teaching these concepts:
   a. alternate interior angles
   b. polygon
   c. polygonal region
   d. polyhedron

5. Give nonexamples which would enable you to draw attention to the parts of each definition.
   a. A square is a <u>quadrilateral</u> with <u>4 right angles</u> and <u>all sides congruent</u>.
   b. A rhombus is a <u>parallelogram</u> with <u>two adjacent sides congruent</u>.
   c. A parallelogram is a <u>quadrilateral</u> with <u>both</u> pairs of opposite sides <u>parallel</u>.
   d. A square is a <u>rhombus</u> with a <u>right angle</u>.

6. It often happens that a student's concept is incomplete or erroneous—or that his idea is all right but his "definition" is incomplete. E.g., "Squares are figures with all sides equal." If a student makes such a statement and clearly means it to be a complete definition, a classmate or the teacher should provide a counterexample, an example or nonexample which shows the incorrectness of the statement. Supply counterexamples for these "definitions."

Counterexample for (a)

   a. "Complementary angles give a right angle."
   b. "A square is a quadrilateral with all sides equal."
   c. "An angle is two rays."
   d. "In perpendicular lines, one goes up and the other goes across."
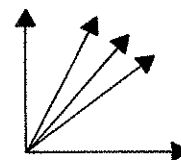
7. (Diagnosis and remediation) Exercise: Circle the larger angle in each part. Abe's work:
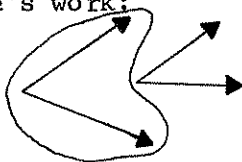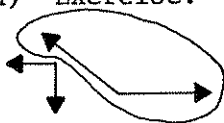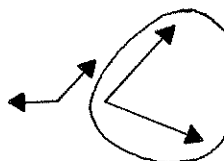
   a.   b.   c.   d.
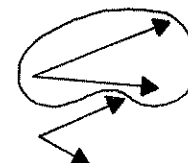
   What might account for Abe's incorrect answers in parts c and d? (Hint: irrelevant property.)
   Suppose you then give Abe the exercise to the right and he chooses the first angle as the larger. Does his response fit your diagnosis of his error? How would you help Abe?
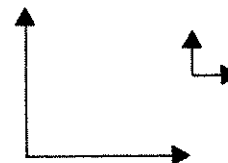
8. If our concrete and pictorial models always incorporate an irrelevant attribute, students may incorporate that attribute into their concept of what we're modeling. The easiest example to point to is our use of regions to model concepts which do not involve a region. Pick the better of the two models for each of the following.
   a. models for a circle: pizza, or rim of pizza pan
   b. models for an angle: cardboard region, or Tinker-Toy model
   c. models for a polygon: cardboard region, or geoboard figures
   d. models for a polygonal region: cardboard cutout, or strings on a pegboard
   e. models for a polyhedron: a drinking-straw skeleton, or a geoblock

9. As in the previous exercise, our models may result in student confusion. How would you avoid these two problems?
   a. Confusing area with perimeter (teacher uses cardboard regions when discussing both).
   b. Not seeing overlapping figures (teacher has cut cardboard models in illustrating supplementary and complementary angles).

10. Mr. Day: "I don't think you really have a concept unless you can put it into words."
    What is your feeling?

11. (Discussion) Teaching which of the following concepts would likely require the
    most examples? Why?
    a. circle, isosceles triangle, equilateral triangle, regular polygon, pentagon
    b. perpendicular lines, altitude of a parallelogram, altitude of a triangle,
        altitude of pyramid
        (Possible criteria: complexity, familiarity, earlier related work. One choice:
        a. regular polygon, b. altitude of triangle or parallelogram, whichever is
        encountered first.)

12. Consider some concept your students seem to have trouble with (perhaps square
    root or irrational number, for example). How could the guidelines be applied
    to teaching that concept?

13. Informal language can sometimes lead to a misconception. How could these phrases
    be misinterpreted?
    a. improper fraction
    b. reduced fraction

14. Research indicates that we should leave as many examples and nonexamples visible
    as we can when we are teaching a concept (be sure the nonexamples are labeled
    as such). This helps students to compare attributes in the samples. What are
    some ways of preserving your examples and nonexamples when . . .
    a. you have just one transparent geoboard for the overhead projector?
    b. your supply of geostrips is very limited? (See *Geostrips* in the Polygons
        subsection.)
    c. you have very little free chalkboard space?

15. Plan how you could use the following ideas to diagnose your students' background
    in geometry or to allow them some exploration at a nonverbal level.
    a. *Sort Out*
    b. *As Easy as 1, 2, 3*
    (Both are in the Polygons subsection.)

16. This exercise illustrates that an idea can be formed without a definition or
    verbal label. (Some psychologists use the word "concept" to include the verbal
    label.)
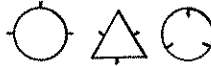
    a. These are   These   Which of
        examples.   are not.   these are?
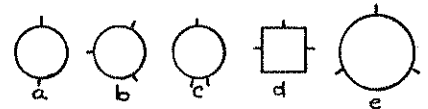
    b. A definition <u>can</u>, of course, add precision to a concept. And a verbal label
        <u>can</u> make communication easy. Suppose the concept exemplified above is called
        an uggle, defined as follows: An uggle is a figure made up of a 7 mm circle
        with 1 mm line segments on extensions of diameters and at points which tri-
        sect the circle. Would you change your answers in part <u>a</u>?

17. (Outside reference) Ausubel and Robinson [1969] distinguish between concept
     formation and concept assimilation. The major difference is the starting point;
     here is a simplified version:
           concept formation: examples, nonexamples——▶idea (——▶perhaps definition)
           concept assimilation: definition——▶idea (——▶examples, nonexamples may be
                                  needed)
     (In either case, the concept name may occur early.)
     Study some of your lesson plans to see whether you tend to favor one form. Does
     your textbook use one form more than the other?

18. (Outside reference) Henderson [1970] gives a detailed analysis of the moves
     involved in teaching concepts. Audiotape the part of one of your classes in
     which you develop a concept, read Henderson's chapter, and then analyze your
     moves in the audiotape. Is there anything you might do differently next time?

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This chapter includes a report of interesting work dealing with varying examples and with teaching geometric terms which have an everyday meaning ("vertical," "adjacent," for example).

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

GOALS THROUGH DISCOVERY LESSONS

DISCOVERY--WHAT IS IT?

Like love, discovery is a many-splendored thing. "The" discovery method is unfair phraseology; "a" discovery approach is accurate. Most teachers would not quibble with defining discovery like this:

Students have discovered a result

--when they had no sure-fire algorithm to find the result, but

--they did find the result, without being told or finding it in an outside source.

The sequence of events in a discovery lesson is usually based on the scientific method, without the steps being enumerated or memorized:

1. Problem defined.
2. Data gathered, processed, organized, analyzed.
3. Conjecture formed.
4. Conjecture tested further.
5. Conjecture (possibly) verbalized.

The key difference in uses of the word "discovery" lies in the nature and extent of the teacher guidance at the steps of the sequence, most often at step 2 in mathematics lessons. The diagram below shows some broad categories of approaches.

	Pure discovery	Degrees of guided discovery		Pure exposition
Teacher role:	Does nothing (literally)	States problem	States problem, gives guidance	Does everything
Student role:	Defines problem, attempts solution	Attempts solution	Following guide, attempts solution	Listens (?), takes notes (??)
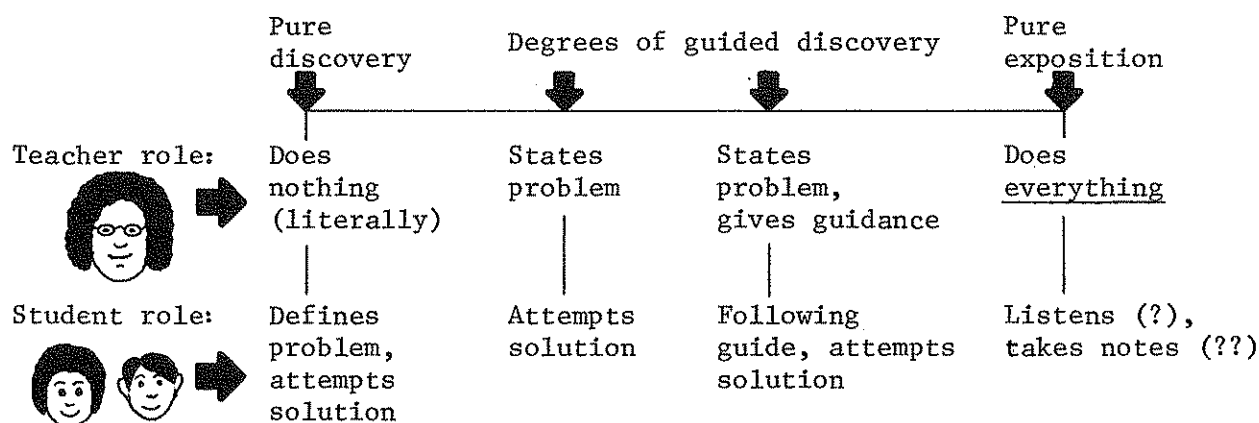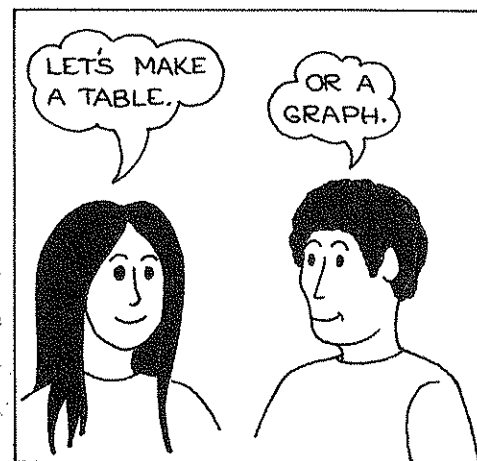
The two extremes seem to have little to offer in the middle school classroom, except in very small doses. Hence, our use of "discovery" here will actually be short for "guided discovery."

Excerpts from four lesson plans on Euler's formula--

Mr. McIntosh: . . . Hold up regular polyhedra; point out faces, vertices, edges; write V + F = E + 2 on board; check with a couple of the solids, . . .

Ms. Winesap: . . . Use the page *Vertices, Faces and Edges* but add V + F column, E + 2 column, . . .

Miss Jonathan: . . . Use the page *Vertices, Faces and Edges* in small groups (copy for each stu-dent). Will need 6 sets of solids . . .

Mr. Applebaum: . . . Say nothing. Write on board, "Is there a relationship among V, F and E for regular polyhedra?" Pass out solids, sug-gest they trade after studying. Let work 10-15 minutes, then perhaps small-group it. Remind Jannetta and Cleveland not to shout out the answer.

Mr. McIntosh's lesson (so far) is <u>not</u> a discovery lesson, whereas the other three are. Ms. Winesap may be dealing with students who are slower or are inexper-ienced with looking for patterns. Miss Jonathan may be using a discovery lesson to promote student-student interaction. Mr. Applebaum's class apparently is very exper-ienced with discovery approaches (and presumably familiar with the vocabulary and no-tation). Depending on the learning from earlier, more guided lessons to carry over, he has defined the problem but has given his students <u>no</u> advice on how to proceed.

When you use a discovery approach with your class, you will have to assess their reactions to discovery approaches. As your class experiences success with <u>guided</u> dis-covery lessons, you can try to offer less guidance. Many of the discovery lessons in this resource can be altered to fit the background of your class, as the teachers have done above. For another example, see *Pythagoras on the Geoboard* in AREA & VOLUME: Pythagorean Theorem. (The pages from this resource which incorporate some degree of discovery are listed starting on page 14.)

DISCOVERY--WHAT GOOD IS IT?

Let's dispose of one thing: guided discovery lessons are not intended to add new facts to the <u>world</u>; the knowledge <u>will</u> be new to the <u>learner</u>. And, if transmitting knowledge were the only criterion, discovery lessons would be close to the bottom of the list of fast strategies. Discovery takes time. Even staunch advocates of discovery acknowledge that we can't teach <u>everything</u> by discovery if we are to "cover" the usual body of material. But discovery approaches may offer unique ways of reaching some goals. That is, discovery lessons may produce "by-products" more valuable than the particular result discovered. Here are some sample claims for discovery lessons.
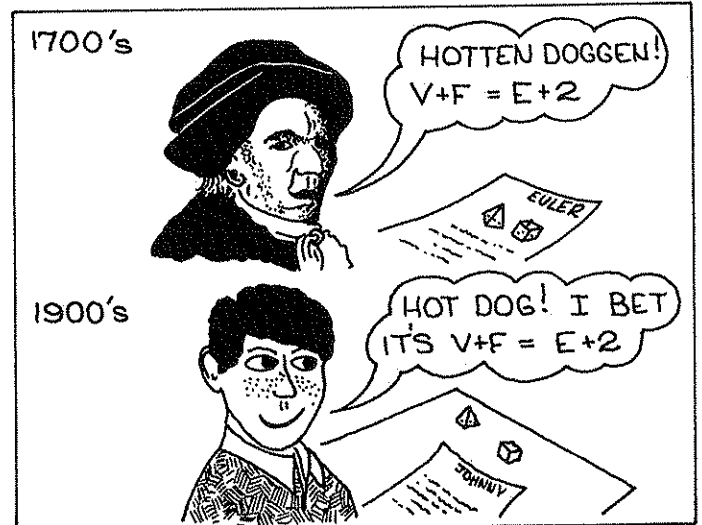
1. <u>Discovery lessons establish a different atmosphere</u>. The student is <u>not</u> dependent on the teacher to "show-and-tell" everything as the only source of knowledge in the classroom. The students themselves are actively--and often excitedly--involved in finding out results for themselves. Bruner feels that mastery of a field includes ". . . the development of an attitude toward learning and inquiry, toward guessing and hunches, toward the possibility of solving problems on one's own." [1963, p. 20] Discovery lessons afford opportunities for these attitudes to develop.

2. <u>General problem-solving strategies can be given explicit attention</u> as a result of discovery lessons. For example, Mr. Applebaum's students surely will look at specific cases, tabulate the results and look for a pattern--all valuable activities in general problem solving. Our students will be functioning citizens in the year 2000. Since we don't know what problems they will be facing in 1984, let alone 2000, perhaps the best things we can supply them with are some general approaches to solving problems. (See *Problem Solving* in the TEACHING EMPHASES section for more on general problem-solving strategies.)

3. Small-group collection and organization of data and conjecturing about the results give students <u>the chance to hear the thinking of peers, to contribute to a collective effort, and to practice communication skills</u>. The students will receive valuable reinforcement from the peer discussions and may be more willing to offer their ideas in a small group.

4. Many discovery lessons disguise drill with basic skills like computation, measurement, and construction. These skills are the tools for tackling the "larger undertaking" and are not the primary focus, yet the students are practicing them. Discoveries are not made in a vacuum, and drill is often involved in generating the data to be analyzed. For examples, see *Part of a Line* in LINES, PLANES & ANGLES: Lines (practice in mental arithmetic); *A Pair of Angles in the Arc* in the Circles subsection (drill with protractor); *Simson's Line* in POLYGONS & POLYHEDRA: Polygons (construction practice).

The four points above could be joined by many others; articles by Davis [1966, 1967] and Bruner [1961] list additional possible by-products of discovery lessons. For instance, Bruner [1961] identifies these further advantages of discovery lessons:

(a) The content learned is more "intellectually potent." Since the student has been actively involved in figuring out the result, it should be more meaningful than a result which has been only memorized.

(b) Similarly, the student's active involvement should give longer retention or, since the student figured out the result once, the potential to rediscover a hazy result.

(c) Since discovery lessons allow students to learn things on their own, this greater control may be quite motivating and lead to a shift from learning for extrinsic rewards (like grades) to learning for intrinsic rewards (like this increased sense of intellectual power.

Limitations of discovery approaches. The time drawback was mentioned above, although avid practitioners claim that the students get faster and faster as they get more experience. Ausubel [1968] points out that discovery can be "rote discovery;" for example, the student may be given so much guidance that he could fail to arrive at the result only by falling asleep or breaking his pencil.

Some students are slower than others. One way to handle this to to have different groups of students working on different lessons with differing amounts of guidance. The next day, say, you could let the students explain their lessons to each other.

As with most "good" techniques, discovery lessons may not fit all students. At first, some may not be able to function well without the usual teacher direction and will need constant assurance or additional hints. The impulsive student may jump to invalid conclusions much of the time. Although there is no research evidence, discovery lessons may help such students become less teacher-dependent or less impulsive.

A final issue lies in the <u>choice of content</u> to be discovered and is particularly important for out-of-class discovery lessons. Quite often additional data or a new case may be needed. Some think that the <u>student</u> should generate this new information. Doing this tests whether the student is overgeneralizing by applying the conjecture to a case it doesn't cover. Some discovery lessons (e.g., *Some Can, Some Can't* in the Circles subsection) may involve arriving at a concept by looking at examples and non-examples; for such content the student <u>cannot</u> generate a new case and test it without appealing to an outside authority. On the other hand, statements like Euler's formula present no problems since the student <u>can</u> easily produce some solid figure to test. Hence, some teachers avoid out-of-class discovery assignments when the students cannot produce new data for themselves.

DISCOVERY LESSONS--HOW DO I DO THEM?

Studying the discovery lessons throughout this resource will give you several ideas at the Ms. Winesap/Miss Jonathan level of guidance. In addition, you can often spot places in the course of other lessons where discovery-type activity can be worked in ("Oh, are you suggesting a pattern here?" "Notice anything about these four examples?" "How would you test that idea?"). For example, the extension suggested in *The Plane Truth* in LINES, PLANES & ANGLES: Planes illustrates building a discovery question into a teacher-led discussion. See number 11 on page 11 for sample teacher comments which enable us to teach without telling everything.

Here are some points to keep in mind when planning a discovery lesson.

<u>Have the objectives clearly in mind</u>. These usually fall into two areas: The specific <u>content</u>, and exposure to the expected or planned <u>problem-solving strategy</u>. For example, one teacher's objectives for the *Vertices, Faces and Edges* lesson might have been "The student practices vocabulary in deriving Euler's formula" and "The student seeks a pattern in a table of data." Of course, some topics do not lend them-

selves to discovery lessons, and a particular problem-solving strategy may not "fit" a particular problem. The problem can be stated directly (see *Vertices, Faces and Edges*, problem 2) or by means of a "target task" [Wills, 1970, p. 282]: "What is the sum of the measures of the angles of a 102-gon?" or "How many segments can you get from 1000 points (no 3 collinear)?" Let us quickly note that

ALTERNATE FORM FOR TARGET TASK FOR <u>CHORDS IN CIRCLES</u> (CIRCLES SECTION)
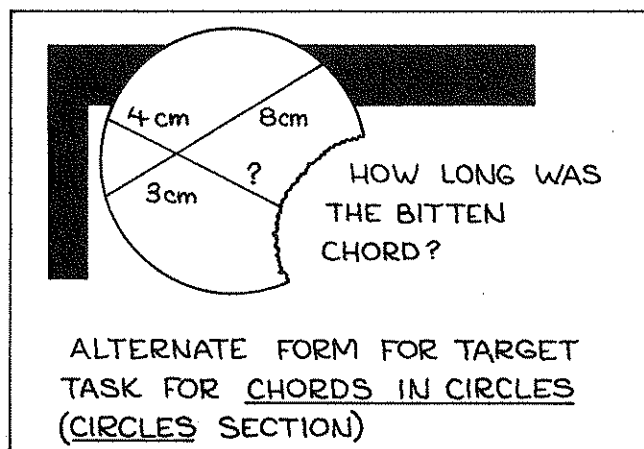
the first time you use a target task, be sure that the students don't leave the room
thinking they've got to draw a polygon with 102 sides and then measure all 102 angles!
As you know, many students need considerable guidance with something new.

Decide on the degree of guidance. Mr. Applebaum's approach—just state the
problem—might have more success after the students have had exposure to problem-
solving strategies in, say, the context of guided lessons like those in this resource.
As one teacher put it, "Don't expect the kids to discover Pluto if they can't see the
moon!" After experience and after attention to strategies, less guidance and more
student generation of data are possible and desirable. Is the teacher going to be
available to supply the data in the year 2000?

How will the students test their conjectures? There are at least these choices:

(a) have the student test the conjecture on a new example, either supplied
by you or generated by the student;

(b) have the student ask you (why is this not in the full spirit of a dis-
covery approach?);

(c) use a target task (see above) and have the answer to the target task
recorded in a specified place;

(d) compare with the conjectures of other students (be sure the students
know that everyone could be wrong, even if they all have the same conjecture).

Plan the follow-up carefully. It is during the follow-up that attention can be
focused on the strategies that were used; most often these will be "look at several
cases," "organize the data," "look for a pattern" or "think of a simpler problem."
(See the *Problem Solving* teaching emphasis for a list.) Many students won't learn
these by osmosis but will need specific instruction and practice on making a table,
generating examples, seeking patterns. After some experience with strategies, a
follow-up question like "What strategy was used here?" is an appropriate reminder
that attention to strategies is an important concern. Quite often the follow-up can
include an extra or optional activity (these are especially useful for fast finish-
ers). For example, *Vertices, Faces and Edges* could be followed by "Do you think this
formula works for polyhedra that we haven't considered?" or by "See whether it works on
the soma cube pieces" or by *Euler's Formula Again* in POLYGONS & POLYHEDRA: Polyhedra.

At some time be sure to establish the discovered result and the major points of
the lesson—i.e., establish "closure." The desired concept or generalization may be
slow in coming for some students. We should not leave them dangling indefinitely.
A sense of success is important. One research finding tentatively indicated that a
lack of success at a discovery task (without closure) may inhibit later discovery
performance. [Scandura, et al., 1969]

DISCOVERY LESSONS--OTHER POINTS

1. <u>Ease in.</u> Students who have always had the teacher show them what to do will flounder at first. One of the best lessons students may learn is not to be so dependent on the teacher. Some students may try to lead you into making the discovery for them; resist! To give some experience at forming conjectures from numerical data you might play "What's My Rule?" (leader thinks of a rule like "add 8," players give "input" numbers, leader gives "output" numbers, players are to guess the rule).
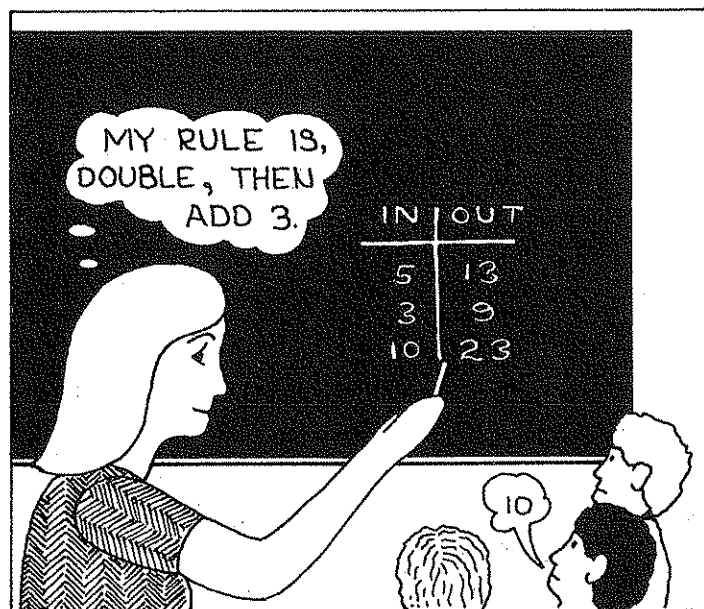
In early discovery lessons be ready to give extra guidance and encouragement, plan to give attention to strategies . . . and don't expect too much. There usually aren't any miracles the first few times around.

2. "Conjecture" is a useful word, but "<u>educated guess</u>" or "prediction" may be more useful at times. You will want to establish the tentative nature of the conjectures. Follow up an incorrect guess to illustrate how to test a guess.

3. Data organized in the natural 1, 2, 3, . . . order make it easy for students to predict the <u>next</u> line from the case(s) directly above (vertical prediction) without even looking at the "input" number. In most cases, however, we are interested in predicting <u>from</u>

in	out
1	→ 7
2	→ 15
3	→ 23
4	→ 31
5	→ 39
6	→ ?
⋮	
100	→ ?

the input number (<u>horizontal prediction</u>) since that allows us to <u>skip</u> lines. The tendency to predict vertically <u>could</u> be avoided by arranging the data in random order (which happens naturally in most "What's My Rule?" games), but since the natural order is more desirable in the long run, some remarks or some horizontal arrows may establish the search for horizontal prediction.

4. <u>Be open to the student's conjectures.</u> Give constant encouragement. Compare the teacher statement "Does it check in all the cases?" with "No!" Supplying new cases or shunting the judgmental role ("What do you think of Denise's idea?") can help you maintain a low profile as the only arbiter of correct/incorrect.

5. Have some way of keeping the fast finisher from giving the discovery away. The most common ways are whispered conferences with you or checking the answer to a target task or a new example. You might have an early finisher prepare a new example to present to the class. Using different lessons with different groups has already been mentioned.

6. A student may get a mind set on a table of data and not be able to form a conjecture or alternate conjectures if a first idea does not work. Depending on the situation and the student, you might (a) suggest that he "sleep on it," pointing out that sometimes people get so set on one idea that it's hard to get away from that idea or (b) give him really big hints ("What is V + F here? How does that compare with E? V + F here? E? . . .).

7. Most middle school discovery lessons will involve looking at specific cases and using them to guess a general result—that is, inductive reasoning. IMPORTANT: *Students must learn that inductive reasoning is not 100% reliable.* Just because the cafeteria has served pizza every Friday so far does not assure they will next Friday. Students must also learn the danger of jumping to a conclusion on the basis of only a few cases. See number 9 on pages 10 and 11.

8. Students may not be able to verbalize their discoveries very well. There is agreement that a student can indeed have a good idea but be unable to put it accurately into words ("nonverbal awareness"). Trying new examples or a target task can give evidence on whether the student's idea is all right without requiring the verbalizing of the conjecture. Asking for a verbalization of what should be an uncomplicated result (after evidence that the result has been reached) is a reasonable request and may fall under an objective like "can communicate mathematical ideas." If a result is hard to put into words, you might stick to nonverbal evidence.

9. Make loud and clear that the problem-solving *strategies* are important. Point out that how the students approach a problem may be the thing of greatest value. As you know, a quick "answer" may be the important thing to the students. One fast way to communicate that strategies are important is to put a question about strategies on a quiz!

$?\ ?\ ?\ ?\ ?$

1. (Discussion) Bruner seems to believe strongly in the motivating effects of discovery: "I have observed a fair amount of teaching in the classroom: not much, but enough to know that a great deal of the daily activity of the student is not rewarding in its own right . . . . It is not surprising then that it is necessary to introduce a series of extrinsic rewards and punishments into school activity—competition, gold stars, etc.—and that, in spite of these, there are

still problems of discipline and inattention. Discovery, with the understanding and mastery it implies, becomes its own reward, a reward that is intrinsic to the activity of working." [1960, p. 613]
Describe your feelings about and experiences with intrinsic and extrinsic rewards with middle school students.

2. Give possible times, if any, when each of the following might be used with middle school classes.
    a. pure lecture   b. pure discovery

3. (Discussion) Should small groups for discovery lessons be made up of students of similar abilities or of widely different abilities? Why?

4. Write target tasks for each of the following:
    a. *One Way Ray* in LINES, PLANES & ANGLES: Lines

    c. *Sum Thing* in POLYGONS & POLYHEDRA: Polygons
    d. *The Lost Chord* in the Circles section

5. Write a discovery lesson . . .
    a. based on *In 'N Out Regions* in the Circles section, but with more guidance toward the general formula. (Note that this formula is very difficult to figure out without guidance but that "vertical prediction" is relatively easy.)
    b. based on the second part of *Are You Right All the Time?* (Circles subsection) but with less guidance.
    c. based on idea 3 in *Puzzling Pennies* (Circles subsection) and using a target task. As in number 9,   there are reasonable but incorrect patterns suggested by simpler cases.

7. It is handy to have a few samples of problems which
    yield data that do _not_ seem to suggest any reason-
    able conjecture. For example, no one knows a for-
    mula for predicting the number of different
    polyominoes of n squares. Plan how you could use
    this information with a class. (See *Hexiamonds
    vs. Polyominoes* in POLYGONS & POLYHEDRA: Polygons.
    Is there a formula for n-iamonds?)

-ominoes	number
1	1
2	1
3	2
4	7
5	12
6	35
7	108

8. Discovery lessons in which time restricts each student to testing only one new
    example gives an opportunity to bring up the risk associated with basing a con-
    jecture on one example. Explain how you might make this point in using one of
    the following: *Morley's Discovery, An Equi-Meeting,* or some part of *Special
    Lines and Segments in Triangles* (all in POLYGONS & POLYHEDRA: Polygons).

9. The first few cases in each of these situations invite a reasonable but erroneous
    conjecture. Predict the next case in each and then test it. What point about
    inductive reasoning could be made in class with such examples?
    a. What is the greatest number of pieces you can get with straight cuts across
        a pizza (size of pieces doesn't matter)?
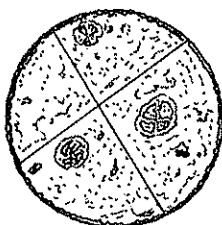
# of cuts	maximum # pieces
0	1
1	2
2	4
3	?

    b. Does the formula (n x n) - n + 41 always give prime numbers?

n	(n x n) - n + 41
1	41 (prime)
2	43 (prime)
3	47 (prime)
⋮	⋮
40	1601 (prime)
41	?

    c. Suppose you cut a large piece of cheese with straight cuts. What is the
        greatest number of pieces you can get with 5 cuts? (Polya)

# of cuts	maximum # of pieces
0	1
1	2
2	4
3	8
4	?
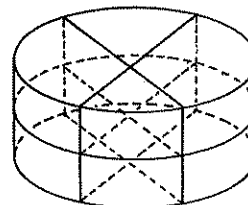
3 CUTS

d. Target task: $1111111111^2 = ?$

$$1^2 = 1$$
$$11^2 = 121$$
$$111^2 = 12321$$
$$1111^2 = 1234321$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$111111111^2 = 12345678987654321$$

e. $\dfrac{1}{2} - \dfrac{1}{3} = \dfrac{1}{6}$ 　　　　$\dfrac{7}{3} - \dfrac{7}{4} = \dfrac{7}{12}$ 　　f. The pattern from idea 3 in *Puzzling*

$\dfrac{4}{9} - \dfrac{4}{10} = \dfrac{4}{90}$ 　　　$\dfrac{3}{2} - \dfrac{3}{4} = ?$ 　　　　　 *Pennies* (in the Circles subsection).

$\dfrac{5}{7} - \dfrac{5}{8} = \dfrac{5}{56}$ 　　　[Brown, 1971]

g. $\dfrac{1}{6} = \dfrac{1}{7} + \dfrac{1}{42}$ 　　　$\dfrac{1}{8} = \dfrac{1}{9} + \dfrac{1}{72}$ 　　h.

$\dfrac{1}{5} = \dfrac{1}{6} + \dfrac{1}{30}$ 　　　$\dfrac{1}{4} = \dfrac{1}{6} + ?$

93	39
x26	x62
32	23
x46	x64
82	28
x14	x41
47	74
x53	x35

10. (Discussion) Davis [1966] uses the term "torpedoing" to describe giving the
    students the opportunity to make reasonable false conjectures (as in number 9).
    Besides making students less trustful of inductive reasoning, Davis feels that,
    having made a conjecture that "almost" worked, the student will be spurred on
    to try to find one that does.
    a. Discuss this idea in terms of stronger students.
    b. Discuss this idea in terms of weaker students.

11. Here is a list of questions used in discovery lessons. [Johnson and Rising, 1972,
    pp. 180-181] Think back over, or listen to a tape of a recent lesson of yours
    to see whether you are using similar questions in your daily teaching.

    Give me another example.　　　　　　That seems to work. Will it always?
    Do you believe that, Bill?　　　　　Have we forgotten any cases?
    Will that work with fractions, too?　What do you mean by that?
    How do you know that?　　　　　　　Say that another way.
    Are you sure? Let's test a zero.　 How can we simplify this?
    Can anyone find a case for which　　Can you make a rule that a sixth grader
        John's rule doesn't work?　　　　　could follow?
    Alice, can you convince this　　　　Why do you and June disagree?
        majority that they're wrong?

12. (Discussion) The possible advantages of discovery lessons cited by Bruner
     (p. 4) may seem too "ivory-towerish." What does your experience with discovery
     lessons suggest about these advantages with middle school students?

13. (Discussion) Suggest some geometry results that do not seem to fit a develop-
     ment by discovery. Does everyone agree about the suggestions?

14. What sort of question could be asked about strategies on a quiz? Here are a
     couple: "What strategy did you use in problem 4?" "What are some ways to
     organize information?"

15. (Review) Outline a series of activities involving circles which would take
     into account van Hiele's first two levels of thinking (see *Planning Instruction
     in Geometry* in the section LINES, PLANES & ANGLES).

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    *Drawn from a book written a half-century ago, this article describes discovery
    teaching. Like the Jones article, it shows that discovery methods are not an
    innovation.*

CLASSROOM MATERIALS INVOLVING SOME DEGREE OF DISCOVERY

LINES, PLANES & ANGLES:

Lines

PART OF A LINE

GEOBOARD I

LINE SEGMENT GAMES FOR TWO

ONE WAY RAY

WHAT'S MY LINE?

BE BOLD---FOLD!

GEOBOARD II

Planes

THE PLANE TRUTH

PICTURE THIS: $\square = 3\triangle + 2$

Angles

ANGLES, NOT ANGELS

ARE YOU CONVINCED?

I'M SEEING STARS

ANGLE WRANGLE

Symmetry and Motions

A MIRROR REFLECTION

ANGLES AND FOLDS

POLYGONS & POLYHEDRA:

Polygons

THESE ARE. THESE ARE NOT.

HEXIAMONDS VS. PENTOMINOES

TESSELLATIONS

THE ANGLES IN TRIANGLES

SUM THING

WHEN IS A TRIANGLE ACUTE, RIGHT OR OBTUSE?

SIDE, SIDE, SIDE.   ANGLE, ANGLE, ANGLE.

SPECIAL LINES AND SEGMENTS IN TRIANGLES

EULER FOUND

NAPOLEON'S IDEA

AN EQUI=MEETING

MORLEY'S DISCOVERY

SIMSON SAYS

I DON'T BELIEVE IT!

INTERIOR ANGLES OF A POLYGON 1

INTERIOR ANGLES OF A POLYGON 2

DIAGONALS IN POLYGONS

PATTERN BLOCKS II

PATTERN BLOCKS III

REFLECT ON THIS

THE TURNING POINT

MEDIAN OF A TRAPEZOID

STRICTLY SQUARESVILLE

SQUARESVILLE SUBURBS

Polyhedra

    SURVEYING SOLID SHAPES

    YOU DECIDE..

    GEOBLOCKS II

    FOLD-UPS

    DO YOU KNOW THAT

    VERTICES, FACES AND EDGES

    HOW MUCH IS LOST?

CURVES & CURVED SURFACES:

    Circles

        PUZZLING PENNIES (number 3)

        THE LOST CHORD

        A STARTLING DISCOVERY

        IN 'N' OUT REGIONS

        ARE YOU RIGHT ALL THE TIME?

        SOME CAN, SOME CAN'T

        CHORDS IN CIRCLES

        A PAIR OF ANGLES IN THE ARC

        FINDING THE RIGHT ANGLE

        SPECIAL QUADRILATERALS

    Other Curves

        SPIROLATERALS

SIMILAR FIGURES:

    HOW DOES YOUR BILLIARD TABLE SIZE UP?

    ENLARGEMENTS OF ENLARGEMENTS

MAKE IT EASY ON YOURSELF

PROPORTION-ALL

SPLITTING ANGLES

PAIRING UP

AREA & VOLUME:

Perimeter

PERIMETER PATTERNS OF POLYGONS

Area

AREA FORMULAS ON THE GEOBOARD

BE PICKY ABOUT YOUR GEOBOARD

$2 \times 2 = 4$

BIG FOOT

A SHEEPISH PROBLEM

LOTS OF LITTLE ONES

Pythagorean Theorem

PYTHAGORAS ON THE GEOBOARD

Surface Area

THE SIX FACES OF C'S

STAIRCASES

SPACE STATIONS

INK YOUR PAD

THE CUBE PAINTER RETURNS

PAINT-LESS

CHANGING SURFACE AREAS 1

CHANGING SURFACE AREAS 2

Volume

THE PAINTED CUBE

VOLUME WITH CUBES

LAYER UPON LAYER

VOLUME WITH GEOBLOCKS

EUREKA, I'VE FOUND IT!

NESTED SPHERES

CHANGING VOLUMES 2

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

QUESTIONING

> *Questioning is the heart of teaching.*
> Hoover [1973, p. 78]

Questioning certainly is one of our most versatile and most used tools. Questions[*] can be used . . .

— to motivate a topic: How do photographers make enlargements of photographs? How could you make an enlarged drawing of your hand?

— to challenge: Can you calculate the length of a diagonal of this solid cube?

— to provoke student interaction: Is your method the same as your neighbor's?

— to get students to evaluate: How do you think your method would work on other problems?

— to focus on process: What approach did we use on this problem?

— to guide: What was the scale factor in this drawing?

— to diagnose: How did you get that, Milly?

— to review or summarize: What were the big ideas from yesterday?

— to evaluate: Everyone draw a diagram explaining what the Pythagorean theorem says.

— to communicate interest in a student: Did you watch the game last night, Gail?

— to encourage exploration: Does the center of the enlargement have to be outside the figure?

— to invite student questions: What question does this information make you think of?

— to retrieve wandering attention: Willy . . . $3^2 + 4^2 = x^2$, would you take it from there, please?

— to enhance transfer: How could you use the result in a situation like this one?

This section summarizes some areas of concern about classroom questioning and reviews some "dos" and "don'ts" for using this valuable teaching device. Your primary aim might be to consider *how to use questions to promote learning as well as to evaluate whether students know specific information.*

---

[*]The term "question" will be used to include instructions like "Figure out the length of the hypotenuse" as well as interrogative forms like "What is the length of the hypotenuse?" Procedural questions like "When is the assembly?" or "Who's absent?" are not included.

QUESTIONS AT DIFFERENT LEVELS

To add a little substance to the modifiers "lower level" and "higher level" for types of questions, consider the outline to the right. "Lower-level" questions are those which call for exercise of memory only. They involve recall or recognition of specific facts, cases or routine procedures ("What is 9^2?" "What is the sum of the degree measures of the angles of a triangle?" "Make an enlargement with the scale factor = 2").

"Higher-level" questions, on the other hand, call for more than memory. Here are some examples.

Lower level

  Recall/recognition

Higher levels

  Represent/interpret information

  Explain
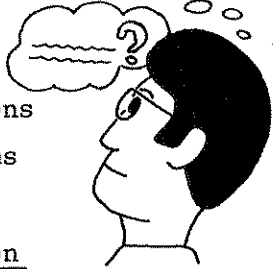
  Analyze, interrelate and apply information

  Open search

    --for solutions

    --for problems

Opinion/evaluation

Represent/interpret information   "Make a graph for those data." "What scale factor was used in that reduction?"

Explain   "How can you tell whether you can use the Pythagorean theorem or not?" "How can you tell the difference between a 'shrink' and an enlargement?"

Analyze, interrelate and apply information   "What pattern do you see?" "How is an enlargement like a shrink?" Word problems which require the students to analyze the data and interrelate them so that they will know what equation or operation to use involve questions of this sort.

Open search   These two sub-levels exist within open search:
    (a) The students are exposed to a well-defined situation never encountered before but for which they must find a solution.
    (b) The students are exposed to a non-structured situation and invited to make their own investigations.

The extensions in *How Does Your Billiard Table Size Up?* (in SIMILAR FIGURES) give examples of (a) as would the question, "What might it mean if the scale factor is negative?" (before *A Negative Feeling* in SIMILAR FIGURES). Handing a student a pantograph without instruction and saying, "See what you can find out about this," would be an illustration of (b). Telling students to investigate Pythagorean triples would be another example.

There seems to be a separate category not covered in the above. Questions in which we invite the students to give their opinions or evaluations do seem to have merit. "Which way do you like to make enlargements best?" (probably followed by

"Why?") and "Why do you think this fact is worth having a special name--the Pythagorean theorem?" are samples of what might be called opinion/evaluation questions.

Why should a teacher be concerned about these levels? "When a teacher asks a question, he is giving a student the opportunity to use his mind." [Carin and Sund, 1971, p. 2] Critics, however, point out that we may be giving students the opportunity to use only part of their minds--the part devoted to memory. From classroom observations, it appears that perhaps more than 80% of teachers' questions call for recognition or recall types of thinking. [Hoetker and Ahlbrand, 1969; Gall, 1970] The lament is not that low-level questions are being asked; rather it seems so few questions are asked that "make you really think"--i.e., higher-level questions. In one study with geometry teachers, the researcher identified only 24 questions out of 1841 (1.3%) as being "higher-level" questions! [Friedman, 1973--his definition of "higher" may have differed from ours.] Contrast these two sets of hypothetical questions:

Questions from
Mr. Denson

"What is the name of the theorem?"

"So 3^2 + what = h^2?"

"Sara, what is 8^2?"

"The kind of triangle this works on is a . . .?"

"Okay, Jack, what is the hypotenuse?"

Questions from
Ms. Jackson

"What did you find out about Pythagoras?"

"What equation would you write?"

"How would you estimate $\sqrt{71}$? . . . Sam?"

"When would the Pythagorean theorem come in handy?"

Notice that Mr. Denson's questions could be answered by "low-level" thinking. We must acknowledge that students should be able to retrieve facts from their memories. However, as Hoover notes, ". . . concentrating on memory neglects other intellectual processes learned through practice." [1973, p. 78] Do Ms. Jackson's questions require anything more than memory? It is impossible to say for certain without knowing what has gone on in her class, but it appears likely that her questions could require processing of information in addition to retrieval of information.

Although it takes work and preparation, teachers can come up with higher-level questions. Concern with questions is not a recent fad: Smith and Reeves recommended that a "good lesson plan should contain about eight or ten of the most searching and stimulating questions that the teacher can ask in connection with the subject under

consideration." [1927, p. 292] This may seem unduly ambitious, but it indicates the importance that has been attached to classroom questions for at least 50 years.

What can a teacher do who wants to increase the number of higher-level questions asked?

1. Devote time to thinking about (8 or 10?!) questions. Questions like "Does this work for other figures?" (see *Look Alike Figures* or *Splitting Sides* in SIMILAR FIGURES) or "Does this happen if the figures are not similar?" (see *Pairing Up* in SIMILAR FIGURES) are not hard to generate with a few seconds' thought.

2. Sanders [1966, p. 157] recommends that teachers immerse themselves in a topic before planning even a single question for a unit on the topic. He feels that to teach on the memory level, not much more than is in the textbook is needed, but that much more background is needed to ask higher-level questions.

3. A third way is to focus on a classification of levels of questioning (like the one above). Most books about questioning are organized in this way (Sanders, 1966; Carin and Sund, 1971; Hunkins, 1972; see also Schmalz, 1973). Quite often the same classification system that is used for objectives and/or formal evaluation is used. For example, the section *Broad Goals and Daily Objectives* in Ratio, Proportion and Scaling, Mathematics Resource Project, mentions the facets in Figure 1. Study of a category like "open search" could give a teacher some ideas for open search questions.

OPEN SEARCH	
GENERALIZATIONS, ALGORITHMIC THINKING	ATTITUDES, FEELINGS, APPRECIATIONS
RECALL, RECOGNITION	

FIGURE 1
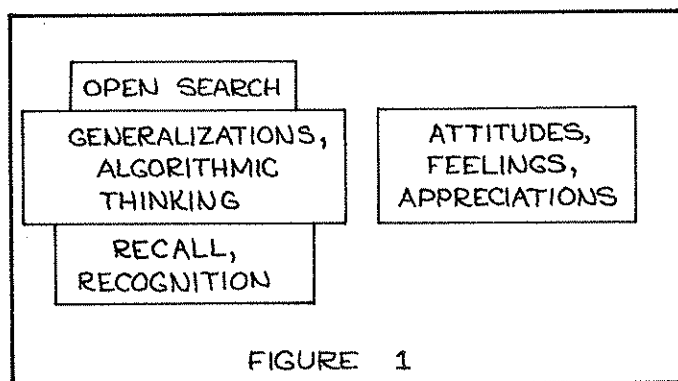
GETTING STUDENTS TO ASK QUESTIONS

Open a first or third grade to questions, and you will be deluged; fifth-graders say nothing . . . Curiosity, questions, speculation--these are outside school, not inside. Holt [1964, p. 157]

How many times have you been witness to a classroom situation in which a student is ridiculed because of a "silly" question? How many times have you seen a student virtually destroyed verbally because he has asked a question that was just answered by the teacher the moment before? Fremont [1969, p. 69]

Perhaps one of the greatest condemnations of the teaching profession as a whole is that we stifle student comments and questions. In many classes it is hard work to get students to be confident enough to express their feelings. This is a result of previous exposure to teachers who were not good listeners and who did not have sufficient confidence in their ability to respond to these comments in a satisfactory way. It takes an extremely confident person and a relaxed atmosphere in the classroom for meaningful student participation in the teaching-learning process. Bassler and Kolb [1971, p. 206]

1. "If we are genuinely convinced that it is imperative for students to feel free to ask questions, then we must maintain an atmosphere conducive to questioning, an atmosphere that will allow each child to feel completely free to ask whatever may be of concern to him. One sure way to help develop such a climate is to honor each and every question posed—no matter how often repeated and no matter how minute a point may be involved. Make the questioner feel good for having asked and everyone in your room will feel equally free to pose any question without the fear of the question being silly or holding the class back." [Fremont, 1969, p. 69]

2. Perhaps we do not reward question-asking enough. Ausubel [1968] says that the apparent decline in student curiosity may be due to the emphasis in schools on extrinsic incentives—e.g., grades, teacher commendation or other reinforcers external to the student. This emphasis causes a decline in the asking of questions; most "rewards" in school come from the answering of questions. Indeed, one effort in England to foster question-asking bewildered the students: "Questions are not answers and it is answers that count!" [Robinson, 1974, p. 415] Perhaps we should unearth some old saws or invent new ones like "A good question is worth a thousand answers." One of the potentials of discovery lessons, particularly open-ended ones, is to keep alive (or revive) student interest in inquiry. To ask "Will my idea work?" and then to find out that it does can give a sense of competence that is reward in itself. (See *Goals Through Discovery Lessons* in the CURVES & CURVED SURFACES section.)

3. Most teachers would be delighted to get more student questions and student-student interaction (about mathematics). One way is to try to maintain a "low profile" as the authority figure in the classroom. You might acknowledge a student question and refer it to the
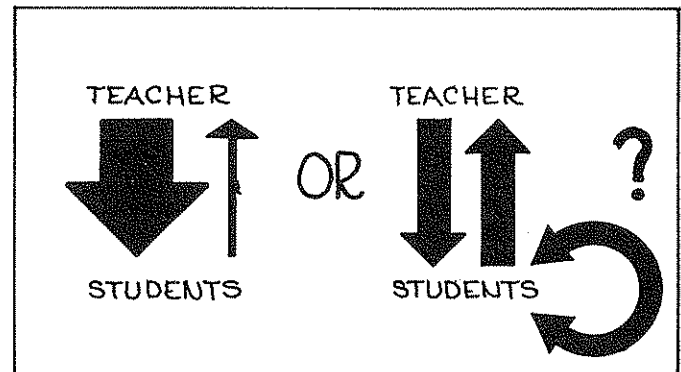
rest of the class ("Good question, Sam. Anyone have an idea?"). Or, ask the class to evaluate a non-trivial student response rather than do it yourself—this technique also communicates that you _expect_ them to evaluate work by means other than appealing to authority. Small groups in laboratory lessons might also foster more student-to-student questioning.

4. If we want the students to ask questions, we will have to give them opportunities. After some introductory work with similar figures, we could ask, "What do you think might be some interesting questions to ask about similar figures?" or more directly, "Write down two questions about similar figures that we could investigate." Early questions are not likely to be heartening ("What good are similar figures?" "Will they be on the test?") but repeated emphasis on "What question does this make you think of?" may eventually reach some students who will spontaneously start asking "Does this always work?" or "What happens if . . .?" questions. Perhaps one of the most important lessons for students is that asking questions enables them to exert some control over what they learn.

5. Perhaps we can influence the _quality_ of our students' questions by our own questions. One study showed a very high correlation (.90) between the level of _teacher_ questions and the level of _student_ questions. [Davis and Tinsley, 1967]

> _Interviewer: Do you think that raising questions is just as important as knowing how to solve them?_
>
> _Jean Piaget: Yes, I do._
>
> _I: Would there be any way of facilitating a child's tendency to ask questions? Or do you simply try not to kill it?_
>
> _P: You could facilitate it mainly through having a multitude of materials available that raise questions in a child's mind without suggesting the answers._ [Piaget, 1973, p. 23]

THREE RESEARCH FINDINGS

1. It may be that less quantity is related to greater quality. Kleinman [1965], in a small study with junior high science teachers, found that the teachers who asked the greater percent of higher-level questions were the ones who asked _fewer_ questions. (However, see number 14 on page 13.)

2. Here is a sobering finding, at least for the English teachers. Hoetker and Ahlbrand [1969] cite one study with junior high English teachers in which the teachers gave _wrong information_ in their responses to 15% of the student questions! Might this account for the scarcity of student questions? What would a similar study with

mathematics teachers show? *There is certainly nothing wrong with admitting that you don't have an immediate answer.* Try "Good question, Liz. I don't have a good answer right now. Let me see what I can find out tonight." Or see whether the class has any ideas or, depending on Liz and the situation, ask Liz to research the question-- "Excellent question, Liz, but I don't know. The answer would be valuable information to the whole class. Would you and Beth see what you can find and report to us tomorrow?" Or, "let's spend ten minutes of class time to see if we can figure it out." Let us teach with an open mind--we can learn by asking questions, just like our students, and by working with them to find answers we do not know.

3. Rowe [1969] notes that even veteran teachers allow an average of only <u>one second</u> for a student to start to answer. Her work with teachers to extend their "wait-times" to 5 seconds or preferably longer gave these results:

WHAT'S YOUR WAIT-TIME?

A ONE AND
A TWO AND
A THREE AND
. . . . .
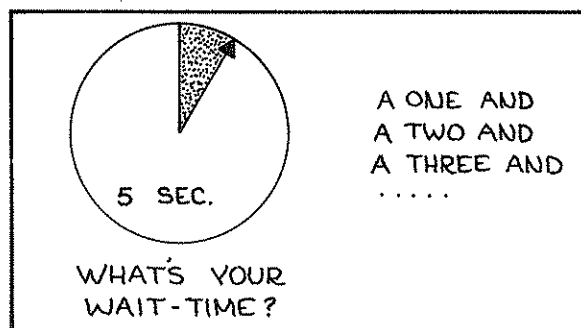
--students gave longer answers and fewer "I don't know" responses

--there was more student discussion of answers

--there were <u>more</u> student questions

--the teachers were more flexible in treating students' responses (researchers have observed that in rapid-fire questioning settings, teachers seem to look for a particular response and reject others; with a moment's thought, you often can see that a student's "wrong" answer is actually correct for a different interpretation of the question)

--there was a greater variety in the levels of teacher questions

--teachers changed their expectations of certain students (Teachers unfortunately seem to wait <u>less</u> time for students judged to be weak; in Rowe's study, waiting longer got more and better responses from these "weak" students. See the section *Student Self Concept* in <u>Number Sense and Arithmetic Skills</u>, Mathematics Resource Project.)

It could be that our pacing of questions is too fast for many students; slower students (by definition) need more time. There is also a danger of judging whole-class progress from the quick responses of a handful. Waiting a few seconds without calling on anyone won't seem strange if you let

QUESTION   WAIT   RESPONSE   WAIT   MORE TALK
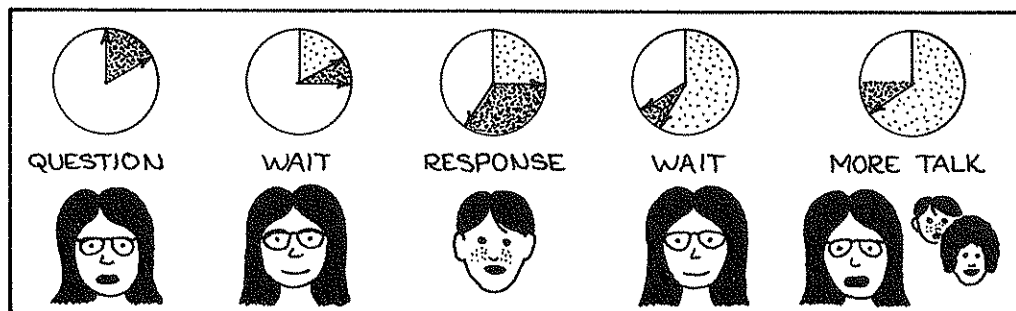
students know that you do not expect an immediate answer (and that you don't want someone to blurt out an answer before everyone has had time to think).

65

On the other side of the response, Rowe [1969] found the same average of only about one second _after_ a student's response before teachers repeated or rephrased the answers or asked another question. We _teachers_ often do not need a few seconds to think about a student's response or question, but _the rest of our students might._ Perhaps we should rehearse our nonverbal ways of communicating "I'm thinking" for use during post-response wait-time.

TRICKS OF THE QUESTIONING TRADE

Here are some aspects of getting, asking and handling questions that may or may not have occurred to you.

1. Sanders [1966] suggests that the _use of an original source_ or the "real" thing rather than a simplified drawing or description generates more student questions. Although Sanders' book is oriented toward social studies, his suggestion probably has merit in the mathematics classroom. Textbook material often cuts away extraneous but provocative material. So why use a simplified textbook map if you can get copies of "real" ones? Why look at a picture of a pantograph when you can use one?

2. Carin and Sund [1971, p. 3] recommend the _use of personal questions_ to show students that you are interested in them and to get information about their interest and backgrounds: Where were you born? Who is your favorite singer? Have you seen a good movie lately? What did you do this weekend? What do you want to become? How do you feel about that?

3. _Spread the questions around._ Call on nonvolunteers, unless you _know_ the student has no idea. Calling only on volunteers may allow a few students to dominate the student end of the class discussion and lets some students take a passive, non-contributing role.

4. To establish the sort of _accepting climate_ that Fremont refers to, some writers even go so far as to advise, "Teachers should avoid making judgments on and correcting faulty answers." [Carin and Sund, 1971, p. 15] Follow an incorrect answer or a blank look with another question that gives a hint.

WHAT IS THE SCALE FACTOR ?

?

HOW DO YOU GET THE SCALE FACTOR.

OH! ITS 3.

TEACHER   STUDENT

Depending on the class "personality," you might also choose to refer a faulty response to the class for reaction.
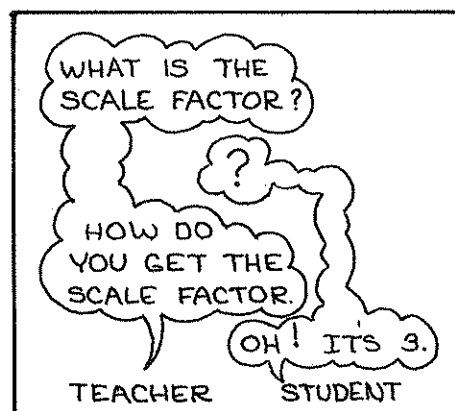
5.   Study your questions for their "tone." Are you unintentionally communicating low expectations, for example? Compare a not-too-hopeful "Was anyone able to get number 8?" with an enthusiastic "Number 8 . . . Kris?"

6.   Watch the students for visual feedback on the reaction to your questions. The few volunteers may be in good shape, but if half the class looks puzzled, you may want to rephrase the question. Similarly, listen to students' responses. Don't reject a response just because it is not the one you were expecting. Quite often an incorrect response can be refined to a correct one. An offbeat answer may lead to a worthwhile digression or provide an alternate route to some objective. Or perhaps the student has even discovered a novel approach, a fact which you would wish to point out for its impact on the class.

STOP
LOOK
LISTEN

Do be certain that you understand what the student is asking. Often students phrase questions in their frames of reference and we interpret the questions in ours. A few seconds' thought devoted to "What is this student really asking?" might help us avoid a lengthy discourse on something other than what the student was asking about.

7.   Johnson and Rising [1972] list some questions from discovery lessons. They note that questions like those below can encourage students, "are equally applicable to right and wrong answers," ". . . treat each student as a partner in the learning process," and ". . . encourage interaction between students." [p. 181]

Give me another example.
Do you believe that, Bill?
Will that work with fractions, too?
How do you know that?
Are you sure? Let's test a zero.
Why do you and June disagree?

Alice, can you convince this majority that they're wrong?
That seems to work. Will it always?
What do you mean by that?
Say that another way.
How can we simplify this?

8.   Give some sort of response to every answer. Acknowledging correct answers gives reinforcement and shows the students that you listen to and value responses. When a student does not answer at all, you might rephrase the question, present more information or say you'll come back to the student later (if you think all he needs is a little more time and fewer eyes staring at him). When a student gives a wrong (or partly wrong) answer, it is a bit more of a challenge. You can sometimes identify with the error ("I get that mixed up myself if I'm not alert") or give a qualified okay ("In some instances that's correct, but is it always so?") or ask for a class

evaluation (this last gambit should not be reserved exclusively for use with incorrect answers or it becomes an immediate cue that the answer is wrong). Carin and Sund [1971, p. 35] advise above all against using sarcasm or making a personal attack or an accusation ("Well, didn't do your homework, did you?"). Worst of all might be <u>saying nothing at all</u>!

9. Here is a list of practices to avoid for one reason or another. (Like most recommendations, the practices <u>might be appropriate in some situations</u>.)

--Avoid answering your own questions. If you get nervous when no student volunteers right off, a little more wait-time or a rephrasing of the question may get results.

--Avoid repeating your questions. Always repeating the question may breed inattention to the first statement of a question.

--Avoid interrupting a student response.

--Avoid (a) too many questions which can be answered with "yes" or "no" and (b) "fill-in-the-blank" questions like "Figures which have exactly the same shape are called . . . ?"

--Avoid repeating a student's response. If you feel that the answer wasn't loud enough for all to hear, ask the respondent to repeat the answer by speaking a little more loudly or by facing more of the class.

--Avoid always following a predictable pattern in calling on students (e.g., going down a row).

--Avoid naming the student and then asking the question. Unnamed students can then "tune out." Ask the question, pause and then call on someone. (One exception occurs when you wish to draw an inattentive student back "into" the class. In such cases you might also choose a rather easy question; there is not much benefit in causing the student discomfort or forcing a belligerent "I don't know"; he will likely know the real reason you called on him.).
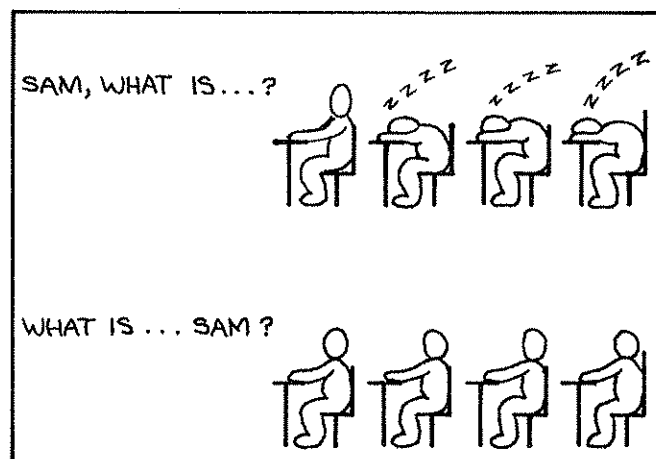
SUMMARY CHECKLIST

1. Have I been asking higher level questions?

2. Have I fostered an atmosphere that encourages student responses <u>and</u> student questions?

3. Am I allowing enough time for students to consider questions and give their answers?

4. Have I brushed up on my background lately so that I can field almost any question and can think of good leading questions?

5.   Do I react in a positive manner to student questions that I cannot answer?

6.   Am I helping to build my students' self-concepts as we interact in questioning situations?

? ? ? ? ?

1.   Evaluate the following questions. Which ones would you consider to be higher-level questions?
     a.   "Kay, state the Pythagorean theorem."
     b.   "The scale factor is . . .?"
     c.   "What happens if the center of the enlargement is moved?"
     d.   "How would you make an enlargement of a 3-D figure like a cube?"
     e.   "How do you know?"

2.   (Discussion) Think of a variety of questions to ask when using these pages from SIMILAR FIGURES.
     a.   *4 to Make 1*
     b.   *Bigger than Life*
     c.   *Similar Figures and the Pantograph*

3.   (Discussion) Why are the "do-nots" on page 10 considered to be, in general, poor practices?

4.   Background: Students have been multiplying fractions, each of which is less than 1. Write two lower-level questions and two higher-level questions to ask the students.

5.   (Discussion) What might be the consequences of each of these practices?
     a.   Asking higher-level questions during class but never on an examination.
     b.   Asking higher-level questions on an examination but not during class.

6.   Record a class session on tape.
     a.   Time your "wait-times" while waiting for students to respond to questions and after students complete their responses. Keep the records of wait-times separate for lower and higher-level questions. Compare your wait-times for "weaker" and "stronger" students (on questions at the same level of difficulty).
     b.   Note how you respond to student questions and to their answers to your questions. (Interrupting a student and repeating a student's response are fairly common; check yourself to see whether you do either.)
     c.   Are you enthusiastic when you ask questions?

7.   Some teacher remarks from a discovery lesson [Johnson and Rising, 1972, pp. 181-182] are given below.

     "Oh?"   "That's good . . . . Can anyone say
     "I see."   this another way?"
     "Do you agree, Jane?"   "Can anyone think of a case where
     "Do you think it will always   this wouldn't hold true?"
       work   "Aha!"
     "Why?"   "Can anyone suggest an example when
                                                we might run into trouble?"

     a.   List a few more teacher remarks of the same nature.
     b.   Which of these remarks do any teaching?

8. (Discussion) Wesley, in a social studies methods book, says, "The question is
    a natural expression of the thinking mind. The teacher who does not receive a
    number of unsolicited questions should seriously examine his methods." [1937, p.
    489]
    a. Do you agree with Wesley?
    b. Is it easier for students to ask questions in some subject areas than in
        others?
    c. These might be reasons for students not asking questions: they don't know
        what they don't understand, asking is tantamount to admitting ignorance, the
        teacher may respond negatively, it is embarrassing to struggle to ask a clear
        question while everyone is looking, the teacher may ask questions in return.
        What might be other causes for students not asking questions? What do these
        say to us?
    d. Outline how you might get students to ask more questions during the next week.

9. Get a record of the questions you ask during a class period (ask a colleague or a
    free student, or tape record the lesson and write the questions down as you lis-
    ten to the tape). Sanders [1972, p. 267] suggests looking at your questions with
    these four points in mind:
    a. What variety of teaching was achieved?
    b. What opportunities for good questions were missed?
    c. Were the questions appropriate for the class in terms of difficulty and
        interest?
    d. What improvements could be made in wording questions?
    And you might also note . . .
    e. What good questions did you ask? Why were they good?
    f. Were all your questions answered by only a few students?
    g. Did you call on students in several areas of the room?
    h. Did you have a sampling of questions from higher as well as lower levels?

10. Evaluate these as possibilities for Piaget's "multitude of materials" (p. 6).
    a. *Comparing Similar Solids* (in Similar Figures)
    b. *Pythagoras and Tangrams* (in AREA & VOLUME: Pythagorean Theorem)

11. One way to get students in the habit of asking questions is to require them to
    make up questions on reading material and answer them. Frase and Schwartz [1975]
    report two studies with older students in which such question-asking resulted in
    greater recall. Try this with a class for a month to see if you think it helps.

12. As noted above, Fremont suggests that we accept all student questions. How would
    you respond to these questions?
    a. "What page are we on?" (just as you have nearly finished explaining what you
        want the students to look for in a textbook assignment)
    b. "Are we going to have a quiz tomorrow?" (right after you announce that there
        will be a quiz tomorrow)
    c. "Why do we have to do this stuff?" (after you have cited several applications
        where the topic is used)
    d. "What is the date?" (when you write it on the blackboard each day)
    e. "Why can't we leave class a couple of minutes early?"

13. Behavior modification for teachers?! Sanders [1972] reports an experiment in which an electronic device placed in a teacher's ear beeped when the teacher asked a higher-level question (the device was controlled by a researcher). Even though the teacher had no instruction in levels of questioning and was told only that the study concerned the way children think, "a marked change in the desired direction" of questioning level was achieved! Suggest some alternate ways to help teachers ask higher-level questions without using a beep-in-the-ear method.

14. (Research interpretation) Recall the finding that teachers who asked the greater percent of higher-level questions were the ones who asked _fewer_ questions. [Kleinman, 1965] Evaluate the following "interpretations."
    a. If you plan to ask more higher-level questions, you will ask many fewer lower-level questions.
    b. If you ask fewer questions, more of your questions will be higher-level questions.
    c. If you ask higher-level questions, there is less time for other questions.

15. Evaluate: "Using open-ended questions that go unanswered until the next day is an effective technique to introduce new concepts and to generate interest for the next day's class." [Johnson, 1973, p. 15]

    Give an example.

16. Considerable work on questioning seems to have been done in science and social studies teaching. You might like to talk to a teacher or two in these areas to see what their ideas on questioning are.

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GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

TEACHER SELF-EVALUATION

The fact that you are using this resource means that you are already interested in enriching your teaching practices. This section outlines some areas in which to evaluate yourself and, if you feel it is needed, to improve yourself as a teacher.
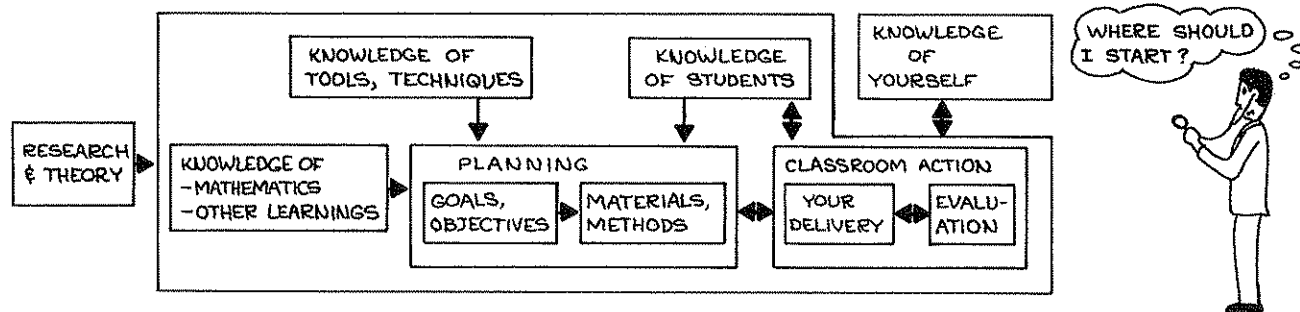
KNOWLEDGE OF STUDENTS

You may have shared a weary laugh at the jest, "Teaching would be fun if it weren't for the students." Yet, if you did not like young people, it is not likely that you would have become a teacher. Those amazing, exasperating, maturing, immature, hard-working, lazy, inspired, indifferent, kind, cruel creatures—how fascinating they are! Their marvelous diversity, their multitude of motives, their ways of thinking and non-thinking—what a challenge! If only it were the 22nd century, say, and we knew them better and how their minds work. The youth of psychology as a science and the difficulty of experimenting with humans mean that we do not know very much about teaching as a science. But progress is being made. What can we do until the breakthroughs in educational psychology occur? One thing, of course, is to keep up with psychological findings and theories which might give insights into the student's "life space" or the functioning of a student's mind. Topics from developmental psychology, social psychology and educational psychology are scattered throughout these resources (e.g., *The Teaching of Concepts* section) and may give you an orientation for further study. Observing students in your class, in other classes, in extra-curricular affairs and in out-of-school settings is fun and may reveal something worthwhile. What library book is Franklin reading? Who are Donna's friends? Does Liz act the same way in Ms. Mallory's class? (You may find Dreikurs, 1968; Crosswhite, et al., 1973; and Purkey, 1970, to be provocative reading.)

Some feedback from students is automatic through their performances in class and on tests. Every teacher is also aware of the attitudes and interests that students reveal in class and in outside pursuits. To gain more information (and to communicate concern about the students' views), many teachers periodically have students fill out questionnaires or complete open-ended phrases.

> If I were the teacher, _____ . . .
>
> The thing I like least about mathematics
> class is _____ . . .
>
> To help everyone learn mathematics, I
> would _____ . . .
>
> ---
>
> Rate this class compared to others. 10
> is highest and 1 is lowest.
>
> The lessons are interesting. _____
>
> I know what is expected of me. _____
>
> This subject is important. _____
>
> .
> .
> .

KNOWLEDGE OF
— MATHEMATICS
—OTHER LEARNINGS Your use of this resource shows that you know that mathematics
involves more than computational skills. Mathematics is a continually growing field
of study; one estimate is that of all the recorded mathema-
tical knowledge, more than half of that knowledge has been
developed in this century! Some of the important topics find

their way into the curriculum, of course. Transformation geometry, computer lan-
guages (about 30 years old), probability and statistics are some "new" topics cur-
rently seeking their places in school mathematics. How can one keep up-to-date?
The teacher commentaries and the sections of mathematical content of these resources
give a start. Inservice or summer courses, journal articles, sessions at mathematics
conferences, professional organizations, self-study or study of experimental curricula
may also help to fill in gaps or to suggest new alternatives. Particularly knowledge-
able faculty members would likely be willing (and flattered) to share their expertise
in a regular faculty seminar. There is a great deal of satisfaction in knowing that
you understand what you are trying to teach.

How about non-content areas of learning? Values clarification—what is it?
Can/should I do anything about it in my mathematics classroom? What is this talk
about "humanism" in education? How does career education relate to my classes? As
before, I might plan some time to read journals (like the April, 1976 School Science
and Mathematics issue on career education), attend professional meetings, talk with
colleagues, take a "Recent Developments in Education" course, browse in the faculty
professional book collection with books like that by Kirschenbaum and Simon [1973].

| KNOWLEDGE AND USE OF TOOLS, TECHNIQUES |

Am I using enough concrete aids? Have I given discovery lessons a fair try? Is an overhead projector worth the trouble of checking out?--perhaps I should try it. How can I get better use from those enrichment booklets in the library? Can I find some sort of game to practice mathematics vocabulary? Is there a good film on geometry? Am I really getting good results with my homework assignments? Is one textbook enough for all our seventh graders? That's how I taught it last year--is there another way? Have I done anything besides lecture lately? What was that model Dan used to show the Pythagorean theorem in his class? . . .

(NCTM yearbooks and pamphlets can give many ideas in these areas.)

| GOALS, OBJECTIVES |

If you are using this resource, you must have evaluated the frequent lick-and-a-promise treatment of geometry--several words and a few formulas--and found it wanting. That is, some new or unmet goal was not being properly served. It is valuable to reflect on what we should be trying to achieve in middle school and in middle school mathematics. Times change, and new knowledge is discovered. (See *Broad Goals and Daily Objectives* in the resource Ratio, Proportion and Scaling of the Mathematics Resource Project.)

| CLASSROOM ACTION |

A teacher naturally thinks about what went on in class. On occasion it can be helpful to have a more complete record to study carefully, particularly if the teacher is uneasy about one class or about one aspect of classroom climate. Perhaps the simplest thing to do is to audio-record a class session and listen to it for ratio of teacher-talk to student-talk, tone of voice, types of questions, handling of student responses, possibly annoying speech habits (constant "O.K.'s," "you know's," "right's," and "uh's"), clarity of explanations, source and type of student questions, . . . A tape recorder is usually available and can be relatively unobtrusive.

MAYBE I SHOULD WORK ON MY QUESTIONING TECHNIQUES.
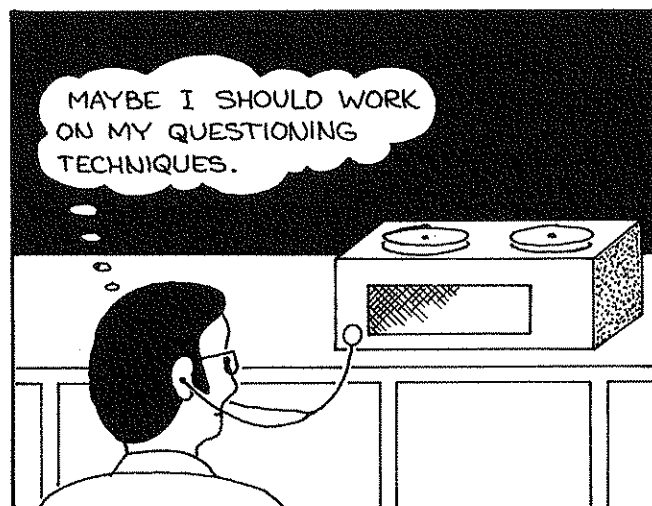
A more ambitious undertaking--but one which gives a much more complete record-- is to _videotape_ one of your classes. You should plan more than a one-day try since it may be difficult for you and the students to ignore the camera and act naturally. Perhaps you and another teacher can trade off taping each other. In a pinch, you can learn to hook up the equipment (it is easy) and a student can be the cameraperson.

Viewing the first tape may result in noticing only physical appearances--is _that_ what my friendly smile looks like? Was my hair _that_ messed up? On a second viewing or on looking at subsequent tapes, you will be more likely to notice other things: Do I really stand in front of what I'm writing on the board? My boardwork _is_ really neat looking. Don't I _ever_ look at the first row? I handled Beth's question very well. Can't I give more positive reinforcement to Willie? Why did I erase the board so fast? Does my "body language" keep some students from responding? Am _I_ talking too much? Was _that_ my only reaction to Jay's excellent answer? _That_ was a nice smile--I'll bet Hazel really felt good when she saw it. Did _anyone_ hear the assignment? Do I move around to all areas of the room during seatwork? How else could I have handled Diana's question? Was that my "explanation"? . . .

You may choose to ask a trusted colleague to view the playback and offer comments. Viewing a videotape can be humbling (one writer calls it "self-confrontation") since you may tend to notice shortcomings more than strong points; resolve to give yourself some _positive_ reinforcement! For improvement, it helps to have some narrow focus on a single area (e.g., questioning, positive reinforcement, use of examples), especially if _you_ have some specific behaviors in mind and _want_ to improve. [Salomon and McDonald, 1970]

For some reason we do not make use of an excellent and available source of people who often know what to look for in a classroom: _our fellow teachers_. Do you ask someone to read over your quizzes occasionally? Have you ever invited someone to visit your class and critique it? Have you ever visited someone else's class (even in a different subject) to observe teaching techniques, particular students, classroom atmosphere? Even in a team-teaching situation, there seems to be reluctance to invite or offer an evaluatory opinion. Perhaps it is a

matter of too little time, or our egos are too fragile, or we do not want to hurt someone's feelings. On the other hand, a colleague with an interest in some form of classroom interaction analysis (e.g., Flanders, 1970) might be willing to provide an objective report for one of your class sessions. Some departments arrange a rotating visitation schedule; host teachers naturally do their best, so the visiting teacher can perhaps pick up new techniques or ideas. A post-visit chat may or may not be part of the procedure.

KNOWLEDGE OF YOURSELF   Let's (gratefully) face it--no one is or ever can be perfect. Even the "master teachers" in your school (you may be one of them) have ideas, class sessions or even days that just do not turn out right. What we want is to have fewer of those "down" periods. When it is a matter of background or planning, we have something relatively easy to work on. If

our attitude or out-of-school demands (or the students' outlooks) are the trouble, acknowledging that fact may help in trying to change things or compensate for them. "If I'm not charismatic or witty or eloquent or one of those really nice people that most everyone likes and gets along with, that's all right. I'll just have to build on the things I am and the things I can do."

However, "to thine own self be true" does not mean to use "it's not me" as an excuse. Do I avoid discovery lessons because they "don't work for me"--or because my background is so weak I can't tell how to react to an unexpected student idea? I can do something about my background. Do I spend little time thinking about higher level questions because "the kids can't answer them"--or because I've thought about how hard it is to time the questions and give hints and suggestions to guide the students? I can do something about my planning.

RESEARCH & THEORY   By analyzing research studies, Rosenshine and Furst [1971] have identified ten areas of teaching behavior related to student achievement. The list cannot be regarded as the last word for two reasons: (1) The studies ranged over all grade levels, so possibly important areas for middle school teachers (e.g., perhaps warmth) may not have shown up at enough grade levels to be included in the general

list, and (2) the research designs, the measurements, the samples, or the analyses of many of the studies have been questioned (see Heath and Nielson, 1974). Here are the ten areas.

1. <u>Clarity of presentation</u>. The points made in the classes of effective teachers were clear. The teachers had enough background to answer questions intelligently. The level of presentation was "just right" most of the time. The lessons were well organized.

2. <u>Variability</u>. The teachers used a variety of teaching procedures, types of evaluation, instructional materials and teaching devices (as is possible with ideas and activities from this resource). The teachers were flexible in procedure, that is, appeared to change plans to follow up, or to allow the class to follow up, unanticipated student responses or questions.

FACTORS IN TEACHER EFFECTIVENESS

I'M NOT SURE . . .

Clarity ✓
Variability ✓
Enthusiasm ?
Task—orientedness
Opportunity

Use of student ideas
Harsh criticism (avoid)
Structuring remarks
Many levels of questions
Probing

3. <u>Enthusiasm</u>. The effective teachers were involved, excited and interested, and communicated these attitudes through movements, gestures, voice inflections, . . .

> *Teaching obviously has much in common with the theatrical art. For instance, you have to present to your class a (topic) which you know thoroughly, having presented it already so many times in former years in the same course. You really cannot be excited . . .—but, please, do not show that to your class; if you appear bored, the whole class will be bored. Pretend to be excited . . . , pretend to have bright ideas when you proceed, pretend to be surprised and elated . . . <u>You should do a little acting for the sake of your students who may learn, occasionally, more from your attitudes than from the subject matter presented.</u>*
> (Polya, 1965, p.101, emphasis added)

4. <u>Task-oriented and/or business—like behavior</u>. The teachers with greater student achievement were described as responsible, steady or poised. They were systematic and learning oriented and encouraged their students to work hard.

(The following five areas were less strongly supported by the research studies.)

5. <u>Opportunities for students to learn the material</u>. "That's obvious!" you may say. But, believe it or not, some teachers completely <u>skip</u> important aspects like

word problems ("the kids don't do well on them"), geometry ("I don't know much about geometry") or even long division ("I don't understand it myself")!

6. <u>General indirectness and use of student ideas</u>. This area includes praising or encouraging student responses, accepting students' feelings and in particular acknowledging and building on their ideas. Doing these things, of course, means giving the students chances to respond and to offer their feelings and ideas. How can this be done in the mathematics classroom?

7. <u>Negative effect of harsh criticism</u>. You probably do not like harsh criticism either. It seems to be only severe or perpetual criticism that interferes with achievement. [Rosenshine and Furst, 1971, p. 51] <u>Mild</u> forms of criticism or disapproval ("No, that's not the way") or control statements ("Okay, you guys, let's pay attention now") are all right; ". . . there is no evidence to support a claim that teachers should avoid telling a student he is wrong or should avoid giving academic directions." [p. 51] Thank goodness!

8. <u>Use of structuring remarks</u>. This means that the teachers provided overviews (or summaries) of what was to happen (or had happened). These could well occur several times within a lesson as well as at the start and end. In addition, the teachers used verbal "markers" like "This is important," or "Watch this carefully," or "Here is the main idea."

9. <u>Use of many levels of questioning</u>. Asking more than "What is the answer?" questions can alert the students to your expectations about important sorts of goals like those having to do with understanding why or problem solving processes. As *Questioning* in the section SIMILAR FIGURES suggests and as the research indicates, using several "levels" of questions may be productive.

10. <u>Probing</u>. This area "refers to teacher responses to student answers which encourage the student (or another student) to elaborate upon his answer." [Rosenshine and Furst, 1971, p. 51] A teacher response which invites the student to clarify, explain, interpret or generalize his remark would fall into this area.

> *. . . To be a professional means to accept responsibility . . . responsibility for actions and for results. It is to act in the best interests of those served . . . to help them grow rather than shrivel. When we accept the responsibility for professionally influencing the lives and actions of other people, we must do all we can to make that influence positive rather than negative. When we accept the money and the trust of the community, we must accept not only the responsibility for sending our students away with as much knowledge and skill as is within our power to give them, but also for sending them away with the ability and the inclination to use those skills to help themselves and others.*
> [Mager, 1968, p. 99]

? ? ? ? ?

Instead of a list of exercises to read through, we thought you might prefer a period of introspection—or rest!

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> *The articles in this special issue focus on career education.*

Some Organizations and Journals of Special Interest to Mathematics Teachers

Association of Teachers of Mathematics, Market Street Chambers, Nelson, Lancashire, BB9 7LN, England. Memberships in the USA may be obtained through Cuisenaire Company of America, Inc., 12 Church Street, New Rochelle, New York 10805. Journal: Mathematics Teaching

The National Council of Teachers of Mathematics, 1906 Association Drive, Reston,
     Virginia 22091. Besides journals and yearbooks, NCTM publishes a wealth of
     pamphlets for mathematics teachers.
     NCTM Journals: The Arithmetic Teacher, The Mathematics Teacher, Journal for
               Research in Mathematics Education
     Sample Yearbook Titles: The Slow Learner in Mathematics, 35th Yearbook, 1972
                    Instructional Aids in Mathematics, 34th Yearbook, 1973
                    Geometry in the Mathematics Curriculum, 36th Yearbook,
                    1973
                    Measurement in School Mathematics, 1976

School Science and Mathematics Association, Inc., Lewis House, P.O. Box 1614,
     Indiana University of Pennsylvania, Indiana, Pennsylvania 15701
     Journal: School Science and Mathematics

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

VISUAL PERCEPTION

WHAT IS VISUAL PERCEPTION?

Sight is a marvelous and unique sense. We use it to interpret our world. We rely on our sight to guide us as we move about, to help us recognize people and surroundings, to enable us to read and write. We take our ability to interpret what we see for granted. However, the ability to see and interpret—visual perception—is *not* intuitive and instantaneous; it is learned. Marius von Senden (1960) writes about the case histories of people blinded by cataracts since birth. When the cataract operation became safe and perfected in the 1950's, these people were given the operation. Before the operation the blind patients could identify objects only by touch, taste, smell and/or sound. After the operation the newly-sighted patients were asked to identify the same objects using only their sight. The patients had no idea what the objects were. To them the world was a blur of color and movement. Words, such as cube and sphere, had no "visual" meaning to them at all, yet the cube and sphere were easily identified when touched. Before the newly-sighted patients could use their sight, they *had to* experiment with their environment visually. They had to learn to interpret what they saw. As their visual perception developed, their senses of touch, hearing, taste and smell worked together with sight to create a three-dimensional world of reality.

In a similar manner we need to interpret what we see as we learn about geometric shapes and concepts. Visual perception plays a vital role in the understanding of geometry. If the students are having trouble with visual perception, they will probably experience frustration and failure when trying to recognize pictures and patterns in geometry. For example, if middle school students are given an illustration like Figure 1 and asked to identify as many triangles as they can, then the teacher is assuming the students can find the triangles and write them as $\triangle$ ABG, $\triangle$ GHJ, $\triangle$ HIJ, etc. Chances are some students will *not* see $\triangle$ ACH or $\triangle$ BJC. Perhaps the illustration will make no sense at all if the middle

Figure 1.

school student has had no past experience with such geometric drawings. The same illustration can be given to a high school geometry student who will not only identify *all* the triangles, but also the quadrilaterals, the pentagons, the parallel line segments, the transversals and a number of geometric relationships that are
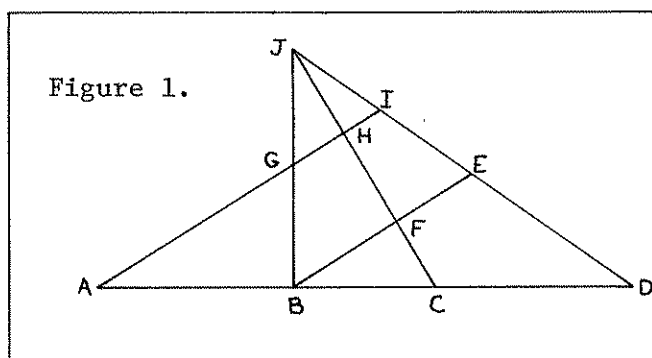
"visible." The reason a high school student can do this (and not all can, by any means) is because "vision is learned, and more particularly, that the process of visual perception is learned." [Buktenica, 1968, p. 37] In order to identify visual problems and aid the development of each student's visual perception, it is helpful to look at the various aspects of visual perception.

IMPORTANT ASPECTS OF VISUAL PERCEPTION

Visual perception is "the ability to recognize and discriminate visual stimuli and to interpret those stimuli by associating them with previous experiences." [Frostig, 1966, p. 8] It can be broken down into various aspects or categories. These perceptual categories are "somewhat arbitrary" and have been identified "as a result of observing children having handicaps or disturbances in these areas." [Buktenica, 1968, p. 34] "There is obvious overlapping of the categories, and training in one area might clearly effect training in other areas." [Buktenica, 1968, p. 38]

"Visual-motor coordination is the ability to coordinate vision with movements of the body or parts of the body." [Frostig, 1966, p. 8] The student who is so busy concentrating on simple motor skills and movement often has trouble thinking about anything else. For example, if a student is having difficulty using a ruler and pencil to draw a straight line or using a compass to draw circles, chances are that he will *not* perceive or understand ideas or concepts at various levels of abstraction. "When the coordinated movement becomes habitual, the child will be able to move through many and varied learning experiences and give his whole attention to the act of learning, for he will no longer need to mentally control and direct his movements." [Chaney and Kephart, 1968, p. 94] If coordination of eye-body movement is not developed properly, it will be reflected in our perception of external objects.

Figure-ground perception is the visual act of distinguishing foreground from background. For example, figure-ground perception enables one to identify a specific figure in a picture (Figure 2). As we focus our attention on a figure, we must cut off the extraneous things surrounding it and not be distracted by irrevelant visual

stimuli. The student who is looking for a five-pointed star in a more complex design must scan the design and disregard irrelevant shapes while looking for the star in the midst of a large area with similar shapes. "Involved in this process is the differentiation of an object, shape or form from its background." [Cunningham and Reagan, 1972, p. 42]

Perceptual constancy is the ability to recognize an object out of its original context or from a different viewpoint. "We must teach children to recognize that an object is the same shape, size, texture, etc. in spite of the fact that the object might be incomplete or distorted as a result of viewing it in a strange context or from an unusual angle." [Buktenica, 1968, p. 58] Once we are aware of the "constancies" of our environment, they serve to stabilize the perception of the visual world. For example, when we visually perceive a football field, we know it is rectangular, yet we rarely "see" it as a rectangle (Figure 3). Perceptual constancy helps us in adjusting to our environment.

Position in space is determined by the relationship of an object (i.e., how far away it is, how large it is) as compared to and perceived by the observer. Spatially a person is always the center of his own world. He must perceive the proper position of things in his environment in relation to himself. For example, students may experience reversals in

Figure 2.   Figure-ground Perception.
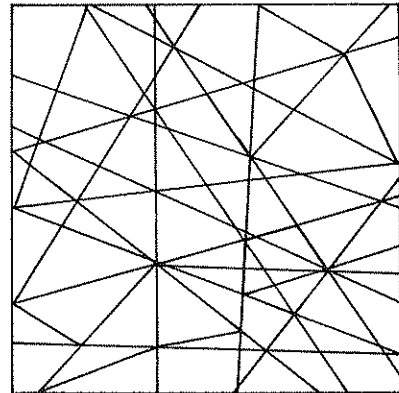
Find the 5-pointed star.

Figure 3.   Perceptual Constancy
A FOOTBALL FIELD FROM DIFFERENT ANGLES.

We think "rectangle."
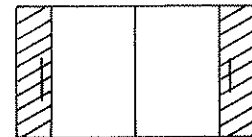
We see . . .
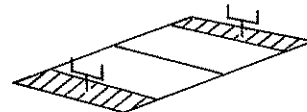
a parallelogram . . .
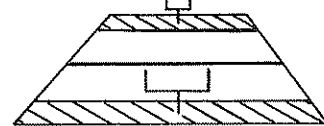
a trapezoid . . .

a quadrilateral.
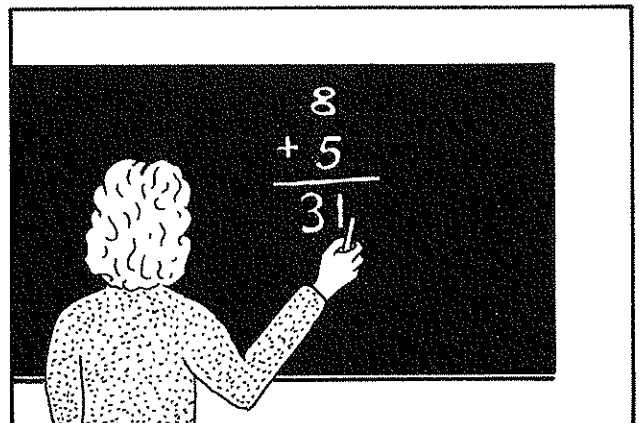
reading (was for saw), arithmetic (24 for 42) and handwriting (ƨ for s, b for d, ɘ for e).
Perception of position in space is important to the student if failure at academic
tasks and lack of coordination are to be overcome and/or evaded. (Frostig, 1966)

"Perception of spatial relationships is the ability of an observer to perceive
the position of two or more objects, both in relation to himself and in relation to
each other." [Frostig, 1966, p. 12] For example, the student constructing a scale
drawing of a three-dimensional building has to perceive the position of its components
in relation to each other. He can then map the position of the components on paper.

Visual-spatial operations require a strong sense of body orientation. Students
learn spatial or depth cues; they can then estimate and judge the distances of
things around them. Through previous experiences and interactions with the environ-
ment, the students will unconsciously create the spatial image of an object. Soon
the impression they gain from looking at an object is more accurate and certain than
if they only touched it with their hands (Figure 4). Indeed, seeing is an *active*
rather than passive experience. (Wilman, 1966)

Figure 4.   Spatial Relationships.

(a)   (b)   (c)   (d)

This drawing   One of these drawings shows the same object viewed from a
represents a   different position. Which is it?
solid object.
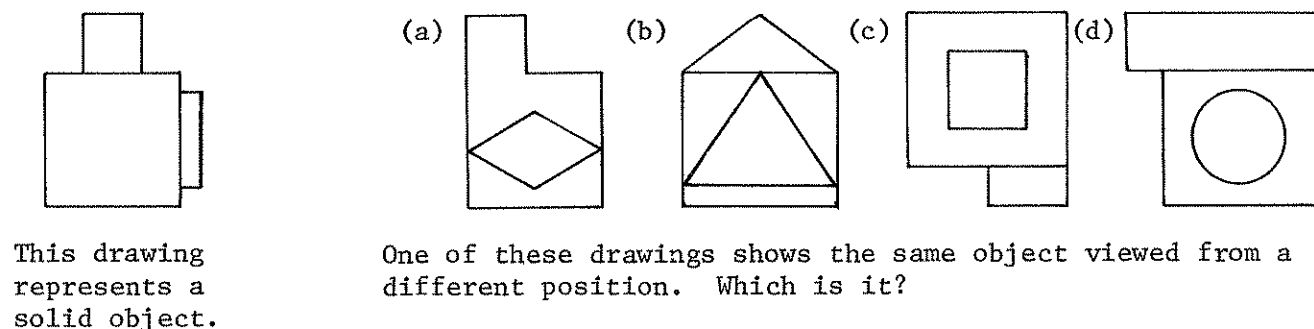
Visual discrimination is the ability to distinguish similarities and differences
between objects, as the matching of items depending on a given similarity. Initially
students may start with concrete models and actual objects. For example, by matching,
sorting and arranging Attribute Blocks, students are learning visual discrimination
in relation to color, shape, size and thickness. Later, the students can progress to

pictures and abstractions
(Figure 5) as they develop
their skills in visual dis-
crimination. They need to
learn to make visual and
verbal comparisons of things
they see.

Figure 5.   Visual Discrimination.
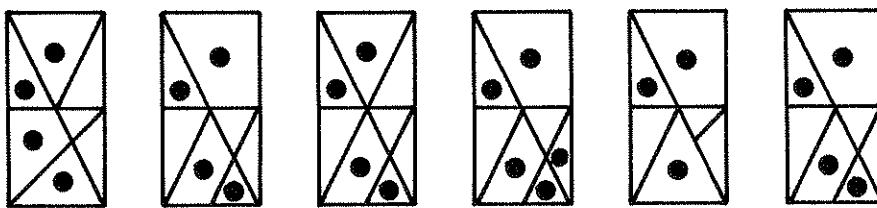
TWO of these figures are exactly alike. Which ones?

Visual memory is the ability to recall accurately an object no longer in view and to relate its similarities and differences to other idems in view or not in view. "There are marked individual differences in the capacity for mental imagery. Observers endowed with a so-called 'photographic memory' find the pictures flash upon the inward eye with all the clarity of Wordsworth's daffodils, while others are hardly capable of any visual experience other than of things physically present." [Wilman, 1966, p. 53] We retain images of objects as we have perceived them through our eyes. It is known that, on the average, we can remember only a small number of un-related items--about five to seven things. In order to remember a larger number of items, we must commit the things to a more permanent storage through abstraction and symbolic thinking. (Hochberg, Illusion in Nature and Art, 1973)
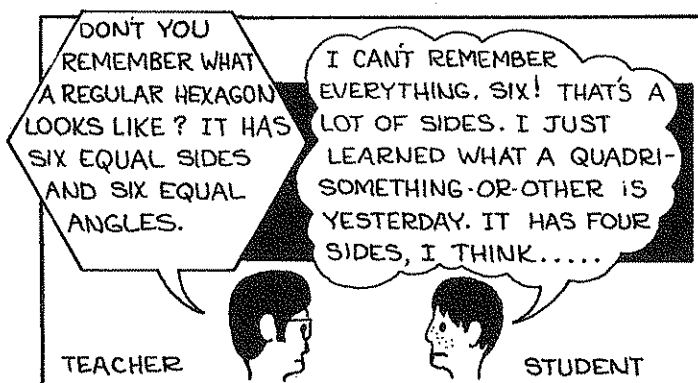
The aspects of visual perception often affect student performance on various geometry problems. Deficiencies in visual perception may influence the student's ability to learn geometry; hence, it is important to keep these aspects in mind as we teach geometry.

VISUAL PERCEPTION IN GEOMETRY

Geometry deals with the properties and relationships of points, lines, angles, surfaces and solids. There are many concepts in geometry that cannot be recognized or understood, unless the students can visually perceive examples and identify figures and/or properties by associating them with previous experiences. Many geometrical concepts require a visual interpretation as problems are viewed or drawn on paper. Here are a few examples.

Look at the similar triangles in Figure 6(a). One is larger than the other; they have the same shape and con-gruent angles. The triangles in Figure 6(b) are also similar, but the triangles are oriented so a visual matching of shape and angles is not so obvious. A student with well-developed visual per-ception could recognize the similarity.

Figure 6. Similarity.

(a)

(b)

89
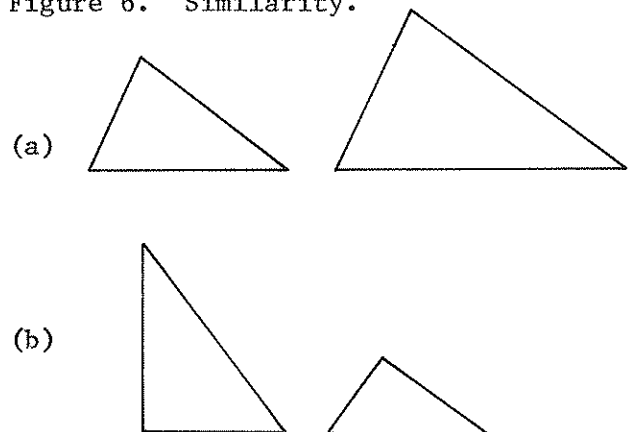
property nonetheless. The student also needs to possess perceptual constancy--the triangle is the same shape and size no matter how it is rotated or translated or reflected in the plane (Figure 7).

Figure 7. Rotations.
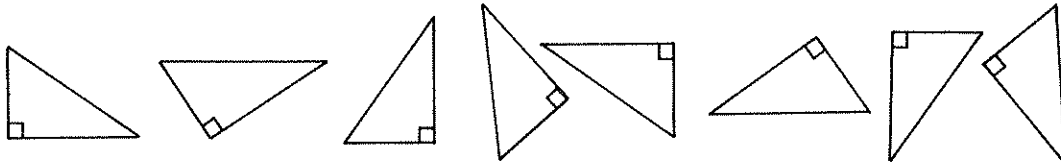
Sometimes the student may be asked to examine the cross section of an object (Figure 8) or the location of places in relationship to himself (Figure 9). Showing students how to interpret "standard" pictures of geometric figures can help them perceive various problems (Figure 10).

Figure 8. Orthographic Imagination--
Viewing a Cross Section
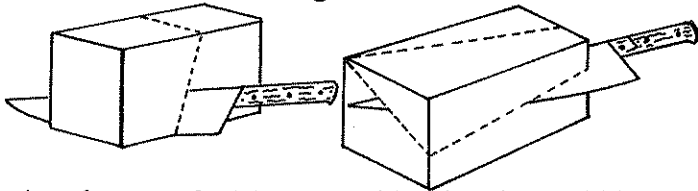
As these cuboids are sliced, what will their cross sections look like?

Figure 9. Geographic Symbols
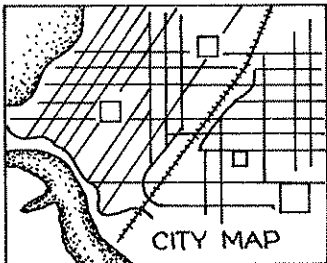
Locate your home, school, favorite restaurant and other important places on a city map.

Figure 10. Solids

(a)
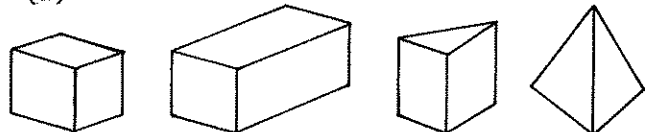
(b)
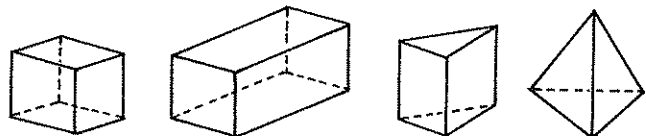
Name the three-dimensional figures that these two-dimensional drawings represent. What 2-D figures make up each 3-D representation? How do the dotted lines help?

Students need to develop fine visual discrimination to interpret geometric drawings. Otherwise, they will not be able to pick out details from drawings or illustrations. For example, fine visual discrimination is necessary in order to identify whether a closed four-sided geometric figure is a square, a rectangle, a parallelogram, a rhombus, a trapezoid, or just a quadrilateral.

Geometry vocabulary also plays an important role in helping students to identify figures and relationships, to organize their experiences, and to develop the ability to think abstractly about geometric ideas and concepts. The word *rectangle*, for example, probably brings to mind a four-sided geometric figure with four right angles and opposite sides congruent. The acquisition of the word allows the student to use the concept in future planning and enables him to communicate it. (Frostig, 1966)

Students' visual understanding will mature at different rates. To illustrate the visual understanding of a geometric figure, consider a *rectangle* and how the student learns about it visually. First, the student learns to recognize any rectangle and differentiates it from other geometric figures. Next, the student learns to trace rectangles; then he learns to copy them. He learns to copy by looking at various rectangles, but later the student can draw them from memory. At this point the student can spontaneously draw any rectangle and reproduce it so it looks like a rectangle each time it is drawn from memory. (Buktenica, 1968)

Students will NOT always understand a concept just because the picture "illustrates it." The picture may well illustrate it to the teacher, but not to the students, unless they are taught what to see. For example, the students must have their *attention* directed to rectangles (Figure 11) if that is what they are searching for in a geometric drawing. "Attention is influenced by motivation--we perceive to a great extent what we want to perceive. It is determined by past experience." [Leibowitz, 1965, p. 29] If the students know what rectangles look like, they will be able to find them. While at the same time they are looking for rectangles, they may fail to notice the triangles in the same geometric drawing because that is NOT what they are looking for. They see what they are trying to see. If the students are not familiar with rectangles or triangles, the geometric drawing may be relatively meaningless.
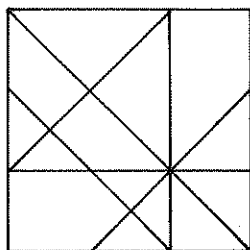
Figure 11.

Since drawings, illustrations, three-dimensional models (such as polyhedra), two-dimensional figures (such as polygons), and other visual media are used to teach geometry, it is helpful if the teacher provides accurate drawings, constructions and visual aids for the geometry students. Mathematics teachers have been observed to draw perpendicular lines, circles, triangles, etc. with no apparent attempt to make them visually accurate. Instead the student is asked to *imagine* that:

1. These lines are perpendicular.

2. This is a circle with its radius.

3. This is an equilateral triangle.

"Imagining" geometry concepts is fine for persons who already know what the concepts are, but students need to learn the concept from *accurate* drawings, models, cutouts, and other concrete/semiconcrete aids before they can be expected to "imagine."

The student's visual perception and understanding of geometry can be improved by various training programs. These programs take into account many of the aspects of visual perception; they also use many geometric examples to develop the student's ability to interpret a drawing.

VISUAL PERCEPTION TRAINING PROGRAMS

Programs are available that can be used with students to develop and improve their visual perception. One of the best known is the Frostig Program (Frostig, 1964) that stresses improvement in five of the areas of visual perception: visual-motor coordination, figure-ground perception, perceptual constancy, position in space, and spatial relationships. In Handbook of Visual Perceptual Training [Cunningham and Reagan, 1972], a training program is outlined with references to other programs and materials that would be helpful in teaching visual perception.

It appears that visual perception skills and geometry concepts can be learned simultaneously, since geometry requires that the student recognize figures, their relationships and their properties. Informal geometry could easily be taught and included with a visual perception training program so as to improve students' visual perception. Many of the pages in the Mathematics Resource Project series require identification of geometric figures and shapes in two and three dimensions as they are rotated, translated and/or reflected. The pages also offer many diagnostic activities and ways to evaluate the visual problems that a student may have.

SUMMARY

1. Visual perception—the ability to see and interpret our environment from our previous sensory experiences—is *learned* (and hence can be taught).

2. Visual perception plays a vital role in the understanding of geometry.

3. There are many aspects or categories of visual perception: Visual-motor coordination, figure-ground perception, perceptual constancy, position in space, spatial relationships, visual discrimination and visual memory are some of the important ones.

4. Many geometry concepts require visual interpretation; hence, students need to develop fine visual discrimination to interpret geometric drawings.

5. Geometry vocabulary is important in naming or identifying visual experiences.

6. Students' visual understanding will mature at different rates.

7. Students will *not* always understand a concept just because the picture *illustrates it*.

8. Geometric drawings and illustrations must be clear and accurate.

9. Various training programs are available which can be used to emphasize visual perception and enhance the teaching of geometry.

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EXAMPLES OF VISUAL PERCEPTION ACTIVITIES
IN THE CLASSROOM MATERIALS

I.   Two-Dimensional Visual Patterns

With practice, students can learn
to examine each figure closely and
sketch the reflection of it free-
hand.
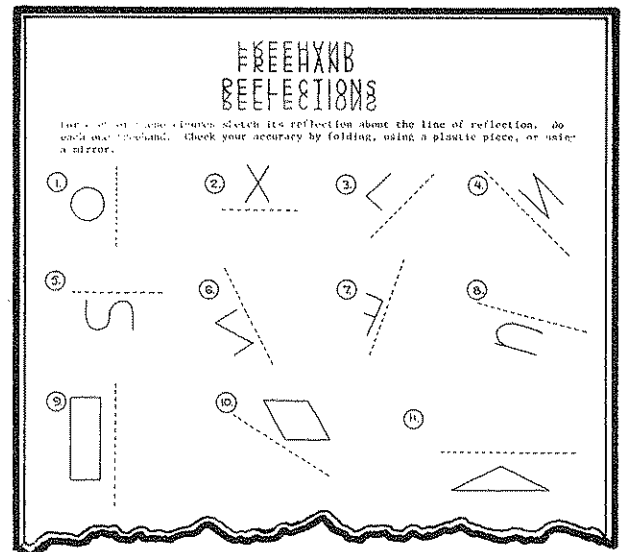
II. Three-Dimensional Visual Patterns

Slicing a 3-D object will result in a cross section. What will the cross section look like? Will it be a square, an equilateral triangle, a regular hexagon? Our visual imagination can help us identify a cross section only if we have learned to perceive 3-D objects through previous visual experiences.

CROSS SECTIONS

1) The plane is cutting the square pyramid to get a cross section that is a square. →→ cross section

This plane gives a cross section that is a kite.

What will this cross section show? _____

2) Name the shape of each of these cross sections of an octahedron.

The Soma Cube is a 3-D puzzle of seven pieces. The students can learn to recognize these pieces in different orientations.

SEEING IT LIKE IT IS

Materials needed: A set of Soma cube pieces

Activity:

A) Place the pieces in front of you in this order.

1)   2)   3)   4)   5)   6)   7)

B) These pieces have been rearranged. Write the number of the piece shown.

C) Only one view is shown. Which pieces could it be? Each one has more than one answer.

Cubes are commonly used as dice or building blocks for volume activities. Here we see that the faces of a cube can present many spatial puzzles to be solved visually.

SEVERAL VIEWS OF CUBES

Below are six activities with cubes and nets that involve visual perception. Each idea could be expanded into a student page.

The six views below are of the same cube.

What figure is opposite each of the following:

△ = _____
▲ = _____
□ = _____
○ = _____
■ = _____
● = _____

A cube can occupy a space in 24 different ways. Here are five ways.

Some other ways are shown below. Can you fill in the missing letters? (Be sure they are positioned correctly.)

Make a cube to check your answers.

Make a cube to help you.

Each net represents an unfolded cube with designs on several faces. Four cubes are pictured below each net. Match the net with the correct cube.

a)

b)

c)

Suppose a hole is drilled through a cube:

Which of these nets show the result?

Experiment with various shaped holes.

Select the nets which describe the cube.

The nets below can be used to form a cube. On each net the back face and the bottom face are marked.

1) Mark the remaining four faces on the net: front, left, right, top.

2) Check each answer by cutting out the net and making the cube.

If a cube was made from the net below it would spell CUBE around four of its faces.

C U B E

Letter the nets below so that each spells CUBE when assembled.

Cut out the nets to check your answers.

III. Recognizing Shapes and Space Figures

Making 3-D models from the 2-D representations (multi-view orthographic drawings) tests the students ability to perceive spatial relationships and construct in three dimensions that which is seen in two dimensions.

Prism or pyramid? It is not easy for students to tell which is which unless they have had the experiences of touching and seeing examples of many objects--prisms, pyramids and other space figures.

Making designs with hexiamonds can provide students with practice in visualizing and recognizing spatial patterns.

Some similar figures have a special
property. Students can discover
which figures will work by construct-
ing their own patterns and experiment-
ing with the shapes.

TRICKIE CUBES

Materials Needed: A set of cubes

Activity: Make each of these models with your set of cubes. On another sheet of
paper write the number of cubes needed to make each model.

Using a 2-D paraline drawing, the students are asked to construct the corresponding 3-D object. These exercises help the student to learn to interpret the 2-D representations of 3-D objects—an important skill in visual perception.

IV.   Understanding a Concept

BUDDIES

The two figures to the right have exactly the same shape. Figure VWXYZ is similar to figure ABCDE.

Each angle of figure VWXYZ can be matched with a congruent angle of figure ABCDE as shown.

∠V and ∠A are corresponding angles.

VZ and AE are corresponding sides since they match when the angles are paired.

Four more pairs of similar figures are drawn to the right. Use them to answer the questions below.

The letters in the blanks should answer the riddle below.

WHAT DID THE FOUR 90° ANGLES HAVE IN COMMON?

1) Angle D corresponds to angle ____.
2) Side X'Y' corresponds to side B____.
3) Side D'A' corresponds to side G____.
4) Angle P corresponds to angle ____.
5) Side B'C' corresponds to side K____.
6) Angle Q corresponds to angle ____.
7) Side JR corresponds to side C____.
8) Angle D' corresponds to angle ____.
9) Angle T' corresponds to angle ____.
10) Side EF corresponds to side D____.
11) Angle M corresponds to angle ____.

What are corresponding angles? Can corresponding sides also be found? Learning the concept of corresponding parts comes through visual experiences with similar figures.

TANGENTS

I Have students:

1) Mark a point on a sheet of plain paper.

2) Take a straightedge and place one edge through the point.
Draw a line along the other edge.

3) Repeat this several times using the same point.

4) Continue drawing straight lines until they cannot find space for any more.

What shape do students see?

All lines drawn are called tangents to the circle they form. All touch the circle but none go inside it.

How many tangents are there to one circle?

II Have students explore all the possible arrangements for two circles of different size. Templates can be used to sketch the arrangements, or the outlines of two coins can be traced.

NO COMMON POINTS   ONE COMMON POINT   TWO COMMON POINTS

III For each arrangement have students discover the maximum number of common tangents for both circles. Suggest that they slide their ruler around and try to position it so that one edge touches both circles at the same time.

EXACTLY FOUR COMMON TANGENTS | EXACTLY THREE COMMON TANGENTS | EXACTLY TWO COMMON TANGENTS | EXACTLY ONE COMMON TANGENT | NO COMMON TANGENT

Find the different ways the figures will fit together. In order to fit the various polyhedra into a given place or hole, the students must work with 3-D models to understand the idea.

LAYER UPON LAYER

A) How many unit cubes would be needed to fill this box?

To find out:

1) How many unit cubes would be needed to cover the base? _____

2) How many layers would it take to fill the figure? _____

3) The figure has a volume of _____ cubic units.

Tangents are involved in various theorems in geometry. The concept of a tangent can be learned through seeing tangent lines--arranging them, sketching them, discovering how they relate to the circles and other lines.

BRICKS AND POLYHEDRA

1) Mac, a mathematician, was building a brick fence along his property line. In how many different ways can Mac place a brick on the fence? _____

2) Suppose the brick was decorated with a wedge-shape on opposite sides.

a) How many ways can the brick be placed so the wedge is pointing to the right? _____, to the left? _____, in either direction?

3) By the gate Mac wants to create an unusual effect and use cubical bricks. In how many different ways can he place one of these bricks? _____

4) How many ways can the door knob be placed in the square hole? _____

5) How many ways will each polyhedron fit snugly into the hole below it?

TRIANGULAR PRISM | SQUARE PYRAMID | REGULAR TETRAHEDRON | REGULAR OCTAHEDRON | 14-SIDED POLYHEDRON

What is volume? How can volume be understood through 2-D drawings? The concept of volume is often abstracted on paper instead of being investigated in its 3-D "concrete" existence. Students need to visually experience both the concrete and abstract representations of volume.

YOU CANT BUY
PARTS FOR IT

What are the various angles formed from the intersection of a transversal with two parallel lines? Can the angles be identified? Fine visual discrimination is needed as students learn to recognize different kinds of angles.
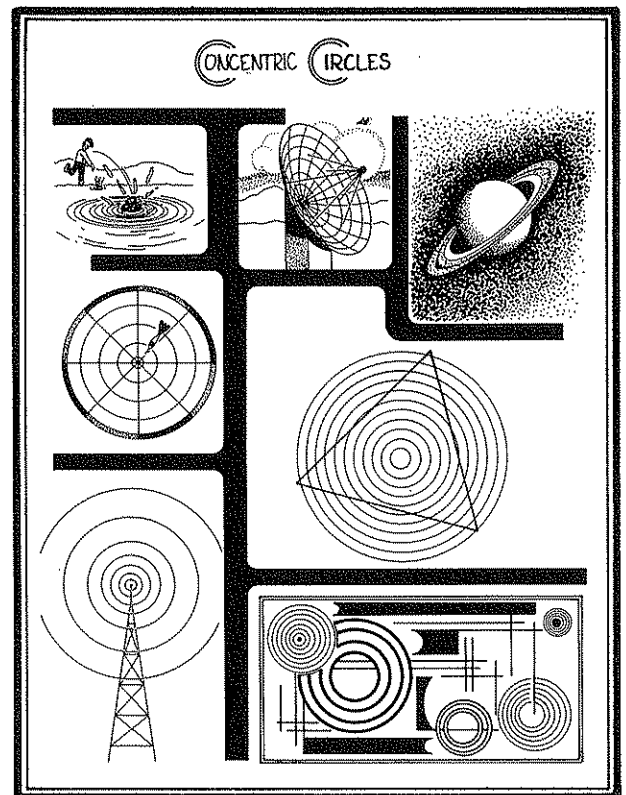
What are concentric circles? Where do they occur in nature? What human inventions use concentric circles? Make a bulletin board display to answer these questions.

SOME CAN,
SOME CAN'T

What is an inscribed polygon? What kinds of polygons can be inscribed in circles? Students can create their own examples and test their conjectures by drawing pictures.

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

GRAPHIC REPRESENTATION

WHAT IS GRAPHIC REPRESENTATION?

"Hey, Jim, come on out to my house for dinner Friday night."

"Sure, how do I get there?"

"Here, let me draw you a map."

Sound familiar? From time to time we make a freehand sketch of our location because a verbal description is not enough. We want the person to visually perceive the situation, hence we draw a map. We communicate through graphic representation-- drawings to be perceived and understood by ourselves and others.

Drawings help us represent ideas, concepts and actual objects. Verbal description is one form of communica- tion; but too often our words go in one ear and out the other. We often learn by seeing. As a result, graphic representation plays an important role in expressing our thoughts and ideas.

Can you help me with this mathematics problem? It says that I have parallel lines cut by a transversal, and a pair of alternate interior angles measure 60°. What does that mean? How do I find the other angles?

Well, first of all, can you draw a picture of it?

Yes, that would help!

STUDENT

TEACHER
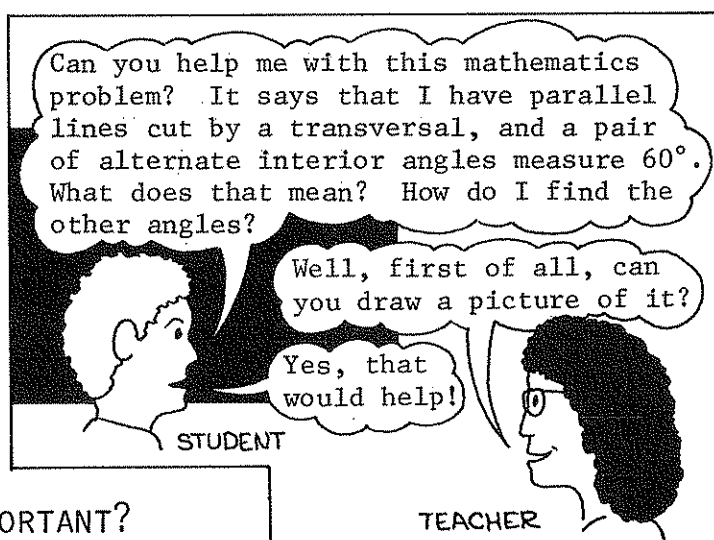
WHY IS GRAPHIC REPRESENTATION IMPORTANT?

Graphic representation, like the written word, allows us to communicate our ideas to others and sometimes even to ourselves. We often use geometric drawings to illus- trate a concept or relationship. For example, the Pytha- gorean theorem is illustrated at the right. Sometimes graphic representation of a problem can help one find the solution, or at least approximate it. Organizing our thoughts into a diagram or sketch is a natural step in the problem-solving process. For example, try this story problem without drawing a picture: George went due West for 8 blocks, North for 3, and East for 4. On the beeline, how far was he from his starting place? It is most helpful to make a sketch to solve this problem. Some students might not think of making a sketch; others might have difficulty making an accurate sketch.

$$a^2 + b^2 = c^2$$

Students learn visual perception; they must also learn graphic representation. Visual and graphic thinking are so interrelated in their development that often one progresses with the other. The identification of drawing with art keeps many people from realizing that graphic representation is present in all subject areas. In
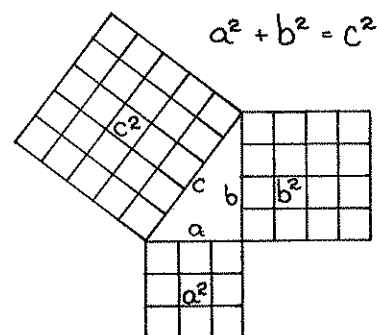
particular, the study of geometry repeatedly calls upon the student to understand
ideas, concepts and relationships by drawing representations of them. Pictures and
drawings are also necessary when studying fractions and scaling. (See the resource
books Number Sense and Arithmetic Skills and Ratio, Proportion and Scaling, Mathema-
tics Resource Project.) Much of what we do see in and out of school revolves around
visual thinking and graphic communication. Let's see what we can do to help our
students learn graphic representation.

DRAWING IN GEOMETRY

"Skill in drawing is often dismissed as a happy gift of coordination of hand and
eye, incapable or unworthy of being learned or taught, and having little to do with
the intellectual process." (Lockard, 1968, p. 5) But, drawing is important in geometry,
and it requires more mind than hand and eye. The geometer's main reason for learning
to draw is "to possess a tool with which he can test his first conceptual ideas, dis-
card, refine them and finally present them to others." (Lockard, 1968, p. 5) We as
teachers seek to present the language of geometry, the figures and illustrations, and
the practical substance of graphic representation for our students to use.

Locating Points in Drawing

Knowing where to locate points can
often help when constructing a picture
of an idea. For example, find the mid-
points of the sides of a quadrilateral
and join them with line segments
(Figure 1). Is the inside quadrilater-
al always a parallelogram?

Figure 1. Quadrilateral and parallelogr

Figure 2. Points and line segments

2   3   4   5   . . . . .   n?
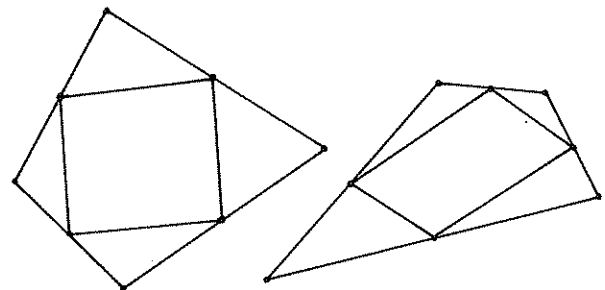
Or, given n non-collinear points, how
many line segments must be drawn so
that each point is connected to every
other point (Figure 2)?

One can also learn to use points to make three-dimensional drawings.

Showing points by dots is important in drawing because we use them for identification, emphasis or impact. The following examples illustrate some important points in geometry.

MIDPOINT   M   VERTEX   PERSPECTIVE POINTS   INCENTER   CENTER
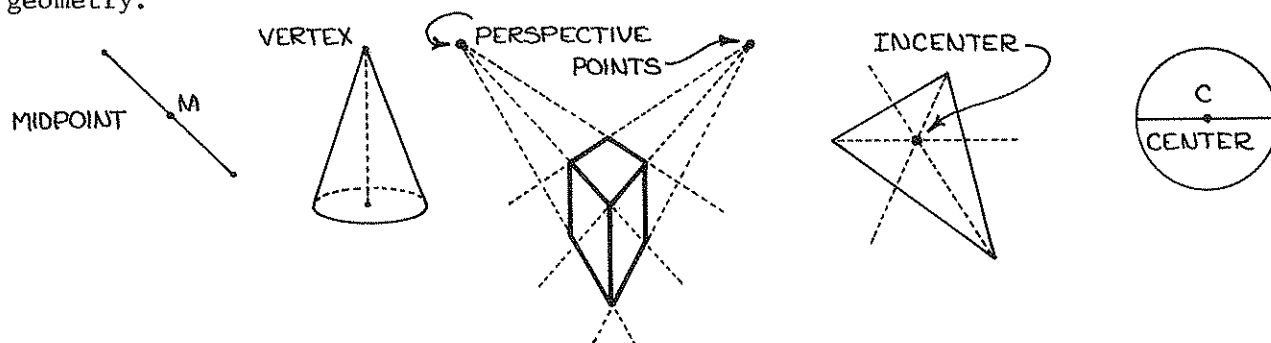
Using Grid Paper

Various kinds of grid paper can be used to aid geometric drawing. Dot paper is often used with a geoboard so students can record their figures and designs (Figure 3). Squared paper is advantageous when working with the area of geometric shapes. For example, draw several rectangles that have an area of 12 squares (Figure 4). In algebra, lines, inequalities, and absolute value functions are plotted on a squared grid. Line and bar graphs are often placed on grid paper. Isometric grid paper can be used to make three-dimensional drawings or create specific figures and tessellations (Figure 5).

A helpful book that contains many types of grids is Math Activity Worksheet Masters by Stokes and Laycock.

Representing Three-Dimensional Objects

One of the more difficult techniques to learn is making a two-dimensional representation of a three-dimensional object. There are at least three ways to do this.

Figure 3. Dot paper
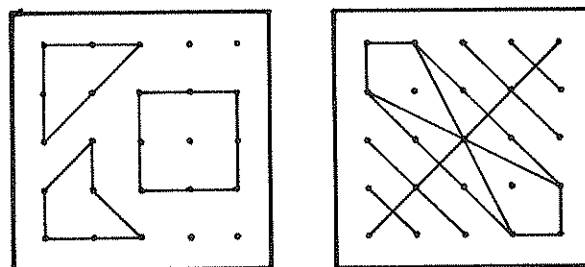
Figure 4. Rectangles of area 12
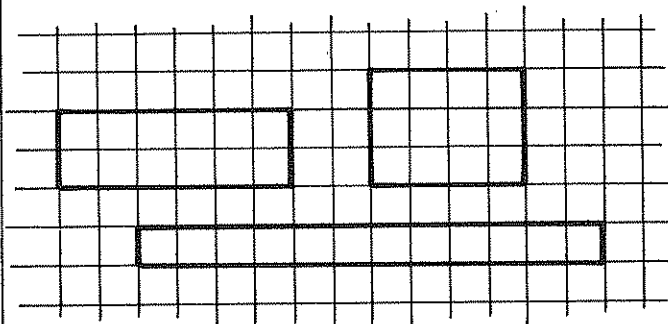
Figure 5. Isometric grid paper
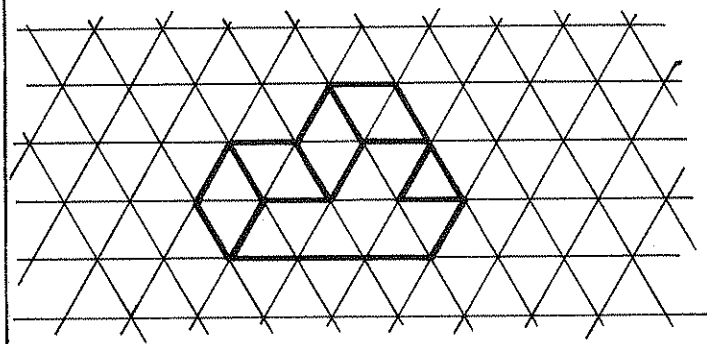
The <u>multi-view ortho-graphic</u> is a group of drawings where each drawing shows a single face of a 3-D object as it would appear if the observer could look "straight" at the face (Figure 6).

Figure 6.   Multi-view orthographic drawings

FRONT   SIDE   TOP   BOTTOM   OBJECT
(represented in a paraline drawing)
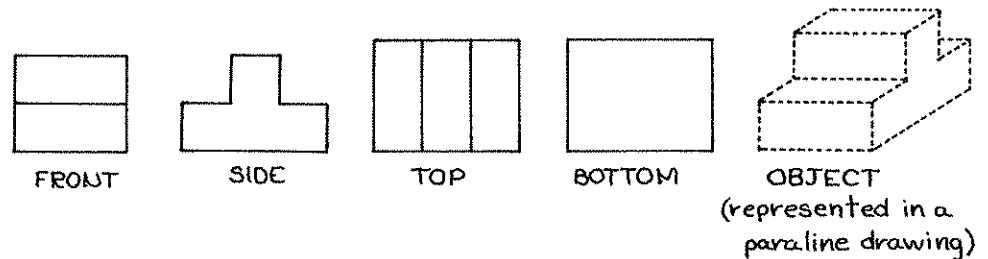
Although multi-view orthographics are useful, especially in drafting and architecture, <u>paraline drawings</u> (Figure 7) are easier to interpret as representations of three-dimensional objects.   There are three kinds of paraline: isometric, oblique and dimetric.   They have the common characteristic of showing three adjacent faces of an object in a single drawing.

Figure 7.   Paraline drawings

a) <u>Isometric</u> All the angles are at 120° from the point viewed.   (This type of drawing is usually made on isometric grid paper.)
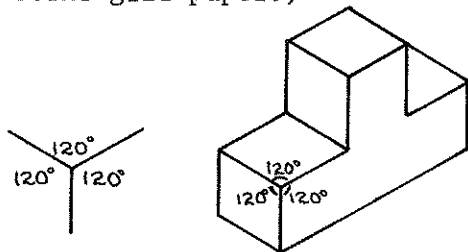
b) <u>Oblique</u> One of the angles is a right angle, so one side is seen perpendicular to the observer.
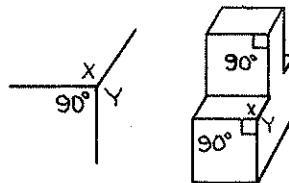
c) <u>Dimetric</u> Two of the angles are of equal measure, but the third angle is different.
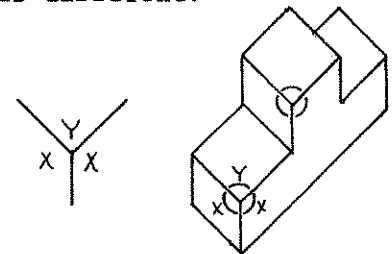
"<u>Perspective drawing</u> gives the truest picture of the object, but is the most difficult type of drawing to construct.   Since the more distant parts of the object look smaller, few measurements can be made directly to scale on the drawing." (Martin, 1962, p. 40)   Figure 8 shows an example of two-point perspective and one-point perspective.

Figure 8.   Perspective drawings
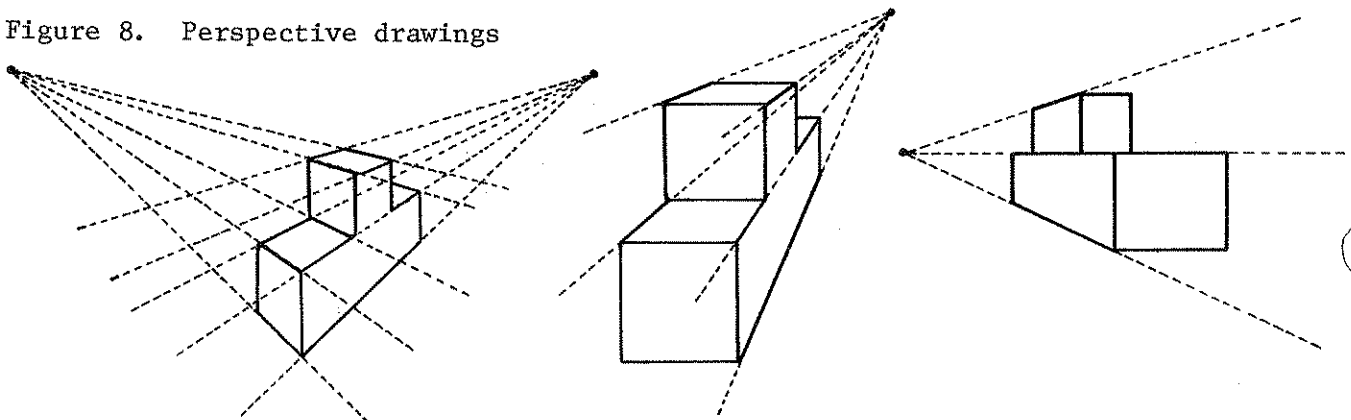
Additional drawing techniques are needed for the representation of curved objects. For example, a picture of a sphere can be drawn by the method illustrated in Figure 9. Other examples are given in Figure 10. Students may learn to use shading or shadows, heavy solid lines or light dotted lines, points for emphasis and other graphic techniques. By giving the students a two-dimensional figure and asking them to make it look like a three-dimensional object, the result will often show various techniques as well as deficiencies in drawing.

Figure 9. Circle to Sphere

2-D   TO   3-D
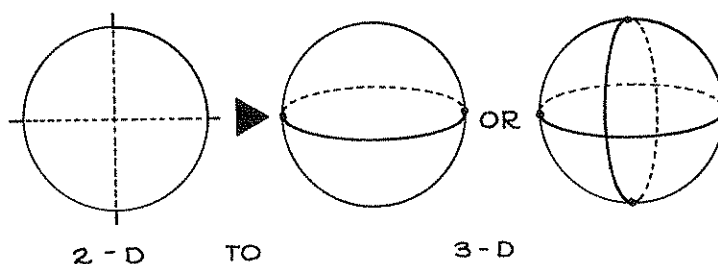
Figure 10. 2-D figure to 3-D object
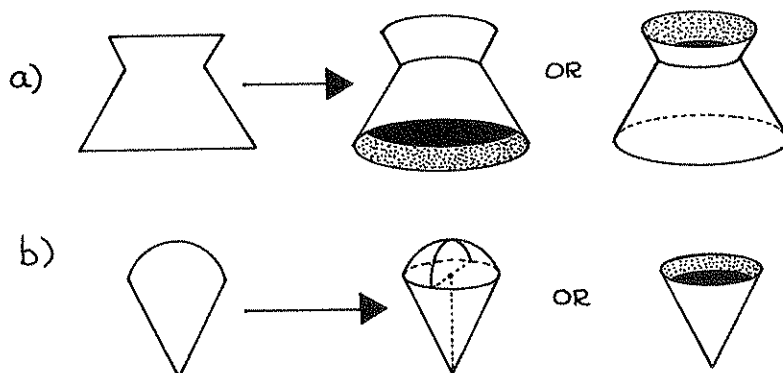
Interpreting Drawings and the Objects to be Drawn

When using pictures or illustrations in geometry, one needs to be aware of graphic representations that may be ambiguous and misleading. "Exposure to mediocre textbook illustrations, catalogues, books, posters and similar products of visual ineptness often test the ability of the student to learn a concept or idea in spite of the drawing, not with the help of it." (Arnheim, 1969, p. 312) Teachers need to be careful when making drawings. The pairs of drawings below show some contrasting representations. Which drawings are clearest?
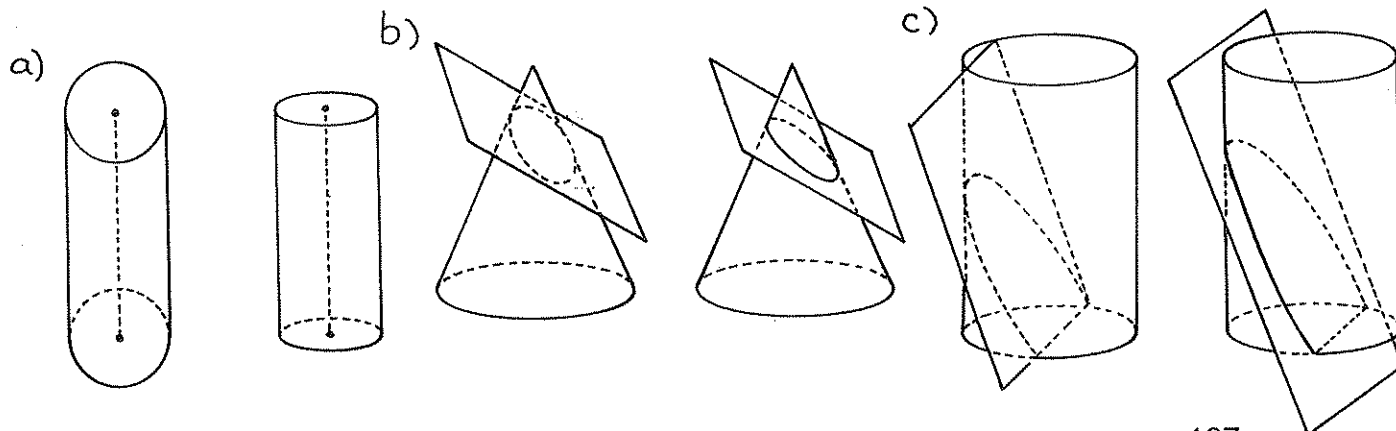
Another point to consider with drawings in geometry (and other subjects as well) is whether the visual complexity of the object to be drawn can be grasped by the student. A student's drawing will be an image of what he sees and understands. "Learning to draw is really a matter of learning to see—to see correctly—and that means a good deal more than merely looking with the eye." (Nicolaïdes, 1975, p. 5) This again emphasizes the importance of visual perception (See *Visual Perception* in the TEACHING EMPHASES section and *Planning Instruction in Geometry* in LINES, PLANES & ANGLES.). Students might have more success making drawings of objects if they have handled the objects and had a chance to discuss what they see.

TOOLS FOR DRAWING IN GEOMETRY

The basic tools used for drawing in geometry are the protractor, the compass, the straightedge and the pencil. Students can also use colored pencils and pens, rulers, templates, T-squares, special drawing pens and pencils and other graphics materials for the more elaborate drawings they do. The overhead projector and the opaque projector can be used to make enlargements of geometric designs and illustrations. The students will need to develop skills in using these tools. Instruction in the techniques of handling and using the tools will help the students increase their skills. Encourage students to make their own designs for practice.
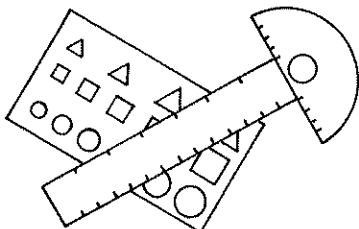
TEMPLATE & T-SQUARE

The following sections elaborate on the use of the basic tools: the protractor, the compass and the straightedge.

The Protractor

Protractors are used to draw angles as well as measure angles. Unfortunately, many students do not know how to read the scales marked on a protractor. Initially, it might be helpful to make a transparency of protractors; each transparent protractor is marked off every 5 or 10 degrees without the number of degrees written by each mark (Figure 11). The student counts by tens or fives to find the number of degrees in the angle being measured. (See the transparency master on e page 18.) Eventually, the student will learn to read the correct scale on commercial protractors.

Figure 11. Unnumbered protractor

To use the protractor to draw an angle requires practice.   There are definite
steps to be learned (Figure 12).   Show the students how to place the protractor and
locate the desired measure of the angle.   (See *Professional Protractors, Protractors
and Polygons* and *I'm Seeing Stars* in the ANGLES section for skill building ideas.)
The overhead projector is a marvelous audio-visual aid to use when demonstrating how
to measure or draw angles.

Figure 12. How to draw a 75° angle		Count in this direction to 75°.	
Draw one ray of the angle.	Place the protractor so the center of the protractor is at R and the zero degree line of the protractor is directly on top of the ray.	Mark off a point at 75° from the ray.	Remove the protractor and draw a ray from R through the point to form a 75° angle.
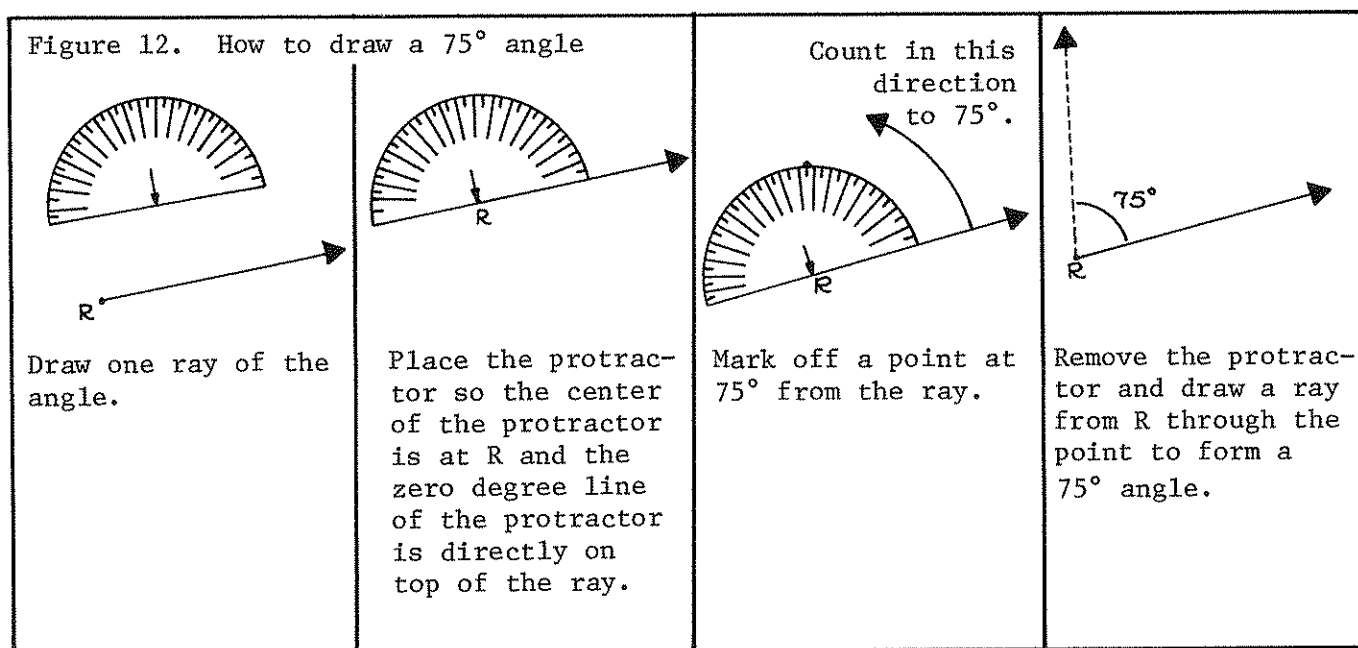
Compass-Straightedge Constructions

Geometric constructions are still a part of a typical high school geometry
course.   They are also contained in many 6th, 7th and 8th-grade textbooks.   Many stu-
dents enjoy working with the simple construction tools in geometry.   (See pages 15-16
for the basic straightedge-compass constructions.)

Before straightedge and compass can be used to "bisect an angle" or "construct
a perpendicular," skills in the use of these instruments must be developed.   Although
some students might have difficulty with their straightedges slipping, it is the
compass that causes the most problems.   Learning to keep the point of the compass
stationary while the pencil glides smoothly on the paper can be a challenge.   Here
are some suggestions you might like to pass on to your students.

a)   Buy a good quality compass.

b)   Be sure your pencil point is sharp.

c)   Be sure the pencil point and the compass point meet when the compass is
together.

d)   Be careful not to change the compass opening as you make an arc.

e) Make construction arcs lightly; do not erase them unless you are making an art design.

f) Use paper or cardboard under your paper so the point can "dig in." This also protects the desk or table top.

Making circle designs like those in Figure 13 can help build confidence and skills in using a compass. Students can create their own designs for a bulletin board. Compasses can be made available for students to practice with in their spare time. Since success and creativity are involved, students usually find these constructions enjoy-

Figure 13. Circle designs

able. At first they will find it easier to make larger designs, but as their skills increase they can make their designs more intricate and smaller. These skills will be useful when students construct geometric figures in late middle school or in a geometry course.

Copying designs requires analysis of the design. How was it constructed? Is it symmetrical? Was the radius of the circle used to mark off six equal arcs? The page *Art Inside the Circle* in the CIRCLES subsection suggests various designs. The book Creative Constructions by Dale Seymour and Reuben Schadler gives many more worksheets and designs which your students might enjoy.

When introducing geometric constructions to students, it is helpful to have a large compass to use at the chalkboard. If a large compass cannot be obtained, a string tied around a piece of chalk will work. With a string compass, a right-handed person should start at about 8 o'clock, go clockwise and turn the left thumb (at the center) as the circle is drawn.

Demonstrating constructions on an overhead is also helpful (Figure 14). A regular compass can be used with an overhead pen substituted for the pencil. Place a small piece of masking tape on the transparency plastic to provide a non-slip surface for the compass point. By attaching the overhead pen to the compass (with tape or a rubberband), circles can be drawn

Figure 14. Using construction tools on the overhead projector.
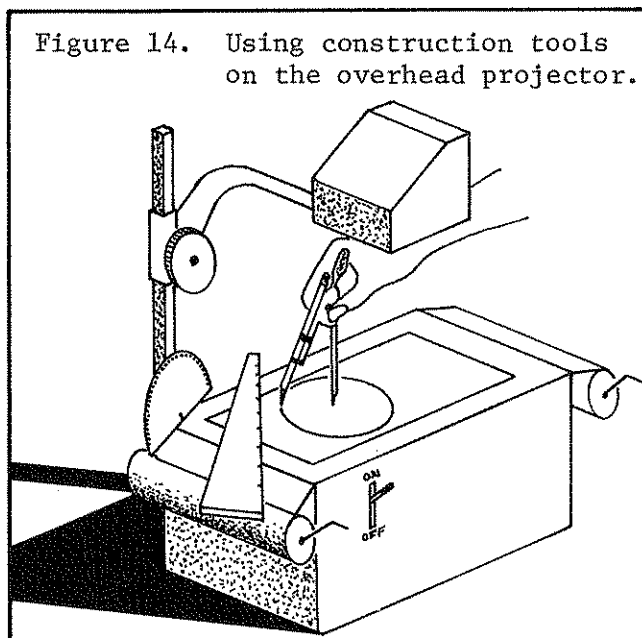
without slipping. It is also convenient to have a transparent ruler and protractor with clearly marked intervals and numbers. The students can view the teacher's hands and movements during a construction demonstration and imitate the motions. Needless to say, the teacher may need to practice making circles and designs before demonstrating such skills to the students. Setting a good example is important. Construct as you want your students to construct—carefully, neatly, skillfully.

The Graphic Computer

Our technological advances have brought us the graphic computer (Figure 15). It has become a valuable tool for visual and graphic thinking. "An interactive graphic computer will allow the visual thinker to manipulate graphic imagery in space and time, to have access to a vast visual computer memory, to decrease his involvement in routine visualization tasks, and most important, handle more complexity faster." (McKim, 1972, p. 32) The graphic computer is used in architectural design, graphic design, scientific experiments, geographical mappings, education, research and many businesses and academic disciplines.

Figure 15.   The Graphic Computer
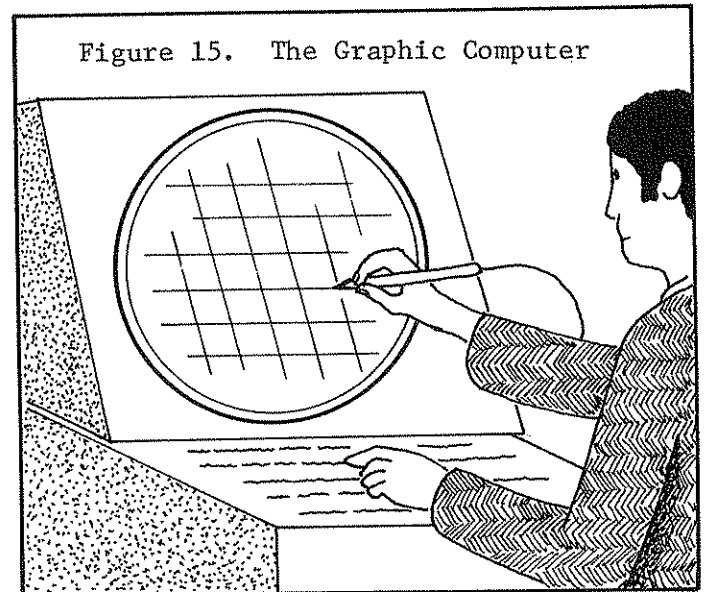
GRAPHIC REPRESENTATION REVISITED

Graphic representation is a language, a way of communicating. Many people doodle and make rough sketches, not intending to express or even communicate; it is a kind of graphic talking to oneself. However, graphic representation communicates; it is the written or pictorial version—the counterpart—of visual perception. We perceive and interpret what we see according to previous experience. We are visual thinkers. Through visual–motor coordination we attempt to draw what we see or think, knowing that our drawing will be a representation of our visual experience. The keenness of our visual perception and our ability to draw will influence the accuracy and meaning of each graphic representation we make. This is continuous interaction

Figure 16.   Interaction
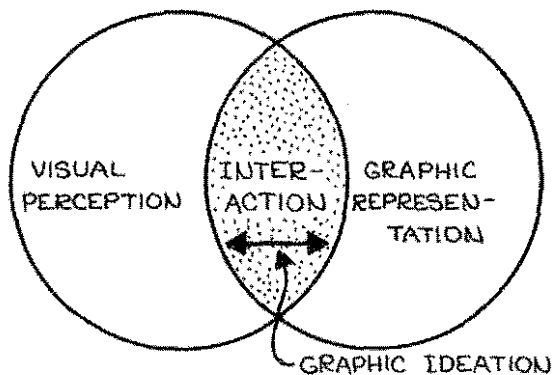
between visual perception and graphic representation (Figure 16). This is called graphic ideation--visually and graphically talking to oneself. The visual thinker must first discover an idea worth communicating. Graphic representation then follows as one seeks to communicate through sketches, charts, diagrams, drawings, illustrations, etc. It is important to remember that the drawing of an object (or idea) is not the object itself. A graphic symbol is always less than what it represents; "the word is not the thing," nor is the graphic symbol.

(McKim, 1972) Sometimes a given graphic symbol represents only ONE of many ways to view a geometric idea. If our idea is "congruent triangles," then we can illustrate the idea in various ways (Figure 17). Once our visual memory recalls what "congruent triangles" are, we can communicate the idea to others through graphic examples.

Basically, there are two kinds of graphic representations: the representations of real or concrete objects and the representations of abstract ideas. This is displayed in Figure 18. The most concrete of graphic languages are three-dimensional models and rough three-dimensional mockups. (A mockup

Figure 17.   Pairs or groups of congruent triangles

is a full-size scale model of an object usually made of soft, inexpensive material like cardboard or styrofoam.) In geometry we construct three-dimensional models of the Platonic solids and other polyhedra. Sometimes designers and engineers make three-dimensional mockups of machines, furniture and other creations so the appearance and design can be evaluated and easily modified.

Figure 18.   GRAPHIC
              ABSTRACTION
              LADDER
charts
graphs
diagrams
schematics
compass-straightedge constructions
two-dimensional geometric drawings

abstraction

Abstract Graphic
Languages

maps and scale drawings
multi-view orthographic drawings
paraline drawings
perspective drawings
rough three-dimensional mockups
three-dimensional models

Concrete Graphic
Languages

reality

Another important way of representing concrete or real objects is through drawings and projections. In previous paragraphs, multi-view orthographic, paraline and perspective drawings were presented. These types of drawings are "important to many persons of greatly varied interests. Some are interested in architecture, city planning, engineering, commercial designing, interior decorating or industrial designing. These people need a general knowledge of all methods of representing objects by drawings in order to be able to select the best type of drawing for each purpose." (Martin, 1962, p. ix)

Maps and scale drawings are representation of real things like cities, buildings, cars, our solar system, farm land and so forth. Geographers, architects, surveyors, scientists and many other people use, construct and make maps and scale drawings.

As we move on up the ladder of graphic languages, we enter the realm of the abstract. "Abstract graphic languages encode abstract ideas, not concrete things." (McKim, 1972, p. 128) Much of what we learn in geometry is abstract. Yet geometric ideas like point, line and angle are discussed and studied through two-dimensional

drawings. We use the compass, protractor and straightedge to construct the drawings
of our geometric ideas. We often just sketch points, lines and angles freehand to
supplement our speech in describing the ideas. The drawings give more precise des-
criptions of our thoughts which permit little variation in interpretation.

There are other abstract graphic languages which help us explain and understand
abstract ideas. Schematics are used to represent various designs such as the orbits
of the planets, the cross section of a polyhedron, the framework of a building or
the molecular structure of an element. Diagrams such as an electrical circuit or a
sociogram help us interpret what is happening in our work. The Venn diagram can be
used to clarify relationships and concepts (Figure 19). Line graphs, bar graphs and
circle graphs are pictures of statistical data (Figure 20). Conclusions and inter-
pretations can be made from the "graphic" data that would be nearly impossible other-
wise. Charts organize data and illustrate movement from one step to another. For
example, we use charts to show classification or stucture within an organization.
The flowchart is often used to plan a logical procedure for problem solving. All of
these abstract graphic languages enable us to organize and communicate our thoughts
and ideas.

Figure 19. Venn Diagram

PARALLELOGRAMS

RHOMBUSES   SQUARES   RECTANGLES

Figure 20. Statistical Graphs

TEMP   CARS

DAY   LINE   COUNTRY   BAR   CIRCLE

TRANSPORTATION 10%   FOOD 20%   MISC. 30%   RENT 25%   15% SAVINGS
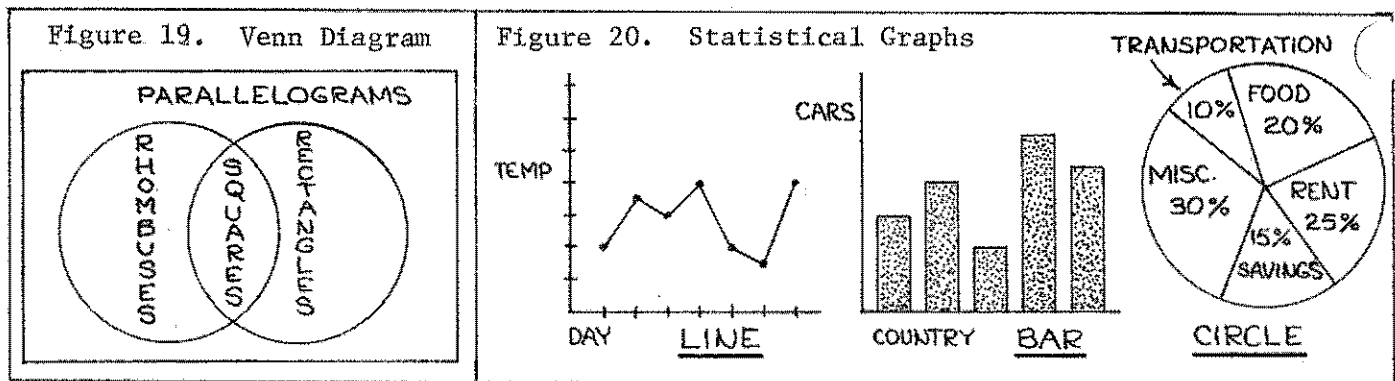
A PICTURE IS WORTH A THOUSAND WORDS

"The impulse to draw is universal in young children despite the common scarcity
of parents who draw." (McKim, 1972, p. 25) Children enjoy drawing. They color; they
trace; they scribble. As children learn motor-sensory control of their bodies, their
hands enable them to perform fine motor coordination skills such as writing and draw-
ing. The emphasis often falls on writing, but writing is quite different from gra-
phic communication. Pages and pages of words cannot describe a tree or a building or
an animal or even some ideas as well as skillfully-drawn pictures. "Almost everyone
learns to read and write in our society; almost everyone can also learn to draw. No
more habitual disclaimers about lack of artistic talent." (McKim, 1972, p. 49)

SUMMARY

1. Graphic representation is a language—a way of communicating through drawings.

2. Graphic representation is often used to illustrate a concept or relationship, to help in problem solving and to enhance the understanding of ideas.

3. Drawing is important in geometry.

    a) Knowing where to locate points can often help when constructing a picture of an idea.

    b) Various kinds of grid paper can be used to aid geometric drawing.

    c) We can represent three-dimensional objects through multi-view orthographic drawings, paraline drawings and perspective drawings.

    d) Students must be taught to interpret drawings and the objects to be drawn.

4. The basic tools used for drawing in geometry are the pencil, the straight-edge, the compass and the protractor. The teachers and students need to develop skill in using these tools.

5. There are various levels of graphic representation ranging from concrete to abstract.

6. Almost everyone can learn to draw and communicate graphically.

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     State University, 1967.

BASIC COMPASS-STRAIGHTEDGE CONSTRUCTIONS

The following diagrams explain how to do the basic constructions. Inscribing regular polygons is covered in the student materials in POLYGONS & POLYHEDRA. The grid preceding each subsection can help you find activities involving geometric constructions.

I. Copy a line segment.

To copy $\overline{PQ}$ on ray r...	... set the compass legs on P and Q.	Mark off the same distance on r.	$\overline{P_oQ_o}$ is congruent to $\overline{PQ}$.

II. Copy an angle.

To copy $\angle C$ at H_o ...	... make arcs with the same radius.	Then copy the length.	Draw $\overline{H_oJ_o}$.	$\angle H_o$ is congruent to $\angle H$.
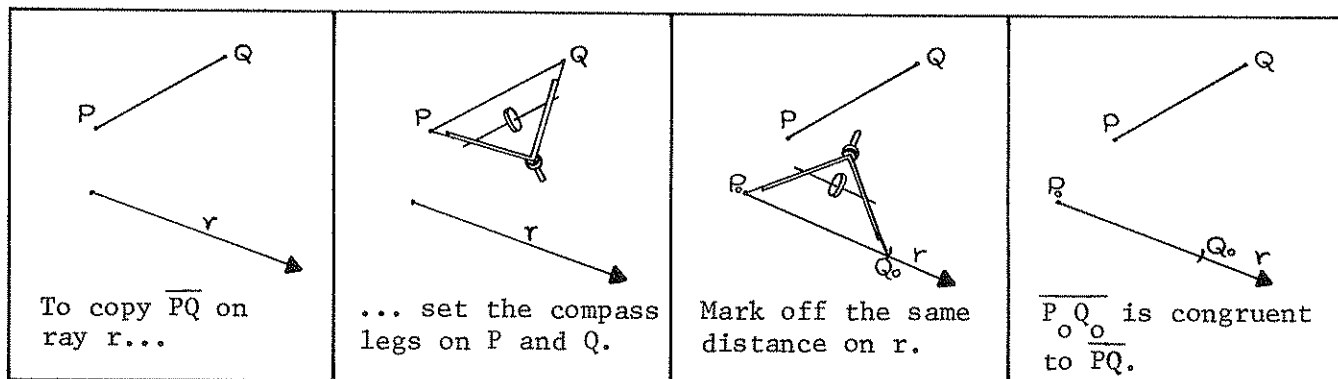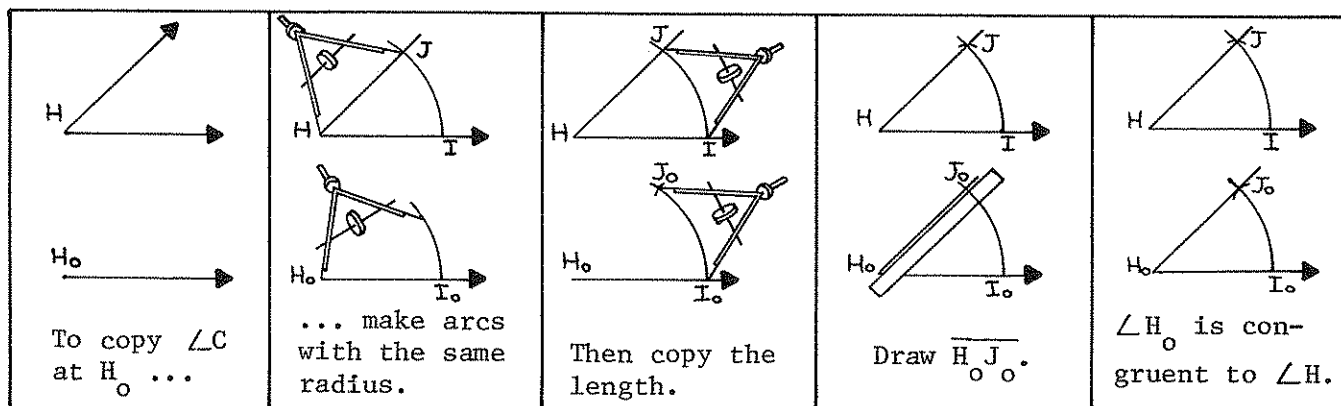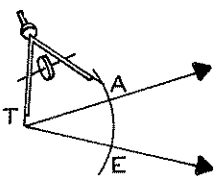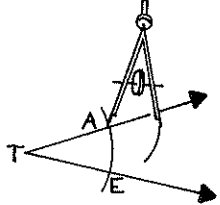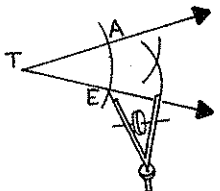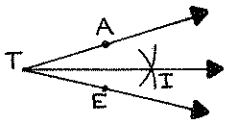
III. Bisect a segment. (Bisect means divide in half.)

To bisect $\overline{RS}$ ...	... draw an arc from R.	Use the same radius from S.	Midpoint M.
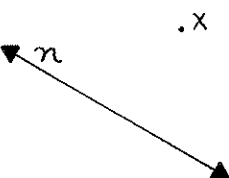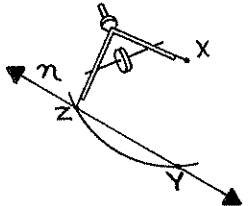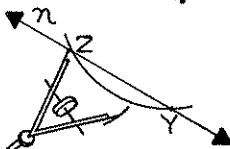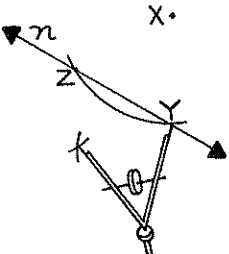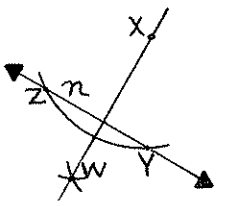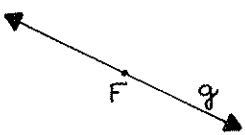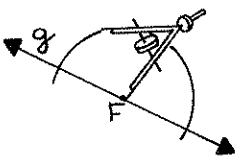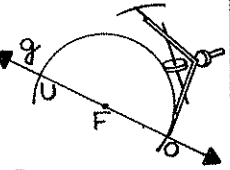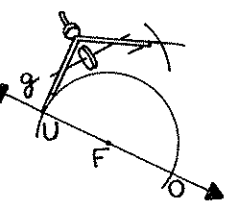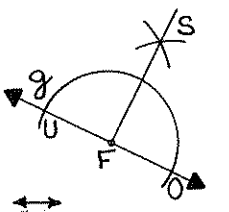
IV. Bisect an angle.

To bisect ∠T ...	... draw an arc from T.	Then draw an arc from A.	Use the same radius from E.	$\overrightarrow{TI}$ bisects ATE.
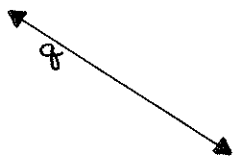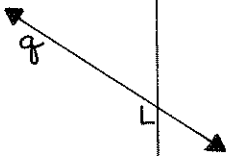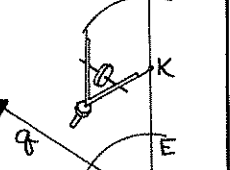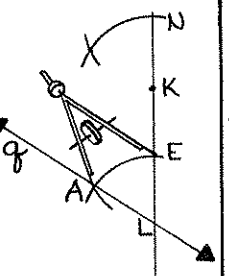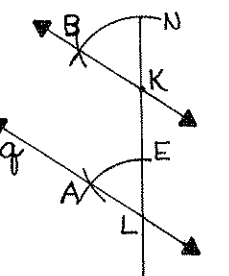
V. Construct a perpendicular from a point to a line.

To construct a line perpendicular to n through x ...	... draw an arc from x that meets n twice.	Draw an arc from z.	Use the same radius from y.	$\overleftrightarrow{XW}$ is perpendicular to line n.

VI. Construct a perpendicular to a line through a point on the line.

To construct a line through F perpendicular to g ...	... from F draw an arc that meets g twice.	Draw an arc from O with a greater radius.	Use the same radius from U.	$\overleftrightarrow{FS}$ is perpendicular to line g.
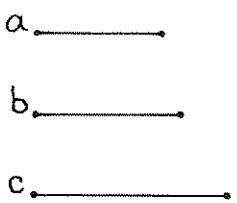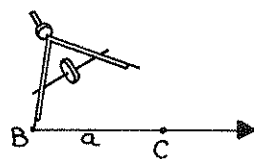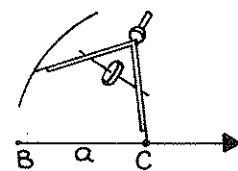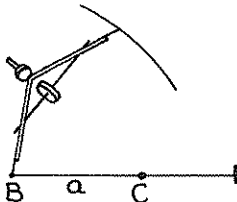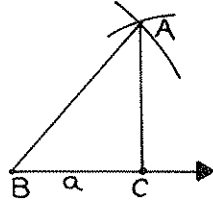
VII. Construct a line parallel to a line, through a point.

To draw the line through K parallel to q ...	... draw any line through K crossing q.	Draw an arc from K. Use the same radius from L.	Use the radius EA to make an arc from N.	$\overleftrightarrow{KB}$ is parallel to line q.

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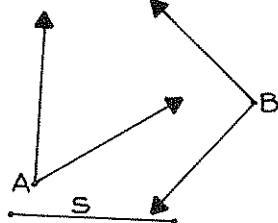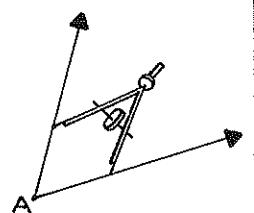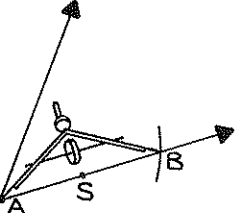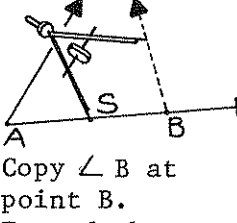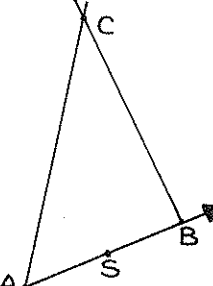
CONSTRUCTING TRIANGLES WITH GIVEN SIDES AND/OR ANGLES
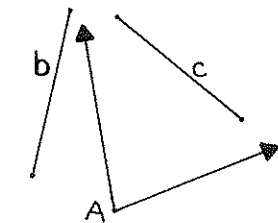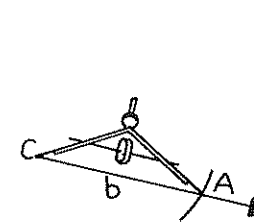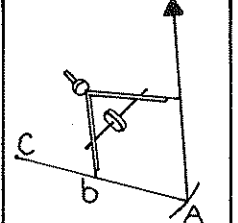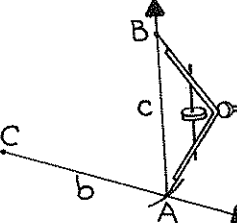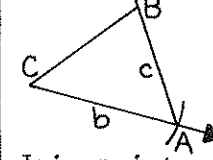
1. Side-Side-Side (SSS Construction)

To make a triangle from these three sides ...	... copy a side, say a, and label the endpoints B and C.	Draw an arc with radius the length of side b from point C. Then ...	... do the same for the length of side C from point B.	The intersection of the arcs is the third vertex A of the triangle.
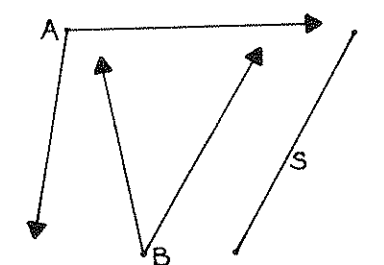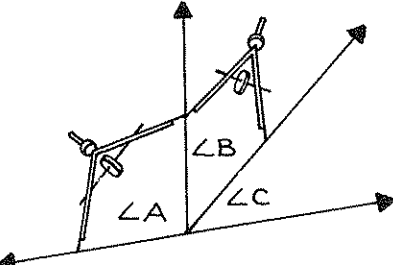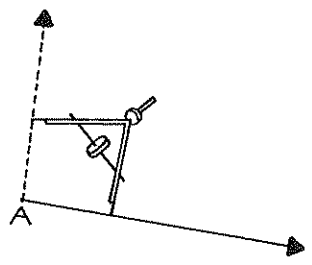
2. Angle-Side-Angle (ASA Construction)

To make a triangle from these ...	... copy ∠A.	Copy side s.	Copy ∠B at point B. Extend the rays until they meet at point C.	

3. Side-Angle-Side (SAS Construction)

To make a triangle from these ...	... copy side b.	Copy ∠A at point A.	Copy side c from point A.	Join points B and C with a segment to complete the triangle.

4. Angle-Angle-Side (AAS Construction)

To make a triangle from these ...	... draw a straight line and copy ∠A and ∠B to find ∠C.	Now proceed as in the ASA construction. Copy ∠A, copy side s, copy ∠C, etc.
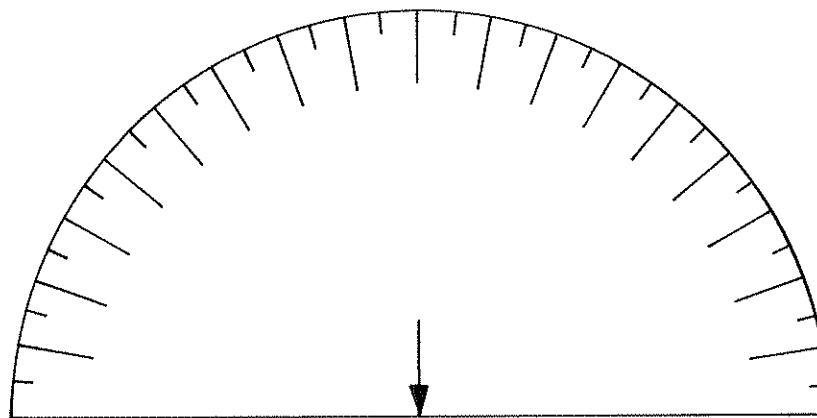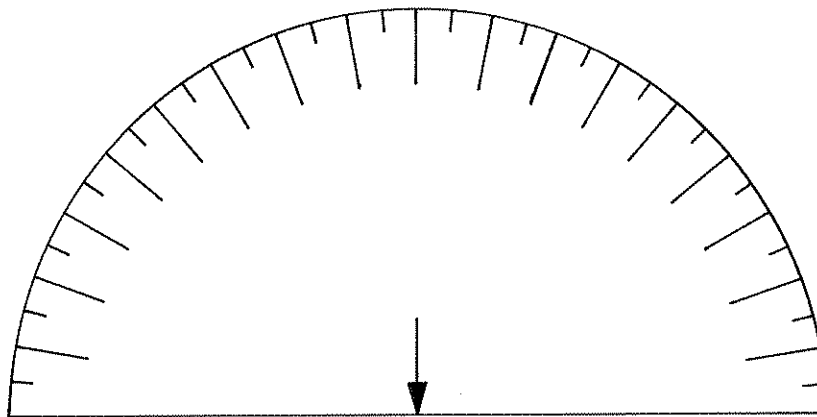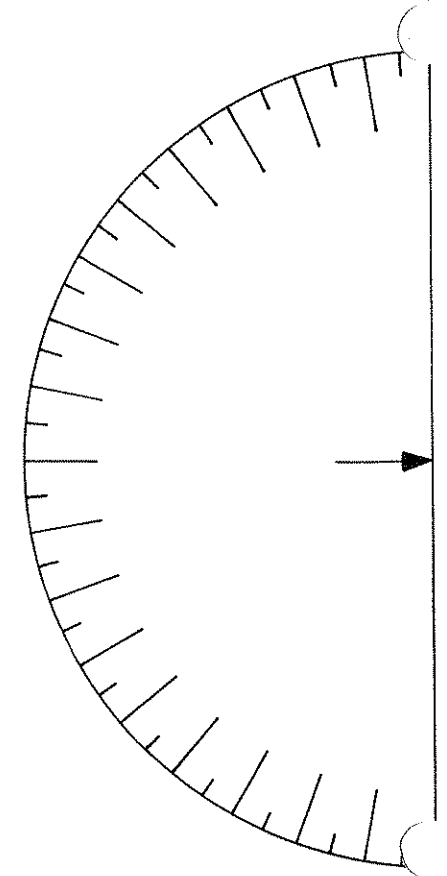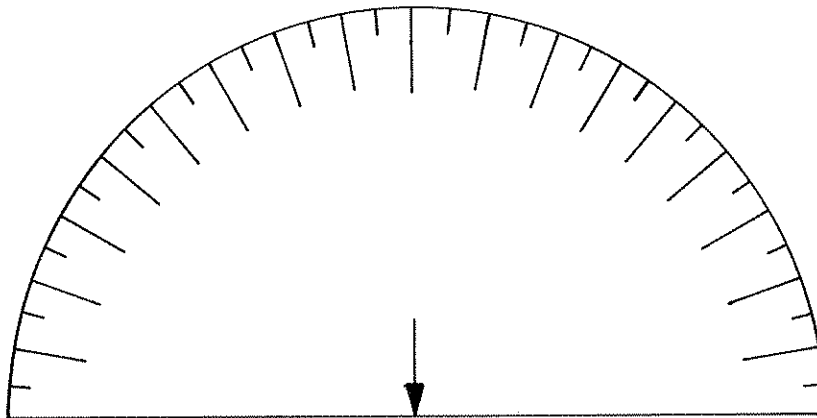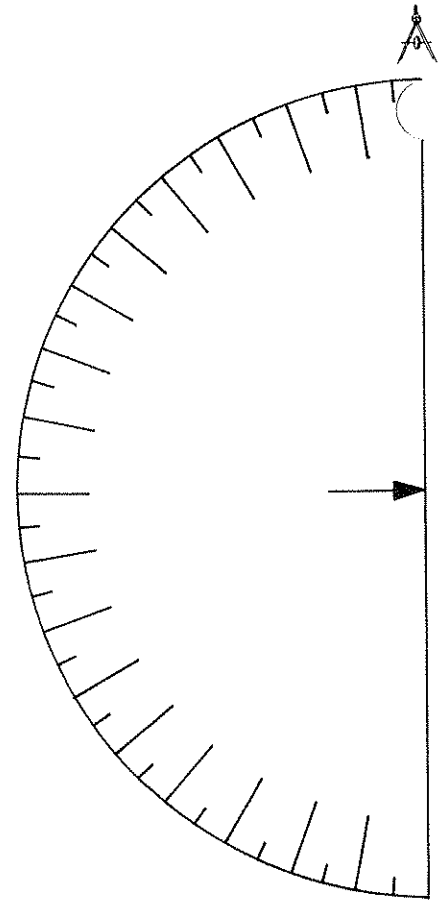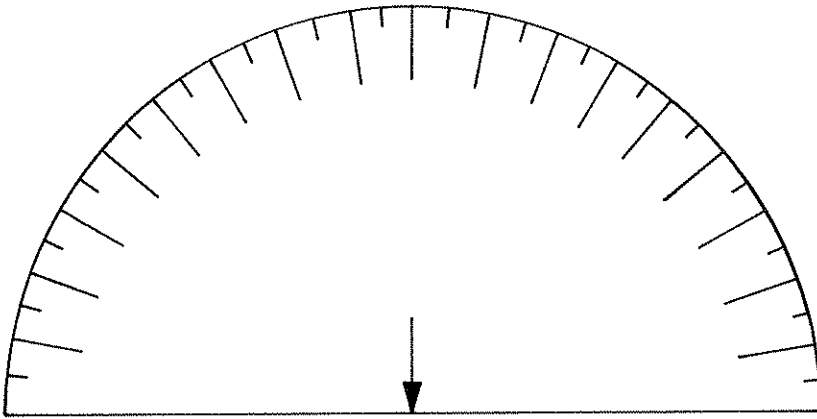
EXAMPLES OF GRAPHIC REPRESENTATION ACTIVITIES
IN THE CLASSROOM MATERIALS

I. Grid Paper

Grid paper can be used for geometric drawings in may ways. It is convenient, time-saving and it provides a base from which to record and represent geometric figures, objects or pictures.

Squared paper can be used to make designs and reflections as well as to graph points and lines on the Cartesian coordinates, to make statistical graphs or to enlarge/reduce figures. Students should be encouraged to create their own designs and reflections.

GRID REFLECTIONS
Draw the reflection of each design across the given line.

TWO DIMENSIONAL REPRESENTATIONS
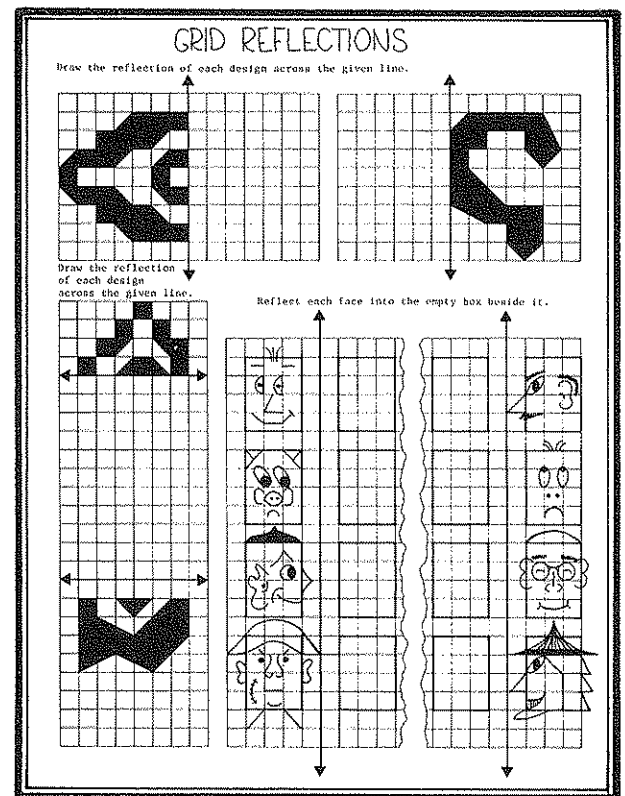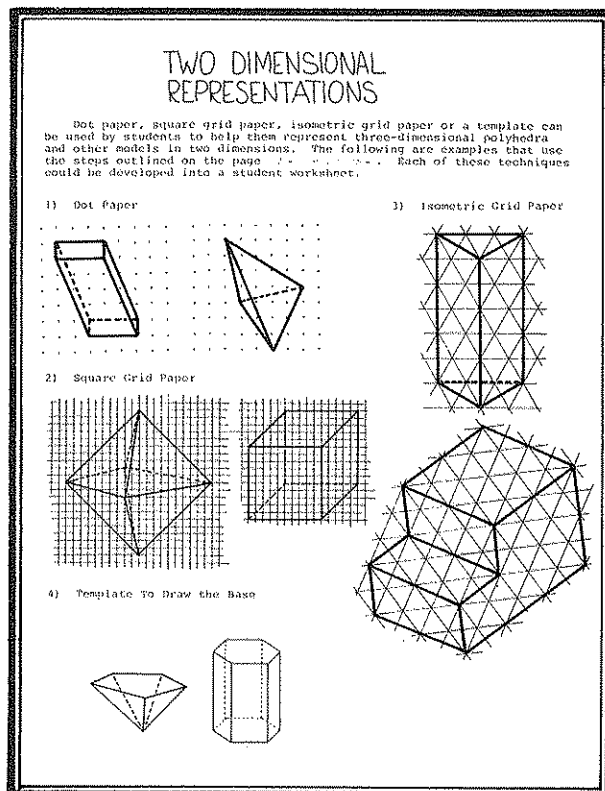
Making paraline drawings of 3-D objects is often done on dot paper, square grid paper or isometric grid paper.

II. Instruments and Tools

Drawing in geometry often requires the use of certain instruments and tools such as the template, the pantograph, the compass, the protractor and the straight-edge. Constructing angles, polygons, circles and other geometric figures can be done accurately and skillfully with practice.

ART INSIDE THE CIRCLE

Materials Needed: Compass, straightedge, colored pens or crayons

1) Have your students inscribe a regular hexagon and the two equilateral triangles, then draw the diagonals and other segments to get the pattern shown at the right.

2) Have them discover how the designs below were made.

a)   b)   c)
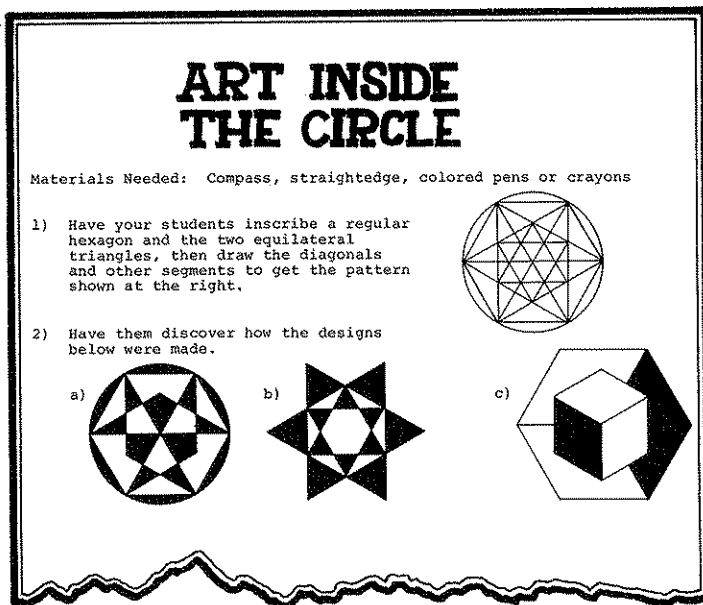
Designs made with compass and straight-edge can create a colorful bulletin board display.

Templates can play an active role in geometric drawing. Getting the proper perspective is important when representing a 3-D object in a 2-D drawing. Templates provide consistent, easy-to-use forms of similar figures.

Splitting Angles uses angle bisection to initiate a discovery activity.

This problem-solving venture incorporates the construction of stars with discovery of a formula for the number of points. Success with this activity will rely on the students motor skills and their ability to draw accurately.

III.   Sketching

   Making a sketch related to a word problem, idea or object often clarifies a situation and leads to a solution of a problem or the grasp of a concept.
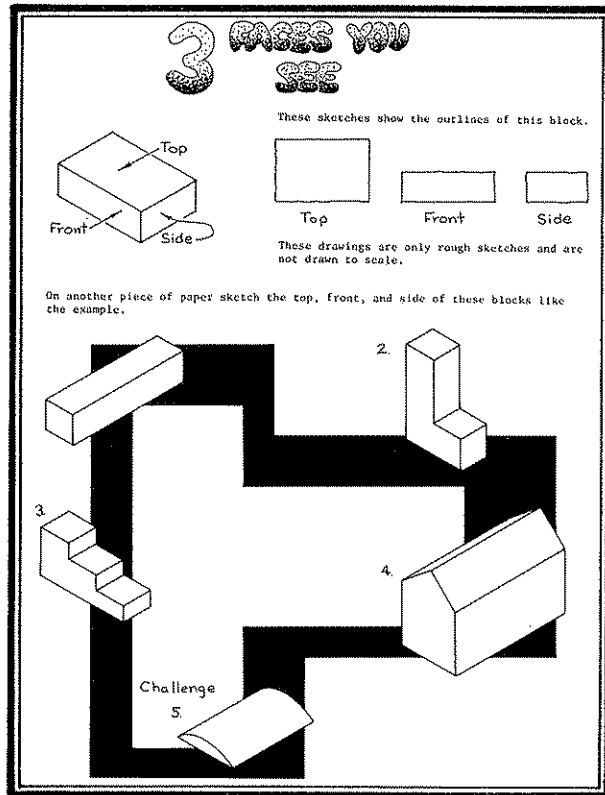
Multi-view orthographic sketching identifies the different positions from which an observer can view an object "straight on."

This game requires that players think ahead before sketching their moves.

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

CALCULATORS

RATIONALE

As early as the seventh century B.C. the counting board or abacus was invented and used for simple whole number computations. Merchants and traders of ancient times probably would have found the abacus cumbersome to carry around in their back pocket. If they were alive today, they could not only have a calculator in their pocket, but they might have a computer terminal in their briefcase! Electronic calculators are one of the hottest selling items around the world. They give instantaneous, effortless answers to many computations. They are small, quiet and inexpensive.

Using a calculator is relatively easy. You push a few buttons in sequence and "Voilà!" the keyboard display flashes the answer. "Most of us have so far explored numberland by the very laborious, manual route. The hand calculator lets you travel by automation, and explore far afield effortlessly." [Wallace Judd] Paper and pencil calculations are often slow, inaccurate and tiresome. Interest and enthusiasm for mathematics is often killed by such drudgery. The calculator becomes a fantastic tool that frees us to do investigations and problem solving. Its speed allows us to keep pace with our racing minds as we search for solutions, conjectures, and more questions.

The electronic calculator is NOT a fad; it is here to stay. Like the radio and television, soon everyone may own one (or two or three). The calculator is bound to change our way of life just as other advances in technology have. Already educators are arguing about the use of the calculator in the mathematics classroom. Should the calculator be used when teaching arithmetic skills in elementary schools? Will children need to memorize addition and multiplication facts if they learn to compute using a calculator? Will senior high students need to learn how to use logarithmic tables, or should they use an electronic calculator instead? In other words, the whole mathematics curriculum from kindergarten through college will need to make serious adjustments to account for the use of the electronic calculator. Because the calculator is becoming available to all members of our society, including children, educators will need to decide how electronic calculators fit into the school curriculum.

Recently, pocket or desk calculators have been used in mathematics classrooms to motivate students and expand their ability to solve "messy" real-world problems (i.e., stock investments, tax forms, interest on car payments, pollution controls). The calculator provides the immediate feedback of answers and a problem-solving flexibility that encourages the student to become involved in complex computations.

Perimeter, area and volume problems often involve calculations with large decimal numbers. For examples, a real estate salesman, a curtain saleslady and a driveway paver all deal with estimating costs regarding the amount and quality of their product. Each salesperson uses a calculator to give the customer immediate estimates and price comparisons. Students can also use the calculator to do estimates and comparisons as they work with applications of area and volume. Various problems posed about similarity and proportions, circles and spheres, polygons and polyhedra can initiate many complicated numerical computations that, without a calculator, would become excessive and tedious for students to solve.
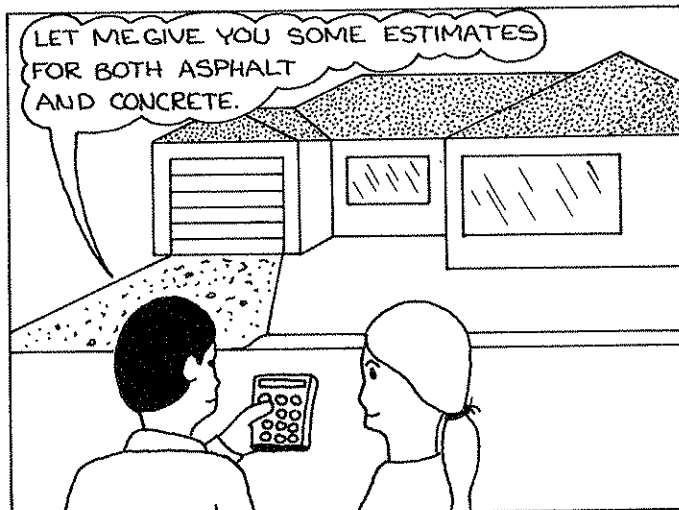
However, one needs to be careful! Most calculators do not retain and display all the numbers or operations entered. If wrong numbers are entered or operations are entered in the wrong order, the incorrect answer must be recognized by the student. To tell a reasonalbe answer from an unreasonable one, a student needs to know how to compute using the basic arithmetic facts, how to round numbers, how to estimate and approximate answers, and how to place a decimal point. Arithmetic skills and number sense are very important if the hazards of a calculator are to be avoided. The calculator does not replace thought processes. It is a tool that saves time and energy and frees us to think and do mathematics above the computational level.
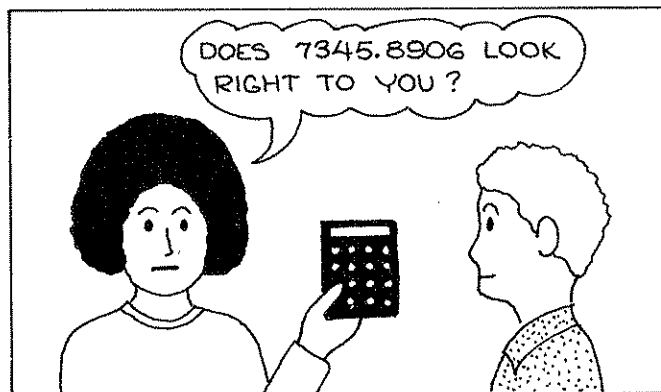
SUMMARY

I. Calculators fit into the classroom in different ways:

    1. Non-electronic calculators (abacus, etc.)

        a. teach concepts in counting, place value, and arithmetic computations, and

        b. demonstrate algorithms for solving computational problems.

2. Electronic calculators free the students from tedious pencil and paper cal-
    culations.   They allow the student to . . .
    a. speed up lengthy calculations, and
    b. investigate and work on mathematical problems and applications that would
        otherwise involve long, unmanageable calculations.

II. The teacher can prepare students for electronic calculators by . . .
    1. emphasizing estimation and approximation skills which are vital in checking
        answers and placing the decimal point correctly.
    2. teaching the student to determine the reasonableness of exact answers by
        approximate calculations.
    3. introducing situations and problems where the hand calculator is an obvious
        aid to cumbersome, time-consuming calculations.
    4. asking students what types of mistakes can be made while using the calculator.

III. Teachers can prepare themselves for using the electronic calculator in instruc-
    tion by . . .
    1. experimenting with it themselves.   (Let the students see the teacher using a
        calculator.)
    2. reading current periodicals and checking the mathematics publication companies
        for new "calculator" books.   (There is currently no standard body of knowl-
        edge about how to use a calculator in the classroom.)
    3. having an open mind about the use of the calculator before deciding that the
        calculators will be a "cure-all" to teaching computation, or that they should
        be banned from the mathematics curriculum.

Selected Sources for Calculators

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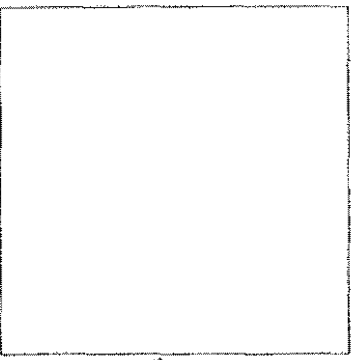
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EXAMPLES OF CALCULATOR ACTIVITIES IN THE CLASSROOM MATERIALS

Finding the area of a circle involves the multiplication of complex decimal numbers like π. The electronic calculator frees the student to look carefully at the problem and explore possible ways of solving it rather than draining the student's energy through tedious, frustrating calculations.

Measurements in millimetres must be taken. Once the data is recorded in a table, the calculator can be used to perform the calculations necessary for finding the decimal approximation of π.

Given three lengths for the sides of a triangle, determine whether the lengths will form a triangle. Use a calculator to find patterns and make comparisons.

A SPIRAL IN A GOLDEN TRIANGLE

A golden triangle is one of the triangles found in a regular pentagon by connecting the end points of a side to the opposite vertex. (See △ABC.) It is an isosceles triangle with angles of 36°, 72° and 72°. The ratio of BC to AB is the golden ratio which is approximately .618.

Beginning with a golden triangle ABC, a series of similar golden triangles can be constructed with these steps.
1) Bisect angle B.
2) Label the new point D.
3) Triangle BCD is a golden triangle.
4) The ratio of CD to BC is approximately .618.
More golden triangles can be constructed by bisecting in order angle C, angle D, etc.

As a suggested activity have students:

A) construct a golden triangle with a base of 15 centimetres. This fits on a 8½ x 11 sheet of paper if the base is drawn near the bottom.

B) construct a series of similar golden triangles.

C) measure to the nearest millimetre AB, BC, CD, DE, EF, FG, ...

D) use a calculator to find the ratios BC:AB, CD:BC, DE:CD, EF:DE, FG:EF, ... The ratios should all approximate the golden ratio.

E) draw a smooth curve through the vertices A, B, C, D, E, F, ..., to form a spiral.

Finding the volume of a sphere $(V = \frac{4}{3}\pi r^3)$ can be easily accomplished with the aid of a calculator.

WHEN YOU'RE HOT YOU'RE HOT

All hot water heaters in Cubesville are built in the shape of cubes. Cid needs to buy a new one and wants to buy the most efficient size in order to save energy. The salesman told Cid that the surface area of the heater allows the heat from the water to escape and the most efficient size is one with the smallest surface area per litre of water. Complete the chart below to help Cid make a decision on the most efficient of the four hot water heaters to buy.

REMEMBER: 1000 cm³ = 1 LITRE

	1	2	3	4
LENGTH OF EDGE (cm)	40 cm	60 cm	80 cm	100 cm
SURFACE AREA (cm²)				
VOLUME (cm³)				
VOLUME (LITRES)				
SIMPLIFIED RATIO OF SURFACE AREA (cm²) TO VOLUME (LITRES)	:	:	:	:

1) Which water heater is the most efficient? _____ Can you explain? _____

2) What other factors should Cid consider before he buys a water heater? _____

3) Could there be a hot water heater with a simplified ratio of 1 cm²:1 litre? _____ What would be the length of an edge? _____

4) Water has a mass of about 1 kilogram per litre. How much mass will there be for the water in heater #4? _____

Use the calculator to find the ratios of the lengths of the triangles' sides. The ratios approximate the golden ratio 0.618.

NESTED SPHERES

I. A sphere which is nested inside a cube has its center at the center of the cube and is tangent to all six faces.

What is the diameter of the sphere? _____ cm

What is the radius of the sphere? _____ cm

What is the volume of the sphere? _____ cm³

VOLUME OF A SPHERE CAN BE FOUND BY USING THE FORMULA $V = \frac{4}{3}\pi r^3$

II. Imagine making three slices with a knife as in the picture.

How many smaller congruent cubes are formed if the cuts pass through the midpoints of the edges? _____ As in part (I) spheres are to be nested in each of the smaller cubes. What is the diameter of one of the smaller nested spheres? _____ cm Its radius? _____ cm Its volume? _____ cm³ What is the total volume of the eight smaller spheres? _____ cm³

III. Making more equally spaced slices produces more and smaller cubes. A sphere can be nested in each cube. Imagine making the number of cuts listed in the table. Complete this table. Do you see anything special in the table?

NUMBER OF EQUALLY SPACED CUTS	TOTAL CUTS	NUMBER OF CONGRUENT CUBES	VOLUME OF ONE NESTED SPHERE	TOTAL VOLUME OF ALL NESTED SPHERES
(PART I ABOVE) 0	0	(THE ORIGINAL) 1		
(PART II ABOVE) 1	3			
2	6			
3	9			

When doing large number computations and comparisons, the calculator provides quick, accurate answers that facilitate the problem-solving process.

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

APPLICATIONS

RATIONALE

Over 2000 years ago man developed number symbols, arithmetic calculations and geometry to describe and record real-world happenings. Mathematics was used to solve the problems of merchants, scientists, builders and priests.

About 600 B.C. Greek mathematicians took a different approach. They began studying numerical patterns and geometry for their aesthetic qualities. Mathematics became an intellectual exercise with no necessary applications in mind. The development of mathematics was soon traveling in two directions: practical or applied mathematics, originating from the Egyptians, and "pure" mathematics, originating from the Greeks.

Practical and "pure" mathematics are not always separable. One often inspires and directs the other; they become interwoven. As a result, applications of mathematics fall into three categories:

1) applications to real-life situations such as business, finance, sports, polls and census taking, carpentry, building and road construction, graphic design and advertising.

2) applications to other disciplines (i.e., science, music, art, astronomy, geology, aeronautics).

3) applications to other branches of mathematics (problem-solving activities in the realm of "pure" mathematics).

The Egyptians, for example, were interested in learning as much as they could about their environment and how to control it. Today we are also curious about the rapidly changing environment we have created. Because of the complexity of our culture and its emphasis on technology, mathematics is very important to us in our jobs, in our daily living and in our future.

We face many problems in our daily living. Since all problems require the collection of information before solutions can be found and analyzed, mathematics is often a helpful tool in solving problems; yet few people relate mathematics to real-life situations or real-life situations to mathematics.

Many teachers have neglected to teach applications of mathematics, yet educators and the public agree that applications of mathematics are very important and should be taught in the mathematics classroom. Society is demanding accountability and relevancy in our education system. Students need ample opportunity to experience mathematics in a practical sense so that they will be better equipped to apply it as adults.

Applications should include appropriate topics and activities. Here are a few questions to consider when choosing an application of mathematics:

a) Is it interesting to the students and the teacher?

b) Does it start at the appropriate skill level?

c) Does it extend and develop the computational and/or problem-solving skills of the students?

d) Does it include topics, skills or ideas which might help the students contribute to society and deal with real-life situations?

e) Could it be done as a laboratory activity?

f) What concepts does it employ and develop?

g) How much time would it take to teach?

h) What equipment and materials are needed or available?

SUMMARY

1. Applications of mathematics fall into three categories:

   a) applications to real-life situations

   b) applications to other disciplines, and

   c) applications to other branches of mathematics.

2. Down through the centuries, mathematics has been a useful tool for solving real work problems and analyzing our environment.

3. Even though many teachers have neglected to teach applications of mathematics, our complex society demands that public education teach practical mathematics and problem-solving techniques.

4. Mathematics can be used to solve problems in the real world and in other disciplines.

5. Applications to real-life situations and other subject areas (i.e., physics, social science, economics, art, music) make abstract mathematics more meaningful and understandable.

6. Applications should include appropriate, interesting topics and activities for students and teachers.

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EXAMPLES OF APPLICATIONS IN THE CLASSROOM MATERIALS

I. Applications to Real-Life Situations
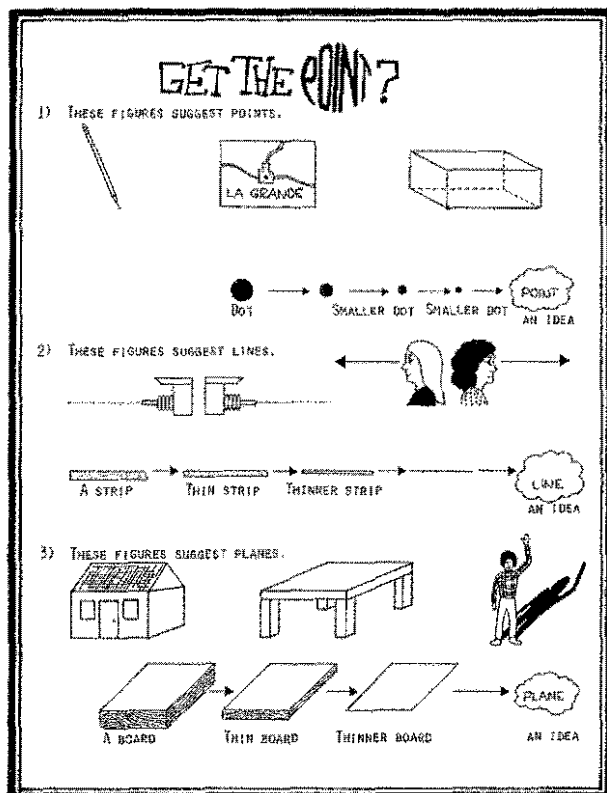
There are many ideas or objects in our everyday settings that help express the meanings of point, line and plane.

In this activity a student uses angles to find the bearing of various objects in relationship to the student's location.
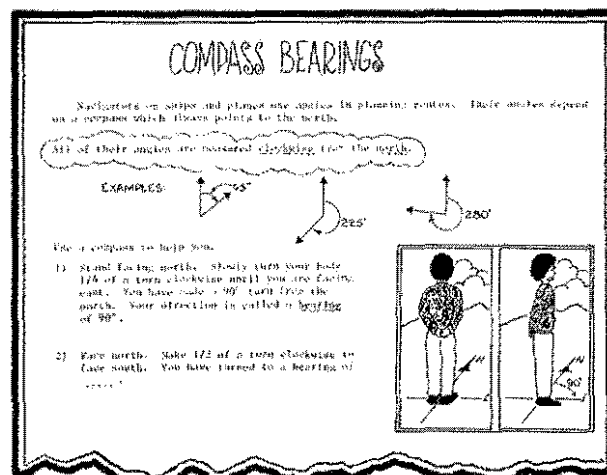
Whether we are putting away left-over food in a plastic container or using a monkey wrench to tighten the nut and bolt on a bike seat, we run across many shapes, containers and tools that require us to make decisions about fitting things into place.

PACKING CIRCLES

I   Students need approximately 36 pennies or disks all the same size. Provide each with several differently shaped regions. The students are to cover the surfaces with the disks. Can the disks be arranged in more than one way?

Circular Regions   Rectangular Regions   Triangular Regions

II Students should discover two ways to pack circles. In each arrangement how many circles are tangent to each circle?

In which arrangement are the circles packed so that less space is wasted?

FIGURE 1   FIGURE 2

III When fragile glasses are packed into a crate, stiff cardboard is needed between them. Supply students with the two types of circle arrangements. Have them mark both arrangements to show where the cardboard would be placed.

Ask what kind of polygons are formed by the separators.

FIGURE 3   FIGURE 4

Why is square packing often used when separators are needed for fragile goods and hexagonal packing is used when separators are not needed?

IV Ask students to look for some examples of circle packing. Have them make drawings and explain the type of packing used for:

a) bottles in a crate
b) box of school chalk
c) cigarettes in packages and tins
d) boxes of eggs
e) boxes of oranges, apples, grapefruit
f) tread on some kinds of tennis shoes

WHAT TYPE OF PACKING DO BEES USE IN CONSTRUCTING A HONEYCOMB?

Students find out which size soft drink gives them the most for their money.

MOM, DAD AND THE KIDS

I   How many different ways can the lids be placed on these containers?

1) Rectangular bread container _____
2) Square sandwich container _____
3) Circular cake container _____

II How many different ways can the hex wrench, the wrench or the pliers be placed on the appropriate nut?

1)   2)   3)

The kids just got a new game where they try to see who can put the pieces of the game into the appropriate hole in the shortest amount of time.

III 1) How many different ways can each piece be placed in the game board so that the shaded side is showing?

a)   b)   c)   d)   e)   f)   g)   h)   i)   j)

2) How many different ways can each piece be placed in the game board if it doesn't matter if the shaded side is up?

a) _____   b) _____   c) _____   d) _____   e) _____
f) _____   g) _____   h) _____   i) _____   j) _____

Packing items of various shapes can present the problem of wasted space. Students can discover ways of packing these shapes to attain less wasted space and a convenient assembly of items into crates.

Cheapest Drink

GET – Box of assorted drinking cups from local restaurants for this year

1. Which restaurant sells the cheapest large size soft drink? Medium size soft drink? Small size soft drink?

2. Which size drink gives you the most for your money? Make a guess. Record your guess.

3. Now think of a way to actually find out which size soft drink gives you the most for your money.
   Use your procedure to find the answer.
   Record your data in a table.
   Which size was the best buy? Record.

Pizza Parlor

GET — Cardboard pizzas
Pizza menu

1. Study the menu.
   a. Which kind of pizza would you like to order? How much does each size of this pizza cost?
   b. Look at the prices of some other pizzas. When you buy larger sizes, do their prices seem to increase in a reasonable way? Explain.

2. Find the most expensive pizza on the menu. Which pizza size do you think gives you the most for your money?
   a. Think of a way to solve this problem. Then talk with your teacher about your plan.
   b. Solve the problem. Show your steps so that they can be understood.

To determine the amount of money a newspaper receives from its commercial ads, the students approximate the area of the advertising sections and multiply by the price per square centimetre.

Does the largest size pizza really give you the most for your money? Students compare the areas of the different-sized pizzas with their prices to determine the best buy.

SURVEYING THE SITUATION

Often a surveyor finds it impractical to measure distances directly with a tape measure. They find these distances indirectly by using a transit and similar triangles.

To measure the distance across a wide river a surveyor sets up a base line along the river bank, marking the end points with stakes A and B. The transit (an instrument equipped with a telescope and protractor) is placed over stake A and the surveyor turns the telescope to see stake B. By rotating the telescope again to sight a landmark (a tree or rock) across the river, the surveyor measures the angle at A. The transit is then placed over stake B and a similar measurement is made.

On paper a triangle similar to the one above is drawn. The length of A'B' determines the similarity ratio or scale factor of the drawing. A convenient scale to use is 1 cm:10 m. Since $\angle A' \cong \angle A$ and $\angle B' \cong \angle B$ the scale drawing is similar to the original triangle.

By measuring TR on the scale drawing and setting up a proportion the corresponding measurement in the original triangle can be found. TR = 7 cm so 7 cm:distance across the river = 10 cm:100 metres. Thus the distance across the river is 70 metres.

Students can use the method outlined above to measure distances indirectly or to make a scale drawing of a field or playground.

If a commercial transit is not available students can assemble the homemade transit pictured to the right.

IT ALL 'ADS' UP

Materials: Newspaper

1) Look for examples of large, medium and small size commercial advertisements in your newspaper. Select five of the most interesting. Find and record the width and length of each ad to the nearest half centimetre.

AD	WIDTH	LENGTH
# 1		
# 2		
# 3		
# 4		
# 5		

SAVE $2.00
10-gal. All-Glass Aquarium
Regular $7.99
$5.99

2) Find the perimeter and area of each ad.

AD	PERIMETER (cm)	AREA (cm²)
# 1		
# 2		
# 3		
# 4		
# 5		

3) If you were to sell your five advertisements for $.50 per square centimetre, how much money would you receive? _____

Distances can be measured indirectly using surveying devices. The surveying method involves the use of transit and similar triangles.

MAKE A DIP STICK

This activity calibrates a dowel which can then be used as a dipstick to find the amount of liquid in a container.

Equipment: Eight to ten containers approximately the same height but having different shapes, e.g., detergent bottle, starch bottle, pop bottle, catsup bottle, milk carton, vase, bubble bath containers

Eight to ten thin wooden dowels

Eight to ten graduated cylinders that measure in ml (medicine cups from a hospital work nicely)

1. (a) Use an irregularly shaped bottle for a classroom demonstration. Let the students make conjectures about where the marks will appear. Pour 50 ml of water into the bottle. Carefully lower a thin dowel into the bottle until it touches the bottom. Lift the dowel out and mark the water level. Repeat the procedure until the bottle is full.

   (b) The dowel is now calibrated to measure the amount of liquid in the bottle to the nearest 50 ml.

   (c) Discuss how the spacing of the marks is related to the shape of the bottle.

II. Applications to Other Disciplines

Students will need to be geared up for this activity. Perhaps a class discussion and demonstration would clarify how gears work to bring about motion in opposite directions at varying speeds.

How does the gas station attendant measure the fuel in the station's tanks? Students calibrate a dowel which can then be used as a dip stick to measure the volume of fluid in a container.

GROOVING AROUND

1) Gear A turns in a clockwise direction. In what direction do you think gear B will turn?

2) When gear Y makes one complete turn, how much of a turn will gear X make? _____

   How might these gears be used in counting? _____

   Can you think of any measuring devices around the home that use this idea? _____

3) If gear A makes three turns how many turns will gear B make? _____ gear C? _____

   Suppose gear B is removed and gear C touches gear A. How many turns will gear C make when gear A is turned three times? _____

   Does gear B make a difference in the ratio of turns for A to C? _____

4) 48 TEETH / 16 TEETH / 32 TEETH / 8 TEETH

   In the diagram the two dotted gears are joined together so that they turn at the same speed. The other two gears turn separately.

   How many times does the 16-teeth gear turn for every turn of the largest gear? _____

   Can you think of something you look at often which needs two gears that turn in this ratio? _____

5) Sometimes gear wheels are at right angles to each other. Why are the gear wheels arranged this way? _____

   Can you find some examples of these gears in your home? _____

   If gear A has 30 teeth and gear B has 50 teeth, how many times will gear B turn when A makes 5 complete turns? _____

Many companies and businesses are readily recognized by their trademark(s). The graphic designs of these trademarks are often symmetrical and exhibit a number of fundamental geometric shapes.

In this activity students compare an
object with its projected image. They
can measure the corresponding sides
and angles.

REFLECT ON THAT IMAGE

Materials Needed: Concave mirror, candle, 4 cm x 4 cm pieces of glass or plastic, clay

The type of mirror needed for your investigation is a concave mirror. An ordinary magnifying cosmetic or shaving mirror will work.

1) Find the focal length of the mirror. If the sun is out you can reflect the sun's rays to a fine point on a piece of paper. Use caution! If you have a good mirror it will have the same effect as a magnifying glass and may give a very hot focal point. If the sun is not out use any object that is a long way off and project its image onto a piece of paper. The focal length is _____ cm.

THE POINT WHERE ALL THE SUN'S RAYS ARE FOCUSED IS THE FOCAL POINT. THE DISTANCE BETWEEN THIS POINT AND THE MIRROR IS THE FOCAL LENGTH.

2) Describe images.

   a) Put a candle in front of a mirror at a distance slightly greater than the focal length. Using a piece of paper as a screen find the image (see the figure to the right). Describe the image using general terms (bigger, smaller, same shape, etc.)

   IMAGE OF CANDLE FLAME ON A PIECE OF PAPER

   THE OBJECT MUST BE AT A DISTANCE FROM THE MIRROR GREATER THAN THE FOCAL LENGTH

   b) Move the candle about 2 metres from the mirror. Find the image with the paper. Again describe the image in general terms.

III. Applications to Other Branches of Mathematics

A CAN CAN DO IT

I   Place a 3 lb. coffee can on a table so that the bottom faces the class. Tape a marker on the rim of the can.

Question: What sort of a path will the marker make if the can is rolled along the edge of the table?

Encourage students to make predictions. Each guess could be sketched on the chalkboard. Roll the can and ask students to refine their guesses.

II The path can be drawn by using the coffee can. Loop a string around the can and fasten it to the ends of a board to keep the can from slipping as it rolls. Tape a pen to the inside of the can. Fasten a sheet of paper to the wall. When the can is rolled along the wall the pen will draw the path on the paper. The path, called a cycloid, was first conceived by Mersenne and Galileo Galilei in 1599. (See the diagram on the next page.)

The rolling motion of a can can serve
as a point of interest when studying
about a cycloid curve.

A CASE OF COLOR
(CONTINUED)

4) Here are two maps that need only two colors. Color them.

5) Color the map below. How many colors did you use? _____ Compare your answer with a friend.

6) What are the fewest number of colors needed to color the maps below? Write your answer next to each map after coloring.

Map coloring is an application of the
unproved theorem: Four colors are
enough to color any map!

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What is a catenary curve? The students
study this common curve by working
through the activity questions.

HANGING TOGETHER

Telephone wires or electric cables usually hang in
a special curve between the posts that support them.

This curve is called a catenary (kat'-ə-ner-ē).
Catena is Latin for chain. A catenary is the
curve in which a looped chain hangs.

The shape of the Gateway Arch in St. Louis, Missouri,
is a catenary. (The construction of the arch could
be a topic for student investigation.)

Students can make a graph of a catenary by the following method.

1) Fasten a large sheet of butcher paper to the
    wall.

2) One student holds a chain in front of the
    paper and a second student sketches the curve.

    Different catenaries can be sketched by vary-
    ing the position of hands. Students can
    investigate holding chains of different
    lengths or chains of differently sized links.

3) The following points approximate the graph of
    a catenary when graphed on cm grid paper where
    each unit represents 3 cm.

    A ————————▶ 3   cm
    Z and B ——▶ 4.6 cm
    Y and C ——▶11.3 cm
    X and D ——▶ 30   cm
    W and E ——▶ 81   cm

    Students can test the graph by holding a
    chain in a suitable position.

    Students may think
    that a catenary
    and a parabola are
    the same curve.
    They look alike
    at the vertex but
    a parabola opens
    more rapidly.

CATENARY   PARABOLA   VERTEX

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

PROBLEM SOLVING

Teaching for effective problem-solving behavior is itself a challenging problem. Providing instruction in problem-solving processes is an important goal of mathematics education. This section considers some aspects of problem solving and suggests one way to go about teaching for more effective problem solving.

WHAT IS A PROBLEM?

Every mathematics textbook includes activities labeled EXERCISES or PROBLEMS. There is often a distinction made between those two words. Consider the following:

    a. A polygon has several 90° and 100° interior angles. There is a total of 750° in all the angles of these two sizes. How many angles of each of these sizes are there in the polygon?

    b. A polygon has several 90° and 100° interior angles. There are five 90° angles and three 100° angles. What is the total number of degrees in all these angles?

The number of degrees in (b) is simply calculated:

$$5(90) + 3(100) = \text{total number of degrees in the angles.}$$

The first example, however, is not quite so straightforward. (How would you try to solve it?) The degree of straightforwardness can be used to distinguish between problems and exercises.

The essential difference between a problem and an exercise is that for a problem there is no fact or algorithm readily accessible to the problem solver to assure a solution. He must think of a way to put together all the information at his disposal to achieve the desired outcome. Since different people have different backgrounds (and memories), what is a problem for one person may be an exercise for someone else or for the same person at a later time. For example, consider the following:

What is the sum of the angles of a convex polygon of 20 sides?

For a student meeting the question for the first time it is certainly a PROBLEM. But the solution may provide the student with a generalization (stated as a formula) that can be used to find the sum of the angles of any convex polygon in the future. All such questions encountered later will be EXERCISES.

WHY ATTEMPT TO TEACH PROBLEM SOLVING?

One objective of education is to develop reasoning skills in students to prepare them better for life in the real world--and no one will argue with the statement that the real world is full of problems to be solved. Yet mathematics students often feel helpless if they forget a formula or a method of solution to a "type" problem. Once outside the classroom, students may not be able to use the algorithms they have "memorized" because they do not recognize when the algorithms can be applied. When they come across ambiguous, disorganized situations, they may not know how to proceed. Students who are accustomed to using a problem-solving approach do know how to begin the search for a solution. Hence, it seems that teachers should pose and provide settings that have no obvious method or algorithm to follow in reaching a solution--that is, teachers should supply experience with problems.

WHY ATTEMPT A VARIETY OF PROBLEMS?

George Polya, in his book How to Solve It, suggests that one becomes a good problem solver by solving a variety of problems. This philosophy, based on such old proverbs as "practice makes perfect" and "I do and I understand," implies that there is a certain amount of skill required for effective problem solving. There is an analogy that may be drawn between the artist at solving problems and another artist-- the musician, the dancer or the athlete. In each case perfection of performance depends on the ability to use certain techniques which must be exercised continually. This is clear in the case of the performing artist, but what are the techniques re- lated to mathematical problem solving? Several general problem-solving techniques are given and illustrated in the following pages. In addition to these general tech- niques and obvious ones like the four basic operations, the problem solving "box of tools" must include lots of specific concepts and facts like these: the sum of the degree measures of the angles of a triangle; the formulas for the area of a triangle, a square, a circle; the Pythagorean theorem; etc. (Some of these facts might very well be established through a problem-solving approach!) By solving a variety of problems students also develop a sense of appropriate techniques to use in attacking new prob- lems. Solving a variety of problems, particularly if they are well chosen sequences of problems, will provide students with experiences that develop techniques necessary to become proficient. Student success will encourage positive attitudinal results and the motivation to attempt more difficult problems.

HOW DO SUCCESSFUL PROBLEM SOLVERS ARRIVE AT SOLUTIONS?

Now that we have looked at the What and the Why's, we come to the heart of the matter--the How. Just how does a good problem solver solve a problem? If there is no algorithm at hand to lead to the solution of a problem, is there any set of "help- ful hints" to serve as a guide? Recent research in problem solving has been concerned with just that--HEURISTICS--"rules of thumb" or techniques that aid a problem solver in reaching the goal. A heuristic differs from an algorithm in that there is no assurance of success if it is used. Polya suggests several heuristics in the form of questions for problem solvers to ask themselves. He classifies these questions into four categories parallel to what he describes as the four phases in the solution of a problem. These phases include:

- *UNDERSTANDING THE PROBLEM*
- *DEVISING A PLAN*
- *CARRYING OUT THE PLAN*
- *LOOKING BACK*

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Polya further suggests that <u>experience in solving problems</u> and <u>experience in observing problem solvers</u> at work must be the basis on which heuristics are built. Let us, then, look at some problems to get the flavor of Polya's approach and to illustrate the use of heuristics. (A list of several heuristics is given following the references.)

PROBLEM 1

A number represented by the pattern in each picture below is called a triangular number. (Since the pattern can be built up gradually, we can speak of the <u>first</u> triangular number, the <u>second</u>, etc., as labeled below.) Find the 50th triangular number.

```
                              .
                    .        . .
          .        . .      . . .
.        . .      . . .    . . . .     etc.
1         2         3         4
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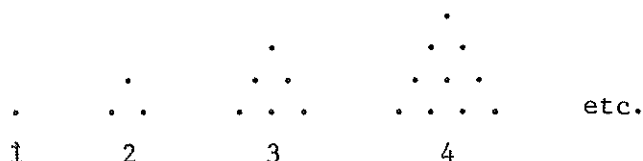
■ *UNDERSTANDING THE PROBLEM*

I must find the 50th triangle formed by repeatedly adding a line of dots to the preceding triangle.

■ *DEVISING A PLAN*

● *Can I Find a Pattern?*

Each triangle is formed by adding another row of dots to the preceding triangle. In fact, the nth triangular number has n dots in its last row. One obvious (and valid!) solution, therefore, would be to draw the dots until a triangle with fifty dots in the last row is formed, and then count the dots. There must be an easier way!

● *Organize the Data into a Table*   ● *Look for a pattern*

n (# of triangular number)	T (triangular number)
1	1 ⟩ + 2
2	3 ⟩ + 3
3	6 ⟩ + 4
4	10 ⟩ + 5
5	15 ⟩ + ?
.	.
.	.
.	.

Once again, since we did find a pattern we could fill in the table until n = 50 is reached. This is easier than pecking out dots, but perhaps we can get even more efficient. Suppose we could find a relationship between n (the number of the triangular number) and T (the triangular number). Let's try.

n	T
1	$1 = \underline{1} \times 1$
2	$3 = \underline{2} \times 1\frac{1}{2}$
3	$6 = \underline{3} \times 2$
4	$10 = \underline{4} \times 2\frac{1}{2}$

That looks promising. Suppose we rewrite $1\frac{1}{2}$ as $\frac{3}{2}$ and $2\frac{1}{2}$ as $\frac{5}{2}$. Our table now becomes:

n	T
1	$1 = 1 \times 1 = 1 \times \frac{2}{2}$
2	$3 = 2 \times \frac{3}{2}$
3	$6 = 3 \times 2 = 3 \times \frac{4}{2}$
4	$10 = 4 \times \frac{5}{2}$

■ *CARRYING OUT THE PLAN*

From there we can predict that:

20	$210 = 20 \times \frac{21}{2}$
.	.
.	.
.	.
and 50	$1275 = 50 \times \frac{51}{2}$

■ *LOOKING BACK*

● *Can I Generalize From the Data?*

We might conjecture that for any n:

n	$\frac{n(n+1)}{2}$

We not only have a reasonable idea for the 50th triangular number, we may have a correct generalization (formula) for finding <u>any</u> triangular number! And remember the facts that should be in a "box of tools" for solving problems? Recognizing the triangular numbers:

1, 3, 6, 10, 15, . . .

and their general form, $\frac{n(n+1)}{2}$, can be a handy tool for future work in problem solving. For example, triangular numbers appear in *Part of a Line*, *Geoboard I* and *II* (all

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in LINES, PLANES & ANGLES: Lines), *Angles Not Angels* (in LINES, PLANES & ANGLES: Angles), and *The Last Chord* (in CURVES & CURVED SURFACES: Circles). By the way, we've talked about <u>triangular</u> numbers. ● *Can you suggest other related problems* (with <u>square</u> or <u>pentagonal</u> numbers perhaps)?

Let us summarize the heuristics we used as we carried out Polya's steps:

- ● Can you find a pattern?
- ● Organize the data into a table.
- ● Can you generalize from the data?
- ● Is there another way to solve the problem?
- ● Can you think of a new, related problem?

The next problem solution emphasizes the <u>Devising a Plan</u> and <u>Carrying Out the Plan</u> steps and introduces two more heuristics.

PROBLEM 2

What is the sum of the interior angles of an eight-sided convex polygon? A fifty-sided convex polyton?

● *Would a Figure Help?*

An eight-sided convex polygon is not too difficult to sketch (the fifty-sided polygon is a different story!) (Fig. 1) We <u>could</u> use a protractor, but *is there another way to solve the problem?*
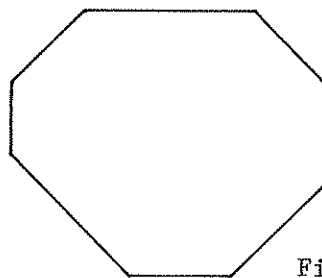
Figure 1

● *Have I Seen a Similar Problem Before?*

The only information in our "box of tools" related to interior angles of polygons might be the theorem that the sum of the measures of the interior angles of a triangle is 180°. Now can we relate this to our figure? . . .
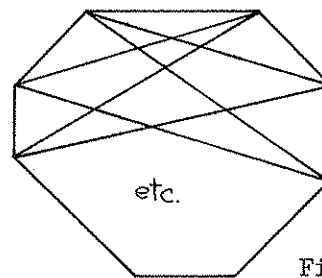
Figure 2

Drawing diagonals of the polygon (Fig. 2) divides it into triangles--but too many are overlapping--we need <u>non-</u>overlapping triangles whose interiors <u>cover</u> the polygon completely; one way to do this is to draw all possible diagonals from one vertex. (Fig. 3)
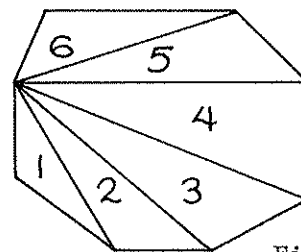
Figure 3

These 6 triangles completely "cover" the polygon and have all their angles included in the angles of the polygon. The sum of the angles of each triangle is 180°. Therefore, the sum of the measures of the interior angles of the eight-sided polygon is 6 x 180° = 1080°.

Now, how about the fifty-sided polygon?

Have I Ever Seen This or a Similar Problem Before?

Certainly the problem is similar to that of finding the sum of the interior angles of an eight-sided convex polygon. But, sketching a figure would not be easy! So, let's try another approach. In Problem 1 we solved a problem that was similar in that we were looking for an nth term. How was that problem approached? We

--looked at simpler cases,

--organized the data into a table,

--searched for a pattern,

and

--tried to generalize.

If we use that procedure here (taking the hint from the solution of the first part of the problem), we find:

3-sided convex polygon
Sum of measure of inter-
ior angles = 180°
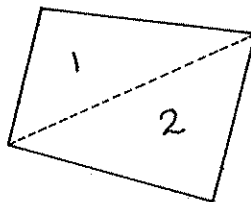
4-sided convex polygon
Sum of measures of inter-
ior angles = 2 x 180°
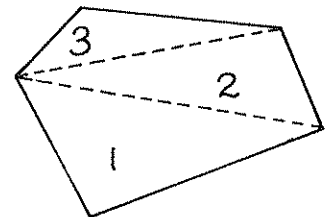
5-sided convex polygon
Sum of measures of inter-
ior angles = 3 x 180°

Organizing these data into a table gives:

# of sides of convex polygon	Sum of measure of interior angles
3	1 x 180°
4	2 x 180°
5	3 x 180°
6	.
7	.
8	6 x 180°
.	.
.	.
.	.
50	?

A mental picture of what is happening indicates that the number of triangles formed is always two fewer than the number of sides of the polygon; therefore, the sum of the interior angles of our fifty-sided convex polygon is probably

$$48 \times 180° = 8640°$$

In fact, for an n-sided convex polygon, the sum of the interior angles will always be

$$(n - 2) \times 180°$$

since a triangle is formed by each segment drawn from one vertex to each of the others except the two adjacent vertices. That gives us a generalization (formula) to add to the "box of tools." (See *Interior Angles of a Polygon 2* for an alternate solution; *Interior Angles of a Polygon 1* uses the development above. Both are in the Polygons subsection of POLYGONS & POLYHEDRA.)

In these two example problems we have seen the importance of a well-stocked "box of tools" for problem solving. Specific facts, computational skills and in particular, several heuristics organized around Polya's steps helped us find the solution to each problem.

HOW DOES A TEACHER TEACH PROBLEM SOLVING?

Now that we've considered how successful problem solvers arrive at solutions, our next task is to plan instruction to help our students become more successful problem solvers. Again, we look to Polya for direction. His suggestion that problem solving is best learned by solving problems and by observing others doing so serves as the basis for the following four-step technique for teaching problem solving.

> STEP 1: Discuss the notion of PROBLEM and the problem-solving
> process. Emphasize the fact that the process (or how
> the solution is arrived at) is as important as the
> solution itself. The importance of the process must be
> a daily emphasis, not just a one-shot pep talk.

> STEP 2: Emphasize Polya's four phases in the solution of a
> problem.

>> a. *UNDERSTANDING THE PROBLEM:*
>> Spending time trying to understand what is given and
>> what is asked for before trying to apply a formula or

algorithm can prevent erroneous starts. Better yet,
it could lead to insights about relationships among
the data that would result in "neat" original solu-
tions.

b. *MAKING A PLAN:*

Preparing the plan of the solution is a very impor-
tant aspect of the solution process. Too often,
students rush into combining data without any
rationale for the operations. Actually plotting
the course of an action before taking it will make
the problem solver aware of relationships between
the data and the goal, and help prevent super-
fluous operations.

c. *CARRYING OUT THE PLAN:*

The actual mathematical operations should be per-
formed only <u>after</u> the problem solver knows the
direction toward the solution and why the opera-
tions will help him reach the goal. It is during
this phase that computational accuracy should be
emphasized, but not to the point of implying that
computation is the most important aspect of prob-
lem solving.

d. *LOOKING BACK:*

This is the most neglected phase in the problem-
solving process. It includes not only checking,
but also studying the solution and the process
to determine if another solution is possible, if
another way to solve the problem may be found and/
or if other similar problems could be suggested
from the given one. The emphasis on looking back
can be a great help in the future when new problems
are encountered. (It is, however, a very difficult
habit to instill in young problem solvers!)

Each of these phases should be discussed and repeatedly
illustrated by solving some not-too-difficult, yet
challenging, problems with the class.

STEP 3: Build a list of heuristics. Such a list might include questions that the students can ask themselves as they are trying to solve a problem, as well as suggestions for paths to follow. This can be done by "modeling" solutions to problems as we did in the preceding section or by having the students make suggestions after having tried some problems on their own. A typical set might include the heuristic questions or suggestions summarized on page 20.

STEP 4: Solve a variety of problems. Search through supplementary texts, workbooks and periodicals for interesting problems. Set up a "problem file" of related problems that could be used in sequence. Do a problem every day, if possible, and assign longer "brain teasers" as special projects or for group solution. Problem solving shouldn't be left for "If time permits"--it is a vital part of every mathematics program.

In the beginning the teacher should realize that most students are NOT experienced problem solvers. They may become frustrated quickly and tend to give up easily. They often make hasty, incorrect conjectures and fail to check the reasonableness of their answers. They lack a knowledge of problem-solving techniques and the ability to use them. Some students have not acquired the necessary computational skills or reading/comprehension skills needed to carry out or initiate the problem-solving process.

No teacher or student has to memorize Polya's four steps and his list of "things to try," but there are specific skills from the list that can be the focus of a lesson. *What's My Line* from the Lines subsection in LINES, PLANES & ANGLES is a good introductory problem for students who have little confidence in their ability to tackle a problem-solving situation. The activity gives the teacher an opportunity to guide the student through "things to try" and finally arrive at a generalized solution. Other introductory activities include   *In 'N' Out Regions* from the Circles subsection in CURVES & CURVED SURFACES;

and *Sum Thing* from the Polygons subsection of POLYGONS & POLYHEDRA. Each of these can be used to illustrate some of the specific problem-solving suggestions discussed earlier.

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The next pages are devoted to five more sample problems, which illustrate various heuristics or other aspects of problem solving. Problems 3 and 4, for example, show how the result from one problem can be used in another problem. (Problem 4 also illustrates how looking for another solution may result in a "neat" solution.) If you are quite familiar with problem solving, you may wish to skip ahead to the closing remarks of this section.

PROBLEM 3

What is the area of the largest triangle that can be inscribed in a semicircle of radius 6 cm?

● *Would a Diagram Help?*
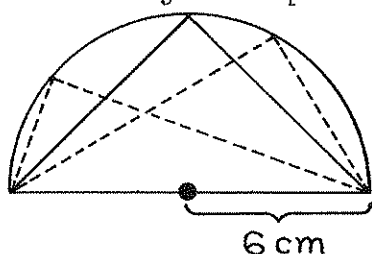
The area of a triangle is one-half the base times the height. If we use the diameter of the semicircle as the base of the triangle, we see that the triangle with the largest area would be the one with the greatest height to the diameter. Sketching several cases (as in the diagram) suggests that the greatest height would be along a radius of the circle.

So, $A_{triangle} = \frac{1}{2} bh = \frac{1}{2} (12 \times 6)$
$$= 36 (cm^2)$$

PROBLEM 4

A circle of greatest area is cut out of a 4 cm square of material. A square of greatest area is then cut out of the circle. How much material is wasted?

● *Would a Diagram Help?*

Both shaded regions show the wasted material.
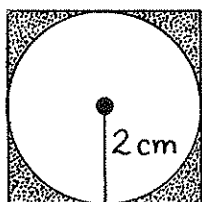
Figure 1
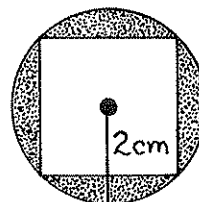
Figure 2

We need to find the wasted area in each case. This can be done by finding the areas of the square and circle, and subtracting:

In Figure 1: $A_{square} = s^2 = 4^2 = 16$

$A_{circle} = \pi r^2 = \pi \times (2)^2 = 4\pi \approx 12.56$

Waste $\approx 16 - 12.56 = 3.44$ (cm^2)

Finding the areas in Figure 2 is slightly more complex:

● *Have I Seen a Similar Problem Before?* ● *Can I Alter the Diagram?*

If a radius in Figure 2 is drawn from the vertex of the square and another extended to form a diagonal of the square, the resulting figure is Figure 3. Now, the area of this square can be found by doubling the area of the isosceles triangle (as found in Problem 3 above). From Figure 3, then:

Figure 3

$A_{circle} = \pi r^2 = \pi \times 2^2 = 4\pi \approx 12.56$

$A_{square} = 2 \times \frac{1}{2}b \times h = 2 \times \frac{1}{2}(4 \times 2) = 8$

$\therefore A_{waste} \approx 12.56 - 8 = 4.56$ (cm^2)

So, the total waste $\approx 3.44 + 4.56 = 8$ (cm^2).

● *Is There Another Way to Solve the Problem?*
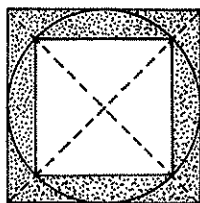
Figure 4

That was a pretty complicated solution--it could be simplified by using the diagram effectively. Superimposing Figure 1 on Figure 2 (see Figure 4) shows us that the waste is the difference in area between the big and the small squares. As we found in the first solution, the area of the first square is 16 cm^2, and that of the second square is 8 cm^2; therefore, $A_{waste} = 16 - 8 = 8$ (cm^2).

Or . . .

A really "neat" solution results from changing the position of the square inscribed in the circle to the new position illustrated in Figure 5. Considering this
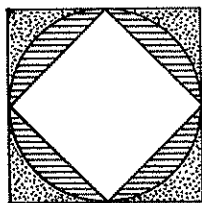
Figure 5
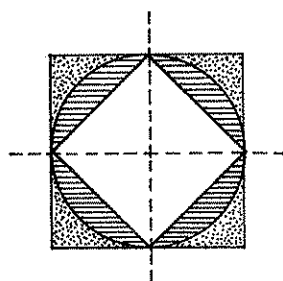
Figure 6

new figure with its shaded portions suggests that the wasted area is exactly one-half of the area of the original square. This becomes clear if, as in Figure 6, two of the lines of symmetry of the square are drawn.

Thus, the wasted area is $A_{waste} = \frac{1}{2}(s^2)$

$$= \frac{1}{2}(4^2) = 8 \ (cm^2).$$

This third solution was produced with only one computation. The diagram helped to produce a very "economical" solution.

The next two problems illustrate further the very useful heuristic, Have I Seen a Similar Problem?

PROBLEM 5

A 25-foot ladder is leaning against a building so that the foot of the ladder is 7 feet from the building. If the top of the ladder slides 4 feet, how far is the foot of the ladder from the building in its new position?

● *Would a Figure Help?*
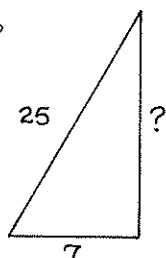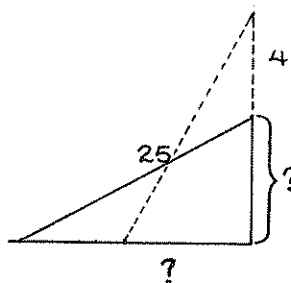
First Position   Second Position

● *Have I Seen a Similar Problem Before?*

Although this could be a pretty difficult problem, it reduces to a very simple one if my "box of tools" contains a working knowledge of the Pythagorean theorem (or of Pythagorean triples--see the teacher commentary to the Pythagorean Theorem subsection in AREA & VOLUME). The Pythagorean theorem and some calculations ($25^2 = s^2 + 7^2$; $s^2 = 576$; $s = 24$) tell me that in the first position the height of the ladder is 24 feet and, therefore, in the second position the height of the ladder is 20 feet.
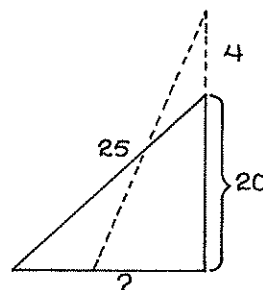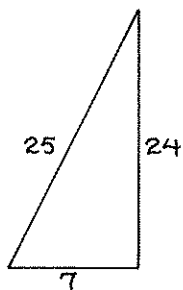
Back to the "box of tools"--$25^2 = 20^2 + x^2$; $x^2 = 225$; $x = 15$. So, the distance from the foot of the ladder to the wall is 15 feet in the new position.

PROBLEM 6

Start anywhere on a coordinate grid and call your starting point K. Move to the right 4 and **up** 3. Label the point A; move left 1 and up 3, label the point R; move left 1 and down 1, label the point E; finally, move 5 to the left and label the last point N. Connect the points in order to form a polygon. Find the area of KAREN.
(☐ = 1 sq. unit.)

● *Would a Diagram or Figure Help?*
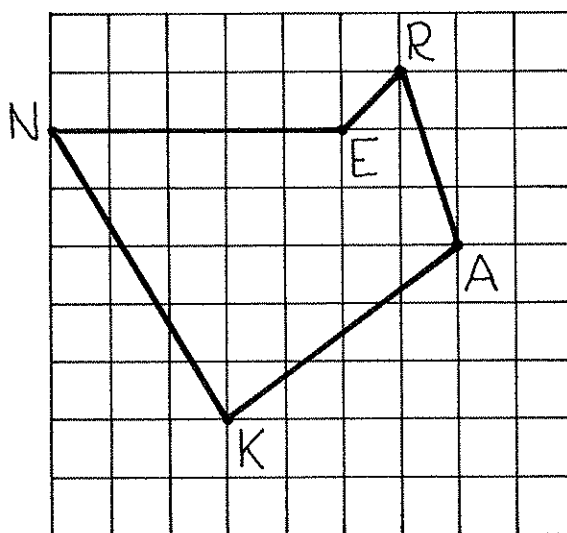
Now, this polygon doesn't look like anything we've found an area for before. But we do know how to find areas of triangles—-so if we could break up the polygon into triangles, find the areas and the sum of these areas, we could find the desired result. This is difficult because so many sides of the triangles we could form are not horizontal or vertical; therefore, the altitudes would be difficult to calculate. Let's try another approach—-

● *Have I Seen a Similar Problem?*

The figure looks like something we might form on a geoboard with a rubber band—-how do we find areas of irregular figures on the geoboard? One method is to form a rectangle touching as many nails as possible around the figure and then to subtract the waste! (See *The Rectangle Method* in the Area subsection of AREA & VOLUME.)

Applying this technique we have:
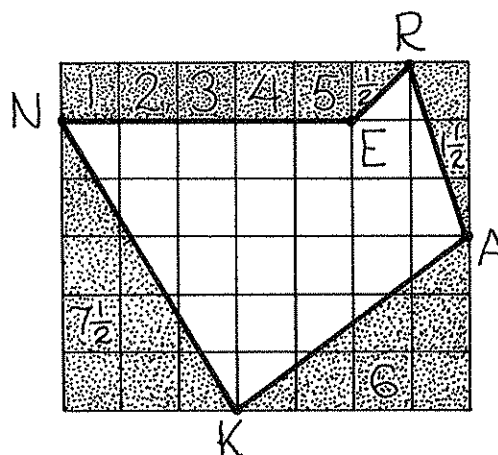
The area of the rectangle is 6 x 7 or 42 sq. units. The area of the shaded portion by "geoboard geometry" is $5\frac{1}{2} + 1\frac{1}{2} + 7\frac{1}{2} + 6 = 20\frac{1}{2}$ sq. units, so the area of KAREN is $21\frac{1}{2}$ sq. units! (For a check or an alternate solution, see *Be Picky About Your Geoboard* in the Area subsection of AREA & VOLUME.)

FINALLY, PROBLEM 7 (AND TWO ADDITIONAL HEURISTICS)

Suppose we are given 200 matchsticks and asked to build stairs like those pictured below. What is the greatest number of <u>squares</u> that can be formed if the configuration consisting of the maximum number of squares is constructed from the 200 matchsticks?

| 1st | 2nd | 3rd | etc. |

● *Do I Understand the Problem?*

We have 200 matchsticks with which to build the stairs. We might use them all, we might not. These stairs are formed by adding a number of squares corresponding to the next consecutive digit to the preceding set of stairs. We are trying to find the number of squares formed if we use as many matchsticks as possible. (With so many matchsticks we probably prefer some approach other than actually building the stairs.)

● *Can I Organize the Given Information Into a Table?*

Stairs with 1 square on a side use 4 matchsticks and form 1 square.

Stairs with 2 squares on a side use 10 matchsticks and form 3 squares.

Stairs with 3 squares on a side use 18 matchsticks and form 6 squares.

.
.
.

So, we can form the following table:

# of squares on a side	# of matchsticks used	# of squares formed
1	4	1
2	10	3
3	18	6
.	.	.
.	.	.
.	.	.

● *Can I Find a Pattern?*
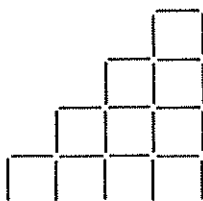
The last column looks like the beginning of the se-
quence of triangular numbers discussed earlier. If we
● draw the figure using 4 squares on a side, we find that
there are, indeed, 10 squares. In fact, the dots in the
triangular numbers figure are replaced by the squares in
this problem.

● *Can I Find a Pattern for the Number of Matchsticks?*

# of squares on a side	# of matchsticks	# of squares formed
1	4 ⟩ 6	1
2	10 ⟩ 8	3
3	18 ⟩ 10 ?	6
4	? ⟩ 12 ?	10
5	?	15

To go from 1 square to 2 squares on a side we added <u>6</u> matchsticks; to go from 2
squares to 3 squares on a side we added <u>8</u> matchsticks; therefore, we conjecture that
to go from 3 squares to 4 squares on a side we would add <u>10</u> matchsticks. This is
verified by actual count of 28 matchsticks in the above figure. We now have patterns
for the number of matchsticks used and the number of squares formed. But how can we
find how many squares would be formed using 200 matchsticks?

● *Can I Work Backwards?*

<u>If</u> we could find

   1. a relationship between the number of squares on a side and the number
       of matchsticks used,

   2. a relationship between the number of squares on a side and the total
       number of squares formed

   and

   3. the largest number of squares on a side that could be formed using the
       200 matchsticks,

156 we could easily find the desired number of squares.

Let's look at our table again:

# of squares on a side	# of matchsticks used	# of squares formed
1	4	1
2	10	3
3	18	6
4	28	10
5	40	15
6	54	21
.	.	.
.	.	.
.	.	.

One possible solution would be to fill in the table until the number of matchsticks used exceeds 200, and then to find the corresponding number of squares. That would be <u>one</u> method of solution. But . . .

● *Can I Find a Generalization from the Table?*

Let's work with the number of matchsticks first (● solve part of the problem). We might try to write the number of matchsticks as a product of factors:

# of squares on a side	# of matchsticks
1	4 = 1 x 4
2	10 = 2 x 5
3	18 = 3 x 6
4	28 = 4 x 7
.	.
.	.
.	.

This shows us that the number of matchsticks is the number of squares on a side (n) times three more than that number (n + 3). So we have:

# of squares on a side	# of matchsticks
1	4 = 1 x 4
2	10 = 2 x 5
3	18 = 3 x 6
.	.
.	.
.	.
n	n(n + 3)

Therefore, n(n + 3) matchsticks are needed to complete a set of stairs n squares on a side. Since we have 200 matchsticks, we need to find what number, multiplied by 3 more than itself, will give us a product closest to but not exceeding 200.   157

● *Can We Make an Intelligent Guess?*

      10 x 13 = 130   too small!

      20 x 23 = 460   too large!

Let us try to "zero in" on the correct number of matchsticks by ●successive approximation trial and error:

      11 x 14 = 154   too small

      15 x 18 = 270   too large

      13 x 16 = 208   <u>just</u> too large

      12 x 15 = 180

So, 180 is the maximum number of our matchsticks that can be used, and the stairs will have 12 squares in the tallest side. Now we can easily find the total number of squares; since they represent the triangular numbers:

# of squares on a side	# of matchsticks used	total # of squares (triangular #)
1	1 x 4	1
2	2 x 5	3
3	3 x 6	6
.	.	.
.	.	.
.	.	.
n	n(n + 3)	$\dfrac{n(n + 1)}{2}$

Therefore, the number of squares is the 12th triangular number, or $\dfrac{12(13)}{2} = 78$.

● *Can You Find Another Way to Solve the Problem?* ● *Can You Think of Another, Similar Problem?* (Perhaps in three dimensions using cubes.)

WHY TEACH PROBLEM SOLVING?--A FINAL ARGUMENT

Moise bemoans what seems to be so many students' view of problem solving: ". . . problem solving is not a process by which one ascertains the truth. Rather, it is a process by which one gets the answer in the back of the book by a sequence of steps, each of which has been authorized by the teacher." (SIAM News, February, 1975) Indeed, too many mathematics assignments do require only rote procedures to be followed while finding the same answer as the "answer in the back of the book." This is really drill and practice, not problem solving, and the students are doing exercises, not problems.

If our students are to become independent thinkers and problem solvers, it is important that we give them many situations which cannot be routinely solved. It is important that we as educators provide guidance and examples that involve a variety of problem-solving techniques. Problem solving is a process of thinking that "emancipates us from merely routine activity."

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HEURISTICS FOR PROBLEM-SOLVING

UNDERSTANDING THE PROBLEM:

State the problem in your own words.

What are you trying to find out? What is the unknown?

What relevant information do you get from the problem? What is given?

Is there any information that is not needed to solve the problem?

Are there any diagrams, pictures or models that may provide additional information about the problem?

Can you try some numerical examples?

Is it possible to recreate, act out or make a drawing of the problem?

Can you make an educated guess as to what the solution(s) might be?

DEVISING A PLAN:

Make a diagram, number line, chart, table, picture, model or graph to organize and structure the data.

Guess and check. Organize the trial and error investigations into a table.

Look for patterns.

Translate the phrases of the problem into mathematical symbols and sentences. Can you write an equation?

Try to solve one part of the problem at a time (i.e., break the problem into cases)

Have you worked a problem like this before? What method did you use?

Can you solve a simpler but related or analogous problem?

Keep the goal in sight at all times. Can you work backwards?

CARRYING OUT THE PLAN:

Keep a record of your work.

Perform the steps in your plan; check each step carefully.

Complete your diagram, chart, table or graph.

Follow patterns; organize and generalize them.

Compare your estimates and guesses with your work.

Solve the mathematical sentence; record the calculations and answer.

Work out any simpler but related or analogous problems. Compare the solutions.

LOOKING BACK:

Can you check your result? Is the answer reasonable?

What does the result tell you? What conclusions can be made?

Is there another solution? Is there another way of finding the answer?

Make up some problems like the one you worked. Is there a rule or generalization that can be used to solve similar problems?

What method(s) helped you get the answer(s)?

EXAMPLES OF PROBLEM-SOLVING ACTIVITIES IN THE CLASSROOM MATERIALS

I. Manipulatives and Models

Manipulatives and models can improve the understanding of the problem. They provide a physical representation of the situation, creating visual and physical feedback that is often necessary in the search for a solution.

By building a model from cubes, the students can gather data about the model and organize it in a table. By examining the table for patterns, a formula can be discovered for the surface area of each "space station."

Paper-folding patterns can generate a predictable result--a geometric shape.

The geoboard is an excellent manipulative to use for posing problems on perimeter, area, line segment lengths, similar figures and other geometric concepts. Students record their findings on dot paper and in a table. Soon numerical patterns appear and a general formula can be discovered.

GEOBOARD I

You need a geoboard, rubber bands, and dot paper.
Use dot paper to record your results for questions 1, 2, 4, & 5.

6) Use only the bottom row of nails on the geoboard. How many line segments can you make with

1 nail _____
2 nails _____
3 nails _____
4 nails _____
5 nails _____

10 nails _____

Look for a pattern.

9) How many line segments of different length could you find on the following geoboards?

DIMENSIONS OF GEOBOARD	NUMBER OF SEGMENTS
1×1	
2×2	
3×3	
4×4	

BE PICKY ABOUT YOUR GEOBOARD

On a geoboard the area of a figure with its vertices located at nails can be determined by Pick's formula (shown to the right). B represents the number of nails on the boundary of a figure and I represents the number of nails in the interior of the figure. You will probably want to try the formula with several simple figures.

PICK'S FORMULA

$$AREA = \frac{B}{2} - 1 + I$$

$B=4, I=1,$
$AREA = 2$ SQ. UNITS

By using carefully designed activities (see pages 3 and 4 of this lesson) it is possible for students to discover Pick's formula. Lessons can be found in almost any material that includes the geoboard. See especially the May, 1974, issue of The Mathematics Teacher and Math Workshop, Level F, a textbook published by Encyclopedia Britannica Press, Inc.

As shown on page 3 one approach is to first have students discover that all 3-nail figures, 4-nail figures, etc., with no interior nails, have constant areas.

NAILS ON BOUNDARY	3	4	5	6	7	8	9	10
AREA	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

You may want to leave the discovery completely open, guide the students by providing hints on a chart (see #6 on page 3), or give the formula and have students use several figures to verify the formula.

On page 4 students now investigate the effect of including interior nails in the figure. The chart below shows the results of making 3-nail figures with varying numbers of interior nails.

NAILS IN INTERIOR	0	1	2	3	4	5
AREA	$\frac{1}{2}$	$1\frac{1}{2}$	$2\frac{1}{2}$	$3\frac{1}{2}$	$4\frac{1}{2}$	$5\frac{1}{2}$

For 4 nails on the boundary the chart looks like this.

NAILS IN INTERIOR	0	1	2	3	4	5
AREA	1	2	3	4	5	6

SUM THING

1) Use your scissors to cut out these strips.

IN MAKING A TRIANGLE WITH STRIPS BE SURE THE ENDS MEET

2) Try to make a triangle with each set of strips. Draw a circle around the lengths which do make a triangle.

a) 2, 3, 4
b) 2, 3, 5
c) 2, 3, 6
d) 2, 3, 7
e) 2, 3, 8
f) 2, 4, 5
g) 2, 4, 6
h) 2, 4, 7
i) 2, 4, 8
j) 2, 5, 6
k) 2, 5, 7
l) 2, 5, 8
m) 3, 4, 5
n) 3, 4, 6
o) 3, 4, 7
p) 3, 4, 8

3) If two sides are 2 units and 3 units, what is the longest the third side can be? _____

4) If two sides are 3 units and 4 units, what is the longest the third side can be? _____

5) Complete this table.

USE YOUR ANSWERS FROM PROBLEM #2

Length of sides	Sum of two shortest sides	Length of longest side	A triangle? Yes or No
2, 3, 4	5	4	Yes
2, 3, 5			
2, 3, 6			
2, 3, 7			
2, 3, 8			
2, 4, 5			
2, 4, 6			
2, 4, 7			
2, 4, 8			
2, 5, 6			
2, 5, 7			
2, 5, 8			
3, 4, 5			
3, 4, 6			
3, 4, 7			
3, 4, 8			

6) Look at the table. What conclusion can you make about the lengths of the sides of a triangle?

Strips of paper are used to form triangles. The length of the sides can be organized in a table. The students can then determine which set of three given strips will make a triangle.

II. Drawings, Diagrams and Pictures

Graphic representation allows us to communicate various ideas or concepts and to represent actual objects. We often use geometric drawings to illustrate a concept or relationship. Organizing our thoughts into a diagram or sketch is a natural step in the problem-solving process.

THE PIT AND THE POOL

Find the volume of each of these figures.
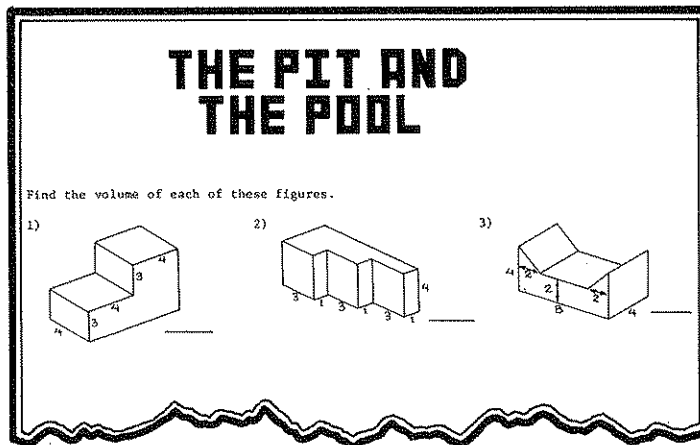
From the picture and the given measures of height, width and length the students find the volume of each figure. The models can be constructed if the student is having trouble visualizing the figure from the drawing.

Look for patterns in each picture to solve the problem.

HOW MUCH IS LOST?

Vertex A of this cube is surrounded by three right angles. What is the total of the measures of these three angles? _____

If the three right angles were laid out flat, it could look like this.

When laid out flat there are 360° about vertex A, but the three right angles measure only 270°. The angle loss about vertex A is 90° (360° - 270°).

What is the angle loss at vertex B? _____

What is the angle loss for all eight vertices of the cube? _____

Fill in this chart for each of the regular polyhedra.

	Number of vertices	Angle loss at each vertex	Angle loss for the polyhedron
Tetrahedron			
Cube	8	90°	720°
Octahedron			
Dodecahedron			
Icosahedron			

Investigate the angle loss for each of the semi-regular polyhedra described on the page *Polyhedra*.

A BAKER'S DOZEN

Find the volume of each box even though part of the box is hidden. Write the letter above the correct answers in the code below.

ALL DIMENSIONS THE SAME.

4 LAYERS

Understanding the drawings is of utmost importance if a pattern is to be seen and data recorded correctly in the chart.

First, a picture is drawn. Then by making a table and looking for patterns, a general formula is discovered that can be used to predict the sum for any n-sided polygon.

How do perimeters or circumferences of similar figures compare? The use of pictures and tables of recorded data help students to see patterns and relationships.

A table of organized data included with drawings of polyhedra makes it possible to find patterns and generalize the results into Euler's Formula $V + F = E + 2$.

III. A Variety of Problem-Solving Techniques

A number of steps can be carried out in the problem-solving process.

What are similar figures? What patterns are found when corresponding sides and angles are compared? The relationships of corresponding parts can be discovered through measurement and data organization.

Each of these problems can provide a challenge for the students. Various problem-solving strategies can be used to find the solution(s): make a diagram or picture, look for patterns, guess and check.

Drawings and charts of organized data make it possible to discover patterns which eventually lead to a general solution of the problem.

PERIMETER PATTERNS OF POLYGONS

EXAMPLE:
- 1 TRIANGLE PERIMETER OF 3
- 2 TRIANGLES PERIMETER OF 4
- 3 TRIANGLES PERIMETER OF 5
- 4 TRIANGLES PERIMETER OF 6

# OF POLYGONS	1	2	3	4	5	10	23		93		n
PERIMETER	3	4	5	6	7	12		58		108	

1) DRAW MORE POLYGONS TO HELP YOU.

# OF POLYGONS	1	2	3	4	5	10	20		53		n
PERIMETER	4							68		202	

2)

# OF POLYGONS	1	2	3	4	5	10	20		54		n
PERIMETER								140		784	

3)

# OF POLYGONS	1	2	3	4	5	10	20	39			n
PERIMETER									143	452	

A STARTLING DISCOVERY

I   Each circle below has 12 division points.

a) Join *each* point to every second point. (1 to 3, 3 to 5, ... and so on. Also 2 to 4, 4 to 6, ...)

b) Join *each* point to every third point. (1 to 4, 4 to 7, ... and so on. Also 2 to 5, 5 to 8, ... and 3 to 6, 6 to 9, ...)

c) Join *each* point to every fourth point.

d) Join *each* point to every fifth point.

How does the last figure differ from the first three? _____
Use the drawings to help you complete the table.

NUMBER OF DIVISIONS BETWEEN POINTS JOINED	NUMBER OF INSCRIBED POLYGONS FORMED (P)	NUMBER OF SIDES ON EACH POLYGON (S)	PRODUCT (P × S)
a) 2	2	6	
b) 3			
c) 4			
d) 5			

Only the last figure can be drawn without lifting the pencil from the paper.

What property do the numbers 2, 3, 4 have that 5 doesn't that might help to explain this? _____

By following a pictorial pattern and a numerical pattern, the perimeter pattern for each polygon can be generalized into an algebraic formula.

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

ESTIMATION and APPROXIMATION

RATIONALE

Why estimate and approximate? Why be concerned with educated guesses or a process to improve the accuracy of an educated guess?

In their daily lives people encounter situations where it is convenient to make quick estimates or approximations. A life and death estimation is made when a person decides if it is safe to cross the street or if a car or bike can be stopped in time. Before anyone can make an estimation that is more than a guess, it is necessary to have many experiences in seeing and interpreting the environment. Classroom experiences can be set up to help <u>develop</u> measuring skills, visual perception, an understanding of area and volume, and a number sense for large and small numbers. Many estimations require familarity with the common units for measures of length, weight, time, area, volume, cost and so on. If we are concerned about students having the ability to estimate distances, areas and volumes and approximate any given calculations involved, then we need to work on such things as: rounding results, finding upper and lower limits where the answer is somewhere in between, reasonableness of answers, understanding the units of measure, and making scale models or drawings.
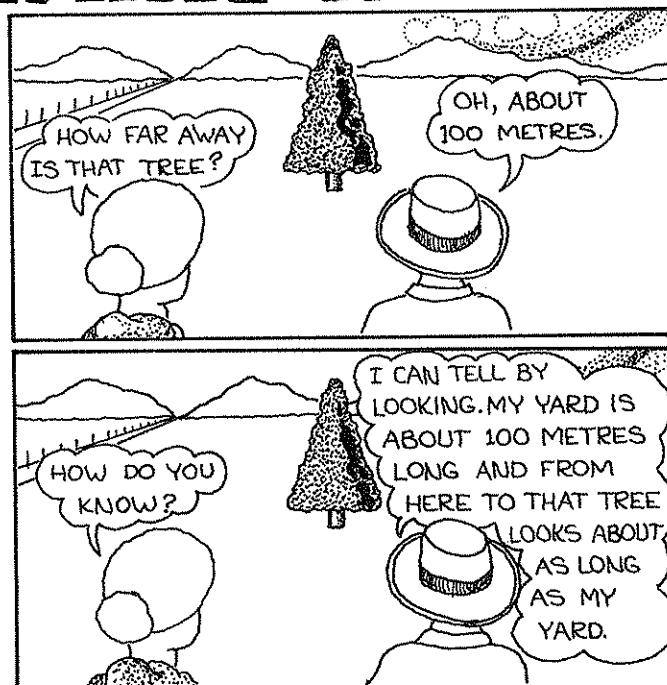
We use approximations every time we measure. A tree's height might be given as 6 metres, but its height might be closer to 603 cm than to 600 cm. Rigorous and precise measurements are often not necessary, and they can obscure the situation. For example, it is an average of 149,497,892 kilometres to the sun from the earth; it takes 8 minutes 20 seconds for the sun's light to reach the earth. This could be expressed in approximate terms which would be easier to grasp and remember. The sun is <u>about</u> 150 million kilometres from the earth; it takes <u>about</u> 8 minutes for the light to reach us.

We make many educated guesses every time we . . .

    a) put food away in containers. (How much will each container hold? How many quarts can be canned from ten pounds of cherries?)

    b) buy paint, wallpaper or carpet to redecorate a room. (How much surface area for the walls, ceiling or floor?)

c) walk or drive to reach a destination at a certain time. (How long does it take to walk ten blocks? What will be the arrival time?)

The reasonableness of our estimated results may mean a difference of time and money to each of us, whether it be in buying construction materials so there is little waste or leftovers, in making travel plans or in planning the amount of food needed at a party.

To quickly check the reasonableness of an approximation it is helpful if a person has already developed perceptual skills, measuring skills and arithmetic skills. These include the ability to . . .

a) size up a situation visually. (Looks like I'm about two-thirds done painting the house. Two more gallons should finish the job.)

b) detect optical illusions. (Various arrangements of pictures on the wall may appear to leave different amounts of uncovered surface area even though the conservation of area principle holds true.)

c) measure with and to visualize units such as centimetres, litres, square metres, cubic centimetres, and so on.

d) perform single-digit operations accurately. (9 million x 7 million requires 9 x 7 = 63.)

e) multiply and divide by powers of ten. (Knowledge of exponential notation may be helpful in more complex approximations.)

f) perform operations with multiples of powers of ten--mentally, if possible.

g) use proportions, inequalities and other relationships.

h) round whole numbers and decimals to one or two significant digits.

Here is a problem that illustrates some of these points: How many litres of oil would it take to fill the Alaskan pipeline? (See *A Variety of Volume Vexations* in the Volume subsection of AREA & VOLUME.)

The pipeline could easily be about 5000 km long. Say it has a 20 cm inside diameter; then we use the formula for the volume of a cylinder: $V = \pi r^2 h$ and hence, we get approximately 3.14 x 400 x 500,000,000 = 3.14 x 200,000,000,000 = 628,000,000,000 cm^3 = 628,000,000 litres.
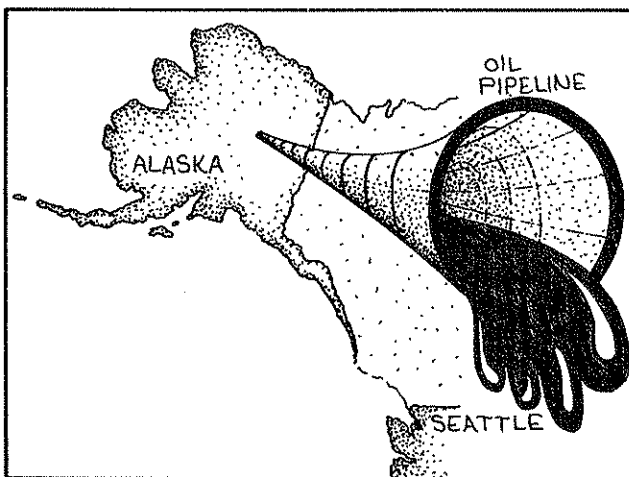
There is much to be said for knowing when to estimate and when to approximate, and when to use an exact answer. The use of estimation and approximation should help all persons to deal with exact numbers, understand and perform operations with numbers arising from measurement, deal comfortably with numbers through approximate calculations and rounding off, and in general develop a number sense. Finally, it would seem most worthwhile if teaching the techniques of estimation and approximation helped students realize that often a reasonable approximation is just as acceptable as an exact answer.

SUMMARY

These are the key questions and points to be considered when teaching estimation and approximation:

1.   When do we need to estimate and approximate to find a rough answer?

2.   When do we need precise answers?

3.   We often estimate "how many" (e.g., length in cm, floor tiles, people, items) or "how much" (e.g., money, volume of air and water).

4.   We often estimate the dimensions, capacity or amount of something we would measure.   (Measurements are always approximate.)

5.   Problem-solving and computation is aided by the use of estimation and approximation to . . .

    a) find lengths, areas or volumes of things that help us "size up" the situation.

    b) check the reasonableness of answers.

    c) narrow the scope of our investigations or put upper and lower limits on the range of the answer.

    d) simplify computations.

6.   The students need a sound background in perceptual skills, measuring skills, arithmetic skills and number sense.

Selected Sources for Estimation and Approximation

Garvin, Alfred D.   Shortcuts, Checks and Approximations in Mathematics.   Portland, Maine: J. Weston Walch, 1973.

Herrick, Marian, et al.   Mathematics for Achievement/Individualized Course 2, Book 5.   New York: Houghton Mifflin, 1972.

Mathex Book 5 Measurement and Estimation.   Chicago: Encyclopedia Britannica, 1970.

Peas and Particles (Teacher's Guide).   Elementary Science Study.   New York: Webster/ McGraw-Hill, 1969.

EXAMPLES OF ESTIMATION AND APPROXIMATION ACTIVITIES
FOUND IN THE CLASSROOM MATERIALS

Containers come in all sizes and shapes. We use them for storage of food, liquids, small items and many other things. We often estimate how much a container will hold when we want to fill it. It is easy to make a mistake in estimation. The result is an overflowing or partially-filled container.

METHODS FOR FINDING VOLUMES

Materials: 8 to 10 commercial containers, some about the same approximate size; e.g., oatmeal and other dry cereal boxes, paper cups, empty cans and bottles.
Filler Material: Sand, cornmeal, puffed rice, etc.

I   Comparing Volumes
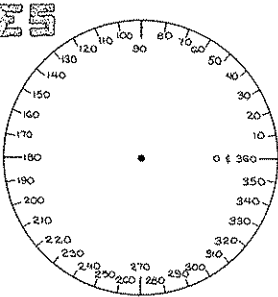
A) Have students guess which container holds the least; the most. Have them sort their guesses from least to greatest volume.

B) Suggest that students use the filler material to check their guesses. Have them decide on a method. One possibility is to fill the predicted largest container and pour from one container to the next. Students can discuss their methods.

ANCIENT ANGLES

A long time ago the Babylonians, who thought there were 360 days in a year, drew an angle that was 1/360 of a circle. This was their unit angle. It is now known that there is a little more than 365 days in a year but the Babylonian unit angle is still used as the basis for measuring angles or turns in a circle. The unit angle has a measure of 1 degree (1°).
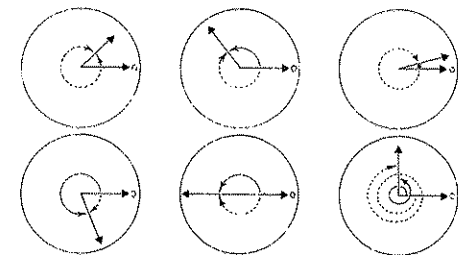
Imagine two arrows pointing towards 0°.

Figure 1 shows a counter clockwise turn of 90° or a clockwise turn of 270°.

Figure 2 shows a counter clockwise turn of 225° or a clockwise turn of 135°.

FIGURE 1

FIGURE 2

Use the picture at the top as a guide and approximate the amount of turn shown in each figure. Give both a counter clockwise and a clockwise amount.

BE CAREFUL

Being able to approximate the measure of an angle comes from experience in measuring angles with a protractor. Most people quickly learn to recognize a 90° angle, but we can also learn to approximate the measures of other angles.

Our experience and visual skills can be put to the test when a tiling job requires a careful eye and a reasonable approximation.

Approximations can sometimes be made by finding an upper and lower limit where the exact answer lies somewhere in between. Then, by averaging the two bounding limits, a close answer is found. Approximating by this method can be quite accurate since the upper and lower limits can often be adjusted closer and closer to each other.

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If we estimate the area of a curved
figure, it is easy to follow up our
guess by an approximation. Without
advanced mathematics, the measurement
of a curved figure's area is nearly
impossible without using square grids
to help us find a reasonable approxi-
mation.

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

LABORATORY APPROACHES

Like "modern math" or "discovery learning" or "individualization," the phrase "laboratory approach" means different things to different people. Although the phrase may conjure up visions of manipulative materials, perhaps the most commonly agreed-on characteristic is that <u>in laboratory approaches the emphasis is on learning-by-doing</u> as opposed to learning-by-listening. The important feature is that the student is an active participant rather than a passive receptor. Often this involvement <u>is</u> accomplished through the use of a manipulative.

WHAT IS A LABORATORY ACTIVITY?

A laboratory activity is a task or mathematical exercise that emphasizes "learning by doing." Two examples are given below.

a) Jamie and Pat are working with an activity card. Their teacher has supplied them with tagboard strips for the activity. They have made triangles, squares and other polygonal shapes from the strips and have discovered that only the triangle is rigid. They have copied the table on their papers and are trying to complete it, but they are having trouble with the decagon. They look up the word 'decagon' and find that it means a ten-sided polygon. Jamie remembers to look for a number pattern in the table, and decides that eight strips are needed to make a decagon rigid. Having finished the activity, they return the activity card and strips to a manila envelope and hand their papers to their teacher.

b) A class was challenged to find all the pentom-
    inoes (figures made with five adjacent squares)
    and the hexiamonds (figures made with six adja-
    cent triangles). They were given squared and
    triangled grid paper on which to record their
    shapes. The students worked individually for
    twenty minutes and then shared their findings
    in small groups. The teacher helped the class
    draw the shapes on the poster shown to the
    right until all the different shapes had been
    recorded.
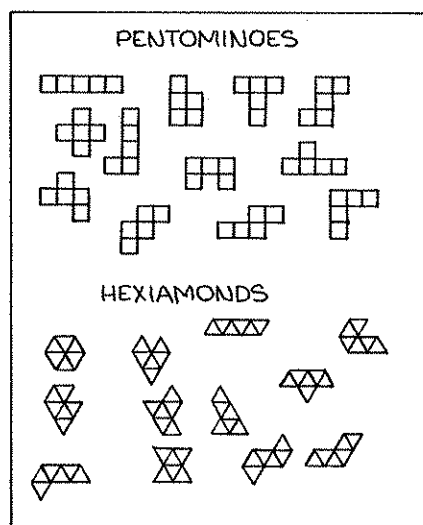
The next day the class used
their sets of hexiamonds and
pentominoes. Students worked
individually on this investiga-
tion and pooled their results.
They discovered which shapes
folded into open boxes and tetra-
hedra by cutting out the shapes
and actually doing the folding.

     The two lab activities above involved
students in active learning. The teacher pro-
vided the activity or the challenge and then
became a resource person. Students explored
the activities at their own pace and in their
own way. The activities provided opportuni-
ties for students to use problem solving
processes--organizing information into a ta-
ble, looking for patterns, making predictions
(predicting which hexiamonds will fold into tetrahedra) and checking predictions
(cutting out the hexiamonds and folding them). Cooperation among pairs or groups of
students was encouraged and each student had a chance for success.

Lab activities can vary greatly in form from those described above. Active learning can be accomplished through a game, a puzzle, a paper and pencil exercise, a set of manipulatives with a task card, or an experiment using apparatus and instruments to take measurements. A game like *Crossroads* from the Planes subsection involves two students, a game board, some markers and tiles, and strategic moves on a coordinate system. A challenging puzzle might require a student to apply several problem-solving techniques. A lab activity could use Cuisenaire rods to illustrate volume concepts, tiles to introduce area, wooden cubes to demonstrate spatial relationships, or paper strips to make flexagons.

Laboratory activities can directly involve students in "hands-on" assignments, often with group participation. Manipulative objects used in the activities often provide physical models that can introduce or clarify a mathematical concept to the student. There are also experiments which can be performed to take measurements and gather data. Students learn how to use certain equipment, tools and techniques in their search for solutions. Lab activities encourage the student to take an active role in learning mathematics rather than the passive role of "you teach me."

ORGANIZATION OF LABS

Collecting Equipment

A collection of lab materials is necessary for lab activity sessions. Items can be made, gathered or purchased. Below are three lists of suggested materials and manipulatives that have been used in various lab activities.

COMMON ITEMS

--Adhesives: tapes, glue

--Coloring Materials: pens, pencils, chalk, crayons, paint

--Fasteners or Binders: string, nails, pins, rubber bands, staples, wire

--Hard and Soft Wood: blocks, cork, boards, pegs, tagboard, toothpicks

--Paper and Write-on Materials: cardboard, construction paper, grid paper, letter
   stencils, plastic sheets, road maps, waxed paper, index cards, tracing paper

--Miscellaneous: measuring cups and spoons, modeling clay, needles, calendars, boxes,
   bottles, cans, cartons, cups, mirrors, catalogs, almanac, restaurant menus,
   cylinders, spinners, dice, playing cards

TOOLS AND INSTRUMENTS		*MANIPULATIVES*
hammer	tape measures	Cuisenaire rods
hand-calculators	metre sticks & rulers	geoboards
paper cutter	metric weights	Soma® cubes
paper punch	balance scales	attribute blocks
paper stapler	stop watch	tangrams
scissors	thermometer	colored wooden cubes
straightedges	T-square	linking cubes
protractors	templates	geoblocks
compasses	clock	pattern blocks

The Mathematics Laboratory

The math lab is an _environment_ that provides for active learning and encourages active participation. In terms of physical organization, three basic kinds of mathematics laboratories are most often discussed.

1) A decentralized laboratory--a self-contained set of lab materials stored in the teacher's classroom and readily available for the students to use.

2) A rolling or movable laboratory--a set of lab materials placed on a cart, stored in a central location, and wheeled from classroom to classroom as needed.

3) A centralized laboratory--a room especially designed (or adapted) and equipped for use as a permanent math lab. Classes are usually brought into the lab room on a rotating schedule that allows each mathematics class to use the lab materials several times a week as needed.

For most schools, the decentralized laboratory is the most practical and desirable math lab. Lab materials can be collected and organized at a modest rate as they are constructed, donated or purchased.

Eventually a set of lab materials will grow to a size large enough to be quite versatile. The classroom environment needs to be versatile as well. Flat tables, bookcases, movable carts and other furniture can be added to provide work areas for the students and storage space for the lab activities.

Lab materials may be packaged for student use in boxes, manila folders or envelopes. For example, a task card may be placed in a shoe box along with grid paper, scissors and several crayons that are required for the activity. Teachers may choose to package the materials with an activity card, or they may decide to have the students get needed materials for an activity from organized storage shelves. It is important that the students learn how to obtain the necessary lab items, how to read and follow the directions on a lab card, and how to cooperate and share their ideas.

Organization of the Students

There are various ways of organizing the students for math lab participation. If a class is inexperienced in laboratory methods, whole class or large group activities might be easier to initiate. Other activities may be done in small groups where the students rotate among several lab stations. (See *Teaching Via Laboratory Approaches* in the resource Mathematics in Science and Society.) Often pairs of students work well; the teacher can carefully select the partners or, in some cases, allow the students to choose partners. Once the students understand the basic lab procedures, each pair can do a different lab activity. This cuts down the competition with classmates and allows the pair to make decisions and solve problems at their own rate.

Content of Lab Activities

Lab activities can be organized around a mathematical topic, say volume. The materials such as Cuisenaire rods, wooden cubes and other items are collected for each volume activity. Once the materials have been selected, they can be organized into lab packages. The class can then be divided into pairs or groups for the lab sessions. In this way topical units can be presented through a laboratory approach.

Another way to develop a number of lab activities is to select ideas or concepts that are all different. Each lab activity is unique and self-contained. For example, a series of ten lab activities could include the exploration of ten different topics, one topic per activity. One program which is organized this way is MathLab--Junior High (McFadden et al., 1975).

The Role of the Teacher

The teacher's role changes from one of disseminator of knowledge to resource person and moderator. The students do the lab activities; the teacher watches and encourages inquiry and independent thinking. Using a laboratory approach requires considerable preparation and forethought. The teacher needs to find, organize and store lab materials for easy use; tell students where lab materials are, what to do with them and how to schedule their use; carefully prepare task cards or directions for the lab activities; instruct students in problem-solving methods of attack and investigation; interact enthusiastically with students and share in their experiences; and evaluate each student's attitudes, work habits and accomplishments.

GETTING STARTED

There are many ways to implement the lab approach. The descriptions below provide several suggestions to consider when starting to use a laboratory approach.

Mr. Langford has a class of thirty seventh graders. He was not sure about using lab materials, so he decided to start small. He set up an "activity corner" in the room. Three lab cards with the necessary equipment (e.g. in this case, squared paper, ceramic tile, measuring tape, metric wheel) were set up in the "activity corner." Each day of one week a different group of six students was allowed to work in pairs using the lab materials. The rest of the class worked on related paper and pencil exercises. All week was spent on the study of area. All thirty students had a chance to do the lab activities, and the activities integrated well with the week's mathematics concept of area. Mr. Langford wants to collect or write task cards that mix well with his established curriculum. Later, he might try other ways of using the lab activity cards.

Ms. Wilkins decided to assign each Friday as a "lab day" for her eighth-grade class of 28 students. She had watched several classes using a "lab day" once a week and decided to try it herself. She prepared two sets of seven lab cards covering seven different mathematical topics. Each student was assigned a partner, and the pair worked together for each of the seven "lab days." For seven weeks the students rotated to a new lab activity each Friday. They were asked to keep a record of their results and follow the planned rotation schedule. Ms. Wilkins found that this seven-week period with one "lab day" a week coincided well with the nine-week term. She developed a second set of lab materials for another seven weeks. This time there were 14 task cards put into 14 shoe boxes along with manipulatives, paper, or other materials needed for each activity. Each card treated the topic of measurement and contained various levels of abstraction and enrichment options for the students.

Mr. Jeffreys and Ms. Slone had adjoining sixth-grade rooms. They had been team teaching a number of units in mathematics. They decided to try the lab approach for their unit on polygons and polyhedra. They made or purchased geoboards, metric geoblocks and other equipment. Mr. Jeffreys and Ms. Slone picked out ten activities from the POLYGONS & POLYHEDRA section. One activity was used each day for two weeks. Often the class was divided into groups of 3 or 4 students to compare their data and discuss the activity.

The above are examples of teachers who were willing to support an active approach to learning. They prepared for using the lab approach by collecting and organizing materials and deciding on the content of lab activities.

Initially, when selecting material and equipment to use in the math lab, find readily available materials in the school. As time goes on, you will be able to buy, make or scrounge other materials as they are needed for particular activities.

Pages from the resource Geometry and Visualization that are marked with the lab symbol, the problem-solving symbol and the graphic representation symbol may include ideas for lab activities. The lab symbol itself mainly flags those pages which use a manipulative. Lab activities can be found in the resources Number Sense and Arithmetic Skills, Ratio, Proportion and Scaling, and Mathematics in Science and Society. Ideas for laboratory activities can be found in any of the sources listed on pages 8 and 9. Many periodicals (such as The Arithmetic Teacher or The Mathematics Teacher) include sections in each issue which contain ideas for activities that require a minimum of preparation and materials. Notice the interests of the students. Be creative and use your own ideas or their ideas as a source of lab activities. Discuss and exchange ideas about math labs with other teachers.

Begin with a lab activity that everyone can do at the same time. Later on, the students can separate into pairs, groups or small teams. Experiment with the size and the make-up of the groups. In the beginning it is a good idea to provide activities where each group member has a specific role. Have a specific objective(s) in mind for each activity, and have a clear idea of its mathematical content. Go through the lab activity to find what background concepts or skills the students will need to tackle it. Check for any difficulties the students might encounter as they do the activity.

Start small--in no way can most teachers and students survive a complete change of program. Students who have become passive learners need time to adapt to the role of active learners. The students need to develop inquisitive attitudes that motivate them to keep at a problem and not give up. They need supervision and guidance from the teacher as they learn to function in the lab environment. Eventually, the students should be able to select materials for each lab activity and return materials to the proper storage area when finished.

Plan for Evaluation

"Teacher evaluation of pupil progress should take two forms: (1) evaluation of written records, i.e. record papers, and (2) assessment of pupil competency based on observation and interaction with the youngster as he works. In both cases, the emphasis is on the progress of the individual in solving a given problem using his own particular talents and capabilities. Obviously, no evaluation is quite so valuable as that done first-hand. A laboratory approach offers unique opportunities to assess

understandings and competencies through observation and discussion with the pupil as
he completes his assigned task. In evaluating written records, provide positive rein-
forcement for carefully completed recordings. Encourage completeness of answers,
keeping in mind that the record paper is primarily a communications device. If each
pupil keeps a folder of his completed record papers, he can note his own improvement
in recording throughout the school year." (McFadden et al., Program Teacher Commen-
tary on using MathLab, 1974, p. 5)

SUMMARY

The laboratory approach is a philosophy which emphasizes "learning by doing" and
breaks away from formal teaching methods. "It is a system based on active learning
and focuses on the learning process rather than on the teaching process." [Kidd, et al.,
1970] Experiences are devised to help the student learn mathematics by seeing, touch-
ing, hearing and feeling. An environment--the math lab--emerges where the teacher
and the students work and communicate with each other to plan activities and learn by
doing. At the level of their abilities and interests, the students discover rela-
tionships and study real-world problems which utilize specific mathematical skills.

A laboratory approach breaks the monotony of straight textbook teaching. It
extends and reinforces the students' understandings and skills while providing back-
ground experiences for later development of abstract concepts. It also offers a
unique, concrete way to learn mathematics. A laboratory approach can be integrated
into the classroom and used along with, not in place of, many other equally valuable
teaching strategies.

Lab activities help to eliminate the narrow one-method syndrome too often charac-
teristic of mathematics classes. A variety of methods of attacking a problem can be
explored. Open-ended activities encourage students to make discoveries, formulate
and test their own generalizations (i.e., problem solving). Lab assignments can be
used to challenge the students by providing them with opportunities for developing
self-confidence, habits of independent work, and enjoyment of mathematics. The
relaxed atmosphere can encourage student involvement and positive attitudes toward
mathematics. By direct observation, the teacher can assess the student's skill in
problem solving and computing while the student's attitude and work habits can also
be evaluated.

Selected Sources for Laboratory Approaches

The Arithmetic Teacher. Reston, Virginia: The National Council of Teachers of
    Mathematics.

EXAMPLES OF LABORATORY ACTIVITIES IN THE CLASSROOM MATERIALS

I. Manipulatives and Equipment

TANGRAM CONSTRUCTION

A) The pieces of the Chinese Tangram puzzle can be constructed as a compass and straightedge activity. A chalkboard or overhead demonstration or an audio tape recording could be used to convey the instructions.

  1) Construct a 10 cm square. Label it ABCD.

  2) Draw diagonal $\overline{AC}$.

  3) Bisect $\overline{AB}$. Label the midpoint E.

  4) Bisect $\overline{BC}$. Label the midpoint F.

  5) Draw $\overline{EF}$.

  6) Bisect $\overline{EF}$. Label the midpoint G.

  7) Draw $\overline{DG}$.

  8) $\overline{DG}$ intersects $\overline{AC}$. Label the intersection H.

  9) Bisect $\overline{AH}$. Label the midpoint J.

  10) Draw $\overline{EJ}$.

  11) Bisect $\overline{CH}$. Label the midpoint K.

  12) Draw $\overline{GK}$.

  13) Label the seven pieces like this:

  △CDH I   △AEJ V
  △ADH II   △CHK VI
  △BEF III   ▱CFGK VII
  ▱EGHJ IV

This Chinese Tangram puzzle can be used to teach area, perimeter, similarity and congruence.

The circular geoboard can be used to teach symmetry, rotations, inscribed polygons, area, perimeter and circumference. This manipulative is also a helpful tool in problem solving.

POLYGONS ON A CIRCULAR GEOBOARD

Materials Needed: Circular geoboard, rubber bands

Activity: Use rubber bands for these exercises. You can stretch one rubber band for each polygon or use one rubber band for each side.

1) Which of these polygons can be inscribed in the large circle?

  a) square
  b) regular octagon
  c) scalene triangle
  d) regular pentagon
  e) pentagon
  f) isosceles triangle
  g) equilateral triangle
  h) regular hexagon

2) Are there any of the above polygons that cannot be inscribed in the small circle?

PATTERN BLOCKS III

1) Use several squares.

2) Use four squares to fit around a point.

3) If there are 360° around a point, how many degrees are there in one angle of the square? _____

4) If a square has four congruent angles, how many degrees are in the sum of the angles of the square? _____

5) Use several of the triangles.

6) Fit the triangles around a point.

7) If there are 360° around a point, how many degrees are there in one angle of the triangle? _____

8) Since this triangle has three congruent angles, how many degrees are in the sum of the angles of the triangles? _____

Pattern blocks are a handy manipulative to use when teaching about angles and the sum of the angles of various polygons.

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This activity has the students find the axes of rotational symmetry of the Platonic solids. Polyhedral models and wire are used to provide concrete manipulatives which help the students visualize the concept.

Geometric solids have various properties that can be explored by the students in a laboratory activity.

A Lake and Island Board is a manipulative that can be used to teach perimeter, area and volume relationships.

The faces, edges and surface area of
geoblocks can be investigated by
covering a block with aluminum foil
(a shell) and cutting the shell at
appropriate places so it will flatten.

II.   Grid Paper and Designs

GEΔ-BLΔⳞKⳞ I

Readiness Activities:

1) Build some shapes with the blocks.

2) Find all the blocks that have at least one triangular face. How many differently shaped triangular faces can you find?

3) Try to find a block with no rectangular (or square) faces.

4) Build a larger block with two or more smaller blocks.

5) Decide how many blocks of one shape are needed to fill a box.

6) Feel a block without looking at it, then draw a sketch of the block.

7) Build a shape with the blocks. Draw a sketch of top, front and side views. Give your sketches to a friend. Can he build the same shape by looking at your sketches?

Students use square grid paper to design
patterns for constructing a cube. They
then cut and fold the patterns to see if
a cube can be made.

YOU'VE GOT ME IN STITCHES

Line or string designs are patterns made by joining one set of points to another set with straight line segments. The patterns create an illusion of curved lines even though straight line segments are used. By varying the selection of points different patterns can be formed.

JOIN A TO A, B TO B, C TO C, ETC.

An easy way to begin is to join equally spaced points on both sides of an angle (#1).

Increase the number of points on a side (#2).

Vary the size of the angle (#3 and #4).

Use different units for spacing points on each side of the angle (#5).

SHADE REGIONS FOR SPECIAL EFFECT.

Combine several angles (#7, #8, and #9).

TO SEW THE DESIGNS USE SQUARES OF COLORED RAILROAD BOARD. PUNCH HOLES ABOUT ½ cm TO 1 cm APART WITH A FAT NEEDLE. USE ORDINARY SEWING THREAD TO SEW THE DESIGN THROUGH THE HOLES. MASKING TAPE CAN BE USED TO FASTEN LOOSE ENDS.

THE SIDES OF THE ANGLE DON'T NEED TO BE DRAWN, ESPECIALLY IF THE PATTERN IS STITCHED.

Making line or string designs can motivate students as well as give them practice in visual-motor coordination.

TESSELLATIONS
(PAGE 3)

2) Have students discover which regular polygons of two or more kinds can tessellate a plane if the arrangement of corners is the same at every vertex—a semi-regular tessellation.

a) First have students mark a point and try to surround it with squares and triangles. (Since the arrangement of angles must add to 360° students can conclude that only 3 triangles and 2 squares will work. $3 \times 60° + 2 \times 90° = 360°$.)

There are two possible arrangements: The first is described as 3-3-3-4-4; the second as 3-3-4-3-4.

Each arrangement will make a different tessellation. Note that each vertex has the same arrangement.

3-3-3-4-4   3-3-4-3-4

b) By trying to fit the tiles around a point and then continuing the arrangement students can discover that there are only eight semi-regular tessellations. The other six are illustrated below.

3-6-3-6   3-3-3-3-6   4-4-6-4   3-12-12

4-8-8   4-6-12

Note: The three regular tessellations in #1 could be described as 3-3-3-3-3-3; 4-4-4-4; 6-6-6.

3) An unlimited number of tessellations can be formed from regular polygons if it is not required that all vertices be the same. Encourage students to create their own designs. (Be sure there are no spaces between the tiles and no overlapping.) If the designs are drawn on paper, students could look for the minimum number of colors needed to shade the design (see the section LINES, PLANES & ANGLES) and the possible ways to vary the colors.

Placing regular polygons to form tessellations can reveal geometric patterns and designs.

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III. Miscellaneous Laboratory Activities

Enlargements of figures can be made using this rubber band method. The activity requires a steady hand and a keen eye.

A SNAPPY SOLUTION TO SIMILAR FIGURES

Materials Needed: Several identical rubber bands, a thumbtack, a centimetre ruler, butcher paper, large table.

Activity: Loop two identical rubber bands together to form a knot in the middle.

I   To make an enlargement of triangle ABC on the butcher paper:

1) Pick a point P so the distance from P to A is longer than the length of a rubber band.
2) Hold one end of the rubber band on point P with your thumb or use a thumbtack. Be sure to place tape over the thumbtack to secure it.
3) With a pencil in the other end, stretch the rubber bands until the knot is over A. Mark a dot with the pencil and label the dot A'.
4) Repeat step 2 with the knot over B to find B', then over C to find C'.
5) Connect A', B' and C'.

6) Measure the lengths of the sides of the two triangles.

A'B' _____   B'C' _____   C'A' _____
AB _____   BC _____   CA _____

7) How do these lengths compare?

A'B' and AB _____
B'C' and BC _____
C'A' and CA _____

8) In each case is the first length about twice as long as the second length? _____
The rubber bands have helped you to make an enlargement of the triangle using a scale factor of 2.

9) Draw another figure and make an enlargement of it using a scale factor of 2.

PIE ARE ROUND, CORNBREAD ARE SQUARE

I   Cut a large circular disk from tagboard. A radius of 20 cm is convenient for a bulletin board display; a radius of 5 cm is handier for an overhead demonstration.

II Draw line segments to divide the disk into 12 congruent sectors. This can be done by drawing 30° central angles or by inscribing a regular 12-gon as described in *Inside the Circle 1* in the Circles subsection.

III Cut along one diameter to make two semicircular regions.

IV Make cuts along the line segments drawn in II, but be careful not to cut through to the original circle.

V   Fan out each semicircular region and fit them together as shown. This figure is very much like a parallelogram.

VI The area of a parallelogram is base times height.

VII The area of the "parallelogram" to the right is

AREA = b × h

$\underbrace{\text{BASE}}_{\frac{1}{2} \times \text{CIRCUMFERENCE}} \times \underbrace{\text{HEIGHT}}_{\text{RADIUS}}$ =

$\frac{1}{2} \times 2 \times \pi \times \text{RADIUS} \times \text{RADIUS}$ =

$\pi \times \text{RADIUS} \times \text{RADIUS}$ =

$\pi \times r^2$

CIRCUMFERENCE EQUALS $2 \cdot \pi \cdot$ RADIUS

THE HEIGHT IS THE RADIUS OF THE ORIGINAL CIRCLE.

THE BASE IS ONE-HALF OF THE CIRCUMFERENCE OF THE CIRCLE.

The area of a circle can be determined by sectioning the circle and arranging the parts into a parallelogram.

Given a fixed length of fencing, students are to determine the rectangular-shaped pen that encloses the greatest area. They then investigate other shaped pens.

It's magic! Many students will be fascinated with these tricks. A few students may want to put on a Magic Show for the class.

LARGEST CONTAINER

This activity will help students see that containers which have the same surface area do not necessarily have the same volume.

I   Tall containers - vs - short containers

Using two rectangular pieces of tag board (shown below) students can roll up two cylinders each having the same surface area (in this case 15 cm x 24 cm or 360 cm^2).

AN EXTRA CM ON THE END TO USE AS A TAB MAKES THE MODEL EASIER TO ASSEMBLE

Students can study, predict and record information concerning the cylinders. Some suggestions about the surface area and volume might key them in on critical information. Have students check their predictions. See *Methods for Finding Volume* in this section for a variety of techniques to use.

II Changing the number of sides.

To assemble the models below students will need four 15 cm x 25 cm pieces of tag board. The first piece is to be rolled into a right circular cylinder having a height of 15 cm and a circumference of 24 cm. The remaining three are marked in the appropriate places and folded into the patterns shown.

Students can arrange the containers from least to greatest volume. Have them check their guesses and measure the volume of each by using the methods in *Methods for Finding Volume*.

Students discover that containers having the same surface area do not necessarily have the same volume.

ANGLES & FOLDS

Materials: Scrap paper, scissors, straightedge, protractor

Have your students:

1   Draw 2 lines across their papers intersecting at a 60° angle. Accuracy is important. See figure 1.

2   Sketch an irregular figure keeping away from the intersection point. See figure 2.

3   Carefully fold the paper on one of the lines and cut along the outline of the figure. Turn the paper over and cut along the outline that remains. See figure 3.

4   Open the paper and observe the symmetry.

5   Fold along the other line and cut away the portion that is not doubled. Turn the paper over and cut off the undoubled part.

6   Open and observe.

Using a protractor and some paper-folding techniques, the students can discover many surprising geometric shapes.

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

SECONDARY EMPHASES

There are many important topics and relations (threads) in geometry. Some of these topics (polygons, circles, area, . . .) and relations (similarity) have been used to form an organization for this resource. Others occur in activities throughout the resource. This section gives a brief overview of several of the threads which appear in the activities. A listing of related classroom pages is given after each overview. The lists of pages could be used to pull together activities with a common theme or to provide guidance for the reader who wants to explore a thread.

The threads chosen are not necessarily the most important (for example, the relation of congruence is not included). More information about other relations, topics and threads can be found within the MATHEMATICAL CONTENT section.

The threads discussed are:

      STRAIGHTEDGE AND COMPASS CONSTRUCTIONS
      TOPICS IN TOPOLOGY
      SYMMETRY AND MOTIONS
      COORDINATE GEOMETRY
      MAXIMUM AND MINIMUM QUESTIONS
      LIMITING PROCESSES
      INVARIANCE

STRAIGHTEDGE AND COMPASS CONSTRUCTIONS

Over two thousand years ago, Greek mathematicians found that they could do many constructions by using only two instruments: a straightedge for drawing line segments and a compass for drawing circles. Determining which figures can be constructed with only these two tools became an intellectual challenge, and the Greeks discovered that some constructions appeared impossible. The following are three famous construction problems which the Greeks could not solve using only a straightedge and compass.

1) Can every angle be divided into three congruent angles (trisected)?
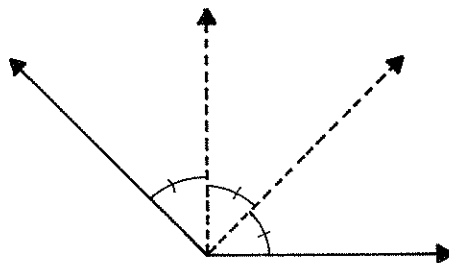
2) Can a square be constructed so its area equals that of a given circle?

AREA (CIRCLE) = AREA (SQUARE)
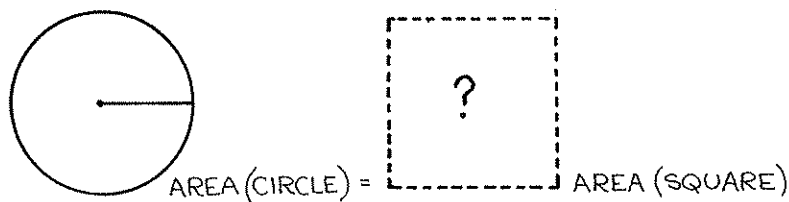
3) Starting with any cube, can the edge be constructed of a second cube that has double the volume of the original cube?

VOLUME = V   VOLUME = 2V

These constructions are impossible with compass and straightedge. In 1837 Peter Wentzel proved that it was impossible to trisect an arbitrary angle with only a straightedge and compass; however, some people still try to show it can be done. (An interesting observation: If an infinite number of bisections were possible, any angle could be trisected. See *Are You a Third of the Way There Yet?* in the Lines subsection.)

Another problem the Greeks undertook was that of constructing regular polygons with only a straightedge and compass. It was not until Carl Friedrich Gauss (1777–1855) studied this problem that it was completely resolved. Gauss determined the values of n for which regular n-gons can be constructed.
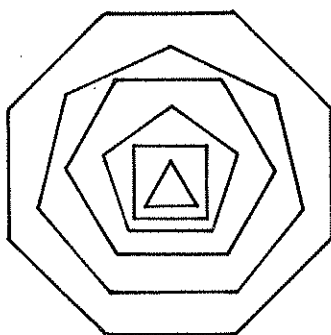
Regular Polygons	Can it be constructed?
Triangle	Yes
Square	Yes
Pentagon	Yes
Hexagon	Yes
Heptagon	NO
Octagon	Yes
.	.
.	.
.	.

Constructions are still a part of the geometry curriculum. Students often enjoy them, but some students have difficulty using the tools and doing the basic constructions. Hints are given in the *Graphic Representation* teaching emphasis for helping students use a straightedge and compass. The steps for the basic constructions are also given there.

CLASSROOM MATERIALS INVOLVING STRAIGHTEDGE AND COMPASS CONSTRUCTIONS

LINES, PLANES & ANGLES

Lines

YOU'VE GOT ME IN STITCHES

Angles

ANGLE WRANGLE

Symmetry and Motions

WHERE IS THAT LINE?

POLYGONS & POLYHEDRA

Polyhedra

HOW WELL DO YOU STACK UP?

CONSTRUCTING NETS OF IRREGULAR TETRAHEDRA

Polygons

TANGRAM CONSTRUCTION

WHEN IS A TRIANGLE ACUTE, RIGHT OR OBTUSE?

SIDE, SIDE, SIDE/ANGLE, ANGLE, ANGLE

SPECIAL LINES AND SEGMENTS IN TRIANGLES

AN EQUI-MEETING

MORLEY'S DISCOVERY

SIMPSON SAYS

EULER FOUND

NAPOLEON'S IDEA

CURVES & CURVED SURFACES

    Circles

        CIRCLE MAKING METHODS

        ENCOMPASSING CIRCLES

        INSIDE THE CIRCLE I

        INSIDE THE CIRCLE II

        INSIDE THE CIRCLE III

        ART INSIDE THE CIRCLE

        EASTER EGGS

        CIRCLE ART

        CIRCUMSCRIBING A TRIANGLE

    Other Curves

        ROLLERS WITH CORNERS

        A SPIRAL IN A GOLDEN TRIANGLE

SIMILAR FIGURES

        MAKE IT EASY ON YOURSELF

        DIVIDE ME UP

        SPLITTING ANGLES

        SOME RECTANGLES ARE GOLDEN

        MEANS ARE MEANINGFUL

    Area

        LOTS OF LITTLE ONES

Readings and References

Courant, R. and Robbins, H. <u>What is Mathematics?</u> New York: Oxford University Press, 1941.

The question of constructibility with straightedge and compass and with compass alone is considered in chapter 3. The reading requires some mathematical familiarity with the topic.

Eves, Howard. <u>An Introduction to the History of Mathematics</u>. New York: Holt,
Rinehart and Winston, 1961.

   *Chapter 4, pages 77 to 109, of this excellent history book is
   devoted to the three famous impossible problems.*

Stein, Sherman K. <u>Mathematics; The Man Made Universe</u>. San Francisco: W.H. Freeman
and Company, 1969.

   *One chapter of this textbook for the general high school or
   college student is devoted to constructions. The author uses an
   outline of a theorem-proof method to explain it is impossible to
   trisect a 60° angle with straightedge and compass.*

TOPICS IN TOPOLOGY

How is a doughnut like a coffee cup? In what way is a triangle like a circle?
Can a highway inspector always find a path over a network of roads so that he tra-
vels each road exactly once? How can it be determined which points are inside a
simple closed curve? What are the properties of a one-sided surface? These ques-
tions and others like them are studied in topology. Notice that none of the questions
involve measurements. Topology is the study of properties of geometric figures that
remain the same (invariant) even when the figure is stretched or distorted.

A doughnut is like a coffee cup because a doughnut shape can theoretically be
reshaped into a coffee cup shape without cutting the surface or joining any new
points together. Think of taking a piece of clay in a doughnut shape and changing
it like this:
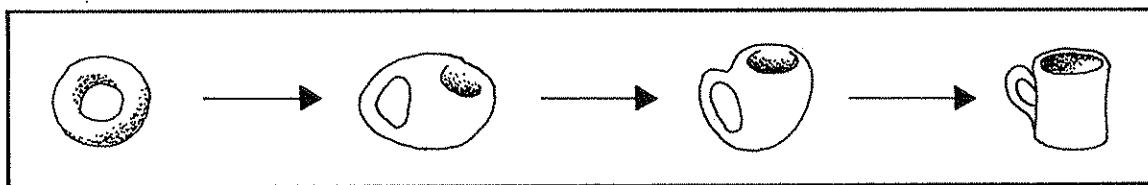
The doughnut or torus and coffee cup belong to the same class. Notice that each has
one hole. Spheres, cubes and holeless blobs belong to another class. They can be
changed into each other in a similar manner.*

Figures in a plane can be grouped in a similar way.
Any simple closed curve can be deformed without cutting
or joining new points into any other simple closed
curve. What properties of a simple closed curve remain
the same even if the plane is stretched or shrunk like
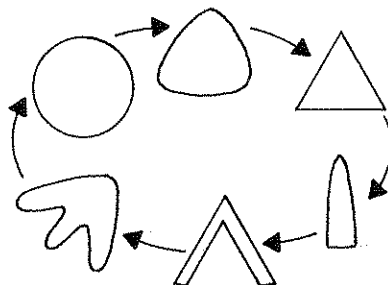
*A technical term for the class to which both the torus (doughnut shape) and coffee
cup belong is genus 1. The sphere, cube and holeless blob belong to genus 0.

a sheet of rubber? Your students might like to see what happens when they draw a
figure on an old balloon and stretch the balloon as shown below.
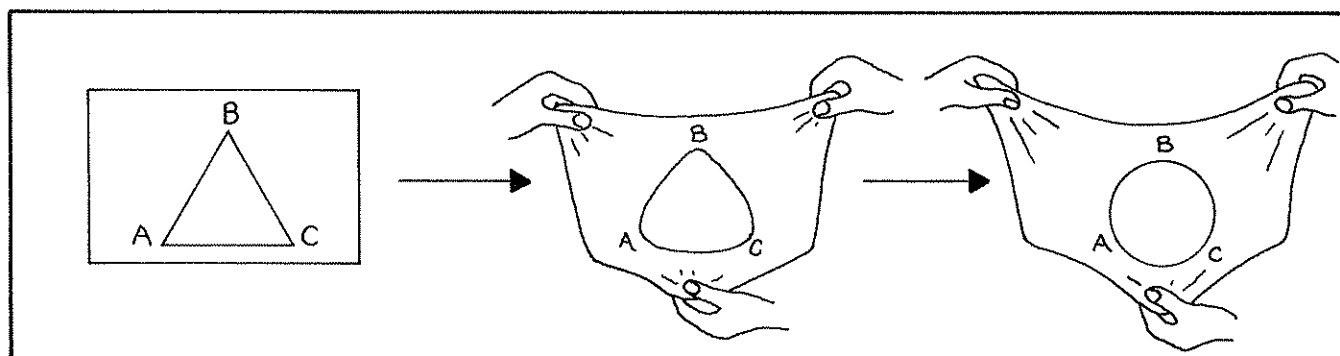
Each of the curves still separates the plane into two regions: the set of points
inside the curve and the set of points outside the curve. The points on the curve
are still in the same order.

Some simple closed curves look complicated. Is
point A inside or outside the curve? Since any line
from A to the outside of the curve <u>crosses</u> the curve
an odd number of times, point A is inside the curve.
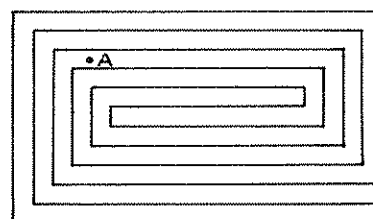
The problems of the highway inspector and the traveling salesman are studied
in topology. To be thrifty, a highway inspector wants to find a route that passes
over each section of highway exactly once, while a
traveling salesman wants a route which passes through
each town exactly once. To study these problems, a
network of the highway system can be drawn. The dots
or vertices represent the towns and the segments or
arcs represent roads. Since distances are not the
issue, the network does not have to be a scale drawing
of the original highway system; only the way the towns
are joined is important. If you will try to find paths
on the network at the right, you will find a solution
for the traveling salesman but no solution for the highway inspector. The
classroom page *Schlegel and Hamilton* in the Polyhedra subsection explores other
versions of these problems.
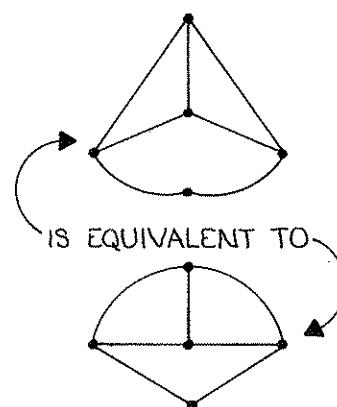
The intersections of a network are usually called vertices, the arcs are called edges and the regions are sometimes called faces. The network shown above has 5 vertices, 7 edges and 4 faces. Notice that the network separates the plane into 4 regions, one of which is outside the network. An interesting relationship exists among the number of edges, vertices and regions of a network drawn on a particular surface. This relationship is explored for networks on polyhedra in *Euler's Formula Again* in the Polyhedra subsection.

Most surfaces have two sides. The two sides could be distinguished by painting them different colors which do not touch except at the edges of the surface. A sheet of paper could be red on one side and white on the other. The surface of a football could be tan leather on the outside and black laytex on the inside. A surprising discovery of a one-sided surface was made by A. F. Moebius, a German mathematician and astronomer in the 1800's. He discovered that putting a half twist in a rectangular strip of paper and joining its ends produced a one-sided surface. This surface cannot be painted with two colors unless the two colors meet on the surface. The properties of such surfaces are explored on the pages *Two-Faced? Never*! in the Curved Surfaces subsection.
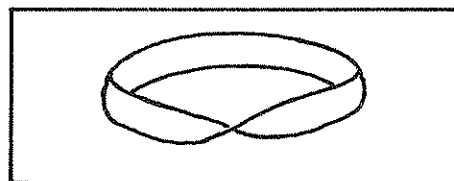
There are many other interesting topics in topology. Among them are map-coloring, the study of knots and various tricks like removing your vest without removing your coat. These ideas are pursued on the resource pages and references listed below.

CLASSROOM MATERIAL INVOLVING TOPICS IN TOPOLOGY

LINES, PLANES & ANGLES

Lines

TAKE A ROUND TRIP

POLYGONS & POLYHEDRA

Polyhedra

GEOBLOCKS II

DO YOU KNOW THAT?

<div style="text-align:center">

VERTICES, FACES AND EDGES

EULER'S FORMULA AGAIN

THEY CAME BY TWOS

SCHLEGEL AND HAMILTON

CURVES & CURVED SURFACES

Curved Surfaces

TWO-FACED? NEVER!

TRICKS WITH TOPOLOGY

</div>

Readings and References

Barr, Stephen. Experiments in Topology. New York: Thomas K. Crowell Co., 1972.

>A paperback suitable for teachers describing experiments with Moebius bands, Klein bottles, projective planes and map-coloring.

Bergumini, David. Mathematics. New York: Time Inc., 1970.

>A book in the Life Science series with sixteen pages with large colored pictures devoted to topics in topology.

Courant, R. and Robbins, H. What is Mathematics? New York: Oxford University Press, 1941.

Fadiman, Clifton. Fantasia Mathematica. New York: Simon and Schuster, 1958.

>A paperback collection of stories and diversions, six of which involve topics in topology, which would make enjoyable readings to the class or for individual students.

Froman, Robert. Rubber Bands, Baseballs and Doughnuts. New York: Thomas Y. Crowell Company, 1972.

>A library book for elementary students.

Johnson, Donovan A. and Glenn, William H. Topology, The Rubber-Sheet Geometry. London: John Murray.

>A pamphlet of interest for middle school or high school students or teachers. This has also been published by Dover Publications as Part V of Exploring Mathematics on Your Own.

Kasner, Edward and Newman, James. Mathematics and the Imagination. New York: Simon and Schuster, 1967.

>A delightful paperback book, one of whose nine chapters is "Rubber Sheet Geometry."

SYMMETRY AND MOTIONS

(Symmetry and motions are discussed in the commentary to the Symmetry and Motions subsection.)

CLASSROOM MATERIAL INVOLVING SYMMETRY AND MOTIONS

LINES, PLANES & ANGLES

Lines

YOU'VE GOT ME IN STITCHES

Angles

I'M SEEING STARS

Symmetry and Motions

(The entire subsection)

POLYGONS & POLYHEDRA

Polyhedra

SEEING IT LIKE IT IS

TWO COLOR PROBLEMS

SEVERAL VIEWS OF CUBES

FOLD-UPS

BRICKS AND POLYHEDRA

MOM, DAD AND THE KIDS

PLANES OF SYMMETRY

AXES OF SYMMETRY

Polygons

THE PERPLEXING PENTOMINOES

HEXIAMONDS VS. PENTOMINOES

REPEATING SHAPES

TESSELLATIONS

TESSELATIONS WITH TRIANGLES

THUMB-TACKTICS

REFLECT ON THIS

THE TURNING POINT

CURVES & CURVED SURFACES

Curved Surfaces

A REVOLVING SITUATION

Circles

A PENNY FOR YOUR THOUGHTS

Readings and References

Weyl, Hermann. "Symmetry." The World of Mathematics. Vol. 1. Edited by Newman,
    James R. New York: Simon and Schuster, 1956.

    *This delightful 54-page section on symmetry includes many
    excellent photographs showing symmetry in nature, art and
    architecture.*

COORDINATE GEOMETRY

The idea of using ordered pairs of numbers (coordinates) to locate points on a
plane was used by René Descartes (1596-1650). The technique was extended to three-
space by Fermat (1601-1665). This simple idea brought algebra and geometry together
so that it is possible to draw a geometric curve to represent algebraic equations or
to determine equations for a curve. Coordinate geometry is a powerful tool and an
important basis of calculus.

Several different coordinate systems are possible. The rectangular (Cartesian)
coordinate system and the polar coordinate system are shown on the next page.

A GRAPH OF
Y = 2X - 2

(2,2)

(1,0)

(0,-2)

(-1,-4)

(4,45°)

(3,180°)

(1,0°)

(2,270°)

(3,315°)
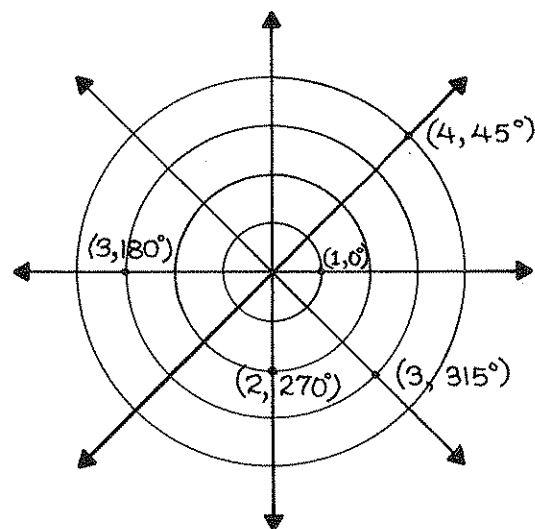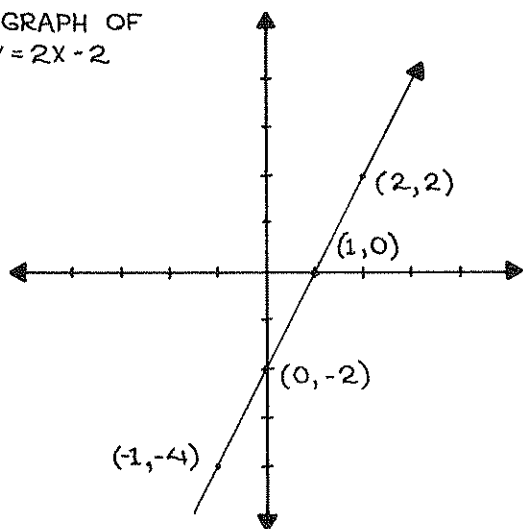
Both of these systems are used in the classroom pages listed below.

CLASSROOM MATERIAL INVOLVING COORDINATE GEOMETRY

LINES, PLANES & ANGLES

Planes

CROSS ROADS

COORDINATE TIC-TAC-TOE

IT'S STARTLING!

ARE YOU CLUMSY OR COORDINATED?

A LONG TIME AGO

IT'S A DOG'S LIFE

A MAZE YOUR FRIENDS

PICTURE THIS

WE ALL LINE UP

Symmetry and Motions

A SHIFT IN THE RIGHT DIRECTION

POLYGONS & POLYHEDRA

    Polygons

        TANGRAM CONSTRUCTION

CURVES & CURVED SURFACES

    Circles

        CONCENTRIC TIC-TAC-TOE

    Other Curves

        HANGING TOGETHER

        ARCHIMEDEAN SPIRALS

SIMILAR FIGURES

        GRID PAIRS

AREA & VOLUME

    Area

        THE SECRET AREA

    Pythagorean Theorem

        PYTHAGORAS ON THE GEOBOARD

Readings and References

Bell, E. T. Men of Mathematics. New York: Simon and Schuster, 1965.
Chapter 3 is about Descartes; Chapter 4 is about Fermat.

Eves, Howard. An Introduction to the History of Mathematics. New York: Holt, Rinehart and Winston, 1961.

Chapter 10 discusses the development of analytic (coordinate) geometry.

MAXIMA-MINIMA QUESTIONS

    Have you ever tried to determine the rate at which your car gives maximum mileage or to minimize the distance of a trip? Many of our daily activities involve

maximizing or minimizing some quantity or quality. Maximum and minimum type ques-
tions are also important in many branches of mathematics. What is the shortest path
between two points on a sphere? (What path would be shortest when flying from New
York to London?) What is the maximum area of a figure with a given perimeter?
Geometry is an excellent source for maximum-minimum activities which are within the
realm of middle school mathematics. One example is given below. Notice the problem-
solving strategies which could be worked into this example. (Trial and error,
organizing information into a table, making a graph and varying the data to make
new problems.)

Example:

What size corners should be
cut from this sheet to make
a box of maximum volume?
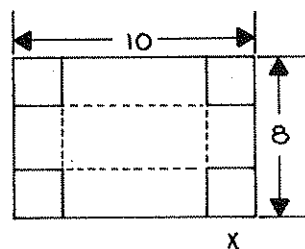
Students might try cutting first, then use some arithmetic to find volumes for
various values of x. Graphing might also be used:

X	L	W	V
0	10	8	0
1	8	6	48
2	6	4	48
3	4	2	24
4	2	0	0
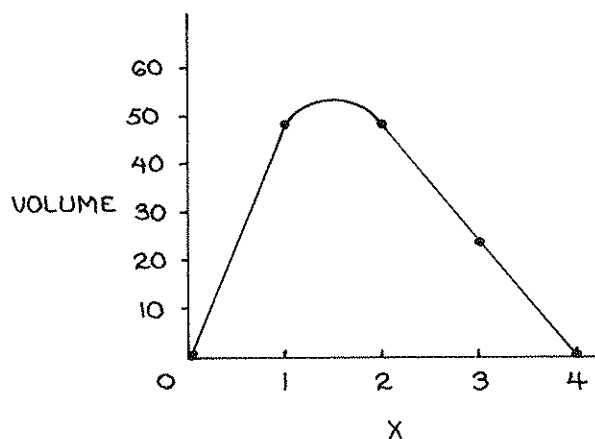
Is the maximum volume at $x = 1.5$? A calculator can be used to try $x = 1.25$,
$x = 1.5$ and $x = 1.75$. (The curve is $\underline{not}$ symmetrical and students will obtain an
approximate answer.)

What happens if the original shape is changed to a 10 by 10 square? A 10 by
20 rectangle?

CLASSROOM MATERIAL INVOLVING MAXIMA-MINIMA QUESTIONS

LINES, PLANES & ANGLES

Lines

PART OF A LINE

GEOBOARD I

TAKE A ROUND TRIP

ONE WAY RAY

WHAT'S MY LINE?

BE BOLD . . . FOLD

GEOBOARD II

POLYGONS & POLYHEDRA

Polyhedra

GEOBLOCKS II

THEY CAME BY TWOS

CURVES & CURVED SURFACES

Circles

IN 'N' OUT REGIONS

AREA & VOLUME

Perimeter

ARRANGING SQUARES

Area

FRAMED

HOW MANY CAN YOU FIND?

BIG FOOT

A SHEEPISH PROBLEM

PIZZA PARLOR

AREA PROBLEMS TO ATTACK

Surface Area

THE CUBE PAINTER

PAINT-LESS

Volume

LAKE & ISLAND BOARD

MAXIMUM VOLUME

LARGEST CONTAINER

CHEAPEST DRINK

Readings and References

Beckenbach, Edwin and Bellman, Richard. An Introduction to Inequalities. New York:
     Random House, Inc., 1961.

*Chapter 8 of this paperback applies algebraic inequalities to
maximization and minimization problems.*

Courant, R. and Robbins, H. What is Mathematics? New York: Oxford University Press,
     1941.

*Chapter 7 is entitled "Maxima and Minima." Included in the
chapter is a discussion of the least squares method, Steiner's
problem and soap film experiments which demonstrate surface of
least area.*

Kazarinoff, Nicholas D. Geometric Inequalities. New York: Random House, Inc.,
     1961.

*High school algebra, plane geometry and a touch of trigonometry
are used to explore many geometric maxima and minimum problems in
this paperback.*

MacPherson, Eric D. The Themes of Geometry. (Mimeographed.) For further information
     write: Eric D. MacPherson, Dean, Faculty of Education, University of Manitoba,
     Winnipeg, Manitoba, Canada R3T 2N2.

*This book, which will be published soon, suggests that informal
geometry be organized around various themes--one of which is maxi-
ma and minima.*

LIMITING PROCESSES

Limiting processes can be used to find areas, perimeters and volumes. For example, the circumference of a circle can be approximated by placing six equally-spaced marks on a circle, drawing the six chords connecting the points and adding the lengths of the chords. A better approximation results when twelve equally-spaced points are used. As more points are added, each sum becomes closer to the circumference, and so we say the limit of the sums of the lengths of the chords is the circumference of the circle.
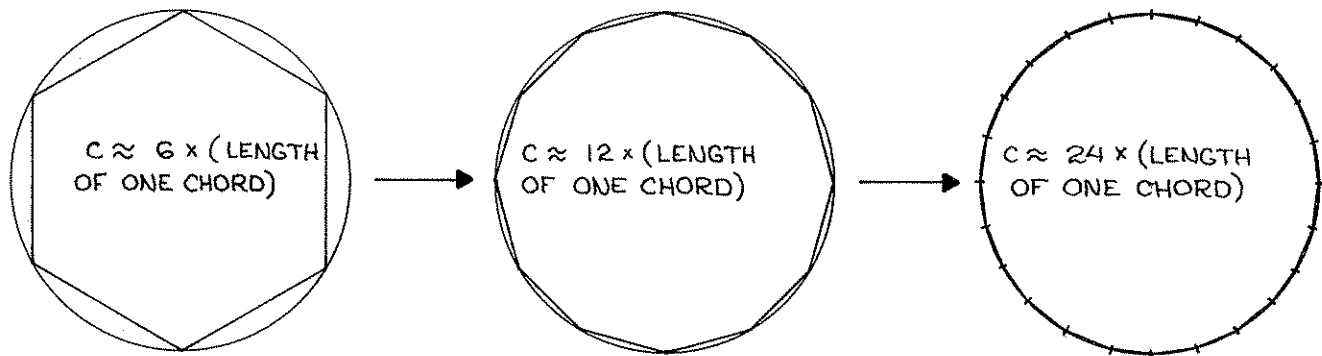
There are many interesting limit questions which arise in geometry. Here are a few.

- A sequence of regular n-gons are drawn. An equilateral triangle, a square, a regular polygon, . . . What shape do the regular n-gons approach as n gets very large?

- A sequence of squares is drawn, each new square inside the previous square as shown. What is the limit of this sequence of squares?
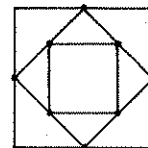
- A sequence of cones is made from 20-cm-in-diameter circular pieces of paper. The first cone is made from $\frac{1}{2}$ of a circular piece, the second is made from $\frac{2}{3}$ of a circular piece, the third from $\frac{3}{4}$, . . . What is the limit of the volumes of the cones as the fraction of a circular piece gets closer and closer to 1? What would the limit be as the fraction decreases to 0 ($\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, . . .)?
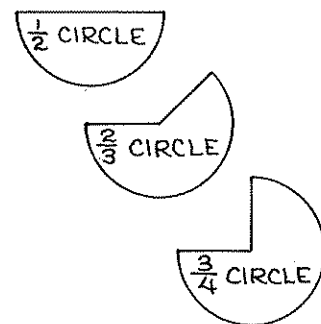
- Suppose two tacks and a piece of string are used to draw an ellipse. (See *Methods for Drawing Ellipses*.) Now, suppose the tacks are moved closer together and another ellipse is drawn with the same string. If this process is continued,

what shape does the sequence of ellipses approach? What happens if the tacks are
moved as far apart as possible? If the tacks are kept in one place, what happens if
the piece of string becomes as short as possible? What happens if the piece of string
becomes very long?

CLASSROOM MATERIAL INVOLVING LIMITING PROCESSES

LINES, PLANES & ANGLES

Lines

GET THE POINT?

ARE YOU A THIRD OF THE WAY THERE YET?

POLYGONS & POLYHEDRA

Polygons

EXTERIOR ANGLES OF A POLYGON

CURVES & CURVED SURFACES

Other Curves

POLYGONAL SPIRALS

AREA & VOLUME

Perimeter

AROUND THE BLOB

INSIDE AND OUTSIDE A CIRCLE

Area

AREAS OF BLOBS

PIE ARE ROUND, CORNBREAD ARE SQUARE

Volume

LARGEST CONTAINER

INVARIANCE

A property, ratio or relationship which remains the same even when other vari-
ables are changed is called an invariant. In geometry there are many important

invariants: For example, regardless of the size of a circle, the ratio of the cir-
cumference to the diameter is π; any triangle has an angle measure sum of 360°; the
ratio of the diagonal of a square (any size) to a side is always $\sqrt{2}:1$.

Here are some additional invariants which are worked into activities in this
resource.

- The sum of the measures of the exterior angles of a polygon is 360°.

- The Pythagorean theorem holds for every right $\triangle$.

- The altitudes of a triangle meet at a point (no matter what the shape of the
  triangle).

- The medians of a triangle meet at a point.

CLASSROOM MATERIAL INVOLVING INVARIANCE

POLYGONS & POLYHEDRA

Polyhedra

DO YOU KNOW THAT

VERTICES, FACES AND EDGES

HOW MUCH IS LOST?

Polygons

THE ANGLES IN TRIANGLES

SPECIAL LINES AND SEGMENTS IN TRIANGLES

DIAGONALS IN POLYGONS

INTERIOR ANGLES OF A POLYGON 1

EXTERIOR ANGLES OF A POLYGON

INTERIOR ANGLES OF A POLYGON 2

CURVES & CURVED SURFACES

Circles

ARE YOU RIGHT ALL THE TIME?

A PAIR OF ANGLES IN THE ARC

SIMILAR FIGURES

WHERE'S THE POINT?

AREA & VOLUME

Area

ARE SQUARES LARGER?

SOMETHING FOR NOTHING

THE WIDE OPEN SPACES

LOTS OF LITTLE ONES

Pythagorean Theorem

PYTHAGORAS SQUARED

PYTHAGOREAN THEOREM

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

LINES, PLANES & ANGLES

Points, lines, planes and angles are represented abundantly in our environment. The flat surface of a roof represents part of a plane, and straight lines mark the edges of buildings, roads and windows. The concepts of straight lines and flat surfaces are often taken for granted, but these concepts are not so natural for some people. The two sketches below contrast the visual impact of an isolated native village to a typical city scene. Children growing up in such a village might have few experiences for forming concepts of straight lines, angles and flat planes; but they might have more background for concepts involving curves and curved surfaces. An interesting account of geometric form in African architecture is given in the book Africa Counts by Claudia Zaslavsky. More information on visual perception in other cultures can be found in the chapter "Illusion and Culture" in the book Illusions in Nature and Art by R.L. Gregory and E.H. Gombrich.

NATIVE VILLAGE IN KENYA

CITY SCENE FROM NAIROBI, KENYA
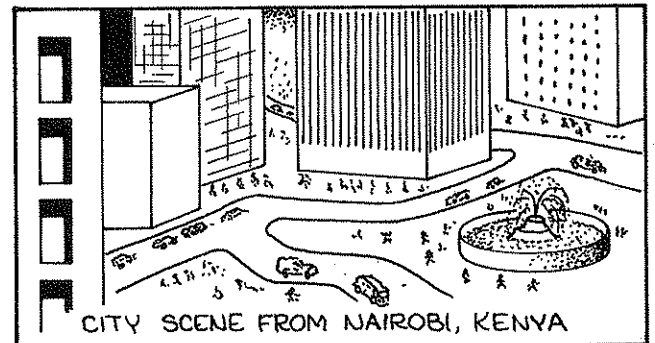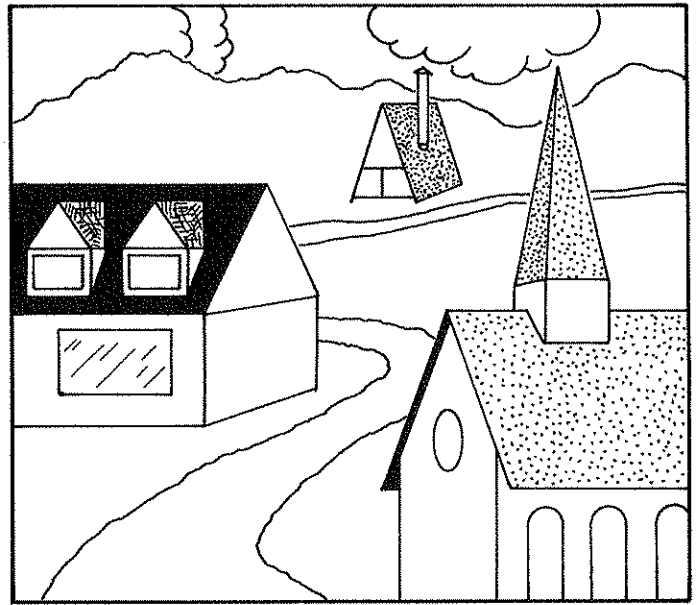
In most instances a child entering school has already had many experiences with objects having flat faces and straight edges. Some ideas of space and geometry are already formed. The child knows an object remains the same even when a change in position makes it appear smaller or gives a different perspective. Small children will spend hours arranging and stacking piles of blocks or cardboard cartons. These informal experiences are continued in the primary and intermediate

211
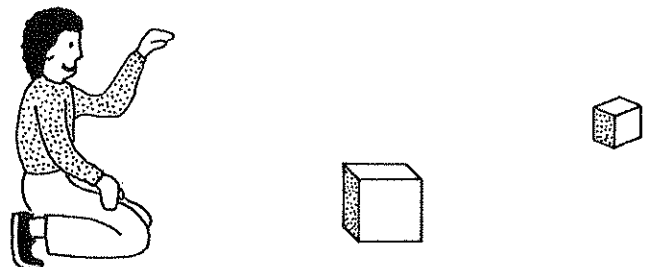
grades when the students are encouraged to sort objects that roll, to count the number of corners, edges or faces, and to fit objects together to make other shapes. These experiences with solid shapes form the basis for understanding points, lines, angles, planes and curved surfaces. All of these informal experiences are important in their own right, but they are also a helpful basis for abstraction and symbolization in the later grades. Some current research has been directed toward the learning of geometric concepts. This is discussed in *Planning Instruction in Geometry* in this section and *The Teaching Of Concepts* in POLYGONS & POLYHEDRA. You might keep in mind that there is no one approved scope and sequence for formal geometry. Even though this resource begins with lines, planes and angles, you might prefer to start with activities involving polyhedra, cones or spheres. Of course, if the activity with polyhedra involves angle measure, you would want to be sure your students have the prerequisite skills.

MOTIVATION AND INVOLVEMENT WITH GEOMETRY

Geometry provides an opportunity to genuinely involve your students. The visual impact of many geometric activities creates interest almost immediately. The handling, examining and making of solids encourages both hands and minds to "get in on the action."

Geometric Designs

Your students would probably enjoy making colorful line designs with needle and thread or constructing designs with straightedge and compass. Here is a chance for every student to succeed, to be creative and to learn some geometric concepts. Decorating the classroom or a hall bulletin board with these designs can help students feel they belong and have contributed. The activities on creating symmetrical shapes or designs can also make attractive displays. Colored pens and pencils can enhance geometric designs, and students enjoy using them. The class
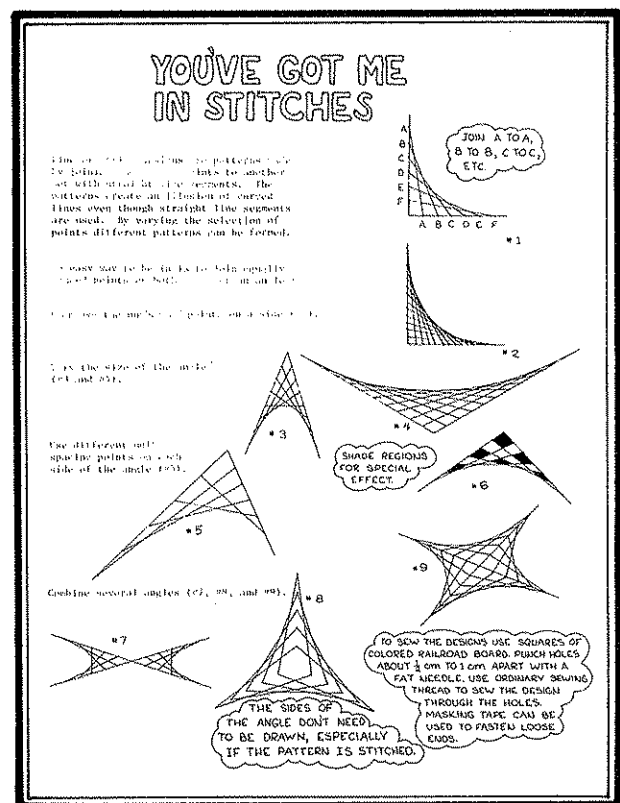
artists might like to work on a combination art and geometry project. You will find
ideas on art and geometry in the classroom materials and commentaries of this
resource. In addition, there are many books available for such ideas. These are
given in sources on the classroom pages of this resource and in the annotated
bibliography.

Manipulative Aids for Learning Geometry

A variety of tools and aids are
fun to use in geometry. These aids
help in measuring, drawing, represent-
ing and understanding geometric shapes
and ideas. Students might like to
make their own simple templates so
their drawings can be more exact.
Mirrors are always fascinating and
they become an important aid in the
study of symmetry and motions. While studying symmetry, you might give the students
a message or assignment that cannot be read unless it is reflected in a mirror.
Don't be surprised if they write you a mirror answer in return! The *Graphic Repre-*
sentation teaching emphasis and later parts of this commentary give ideas for using
drawing tools to make graphic representations.
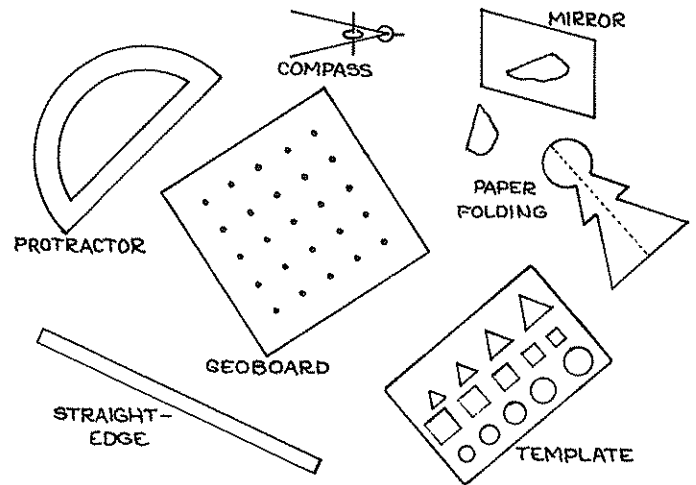
Geometry in the World Around Us

When studying geometry, we can continually use examples from our sur-
roundings. Representations of perpendicular and parallel lines occur in
buildings, roads and even plaids. A group of enthusiastic scouts or a stu-
dent who has a relative in the Navy might find out if the Navy still uses
the method of semaphore signals or other interesting information related to

symbolic communication. Hunting for real-world examples of geometry concepts and
sharing them with the class can be enjoyable for some students.

Humor and Creative Writing

Optical illusions are always fun to puzzle over and they can be worked into
geometry lessons (see *Perception Deception* in the Lines subsection). An idea pre-
sented by Father Bezuszka at an NCTM conference could be a way to encourage class
jokesters to contribute to the class. Present a series of optical illusions similar
to those shown below.

Notice how the vertical line seems to be longer than the horizontal line.	Notice how the circles appear to be of different sizes.	Notice how dots appear and disappear when you stare at the figure.
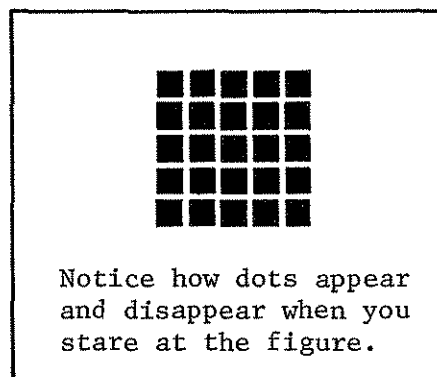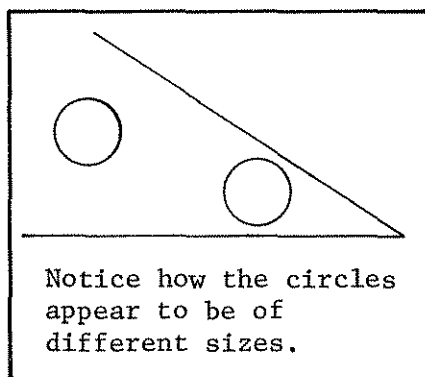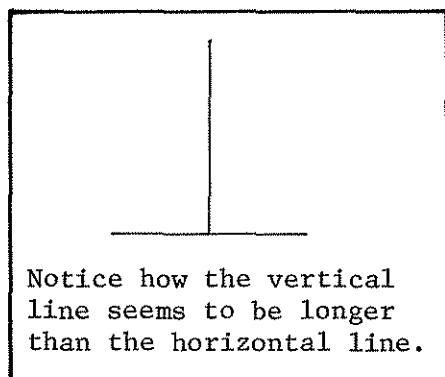
With encouragement, clever students could prepare a set of cards which parallels the
illusions above.

Notice how the vertical line seems to be longer than the horizontal line.	Notice how the circles appear to be of different sizes.	Notice how the shape seems to disappear when you stare at it.
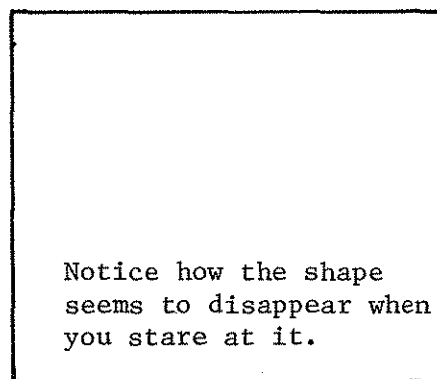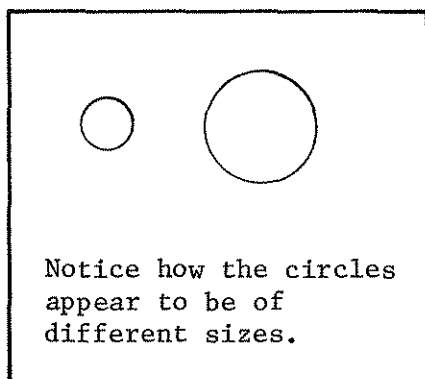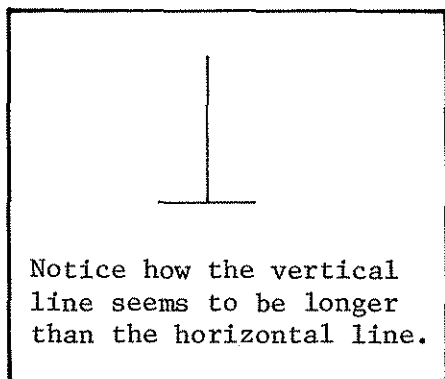
This activity with visual illusions could create a lot of fun for the student, the
teacher and the whole class.

Another teacher read her class a book called <u>Flat Stanley</u> by Jeff Brown. It
was about a boy who became two-dimensional when a bulletin board fell on him. After
various adventures he was blown up again with a bicycle pump. She then asked her
class to think about the points, lines and planes in their mathematics books and
write similar stories. One such story is reproduced on the following page.[*] Your
students might also enjoy writing stories about different dimensions. They can find

[*] Emma C. Carrol. "Creatamath, or--geometric ideas inspire young writers." <u>Readings</u>
<u>in Geometry from the Arithmetic Teacher</u>. Washington, D.C.: National Council
of Teachers of Mathematics, 1970, p. 45.

more ideas in books like <u>Flatland</u> by
Edwin A. Abbott or <u>Sphereland</u> by
Dionys Burger. These stories and the
subsequent discussion can help students
verbalize ideas about dimensions in
space. Other books you and your stu-
dents might enjoy are <u>The Dot and the</u>
<u>Line</u> by Norton Juster, <u>An Adventure in</u>
<u>Geometry</u> by Anthony Ravielli and <u>Points</u>,
<u>Lines and Planes</u> by Marnie Luce.

A One-D Boy, by Bob

*One day, Jim and his brother were
playing with his tape recorder. Sud-
denly, Jim got caught and he came out
1D! He cried and cried but he could
find no way to become 3D again.*

*It wasn't very good to become a jump
rope for girls and that is exactly
what happened to Jim! He was pleased
when some boys rescued him and decid-
ed to use him for a kitestring.*

*"How lucky can a boy-kitestring
get?" said Jim, for as soon as he was
high in the air, he opened his mouth,
the wind sailed in and Jim became 3D
again.*

VISUAL PERCEPTION AND GRAPHIC REPRESENTATION

Some of your students might have
difficulty interpreting shapes and
patterns. These students might not be
able to copy a figure shown on the
chalkboard much less use the relation-
ships involved. Various types of acti-
vities can be devised to diagnose such
problems in visual perception. Could
your students easily decide which
figure on the right is most like the
figure on the left of the dashed line?
If not, they might have trouble with the
geometric diagrams in books and on the
board.

Some students have difficulty drawing and copying. These skills must be
learned and practiced. Dot or grid paper can help students make a copy of a
figure. The student who has difficulty with visual perception might need help
getting started. Moving a finger of one hand over part of a figure and then
following the same movements with a pencil in the other hand can help a student
know where to draw. Activities requiring students to copy designs on grid paper
can be found in the SIMILAR FIGURES section.

In geometry lessons we sometimes ask students to copy a figure and then to determine various things about the figure. Students might need practice copying

figures without the intrusion of additional questions. The Z-like figure to the right has some line segments which are parallel and other line segments meeting at right angles. You may want students
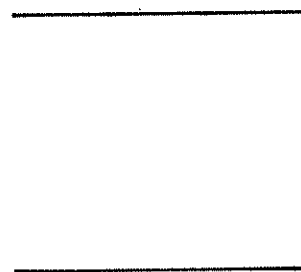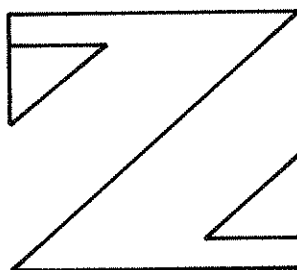
to identify some of the pairs of parallel or perpendicualr segments after they copy the designs.

Visual perception is involved in many simple games and activities. The game Sim from *Line Segment Games for Two* requires each player to visualize triangles which can be formed by joining dots. Each player must plan ahead to avoid forming a triangle while trying to force the other player to form one. After several moves it becomes more difficult to recognize the possible triangles. A student with good visual perception might have a better chance to win this game.
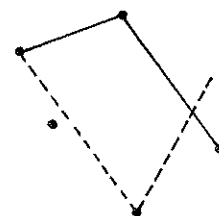
Often it is hard to recognize figures when they are tilted or oriented different- ly from the way we usually represent them. Showing figures in different orientations is very important when teaching a concept. This is discussed further in *The Teaching of Concepts* in POLYGONS & POLYHEDRA. The student pages *Comparing Angles 1* and *2* and *Make the Right Fold*

HOW MANY RIGHT ANGLES CAN YOU FIND?

HMM··· IF ⌐ IS A RIGHT ANGLE IS ⌐ A LEFT ANGLE?

ask students to identify congruent, acute, right and obtuse angles when they are shown in different positions.

The ability to visualize a path or figure in the mind without making the actual drawing is often helpful. Which of the pairs of lines shown below look like they will intersect?
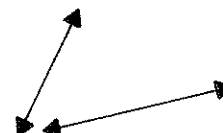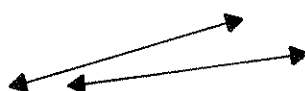
To answer this question a student could extend the sketched lines, but the student who can "tell by looking" that the second and third pairs of lines will intersect, is making use of his visual perception. The pages *But What Do We Have in Common?* from the Lines subsection and *Crossing Over* in the Angles subsection involve questions like those discussed above. The TEACHING EMPHASES symbol (👁) is placed in the upper left corner of classroom pages to point out more pages involving visual perception. The *Visual Perception* teaching emphasis has an overview of visual perception along with highlights of related student pages.

CONSTRUCTIONS WITH A STRAIGHTEDGE AND COMPASS

Before attempting specific constructions with a straightedge and compass, students need to become somewhat skilled in their use. The *Graphic Representations* teaching emphasis gives hints on how to use a compass, ideas that you can use in demonstrating constructions and directions for all the basic constructions.
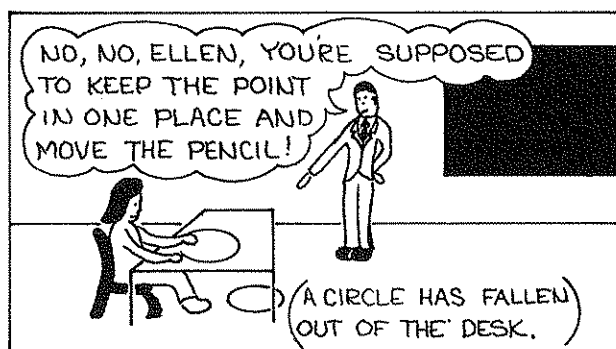
Students will need to understand that an unmarked straightedge is useful only for drawing pictures of line parts. The compass is used to mark equal lengths as well as to construct circles and arcs of circles.

Constructing a Line Segment Congruent to a Given Line Segment

This basic construction will be used in many constructions, so you might want to devise activities which give students practice in this skill. You could provide a page of irregular polygons and ask students to construct for each polygon a line segment whose length is the perimeter of the polygon. Two possible answers are given below:
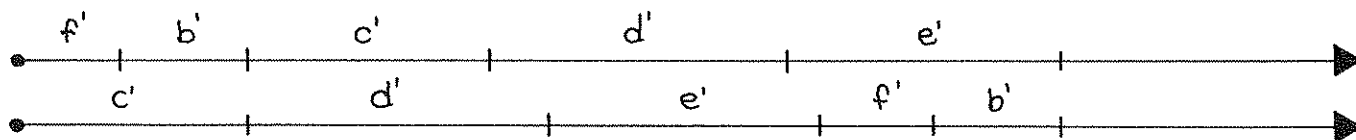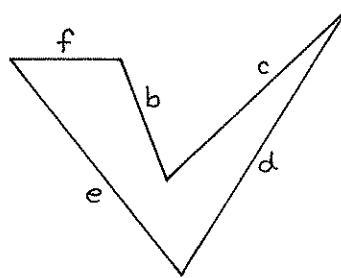
Students can compare their constructions. They might be surprised to discover that their final line segments are the same length even though they did not copy the sides of the polygon in the same order.

Constructing an Angle Congruent to a Given Angle

This construction involves more steps and so it is harder. It is also more difficult to see why the construction works. In answer to the question, "Why are the two angles congruent?" students might answer, "Because using the compass made sure they open the same amount." This is acceptable at this informal level.

Most construction directions show an acute angle being constructed on a line parallel to one of its sides. Students will probably have more difficulty constructing an obtuse angle on a line not parallel to a side. Could your students construct an angle congruent to angle 2 at point T? At point R? At point U?
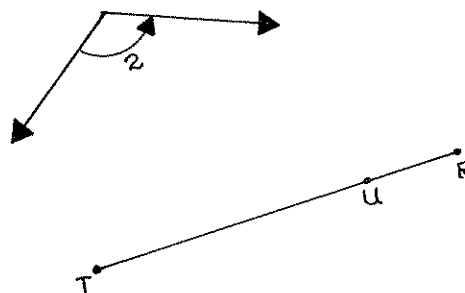
The activity card *Angle Wrangle* in the Angles subsection asks students to construct specified angles and line segments. If they do their constructions accurately they will obtain regular polygons. Another page which you might want to adapt to compass and straightedge constructions is *I'm Seeing Stars* from the Angles subsection. With the angles and line segment below, a star can be constructed.
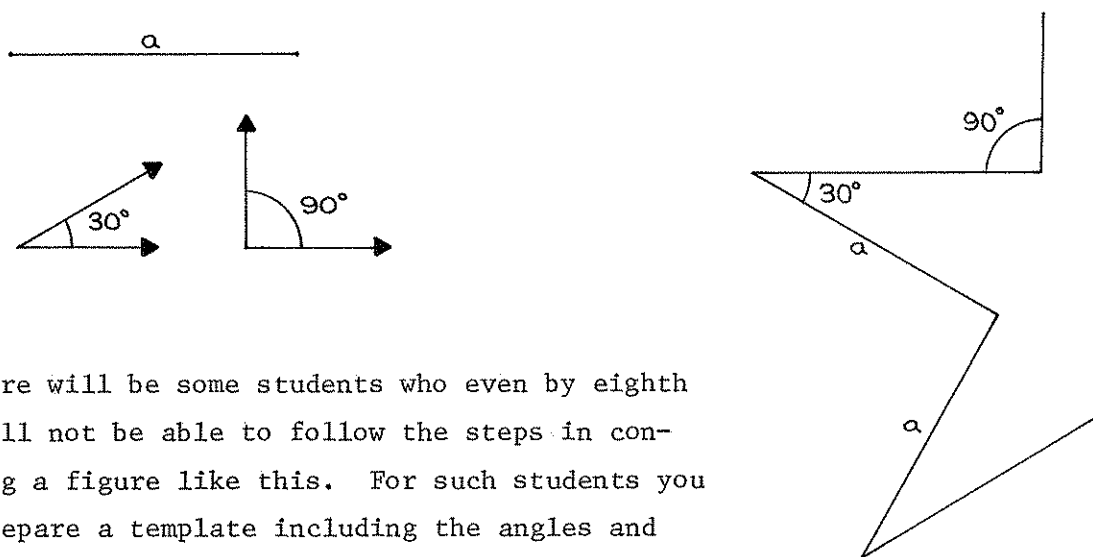
There will be some students who even by eighth grade will not be able to follow the steps in constructing a figure like this. For such students you might prepare a template including the angles and line segment.

A 3 by 5 index card with a 30°, 60°, 90° triangle cut from the center would do nicely. The line segment could be one of the sides of the triangle. After learning how the angles and line segment fit together, some students might want to try the construction again.
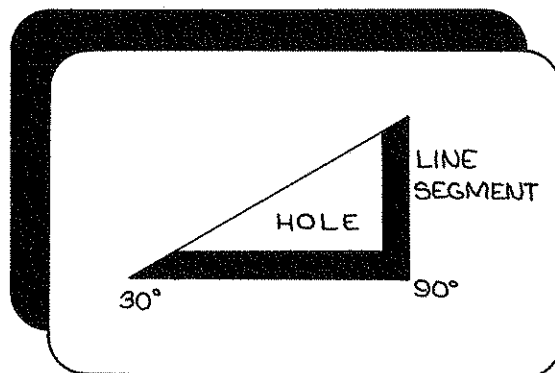
Other Constructions with Angles and Lines

The steps for bisecting an angle, constructing a perpendicular and constructing a line parallel to a given line are given in the *Graphic Representation* teaching emphasis. You might want to demonstrate these constructions and provide skill-building activities similar to those suggested above. The page *Where's That Line* in the Symmetry and Motions subsection can be used for skill building in constructing perpendicular bisectors. Many of the pages in the Polygons subsection and the Circles subsection also provide skill building with a compass and straightedge. Activities involving compass constructions are listed in the SECONDARY EMPHASES section.

PROBLEM SOLVING

The activities in this resource offer many opportunities to stress important problem-solving strategies. One strategy for solving problems is to guess and check.

The guess and check strategy can lead to a more organized search. Another important strategy is to organize the information. Many of the activities in this resource have tables and charts which help students organize information. You might want to adapt some of these activities to open-search questions for students as they become more proficient in solving problems. (See *Goals Through Discovery Lessons* in the CURVES & CURVED SURFACES section and *Questioning* in the SIMILAR FIGURES section.) One

activity which gives students a table in which to organize their information is *Part of a Line* from the Lines subsection. Students are challenged to predice the number of line segments needed to connect ten non-collinear points, two at a time. They can look for a pattern (another strategy) in the table and use that pattern to predict the number of segments needed. When similar problems are posed (see *One Way Ray, What's My Line* and *Be Bold . . . Fold* in the Lines subsection and *Angles Not Angels* in the Angles subsection), students can be encouraged to think, "Have I seen a problem like this before?" Some students might remember to make a table starting with small numbers and to look for patterns.

When you choose pages from this resource, keep in mind that problem-solving processes should be stressed. It is not really important that students know how many line segments are needed to connect ten points, but the processes of organizing information, searching for patterns, recognizing related problems, . . . are very important. The *Problem Solving* teaching emphasis gives more background on such strategies and shows many highlights from student pages.

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

CONTENTS

LINES

Lines are abstract concepts, yet we represent them in many ways. The folded edge of a piece of paper or the marks made on paper with a pencil and straightedge are two common ways of representing lines; however, a line does not stop at the edge of a piece of paper but extends indefinitely in both directions. (See *Get the Point?*)
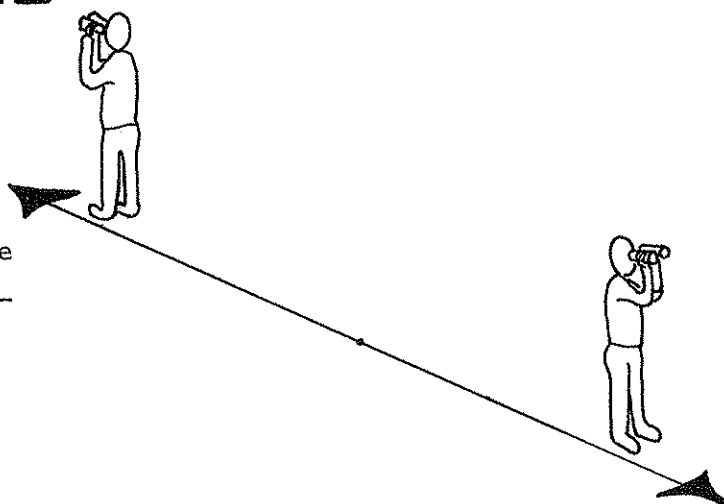
We often say the shortest path between two points lies on a straight line and that any two distinct points determine a line. The shortest path between two points depends on the surface. For example, the shortest path between two points on a sphere lies on a great circle. Great circles on a sphere are somewhat analogous to lines on a plane.
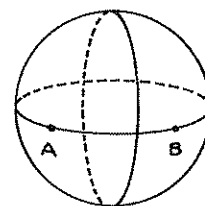
Lines are very important in the study of space, so there are many terms to describe parts of lines, relative positions of lines and relations of lines.

PARTS OF LINES

Each line contains points, line segments and rays. Each point separates a line into two parts. A point and the part of the line on one side of the given point form a ray. (For a student page on rays and notation for rays see *One Way Ray*.) You might want to use a flashlight or a picture of a laser beam as a physical model of a ray. A line segment consists of two points of a line and all the points between the two points. (See the page *Part of a Line*.)
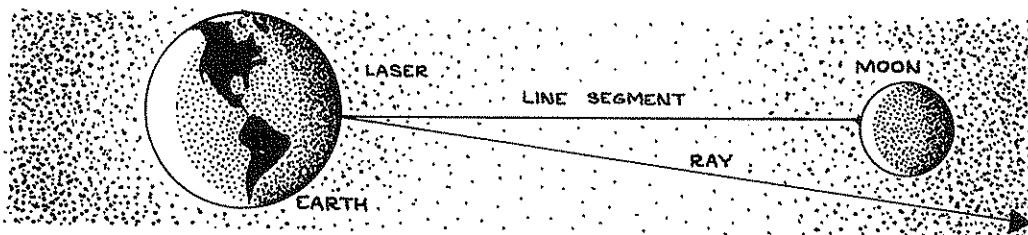
Many geometric objects are composed of rays and line segments. Two rays which share a common endpoint form an angle. A triangle is composed of three line segments. The skeletons of cubes, prisms and other polyhedra are composed of line segments. Line segments can make beautiful designs. Some of these are shown on the page *You've Got Me in Stitches.*
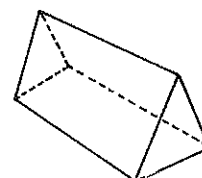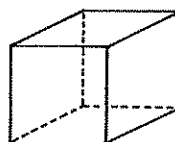
THE POSITION OF LINES RELATIVE TO AN OBSERVER

Even though we know we live on a sphere, it is usually practical to talk and act as though we live on a flat plane. Look at the definitions to the right. Can a line be horizontal and vertical at the same time? Can you find any lines which can never be called horizontal or vertical? If a line is oblique for your position, must it be oblique for everyone else?

A line running from left to right and parallel to the surface of water at rest is called a horizontal line.

A line straight up and down and forming right angles with a horizontal line is called a vertical line.

An oblique line is neither vertical nor horizontal.
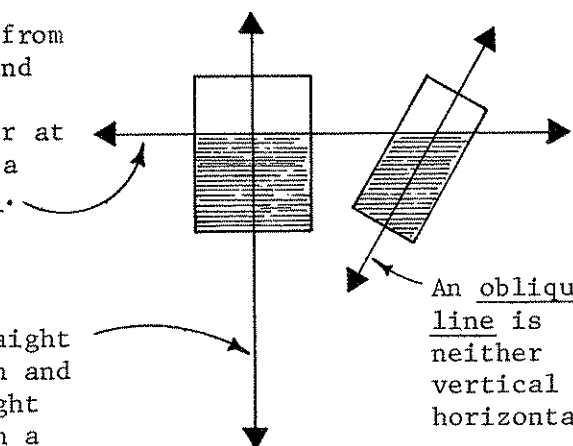
The diagram to the right might help with these questions. You would probably label line ℓ horizontal if you were at the North Pole, vertical if you were at point X, Africa, and oblique if you were on the Aleutian Islands.
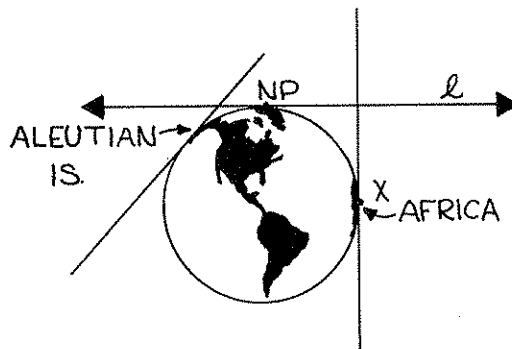
The terms vertical, horizontal and oblique are everyday terms which we use to communicate ideas. They are not specifically defined geometric terms, but they are very useful and generally not confusing. We can recognize a line described as "vertical" even when it is represented on a piece of paper which is lying flat, not up and down. We need words such as these to describe real world objects.

> Could you please get me a dress with fewer horizontal lines? Horizontal lines make me look fat!

> That's more like what I had in mind!
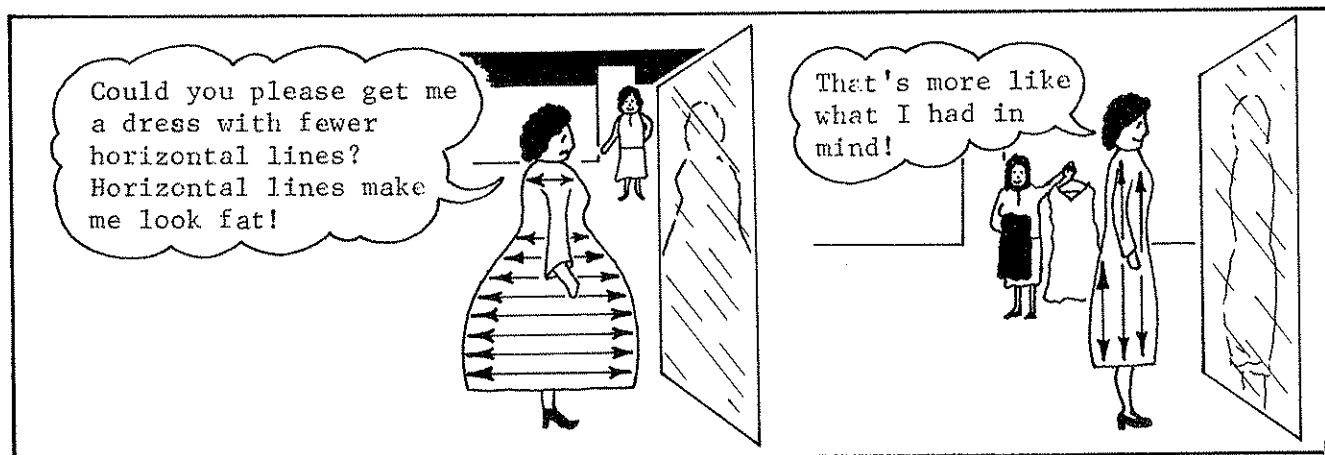
RELATIONS BETWEEN LINES
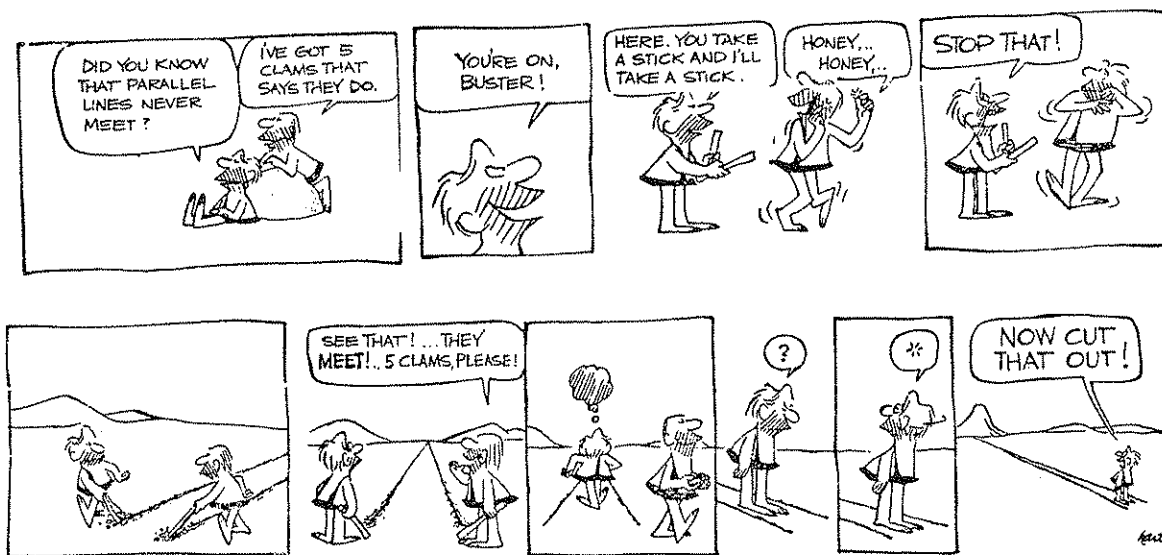
B.C. by permission of Johnny Hart and Field Enterprises, Inc.

Do parallel lines ever meet? No, by definition parallel lines lie in the same plane and never meet, but this is a mathematical idea, not two stick marks on the earth's surface as described in the cartoon above. The paths in the cartoon seem to meet on the horizon, but this is a visual illusion. We use such things as railroad tracks for models of parallel lines. Such models help students understand parallelism. Here too, we often make the assumption that the tracks are straight and lie on a Euclidean (flat) plane, but we know the earth is a sphere.
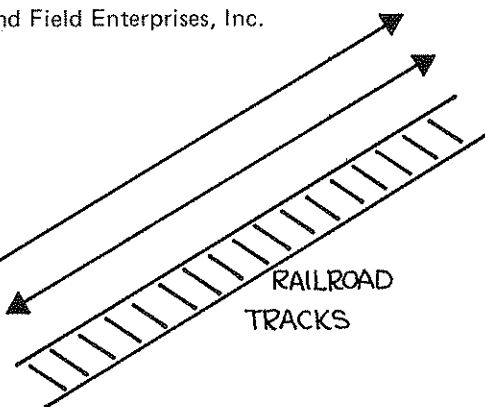
Intersection and perpendicularity are two important relations of lines. To intersect means to have one or more points in common. Two lines of a Euclidean plane can intersect in at most one point. The idea of intersecting lines is included in *Be Bold--Fold, Impossible? or Improbable?* and *But What Do We Have in Common?*
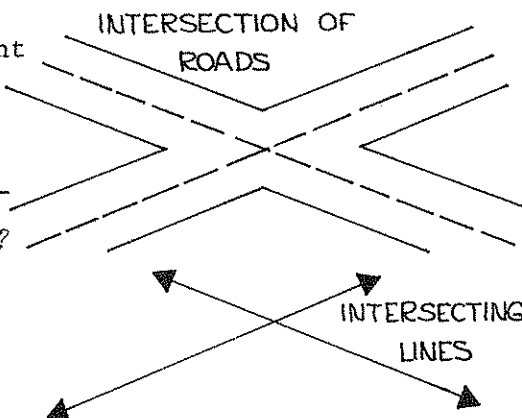
Perpendicular lines intersect and form four congruent angles. Most of the models of perpendicular lines involve horizontal and vertical lines. You will want to be sure your students can recognize perpendicular lines when both of the lines are oblique. Some real world examples are: a square picture hung by a vertex ⬦ , roofs with a 45° slant ⌂ and braces in gates or on buildings ⊠ .

PERPENDICULAR LINES

TELEPHONE POLE
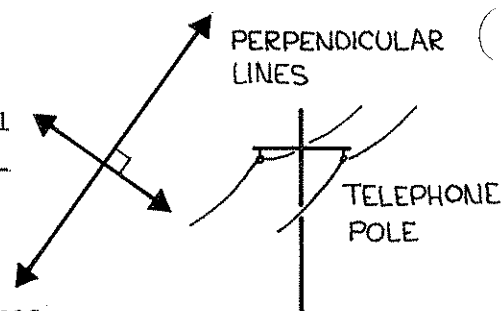
Another relation between lines is skewness. Two lines which are not in the same plane and do not intersect are called skew lines. Examples of skew lines can be found in the edges of any rectangular room. Rods or pencils can be used to represent the various possible relations of lines.

SKEW LINES
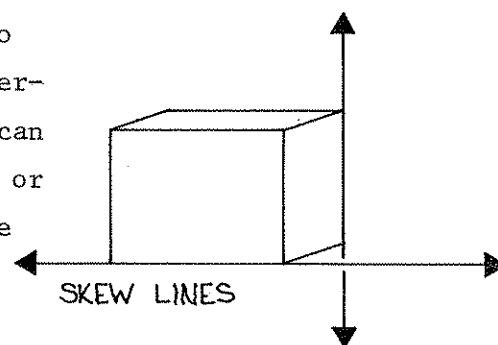
As you teach your students about lines, you might consider the following questions. How do students think of lines? Do they think a line stops at the edge of the paper? Does a diagram like ⟍⟋ represent two lines that intersect? Do they say a line "goes to infinity" and think infinity is a place? Many students have thoughts like those given above, even though they do not say them aloud. Students can gradually expand their understanding of the concepts of points, lines, planes and angles through teacher-given examples and class discussions.

GET THE POINT?

1) THESE FIGURES SUGGEST POINTS.

LA GRANDE

DOT → **SMALLER DOT** → **SMALLER DOT** → **POINT** AN IDEA
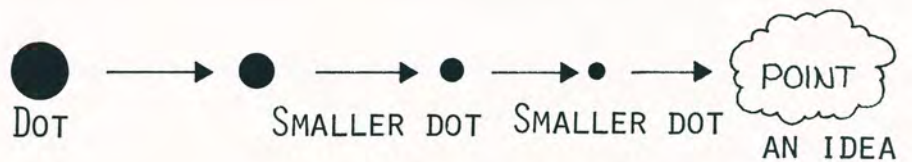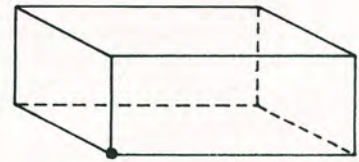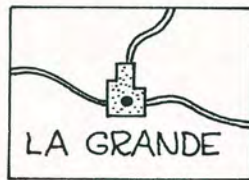
2) THESE FIGURES SUGGEST LINES.

A STRIP → THIN STRIP → THINNER STRIP → → → **LINE** AN IDEA
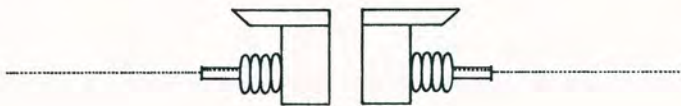
3) THESE FIGURES SUGGEST PLANES.

A BOARD → THIN BOARD → THINNER BOARD → **PLANE** AN IDEA

PART OF A LINE

A ———— B
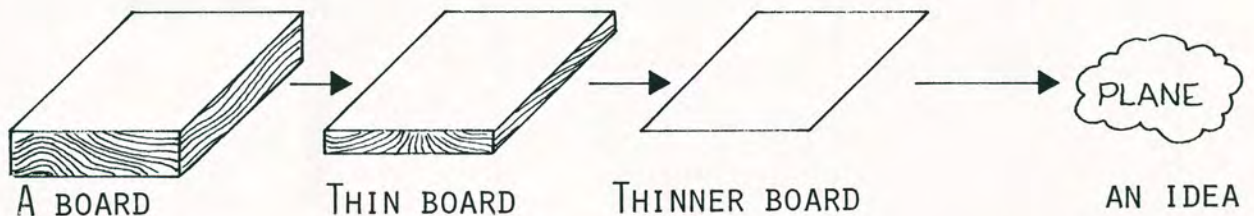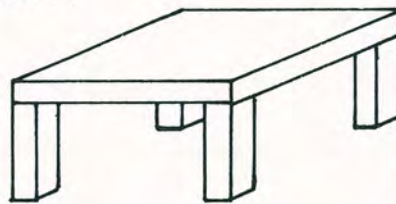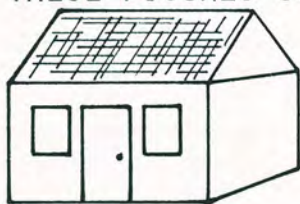
This is <u>line</u> <u>segment</u> AB. It can also be called line segment BA.

A and B are called the end points of line segment AB.

A) Find five examples of line segments in your classroom.

1) _____   2) _____

3) _____   4) _____

5) _____

B) In the diagram to the right one line segment is <u>line</u> <u>segment</u> GE.

H G   F   E

Name four other line segments in the diagram.

1) _____   2) _____   3) _____   4) _____

C) If some points are arranged so that no three are in a line, what is the maximum number of different line segments that can be drawn to connect each point to every other point?

2 POINTS	3 POINTS	4 POINTS	5 POINTS	6 POINTS
• •				
	3			

D) Can you predict the number of segments that will connect 7 points?

Make 7 points and check your prediction.

E) Predict for 10 points.

LINE SEGMENT GAMES FOR TWO

I Claim A Square

Materials: Dot paper and pencils

Rules: Players take turns joining
2 dots across or down with
a line segment.
If a square is completed
the player claims it by
writing his or her initial
in it. The player then
gets another turn.

Object: The player who claims the
largest number of squares
wins.
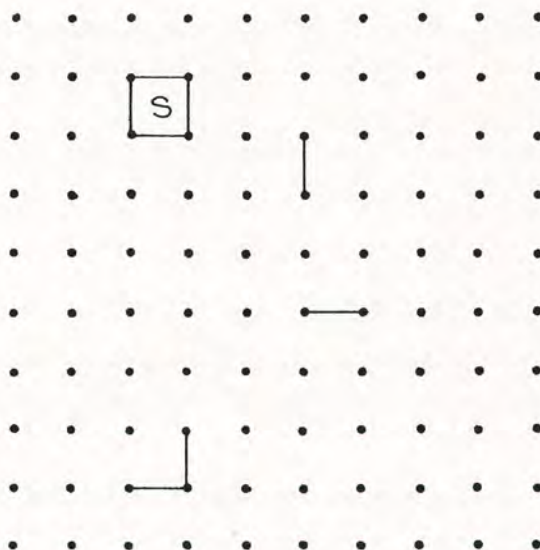
> VARY THE SIZE OF THE LATTICE.
> IS THERE A WINNING STRATEGY? IS IT BETTER TO MOVE FIRST OR SECOND ?

II Hit and Run

Materials: 5 nail by 5 nail geoboard or dot paper
Small red and blue rubber bands

Rules: Players take turns joining
2 nails across or down (not
on a slant) with rubber bands.

Red tries to connect the two
red sides of the geoboard;
blue the other two sides.
Paths may cross each other.

Object: The first player to build a
path from one side to the
opposite side wins.

BLUE

RED

RED
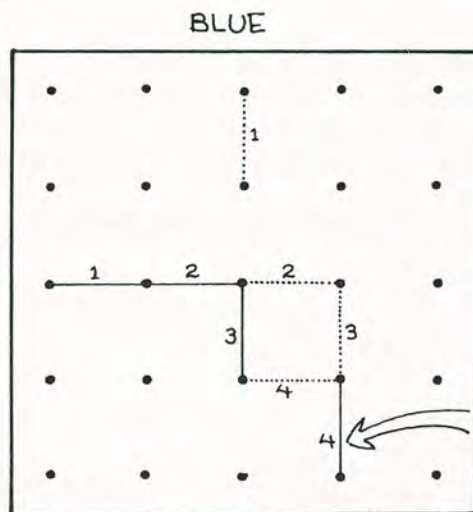
BLUE

> KEY : RED'S MOVES ARE SOLID, BLUE'S ARE DOTTED.

> RED MOVES TO BLOCK BLUE'S PATH.

III Sim

Materials: Six points (no three collinear)

Rules: Two players, using different colored
pens, take turns drawing line segments
to connect pairs of points. One
player uses solid line segments; the
other dotted line segments.

Object: The first player forced to form a
triangle (all 3 sides must be solid
line segments or all 3 dotted) loses.
Only triangles whose corners
are among the six starting
points count.
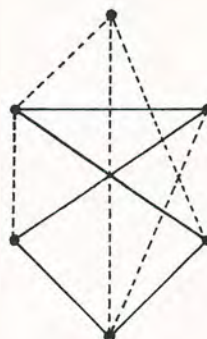
> NEITHER PLAYER HAS LOST YET.

GEOBOARDS

Geoboards are commonly used in geometry for finding area and comparing geometric shapes. The geoboard is used also as a model for fractions.

Geoboards can be purchased from Scott Resources or from Walker Educational Book Corporation.

Construction Information:

Have your woodshop take a 4' x 8' x 5/8" piece of plywood or particle board and cut out thirty-six 10" x 10" squares. Sand sides and edges, and spray the top with dark paint.

Provide each student in your class with:

A hammer (try to borrow from the school shop).
A 10 x 10 board.
Twenty-five 3/4" round-headed nails (brass escutcheon pins work well).
A pattern sheet with 25 dots.

Have the students center the pattern sheet on the board and hammer the nails through the dots. Be sure the nails are pounded in far enough to be firm. They should all be the same height. (You could use a spacer, empty bobbin, or large nut to place around the nail when hammering.)

When all 25 nails are in, have the student tear off the sheet.

DITTOED PAPER
WITH DOTS 2" APART.
(ARRAY OF 25 DOTS).

TAPE PAPER ON
THE BOARD.

USE THE PAPER PATTERN OF
DOTS TO PLACE THE NAILS.
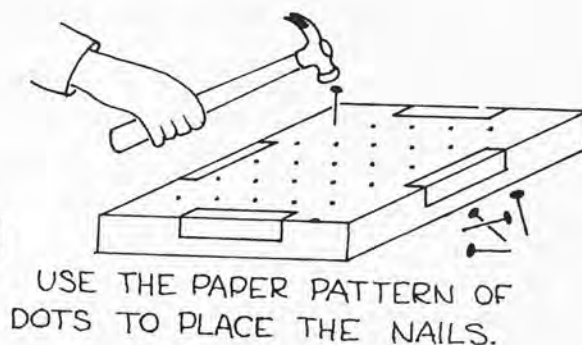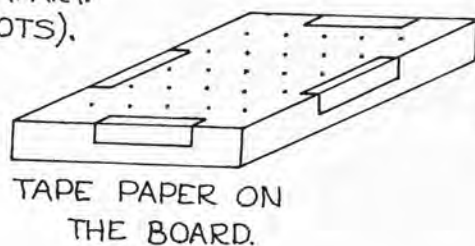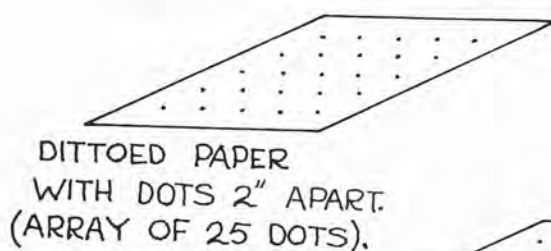
Each student needs a geoboard, about ten colored rubber bands, a supply of dot paper and a pencil to record the results. Sometimes it is helpful to have students work in pairs or small groups.

GEOBOARDS
(CONTINUED)

Readiness Activities:

     The first time students work with geoboards, let them experiment with them. Free play is very important. Students need a few minutes to make designs with rubber bands and discover some patterns on their own. After 5 or 10 minutes, direct the students by having them:

1) Copy some designs on their geoboards as shown below.
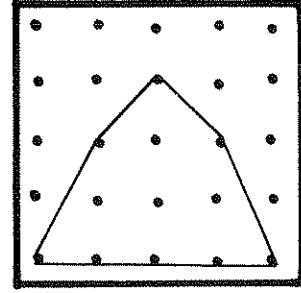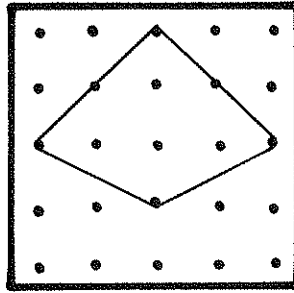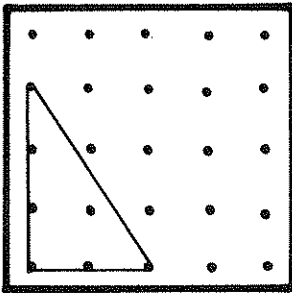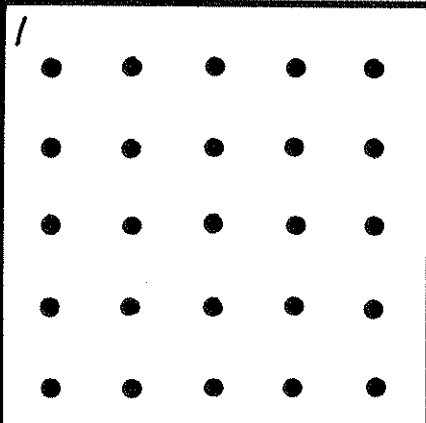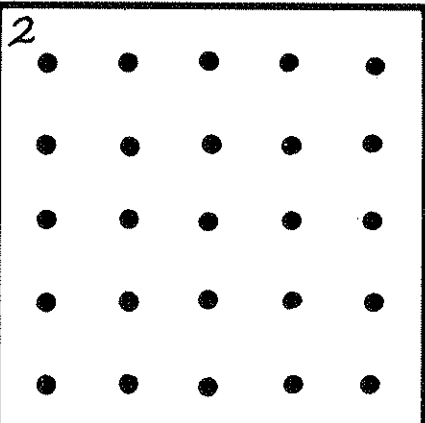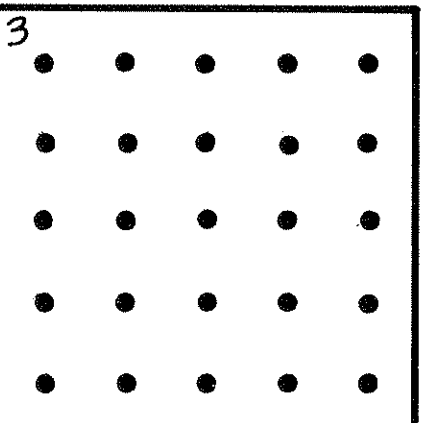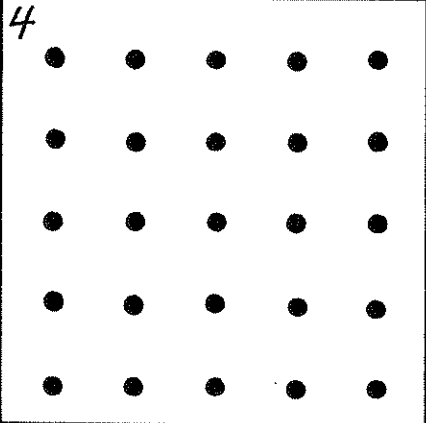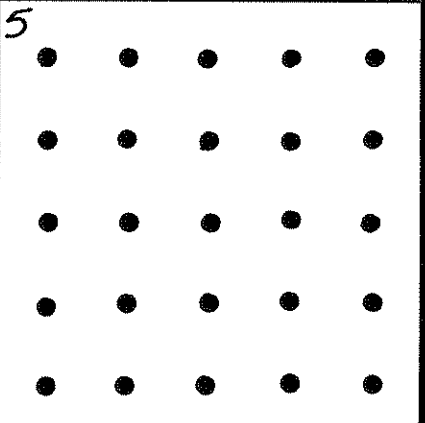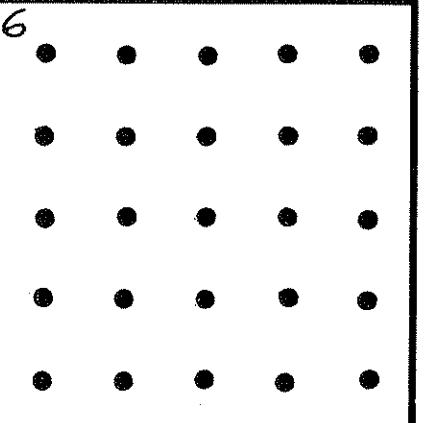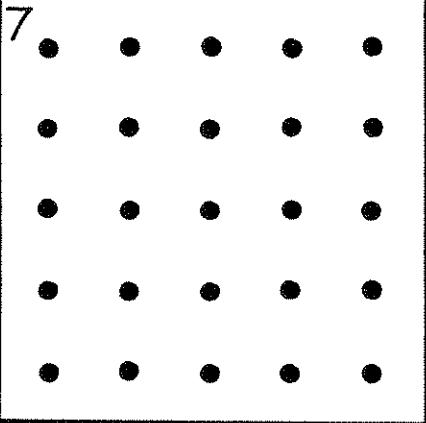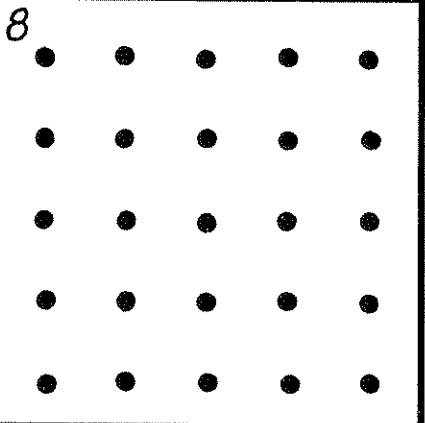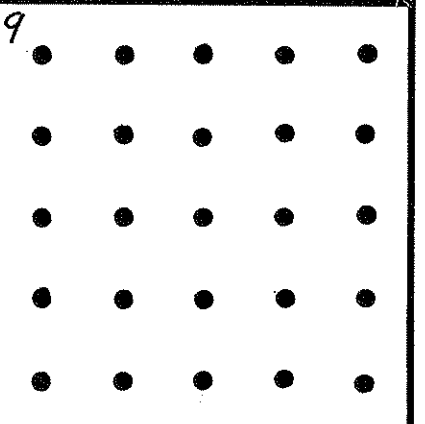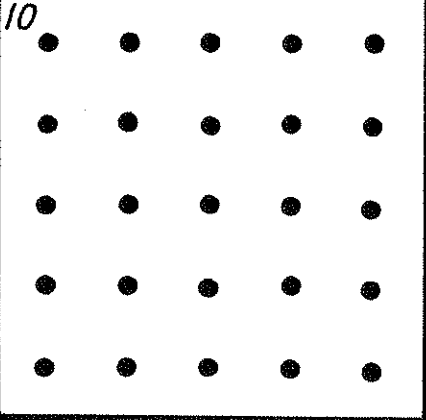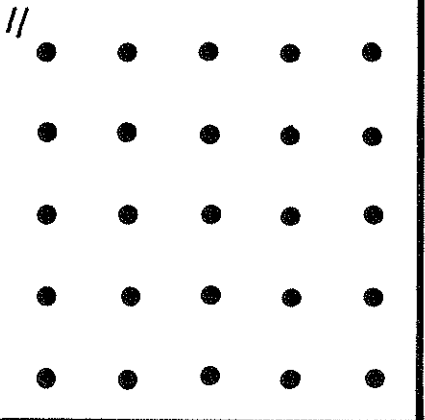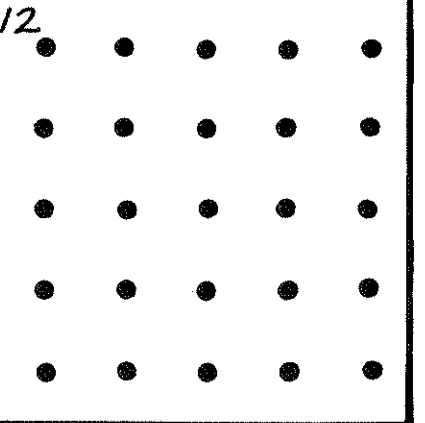
2) Make a figure resembling a STOP sign.

3) Make a four-pointed star.

4) Make different letters of the alphabet.

5) Make the number 471.

6) Make the largest five-digit number you can.

7) Make the smallest square you can.

8) How many of these squares are there on your geoboard?

9) Encircle eight of these squares with a rubber band.

10) Encircle two of these squares.

11) How many of these squares are encircled in each diagram below?
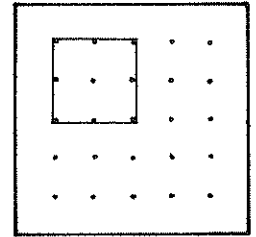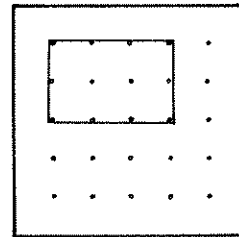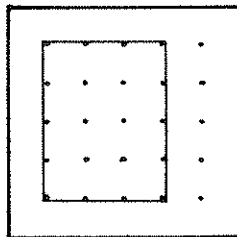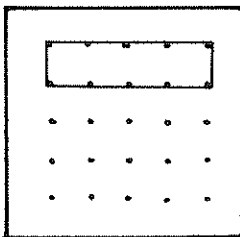
NOTE: Students usually need some practice in recording results. They can record some or all of their answers on dot paper. (See next page.)

GEOBOARD RECORD SHEET

1	2	3
4	5	6
7	8	9
10	11	12

GEOBOARD I

You need a geoboard, rubber bands, and dot paper.
Use dot paper to record your results for questions 1, 2, 4, & 5.

1) Make the shortest line segment possible.

2) Make the longest line segment possible.

3) Can you make a line segment with only one end point?

4) Make line segments of seven different lengths.

6) Use only the bottom row of nails on the geoboard. How many line segments can you make with

•	1 nail	_____
•—•	2 nails	_____
•—•—•	3 nails	____3____
•—•—•—•	4 nails	_____
	5 nails	_____
• • • (vertical)		
	10 nails	_____

Look for a pattern.

7) Use nine nails on your geoboard as shown below.
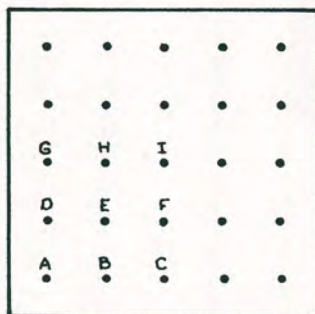
Find all the line segments you can that are congruent to $\overline{BH}$.

Find segments that are congruent to $\overline{GF}$.

8) Find five segments of different length and list them below. Use A as one of the end points each time.

9) How many line segments of different length could you find on the following geoboards?

DIMENSIONS OF GEOBOARD	NUMBER OF SEGMENTS
1×1	0
2×2	
3×3	
4×4	

233

GET YOURSELF AN EYEFUL

IN EACH BOX, MARK THE LONGEST PATH WITH AN **L**, THE SHORTEST WITH AN **S**.
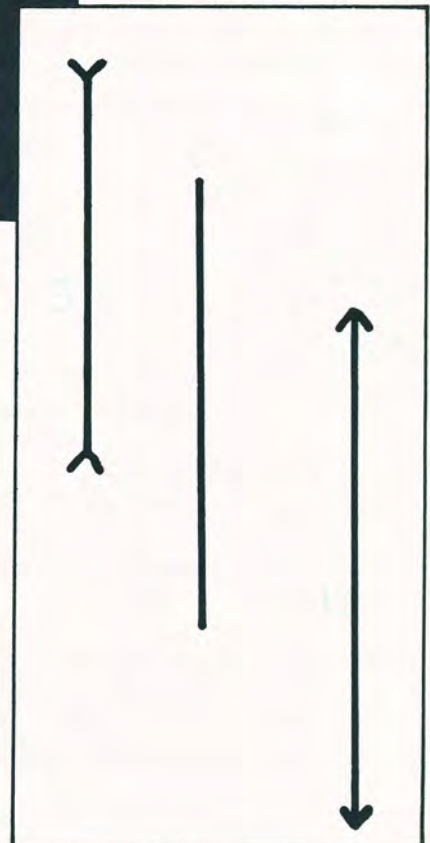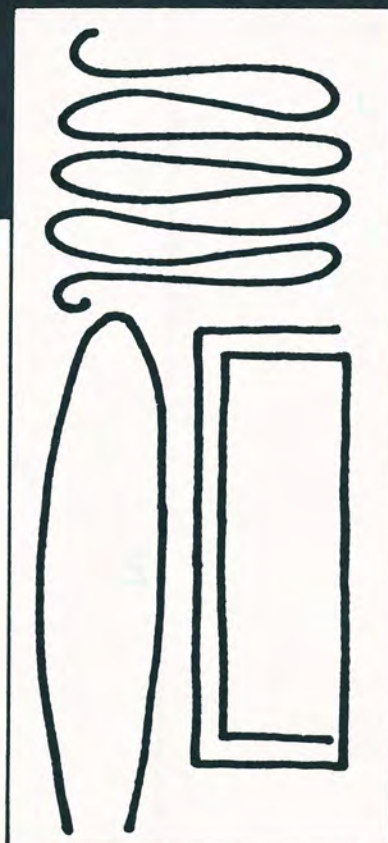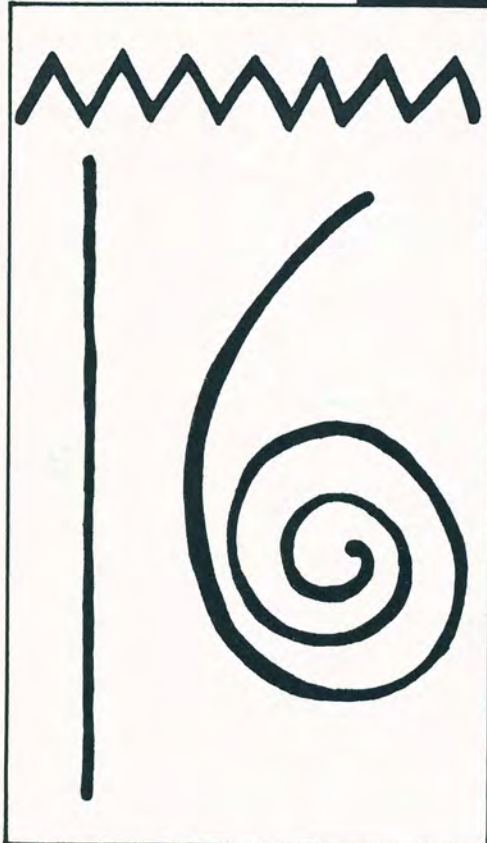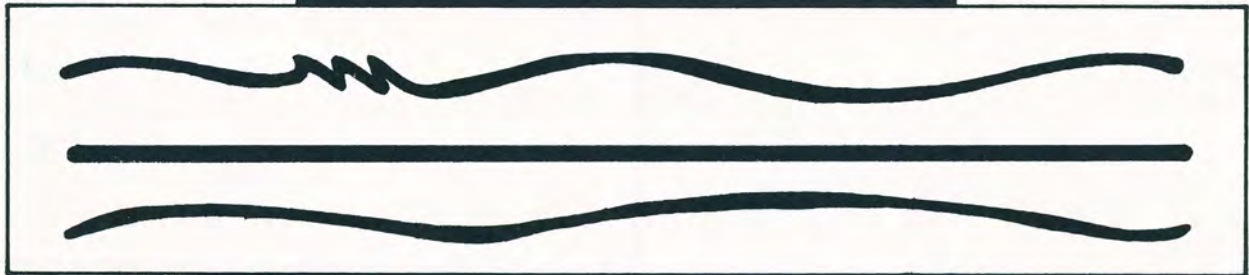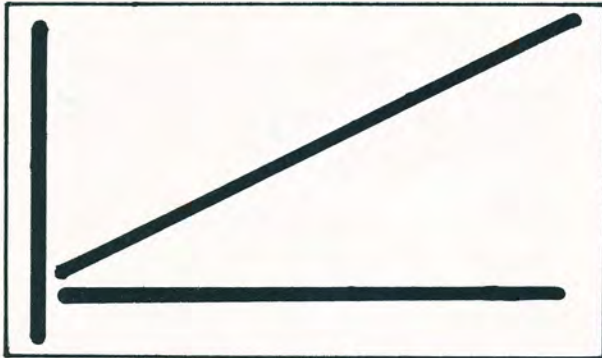
PERCEPTION DECEPTION
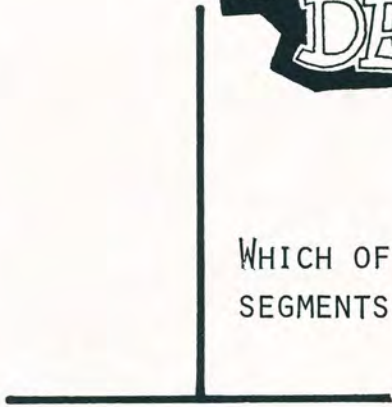
1. WHICH OF THESE TWO SEGMENTS IS LONGER?

2. SHADE IN THE FIGURES.
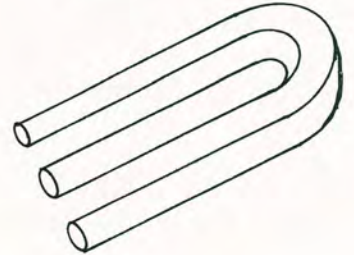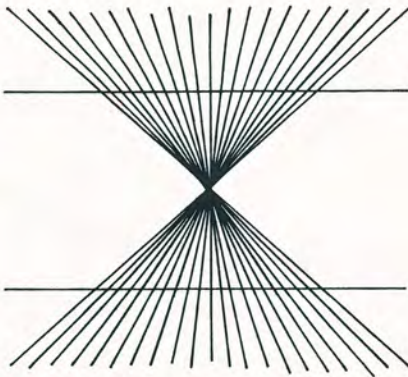
3. ARE THE HEAVY LINES STRAIGHT OR CURVED?

4.
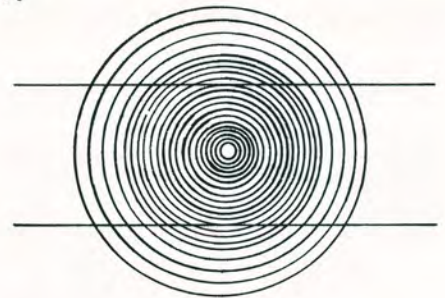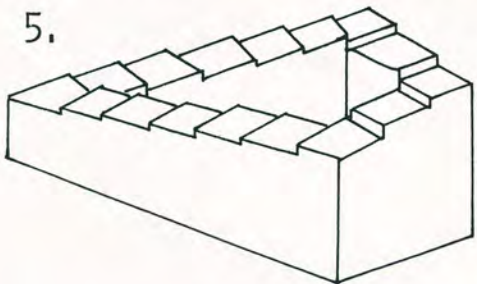
5. PLACE AN X ON THE HIGHEST STEP.
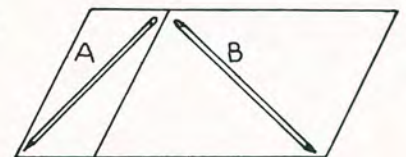
6. WHICH PENCIL IS LONGER?

A   B
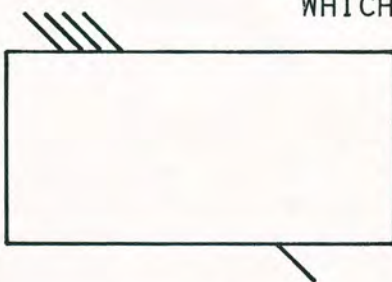
7. WHICH SEGMENT IS CONTINUED?

8.
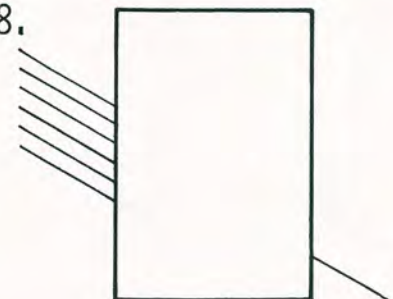
OPTICAL ILLUSIONS SHOW THAT VISUAL PERCEPTION CAN BE VERY INACCURATE. HOW ARE VISUAL ILLUSIONS USED AROUND US?

YOU'VE GOT ME IN STITCHES

Line or string designs are patterns made by joining one set of points to another set with straight line segments. The patterns create an illusion of curved lines even though straight line segments are used. By varying the selection of points different patterns can be formed.

JOIN A TO A, B TO B, C TO C, ETC.

#1

An easy way to begin is to join equally spaced points on both sides of an angle (#1).

Increase the number of points on a side (#2).

#2

Vary the size of the angle (#3 and #4).

#3

#4

Use different units for spacing points on each side of the angle (#5).

SHADE REGIONS FOR SPECIAL EFFECT.

#6

#5

#9

Combine several angles (#7, #8, and #9).

#8

#7
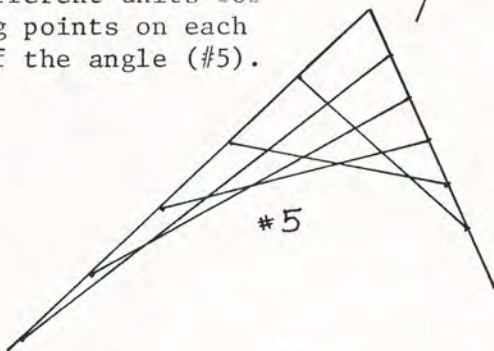
THE SIDES OF THE ANGLE DON'T NEED TO BE DRAWN, ESPECIALLY IF THE PATTERN IS STITCHED.

TO SEW THE DESIGNS USE SQUARES OF COLORED RAILROAD BOARD. PUNCH HOLES ABOUT ½ cm TO 1 cm APART WITH A FAT NEEDLE. USE ORDINARY SEWING THREAD TO SEW THE DESIGN THROUGH THE HOLES. MASKING TAPE CAN BE USED TO FASTEN LOOSE ENDS.

IDEA FROM: *Line Designs*

YOU'VE GOT ME IN STITCHES

(CONTINUED)

The angles in polygons
make beautiful designs.
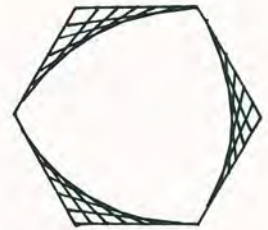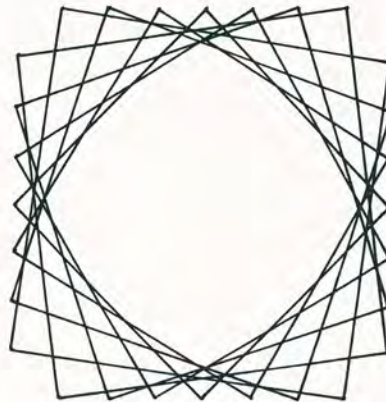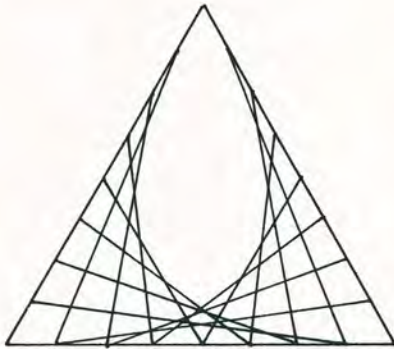
Line or string designs
can also be done on circles or curves.

EVERY OTHER
POINT IS
JOINED

EVERY
4th POINT IS
ALSO JOINED

Angles and circles can
be combined to make
more complicated designs.
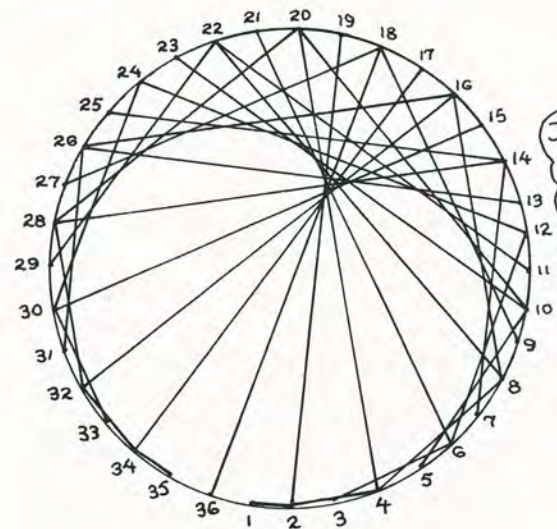
JOIN EACH
POINT TO
ITS
DOUBLE.

Have students experiment and
create their own variations.

To obtain a three
dimensional effect
drive nails at
each point and
repeat the pattern
several times by
wrapping thread
at varying heights
on the nails.
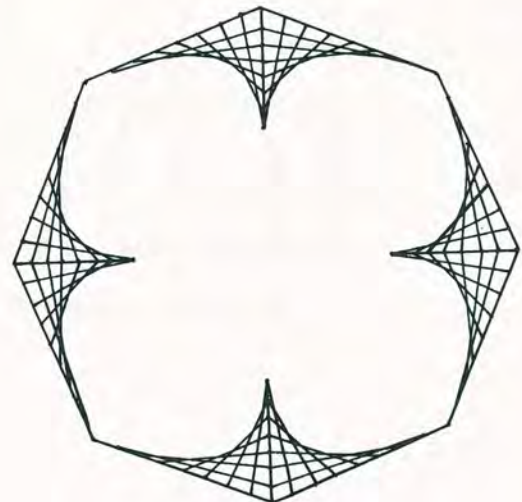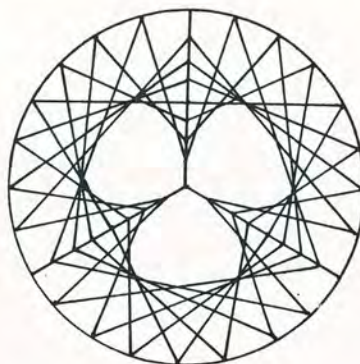
SEGMENT SOLUTIONS

(1) Without lifting your pencil connect each of the nine points using exactly four line segments. Do not retrace segments.

(2) Without lifting your pencil connect each of the sixteen points using exactly six line segments. Do not retrace segments.

3 Draw any four points that are not collinear. Try to draw a square that has one point on each side.
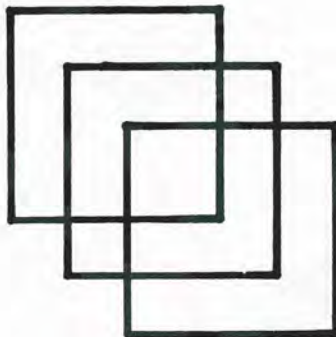
(4) Trace these three squares. Do not lift your pencil. Do not cross any segments or retrace any segments.

TAKE A ROUND TRIP

Antsburg, Boogerville, Cootytown, Dumpton, Evryboro, and Fleaport are six towns joined by a number of bus routes. The chart shows which pairs of towns are connected by a <u>direct</u> bus route and the fare for each.

Direct Route	Fare
Antsburg to Boogerville	$5.00
Boogerville to Cootytown	$3.00
Cootytown to Dumpton	$5.00
Dumpton to Evryboro	$5.00
Evryboro to Fleaport	$3.00
Fleaport to Antsburg	$5.00
Boogerville to Dumpton	$7.50
Boogerville to Fleaport	$3.00
Cootytown to Evryboro	$2.50
Cootytown to Fleaport	$2.50

1) Mark and label six points to represent the towns.
    Join them by line segments to represent each direct
    bus route.

2) Suppose you want to make a round trip that begins and ends at Antburg and that
    passes through each other town only once. How many different round trips are
    possible? _____
    Describe the trips: A ➔__ ➔__ ➔__ ➔__ ➔__

3) Which is the cheapest round trip? _____

4) Can you draw a map of the towns so that no two of the direct bus routes cross each
    other? _____

REMEMBER: THE DIRECT ROUTES
ARE STRAIGHT LINE
SEGMENTS.

ONE WAY RAY

ONE WAY

A)

R   S   T
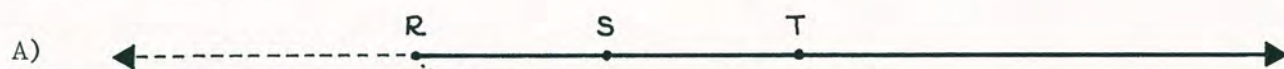
From here on is ray RS or $\overrightarrow{RS}$.

ray CD

ray WV

ray AB

P   Q   R

ray RP
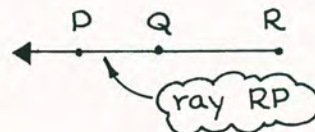
1) Can the top ray be called ray RT? Ray ST? Ray SR? For each case answer why or why not. _____

B) Find an example of a ray in your classroom. _____

C) Name four different rays in the diagram to the right.

R
V
S
T
U
W
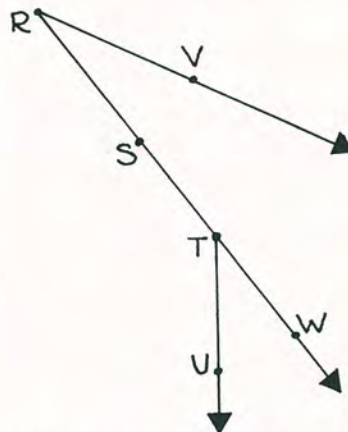
1) _____
2) _____
3) _____
4) _____

D) If some points are arranged so that no three are in a line, what is the maximum number of rays that can be drawn using the points as end points for the rays?

2 POINTS	3 POINTS	4 POINTS	5 POINTS	6 POINTS
•----→ ←----•	• • •	• • • •	• • • • •	• • • • • •
2				

E) Predict the number of rays for 7 points.

F) How do these numbers compare to the number of line segments connecting the points? Explain.

WHAT'S MY LINE

> Points are named with capital letters

A•

1) Use a straightedge to draw four different lines through point A. How many different lines can you draw through point A? _____

2) Draw a line through both points B and C.
How many different lines can you draw through points B and C? _____

B•   •C

3)

R• ──── S ──── V•
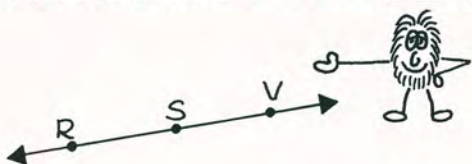
> This line has <u>six</u> names. $\overleftrightarrow{RV}$ and $\overleftrightarrow{VS}$ are two of them can you write the other four?
> a) _____ b) _____ c) _____
> d) _____

4)
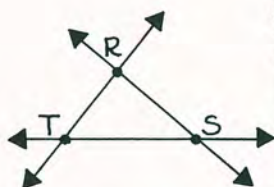
> Three points can determine three lines.

P•
Q•
R•
S•

Draw all the lines through pairs of points. Four points can determine _____ lines.

5) Draw a set of five points, <u>no three on the same line</u>. Draw all the lines through pairs of points.

6) Repeat for 6 points and 7 points. Complete the chart below.

POINTS	••	•••	••••	•••••	••••••	•••••••
LINES	1	3				

7) Look for a pattern in the chart. Use it to predict the number of lines for eight points _____ ; nine points _____ . Check your guesses with drawings.

8) Challenge: Five people meet at a party. If each of the five people shakes hands once with each of the other four, how many handshakes will there be?

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BE BOLD... FOLD!

Materials: Waxed paper or blank white paper

1) Fold your paper to make a straight line.(Unfold.)

2) Fold again to make a new line that
   crosses the first. (Unfold.)

Be sure
to crease
the fold.

3) Fold again to make a third line that crosses the first two at different points.(Unfold.)
   The maximum number of crossings for three lines is _____

4) Repeat by creasing to make a fourth line (that crosses at different points).
   The maximum number of crossings is _____

5) Repeat for five lines and complete the chart below.

6) Use the chart to help you predict the maximum number of crossings for six lines _____
   seven lines _____   Check your guesses by making folds.

NUMBER OF LINES	1	2	3	4	5
MAXIMUM NUMBER OF CROSSINGS	O	1			

---

1) Fold your paper to make a straight line.(Unfold.)

2) Fold again to make a second line on your
   paper.

One line
divides the
paper into
two regions.

3) How many regions are formed? _____
   Could there be more than one answer? _____
   How must you fold the paper to get as many regions as possible?

   _____

4) What is the smallest number of lines needed to give eleven regions? _____

5) What is the smallest number of lines needed to give twenty-three regions? _____

BUT WHAT DO WE HAVE IN COMMON?

1 NAME THE POINT THAT IS ON BOTH LINES.
(YOU MAY HAVE TO EXTEND THE LINES.)

_____   _____   _____   _____   _____
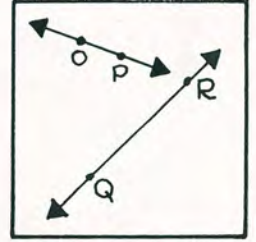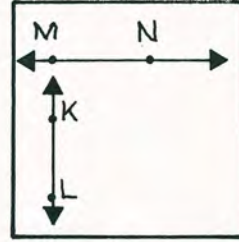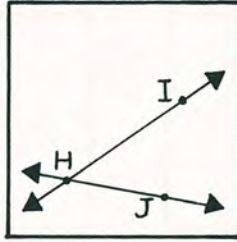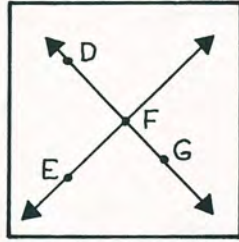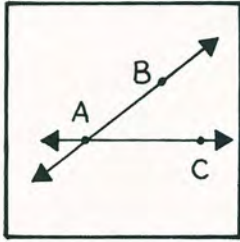
2 IF TWO LINES HAVE A POINT IN COMMON THEY ARE CALLED INTERSECTING LINES.
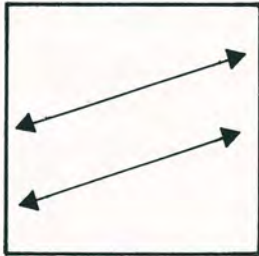
WILL THESE LINES INTERSECT? _____
(YOU MIGHT WANT TO EXTEND THE LINES.)

LINES WHICH DO NOT INTERSECT ARE
CALLED PARALLEL LINES.

3 LABEL EACH PAIR OF LINES AS PARALLEL OR INTERSECTING.
(YOU MAY HAVE TO EXTEND THE LINES.)

_____   _____   _____   _____   _____
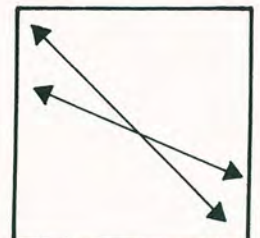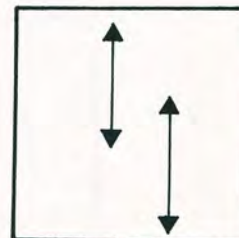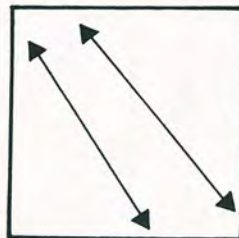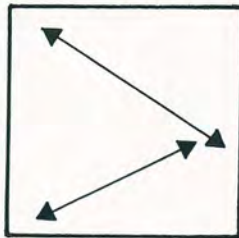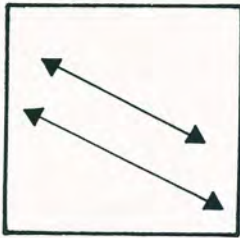
4
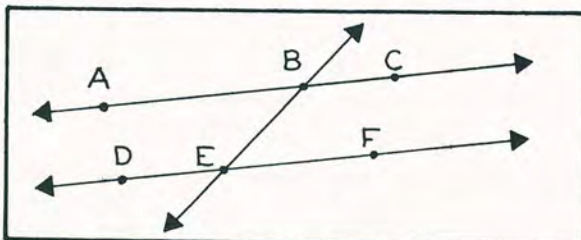
NAME TWO LINES THAT ARE PARALLEL.
_____ , _____

NAME THE LINE THAT INTERSECTS TWO
LINES. _____

NAME TWO POINTS OF INTERSECTION.
_____ , _____

LINE BE IS CALLED A TRANSVERSAL.

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IMPOSSIBLE? OR IMPROBABLE?

Materials: You may wish to use straws to help you complete this worksheet. You may want to flatten the straws.

Sketch an example in each box. Are some impossible? Some may be done several different ways.

1 TWO LINES

INTERSECTION POINTS :

| NONE | ONE |

2 THREE LINES

INTERSECTION POINTS :
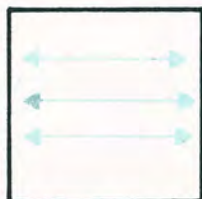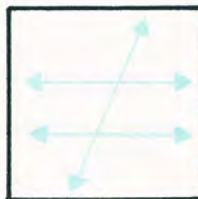
| NONE | ONE | TWO | THREE |

3 FOUR LINES

INTERSECTION POINTS :

| NONE | ONE | TWO (IMPOSSIBLE) | THREE |

| FOUR | FIVE | SIX | SEVEN (IMPOSSIBLE, MAXIMUM IS SIX.) |
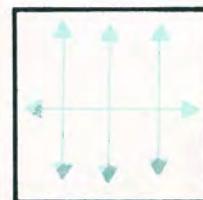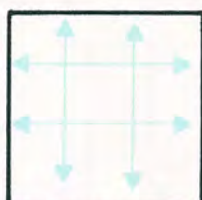
4 FIVE LINES

INTERSECTION POINTS :

| NONE | ONE | TWO (IMPOSSIBLE) | THREE (IMPOSSIBLE) | FOUR |

| FIVE | SIX | SEVEN | EIGHT | NINE | TEN |

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GEOBOARD II

1) Make two line segments on the geoboard so that together they touch 9 nails.

Examples:
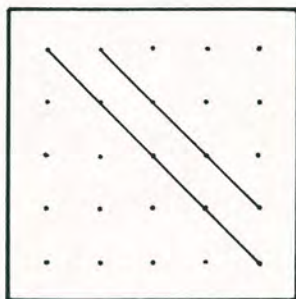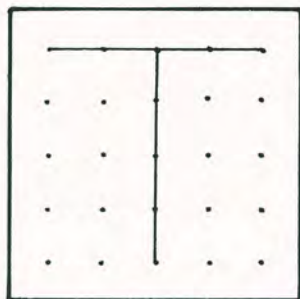
   Make at least five different arrangements. Record each arrangement on dot paper.

2) Which of your figures in number 1 show line segments that are:

   parallel?

   intersecting?

   perpendicular?

   congruent?

3) Which of your figures in number 1 show angles which are:

   right?

   acute?

4) Try to make a line segment for which you can not make a parallel segment.

5) Try to make a line segment for which you can not make a perpendicular segment.

6) Try to make two line segments that if extended would intersect off the board.

7) Try to make two line segments that intersect in more than one place.

8) Two different line segments have at most one point of intersection. Experiment with three line segments on your board. What is the largest number of intersections for three segments? _____
Experiment with four segments, five segments.

9) Record your results in this table.

# OF SEGMENTS	# OF INTERSECTIONS
2	1
3	3
4	6
5	10

   Can you use the table to predict how many intersections there could be for 10 segments? _____

ARE YOU A THIRD OF THE WAY THERE YET?

Students can approximate a third of a strip of paper by a sequence of folds that bisect parts of the paper. Demonstrate the folding process to the students by using a strip of waxed paper on the overhead.

1) Each student needs a piece of adding machine tape 60 cm long.

2) Fold the tape in half, crease, and reopen it. Measure to the nearest centimetre from A to the fold and record.

3) Fold the first half in half, measure from A to the new fold and record.

4) Continue making folds as shown in the diagrams. Make 6 folds. Each time measure from A to the new fold line and record.

MARK AN "A" IN THIS CORNER *FOLD #1*
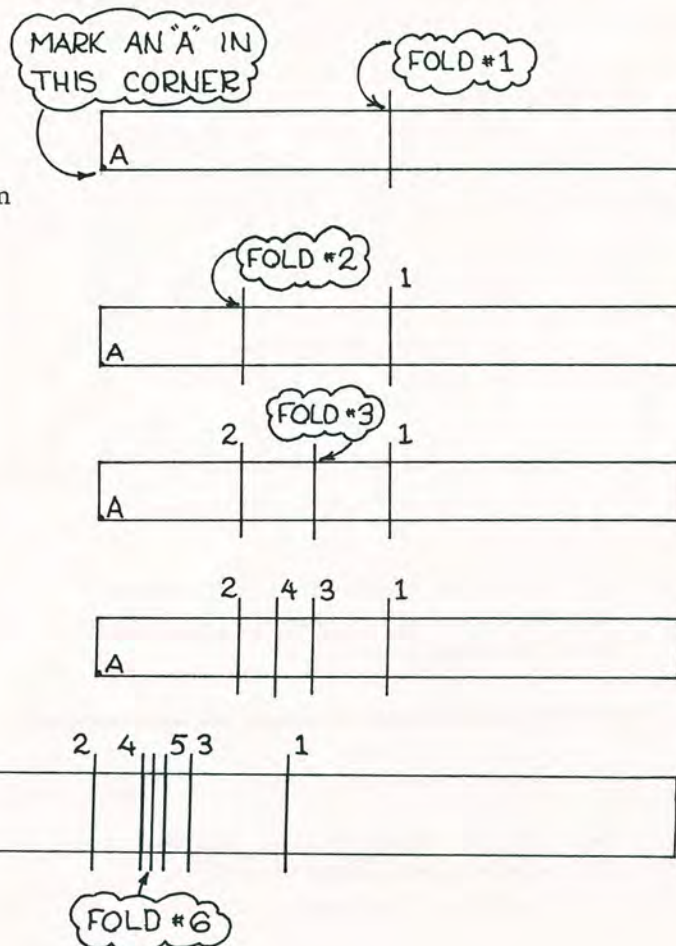
FOLD #2

FOLD #3

FOLD #6

Length from A to fold #1	30 cm
Length from A to fold #2	15 cm
Length from A to fold #3	23 cm
Length from A to fold #4	19 cm
Length from A to fold #5	21 cm
Length from A to fold #6	20 cm

The numbers alternate, first being greater than 20, then less, but getting closer to 20. The folds have a similar pattern.

5) Compare the last length in the table to the original length of the tape: 20 cm to 60 cm. The relationship is 1 to 3.

Extensions:

a) Have students repeat the process on tape of a different length. Does the 1 to 3 relationship occur?

b) Have students select the second half of the tape on which to alternate folds. They should discover the 2 to 3 relationship.

IDEA FROM: *The Themes of Geometry*

Permission to use granted by Eric D. MacPherson

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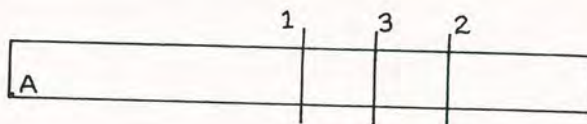
ARE YOU A THIRD OF THE WAY THERE YET? (CONTINUED)

Extensions (continued):

c) The same method could be applied to an angle. Investigate constructing angle bisectors.

THIS ANGLE IS ALMOST $\frac{1}{3}$ OF $\angle ABC$
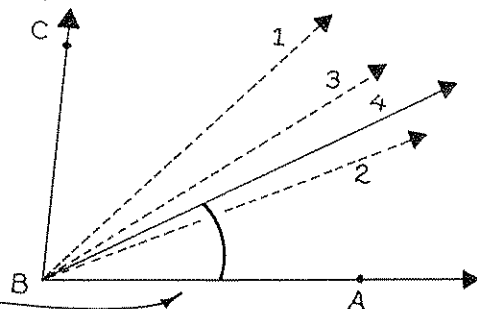
PROOF OF THE METHOD (FOR THE TEACHER)

1) Let AB = 1 unit

2) By making successive folds we generate the series

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \ldots$$

THE ARROWS SHOW THE AMOUNT WE ADD OR SUBTRACT.
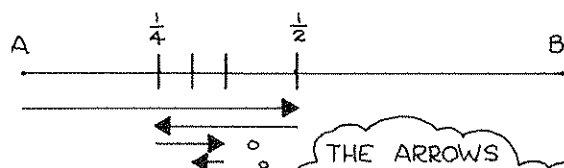
3) $$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots$$

4) That is: If we fold 2 times we have

$$\frac{1}{2} - \frac{1}{4} = \frac{1}{4} = \frac{1}{2^2}$$

4 times
$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} = \frac{1}{4} + \frac{1}{16} = \frac{1}{2^2} + \frac{1}{2^4}$$

2n times
$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots = \frac{1}{2^2} + \frac{1}{2^4} + \ldots + \frac{1}{2^{2n}}$$

5) We let $X = \frac{1}{2^2} + \frac{1}{2^4} + \ldots + \frac{1}{2^{2n}}$ and try to show $X \approx \frac{1}{3}$.

6) Multiply by $\frac{1}{2^2}$ to get $\frac{1}{2^2} X = \frac{1}{2^4} + \frac{1}{2^6} + \ldots + \frac{1}{2^{2n+2}}$

$$\frac{1}{2^{2n+2}} = \frac{1}{2^{2n}} \cdot \frac{1}{2^2}$$

7) Subtract: $$X = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \ldots + \frac{1}{2^{2n}}$$

$$-\frac{1}{2^2} X = \qquad \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \ldots + \frac{1}{2^{2n+2}}$$

$$X - \frac{1}{2^2} X = \frac{1}{2^2} - \frac{1}{2^{2n+2}}$$

THE OTHER TERMS SUBTRACT OUT.

8) Factor to get $X(1 - \frac{1}{2^2}) = \frac{1}{2^2}(1 - \frac{1}{2^{2n}})$

9) Divide by $(1 - \frac{1}{2^2})$ to get $$X = \frac{\frac{1}{2^2}(1 - \frac{1}{2^{2n}})}{1 - \frac{1}{2^2}}$$

10) When we make many folds n gets large and so $\frac{1}{2^{2n}}$ gets close to zero.

So, $$X = \frac{\frac{1}{4}(1 - \frac{1}{2^{2n}})}{\frac{3}{4}} \approx \frac{\frac{1}{4}(1 - 0)}{\frac{3}{4}} = \frac{1}{3}$$

IDEA FROM: *The Themes of Geometry*

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

CONTENTS

LINES, PLANES & ANGLES: PLANES

PLANES

We tend to ignore the curvature of the earth and view our surroundings as though they were built on a flat surface--a plane. The surfaces of our walls, floors and desk tops are portions of planes and can be used to introduce the concept of a plane. Shadows on flat surfaces can also be used to help explain a plane. The shadows can be moved about to show that a plane has no boundary, and the lack of thickness of a shadow can be related to the two-dimensional aspect of a plane. The page *Get the Point* in the Lines subsection has more suggestions for introducing the concepts of points, lines and planes.
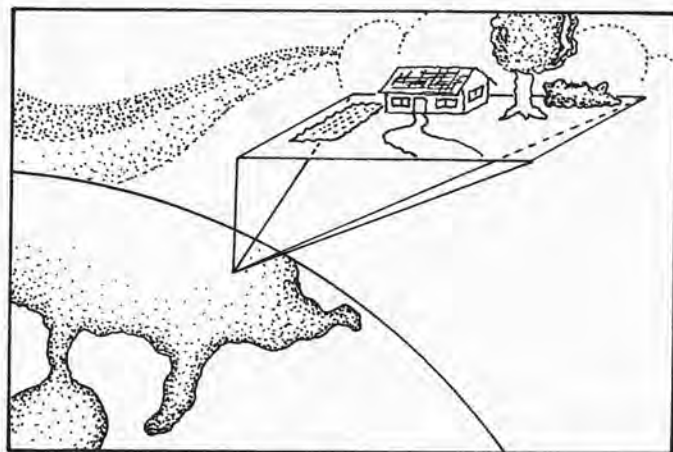
The idea of a plane can be enhanced by imagining the life of two-dimensional creatures. These 2-D creatures must stay on a plane at all times. They can rotate and slide around, but they cannot flip over since that would require a third dimension.

Some Flatland Creatures
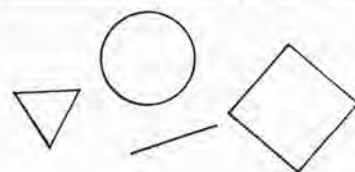
An interesting discussion can be provoked by posing the questions: How could you build a pen for a being who lives on a plane? How could you build a pen for a being who lives on a line? In 3-space?
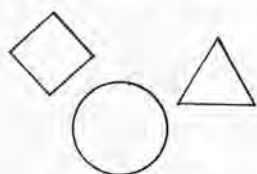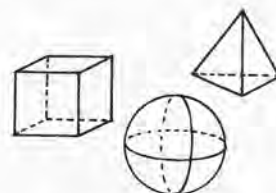
Pens in 2-D.   A pen in 1-D.   Pens in 3-D.

Your students might enjoy reading the book <u>Flatland</u> by Edwin Abbott or <u>Sphereland</u> by Dionys Burger. Both books are fantasies about two-dimensional beings. Included are dream trips to a one-dimensional space (Lineland) and visits from a three-dimensional being--a sphere. An example of a related student-written story is given in the main commentary to LINES, PLANES & ANGLES.

To demonstrate that three points determine a plane, have a student try to balance a stiff piece of cardboard on a pencil point without moving the pencil. Next, have two students
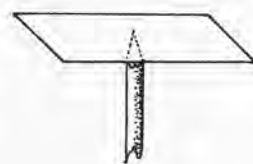
try to balance the cardboard with two pencils. Three pencil points arranged in a noncollinear fashion are needed before the cardboard will balance. The three pencil points and the cardboard demonstrate that three points determine a line. Another approach to this idea is given on *Three's Company, Four is a Crowd*. You might want to try a similar demonstration with a metre stick and a point to show that a line and a point determine a plane.
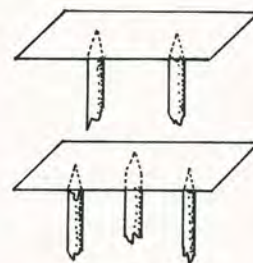
By observing how the flat surfaces of walls and ceilings meet, students can conclude that two planes intersect in a straight line. A thin, open book can also be used as a model for plane intersections. You might want your students to investigate the different ways three planes can intersect and how three planes can separate space. The models below are from the page *The Plane Truth*. How many of these arrangements can your students find?

Three planes can separate space into:
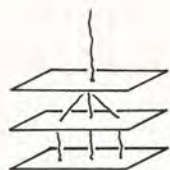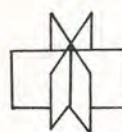
| 4 regions | 6 regions | 6 regions | 8 regions | 7 regions |
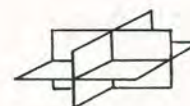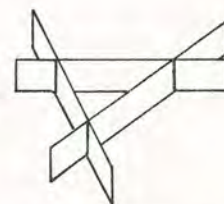
Using parallelograms to represent planes as in the sketches above might be confusing to some students. They might think the plane "stops" at the edge of the parallelogram. Class discussions should help students realize that a plane extends beyond a table, wall or cardboard surface.

LOCATING POINTS ON A PLANE

Probably some of your students have played the game Battleship. This game uses a labeled grid to name points on a flat surface. The battleship grid has the rows labeled with letters and the columns with numbers. The games *Cross Roads* and *A Maze Your Friends* in this resource are somewhat similar to Battleship and can be used to practice naming points with ordered pairs.

When we plot points on a plane in a mathematics class, the grid usually has both rows and columns labeled with numbers. Some of the difficulties students have in using ordered pairs are caused by their misconceptions about number line concepts. (See the commentary to Number Lines in the Concepts subsection of WHOLE NUMBERS in the resource Number Sense and Arithmetic Skills.) Here is an exercise with the responses of two students. Can you determine their error patterns?
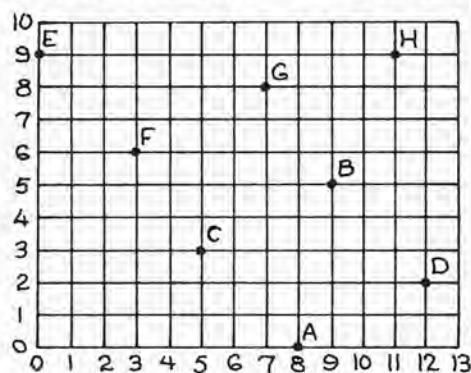
	Holly		Hans	
1. 3 , 6	Point F	√1. 3 , 6	Point A
2. 0 , 9	Point E	√2. 0 , 9	Point B
3. 9 , 5	Point B	√3. 9 , 5	Point C
√4. 13 , 3	Point D	4. 12 , 2	Point D
√5. 6 , 4	Point C	√5. 0 , 10	Point C
√6. 12 , 10	Point H	√6. 4 , 7	Point H
√7. 8 , 9	Point G	√7. 7 , 9	Point G

Holly was doing fine until she started giving ordered pairs for the points, then she started counting the lines instead of the spaces. Hans has two errors. Since the fourth point was D, he assumed the points were being considered in alphabetical order, so he filled in A, B and C. Notice he then gave ordered pairs for E, F and G. He sometimes counted the spaces and sometimes counted lines. A quick review of how ordered pairs are used to name points should correct the errors of both students.

One way to introduce the use of ordered pairs (coordinates) to name points is with the game *Coordinate Tic-Tac-Toe*. In this game students discover the rule for labeling points on a coordinated grid. No talking is allowed within the teams so students will be able to discover the rule at their own rates.

Pages like *A Long Time Ago* and *It's a Dog's Life* can provide motivation for plotting points. The resulting pictures provide means for self-checking.

Once students have the skills in graphing points from ordered pairs, they can begin to make a graph for simple equations like $y = 2x + 3$. Classroom

hints and teacher background for this kind of graphing on a plane is given on the page *Picture This:* $\Box = 3\triangle + 2$. Other activities in this resource which use ordered pairs to locate points on a plane are listed in the SECONDARY EMPHASES section under the heading Coordinate Geometry.

THREE'S COMPANY
FOUR IS A CROWD

Make two models; a three-legged table and a four-legged table. Be sure each model has two legs that are not the same length. (The models could be made from cardboard and dowel rods.)

Place the models on your desk and ask: "Why will the four-legged table rock while the three-legged table is steady?"

To help students decide:

1) Stand a dowel rod or pencil with a sharp point upright in clay. Perforate a sheet of tagboard and place the hole over the pencil point. Tilt the tagboard in different directions. Since each position of the tagboard represents a different plane that contains the pencil point, students can see that one point does not determine a unique plane.

2) Repeat the demonstration with a second pencil to show that two points do not determine a unique plane.

3) Repeat the demonstration with three pencils arranged so that they are not in a straight line. Students can see that three (non-collinear) points do determine a plane. [Will <u>any</u> group of three points determine a plane? Be sure students realize that the three points must be non-collinear.]

4) Repeat the demonstration only this time use four pencils, no three of which are in a straight line. Four points, no three of which are collinear do not <u>always</u> determine a unique plane. Actually four planes are determined, one by each group of three points.
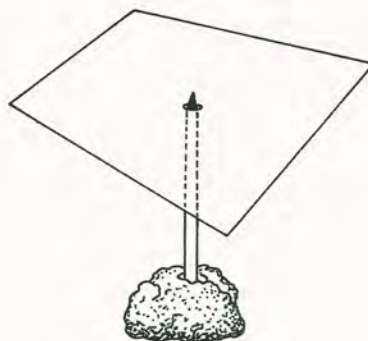
To show that this situation occurs invert the two tables on your desk and place a piece of paper on the three legs of the three-legged table to reinforce that they determine a plane. Demonstrate on the four-legged table that it is possible to support four pieces of paper, (each group of three legs determines a plane.)
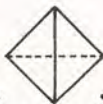
Exhibit a model of a tetrahedron. Identify the vertices as four points not on the same plane. Place a piece of paper against each face to demonstrate again that four non-coplanar points determine 4 planes.

Extension: Will any arrangement of four points <u>always</u> determine one plane?

FOUR POINTS
OF WHICH THREE
ARE COLLINEAR.

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In the following activity students investigate how two, three or four planes can be arranged in space.

Exhibit a model of a tetrahedron. Place pieces of tagboard against two faces to show that two planes intersect in a line. The edge of the tetrahedron is part of this line.

Use the two pieces of tagboard to further demonstrate that two planes are either parallel or they intersect in a line. The number of regions in space formed by each arrangement could be discussed. Three regions are formed by two parallel planes; four regions are formed by two intersecting planes.

A model of two intersecting planes can be easily constructed:

CUT SLITS
FIT THE PAPERS TOGETHER

Ask the class to give examples of two intersecting planes (adjacent walls in the room); two parallel planes (opposite walls in the room).
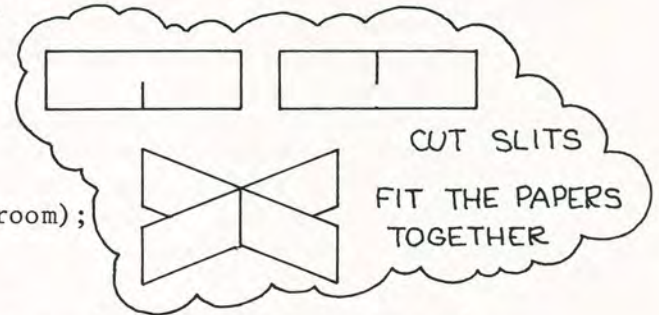
Pose the question: "Suppose we have three planes. (Exhibit three pieces of tagboard.) How can they be arranged?"

As students express possibilities, exhibit models.

Models:
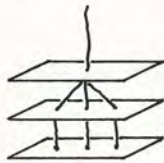
1) None of the planes intersect.
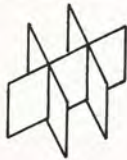
2) Two planes do not intersect. The third plane intersects both of them.
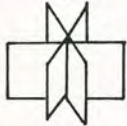
3) All three planes intersect in a single line.
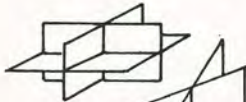
4) All three planes intersect at a single point

5) All three planes do not intersect, but each pair does.

Ask the students to give real world examples of each possibility.

Extension: How can four planes be arranged? How many regions are formed?

CROSSROADS

Players: 2

Materials: Gameboard; 2 markers; 25 number tiles:
3 each of +10, +5, +2, +1 (green),
2 each of -10, -5, -2, -1 (red), 5
yellow unnumbered tiles each having a
value of +10.

Procedure: Randomly place the number tiles
on the dots on the gameboard.
Player #1 positions his marker
on an arrow to indicate any
column. Player #2 places his
marker to indicate any row and
takes the number tile from the
intersection of "crossroads" of
the row and column.

CROSSROADS

SECOND PLAYER MOVES VERTICALLY
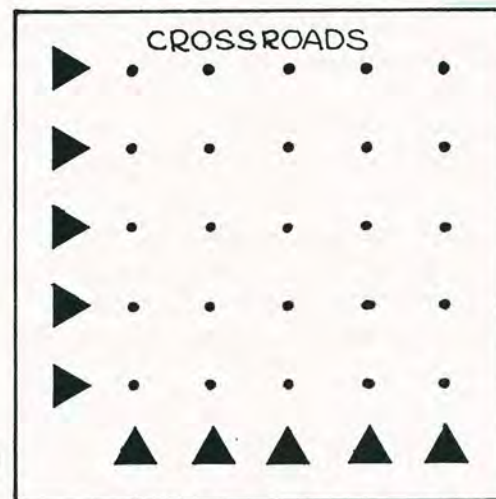
FIRST PLAYER MOVES HORIZONTALLY

Play continues with player #1 moving to a <u>different</u> column and taking the
indicated number tile. To avoid taking a negative number, players may replace
a yellow tile from their winnings on an empty dot when it is their turn.

If a player has no play but to indicate an empty dot, he must replace a
yellow tile. If he does not have one the game is over.

Winner is the player with the highest total of his number tiles----sum the
positive and negative values.

□ = A YELLOW TILE

PLAYER #1's FIRST MOVE

PLAYER #2's FIRST MOVE.
PLAYER #2 TAKES THE □ TILE WORTH +10.

PLAYER #1's SECOND MOVE. PLAYER #1 TAKES THE +1 TILE

Variations: 1) An easy game can be made by using 25 green tiles, all with low
positive values.

2) A medium-hard game could use 15 green tiles (3 of +10's, 4 each of +5,
+2, +1) and 10 red tiles (3 each of -10, -5, and 2 each of -2, -1).

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COORDINATE TIC-TAC-TOE

Materials: Chalkboard or overhead projector with grid

Objective: To discover how to locate points on a grid from number pairs

Directions:

1) Divide the class into two teams, the X's and the O's.

2) Tell the class that this is a game of Tic-Tac-Toe and that a team must get five X's or five O's in a row to win.

3) Play begins by a member of the X-team giving two numbers, each less than or equal to 10. You record these in a table and place an X on the grid in the correct spot. The O-team then gives two numbers and the game progresses.

4) Once a number is given it cannot be changed. The student giving the number pair cannot receive help from the rest of the team. (You might decide which team member responds so that those who are nervous can move early in the game.

5) If the student gives a number pair previously used or beyond the limits of the grid the team loses that turn.

Variation:

To introduce the activity use a grid without axes. Play the game by having each team member place an "X" or "O" on the grid when it is his turn.

Later, position axes on the grid and tell the class that you have become the official recorder. Ask them to devise a method to tell you how to mark their moves. You should receive suggestions like "Right two, up three" or "Up six, to the left seven."

After introducing ordered pairs, the game can be played for additional practice in graphing ordered number pairs.
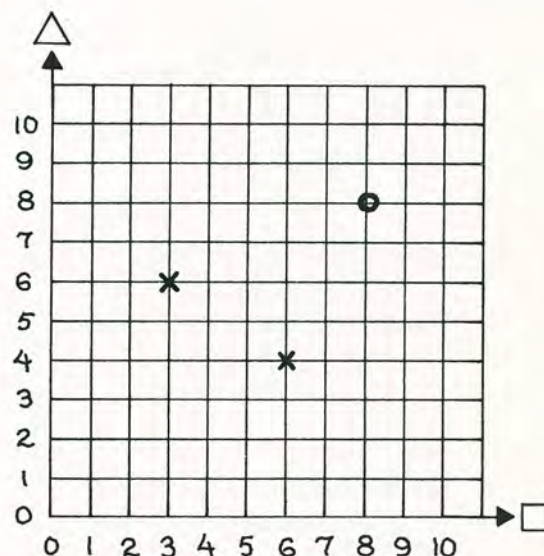
	□	△
X	3	6
O	8	8
X	6	4
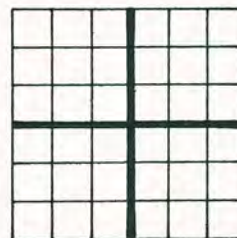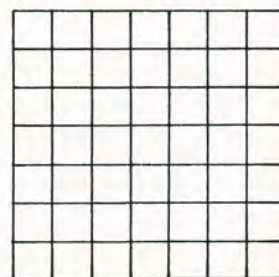
SIGNED NUMBERS COULD BE USED AS COORDINATES ON THIS PAGE.

IT'S STARTLING!

We go together to help locate points on a grid.

VERTICAL

7
6
5
4
3
2
1
0

0 1 2 3 4 5 6 7

HORIZONTAL

V

8
7
6
5
4
3
2
1

1 2 3 4 5 6 7   H

(3,5)

(5,3)

(2,2)

O
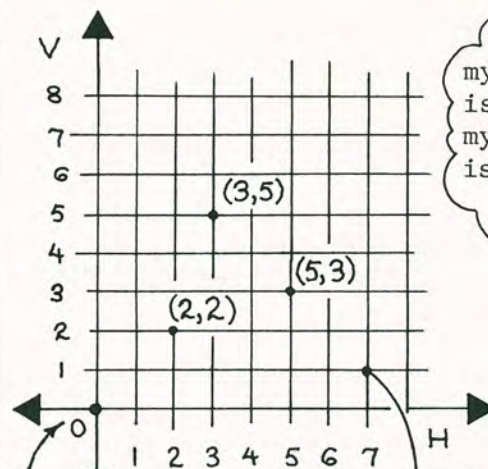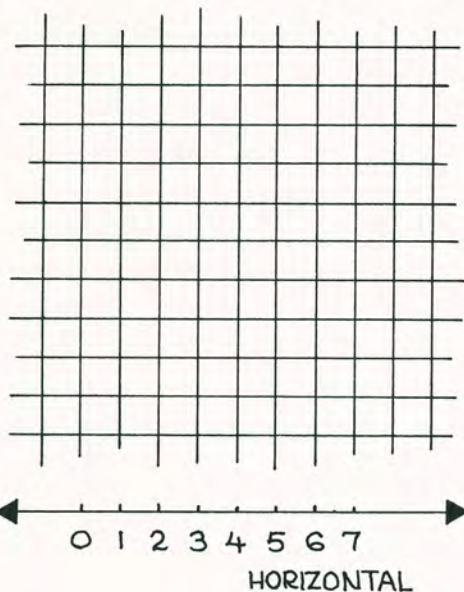
I'm (5,3) my H name "5" is written first; my V name "3" is written second.

Who am I? (__,__)

I'm (0,0) and I'm also called the origin.

1) On the grid paper to the right mark points at these locations:

(H,V)

A) (7,7)

B) (10,7)

C) (8,5)

D) (9,2)

E) (6,4)

F) (3,2)

G) (4,5)

H) (2,7)

I) (5,7)

J) (6,9)

Number the H and V lines to help you locate the points.

2) Now connect A to B, B to C, C to D ..., I to J, J to A by drawing line segments.

V

H
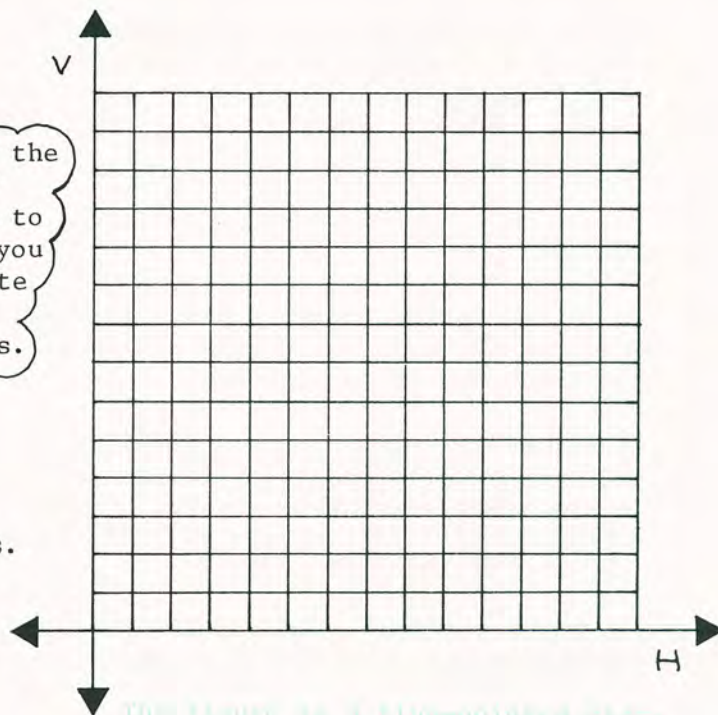
ARE YOU CLUMSY ᴏʀ COORDINATED?

1) Give the coordinates for each point:

T = (3,4)   R = _____   S = _____

(HORIZONTAL) (VERTICAL)

THE COORDINATES OF A POINT ARE THE TWO NUMBERS WHICH LOCATE IT ON THE GRID.

2) Give the coordinates of the end points of segment AB. (__,__), (__,__)
Give the coordinates for three more points on line segment AB.

_____, _____, _____

All the points on line segment AB have coordinates with what second number? _____

3) Give the coordinates of the end points of segment EF. _____, _____
All the points on line segment EF have coordinates with what first number? _____

4) Graph the point (5,4).
All the points directly right of (5,4) have coordinates with what second number?

_____

All the points directly above (5,4) have coordinates with what first number? _____

5) Give the coordinates of the midpoint of each segment:
$\overline{AB}$ _____ $\overline{EF}$ _____ $\overline{RS}$ _____

6) Suppose F is the midpoint of a segment with one end point at E. Write the coordinates of the other end point of the segment. _____
Graph the point.

7) Give the coordinates of the midpoint of the segment whose end points have coordinates (1,1) and (7,7). _____

8) Does point (2,5) lie on line XY? _____
Write the coordinates of three points that lie on $\overline{YY}$. _____, _____, _____
How is the second number related to the first in each pair? _____
Do you think (7,13) would lie on line XY? _____

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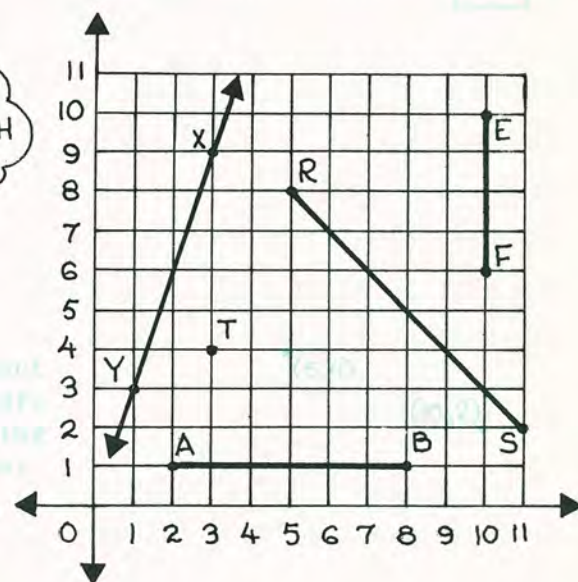
A LONG TIME AGO

1) Locate each group of points on the grid and connect them in order with straight line segments. Do not connect the last point of each group with the first point of the next group.

  I. (6,24), (7,24).

 II. (9,12), (6,12), (8,16).

III. (9,21), (10,22), (10,23), (9,23).

 IV. (5,23), (3,23), (1,22), (1,21), (4,20½).

  V. (5,27), (6,27), (7,26).

 VI. (3,13), (1,12), (3,11), (6,11), (6,3), (2,3), (1,2), (8,2), (8,16).

VII. (10,15), (10,11), (11,10), (11,9), (10,8), (8,11), (6,11).

VIII. (11,22), (11,20).

 IX. (11,24), (12,21).

  X. (12,25), (13,22).

 XI. (3,20), (4,20), (6,19), (6,18), (3,13), (4,12), (9,18), (13,12).

XII. (7,27), (8,26), (12,26), (13,25), (14,23).

XIII. (4,2), (3,1), (10,1), (10,8).

XIV. (5,24), (3,25), (2,24), (3,23).

 XV. (10,21), (9,19), (9,18).

XVI. (10,12), (14,12), (13,11), (10,11).
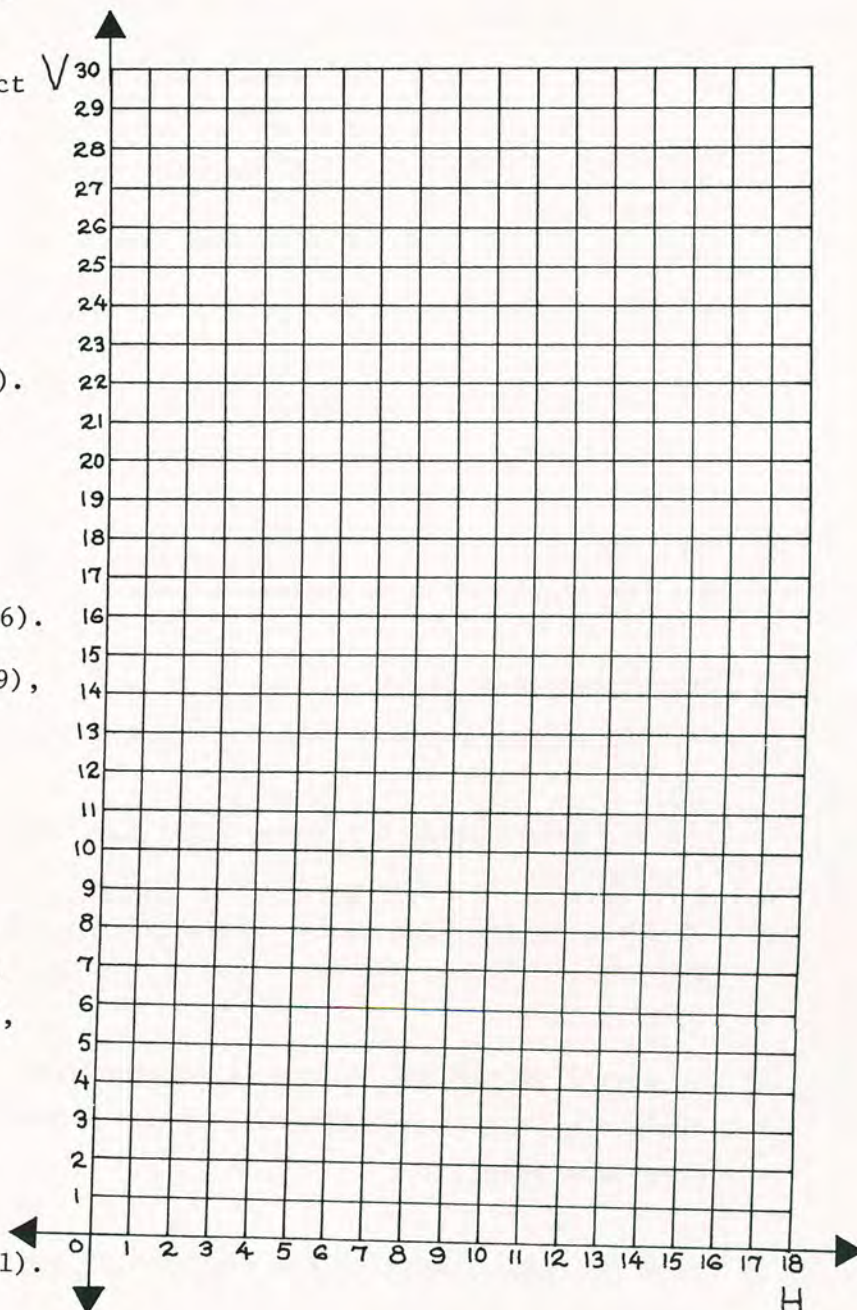
2) On grid paper draw your own picture with points and line segments. Use ordered pairs to describe the picture. Give the description to a friend and see if she can graph your picture.

IDEA FROM: *Mathematics A Human Endeavor*
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IT'S A DOG'S LIFE

Locate the following sets on the graph below and connect them in order with straight line segments. Do not connect points in different sets.

A) (5,4)   C) (7,14)   F) (10,23)   H) (9.5,12.5)   J) (9.5,14)   K) (3,23)   M) (13,21)
   (5,0)   (7,13.5)   (10,21)   (11.5,12.5)   (10,15)   (1,23)   (13,23)
   (15,0)   (7.5,13.5)   (12,21)   (11.5,13)   (11,16)   (1,22)   (15,23)
   (15,4)   (7.5,14)   (12,23)   (12,13)   (12,16)   (3,22)   (15,22)
   (3,4)   (7,14)   (10,23)   (12,12)   (13,15)   (3,21)   (13,22)
   (5,12)   (13.5,14)   (1,21) N)   (18,23)
   (15,12)   D) (9,12)   G) (8.5,18)   I) (6.5,12)   (13.5,13) L)   (4,21)   (17,22)
   (17,4)   (9,14)   (7.5,18)   (5.5,13)   (14,13)   (4,23)   (16,23)
   (15,4)   (9.5,14)   (7.5,18.5)   (5.5,14)   (14,14)   (6,21)   (17,22)
                     (9.5,12)   (8.5,18.5)   (6.5,15)   (14.5,15)   (6,23)   (17,21)
B) (7,12)   E) (7,23)   (8.5,18)   (7,16)   (15,15)
   (7,9)   (9,23)   (9,17)   (7,17)   (15,12)
   (7.5,8.5)   (9,21)   (9,14)   (7.5,18)
   (8,9)   (7,21)
   (8,12)   (7,23)

IDEA FROM: *Graphiti*

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A MAZE YOUR FRIENDS

Players: 2 people or 2 groups

Materials: 2 geoboards, rubber bands, dot paper

Procedure: Each player constructs a maze with rubber bands of one color on her geoboard and conceals it from her opponent.

Each player tries to get through the other's maze by describing a line segment path from (0,0) to (N,N) that does not cross or touch any rubber bands of the maze.

GOAL

START) PLAYER # 2's MAZE
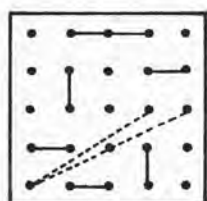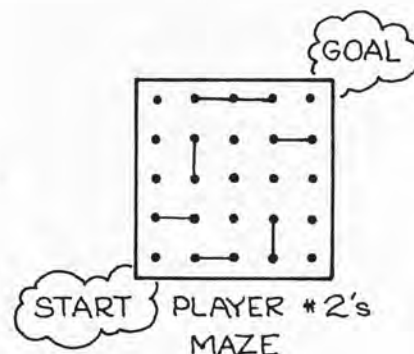
PLAYER #2's MAZE

O.K. FIRST MOVE FOR PLAYER # 1 WOULD BE (2,1), (3,2) OR (4,2).

To begin, player #1 gives the coordinates of a point on the geoboard. Player #2 checks to see if the line segment joining this point to (0,0) crosses or touches her maze by constructing it on the geoboard. (Be sure to use a rubber band of different color.) If the segment is O.K., the rubber band remains. If not it is removed. Players should record their O.K. moves on dot paper. Play alternates and player #2 names a point. At each successive turn the proposed segment is constructed from the end point of the last O.K. segment.

The winner is the first player to travel through the other's maze.

Suggestions:

(1) To introduce the game, make a maze and play while the class looks at the maze. Explain why various moves can or can't be made.

(2) For the next game, show the class the maze and then hide it. The class works together to find a path. Once the method is understood, the game can be played by groups or individuals.

For more drill and practice in graphing and locating points on a grid the game of "Battleship" can be played. Adapt the game so that the ships are plotted on grid points rather than on the coordinate squares.

PICTURE THIS: $\square = (3 \times \triangle) + 2$

In coordinate geometry a line can be described by an equation and likewise certain equations generate number pairs which give a line. By investigating the graphs of linear equations one can relate the position of the line to certain parts of the equation. One can then describe the graph of any linear equation without actually drawing it.

For example the equation: $\square = (3 \times \triangle) + 2$ generates ordered pairs that lie on a line passing through (0,2) with slope 3.

$\triangle$	$\square$	
0	2	
1	5	3
2	8	3
3	11	3
.	.	
.	.	

THE DOTTED LINE HAS A SLOPE OF 3: RISE OF 3 RUN OF 1

(2,8)

(1,5)

(0,2)
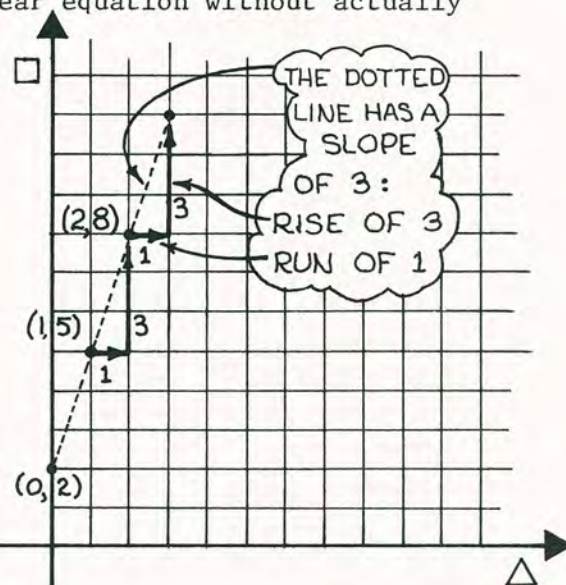
I 1) To acquaint students with these relationships begin by having them investigate the number pattern $\square = 2 \times \triangle$.

   a) First have them use the pattern to complete a table.

   b) Then graph the number pairs ($\triangle$, $\square$) on a grid.

   c) Ask students what they notice. Students should reply that the points lie on a line.

$\square = 2 \times \triangle$

$\triangle$	$\square$
0	
1	
2	
3	
4	
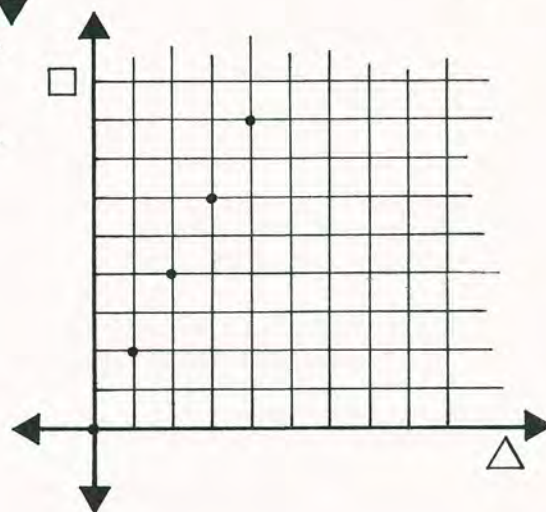
   2) Have students investigate these number patterns in a similar manner:

   a) $\square = 3 \times \triangle$

   b) $\square = 4 \times \triangle$

   c) $\square = 5 \times \triangle$

   3) Ask students to predict what the graph of $\square = 10 \times \triangle$ would look like.

   4) To give students an intuitive feeling for slope, place a graph of $\square = 2 \times \triangle$ on the overhead and ask students to observe the "move" as you travel from (0,0) to (1,2); from (1,2) to (2,4).

The "move" is over 1 up 2.

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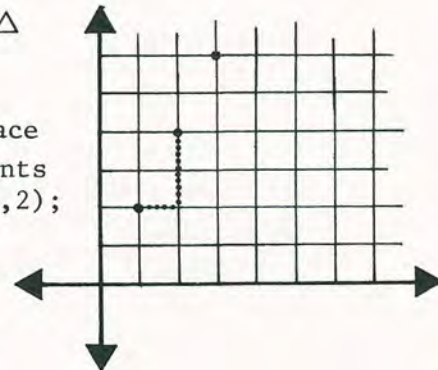
PICTURE THIS: $\square = (3 \times \triangle) + 2$

(CONTINUED)

5) Have them discover the "moves" on the other patterns.

   a) $\square = 3 \times \triangle$ b) $\square = 4 \times \triangle$ c) $\square = 5 \times \triangle$

6) To explore the effect of multiplying by a fraction use the following patterns to construct tables and then graph the number pairs:

   a) $\square = \frac{1}{2} \times \triangle$ b) $\square = \frac{1}{3} \times \triangle$ c) $\square = \frac{1}{4} \times \triangle$ *DESCRIBE THE "MOVE" ON THE GRAPH.*

THE RELATION IS EASIER TO GRAPH IF $\square$ NUMBERS ARE NOT FRACTIONS

$\triangle$	$\square$
0	⊙
2	⋮
4	
6	
8	

"MOVE" IS OVER 2, UP 1.

If signed numbers have been introduced the grid can be extended,

and these patterns can also be explored. $\square = -3 \times \triangle$
$\square = -\frac{1}{5} \times \triangle$
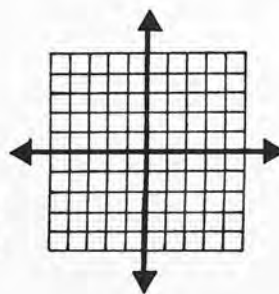
II Students can then investigate the result of adding a constant to a first number to get the second.
i.e. $\square = \triangle + 2$.

1) By constructing tables and graphing the ordered number pairs students can explore this sequence

   a) $\square = \triangle + 1$ b) $\square = \triangle + 2$ c) $\square = \triangle + 3$ d) $\square = \triangle - 1$

2) On the overhead, lay the graphs of the four equations on top of each other so that the axes match.
   Ask students what they notice.   (A family of parallel lines.)

3) Students are ready to investigate this sequence:

   a) $\square = 2 \times \triangle$ b) $\square = (2 \times \triangle) + 1$ c) $\square = (2 \times \triangle) + 2$

III Students should get an intuitive feeling that the multiplier $\square = c \times \triangle$ rotates the graph of the basic equation $\square = \triangle$ and that the number added $\square = \triangle + d$ slides the graph of the equation $\square = \triangle$ up or down.
A bulletin board display could be made of student graphs.

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WE ALL LINE UP

1) Use the number pattern $\square = 3 \times \triangle$ to complete the table.

Graph the number pairs on the grid below in blue.

△	□
0	○
1	
2	
3	
4	

2) Use the number pattern $\square = (3 \times \triangle) + 2$ to complete the table.

Graph the number pairs on the grid below in black.

△	□
0	2
1	
2	
3	
4	

3) Use the number pattern $\square = (3 \times \triangle) + 4$ to complete the table.

Graph the number pairs on the grid below in red.

△	□
0	
1	
2	
3	
4	

4) Use the number pattern $\square = (3 \times \triangle) + 6$ to complete the table.

Graph the number pairs on the grid in green.

△	□
0	
1	
2	
3	
4	

5) See if you can sketch the graph of $\square = (3 \times \triangle) + 8$ on the grid without making a table.

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GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

CONTENTS

LINES, PLANES & ANGLES: ANGLES

ANGLES

There are two views of angles which middle school students are exposed to. The first is the static view of angles that was introduced (at least informally) in elementary school and the second is the dynamic concept of angles which involves rotations.

THE STATIC VIEW

The understanding of an angle begins with experiencing corners of polygons which may be faces of a polyhedron. Children in the primary grades might use the words "square," "sharp" or "blunt" to describe such corners. Through experiences with solids and flat shapes, children first learn there are angles which are more open or less open than right angles.

a "square" corner

a "sharp" corner
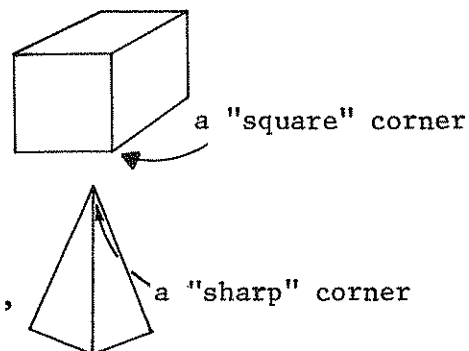
At this stage an angle is seen as a corner. A right angle can be represented by the corner of a rectangular piece of paper, or it can be formed by folding a paper and then folding again as shown to the right. (A different way to fold a right angle is given in *Make the Right Fold*.) Acute, right and obtuse angles can be related to corners which "fit inside," "fit on top of" or "contain" the folded corner. A straight angle can be represented by two right angles which have been joined as shown below.

FOLD

FOLD

ACUTE ANGLE   RIGHT ANGLE   OBTUSE ANGLE   STRAIGHT ANGLE
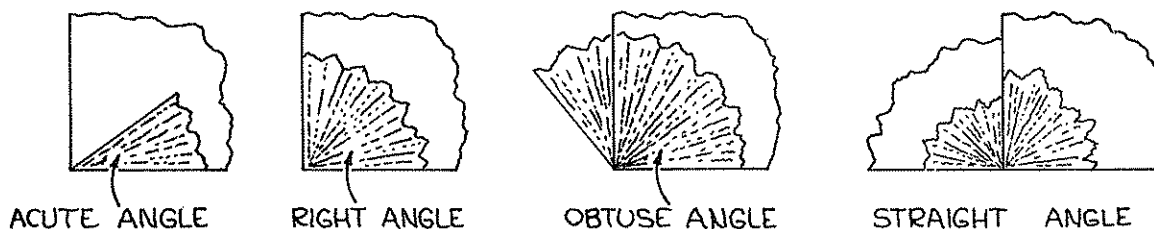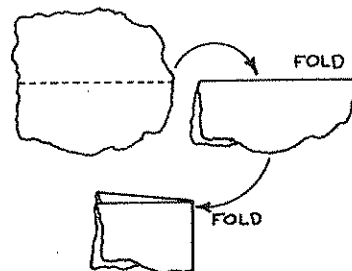
In middle school the concept of angle becomes more refined. (See *Angles not Angels*.) An angle is the union of two rays with a common end point. This is represented on paper with the arrows as shown below. Some students might have difficulty with this notion since they may see angles as the interior region of the angle rather than the two rays themselves. The idea of comparing angles by the amount of opening is still used much as described above. See *Comparing Angles 1* and *Comparing Angles 2*.

SIDE

VERTEX
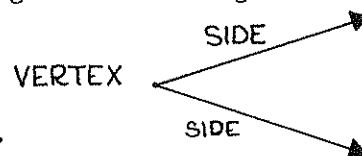
SIDE

THE DYNAMIC VIEW

Viewing an angle as obtained by rotating one of its sides is much different from viewing it as a static pair of rays. In the dynamic view, when one of the sides is held in a set position and the other is allowed to rotate in a plane, the angle is observed as changing. A model of this can be made with two strips of cardboard and a fastener. Strips from erector sets can also be loosely bolted together for a model.

The diagrams at the right relate the words acute, right, obtuse and straight to turns or rotations of a side of an angle. Notice that the stationary side is shown as a horizontal ray pointing to the right. This has been a standard choice but it is not necessary. The angle shown below is a right angle, but it has no horizontal side. The arrow designates the stationary and rotating sides.
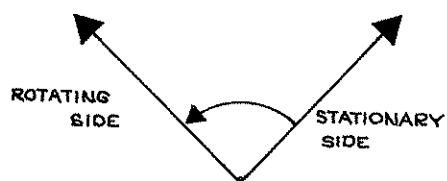
AN ACUTE ANGLE IS LESS THAN $\frac{1}{4}$ TURN

A RIGHT ANGLE IS $\frac{1}{4}$ OF A TURN

AN OBTUSE ANGLE IS MORE THAN $\frac{1}{4}$ TURN AND LESS THAN $\frac{1}{2}$ TURN

A STRAIGHT ANGLE IS $\frac{1}{2}$ TURN

A REFLEX ANGLE IS MORE THAN $\frac{1}{2}$ TURN AND LESS THAN A WHOLE TURN
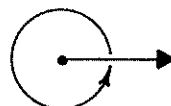
A COMPLETE TURN

MORE THAN 1 TURN

ROTATING SIDE

STATIONARY SIDE
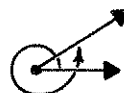
Usually counterclockwise rotations are assigned positive measures and clockwise rotations are assigned negative measures. This distinction does not seem necessary until the study of trigonometry and it might cause confusion. In everyday usage we assign a positive measure to an angle regardless of its direction of rotation. This is also true in determining bearings as described in *Compass Bearings.*

MEASURING ANGLES

The measure of an angle can be viewed as describing how "open" its sides are (static) or how much one side has been rotated from the other side (dynamic view).

Static

To decide how open an angle is, a circle could be cut into congruent pie shapes. If 5 of these pie shapes exactly cover the interior of an angle, the angle would have measure 5.

CALL THESE BLOPS

THE MEASUREMENT OF THIS ANGLE IS 5 BLOPS.
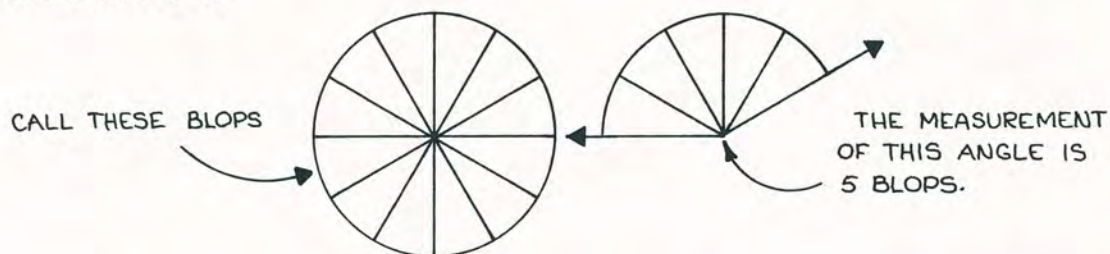
Thousands of years ago the Babylonians decided to split the region of a circle into 360 congruent pie shapes which they called "degrees." (See *Ancient Angles*.) We still use this notation today. You might like to give your students some practice in measuring angles in nonstandard units. The page *An Angle Measure* gives some ideas on student-developed units. This practice in nonstandard units can help build measurement concepts and can keep the students' ideas on angle measure more flexible.

Dynamic

To decide how much one side has been rotated we can compare it to quarter turns, half turns and whole turns.

$\frac{1}{8}$ TURN $\frac{3}{8}$ TURN $1\frac{1}{4}$ TURN
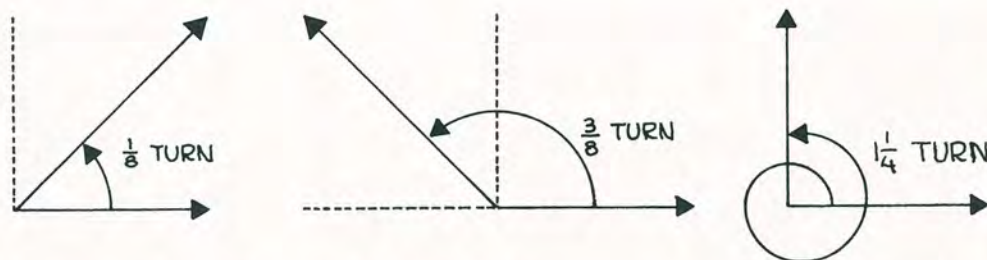
Usually we replace the word "turn" with a number and a unit. If a complete turn is replaced by 360°, the angles shown above would measure 45° (from $\frac{1}{8}$ x 360°), 135° (from $\frac{3}{8}$ x 360°) and 450° (from 360° + 90°).

ANGLES, NOT ANGELS

TWO RAYS WITH THE SAME END POINT FORM AN ANGLE
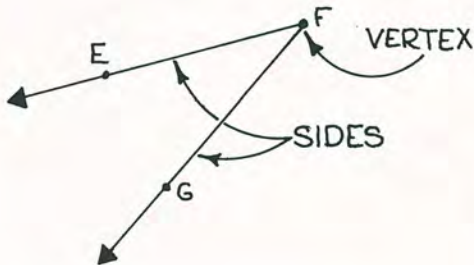
This angle can be called angle EFG or angle GFE. The letter naming the vertex is always in the middle.

∠EFG   OR   ∠GFE
VERTEX   VERTEX
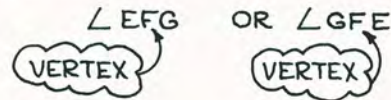
If there is no confusion the letter naming the vertex can be used alone.   ∠F

A) Find some examples of angles in your classroom.

_____
_____
_____
_____
_____

B) Name four different angles in this diagram. Use the 3-letter names.

1) _____
2) _____
3) _____
4) _____
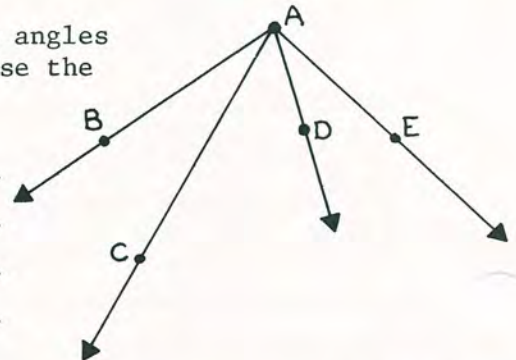
C) How many different angles are formed using these rays?

D) The angel below has been drawn with many angles. Can you draw a simpler angel with fewer angles?

2 RAYS	3 RAYS	4 RAYS	5 RAYS	6 RAYS
	3			
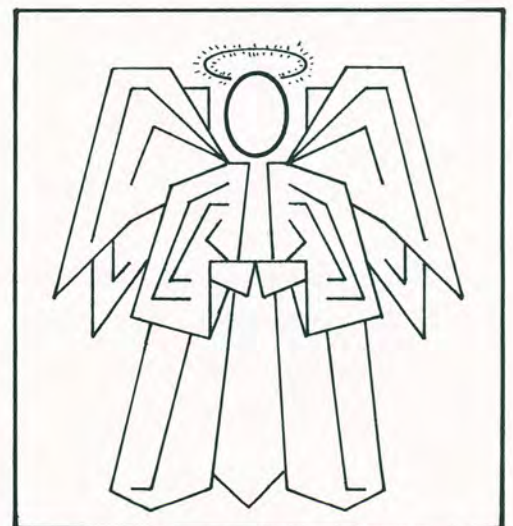
Predict the number of angles formed using 7 rays. _____
Check your prediction.

Predict the number of angles formed using 10 rays. _____

ALL CORNERS AREN'T SQUARE

People who make their living by tiling floors often have to cut a tile to make it fit a corner. Draw a line from each marked corner of a tile to the corner it will fit.

1) TILE   CORNER
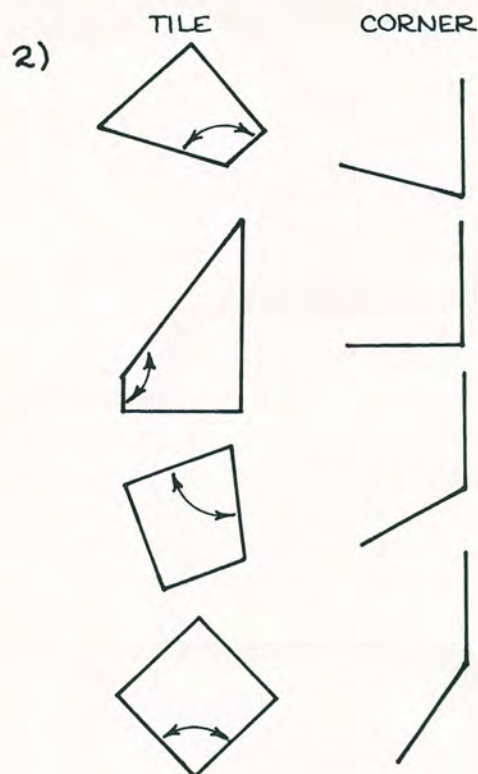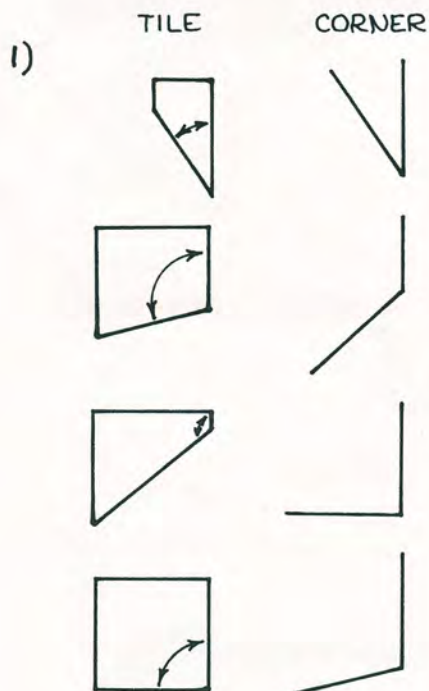
2) TILE   CORNER

A tiler uses a tool like a compass to trace the corner and mark a line on the tile. When the tile is cut it will fit into the corner. Sketch lines on the tiles below to show where the tiles could be cut to fit into the corners.

3) CORNER   TILE

a)

b)

c)

d
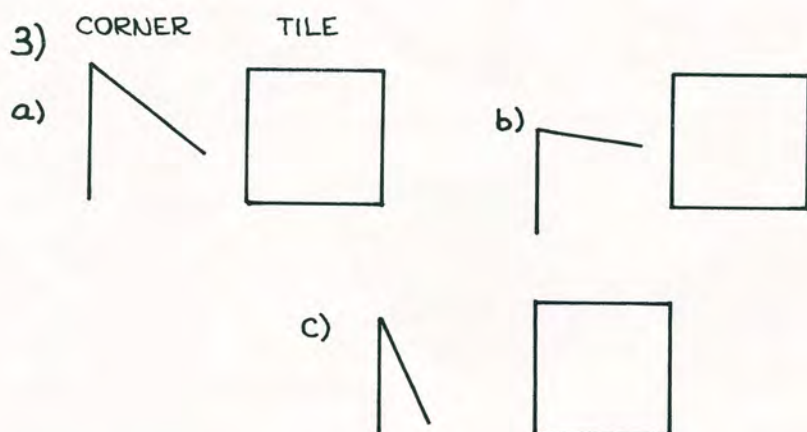
COMPARING ANGLES 1

1) Use the letters to put the openings between the clockhands in order from smallest to largest.

_____ _____ _____ _____ _____
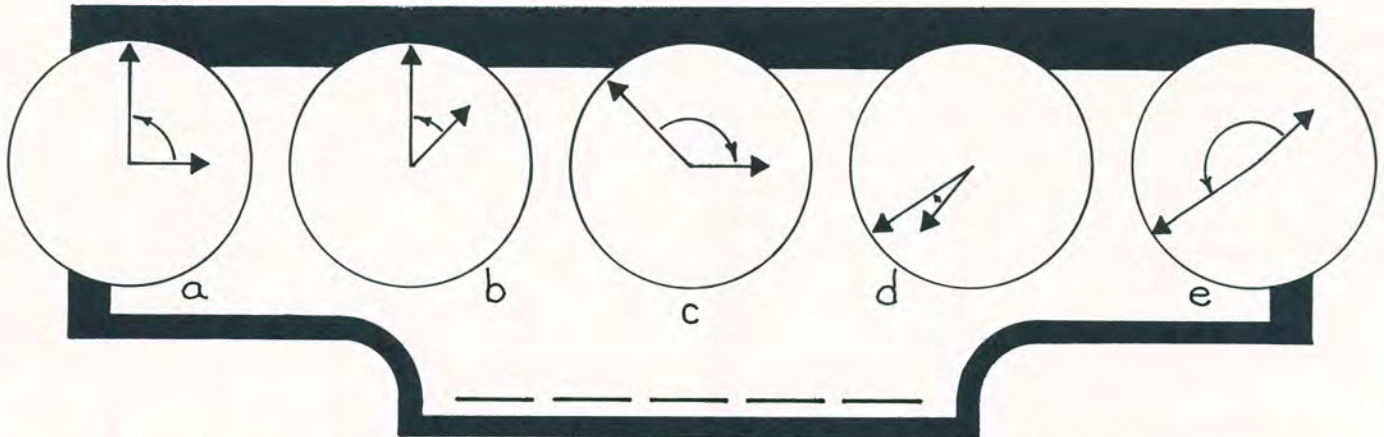
2) a) Which piece of pizza is bigger than (a) but smaller than (c)? _____
    b) Which piece of pizza is smaller than (d) but bigger than (b)? _____
    c) Which piece of pizza is 1/4 of a whole pizza pie? _____
    d) Which 2 pieces of pizza together would make about 1/2 of a whole pizza pie?
       _____ , _____
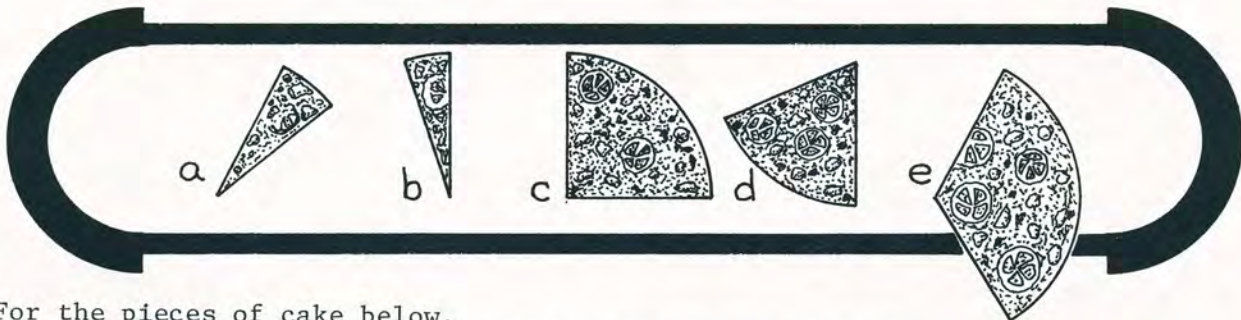
3) For the pieces of cake below,
    a) Which piece is smaller than (b) and bigger than (c)? _____
    b) Which piece is a little smaller than 1/2 of a whole cake? _____
    c) The difference between which 2 pieces is about 1/4 of a whole cake? _____
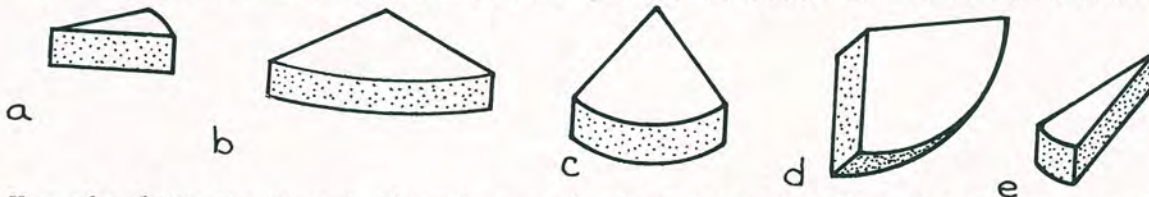
4) Use the letters to put the pieces of pizza in order from largest to smallest according to the opening between the edges.

_____ _____ _____ _____
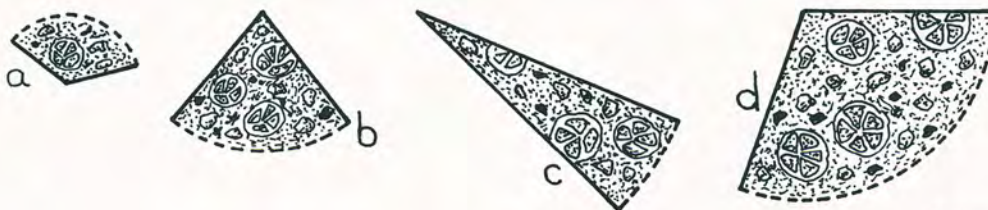
COMPARING ANGLES 2

1) CIRCLE THE LARGER OF EACH PAIR OF ANGLES.

    ⓐ   ⓑ   ⓒ
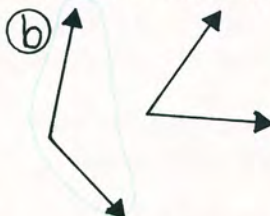
2) CIRCLE THE SMALLER OF EACH PAIR OF ANGLES.

    ⓐ   ⓑ   ⓒ
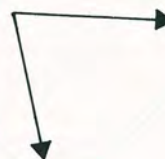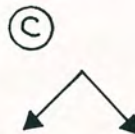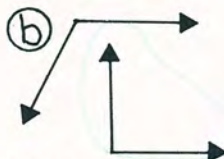
3) CIRCLE THE TWO ANGLES THAT SHOW THE SAME SIZE.

    ⓐ   ⓑ   ⓒ
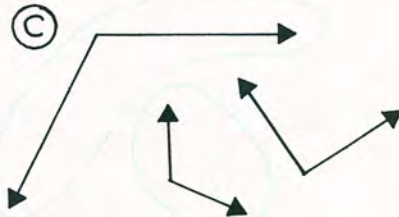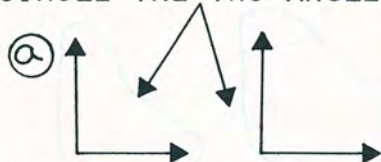
4) FOR THE ANGLES SHOWN BELOW

   A) WHICH ANGLE IS SMALLER THAN   E? ____d____
   B) WHICH ANGLES ARE LARGER THAN   F? ____ , ____
   C) WHICH ANGLE IS SMALLER THAN   B AND LARGER THAN   F? ____
   D) WHICH ANGLE IS SMALLER THAN   A AND LARGER THAN   D? ____

   a   c   b   d   e   f
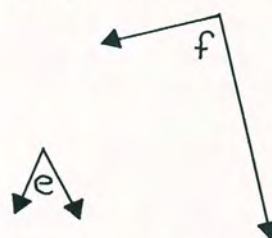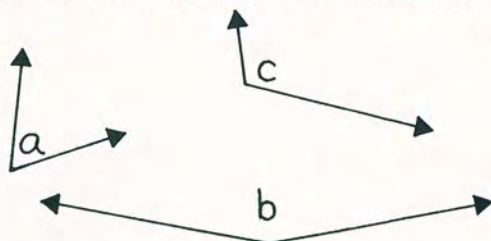
5) SKETCH A RAY TO COMPLETE AN ANGLE THAT HAS THE SAME SIZE AS THE
   GIVEN ANGLE.

    ⓐ   ⓑ   ⓒ

MAKE THE RIGHT FOLD

An easily recognized and useful angle is the right angle. Students can make models for right angles by folding paper. Two methods are shown below. These models can be used to determine whether an angle is right, acute, or obtuse. See *An Acute Case of Obtusity*. The models can also be used to sketch a right angle wherever needed.

A) 1) Have students fold and crease a piece of paper. The fold can be made anywhere.

    2) Then fold again so the first fold is placed on top of itself.

    3) The right angle occurs at the point of intersection of the two folds. The paper could be unfolded to show four right angles at this point.
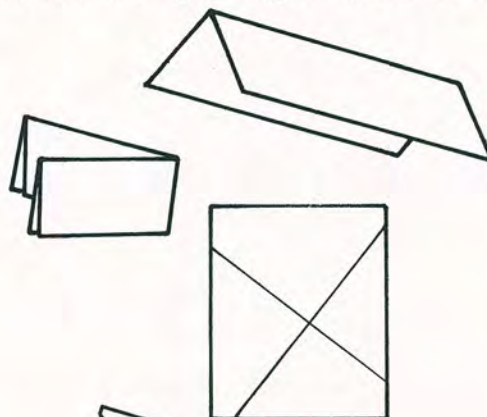
B) 1) Use a strip of paper with at least one straight edge. Fold the right side down at any angle. Crease.

    2) Fold the left side down so the straight edges meet. The angle formed will be a right angle.
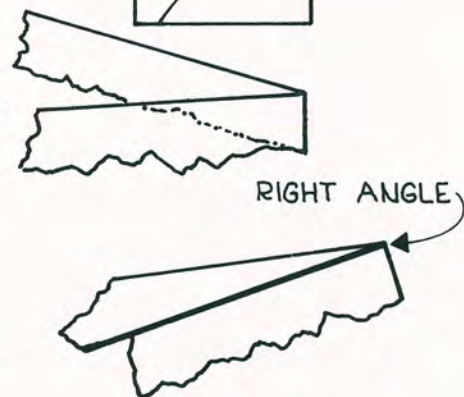
RIGHT ANGLE

C) Having made models of right angles students can use them to check right angles. Several ideas are given below.

    1) Which of these are right angles?

    2) Name the right angles in this model.

    3) Mark the right angles in this design.
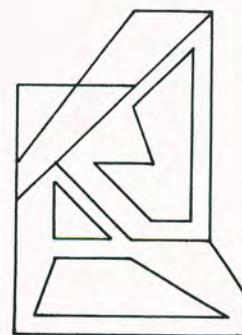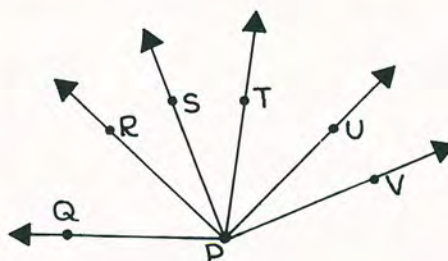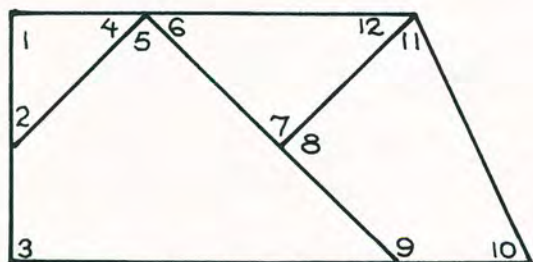
AN ACUTE CASE OF OBTUSITY

The concept of acute angles being smaller than right angles and obtuse angles being larger than right angles can be demonstrated using paper folding. Have students fold a piece of paper twice to show intersecting lines. You can demonstrate at the overhead using a thin piece of transparency material or wax paper. Unfold the paper and label the angles 1, 2, 3, and 4. Using a right angle model from *Make the Right Fold* (or a corner of a piece of paper) label each angle as smaller than or larger than a right angle.
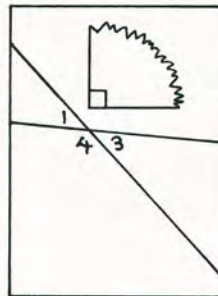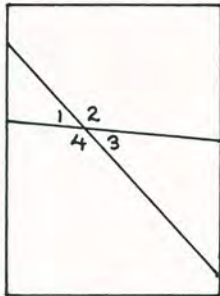
∠ 1 smaller

∠ 2 larger

∠ 3

∠ 4

Repeat with two other lines and ask students if they can find an example where the two lines don't form angles that are smaller than or larger than a right angle.

At this time the words acute and obtuse can be introduced. The following exercises are examples of skill building activities that could be used. You may want to have students make a guess about the angle and then check it with a right angle model.

a) Circle the acute (obtuse) angles.

1)   2)   3)   4)
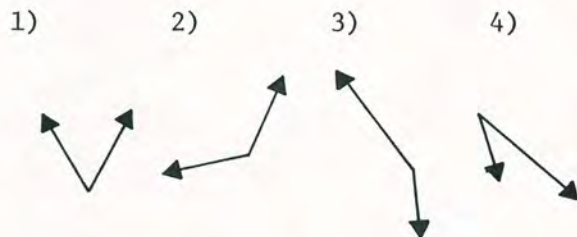
b) Name two acute angles in this figure.
Name two obtuse angles in this figure.
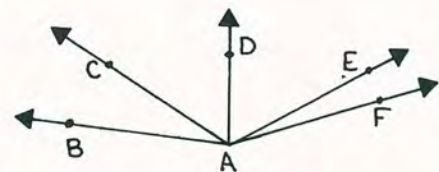
c) In this polygon circle the lettered angles if they are acute angles. Put an X on the letters of obtuse angles.
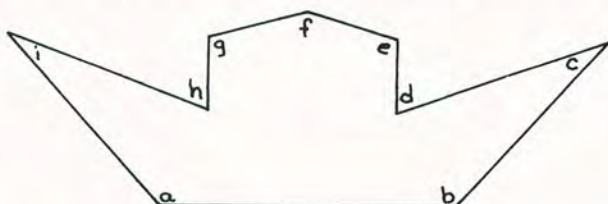
d) Find examples of angles in your classroom that are either acute or obtuse. Check each with a right angle model.

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A COMPLEMENT FOR YOU

1) Draw a triangle with one right angle.
   It can be any size that you want.

2) Label it BAC with the right angle labeled A.
   Carefully cut out the triangle.

3) Fold so that vertex B fits on vertex A.

4) Fold so that vertex C fits on vertex A.

5) Did angle B and angle C exactly cover angle A?

6) Try it again with a different right triangle.
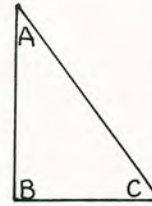
> Two angles that will fit together to exactly cover a
> right angle are called complementary angles.
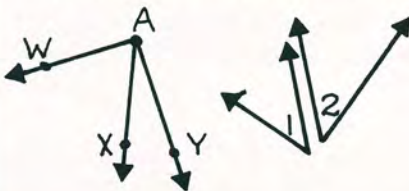
COMPLEMENTARY ANGLES	NOT COMPLEMENTARY ANGLES

7) Sketch a complementary angle for each of these angles.

   a)   b)   c)   d)
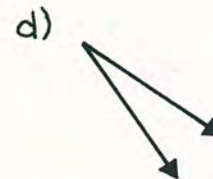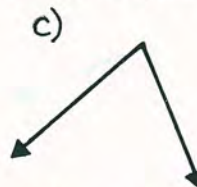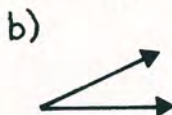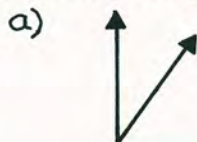
8) Match each angle with its complementary angle. You may
   cut them out and fit them on the given right angle.
   Record your answers in the table to the right.

Complementary Angles

1) _____ _____
2) _____ _____
3) _____ _____
4) _____ _____

RIGHT
ANGLE
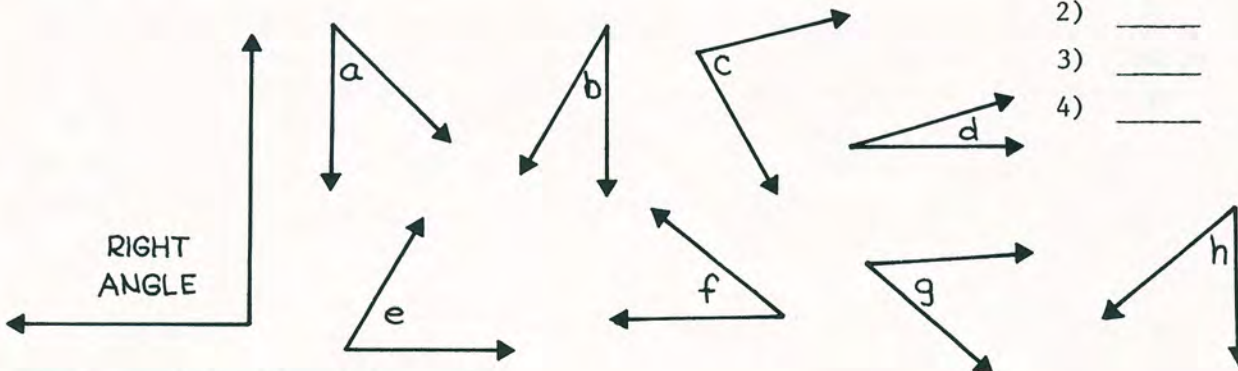
ANGLE WRANGLE

Equipment: Compass and straightedge

Activity: Each of the boxes below shows an angle and a line segment. You are to use
the angle and the line segment to construct a figure which has all congruent
angles and all congruent sides. Such a figure is called a regular polygon.
Start with box A and do these things:

1)   On another sheet of paper construct an angle congruent
to the angle shown in the box.

2)   On the sides of your angle mark off line segments
congruent to the line segment given in the box.

3)   Construct another angle congruent to the angle in the
box at one of the new end points.

For A your picture should look like this.

4)   Repeat 2 and 3 as often as necessary.

5)   What happens? Compare your work with
someone else's. Did they get the
same figure?

Does your figure look something like this after step 3?
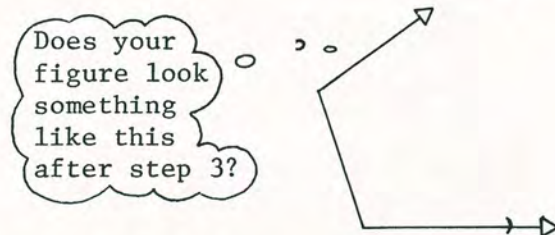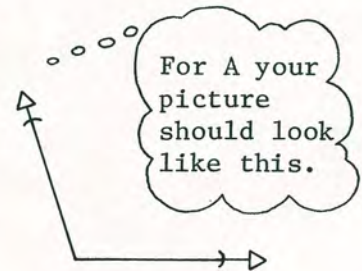
6)   What do you think will happen with box B, C and D? Try them and find out.

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CROSSING OVER

1 WHICH OF THESE PAIRS OF LINES APPEAR TO BE PARALLEL? _____

A   B   C   D
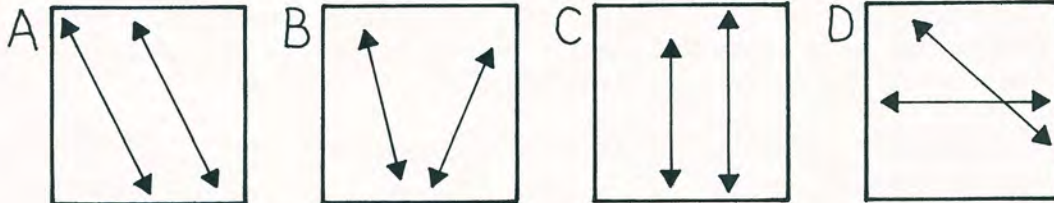
2
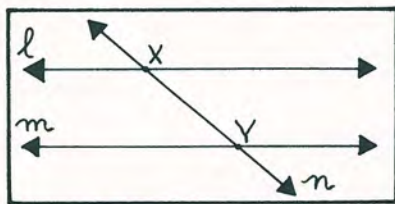
LINES l AND m ARE PARALLEL
LINES, n INTERSECTS THEM
BOTH. IT IS CALLED A
TRANSVERSAL. THE TRANSVERSAL
INTERSECTS LINE l AT POINT _____
AND LINE m AT POINT _____.

3
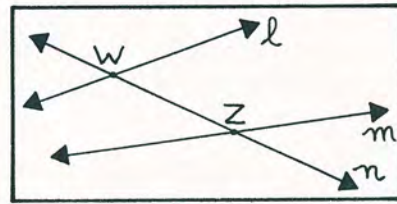
LINES l AND m ARE NOT
PARALLEL. LINE n IS A
TRANSVERSAL BECAUSE IT
INTERSECTS EACH LINE IN
A DIFFERENT POINT. THE
TRANSVERSAL INTERSECTS
LINE l AT POINT _____
AND LINE m AT POINT _____.

4 NAME A TRANSVERSAL OR WRITE "NONE" IF THERE IS NOT ONE IN THE
DIAGRAM.

A   B   C   D

_____   _____   _____   _____
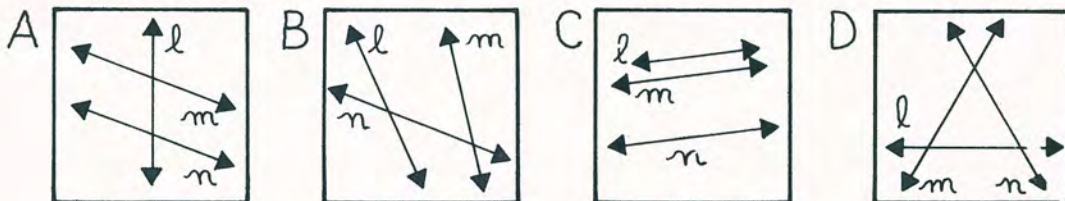
5 DRAW A TRANSVERSAL FOR EACH PAIR OF LINES BELOW.

A   B   C   D

280
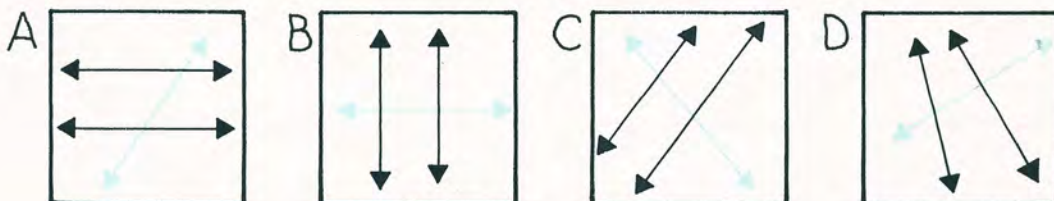
MATCH UP!

Match the phrases that mean the same by placing each letter in a blank by a number. Some blanks will have more than one letter.

_____ 1) Supplementary angles

_____ 2) Complementary angles

_____ 3) Two parallel lines

_____ 4) Two perpendicular lines

_____ 5) Parallel lines cut by a transversal

_____ 6) Collinear points

_____ 7) Concurrent lines

a) Points which lie on a straight line

b) A line intersecting parallel lines

c) Lines intersecting at one point

d) Two angles whose measures have a sum of 180°

e) Two angles which can be joined to form a right angle

f) Two lines on a plane which do not intersect

g) Two angles whose measures have a sum of 90°

h) Two lines meeting at 90°

i) Two intersecting lines forming 4 equal angles

j) Two angles which can be joined to form a straight angle

Match each phrase 1 to 7 and a to j with a diagram by placing each number and letter in an appropriate box.

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In this activity students are led to discover relationships between angles that are formed by two lines and a transversal. Students are given a sheet with figures--examples shown below.

Students are directed to compare two angles in one of the following ways.

1. Cut out one or two of the angles in each figure and compare them with the remaining angles.

2. Fold and cut angles from another piece of paper to use to compare.

3. Measure the angles with a protractor.

FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4
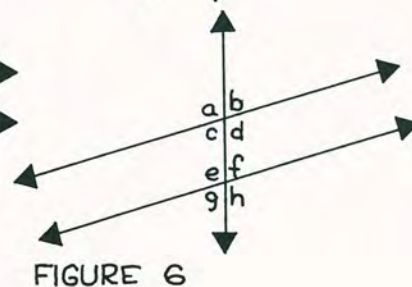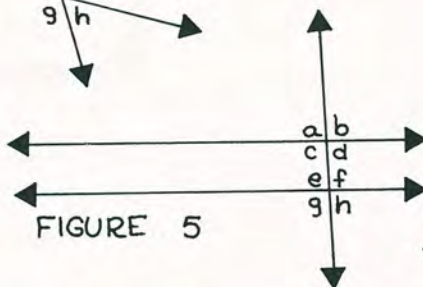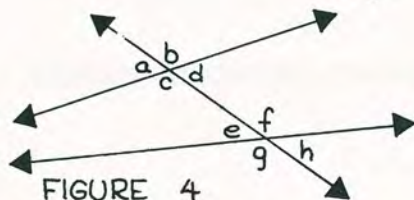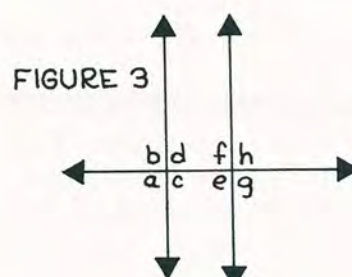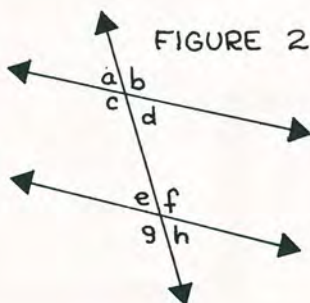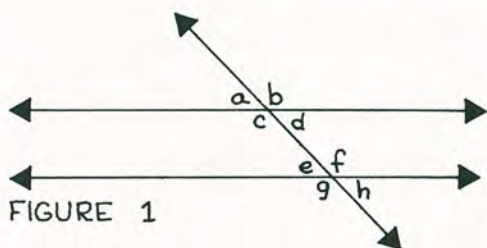
FIGURE 5

FIGURE 6

Questions like these could be asked.

1. In which figures are all the lettered angles congruent?

2. In which figures are $\angle$a, $\angle$d, $\angle$e, $\angle$h all congruent to each other?

3. In which figures are $\angle$b, $\angle$c, $\angle$f, $\angle$g all congruent to each other?

4. In figure 4 which angles are congruent?

5. Why aren't more angles congruent in figure 4?

6. Why are so many angles congruent in figures 3 and 5?

Names of various pairs of angles could be introduced at this point.

REPETITIOUS ANGLES I
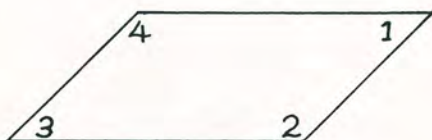
1 Do any of the four angles appear to be the same size? _____

Trace the figure. Fold or cut it to show that $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$.

2 The figure above was used four times to make the pattern to the right. The edges form two intersecting lines.
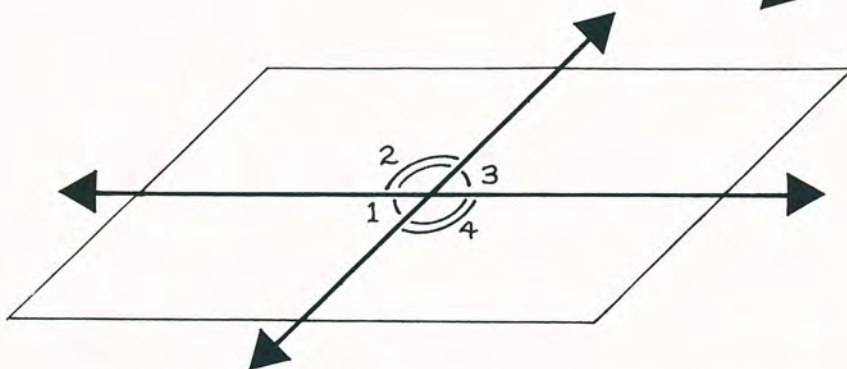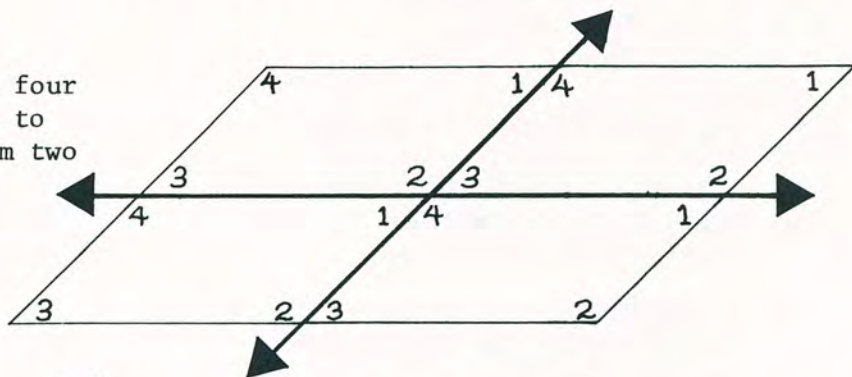
3 Look at the drawing on the left. Angles 1 and 3 formed by the two intersecting lines are called vertical or opposite angles. They are the same size.

What do you know about angles 2 and 4? _____

4 Next to each figure list all the pairs of vertical angles you can find for each figure.

FIGURE A
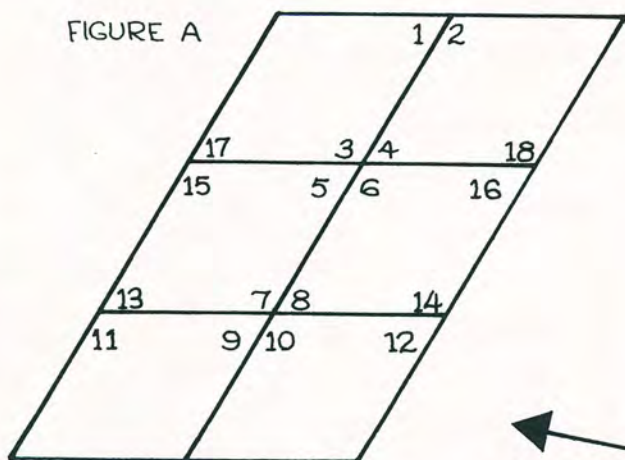
FIGURE B
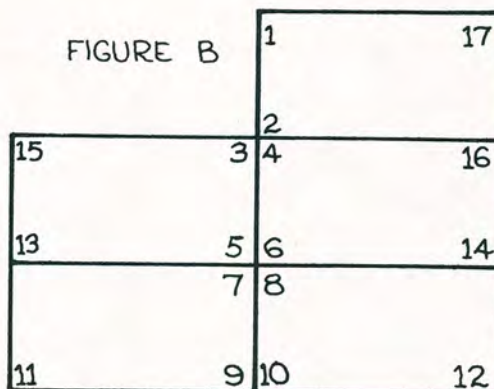
FIGURE C
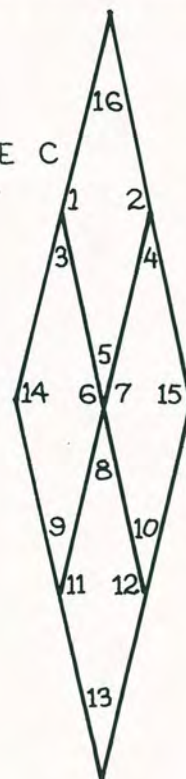
FIGURE D
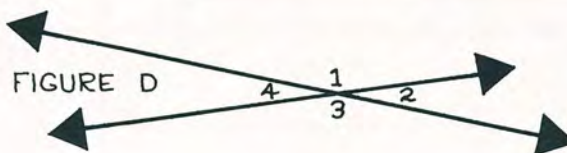
REPETITIOUS ANGLES II

An activity similar to *Repetitious Angles I* can be done to show congruence of various angles formed by two parallel lines and a transversal.

a) Take any parallelogram. Show that $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$.

b) The parallelogram above was used six times to make the pattern to the right. The edges form two parallel lines and a transversal.

c) Angles 1 and 3 indicated with ★ are called alternate interior angles.

d) Angles 3 and 1 indicated with ■ are called alternate exterior angles.

e) Angles 4 and 4 indicated with ▲ are called corresponding angles.
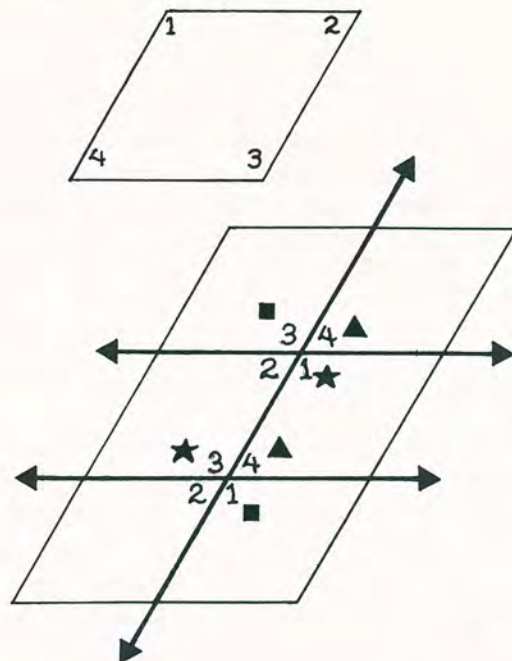
(a) and (b) demonstrate that, given two parallel lines cut by a transversal, the angle pairs in (c), (d) and (e) are congruent.

Students could then make their own tessellations, or be given some like the ones below, and identify congruent angles for each figure.
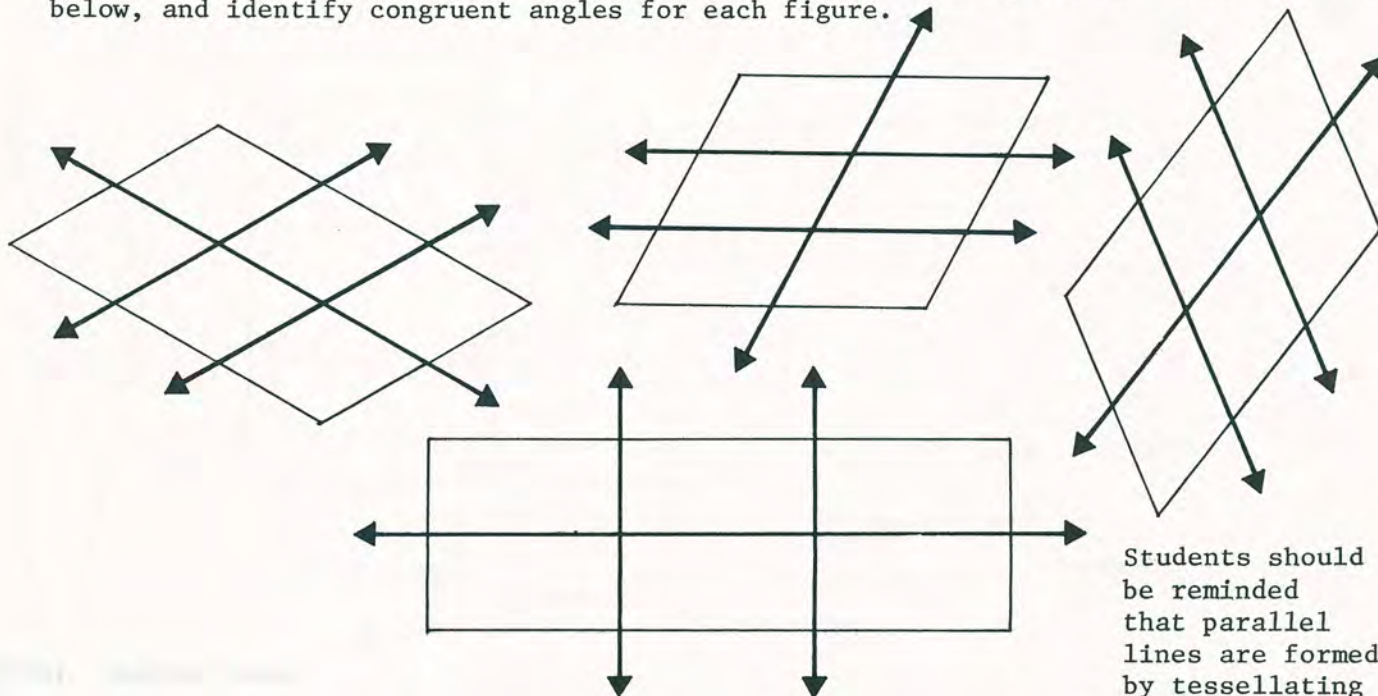
Students should be reminded that parallel lines are formed by tessellating a parallelogram.

KEEP YOUR EYE ON THE ANGLE
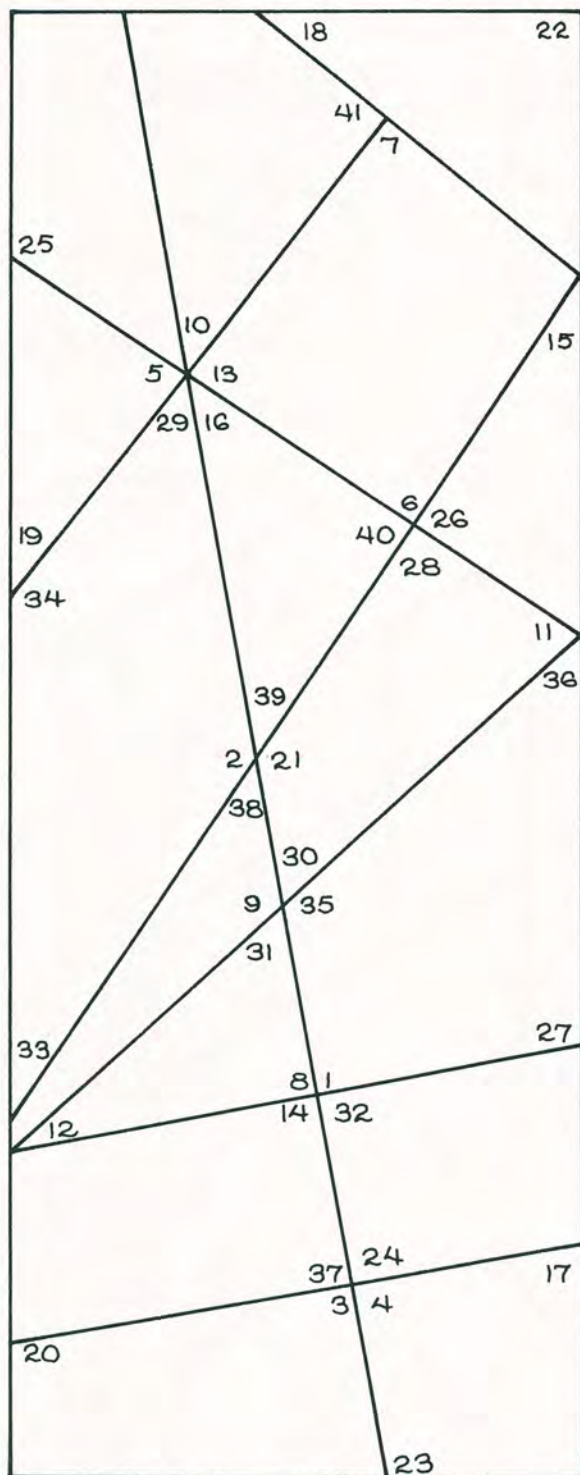
For each exercise try to find a vertical angle. Circle the correct answer in the chart. If no vertical angle is shown or if the correct answer is not given circle "none." Fill in the letters below to form a message.

Diagram labels: 18, 22, 41, 7, 25, 15, 10, 5, 13, 29, 16, 6, 26, 40, 28, 19, 34, 11, 36, 39, 2, 21, 38, 30, 9, 35, 31, 33, 8, 1, 14, 32, 12, 27, 37, 24, 3, 4, 17, 20, 23

		∠33	∠14	∠32	NONE
①	∠1 →	J	O	G	I
②	∠2 →	∠9 Z	∠39 H	∠21 C	NONE Y
③	∠3 →	∠37 K	∠24 A	∠4 X	NONE M
④	∠4 →	∠17 L	∠32 C	∠24 J	NONE T
⑤	∠5 →	∠10 I	∠13 R	∠19 A	NONE N
⑥	∠6 →	∠28 U	∠7 D	∠15 F	NONE H
⑦	∠7 →	∠10 A	∠41 P	∠22 K	NONE I
⑧	∠8 →	∠32 L	∠31 N	∠12 B	NONE G
⑨	∠9 →	∠38 R	∠35 G	∠30 L	NONE T
⑩	∠10 →	∠7 Q	∠13 B	∠29 N	NONE U
⑪	∠11 →	∠36 F	∠28 C	∠26 S	NONE E
⑫	∠12 →	∠33 D	∠8 E	∠31 V	NONE S
⑬	∠13 →	∠5 V	∠29 M	∠16 E	NONE W

__ __ __ __ __ __ __ __   __ __ __ __ __ __
13 11 5 4 7 2 3 8   3 10 9 8 11 12

__ __ __   __ __ __ __ __ __ __ __ __
3 5 11   2 1 10 9 5 6 11 10 4

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YOU CAN'T BUY PARTS FOR IT

VERTICAL

$\angle 1 \cong \angle 3$

AND _____ _____
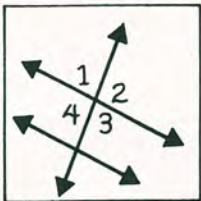
ALTERNATE INTERIOR
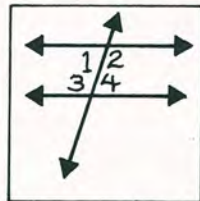
$\angle 1 \cong \angle 4$

ALTERNATE EXTERIOR
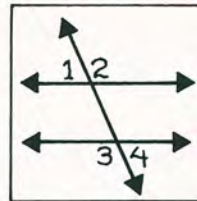
$\angle 2 \cong \angle 3$

CORRESPONDING

$\angle 7 \cong \angle 5$ $\angle 3 \cong \angle 7$

_____   _____
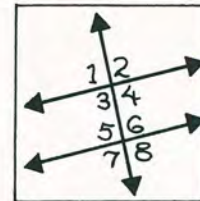
Cross out letters next to pairs of angles that are congruent. The remaining letters can be arranged to form two words.

Y̶ ∠1 , ∠2
Ø̶ ∠3 , ∠4
U ∠5 , ∠6

C ∠7 , ∠8
A̶ ∠9 , ∠10
N̶ ∠11 , ∠12
T ∠13 , ∠14

B̶ ∠15, ∠16
U̶ ∠17, ∠18
Y ∠19, ∠20

P ∠A , ∠B
A ∠C , ∠D
R̶ ∠E , ∠F
T ∠G , ∠H
S ∠I , ∠J

F ∠K , ∠L
O ∠M , ∠N
R ∠O , ∠P

I̶ ∠Q , ∠R
T ∠S , ∠T

THE LINES THAT LOOK PARALLEL ARE PARALLEL.
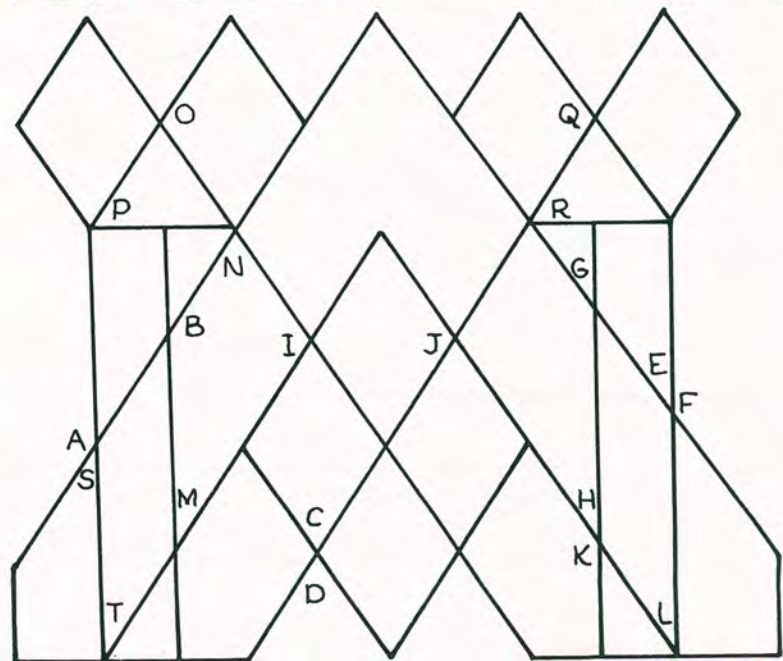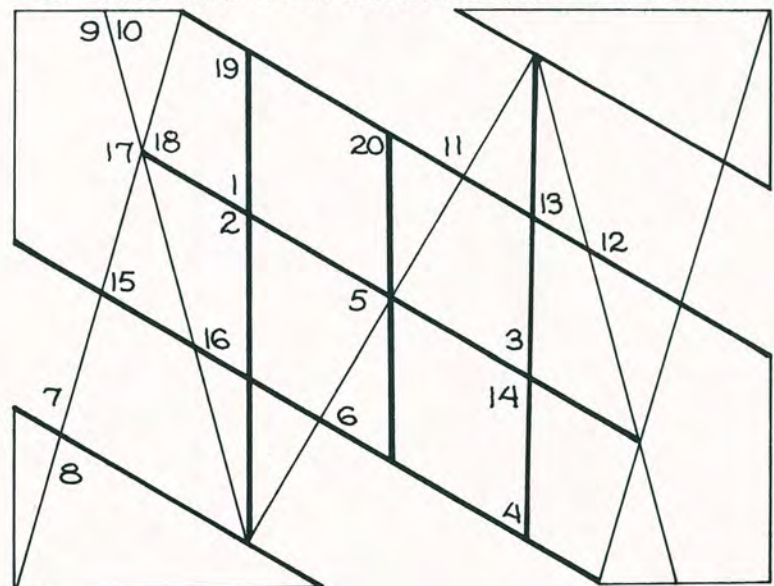
Y _ _ _   B _ _ _ _ _

AN ANGLE MEASURE

Angle measurement can be introduced by having students make their own protractors. Let each student draw his own unit angle. You might suggest that it be a small angle. This angle is to be marked off on a semi-circular shaped piece of tracing paper or transparency material. Both work well because students can see the angles and read the angle measures when the homemade protractor is placed on top of an angle. To save time the semi-circles could be cut out beforehand. Be sure to mark a dot at the midpoint of the diameter of the semi-circle.

a) Demonstrate the steps on the overhead as students work at their desks.
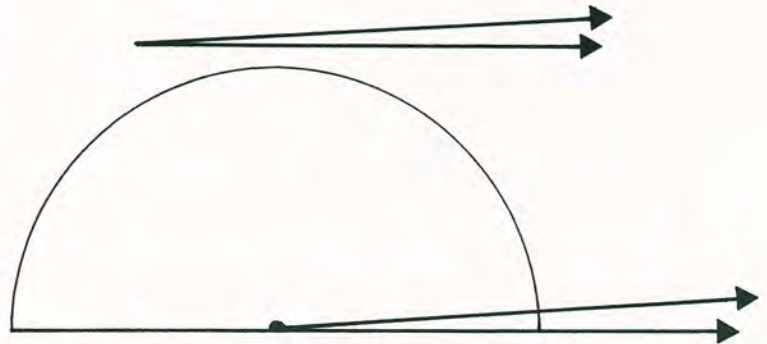
b) Have students place the tracing paper over the unit angle with the dot on the vertex of the angle and the lower side along the edge of the paper. Stress proper positioning.

c) The tracing paper can then be rotated around until the entire semi-circle is marked. Some students may have a part of a unit angle for the last division.
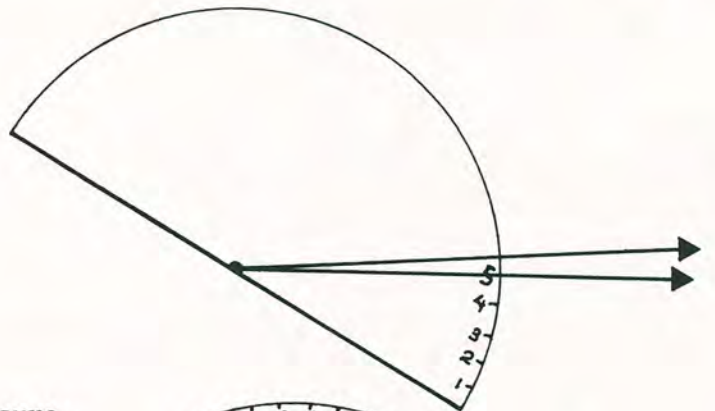
d) Provide students with angles to measure. Some of the angles could be the interior angles of a polygon. An angle's measure is easier to read if the sides of the angle extend beyond the edge of the protractor. Stress proper positioning of the protractor to get an accurate reading. Show that angle size can't be read directly by a random positioning of the protractor.
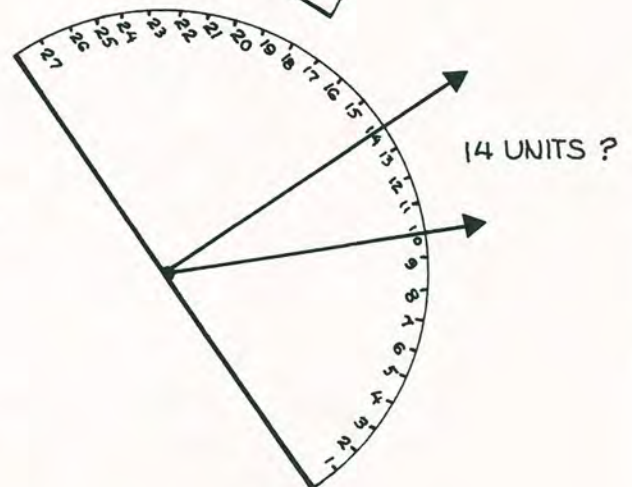
e) Have students compare their answers for the same angles. Are the angle measures the same. Discuss why or why not.

14 UNITS ?

ANCIENT ANGLES

A long time ago the Babylonians, who thought there were 360 days in a year, drew an angle that was 1/360 of a circle. This was their unit angle. It is now known that there is a little more than 365 days in a year but the Babylonian unit angle is still used as the basis for measuring angles or turns in a circle. The unit angle has a measure of 1 degree (1°).

Imagine two arrows pointing towards 0°.

Figure 1 shows a counter clockwise turn of 90° or a clockwise turn of 270°.

Figure 2 shows a counter clockwise turn of 225° or a clockwise turn of 135°.

FIGURE 1

FIGURE 2

Use the picture at the top as a guide and approximate the amount of turn shown in each figure. Give both a counter clockwise and a clockwise amount.

BE CAREFUL

PR**O**FESSIONAL TRACTORS
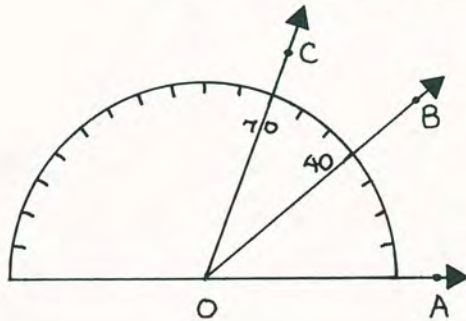
∠COA measures 70°. ∠BOA measures 40° so ∠COB measures 70° − 40° or 30°.

∠AOG measures _____°.

∠BOD measures _____°.

∠AOF measures _____°.

∠EOF measures _____°.

∠FOG measures _____°.

∠AOB measures _____°.
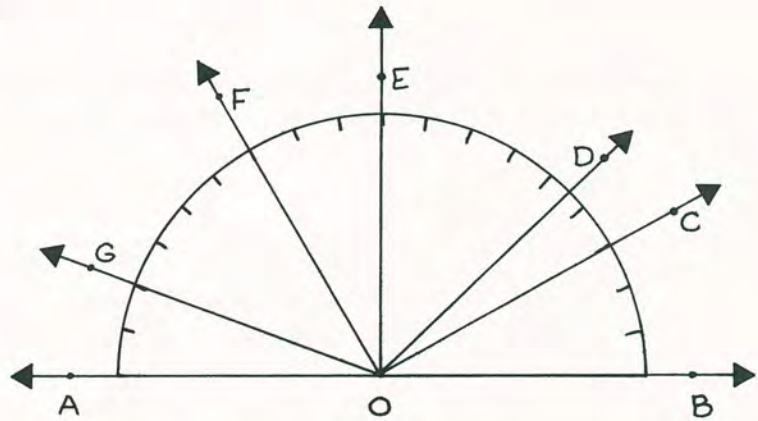
The measure of ∠EOF plus the measure of ∠FOG is the same as the measure of ∠ _____ or _____°.

The measure of ∠BOD minus the measure of ∠BOC is the same as the measure of ∠ _____ or _____°.

The measure of ∠COF minus the measure of ∠COD is the same as the measure of ∠ _____ or _____°.

The sum of the measures of ∠COD, ∠DOE, and ∠EOF is the same as the measure of ∠ _____ or _____°,

The measure of ∠AOF plus the measure of ∠COF minus the measure of ∠COD is the same as the measure of ∠ _____ or _____°.

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PROTRACTORS and POLYGONS

The <u>protractor</u> is a tool used to measure angles. On the protractor to the right one ray of the angle gives two numbers--50 and 130. Which is correct? Since the angle is an acute angle (smaller than a right angle), the correct answer is 50°.
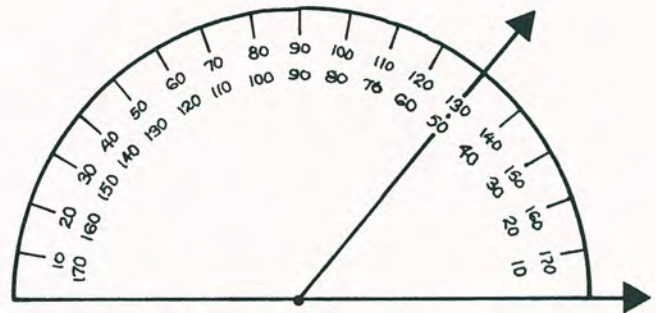
For the polygons below name each angle as acute, right or obtuse. Use a protractor to find the measure of each angle, and then find the sum of the angle measures.

∠1 _acute_   _50°_
∠2 _____   _____
∠3 _____   _____

SUM   _____

∠4 _obtuse_   _____
∠5 _____   _____
∠6 _____   _____
∠7 _____   _____

SUM   _____

∠8 _____   _____
∠9 _____   _____
∠10 _right_   _____
∠11 _____   _____
∠12 _____   _____

SUM   _____
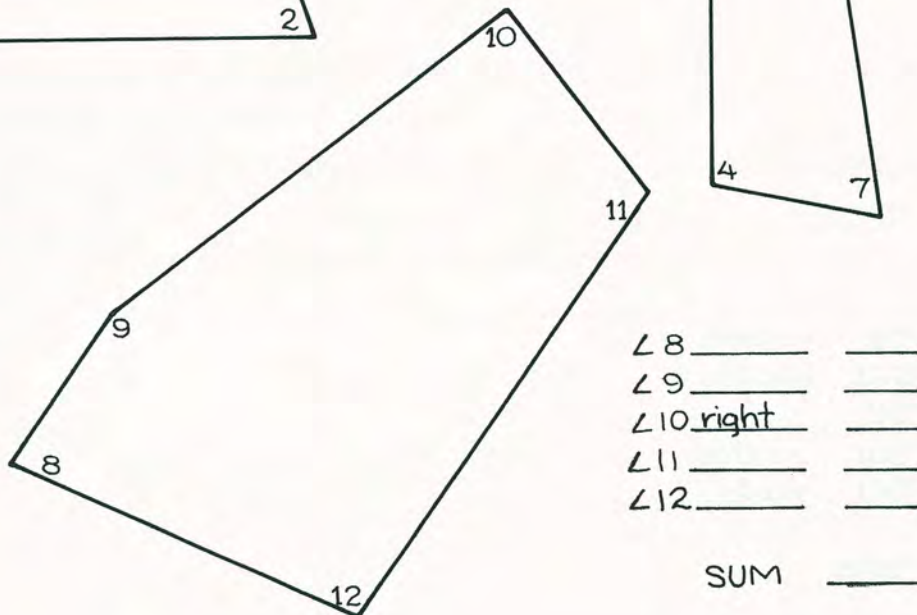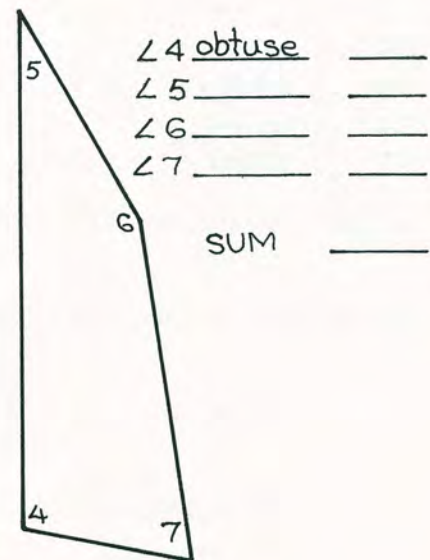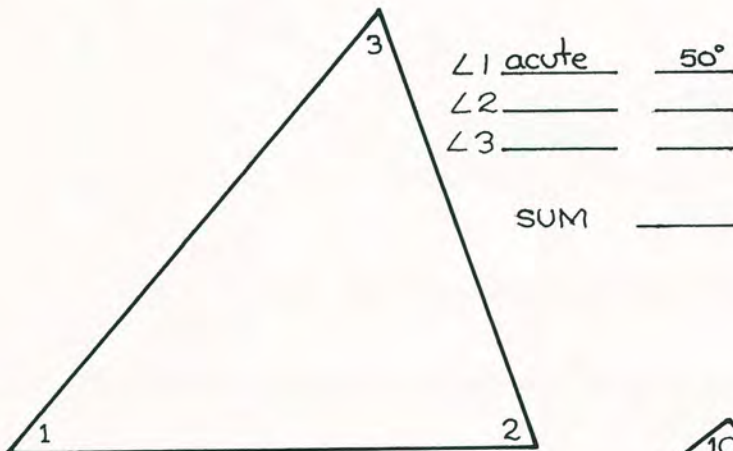
I'M SEEING STARS

Materials needed: Protractor and metric ruler

Activity:

1. Make a star that has
   a) sides that are 5 centimetres long
   b) outer angles that each measure 80°
   c) point angles that each measure 20°
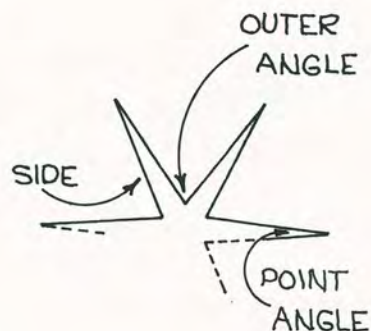
How many points does your star have? _____

Record all the information in the table below.

2. Make a star that has sides of 5 centimetres, outer angles that measure 30°, and point angles that measure 90°. Record in the table.

3. Make a star that has sides of 5 centimetres, outer angles that measure 110°, and point angles that measure 20°. Record in the table.

4. Make a star that has sides of 5 centimetres, outer angles that measure 120°, and point angles that measure 30°. Record in the table.

	Number of Points	Measure of Outer Angle	Measure of Point Angle	Outer Angle minus Point Angle	360 ÷ (column 4 number)
1.		80°	20°		
2.					
3.					
4.					

Predict the number of points that each of these stars would have.

5. Outer angle measures 92°; point angle measures 20°. _____

6. Outer angle measures 102°; point angle measures 30°. _____

7. Outer angle measures 140°; point angle measures 20°. _____

8. Outer angle measures 150°; point angle measures 30°. _____

Give the angles needed to make these stars.

9. 8 points, outer angle measures 65°, point angle measures _____.

10. 10 points, outer angle measures _____, point angle measures 30°.

Could you make a perfect star that has 7 points?

COMPASS BEARINGS

Navigators on ships and planes use angles in planning routes. Their angles depend on a compass which always points to the north.

All of their angles are measured <u>clockwise</u> from the <u>north</u>.

EXAMPLES: 45° 225° 280°
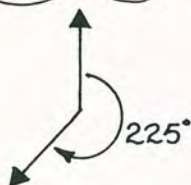
Use a compass to help you.

1) Stand facing north. Slowly turn your body 1/4 of a turn clockwise until you are facing east. You have made a 90° turn from the north. Your direction is called a <u>bearing</u> of 90°.

2) Face north. Make 1/2 of a turn clockwise to face south. You have turned to a bearing of
_____.

3) Face north. Make 3/4 of a turn clockwise to face west. You have turned to a bearing of
_____.

4) Pick out four objects in your classroom. Face north; then turn clockwise until you are facing the object. Write the approximate bearing for each object.
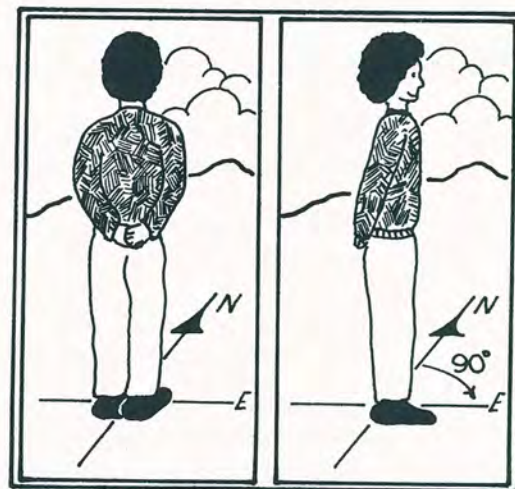
	Object	Bearing
a)		
b)		
c)		
d)		

5)
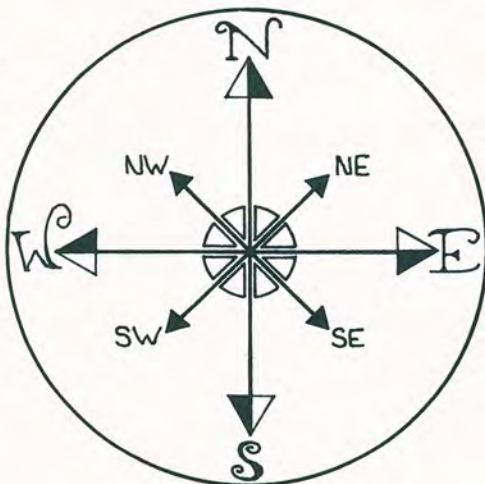
Use the picture of the compass and write the bearings represented by each of the directions.

a) NE

b) S

c) E

d) SW

e) SE

f) W

g) NW

CAN YOU DIG IT

Percy, the pirate, buried his treasure under an old oak tree. He left the code below for anyone that wanted to discover where the treasure is buried. Crack the code and, by starting at the last board on the boardwalk, find the tree that is guarding the treasure.

MY BARE RING WILL SHOW YOU
WHERE TO FIND THE TREASURE.

(4 cm, 070°) → (5 cm, 280°)

(2 cm, 300°) → (4 cm, 075°)

(4 cm, 345°) → (2 cm, 215°)

(3 cm, 335°) → (7 cm, 095°)

(3 cm, 285°) → (6 cm, 055°)

Percy, the Pirate.

START

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

CONTENTS

SYMMETRY AND MOTIONS

SYMMETRY

The study of symmetry begins in the early grades. Paper is folded in various ways and shapes are cut from the folded paper. In this way, children make hearts, "snowflakes" and paper dolls. The fold lines represent lines of symmetry. The heart has one, the "snowflake" shown has four and if the strip of paper dolls were to continue in both directions without stopping, it would have an infinite number.

Line Symmetry

There are many ways to create shapes with lines of symmetry. One simple way is to fold a piece of paper in half and tear a shape from the folded edge. The two resulting shapes each have a line of symmetry. Many other ideas are found on the page *Symmetry*. In a shape with line symmetry, each point on one side of the symmetry line corresponds to a point on the other side which is the same distance from the line. To help students understand this, shapes can be cut from grid paper. The squares give a quick way for determining distances. Students might like to color some of these grid designs. Can they color a design so it loses its symmetry if color is considered? Can they color a complicated design on one side and copy it on the other so the design stays symmetrical? The student pages *Flags of the World*, and *Trade Marks* asks students to find lines of symmetry in a figure. Other activities involving lines of symmetry can be found in the Polygons subsection.

Shapes which have two or more intersecting lines of symmetry can also be made by folding. It is interesting that in a shape with <u>exactly</u> two lines of symmetry, the two lines must be perpendicular. Your students might like to test this. They can cut a shape with one line of symmetry, <u>open it</u>, make a new fold and cut away the excess paper from both sides. They will find their shape is no longer symmetrical with respect to the first line unless the folds are perpendicular. An activity which develops this idea farther is *Angles and Folds*.

FIRST FOLD

SECOND FOLD
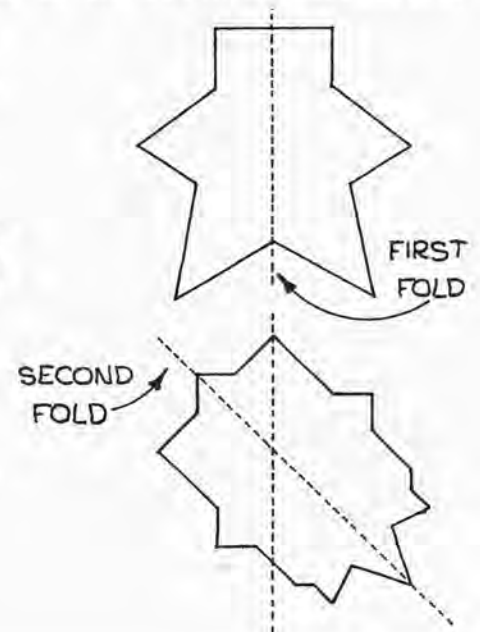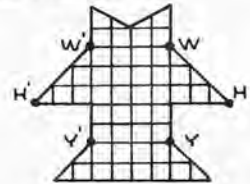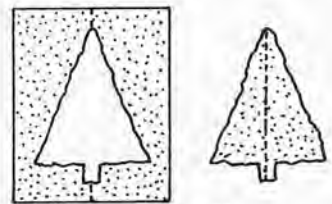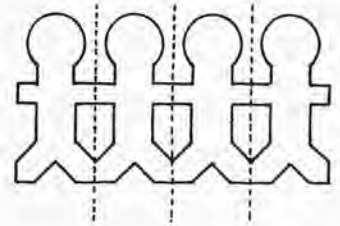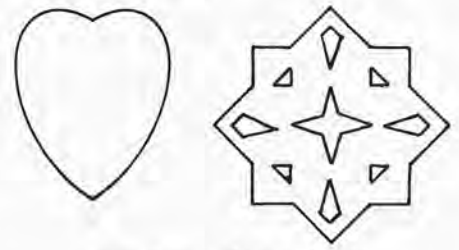
Rotational Symmetry

Figure A at the right with two perpendicular lines of symmetry also has rotational symmetry. If it is turned 180° about center point P, it will match the original shape. If it is turned 360°, each point returns to its original position. (Note: Any point on Figure A can be reflected across center point P to another point on Figure A. For example, M can be reflected across point P to M'. Notice that M and M' are the same distance from point P. Any figure for which this is true is said to have point symmetry. Figures A and C on this page have point symmetry; the other figures do not.) Figure B can be turned any multiple of 120° about point P and still "fit" on its outline. What is the least positive angle through which the snowflake can be turned before it matches itself? What other angles work? Two different turns less than or equal to a full turn of 360° rotate Figure A onto itself. There are 3 possible turns for Figure B and six for Figure C. The student pages *Pattern Blocks III* and *The Turning Point* in the Polygons subsection provide some introduction and practice of these ideas.

Often it is easier to decide how many different turns (less than or equal to a complete turn) will rotate a figure onto itself than to determine the angles of rotation. For the five-pointed star at the right, your students might be able to determine that there are five possible turns about point P and yet be unable to determine the angles of rotation. You could show the five-pointed star on the overhead with rays from the point of rotation as shown at the right. All five angles are equal so each must measure $\frac{360°}{5}$ or 72°. The angles of rotation are 72°, 144°, 216°, 288° and 360°. What are the angles of rotation for a six-pointed star? For a regular polygon with 3, 4, 5 or 6 sides?

FIGURE A

FIGURE B

FIGURE C
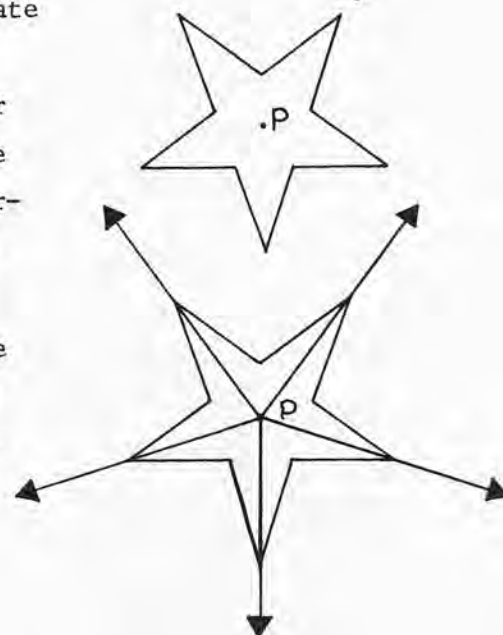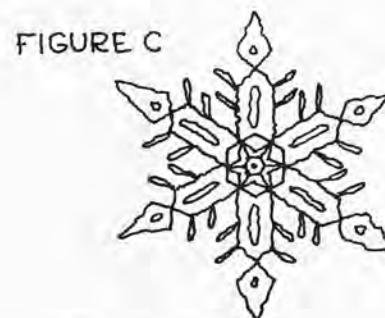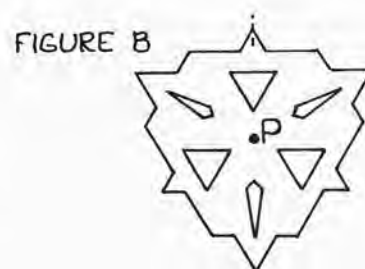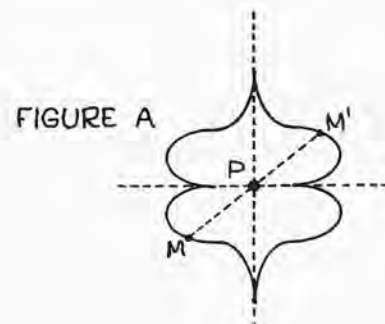
Multiple spirals as shown to the right are interesting shapes which have rotational symmetry but no line symmetry. The first has five different turns, the second has eight. If the two patterns are superimposed, how many turns are there for the new shape? The first shape needs to be rotated a multiple of $\frac{360°}{8}$ or 45° to "fit". A rotation for the combined patterns must be both a multiple of 72° and of 45°. The least common multiple of 72° and 45° is 8 x 9 x 5 or 360°, so only a complete turn rotates the figure onto itself. Interestingly, those kinds of multiple spirals with number of turns 3, 5, 8, 13, 21, 34, . . . (the Fibonacci sequence) occur frequently in nature. Sunflowers and the centers of daisies have the spirals superimposed as shown above. You and your students might enjoy the colored picture of a daisy spiral on page 93 of Mathematics from the Life Science Library. The basic spirals for the daisy center are also sketched on that page. A challenging question for investigation is, "For which pairs of superimposed spirals are there two or more angles of rotation less than or equal to 360°?"

Do figures with rotational symmetry have to have line symmetry? You could challenge your students to find or make figures with rotational symmetry but with no lines of symmetry. One of the objects at the right has both rotational and line symmetry, one has only line symmetry (except for the trivial 360° rotation) and one has only rotational symmetry. Can you identify them? You might like to develop an activity card or transparency which asks students the same type of questions.

HORSESHOE CRAB

FLOWER

PROTOZOA
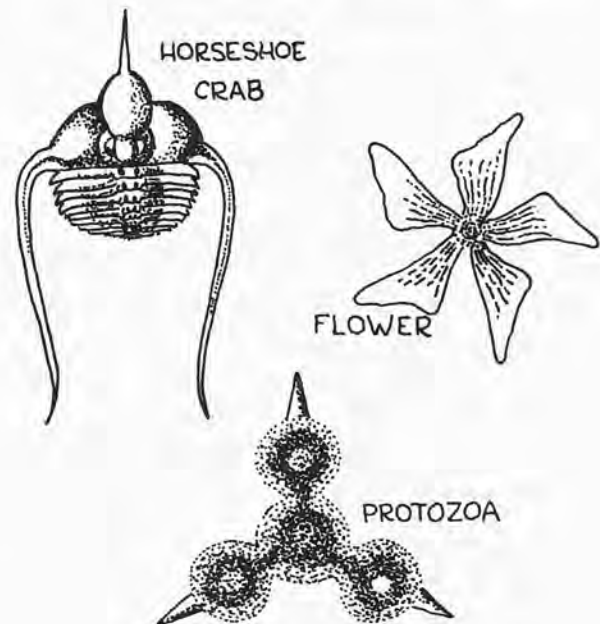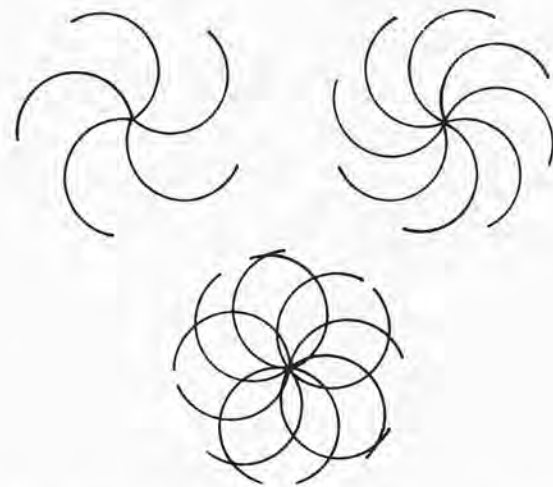
MOTIONS

Three types of elementary motions of geometry are reflections (flips), rotations (turns) and translations (slides). Each of these has been applied to the ⌐F below.

REFLECTION (FLIP)
ACROSS LINE ℓ.

ROTATION (TURN)
ABOUT POINT P.

TRANSLATION
(SLIDE)
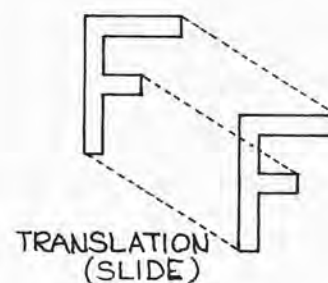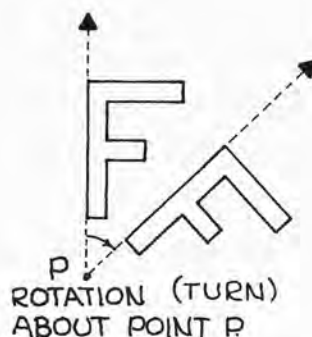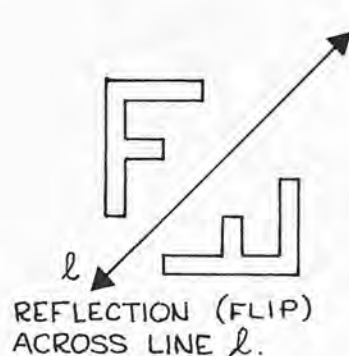
Reflection

Folding and cutting a figure with line symmetry is much like reflecting a shape across a line. (See *Reflection Methods*.) Each point P has a reflection point P' which is the same distance from the axis of reflection. This axis is the perpendicular bisector of line segment PP'. The student page *Reflections* : ƨnoᴉƚɔɘlʇɘЯ asks students to decide if one figure is a reflection of the other. The students could work the page informally by using a mirror, mira (a manipulative available from several commercial sources) or a piece of sheet plastic. More formally, they could use a ruler and a right angle to check for equal distance and perpendicularity. It is best to give students some experience with reflection instruments such as mirrors, miras or plastic pieces before asking them to use only visual means of recognizing reflections.
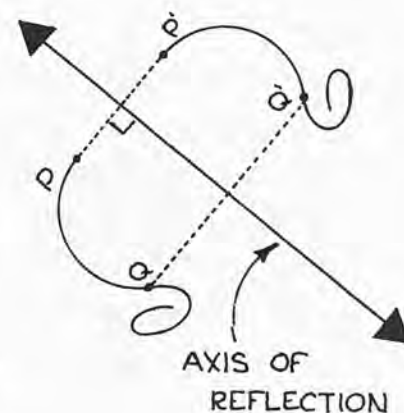
After students have learned to recognize the reflection of a shape, they can try various methods for drawing reflections. You might want to start students with grid paper reflections. Some fairly simple designs can be reflected <u>left and right</u> across a vertical line.

AXIS OF
REFLECTION
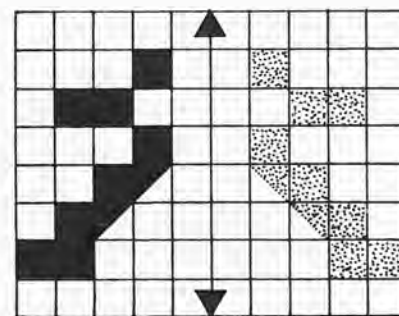
Some of the reflections problems could include reflections across horizontal lines (both up and down). (See *Grid Reflections*.) Reflecting across an oblique line is usually a challenge. Another challenge is to draw freehand reflections as on the page *Freehand Reflections*. Some of your students might be frustrated with designs that are too complicated. You can make simple designs for these students. Others might go on to create their own imaginative designs. Colored pens or pencils add much more visual impact and the students' work could make a nice display.

It is not too difficult to determine an axis of reflection (or line of symmetry) for Figure A, but it is more difficult to determine a line so Figure B' is a reflection of Figure B across that line. The student worksheet *Where's That Line?* has students determine the line of symmetry for two figures. For some students or classes it might be best to have them simply sketch the line of symmetry where they think it would occur. Those students who have been introduced to constructions could pick a pair of corresponding points P and P', draw line segment PP' and construct its perpendicular bisector. Such a page could provide a motivation for practice in constructing perpendicular bisectors, a necessary skill for many discovery activities in triangles.
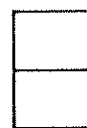
FIGURE A

FIGURE B

FIGURE B'

Rotations

Rotating a figure about a fixed point can be illustrated with the aid of an overhead projector. First draw a simple shape like the S shown to the right on a transparency. Choose and label a point on the transparency. Take a second transparency, place it over the first and trace the shape and the point. With both
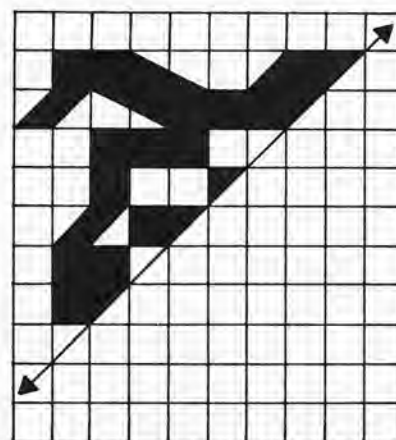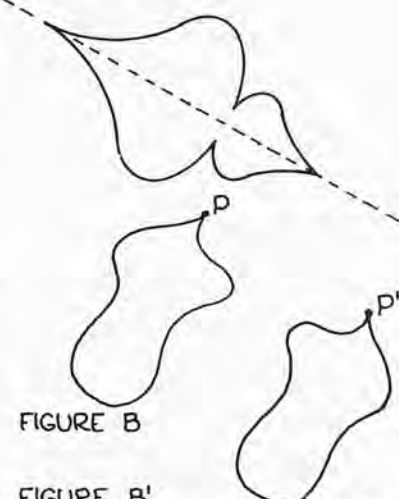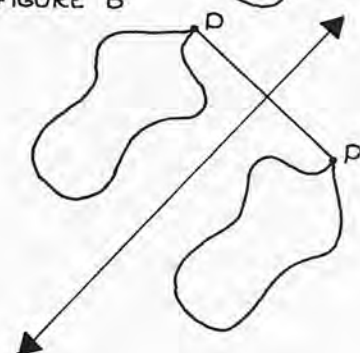
P.

transparencies on the overhead projector,
turn only the top transparency while
keeping the points superimposed. Stu-
dents will be able to observe the shape
rotating about the chosen point. Any
one of the rotated shapes is called a
rotation of the original shape about
point P. Students can probably observe
the circular pattern being created by
the rotations. They can try rotating
one figure about several different
points and comparing the results.

Translations

     Every point in the figure KITE at the right corres-
ponds to a point in figure K'I'T'E'. The distance
between corresponding points is constant and the lines
containing pairs of corresponding points are parallel.
The figure looks as though it has been "slid" to the
lower right. K'I'T'E' is called a translation of
figure KITE.

     The pages in this section which cover translations
do so with the aid of grids. (See especially *A Shift
in the Right Direction*.) It is fairly simple for stu-
dents to "move" the irregular octagon to the right six
units in a horizontal direction when a background grid
is provided.
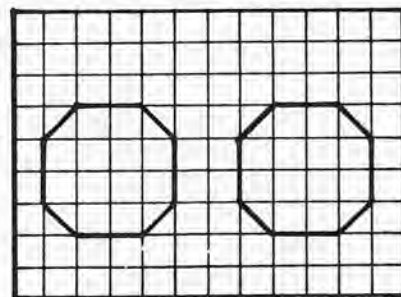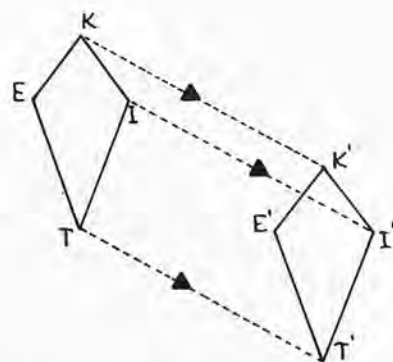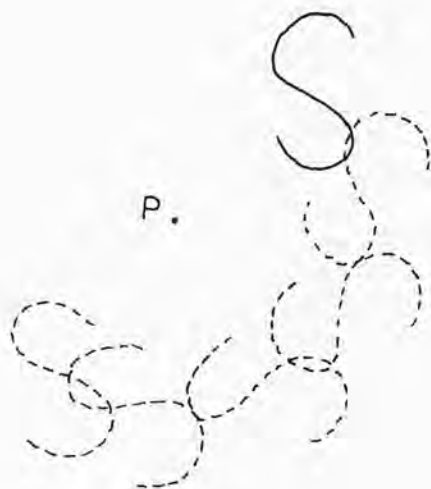
CONCLUSION

     Symmetry and motions are important topics in geometry and students usually enjoy
the related informal activities. The important relation congruence is closely tied
to motions and under the more formal name of transformations, motions have become
an important part of many high school geometry courses. Symmetry is of interest
because of its importance in art, nature and geometry. Besides the symmetry of two-
dimensional shapes discussed here, the symmetry of some three-dimensional shapes is
covered in other parts of this resource. Other activities involving symmetry and
motions are listed in the SECONDARY EMPHASES section.

SYMMETRY

To introduce the concept of line symmetry students could be given the opportunity to create symmetrical figures. Several ideas and selected sources are given below.

1) Have students fold a piece of paper. Make a random cut through the folded paper and a symmetrical figure will be formed.
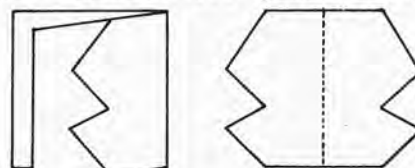
2) Students could first draw a pattern along the fold line and then cut along the outline to make the figure. Make a valentine using this method.
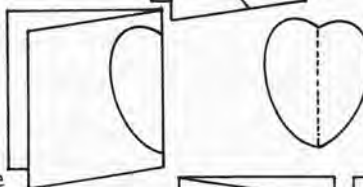
3) After folding the paper students could use a pin or compass point to punch tiny holes through both pieces of paper. Unfolding the paper will reveal a symmetrical figure. (See Mirror Magic, by Janet S. Abbott.)
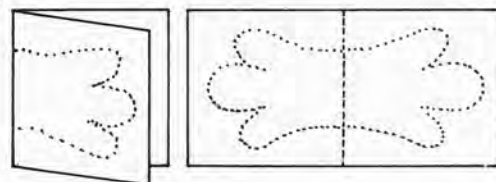
4) Students can mark points on a grid paper. By using a line of symmetry the images of the points can be marked. Then connecting the points in order and the images in order will produce a symmetrical figure. (See Reflections and Rotations, Oakland County Mathematics Project.)
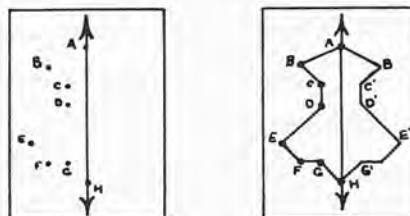
5) Students can fold a piece of paper and then place some spots of ink between the two pieces. By pressing the paper together an ink blot will be formed that will be symmetrical. (See The School Mathematics Project, Book A.)
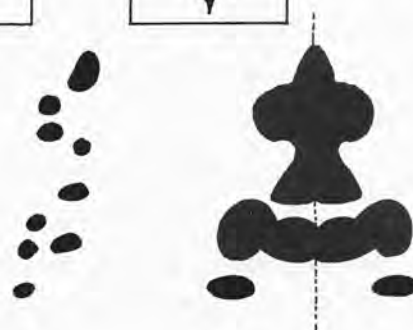
6) Place a piece of carbon paper face down on top of a sheet of paper. Fold the paper so the carbon is inside. Use the fold as an edge and draw a design on the outside of the paper. Unfold the paper and remove the carbon paper. A symmetrical figure can be seen on the inside of the paper.
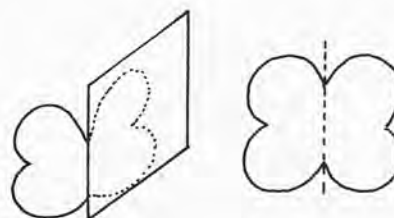
7) A transparent piece of plastic (or the commercial product Mira) can be placed next to a figure and the reflection of the figure can be drawn by looking through the plastic and tracing the image. (See Mira Math for Elementary School.)
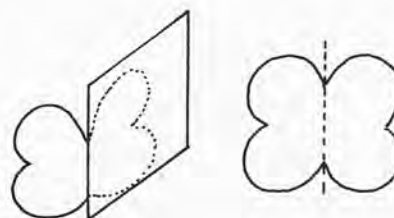
These methods help students gain an intuitive feeling for line symmetry. Do you have other ways of having students make line symmetric figures?

Many companies have trademarks that have lines of symmetry. Several are shown below. Many other examples can be found in newspapers, magazines, and the yellow pages.
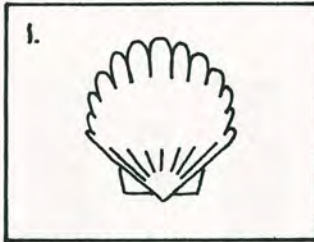
1.	2.	3.	4.
Shell Oil Co.	The Bell System	Blue Cross of Oregon	Metropolitan Life Insurance
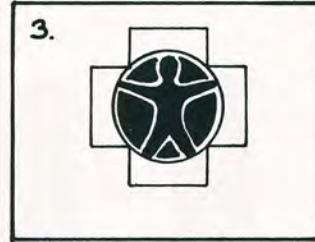
5.	6.	7.	8.
Volkswagen of America	Chrysler Corporation	International Harvester	American Automobile Association
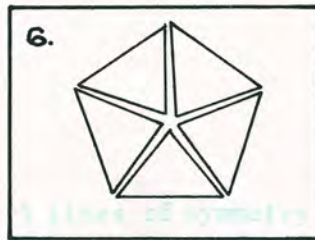
9.	10.	11.	12.
American Airlines	The Yellow Pages	Chevrolet	Safeco Insurance
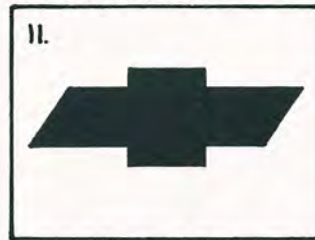
Activities for the students might include:

1) A worksheet where students decide if a trademark has a line(s) of symmetry and then sketch the line(s) of symmetry on the trademark.

2) A cut and paste activity where students search newspapers, magazines or the yellow pages to find examples. If point, line and plane symmetries have been studied the examples might be grouped according to each type of symmetry.

For any of these activities students could sketch the line(s) of symmetry on the trademarks. Non-examples of line symmetry such as the Chevrolet trademark (point symmetry), the Safeco Insurance trademark (one part has line symmetry and one part has point symmetry), or trademarks with no symmetry could be included.

Two excellent sources of trademarks are <u>Trademarks:A Handbook of International Design</u> by Peter Wildbur and <u>Trade Marks & Symbols</u>, Volume 2 by Yasaburo Kuwayama, both published by Van Nostrand Reinhold Co.

FLAGS OF THE WORLD

UNDER EACH FLAG WRITE THE NUMBER OF LINES OF SYMMETRY THE FLAG HAS.
THEN SKETCH THE LINES OF SYMMETRY ON THE FLAGS.

ALBANIA	BARBADOS	BURUNDI	CANADA
1	1	1	1

GREAT BRITAIN	HONDURAS	INDIA	IRAQ
2	1	2	1

JAMAICA	KOREA (SOUTH)	MAURITANIA	PANAMA
2	NONE	1	NONE
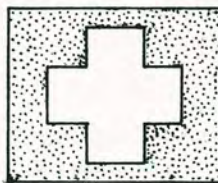
SWITZERLAND	YUGOSLAVIA
2	1

LOOK IN AN ENCYCLOPEDIA AND FIND OTHER FLAGS THAT HAVE LINES OF SYMMETRY.
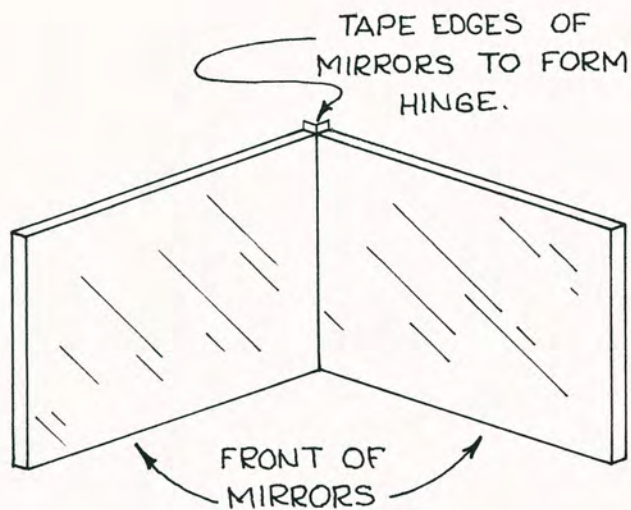
TAPE EDGES OF
MIRRORS TO FORM
HINGE.

FRONT OF
MIRRORS

CLASS ACTIVITY

Materials: hinged mirrors; colored pieces;
             protractor

1) Place your mirrors so that they line up
    with the angle drawn below. Put some
    colored pieces between the mirrors. How
    many lines of symmetry does the resulting
    design have? ___4___   Move the pieces
    around. How many lines of symmetry does
    the new design have? ___4___   Will moving
    the pieces ever change the number of lines
    of symmetry? ___No___   What is the measure
    of the angle? ___45°___
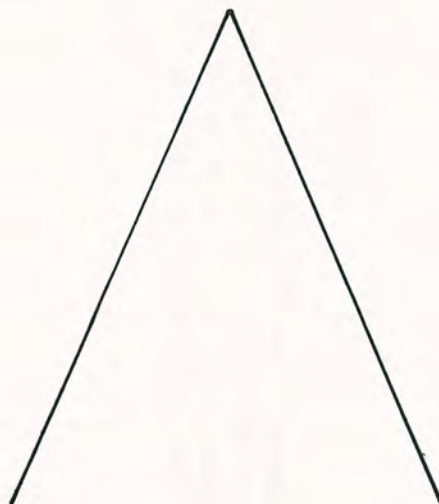
2) Measure the following angles and use your mirrors to determine how many lines of symmetry they generate. Record your results in the table. Use this information to fill in the rest of the table.

ANGLE	NUMBER OF LINES OF SYMMETRY
180	1
90	2
60	3
45	4
36	5
30	6
22.5	8
20	9
10	18
$\frac{1}{2}$	360
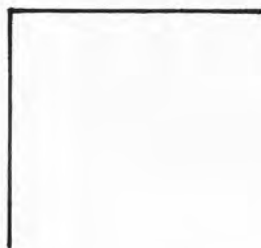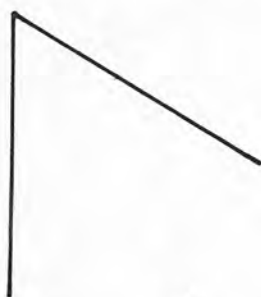
PREDICT FOR THESE TWO ANGLES { 10, $\frac{1}{2}$

CLASS DISCUSSION

1) If you wanted 12 lines of symmetry, how would you determine the size of the angle between your mirrors?

2) If you wanted a design with N (where N can be any number) lines of symmetry, how would you determine the size of the angle between your mirrors?

3) Describe what happens to the line symmetry when the mirrors are placed at an angle of 85°.

ANGLES & FOLDS

Materials: Scrap paper, scissors, straightedge, protractor

Have your students:

1   Draw 2 lines across their papers
    intersecting at a 60° angle.
    Accuracy is important. See figure 1.

2   Sketch an irregular figure keeping
    away from the intersection point.
    See figure 2.

3   Carefully fold the paper on one of
    the lines and cut along the outline
    of the figure. Turn the paper over
    and cut along the outline that
    remains. See figure 3.

4   Open the paper and observe the
    symmetry.

5   Fold along the other line and cut
    away the portion that is not doubled.
    Turn the paper over and cut off the
    undoubled part.

6   Open and observe.
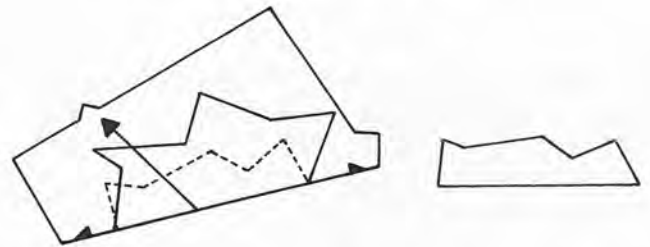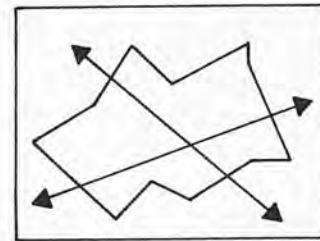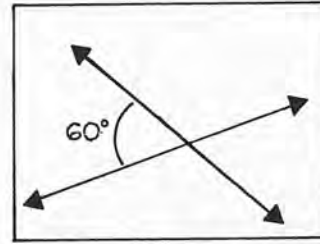
Ask your students how many times they think they will be able to keep folding and
cutting off excess paper. Have students keep track of the number of folds.

7   Continue folding and cutting until you are finished.

This should take a maximum of three folds. If it takes more the student probably
didn't fold carefully, cut away on both sides, or draw the angles carefully. To
decrease the chances of error you might want to ditto off the angles on scrap paper or
even ditto off a figure and angle the first time the students try it. This way they
should all come out with the same figure after folding and cutting.

The steps could be repeated with a 45° angle. This should take a maximum of four
folds. Students could then predict the maximum number of folds for a 90°, 20°, 30°,
10° angle.

Setting up a table would help.

ANGLE	MAXIMUM NUMBER OF FOLDS
45°	4
60°	3
90°	.
°	.
°	.
°	.

Two figures can also have a line of symmetry--if one figure <u>exactly fits</u> onto the other figure when it is "flipped" across the line of symmetry.
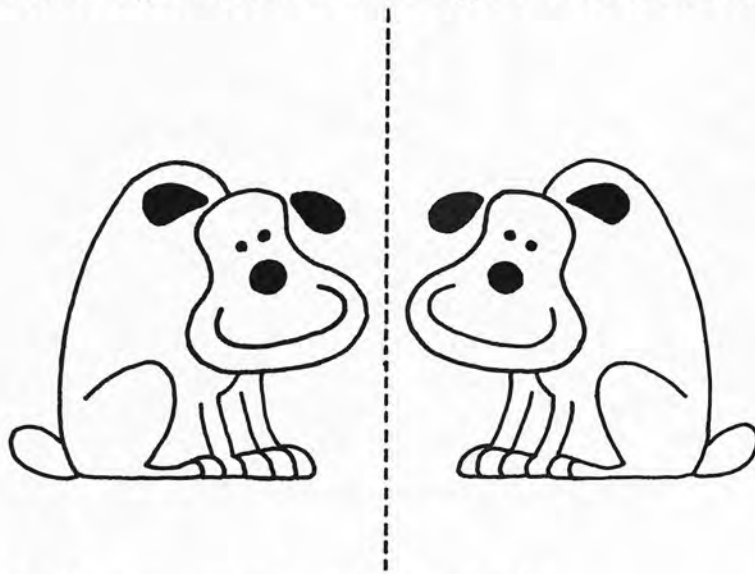
If the dogs exactly fit when folded over the line, one is called a reflection of the other. The line of symmetry is also called a line of reflection.
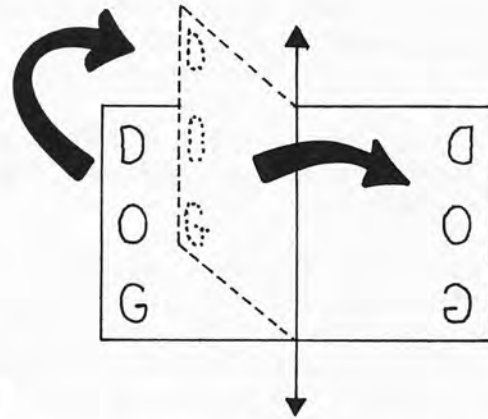
Imagine the word "dog" being "flipped" across the dotted line. Would it exactly fit the other "dog"? Try it by folding the paper, using a piece of plastic, or using a mirror.

For each pair of figures decide if one is a reflection of the other. You may use a piece of plastic or a mirror, or fold the paper to help you decide.

1.

3.

4.

308

5.

6.

7.

8.

9. M M
A A
T T
H H

10. bid | bid

11.

12.

13.

14.

REFLECTION METHODS

A) A method for drawing reflections uses grid paper. See *Grid Reflections*.

B) Two other methods use tracing paper.

1) Have students trace the figure and the line of reflection on tracing paper. Then fold the paper along the line of reflection with the figure on the outside. Tracing over the figure produces the reflection which can be seen by unfolding the paper.
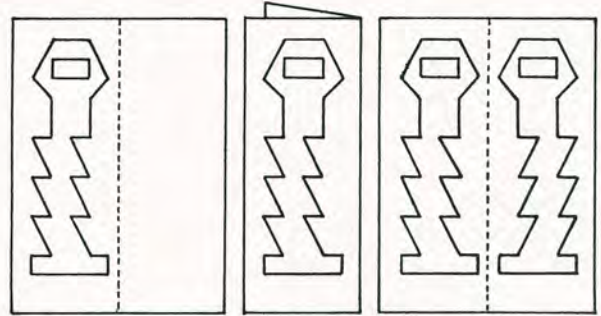
2) Mark a reference dot on the line of reflection. Place tracing paper over the figure and trace it, the line of reflection, and the reference dot. Flip the tracing paper across the line of reflection and line up the lines and the reference dots. Retrace the figure pressing down hard to leave an impression on the original paper. Remove the tracing paper and mark over the impression.
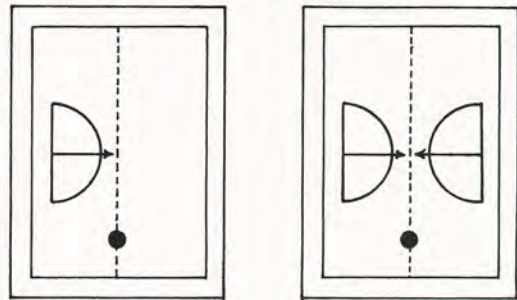
C) Fold the paper along the line of reflection. Then using a pin or compass point punch tiny holes through the vertices of the figure. Unfold the paper and connect the pin holes to produce the reflection. More pin holes will be necessary if the figure contains curved lines.
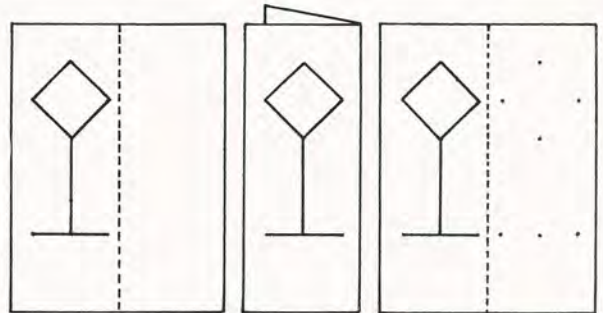
D) Place a piece of plastic along the line of reflection perpendicular to the paper. Look through the plastic and with a pencil carefully trace the image as it appears through the plastic. Straight line figures can be done by marking the vertices and then connecting the points.
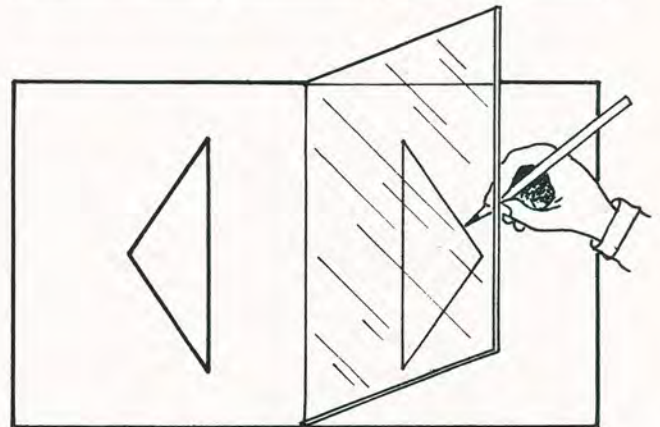
IDEA FROM: *Mira Math for Elementary School*
Permission to use granted by Creative Publications, Inc.

GRID REFLECTIONS

Draw the reflection of each design across the given line.
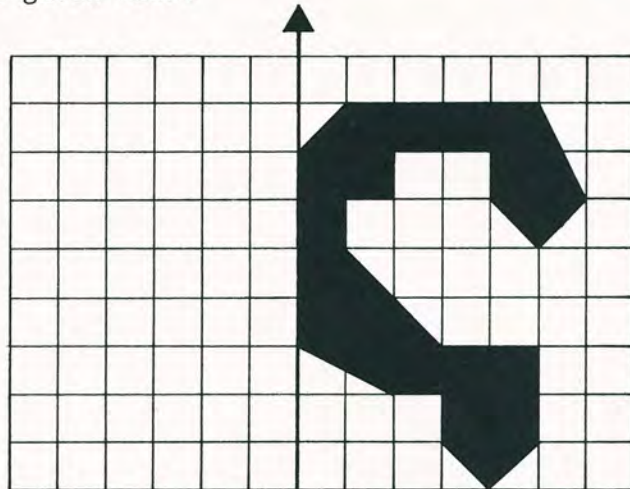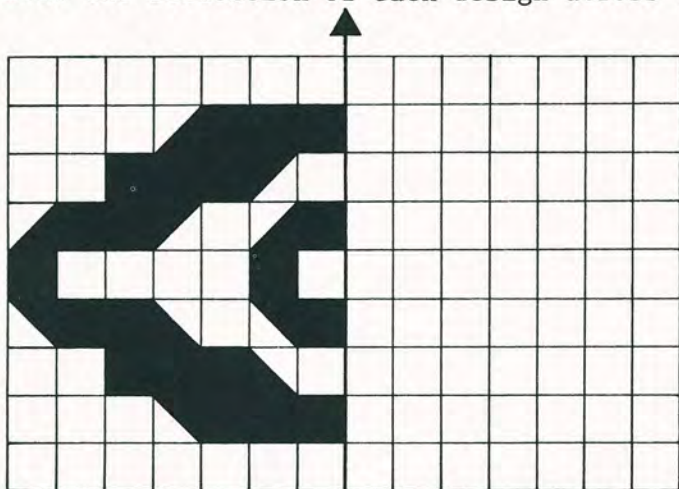
Draw the reflection of each design across the given line.
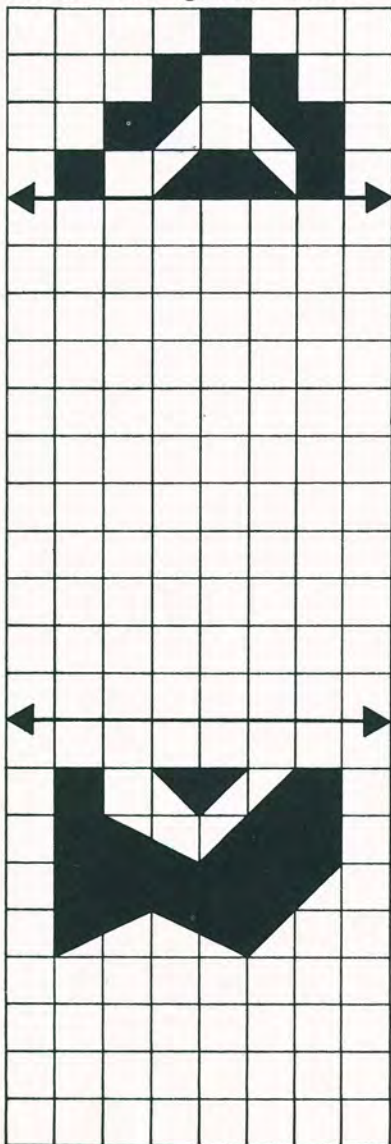
Reflect each face into the empty box beside it.
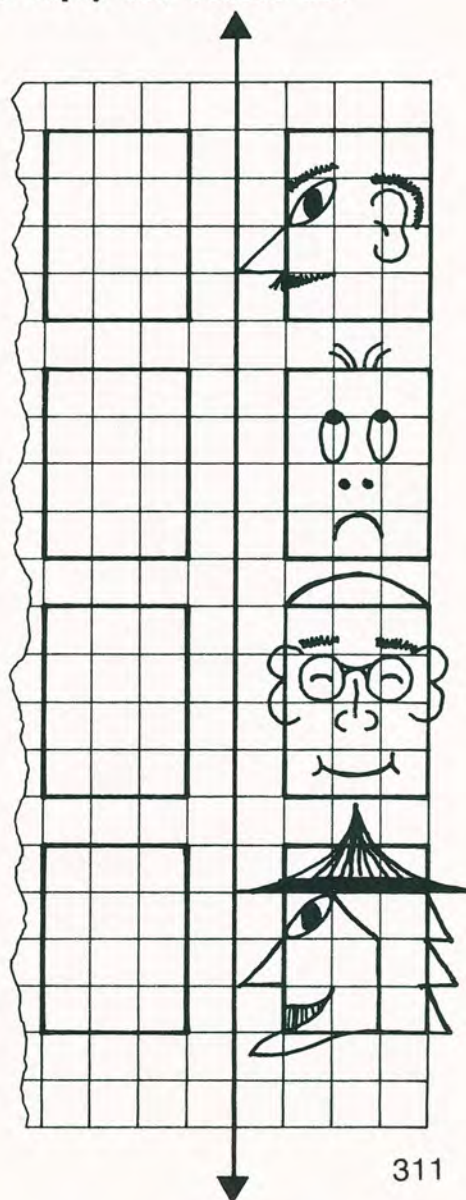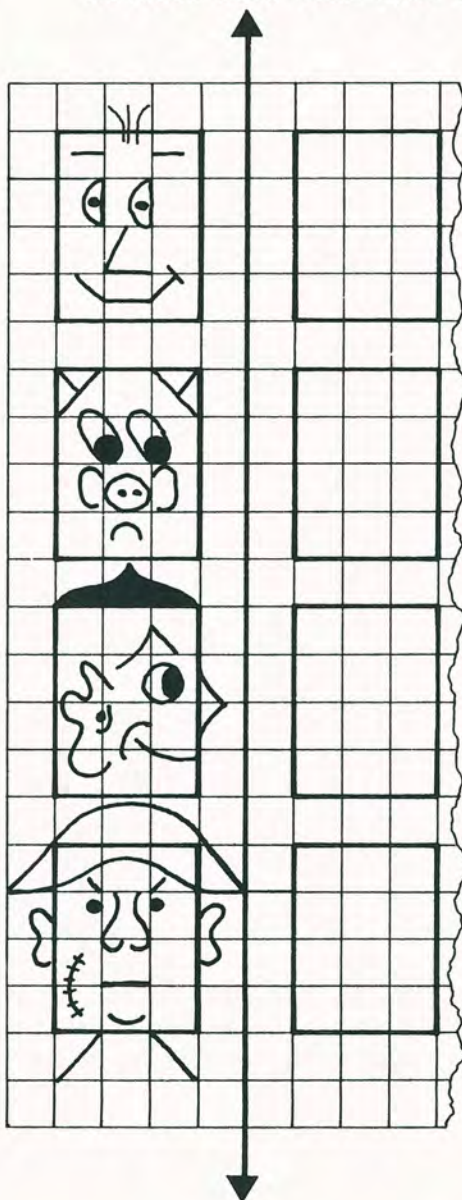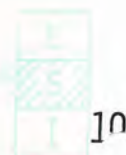
For each of these figures sketch its reflection about the line of reflection. Do each one freehand. Check your accuracy by folding, using a plastic piece, or using a mirror.

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

11.

12.

13.

14.

15.

16.

17.

18. Write the reflection of your name. Try it with printing and with cursive writing.

IDEA FROM: *The School Mathematics Project,* Book C

WHERE'S THAT LINE?
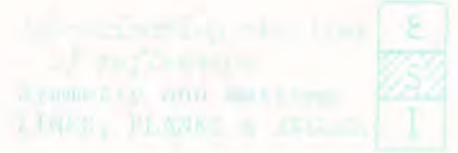
Approximate the line of reflection for each pair of figures

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WHICH WAY WILL THE ARROW POINT?

Suppose a piece of carbon paper was placed carbon side down on a sheet of paper and the paper was folded in half with the carbon inside and then in half again. By pressing hard with a pencil or pen the arrow shown in the diagram will be drawn in all four sections of paper. Choose the view you would see if you made the given mark, unfolded the paper and removed the carbon paper.
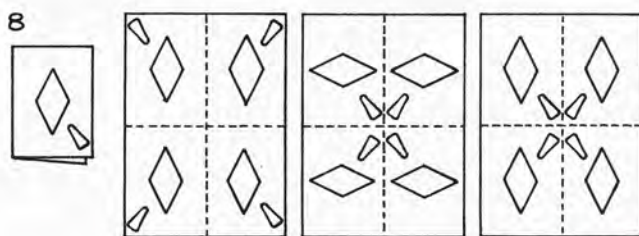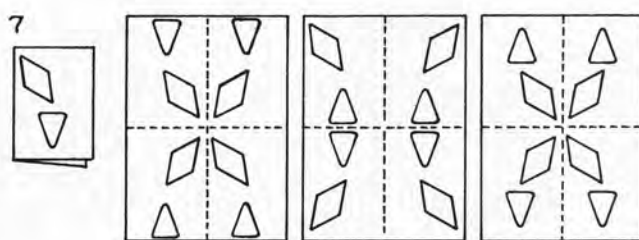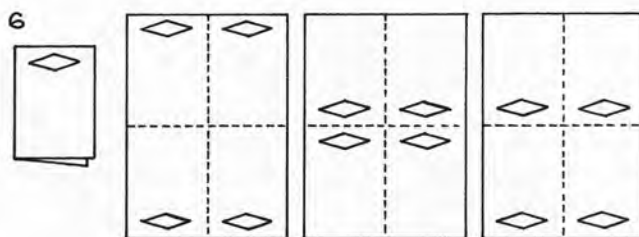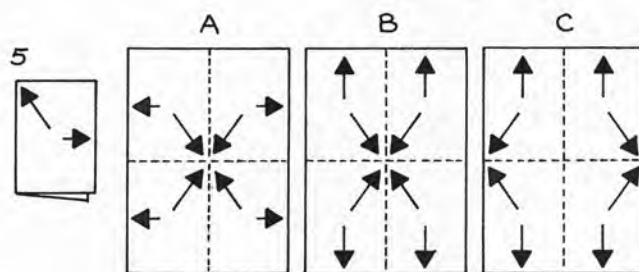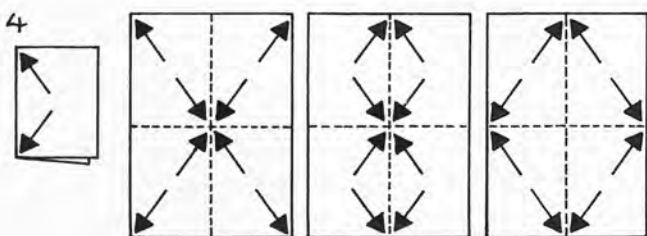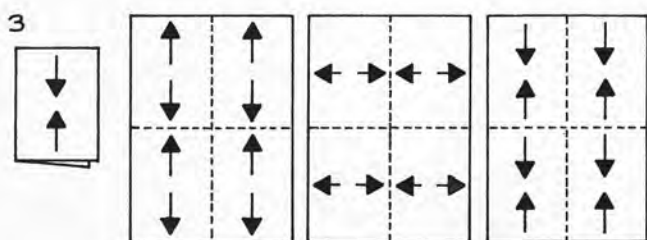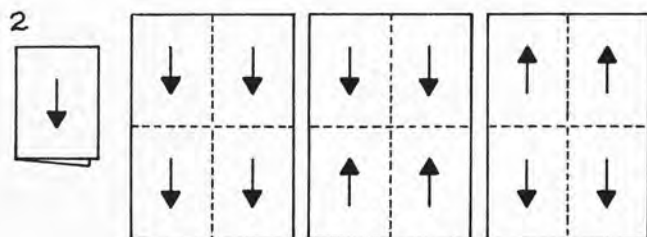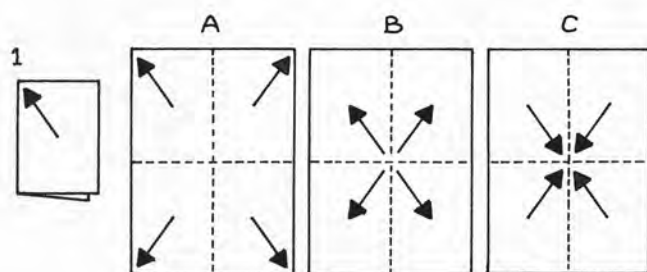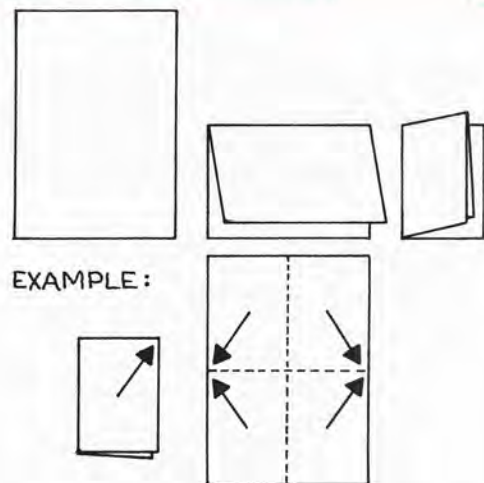
EXAMPLE:

IDEA FROM: *Seeing Shapes*

A SHIFT IN THE RIGHT DIRECTION

One approach for introducing translations or slide motions in geometry is through the use of grid paper.

1) First have the student investigate the effect of moving a set of points either horizontally or vertically. The new figure should not overlap the original.

For example: Move each point 6 units to the right.
Is the new figure a cube?
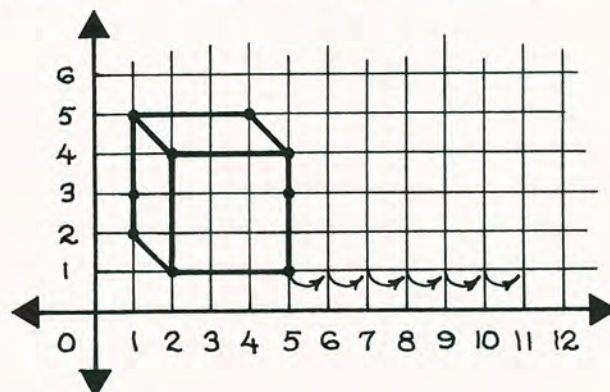
2) Next the student can perform a horizontal or vertical move that does produce a new figure which overlaps the old.

Points on the figures are emphasized and the instructions refer to moving points on the figure rather than just moving the figure.

Can you show on the graph where the bird will be if each point is moved 4 units up?
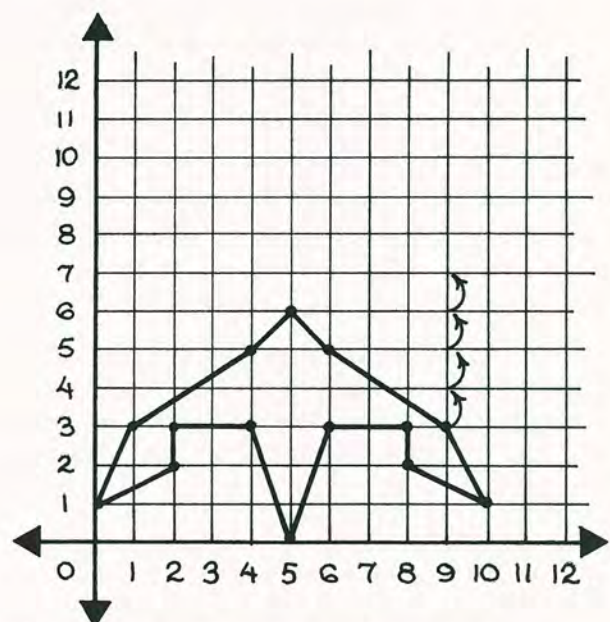
3) The student can examine pictures and determine if they do illustrate the given translation. Select examples and non-examples.

Every point was moved 3 to the right.
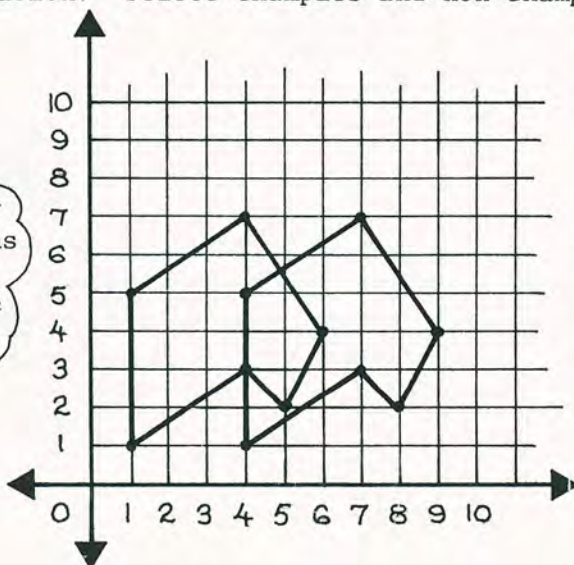
Each point was moved 5 units down.

315
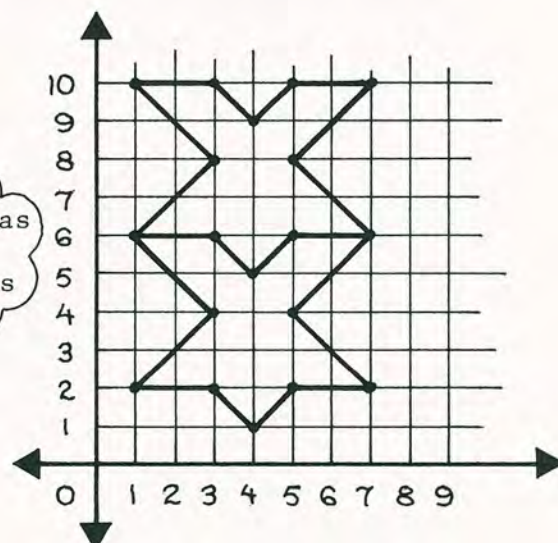
A SHIFT IN THE RIGHT DIRECTION

(PAGE 2)

4) Students can then perform moves that involve both a horizontal and vertical shift.

Move every point 4 right and 3 up.

Move every point 6 right and 0 up.
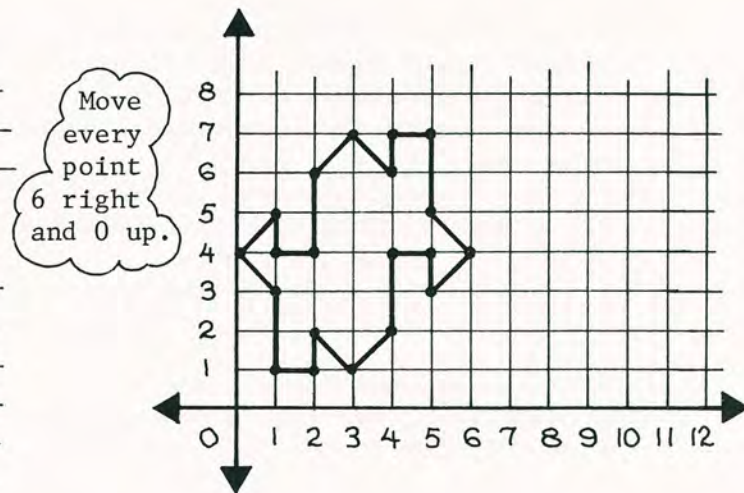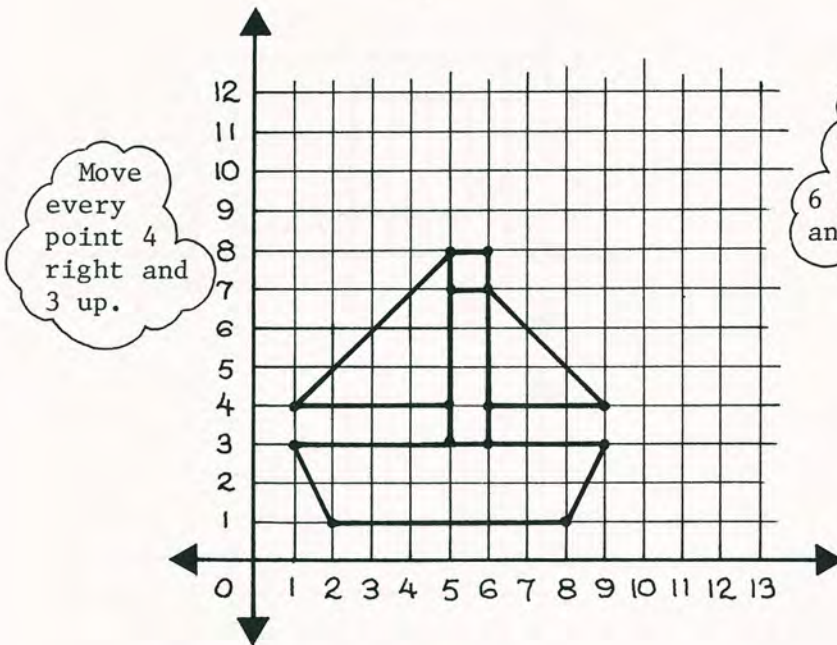
Discussion questions might include:

    a) Is the ship larger or smaller after the move?
    b) Is the ship a different shape after the move?

5) Examples that show a figure before and after a move can be given. Students are asked to describe the move.

Every point was moved right ____ and down ____. If the point (9,10) is shifted with this move, what is its location? (__,__)

BEFORE

AFTER
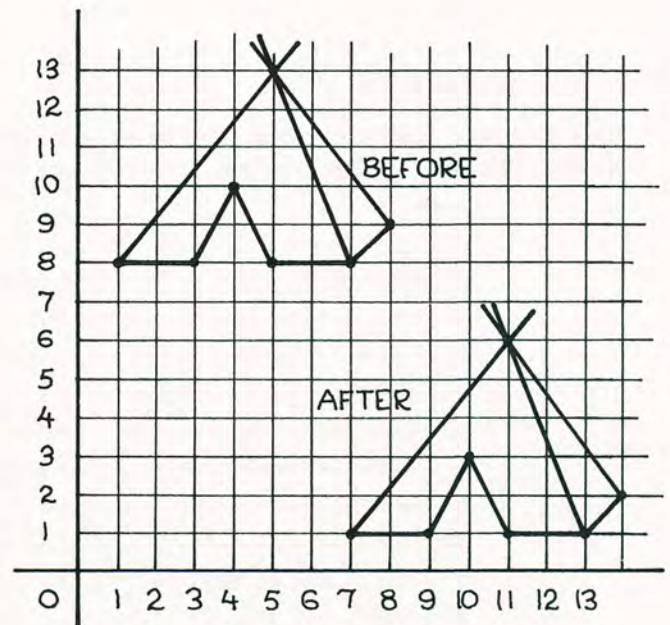
A SHIFT IN THE RIGHT DIRECTION

(PAGE 3)

6) "Point movers" can then be introduced.

> This arrow shows both the direction and the distance of the translation.

> A right 2 up 3 "point mover "

3

2
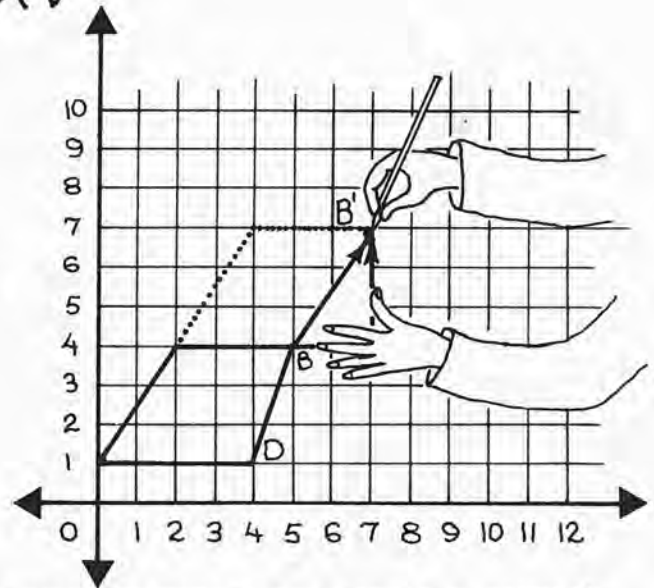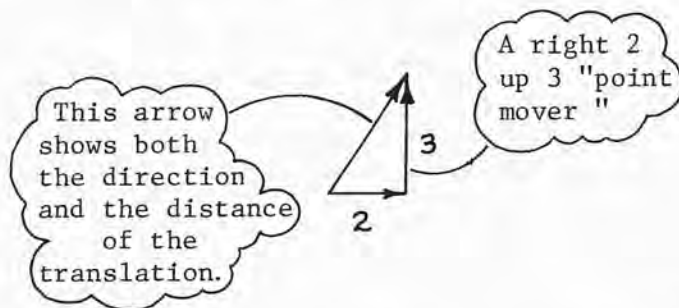
7) Have students design "point movers" for various translations and then have them use the movers to translate several figures. Any point on a figure can be conveniently translated instead of just the grid points.

8) Students can then be asked to design a figure on graph paper and show its new position after a translation.

    a) right 2, up 6   b) right 4, down 2   c) left 6, down 1

Students can give their figure and translation instructions to a friend to do.

9) Students can now be introduced to the arrow (without the point mover attached) to describe a translation.

> Arrows that fit nicely on the grid paper should be used first.

> Draw the image of the line segment after the translation.

IMAGE
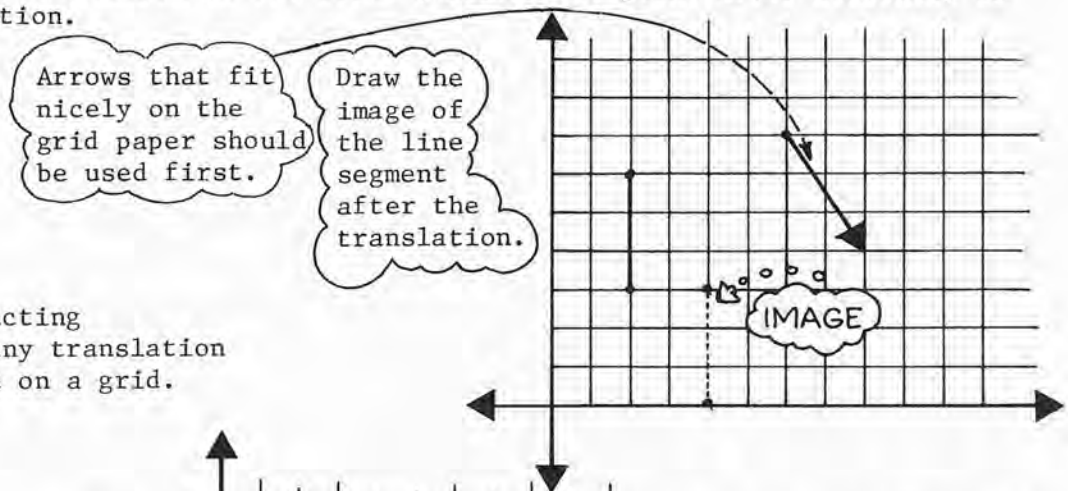
However by constructing a "point mover," any translation can be easily done on a grid.

> Use the corner of a piece of paper to make the "point mover."

A

A'
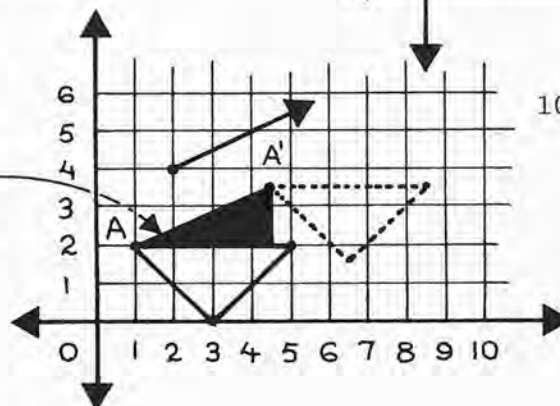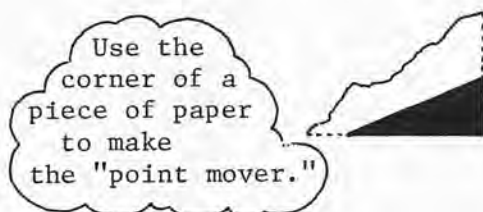
10) Students may use a sequence of translations on a figure to create an illusion of motion.

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GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

POLYGONS and POLYHEDRA

We are apt to see polygons and polyhedra almost anywhere in the world around us. Nature consistently produces crystals in the shape of polyhedra and snowflakes which fit nicely into hexagons (polygons with six edges). Some insects have eyes which are faceted by hexagonal shapes, and bees build their honey comb in the shape of hexagonal prisms. Many tiny sea creatures have a basic polyhedral shape with appendages radiating from that shape.

SNOW FLAKE

KITE

A TINY SEA CREATURE

HONEYCOMB

ABSTRACT ART
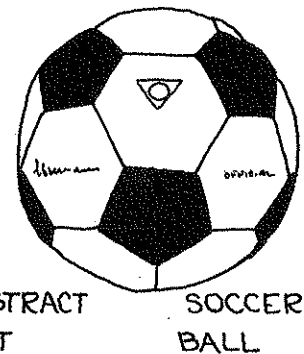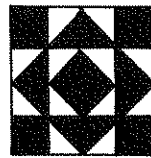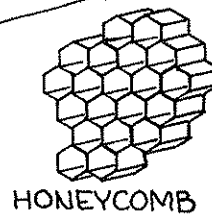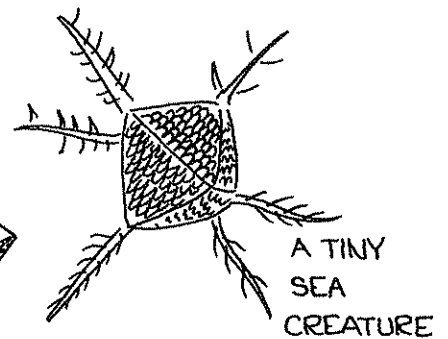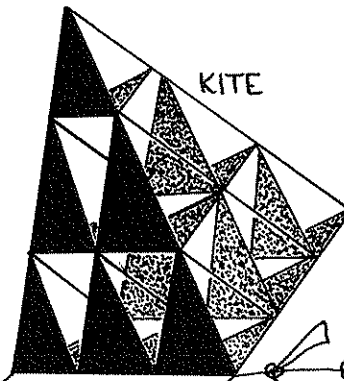
SOCCER BALL

Humans have chosen to use polygons and polyhedra as basic units of design. The design might be artistic as in abstract art or it might be a way to make a soccer ball out of pieces of leather. The tetrahedron (a polyhedron with four triangular faces) has appeared in the design of satellites and kites.

USING REAL-WORLD EXAMPLES

Using examples of polygons and polyhedra from the real world can make lessons more enjoyable and meaningful. Do your students view hexagons as shapes which occur only in mathematics class? You could help them see that the hexagon is a shape used by nature and humans and that it is important in such fields as mathematics, art, engineering, chemistry and biology. Several activities involving real-world examples are suggested on the next page. The meanings of names and terms connected with polygons and polyhedra can be found in the *Geo Glossary* at the end of this resource. Many terms are also explained directly on the classroom pages and also in commentaries preceding subsections of the resource.

WHO CARES ABOUT HEXAGONS? I'LL NEVER SEE THEM OUTSIDE OF THIS CLASS ANYWAY.

319
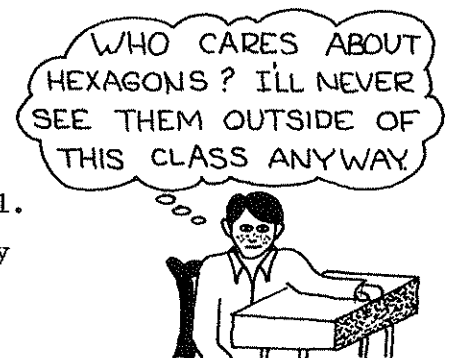
● Take slides of objects which have polygonal or polyhedral shapes. You could use pictures in books to supplement the pictures you take of actual objects. Have students identify the shapes used in the objects. They might like to discuss why the particular shapes were used. (Of course, some of these questions become difficult to answer.) If you are not a camera bug, you might find a student, another teacher or teacher's aide to take the pictures.

● Have students bring newspaper or magazine clippings which use polygons and polyhedra. Use these to make a growing bulletin board display. The page *Trademarks* in the Polygons subsection suggests that students look for shapes in trademarks shown in the yellow pages.

● Let your students design a kite in the shape of a tetrahedron. How strong is it? Is it easy to fly? What would be the advantages for using a tetrahedral shape for a house? The disadvantages?

Here is a picture of the tiniest satellite designed in the world. It is about 13 cm on an edge. No matter how it tumbles in space, it will have one or more sides toward the sun to absorb energy and one or more sides away from the sun for cooling. Can you tell me what shape it is?
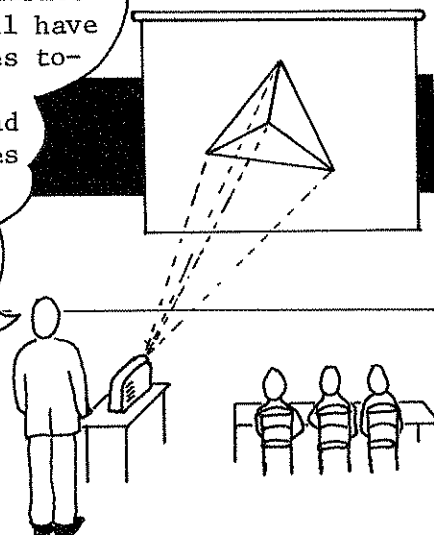
A picture of this satellite model can be found in <u>Mathematics, A Human Endeavor</u> by Harold Jacobs.

● Have your students grow some crystals. As a first attempt you might like to try alum crystals. A growing alum crystal might start in the shape of a cube and become transformed into the shape of an octahedron. You could ask a science teacher for specific material and suggestions. Additional sources and ideas for growing crystals are given in *Space-Filling Forms* in the Polyhedra subsection.

HANDS-ON AND MINDS INVOLVED

Geometry offers a wealth of oppor-
tunities for involving students through
hands-on activities. Tangrams, pento-
minoes, hexiamonds and tessellations
offer students a chance to experiment
with arranging polygons to form pat-
terns. The page at the right shows the
twelve hexiamonds, but it suggests that
students find as many different hexia-
monds as they can by themselves. The
class can work on finding the shapes
in spare time at home or at school for
a week. As new shapes are found they
can be traced on a poster. Here is a
chance for success for the student who
doesn't care for numbers or reading
mathematical problems. What fun to
find the hexiamonds by yourself or to
be the only one to find a certain shape!

Similar opportunities for success are possible in the activities *Sort Out,*
Poly Art,   *Figure It Up,*   *Thumb-Tacktics, Repeating*
Shapes and *Tessellations* from the Polygons subsection. Giving opportunities for
success is important, but the activities listed above involve many other important
ideas in geometry. Some of these are symmetry, visual perception, shape differen-
tiation and problem solving. You might also want to use *Repeating Shapes* and
Tessellations as readiness activities for area concepts.

When teaching content involving polygons and polyhedra there usually are several
modes of presentation available. For example, you could tell your students there are
only five regular convex polyhedra possible and show pictures of them (see *Only Five*
of Them). On the other hand you might want them to discover this themselves through
experimentation and class discussions. If you have never had the chance to make
polyhedral shapes you might find it just as enjoyable and surprising as your students
do.
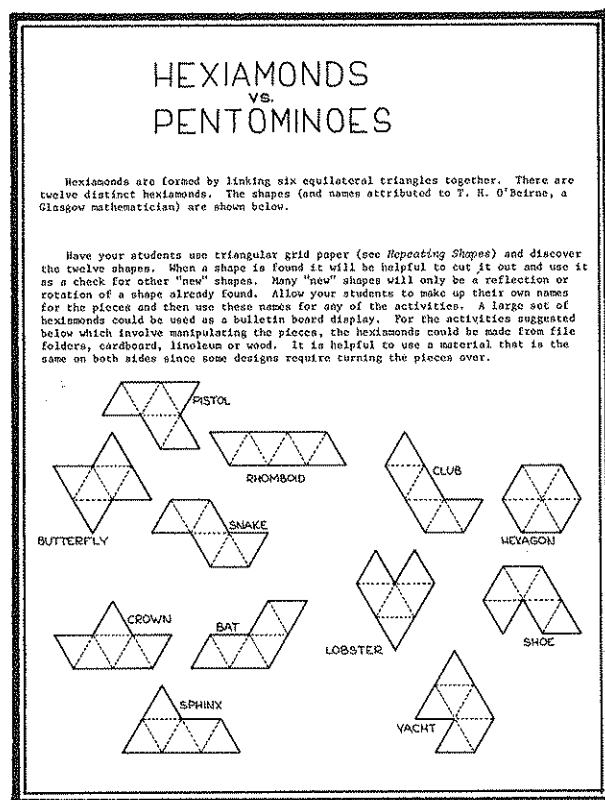
A turned off sixth grade class became interested in the project of making a model of a soccer ball from railroad board shapes and rubber bands (see *Band Together* in the Polyhedra subsection). They cut, trimmed and folded the flaps for the faces and began to assemble the "ball." When it was nearly finished, they ran out of faces and decided they needed to make 3 more white faces and one black face. Alas! They found their newly cut shapes made the top look squashed. Why wasn't it round like the rest? After checking over the ball, they saw that the last shapes were wrong--they had mistakenly cut white pentagons instead of hexagons and a black hexagon instead of a pentagon! After cutting new faces, the ball fit perfectly, but the students had many questions. "Why didn't it go together with the first shapes? What shapes will 'go together'? How can we tell what faces will fit together into a ball shape?" Their teacher took this opportunity to introduce the experiments with shapes given in *Band Together*. Soon polyhedra were hung from the ceiling and students were busy with investigations similar to those described in *Do You Know That . . ., Polyhedra* and *How Much is Lost?*

The pages of this resource give many options for you to choose from. Some will give students pencil and paper work involving compass and straightedge discoveries, visual perception or drawing. Others are centered around discussion or hands-on material. Varying the activities might prove interesting for you and your students.
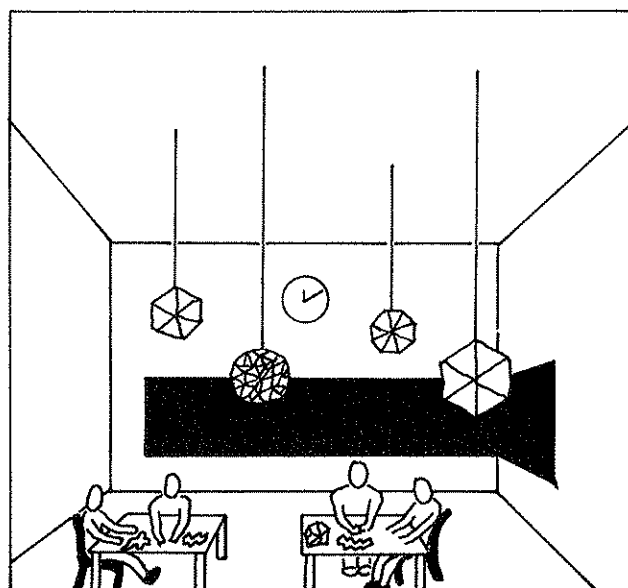
HOW DOES SHAPE AFFECT STRENGTH AND RIGIDITY?

Polygons made from straws, erector sets, geostrips or D-stix (see *Constructing Polyhedra Models*) can be used to discover which polygons are rigid. Be sure the joints are flexible so the rigidity is determined by the shape of the polygon. Students will discover that the triangle is rigid but that other polygons can be deformed.
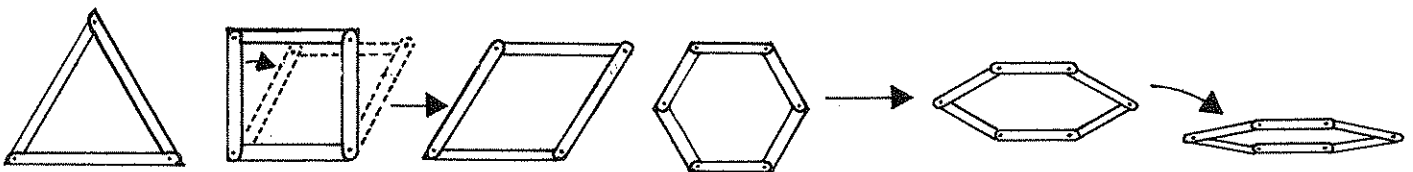
The fact that many shapes are not rigid is also useful. One example of this use is in the expansion gates used in doorways to keep toddlers out of dangerous places.
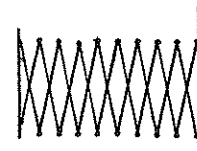
After discovering that the triangle is the only rigid polygon, students can investigate the number of additional strips (diagonals) needed to make a polygon rigid.
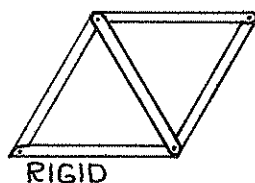
RIGID
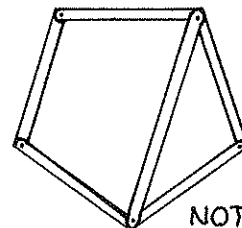
NOT RIGID –
2 DIAGONALS
ARE NEEDED.

Even though only one diagonal is necessary to make a quadrilateral rigid when it is lying flat (contained in a plane), one additional side is not enough to make a 4-sided shape rigid in 3-space. Have your students experiment with straws joined with hair pins or elastic thread (see *Constructing Polyhedra Models*). How many braces are necessary to make an n-sided polygon into a rigid 3-dimensional object? A table can be made for the results. Is it necessary for all the faces of a rigid body to be triangles? The pentagon at the right was changed to a 3-dimensional skeleton by the addition of 3 braces. Is it a rigid shape if it is suspended by one vertex?
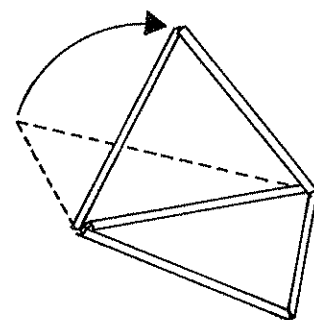
NUMBER OF SIDES	NUMBER OF BRACES
3	0
4	2
5	
6	

5 SIDED POLYGON
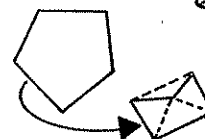
Only three of the five convex regular solids are rigid. The cube and the dodecahedron are easily deformed. How many braces are necessary to make a cube rigid? Can your class find out? Your class might enjoy building bridges from D-stix, straws, toothpicks or erector sets to compare strengths of materials and rigidity of designs.
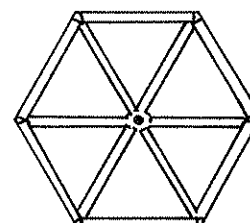
A TOTALLY COLLAPSED CUBE.

Since triangles and tetrahedra are rigid, they are used in the design of many structures. Triangles are even more noticeable with the rising popularity of geodesic domes for buildings as well as playground equipment. The tetrahedron is being used as a basic shape for objects from kites to houses. The tetrahedron has even been suggested by Buckminster Fuller as a model for a floating city two miles on an edge and housing a million people.
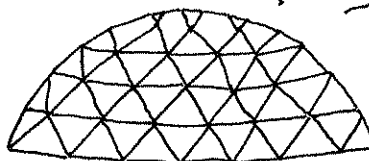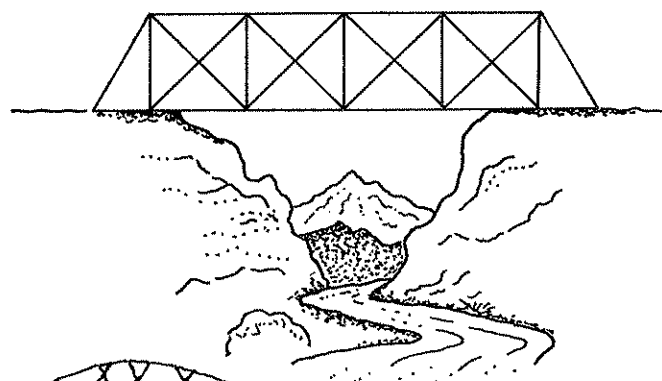
If triangles and tetrahedra are so important to rigidity, why do we build most of our homes and funniture in box-like shapes? Utilization of space is one reason. The other is that there are triangles hidden in the rectangular sides of buildings and furniture. The rectangular pieces of plywood used on the sides of buildings have the strength of triangles because their interior regions are wood. Buildings made with boards are often seen in a state of collapse. A brace might be used to keep the building from falling over. This brace mades a triangle with the building and ground to provide some rigidity.
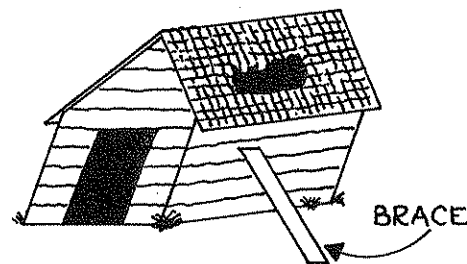
VISUAL PERCEPTION AND GRAPHIC REPRESENTATION

We often use two-dimensional pictures to represent three-dimensional objects. This can be very confusing. Look at the intersecting lines on the faces of the cube at the right. Which pairs of lines are perpendicular? If the lines were interpreted as being on the plane of the paper, we would say angle 1 measures 90° but angle 2 does not. When we see that the lines are on the cube and the cube is shown in perspective, angle 2 measures 90° but angle 1 does not. Some of your students might see in three-dimensional perspective when you want them to be thinking two dimensions and vice versa. *Repetitious Angles II* in the Angles subsection shows several tessellations (tiling patterns). It would not be difficult to interpret them as one tessellation in different perspectives.
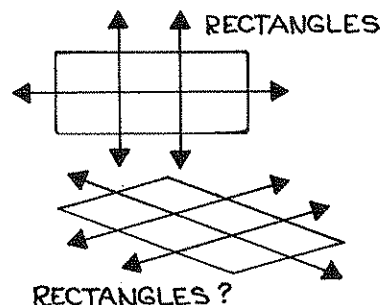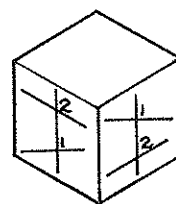
Some drawings can be interpreted in several different ways. The picture at the right can be seen as a tessellation of parallelograms, as a set of cubes (or steps) whose black faces are toward the viewer, or as a set of open rooms whose floors are black and walls are white. The different interpretations for drawings make delightful optical illusions which are fun for viewers and artist alike. Your students might be interested in The Graphic Works of M. C. Escher, and they probably would enjoy coloring or painting optical illusions from The Visual Illusions Coloring Book, by Spyros Horemis.

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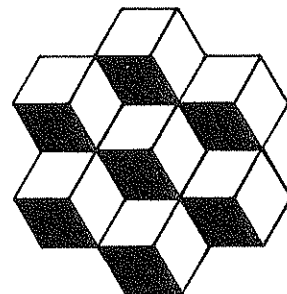
Because drawings and objects can be interpreted and seen in different ways, it might be that your students see things differently. Here are some interesting experiments to try with your class.

● Pick out a ceiling or floor tile at the front of the room. Ask students to draw exactly what they see. Many of them will draw a square even though they see the shape of a quadrilateral that is not square. When asked why they drew a square, students might reply, "because all the tiles are square."
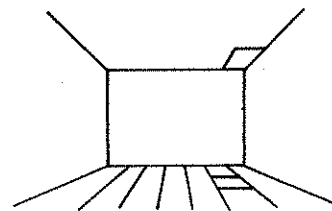
● Place a cube squarely in front of each student. Ask them to draw exactly what they see. If they close one eye they will see a variation of figure a but they might draw any of the other shapes shown in figure b. Have them turn the cube 45° and draw what they see. Now their view (but probably not their drawing) will be similar to figure c.

FIGURE a

FIGURE b
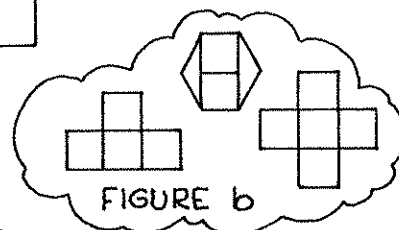
FIGURE c

DIFFERENT VIEWS OF A SQUARE

● Make a square region from stiff cardboard. Have your students sketch what they see when the cardboard is held in different orientations in front of them.

With these activities there is no need to judge a student's drawing as right or wrong, but it is interesting for both teachers and students to discover that people use different drawings to represent the same thing. To help your students develop more awareness and ability in the area of graphic representations, you might try the following activities.

● Make a series of cards which display pictures of solids. Have your students hold a solid in the same position as the picture of the solid shown. Here is a chance for you to observe problems your students have in connecting pictures with three-dimensional objects.
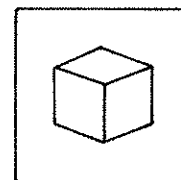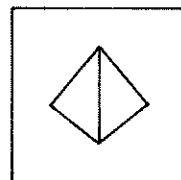
Develop a page which asks students to identify possible views of two- and three-dimensional figures. Can students demonstrate how the shape should be held to obtain that view?

The page *Shadow Stumpers* in the Similar Figures section should initiate much discussion among students. This activity offers a chance for students to invent their own questions. Encourage them to make up sensible questions based on the figures and the shadows. Students might ask questions like this: What kinds of shadows could be cast if the light source can be in any position relative to the object? Is there any way the parallelogram could be the shadow for a card? book? glass? What kinds of shadows occur during an eclipse? Varying the question to invent an original problem is one of the higher levels of problem solving. Here is an opportunity for you to encourage this kind of thinking and to emphasize visual perception as well.

WHICH OF THESE COULD BE A VIEW OF A SQUARE ?

WHICH OF THESE COULD BE A VIEW OF A CUBE ?
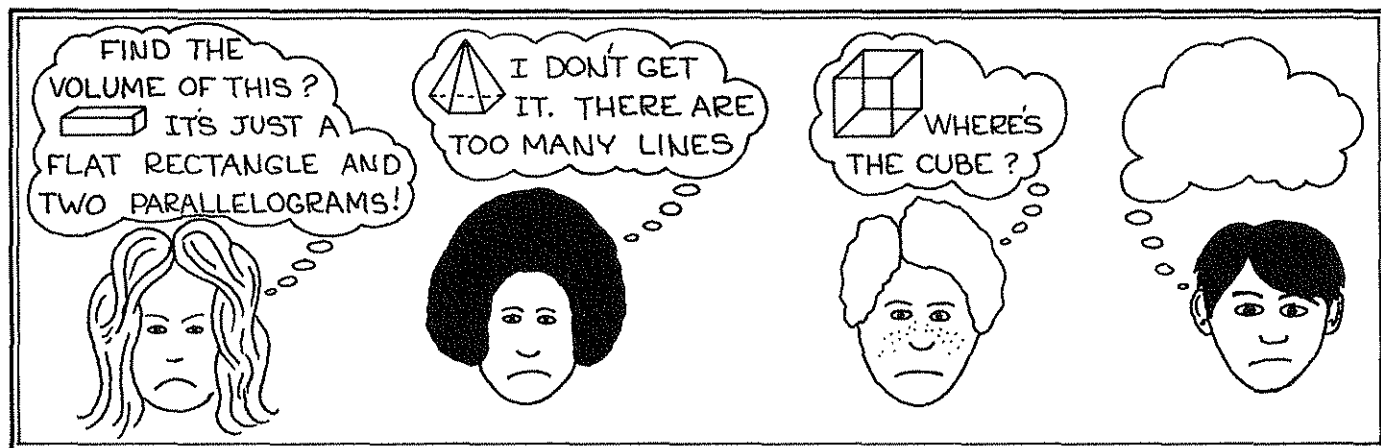
As you work with your students try to hear what they are thinking. Do they really see the same things you do? Do the sketches in the book and on the board make sense? How do they interpret two-dimensional drawings of three-dimensional objects?

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

CONTENTS

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POLYHEDRA

POLYHEDRA MODELS

Polyhedra seem to have a visual and tactile appeal to all ages. Young children build with blocks while adults use polyhedral shapes for objects as varied as dice and lighting fixtures. The study of polyhedra is interesting, and it can help us understand many shapes in our environment. Your students will receive more enjoyment and understanding from a unit on polyhedra if they have actual models to explore. Wood, plastic or colorful paper models can be obtained from commercial publishers, or your class can make the models. Sources for commercial models are given on the page *Surveying Solid Shapes*. Hints for constructing shell models and skeletal models are given in *Constructing Polyhedra Models, Folding Fun,* and *Band Together.*
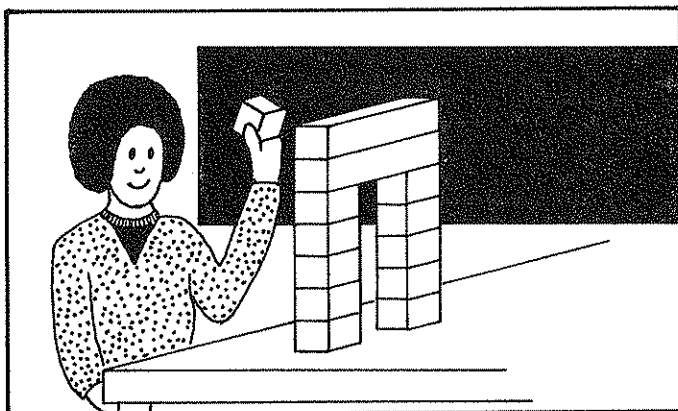
Activities in which students classify or describe solid shapes can be used to review or introduce one- and two-dimensional concepts. (See *Planning Instruction in Geometry* in the POLYGONS & POLYHEDRA section.) The page *Surveying Solid Shapes* suggests a variety of activity stations which might be set up in your classroom. As students are working on these basic activities they will probably be using words like square, corner, round and pointed. At this time you could informally review or introduce some terms which students will need to know. "The point at the corner of a polyhedron is called a vertex . . . . The line segment when two faces meet is called an edge . . . . This block shape is called a cube; each of its faces has the shape of a square. Do you see the difference between a square and a cube? . . ."
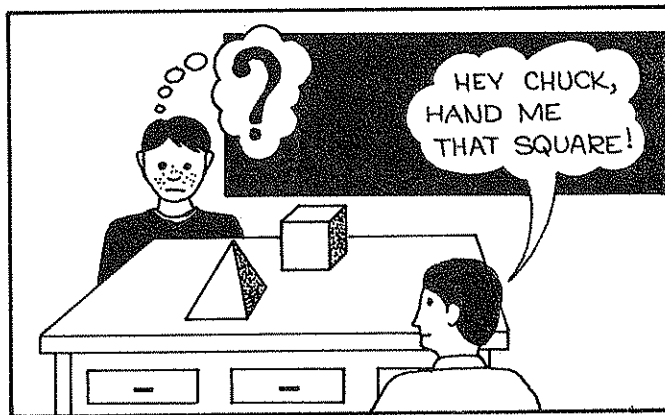
The faces of solid shapes represent portions of planes. "Do you see that these two faces meet at an edge of the polyhedron? They are portions of two planes intersecting in a straight line . . . . These three faces meet at a vertex. Each of the faces is part of a plane. The three planes intersect at the same point." Dihedral angles (space angles formed by two intersecting planes) could also be introduced informally by using solid polyhedra.

Since the faces of a skeletal polyhedral model are open, students might be able to give more attention to edges as line segments and vertices as points. The use and construction of both shell and skeletal models will give students broader understanding on which to base future work with polyhedra.
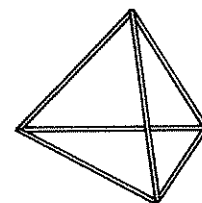
In addition to providing first-hand knowledge about polyhedra, hands-on experience with polyhedra models can provide the concrete background for pictorial representation of specific polyhedra. If the diagram to the right is supposed to bring to mind a pyramid whose base is a square, it would be beneficial to have handled and viewed such a pyramid. There is evidence that the meaning for such "pictures" or graphic representations needs to be taught. (See *Visual Perception* in the TEACHING EMPHASES section, *Planning Instruction in Geometry* in the POLYGONS & POLYHEDRA section, and the main commentary to POLYGONS & POLYHEDRA.)
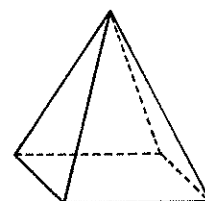
Actual models give meaning to pictorial representations and they are even more beneficial in adding meaning to names of polyhedra. How do you remember meanings for the words prism and pyramid? Does the word "prism" make you think of an experiment in a science class when light rays were passed through a glass prism? Do you associate the word "pyramid" with the many representations you have seen of the Great Pyramids of Egypt? Students need to form these (and other) associations too. When using a page like *You Decide . . .,* you might want to be sure students associate the drawings with the correct 3-dimensional objects. Using a page like *You Decide . . .* previous to a page like *Prisms & Pyramids* will give students a chance to formulate their own understanding of prisms and pyramids before a more structured description is given.

YOU DECIDE...

GRAPHIC REPRESENTATIONS

Drawings of three-dimensional objects seem especially hard for students to make. Many people who can make a drawing of a rectangular box as shown at the right cannot remember how they learned to make such representations. Others remember from a drafting class in high school and still others do not know how to make such a drawing at all. *Graphic Representation* in the TEACHING EMPHASES section gives background and suggestions for helping your students make graphic representations.
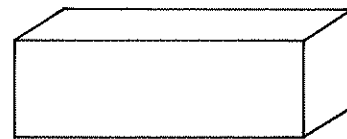
There are different procedures for drawing a rectangular box. One of these methods is shown below:
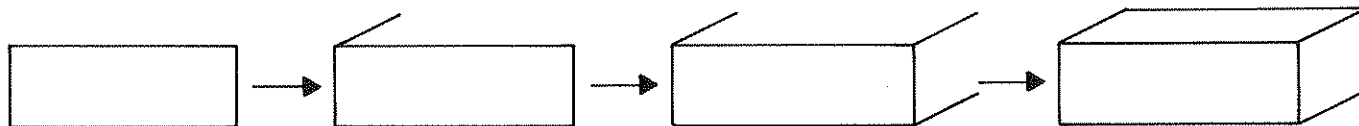
Or you might try the procedure given on the page *Two-D or Not Two-D*. Here students are asked to draw two faces of the shape and connect corresponding vertices. The hidden edges might be dotted in or not sketched at all.
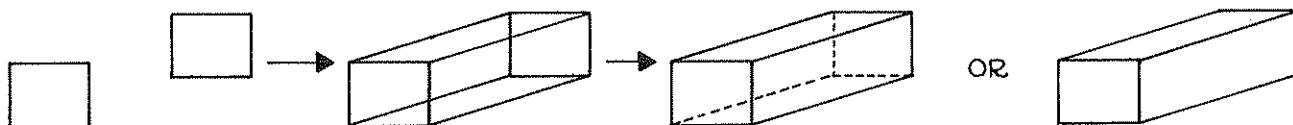
It is not easy to figure out how to make a drawing of a three-dimensional T-shape or how to make a drawing of a prism. The pages *Block It!* *(1 and 2)* and *Two-Dimensional Representation* give ideas for helping your students learn to make such drawings. After students have had some of these experiences, you might want to make enlargements or reductions of their drawings. For ideas on this see the SIMILAR FIGURES section or the Making a Scale Drawing subsection in the resource Ratio, Proportion and Scaling.
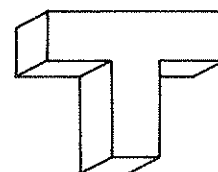
SHELLS AND NETS OF POLYHEDRA

After experimenting and making models of polyhedra you might want to raise questions like, "What does a folded-down tetrahedron look like?" or "What flat shapes will fold into cubes?" You could introduce these ideas with an activity like

Geoblocks I. Students are asked to cover solid shapes with tin foil and cut the tin foil covering or <u>shell</u> at enough edges so the shell will lie flat. Now students will have a shape which can be folded into the shell of a polyhedron. Surprisingly there is a relationship between the number of cuts necessary and the number of faces and edges of the polyhedron. Notice that the page to the right uses terminology like "square pyramid." You might decide to number the blocks instead of using these specific terms, or place a gummed label on each block with the name of that solid. Otherwise, a quick review of terminology (see *Do You Know That*) or easy access to a glossary of terms would probably be necessary (see *Geo Glossary* at the end of this resource).

The page *Fold Ups* has students determine which arrangements of six squares can be folded into a cube. Discovering the shapes that fold into a cube can be challenging for anyone. The activities *The Perplexing Pentominoes* and *Hexiamonds vs. Pentominoes* in the Polygons subsection could also be used in a unit with *Fold Ups*.

Nets are planar diagrams of line segments representing the edges of flattened shells of polyhedra. The page *Several Views of Cubes* gives six activities with cubes and nets. These activities can be used to diagnose or build skills in visual perception. Here is one of the activities:

GEΔ-BLΔ⌐Kⴽ I

Geoblocks can be used to develop methods of classifying polyhedra, to practice visual perception and to develop concepts of similarity, volume and surface area.

Geoblocks are wooden blocks cut into a wide variety of shapes and sizes. The blocks are available in both metric and English dimensions. The particular sizes and shapes are very useful for illustrating properties and showing area and volume relationships of polyhedra.

Geoblocks were developed by the Elementary Science Study of Education Development Center, Inc. Some sources for Geoblocks are Creative Publications, McGraw-Hill Book Company and Selective Educational Equipment (SEE).

Readiness Activities:

1) Build some shapes with the blocks.

2) Find all the blocks that have at least one triangular face. How many differently shaped triangular faces can you find?

3) Try to find a block with no rectangular (or square) faces.

4) Build a larger block with two or more smaller blocks.

5) Decide how many blocks of one shape are needed to fill a box.

6) Feel a block without looking at it, then draw a sketch of the block.

7) Build a shape with the blocks. Draw a sketch of top, front and side views. Give your sketches to a friend. Can he build the same shape by looking at your sketches?

THIS SHAPE FOLDS INTO A CUBE.
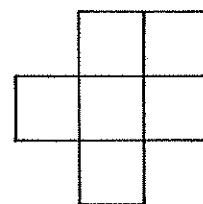
THIS SHAPE WILL NOT FOLD INTO A CUBE.

SUPPOSE A HOLE IS DRILLED THROUGH A CUBE.

WHICH OF THESE NETS SHOW THE RESULT?

a)   b)   c)   d)

If your students are skillful at imagining the nets being cut and folded into shells of cubes, they can probably see that c and d have the necessary holes on opposite faces.

Facts About Polyhedra

There are many facts and relations about polyhedra that can be taught to middle-schoolers. Some of them are presented in the resource in a particular way: activity cards, worksheets, teacher directed activities, ... You will probably want to adapt many of these ideas to suit the background of your students and to fit the modes of presentation which are most successful for you. Here is an overview of some of these polyhedra-based ideas.

Do you want your students to
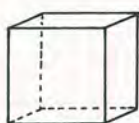
a) relate the number of faces, edges and vertices of a polyhedron by the formula $V + F = E + 2$? Try a guided discovery lesson (*Vertices, Faces and Edges*). Note: Several ways to use this page or idea are discussed in *Goals Through Discovery Lessons* in the CURVES & CURVED SURFACES section. There is related material on networks in the commentary and classroom pages of the Networks and Regions subsection.

b) be able to explain why there are only five regular convex polyhedra? Demonstrate the possible angle arrangements at the vertices with paper models (*Only Five of Them*).

c) experiment with joining the "midpoints" of faces of a polyhedron to produce another polyhedron? Use pictures or have your students make models from straws or clear plastic forms and yarn (*They Came by Twos*).
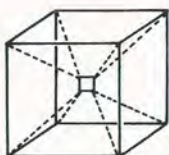
d) be amazed at the appealing geometric forms produced by the forces of nature? Dip polyhedral skeletons in soap solution and show them the surprising arrangement of soap films.
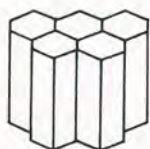
e) discover that some polyhedra can be used to "fill space" but others cannot? Have them examine packing cartons, try to fit tetrahedra together with no spaces left or look at pictures of Devil's Post Pile (*Space-Filling Forms*). Note: Since the ideas are related, students could investigate how polygons fill a plane (tessellations) and the packing of circles and spheres during a unit with space-filling polyhedra. See the Polygon subsection and the Circle subsection.
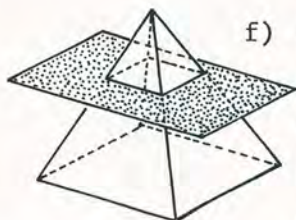
f) visualize or predict cross sections of polyhedra? Let students touch and see the solids; then have students imagine or draw the intersection with a particular plane (*Cross Sections*). Note: An activity on cross sections of cones, cylinders and spheres can be found in the Curved Surfaces subsection.

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SURVEYING SOLID SHAPES

If students have had limited experience with solid shapes you might set up activity stations that require the students to handle solids and think about their properties. A variety of shapes should be used. The collection should include spheres, cylinders, cones, cubes, rectangular prisms, triangular prisms, tetrahedra, square pyramids, octahedra, dodecahedra and icosahedra.
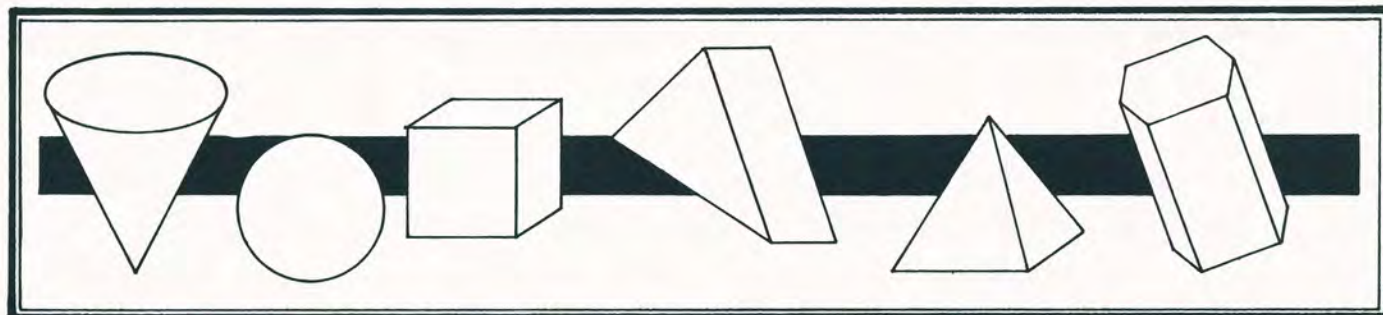
The following stations are suggested. Instead of writing the solids at each station, you could provide a table for the students to use as a check list.

> Find all the solids that roll easily. List them. Test each solid to be sure.

> List all the solids with corners. List the number of corners for each one.

> List all the solids with straight edges. For each solid count the edges and record the number.

> List all the solids which have curved surfaces. Circle those that also have a flat surface.

> List all the solids with flat surfaces. For each solid count the number of flat surfaces and record the number.

> List a solid that has more than 4 flat surfaces.
> List the solids that have fewer than 6 edges.
> List the solids that have more than 6 vertices.

> Which solids roll in a straight path no matter where they are started on their surface?
> Which solid can roll in a straight path, but will not roll if placed on its flat surface?
> Which solid rolls in a circular path?

> List a solid for each property
> a) 2 flat surfaces
> b) twelve edges
> c) 1 curved and 1 flat surface
> d) no edges at all

GEΔ-BLΔCKS I

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3) Try to find a block with no rectangular (or square) faces.

4) Build a larger block with two or more smaller blocks.

5) Decide how many blocks of one shape are needed to fill a box.

6) Feel a block without looking at it, then draw a sketch of the block.

7) Build a shape with the blocks. Draw a sketch of top, front and side views. Give your sketches to a friend. Can he build the same shape by looking at your sketches?

FOLDING FUN

Students will need 8 1/2 x 11 rectangular sheets of paper and scissors. Glue or tape is necessary to secure the solids. Thin, colored transparency material can be used on an overhead projector as a demonstration model.

A regular tetrahedron

a) Fold a piece of paper lengthwise.

b) Fold a bottom corner to the center line so the crease goes through the other bottom corner.

c) Fold the right edge over so it is on top of the crease in (b).

d) Unfolding reveals a triangle. Cut out the unshaded triangle.

e) Fold each vertex to the midpoint of the opposite side.

f) You should have this.

g) Fold each edge to the parallel fold line. (It looks like a boat.)

FOLD LINE FOR THIS

h) Unfolding reveals this.

i) Mark the solid lines as shown and then cut on the solid lines.

j) Fold the figure to get a tetrahedron.
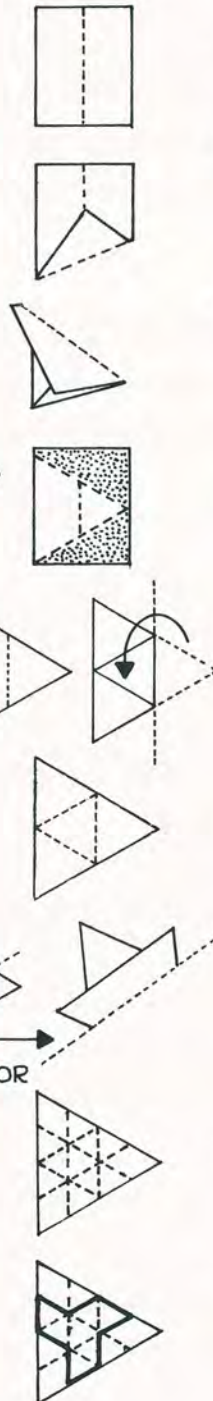
A cube

a) Fold the bottom edge over to the side edge and crease.

b) Cut off the excess at the top to get a square.

c) Fold the other diagonal.

d) Fold each corner to the center and crease.

e) Fold each corner to the third line.

f) You should have this.

g) Fold each corner to the first line to get this.

h) Mark the solid lines as shown and then cut on the solid lines.

i) Fold the figure to get a cube.
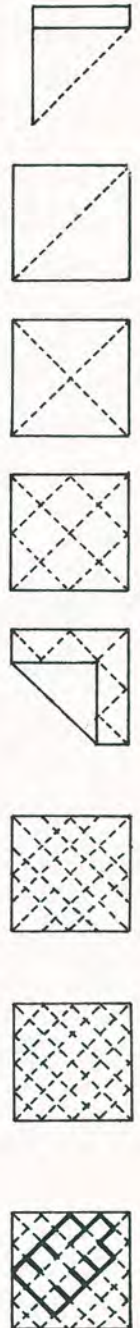
CONSTRUCTING
POLYHEDRA
MODELS

The study of polyhedra can be greatly enhanced by having students construct models. The following are suggestions of various methods and materials that can be used.

Skeletal Models:

1) Drinking straws and

a)

hair pins

b)

pipe cleaners

c)

bent paper clips

d)
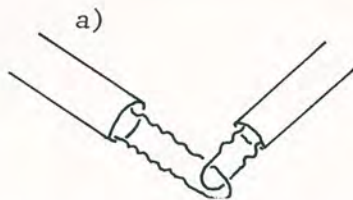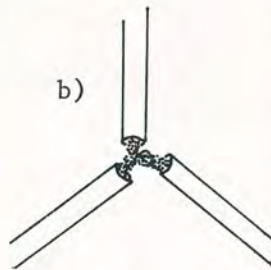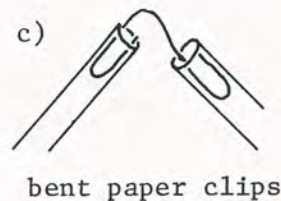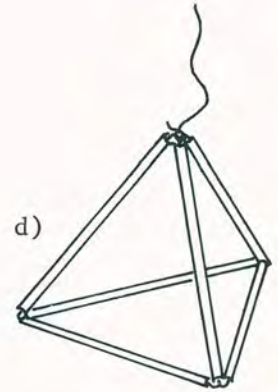
string or thread

A large model can be made from golf club tubes and rope.

2) D-stix (MANUFACTURED BY GEODESTIX P.O. BOX 5179, SPOKANE, WA 99208) or Super Structures™ (MANUFACTURED BY SYNESTRUCTICS, INC. CHATSWORTH, CA 91311)
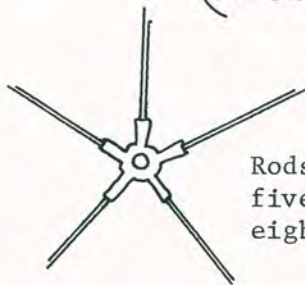
Rods that fasten into five-sleeve, six-sleeve or eight-sleeve connectors.
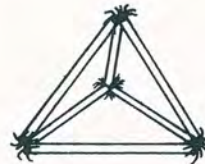
Spokes that fit on The Universal Node™.

3) Pipe cleaners twisted together at the ends.

4) Toothpicks, stirsticks, pick-up sticks, etc., held together with

a)

glue

b)
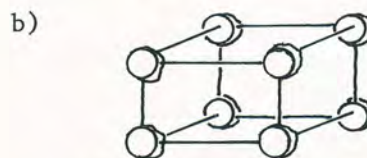
miniature marshmallows

c)

clay

CONSTRUCTING
POLYHEDRA
MODELS (CONTINUED)

5) Erector Set parts or Tinker Toys for many rectangular prisms.
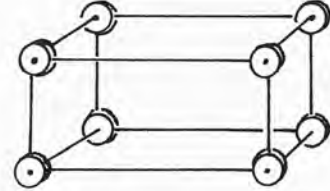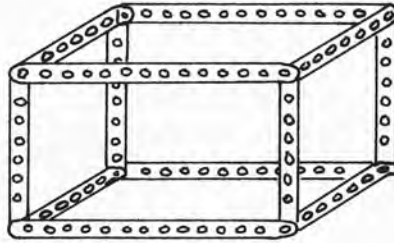
Shell Models

1) Cut out and paste
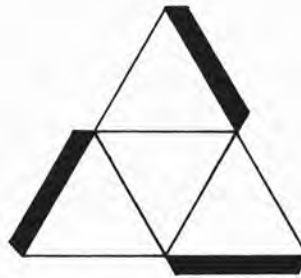
2) Geo-Rings and staples

See Geo-Ring Polyhedra by
Linda Silvey, published by
Creative Publications, Inc.
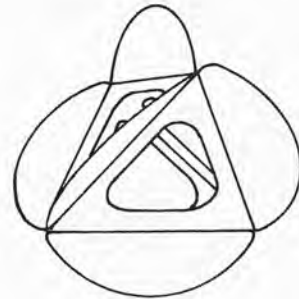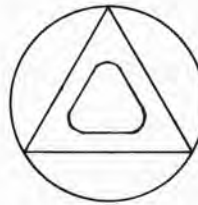
3) Panel Pieces and rubber bands

See Math Projects : Polyhedral Shapes
by Bassetti, Ruchlis and Malament,
published by Book-Lab, Inc.
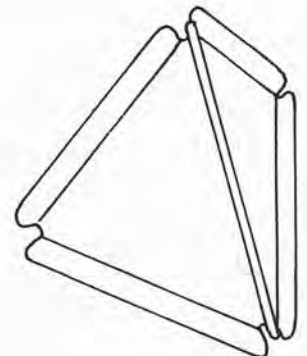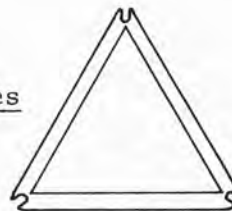
BAND TOGETHER

These pages give patterns and directions for making shapes which can be assembled into polyhedra. The shapes for each polyhedron are joined by rubber bands so the polyhedron can easily be taken apart for compact storage. Students can help prepare the shapes. You will want to make at least 30 triangles, 20 squares, 12 pentagons and 8 hexagons. The exact number and type of polygons needed for a particular polyhedron can be obtained from the pages *Polyhedra*. Making all of the triangles one color, squares another, ... will result in very attractive polyhedra. Suggestions for using the set of shapes are given on page 3.

REGULAR HEXAGON

PATTERNS

REGULAR PENTAGON

HINTS FOR PREPARING SHAPES

1) Using the patterns draw as many shapes as you need on railroad board. You might want to trace around a cardboard pattern.

2) Cut out the shapes leaving the corners "sharp" as shown.

3) Use a paper punch to form each rounded hole.

PUNCH

4) Cut the excess away with scissors.

5) Using a straightedge, score each flap with a scissors edge, from center hole to center hole.

SCORE AND FOLD

6) Fold up on each score line.

7) Join shapes by placing two folded edges together and wrapping a rubber band around them.

RUBBER BAND

RECTANGLE

REGULAR OCTAGON

SQUARE

EQUILATERAL TRIANGLE

Suggestions for using the shapes.

1) Students can make a "corner" using four triangles as shown.
They can continue adding triangles so each "corner" will
have exactly four triangles. What kind of figure is formed? (A regular octahedron)
Is there only one figure possible when four triangles meet at each corner? What kind
of figure results when three triangles (five triangles or six triangles) meet at each
corner? (Respective answers: a regular tetrahedron, a regular icosahedron and a
flat tessellation of the plane)

2) Students can see what kinds of polyhedra can be made using only triangles, only
squares or only pentagons. They can then explore possibilities of mixing two kinds
of shapes. Is it possible to make a polyhedron having two triangles and one square
meeting at each corner? What if two triangles and two squares meet at a corner? Is
the result different if the two triangles and two squares are arranged like figure A
instead of like figure B?
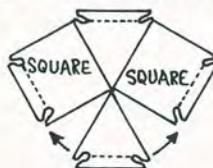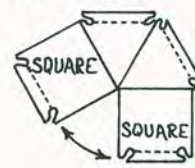
FIGURE A   FIGURE B

3) Notation is often helpful. Designate four triangles meeting at a corner as $\frac{3\,|\,3}{3\,|\,3}$.
The 3 means a triangle. Using this notation the corners made from two triangles and
two squares above would be designated by $\frac{4\,|\,3}{3\,|\,4}$ and $\frac{3\,|\,3}{4\,|\,4}$ respectively. Can a
polyhedron be formed with all its corners described by $\frac{6\,|\,4}{6}$? (That is, two
hexagons and one square meeting at each vertex.) What about $\frac{8}{8\,|\,3}$? $\frac{6}{6\,|\,3}$?
Find a corner that won't work.

4) Students can look at pictures of polyhedra and assemble models that match the
pictures.

5) The models made with these shapes are very attractive if they are hung from the
ceiling. Students might enjoy making a "soccer ball" using 12 black pentagons and 20
white hexagons. The resulting ball will be about 50 cm in diameter.

IDEA FROM: *Math Projects: Polyhedral Shapes*
Permission to use granted by Book-Lab, Inc.

SEEING IT LIKE IT IS

Materials needed: A set of Soma® cube pieces

Activity:

A) Place the pieces in front of you in this order.

1)   2)   3)   4)   5)   6)   7)

B) These pieces have been rearranged. Write the number of the piece shown.

___   ___   ___   ___   ___   ___   ___
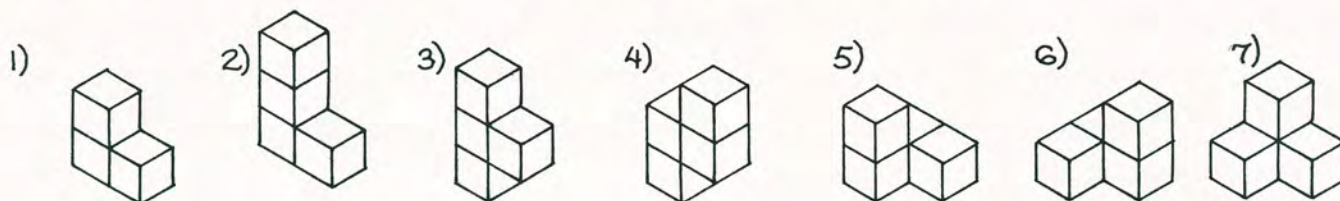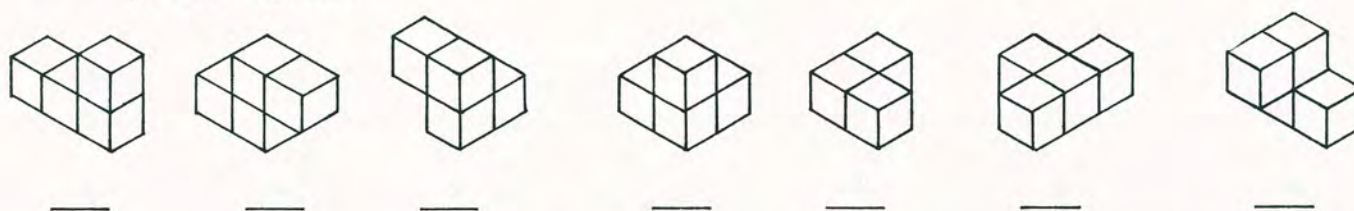
C) Only one view is shown. Which pieces could it be? Each one has more than one answer.

_____   _____   _____
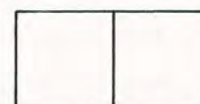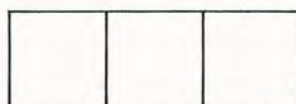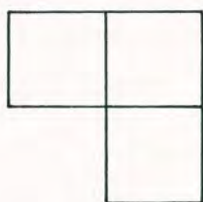
D) Piece 2, shown to the right, can be placed to show 2 cubes on the bottom, 1 cube in the middle and 1 cube on top. Call this 2, 1, 1. Call this 2, 1, 1.

1) Can piece 3 show 1, 1, 2? _____
2) Can piece 7 show 1, 3 (1 cube on bottom, 3 cubes on top)?
3) Can piece 5 show 2, 2? _____
4) Can piece 1 show 1, 1, 1? _____
5) Can piece 4 show 2, 2? _____
6) Can piece 6 show 3, 1? _____

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Materials needed: A set of cubes

Activity: Make each of these models with cubes. On your paper draw a sketch of each model that shows the top, front, and side views.

Example:

Top

Side

Front

Top   Front   Side

1.

2.

3.

4.

5.

6.

7.

2 cubes in this row

3 cubes in this row
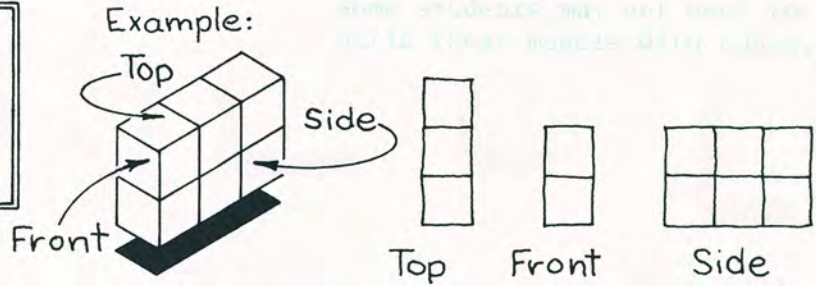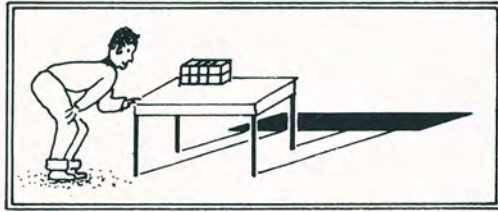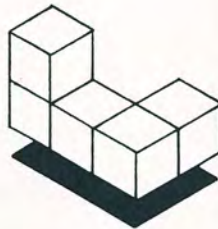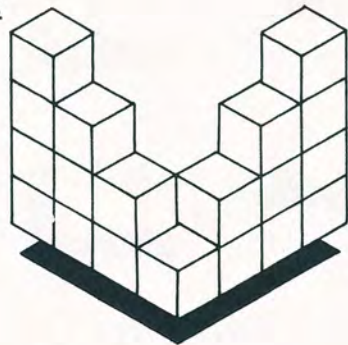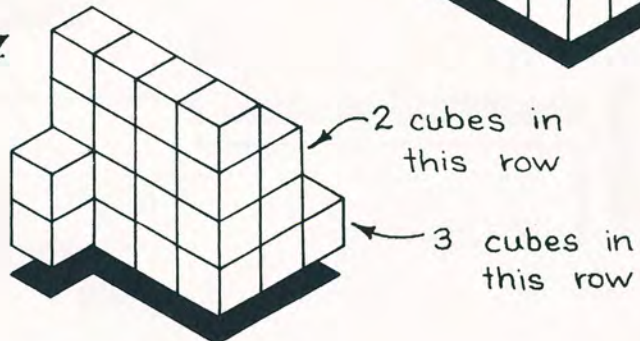
HOW WELL DO YOU STACK UP THIS TIME?

Materials needed: A set of cubes
Activity: Use the three views. First, estimate the number
of cubes needed and then build the model.

Example:

Top   Front   Side

It helps to do the top view first.

I guess 16 cubes.
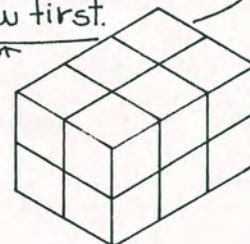
	Top	Front	Side		Top	Front	Side
1.				6.			
2.				7.			
3.				8.			
4.				9.			
5.							

10. Challenge

3 FACES YOU SEE

These sketches show the outlines of this block.

Top
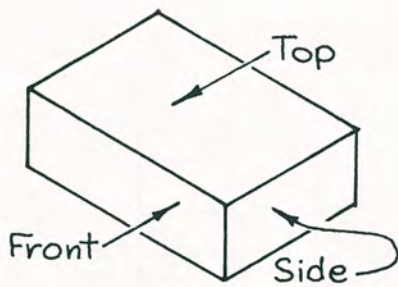
Front

Side

Top   Front   Side

These drawings are only rough sketches. and are not drawn to scale.

On another piece of paper sketch the top, front, and side of these blocks like the example.

1.

2.

3.

4.

Challenge

5.
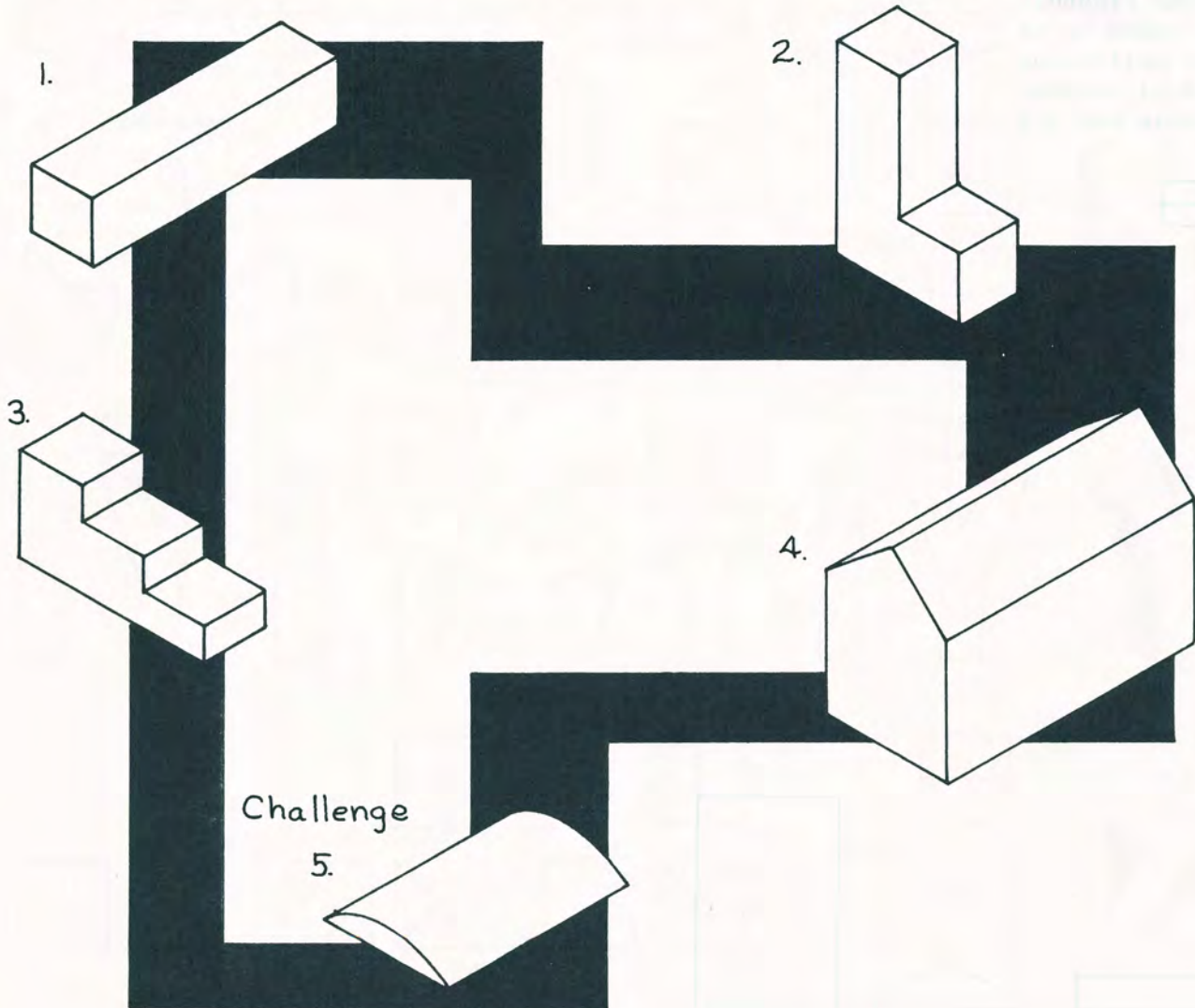
3 FACES YOU SHOULD HAVE SEEN

Circle the letter that shows the correct top, front, and side views.
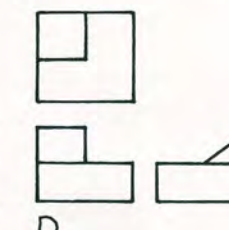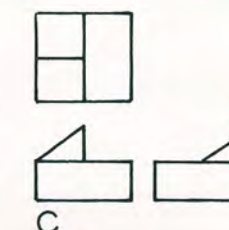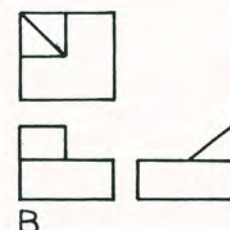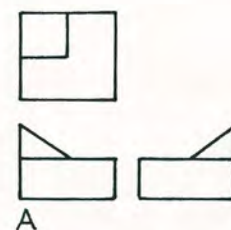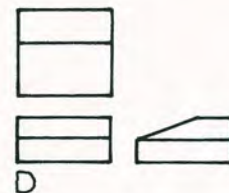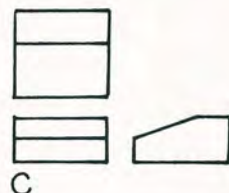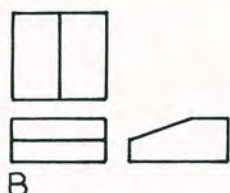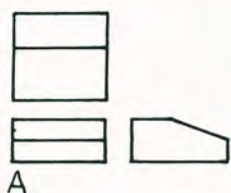
Dotted lines stand for edges hidden by the view.
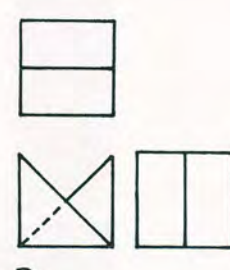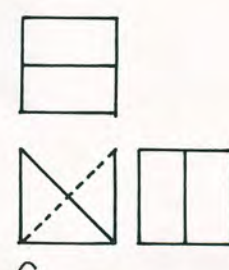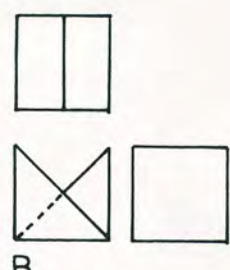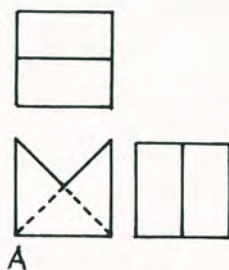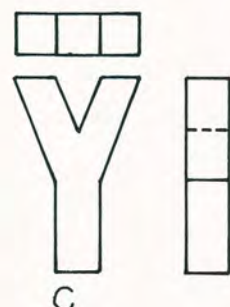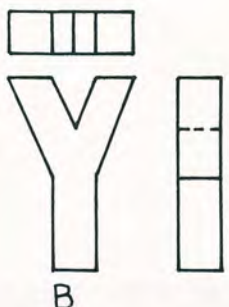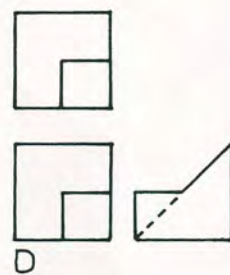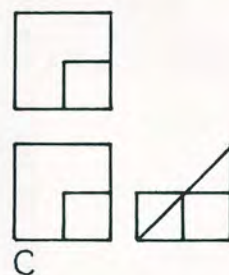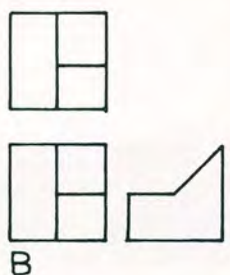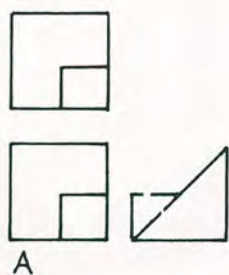
Materials Needed: Metric ruler

Activity: Drawing top, front and side views

Example:

a.

Top

Front   3 cm   Side   2 cm   1.5 cm

Top   1.5 cm   3 cm

Front   2 cm   3 cm

Side   2 cm   1.5 cm

b.

.5 cm   .5 cm   1 cm   2 cm   1 cm

c.

1.5 cm   2.5 cm   2 cm

d.

1 cm   1 cm   1 cm   1 cm   3 cm   1.5 cm
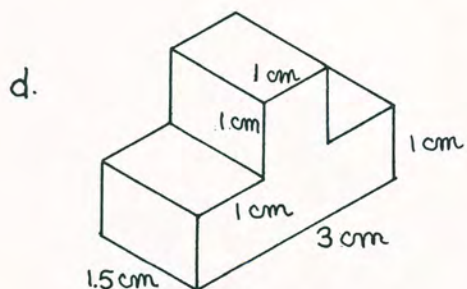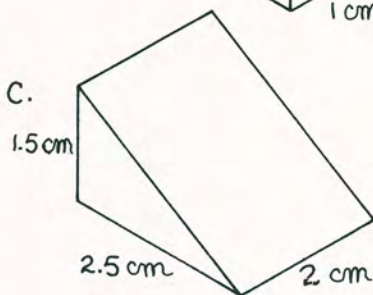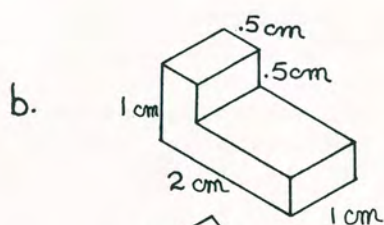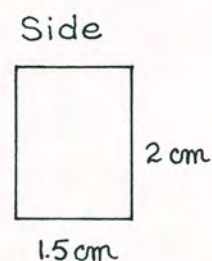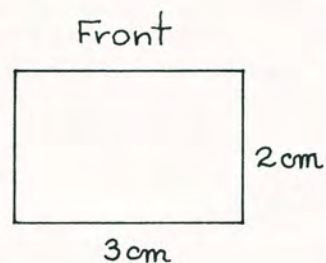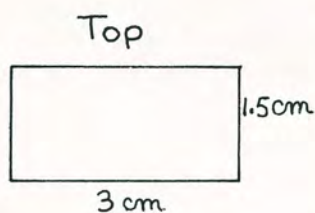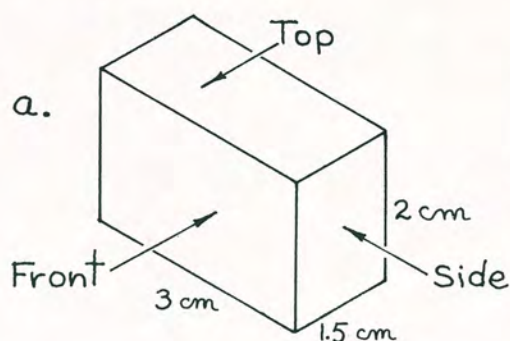
SOLIDS, SHELLS and NETS

1) Provide each student with a box. (Students could help collect cereal, soap, rice, etc., boxes prior to the activity.) Each student cuts along as few edges as possible so the container can lay flat in one piece. Students then compare nets. How do they differ? How are they the same? Have students sketch their nets and then tape the containers back together. By cutting the container's edges again but varying the cuts students see if they can make a differently shaped net.
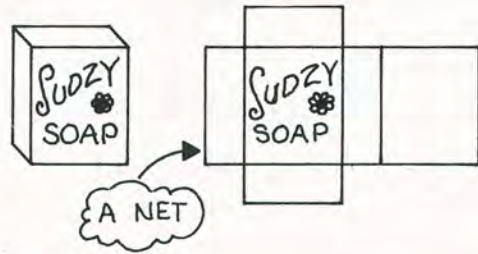
2) Collect enough solid polyhedra for each member of the class to have one. The collection should include tetrahedra, square pyramids, cubes, rectangular prisms and triangular prisms.
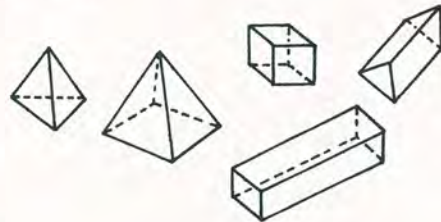
Each student chooses a solid and makes a net by rolling the solid from face to face on poster paper and tracing each face. Each face should be marked after it is traced to avoid tracing the same face twice. After drawing the net, students describe (name) the faces and then try to assemble the net into the shell of the solid. Students trade to get a different solid and repeat the activity. Compare the nets of each solid by having students sketch their nets on the chalkboard. Is it possible to have more than one net for each solid?

3) Play this game in pairs. Make a die with the numbers 3, 3, 4, 4, 4, 5 on the faces. Use the patterns to the right to make a set of triangles, squares and pentagons. Each player rolls the die in turn and takes a shape with the same number of edges as the number rolled. She then tries to arrange her shapes into sets that will form the net of a solid shape. When a net is completed, the player receives a score equal to the number of plane shapes used in the net.
The pieces in the completed net are then returned to the pool. At the end of a fixed time period the scores are totaled. The number of unused shapes that each player holds is deducted from her score.

SOLIDS, SHELLS, and NETS
(CONTINUED)

4) Provide the students with nets for various polyhedra. The student

   a) guesses how many cuts were made to form each net from the shell.
   b) lists the edges that will be joined together when each net is assembled, and
   c) checks his answers by cutting out the net and assembling it.
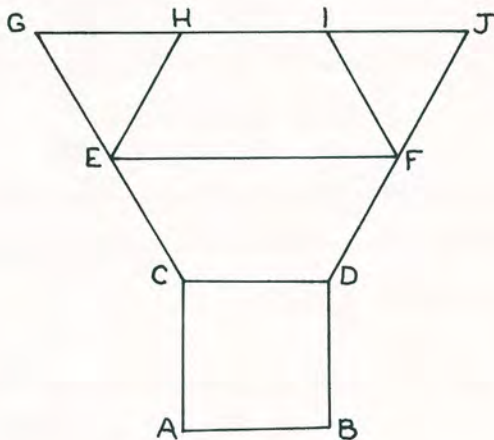
a) Number of cuts = _____.

b) $\overline{GE}$ joins _____.

   $\overline{CA}$ joins _____.

   $\overline{AB}$ joins _____.

   $\overline{DB}$ joins _____.

   $\overline{DF}$ joins _____.

   $\overline{FJ}$ joins _____.

   $\overline{HI}$ joins _____.

TWO OF THESE SOLIDS CAN BE JOINED TO FORM A TETRAHEDRON!
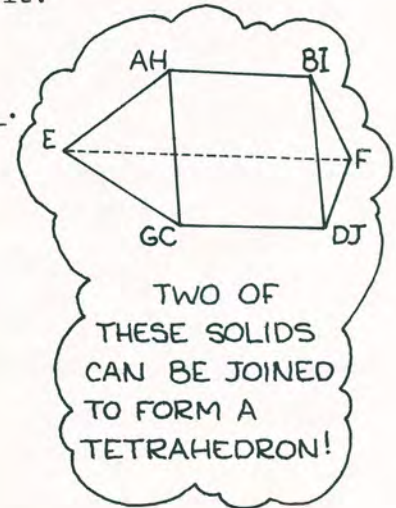
a) Number of cuts = _____.

b) $\overline{AB}$ joins _____.

   $\overline{AE}$ joins _____.

   $\overline{FG}$ joins _____.

   $\overline{IJ}$ joins _____.

   $\overline{JH}$ joins _____.

   $\overline{HD}$ joins _____.
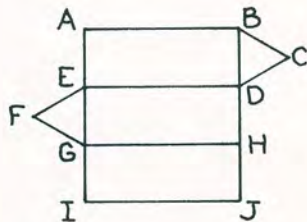
a) Number of cuts = _____.

b) $\overline{AE}$ joins _____. $\overline{NJ}$ joins _____.

   $\overline{DI}$ joins _____. $\overline{KF}$ joins _____.

   $\overline{HL}$ joins _____. $\overline{CG}$ joins _____.

   $\overline{LM}$ joins _____. $\overline{BC}$ joins _____.

   $\overline{MN}$ joins _____.
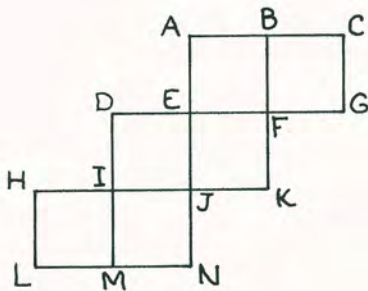
IDEA FROM: *Seeing Shapes*

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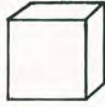
GEΔ-BLOCKS II

Materials: Geoblocks or other
three-dimensional shapes

1) Have students trace the faces
of the blocks on newsprint.

2) They can keep a record of
their results as shown in
the table on the right.

Shapes can be identified
by a sketch, by name or
by numbers placed on the
blocks.
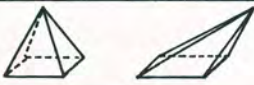
SHAPE	NUMBER & NAME OF FACES
(cube)	6 - SQUARES
SQUARE PYRAMID	1- SQUARE 4- TRIANGLES
# 4	2- SQUARES 4 RECTANGLES

Materials: Geoblocks or other three-dimensional shapes
Aluminum foil - or plain paper

1) Have students cover a block with foil as if they were wrapping a
present

2) Then cut only enough edges to remove the block. Be careful not to
cut off any faces.

3) Cut enough edges to allow the shell
to lie flat. (Shells will probably
not all have the same patterns.
Discuss why the patterns can be
different.)

4) Refold to make the block.

5) Have students develop the table
shown below and look for relationships.
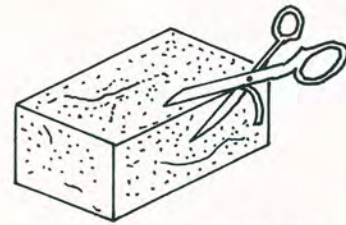
MINIMUM
OF CUTS TO

SHAPE	# OF FACES	# OF EDGES	FLATTEN SHELL
(pyramid / tetrahedron)	5	8	4
TRIANGULAR PRISM	5	8	4
#2 (RECTANGULAR PRISM)	6	12	7

1 + EDGES = FACES + CUTS

MAGIC CUBE

What is magic about this cube?

Here is a pattern for a model of a 3 x 3 x 3 cube. When assembled, it appears to be formed from 27 smaller cubes, each bearing a number from 1 through 27.

			10	24	8						
			23	7	12						
			9	11	22						
10	23	9	9	11	22	22	12	8	8	24	10
26	3	13	13	27	2	2	25	15	15	1	26
6	16	20	20	4	18	18	5	19	19	17	6
			20	4	18						
			16	21	5						
			6	17	19						

1) Cut out and assemble the cube.

2) Find the face with 1 on it. Write the three numbers in each of the two rows that contain the 1. Add each set of numbers. Are both sums the same? Now add the three numbers in each of the four other rows on the same face. Do you always get the same sum?

3) Find the face with 27 on it. Add each of the six rows of three numbers on this face. What do you find?

4) There are 36 different rows of three numbers on the six faces of the cube. Can you find them all? Are all their sums the same?

5) One of the numbers from 1 through 27 is not marked on any face. What number is it? Where is the cube located with this number on it?

6) The missing cube is numbered 14 and it is in the very middle of the 3 x 3 x 3 cube. There are 13 different rows and diagonals in the cube that contain this small middle cube numbered 14. Can you find them all? Are all their sums the same?

7) Why do you think this 3 x 3 x 3 cube is called a magic cube?

TWO COLOR PROBLEMS

I   The checkerboard to the right originally had 64 squares, but two have been removed.
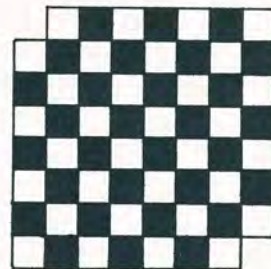
Suppose you have a set of dominoes and each is just large enough to cover two adjacent squares on the board.

1) Since each domino covers only two squares, how many dominoes will you need to cover the board? _____

2) How should they be arranged? Experiment by drawing the checkerboard on graph paper and drawing a line segment through each pair of squares covered by one domino. Do not draw any domino on a slant. Each domino must lie completely on the board.

3) Try to solve the problem on a smaller board. How many dominoes are needed? _____   Can you find an arrangement that will work? _____

4) Use the board in #3 to answer these questions:

    a)   What color were the two squares that were cut off? _____
    b)   How many black squares are on the board? _____   White squares? _____
    c)   Can a domino cover two squares of the same color? _____
    d)   After six dominoes have been placed on the board how many black squares have been covered? _____   White squares? _____
    e)   What color will the remaining two squares be? _____
    f)   Can this board be covered with the dominoes? _____

5) Is it possible to cover the larger checkerboard above with the dominoes? _____

II   Suppose we have a large cube which we cut up into 27 smaller cubes. If we glue the cubes together in pairs we will have 13 pairs with one cube left over. Suppose we throw the extra cube away. Is it possible to put the 13 double-cubes back together to form the original cube with a hole in the center? _____   Pretend we could. Paint the 26 cubes alternately black and white so the assembled cube would look like ⟶
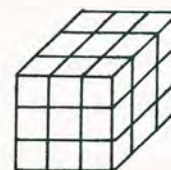
1) How many white cubes are there? _____   Black cubes? _____

2) Each double-cube is made of one white and one black cube. How many of each color were used to form the 13 double cubes? _____

3) Compare your answers to #1 and #2. Do you think you could make the cube with a hole in the center from the 13 double-cubes if you were patient and kept trying different arrangements? _____

CALENDAR POLYHEDRA

1) Have students make the net for a dodecahedron with each edge 5 centimetres long. On each face glue a month from next year's calendar. You can prepare a master of the months of the year by using the shrinks of the months of a calendar. For example, February has small copies of January and March.
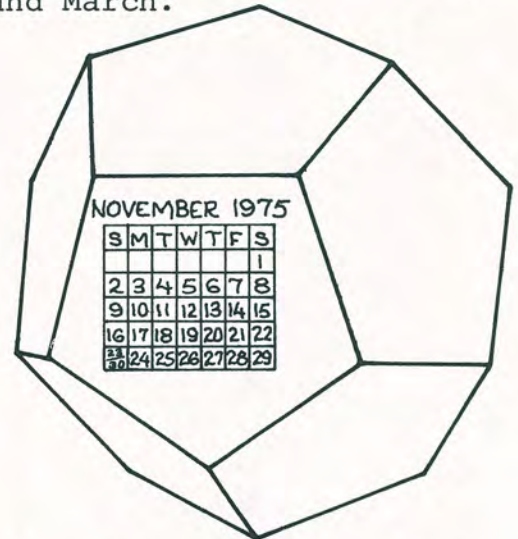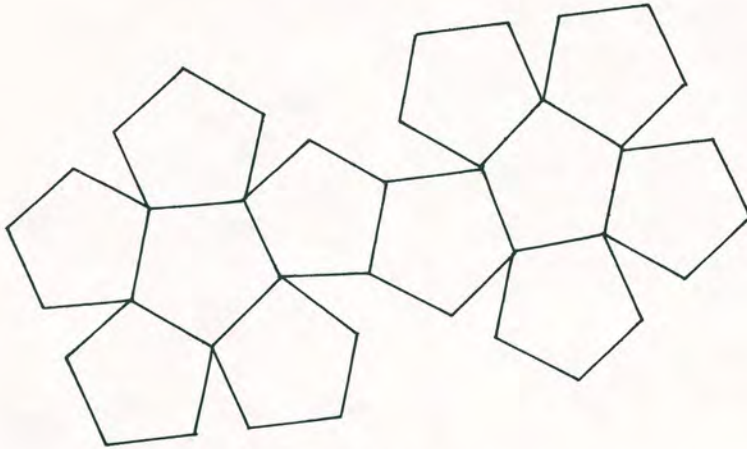
2) Have students make two cubes and place the digits 0 - 9 on the faces so any day of the month can be shown using <u>both</u> cubes. For example, the third day of the month would be shown as .
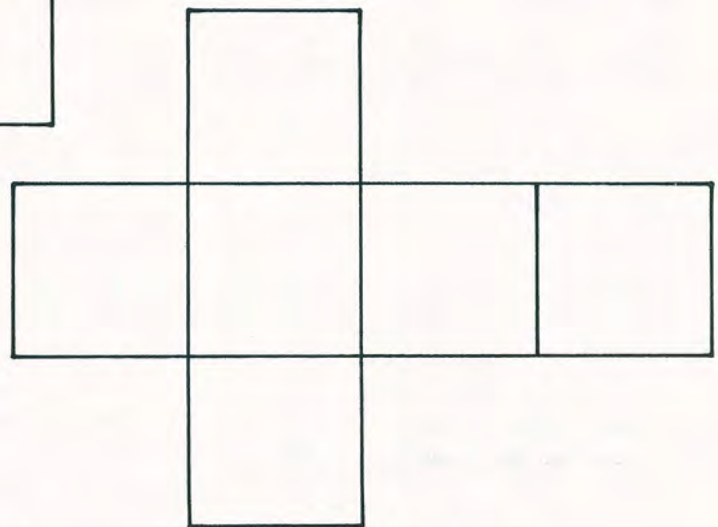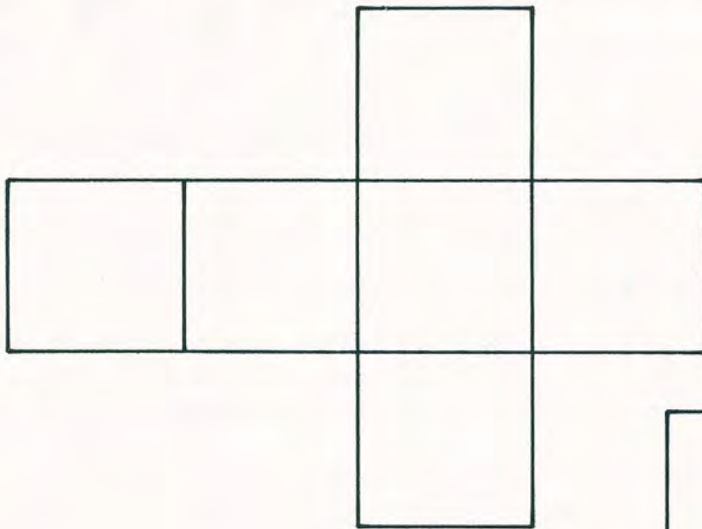
Students could present both of these calendar polyhedra to a friend or parents as Christmas presents.

SEVERAL VIEWS OF CUBES

Below are six activities with cubes and nets that involve visual perception. Each idea could be expanded into a student page.

The six views below are of the same cube.
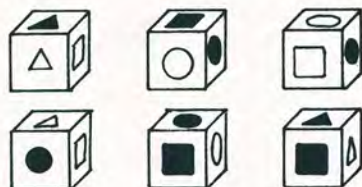
What figure is opposite each of the following:

△ = ___
▲ = ___
□ = ___
○ = ___
■ = ___
● = ___

Make a cube to check your answers.

A cube can occupy a space in 24 different ways. Here are five ways.

Some other ways are shown below. Can you fill in the missing letters? (Be sure they are positioned correctly.)

Make a cube to help you.

Each net represents an unfolded cube with designs on several faces. Four cubes are pictured below each net. Match the net with the correct cube.

a)

b)

c)

Suppose a hole is drilled through a cube:

Which of these nets show the result?

Experiment with various shaped holes.

Select the nets which describe the cube.

The nets below can be used to form a cube. On each net the back face and the bottom face are marked.

1) Mark the remaining four faces on the net: front, left, right, top.

2) Check each answer by cutting out the net and making the cube.

If a cube was made from the net below it would spell CUBE around four of its faces.

| C | U | B | E |

Letter the nets below so that each spells CUBE when assembled.

Cut out the nets to check your answers.

FOLD-UPS

Materials: scissors, graph paper

This pattern

	4		
2	6	5	1
	3		
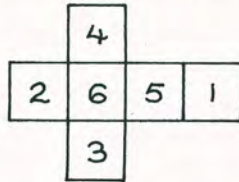
can be folded up to make a cube.
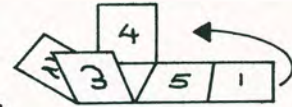
There are other patterns which also can be used to make a cube.

1) Study the patterns below and circle those that you think can be used for a cube.

2) Guess which of the numbered faces will be opposite the face "X". To check your answers cut out each pattern and try to make a cube.

6	5	1	2
	3		X

	5	1	2
4	6	X	

		4	1
	X	5	
2	3		

	4		
5	1	X	6
	3		

4	6	3	
	5	X	2

6	5	1	2	X
		3		

5	X	4
6	3	
	2	

2			
1	X	6	4
5			
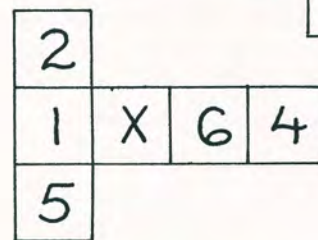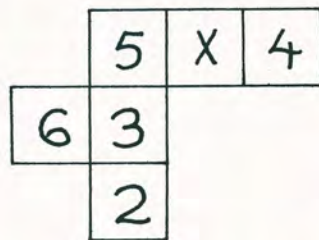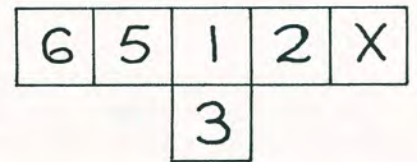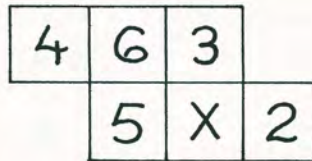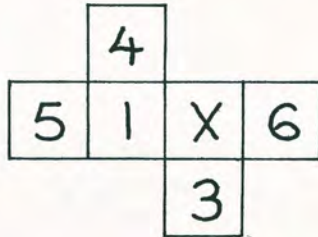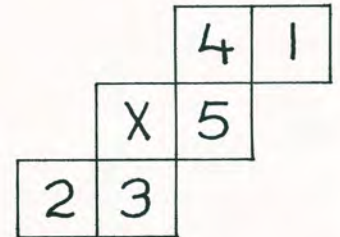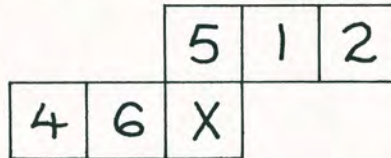
3) Copy those that work on a piece of graph paper. See how many more patterns you can find. Compare with a neighbor.

Patterns that work can be made by rolling a die on paper and tracing each face. Number each face to keep from tracing the same one twice.

1			
2	4	5	9
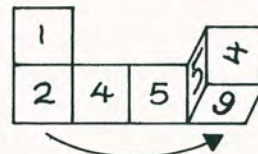
POLYHEDRA

Two types of polyhedra, regular and semi-regular, can be described by the arrangement of the faces at each vertex. In each regular polyhedra all the faces are congruent and the same number of faces meet at each vertex. The five regular convex polyhedra are these:
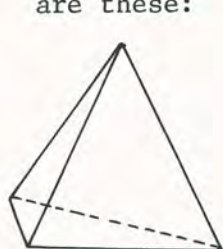
Tetrahedron
3-3-3

3 triangles at each vertex
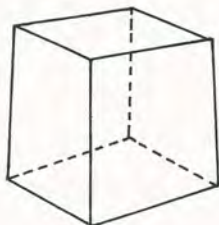
Cube (hexahedron)
4-4-4

3 squares at each vertex
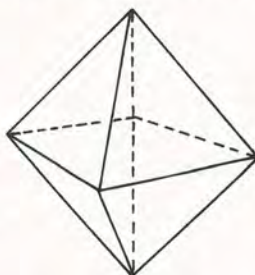
Octahedron
3-3-3-3

4 triangles at each vertex
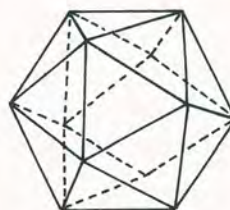
Icosahedron
3-3-3-3-3

5 triangles at each vertex
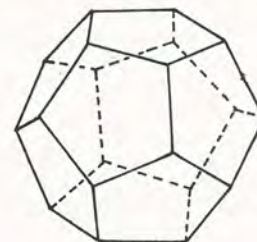
Dodecahedron
5-5-5

3 pentagons at each vertex

They are often referred to as the Platonic solids, named after Plato (350 B.C.), a Greek philosopher, who studied the solids in detail and showed how to construct them in his book Timaeus. The Greeks believed the five solids corresponded to the following elements of the universe: (1) Tetrahedron-fire (2) Hexahedron-earth (3) Octahedron-air (4) Icosahedron-water (5) Dodecahedron-universe.

Another type, semi-regular polyhedra, are solids with faces in the shape of more than one kind of regular polygon, yet the arrangement of faces at each vertex is the same. Among these are the thirteen Archimedean solids identified by the Greek mathematician Archimedes. Most of them can be obtained from the five regular polyhedra by the appropriate cutting of corners. Each Archimedean solid is shown below with the net from which it can be made.

4 triangles
4 hexagons

"Truncated" means "cut off." The truncated tetrahedron looks like a regular tetrahedron whose corners were cut off 1/3 of the way along each side.

Truncated Tetrahedron
3-6-6

1 triangle and 2 hexagons at each vertex

The truncated cube is a cube with the corners cut off 1/3 of the way along each side.
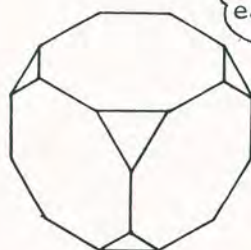
Truncated Cube
3-8-8
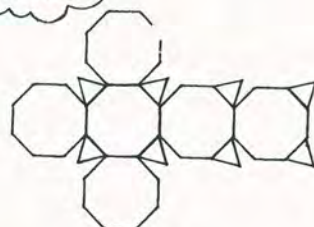
8 triangles
6 octagons

POLYHEDRA
(PAGE 2)
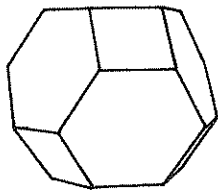
Truncated Octahedron
4-6-6
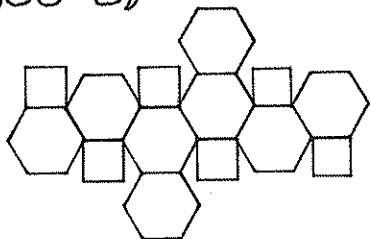
6 squares
8 hexagons

How is the truncated octahedron related to the regular octahedron?
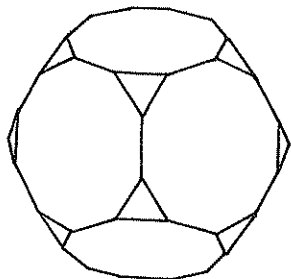
Truncated Dodecahedron
3-12-12
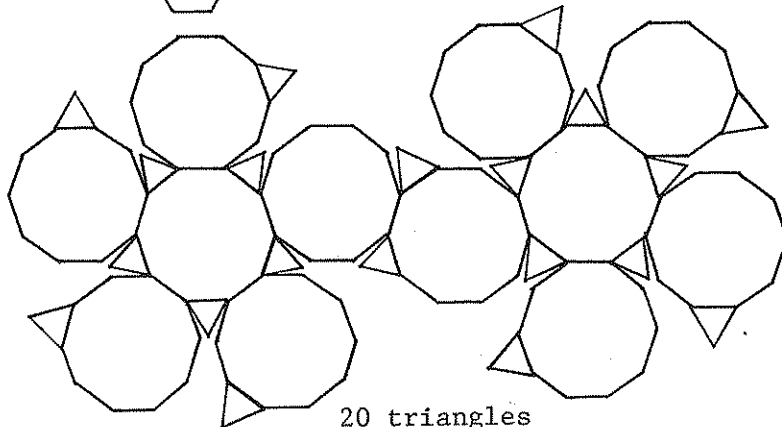
20 triangles
12 decagons
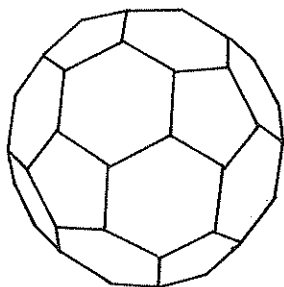
Truncated Icosahedron
5-6-6
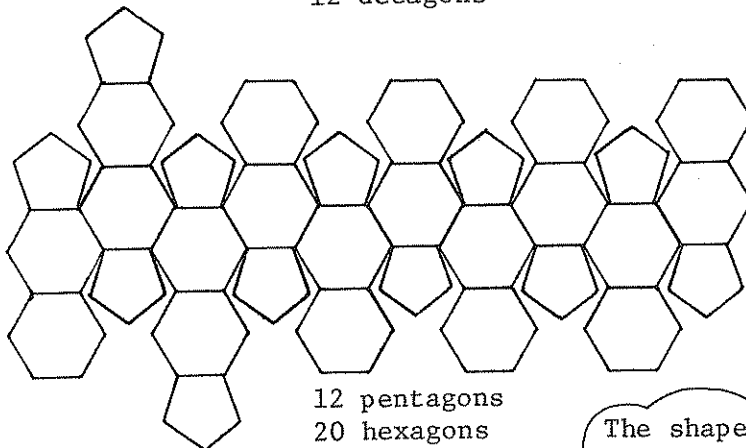
12 pentagons
20 hexagons
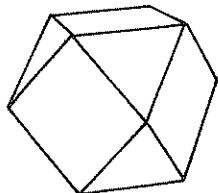
Cuboctahedron
3-4-3-4
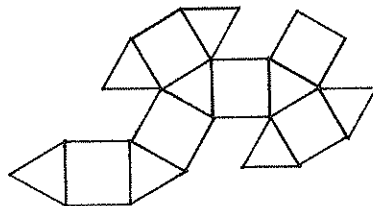
8 triangles
6 squares

The shape is a cube with its corners cut off at the centers of each edge or an octahedron with its corners cut off at the centers of each edge.
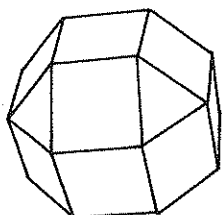
Small Rhombicuboctahedron
3-4-4-4
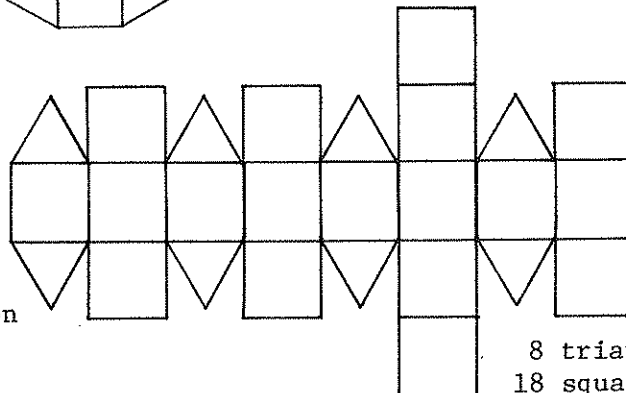
This solid is a cuboctahedron with its corners cut off at the centers of each edge.

8 triangles
18 squares

POLYHEDRA (PAGE 3)
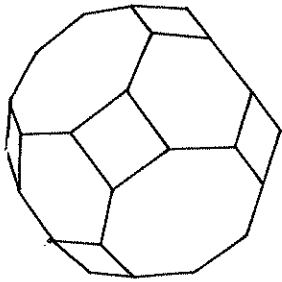
Great Rhombicuboctahedron
or Truncated Cuboctahedron
4-6-8
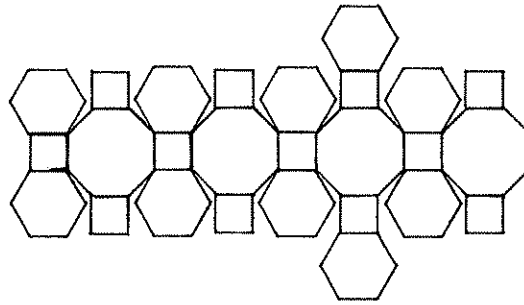
12 squares
8 hexagons
6 octagons

This shape is formed by cutting off the corners of the cuboctahedron 1/3 of the way along each side.
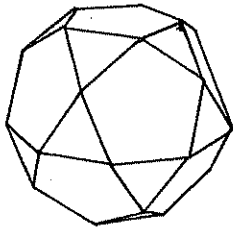
Icosidodecahedron
3-5-3-5
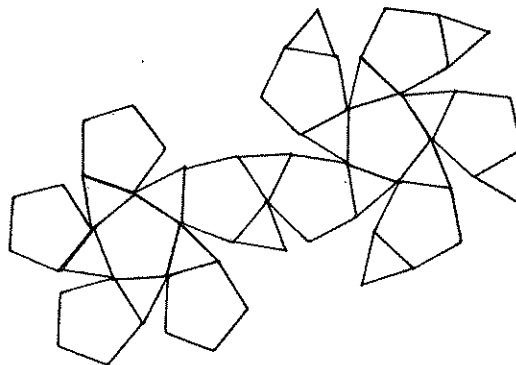
20 triangles
12 pentagons

This shape is formed by cutting off the corners of the dodecahedron 1/2 of the way along each side.
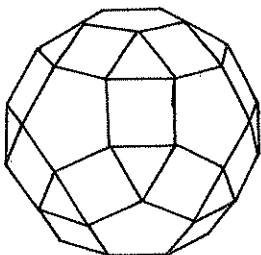
Small Rhombicosidodecahedron
3-4-5-4
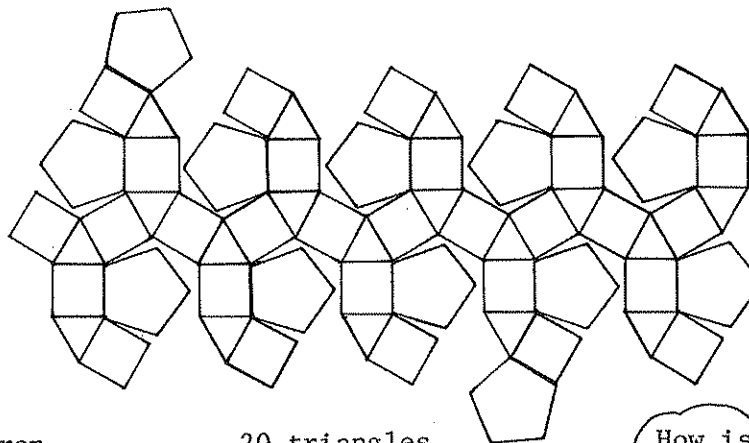
20 triangles
30 squares
12 pentagons

How is this shape related to the icosidodecahedron?

POLYHEDRA (PAGE 4)

Great Rhombicosidodecahedron
or Truncated Icosidodecahedron
4-6-10
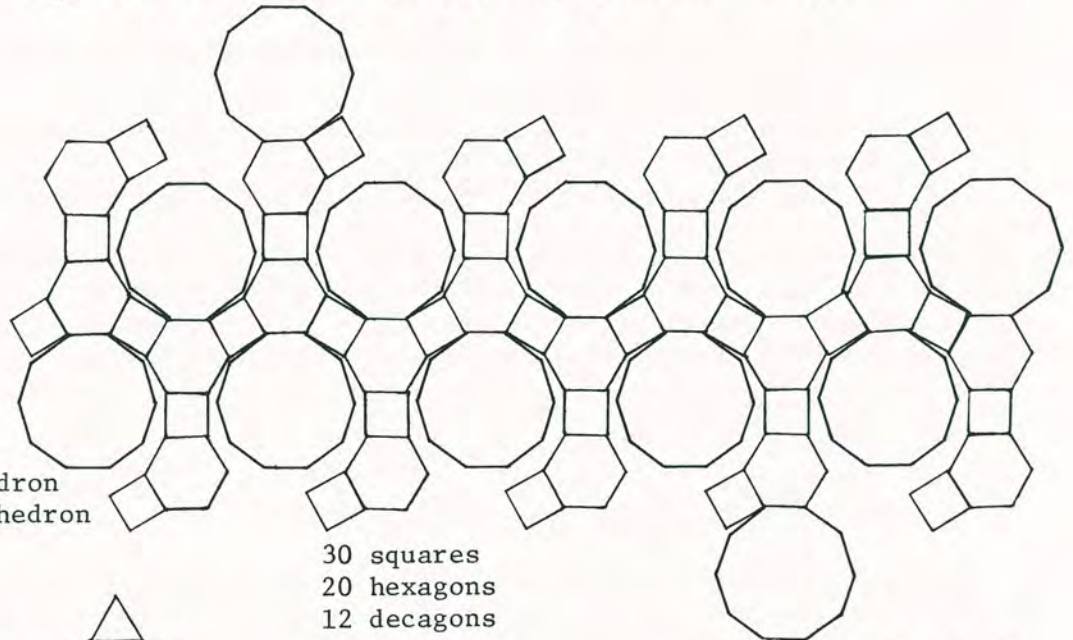
30 squares
20 hexagons
12 decagons

How is this shape
related to the
icosidodecahedron?

Snub Cube
3-3-3-3-4
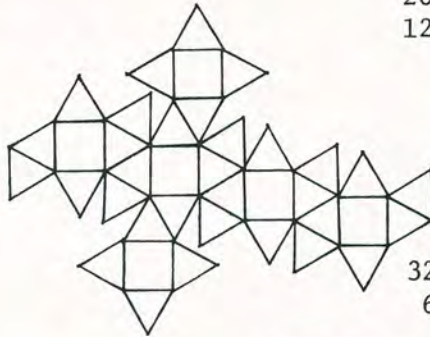
32 triangles
6 squares

Snub Dodecahedron
3-3-3-3-5
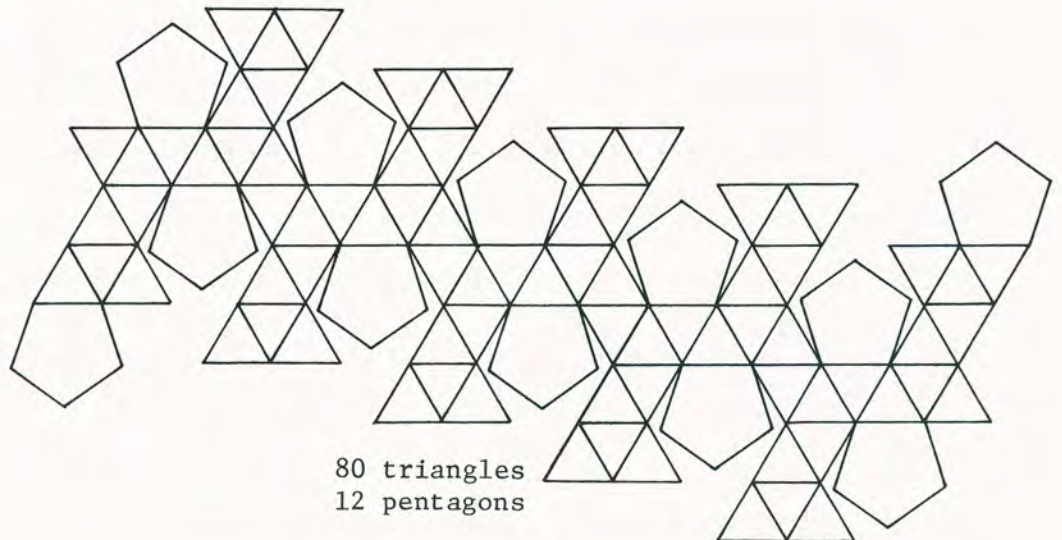
80 triangles
12 pentagons

Two sources that provide patterns
for the solids are

1) <u>Math Projects: Polyhedral
   Shapes</u>, Bassetti, Ruchlis,
   Malament, Book-Lab, Inc., 1968

2) <u>Geo-ring Polyhedra</u>, Linda
   Silvey, Creative Publications,
   Inc., 1972

ONLY FIVE OF THEM

Students can be shown that only five regular convex polyhedra are possible. The "proof" involves two conditions.

1) At any vertex of a convex polyhedron the sum of the measures of the angles is less than 360°.
2) At least three planes intersecting in space are needed to form a vertex.

To illustrate the first condition have students try to fold one of the figures in diagram 1 to make some of the faces of a polyhedra. They soon discover it can't be done without bending or distorting the faces. Have them cut along the dotted line. The polygons will now fold but two of the faces will overlap. See diagram 2.
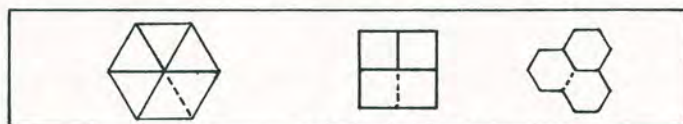
DIAGRAM 1
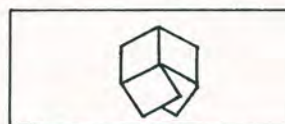
DIAGRAM 2

The second condition becomes clear by folding just two triangles or two squares. No rigid vertex is determined but students should see how another triangle or another square could be added to make a rigid vertex. See diagram 3.
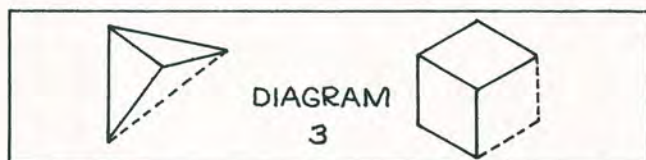
DIAGRAM 3

Since the regular polyhedra have regular polygons for faces, the possibilities can be listed.

1) Three triangles (60° per angle or 180° about the vertex)

2) Four triangles (60° per angle or 240° about the vertex)

3) Five triangles (60° per angle or 300° about the vertex)

4) Three squares (90° per angle or 270° about the vertex)

5) Three pentagons (108° per angle or 324° about the vertex)
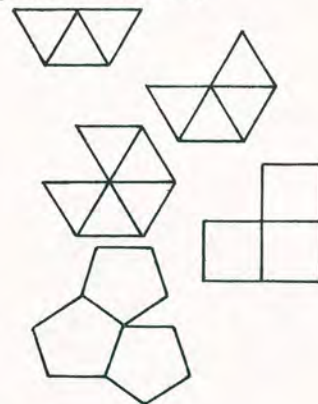
Six triangles, four squares, four pentagons, three hexagons, etc. all contradict condition (1). See diagram 1 above.

The possibilities (1 - 5) are, in fact, the patterns needed for the five Platonic solids, the tetrahedron, the octahedron, the icosahedron, the hexahedron (cube) and the dodecahedron.

PRISMS & PYRAMIDS

1) PRISM
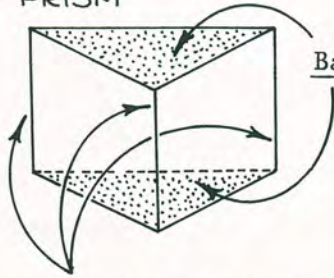
Bases are parallel and congruent.

Lateral edges are parallel and the same length.

3) What shapes can be used for the bases of prisms? _____

4) What shape do the rest of the faces have? _____

5) Give a name to each prism in (2) from the shape of its base.

2) Shade the bases of these prisms.

a

b

c

d

f

---
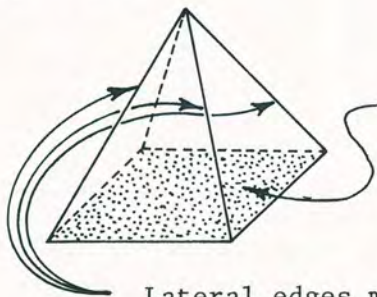
6) PYRAMID

Base

Lateral edges meet at one point.

8) What shapes can be used for the bases of the pyramids? _____

9) What shape do the rest of the faces have? _____

10) Give a name to each pyramid in (7).

7) Shade the base of each pyramid.

a

b

c

d

e

f

365

DO YOU KNOW THAT

I   A pyramid is named by the shape of its base.
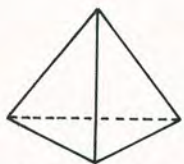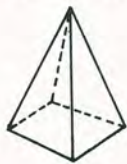
Triangular   Square   Pentagonal   Hexagonal   Octagonal
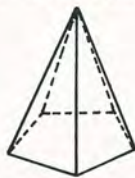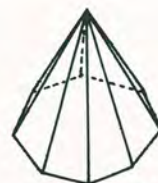
1) Use the drawings to help you complete the table.

Do you agree with the answers on the first line? _____

2) Look for patterns in the table.

Pyramid	Number of sides in base	Number of faces	Number of vertices	Number of edges
Triangular	3	4	4	6
Square				
Pentagonal				
Hexagonal				
Octagonal				

a)   If the base of a pyramid has 10 sides, the number of faces is _____.
b)   If the base of a pyramid has 12 sides, the number of vertices is _____.
c)   If the base of a pyramid has 14 sides, the number of edges is _____.

3)   If the base of a pyramid is a polygon with 20 sides, the number of faces is _____; the number of vertices is _____; the number of edges is _____.

II   A prism is also named by the shape of its base.
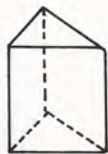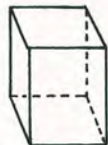
Triangular   Square   Pentagonal   Hexagonal   Octagonal
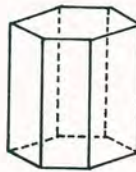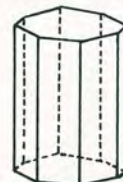
1) Use the drawings to help you fill in the table.

2) Look for patterns in the table.

a)   If each base of a prism has 10 sides, the number of faces is _____.

Prism	Number of sides in base	Number of faces	Number of vertices	Number of edges
Triangular	3	5	6	9
Square				
Pentagonal				
Hexagonal				
Octagonal				
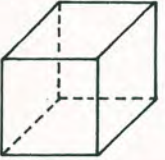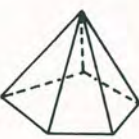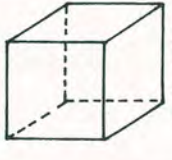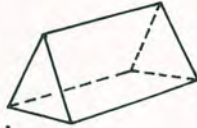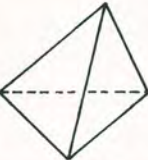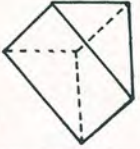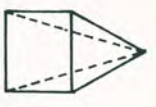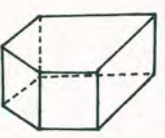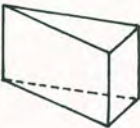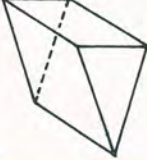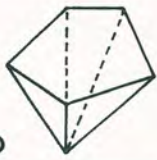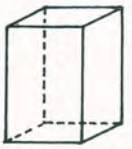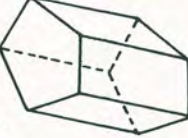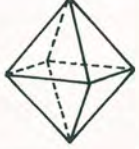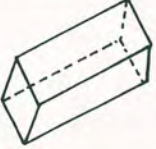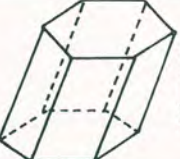
b)   If each base of a prism has 12 sides, the number of vertices is _____.
c)   If each base of a prism has 10 sides, the number of edges is _____.

3)   If each base of a prism is a polygon with 20 sides, the number of faces is _____; the number of vertices is _____; the number of edges is _____.

YOU DECIDE...

THESE ARE PYRAMIDS ⟷ THESE ARE NOT PYRAMIDS

THESE ARE PRISMS ⟷ THESE ARE NOT PRISMS
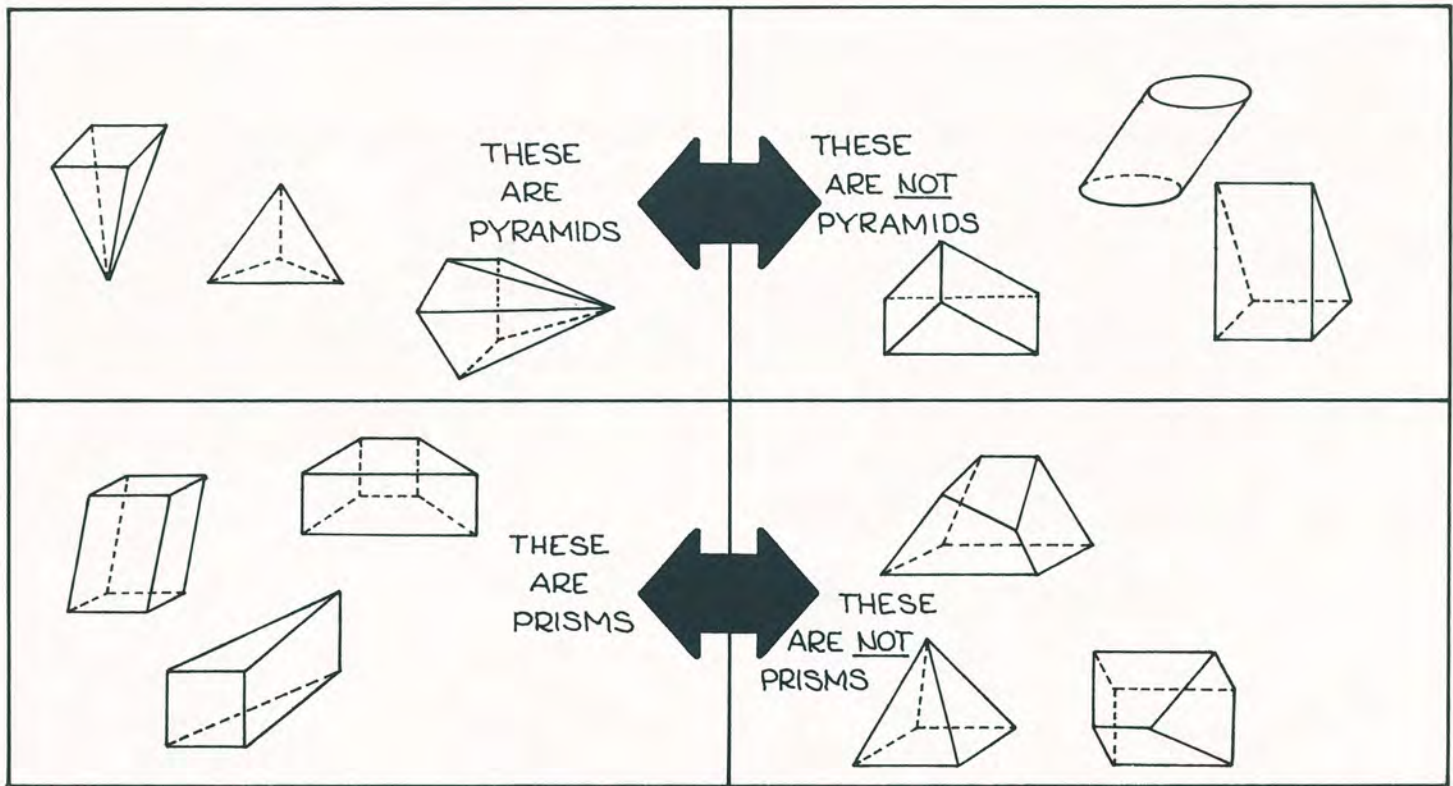
X-out the boxes with incorrect words. Copy in order (from left to right) the letters in the remaining boxes on the blanks below.

C — PYRAMID	D — PRISM	A — PYRAMID	I — PRISM	M — PRISM
U — PRISM	M — PYRAMID	F — PYRAMID	M — PRISM	Y — PYRAMID
, — PRISM	H — PYRAMID	S — PRISM	P — PYRAMID	R — PRISM
D — PYRAMID	I — PRISM	T — PYRAMID	S — PRISM	M — PRISM

RIDDLE: WHAT IS AN EGYPTIAN PYRAMID?

__ __ __ __ __ __ __ __ __ __ __ __ __ __ __ __ __

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TWO-D OR NOT TWO-D

Pictures of polyhedra can be drawn on a flat surface by following these steps.

For Prisms   A)
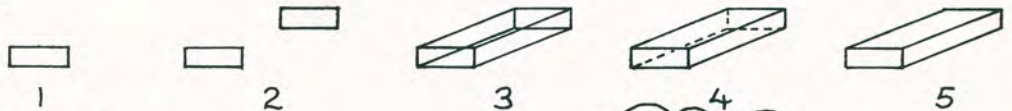
1) Draw a base.
2) Draw the other base.
3) Connect the vertices.
4) Dot the hidden edges,
or 5) leave out the hidden edges.

1   2   3   4   5

THIS VIEW SHOWS THE TOP.

B)

1   2   3   4
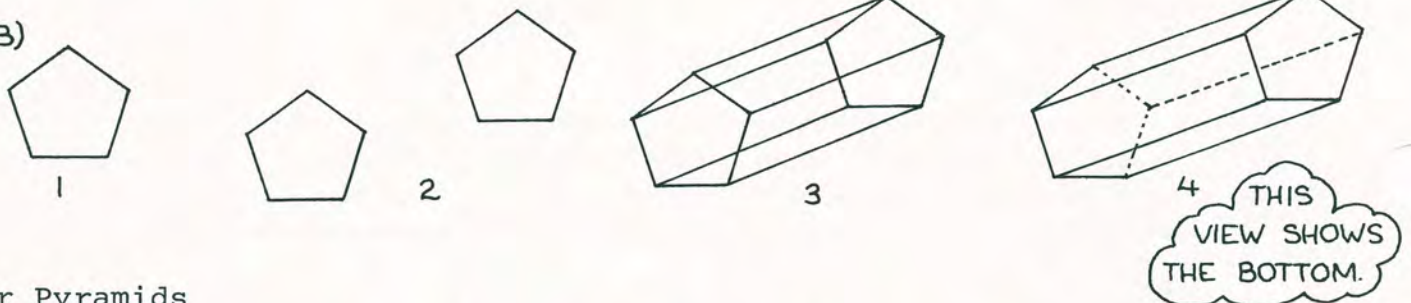
THIS VIEW SHOWS THE BOTTOM.

For Pyramids

1) Draw the base.
2) Draw a dot for the point.   C)
3) Connect the point to the vertices.
4) Dot the hidden edges,
or 5) leave out the hidden edges.   D)

1   2   3   4   5

1   2   3   5
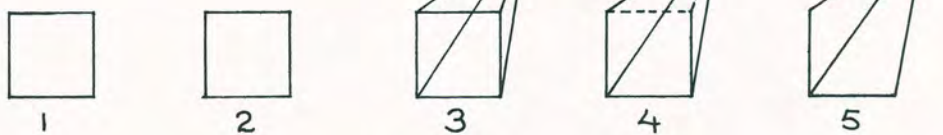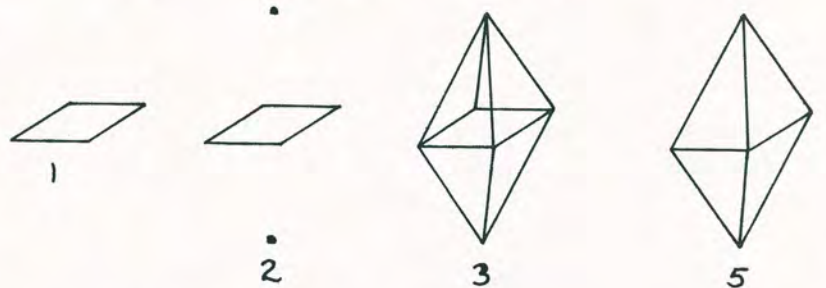
Draw these polyhedra. Grid paper may be helpful.

1) Rectangular prism   2) Hexagonal prism
3) Triangular pyramid   4) Octagonal pyramid

IDEA FROM: *Geometric Excursions*

TWO-DIMENSIONAL REPRESENTATIONS

Dot paper, square grid paper, isometric grid paper or a template can be used by students to help them represent three-dimensional polyhedra and other models in two dimensions. The following are examples that use the steps outlined on the page *Two-D or Not Two-D*. Each of these techniques could be developed into a student worksheet.

1) Dot Paper
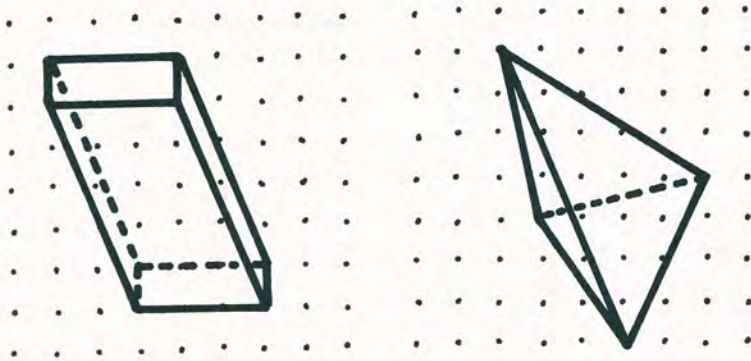
2) Square Grid Paper

3) Isometric Grid Paper
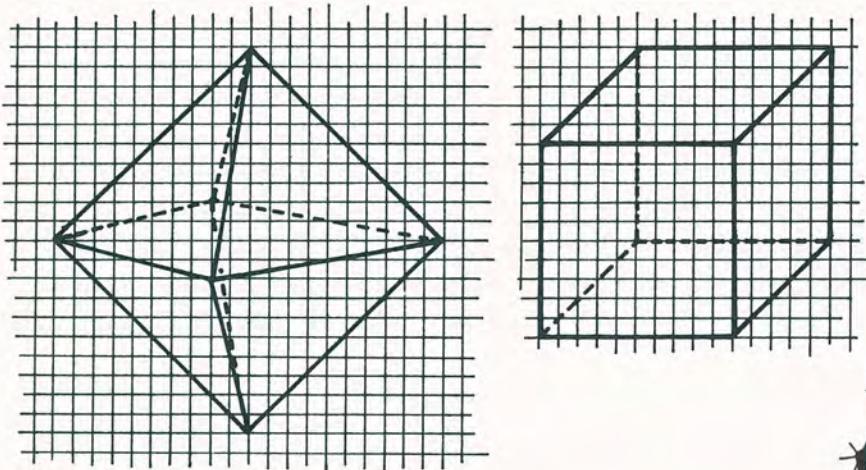
4) Template To Draw the Base
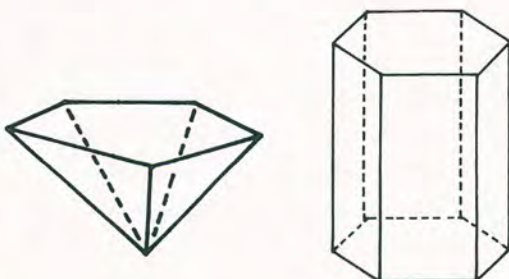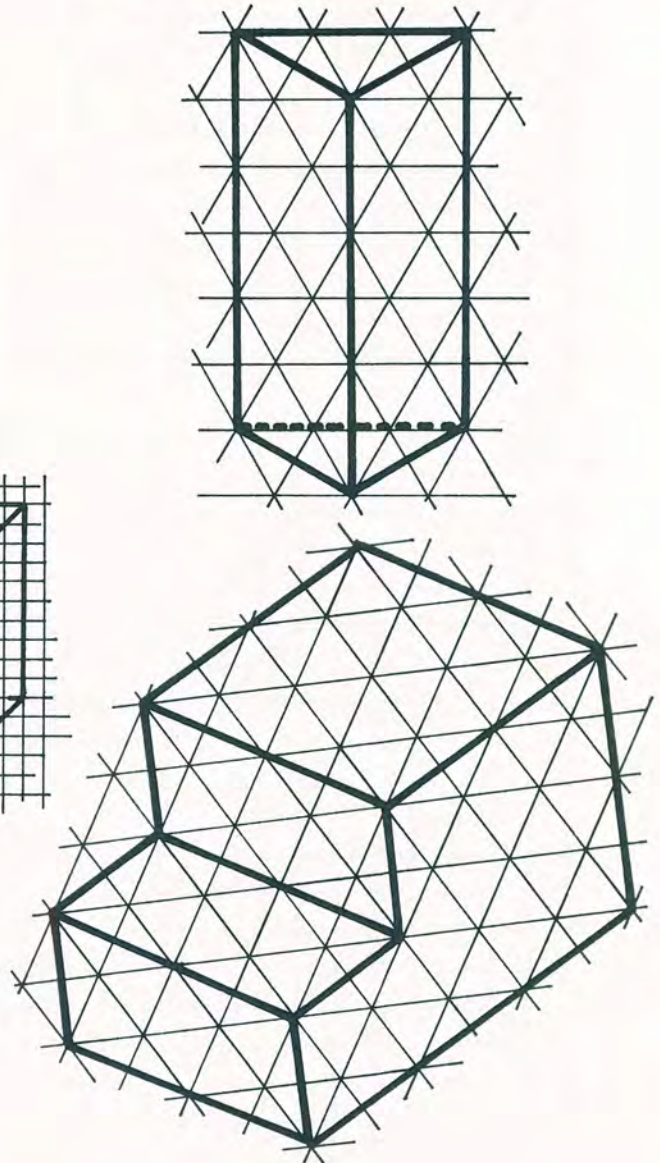
BLOCK IT
1

I   Make a drawing beside each block so there are pairs of identical blocks.
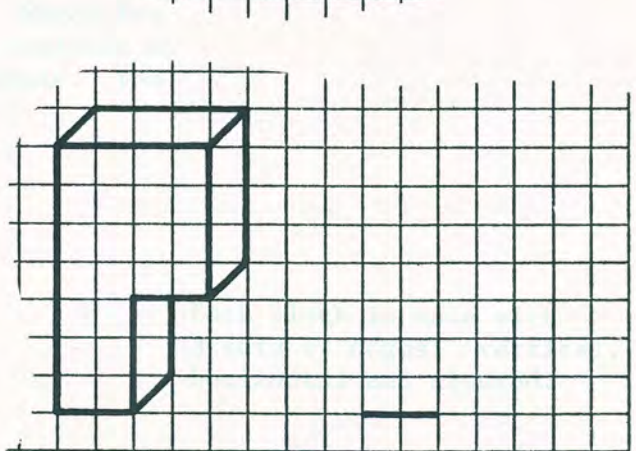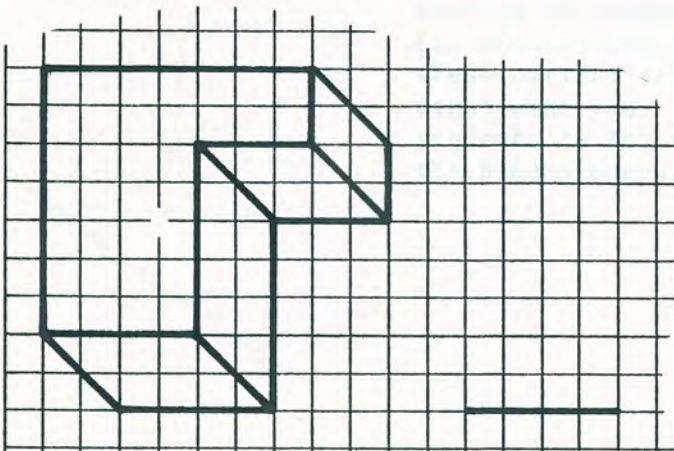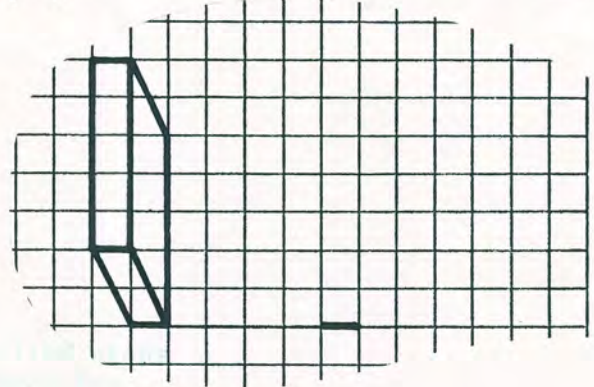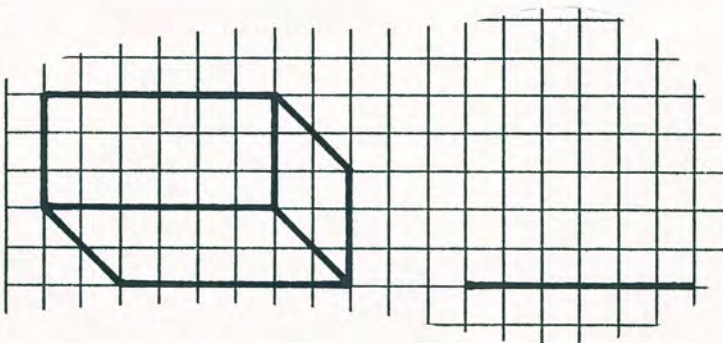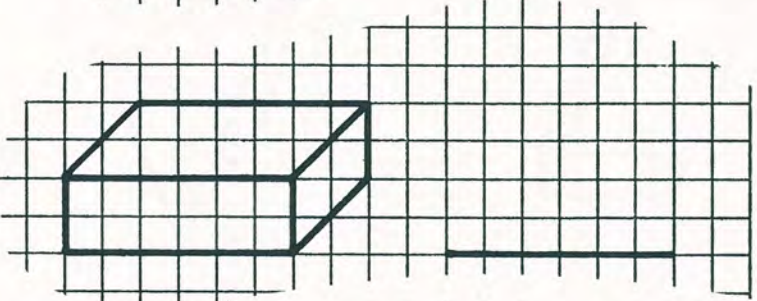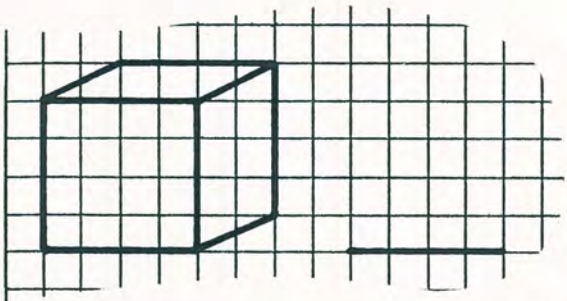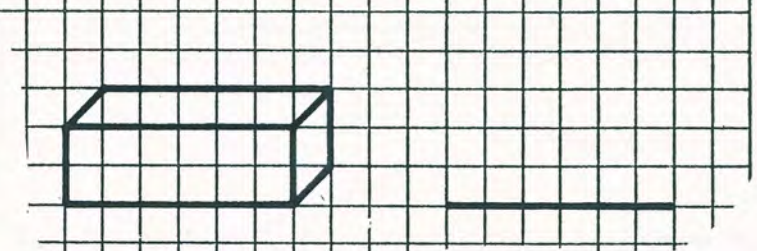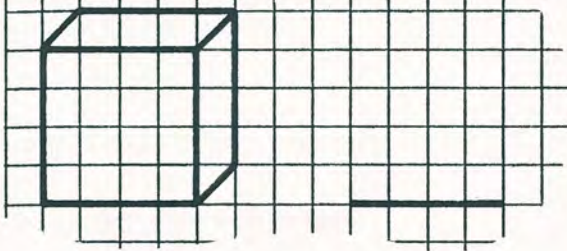
II   What do you notice about the edges of the blocks above?

BLOCK IT 2
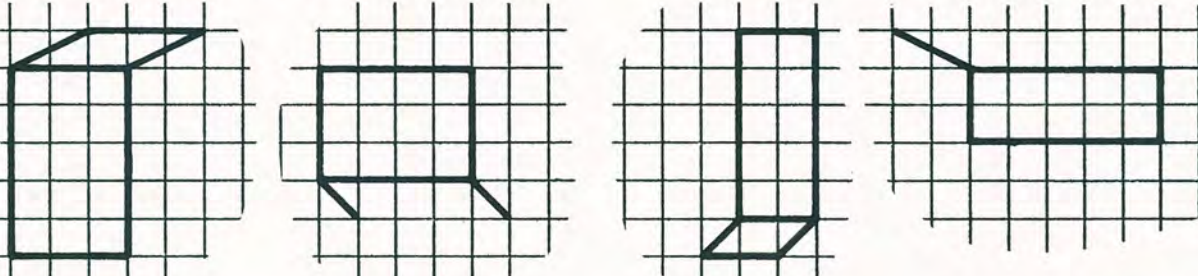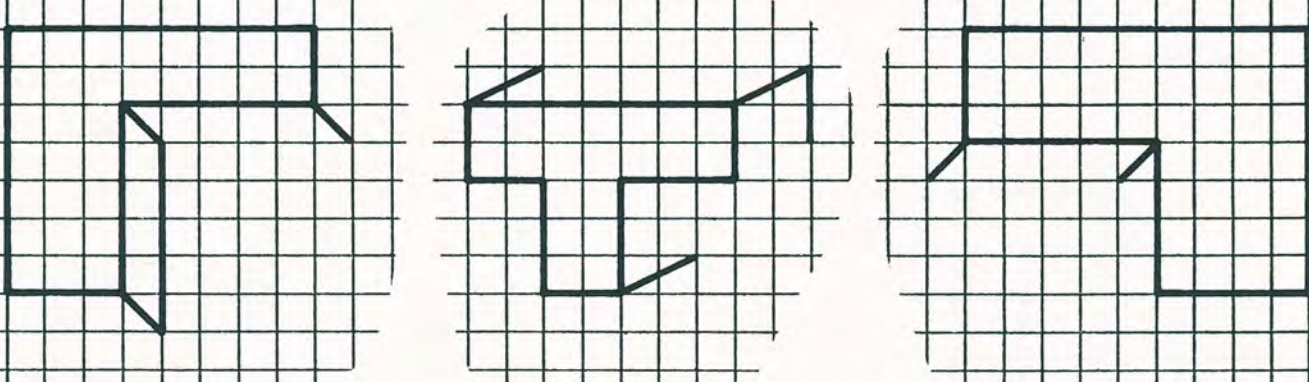
1) This was supposed to be a picture of a cube. What's wrong with it? Can you make a correct drawing here?
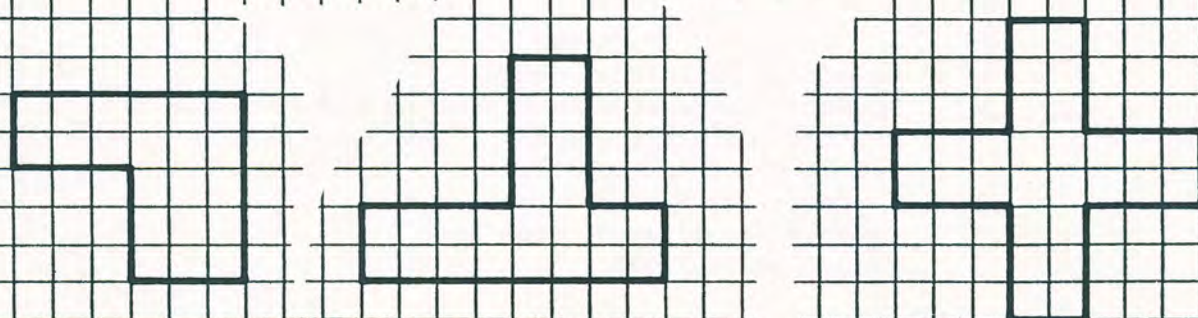
2) Make a correct drawing of this box.

3) There are edges missing in the picture of the rectangular blocks shown below. See if you can find all the missing edges and draw them in. Some of the blocks have a piece cut out.

4) Add edges to these shapes so they look like blocks with smaller blocks cut out.

5) On another sheet of graph paper make some drawings of blocks that look 3-D.

TOO MANY EDGES!

Some pictures of blocks are shown below, but there are too many edges showing! Take a piece of tracing paper and trace only those edges which would show if the blocks were solid.
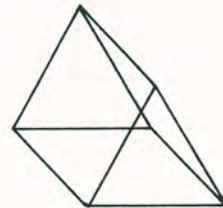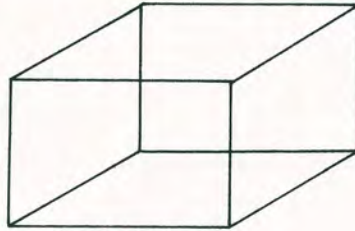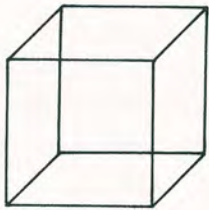
Compare your tracings with those of your classmates. Some of you probably got different figures.
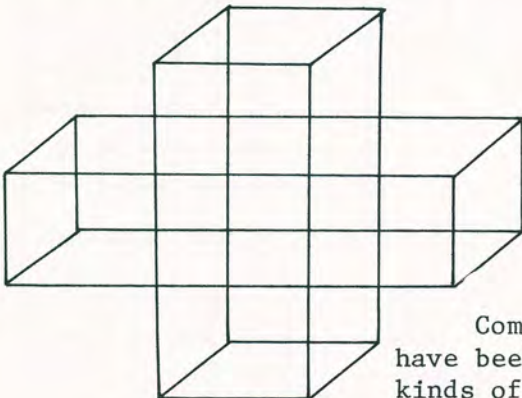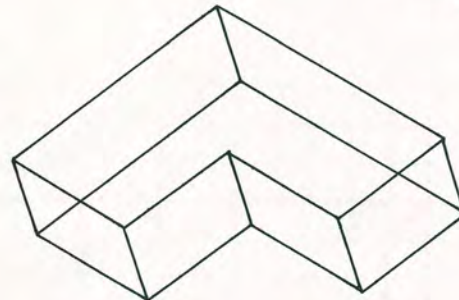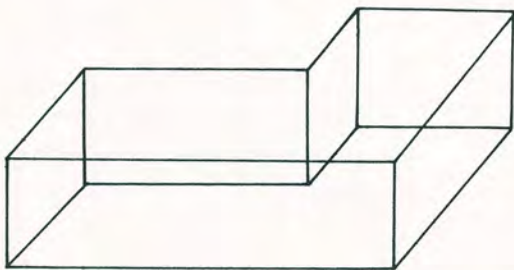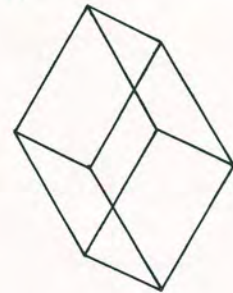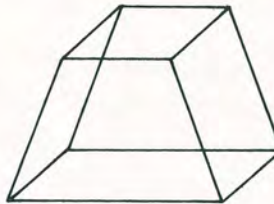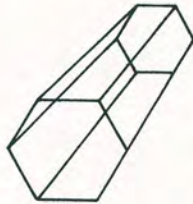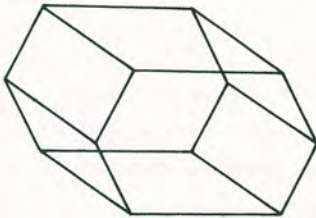
Computer programs have been written so these kinds of pictures with or without the extra edges can be drawn by computers. <u>Geometry</u>, by Harold Jacobs, shows a complicated computer design on page 593.
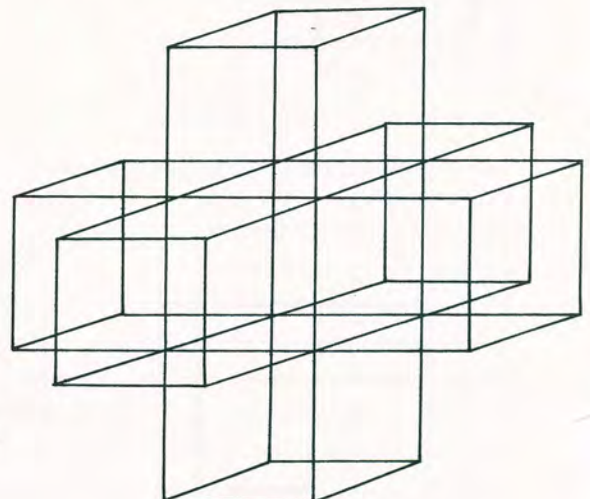
CROSS SECTIONS

1) The plane is cutting the square
   pyramid to get a cross section
   that is a square.
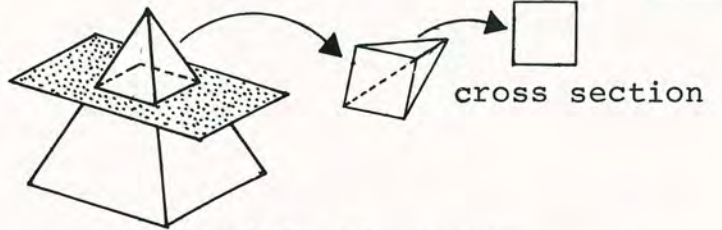
cross section

This plane gives a cross
section that is a kite.

What will this cross
section show? _____
_____
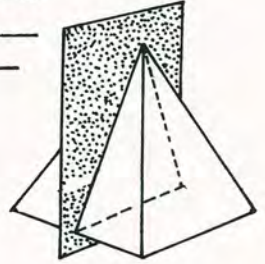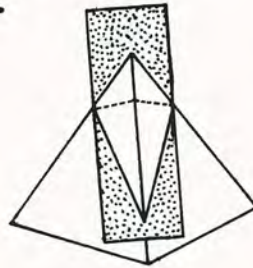
2) Name the shape of each of these cross sections of an octahedron.
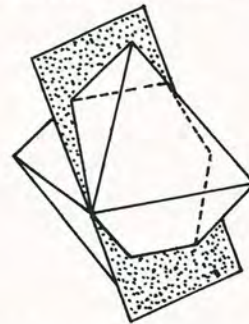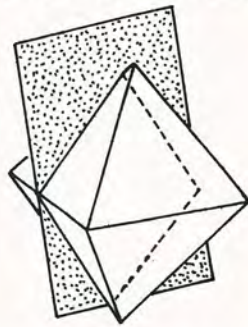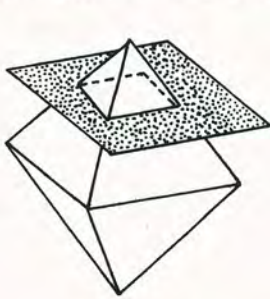
3) Name the shape of each of these cross sections of a cube.
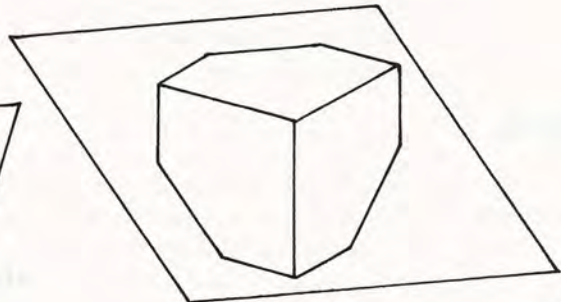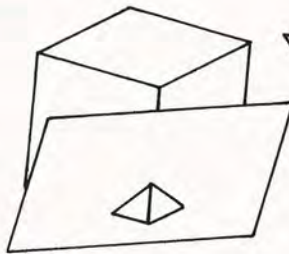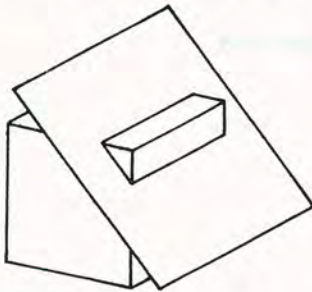
4) Where would you make a
   cut in this triangular
   prism to get a cross
   section that shows
   a) a triangle?
   b) a rectangle?

5) What different
   shapes can you
   get as cross
   sections of the
   tetrahedron?
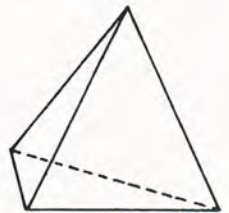
VERTICES, FACES and EDGES

Materials needed: Models of the five regular polyhedra

Activity:

1) Use each model to fill in this chart. A friend can help you keep track of the parts being counted.

Polyhedron	Number of Vertices	Number of Faces	Number of Edges
Tetrahedron	4	4	6
Cube	8	6	12
Octahedron	6	8	12
Dodecahedron	20	12	30
Icosahedron	12	20	30

2) Discover a rule that relates the number of vertices, faces and edges of a polyhedron.

3) If, after studying the chart, you did not discover the rule, trace the dotted lines at the bottom of this page. Read the message by turning the page upside down and not using the wavy lines.

4) Suppose part of the tetrahedron was sliced off. Does the formula still work?

   V   F   E
   6   5   9
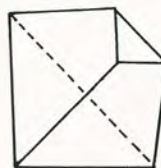
5) If two corners of the cube (hexahedron) are sliced off, does the formula work?

   V   F   E
   12   8   18
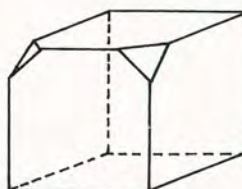
6) If parts of the top and bottom of the octahedron are sliced off, does the formula work?

   V   F   E
   12   10   20
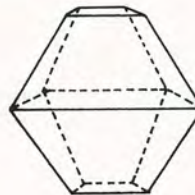
7) The formula was discovered by Leonard Euler in the 18th century. Exhibit "real life" polyhedra to your students and have them verify Euler's formula. Examples might be a picture of a house or Egyptian pyramid, an unsharpened pencil, a milk carton, a 2 x 4 board, an architect's ruler, etc.

TIPS: Activity Card/Worksheet/
        Manipulative

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DECIDE WHETHER OR NOT EACH OF THESE 3-DIMENSIONAL SHAPES SATISFIES
EULER'S FORMULA FOR VERTICES, FACES AND EDGES: $V + F = E + 2$

1)

2)

3)

4)

5)

BRICKS AND POLYHEDRA

1) Mac, a mathematician, was building a brick fence along his property line. In how many different ways can Mac place a brick on the fence? _____

2) Suppose the brick was decorated with a wedge-shape on opposite sides.

   a) How many ways can the brick be placed so the wedge is pointing to the right? _____, to the left? _____, in either direction? _____

3) By the gate Mac wants to create an unusual effect and use cubical bricks. In how many different ways can he place one of these bricks? _____

4) How many ways can the door knob be placed in the square hole? _____

5) How many ways will each polyhedron fit snugly into the hole below it?

TRIANGULAR PRISM   SQUARE PYRAMID   REGULAR TETRAHEDRON   REGULAR OCTAHEDRON   14-SIDED POLYHEDRON

MOM, DAD AND THE KIDS

I   How many different ways can the lids be placed on these containers?

1) Rectangular bread container _____

2) Square sandwich container _____

3) Circular cake container _____

II   How many different ways can the hex wrench, the wrench or the pliers be placed on the appropriate nut?

1) _____

2) _____

3) _____
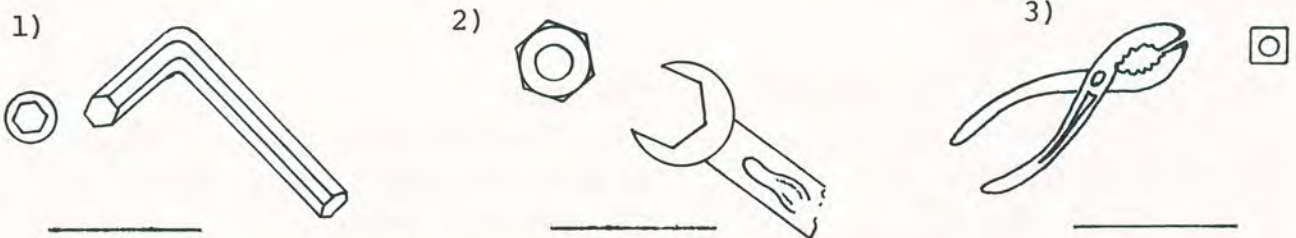
The kids just got a new game where they try to see who can put the pieces of the game into the appropriate hole in the shortest amount of time.

III   1)   How many different ways can each piece be placed in the game board so the shaded side is showing?
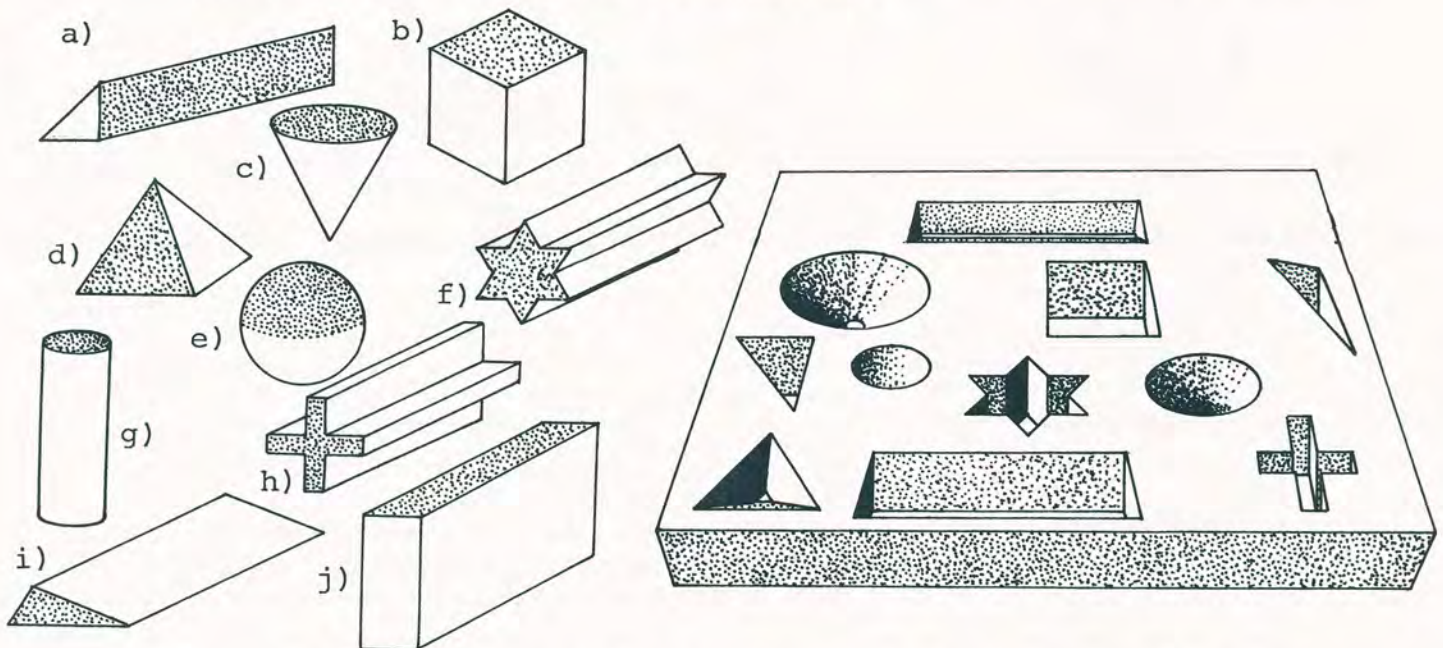
a)

b)

c)

d)

e)

f)

g)

h)

i)

j)

2) How many different ways can each piece be placed in the game board if it doesn't matter if the shaded side is up?

a) _____   b) _____   c) _____   d) _____   e) _____

f) _____   g) _____   h) _____   i) _____   j) _____
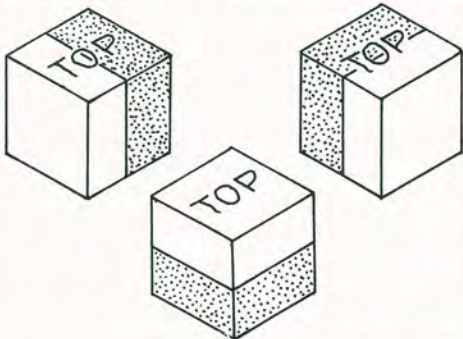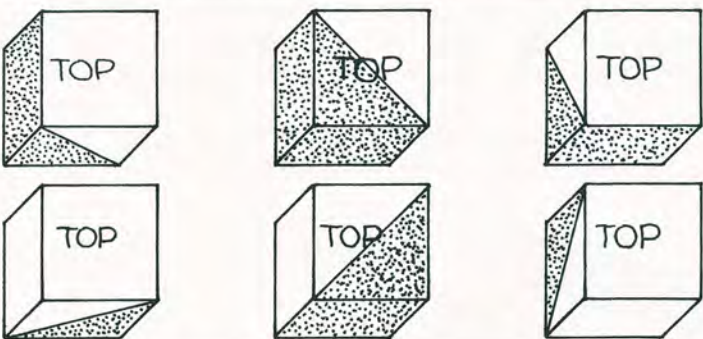
PLANES OF SYMMETRY

Materials: Play-Doh or flour-based clay
            Wire to cut cubes
            Mirrors - if desired

This activity involves finding planes of symmetry in the cube. The students can make cubes from the Play-Doh, then cut through the cubes with the wire, using a sawing motion. Halves of the cube can be placed against a mirror to see if an entire cube appears to be formed.

Students could work together in small groups making examples of as many different planes of symmetry as they can find. You might want to tell them that there are nine in all for the cube.

THE NINE PLANES OF SYMMETRY FOR THE CUBE

PLANES PARALLEL TO OPPOSITE FACES	PLANES PASSING THROUGH PAIRS OF OPPOSITE EDGES

Planes of symmetry for the Platonic solids are summarized below.

Tetrahedron	6	each contains one of the six edges
Cube	9	see above diagrams
Octahedron	9	3 planes, each passing through 2 pairs of opposite vertices and 6 planes, each passing through midpoints of a pair of opposite edges and a pair of opposite vertices.
Dodecahedron	15	each passing through a pair of opposite edges
Icosahedron	15	
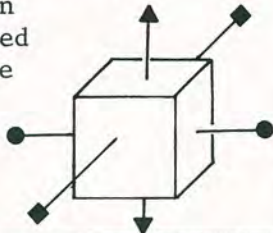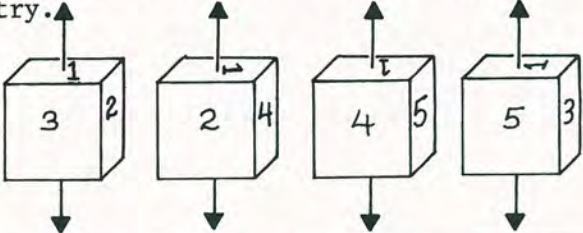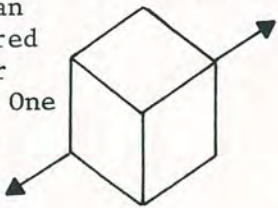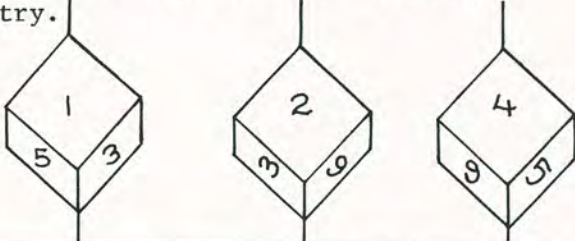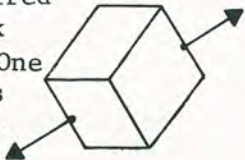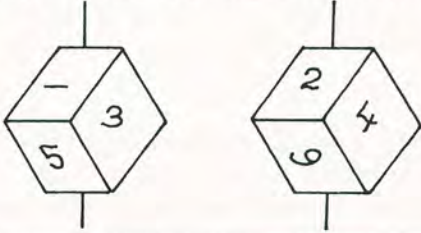
AXES OF SYMMETRY

Materials: Styrofoam or foam rubber cubes and (if desired) models of other Platonic solids
Knitting needles or pieces of wire from coat hangers

This activity involves finding the axes (lines) of rotational symmetry in the Platonic solids. The axes of rotational symmetry of the cube are shown as an example. The axes go through pairs of opposite faces, vertices and edges.

F A C E S	There are six faces. They can be paired in three ways.	Each of these three axes has fourfold symmetry.
V E R T I C E S	There are eight vertices. They can be paired in four ways. One way is shown.	Each of these four axes has threefold symmetry.
E D G E S	There are twelve edges. They can be paired in six ways. One way is shown.	Each of these six axes has twofold symmetry.

Axes of symmetry for the Platonic solids are summarized below.

Platonic solid	Twofold	Threefold	Fourfold	Fivefold
Tetrahedron	3 THREE PAIRS OF NONINTERSECTING EDGES	4 FOUR PAIRS OF A FACE AND ITS OPPOSITE VERTEX	——	——
Cube	6 SIX PAIRS OF OPPOSITE EDGES	4 FOUR PAIRS OF OPPOSITE VERTICES	3 THREE PAIRS OF OPPOSITE FACES	——
Octahedron	6 SIX PAIRS OF OPPOSITE EDGES	4 FOUR PAIRS OF OPPOSITE FACES	3 THREE PAIRS OF OPPOSITE VERTICES	——
Dodecahedron	15 FIFTEEN PAIRS OF OPPOSITE EDGES	10 TEN PAIRS OF OPPOSITE VERTICES	——	6 SIX PAIRS OF OPPOSITE FACES
Icosahedron	15 FIFTEEN PAIRS OF OPPOSITE EDGES	10 TEN PAIRS OF OPPOSITE FACES	——	6 SIX PAIRS OF OPPOSITE VERTICES

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SPACE-FILLING FORMS

I   Show containers used in packing commodities. For example: cereal box, tennis ball can, milk carton, egg carton. Ask students to give some examples. Can they give reasons for the particular shape being chosen? For example, cylinders are often used because they are easy to make and handle. Suggest that they work for a company which has just developed a new breakfast cereal. Have them name the product, design a packet to contain the cereal and give reasons for the shape of the container.

II In *Tessellations* flat shapes were fitted together to cover a plane. Any tessellation of plane figures can be extended to form space-filling prisms.

The Devil's Post Pile National Monument in California is a natural example. Many of the columns have the shape of right regular prisms, most of them are hexagonal and others are pentagonal.
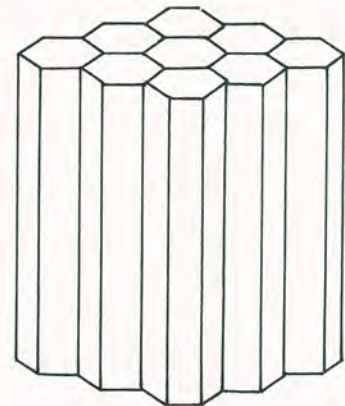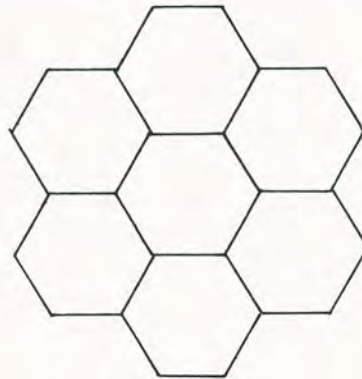
Ask students for other examples of space-filling arrangements of

a)   rectangular prisms

b)   hexagonal prisms

Supply a net for a quadrilateral prism. Have them use their net to make five prisms. Students can then tessellate with the prisms. Ask students to design a net for a triangular-ended prism. Have them use their net to make five prisms. Students can then tessellate with the prisms.
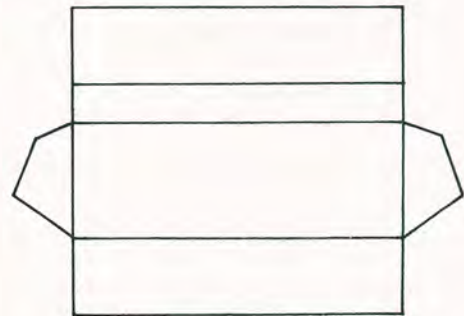
III In groups have students investigate filling space with

a)   cubes
b)   regular tetrahedra
c)   regular octahedra (make 10)

If models are not available, students should make about 20 of each by plaiting or by cutting out nets. The edges of these models should all have the same length.

Students will discover that neither the tetrahedra nor octahedra will fill space by themselves. Have them try using both together. If their arrangement of tetrahedra and octahedra were extended indefinitely what would be the ratio of tetrahedra to octahedra?

SPACE-FILLING
FORMS

Have students investigate to see if other regular polyhedra (dodecahedra and icosahedra) will tessellate space either by themselves or combined with other polyhedra.

Do any of the Archimedian solids fill space either by themselves or combined with any other polyhedra?

IV Have each student make a <u>rhombic dodecahedron</u>. The polyhedron can be constructed by making cubes and pyramids and joining them in the following way:

a)

THE NET OF THE CUBE

Use the net to make a cube.

b)

THE NET FOR ONE OF THE PYRAMIDS.

Have students use the net to make 6 pyramids.
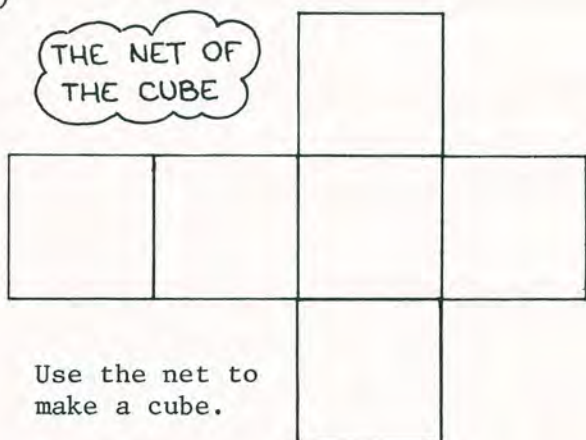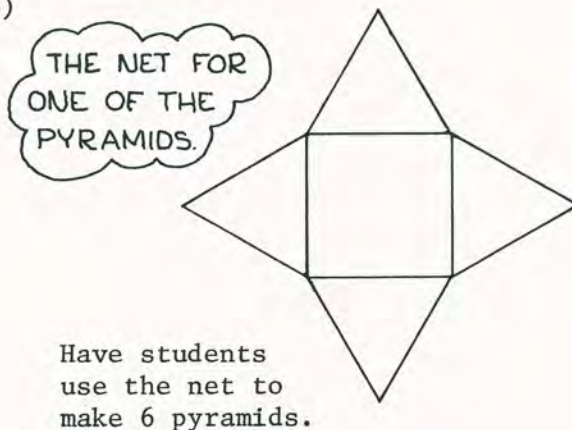
c) Use the cube and 6 pyramids to make the rhombic dodecahedron by attaching one pyramid to each face of the cube. Ask students to think about the cube and explain why the rhombic dodecahedron has 12 faces and why they are all rhombuses.
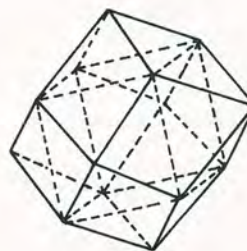
Ask students if they think rhombic dodecahedra will fill space? Have them combine models to check their guess.

V Have students examine crystals of some substances (table salt, alum, sugar, epsom salts, copper sulfate) under a magnifying glass or microscope and describe what they see. In table salt a majority of the crystals are approximately cubes; alum forms crystals in the shape of regular octahedra.

For more information on what crystals are, how to grow them with simple household articles plus the names of a few chemicals you can get from the druggist see: <u>Crystals and Crystal Growing</u>, Alan Holden and Phyllis Singer, Anchor Books, Doubleday and Company, Inc.

THEY CAME BY TWOS

Large models of the regular polyhedra are needed for this demonstration. A knowledge of Euler's formula is also necessary. $V + F = E + 2$ is the form that will probably work best. Explain to your students that you are trying to discover which polyhedra will fit inside other polyhedra so each vertex will touch a face.

1) Show students the model of the tetrahedron.

   a) Have them imagine a center for each face. How many centers (vertices) would there be? _____

   b) If centers of adjacent faces are connected, how many segments (edges) could be drawn? _____

   c) How many faces would a polyhedron have if it has 4 vertices and 6 edges? _____

   d) So another tetrahedron would fit inside this tetrahedron.

Show a graphic, a model or a picture at this time.
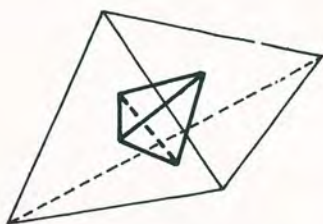
Some excellent pictures can be found in Mathematical Shapshots by H. Steinhaus.

Straw Polyhedra by Mary Laycock describes how to construct duals using straws.

2) Show students the model of the cube.

   a) Have them imagine a center for each face. How many centers (vertices) would there be? _____

   b) If centers of adjacent faces are connected, how many segments (edges) could be drawn? _____

   c) How many faces would a polyhedron have if it has 6 vertices and 12 edges? _____

   d) So an octahedron would fit inside a cube.

Show a graphic, a model or a picture.

(CONTINUED)

A similar development would show a cube would fit inside an octahedron. Likewise the icosahedron would fit inside the dodecahedron and the dodecahedron would fit inside the icosahedron. Both of these would be very difficult to visualize. A graphic, a model or a picture could illustrate these examples.

A chart like the one below can be used to summarize the data and point out <u>duals</u> among the regular polyhedra.

Polyhedron	Number of Vertices	Number of Faces	Number of Edges
Tetrahedron	4	4	6
Cube	8	6	12
Octahedron	6	8	12
Dodecahedron	20	12	30
Icosahedron	12	20	30

The tetrahedron is its own dual.

The cube and octahedron are duals of each other.

The dodecahedron and the icosahedron are duals of each other.

HOW MUCH IS LOST?

Vertex A of this cube is surrounded by three right angles. What is the total of the measures of these three angles? _____

If the three right angles were laid out flat, it could look like this.

When laid out flat there are 360° about vertex A, but the three right angles measure only 270°. The <u>angle loss</u> about vertex A is 90° (360° − 270°).

What is the angle loss at vertex B? _____

What is the angle loss for all eight vertices of the cube? _____

Fill in this chart for each of the regular polyhedra.

	Number of vertices	Angle loss at each vertex	Angle loss for the polyhedron
Tetrahedron			
Cube	8	90°	720°
Octahedron			
Dodecahedron			
Icosahedron			

Investigate the angle loss for each of the semi-regular polyhedra described on the page *Polyhedra*.

IDEA FROM: *The Themes of Geometry*

SCHLEGEL
AND HAMILTON

Imagine a cube made with elastic faces. Remove the top face and stretch the four top edges out and back until they lie flat on the table. The net of the cube in this position would look like the diagram at the right. Notice that one face has "disappeared" but the region outside the square represents that face.
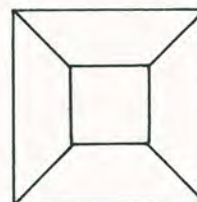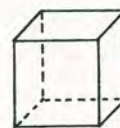
HOW MANY
VERTICES ? _____
EDGES ? _____
FACES ? _____

REMEMBER,
ONE FACE WAS
REMOVED.

This type of drawing is called a Schlegel diagram. It allows you to see all the vertices, edges and faces of a polyhedron at the same time.

Have students identify each of these as the Schlegel diagram of a tetrahedron, octahedron, dodecahedron or icosahedron.
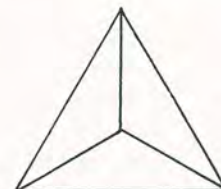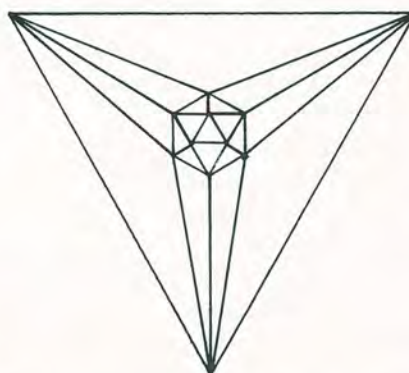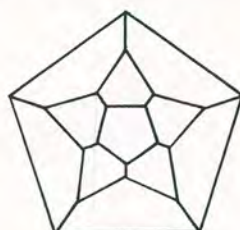
Have students color enlarged Schlegel diagrams using as few colors as possible. Two regions sharing the same edge can not be the same color. Remind students that the outside represents a face and should also be colored. The same coloring pattern could then be used to color the actual polyhedron.

A Hamilton line is a path along the edges that passes through each vertex exactly once. If you have solid models of the regular polyhedra, have students put a pin or tack at each vertex and then see if they can show a Hamilton line with string or yarn. The Schlegel diagram can also be used as a model to find a Hamilton line. For example a Hamilton line for the cube could be the solid line in this diagram. Have students draw Hamilton lines for the regular and other polyhedra.

Unlike the Euler line   there does not exist a general solution for the Hamilton lines of a network.

CONSTRUCTING NETS
OF IRREGULAR
TETRAHEDRA

1) Supply the students with several nets of tetrahedra to cut out and assemble. The edges can be fastened with tape.
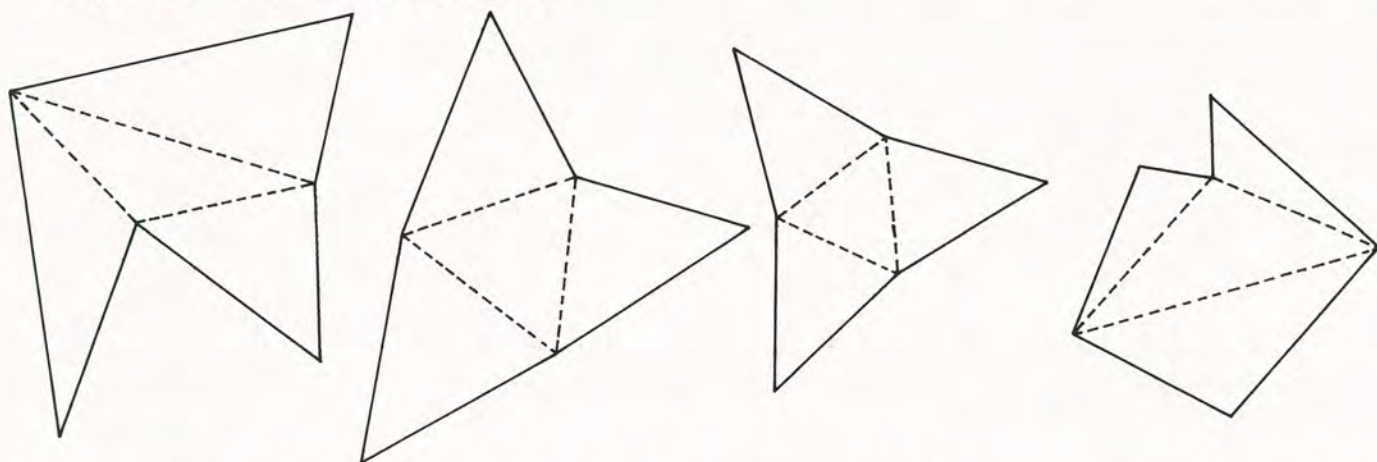
2) Give the students the instructions for constructing the nets. Do the first net as a class activity. As you demonstrate each step have the students repeat the procedure at their desks. Then have the students try one on their own.

a) Draw a triangle of any size and shape.
   Label the longest side a and the other two sides b and c and the vertices A, B, C as shown.
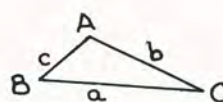
b) Construct a second triangle with base b. Be sure to select the vertex F so that ∠FAC is <u>about</u> the same size as ∠BAC.
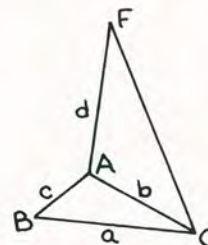
c) Set your compass to copy segment d. With A as the center and this compass setting draw an arc. Select a point D on this arc that lies inside ∠BCA but outside of triangle BAC. Draw triangle ABD.
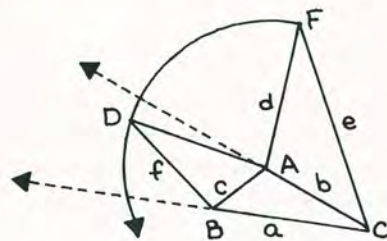
d) Set your compass to copy segment e. With C as the center and this compass setting draw an arc. Now set the compass to copy segment f. With B as the center and this compass setting draw an arc that intersects the previous arc. Label the point E. Draw $\overline{EB}$ and $\overline{EC}$ to complete the net.
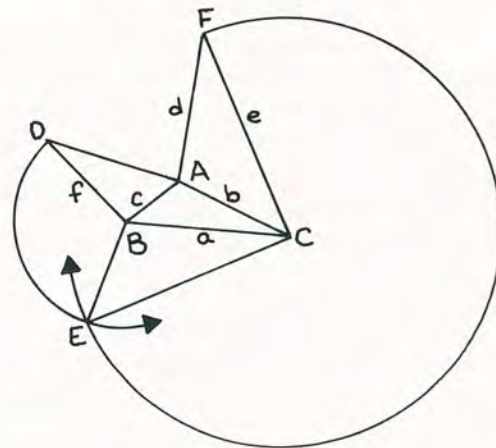
e) Cut out the net and assemble to make a tetrahedron.

COLLAPSOIDS

Collapsible and easily stored polyhedra can be constructed using baseless square pyramids. These nonconvex polyhedra models, invented by Jean J. Pedersen, are called collapsoids.

Instructions for constructing the 12-celled collapsoids:

1) Make 12 baseless square pyramids.
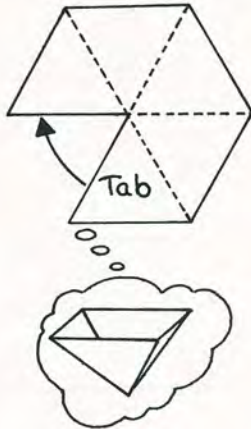
To construct the net of a baseless square pyramid

a) inscribe a hexagon in a circle (The finished collapsoid will have a diameter approximately equal to the diameter of the circle.

b) cut out the hexagon and crease to make six equilateral triangles.

REMOVE EXTRA TRIANGLE

Glue the tab in position to form the pyramid.

2) Cut 16 two-triangle tabs and crease as indicated.

Each tab is used to join two baseless pyramids.

GLUE BOTH UNDERNEATH SIDES

= ONE BASELESS PYRAMID LIKE
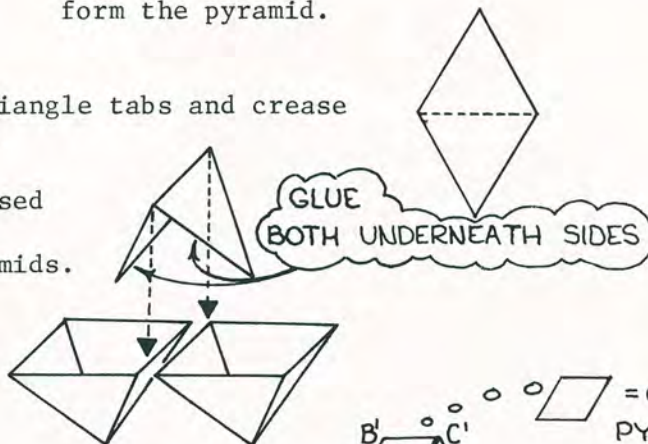
3) Join the twelve baseless pyramids to make a net like:

The heavy line segments represent the 16 tabs.
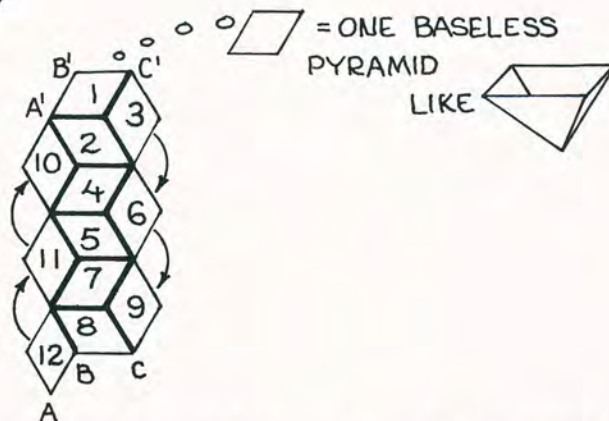
4) The edges of the net can be joined in two ways to form two models of the collapsoid.

a) Join AB to A' B' and BC to B' C' (ignore the arrows on the net diagram.) Secure the edges with paper clips for displaying.

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COLLAPSOIDS

(CONTINUED)

b) Join the edges indicated with arrows on the net (ignore the letters.)
    This model distorts slightly if the open edges are joined only with paper
    clips. To help correct this use the following scheme. You will need 8
    additional tabs.

Glue just this side in place.

Slide paper clip through hole to attach this side.
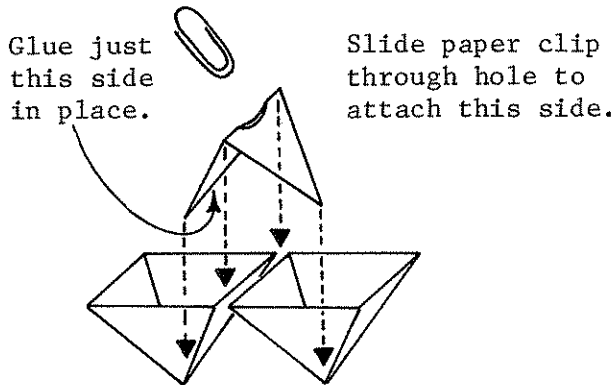
When assembled the edge should look like this.
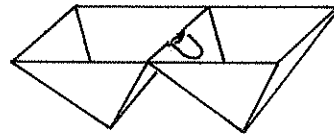
Two Models for 30-celled collapsoids.

MAKE 30.
USE A REGULAR DECAGON FOR A PATTERN.

Tab
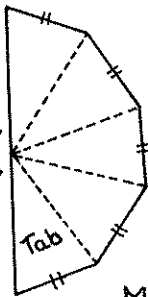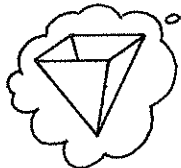
MODEL A

a) Join the 30 baseless
    pyramids to form the
    net.

    Then join AB to A' B',
    BC to B' C', CD to C' D',
    and DE to D' E' (ignore
    the arrows.)

    Edges may be secured
    with paper clips.

MAKE 30

Tab
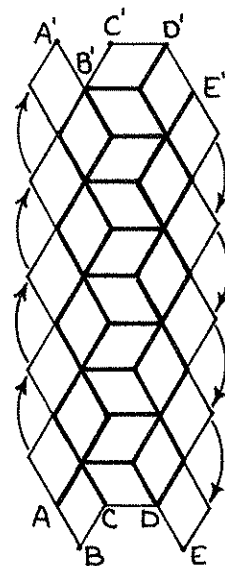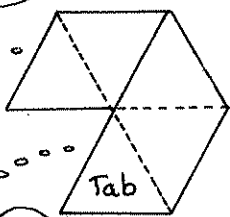
MODEL B

b) Join the 30 baseless
    pyramids to form the
    net.

    Complete the model
    by joining the edges
    indicated with arrows
    (ignore the letters.)

    Secure the edges using the scheme outlined in
    4(b) for 12-celled collapsoids.

For more information on collapsoids and instructions
for constructing 20- and 90-celled collapsoids see:
"Collapsoids," Jean J. Pedersen, The Mathematical
Gazette, 1975, Vol. 59, p. 81-94.

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

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POLYGONS & POLYHEDRA: POLYGONS

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POLYGONS

UNDERSTANDING THE TERMS FOR POLYGONS

What do your students understand about the term "rectangle"? Can they draw a rectangle when asked? Do they always assume a rectangle is longer in one direction than the other; that is, do they understand that a square is a special kind of rectangle? Can they define a rectangle so that only rectangles will fit the definition?
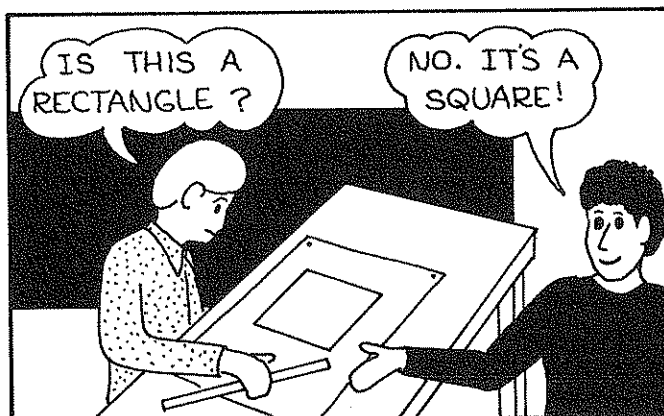
Probably most of your students can draw a rectangle, but some of them might not view squares as special rectangles. This can cause frustration with certain types of problems. Consider the problem: Find a rectangle whose diagonals are perpendicular. The student who has concluded "squares are not rectangles" is less likely to discover a solution since squares are the only rectangles with perpendicular diagonals.

Your students probably know that a triangle has three sides and a rectangle has four sides. Do they also know that "triangle" is the term for a general three-sided plane figure while quadrilateral (not rectangle) is the term for a general four-sided plane figure?

Geometry has the special problem of having a voluminous, special set of terms which are used even in middle school. Surely we want students to understand the meaning of the terms which are used in the classroom and in the problems they work. At the same time we do not want to make vocabulary the sole focus of the learning since the main purpose of vocabulary is to communicate about concepts or ideas. You will find *The Teaching of Concepts* in the section LINES, PLANES & ANGLES especially useful reading for teaching concepts about polygons.

Sorting and classifying objects can be a good way to begin work with polygons. While students are sorting objects by shape, color and other attributes as suggested in the activities *Sort Out* and *As Easy as 1, 2, 3*, you can listen to the words they use and ask if they know names for the shapes. Many students forget words like hexagon and polygon since they are seldom used outside of the classroom.

Using these words in your conversation with students as you walk around the class-room can help students learn the meanings of the words.

Learning names for geometric concepts can be boring and even confusing, but it can also be fun and challenging. Suppose you want your students to understand what a polygon is. You could introduce the term in this way:
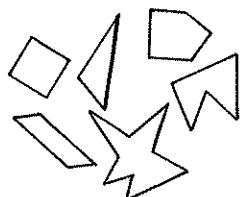
These are polygons.   These are not polygons.   Which are polygons?
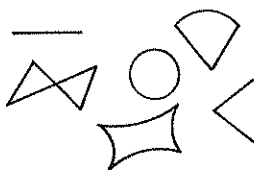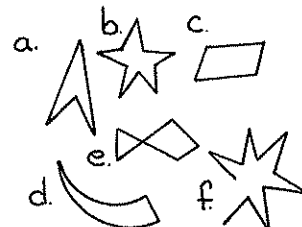
The discussion which follows such an activity could bring out the characteristics of a polygon. Students can determine that all the sides of a polygon are straight line segments, and a polygon consists of segments in a plane with only end points in common. A class discussion can help students see that the figures a, b and c are polygons but that d, e and f are not.

Introducing a term in this manner lets students propose and refine their own definitions, whereas always giving a ready-made definition can allow students to take a passive role and overlook some important aspects.

A similar approach can be used to help students see that the term "convex" is applied to those polygons which do not have "inside corners"—so each segment con-necting interior points is in the interior of the polygon.

These are convex polygons.   These are polygons, but   Which are convex polygons?
                                not convex.
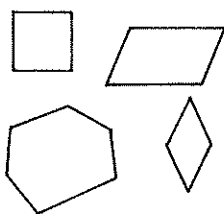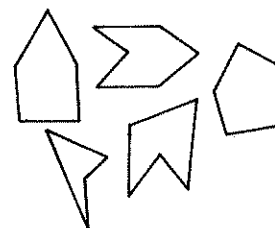
Many more ideas for learning vocabulary and concepts in this way can be found on the page *These Are. These Are Not*. If your stu-dents like the problem solving and challenge of this kind of activity, they might enjoy the book <u>Weeple People</u> by D. Craig Gillespie. You could provide a challenge a week for your students from this book. Students could also make up challenges for the rest of the class.

KANKOS

All of these characters are KANKOS:

Not one of these belongs to the KANKO family.

An activity, emphasizing the characteristics of hexagons does not guarantee that students will remember what a hexagon is, or that they will not regress to an earlier, more narrow understanding of the term. The term hexagon usually brings to mind this figure: ⬡ not anything like ⬠ , ⌄ or ⋁ . Perhaps we intend to say "regular hexagon" when we mean the first shape, but sometimes we drop "regular" in our speech and assignments because clarifying words seems cumbersome and unnecessary. This is a natural abbreviation of our thoughts and words, but it helps if we tell students our intentions. Has the situation below ever happened to one of your students?
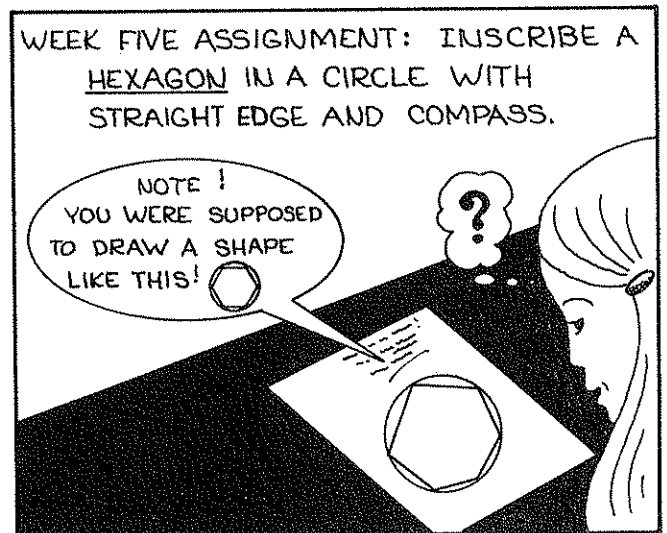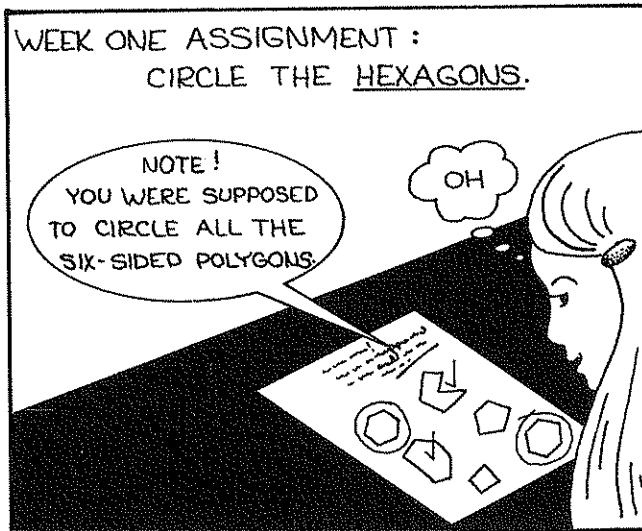
It is helpful to familiarize students with prefixes for geometric words because these prefixes give clues to the meanings of the words. You could put a list of these with their meanings on a poster for easy reference. The page *Pre-fix It* is a crossword puzzle using prefixes of polygons. Students could also make use of an illustrated dictionary of geometry terms such as that given in the *Geo Glossary* at the end of this resource. Your students might want to add more words to the *Geo Glossary* for their own reference.

Prefixes		
	Greek	Latin
one		*uni-*
two		*duo-*
three	*tri-*	*tri-*
four	*tetra-*	*quadr-*
five	*penta-*	*quint-*
six	*hexa-*	*sex-*
seven	*hepta-*	*septa-*
eight	*octa-*	*octa-*
nine		*nona-*
ten	*deca-*	*deci-*
twelve	*dodeca-*	*duodeci-*
twenty	*icosa-*	
100	*hecto-*	*centi-*
1000	*kilo-*	*milli-*

*Centi- and milli- also mean $\frac{1}{100}$ and $\frac{1}{1000}$, respectively.

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The meanings for words can seem arbitrary at times as this quote from Through the Looking Glass illustrates. The game *Polygon Pass-Out* can help students verbalize the characteristics of triangles, parallelograms and other polygons. This game could be used as motivation for learning the meanings of terms used in geometry. The game will also give you a chance to diagnose student understanding of these terms.

> *"I don't know what you mean by 'glory,'"* Alice said.
> *Humpty Dumpty smiled contemptuously. "Of course you don't--till I tell you. I meant 'there's a nice knock-down argument for you!'"*
> *"But 'glory' doesn't mean 'a nice knock-down argument,'"* Alice objected.
> *"When I use a word,"* Humpty Dumpty said *in rather a scornful tone, "it means just what I choose it to mean--neither more nor less."*
> *"The question is,"* said Alice, *"whether you can make words mean so many different things."*
> *"The question is,"* said Humpty Dumpty, *"which is to be master--that's all."*
>
> Lewis Carroll, Through the Looking Glass

PERCEIVING POLYGONS AGAINST A BACKGROUND

A design like that shown at the right can provide practice in visual perception. Can your students pick out a square against the background of lines? How many squares of different sizes can they see? (There are at least eight different sizes.) Can they see the squares in this orientation: ◇ at the same time they see the squares oriented like this: ▢ ? It is rather interesting to notice that the squares fade into the background as you begin to look for triangles or octagons in the design. Some students will have difficulty focusing on a shape and interpreting the rest of the design as background. The *Visual Perception* teaching emphasis gives a discussion of this aspect of visual perception. Differentiating a figure from its background also occurs in the student pages *Figure it Up* and *Repeating Shapes*. You might need to give some students special help in seeing particular shapes in these activities.

Puzzle posters which ask the viewer to find hidden figures, or to find a figure which differs from the rest can be fun. Several large colorful posters are available from Creative Publications, Inc.
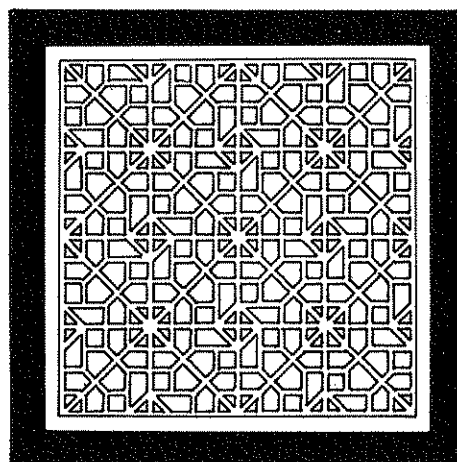
FACTS ABOUT POLYGONS

There are a number of facts about triangles and other polygons that are often examined in middle school geometry courses. They can be taught in a formal way, demonstrated by the teacher or discovered by students through carefully planned lessons. *The Angles in Triangles* gives seven different ways you could explain, demonstrate or guide students to discover that the sum of the angle measures of a triangle is 180°. You might want to use several of these ideas to reinforce the point.

One way to demonstrate that the sum of two sides of a triangle must be greater than the third is to have students try to make triangular models from different length straws or strips. This is developed on the pages *Sum Thing* and *When is a Triangle Acute, Right or Obtuse?*

What does it mean that three sides determine a triangle, but three angles do not? To explain this to your class, hand out the page *Side, Side, Side/Angle, Angle, Angle*. After they have each drawn a triangle whose sides are the length of the given line segments, do they see that all the triangles they made are congruent? When they construct a triangle with three given angles, say 60°, 30°, 90° or those given on the page, the triangles could be of many different sizes. The teaching emphasis *Graphic Representation* contains methods for constructing triangles using a straightedge and compass when various parts of a triangle are given.

Congruence of line segments, angles and polygons is usually introduced in the middle grades. (See *An Exact Fit.*) Students often start studying this concept by tracing or cutting out a shape and moving it to see if it fits another shape. This physical movement corresponds to the geometric motions of reflection (flip), rotation (turn) and translation (slide). Some students might be able to "eyeball" two shapes and decide if they are roughly congruent or not. This can be tricky, however, as things are not always what the eye perceives. The *Visual Perception* teaching emphasis discusses this problem in more detail.
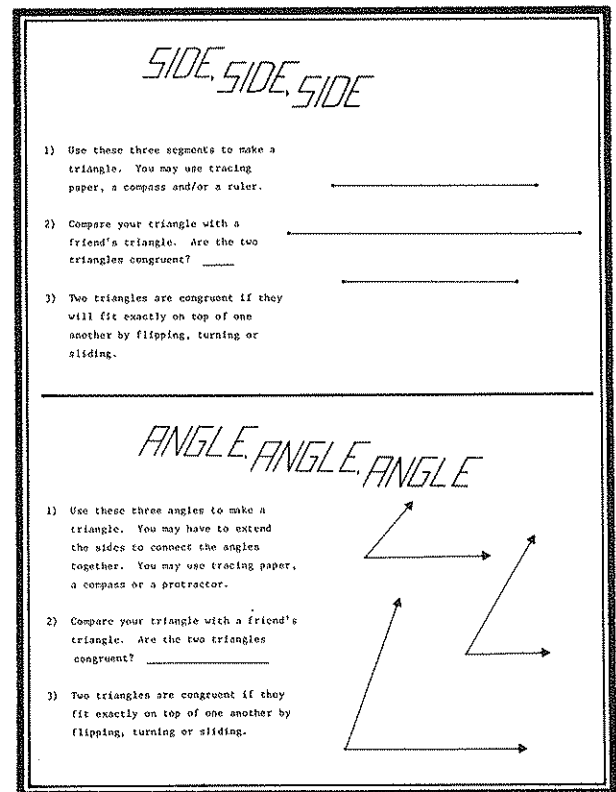
Other activities in this section introduce the sum of the interior or exterior
angles of polygons, the special line segments associated with triangles (medians,
altitudes, . . .) and the relationships involving angles or line segments associated
with polygons. Most of these activities take the form of guided discovery lessons
and could be used as teacher demonstrations, worksheets for the entire class or

enrichment activities for individual
students. Some of the lessons require
a background with straightedge and com-
pass constructions; however, you could
substitute a protractor to bisect
angles or a right angle model for draw-
ing perpendiculars if your students
haven't covered formal constructions.
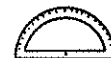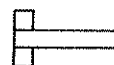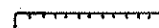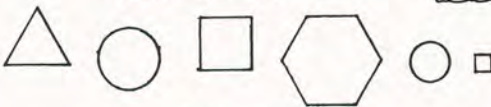
SORT OUT

This activity is an experience in grouping sets of objects into subsets by looking at various attributes of the objects. Suggestions for materials include:

Set 1   Pictures from magazines

Set 2   Attribute blocks

Set 3   Nuts, bolts, buttons, etc.

Set 4   Geoblocks

Set 5   Variety of books, i.e. paperbacks, workbooks, textbooks, etc.

After picking your materials, place each set on a different table. Assign 4 to 5 students to each table. Have the students separate the objects into subsets in any way they wish. They will use attributes like size, color, shape, material, purpose, etc.

Next have the students write down a brief description of how they have separated the objects. Have them leave this description face down on the table. Students will need about 10 minutes to sort the objects and write the description. Objects should be left in the subsets.

Now the groups will rotate to different tables spending about 5 minutes at each table. They will guess how the objects have been divided then check their guess by looking at the description written by the original students. If time allows the group could be encouraged to reorder the objects and write a new description before moving on to the next table.

This is a worthwhile activity to do at the beginning of the year as it will also help familiarize students with some of your manipulatives.

AS EASY AS 1,2,3

This activity is an experience in subdividing a set of geometric figures into smaller related subsets. It would be suitable to use after *Sort Out*. The activity is written for 30 geometric figures but you might like to use fewer figures. Thirty figures and a sample subdivision are included on the next page.

In this activity students will:

1) Subdivide the set of 30 figures into subsets of related figures.

2) Write a description of each subset, so that anyone reading the descriptions could separate the figures into exactly the same subsets.

Groups of 4 to 5 students are given identical sets of 30 randomly numbered figures. The students are asked to separate the figures into subsets of related figures. Some students may sort by number or color (if you make the figures different colors). Others may separate the figures into 4-sided and not 4-sided figures or into curved and not curved figures. Any group finishing early in its classification could be encouraged to develop another system that gives a larger number of related subsets. For example, you could suggest that the group try to find a classification that gives five subsets of related figures. One such system could be figures with 3 sides, 4 sides, 5 sides, 6 sides or others.

Have each group of students write a description of its subsets. For future reference have one student in each group record the number of the figures in each subset. Later in the class period or the next day have pairs of groups trade descriptions. The groups will then try to subdivide the figures into subsets according to the other group's description. Finally have each pair of groups meet to discuss the results.

IDEA FROM: *Lab Geometry*, Teacher's Edition

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AS EASY AS 1,2,3

(CONTINUED)

THIRTY SAMPLE FIGURES

30 11 12

squares

3 5 20
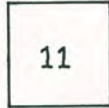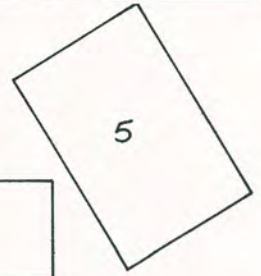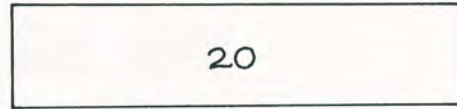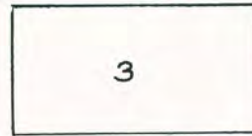
rectangles (not squares)

16 18 2

right triangles

7 14 19

other triangles

13 26 4
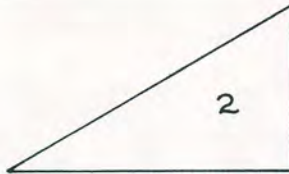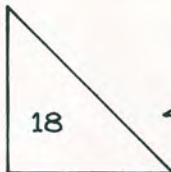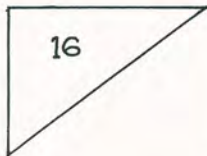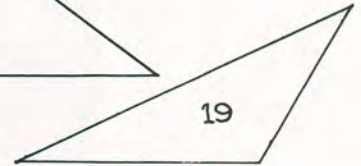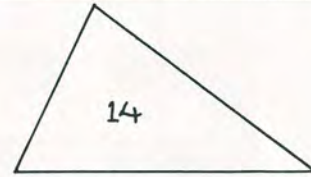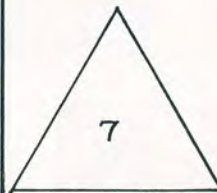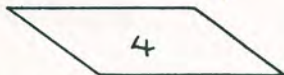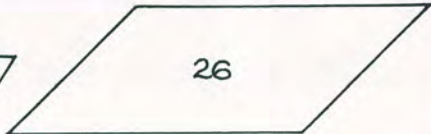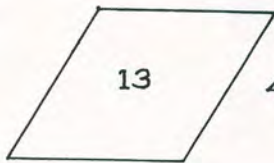
parallelograms (not rectangles)

29 9 24

trapezoids

22 28 8

pentagons

21 10 23

hexagons

1 27 17

curved figures

15 25 6
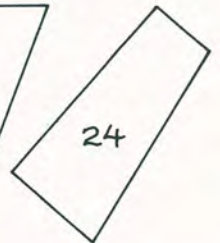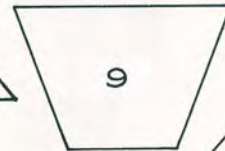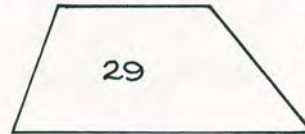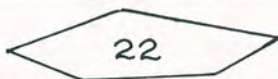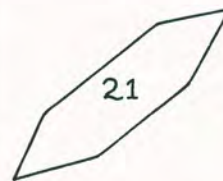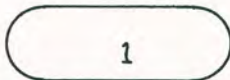
concave polygons

PRE-FIX IT

The crossword grid contains the following answers:

Across
- 2. OCTOPUS
- 4. BISECT
- 5. UNICORN
- 7. BIANNUAL
- 10. UNIQUE
- 11. PENTATHLON
- 14. TRIO
- 15. DECADE
- 16. QUADRUPED
- 17. BIKINI
- 18. QUADRAPHONIC
- 19. CENTURY
- 20. PENTAGON

Down
- 1. DECEMBER
- 3. TRIPLETS
- 6. MONOLOGUE
- 7. BICYCLE
- 8. NONAGENARIAN
- 9. MONOCLE
- 12. OCTAVE
- 13. QUADRUPLE
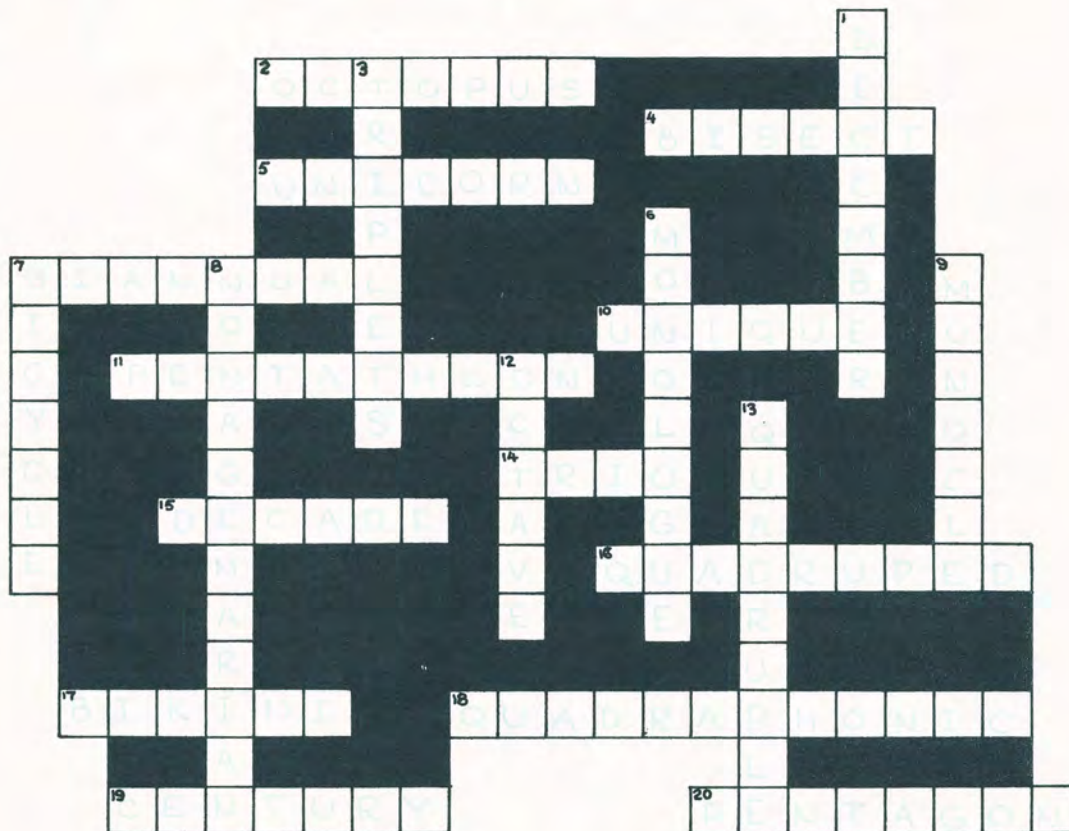
ACROSS

2. Marine animal with eight arms
4. To divide into two equal parts
5. Legendary animal with one horn
7. Means twice a year
10. One of a kind
11. Five-event athletic contest
14. Singing group with three people
15. Ten years
16. An animal with four legs
17. A two-piece bathing suit
18. A sound system with four channels
19. One hundred years
20. A five-sided building in Washington D. C.

DOWN

1. The name of a month that <u>used to be</u> the 10th month
3. Three babies born at one birth
6. A speech by one actor
7. A vehicle with two wheels
8. A ninety year old person
9. An eyeglass for one eye
12. A musical interval of eight notes
13. To make four times as much

TRADEMARKS

Businesses often use geometric shapes as symbols to identify their products. These symbols are called trademarks.

Here is the American Red Cross trademark.

This is the symbol for ecology.
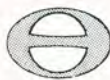
Below are some common geometric shapes. Write the name of each shape under its picture.
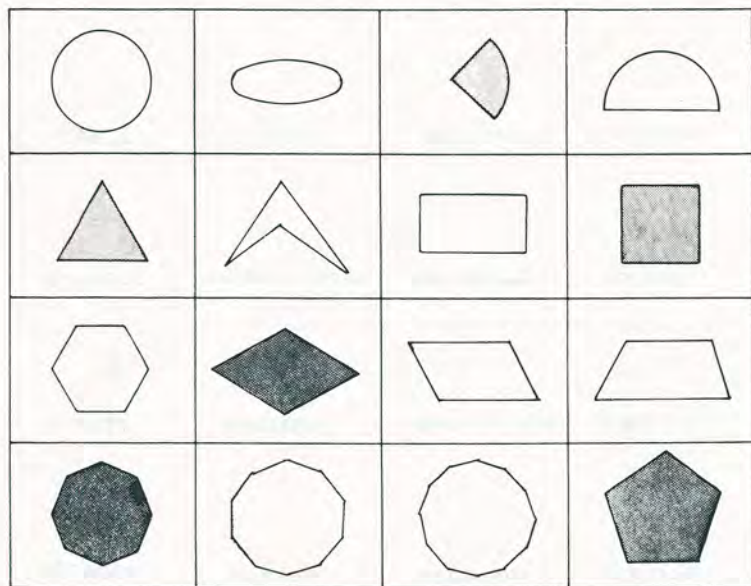
Find trademarks which use shapes like these. Cut and glue them in each space or on the back of this page.

Teacher's page: Trademarks

Mathematical topics:
  vocabulary
  congruency
  shape recognition
  non-metric geometry

Materials needed:
  glue
  scissors
  newspapers
  phone books (yellow pages)
  magazines (Time, Newsweek, Fortune, Sports Illustrated, Scientific American, Sunset)

Comments:
  Trademarks provide both motivation and application for geometry. Students of all abilities can recognize some shapes in emblems. They enjoy the hunting and finding process and also seeing the emblems again. Transfer occurs with open-ended activities like these. Later most students will notice how often advertisers use geometric shapes.

  Students will have difficulty finding all the shapes. Have them find as many as possible or try to complete a row, column or diagonal for "Trademark Bingo"

Extensions:
  Are the trademarks on billboards and in windows the same size as those in magazines? How are they related? (Discuss ratio and proportion, scale drawings.)

  Many trademarks contain more than one shape. What other shapes can they find in their trademarks?

  Look for shapes along the highway. Certain signs represent specific types of information or warnings. See your state's driver's manual for some of these shapes:

Further sources:
  Wildbur, Peter. Trademarks: A Handbook of International Design. New York: Van Nostrand Reinhold Co., 1966.

DON'T GET STUCK ON THIS

Shade the whole box if it has an INCORRECT figure. The shading will cause a "desert" shape to occur.

QUADRILATERAL ➤

RHOMBUS ➤

PENTAGON ➤

HEXAGON ➤

SQUARE ➤

PARALLELOGRAM ➤

RECTANGLE ➤

TRAPEZOID ➤

GEO-GIN

<u>Players</u>: 2 to 4

<u>Materials</u>: White poster board or card stock; 4 different colored pens

<u>Constructing the game</u>: Cut forty-eight 6 cm by 10 cm rectangular cards from white poster board or use commercial card stock. Mark the top of each card by writing a "T" in the upper left corner. Separate the cards into three sets of 16 each. On each card of one set draw a circle (4 cm in diameter); on each card of the second set draw a 4 cm square; and on each card of the third set draw a rhombus with 4 cm sides. Divide each figure into four equal parts as shown.
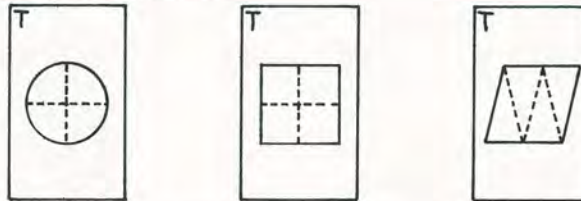
Separate the circle cards into four groups of 4 each. Select a different colored pen to use for each group of circle cards and color one-fourth of the circle.

A sample group for one color is shown.
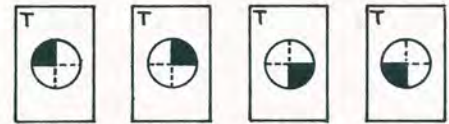
Do the same for the square and rhombus cards. You can decorate the backs of the cards to make the deck look attractive.

<u>Playing the game</u>: The deck is shuffled and eight cards are dealt to each player. The dealer turns over the top card and places the remaining cards face down on the table. The object is to obtain two complete figures. A complete figure consists of four cards of the same figure with a different portion of each figure colored--all four parts would fit together to make a completely shaded figure. The cards must be all turned so that the "T" is in the upper left corner. To make a completely shaded figure various colors can be used.

Play begins as the player left of the dealer decides whether to use the card turned face up or to draw the top card on the face-down pile. For each card added to a hand the player must discard one. After a player has discarded, the player on her left then decides whether to use the last card discarded or the top card in the face-down deck. Play continues clockwise until one player wins by getting two complete figures in his hand. The winner calls out "Geo-gin" and lays down his hand. He scores 10 points for winning and an additional 5 points for each figure he completes using only one color. Any other player that has a completed figure in her hand in <u>one color only</u> scores five points.

The game continues until one player obtains 50 points or until a time limit on the game has been exceeded.

Strictly Squares Ville

I'm Sid the Census Taker. I have to count the squares of all sizes in each town. Can you help me finish the census?

Onesville

Twosville
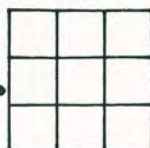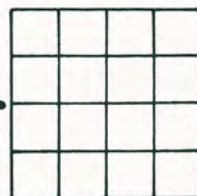
Threesville

Hmmm - Threesville has nine 1 x 1 squares, four 2 x 2 squares, and one 3 x 3 square.

Foursville

Fivesville

SID'S CENSUS LIST

	SIZES OF SQUARES					SQUARE POPU-LATION
	1×1	2×2	3×3	4×4	5×5	
ONESVILLE	1					1
TWOSVILLE	4	1				5
THREES...	9	4	1			14
FOURS...	16	9	4	1		30
FIVES...	25	16	9	4	1	55
⋮						

Can you predict the square population for Sevensville? For Tensville?

IDEA FROM: *Problems—Green Set,* Nuffield Mathematics Project

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SQUARESVILLE SUBURBS

How many triangles can you find in this figure? Don't forget to look for different sizes!
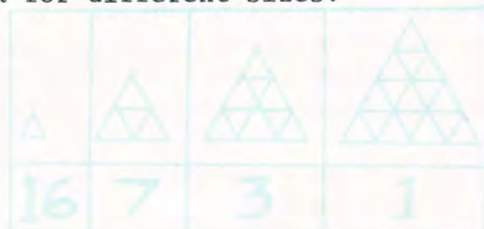
| 16 | 7 | 3 | 1 | = 27 |
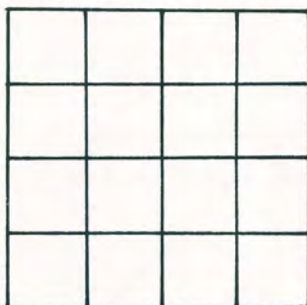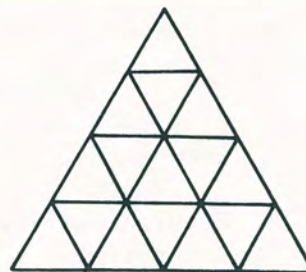
How many rectangles can you find? Make a chart. Remember, a square is a rectangle too.

| 1x1 | 1x2 | 1x3 | 1x4 | 2x2 | 2x3 | 2x4 | 3x3 | 3x4 | 4x4 | = 100 |
| 16 | 24 | 16 | 8 | 9 | 12 | 6 | | | 1 | |

How many of this shape ⌐ are in this figure? 30

How many are there of this shape ⊤ ? 24

How many cubes are needed to build this staircase? 1+2+3+4+5 = 15

How many cubes would be needed for a staircase with ten steps? 1+2+3+4+5+6+7+8+9+10 = 55

407
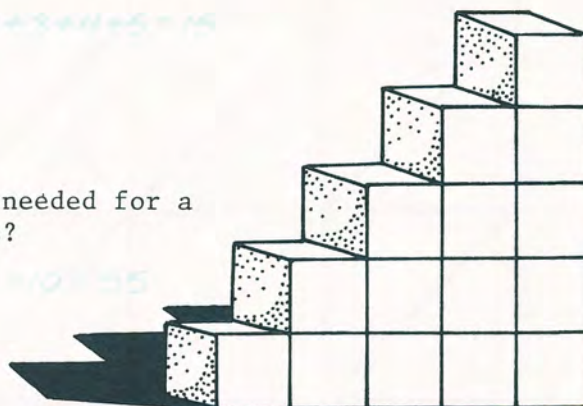
FIGURE IT UP

Needed: Two players
         One playing board
         One reference sheet on geometric shapes (see next page.)

Directions: Players take turns shading any small triangle.

When a shape (see reference sheet next page) is completed by a player he receives points if the shape covers points on the playing board.

Figures may overlap and new figures may be built from figures already there.

A player may claim points for only one figure each turn and he must have completed the figure that turn.

A player must also state the name of his figure and trace it for his opponent.

The game ends when the board is completely covered.

The winner is the player with the most points.

PLAYING BOARD

IDEA FROM: *Fun With Numbers*
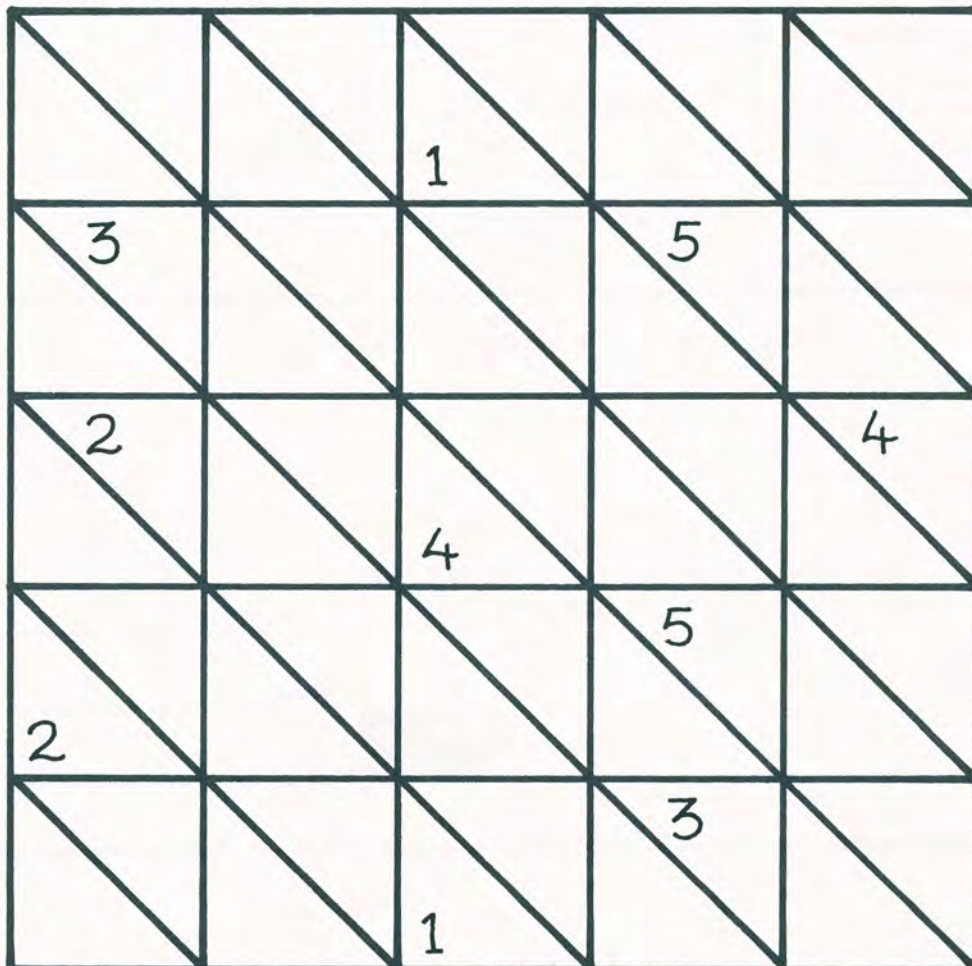
FIGURE IT UP

(CONTINUED)

Reference Sheet

You may make the following figures:

TRIANGLES – They must cover at least 4 but not more than 25 of the small triangles on the playing board.
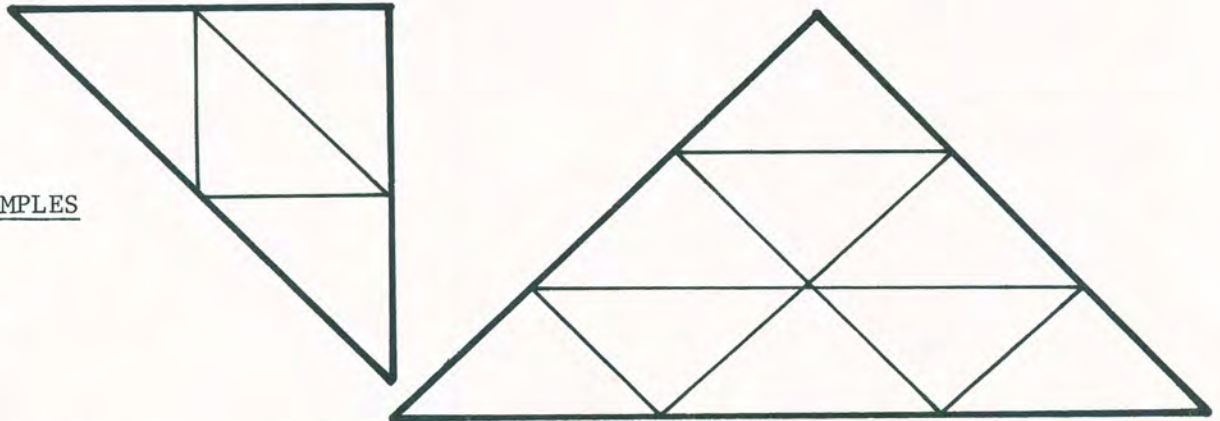
EXAMPLES

QUADRILATERALS – They must cover at least 4 but not more than 24 of the small triangles on the playing board.

EXAMPLES
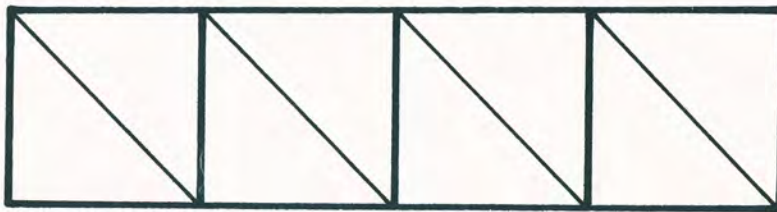
RECTANGLE
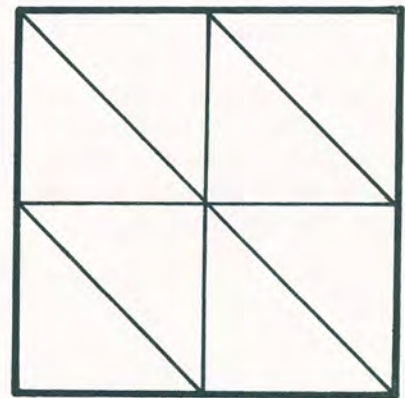
SQUARE

PARALLELOGRAM

TRAPEZOID

TRAPEZOID

IDEA FROM: *Fun With Numbers*

Permission to use granted by San Francisco Unified School District
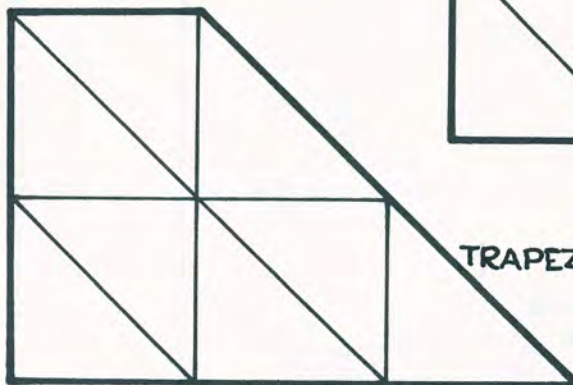
POLY-ART

Use a plain piece of paper 8 1/2" X 11". Fold the paper in any way so that the corners do not meet. Crease the fold firmly. Fold 3 more times.

Open the paper.

Color all the triangles yellow.

Color all the quadrilaterals red.

Color all the pentagons blue.

etc.
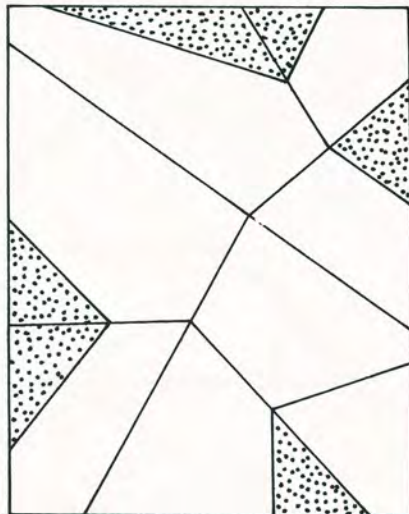
NESTING

Draw a regular polygon. Find the midpoints of each side. Connect them. Find the midpoints of the new sides. Connect them, etc.
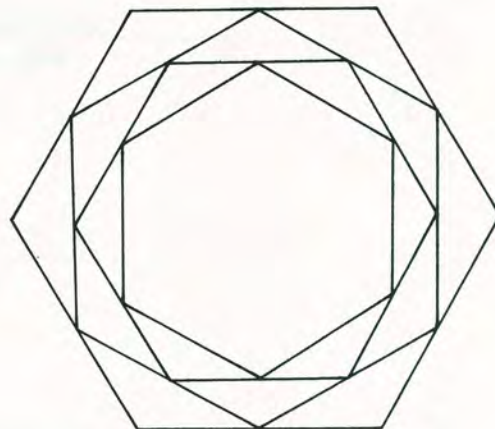
SPIRALS

Draw a regular polygon. Find 1/6 (or 1/5, 1/4, 1/3) of each side as shown below. Connect these points. Find 1/6 of each of the new sides. Connect the points, etc.
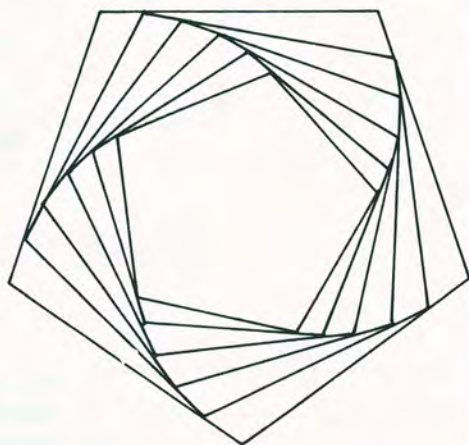
SQUARES

Use different colors to cut out a 10 cm X 10 cm square, a 9 cm X 9 cm, 8 cm X 8 cm . . . . 1 cm X 1 cm. Arrange the ten squares into different patterns.
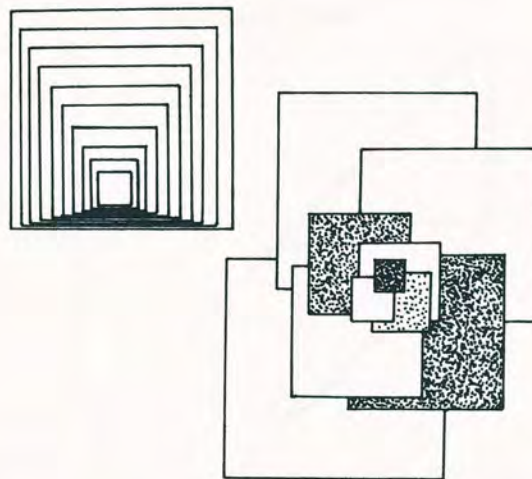
POLYGON PUZZLES

The geometric puzzles below could be enlarged and cut out of tagboard for a permanent puzzle. Place them in separate envelopes with instructions to arrange the pieces into a shape (parallelogram, trapezoid, etc.) and then into a square. Making each puzzle a different color will simplify putting the pieces back. Students enjoy signing their name on the back of the envelope of each puzzle they complete.
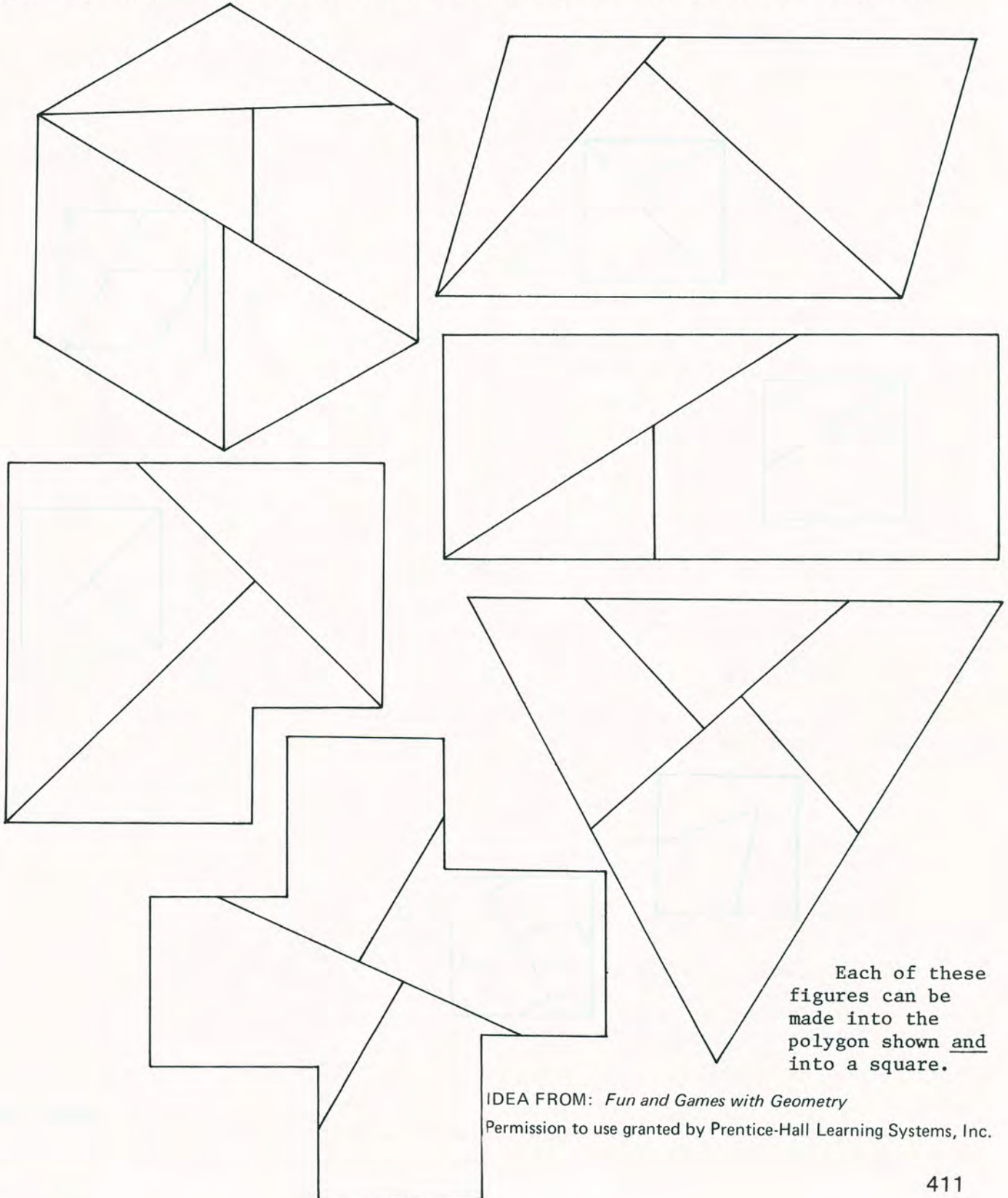
Each of these figures can be made into the polygon shown <u>and</u> into a square.

TANGRAM CONSTRUCTION

A) The pieces of the Chinese Tangram puzzle can be constructed as a compass and straightedge activity. A chalkboard or overhead demonstration or an audio tape recording could be used to convey the instructions.

1) Construct a 10 cm square. Label it ABCD.

2) Draw diagonal $\overline{AC}$.

3) Bisect $\overline{AB}$. Label the midpoint E.

4) Bisect $\overline{BC}$. Label the midpoint F.

5) Draw $\overline{EF}$.

6) Bisect $\overline{EF}$. Label the midpoint G.

7) Draw $\overline{DG}$.

8) $\overline{DG}$ intersects $\overline{AC}$. Label the intersection H.

9) Bisect $\overline{AH}$. Label the midpoint J.

10) Draw $\overline{EJ}$.

11) Bisect $\overline{CH}$. Label the midpoint K.

12) Draw $\overline{GK}$.

13) Label the seven pieces like this:

△ CDH	I	△ AEJ	V	
△ ADH	II	△ GHK	VI	
△ BEF	III	▱ CFGK	VII	
▢ EGHJ	IV			
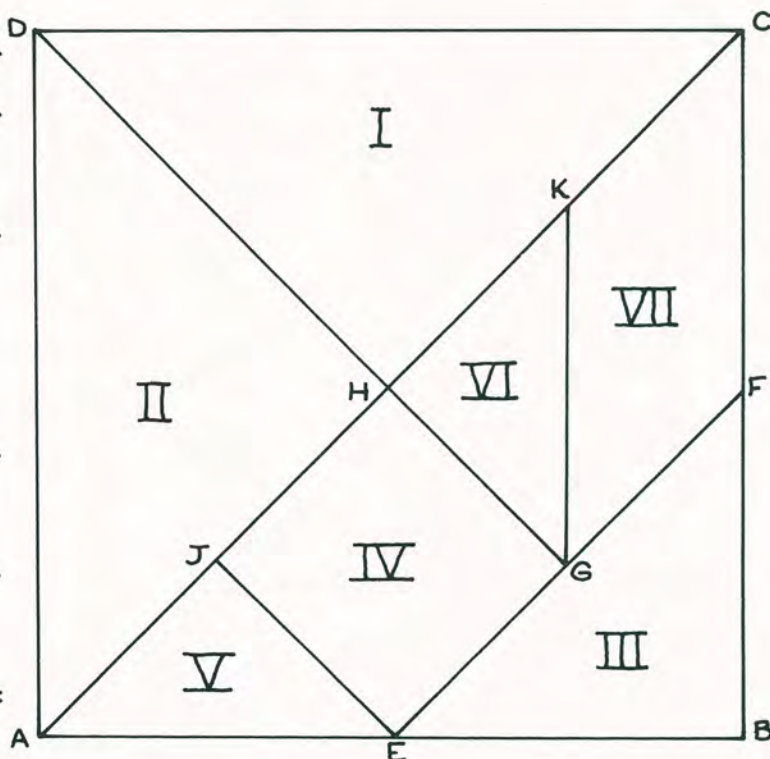
B) The Tangram pieces can be made using a coordinate plane. This construction forms a square 8 units on a side.

1) Plot the points: A(-4,-4) B(4,-4) C(4,4) D(-4,4) E(0,-4)
F(4,0) G(2,-2) H(0,0) J(-2,-2) K(2,2).

2) Draw the line segments:
$\overline{AB}, \overline{AC}, \overline{AD}, \overline{BC}, \overline{CD}, \overline{DG}, \overline{EF}, \overline{EJ}, \overline{GK}$.

3) Label the pieces with Roman numerals as in part (A) and cut them out.

4) For practice in graphing points in various quadrants the Tangram could be translated by changing the coordinates of the points.
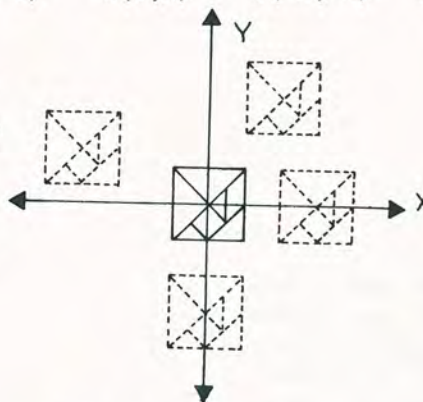
IDEA FROM: *Mathex,* Junior-Geometry, Teacher's Resource Book No. 9, and *Pic-A-Puzzle*

TANGRAM CONSTRUCTION
(CONTINUED)

A Tangram with different pieces can be constructed in a similar manner.

C)
1) Construct a 10 cm square. Label it MNOP.

2) Find the center of the square. Label it Q.

3) Draw $\overline{PQ}$ and $\overline{OQ}$.

4) Bisect $\overline{MN}$, $\overline{NO}$, $\overline{OP}$, and $\overline{MP}$. Label the midpoints R, S, T and U respectively.

5) Draw $\overline{QR}$ and $\overline{QS}$.

6) Bisect $\overline{PQ}$ and $\overline{OQ}$. Label the midpoints V and W respectively.

7) Draw $\overline{TV}$ and $\overline{TW}$.

8) Draw $\overline{RU}$.

9) Label the seven pieces like this:

☐ RNSQ   I   ☐ QWTV   V
△ OQS   II   ⟋ PQRU   VI
△ OTW   III   △ MRU   VII
△ PTV   IV
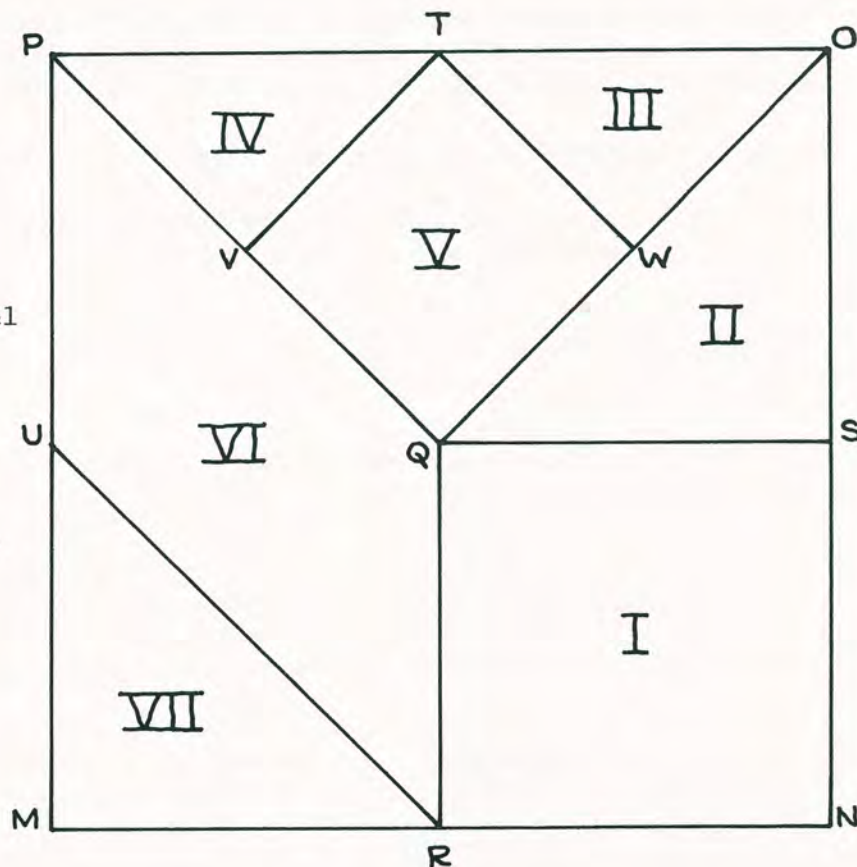
D) These Tangram pieces also can be made using a coordinate plane.

1) Plot the points: M(2,3)   N(10,3)   O(10,11)   P(2,11)   Q(6,7)   R(6,3)

S(10,7)   T(6,11)   U(2,7)   V(4,9)   W(8,9).

2) Draw the line segments: $\overline{MN}$, $\overline{MP}$, $\overline{NO}$, $\overline{OQ}$, $\overline{OP}$, $\overline{PQ}$, $\overline{QR}$, $\overline{QS}$, $\overline{RU}$, $\overline{TV}$, $\overline{TW}$.

3) Label the pieces with Roman numerals as in (C) and cut them out.

4) The outline of the Tangram can be translated into many different positions by changing the coordinates of the points.

413
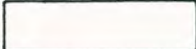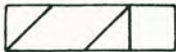
THE TANTALIZING TANGRAM

The following are activities that use the Tangram pieces constructed in part (A) of *Tangram Construction*. Each could be expanded into an entire activity card by including more similar types of exercises. Also similar activities could be developed for the second set of Tangram pieces shown in *Tangram Construction*. Several of these could develop into a continuing activity. For example a large bulletin board or chalkboard display of the chart below could be made and when a student discovers an answer, he could sketch it on the chart and write his name on it.

Materials needed: Tangram pieces, paper for recording answers.

Activity: For each exercise use the Tangram pieces to form the figures and then sketch the answers on your paper.

1) Show that pieces V and VI can exactly cover

    a) IV   b) III   c) VII

2) Show that piece I can be exactly covered by

    a) IV, V, VI   b) III, V, VI   c) IV, V, VI

3) Use pieces IV, V and VI to form a right triangle. Now by moving one piece form a rectangle, then a parallelogram and then a trapezoid.

4) How much of this chart can be done using the Tangram pieces?

NUMBER OF PIECES USED	TRIANGLE	SQUARE	RECTANGLE	PARALLELOGRAM	TRAPEZOID
1					
2					
3					
4					
5					
6					
7					

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THE TANTALIZING
TANGRAM
(CONTINUED)

5) Use all seven pieces to make a representation of each letter of the alphabet and of each digit.

6) Use all seven pieces to duplicate a design. Any resource book on Tangrams will show many figures to duplicate. An interesting problem is to discover how one man can have a foot and another no foot when both are made using all seven pieces.
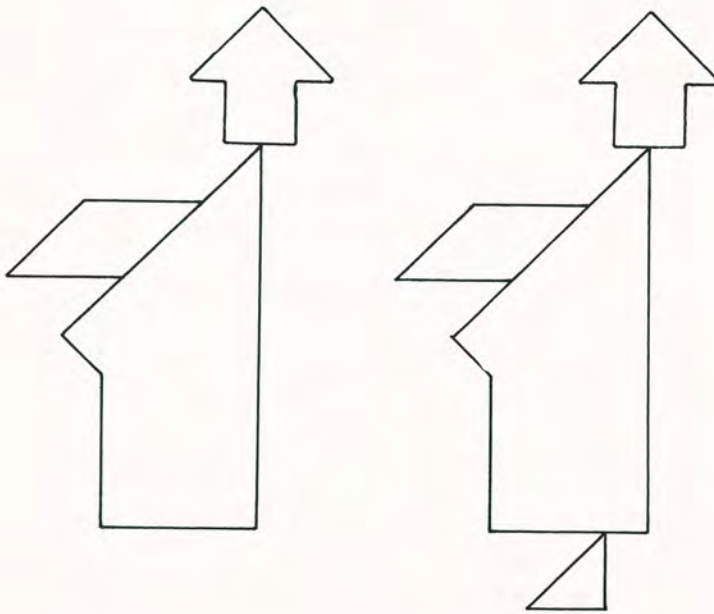
7) Students can create their own designs with the Tangram pieces. They can also draw the outline and challenge other students to fill in the pieces.

For other uses of Tangrams see the FRACTION section in Number Sense and Arithmetic Skills, the SIMILAR FIGURES section in Geometry and AREA & VOLUME in Geometry. Commercially produced Tangrams can be purchased from several companies.

THE PERPLEXING PENTOMINOES

Materials needed: Five squares, 3 centimetres on a side, and centimetre
grid paper or five 1-inch tiles and inch grid paper.

Activity:

1) A pentomino is a pattern made by joining 5 squares together
   so that each shares a common side with another. How many
   different pentominoes do you think there are? _____

EXAMPLES

2) Take the 5 squares and make all the pentominoes that you
   can. Copy each pentomino pattern on the grid paper and
   cut out the shape. If one of the patterns can be turned
   or flipped to exactly fit another one, the two patterns
   are the same pentomino.

These four patterns are
the same pentomino.

3) Check with your teacher
   to see if you have found
   all the pentominoes.

4) Try to arrange the
   pentominoes so that they
   make a rectangle. Do
   not overlap the pieces.
   There are more than 2000
   ways to do this!

5) Play a game using the pentominoes.

   Needed: 2 players
             Game mat is an 8 by 8
             square constructed out
             of the grid paper.
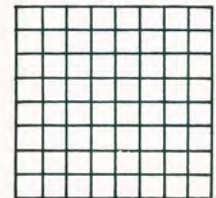
   a) Players alternate picking
      pentomino pieces until all
      the pieces have been selected.

   b) Each player in turn then
      places a pentomino on the mat.
      Play continues until it is
      impossible for a player to
      place on that mat a pentomino
      that doesn't overlap another
      pentomino or lie completely
      on the mat.

   c) The winner is the last person
      to successfully place a
      pentomino on the mat.

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HEXIAMONDS
vs.
PENTOMINOES

Hexiamonds are formed by linking six equilateral triangles together. There are twelve distinct hexiamonds. The shapes (and names attributed to T. H. O'Beirne, a Glasgow mathematician) are shown below.

Have your students use triangular grid paper (see *Repeating Shapes*) and discover the twelve shapes. When a shape is found it will be helpful to cut it out and use it as a check for other "new" shapes. Many "new" shapes will only be a reflection or rotation of a shape already found. Allow your students to make up their own names for the pieces and then use these names for any of the activities. A large set of hexiamonds could be used as a bulletin board display. For the activities suggested below which involve manipulating the pieces, the hexiamonds could be made from file folders, cardboard, linoleum or wood. It is helpful to use a material that is the same on both sides since some designs require turning the pieces over.

PISTOL

RHOMBOID

CLUB

BUTTERFLY

SNAKE

HEXAGON

CROWN

BAT

LOBSTER

SHOE

SPHINX

YACHT
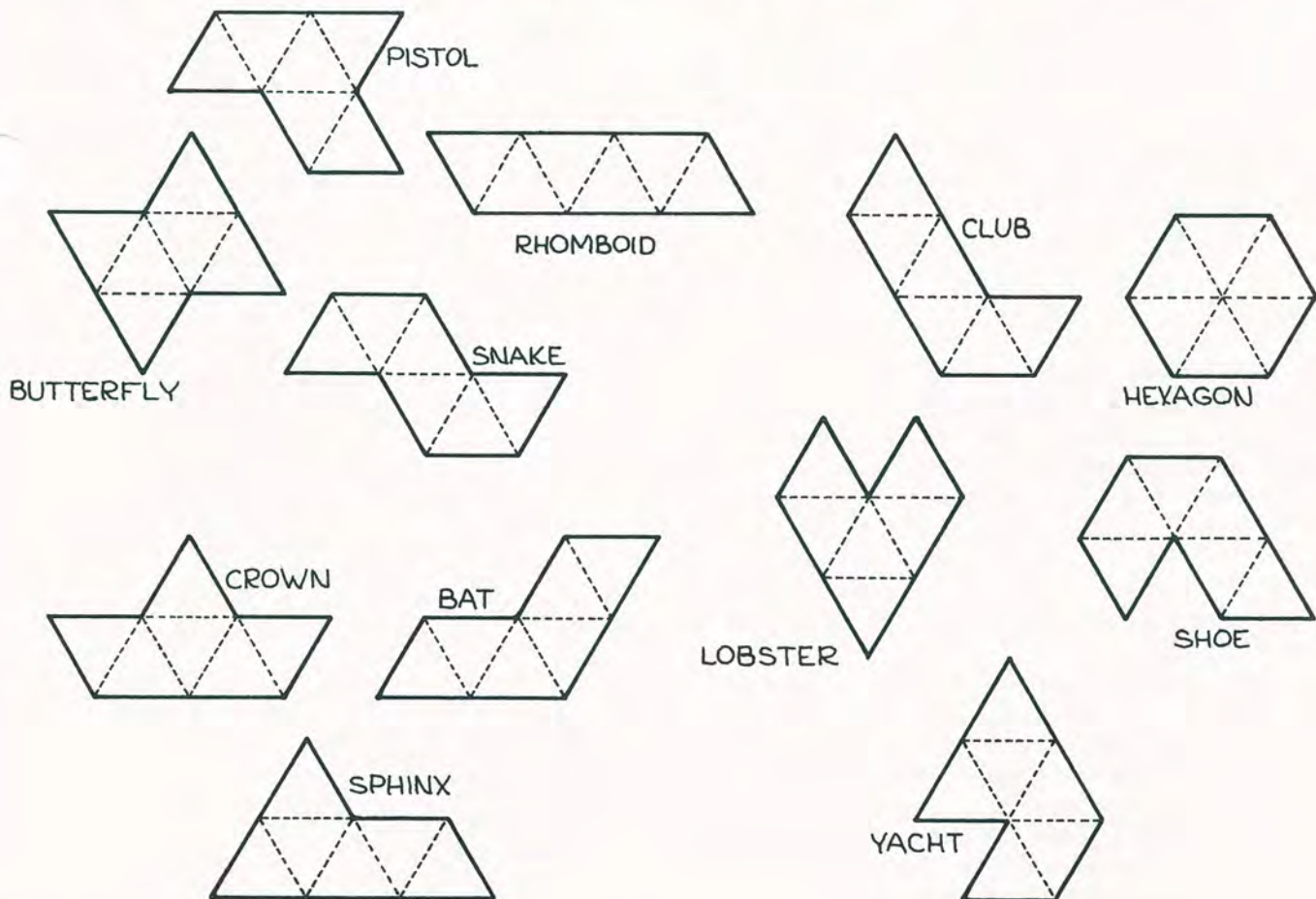
IDEA FROM: *Martin Gardner's Sixth Book of Mathematical Games from Scientific American* and *Art 'N' Math*

HEXIAMONDS
vs.
PENTOMINOES
(CONTINUED)

Activities:

1) Use all of the twelve hexiamonds to make these designs.

RIFLE

BIRD

SNAKE

SAILBOAT   **TUGBOAT**   **ARROW**
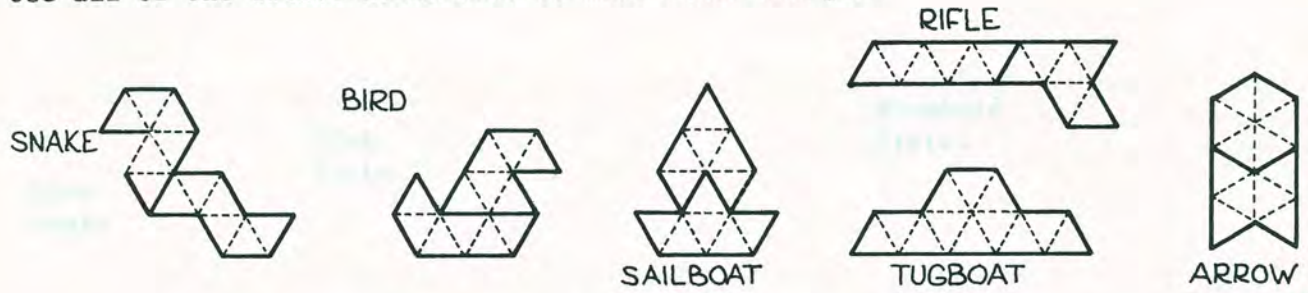
2) Use all of the twelve hexiamonds to make these three pairs of twins.
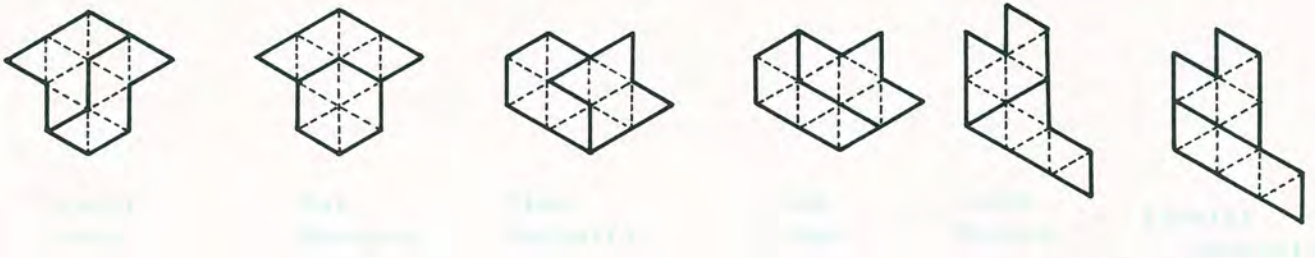
3) Duplications of each hexiamond using a scale of 2:1 can be made using four pieces.

SCALE OF **2:1**

4) A star can be formed using eight of the hexiamonds. Exclude the pieces Bat, Pistol, Yacht and Crown. Students will need an outline of the star to help them fit the pieces.
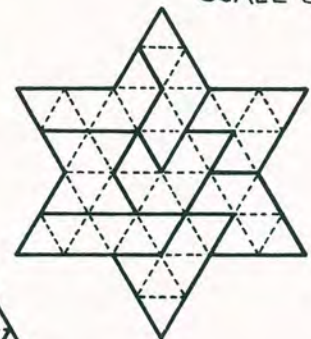
5) A covering game using a 60° diamond can be played.

   a) Two players alternately draw pieces.
   b) Players alternately place a piece on the game mat until no one can play.
   c) The player with the fewest pieces left is the winner.
   d) As of 1961 over 40 ways of using all twelve hexiamonds to completely cover the mat were known.
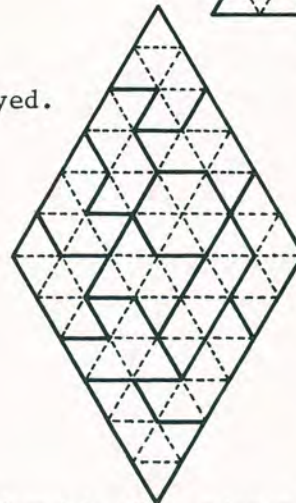
REPEATING SHAPES

Provide each student with an equilateral triangle, 3 cm on a side and a sheet of thin cardboard approximately 20 cm by 30 cm. Ask students to cut as many triangles as possible from their cardboard that are the same size and shape as the triangle.

When students have about thirty triangles ask them to make shapes by placing the triangles together on their desks. Have them share their creations with the rest of the class. Then restrict the shapes by requiring the students to place the sides of the triangles together so that they fit exactly. Some well known basic shapes that can be formed are:

RHOMBUS   TRAPEZOID   PARALLELOGRAM   REGULAR HEXAGON
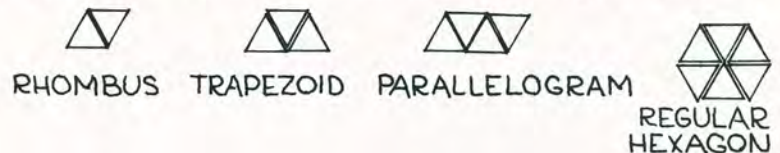
> Point out to the student that our language "triangle," "trapezoid," etc. refers to the outer edge of the figure. The cardboard shape is actually a "triangular" or "trapezoidal" region.

From the basic hexagon a star can be made.

Triangles can be added to the star to make a larger hexagon.
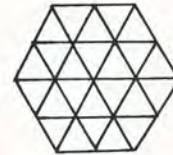
> You may want to assemble one on the overhead.

Ask the students if it is possible to make larger and larger hexagons with smaller hexagons inside them. (The students could count the number of triangles used in each sized hexagon.)

Have students examine the basic 6-triangle hexagon and locate other shapes inside it. For example, there are 6 rhombuses. You may want to demonstrate student responses on the overhead to the rest of the class. Students could discover how many different ways the basic hexagon can be separated into 3 rhombuses.

Is it possible to start with the basic rhombus and make larger and larger rhombuses that are the same shape as the original one? How many triangles in each? Can students see hexagons within the rhombuses? Are there trapezoids inside these?

The same approach could be used if students start with either the basic trapezoid or parallelogram. Students should begin to see that the basic shapes are tied together and that each shape could be repeated infinitely by adding more triangles.

Is it possible to cover the tops of the desks with triangles without leaving spaces? the floor? the whole playground? Could the same be done with a rhombus, a trapezoid or a hexagon?

Supply students with sheets of paper tessellated with equilateral triangles. (See the next page.) Ask them to use colored pencils to design the plan of a tiled floor. The pattern should repeat. Some suggestions:
1) Use only hexagonal tiles.
2) Use only trapezoids.
3) Use only rhombuses.
4) Use hexagons, triangles and rhombuses.
5) Use a mixture of any tiles.

A bulletin board display could be made from the students' tiling patterns.

REPEATING SHAPES
(CONTINUED)
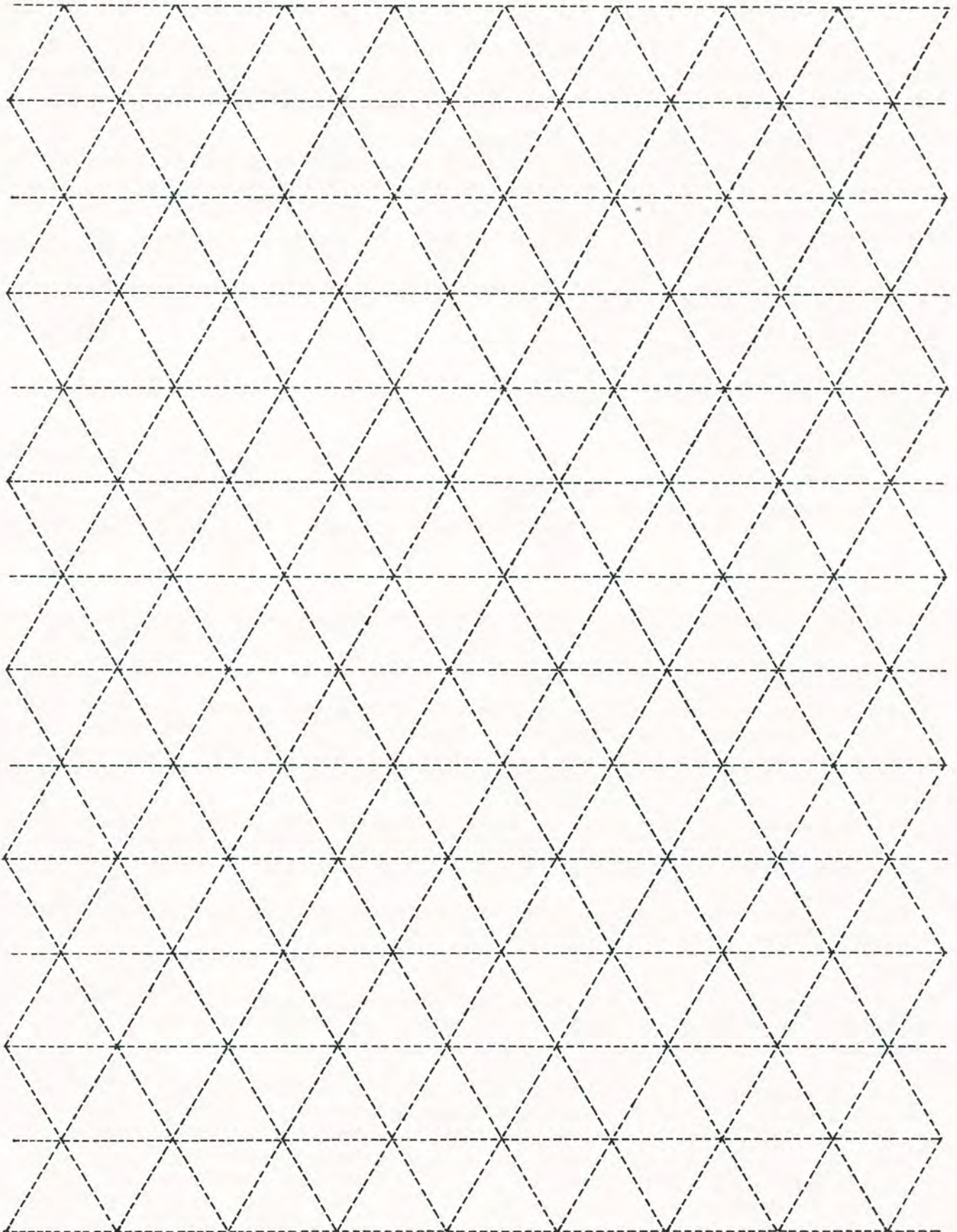
TESSELLATIONS

Patterns which fill a plane with no overlaps or gaps are called tessellations. Tessellation comes from the Latin word *tessellare* which means to pave with tiles. Tessellations can be used to introduce motion geometry, to help develop a feeling for congruency, similarity and pattern,to investigate and apply the angle properties of polygons, to introduce the concept of area and to provide students with the opportunity to improve their competence in geometrical construction. Some student activities and their consequences are outlined below.

As readiness for all tessellation activities show the students samples of tiling patterns. Have them identify the basic shapes in each pattern.

I Investigations with triangles (Students need construction paper and scissors.)

1) Supply students with a right triangle tile. Have them cut out ten triangles having the same size and shape as the tile. Ask them to use their triangles to make a tiling pattern.

Model for right triangle tile. Be sure to make the student's tile larger.
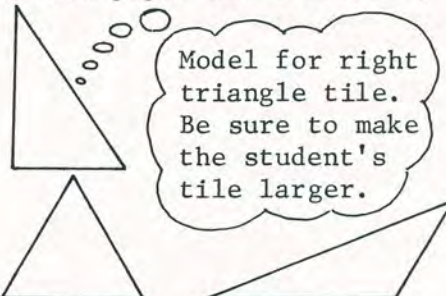
2) Have them repeat experiment #1 for each of these shapes:
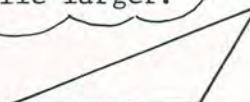
ISOSCELES TRIANGLE   EQUILATERAL TRIANGLE   SCALENE TRIANGLE

3) Have students draw a triangle of any shape they choose. Could they use it to make a tiling pattern? (Students should discover that any .triangle will tessellate a plane.)

II Investigations with quadrilaterals (Students need construction paper and scissors.)

1) Divide the class into groups. Each group of students tries to tessellate with a familiar quadrilateral. Provide patterns for all students to use in cutting out their quadrilaterals. Students in each group can help each other. The groups could share their tessellations with the rest of the class.
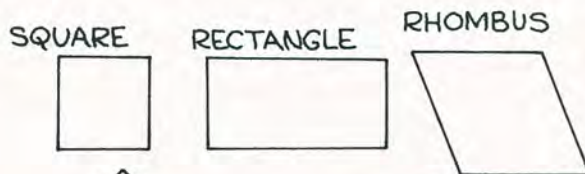
SQUARE   RECTANGLE   RHOMBUS

KITE   TRAPEZOID

2) Supply students with an irregularly shaped quadrilateral to use as a pattern to cut out ten quadrilaterals. Ask them to form a tiling pattern with the shapes.
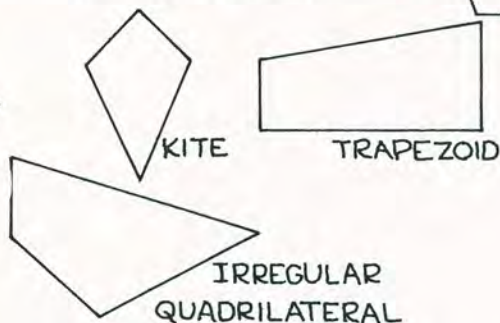
IRREGULAR QUADRILATERAL

Students can be somewhat haphazard in their early attempts to produce tessellations as shown to the right. Shapes can be added without a pattern emerging. To help develop an awareness for an organized approach you could provide a partly drawn pattern and have them continue the pattern.
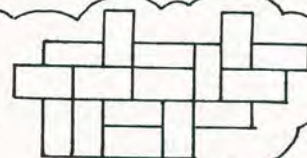
3) Ask students if they think any quadrilateral can be used to make a tiling pattern. Suggest that they choose a "difficult" quadrilateral and check to see if it will tessellate.

It is possible to tessellate any quadrilateral by rotating it 180° about the midpoints of the sides. You don't need to mention this to the students unless they bring it up.

4) Dot paper can also be used to make tiling patterns after students have had experience fitting shapes together. Coloring the pattern should be encouraged since the use of color often makes it easier for students to continue their pattern.
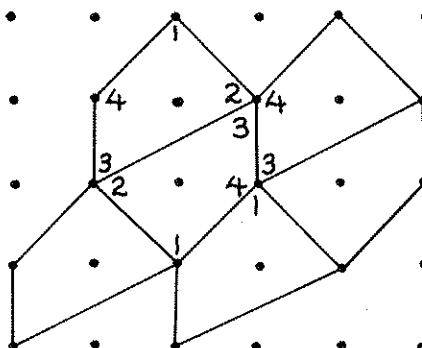
In this tessellation the corners of four quadrilaterals always meet at a point. Each quadrilateral presents a different corner, so $m \angle 1 + m \angle 2 + m \angle 3 + m \angle 4 = 360°$.

5) Ask students if they think every polygon will tessellate. Challenge them to find an example of one that doesn't.

III Investigations with regular polygons

Have students make themselves a set of tiles from construction paper. A pattern for the tiles is provided. Students should cut about 30 triangles, 10 squares and 5 each of the other regular polygons. Students could work in pairs.

1) Allow students to discover which regular polygons will tessellate if only one shape is allowed and the sides coincide. One approach might be:

a) Have them mark a point on their paper and try to surround it with equilateral triangles.

b) Have them repeat for each of the other polygons.

PATTERN FOR REGULAR POLYGONS

c) Complete a table to organize the facts.
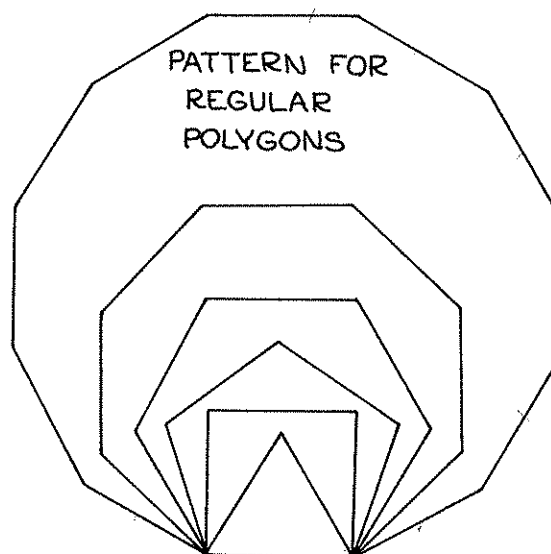
I Polygon	II Number of degrees in each angle	III Does the polygon tessellate?	IV If it tessellates, how many polygons fit around the point?
Triangle	60°	Yes	
Square			

To find the number of degrees in each angle have students first find the total number of degrees in all the interior angles. See Interior Angles of a Polygon 1 in the section POLYGONS & POLYHEDRA. Then divide this total by the number of angles in the polygon.

By relating the numbers in columns II and IV to 360°, the number of degrees in one complete turn, students can discover why only the triangle, square and hexagon tessellate. These three patterns are called <u>regular tessellations</u>.

TESSELLATIONS
(PAGE 3)

2) Have students discover which regular polygons of two or more kinds can tessellate a plane if the arrangement of corners is the same at every vertex--- a <u>semi-regular tessellation</u>.

a) First have students mark a point and try to surround it with squares and triangles. (Since the arrangement of angles must add to 360° students can conclude that only 3 triangles and 2 squares will work. $3 \times 60° + 2 \times 90° = 360°$.)

There are two possible arrangments: The first is described as 3-3-3-4-4; the second as 3-3-4-3-4.

Each arrangment will make a different tessellation: Note that each vertex has the same arrangment.

3-3-3-4-4   3-3-4-3-4

b) By trying to fit the tiles around a point and then continuing the arrangement students can discover that there are only eight semi-regular tessellations. The other six are illustrated below.

3-6-3-6   3-3-3-3-6   3-4-6-4   3-12-12

4-8-8   4-6-12

Note: The three regular tessellations in #1 could be described as 3-3-3-3-3-3; 4-4-4-4; 6-6-6.

3) An unlimited number of tessellations can be formed from regular polygons if it is not required that all vertices be the same. Encourage students to create their own designs. (Be sure there are no spaces between the tiles and no overlapping.) If the designs are drawn on paper, students could look for the minimum number of colors needed to shade the design   and the possible ways to vary the colors.

TESSELLATIONS

(PAGE 4)

4) Students will also discover tessellations that don't require the sides of the polygons to coincide. (Irregular tessellations)
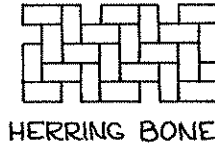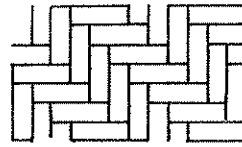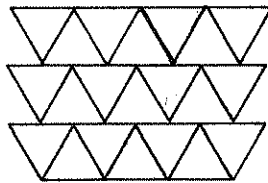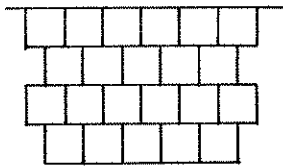
HERRING BONE

IV   Escher – Type Tessellations:

> M. C. Escher (1898 – 1972) was a Dutch artist famous for his tessellations.

Any shape that will tessellate a plane can be transformed into a new shape (by adding and subtracting parts) that will tessellate the plane as long as the area of the new shape is the same as the original.

1) Start with a basic shape that tessellates.

2) Find the midpoints of the sides.

> Be sure the portion is rotated, not flipped.

3) Cut out a portion of the shape between a vertex and midpoint. Rotate the portion and attach it to the same side. (See figure.)

4) Step #3 could be performed on the other sides.

5) The new shape now tessellates. Students can use their artistic talents and imagination to decorate the shape.

> The film: Adventures in Perception is an excellent source of information on Escher's life and his graphics.

> For a more sophisticated approach to Escher tessellations see: "How to Draw Tessellations of the Escher Type," The Mathematics Teacher, April, 1974.
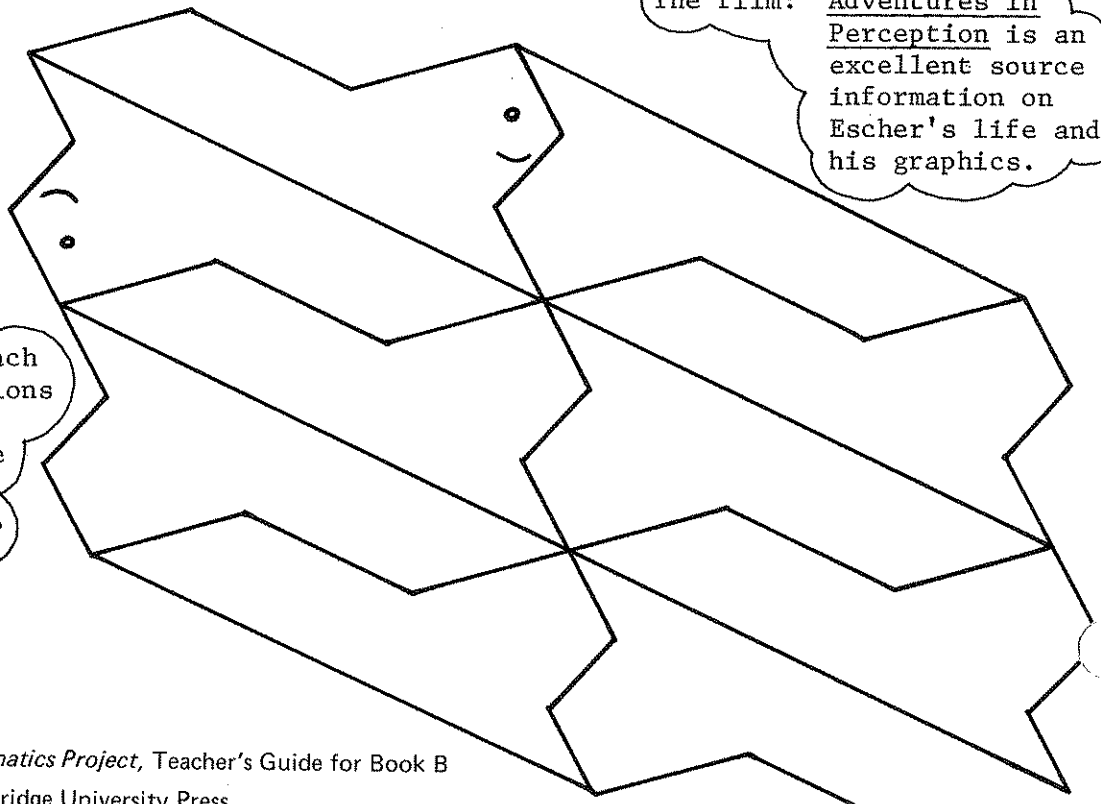
TESSELLATIONS WITH TRIANGLES

A tessellation made with a scalene triangle can be used to illustrate many geometrical concepts. You may wish to make a transparency of a similar pattern for use on the overhead.

ALL LINES PARALLEL TO THIS LINE MAKE A FAMILY.

BASIC TILE

1) Students can identify three families of parallel lines.

2) By rotating and sliding the basic tile to cover other triangles in the tessellation all the angles can be marked. At any vertex in the pattern it can be seen that $m\angle 1 + m\angle 2 + m\angle 3 = 180°$ or that the sum of the angles of a triangle is $180°$.
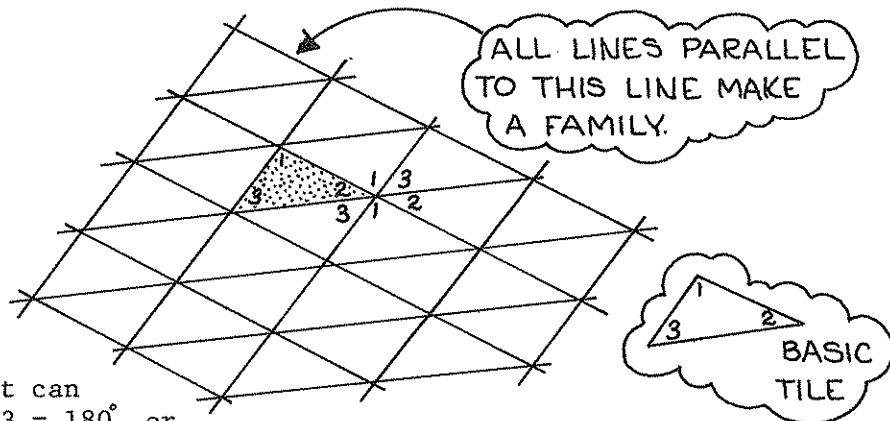
3) By taking any two <u>parallel</u> lines (P_1, P_2) and a transversal (t), students can see that "alternate interior angles" are congruent, and "interior angles on the same side of a transversal are sup-plementary. See *Repetitious Angles II* in the section of LINES, PLANES & ANGLES.

4) Three or more <u>parallel</u> lines (ℓ_1, ℓ_2, ℓ_3, ℓ_4) cut off proportional segments on two transversals (r_1, r_2). That is,
$$\frac{a_1}{b_1} = \frac{a_2}{b_2}.$$
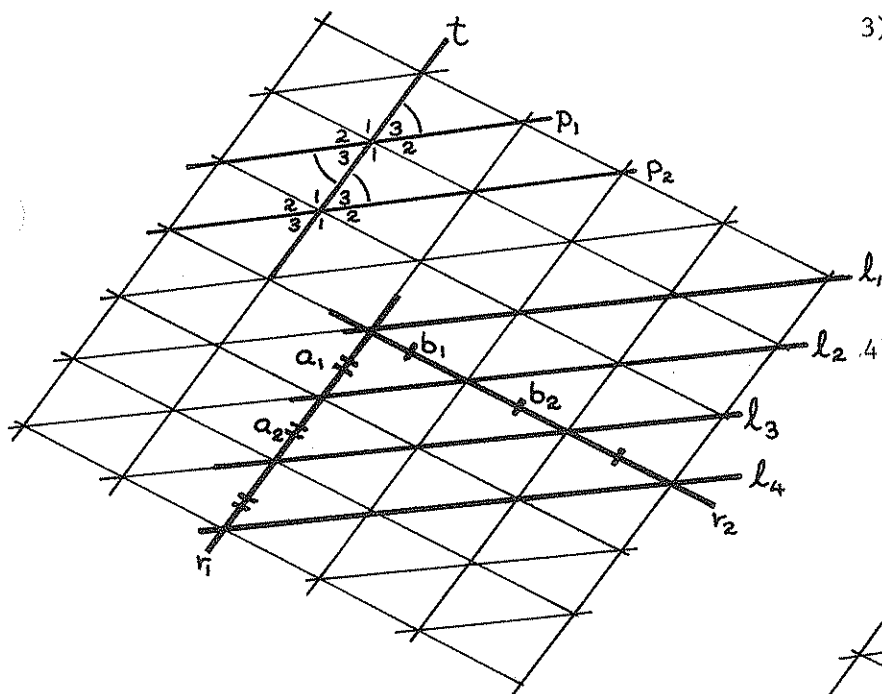
5) A line (ℓ) that passes through the midpoints of two sides of a triangle is a parallel to the third side. Also the line segment joining the two midpoints has half the measure of the third side.

6) Many examples of similar triangles occur in the tessellation.
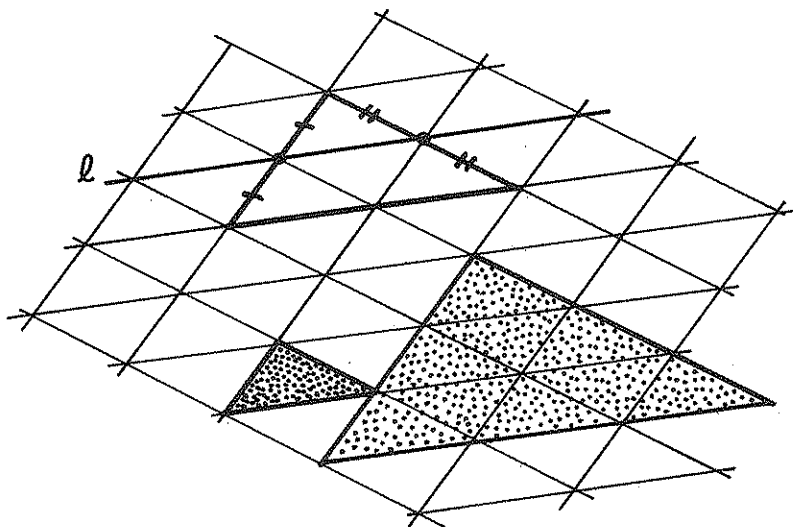
THUMB-TACKTICS

Materials Needed: Nine notecards, five triangular cards, thumbtacks and cardboard

A picture can be held up with four thumbtacks

Two pictures use either six,
seven or eight thumbtacks.
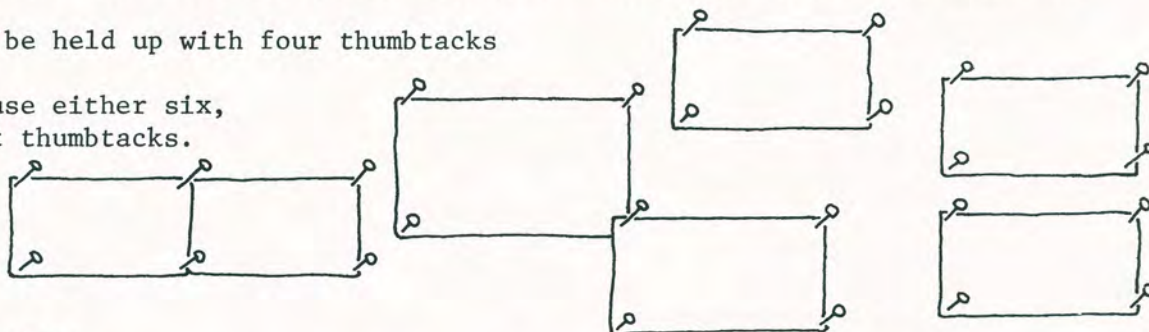
Show how to fasten:

PICTURES	3	3	3	3	4	4	4	4	4	5	5	9	16	25
WITH THUMBTACKS	8	9	10	11	9	10	11	12	13					

Triangular pictures can be held up with three thumbtacks.
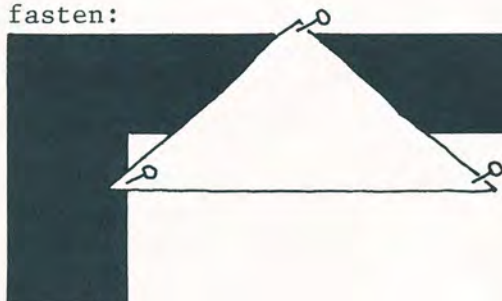
GREATEST NUMBER POSSIBLE

LEAST NUMBER POSSIBLE

Show how to fasten:

TRIANGLES	2	2	2	3	3	3	3	3	4	4	4	5	5	5
WITH THUMBTACKS	4	5	6	5	6	7	8	9			7			9

FEWEST NUMBER POSSIBLE

FEWEST NUMBER POSSIBLE

GREATEST NUMBER POSSIBLE

GREATEST NUMBER POSSIBLE

Show how five triangles and six
triangles can both have the same
"fewest number" of thumbtacks.

7 thumbtacks

IDEA FROM: *Paper Plus*

Permission to use granted by Mrs. Ernest R. Ranucci

TWINKLE, TWINKLE

1) FOLD ANY RECTANGULAR PIECE OF PAPER IN HALF.

2) FIND THE MIDPOINT OF THE BOTTOM EDGE.

3) FOLD THE TOP LEFT CORNER DOWN TO THE MIDPOINT AND CREASE.
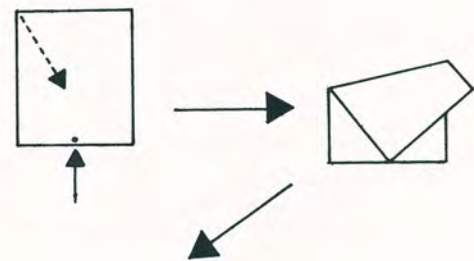
4) FOLD THE BOTTOM LEFT TRIANGLE OVER.
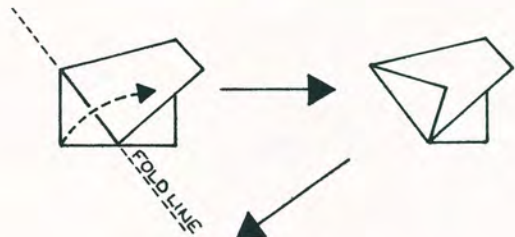
5) FOLD THE TOP EDGE DOWN TO MATCH THE LEFT SIDE EDGE.

6) MAKE A CUT WITH YOUR SCISSORS.

7) UNFOLD THE LEFT HAND PIECE.

8) AFTER YOU HAVE SEEN THE SHAPE REFOLD AND CUT THE PIECE AGAIN.
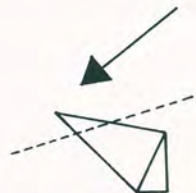
9) EXPERIMENT WITH DIFFERENT SIZED PIECES OF PAPER.

FOLD

BOTTOM

FOLD LINE

FOLD LINE

GEOSTRIPS

Geostrips are manufactured by Invicta Plastics (U.S.A.) Ltd. (Suite 940, 200 5th Ave., New York, NY 10010). They can be used to investigate angles, polygons, rigidity, parallelism, the Pythagorean theorem, etc.

Construction Information:

Geostrips can be constructed from tagboard. The nine different lengths are given below. A set consists of 12-R1's, 12-R2's, 4-R3's, 8 B1's, 8-B2's, 6-Y1's, 6-Y2's, 6-W1's and 6-W2's. This would be enough for six students working in pairs or threes. Measurements are made from center of one hole to center of the other. Width is 2 centimetres.

RED STRIPS

R1 ⊖—4cm—⊖   ○
⊢—— 8cm ——⊣

R2 ⊖——8cm——⊖   ○
⊢————16cm————⊣

R3 ⊖—4cm—⊖   ○   ○   ○   ○   ○   ○   ○
⊢————————32cm————————⊣

BLUE STRIPS

B1 ⊖—5.6cm—⊖   ○
⊢— ≈11.3cm —⊣
B1 IS R1 × √2

B2 ⊖—4cm—⊖   ○   ○   ○
⊢———— ≈22.6cm ————⊣
B2 IS R2 × √2

YELLOW STRIPS

Y1 ⊖—4cm—⊖ ○ ○   ○
⊢——— ≈13.9cm ———⊣
Y1 IS R1 × √3

Y2 ⊖—4cm—⊖   ⊖—4cm—⊖   ○   ○   ○
⊢————— ≈ 27.8cm —————⊣
Y2 IS R2 × √3

WHITE STRIPS

W1 ⊖——8.9cm——⊖   ○
⊢———— ≈17.9 cm ————⊣

W2 ⊖—4cm—⊖   ○   ○   ○
⊢———— ≈20.8cm ————⊣
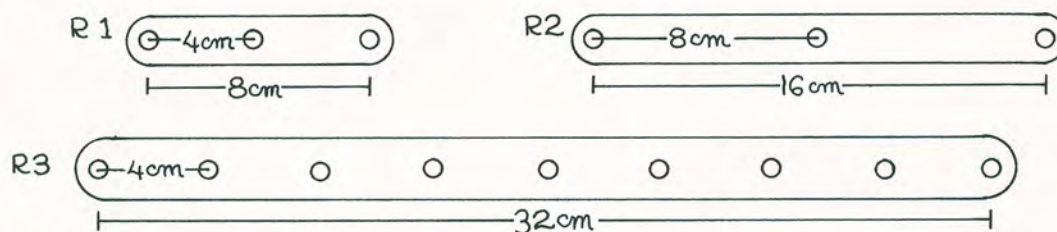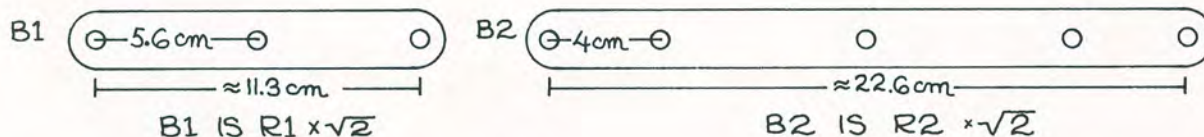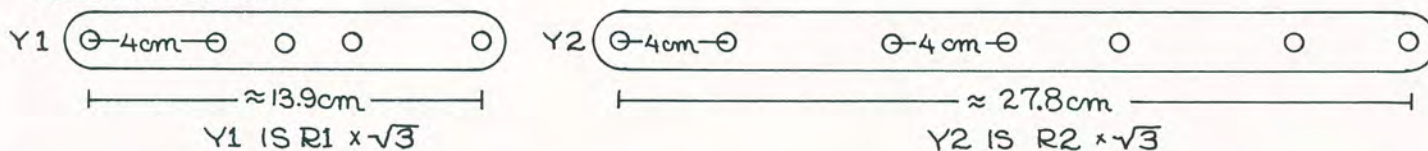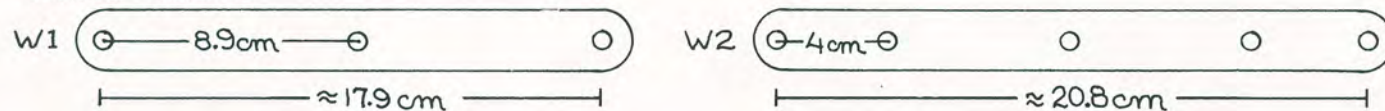
Readiness Activities:

1) Make some letters of the alphabet with the strips.

2) Make two shapes that are exactly the same.

3) Make some geometrical patterns. Here are some ideas.

AN EXACT FIT

The concept of congruence with students in middle schools can be approached by having them fit objects together to see that all parts of the objects exactly fit. Experiences with this concept can be provided by the activities suggested below.

a) Cut out figures

   Given an envelope with several pairs of congruent figures and some with no match, students could match the pairs of congruent figures by placing one on the other and looking for exact fits. Some figures should have only a few parts congruent to encourage students to examine the entire figure.

ENVELOPE
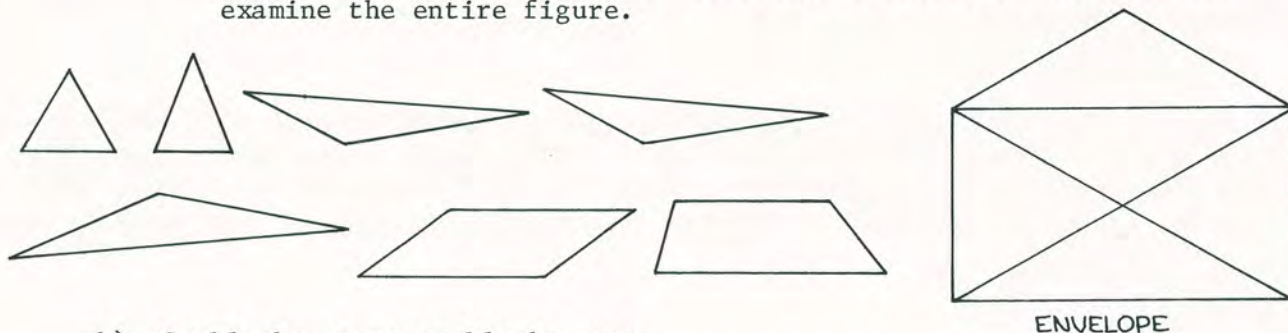
b) Geoblocks, pattern blocks, etc.

   Students can pick out pieces that have the same size and shape. Different sized blocks could be used to find congruent faces.
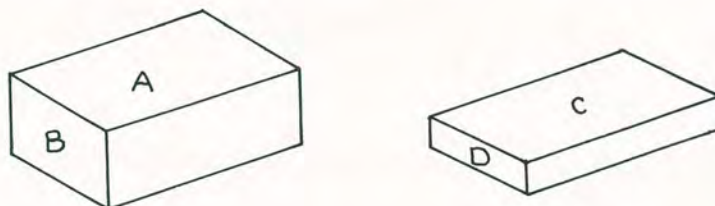
Face A is congruent to face C but face B is not congruent to face D

c) Worksheets

   Given two figures students could use visual perception or tracing paper to determine if the figures are congruent. Some of the figures should be arranged in different orientations to encourage the use of reflections, rotations and translations.

Yes, reflection

Yes, rotation (and translation)

Yes, translation

429
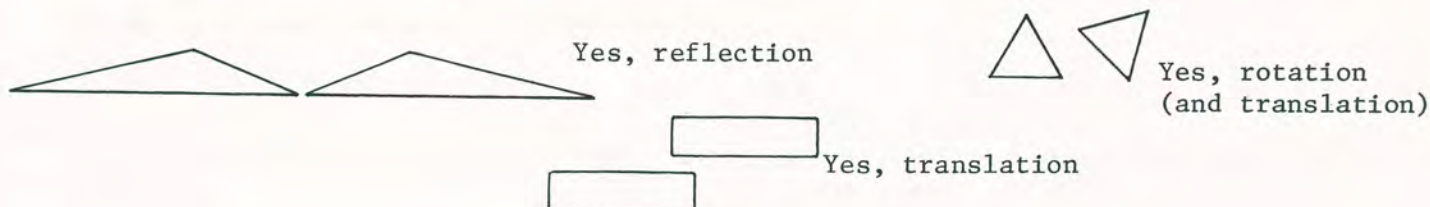
AN EXACT FIT
(CONTINUED)

For triangles knowing that just certain parts are congruent assures that the whole figures are congruent. On the activity card *Side, Side, Side* (SSS) students discover that all triangles formed with three given sides are congruent. Other conditions which assure congruence are: (1) Side, included angle, side (SAS), (2) angle, included side, angle (ASA) or (3) angle, angle, side (AAS).

See the basic compass and straightedge constructions in *Graphic Representation* in the TEACHING EMPHASES section.

On the activity card *Angle, Angle, Angle* (AAA) students discover that three given angles do not necessarily determine congruence. Side, side, angle (SSA) is another case that does not determine congruence. This case is illustrated below using cardboard strips fastened together with brads. Geostrips could also be used (See *Geostrips*).

Given: Side AB
        Side BC
        and ∠CAB
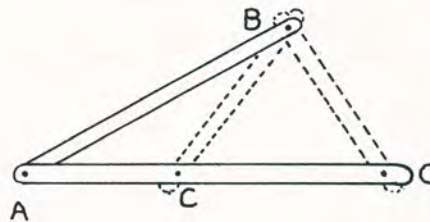
Compass and straightedge construction can also be used to illustrate this concept.

Given: Side AB - 5 cm
        Side BC - 3 cm
        and ∠CAB as shown
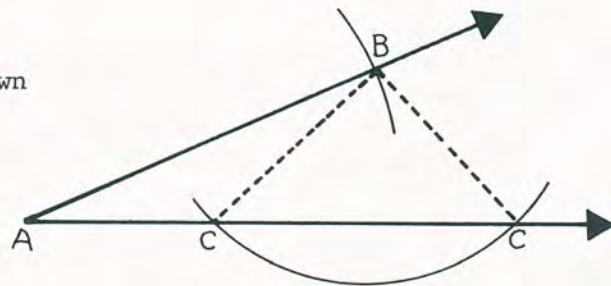
The next page shows two activity cards for (SSS) and (AAA). Similar activities could be developed for the other cases.

DESCRIBING TRIANGLES

Draw a line to match each figure with its description.

a)

all sides have the same length (congruent)

b)

each side has a different length

c)

two sides have the same length (congruent) and the third side has a different length

d)

all angles are acute angles

e)

one angle is a right angle and the other two angles are acute

f)

one angle is an obtuse angle and the other two angles are acute

g)

all three angles have the same measure (congruent)

h)

each angle has a different measure

i)

two angles have the same measure (congruent) and the third angle has different measure

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DESCRIBING TRIANGLES

(CONTINUED)

Answer yes or no for these questions. If you answer yes write the letter of the triangle that shows the properties. If one is not shown draw a triangle that has the properties.

Can a triangle have

		Yes	d
1)	a right angle and no congruent sides?	Yes	d
2)	an obtuse angle and two sides congruent?		
3)	three right angles?		
4)	all acute angles with exactly two of the angles congruent?		
5)	two obtuse angles?		
6)	a right angle and two sides congruent?		
7)	an obtuse angle and no congruent sides?		
8)	all acute angles and all angles congruent?		

Triangles are usually classified in two ways, by the angles and by the sides.

By angles

1) acute . . . all acute angles

2) right . . . one right angle

3) obtuse . . . one obtuse angle

By sides

1) equilateral . . . all sides congruent
   (also all angles congruent)

2) isosceles . . . two sides congruent
   (also two angles congruent)

3) scalene . . . no sides congruent
   (also no angles congruent)

Classify each of the triangles (a) - (i) as either acute, right or obtuse. Then classify each as equilateral, isosceles or scalene.

a) _____ _____   f) _____ _____

b) _____ _____   g) _____ _____

c) _____ _____   h) _____ _____

d) _____ _____   i) _____ _____

e) _____ _____

THESE ARE.
THESE ARE NOT.

1) These are regular polygons. These are not. Mark the regular polygons.
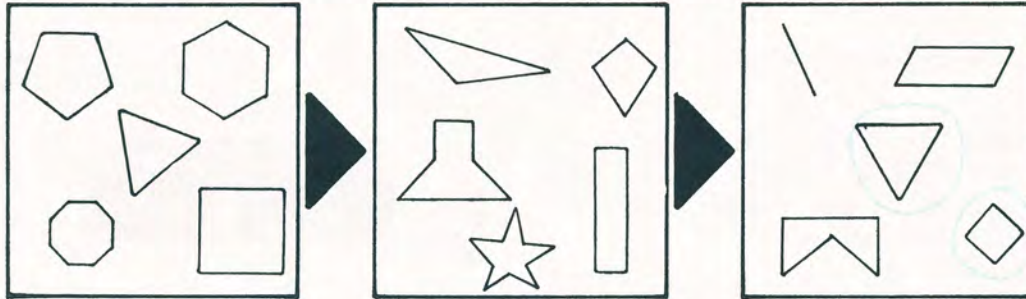
2) These are right triangles. These are not. Mark the right triangles.
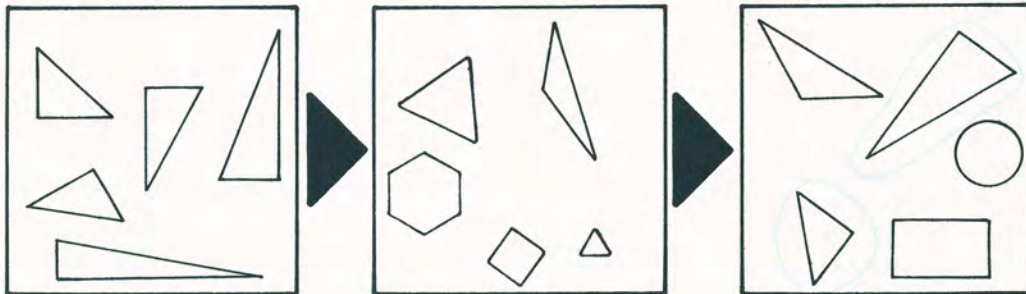
3) These are parallelograms. These are not. Mark the parallelograms.
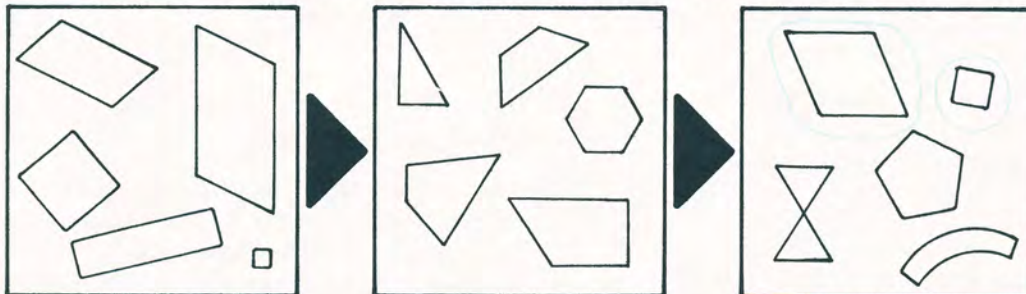
4) These are hexagons.   These are not.   Mark the hexagons.
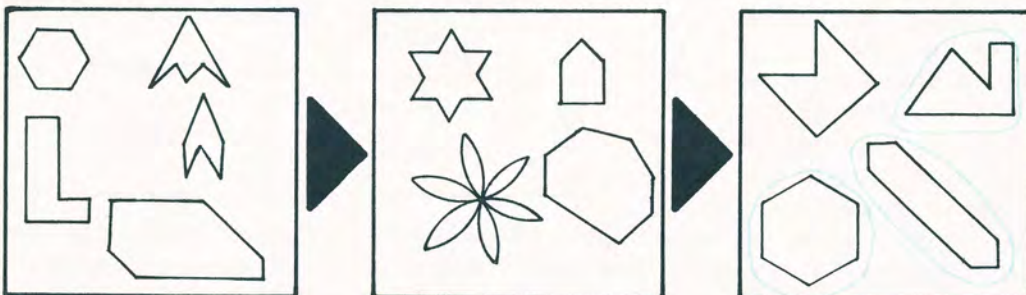
433

POLYGON PASS-OUT

You will need:
1) A deck of cards with geometric figures on them.
2) Two teams.
3) A clue-giver for each team.

Rules:
1) The game is played like Password. The clue-giver for team 1 draws a card and gives a clue to his team. One person on the team tries to guess the geometric figure. If the guess is correct team 1 receives five points.
2) If the guess is not correct the clue-giver for team 2 gives a clue and team 2 makes a guess. A correct guess would score four points.
3) If no correct answer is given go back to team 1, etc.
4) If after 5 rounds, neither team has guessed the figure, the shape is revealed and a new round is begun.
5) Team 2 starts the next round.
6) The team with the highest score at the end wins.
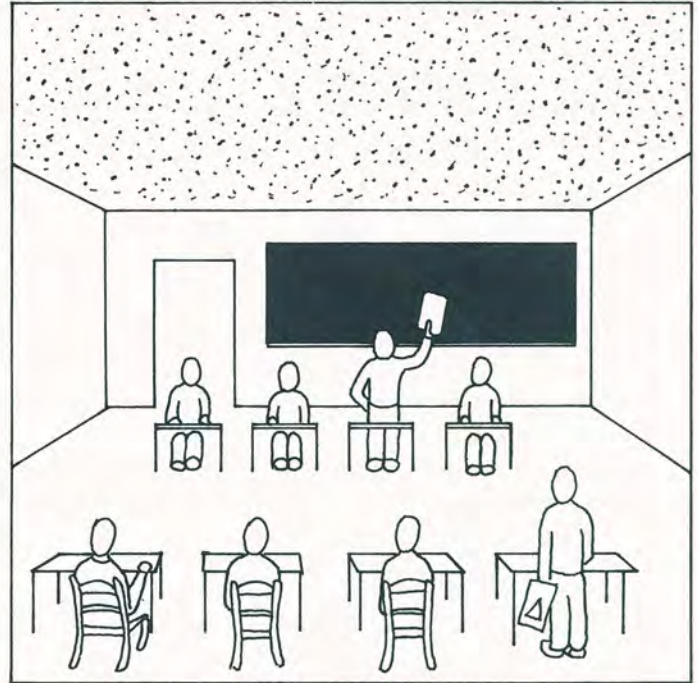
WHAT'S THE DIFFERENCE?

You will need:
1) A set of questions (see samples).
2) Two teams.
3) One person to read the questions.

Procedure:
1) The person reads ONE question. (For example, "What's the difference between a rectangle and a triangle?"
2) Team 1 responds and scores one point if correct. (i.e. A rectangle has four sides; a triangle has three.)
3) Team 2 responds to the same question, but must give another answer to score one point if correct. (i.e. The sum of the angles of a rectangle is 360°-- twice that of the triangle.)
4) Team 1 responds again. (i.e. A rectangle has four interior angles; a triangle has three angles.)
5) Team 2 continues with another answer. (i.e. The triangle is a rigid figure; a rectangle is not.)

6) Play alternates until both teams exhaust their answers and pass in succession.
7) Team 2 starts the next round.
8) The team with the highest score wins.

SAMPLE QUESTIONS

WHAT'S THE DIFFERENCE BETWEEN....
A RECTANGLE AND A TRIANGLE?
A CIRCLE AND A TRIANGLE?
A PARALLELOGRAM AND A SQUARE?
A PARALLELOGRAM AND A RECTANGLE?

WHAT CAN YOU SAY ABOUT....
A TRIANGLE BUT NOT A CIRCLE OR A SQUARE?
A SQUARE BUT NOT A TRIANGLE OR A RHOMBUS?
A ⚡ AND A TRIANGLE?

434   IDEA FROM: *Ideas for Manipulative Materials Elementary Mathematics*

ARE YOU IN SHAPE?

SHADE IN THE PENTAGON TO THE LEFT OF EACH CORRECT DESCRIPTION. USE THE UNSHADED PENTAGONS TO FORM A MESSAGE AT THE BOTTOM OF PAGE TWO. THERE IS ONE FALSE DESCRIPTION IN EACH GROUP.

QUADRILATERAL
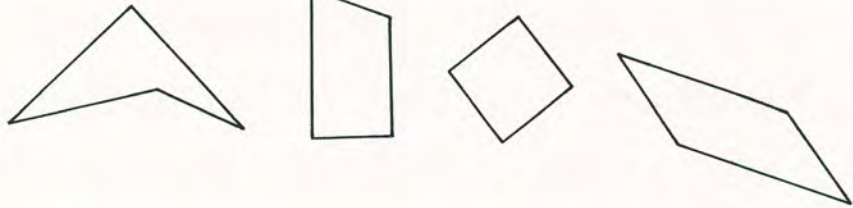
A-1 > A FOUR-SIDED POLYGON.

E-2 > ANYTHING MADE WITH FOUR LINE SEGMENTS.

I-3 > A SIMPLE CLOSED FIGURE FORMED BY FOUR LINE SEGMENTS.

PARALLELOGRAM
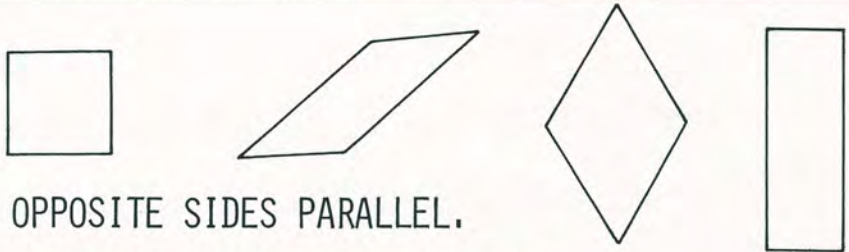
U-4 > A POLYGON WITH OPPOSITE SIDES PARALLEL.

O-5 > A QUADRILATERAL WITH OPPOSITE SIDES PARALLEL.

T-6 > A QUADRILATERAL WITH OPPOSITE SIDES PARALLEL AND CONGRUENT.

N-1 > A FOUR-SIDED POLYGON WITH OPPOSITE SIDES CONGRUENT.

RECTANGLE

S-2 > A POLYGON WITH FOUR SIDES AND FOUR RIGHT ANGLES.

P-3 > A QUADRILATERAL WITH FOUR RIGHT ANGLES.

D-4 > A PARALLELOGRAM WITH FOUR EQUAL ANGLES.

B-5 > A PARALLELOGRAM WITH FOUR SIDES AND FOUR ANGLES.

ARE YOU IN SHAPE?
(CONTINUED)

RHOMBUS
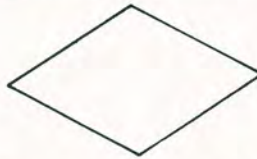
G-6 > A PARALLELOGRAM WITH FOUR CONGRUENT SIDES.

K-7 > A QUADRILATERAL WITH ALL FOUR SIDES CONGRUENT.

O-1 > A QUADRILATERAL WITH OPPOSITE SIDES PARALLEL.

V-2 > A QUADRILATERAL WITH OPPOSITE ANGLES CONGRUENT.

TRAPEZOID
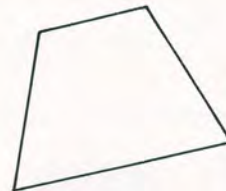
L-4 > A POLYGON WITH FOUR SIDES AND ONLY ONE PAIR OF SIDES PARALLEL.

Y-3 > A FIGURE WITH ONE AND ONLY ONE PAIR OF PARALLEL SIDES.

H-5 > A QUADRILATERAL HAVING ONLY TWO SIDES PARALLEL.

SQUARE

R-2 > A RECTANGLE HAVING ALL FOUR SIDES CONGRUENT.

T-6 > A QUADRILATERAL HAVING FOUR RIGHT ANGLES.

M-1 > A QUADRILATERAL HAVING FOUR CONGRUENT SIDES AND ALL RIGHT ANGLES.

J-3 > A RHOMBUS WITH FOUR CONGRUENT ANGLES.

___ ___ ___   ___ ___ ___ !
 3   1   4   5   2   6

THE ANGLES IN TRIANGLES

"The sum of the measures of the angles of a triangle (180°)" is an important invariant in geometry. Students can discover and illustrate this with many activities.

1) Find the midpoints of the two shorter sides of a triangle. Connect the midpoints. Draw perpendiculars from the midpoints to the third side. Fold on the three lines and the three angles will form a straight angle.
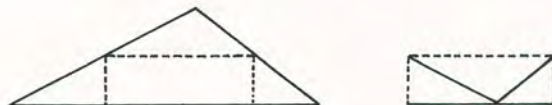
   Note: If an acute triangle is used any two sides may be bisected.

2) In any right triangle the two acute angles will fold to exactly cover the right angle. So the measure of the three angles equals the measure of two right angles.

3) Angles 1 and 2 (the non-adjacent angles) will exactly cover angle 4 (the exterior angle of a triangle) forming a straight angle.
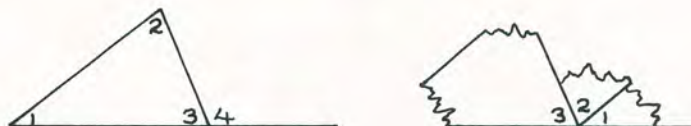
4) The three angles of a triangle can be arranged about a point to form a straight angle.

   a) The Angle Game:

   Two players each draw a triangle and cut off the three angles. Player 1 chooses an angle and places the vertex on the dot. Player 2 plays an angle adjacent to the first angle. Play continues with the winner being the last player to place an angle without overlapping an angle.

   Note: Since the measure of the angles of two triangles is 360°, player two will always win since he will place the last angle.

5) Construct a parallel to a side through the opposite vertex. By noticing the alternate interior angles or by cutting and fitting the three angles together a straight angle is formed.

437
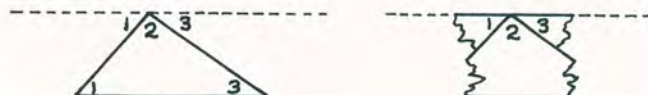
THE ANGLES IN TRIANGLES
(CONTINUED)

6)

Use a
pencil.

Rotate through
an angle.

Slide to the
next vertex.

Rotate again.
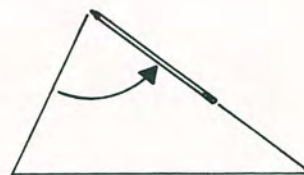
Slide again.
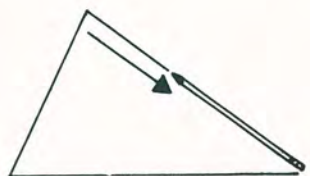
Rotate.
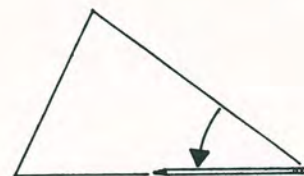
The pencil is pointing in the opposite direction so it has rotated 180°.

7) Use a protractor to find and record the measures of the angles of several triangles.

TRIANGLE	MEASURE OF ∠A	MEASURE OF ∠B	MEASURE OF ∠C	SUM OF MEASURES OF ∠A, ∠B AND ∠C
1				
2				
3				

SUM THING

1) Use your scissors to cut out these strips.

IN MAKING A TRIANGLE WITH STRIPS BE SURE THE ENDS MEET.

The name Cuisenaire and the color sequence of the rods are trademarks of the Cuisenaire Company of America, Inc.

| 2 |
| 3 |
| 4 |
| 5 |
| 6 |
| 7 |
| 8 |

2) Try to make a triangle with each set of strips. Draw a circle around the lengths which do make a triangle.

a) 2, 3, 4   e) 2, 3, 8   i) 2, 4, 8   m) 3, 4, 5
b) 2, 3, 5   f) 2, 4, 5   j) 2, 5, 6   n) 3, 4, 6
c) 2, 3, 6   g) 2, 4, 6   k) 2, 5, 7   o) 3, 4, 7
d) 2, 3, 7   h) 2, 4, 7   l) 2, 5, 8   p) 3, 4, 8

3) If two sides are 2 units and 3 units, what is the longest the third side can be?
4 units

4) If two sides are 3 units and 4 units, what is the longest the third side can be?
6 units

5) Complete this table.

USE YOUR ANSWERS FROM PROBLEM #2

Length of sides	Sum of two shortest sides	Length of longest side	A triangle? Yes or No
2, 3, 4	5	4	Yes
2, 3, 5	5	5	No
2, 3, 6	5	6	No
2, 3, 7	5	7	No
2, 3, 8	5	8	No
2, 4, 5	6	5	Yes
2, 4, 6	6	6	No
2, 4, 7	6	7	No
2, 4, 8	6	8	No
2, 5, 6	7	6	Yes
2, 5, 7	7	7	No
2, 5, 8	7	8	No
3, 4, 5	7	5	Yes
3, 4, 6	7	6	Yes
3, 4, 7	7	7	No
3, 4, 8	7	8	No

6) Look at the table. What conclusion can you make about the lengths of the sides of a triangle?
The sum of the two shortest sides is greater than the longest side.

If students can't answer the questions have them cross out the "no" rows and then look at the numbers left in the second and third columns.

TYPE: Worksheet/Activity Card/ Manipulative

IDEA FROM: *Mathex,* Junior-Geometry, Teacher's Resource Book

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WHEN IS A TRIANGLE ACUTE, RIGHT OR OBTUSE?

1) Suppose a triangle has a perimeter of 15 units. What could the lengths of the sides be if only whole number lengths are allowed?

Write each suggestion on the board, even though the three lengths might not form a triangle. Don't allow students to judge any answers. After all the suggestions have been made, ask the students to pick out the isosceles and equilateral triangles.

2) Have each students use a compass and straightedge to try to draw each of the suggested triangles, or provide the students with 15-unit strips of paper (with units marked) to cut into the lengths listed. (The seven possibilities are: 4,4,7; 5,5,5; 5,7,3; 4,5,6; 1,7,7; 2,7,6; 3,6,6) Students may discover that the sum of the two smaller lengths must be greater than the largest length. A class discussion following the construction activity should help them reach this conclusion.

3) List the seven possibilities in a table and have the students copy the table on their papers. (The last two columns will be filled later.) Have them use their protractors (or "right angle tester") to help classify each triangle as acute, right or obtuse.

PERIMETER OF 15

LENGTH OF SIDES	TYPE OF TRIANGLE		
4,4,7	OBTUSE		

4) Challenge the students to see if they can predict whether a triangle is acute, right or obtuse by knowing only the lengths of the sides.

To help them discover a method have the students draw or construct with paper strips all the possible triangles with a perimeter of <u>16</u> units, measure the angles to decide the type of each triangle, and write the results in a table.

The remaining blank columns in both tables should then be labeled "Sum of the squares of the two shorter sides" and "Square of the longest side." Have the students complete these columns (a calculator could be used) and look for a pattern to help them reach a conclusion.

IF THE SUM OF THE SQUARES OF THE TWO SHORTER SIDES IS GREATER THAN THE SQUARE OF THE LONGEST SIDE THE TRIANGLE IS ACUTE; IF EQUAL, THE TRIANGLE IS RIGHT; IF LESS THAN, THE TRIANGLE IS OBTUSE.

SIDE, SIDE, SIDE

1) Use these three segments to make a triangle. You may use tracing paper, a compass and/or a ruler.

2) Compare your triangle with a friend's triangle. Are the two triangles congruent? _____

3) Two triangles are congruent if they will fit exactly on top of one another by flipping, turning or sliding.

ANGLE, ANGLE, ANGLE

1) Use these three angles to make a triangle. You may have to extend the sides to connect the angles together. You may use tracing paper, a compass or a protractor.

2) Compare your triangle with a friend's triangle. Are the two triangles congruent? _____

3) Two triangles are congruent if they fit exactly on top of one another by flipping, turning or sliding.

PATTERN BLOCKS I

blue
RHOMBUS

natural
NARROW RHOMBUS

yellow
HEXAGON

red
TRAPEZOID

orange
SQUARE

green
TRIANGLE
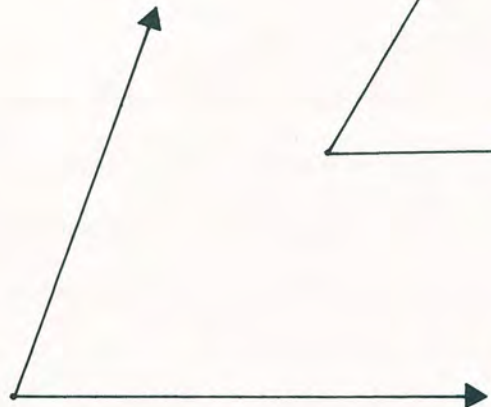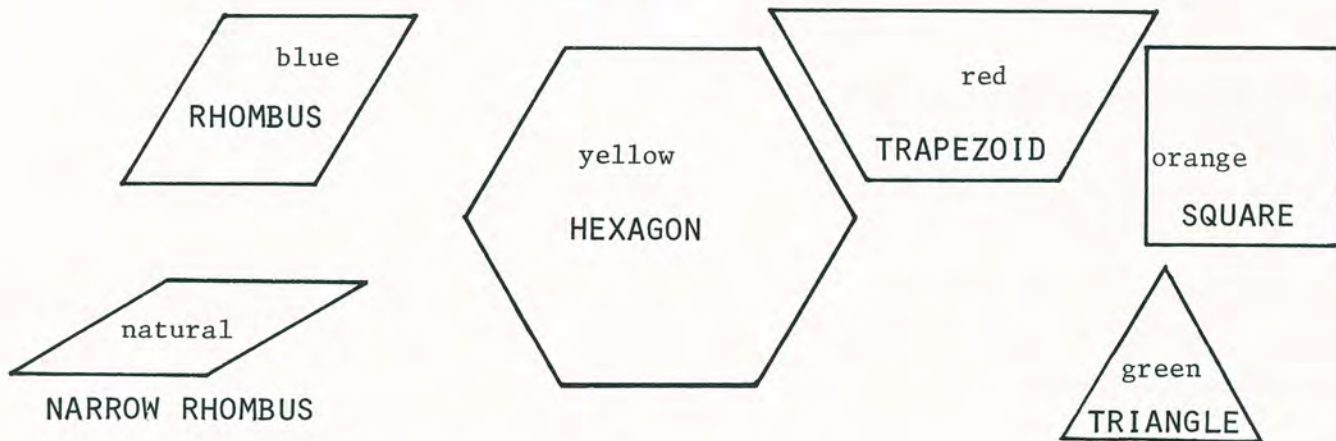
Pattern blocks can be used for place value, fractions, counting, equivalence, addition, sequences, patterns, classification, symmetry, area and perimeter.

Pattern blocks were developed by the Elementary Science Study of Education Development Center, Inc. Some sources for Pattern blocks are found on *Seeing Solid Shapes* in the Polyhedra subsection.

Construction Information:

Pattern blocks are usually painted wooden shapes. A complete set should contain 25 hexagons, 50 natural (wood) rhombuses, 50 blue rhombuses, 50 trapezoids, 25 squares and 50 triangles.

A starter set could be made of laminated colored paper or railroad board. Above is a pattern for each piece. It is important to make them exactly this size. The hexagon pattern can be used to make the trapezoid, rhombus and triangle.

Readiness Activities:

1) Cover the hexagon with as many different combinations of shapes as you can.

2) Can you cover the trapeziod with triangles?

3) What shapes can you make with the narrow rhombuses?

4) Make a picture with several different blocks.

5) Make a big triangle; a big star.

Materials Needed: A set of Pattern blocks

1) Take a square block.

   a) How many sides does it have? _____4_____

   b) How are the sides alike? _____same length_____

   c) How many angles does it have? _____4_____

   d) How are the angles alike? _____same degree measure_____

2) Take a triangle and answer the same questions. _3 congruent sides and 3 congruent angles_

3) Take a hexagon and answer the same questions. _6 congruent sides and 6 congruent angles_

4) How are all three shapes alike? _sides of all three shapes are congruent_

TRIANGLE   SQUARE   RHOMBUS   NARROW RHOMBUS   TRAPEZOID   HEXAGON
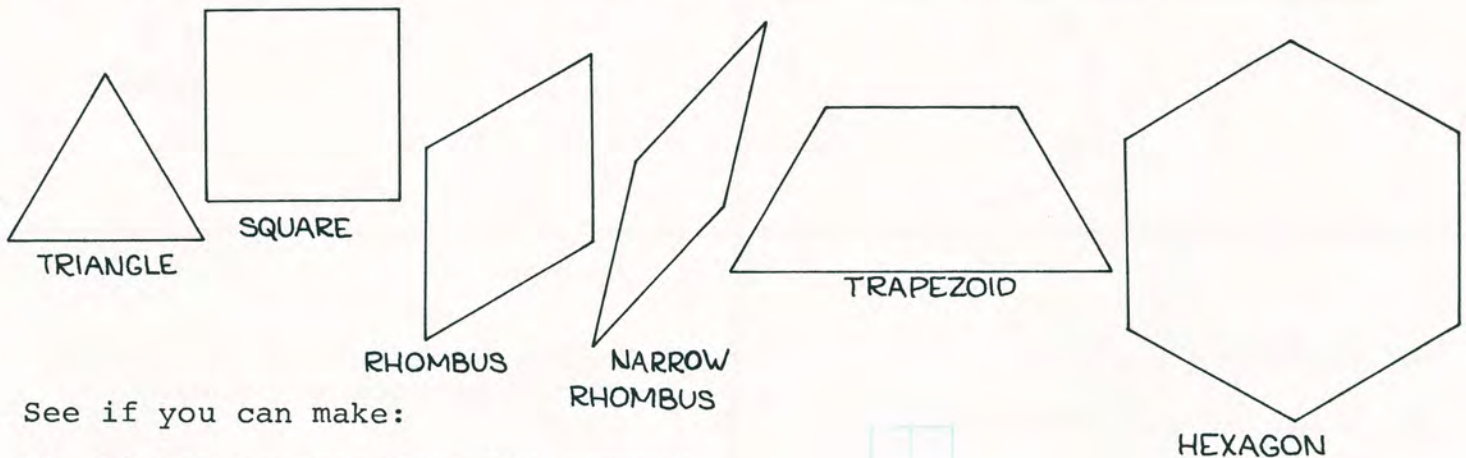
See if you can make:

5) bigger squares from the squares.

6) bigger triangles from the triangles.

7) bigger trapezoids from the trapezoids.

8) bigger rhombuses from the rhombuses.

9) bigger rhombuses from the narrow rhombuses.

10) bigger hexagons from the hexagons.

11) Which one is not possible? ___the hexagon___ Why? _The angle measures will not allow any straight lines to be formed._

PATTERN BLOCKS III

1) Use several squares.

2) Use four squares to fit around a point.
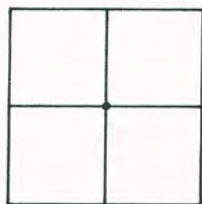
3) If there are 360° around a point, how many degrees are there in one angle of the square? __90__

4) If a square has four congruent angles, how many degrees are in the sum of the angles of the square? __360__
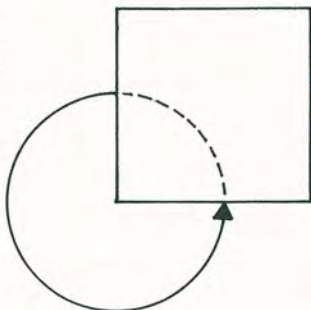
5) Use several of the triangles.

6) Fit the triangles around a point.

7) If there are 360° around a point, how many degrees are there in one angle of the triangle? __60__

8) Since this triangle has three congruent angles, how many degrees are in the sum of the angles of the triangles? __180__
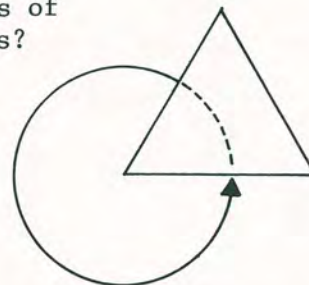
9) Use several hexagons.

10) Repeat the experiments above.

11) How many degrees are there in one angle of the hexagon? __120__

12) What is the angle sum of the hexagon? __720__
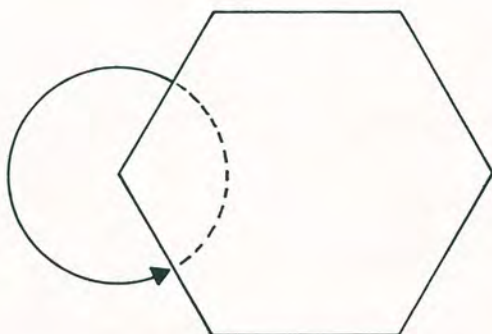
13) Use the information you found in the three experiments to complete this table.

Pattern Block	Number of sides	Angle sum	Measure of one angle
△	3 →	180° →	60°
□	4	360°	90°
⬠	5	?	?
⬡	6	720°	120°

Predict the angle measure for a regular pentagon (a five-sided polygon)? __540°__
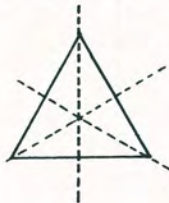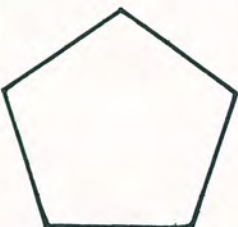
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LINES OF SYMMETRY

1)

POLYGON	Number of sides	Are the sides congruent?	Number of angles	Are the angles congruent?	Trace and cut out each figure. Fold to find all the lines of symmetry. Sketch solutions below.
(triangle)	3	YES	3	YES	How many? 3
(square)	4	YES	4	YES	How many? 4
(pentagon)	5	YES	5	YES	How many? 5
(hexagon)	6	YES	6	YES	How many? 6

2) How many lines of symmetry would a decagon with congruent sides and congruent angles have? _____10_____

3) Which of the following can you make? Yes, Isosceles
   a) A triangle with only one line of symmetry. No
      Exactly two lines of symmetry. Yes, Isosceles trapezoid
   b) A quadrilateral with only one line of symmetry.
      Two lines of symmetry. Yes, rectangle
   c) Pentagons with fewer than five lines of symmetry. Yes, Irregular
   d) Hexagons with fewer than six lines of symmetry. Yes, Irregular
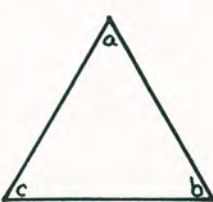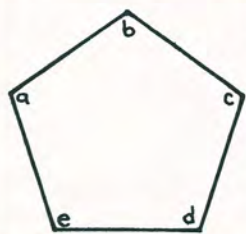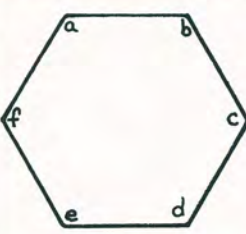
POINT SYMMETRY

1)

POLYGON	Number of sides	Are the sides congruent?	Number of angles	Are the angles congruent?	Trace and cut out each figure. In how many different ways can the figure be turned to fit on its outline? Sketch solutions below.
(triangle)	3	YES	3	YES	How many? 3
(square)	4	YES	4	YES	How many? 4
(pentagon)	5	YES	4	YES	How many? 5
(hexagon)	6	YES	6	YES	How many? 6

2) In how many different ways could a regular octagon be turned and fit on its original outline? _____

3) Which of the following can you make?

   a) A quadrilateral that turns onto its outline exactly two different ways.
      Yes parallelogram, rhombus, rectangle

   b) A quadrilateral that turns onto its outline exactly three different ways.
      No

   c) A triangle that turns onto its outline exactly two different ways.
      No

TYPE: Worksheet/Manipulative

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SPECIAL LINES AND SEGMENTS IN TRIANGLES

Every triangle has special lines and line segments associated with it. Some of these are the three perpendicular bisectors of the sides, the three medians, the three altitudes and the three angle bisectors. By using paper folding, protractor and ruler, or compass and straightedge, students can construct these and investigate their relationships. If the triangle and special lines and segments are drawn in contrasting color the properties are easier to see.

PERPENDICULAR BISECTORS OF THE SIDES

1) Construction:

   a) By paper folding---Fold or draw a large triangle ABC on newspaper, tracing paper or waxed paper. Fold A to B, B to C and A to C and crease each fold.

   b) Compass and straightedge--Construct the perpendicular bisectors of the three segments AC, CB and AB. (See the basic compass and straightedge constructions in *Graphic Representation* in the section on TEACHING EMPHASES.)

THE MIDPOINT OF $\overline{CB}$.

THE PERPENDICULAR BISECTORS OF THE SIDES OF △ABC.
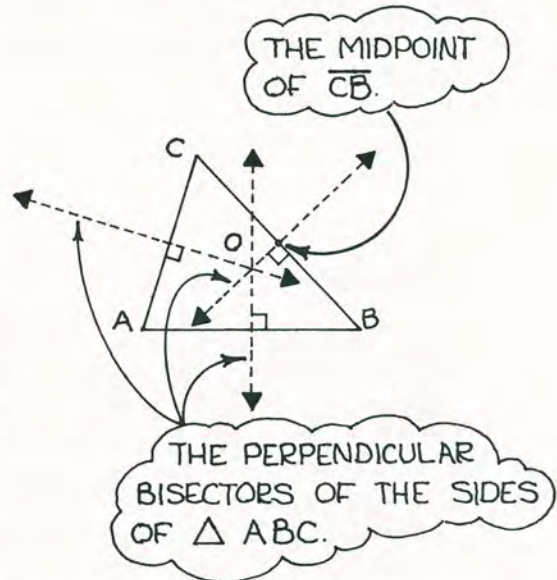
2) Investigations:

   a) The three perpendicular bisectors of the sides will be concurrent. This point is called the circumcenter. (Label it O.) By measuring $\overline{CO}$, $\overline{AO}$ and $\overline{BO}$ students can discover that the point O is equidistant from A, B and C.

   b) If students place the point of their compass at O and make the compass opening equal to CO, they will be able to draw a circle around △ABC that touches △ABC at points A, B and C.

   c) By drawing the perpendicular bisectors of the three sides of various types of triangles students can discover that the circumcenter of any <u>acute</u> triangle always lies inside the triangle; for any <u>right</u> triangle it lies on the longest side (hypotenuse) and for an <u>obtuse</u> triangle it lies outside the triangle.

THE CIRCLE CIRCUMSCRIBES △ABC.

POINT O IS THE CIRCUMCENTER OF △ABC.

447
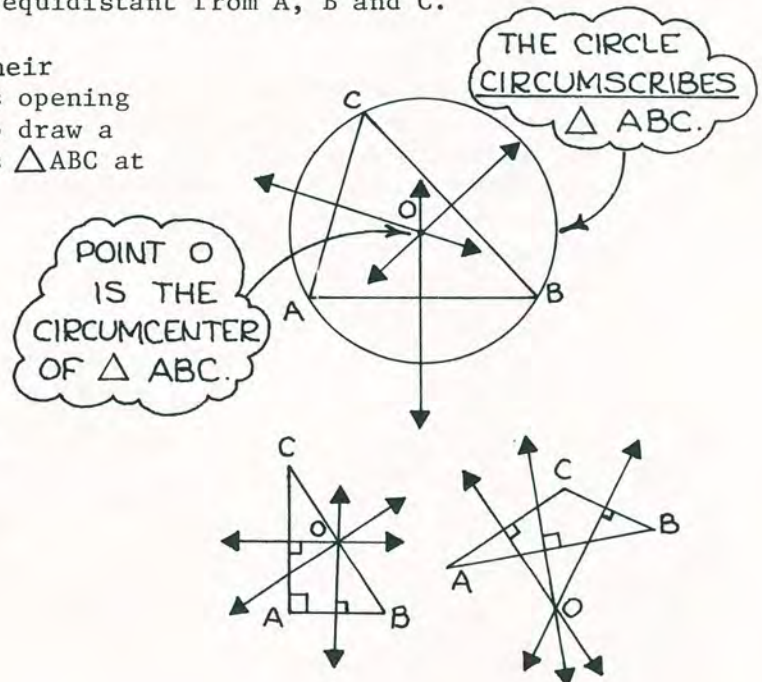
SPECIAL LINES AND SEGMENTS IN TRIANGLES

MEDIANS

1) Construction:

Locate the midpoint of each side of the triangle by measuring with a ruler or constructing the perpendicular bisector of each side. Fold a crease or draw a line segment from the midpoint of each side to the opposite vertex.

THE MEDIANS OF △ ABC.
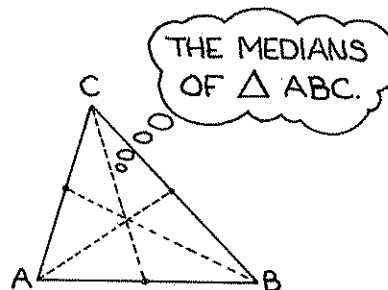
2) Investigations:

a) The three medians will always be concurrent. The point where the medians meet is the centroid. The centroid is the center of gravity or point of balance of the triangle. Students can observe this by cutting a triangular region from heavy paper and balancing it on a pencil whose tip is placed at the centroid. You might have students try to find the point of balance experimentally before discussing the medians and centroid of the triangle.

THE CENTROID

USE THE STICK POINT OF THE COMPASS TO PUNCH A HOLE SO THAT THE TRIANGULAR REGION WILL BALANCE ON THE PENCIL.
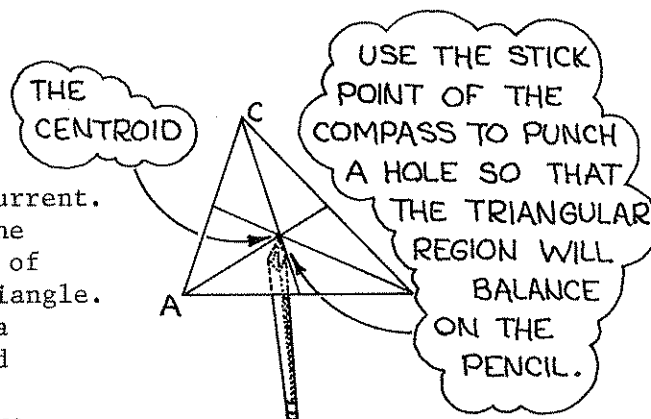
b) By drawing the medians for several triangles students will discover that the centroid always lies inside the triangle. For most triangles the medians and perpendicular bisectors of the sides of the triangle do not coincide. By experimentation students can discover that in an equilateral triangle they do. In every isosceles triangle one median and one perpendicular bisector also coincide.

PERPENDICULAR BISECTOR

*A MEDIAN - - -
A PERPENDICULAR BISECTOR.*

$$\overline{CA} \cong \overline{CB}$$

MEDIAN

448
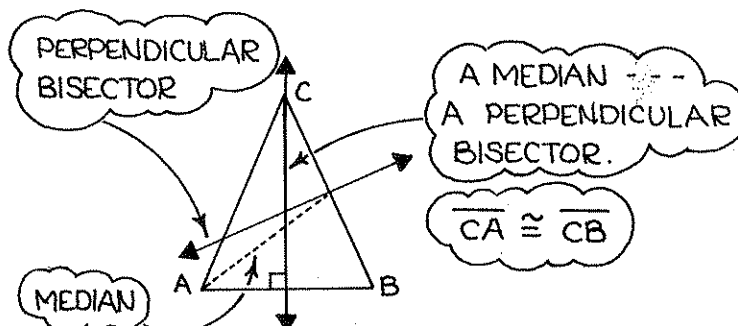
SPECIAL LINES AND SEGMENTS IN TRIANGLES

(PAGE 3)

ALTITUDES

1) Construction:

   a) By paper folding----To construct an altitude from C, fold AB on itself so that the fold line goes through C.

   Repeat the procedure for points A and B.

   b) Compass and straightedge----To construct an altitude from C, construct a perpendicular line segment from point C to the <u>line</u> AB. (See the basic compass and straightedge constructions in *Graphic Representation* in the section on TEACHING EMPHASES.)

   c) A T-square, piece of cardboard or protractor could be used.

2) Investigations:

   a) The lines containing the altitudes are concurrent. The point where these lines meet is the <u>orthocenter</u>. By drawing the altitudes of various types of triangles students can discover that the orthocenter of any <u>acute</u> triangle lies inside the triangle; for a <u>right</u> triangle it is the same point as the vertex of the right angle and for an <u>obtuse</u> triangle it lies outside the triangle. Notice that two altitudes in an <u>obtuse</u> triangle lie outside the triangle and that two sides of the triangle must be extended to construct these altitudes. This can be troublesome for students.

   b) By experimentation students can discover that for most triangles the altitudes are different than the medians. In an equilateral triangle they are the same and in every isosceles triangle one median and altitude are the same.

THE ALTITUDES OF △ ABC.

THE ORTHOCENTER OF △ ABC.

"ORTHOCENTER" IS RELATED TO "ORTHOGONAL" WHICH MEANS MUTUALLY PERPENDICULAR.

THE ALTITUDE FROM C

THE ORTHOCENTER

THE ALTITUDE FROM B.

THE ALTITUDE FROM B

THE ALTITUDE FROM A

THE LINES CONTAINING THE ALTITUDES ARE CONCURRENT.

A MEDIAN AND ALTITUDE

MEDIAN

$\overline{AC} \cong \overline{CB}$

ALTITUDE

449
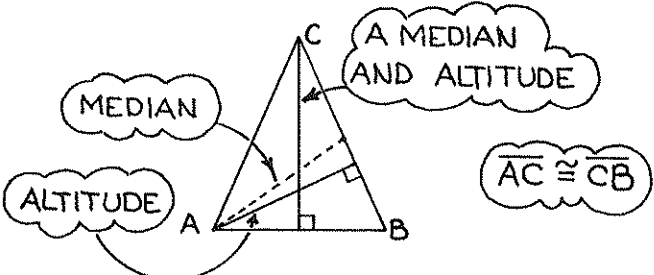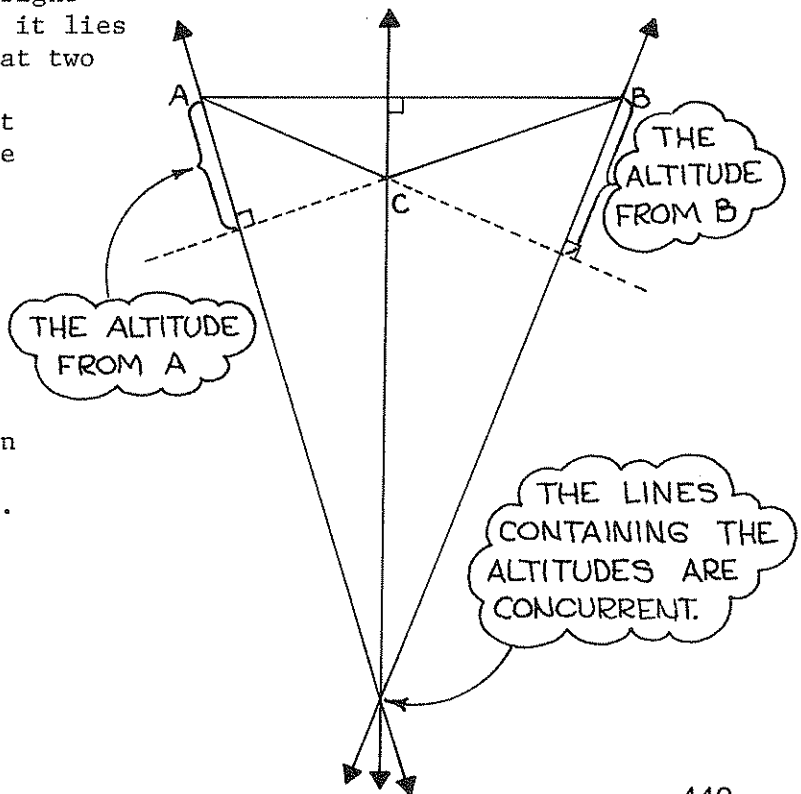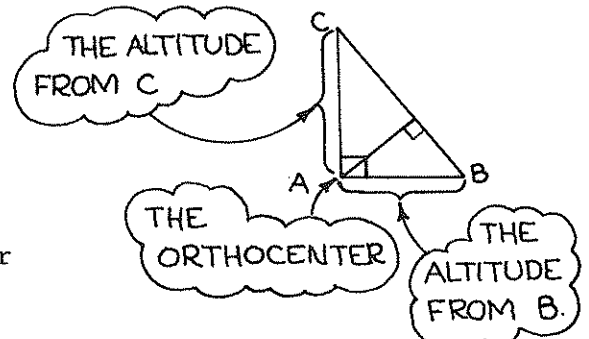
ANGLE BISECTORS

1) Construction:

   a) By paper folding----To construct the angle
      bisector of angle C, fold the paper so
      that segments AC and BC coincide and then
      crease. Repeat the procedure for A and B.

   b) Protractor and straightedge----Measure each
      angle and from the vertex draw the segment
      that divides each angle into two equal
      parts.

   c) Compass and straightedge----Construct the
      angle bisector of each angle. (See the
      basic compass and straightedge constructions
      in *Graphic Representation* in the section on
      TEACHING EMPHASES.)

THE ANGLE BISECTORS OF △ABC.
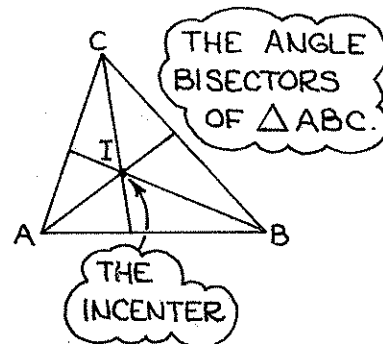
THE INCENTER

2) Investigations:

   a) The angle bisectors of any triangle are
      concurrent and their point of intersection
      is the <u>incenter</u>. The incenter (I) is always
      inside the triangle and is the same distance
      from all three sides of the triangle. By
      placing the stick point of their compass at
      I and making the compass opening equal to
      the distance from I to $\overline{AB}$, students will be
      able to draw a circle inside △ABC that
      touches each side of the triangle in only
      one point.

THE CIRCLE IS INSCRIBED IN △ ABC.

THE DISTANCE FROM I TO LINE AB IS THE LENGTH OF THE PERPENDICULAR SEGMENT FROM I TO $\overline{AB}$.

   b) By experimentation students can discover that for most triangles the altitudes,
      medians and angle bisectors are different segments. However, in an equilateral
      triangle they are the same. Thus an equilateral triangle has three lines of
      symmetry. Students could be challenged to find a triangle where only one
      median, angle bisector and altitude are the same. This occurs in an isosceles
      triangle that is not equilateral. This
      triangle has only one line of symmetry.
      Students can check by folding.
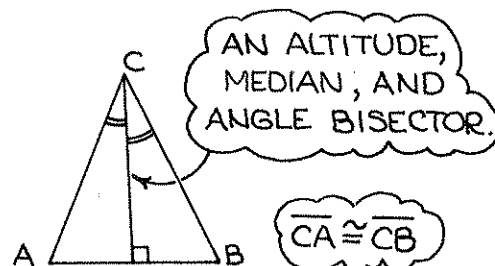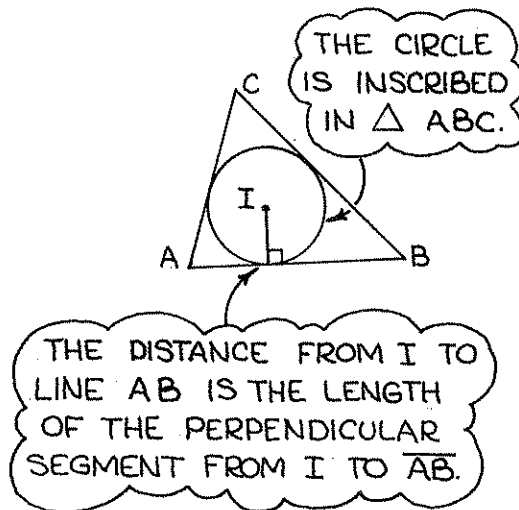
AN ALTITUDE, MEDIAN, AND ANGLE BISECTOR.

$\overline{CA} \cong \overline{CB}$

AN EQUI=MEETING

1) DRAW A TRIANGLE. LABEL IT AS SHOWN.

   Using dotted line segments to draw the triangles will help focus attention on the later line segments $\overline{ZB}$, $\overline{AY}$, $\overline{CX}$.

2) ON EACH SIDE CONSTRUCT AN EQUILATERAL TRIANGLE.

   LABEL THE NEW VERTICES X, Y, Z AS SHOWN IN THE FIGURE.

3) JOIN Z TO B, A TO Y AND X TO C.

4) WHAT DO YOU NOTICE ABOUT $\overline{ZB}$, $\overline{AY}$ AND $\overline{CX}$?

451

MORLEY'S DISCOVERY

Frank Morley (1860 - 1937) discovered something special about triangle. He showed it to his friends and news of it spread to the rest of the mathematical world like wild fire.

See if you can find Morley's discovery.

"$m\angle A$" MEANS THE MEASURE OF $\angle A$

1) In triangle ABC, $m\angle A = 60°$ $m\angle B = 75°$, and $m\angle C = 45°$.

2) Use your protractor and straightedge to draw the angle <u>trisectors</u> of each angle.

TO TRISECT AN ANGLE MEANS TO DIVIDE IT INTO THREE ANGLES OF THE SAME SIZE.

3) Since $\angle B$ measures 60°, if $\angle B$ is trisected each part measures _____°.
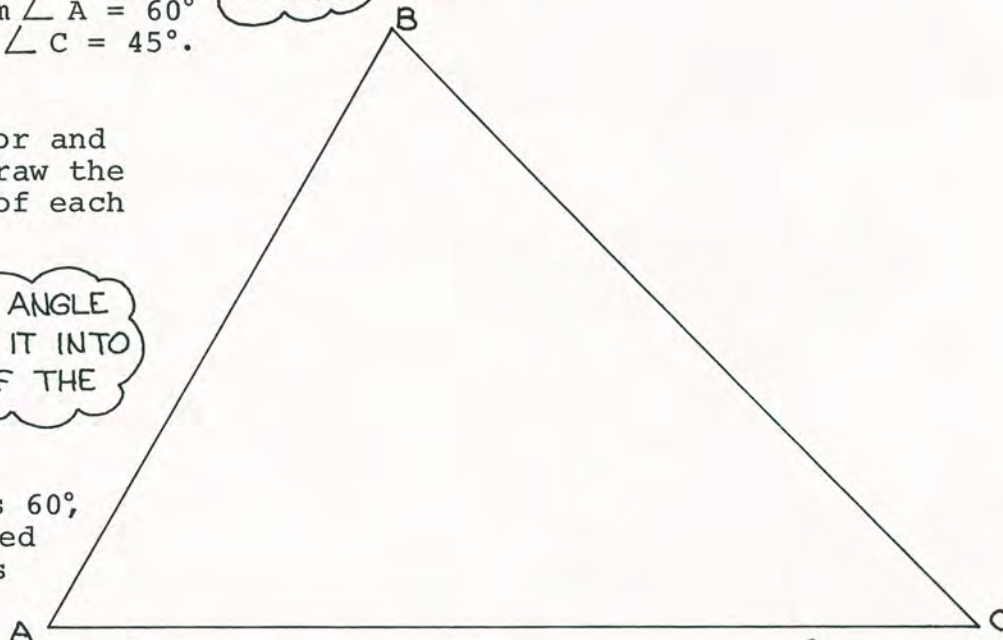
4) On the figures above use X, Y and Z to label the points where the outermost pairs of trisectors meet so that your figure looks like the one to the right.
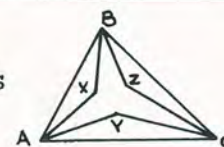
5) Draw $\triangle$ XYZ.

6) Measure the sides of $\triangle$ XYZ to the nearest millimetre and record.

    XY = _____   YZ = _____   XZ = _____

7) Draw a different $\triangle$ ABC where $m\angle A = 120°$, $m\angle B = 45°$, $m\angle C = 15°$. (Make the triangle large.) Repeat the experiment.

8) Draw your own $\triangle$ ABC. Repeat the experiment.

9) In each case what is special about $\triangle$ XYZ? _____

SIMSON SAYS

A

B

C
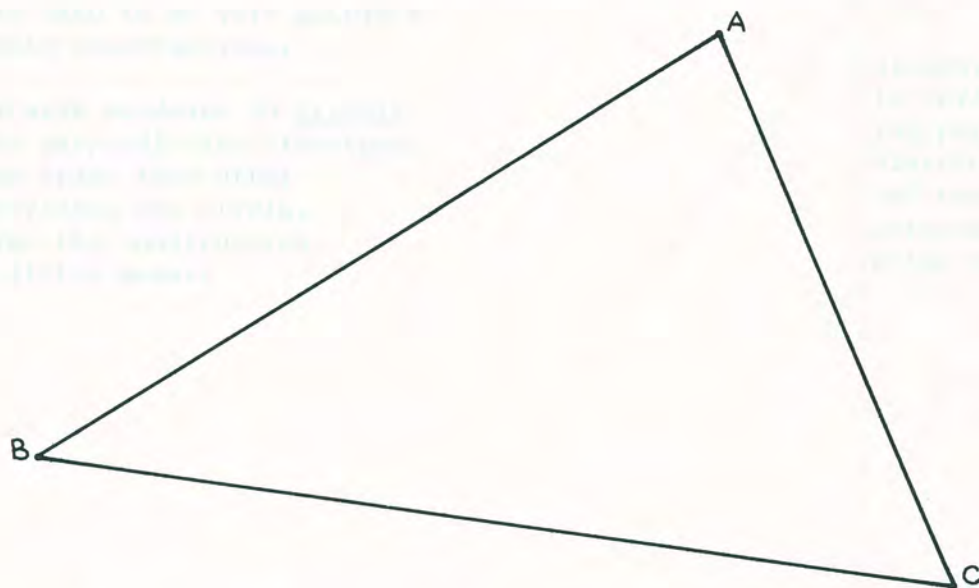
1) Construct the perpendicular bisectors of the three sides of △ABC. Label the point where the perpendicular bisectors meet 0. Place the point of your compass at 0 and make the opening of the compass equal to OC. Draw a circle around △ABC. Your circle should touch the points A, B and C.

2) Pick any point on the circle (except A, B and C) and label it P. From P construct a perpendicular line to AC and label the intersection R. From P construct a line perpendicular to BC and label the intersection S. From P construct a line to AB and label the intersection T.

3) What do you notice about the points R, S and T?

4) Try the experiment on a large obtuse triangle.

5) Do you think this will happen for every triangle?

453

Showing the circumcenter,
    centroid and orthocenter
    are collinear
Polygons
POLYGONS & POLYHEDRA

EULER FOUND....

For information on these constructions see
Special Lines and Segments in Triangles in
the student pages.

TEACHER DIRECTED ACTIVITY

Supply each student with two copies of this page.

1) On one page, have the students construct
   the perpendicular bisectors of the three
   sides of △ABC.
   Label the point where they meet U
   (circumcenter).
   (Students can compare with their neighbor
   to check their drawing. Faster students
   could assist students who need help.)

You may wish to
work through this
activity with the
students on the
overhead or
blackboard as the
students follow.

2) On the same page have them draw the
   three medians of △ABC. Label the
   point where they meet G (centroid).

3) On the second page have the
   students construct the altitudes
   of △ABC.

   Note: The sides AB and CB will
         have to be extended.

   Have them extend the altitudes
   until they meet. Label this
   point H (orthocenter).

Another activity
(Euler's Nine-
Point Circle)
involving these
constructions
can be found in
the November,
1972 issue of
The Mathematics
Teacher. The
article is called
Activities:
Mission-
Construction and
was written by
Charles Allen.

4) By tracing have them copy the
   points U and G onto the second
   page.
   Ask the students if they notice
   anything special about the
   points U, G, H.
   Have them measure segments UG
   and GH and compare them.

   U, G, H are collinear
   and UG:GH = 1:2. The
   line determined by U,
   G and H is called the
   "Euler" line of △ABC.

5) Encourage students to investigate
   this for different triangles.

   The result is true for all
   triangles except the
   equilateral one where U, G
   and H are the same point.

NAPOLEON'S IDEA

> GENERAL, YOU'LL GO DOWN IN HISTORY FOR THIS!

> Mathematics was one of the hobbies of the French general and emperor Napoleon. He was especially interested in geometry.

1) On your paper draw a triangle.
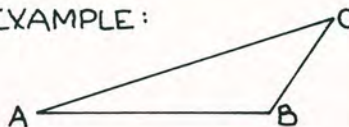
EXAMPLE:

2) On each side construct an equilateral triangle.

EXAMPLE:
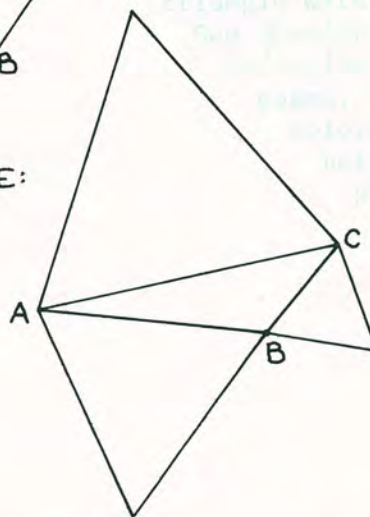
3) Find the centroid of each equilateral triangle. (Construct two medians for each triangle.)

Label the centroids R, S, T as shown in the figure.

EXAMPLE:
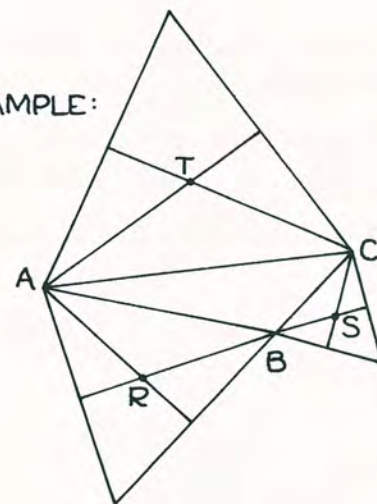
4) Join points R, S, T with line segments to form △RST.
What kind of a triangle is △RST?

5) Napoleon thought this result would occur no matter what triangle you start with. Try the experiment on another triangle to see what you think.

I DON'T BELIEVE IT!

1) In each quadrilateral below measure with a ruler to find the midpoint of each side.

2) Draw line segments to join the midpoints of adjacent sides.

ADJACENT SIDES
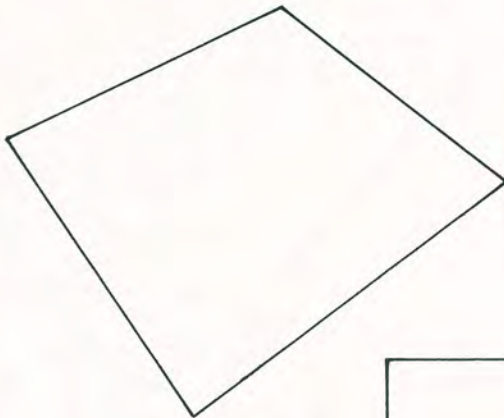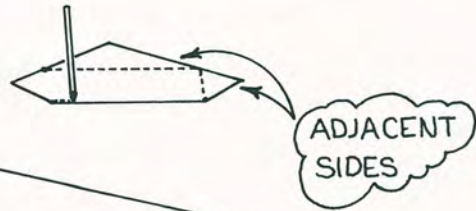
After this activity you may want to have your students investigate which quadrilaterals will produce a square, rectangle or rhombus.
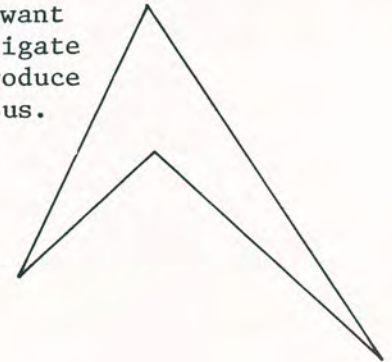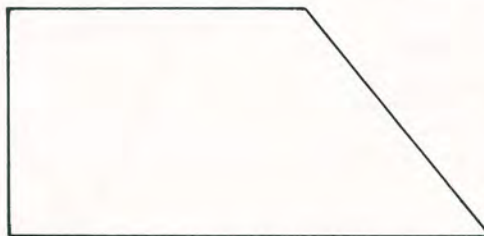
3) What do you notice about the figures you just drew? _____
_____

4) Do you think this discovery is true no matter what quadrilateral you start with? _____ Draw your own quadrilateral and see if it is true.

5) In each quadrilateral below repeat steps #1 and #2.
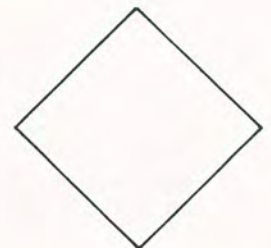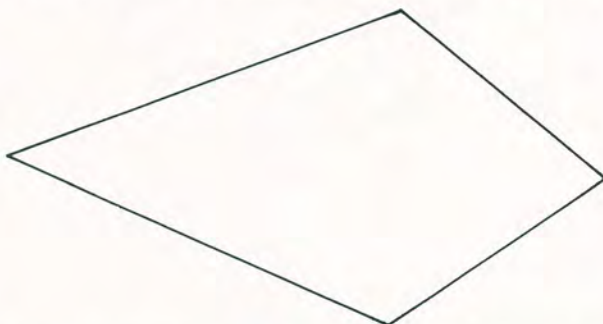
6) a) Cut out the quadrilaterals in #5 and fold each along the line segments you just drew. What do you notice? _____

b) Will this always work? Try a quadrilateral from #2.

c) See if you can draw another quadrilateral which will fold in on itself.

456

MEDIAN OF A TRAPEZOID

I

TRAPEZOID

BASES OF THE TRAPEZOID
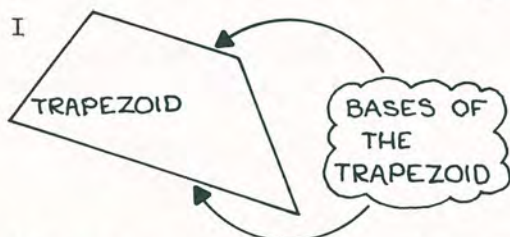
REMEMBER: A TRAPEZOID IS A QUADRILATERAL WITH ONLY ONE PAIR OF OPPOSITE SIDES PARALLEL.

A MEDIAN JOINS THE MIDPOINTS OF THE TWO SIDES WHICH ARE NOT BASES.
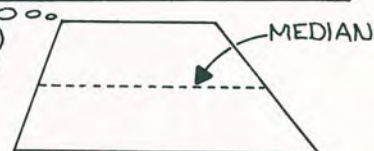
MEDIAN

1) Measure to find the midpoints and draw the median of each trapezoid. $\overline{AB}$ and $\overline{CD}$ are the bases of each trapezoid.

I

II

III

IV

V
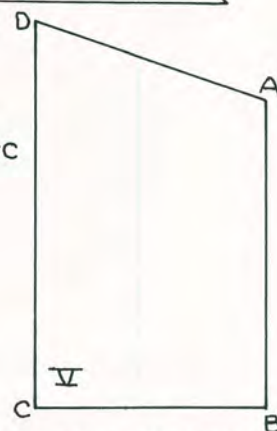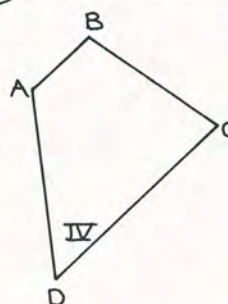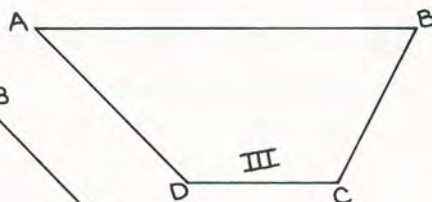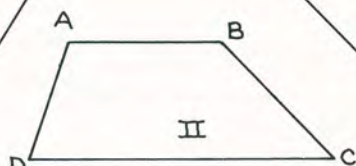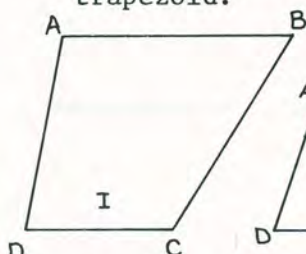
2) Complete the chart. Measure each segment to the nearest half centimetre.

TRAPEZOID	AB	CD	AB+CD	LENGTH OF MEDIAN
I				
II				
III				
IV				
V				

Compare the last two columns in the chart. What do you notice?
_____
_____
_____
_____
_____

Use your observation to predict the length of the median of this trapezoid.

20 UNITS

24 UNITS
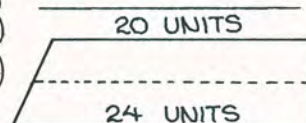
II Extension: Making a Nomograph

Numbers have been marked off on parallel rays $\overrightarrow{AB}$ and $\overrightarrow{DC}$. The distance between each division is .5 centimetres.

1) Find the midpoint of $\overline{AD}$. Label it E.

2) Find the midpoint of $\overline{BC}$. Label it F.

3) Draw $\overrightarrow{EF}$.

4) Starting from E mark divisions every .5 centimetres and number the divisions 2, 4, 6, 8, ... 22.

5) Place your straightedge on the figure so that one edge crosses 1 on $\overrightarrow{AB}$ and 3 on $\overrightarrow{DC}$. Where does it cross $\overrightarrow{EF}$? _____

6) Repeat, but join 6 on $\overrightarrow{AB}$ and 8 on $\overrightarrow{DC}$. Where does the edge cross $\overrightarrow{EF}$? _____

7) If 4 on $\overrightarrow{AB}$ is joined with 5 on $\overrightarrow{DC}$ predict where the edge will cross $\overrightarrow{EF}$. _____

DIAGONALS IN POLYGONS

In a polygon a segment joining any two nonadjacent vertices is called a diagonal.
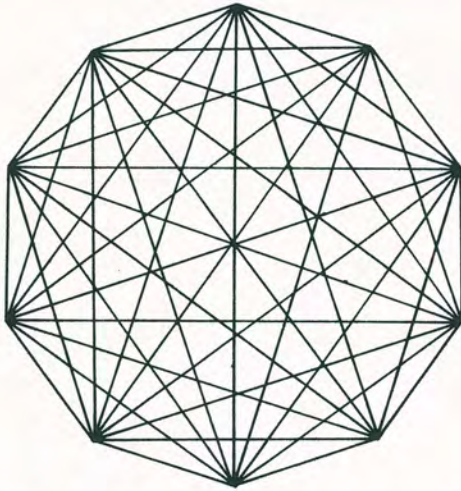
How many diagonals are in this polygon?

More than 10?

More than 25?

More than 100?

Make an estimate. _____

Look at some simpler polygons first.
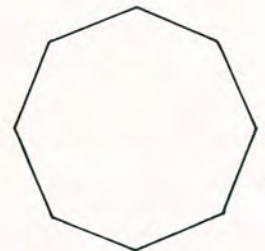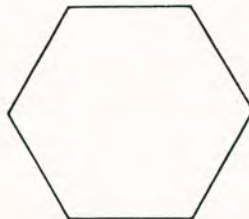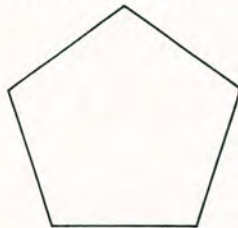
1) Draw all the diagonals for each polygon; then fill in the table.

Number of Sides	Number of Diagonals From Each Vertex	Column 1 X Column 2	Actual Number of Diagonals
4	1	4	2
5	2	10	5
6	3	18	9
7	4	28	14
8	5	40	20

2) Compare the numbers in Column 3 and Column 4. What do you notice? _____

3) Fill in the chart for the 10-sided polygon at the top.

10	7	70	

4) How many diagonals does it have? _____35_____

5) How many diagonals would a 100-sided polygon have? _____4850_____

6) How many sides does a polygon have if it has 170 diagonals? _____20_____

a) There are n-3 nonadjacent vertices.
b) A diagonal must be drawn from each of n vertices.
c) Each diagonal ends on two vertices.

Number of diagonals = $\frac{n(n-3)}{2}$

TYPE: Worksheet/Activity Card

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INTERIOR ANGLES OF A POLYGON 1

What is the sum of the angle measures of each polygon?

A   B   C

THERE'S A HINT BELOW!

D   E   F   G
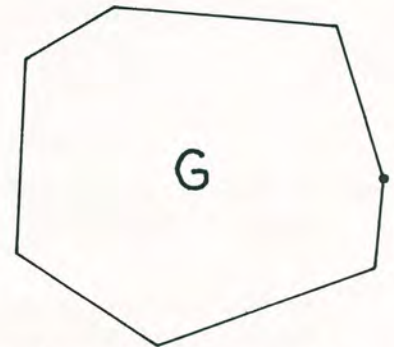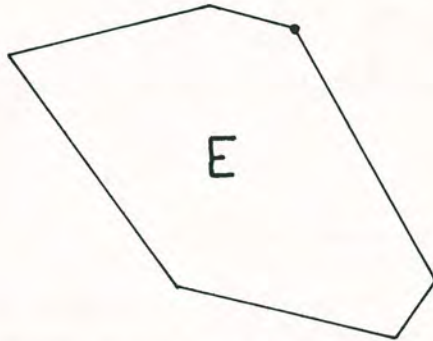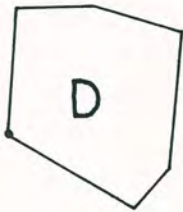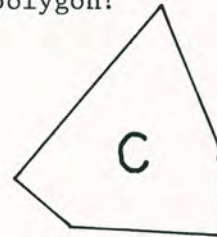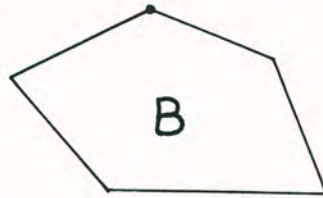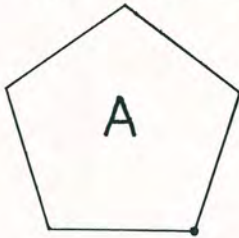
You could use a protractor and measure each angle. That would be easy but would give lots of chances for mistakes. Instead, use a fact you already know.

The sum of the measures of the angles of a triangle is _____.

Hint:

1) For each polygon above draw diagonals from the dotted vertex that will divide the polygon into triangles. Then record the information in the table.

A

THIS IS NOT ALLOWED

Polygon	Number of Sides	Number of Triangles	Sum of Measures
A	5	3	3 x 180° = 540°
B	5	3	540°
C	5	3	540°
D	6	4	720°
E	6	4	720°
F	6	4	720°
G	7	5	900°

2) Predict the sum for an 8-sided polygon. Draw an 8-sided polygon to check your prediction. 1080°

3) Predict the sum for a 12-sided polygon. 1800°

4) Predict the sum for an n-sided polygon.

The sum of the interior angle measures of any convex polygon is 180 (n-2)° where n is the number of sides in the polygon.

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EXTERIOR ANGLES
OF A POLYGON

1) The pencil turning trick can show the sum of the measures of the exterior angles of a polygon is always 360°. Students need to know that rotating the pencil until it is pointing in the same direction involves a 360° turn. Larger polygons drawn on the board or posters can be used for a demonstration.
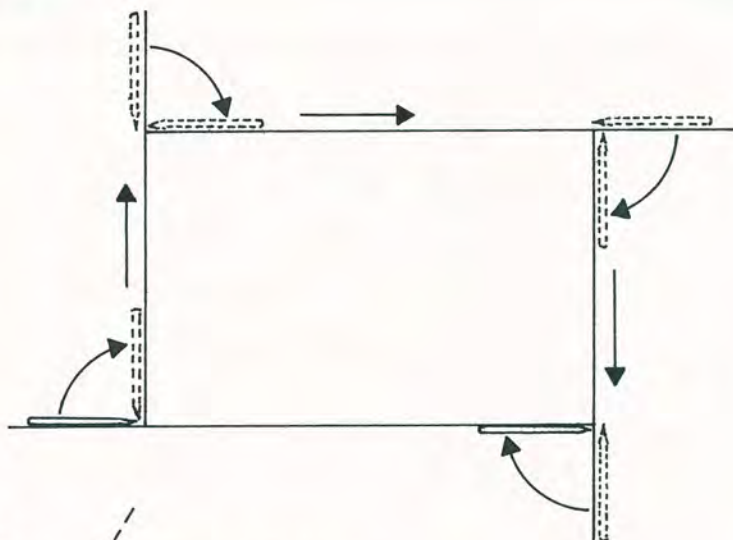
2) A series of similar polygons could be used to illustrate the above concept.

As the similar polygons become smaller and smaller the exterior angles remain the same. The limiting case when the polygon "collapses" to a point produces the angles about a point or 360°.
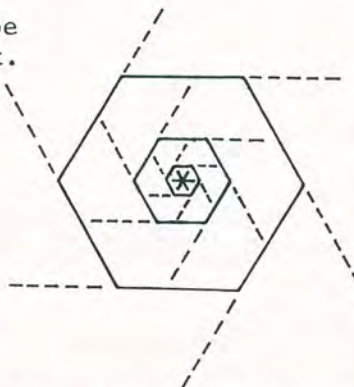
A series of transparent overlays would illustrate this.

3) Students could use a protractor and find the measure of each angle. The sum should be about 360° in each case.
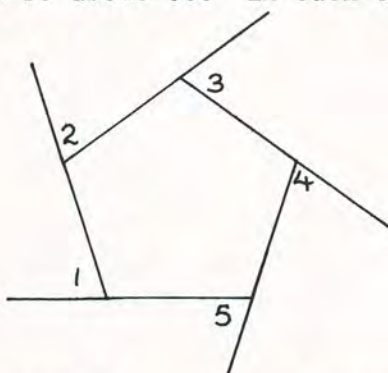
4) Angles 1 - 5 are exterior angles of the polygon. The sum of these angles will be 360°. For this polygon the five exterior angles along with the five adjacent interior angles form five straight angles or 5 X 180° or 900°. So (sum of exterior) + (sum of interior) = 900°. From the previous page (sum of interior) = 3 X 180° or 540°. Thus (sum of exterior) = 900° - 540° or 360°.

In general for any convex polygon with n sides

$$(\text{sum of exterior}) = (\text{sum of straight angles}) - (\text{sum of interior})$$
$$= 180n - 180(n-2)$$
$$= 180n - 180n + 180(2)$$
$$= 180\,(2)$$
$$= 360°$$

You may wish to develop a discovery lesson on exterior angles using the above information

INTERIOR ANGLES OF A POLYGON 2

1) Draw a triangle, a quadrilateral, a pentagon, a hexagon and a heptagon. They may be any size and any shape you want but should be convex polygons.

2) Mark a point E in the interior of each polygon. Connect point E to each of the vertices of the polygon.

3) Fill in this chart with information from the polygons.

	Number of Triangles Formed	Number of Degrees in Triangles Formed	Total Number of Degrees in all the Angles About Point E	Number of Degrees for Angles of the Polygon
Triangle	3		360	
Quadrilateral				
Pentagon				
Hexagon				
Heptagon				

4) Predict the results for a decagon (10-sided polygon). _____

5) Explain in words how you could find the total number of degrees for the angles of any polygon.

6) How do these results compare with the results from *Interior Angles of a Polygon 1*?

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GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

CURVES AND CURVED SURFACES

It was less than five hundred years ago when many people believed the earth was flat and the earth was the center of the universe. Now we know the earth is nearly spherical and the planets travel in elliptical orbits around the sun. Knowledge about curves and curved surfaces has been applied to such diverse areas as astronomy, design of automobiles, the space program, national defense and architecture. Your students might like to start a bulletin board of examples of curves and curved surfaces. Some examples are given above. More ideas are given in *On Being the Round Shape* in the Curved Surfaces subsection, *Concentric Circles* in the Circles subsection and *Hanging Together* in the Other Curves subsection.
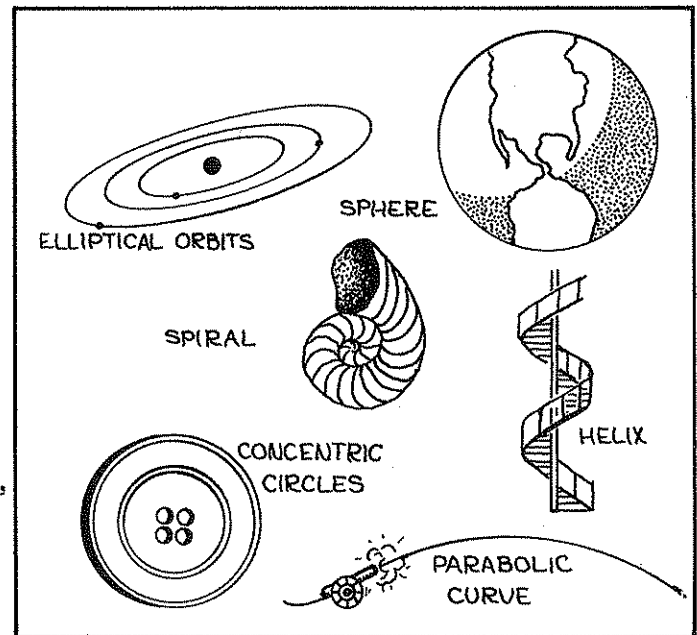
SKILLS AND CONCEPTS THROUGH GEOMETRIC DESIGN

Students usually enjoy making and coloring designs, so why not combine practice in geometric skills with making geometric designs?

● An easy way to begin is with the attractive circle design shown at the right. Students pick a point and draw a sequence of the same sized circles, all of which pass through the chosen point. Students may use a template to draw the circles and shade their design with felt-tip pens. These designs provide practice in motor skills and visual perception. Perhaps they can determine how many circles to draw to avoid having two white crescents side by side.
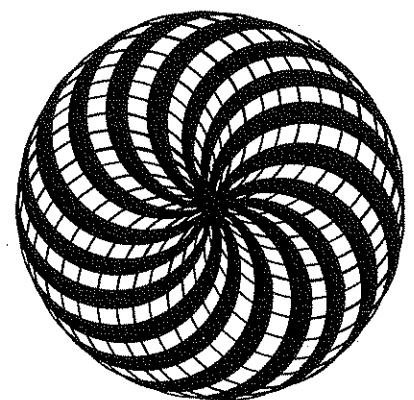
● Designs like those shown below can provide motivation for dividing a circle into 12, 18, 36, ... congruent arcs. Students can use a protractor, or in some cases compasses, to find the congruent arcs. The pages *Inside the Circle I, II* and *III* give methods of inscribing regular polygons and the page *Art Inside the Circle* gives more ideas and sources for circle designs.
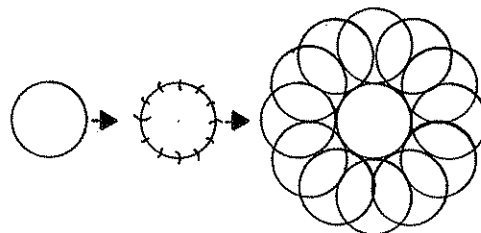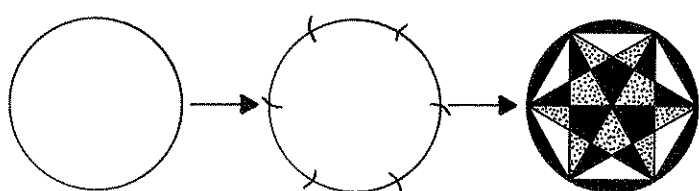
● A circle design like that at the right can form the basis of many tessellations. Students can use compasses to cover a sheet of paper with circles in this pattern. They can join certain points from the pattern to make a tessellation of rectangles, hexagons and several kinds of triangles.
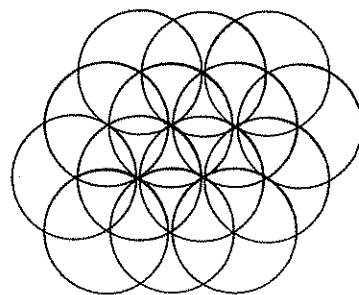
● Circle designs can be extended to an investigation involving multiples of whole numbers. Have students divide a large circle into twelve congruent arcs and label it as shown at the right. What happens when they join all the points representing multiples of two? Multiples of five? Why do such different patterns occur? Can they find a pattern that does not come back to zero? Can they predict the patterns for multiples of 3, 4 and 7? What if each circle were divided into 18 congruent arcs? 36? 40? Students might come up with statements like, "In a twelve circle, multiples of eleven give the same design as multiples of 1, multiples of ten give the same design as multiples of 2 and so on." These designs will make a colorful display if they are colored with felt pens.

MULTIPLES
OF 2

MULTIPLES
OF 5
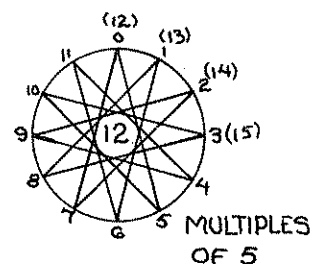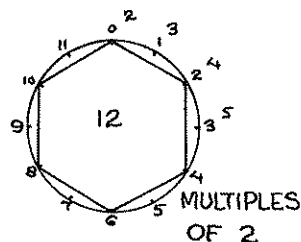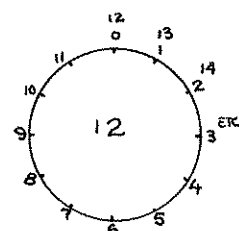
●   An ellipse can also be the basis for a striking design. First students will need to know how to draw an ellipse. Here is an excerpt from the page *Methods for Drawing Ellipses* from the subsection Other Curves.

> Make a loop on both ends of a piece of string. Secure both loops with thumbtacks. With a pencil pull the string tight and carefully move the pencil around, always keeping the string tight. On the chalkboard rubber darts can be used.
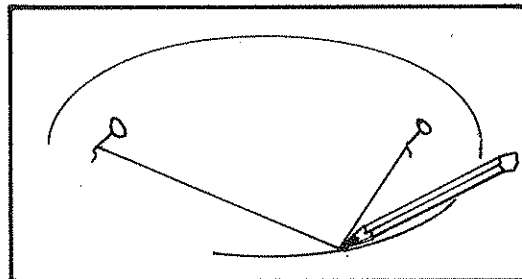
After an ellipse is drawn on a sheet of plain paper, have students choose one of the focal points (tack holes) and use it as a center of a circle which they can draw with their compasses. (See below.) After the circle is divided into 18, 36 or 72 congruent arcs, have them draw lines from the center of the circle through each division point of the circle to intersect the ellipse. The paper pattern can now be clipped to a piece of colored railroad board and a tack used to poke a hole in the cardboard at each intersection point on the ellipse.
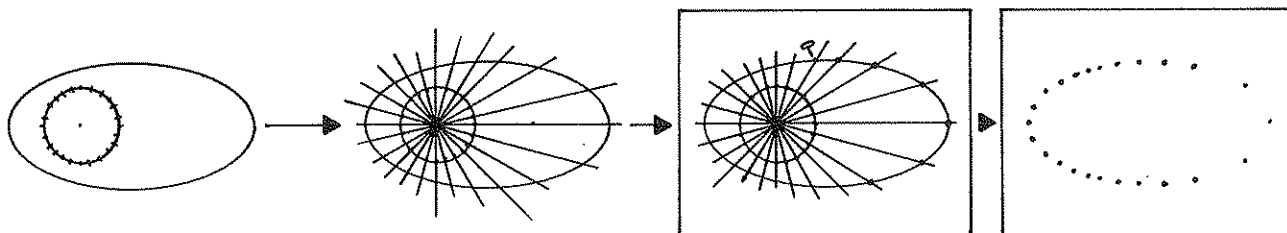
Removing the paper pattern will leave a pattern of holes. Contrasting colored string can be used to make a line design on the railroad board. See *You've Got me in Stitches* in the Lines subsection for basic hints on making line designs. The design will have an elliptical shape on the outside and a circular hole on the inside. Several layers of thread can be used to give a different effect. A variation of this design is to place the circle which is to be divided in the "center" of the ellipse. Another variation is given in *Methods for Drawing Ellipses* in the subsection Other Curves.

    There are many other suggestions for using designs in combination with teaching skills and concepts in the classroom materials. See especially *Polygonal Spirals, More Spiral-Like Designs* and *This Won't Give You Cardioid Arrest* in the subsection Other Curves, and *Circle Art* in the Circles subsection.

SOME CURVED PARADOXES

Have you ever visited a fun house where water seemed to run uphill and the basic laws of gravity appeared to be defied? Here is a demonstration that seems to defy the laws of gravity.

- a. Take two plastic or metal funnels and glue or tape them together.

- b. Cut two strips of cardboard like this:

- c. Join the strips and add a brace to make a form like this:
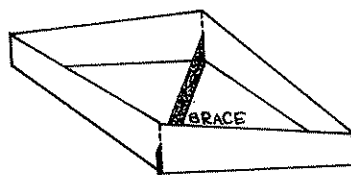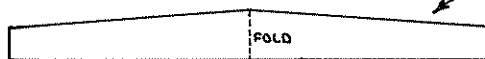
BE SURE THE LENGTH OF THE BRACE IS LESS THAN THE DISTANCE BETWEEN THE NECKS OF THE FUNNELS.

Place the joined funnels on the cardboard strips, halfway to one end. Before you let it go, ask students which way they think it will roll. Surprisingly, it will roll to the center of the strip which looks uphill! Actually, the center of gravity of the funnel is moving downhill because of the shape of the funnel and the widening track. Students might want to try placing a cylinder on the cardboard to see that some shapes will roll as they expected.
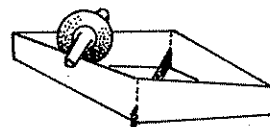
- Here is a simple paradox that will remind students they live on a sphere. A family liked having windows facing the south, so they build a house with windows on all four sides, each window having a southern exposure. How (or where) on earth can this be? A related paradox is the following. A boy on his first bear hunt spotted a bear 100 metres due _west_ of him. Panicking, he ran 100 metres due _north_ before he stopped, pointed the gun due _south_ and shot the bear which had not moved. Based on the information above, what color was the bear? (Answers--At the North Pole and White!)

Your students might enjoy exploring the seemingly paradoxical properties of a Moebius strip as suggested in _Two Faced? Never!_ in the subsection Curved Surfaces. They might also think it is impossible to design a noncircular shape on which a flat surface can roll smoothly. The pages _Rollers with Corners_ in the subsection Other Curves discuss various kinds of curves of constant width which can be used to design rollers. A cardboard model of such a roller could convince students that a smooth ride is possible on these shapes, but that circular wheels are still the best shapes to be mounted on axels.

A NEW WHEEL!

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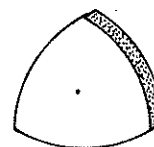
CURVES AND CURVED SURFACES AS PATH OF POINTS

Many curves and curved surfaces can be described as the path of a point satisfying certain conditions. One way to introduce this idea to students is to have them imagine a fly attached to a piece of string. See if they can determine the path of the fly under these conditions:

a.   The other end of the string is fastened to a flat surface and the fly walks on the flat surface, keeping the string taut.

b.   The end of the string is fastened to a flat surface and the fly walks on the surface or flies in the air, keeping the string taut.

c.   The end of the string is held at a fixed position in space.

d.   The end of the string could slide along a square and the fly walks on the flat surface where the square is drawn. The fly stays as far from the square as he can.

e.   The end of the string could slide along the surface of a cube. The fly stays in the air as far from the cube as he can.
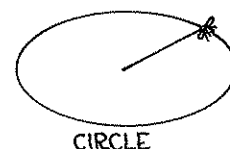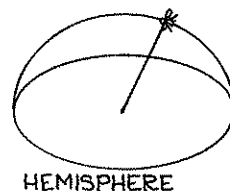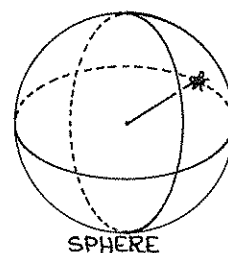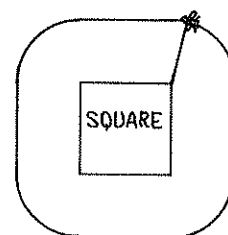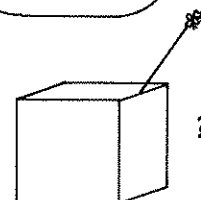
Could your students give the path of a dog whose leash is attached to a 10-metre clothesline? Or the path of a dog whose collar has a rope strung through it with the rope fastened at either end to a stake? The two paths are given below.

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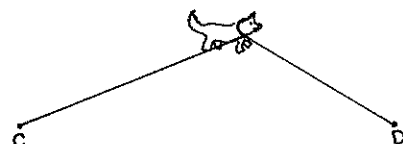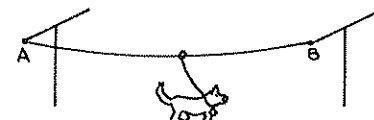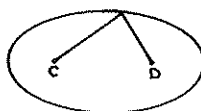
PROBLEM SOLVING

Some very challenging problems are presented in this section. The problem presented in *A Penny for Your Thoughts* in the Circles subsection seems impossible to solve until the strategy simplify the problem is suggested. It is a bit long to reproduce here, but do read and use this activity. Students will enjoy it and you can also use it as a puzzler for other teachers in the faculty lounge.

Another fun question is "What flat shape could be rolled up to make an oblique glass like that shown to the right?" First, a person might try to imagine or guess the shape. Students can be encouraged to check their guesses by sketching the guessed shape, cutting it out and rolling it up. Perhaps this guess and check method might lead them to a more refined guess. Others might abandon trying to visualize the solution in favor of more direct ways of finding a solution. Your students might suggest several ways of finding the shape. Some of these ways are given on *Let's Not Get a Head* in the Curved Surfaces subsection. This is a good activity in which to stress that there are many ways to solve a problem.
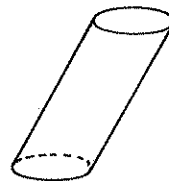
Working backwards is a good strategy to try for the first problem presented in *Puzzling Pennies* in the Circles subsection. The problem is shown below. Starting with the desired result and working backwards to the original arrangement could help some students solve the problem. Making a diagram for each move can help a problem solver keep track of successful moves.
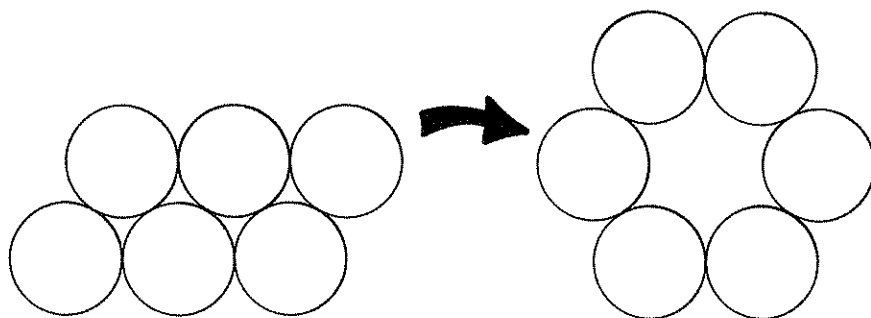
On each move a single coin must be slid to a new position so as to touch two other coins that rigidly determine its new position.

a) Start with six pennies closely
    packed like a parallelogram.
b) In three moves* form a circular pattern so that if a seventh coin were
    placed in the center, the six pennies would be closely packed around it.

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Having students make up their own problems is also a good way to stress problem solving. The page *You Pop the Question* from the Circles subsection has diagrams like that shown to the right. Students are asked to write a question for each diagram that can be solved with the information given on the drawing. Students will have to think through relationships in the diagram before they can make up reasonable problems. An interesting twist to add to this activity would be to have students write a question about the diagram which cannot be answered with the given information. Can other students detect which questions are unanswerable?
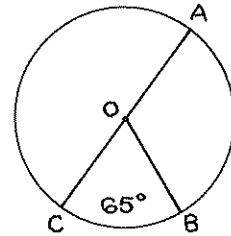
The problems presented in this section involve number patterns, symmetry, visual perception and graphic representation. There are problems which students can solve in different ways and at their own level. Whatever problem-solving activities you choose, try to help students become aware of the strategies they can use to find solutions. Take advantage of situations like those depicted below and perhaps your students will become better problem solvers.

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

CONTENTS

CURVES & CURVED SURFACES: CURVED SURFACES

CURVED SURFACES

Your students have been exposed to many objects with curved surfaces--balls, cones, wheels, ..., but they might not have examined carefully the properties of these curved shapes. An easy way to begin investigating curved solids is to examine how they roll. Does a hard-boiled egg roll in the same path as a spherical ball? Can you predict how a cone will roll? One interesting shape to roll is that of a somewhat flattened solid ball--a round, fat, smooth stone will do. Graphics of the solid and some of its possible paths are given below.

SIDE VIEW :

TOP VIEW :

POSSIBLE PATHS :
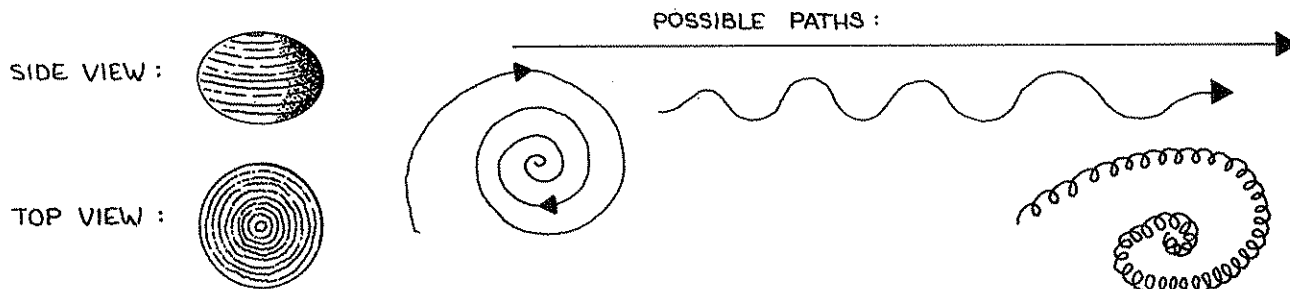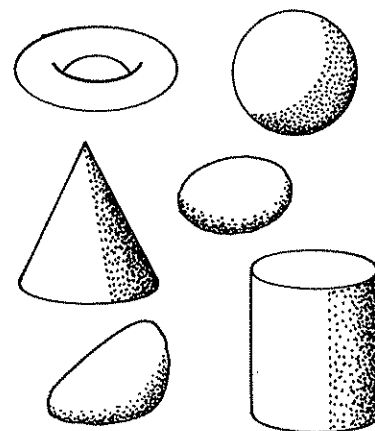
The page *Rollin' Along* gives many more suggestions for having students investigate objects with curved surfaces. You might have students make some curved objects. Curved shapes could also be collected (door knobs, doughnut-shaped toys, balls, cans, cone-shaped water cups), or you could order one of the wooden or plastic sets of solid models which are available from commercial sources.

CROSS SECTIONS OF CURVED SOLIDS

The shadow of a sphere can be a circle or an ellipse, but a sphere viewed from any direction has the outline of a circle. The sun and moon appear as circular regions in the sky. Perhaps this circular view of spheres is why it is difficult for some people to learn the difference in meaning of the words circle and sphere. Another somewhat surprising property of spheres is that all of their cross sections are circles. You might challenge your students to find a flat cross section of a solid sphere that is not a circular region. They could experiment by cutting oranges, old tennis balls, styrofoam balls or balls made from Play-Doh. More ideas on cutting spheres are given in *Is It Plane to You?*
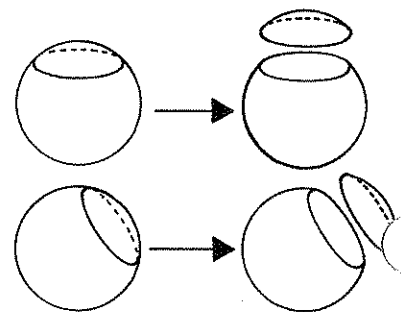
The pages *Plane Intersections* suggest ways to introduce the cross sections of cones and cylinders. Students can cut models and describe the resulting shape, or they might try to predict what a cross section will be and then check their prediction by cutting a model.

These pages on cross sections of curved shapes would go very well with the ideas in *Cross Sections* in the Polyhedra subsection.

INTRODUCING SURFACES OF REVOLUTION
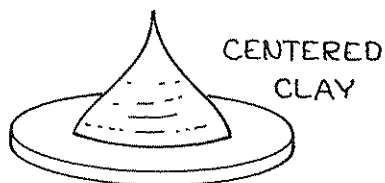
CENTERED CLAY

Have you ever tried to center a piece of clay on a potter's wheel? If you can hold your hands steady while the wheel is turning, you are more apt to be successful at centering clay. When it is centered the mound of clay will look the same on all sides. Its surface represents a surface of revolution. A similar idea is used to form many chair legs, small posts and knobs. Pieces of wood are placed in a lathe. They are revolved until they also have surfaces of revolution.

BLOCK OF WOOD   LATHE PIECE   A TURNED POST
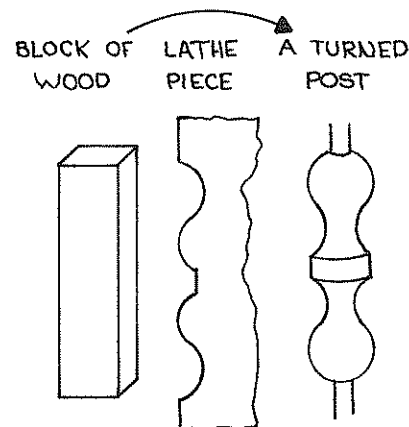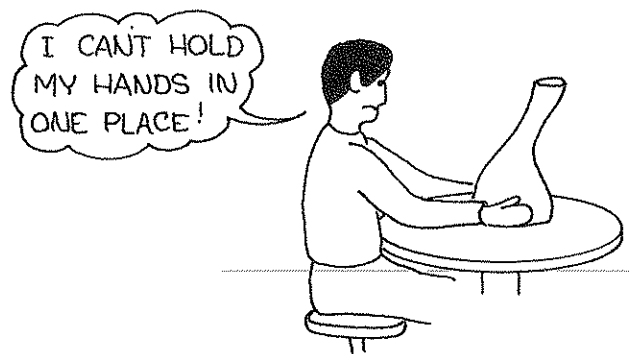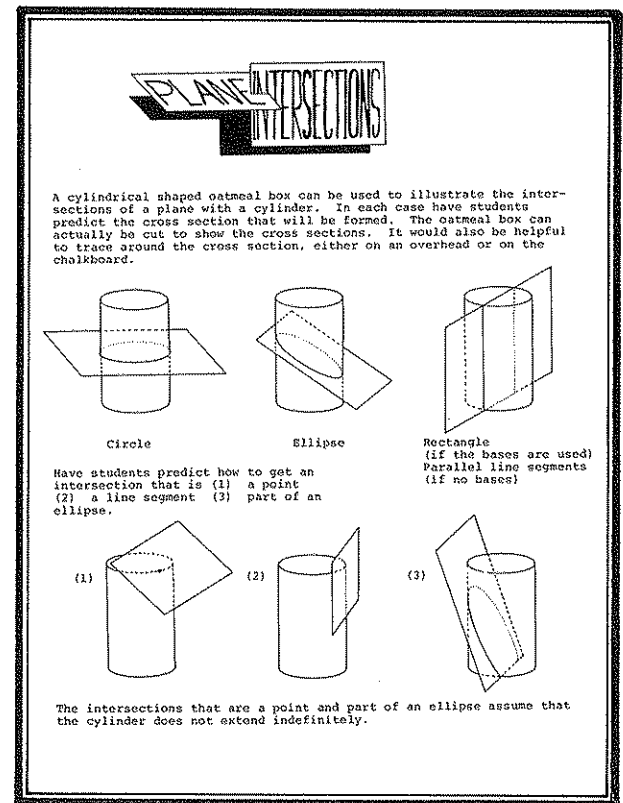
Surfaces of revolution can be demonstrated by spinning flat shapes attached to toothpicks.   Students will have to use their visual perception to identify the surface of revolution. You might also have students visualize the surface that would be formed by revolving a line segment or semicircle around a line.   Ideas are given on the page *A Revolving Situation.*

RESULTS:
A SPHERE!
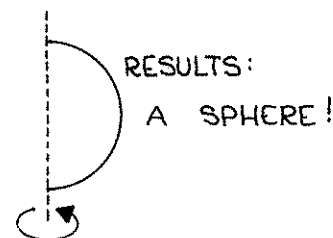
MAKING DRAWINGS OF CURVED SURFACES
Changing a two-dimensional graphic so it looks three-dimensional can be fun. You might want to show your students some ways to do this.

Circle to a Sphere: OR   OR

Triangle to a Cone: OR

Flat Shape into a
    Surface of Revolution: OR

Circle→Short Cylinder
    ⟶ Rectangle:
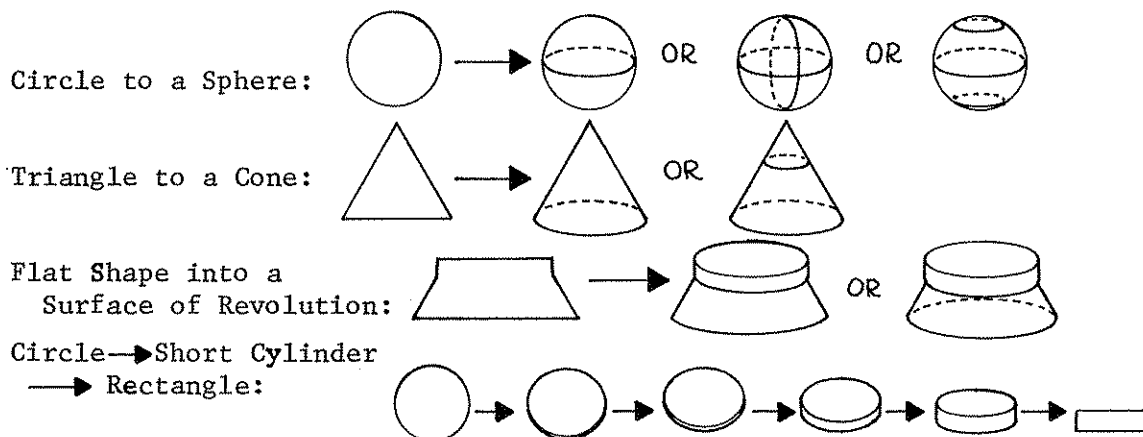
Since cones and cylinders are common curved shapes in geometry, the page *Drawing Ellipses, Cylinders and Cones* gives specific practice in making the graphic representations of them.   The *Graphic Representation* teaching emphasis gives more background information about drawing.

DIFFERENTIATING SURFACES FROM SOLID SHAPES
Although children play with inflated balls, balloons and hollow plastic toys, it might be difficult for them to separate the idea of a surface from the idea of a concrete, solid shape. If you find that your students have difficulty thinking about the surface of a curved shape, you might try some of these activities.

● Cut the peel of an orange into wedges which can be folded away from the meat of the orange. (This might remind students of some planar maps of the earth.) Explain that the orange is like a solid sphere and a spherical surface is somewhat like a peel of an orange, only a surface has no thickness.

● Bring hard-boiled eggs and have students peel some of them in large pieces. The egg shell represents the surface of the egg. Have them look at the inside of the shell. Does it look different on the inside than the outside?

● Bring a few bottles of bubble solution and have students blow some bubbles. The soap film represents a spherical surface. This is a good model since the soap film is very thin and approximates the "lack of mass" of a surface. Your students might try blowing bubbles with dip sticks having triangles or squares on them. Does anything unexpected happen?
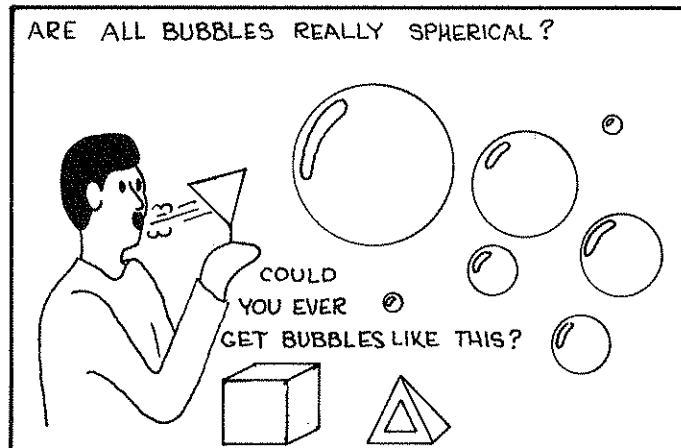
TYPES OF SURFACES

Some surfaces have straight edges, some have curved edges and some have no edges at all. Some surfaces have two sides and some have only one. Students will be able to identify straight or curved edges but they will probably have difficulty with the idea of a one-sided surface. The simplest model of a one-sided surface is the Moebius Strip, a strip of paper which has been given a half-twist before its ends are joined. "Anyone can paint an ordinary paper ring red on one side and green on the other. But, as one mathematician said, 'Not even Picasso could do that to a Moebius band.' If anyone tried, he would only prove that the strip has only one side—on which both colors must meet." (Bergamini, Mathematics, p. 182)

● Show a flat piece of paper labeled like that on the right and ask students to describe points that are on the same side of this surface. Among their responses might be some like these:

"If you can see both letters, they are on the same side."

"If a bug can crawl from one point to the other without falling off, they are on the same side. The bug will fall off if it goes over an edge."
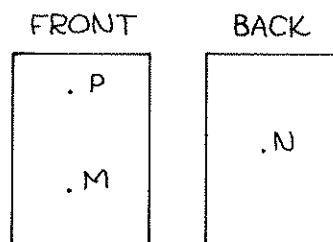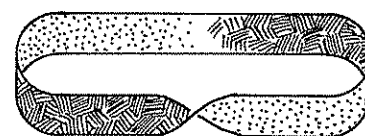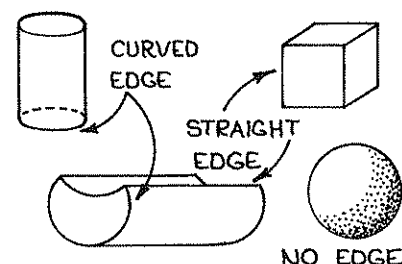
"If a path can be drawn from one point to the other without going over the
edge, then they are on the same side."

Showing an open-ended cylinder and discussing points
P, M and N should help them discard a suggestion like
the first.   If the class accepts either of the other
suggestions, have them make a Moebius strip and mark
two points on the strip that they think would be on different sides.   Show them that
a path can be drawn (or a bug could crawl) from one of the points to the other with-
out going over the edge.   Some students will continue trying pairs of points and
some will try to think up a new description of points on the same side.   Others will
realize they are "stuck" with concluding that all points on a Moebius strip are on
the same side.   Activities with the Moebius strip are discussed in *Two Faced? Never*!
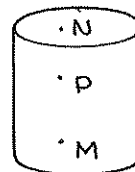
If students like investigating the Moebius strip,
you might let them make models. The surface at the right
can be made as shown below.
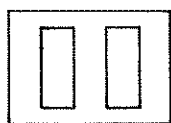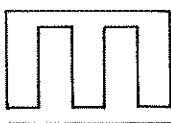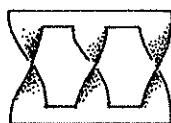
SHEET OF   CUT TWO   CUT AS   HALF TWIST
PAPER   RECTANGULAR   SHOWN   EACH END
                  HOLES   AND GLUE
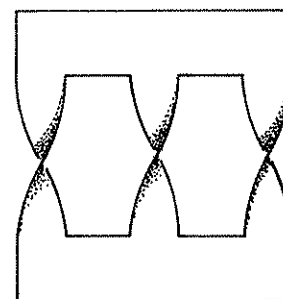
Is this surface one-sided or two-sided?   How many edges does it have?

Exploring curved surfaces should be an enjoyable learning experience for your
class.   The chance to touch, make and investigate these surfaces informally in
middle school can also form a basis for more formal work in high schools.

ON BEING THE ROUND SHAPE

Carousel

Telescope lens

Wheels
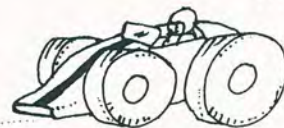
Dials, tuning knobs, and clock faces

Drinking glass
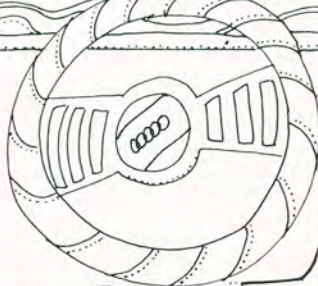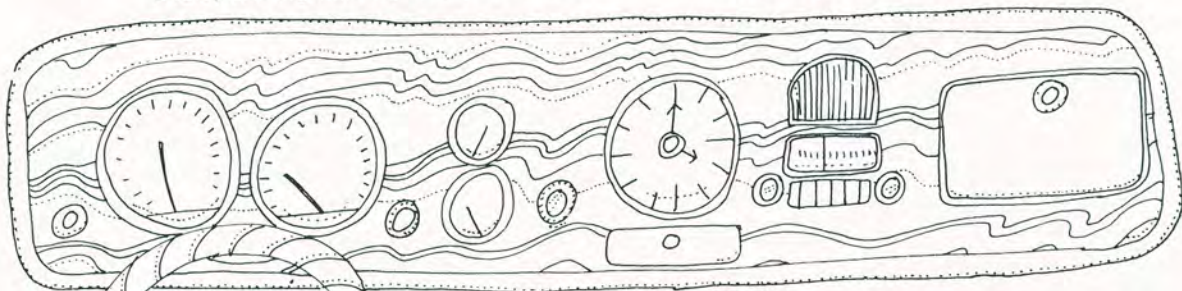
Circles and spheres are useful shapes. Why is each object shown made round or spherical, rather than some other shape?

Gear shift knob

Gum balls

Turtle's shell
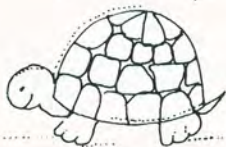
Crystal ball

Balls

Globe

Ice cream scoop

ON BEING THE ROUND SHAPE

(CONTINUED)

Carousel: Every point on the outside edge of a round carousel is always the same distance from the center. This keeps the weight distribution uniform and gives a pleasing effect as the carousel rotates. A square carousel would stick out farther at the corners and would seem to have an uneven motion as it rotated. It would also have more weight distributed along the diagonals.

Telescope lens: The symmetry of the round lens allows light to enter uniformly from all directions.

Wheels: For a smooth ride, a constant distance must be maintained from the axle to the ground. To maintain this distance, the rotating wheel must be round.

Drinking glass: If the glass were square instead of round, liquid would spill from the corners when a person tried to drink from an edge.

Knobs: Tuning knobs and steering wheels are made to rotate in a circle about the center of rotation. A round shape seems most natural. Because of its rotational symmetry, regardless of its position, a round knob is easily grasped and appears balanced to the eye, in contrast to the annoying appearance of a tilted rectangular knob.

Dials and clock faces: The tips of the hands on dials and clocks remain the same distance from their center of rotation. Thus the markings must be about a circle, making a circular face the most natural shape.

Turtle shell: A sphere is a sturdy shape, structurally. Thus a turtle shell is hard to crush and, in addition, holds the greatest amount of turtle for the smallest amount of shell.

Crystal ball: A sphere, having lines of symmetry in all directions through the center, can absorb or emit light (or other waves) from all directions.

Balls: A sphere has rotational symmetry about its center through any angle in any direction and can therefore roll freely. This is an ideal shape for many sports, and it keeps the gum balls from getting stuck in the machine.

Globe: Gravitational forces have made the earth, for which the globe is a replica, roughly spherical.

Ice cream scoop: Because the hemisphere has no corners, the ice cream can slip in and out of this shape fairly easily.

ROLLIN' ALONG

RECTANGULAR PRISM   CONE   CYLINDER
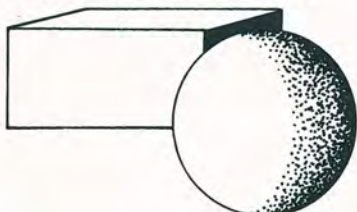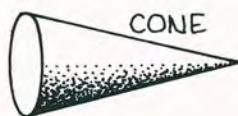
SPHERE   FRUSTUM OF A CONE
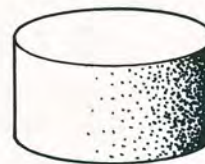
1) a) Which shapes roll in every position? _____
   b) Which roll in some positions? _____
   c) Which never roll? _____

2) a) Roll a sphere on a flat surface.
   b) How much of the sphere touches the surface at any one time? _____
   c) Describe the directions in which a sphere can roll. _____
   d) Describe the path of a sphere when it rolls. _____
   e) Why is a sphere used for the tip of a ball-point pen? _____
   f) Why aren't spheres used for wheels? _____

3) a) Roll a cylinder on a flat surface.
   b) What part of the cylinder touches the surface at any one time? _____
   c) Describe the directions in which a cylinder can roll. _____
   d) What kind of a path does a cylinder make when you roll it? _____
   e) Why are cylinders used for wheels? _____
   f) Which of the five shapes above would be better for a spool of thread than a
       cylinder? _____

4) a) Roll a cone on a flat surface.
   b) How much of the cone touches the surface at any one time? _____
   c) What kind of path does a cone make when you roll it? _____

5) a) Roll a frustum of a cone on a flat surface.
   b) How much touches the surface at any one time? _____
   c) What kind of a path is made? _____

6) a) Investigate the rolliness of two congruent spheres that are stuck
       together.

   b) Investigate the rolliness of two noncongruent spheres that are stuck together.

   c) Investigate the rolliness of three congruent spheres stuck together in various
       arrangements.

7) Some joints in your body need to move in many directions. Your hip joint is an
   example. It is called a ball and socket joint. What other joints move in many
   directions? Which joints move in only one direction?

IDEA FROM: *Rolling*, Topics From Mathematics   479

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IS IT *PLANE* TO YOU?

Materials: Play-Doh or oranges
            Putty knife

The drawings below show a solid sphere being cut by two planes. Four regions are formed.
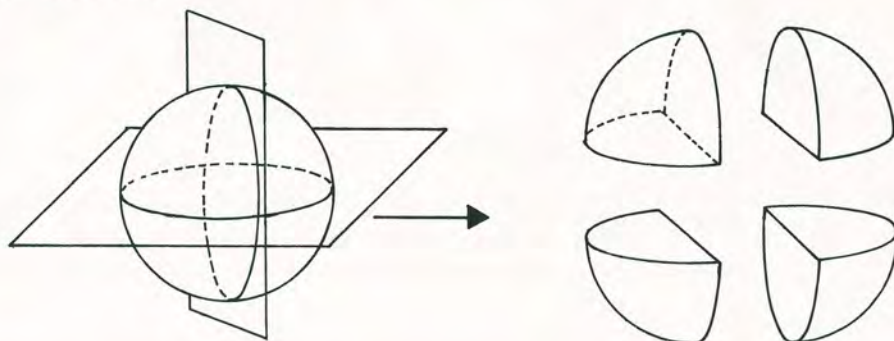
1) Use the Play-Doh to build a model of a solid sphere. See if you can get more than four pieces with two cuts.

2) See how many pieces you can get by making three cuts through a solid sphere.

3) How many pieces are possible with four cuts?

4) The drawings below show two cross sections obtained by cutting through a solid hemisphere with a plane.

What cross sections can be obtained by cutting a solid sphere with a plane?

5) The drawings below show the intersection of a solid sphere and a line.

Describe the intersection of the solid sphere and the line in each drawing.

A cylindrical shaped oatmeal box can be used to illustrate the inter-
sections of a plane with a cylinder. In each case have students
predict the cross section that will be formed. The oatmeal box can
actually be cut to show the cross sections. It would also be helpful
to trace around the cross section, either on an overhead or on the
chalkboard.
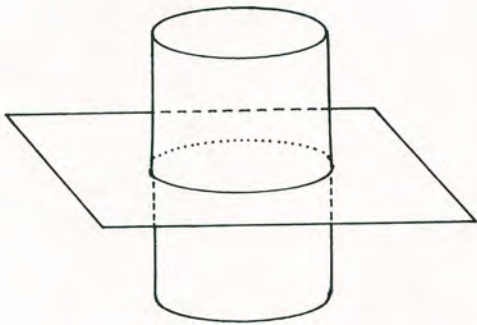
Circle
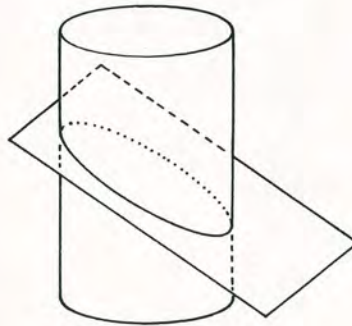
Ellipse
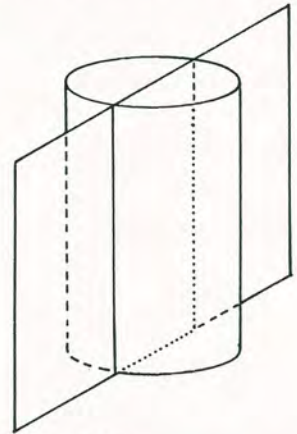
Rectangle
(if the bases are used)
Parallel line segments
(if no bases)

Have students predict how to get an
intersection that is (1) a point
(2) a line segment (3) part of an
ellipse.

(1)

(2)

(3)
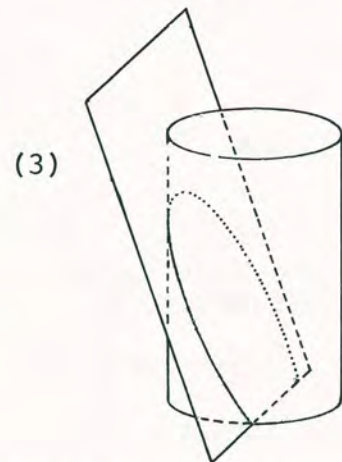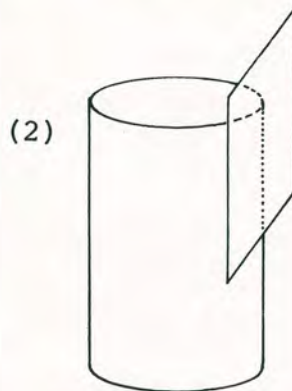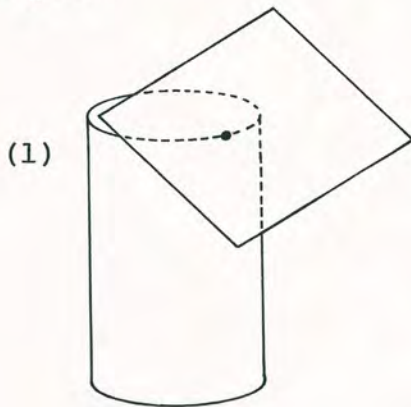
The intersections that are a point and part of an ellipse assume that
the cylinder does not extend indefinitely.

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Similar demonstrations could be developed to illustrate the intersections of a plane and the cone. A cardboard cone used for flavored ice could be used as a model. These intersections assume an infinite cone with no base is being used.
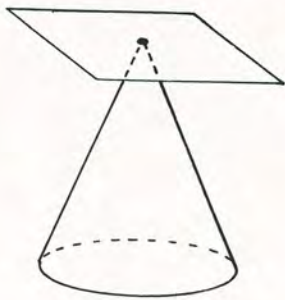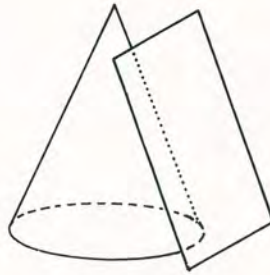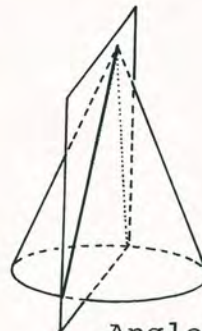
Point   Line   Angle   Circle
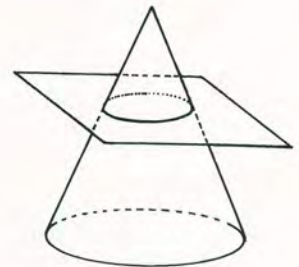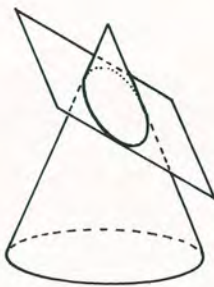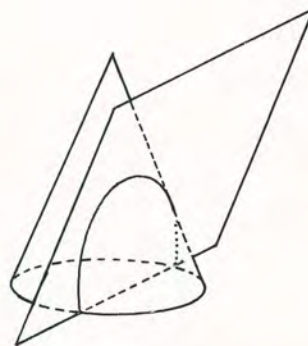
Ellipse   Parabola   Hyperbola
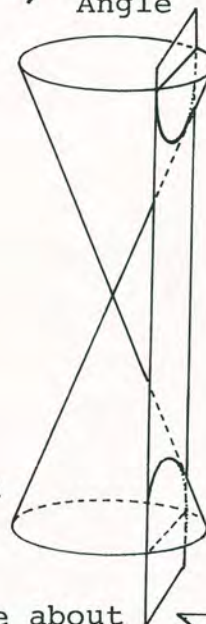
These are some important points to be made about the intersection of a plane and a sphere.

1) All cross sections will be circles.

2) The largest cross section is a great circle.

3) The cross sections could be related to the lines of latitude and longitude on a globe.

4) A plane tangent to a sphere intersects it in one point.
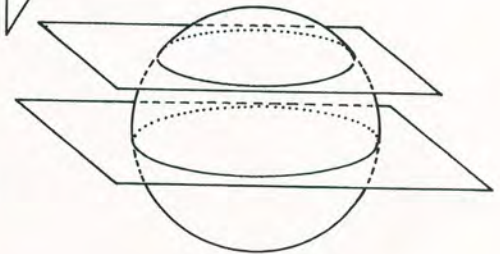
When pouring a carbonated drink into a glass, people often tilt the glass and pour down the side so not too much foam is created. The glass at the right has been invented to solve this problem.

Question: What is the flat shape that could be rolled up to make this shaped glass? Let the students predict the shape and try to make the "glass."

9th Grade Solution:

Make a right circular cylinder and use a pair of scissors to cut across the cylinder. Unfold the middle section and observe the shape. If the cylinder is flattened to make the cut, the shape will only approximate the shape of the glass.
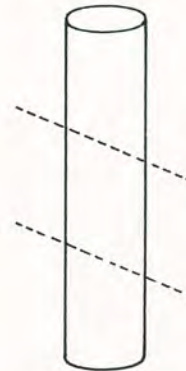
7th Grade Solution:

Wrap a rectangular piece of paper tightly around a candle. With a knife or single-edge razor blade cut through the candle (and paper) with two parallel cuts. Unfold the middle section of the paper and observe the shape. Since the paper was wrapped around the candle several times the shape will be repeated several times. (The edge of the shape happens to be a sine curve!)
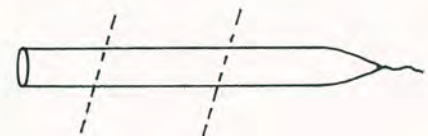
5th Grade Solution:

Make a cylinder and carefully dip it, at an angle, into some liquid solution. Coffee works well. Remove the cylinder and unfold it to observe the shape formed by the coffee stains.

A similar activity can be done with the cone. For the 7th grade solution a model of a cone can be made by packing clay or Play-Doh into a funnel.

A REVOLVING SITUATION

1) If the solid line could revolve around the dotted axis, it would trace out the surface of a cone.
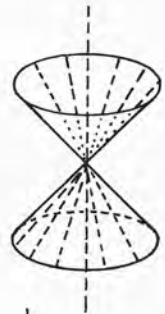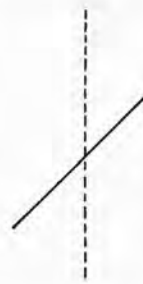
2) How would the surface made by revolving this be different from the surface in (1)?
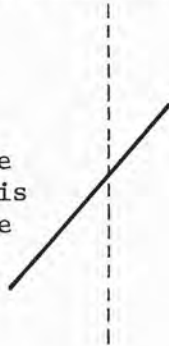
3) What would the surface made by revolving this look like?

4) How would the surface made by revolving this be different from the surface in (3)?
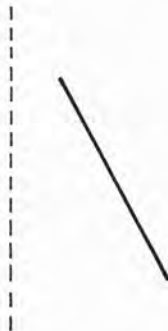
5) What would the surface made by revolving this look like?

If each of the figures below were revolved about the dotted line, the surface of an object found around the home would be formed. Name each object.

6)

7)

8)

9)

10)

11)

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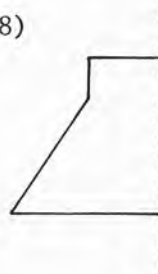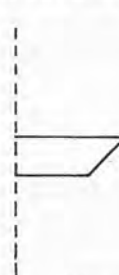
DRAWING ELLIPSES, CYLINDERS AND CONES

By tracing around the inside of a "hole" on this ellipse template you can easily draw ellipses, cylinders and cones.
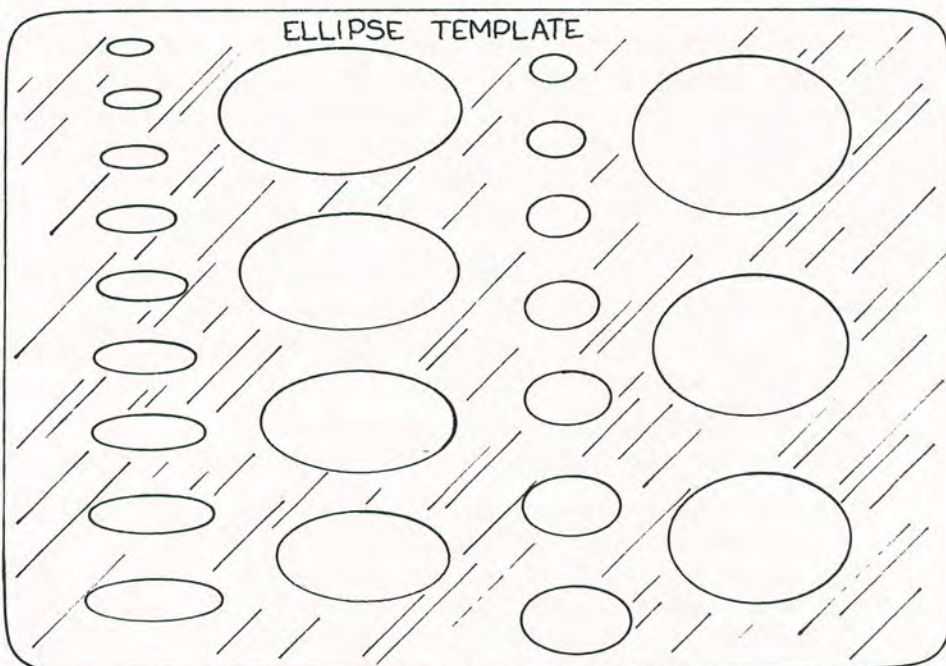
ELLIPSE TEMPLATE

1) Use an ellipse template and draw five different sized ellipses.

2) Which of these looks like the can that tennis balls come in?

_____

A)   B)
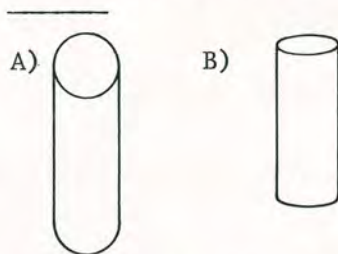
3) Which of these looks like the cone for an ice cream cone?

_____

A)   B)
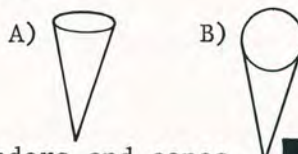
The ellipse template can be used to draw realistic looking cylinders and cones.

4) These steps will help you draw cylinders and cones.

   a) Draw an ellipse for a base.

   b) Draw the sides.

   c) For a cylinder draw the visible part of the other base.

a   b   c
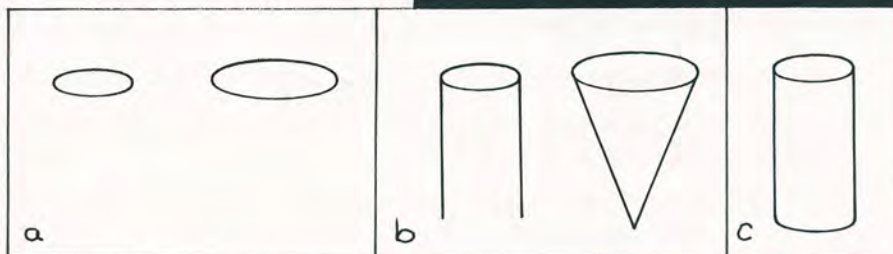
5) Match each of these with the correct picture.

____ a) An upright cylinder viewed from above

____ b) A cone, vertex down, viewed from below

____ c) A cylinder, lying down, viewed from the left

____ d) A cone, vertex left, viewed from the left

____ e) A cone, vertex right, viewed from the left

____ f) An upright cylinder viewed from below
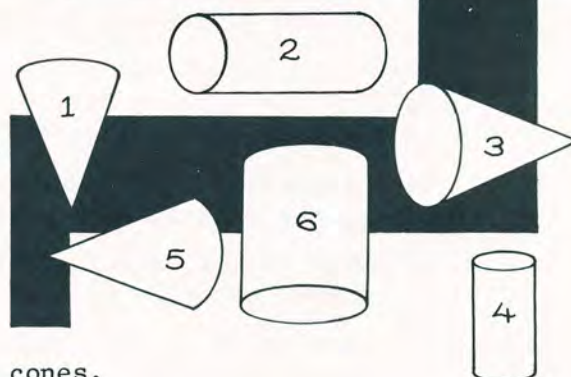
6) Draw four different cylinders and four different cones.

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TWO-FACED? NEVER!

The properties of a Moebius strip are so unusual and unexpected that students enjoy discovering them. A Moebius strip is made by taking a piece of paper (3 cm x 30 cm is a convenient size), making a loop and turning one end over (a half-twist) before taping the two ends together. See the figure to the right.

I   Students can perform the following experiments with a Moebius strip. Each experiment should be done with a different strip.

1) Use your pencil or pen to shade one side of a Moebius strip. How many sides does a Moebius strip have?

2) Cut along a Moebius strip midway between the edges. What is the result?

3) Cut along a Moebius strip one-third of the way from an edge and parallel to an edge. Continue cutting until you return to the same point from which you started. (You will have to cut all the way around the loop twice.) What happens? How do the loops compare in length and width?

4) Cut along a Moebius strip one-fourth of the way from an edge. How does the result compare to the previous one?

5) Predict the result if you cut around a Moebius strip one-fifth of the way from an edge.
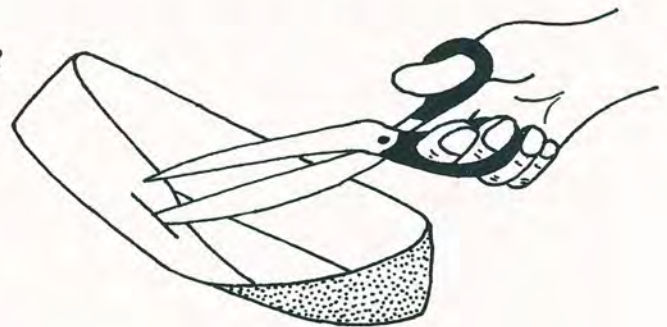
II Students can investigate strips having additional half-twists.

1) Construct a loop by taking a strip of paper (3 cm x 40 cm) and making two half-twists in it before taping the two ends together.

a) How many sides does this loop have? Check by shading one side.

b) Cut along a similarly constructed loop midway between the edges. What is the result?

c) Cut along a similarly constructed loop one-third of the way from an edge. What happens?

d) Predict what will happen if you cut around a similarly constructed loop one-fourth of the way from an edge.

2) Construct four loops by taking strips of paper (3 cm x 60 cm) and making three half-twists in each before taping the ends together. Repeat the investigations of the previous problem on these new loops.

3) Repeat problem 2 except construct the loops by making four half-twists before taping the ends of each strip together.

a) What is the relationship between the number of half-twists in a loop and the number of sides it has?

b) What would be the result if you took a loop with 50 half-twists in it and cut it down the center?

TRICKS WITH TOPOLOGY

Knots

A false knot known as the Chefalo Knot is an example of knots which are used by magicians. It begins as a square knot as in Figure 30a. Then one end is woven in and out as shown in Figure 30b by arrows. When the ends are pulled, the knot disappears.
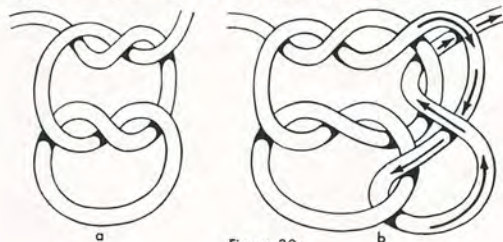
Figure 30

Buttonholing a Friend

Tie a loop of string to a pencil or a short stick. Be sure the loop is shorter than the pencil. Attach the pencil to the buttonhole of a friend's jacket without untying the loop, as shown in Figure 31b. Pull the loop tight, as shown in Figure 31c. Ask your friend to remove the pencil without untying or cutting the loop (or his buttonhole!). If he doesn't see you put it on, he will have a hard time removing it.
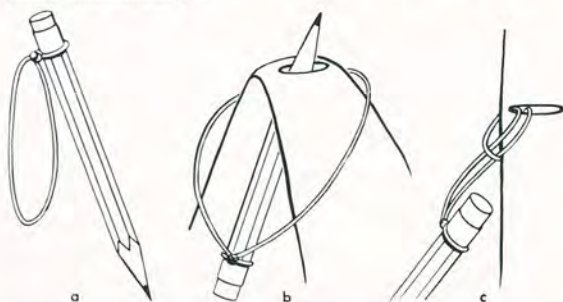
Figure 31

A variation of this puzzle is to loop a string through a pair of scissors and then tie the ends of the string to a large button, as shown in Figure 32. The button must be larger than the opening in the handle of the scissors. The problem is to remove the button from the pair of scissors without untying or cutting the string.
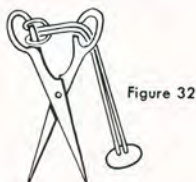
Figure 32

The Ring Puzzle

The three rings pictured below have a strange topological relation. Remove any one ring, and the other two will be found to be free, too. Thus, no two rings are joined, but the three put together are.

Figure 33

Stringing Along

Tie a piece of string to each of your wrists. Tie a second piece of string to each of the wrists of a partner in such a way that the second string loops the first.

The object of this stunt is to separate yourself from your partner without cutting the string, untying the knots, or taking the string off your wrists. This can be done!

Figure 34

Solution: Loop your string under the wrist loop of your partner. Pull your loop over his hand and you will be free.

Buttons and Beads

To make this puzzle you need cardboard, string, two buttons, and two beads. Cut a rectangular piece of cardboard about 1 inch by 6 inches. Cut three small, evenly spaced holes, as in Figure 35.
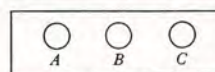
Figure 35

String two large beads on the string. Thread one end of the string through hole A and attach a button larger than the hole. In the same direction, thread the other end of the string through hole C and attach a button as in Figure 36a.
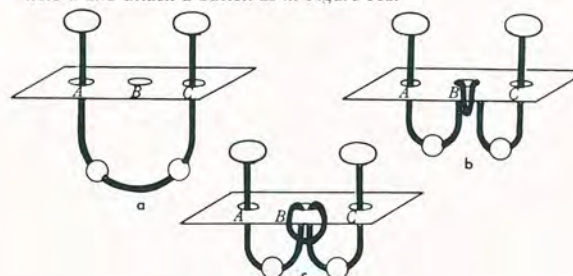
Figure 36

The string is then looped through hole B, as in Figure 36b. To loop it back under itself, as in Figure 36c, the loop is first threaded up in hole A and over the button and then likewise in hole C. The puzzle is now ready for someone to try to undo the loop and get the beads together.

The Paper and String Puzzle

Take a piece of stiff paper and cut it so it measures 6 inches long by 3 inches wide. Now cut two parallel slits half an inch apart in the center of the paper, as shown in Figure 40. Each slit is 3 inches long. Half an inch above the slits, cut a circular hole having a diameter of ¾ inch.

Pass a piece of string about 12 inches long behind the slits and then down through the hole, as shown in the drawing, and tie a large button to each end of the string. Be sure that each of the buttons is too large to pass through the ¾-inch hole in the paper.

Now ask one of your friends to remove the string and buttons from the paper without tearing the paper or taking off either of the buttons. There is little chance of his succeeding unless you show him how.

Figure 40

GEOMETRY AND VISUALIZATION

## Placement Guide for Tabbed Dividers

CONTENTS

CIRCLES

Circles are the most familiar and the most studied of the planar curves. Young children learn to identify circular regions and to draw pictures of circles. Circles were studied formally at the time of the Greeks and they continue to be an important part of geometry courses today. The word circle and its derivations have been incorporated into much of everyday language. Phrases and words like using circular reasoning, going around in circles, circulate, sewing circles, circle the correct answer and circulatory system are common.
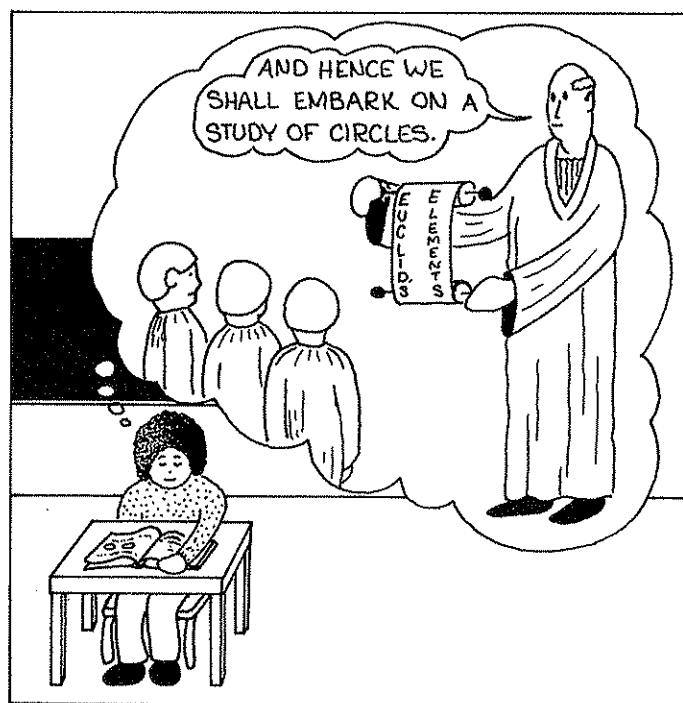
Familiarity with "circle" is helpful, but it can cause a problem. Everyday language employs the word "circle" when referring to either a flat circular disc or to the curved boundary of that disc; however, it is sometimes necessary to use terms which distinguish between these two ideas. Thus, the circle with center P and radius of length r is the set of points in a plane whose distance from P is r. The term circular region is used to describe the part of a plane bounded by a circle.
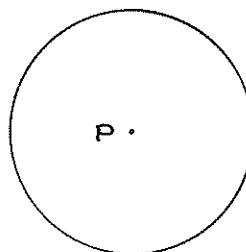
A PICTURE OF A CIRCLE AND ITS CENTER P
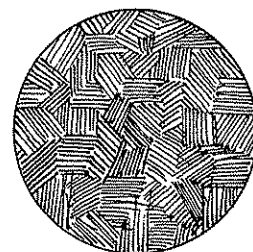
A SHADED CIRCULAR REGION

The circle seems to be the perfect plane figure. It is the most symmetrical plane figure because it has an infinite number of lines of symmetry. Because each point on a circle is the same distance from the center point, the circle is the basic shape used in wheels and gears. Circles are also "curves of constant width" (curves that will remain tangent to two parallel lines when they are rotated), but surprisingly, other such curves of constant width exist; they are discussed in *Rollers with Corners* in the Other Curves subsection.


SOME LINES OF SYMMETRY

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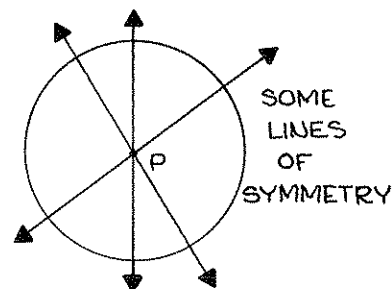
Circles occur in all sizes but only one shape, so all circles are similar. This is not true of ellipses which occur in many sizes and shapes. When several differently-sized circles in a plane share the same center, the circles are called concentric. The page at the right shows many examples of concentric circles. The page could be used on the overhead or bulletin board to motivate a collection of student-found examples. Concentric circles can be used to make a circular coordinate grid. Such a grid is useful for graphing many curves and plotting locations of airplanes as determined by radar. An example of such a grid is given to the right. A discussion of circular coordinate grids can be found in *Concentric Tic-Tac-Toe*. The page *Archimedean Spirals* in the Other Curves subsection uses this type of grid for drawing spirals.

CONCENTRIC CIRCLES

CIRCULAR COORDINATE GRID

$(3, 45°)$

USING MANIPULATIVES TO STUDY CIRCLES

In addition to the traditional store-purchased compass, there are many homemade devices that can be used in the study of circles. *Circle Making Methods* gives nine alternative ways of creating a circular shape with ordinary materials found in a home or classroom. You might want to have your class try some of these methods. A handy device for demonstrating ideas related to circles is a board (shown to the right) with nails arranged in a circle and at a few exterior and interior points of

A LINE THAT TOUCHES A CIRCLE IN ONE POINT IS CALLED A TANGENT TO THE CIRCLE.
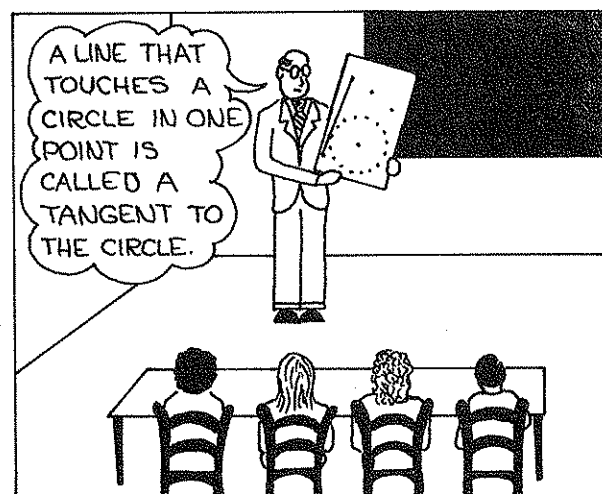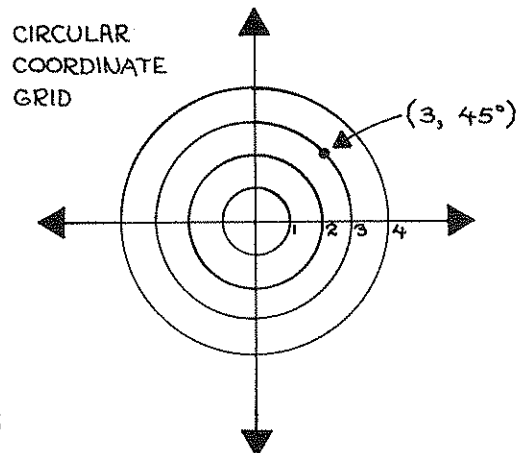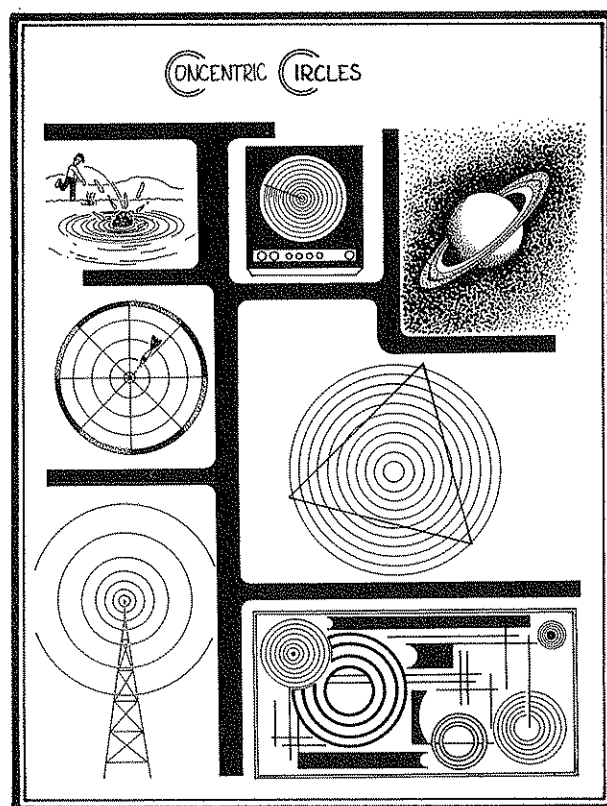
the circle. Directions for making such a board together with suggested concepts that
can be demonstrated on the board are given on the page
The Circleboard.

A related manipulative which can be used by indi-
vidual students is the circular geoboard. There are
several possible designs for such a board. A commonly
used design is shown to the right. A pattern and in-
structions for making circular geoboards are given on
the page *A Circular Geoboard.* With the use of such a
manipulative, students can explore such questions as
"What kinds of triangles can have one vertex at the
center of a circle and the other two vertices on the circle itself?" The investiga-
tion *Circular Intersections* could be adapted to a circular geoboard. For example:
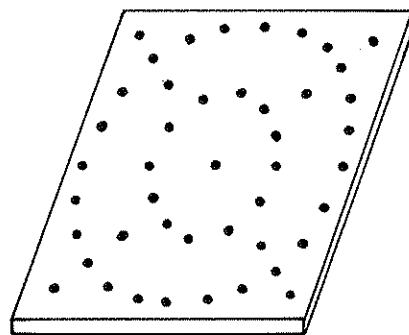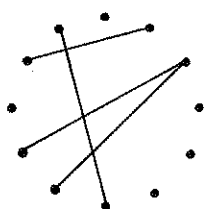
the twelve-point circle of the circular geoboard is separated
into eight regions by four chords (line segments whose end
points are on a circle). Can you use four chords to separate
the twelve-point circle into more than eight regions? More
ideas for using a circular geoboard are found on the pages *Are
You Right All the Time?* and *Polygons on a Circular Geoboard.*

Manipulatives can make the learning of geometry ideas more fun and meaningful.
Why not use them?

SOME IDEAS ABOUT CIRCLES

A topic often covered in a study of circles is
that of inscribing a polygon in a circle by drawing
the polygon inside a given circle so each vertex of
the polygon is on the circle. (See *Some Can, Some
Can't.*) Informally, this can be done with paper folding
as suggested on the page *Folding Polygons in Circles.*
Your students will probably benefit from and enjoy this
paper-folding activity even if you intend to cover the
same concept via straightedge and compass construction
as described in *Inside the Circle I, II* and *III.* A re-
lated idea is to circumscribe a circle around a polygon
by drawing the circle around a given polygon so the cir-
cle passes through each vertex of the polygon. This idea
is used in *Circumscribing a Triangle.*
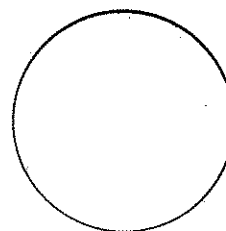
Problem:
Inscribe an
equilateral
triangle in
this circle.
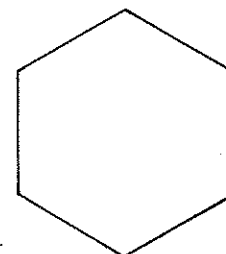
Problem:
Circumscribe
this hexagon
with a circle

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Chords of circles and their relationships are often important in a study of circles. Much work with circles uses those important chords--the diameters. Students can also explore relations involving other chords.

What kind of triangle is formed when line segments are drawn from the endpoints of a chord to the center of the circle? Can students see that two sides of the triangle are radii, so an isosceles triangle is formed?
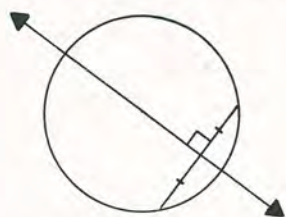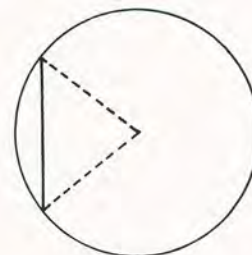
Is there anything special about the perpendicular bisector of a chord of a circle? Will this bisector always go through the center of the circle?

If two nonparallel chords of a circle are given, can the circle be constructed? Does this idea help for circumscribing a circle around a polygon? (Draw the perpendicular bisector of two chords. The bisectors intersect at the center of the circle.)
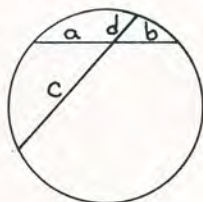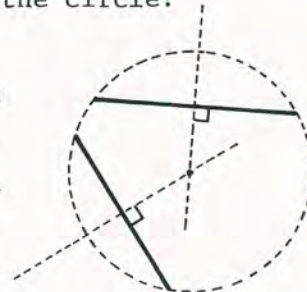
Is there a relation involving the parts of intersecting chords of a circle? (See *Chords in Circles* for an answer.)

Arcs of circles and the angles related to these arcs are also studied. In each diagram to the right angle 1 is an inscribed angle and angle 2 is a central angle. Each pair of angles intercepts the same arc. Whenever this occurs, the measure of the inscribed angle is one-half the measure of the central angle. This relationship is suitable for a student discovery lesson. Students could explore the relationship between inscribed angles and central angles via the circular geoboard, teacher prepared handouts or compass and straightedge constructions. They can compare the angles by tracing, cutting paper patterns or using a protractor. *Arcs, not Arks; A Pair of Angles in the Arc; Arcs I* and *Finding the Right Angle* are student worksheets on arcs and related angles in circles.

PACKING CIRCLES

I   Students need approximately 36 pennies or disks all the same size. Provide each with several differently shaped regions. The students are to cover the surfaces with the disks. Can the disks be arranged in more than one way?

Circular Regions

Rectangular Regions

Triangular Regions

II Students should discover two ways to pack circles. In each arrangement how many circles are tangent to each circle?
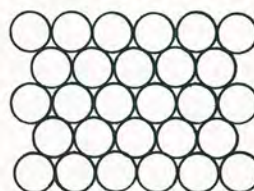
In which arrangement are the circles packed so that less space is wasted?

FIGURE 1   FIGURE 2

III When fragile glasses are packed into a crate, stiff cardboard is needed between them. Supply students with the two types of circle arrangements. Have them mark both arrangements to show where the cardboard would be placed.

Ask what kind of polygons are formed by the separators.

FIGURE 3   FIGURE 4

Why is square packing often used when separators are needed for fragile goods and hexagonal packing is used when separators are not needed?

IV Ask students to look for some examples of circle packing. Have them make drawings and explain the type of packing used for:

a) bottles in a crate
b) box of school chalk
c) cigarettes in packages and tins
d) boxes of eggs
e) boxes of oranges, apples, grapefruit
f) tread on some kinds of tennis shoes

WHAT TYPE OF PACKING DO BEES USE IN CONSTRUCTING A HONEYCOMB?

PUZZLING PENNIES

1)

a) Start with six pennies closely packed like a parallelogram.

b) In three moves* form a circular pattern so that if a seventh coin were placed in the center, the six pennies would be closely packed around it.
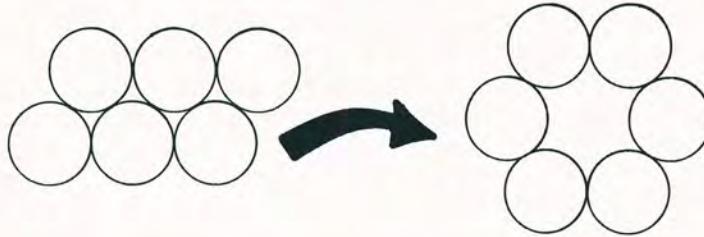
> * On each move a single coin must be slid to a new position so as to touch two other coins that rigidly determine its new position.

2)
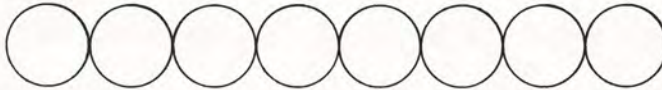
a) Place eight pennies in a row.

b) In four moves* change them into four stacks of two coins each.

c) Suppose two more pennies are added to make a row of ten. Can the ten pennies be double-stacked in five moves?

> * On each move a single penny must "jump" exactly two pennies in either direction and land on the next single penny. The two jumped pennies may be two single coins side by side or a stacked pair.

3)
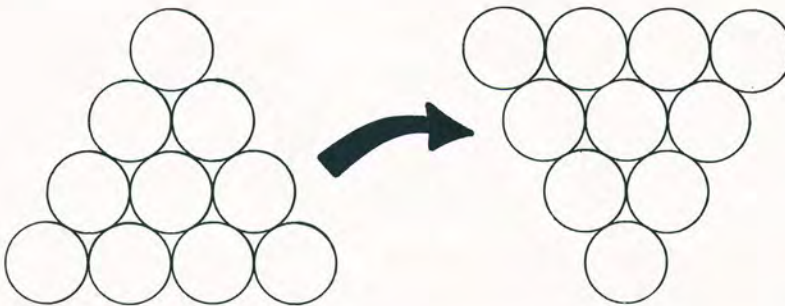
a) Arrange ten pennies to form a triangle.

b) Turn the triangle upside down by sliding one penny at a time to a new position in which it touches two other pennies.

c) What is the minimum number of required moves?

A PENNY FOR YOUR THOUGHTS
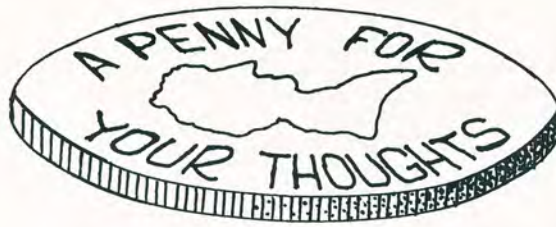
The following problem could be used as an opener for the beginning of school. The activity involves students in discussion and problem solving. All students have an equal chance of succeeding.

The problem: Suppose you and I each have a bag of pennies. We decide to take turns placing a penny on the desk. Each penny must lie completely on the desk, none can overlap, and once a penny is positioned it cannot be moved. The last person to place a penny on the desk wins both bags of pennies.

    a) How would you play to win?

    b) Do you think it is possible to have a winning strategy regardless of the size of the desk? (The desk must be symmetrical--circular, rectangular, square.)

    c) If so, do you want to have the first turn or the second?

    d) Where would you place the first penny?

    e) How would you make each successive play to insure winning?

Solving the Problem

1) Suggest that students solve a simpler problem--reduce the size of the desk.

2) On a one-penny desk what would be the winning strategy? (Go first and place penny on desk.)
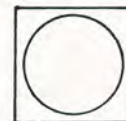
3) Try a larger desk--a two-penny desk. Have students sketch the shape of the desk and experiment to find a winning strategy.
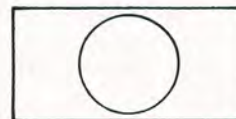
(Go first and place penny on middle of desk.)

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A PENNY FOR YOUR THOUGHTS
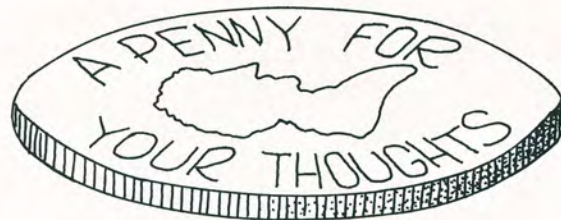
(CONTINUED)

4) If successively larger desks are considered (a 3-penny, 4-penny, 5-penny, etc., desk) students can discover a winning strategy:

a) Go first and place the penny in the center of the desk.

b) Use point symmetry to "copy" each move the second player makes.

Assume that you move first.

YOUR FIRST MOVE

YOUR SECOND MOVE

YOUR OPPONENT'S FIRST MOVE
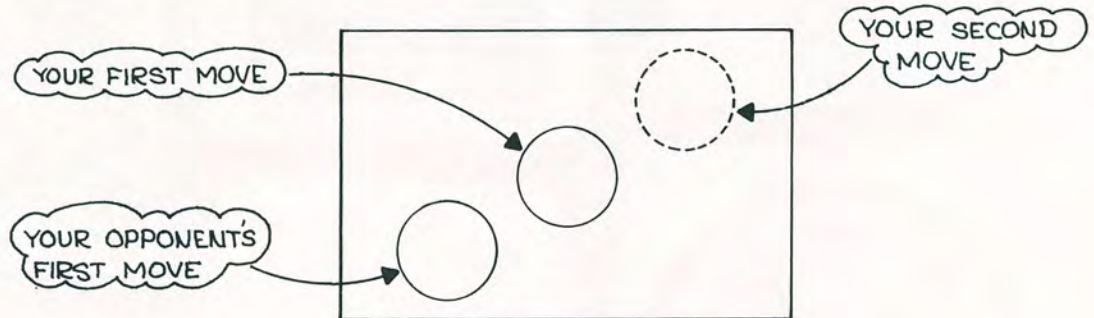
A challenge: Sourdough Sam and Pecos Pete were sitting around a table in the mess hall. "Sam, dig out your silver dollars," ordered Pete. "You and I are going to have a little contest. We'll take turns placing silver dollars around the edge of this table. If you're able to put the last dollar on the edge you get to keep all the money. Otherwise I win. Just to give you a fightin' chance I'll let you start first."

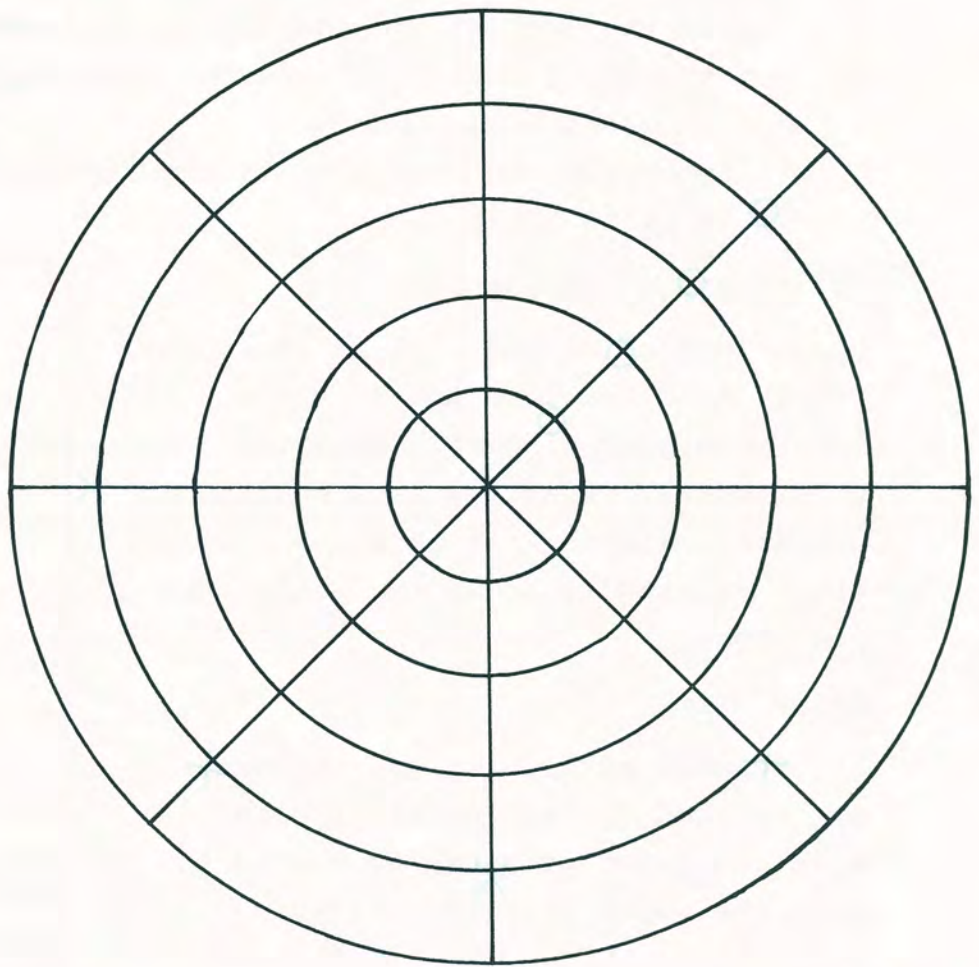If you were Sourdough Sam would you accept the challenge? How would you play to win?

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Concentric Circles

CONCENTRIC TIC-TAC-TOE

PLAYERS: TWO PERSONS OR TWO TEAMS

GOAL: 1) TO PLACE FOUR CONSECUTIVE X'S OR FOUR CONSECUTIVE O'S AROUND A CIRCLE OR ALONG A DIAMETER.

2) MARKS ARE PLACED AT THE INTERSECTION OF THE LINES AND CIRCLES.

3) THE CENTER OF THE CIRCLES MAY BE USED.

CIRCLE MAKING METHODS

Besides the standard method of using a compass, there are other ways of making or drawing circles that may be used and/or may be of interest to your students. A few suggestions for using the compass are given first. Some of the alternative methods could be developed into activity cards or projects.

1) Pointers for using the compass

   a) Use a sharp pencil.

   b) Line up the points of the compass and the pencil

   c) Secure the legs of the compass so they won't slip apart.

   d) Put something under the paper to keep the point from slipping and to protect the desk.

   e) Tilt the compass slightly (in the direction of the rotation) as it is being rotated.

2) String and pencil (chalk)

   Use a loop of string. Place the pencil or chalk in the loop and draw the string taut with your fingers. This is particularly effective as a chalkboard compass. Hint: With your right hand, start drawing at about the 7 o'clock position.
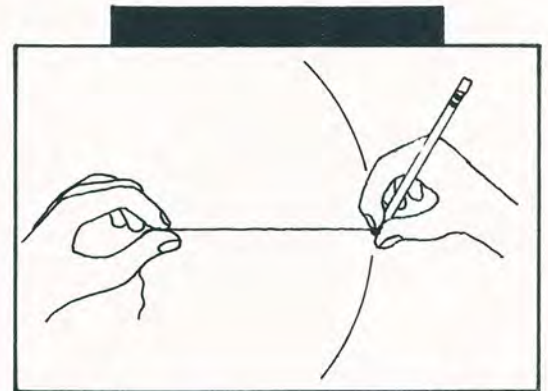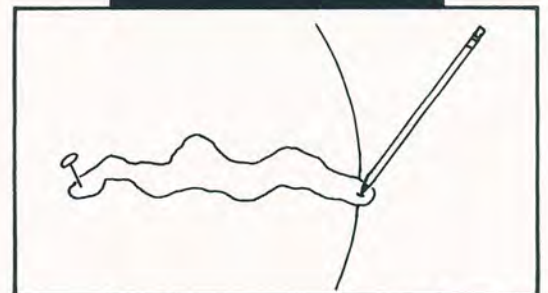
3) Paper clip

   An ordinary paper clip and two pencils can be used to draw small circles. A slight outward pressure is needed to keep the paper clip from slipping.
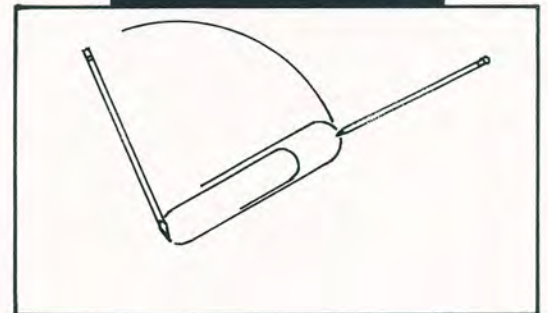
4) Irregular shapes

   Students are often amazed that very unusual shapes can be used to make circles. Use a strip of posterboard cut in an unusual way. Put a pin in one end and a hole in the other end for a pencil.

The following methods use a circular object or figure to draw a circle.

5) Tracing paper

Students trace over a circle found in a book or magazine.

6) Templates, coins, etc.

Coins, cans, bottles, a template with circles, etc., can all be traced around for circles of different sizes.

7) Flashlight

A flashlight held perpendicular to a chalkboard or sheet of paper projects a circle that can be traced.

8) Paper folding or curve stitching

Marking a circle with many uniformly spaced points and then following a pattern of connecting the points with folds or lines, say 1 to 5, 2 to 6, 3 to 7, 4 to 8, etc. will produce a circle within a circle. This makes an attractive curve stitching project. See *You've Got Me in Stitches* in LINES, PLANES & ANGLES: Lines.
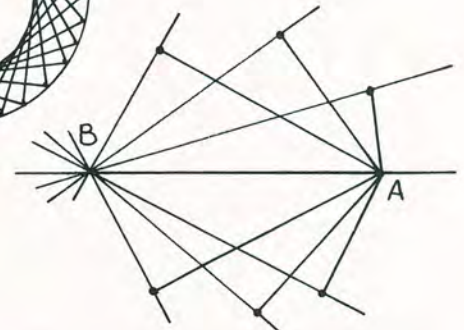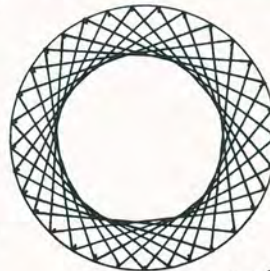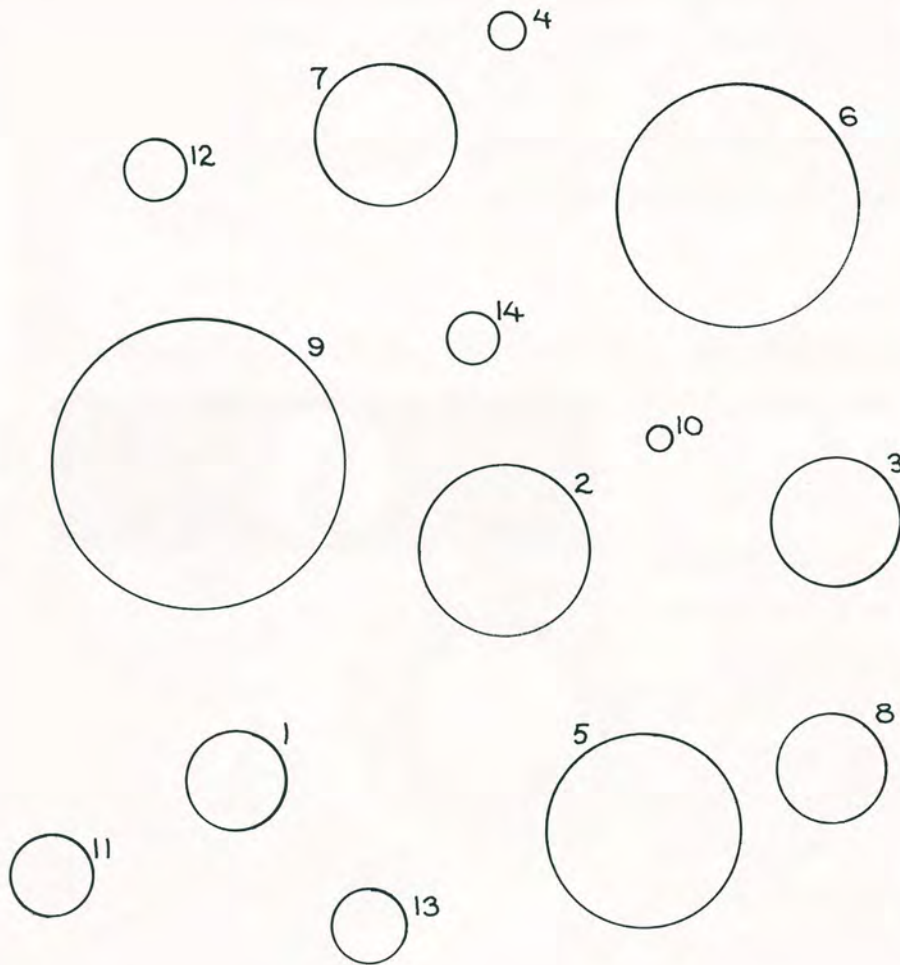
9) Paper folding

Mark a straight line segment on a sheet of waxed paper. Label the end points A and B. Make a crease through A. Make a second crease through B so the first crease lies on itself. If done accurately the angle formed by the two creases is a right angle. Mark the vertex of the right angle. Repeating this procedure many times will yield a series of right angles whose vertices lie on a circle with segment AB as a diameter. Note: This is a difficult folding project.
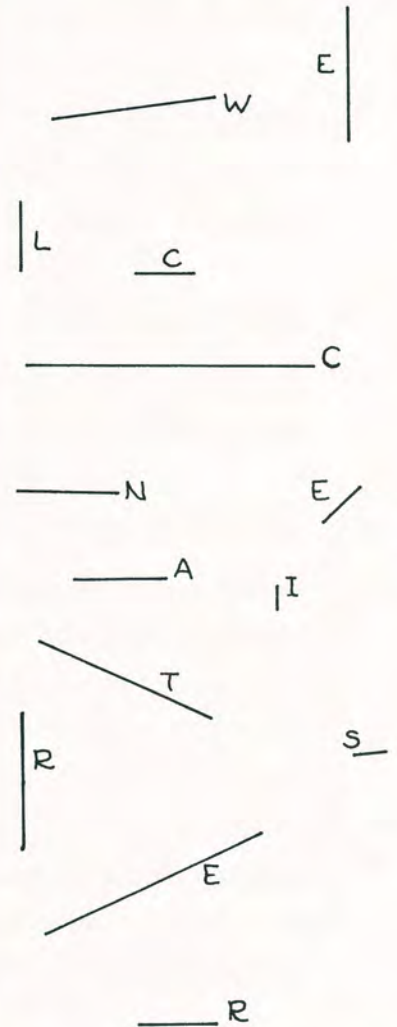
WHAT IS A ROUND-UP?

Match each circle with its diameter. No rulers please. Fill in a letter on each numbered blank provided below.

CIRCLES | DIAMETERS

○4	E
○12 ○7 ○6	——— W
	|L
○14 ○9	—C—
○10	—————————— C
○2 ○3	—— N E/
	—— A |I
○5 ○8 ○1	\T
○11	|R S_
○13	/E
	— R

$\underline{\qquad}$ 1 $\underline{\quad}$2 $\underline{\ }$3 $\underline{\ }$4 $\underline{\ }$5 $\underline{\ }$6 $\underline{\ }$7 $\underline{\ }$8 $\underline{\ }$9 $\underline{\ }$10 $\underline{\ }$11 $\underline{\ }$12 $\underline{\ }$13 $\underline{\ }$14

List the number of each circle in order from smallest to largest.

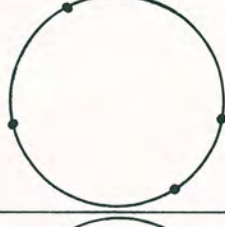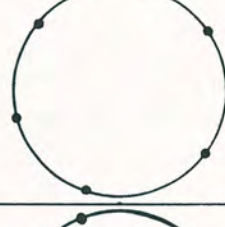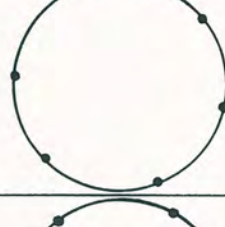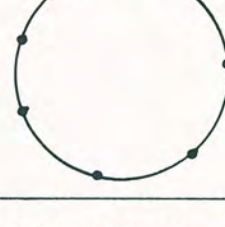— — — — — — — — — — — —

THE LOST CHORD
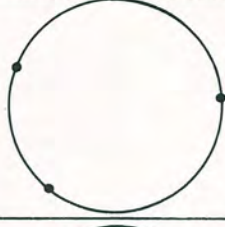
1) Complete the table below.

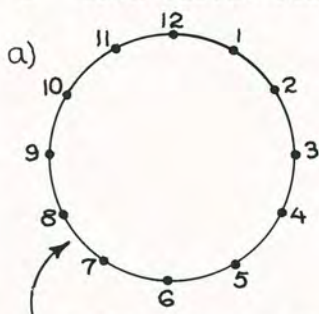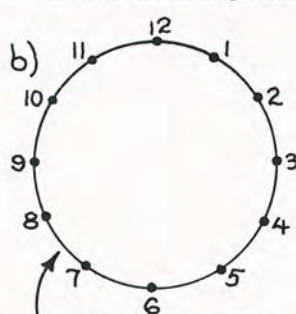| Draw chords to join each pair of points.	Number of points	Number of chords from each point	Number of chords

2) If the pattern above were continued with 8 points identified on a circle and all possible chords drawn, how many chords would come from each point? How many chords would there be in all?

3) What is the relationship between columns 2, 3 and 4? See if this relationship works for 10 points on a circle.
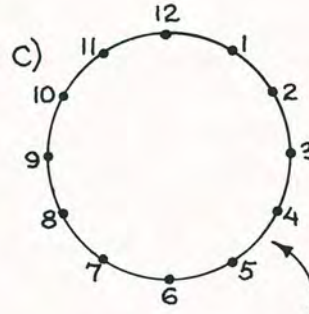
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A STARTLING DISCOVERY

I   Each circle below has 12 division points.
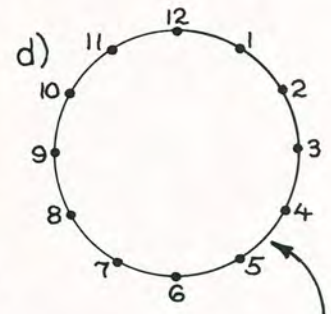
a)

Join <u>each</u> point to every second point. (1 to 3, 3 to 5, ... and so on. Also 2 to 4, 4 to 6, ...)

b)

Join <u>each</u> point to every third point. (1 to 4, 4 to 7, ... and so on. Also 2 to 5, 5 to 8, ... and 3 to 6, 6 to 9, ...)

c)

Join <u>each</u> point to every fourth point.

d)

Join <u>each</u> point to every fifth point.

How does the last figure differ from the first three? _____
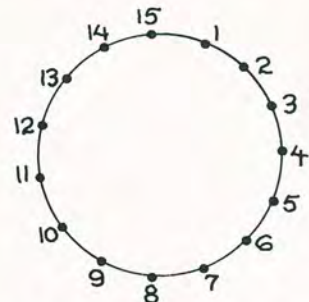
Use the drawings to help you complete the table.

	NUMBER OF DIVISIONS BETWEEN POINTS JOINED	NUMBER OF INSCRIBED POLYGONS FORMED (P)	NUMBER OF SIDES ON EACH POLYGON (S)	PRODUCT (P×S)
a)	2	2	6	
b)	3			
c)	4			
d)	5			

Only the last figure can be drawn without lifting the pencil from the paper.

What property do the numbers 2, 3, 4 have that 5 doesn't that might help to explain this? _____

II   Use what you have learned to predict the table for a circle with 15 division points.

|	NUMBER OF DIVISIONS BETWEEN POINTS JOINED	NUMBER OF INSCRIBED POLYGONS FORMED (P)	NUMBER OF SIDES ON EACH POLYGON (S)	PRODUCT (P×S)
a)	2			
b)	3			
c)	4			
d)	5			
e)	6			
f)	7			

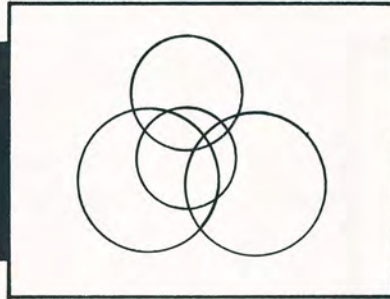Trace this circle 6 times. Make a drawing to check each row in the table.

Which figures could be drawn without lifting the pencil from the paper? _____
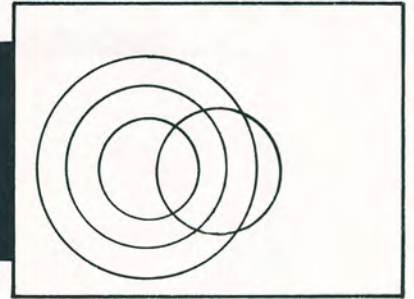
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IN'N'OUT
REGIONS

1) Below are three examples of four circles that intersect and form regions.
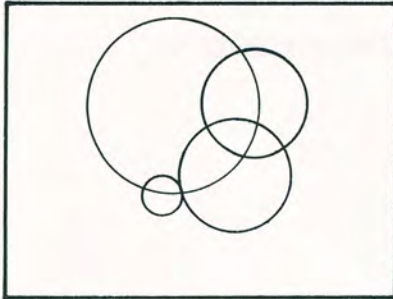
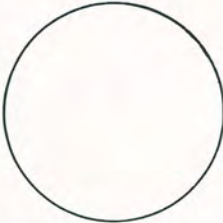| 4 CIRCLES | 4 CIRCLES | 4 CIRCLES |
| 9 REGIONS | 13 REGIONS | 7 REGIONS |

2) What is the <u>maximum</u> number of regions formed by three intersecting circles? _____

3) Can you find the <u>maximum</u> number of regions formed by ten intersecting circles?
    The following six exercises may suggest a pattern.

| I CIRCLE | 2 CIRCLES | 3 CIRCLES |
| _I_ REGIONS | ___ REGIONS | ___ REGIONS |

| 4 CIRCLES | 5 CIRCLES | 6 CIRCLES |
| ___ REGIONS | ___ REGIONS | ___ REGIONS |

CIRCULAR INTERSECTION

Is it possible for three differently sized circles to have no intersection points, exactly one intersection point, exactly two intersection points, exactly three, four, five, six intersection points, more than six intersection points? Use a template or coins to sketch an example in each box below. Are some impossible? Some can be done in many different ways.

NO POINTS   ONE POINT

TWO POINTS   THREE POINTS

FOUR POINTS   FIVE POINTS

SIX POINTS   MORE THAN SIX POINTS   IMPOSSIBLE

TANGENTS

I   Have students:

1) Mark a point on a sheet of plain paper.

2) Take a straightedge and place one edge through the point.

    Draw a line along the other edge.

3) Repeat this several times using the same point.

4) Continue drawing straight lines until they cannot find space for any more.
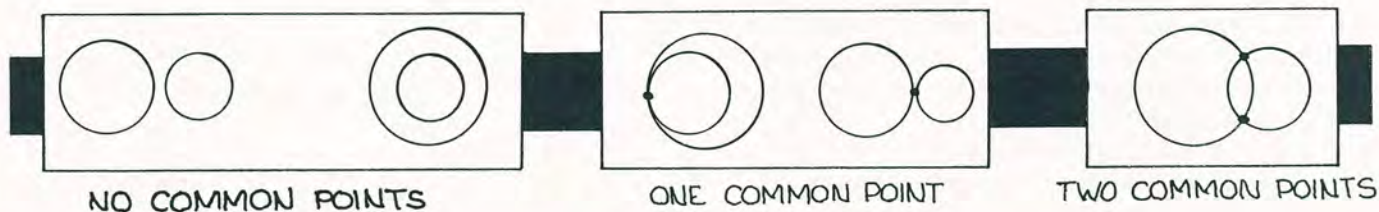
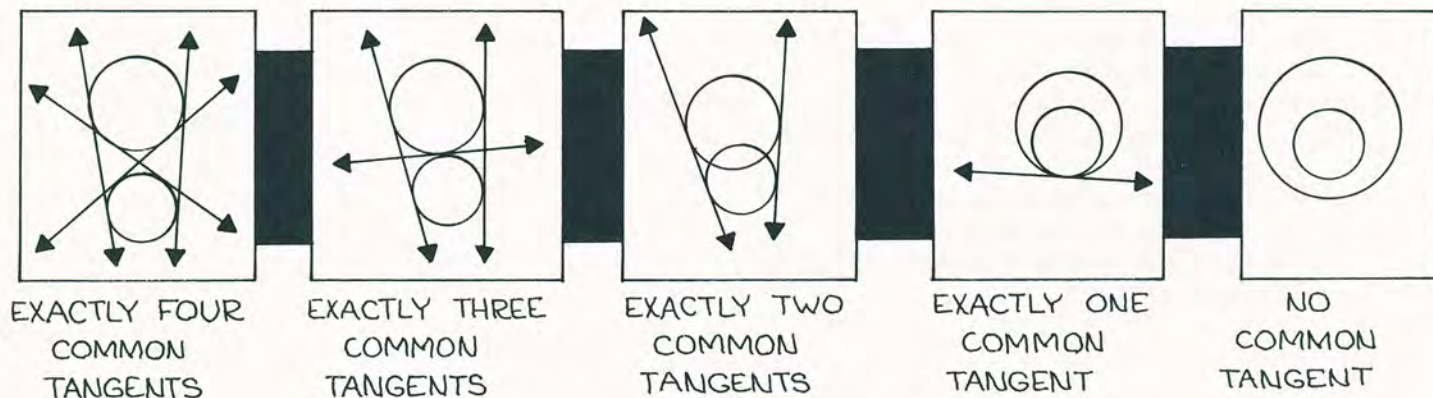What shape do students see?

> All lines drawn are called <u>tangents</u> to the circle they form. All touch the circle but none go inside it.

How many tangents are there to one circle?

II Have students explore all the possible arrangements for two circles of different size. Templates can be used to sketch the arrangements, or the outlines of two coins can be traced.

NO COMMON POINTS   ONE COMMON POINT   TWO COMMON POINTS

III For each arrangment have students discover the maximum number of <u>common</u> tangents for both circles. Suggest that they slide their ruler around and try to position it so that one edge touches both circles at the same time.

EXACTLY FOUR COMMON TANGENTS   EXACTLY THREE COMMON TANGENTS   EXACTLY TWO COMMON TANGENTS   EXACTLY ONE COMMON TANGENT   NO COMMON TANGENT

encompassing
circles

1) Have students:

   a) draw a line and label it AB.
   b) use a compass or template to draw 10 congruent circles tangent to line AB.
   c) connect the centers of those circles with straight lines whenever it is possible to do so without crossing line AB.

   How does the line(s) relate to line AB?

2) Have students:

   a) draw a line and label it CD.
   b) use a compass or template to draw several different size circles tangent to the line at point C.
   c) connect the centers of all these circles with a straight line.

   How does the line relate to line CD?

3) Have students:

   a) use a compass to draw a circle.
   b) now use the compass or template to draw 10 congruent circles tangent to different points of the original circle.
   c) connect the centers of these 10 circles with another circle. (Use the compass.)

   How does this circle relate to the original circle?

4) Have students:

   a) use a compass to draw a circle.
   b) label its center 0.
   c) use the compass or template to draw 5 congruent circles that intersect at point 0.
   d) use the compass to connect the centers of these 5 circles.

   How does this circle relate to the original circle?

THE CIRCLEBOARD
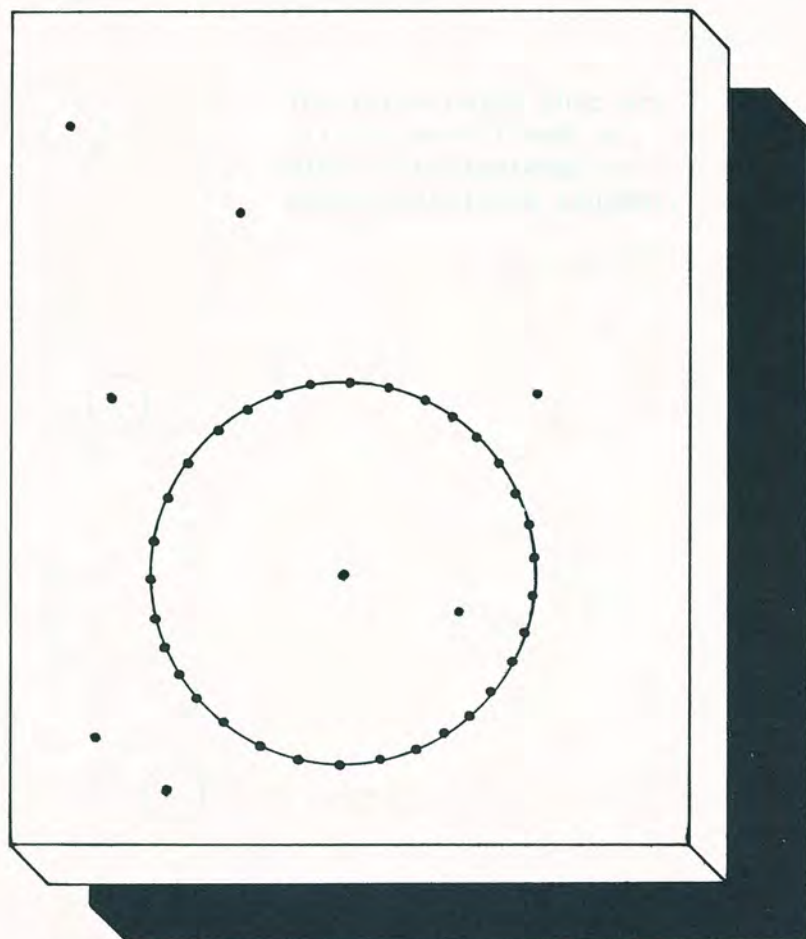
The circleboard can aid students to see properties of and to discover relationships about the circle. Use rubber bands to represent line segments. Students can measure lengths with a ruler and angles with a protractor. A large model can be effectively used for teacher demonstrations.

Use a piece of plywood and draw a circle of appropriate size.

Equally space small nails or brads around the circumference. Put other nails at the center of the circle and at the other places on the board to show the line segments and angles related to the circle.

RADIUS   DIAMETER   CHORD   TANGENT   SECANT

CENTRAL ANGLE   INSCRIBED ANGLE   ARC   SECTOR
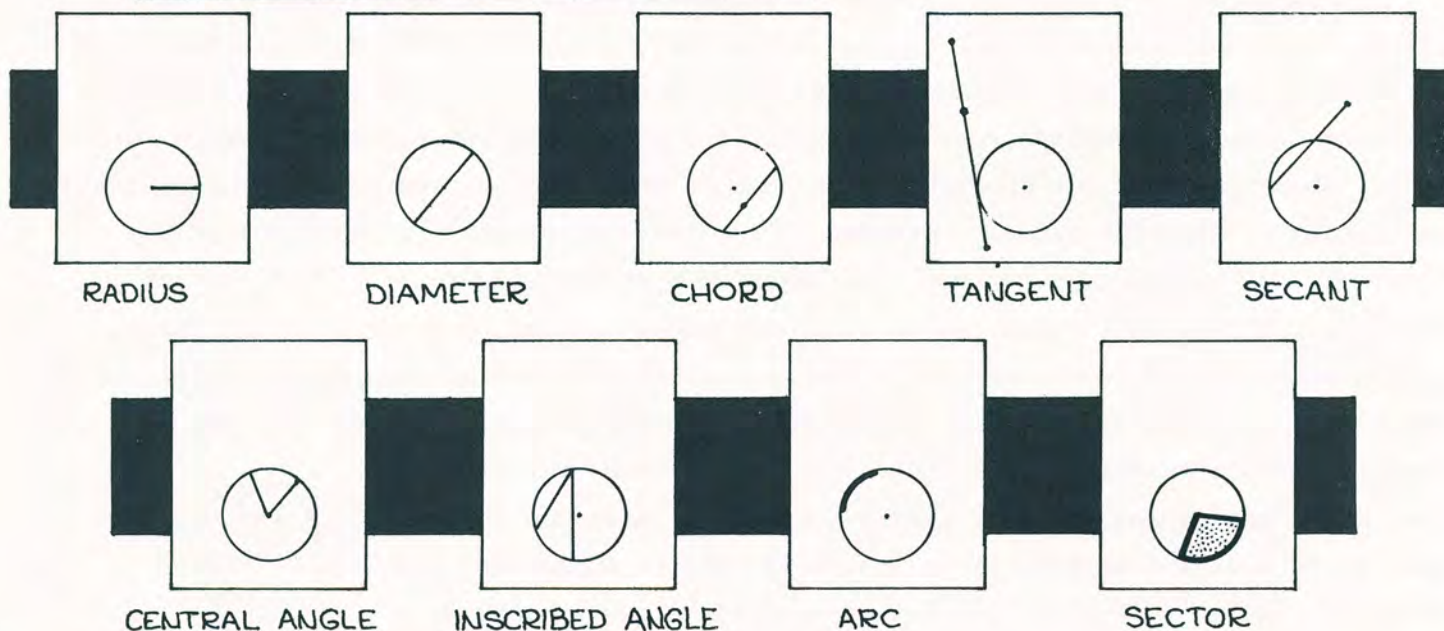
A CIRCULAR GEOBOARD

The pattern to the right can be used to make dot paper and to make a circular geoboard. It has 24 dots in the large circle and 12 dots in the small circle. The number 24 was chosen to allow central angles of 15° and inscribed equilateral triangles, squares, regular hexagons and regular octagons. A 20-dot pattern would allow regular pentagons, five-pointed stars and regular decagons to be inscribed. The 12-dot circle can be used as a model for a clock.

For the dot paper use the pattern of the circular geoboard four times on a ditto master.
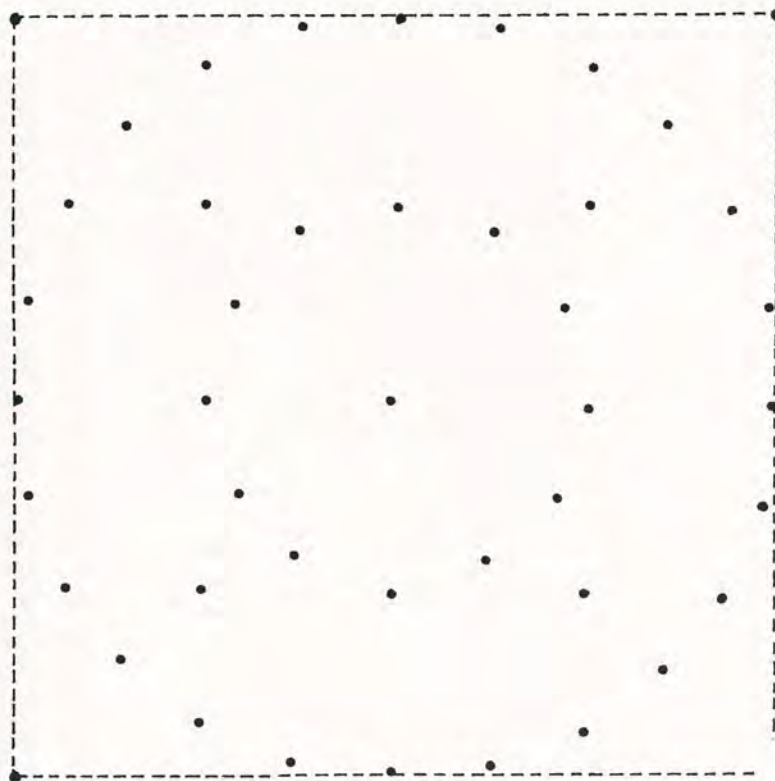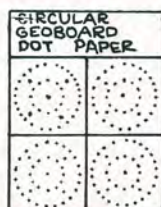
To construct the circular geoboard enlarge the pattern so each length is twice as long. The inner circle will then have a diameter of 10 cm and the outer circle a diameter of 20 cm. The geoboard can be made on a 25 cm by 25 cm piece of 1/2" plywood or similar material. Perhaps your industrial arts department can cut these pieces for you. The upper face can be painted, or contact paper can be used to make the geoboard attractive. Tape the enlarged pattern to the piece of plywood and mark the points where the nails are to be driven. The best type of nail to use is a 3/4" (or 5/8") brass escutcheon nail which can be purchased at most local hardware stores. These nails have rounded heads which hold the rubber bands on and have no sharp edges. After each nail has been started, an ordinary nut, 3/8" high, can be placed over the nail to assure that each nail will be driven into the plywood the same amount. Because the circular geoboard has more nails and the nails are closer together, it is more difficult to construct than the square geoboard. You may want to use older students, math aides or parent volunteers to help with the construction.

ARE YOU RIGHT ALL THE TIME?

Several geometric ideas can be developed using the circular geoboard. A partial list is given below. Can you add to it? A sample activity card is shown that could guide a student through a discovery. For more information about the terms on this page see the Geo Glossary.

A) The number of degrees in an inscribed angle is equal to one-half the number of degrees in the central angle that intercepts the same arc. $(m \angle 2 = \frac{1}{2} m \angle 1)$

B) Inscribed angles intercepting the same arc have the same measure. $(m \angle 2 = m \angle 3)$

C) If two lines intersect, the vertical angles are congruent.

D) The sum of the measures of the interior angles of a quadrilateral is 360°.

E) The opposite angles of a parallelogram are congruent.

F) The number of diagonals in a n-sided polygon is $\frac{n(n-3)}{2}$.
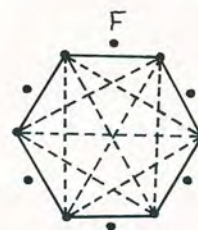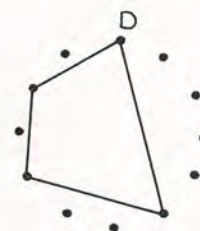
Materials Needed: Circular geoboard, rubber bands, right angle tester

Activity:

1) Make a diameter of the small circle.

2) Attach a rubber band to one end point of the diameter. Stretch it around the next nail on the small circle and then attach it to the other end point of the diameter.

3) Use the right angle tester to determine the type of angle formed at the nail not on the diameter (acute, right, obtuse).

4) Repeat with each nail around the circle.

5) Make a diameter of the large circle and repeat parts 2, 3, and 4.

6) What did you discover?
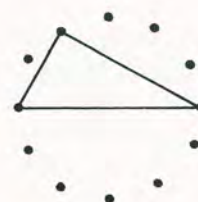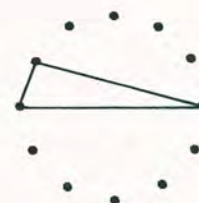
POLYGONS ON A CIRCULAR GEOBOARD

Materials Needed: Circular geoboard, rubber bands

Activity: Use rubber bands for these exercises. You can stretch one rubber band for each polygon or use one rubber band for each side.

1) Which of these polygons can be inscribed in the large circle?

   a)   square
   b)   regular octagon
   c)   scalene triangle
   d)   regular pentagon
   e)   pentagon
   f)   isosceles triangle
   g)   equilateral triangle
   h)   regular hexagon

2) Are there any of the above polygons that cannot be inscribed in the small circle?

3) What other polygons can you inscribe in either of the circles?

Challenge: Use these clues and make these three mystery polygons on your circular geoboard.

Mystery Polygon 1

a)   It has less than four sides.
b)   It is not a scalene triangle.
c)   Its sides are not all the same length.
d)   Two of its sides are along radii of the large circle.
e)   There are six nails on its sides.
f)   There are two nails inside it.

Mystery Polygon 2

a)   It has less than five sides.
b)   It is not convex.
c)   Two of its sides are radii of the large circle.
d)   The two radii form a right angle.
e)   There are six nails on its sides.
f)   There are thirty-eight nails outside it.
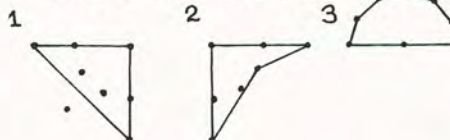
Mystery Polygon 3

a)   It touches eight nails.
b)   One side is a diameter of the small circle.
c)   It is convex.
d)   There are no nails inside it.
e)   The longest side is parallel to the bottom edge of the geoboard.
f)   Flipping it over its longest side is the same as turning it half way around the center nail.

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RIFFLE BOOKS

Motion pictures or teacher demonstrations showing motion often help students see geometric properties. The riffle book is another aid which gives the illusion of motion. By drawing a series of pictures the student can dynamically illustrate many geometric relationships.

Making the Book

1) Each book contains at least twenty small cards.

    3" X 5" index cards work well.

2) The riffle book described illustrates the property that the right angle vertices of all right triangles with the same hypotenuse lie on a semicircle.

    Begin by drawing a line segment (the diameter of a circle) on each card in the same position. (An easy way to locate the same positon is to stack the cards and prick the end points of the line segment with a compass point.) The semicircle should be lightly drawn in.

3) As illustrated, on the first card heavily mark a point for the vertex of the right angle on the semicircle 10° above the diameter. Lightly draw in the sides of the triangle. Mark the right angle.

CARD #1

4) As illustrated, continue this process every 10°.

CARD #2   CARD #3   CARD #4   CARD #5   LAST CARD
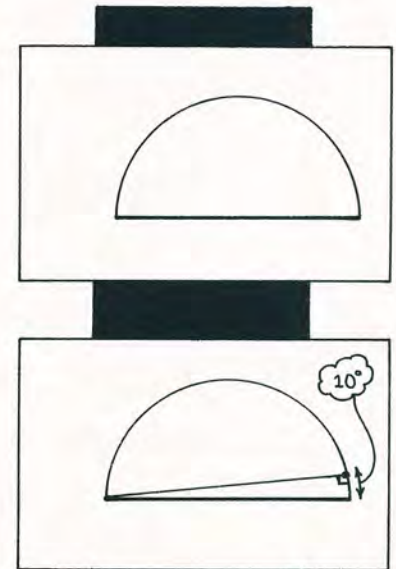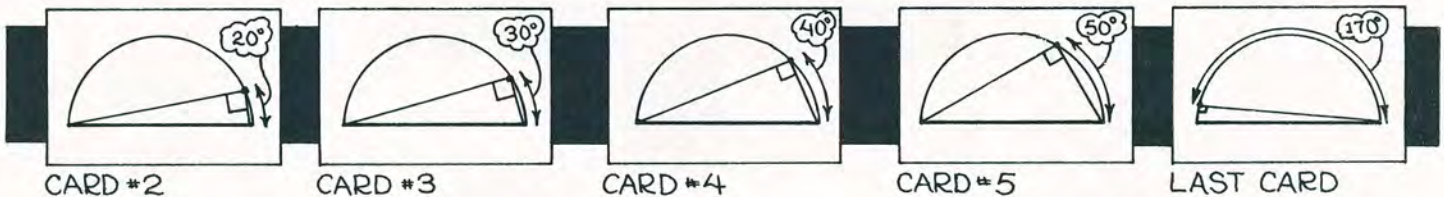
5) Clip all cards in order with one clip or fastener in the upper left corner. Increasing the number of cards makes the motion smoother. You may wish to double the number of cards by placing the vertex at 5° intervals instead of 10°.

6) Riffle books could be used for these properties:

    a) The set of points equidistant from the end points of a line segment is the perpendicular bisector of the segment.

    b) The set of points equidistant from the sides of an angle is the bisector of the angle.

YOU POP THE QUESTION

FOR EACH DIAGRAM BELOW WRITE A QUESTION WHICH YOU THINK COULD BE SOLVED WITH THE INFORMATION GIVEN ON THE DRAWING. YOU MAY LABEL ADDITIONAL PARTS OF THE DRAWING. POINT O INDICATES THE CENTER OF THE CIRCLE.
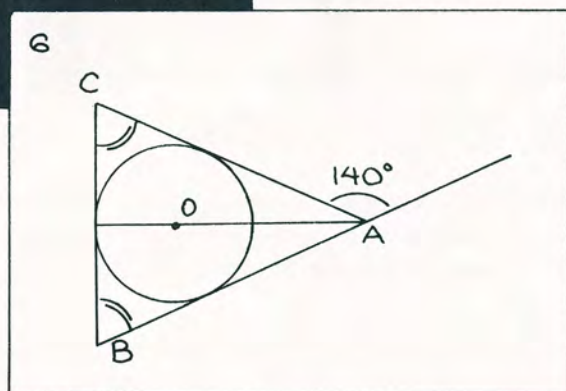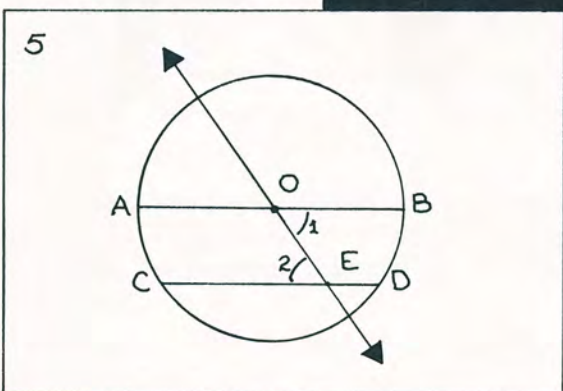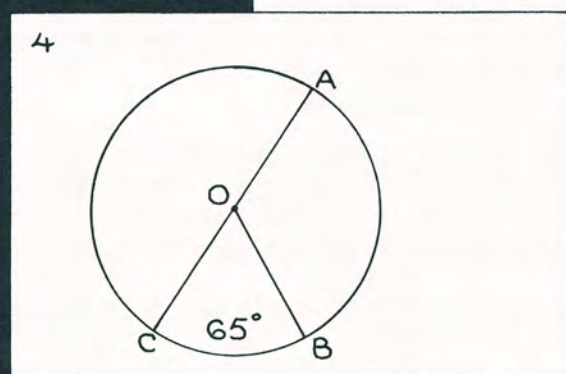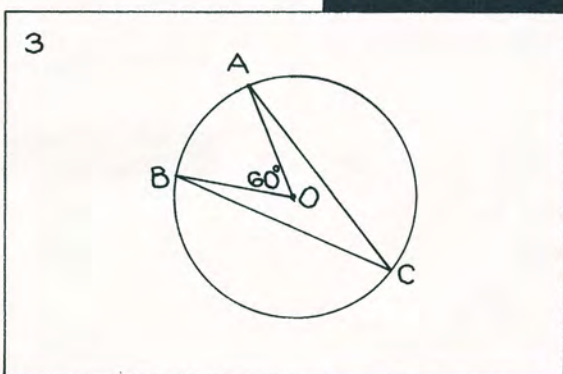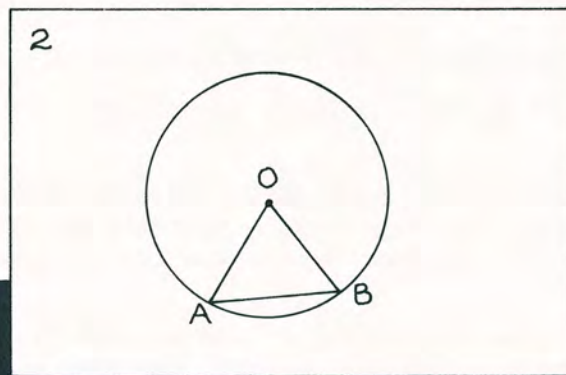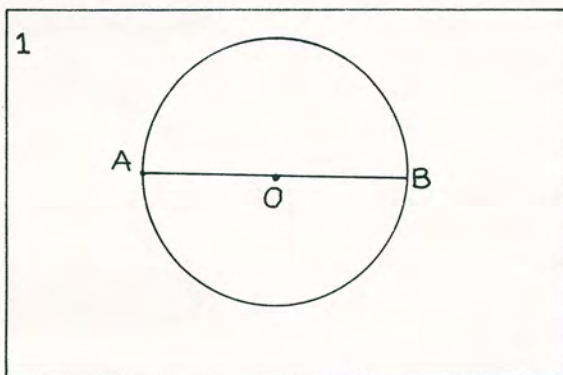
FOLDING
POLYGONS IN
CIRCLES

Each student will need two large circular regions (radius of 5 - 8 centimetres) with the centers marked.

A) The square and the regular octagon:

1) Fold the circle over onto itself and crease. This forms a diameter.

2) Fold again so the crease lies completely on itself. This forms another diameter perpendicular to the first one.

3) Fold again so the two creases lie on top of each other.

4) Unfold and eight points will show on the circumference of the circle. Connect adjacent points to make a regular octagon. Connect every other point to make a square.

B) The equilateral triangle and the regular hexagon:

1) Fold any part of the circular region over so the circle meets the center, and crease.

2) Unfold. The crease line has made two points on the circle. Fold one of these over to meet the center of the circle.

3) Unfold and repeat step 2 until you can see two overlapping triangles.

4) Connect adjacent points to make a regular hexagon. Connect every other point to make an equilateral triangle.

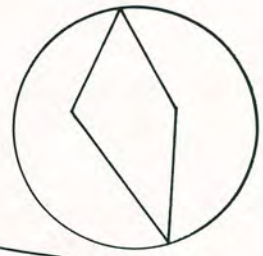5) Fold on every other crease to change the circular region into an equilateral triangle region.
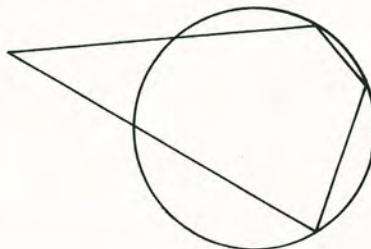
517
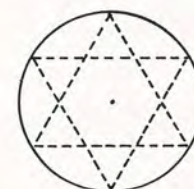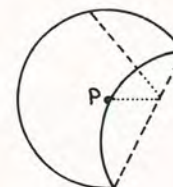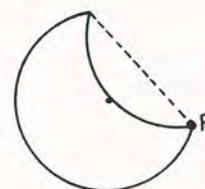
SOME CAN, SOME CAN'T

SOME POLYGONS CAN BE INSCRIBED IN CIRCLES:
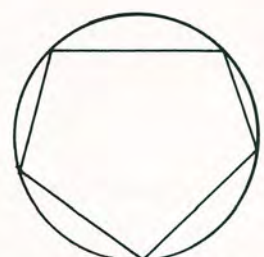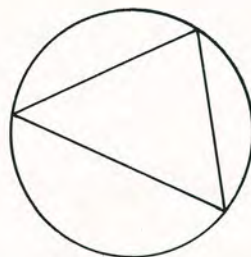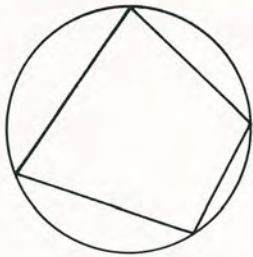
SOME POLYGONS CANNOT BE INSCRIBED IN CIRCLES:
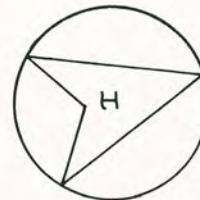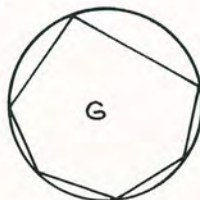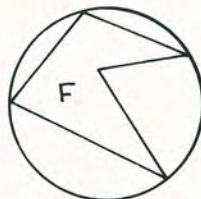
NOTICE: THIS VERTEX IS NOT TOUCHING THE CIRCLE.

WHICH OF THE POLYGONS SHOWN BELOW ARE NOT INSCRIBED IN THESE CIRCLES?

_____

A B C D

E F G H I
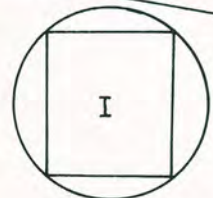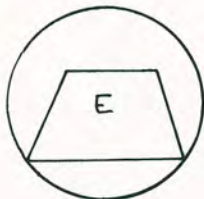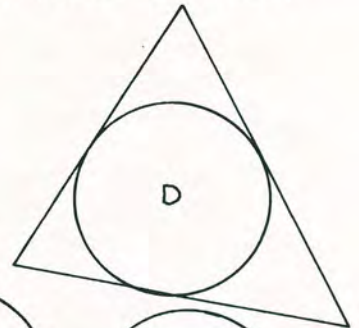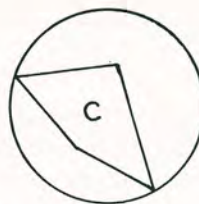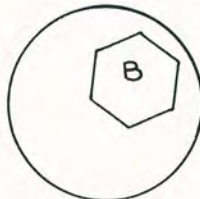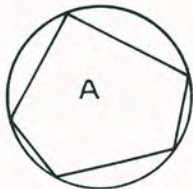
WHICH OF THE POLYGONS DO YOU THINK COULD NOT BE INSCRIBED IN ANY CIRCLE?

_____

INSIDE THE CIRCLE (I)

Materials: Compass, metric ruler, protractor

Activity:

1) Draw a circle with a radius of 5 centimetres.

2) Use the same compass opening to mark 6 points on the circle.

3) Connect the points in order using a ruler.

4) What type of polygon have you drawn? _____
   _____
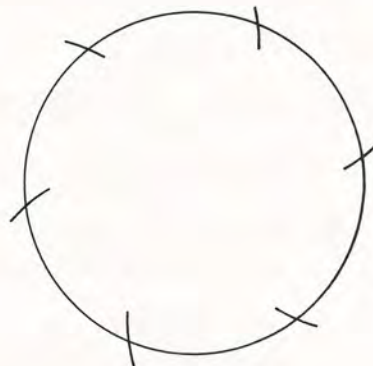
   a) Measure the length of each side of the polygon. What do you notice? _____
      _____

   b) Use the protractor to measure each angle of the polygon. What do you notice?
      _____

> You have drawn a regular hexagon. Since each vertex of the hexagon lies on the circle, the hexagon is <u>inscribed</u> in the circle.

5) Draw another circle the same size and mark it the same way. Connect every other point.

   a) What type of polygon have you inscribed? _____
   b) Measure the sides and angles of this polygon. What do you notice?
      _____

6) Draw and mark another circle. Inscribe a regular polygon with twelve sides. (Hint: Construct the perpendicular bisector of a side of a regular hexagon inscribed in the circle.)

7) Could you inscribe a regular polygon of twenty-four sides? _____

8) What other regular polygons could you inscribe? _____

INSIDE THE CIRCLE

Materials Needed: Compass, metric ruler, protractor

Activity:

1) Draw a circle with a radius of 5 centimetres.

2) Lightly draw a diameter of the circle.

3) Lightly construct another diameter that is perpendicular to the other one.

4) The diameters meet the circle in four points. Connect these points in order.

5) What type of polygon have you drawn? _____ _____

    a) Measure the length of each side of the polygon. What do you notice? _____ _____

    b) Measure each angle of the polygon. What do you notice? _____ _____

> Since the vertices are all on the circle, you have inscribed a square in the circle.

    c) The two diameters are called the _____ of the square.

6) Draw another circle the same size. Inscribe a regular octagon in the circle. (Hint: Bisect the angles formed by the intersecting diameters.)

7) Could you inscribe a regular polygon with sixteen sides in a circle? _____

8) What other regular polygons could you inscribe? _____ _____
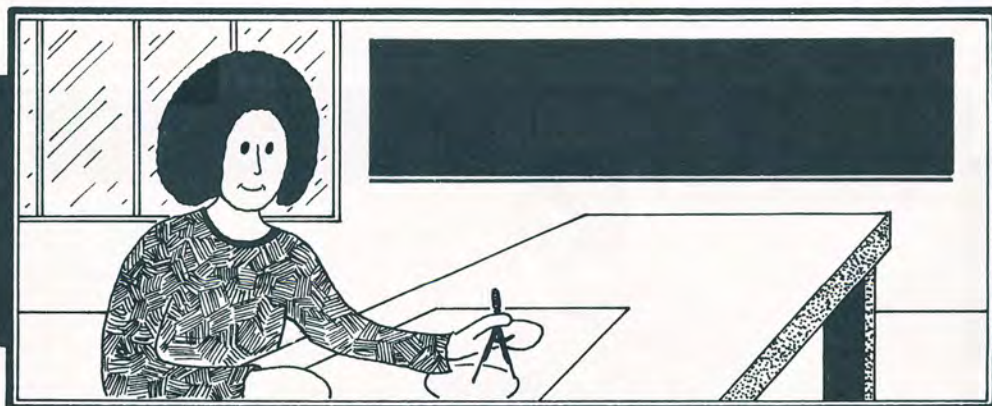
INSIDE THE CIRCLE

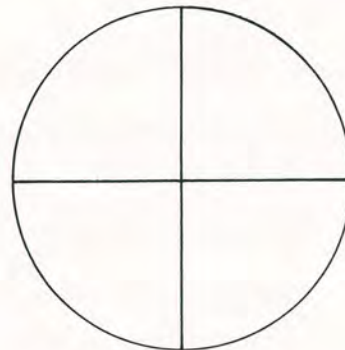Since this construction requires several steps, use great care in each step.

Materials Needed: Compass, metric ruler, protractor

Activity:

1) Draw a circle with a radius of 5 centimetres.

2) Accurately construct two diameters perpendicular to each other.

3) Label the drawing like the figure at the right.

4) Bisect radius OC. Label this point E.

5) Set your compass for the distance between E and B.

6) Put the point of the compass at E and mark point F. (Be careful. F looks like the midpoint of $\overline{AO}$ but it's not.)

7) Set your compass for the distance between F and B. Start at D and mark this distance off around the circle.

8) Connect the points on the circle.

9) What type of polygon have you drawn? _____ Measure to see if it is regular.

10) What shape do you get if you connect every other point? _____

11) Draw another circle. Inscribe a regular decagon in the circle. (Hint: Construct a perpendicular bisector of a side of the polygon you got in question 9.)

12) What other regular polygons could you inscribe using this method? _____
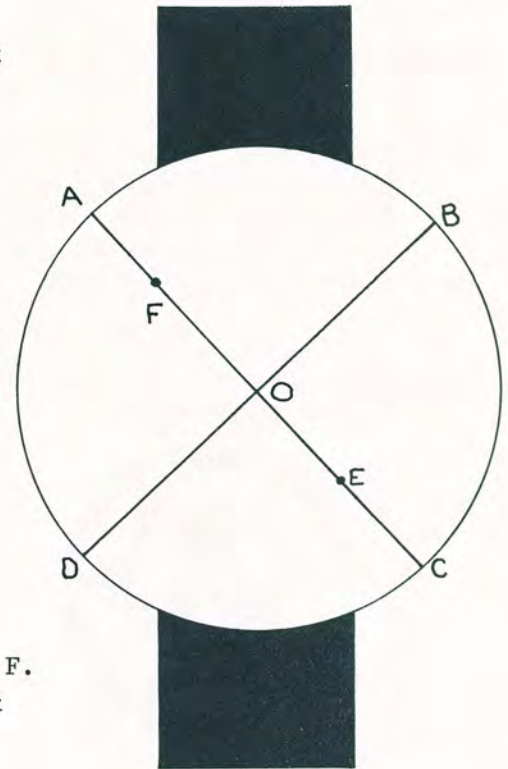
_____

ART INSIDE THE CIRCLE

Materials Needed: Compass, straightedge, colored pens or crayons

1) Have your students inscribe a regular
    hexagon and the two equilateral
    triangles, then draw the diagonals
    and other segments to get the pattern
    shown at the right.

2) Have them discover how the designs
    below were made.

    a)   b)   c)
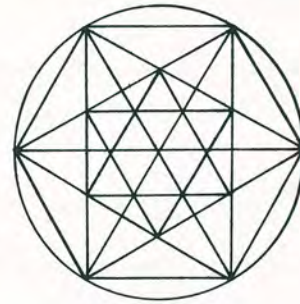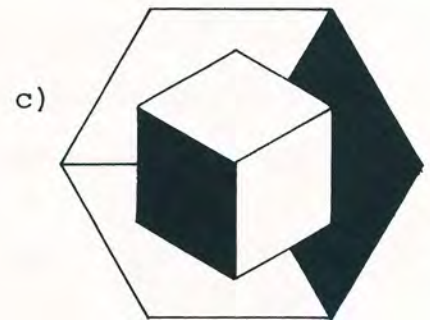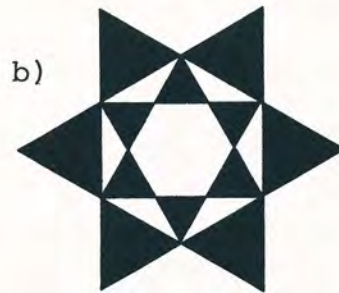
3) Have them create many designs of their own.

4) Students can also use the inscribed square and inscribed octagon
    as a basis for creating designs. Another appealing pattern for
    students is the one created using the inscribed pentagon and
    inscribed decagon. Sample patterns are shown below. Creative
    Constructions by Dale Seymour and Reuben Schadler, published by
    Creative Publications, is a source for many designs using similar
    patterns. Of course, your students will be the richest source of
    designs. An attractive bulletin board can be made from their
    designs.

IDEA FROM: *Creative Constructions*

Permission to use granted by Creative Publications, Inc.
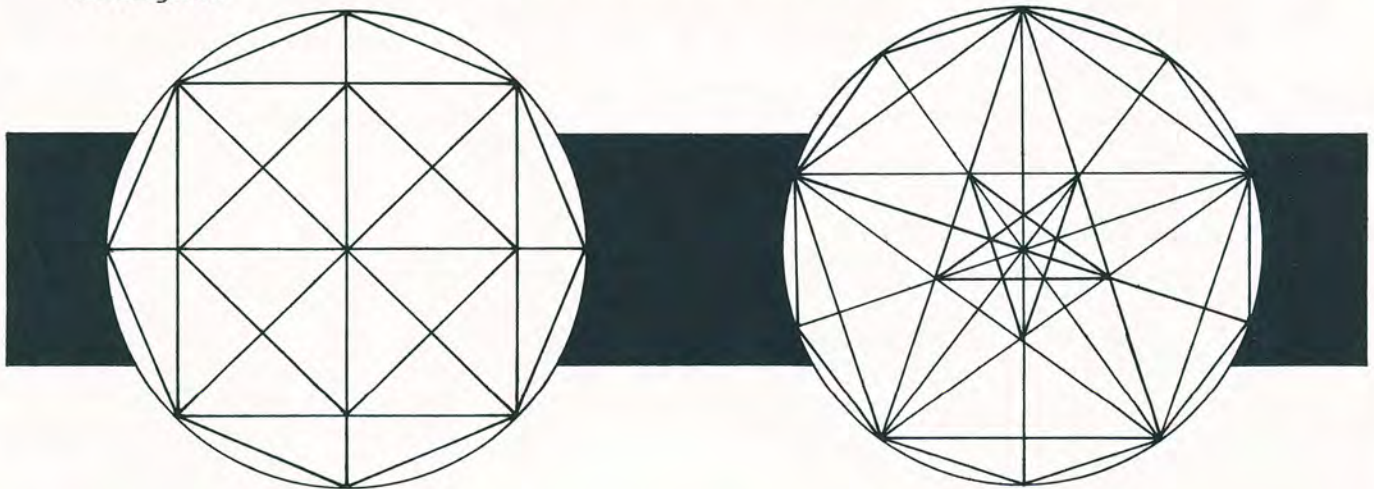
EASTER EGGS

TO CONSTRUCT AN EASTER EGG:

1) Draw a circle. Then
   draw diameter AB.

2) Construct the perpendicular bisector
   of segment AB. Label it CD as shown
   in Figure 1 and be sure it extends
   beyond the circle.

FIGURE 1

3) Make the compass
   opening equal to AB.
   With A as the center
   draw arc BE. With B
   as the center draw
   arc AE.
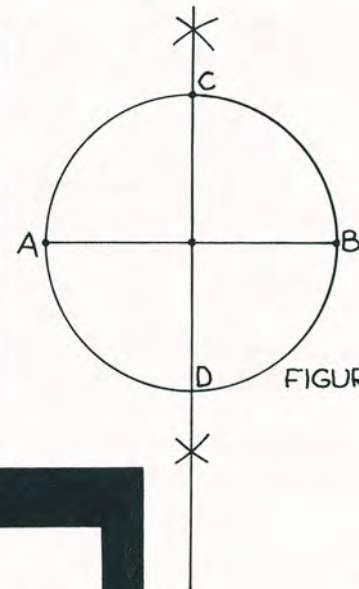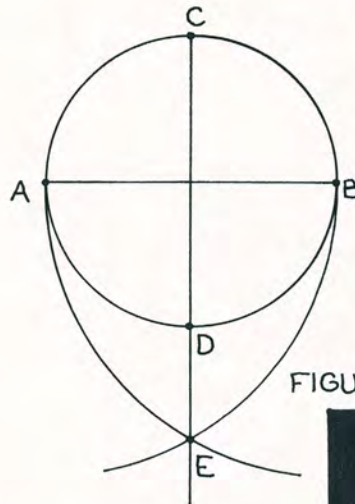
FIGURE 2

4) Draw line segment AF through point D.

5) Make the compass opening equal to DF.
   With D as the center draw arc GF.

6) Decorate your egg.

ARC
GF

FIGURE 3

CIRCLE ART
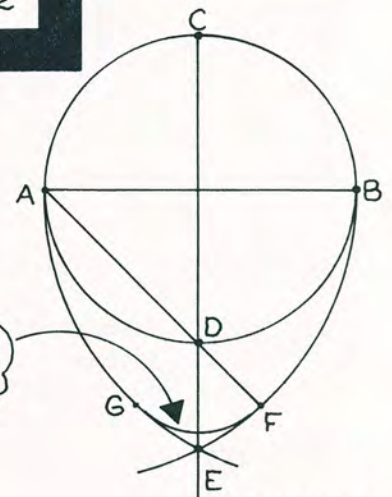
Each figure below is made with sets of circles. Your students might enjoy trying to draw some of them.

1) Draw a line segment and its perpendicular bisector.

2) Draw circles

   a) with centers on the perpendicular bisector

and b) that pass through the end points of the line segment.
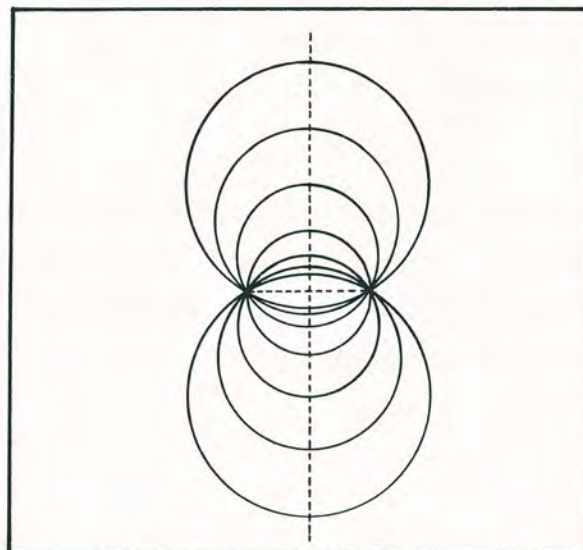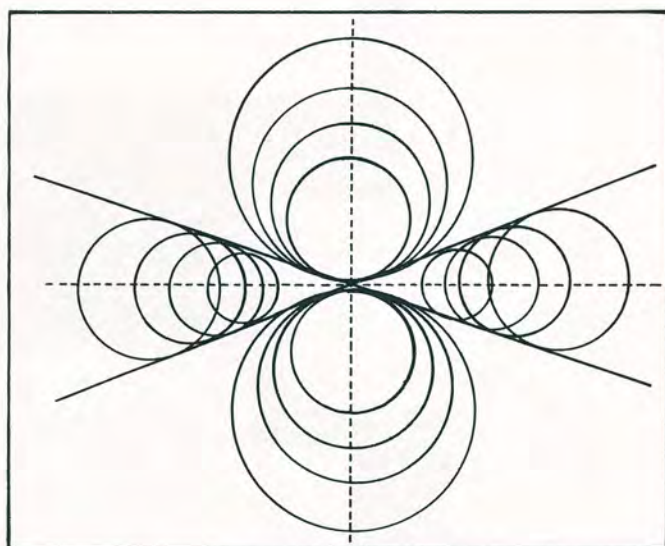
1) Draw two intersecting lines.

2) Draw the angle bisectors of each pair of vertical angles.

3) Draw circles

   a) with centers on an angle bisector

and b) that are tangent to both sides of an angle.

1) Draw a circle.

2) From the center of the circle draw three rays that divide the circle into congruent arcs.

3) Draw other circles

   a) with centers on the original circle

and b) that are tangent to the nearest ray.

CIRCLE ART

(CONTINUED)

1) Draw a circle.

2) Mark a point on the circle.

3) Draw other circles

    a) with centers on the original circle

and b) that pass through the point on the circle.

1) Draw a circle.

2) Mark a point outside the circle.

3) Draw other circles

    a) with centers on the original circle

and b) that pass through the point outside the circle.
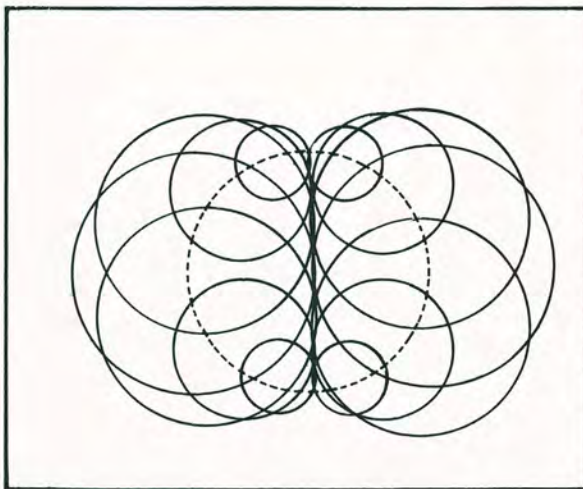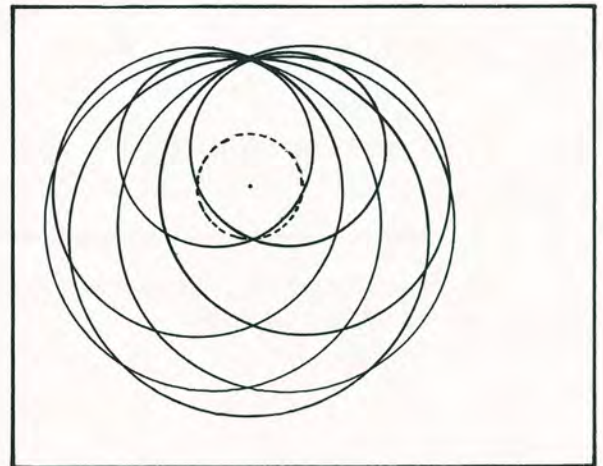
1) Draw a circle and a diameter.

2) Draw other circles

    a) with centers on the original circle

and b) that are tangent to the diameter.

IDEA FROM: *Circles,* Topics from Mathematics
Permission to use granted by Cambridge University Press

CHORDS IN CIRCLES

The chords in these circles are marked in small units. Fill in the blanks for each exercise and try to discover a relationship.

1)

2)

3)
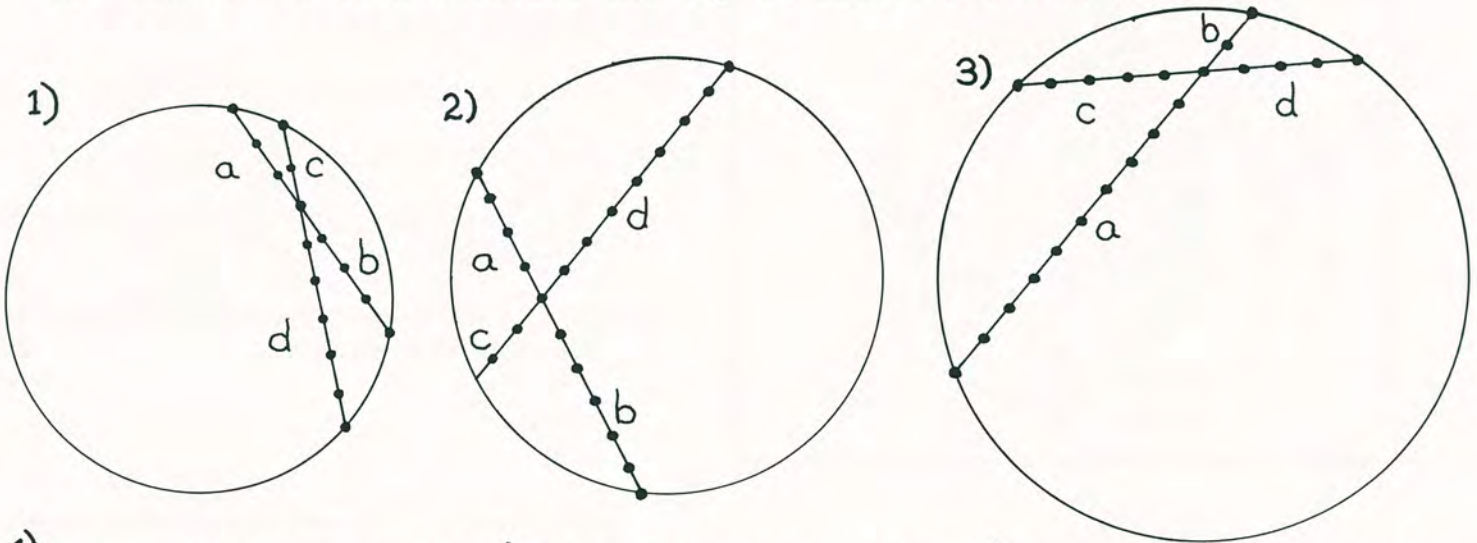
1) a = ___ units, b = ___ units   **2)** a = ___ units, b = ___ units   **3)** a = ___ units, b = ___ units
    c = ___ units, d = ___ units   c = ___ units, d = ___ units   c = ___ units, d = ___ units

What relationship did you find? _____

_____

DID YOU TRY PRODUCTS?

─────────────────────────────────

Measure these intersecting chords to the nearest half centimetre to gather more evidence.

4)

5)

4) a = _____, b = _____
    c = _____, d = _____

5) a = _____, b = _____
    c = _____, d = _____
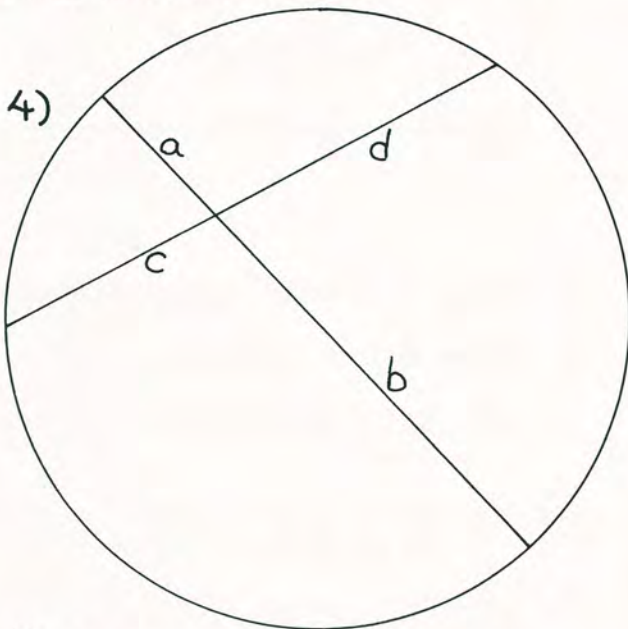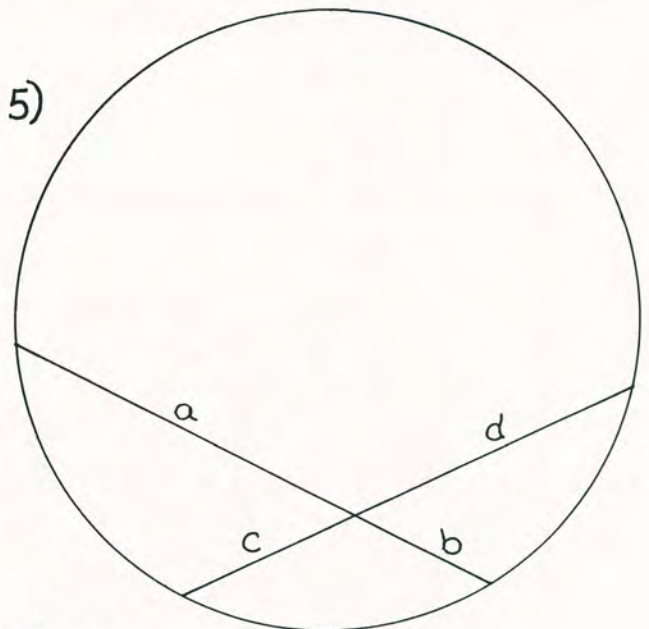
ARCS, NOT ARKS

In a circle there are three kinds of arcs:

Semicircle XYZ. $\overset{\frown}{XYZ}$ is half of a circle.

Minor arc XY. $\overset{\frown}{XY}$ is less than a semicircle.

Major arc YXZ. $\overset{\frown}{YXZ}$ is more than a semicircle.

CENTER C
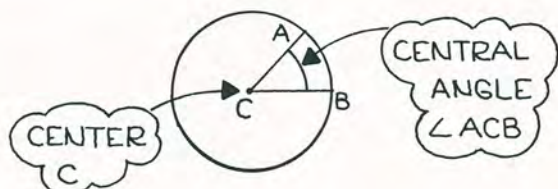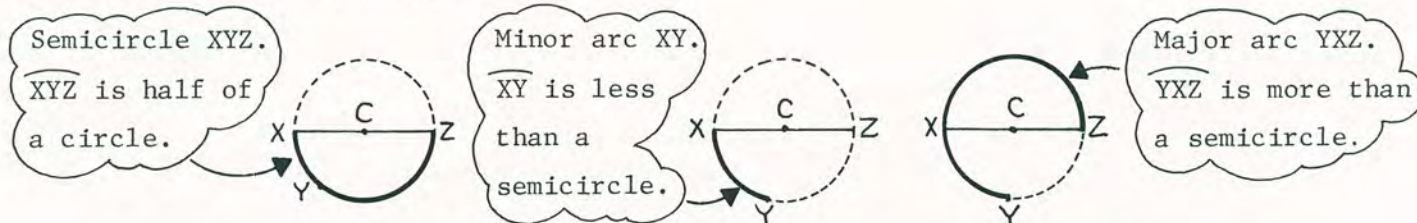
CENTRAL ANGLE $\angle$ACB

The degree measure of $\overset{\frown}{AB}$ = measure of $\angle$ACB.

1) Use the circle to the right in answering these questions.

a) Name two central angles. _____, _____

b) Name two minor arcs. _____, _____

c) Name one major arc. _____

d) Name two semicircles. _____, _____

e) What angle would you measure to find the degree measure of $\overset{\frown}{AB}$? _____

f) Use your protractor to find the degree measures. $\overset{\frown}{BX}$: _____°; $\overset{\frown}{XY}$: _____°; $\overset{\frown}{YX}$: _____°

g) If you know the degree measures of $\overset{\frown}{BX}$ and $\overset{\frown}{XY}$, how can you find the degree measure of $\overset{\frown}{BY}$? _____
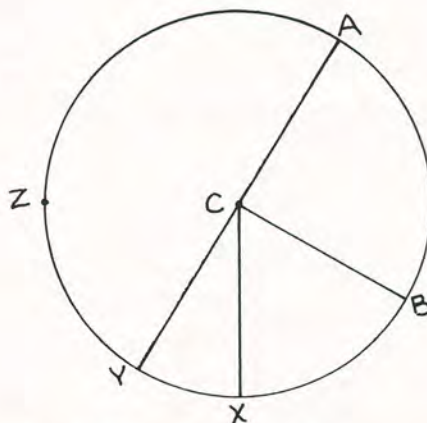
2) Find the degree measure of each arc or angle listed.

$\overline{AB}$ IS A DIAMETER
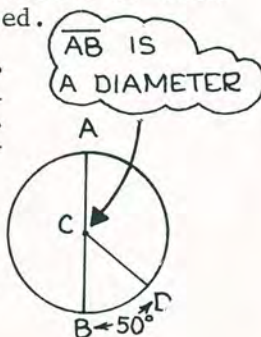
You don't need a protractor for questions 2, 3 and 4.

a) $\angle$BCD: _____°

b) $\angle$ACD: _____°

c) $\overset{\frown}{AD}$: _____°

d) $\overset{\frown}{ADB}$: _____°

e) $\overset{\frown}{BAD}$: _____°

B←50°

3) Find the degree measure of each arc or angle listed.

CENTER

a) $\angle$ACB: _____°.

b) $\overset{\frown}{BD}$: _____°

c) $\overset{\frown}{AD}$: _____°

   $\overset{\frown}{AFD}$: _____°

e) $\overset{\frown}{BFA}$: _____°

f) $\overset{\frown}{DFB}$: _____°

40°
60°
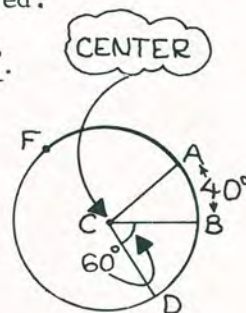
4) Suppose a circle is divided into equal arcs. Find the degree measure of each arc if there are:

a) Three arcs _____   b) Four arcs _____   c) Eighteen arcs _____

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A PAIR OF ANGLES
IN THE ARC

INSCRIBED ∠R CUTS OFF ⌢ST.

For the circles below use your protractor to find the degree measure of the angles and arcs listed. C is the center of each circle.

1)

∠SRT: ___°

⌢ST: ___°
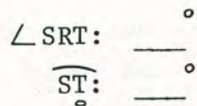
FIND THE MEASURE OF ∠SCT.

2)

∠BAD: ___°

⌢BD: ___°

3)

∠LMN: ___°

⌢LKN: ___°

4)

∠UVW: ___°

⌢UPW: ___°
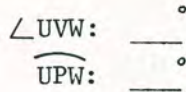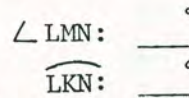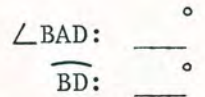
BE SURE TO MEASURE THE MAJOR ARC.

5)

∠SRT: ___°

⌢ST: ___°

How does the measure of each inscribed angle seem to compare to the degree measure of the arc it cuts off? _____

Construct several more examples like these to check your answer.
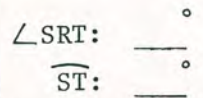
ARCS

1) **NO PROTRACTORS NEEDED FOR THIS ACTIVITY.**
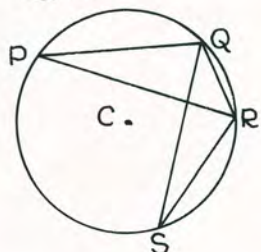
Name the arc cut off by each inscribed angle.

a) ∠P: _____   d) ∠SRP: _____

b) ∠S: _____   e) ∠PQR: $\overarc{PSR}$

c) ∠PQS: _____   f) ∠SRQ: _____

BE SURE TO USE THREE LETTERS IF THE ARC IS A MAJOR ARC.
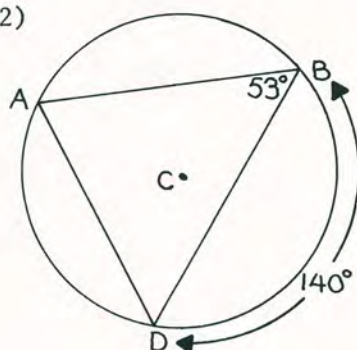
2) m∠B = 53° and m $\overarc{BD}$ = 140°.

Find:

m∠A = _____

m $\overarc{AD}$ = _____

m∠D = _____

m $\overarc{AB}$ = _____
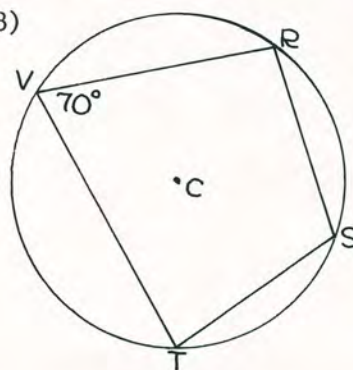
3) m∠V = 70°

Find:

m $\overarc{RST}$ = _____

m $\overarc{RVT}$ = _____

m∠RST = _____
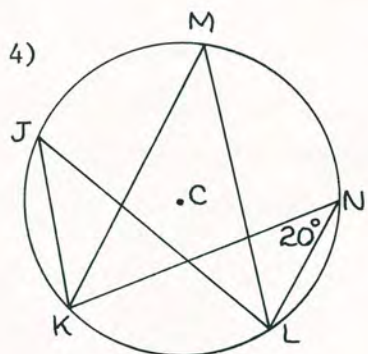
4) m∠N = 20°

Find:

m $\overarc{KL}$ = _____

m∠J = _____

m∠M = _____
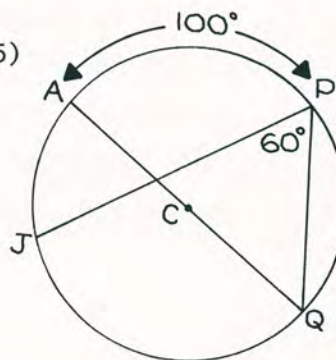
5) $\overline{AQ}$ is a diameter.

m $\overarc{AP}$ = 100°

m∠P = 60°

Find:

m $\overarc{JQ}$ = _____

m∠Q = _____

m $\overarc{PQ}$ = _____

6) Extra for Experts: C is the center of the circle

m∠HJK = 75°, m $\overarc{KM}$ = 60°

Find: m $\overarc{KMH}$ = _____

m $\overarc{HM}$ = _____

m∠HJM = _____

m∠MJK = _____

m $\overarc{JK}$ = _____

m $\overarc{JH}$ = _____
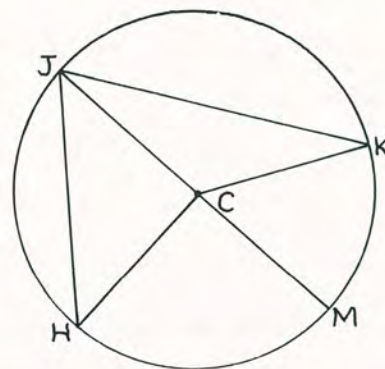
FINDING THE RIGHT ANGLE

I   Look at the diagram to the right. Use a straightedge to draw each line segment given below.

Measure the angles to the nearest degree with a protractor.

a)   Draw $\overline{XA}$ and $\overline{XB}$.   m∠AXB = _____°

b)   Draw $\overline{YA}$ and $\overline{YB}$.   m∠AYB = _____°

c)   Draw $\overline{ZA}$ and $\overline{ZB}$.   m∠AZB = _____°

d)   Draw $\overline{TA}$ and $\overline{TB}$.   m∠ATB = _____°

e)   Draw $\overline{MA}$ and $\overline{MB}$.   Guess: m∠AMB = _____°

Mark a point on the circle and label it R.

Draw $\overline{RA}$ and $\overline{RB}$.   m∠ARB = _____°

AB IS A DIAMETER OF THE CIRCLE.
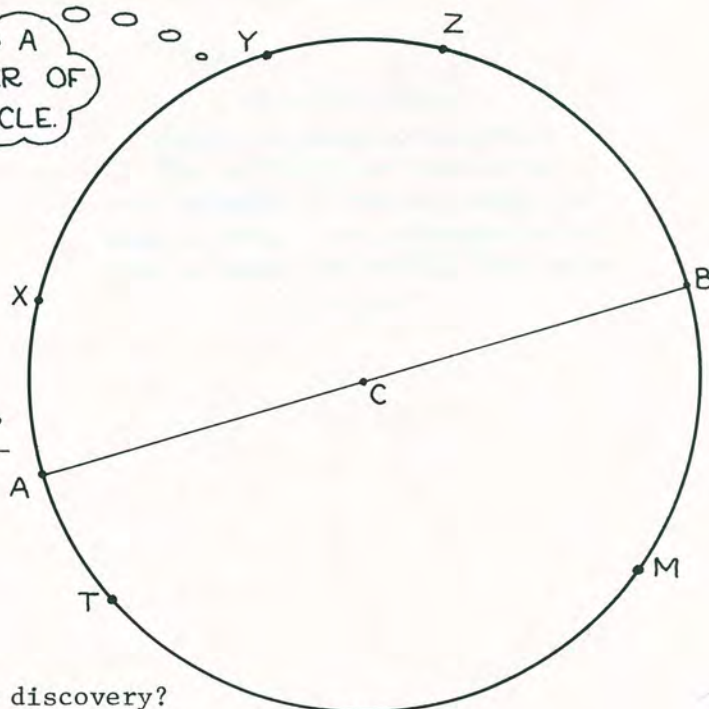
FIGURE 1

Which of the statements below describe your discovery?

CHECK THOSE YOU AGREE WITH.

_____   1)   m∠AXB is greater than m∠AMB.

_____   2)   m∠AYB is half of m$\overset{\frown}{ATB}$.

_____   3)   The arc cut off by ∠AZB is the same as the arc cut off by ∠AXB.

_____   4)   Each inscribed angle in the figure measures 90°.

_____   5)   An angle inscribed in a semicircle is a right angle.

II In the circle above (figure 1) measure these angles:

a)   m∠XAB = _____°   m∠XBA = _____°

b)   m∠YAB = _____°   m∠YBA = _____°

c)   m∠MAB = _____°   m∠MBA = _____°

What is special about each pair of angles?

_____

CIRCUMSCRIBING
A TRIANGLE

I)

1) Locate 3 points on the circle. Label them in order P, Q, R.

2) Draw the chords $\overline{PQ}$, $\overline{QR}$ and $\overline{RP}$.

3) Draw the perpendicular bisector of each chord.

4) Where do all three perpendicular bisectors meet? _____

C IS THE CENTER OF THE CIRCLE.

.C

II)
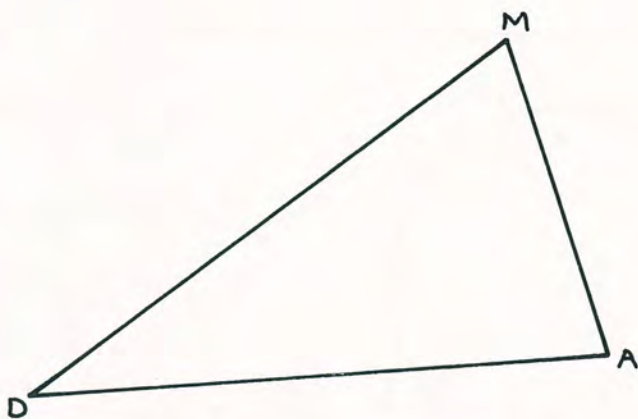
1) The circle through M, A and D got erased. Use steps (3) and (4) above to help you re-draw the circle.

The circle is circumscribed about the triangle.

2) Draw your own triangle. Find the circle that circumscribes it.

3) Challenge: Try to draw three points that cannot lie on the same circle.

M

A

D

SPECIAL QUADRILATERALS

I) Use your protractor to measure the angles in each inscribed quad-
    rilateral below.
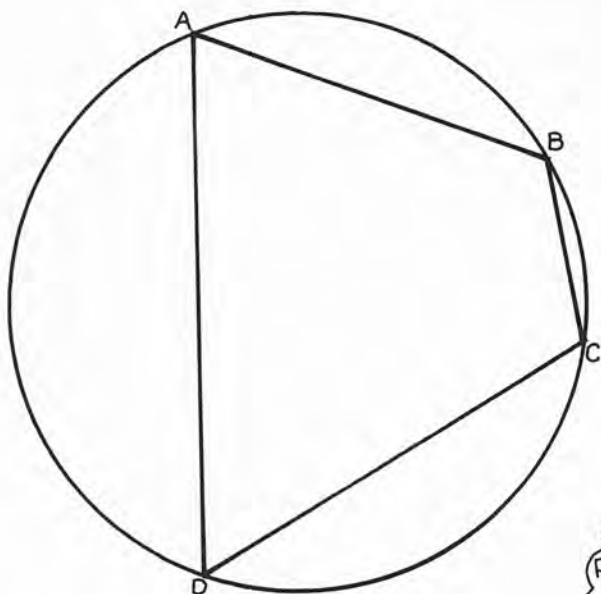
FIND THE
SUM FOR EACH
PAIR OF OPPOSITE
ANGLES.

1) m∠A = ____   2) m∠A + m∠C = ____   3) m∠P = ____   4) m∠P + m∠R = ____

    m∠B = ____   m∠B + m∠D = ____   m∠Q = ____   m∠Q + m∠S = ____

    m∠C = ____   m∠R = ____

    m∠D = ____   m∠S = ____

5) How are the opposite angles related? _____.

II)   III)
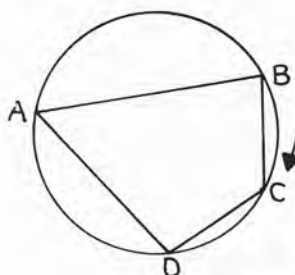
m∠C = 125°

Use your answer
in #5 to predict
the measure of
angle A. ____

CHECK YOUR GUESS
BY MEASURING WITH
A PROTRACTOR.
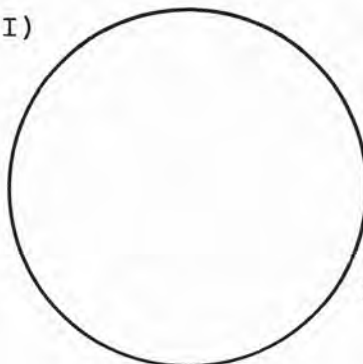
1) Inscribe your own
    quadrilateral in
    the circle.

2) How are the
    opposite angles
    related? ____
    _____

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

CONTENTS

CURVES & CURVED SURFACES: OTHER CURVES

OTHER CURVES

There are many useful and interesting curves other than circles. The familiar bell-shaped curve shown to the right has proved very useful in the application of statistics and probability to economics, sociology, psychology and many other areas. The sine curve and its variations are used to study sound, electricity and other similar phenomena.
A basic sine wave is shown below along with a variation of the sine wave which was produced on an oscilliscope when a tone was played on a trumpet.
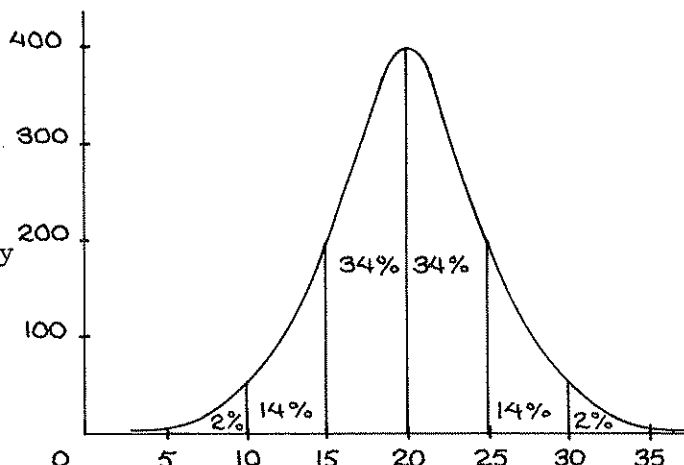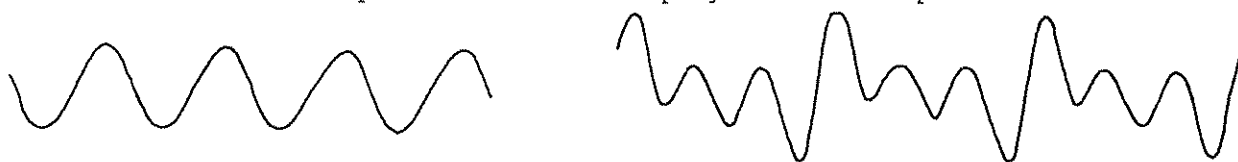
You can find activities about bell curves and sine waves in Harold Jacob's Mathematics a Human Endeavor.

One important use of curves is to give a geometric representation of the relationship between two quantities. If the following data were taken from a student, points representing the data could be placed on a Cartesian coordinate system and a smooth curve could be drawn to connect the points. (Basic graphing of points and relations on a coordinate system is covered in the Planes subsection.)

HOURS SPENT STUDYING FOR CHAPTER TESTS		GRADES ON CHAPTER TESTS
TEST ONE :	1 HOUR	72%
TWO :	$\frac{1}{2}$	63%
THREE :	$2\frac{1}{2}$	87%
FOUR :	0	50%
FIVE :	3	85%
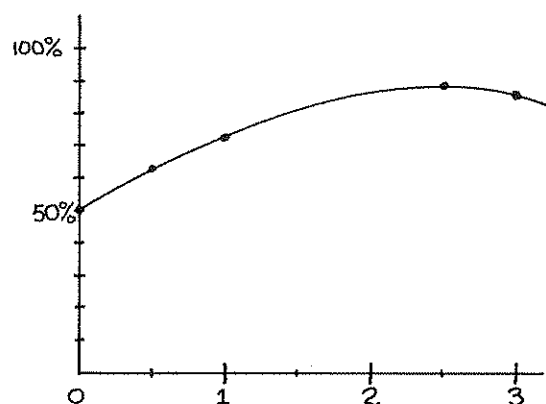
The graph gives an overall visual view of one student's performance and can be used to make generalizations and predictions (none of which is necessarily true) about the relationship of the number of hours spent studying for a test and the test grade.

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Although the curves discussed above are very important in representing relation-
ships, there are other curves which are more closely associated with the subject of
geometry. Among these are the conic sections, cycloids and spirals.

CONIC SECTIONS

The cross sections of a cone are
varied and thus can provide an inter-
esting investigation. Students can
make a clay or Play Doh model of a cone
and cut it with wire to show the several
possible cross sections as described in
Plane Intersections in the Curved Sur-
faces subsection. The conic sections
are the cross sections of a double cone.
The cross sections can be a point, a
pair of intersecting lines, an ellipse,
a parabola or a hyperbola. Students
might confuse a hyperbola with a para-
bola, but the cut for a hyperbola goes
through both halves of the double cone.
Models of the cross sections of **cones**
are available from several commercial
sources. A listing of these sources is
given in *Surveying Solid Shapes* in the
Polyhedra subsection and *Cylinders, Cones
and Spheres* in the Volume subsection.
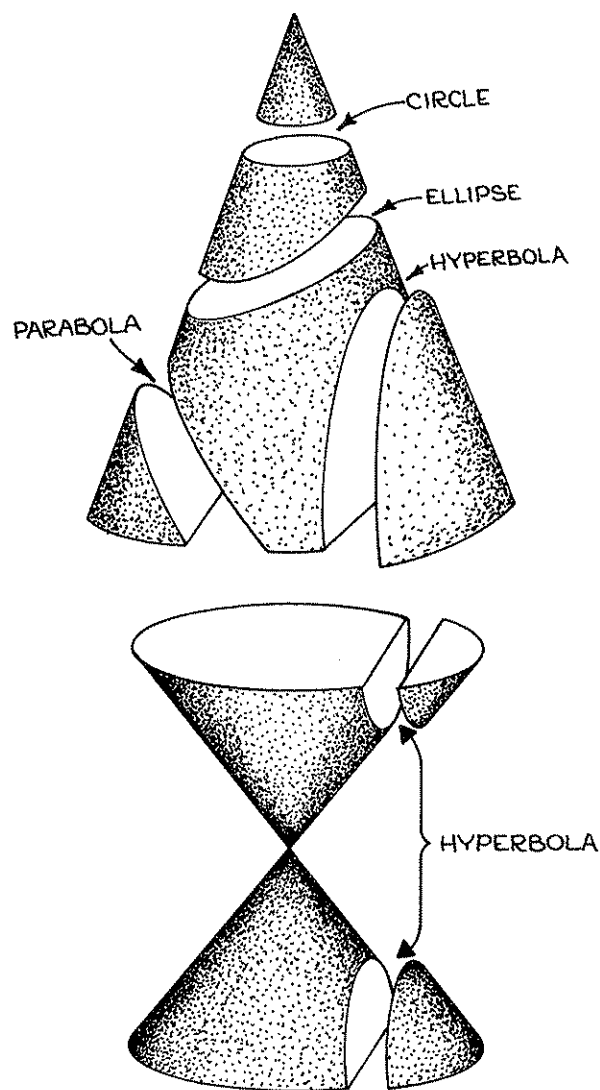
The conic sections--circle, ellipse, parabola and hyperbola--were studied in
ancient times; the Greek, Apollonius is known mainly for writing a book called The
Conics which described and interrelated the properties of these curves. The Greeks
regarded the conics as beautiful and intriguing curves, but they thought they were of
little value in applied mathematics. Today the conic curves are important in astron-
omy, rocketry and the military science of ballistics. Many real-world examples of
these curves are given on the classroom pages of this section. An enlarged collection
of these examples could make an attractive bulletin board for a student project. The
circle, which is really a special ellipse, is such an important conic section that it
is given its own subsection in this resource. Here are a few notes about each of the
other conic sections.

Parabolas

Any free-falling object which has
been thrown or shot into the air will
travel on a path which is approximately
part of a parabola. This fact is one
of the reasons parabolas, their graphs
and their equations are very important
in applied mathematics.

THE PATH OF A BALL IS
PART OF A PARABOLA.
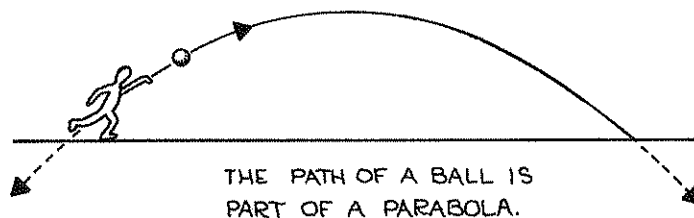
What
kind of path is made by water from a
hose? How can the path be varied? At
what angle to the ground should the hose
be held to shoot water the maximum hor-
izontal distance? What path does a
steel ball make when it is rolled at an
angle across an inclined plane? These
paths are part of parabolas. It can be
pointed out that the curves are smooth
and symmetrical. Some students might
say the curves could be folded along a
line of symmetry so the two "sides" will
match.

Experiments with the water hose or
rubber ball will help students see that the parabolic path will vary if an object is
thrown harder or at a different angle. (A 45° angle gives maximum distance.) Here is
a related investigation with a surprising result. Ques-
tion: Which will hit the floor first--an object dropped
from a table or an object shot horizontally from the
table? Students can flick a coin off a table or file
cabinet at the same time an identical coin is dropped.
Surprisingly, both coins will hit at the same instant.
This is because the same gravitational force is pulling
each coin toward the earth. The horizontal flick of the
coin does not affect the coin's downward movement.

PART OF A
PARABOLA

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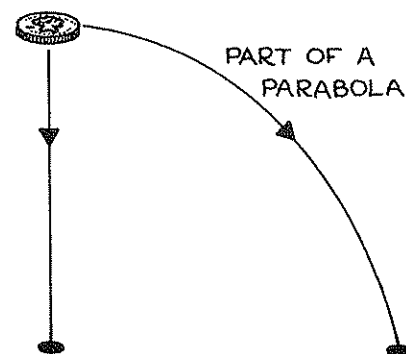
Students will probably think of a parabola as any smooth curve opening up or down. To distinguish parabolas from other curves which look very similar, a more formal definition is needed. You might compare these definitions to those for ellipses and hyperbolas.

A parabola is the path of a point which moves on a plane so it remains equidistant from a fixed point P and a fixed line ℓ. This definition can be difficult to understand; you might want to reserve it for more formal work.
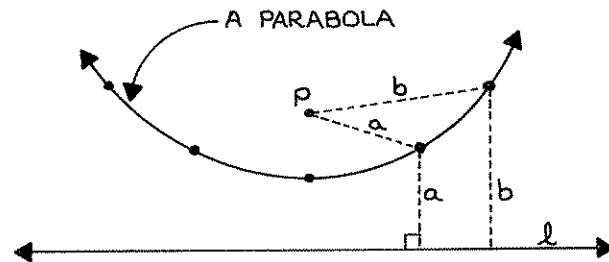
Here is a method for drawing parabolas which your students might enjoy.

a) Make or obtain a page of concentric circles.
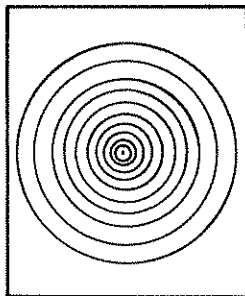
b) Draw a line ℓ at the bottom of another sheet of paper. Choose a point P about 4 cm above the line.
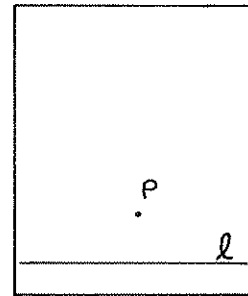
c) Using the page of concentric circles, arrange a circle so it is tangent to line ℓ and passes through point P. Mark its center on the paper.

d) Repeat (c) with several different sizes of circles. Connect the centers with a smooth curve.
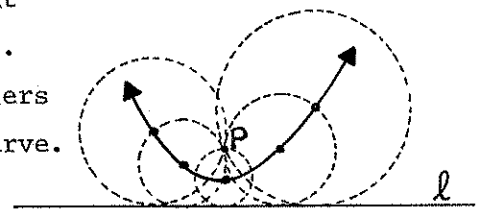
The curve is a parabola. Perhaps your students can see that the center of each circle is the same distance (the radius of the circle) from point P as from line ℓ.

The fixed point in the definition of a parabola is called the focal point. If the parabola is revolved about its axis of symmetry, a parabolic surface is formed. Light rays hitting a mirrored parabolic surface will be reflected to the focal point. A large parabolic mirror can be used to create extremely high temperatures. Parabolic mirrors are also used in the headlights of cars. A bulb can be placed at the focal point and the rays of light are reflected outward parallel to the axis of symmetry. *Reflect on that Image* in the section SIMILAR FIGURES suggests some demonstrations and investigations with a parabolic mirror. MATHEMATICS AND ASTRONOMY and MATHEMATICS AND PHYSICS in the resource Mathematics in Science and Society also give investigations with parabolic mirrors.
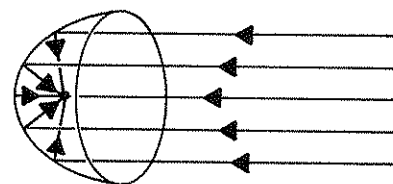
Ellipses

Elliptical shapes can be seen in many places. Any circle seen from an angle appears as an ellipse. The top of the glass shown to the right is drawn as an ellipse, although we interpret it as a circle in perspective Cones and circular cylinders can both be cut to give an elliptical shape. Students might be surprised to see an elliptical design resulting from the paper folding experiment described in *Enveloping Conics*. A similar pattern appears on the pages *Right on Conics*.
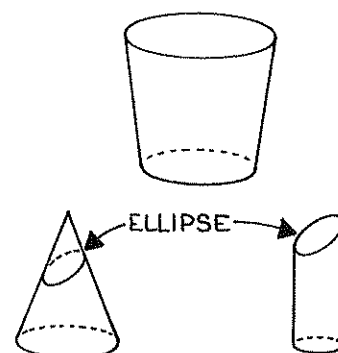
Here is an activity you might try with students. Have them draw a large circle on a piece of paper and pick a point P inside the circle (not the center). Have them mark the center of many circles, each of which passes through point P and is tangent to the circle. (Students can trace the circles from a page of concentric circles.) If the centers of the circles are connected, what curve is formed? What happens if point P is allowed to be the center of the original circle? Variations of this activity
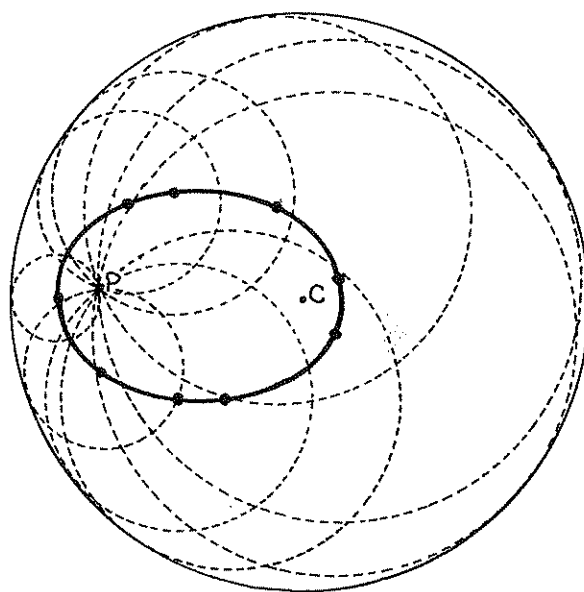
are given under <u>The Parabola</u> and <u>The Hyperbola</u> in this commentary.

An ellipse is usually thought to be an oval shape—a squashed circle. More formally, it can be thought of in several different ways, one of which is given below.

An ellipse is the path of a point which moves on a plane so the sum of the distances of the point from two fixed points is constant. This is the basis for the common method of drawing an ellipse: Attach the ends of a piece of string to two thumbtacks and draw a closed curve, keeping the string taut with a pencil point. The shape of the ellipse can be altered by changing the length of the string or the distance between the two tacks. The thumbtacks are the two <u>fixed points</u> or foci and the length of the string is the <u>constant distance</u>. This definition of an ellipse is used in problem one of *The Pathfinders*.

You might challenge a bright student to explain how this distance definition relates to the "tangent circle method" of drawing an ellipse given on the previous page. Where are the focal points? What is the constant distance? A very clever student might conjecture the foci are at P and C (the center of the large, original circle) and the constant distance is the radius of the large circle. Is this a reasonable conjecture?

Students probably know the orbits of the planets are elliptical. Do they also know the sun is located at a focal point of the elliptical orbits? More information about elliptical orbits is given in ASTRONOMY AND MATHEMATICS in the resource <u>Mathematics in Science and Society</u>.
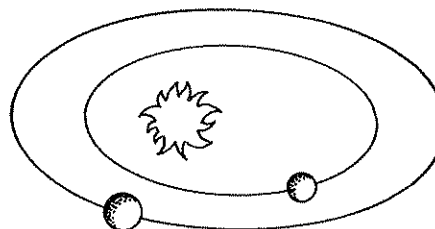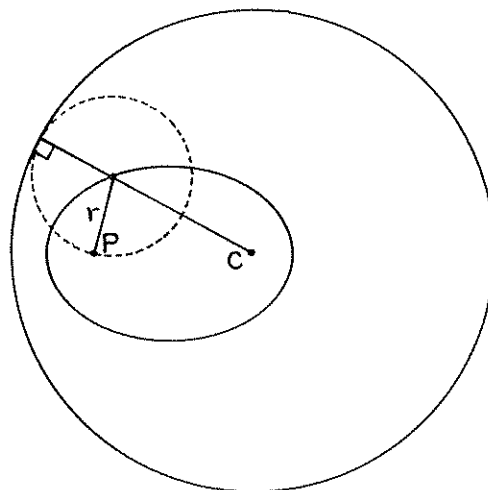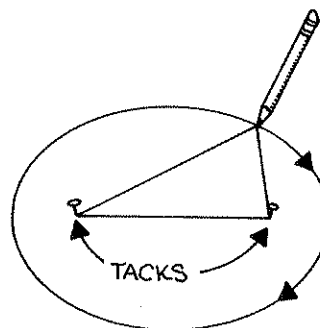
If you have students who enjoy pool, perhaps they would like to learn about elliptical pool tables. The only pocket is placed at one focal point, and the other focal point is marked. If a ball is placed on the marked focal point, where should it hit the side of the table so it will go in the pocket? Surprisingly, the ball will go in the pocket no matter where it hits the side. What direction should a ball not on the marked focal point be hit? The diagrams below show possible paths.
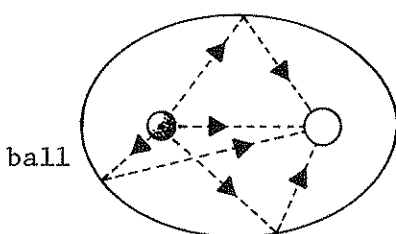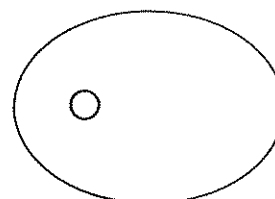
ball

ball

ball

If the ellipse is fairly accurate, it does not matter which direction the ball is hit from a focal point.
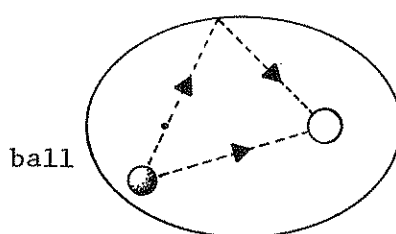
Two possibilities are shown above. Are there others? Hint: Try starting the ball in the opposite direction than shown.
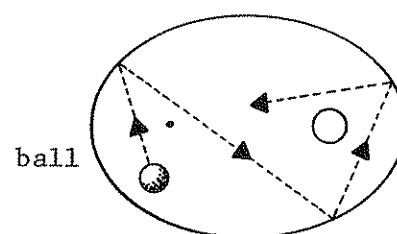
The ball will never pass directly over either focal point.

After discussing the path of a ball in elliptical pool, you might describe the path of sound waves in a room shaped like half of an ellipsoid--an ellipse rotated about the line through the focal points. A person standing at one focal point could hear a whisper at the other focal point. All the sound waves hitting the elliptical sides of the room would be reflected to the listener. A similar design was built into the Morman Tabernacle and a room in the Taj Mahal. The United States capital building and the Museum of Science and Industry in Chicago also have rooms in which this accoustical phenomenon can be experienced.
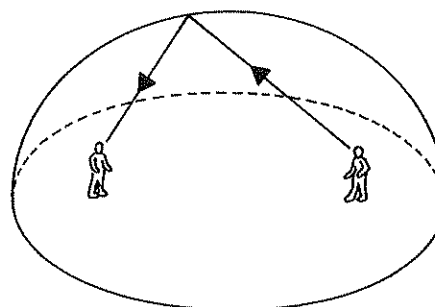
Hyperbolas

A hyperbola is the path of a point which moves on a plane so the difference of its distances from two fixed points is constant. In the diagram to the right $AF_2 - AF_1 = BF_2 - BF_1 = CF_1 - CF_2$. The shape of the hyperbola can be varied by changing the distance between the foci or by changing the constant difference.
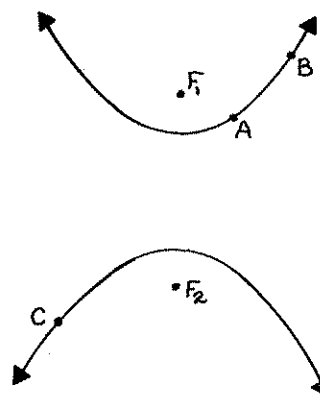
a) Draw a circle with a radius about 2 cm. Label its center C. Pick a point P outside circle C.
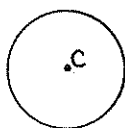
b) Using a page of concentric circles, mark the centers of several circles which are tangent to circle C and pass through point P. Connect the centers in a smooth curve.
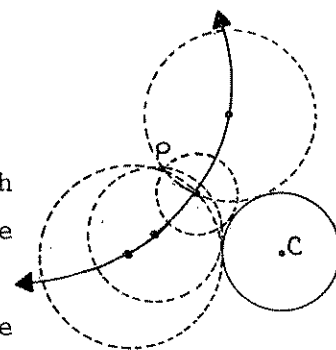
c) Trace circle C and point P on another sheet of paper. Now mark the centers of circles which pass through P, tangent to circle C and have circle C "inside" them. Connect the centers in a smooth curve.
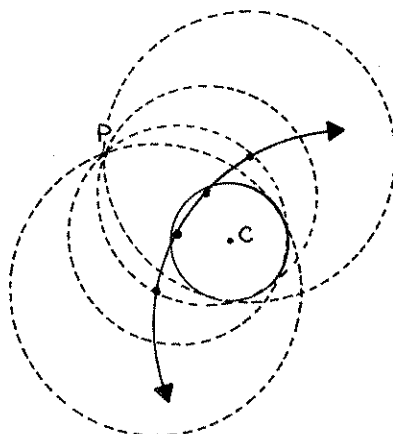
d) Put the two curves together. This is a hyperbola.
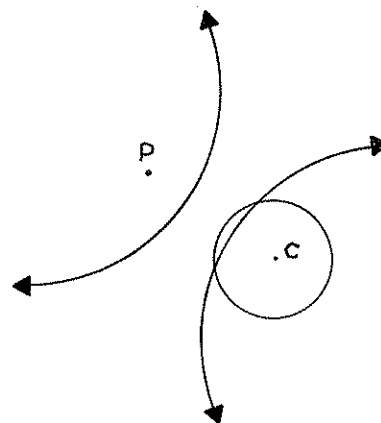
Can any student find the connection between the drawing method given above and the definition given for a hyperbola?

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MORE CURVES AS PATHS OF POINTS

Anyone who has played with a Spirograph (a drawing toy sold commercially) has made curves that can be seen as paths of points. A pen is placed in a hole of a gear and the gear is rotated around another gear or inside a ring gear until a satisfactory design is created. The design at the right came back to its starting point, but not all designs will do this. If you have a Spirograph set available (a student might have one), you could have your class try to predict when a path will return to its starting point. Activity cards on this topic entitled *Poppin' Wheelies in a Ring* can be found in the RATIO section of the resource Ratio, Proportion and Scaling.
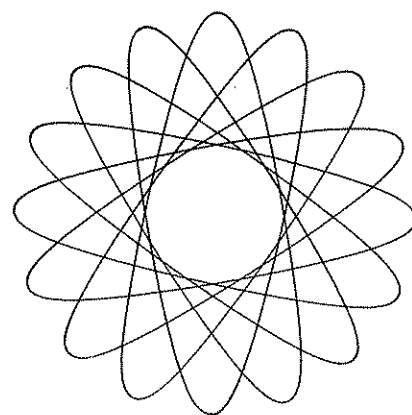
A Spirograph design made on the interior of a 96 tooth gear with the pen in the "12" hole of a circle gear.

The curves shown below were made with a harmonograph, a sketch of which is shown to the right. A different design for a harmonograph is given in PHYSICS AND MATHEMATICS in the resource Mathematics in Science and Society. Many science museums have a similar device available for visitors to use. A pen is suspended from a balance arm, and the apparatus attached to the arm is rotated. The pen then traces a curve on a piece of paper. The weights on the harmonograph can be adjusted to vary the curve produced. It is quite fascinating to watch the curve being produced, and students find it enjoyable to experiment with different motions and arrangements of weights.
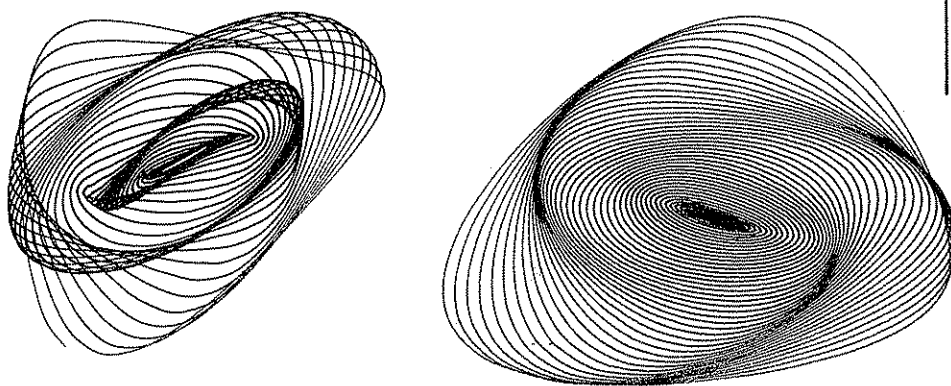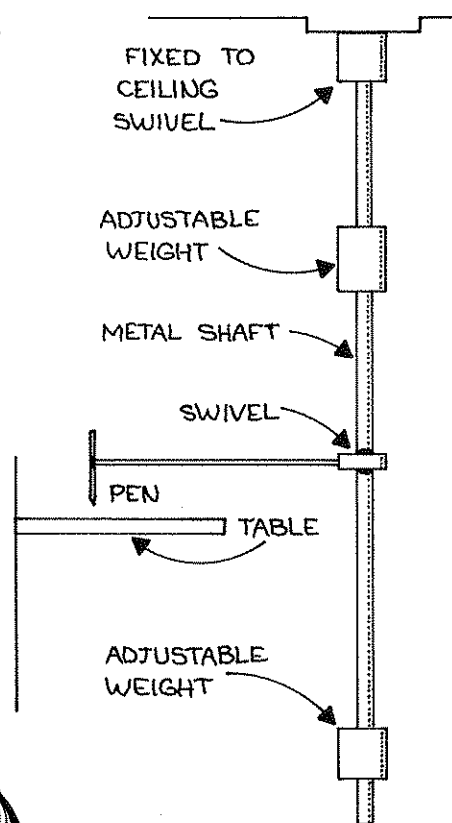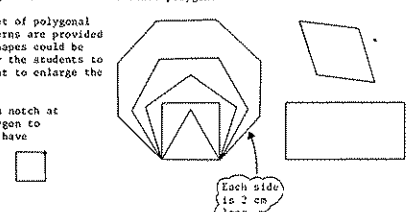
Less complicated curves can be pro-duced in class by tracing the path of a point of a polygonal region as it is rolled along a straight line or around another polygon. The activities suggested in *Revolutionary Curves* (see the reduced page at the right), *A Can Can Do It*, and *Point Out the Path* are good investigations of curves as paths of points. These activities provide a chance for students to predict the path, then test their predictions by tracing the paths. One very interesting curve, the cycloid, is discussed thoroughly in *A Can Can Do It*. Your students might be particularly surprised by the marble experiments described in part V of that activity.

SPIRALS AND HELIXES

A spiral is a plane curve traced by a point which moves around and away from (or toward) a fixed point. We see shapes that look like spirals in the shells of snails, elephant trunks and distant galaxies. A helix is a winding curve on a cylindrical or conical surface. A barber pole has cylindrical helixes, whereas the threads of a screw approximate a conical helix. A collection of pictures could make a nice introduction to the series of student pages on spirals in this resource.

The page *Spirolaterals*
is a simple but fun activity
which begins with a spiral-
like design. Students might
be surprised that repeating
the 1-2-3-4-5 pattern will
create a design which ends
where it begins. Can they
find a pattern which will not
come back to the starting
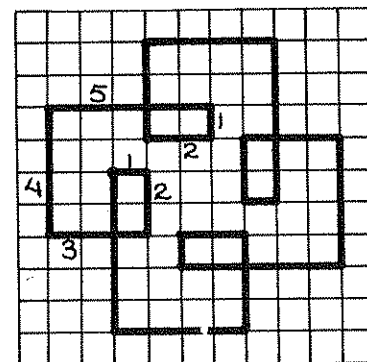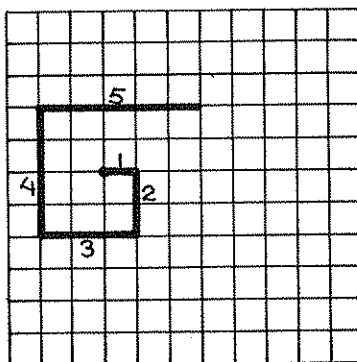
point? Why not try some spirolaterals with your students? This activity is an in-
vestigation which involves little new terminology or involved background.

The page *Archimedean Spirals* has suggestions for simple ways to create true
spiral curves. The page *Polygonal Spirals* can be used as the beginning of an

investigation of spiral-like designs.
The page shows a design based on a
regular octagon. Can a spiral-like
design also be started with a
regular hexagon? (Yes--see the
design at the right.) A square?
Does a regular pentagon work or
does there have to be an even num-

ber of sides? Does the polygon have to be regular? Students can construct a basic
figure and use rulers or approximations to find the midpoints of sides for these
designs.
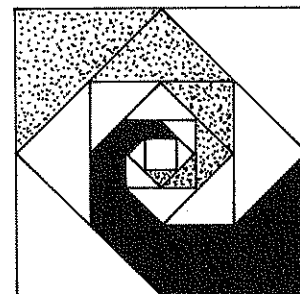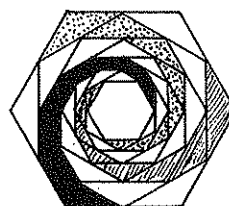
Here is an activity to try with your class. Split the class into four groups.
Have everyone draw a 10 cm segment on their papers. Have Group 1 draw a 9 cm segment
perpendicular to the 10 cm segment at one end point. Similarly, have Group 2 draw a
8 cm segment (the next page shows a reduced version of this), Group 3 a 7 cm segment,
Group 4 a 6 cm segment. Each student then connects the remaining end points with a
dotted guide-line to form a triangle as shown on the next page. Another dotted line
is constructed through the vertex of the right angle and perpendicular to the first
dotted line. A spiral-like design is then formed by drawing perpendicular line seg-
ments at each successive end point. (You will probably have to demonstrate this pro-
cedure at an overhead.)
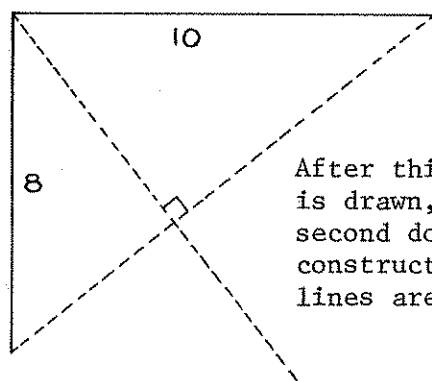
After this dotted line is drawn, then the second dotted line is constructed so the two lines are perpendicular.
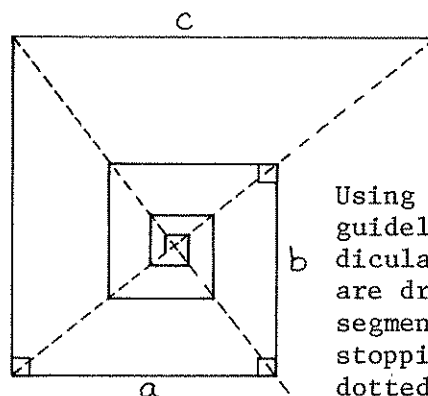
Using the dotted guidelines, perpendicular line segments are drawn. Line segment a is drawn stopping at the dotted line. Line segment b is drawn, etc. . . .

Have students within each group compare their designs. Are they the same? (They should be.) How do the designs in Group 1 differ from those in Group 2, 3 and 4? Students can try other lengths for the line segments. What happens when they start with segments 5 cm and 4 cm (or 20 cm and 16 cm) in length? Does it look like any of the original designs?

If your students have worked with ratios, you might want to include ratios in the above activity. The ratios of successive sides remain constant in any one design. Students could use the constant ratio to construct the design--a calculator would be useful to compute the needed lengths. Full-sized designs based on 10 cm : 7 cm and 10 cm : 9 cm segments can be seen on the page *More Spiral-Like Designs*.

Curves are a rich topic in geometry. The ideas presented here are only a sampling. You or your students might want to read more about curves from the annotated list below.

Further Readings

Gardner, Martin. Martin Gardner's Sixth Book of Mathematical Games from Scientific American. San Francisco: W.H. Freeman and Company, 1963.

The delightful chapter "The Helix" is packed with examples of sprials and helixes from nature and design. Another chapter is devoted to the cycloid. These are suitable reading for an interested student with good reading ability.

Jacobs, Harold R. Mathematics a Human Endeavor. San Francisco: W.H. Freeman and Company, 1971.

Interesting examples of and facts about curves are given in this excellent textbook-resource. Equations are graphed in step-by-step exercises. The normal bell curve, the conics, spirals, cycloids and the sine curve are all covered--see especially chapters 3, 6 and 9.

Steinhaus, H. <u>Mathematical Snapshots</u>. New York: Oxford University Press, 1969.

> *Chapters 10 and 11 of this book show many snapshots which relate to conics, helizes, spirals, sine waves and other curves. The abundance of pictures with a small amount of reading makes this resource book useable for any interested student.*

Valens, Evans G. <u>The Number of Things, Pythagoras, Geometry and Humming Strings</u>. New York: E.P. Dutton and Company, Inc., 1964.

> *The chapters "Stars and Doubled Cubes" and "Geometry for Listening" give some interesting side lights on the conics and spirals.*

An ellipse is a closed curve such that for any point on the curve the sum of its distances from two fixed points (foci of the ellipse) is constant. The ellipse can be represented or drawn in several ways.

A) Make a loop on both ends of a piece of string. Secure both loops with thumbtacks or brads. With a pencil pull the string tight and care- fully move the pencil around always keeping the string tight. On the chalkboard rubber tipped darts can be used.
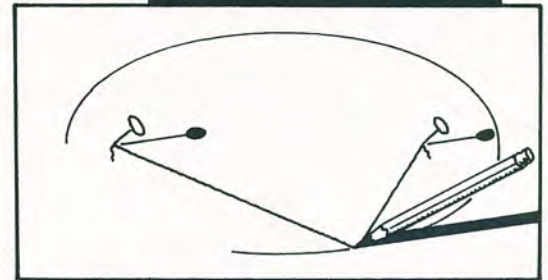
B) Shine a flashlight at the chalkboard at a slight angle and trace around the light.
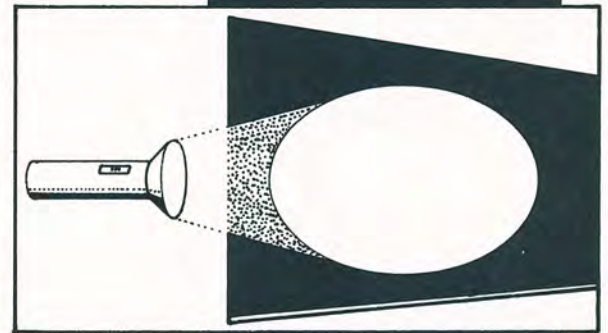
C) A fairly accurate ellipse can be drawn by securing a piece of paper to a cylinder and then using a compass to draw a circle. The "dip" necessary to keep the pencil on the paper draws the ellipse.
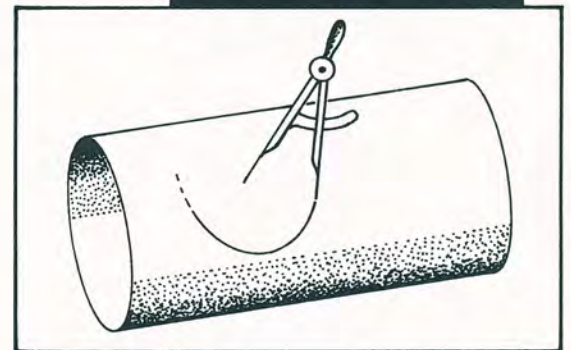
D) An ellipse inside of an ellipse can be drawn (or stitched with thread) by marking evenly spaced points on the ellipse and then joining the points in some pattern. The pattern shown in the example connects each point to the sixth point on the right.
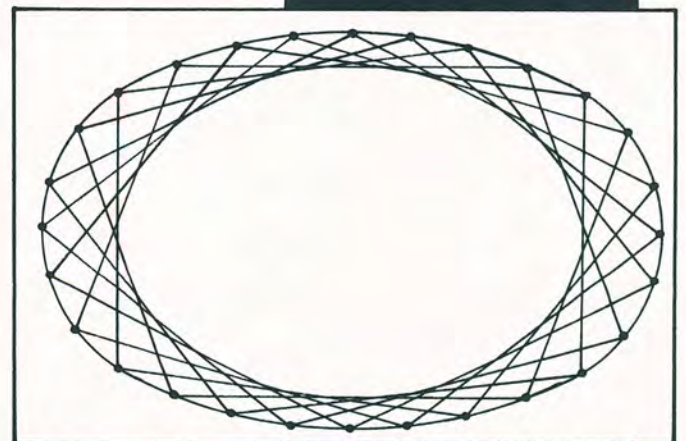
ENVELOPING CONICS

I

1) Get a piece of waxed paper about 30 cm long.

2) Draw a line segment 20 cm long.

3) Mark an end point and mark off every cm on the segment.

4) Put a point A above the midpoint of the segment.

5) Fold the paper so the first cm mark on the segment lies on point A. Crease the fold.

6) Repeat for each cm mark on the segment.

7) What curve does the design make you think of? _____
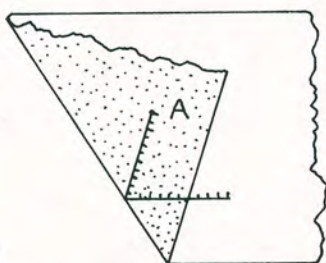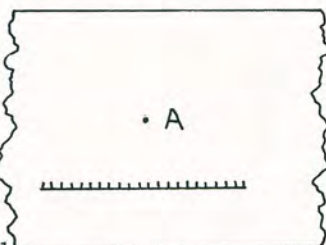
II

1) Use a piece of paper about 30 cm long.

2) Draw a circle with radius 8 cm.

3) Mark off every 15° on the circle.

4) Put a point A inside the circle about half way to the center.

5) Fold the paper so that a 15° mark lies on point A. Crease the fold.

6) Repeat for each 15° mark on the circle.

7) What curve does the design make you think of? _____
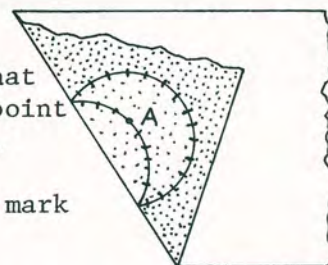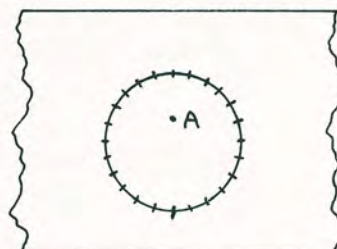
III

1) Repeat steps 1 - 3 from activity II.

2) Put a point A 4 cm outside the circle.

3) Fold each 15° mark on point A. Crease each time.

4) Look at the design.

5) What curve does the design make you think of?
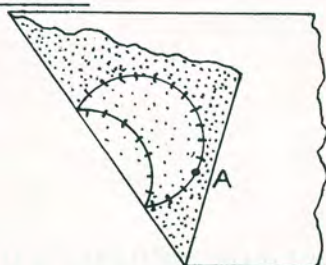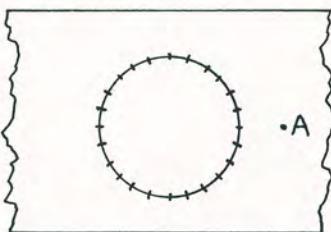
RIGHT ON CONICS

I   Draw a chord that is perpen-
    dicular to each ray where it
    meets the circle.

    One chord is drawn for
    you.

    What curve is suggested by
    your finished design?

A RAY
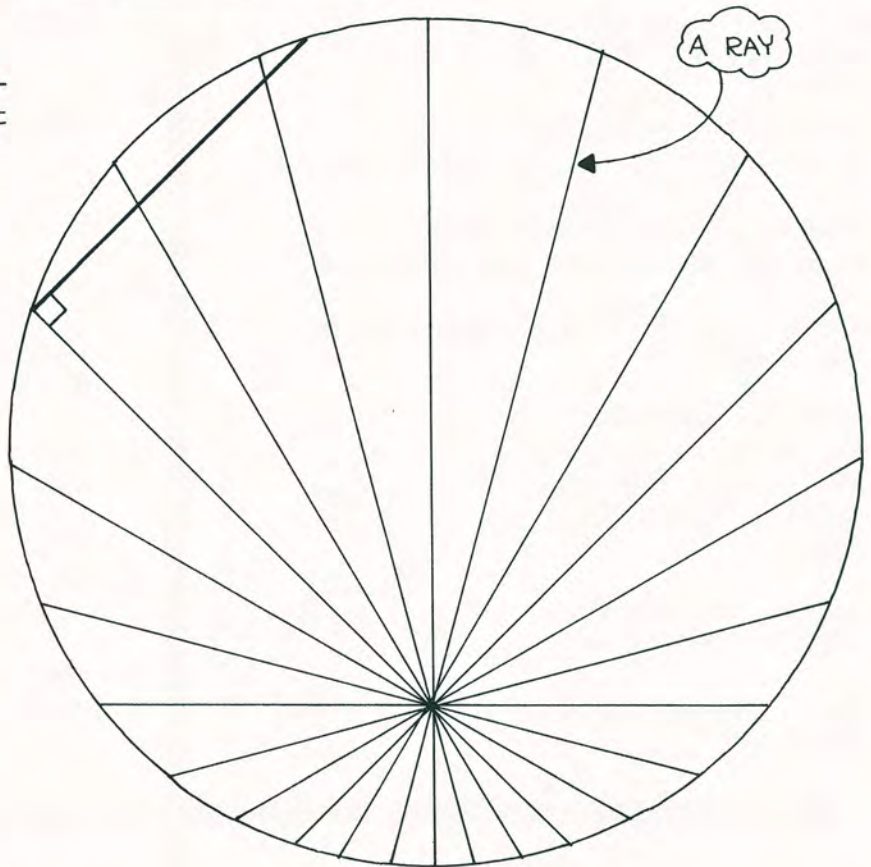
II Draw a line segment that is perpendicular to each ray where it meets the base line.
    One line segment is drawn for you.

    What curve is suggested by your finished design?
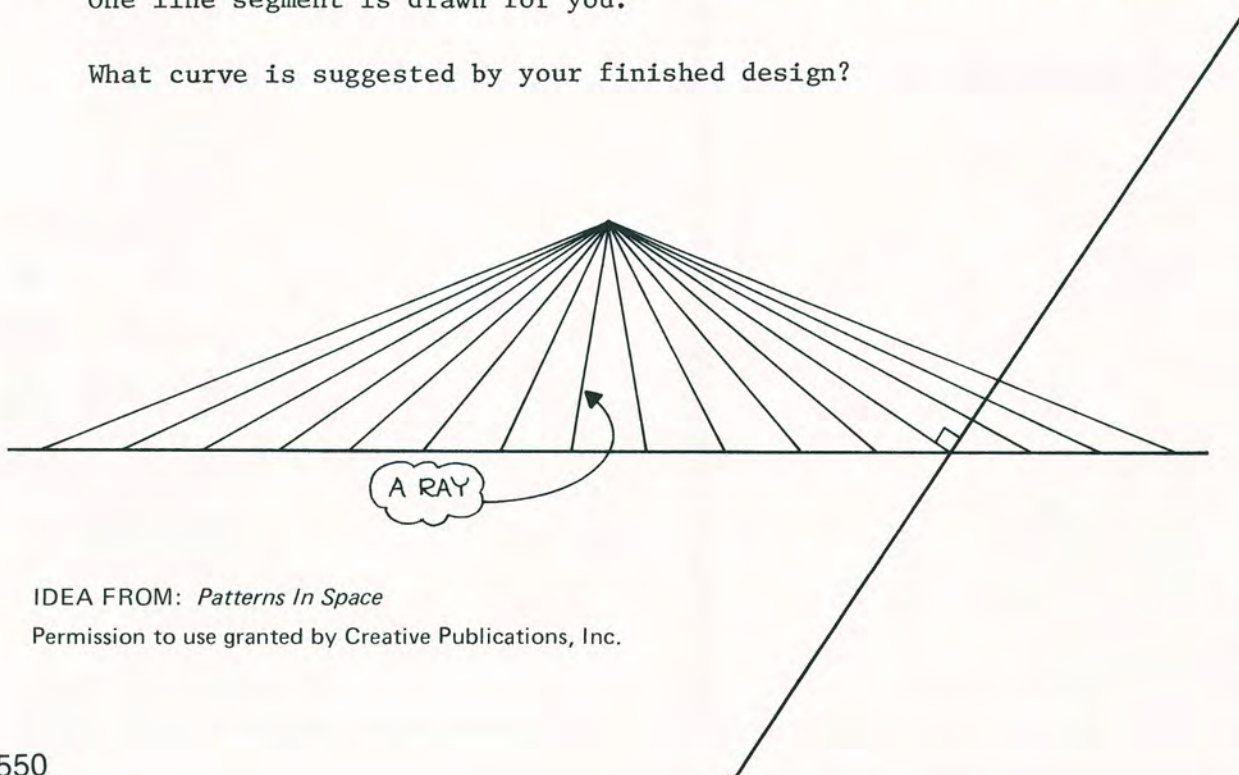
A RAY

IDEA FROM: *Patterns In Space*

THE PATHFINDERS

1) A hypnotized bug must walk from A to B
   by first walking in a straight line to
   a point on the curve and then straight
   to B.

   a) Guess which path is the
      shortest. _____

   b) Measure the lengths of
      the five paths from A
      to B.

      ____ ____ ____

      ____ ____
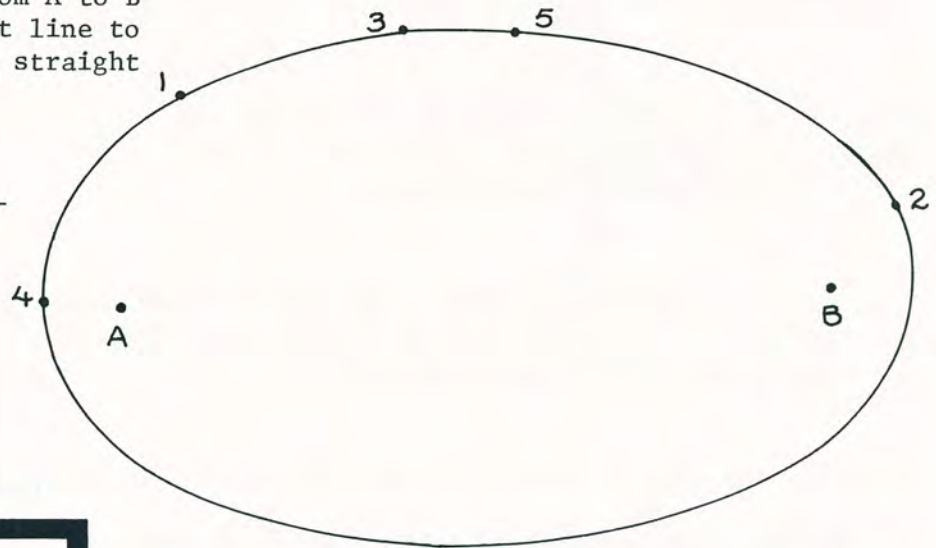
2)
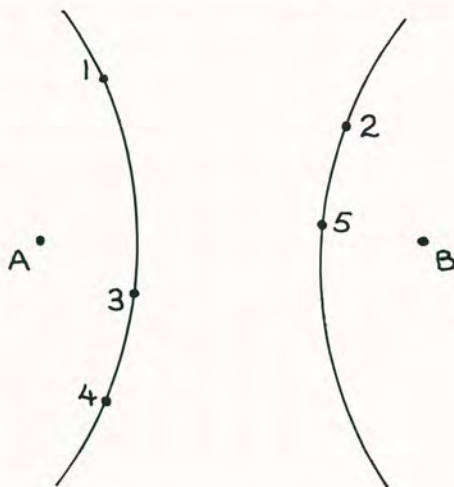
In a contest five wives are at point A and
their husbands are at point B. Couple 1 is
to meet at point 1, couple 2 at point 2, etc.
The winner of the contest will be the couple
that has the smallest difference between the
distances they walked to get to their point.

a) Draw the five paths and predict which
   couple will have the smallest difference.

b) Measure the two segments of each path to
   check your prediction.

BRIDGE

3) While cycling, five bicyclists get caught
   in a storm.
   a) In order to travel the shortest
      distance should each one seek cover
      under the bridge or in the shelter?

      1 ____   2 ____   3 ____   4 ____   5 ____

   b) Measure the two distances for each
      cyclist to check your guess.

HANGING TOGETHER

Telephone wires or electric cables usually hang in a special curve between the posts that support them.

This curve is called a catenary (kat'-ə-ner-ē). Catena is Latin for chain. A catenary is the curve in which a looped chain hangs.

The shape of the Gateway Arch in St. Louis, Missouri, is a catenary. (The construction of the arch could be a topic for student investigation.)

Students can make a graph of a catenary by the following method.

1) Fasten a large sheet of butcher paper to the wall.

2) One student holds a chain in front of the paper and a second student sketches the curve.

   Different catenaries can be sketched by varying the position of hands. Students can investigate holding chains of different lengths or chains of differently sized links.

3) The following points approximate the graph of a catenary when graphed on cm grid paper where each unit represents 3 cm.

    A ——————▶ 3 cm
    Z and B ——————▶ 4.6 cm
    Y and C ——————▶ 11.3 cm
    X and D ——————▶ 30 cm
    W and E ——————▶ 81 cm

Students can test the graph by holding a chain in a suitable position.

Students may think that a catenary and a parabola are the same curve. They look alike at the vertex but a parabola opens more rapidly.

ROLLERS WITH CORNERS

To move a heavy object from one place to another the object is often placed on a flat platform that rests on cylindrical rollers. As the platform is pushed forward the cylinders behind are picked up and put in front again.
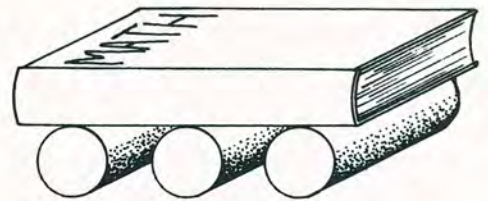
Demonstrate this technique to the students. Dowel rods can be used for rollers; a book for the platform. Students should observe the book traveling smoothly backward and forward on the rollers.

<u>Must rollers and wheels always be circular in shape?</u>

To help answer the question demonstrate the motion of elliptical rollers: Your school shop could make the rollers or you could trace the pattern to the right, cut four ellipses from heavy cardboard and tack them in pairs to the ends of two 16 cm dowel rods.

Place a book on the rollers and roll it back and forth.

In what way does the movement of the book differ from its movement on cylindrical rollers? Why?
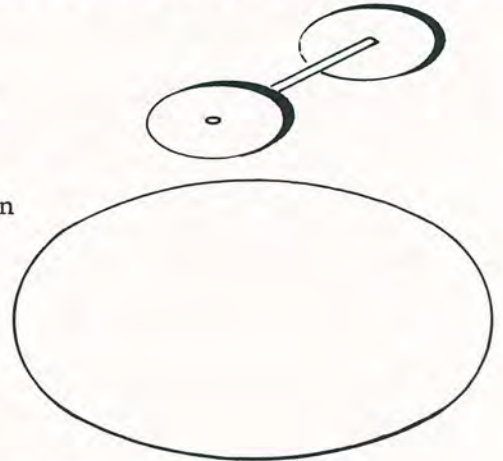
<u>Are cylindrical rollers the only ones that will roll a platform smoothly?</u>

Exhibit rollers made from the shape to the right. Use the same method as suggested for the elliptical rollers.

Ask students to guess the movement of the book. Then demonstrate--the book will travel smoothly backward and forward, but the dowel rod wobbles up and down!

Can you explain why the motion of the book is as smooth as if it were on cylindrical rollers?

(The shape is a curve of constant width---the longest distance across the shape in all directions is the same. If placed between two parallel lines that touch the curve, the curve can be rotated between the lines without changing the distance between them.) Is a circle a curve of constant width? How about an ellipse?

THE CONSTANT WIDTH OF THIS CURVE

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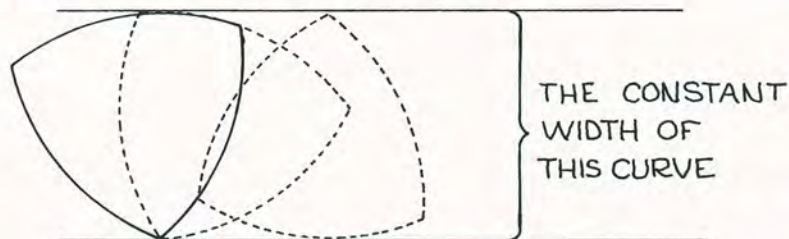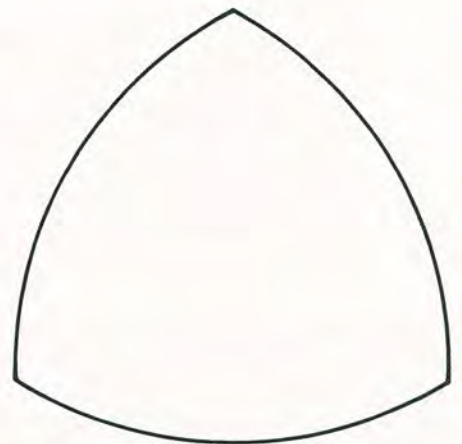
ROLLERS WITH CORNERS
(PAGE 2)

Extensions:

1) The noncircular curve of constant width on the preceding page is called a Reuleaux (roo-low) triangle. To construct this curve draw an equilateral triangle ABC. With the point of the compass at A draw arc BC. In a like manner draw the other two arcs. (The constant width of this curve is equal to the length of $\overline{AB}$.

> The Reuleaux triangle is the curve of constant width that encloses the smallest area for a given width; the circle encloses the largest area.

2) How many curves of constant width are there?

a) An equilateral triangle can be used as a basis for many curves. Extend each side of triangle ABC the same amount. With the point of the compass at A draw arc DI. Widen the compass and draw arc FG. Do the same at the other vertices. (The curve will have a width in all directions equal to the sum of AI and AF.)

b) Any regular polygon (with an odd number of sides) can be used as a basis for a curve of constant width. Use procedures like #1 or #2a. (See the pentagon to the right.)

> With the point of the compass at B draw arc DE.

c) Are curves of constant width always symmetrical?
One way is to start with an irregular star polygon having an <u>odd</u> number of corners and sides of equal length. Place the compass at each lettered vertex and draw an arc to connect the end points of the two segments extending from the vertex. Each arc has the same radius so the resulting curve will have constant width.
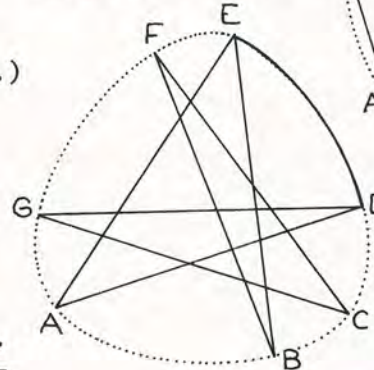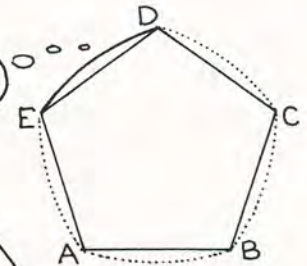
> A seven-point star. From vertex A draw arc ED.

3) Students can construct curves of constant width and test this property by placing them between two parallel lines that just touch the curves. As the curves are rotated they will always touch the lines but not change the distance between them.

4) Curves of constant width can also be placed inside a square so that when rotated they will maintain contact at all times with all four sides of the square.

a) To observe this have students cut a Reuleaux triangle from heavy cardboard and rotate it inside a square hole of proper dimensions (see diagram) cut in another piece of cardboard.

10 cm

10 cm

SQUARE HOLE

REULEAUX TRIANGLE

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ROLLERS WITH CORNERS
(PAGE 3)

b) Place a piece of paper under the square hole and trace the path of one vertex as the Reuleaux triangle turns.

Punch a hole at the center of the triangle and trace its path as you turn the triangle. Is the center fixed?

CENTER OF TRIANGLE

PATH OF CENTER

PATH OF VERTEX
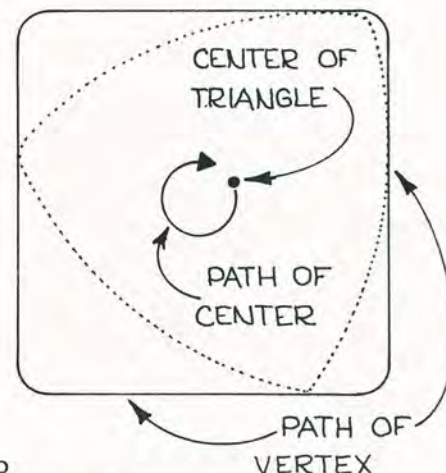
5) Curves of constant width have many mechanical uses.

a) By cutting the corners of the Reuleaux triangle into spikes the triangle can be turned into a bit for drilling square holes!
The Watts Brothers have been manufacturing this tool since 1916.
Because the center wobbles as the drill turns it is not mounted on an axle, but on a specially designed chuck that floats.
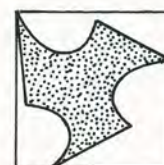
b) The Wankel engine, the rotary engine used in Mazda cars, uses a rotor which is a modified curve of constant width.
The three vertices of the rotor remain in contact with the walls of the housing at all times.

c) Because a manhole cover is a curve of constant width, it can't be dropped through its sleeve into the sewer. Covers could be made in the shape of any curve of constant width.

d) Since the invention of vending machines, coins need to be curves of constant width. Can you explain?

e) In building a "circular" hull for a submarine why can't you test for circularity by just measuring maximum widths in all directions?
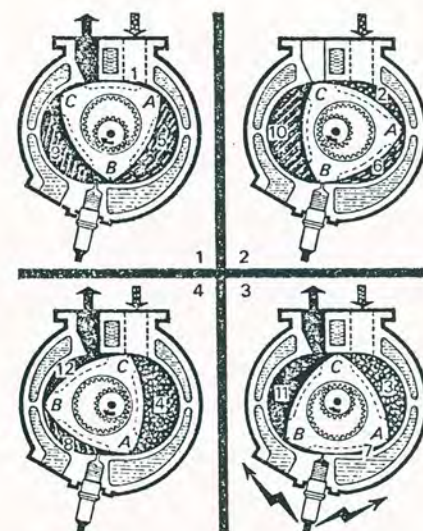
1–4 ▨ Induction   5–7 ▨ Compression
8–10 ▨ Power stroke   11–1 ▨ Exhaust

THE WANKEL ENGINE

6) The film <u>Curves of Constant Width</u> distributed by International Film Bureau could be used to motivate or summarize the topic.

7) The idea of curves of constant width can be extended to three-space with solids of constant width. These shapes will rotate within a cube, at all times touching all six faces of the cube. The sphere is an obvious example. An example is pictured to the right.

This solid is generated by rotating the Reuleaux triangle around one of its axes of symmetry.
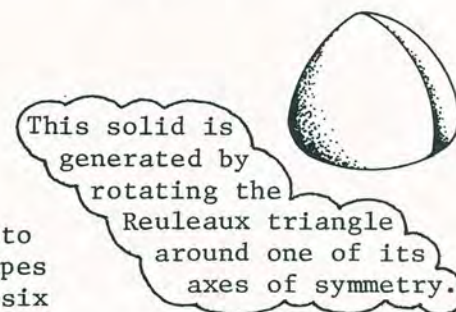
IDEA FROM: *The Unexpected Hanging and Other Mathematical Diversions; Rolling,* Topics from Mathematics; and *The School Mathematics Project,* Teacher's Guide for Book D

REVOLUTIONARY CURVES

The following activities investigate the path formed by a point of a polygon when the polygon is rolled along a line or around another polygon.

Each student needs a set of polygonal regions to roll. Patterns are provided to the right. These shapes could be dittoed on tagboard for the students to cut out. You might want to enlarge the shapes.

Have the students cut a notch at one vertex of each polygon to represent the point or have them glue a tab with a punched hole to the vertex to keep the pencil in place.

> Each side is 2 cm long.
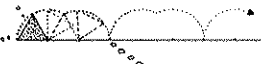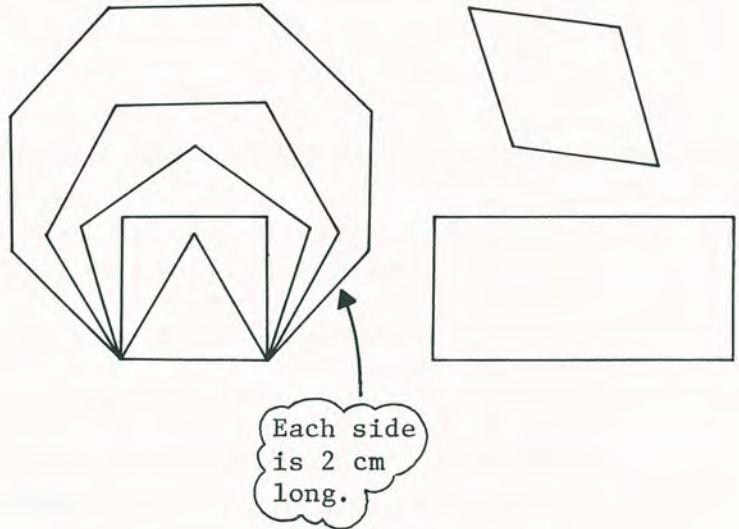
Activities:

1) Investigate the path of the point as the polygon rolls along a line segment.

   a) Students draw a 16 cm line segment and mark points every 2 cm.

> Make dots at the notch as the triangle rolls along. Join the dots with a smooth curve to complete the path.

   b) Students predict what the path of the point will be if the triangle rolls along the line segment. Students then roll the triangle to check their guess.

> Starting position: Notch at end point of line segment.

> Observe: One corner of the triangle is fixed on the line as the triangle rolls.

   c) Students predict the paths for each of the other regular polygons. How do the paths compare? What path would a point on the circumference of a circle make? (See *A Can Can Do It*.)

   d) Students compare the paths of a vertex of a square, rhombus and rectangle.

REVOLUTIONARY CURVES

(CONTINUED)

e) Does the shape of the path change if a point other than a vertex is used? Have students notch the midpoint of one side of each regular polygon, draw the path of the point (notch) as the polygon rolls along the line segment and then compare it to the path formed by the vertex. Investigate using the center of each polygon as the point to be traced.

2) Investigate the path of the point as the polygon rolls around a fixed polygon.

   a) Students predict the path of a vertex of the triangle if the triangle is rolled around a congruent triangle. (Students should draw around the triangle first to have a fixed shape to roll around.) Have them check their guess by rolling the triangle and drawing the path.

   The square is rolled around the square.
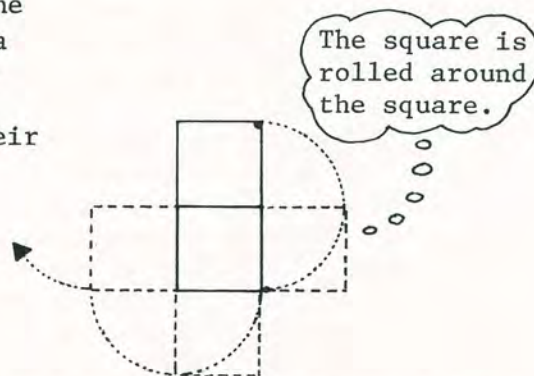
   b) Repeat (a) for each polygonal shape. Have students predict the path before they draw it.

   c) Investigate picking a point other than the vertex to use in tracing a path-- the center or the midpoint of one side.

   d) Investigate rolling the square around the triangle, the pentagon around the hexagon, etc. Can the students predict the path? How many times does the polygonal shape have to roll around the fixed polygon before the path is completed?

The following are the paths of various points that would occur from rolling a square along a line and around a fixed square.

A CAN CAN DO IT

I   Place a 3 lb. coffee can on a table so that the bottom faces the class. Tape a marker on the rim of the can.

    Question: What sort of a path will the marker make if the can is rolled along the edge of the table?

Encourage students to make predictions. Each guess could be sketched on the chalkboard. Roll the can and ask students to refine their guesses.

II The path can be drawn by using the coffee can. Loop a string around the can and fasten it to the ends of a board to keep the can from slipping as it rolls. Tape a pen to the inside of the can. Fasten a sheet of paper to the wall. When the can is rolled along the wall the pen will draw the path on the paper. The path, called a cycloid, was first conceived by Mersenne and Galileo Galilei in 1599. (See the diagram on the next page.)

III Extensions: 1) Suppose a marker is placed on the bottom of the can inside the rim. What will be the path of this marker if the can is rolled along the edge of the table? 2) Suppose a marker is attached to the rim so that it extends beyond the can. What will the path be if the can is rolled along the edge of the table? Will either be the same shape as the curve drawn in II?

MARKER EXTENDS BEYOND RIM

MARKER INSIDE RIM
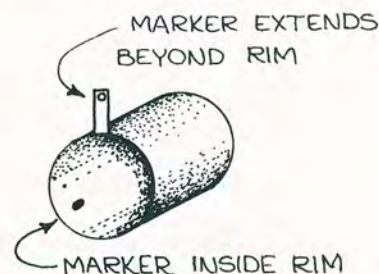
Students should be encouraged to make predictions. The coffee-can device could be used to draw both paths or the students could use the procedure outlined below.

a) Give each student (or have each student construct) a circular disk (radius 1 1/4 inches) with 16 equally spaced points on the rim numbered 0 through 15.

b) On notebook paper have students draw an 8-inch line and mark points every one-half inch. Number the points 0 through 15. The disk can be placed on the line so that the numbers will almost exactly match as the disk is rolled.
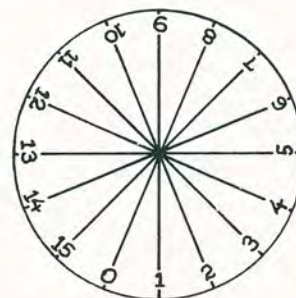
IDEA FROM: *Martin Gardner's Sixth Book of Mathematical Games from Scientific American; Curves and their Properties;* and *Mathematics A Human Endeavor*

A CAN CAN DO IT

(PAGE 2)

c) To sketch the path described in
extension #1, punch a hole in
the <u>middle</u> of the spoke numbered
0. Place the disk on the number-
ed line so that 0 matches 0.
Mark a dot on the paper through
the center of the punched hole.
As the wheel rolls along the
line continue to mark points each
time the numbers on the disk
match those on the line. To com-
plete the path join the dots with
a smooth curve.
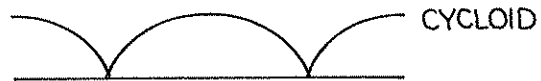
CYCLOID
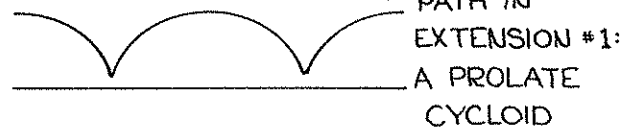
PATH IN
EXTENSION #1:
A PROLATE
CYCLOID
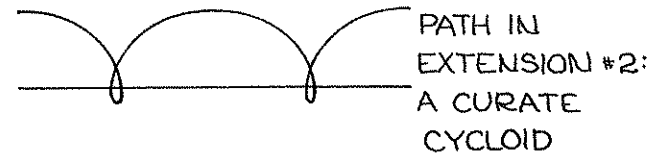
PATH IN
EXTENSION #2:
A CURATE
CYCLOID

d) To sketch the path described in extension #2, tape a small strip with a hole
punched in it to the spoke numbered 0. Roll the disk along the numbered line
and mark dots as before. Join the dots in order with a smooth curve.

On wheels that have flanges, such as the wheels of a train, the points on the
flanges travel a path like that in extension #2. The points actually move
backwards while they make the tiny loop below the level of the path.

IV As the coffee can is rolled steadily along the table does the marker move faster
on the top of the can or on the bottom? Surprise! Relative to the ground the
marker moves faster on the top than on the bottom. To understand this examine
the path of the marker.
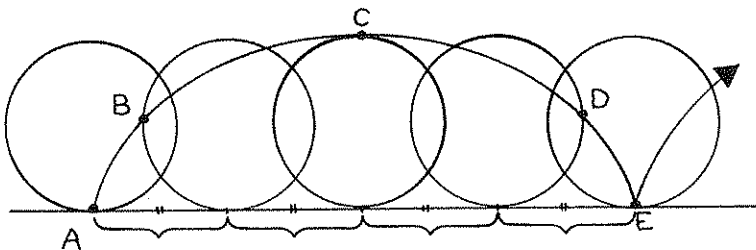
The diagram shows five posi-
tions of the can: starting
position, 1/4 turn, 1/2 turn,
3/4 turn, 1 revolution.

As the can rolls equal distances along the table the marker does not travel equal
distances along the cycloid path. The distance from B to C is much longer than
the distance from A to B so the marker must move faster as it approaches C. The
marker actually comes to a stop when it touches the table.

This phenomenon can be demonstrated by covering the bottom of the coffee can with
white paper that has 16 spokes marked on it. As the can is rolled past your line
of vision fix your gaze on a distant object so that your eyes don't follow the
can. You will find that the spokes are visible only in the lower half of the can
while the upper half is a blur.

Artists have used this fact to indicate motion of wheels that have spokes--they
show distinct spokes only below the axles. Bicycles with reflectors on the spokes
also exhibit this phenomenon. Bring some examples or suggest that students look
for pictures showing this.

IDEA FROM: *Martin Gardner's Sixth Book of Mathematical Games from Scientific American; Curves and
their Properties;* and *Mathematics A Human Endeavor*

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A CAN CAN DO IT
(PAGE 3)

V   The cycloid possesses other interesting properties:

1)   Christopher Wren, a distinguished English architect, showed (in 1658) that the length of the arch (from A to E, see diagram on the previous page) is four times the diameter of the circle.

2)   The area below the arch is three times the area of the circle.

3)   The cycloid is the curve of equal descent. That is, if two marbles were placed on an inverted cycloidal track at different heights and released at the same time they would roll along the track and reach the bottom in the same length of time. (Ignore friction.)

BOTH MARBLES WILL REACH POINT A AT THE SAME TIME
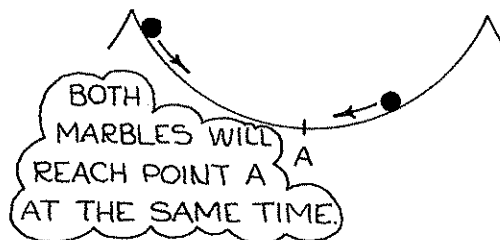
4)   The cycloid is the curve of quickest descent. That is, given two points: A and B (B lower than A but not directly below it), the path from A to B along which a marble can roll, without friction, in the shortest time is an inverted cycloid.
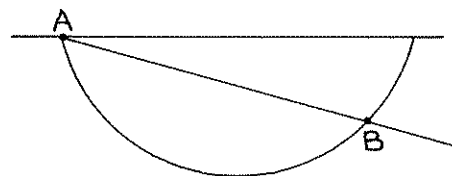
This means that rolling a marble from A to B along a straight ramp will take more time than traveling along the cycloid even though the marble has to roll uphill on the cycloid to reach B!

The property could be demonstrated by bending heavy wire into the shape of a cycloid. Also use wire for the straight ramp. Release two nuts along the wires at point A and observe which reaches point B first. You might design tracks of other shapes to test to convince students that the cycloid is the curve of quickest descent.

5)   The cycloid is the strongest possible arch for a bridge. Many concrete viaducts have cycloidal arches.

SIDES OF GEAR
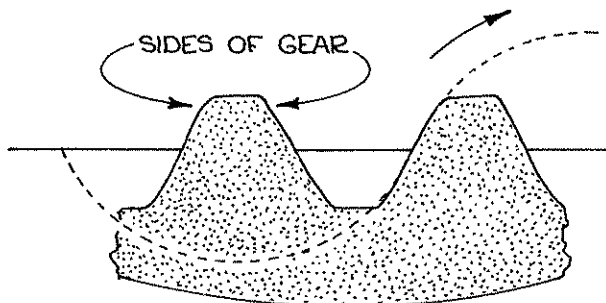
6)   Cogwheels are often cut with cycloidal sides to reduce friction by providing a rolling contact as the gears mesh.

IDEA FROM: *Martin Gardner's Sixth Book of Mathematical Games from Scientific American; Curves and their Properties;* and *Mathematics A Human Endeavor*

POINT OUT THE PATH

I

Two pennies are arranged like this.

The top one rolls halfway around the bottom one without slipping.

Which of the two diagrams to the right shows how the pennies will look? _____

Check your answer by using two pennies. Were you right? _____

#1   #2
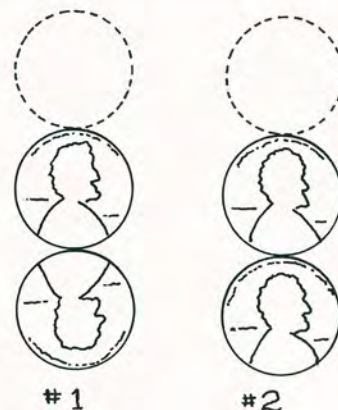
Try the experiment on some of your friends.
Can you explain why some might make the wrong guess?

II As the top penny rolls around the bottom one what kind of a path will the point P trace?

a) To help you sketch the path trace two copies of the disk below on notebook paper and cut out one copy.

b) On the cut-out disk cut a small notch at spoke #0 to represent point P.

c) Place the cut-out disk next to the traced disk so that the notch matches spoke #0.

d) As the disk rolls around mark a dot at the notch each time the numbers on the two disks match.

e) To complete the path join the dots with a smooth curve.

III Sketch the path you think point P will trace if the disk rolls around a circle of twice its diameter.

Check your sketch by using the circle to the right.
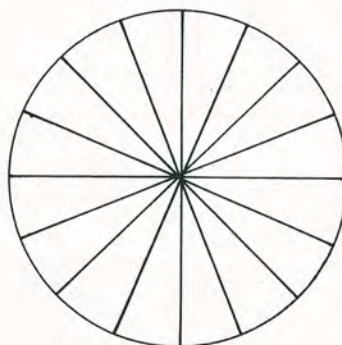
IV Draw the path of point P if the disk rolls around a circle of diameter three times that of the disk.

GROOVING AROUND

1)

Gear A turns in a clockwise direction.
In what direction do you think gear B will turn? _____

2)

When gear Y makes one complete turn, how much of a turn will gear X make? _____

How might these gears be used in counting? _____

Can you think of any measuring devices around the home that use this idea? _____

3)

If gear A makes three turns how many turns will gear B make? _____ gear C? _____

Suppose gear B is removed and gear C touches gear A. How many turns will gear C make when gear A is turned three times? _____

Does gear B make a difference in the ratio of turns for A to C? _____

4)

48 TEETH

16 TEETH

32 TEETH

8 TEETH

In the diagram the two dotted gears are joined together so that they turn at the same speed. The other two gears turn separately.

How many times does the 16-teeth gear turn for every turn of the largest gear? _____

Can you think of something you look at often which needs two gears that turn in this ratio?

_____

5)

A

B

Sometimes gear wheels are at right angles to each other. Why are the gear wheels arranged this way? _____

Can you find some examples of these gears in your home? _____

If gear A has 30 teeth and gear B has 50 teeth, how many times will gear B turn when A makes 5 complete turns? _____

SPIROLATERALS

A) The 1-2-3-4-5 pattern: The figure shows the pattern 1E, 2S, 3W, 4N, 5E. Continue the pattern by moving 1S, 2W, 3N, 4E, 5S, 1W, 2N, etc., until the spirolateral returns to the starting point.

How many times was the pattern 1-2-3-4-5 repeated to complete the spirolateral? _____
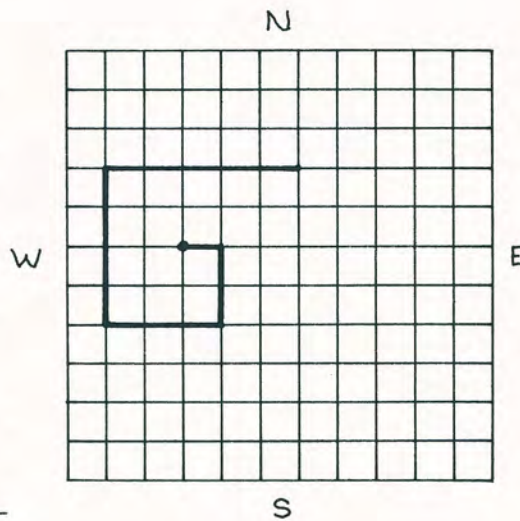
B) Continue the 1-2-3 pattern below to draw a spirolateral.
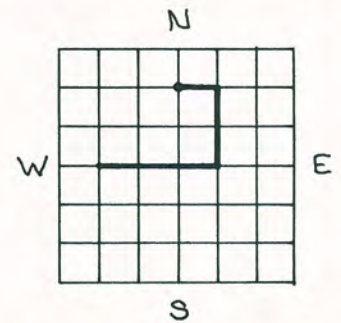
How many times was the pattern repeated? _____

C) Use grid paper to help you fill in this chart.

Number pattern	Number of times pattern was repeated
1	
1-2	
1-2-3	
1-2-3-4	
1-2-3-4-5	
1-2-3-4-5-6	
1-2-3-4-5-6-7	
1-2-3-4-5-6-7-8	

PREDICT THE NUMBER OF PATTERN REPEATS FOR 10 NUMBERS, 15 NUMBERS, 24 NUMBERS.

D) Predict what will happen when you draw a spirolateral using the pattern 5-1-4.

E) Make a spirolateral using your telephone number.

F) Make a spirolateral using any five numbers from the Fibonacci sequence, 1, 1, 2, 3, 5, 8, ...

G) Explain why some spirolaterals return to the starting point and others don't. (Hint: consider E-S-W-N.)

ARCHIMEDEAN SPIRALS

The spiral below is an example of an Archimedean spiral. The movement away from the center is constant for a given amount of rotation. (The spiral below moves out 1/10 inch for each 30° rotation.) This type of path would result if you walked at a uniform speed from the center of a moving merry-go-round towards the outer rim.
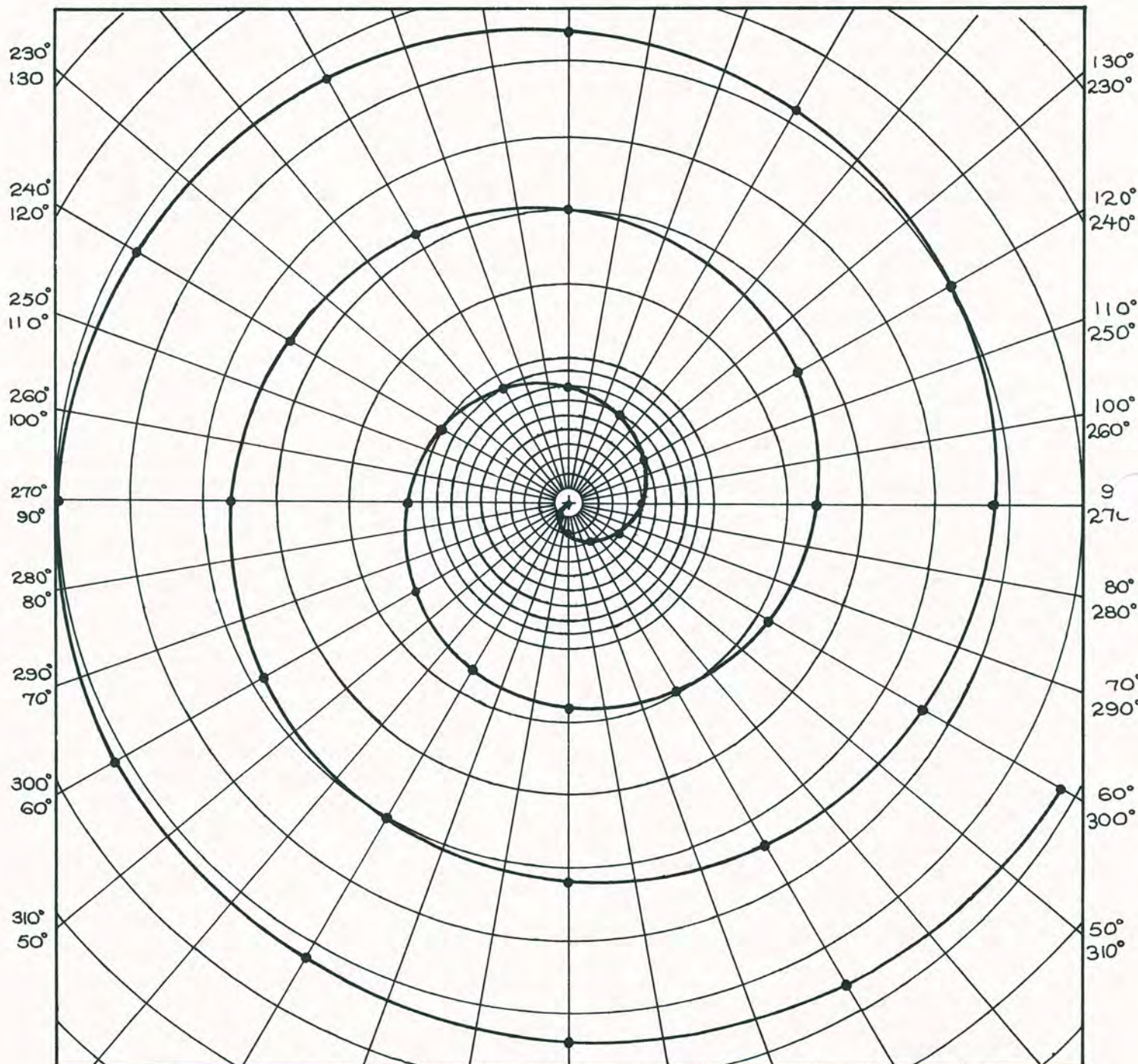
Students can approximate the Archimedean spiral by placing a cardboard disk on a record player and moving a felt tip pen or crayon at a uniform speed from the center to the outside edge. It may be necessary to provide a guide for moving the pen in a straight line as the turning of the disk will tend to pull the pen sideways.

IDEA FROM: *Patterns in Space*

POLYGONAL SPIRALS
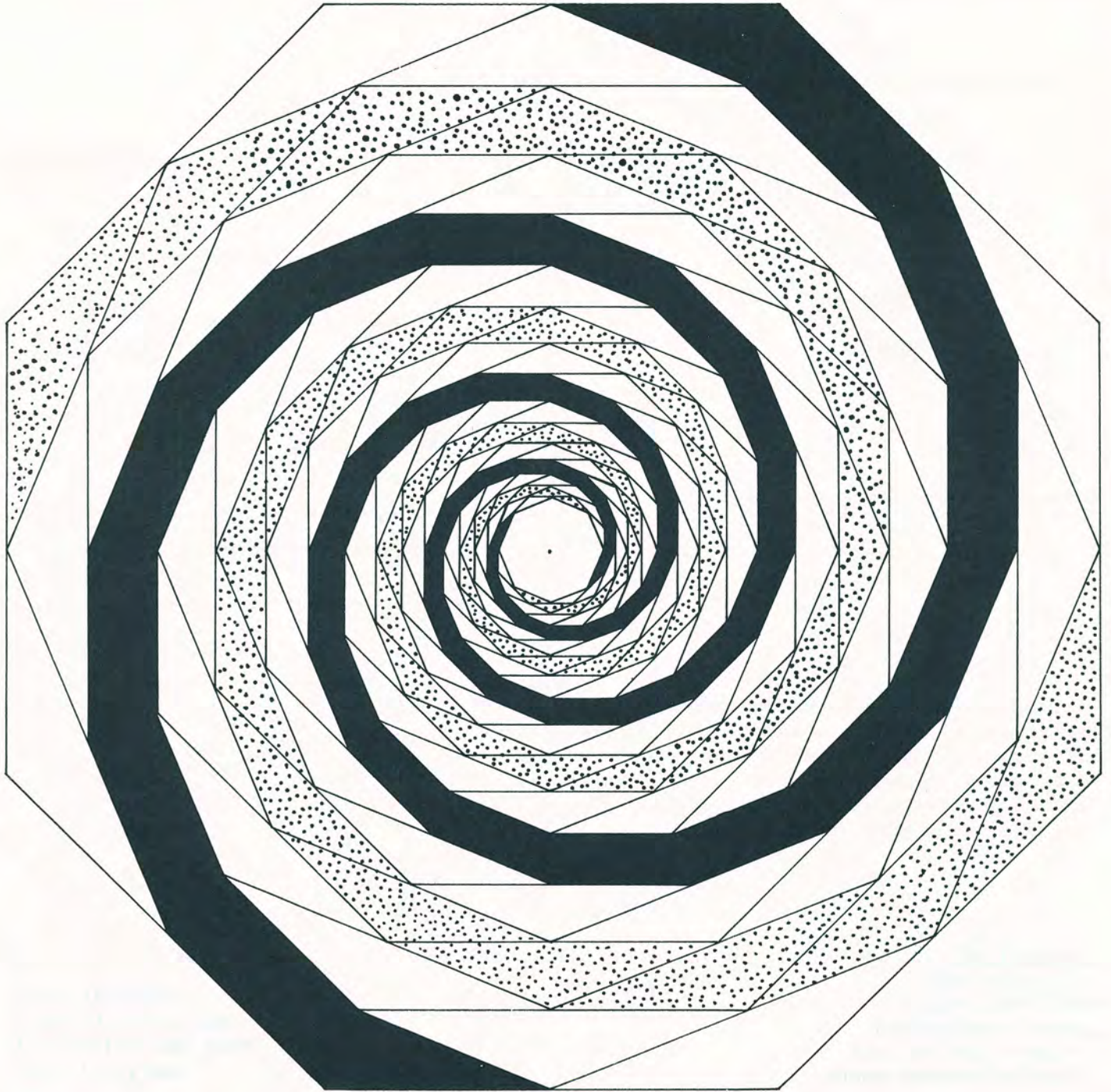
IDEA FROM: *Patterns in Space*

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MORE SPIRAL-LIKE DESIGNS

Two segments, perpendicular to each other and whose lengths are in a given ratio, can be used as a basis for constructing a spiral-like design. In figure 1 the ratio of the lengths is 7:10 (7 cm to 10 cm for the two longest lengths) and in figure 2 the ratio is 9:10.

Figure 1
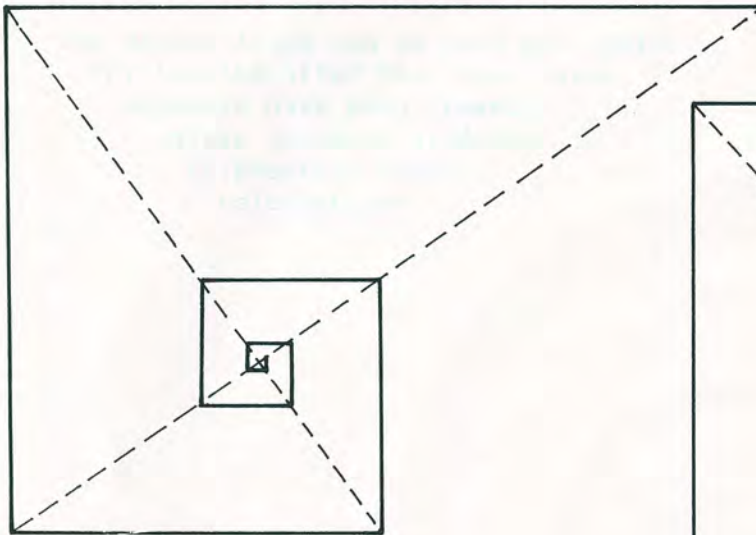
Figure 2
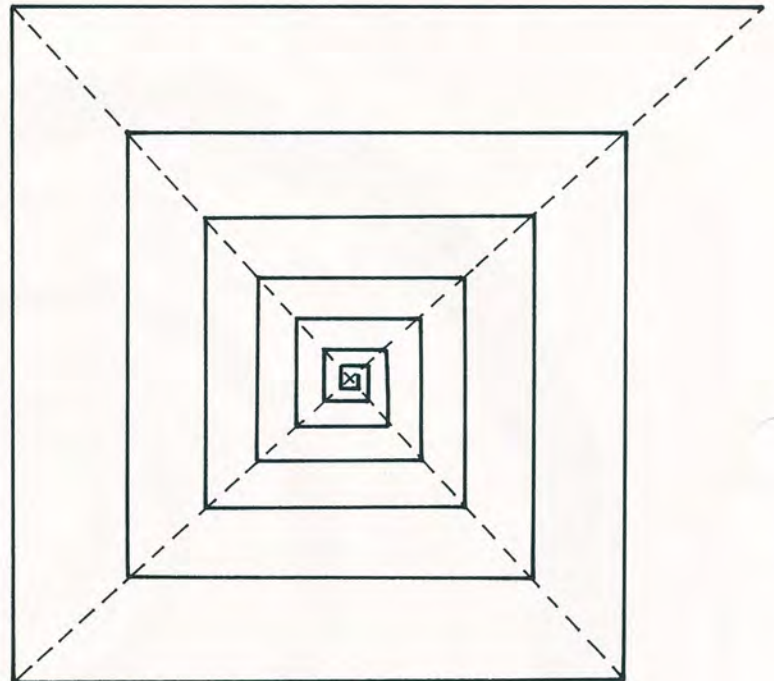
Starting with an isosceles right triangle with legs of 1 unit, a spiral-like series of segments, many of which have lengths that are irrational numbers (numbers which cannot be represented as the quotient of two whole numbers), can be drawn.

$\sqrt{25} = 5$ $\sqrt{3}$ $\sqrt{2}$ $\sqrt{33}$ $\sqrt{16} = 4$ $\sqrt{10}$

IDEA FROM: *Patterns in Space*

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A SPIRAL IN A GOLDEN TRIANGLE

A golden triangle is one of the triangles found in a regular pentagon by connecting the end points of a side to the opposite vertex. (See △ABC.) It is an isosceles triangle with angles of 36°, 72° and 72°. The ratio of BC to AB is the golden ratio which is approximately .618.

Beginning with a golden triangle ABC, a series of similar golden triangles can be constructed with these steps.

1) Bisect angle B.
2) Label the new point D.
3) Triangle BCD is a golden triangle.
4) The ratio of CD to BC is approximately .618.

More golden triangles can be constructed by bisecting in order angle C, angle D, etc.

As a suggested activity have students:

A) construct a golden triangle with a base of 15 centimetres. This fits on a $8\frac{1}{2}$ x 11 sheet of paper if the base is drawn near the bottom.

B) construct a series of similar golden triangles.

C) measure to the nearest millimetre $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FG}$, ...

D) use a calculator to find the ratios BC:AB, CD:BC, DE:CD, EF:DE, FG:EF, ... The ratios should all approximate the golden ratio.

E) draw a smooth curve through the vertices A, B, C, D, E, F, ..., to form a spiral.

THIS WON'T GIVE YOU CARDIOID ARREST

DRAW THE CHORDS CONNECTING
THESE POINTS.

1) 1-19	7) 11-33
2) 7-31	8) 5-27
3) 6-29	9) 15-35
4) 9-32	10) 2-21
5) 17-36	11) 13-34
6) 3-23	12) 4-25

DRAW THE INSCRIBED ANGLES
NAMED BY THESE POINTS.

1) 24-29-3	7) 35-9-14
2) 23-27-35	8) 16-13-7
3) 12-5-30	9) 26-33-8
4) 28-1-10	10) 15-17-18
5) 3-11-15	11) 22-25-31
6) 20-21-23	12) 18-19-20

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GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

CONTENTS

● Have students hold two coins of different sizes in front of one eye as shown to the right. Can they position the coins so the smaller coin visually matches the larger coin? Can this be done so the larger coin is between the eye and the smaller coin? Give the students pairs of polygonal figures. Have them position the figures in front of one eye to see if the pairs will match. Tell them that two plane figures are similar if their outlines can be made to match when they are held parallel to each other in front of the eye. This meaning for simi- larity should help students realize that many rect- angles, triangles and ellipses are not similar, while all circles, squares, regular hexagons and equilateral triangles are similar.

THE COINS ARE HELD PARALLEL TO EACH OTHER.

● The pages *Shadow Stumpers* and *4 to Make 1* are good intuitive introductory activ- ities to figures with the same shape.
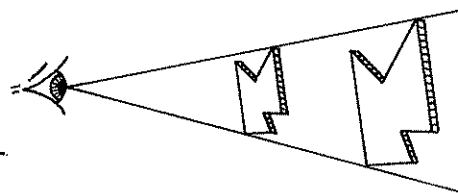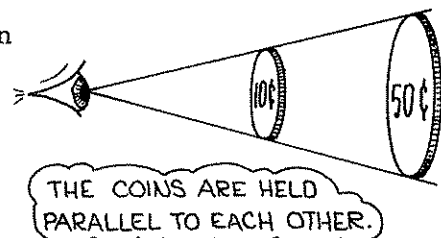
Although many figures can be seen as roughly similar without measuring their cor- responding parts, it is often convenient to measure figures to determine if they are similar. Are the rectangles to the right similar? By measuring in millimetres it can be seen that the length of the second rectangle is twice that of the first rect- angle, but the width is not twice as much. The two fig- ures are not similar. They cannot be made to match in front of the eye. You might want students to measure the pairs of polygons used in the introductory activities described above. Having figures with linear dimensions in whole centimetres which are double or triple the dimensions of similar figures will help students make a comparison. Students might also be encouraged to place the corners of the polygonal pieces over each other so they realize the angles in similar figures must stay the same. These intuitive ideas can be formalized later into ratio notation and condi- tions of similarity.
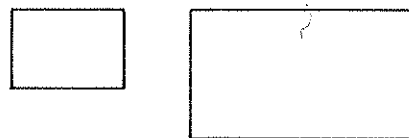
MAKING SIMILAR FIGURES

There are many methods for producing similar figures. Most of the methods for making scale drawings are also appropriate for producing similar figures. Scale drawings often produce similar figures, but there are some exceptions. A scale draw- ing can be made of a car, but the drawing and the car are not similar figures because the car is three-dimensional and the scale drawing is two-dimensional. A three- dimensional scale model of the car would be roughly similar to the actual car. In a

SIMILAR FIGURES

There are many everyday objects which appear to have the same shape but not necessarily the same size. Baseballs, volleyballs and basketballs are of different sizes, but they all have roughly the same shape--that of a sphere. Some model cars, airplanes and boats have the same shape as their full-sized counterparts. More examples are given on *Look Alikes* and *Similar Figures*.

Figures which have the same shape but not necessarily the same size are called similar figures. If two figures are similar, their corresponding angles have the same measure and their corresponding dimensions are in a certain ratio. In the case of the car and the model car shown in *Look Alikes*, if the car is 15 times as tall as the model, it is also 15 times as long and 15 times as wide.

The word "similar" is used in everyday language to mean "having some common characteristic." This meaning of similarity can cause problems with its more precise meaning in geometry. To help avoid this confusion you might try some of these introductory activities while using the words "same shape" as well as "similar."

- Put differently sized prints of the same photograph on a poster or bulletin board (students might donate their school pictures for a short time). Point out that these figures all have the same shape. The pictures are just reductions or enlargements of the original negative. The face in the photograph becomes both wider and larger with each enlargement.

- Use an overhead projector to produce similar figures. (Be sure your screen is perpendicular to the projection rays or the figure on the screen will be distorted.) Students might enjoy using the overhead or opaque projector to enlarge a favorite cartoon or picture.
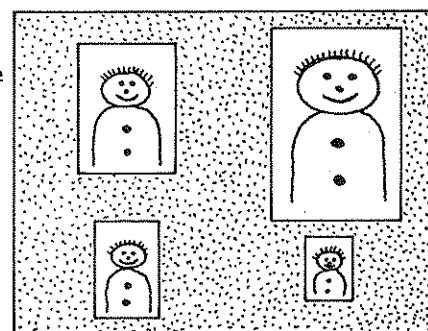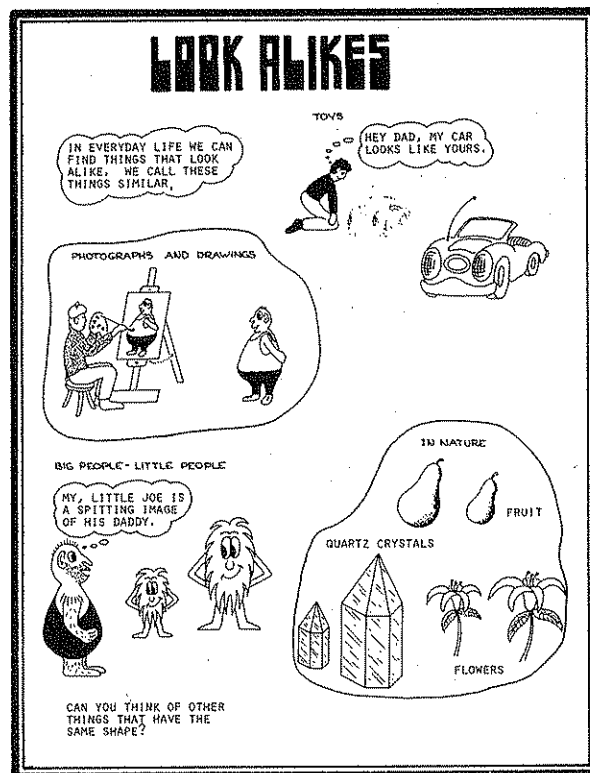
like manner, a map can be a scale drawing of a mountainous region, but the map and the region are not even roughly similar figures. More information and activities on scale drawings can be found in the SCALING section of the resource Ratio, Proportion and Scaling.

Using Grids and Related Ideas to Make Similar Figures

Grid paper of different sizes can be used to make planar figures that are roughly similar. The original can be enlarged or reduced in size simply by changing the size of the grid. See *Navajo Blanket Designs, Patterns for Grids* and *Grid Me Up*. Each square on the original grid corresponds to a square on the reduced grid. This is easy and enjoyable for most students, but some could profit from practice in copying designs onto grids of the same size before attempting to make a smaller or larger design.

This same principle can be applied to some 3-dimensional figures. A shape built from cubes can be enlarged or reduced by using larger or smaller cubes. You might want to develop an activity around this idea, especially if you want to use the more difficult similarity activities suggested below and in *Getting Bigger II.*

Grids with the same sized squares can also be used to draw enlargements or reductions. In the example to the right the edge of one square in the original corresponds to the edge of two squares in the enlargement and to one-half of an edge of a square in the reduction. This method is more difficult for students, but if it is introduced carefully, fifth and sixth-grade students can use it successfully. You might have students mark the edges of grids to correspond to the original grid as shown. Locating particular points and
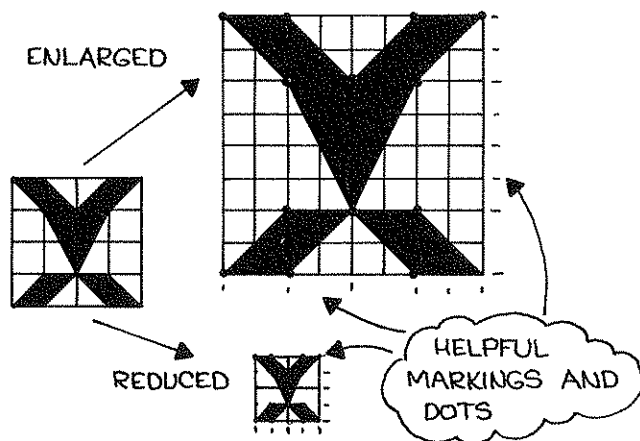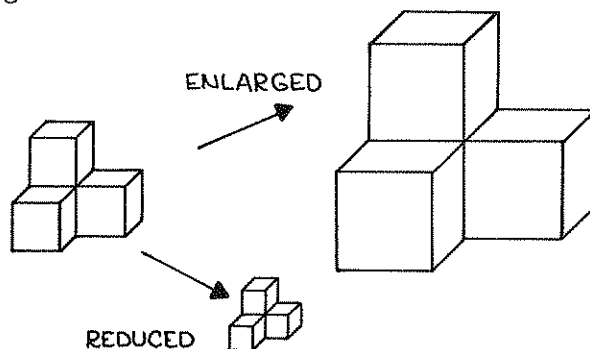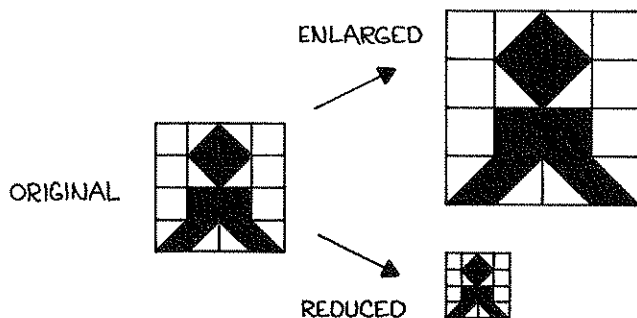
marking them with dots can also help students make accurate similar figures on grids.
Getting Bigger I gives introductory activities for using grids to draw similar fig-
ures. Some of these ideas are applied in *How Does Your Billiard Table Size Up?* The
page *Grid Pairs* gives students a method for drawing similar polygons by doubling or
tripling the coordinates of the vertices.

Similar three-dimensional figures
can also be produced using this princi-
ple. Some students (even fifth and
sixth-graders) quickly catch on to the
method of replacing <u>each</u> little cube
with a larger cube made from eight
little cubes, but others find the pro-
cess quite perplexing. Here are some
of the figures students have been
observed to build when asked to build
a shape similar to the original shown.
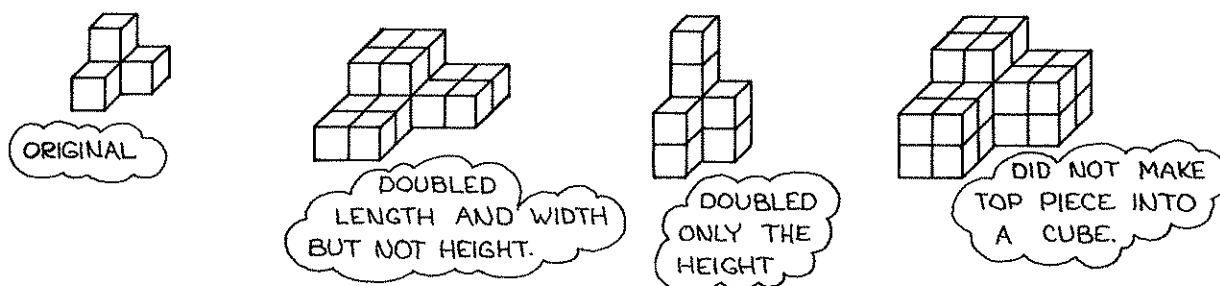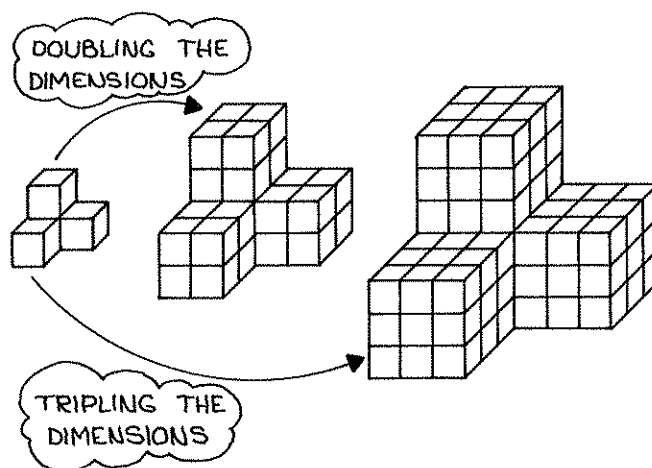
You might want to work with a small group of students, showing them a <u>simple</u> model
and having them build a larger, similar figure using the same sized cubes. Some
suggestions are given on the page
Getting Bigger II. You could also
reverse the procedure by showing stu-
dents a large model and asking them
to build a smaller, similar version.
Be sure you make your original figure
so it is possible to reduce.
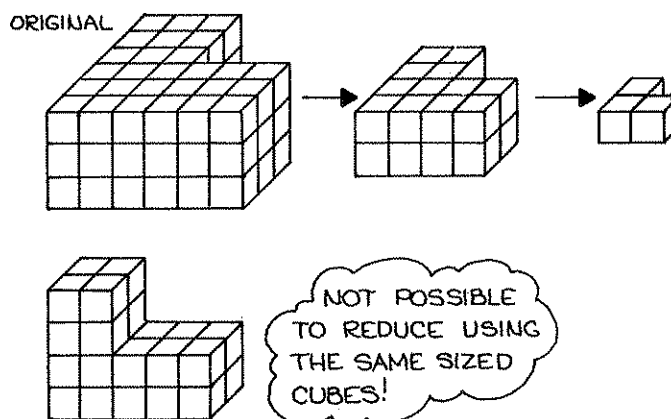
Using Rulers to Draw Similar Figures

Perhaps you have seen a triangular ruler like the one shown here. Rulers like this are used by architects and engineers for making scale drawings. There is a pattern for making an architect's ruler on the student page *Archie Texs' Ruler*. The page should be run on tagboard to make a sturdy ruler. The students are asked to complete the six number lines according to the given scales as shown to the right.

ENLARGEMENT

ORIGINAL

ORIGINAL

REDUCTION

Enlargement. To make an enlargement, measure each side of the original using the 1:1 scale, then reproduce the figure using one of the other scales. Notice that the _units_ change when making the scale drawing, but the number of units read on the ruler stays the same.

Reduction. To make a reduction, measure each side of the original using one of the scales and then reproduce the figure using a scale labeled with a smaller ratio. See the example at the left.

Architect's rulers are useful, but most of our measuring is done with common inch or metric rulers. Using an ordinary ruler with one number line to draw a similar figure involves a different process than using the architect's ruler which has several number lines. Using an architect's ruler with its different scales is much like using differently sized grids. Using an ordinary ruler with one number line is much like using grids of the same size.

To make an enlargement with a centimetre ruler, the <u>number</u> of units for each linear measurement must be multiplied by a constant. 2 cm→4 cm, 5 cm→10 cm, ... In other words, the <u>units</u> stay the same, but the <u>number</u> of units changes.

After some practice in using rulers to enlarge rectangles, triangles and other shapes, your students can try enlarging more difficult patterns. Some suggestions are given in *Constructing Cartons.*   Students might not know the angle measurements must remain the same. Encourage them to trace the angles or to use a protractor.

<u>Using Projection Points to</u>
<u>Draw Similar Figures</u>

The Renaissance painters were interested in depicting the natural world. The specific problem they coped with was that of painting three-dimensional scenes on canvas. The solution was the creation of a new system of mathematical perspective. The most influential of the artists who wrote on perspective was Albrecht Durer. Durer thought of the artist's canvas as a glass window through which the scene to be painted is viewed. From one fixed point lines of sight are imagined to go through the artist's canvas to each point of the scene. This set of lines is called a <u>projection</u>.

Durer's method is very handy for drawing similar figures. In the figure shown to the right, triangle A'B'C' is an enlargement of triangle ABC. To obtain this enlargement, the points A, B and C are projected from Point P so that the points A', B' and C' are twice as far from point P as the corresponding points A, B and C. By this method the sides of △A'B'C' are reproduced twice as long as the sides of △ABC.
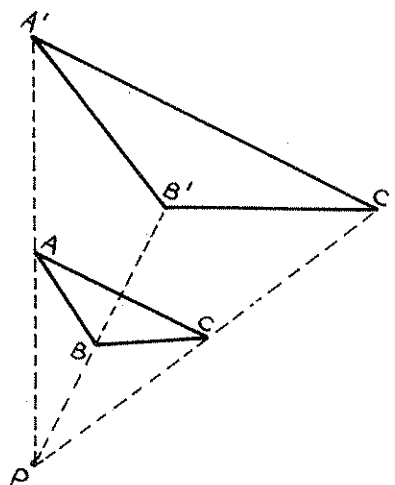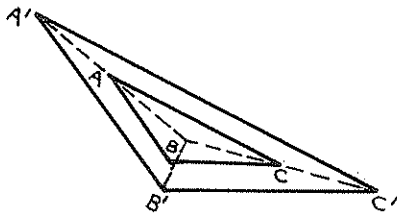
If we place the projection point P inside △ABC as shown at the left and then project the points A, B and C out twice as far from P, we again obtain an enlargement, △A'B'C', in which each side is twice as long as the corresponding side of the original.

Projection points can also be used to produce similar figures which are "upside down." In the example at the right, each point on the original flag has been projected to a point which is twice as far from the point P but in the opposite direction.
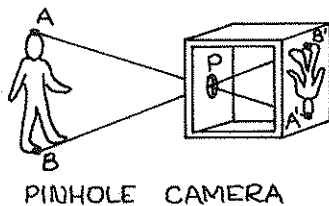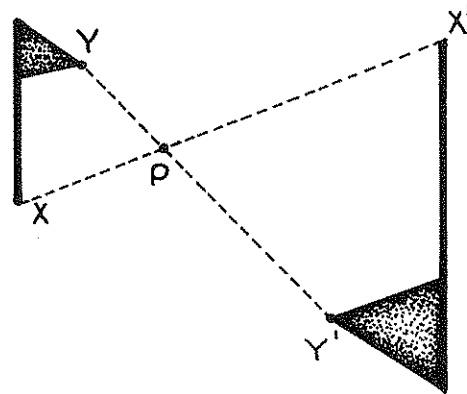
The lenses of our eyes and of cameras invert the images of scenes much like the projection above. The scene is reproduced upside down on the retinas of our eyes and on the film of a camera. Your class might like to make a pinhole camera. You can find plans for such a camera in World Book Enclyclopedia.

PINHOLE CAMERA

The following student pages use projection points with similar figures: *Bigger Than Life, A Shrink, Where's the Point, Enlargements and Reductions, A Negative Feeling* and *Enlargements of Enlargements.*

Using a Pantograph to Make Similar Figures

A pantograph is a mechanical device for enlarging and reducing figures. It can be constructed easily from four strips of cardboard or from an erector set.

These four strips are connected so the strips can move freely. Point P acts like the projection point and should be held fixed. As point D is placed over each vertex of a polygon, a pencil at point A can be used to mark each vertex of the enlargement.

As the new points are found, the ratio PA:PD remains the same. In the picture above PA is twice as long as PD, and we say the scale factor is 2. In order to use the pantograph for a reduction, the pencil should be placed at point D, and point A should be placed over various points of the original.
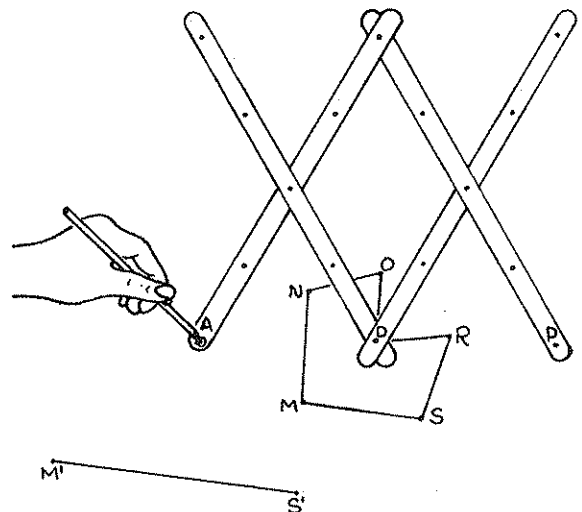
The pantograph on the student page has just one set of holes for enlargements with a scale factor of two.   There are several holes in the arms of the pantograph shown below to allow for different ratios of PA to PD.   The following illustrations show two more settings of the pantograph.
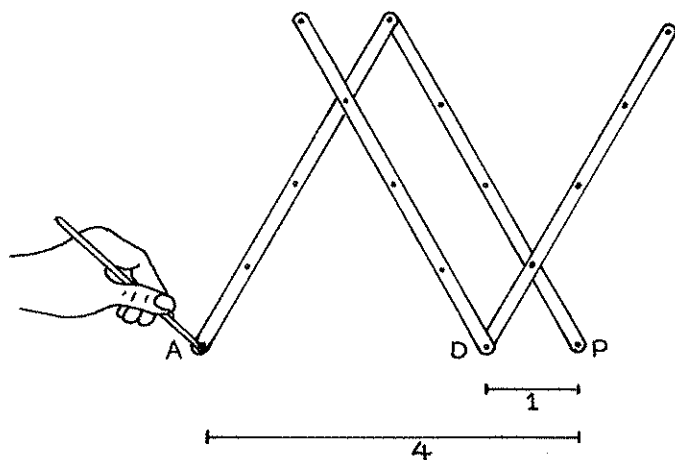
As the pantograph is changed so that D moves closer to P, the ratio $\overline{PA}:\overline{PD}$ gets larger. In this illustration the scale factor is 4.
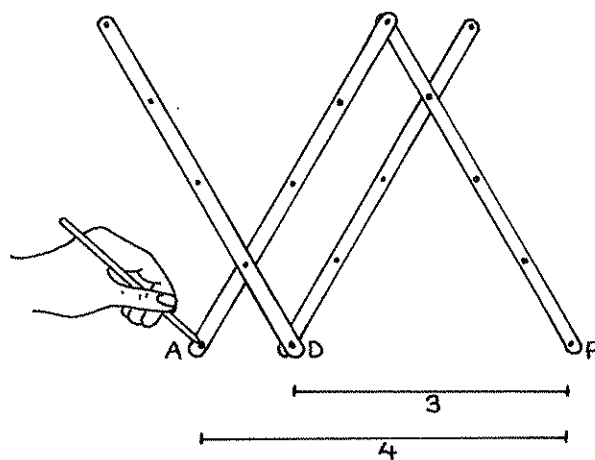
As D moves farther from P, the ratio $\overline{PA}:\overline{PD}$ gets smaller. In this illustration the scale factor is $\frac{4}{3}$.

Pantographs are not always made out of rigid material. Your students might enjoy using the rubber band pantograph described in *A Snappy Solution to Similar Figures*. The teacher page *Similar Figures and the Pantograph* gives more formal teacher background for understanding why the pantograph produces similar figures.

A MORE EXACT TREATMENT OF SIMILAR FIGURES
Although much work can be done with similar figures on an intuitive level, it is sometimes necessary to use more formal sentences such as, "The corresponding angles of similar figures are congruent" or "Measure the corresponding sides of these similar figures. What do you notice about the ratios of the lengths of the corresponding sides?" The idea of corresponding parts of similar figures is essential in much work with similar figures, but many students have trouble determining the pairs

of corresponding sides of two figures which are not oriented the same. You might cut out some congruent shapes and similar shapes to hold in front of the class or to display on an overhead. Have students identify corresponding sides and angles when the shapes are turned or flipped. The page *Buddies* to the right can give some skill-building practice in identifying corresponding sides and angles.

Students can begin to reexamine figures which appear to have the same shape. They can measure corresponding angles and see that they are congruent. They can measure corresponding sides and see the lengths are in the same ratio as suggested in *Geoboard Shapes II,* *Comparing Similar Solids, Reflect on That Image,*   and *Little One-- Big One.* At this point a more exact meaning for "similar" can be verbalized. "Two figures are similar whenever their corresponding angles are congruent and the lengths of corresponding sides are in the same ratio."

After this generalization is clear to students, they can be asked questions like that shown to the right. In answering such questions students will have to remember that the sum of the angle measurements in a triangle is 180°. They will also use proportions to find the missing measures. *Roses are Red . . .* is a worksheet of problems like that shown to the right. On that page students are told

These two figures are similar. Give the missing measurements for the second figure.

each pair of figures is similar. What if the problem were reversed? Can it be determined that two figures are similar or not similar when some of their measurements

are given? Both pairs in the problem
to the right might look similar, but
how can we be sure? *Make it Easy on
Yourself* gives four activities for
determining the similarity of triangles.
The first two activities have students
discover that two triangles are similar
when two angles in one triangle are con-
gruent to two angles in the other (this
is not true for quadrilaterals, etc.).
The third and fourth activities have
students discover that two triangles are
similar if the ratios of the lengths of their corresponding sides are equal. For the
problem above, (a) is a pair of similar triangles because of the pairs of congruent
angles. (b) is not a pair of similar triangles because $\frac{10}{5}$ does not equal $\frac{8}{3}$ or $\frac{7}{3}$.

Which of these are pairs
of similar figures?

(A) 55° 30° 95° 30°

(B) 7 8 10 3 5

You will find many other ideas for exploring similarity in the classroom mater-
ials of this section. Many of the pages are written in the form of guided discovery
lessons with various ratios and ratio relations being stressed.

APPLICATIONS OF SIMILAR FIGURES

Scale drawings (similar figures) are very useful in art, architecture and other
areas. The grid activities in the beginning of this section are examples of ways an
artist might enlarge or reduce a design. Architects need to draw and read the scale
drawings on blueprints. Many patterns for woodshop projects or sewing projects
must be enlarged before they can be used. The grid method is very helpful here,
also. Students can use rulers, protractors, pantographs or opaque projectors to
make larger, similar patterns for line designs, posters or paper airplane patterns.

One important use of similar fig-
ures is making scale drawings to find
distance. (Related student pages are
Measure Up and *Far Out.*) Suppose a
200 m kite string makes a 30° angle
with the ground. How high is the kite?
To solve such a problem we can make a
scale drawing, measure the corresponding

HOW
HIGH ?
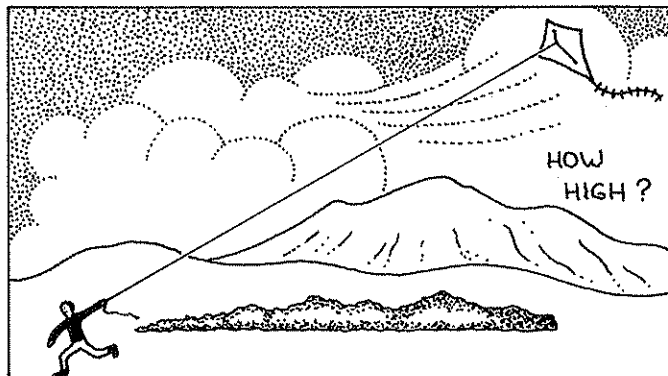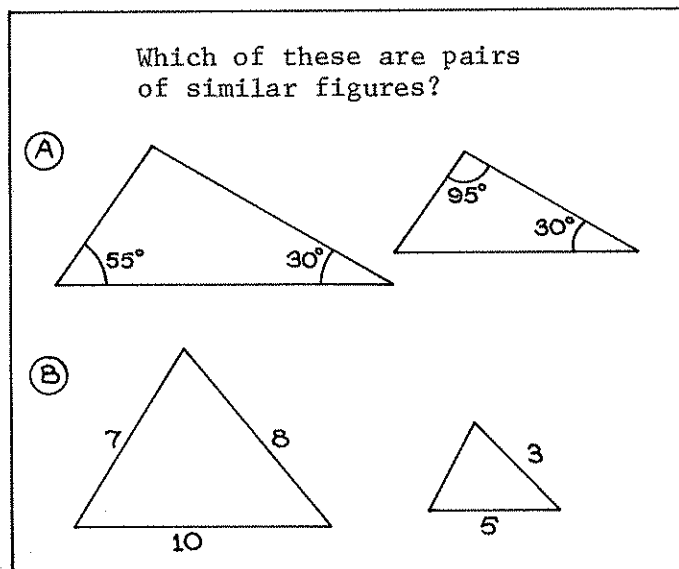
side of the drawing and compute the
approximate height of the kite.   In
the scale drawing to the right, 1 cm
represents 40 m.   The height of the
triangle is 2.5 cm, which represents
100 m.   Thus, the height of the kite is
about 100 m.   If you use this example,
don't be surprised if the kite fliers in your class question the accuracy of the
result.   Kite strings make a curve from the ground to the kite and the "scale draw-
ing" does not take this curve into account.   Here is a good chance to remind students
that the accuracy of mathematics must be checked against the reality of the situa-
tion.   For our purposes, the approximate scale drawing is fine.   A meteorologist
would need a more exact reading of a weather balloon's height.
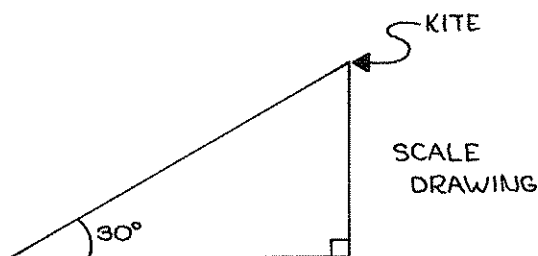
The kite problem involved a 30° right triangle.   The
side opposite the 30° angle was one-half as long as the
hypotenuse.   Will this always be true?   Students can be
encouraged to construct and measure arbitrary 30° right
triangles so they will be convinced of this invariant
relationship.   They could also construct right triangles with one side having length
one-half that of the hypotenuse.   Does the angle opposite this side measure 30°?   Can
students now answer a question like, "A 250 m kite string makes an angle of 30° with
the ground.   How high is the kite?" <u>without</u> making a scale drawing?   That is, can
they see the side of the triangle would be one-half of 250?
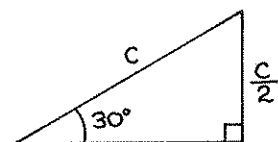
The ratio relationships in a 30° right triangle occur many times in applied
mathematics.   The same is true for the 45° right triangle.   If students work with
these triangles often, they might be able to apply the relationships without resort-
ing to the drawings.   Other right triangles do not have such simple ratios, but
students should realize that the ratios in one 75° right triangle will be the same
as the ratios in the next 75° right triangle.   *Right On Ratios* ask students to explore
ratios in various right triangles.

Hypsometers and transits can be very useful in making scale drawings to measure
distances.   With a hypsometer, outdoor heights can be approximated.   Transits can be
used to measure angles.

Surveying the Situation describes how to make and use a transit.   Some
students will be more interested in making scale drawings if they can use tools like
these.

Similar figures provide an opportunity to allow students to pursue projects of their own choosing. Do you have students who are like those below? Perhaps each group can "do their own thing."

We like art. We'll use the relations in similar figures to enlarge designs.

We like to be outdoors. We choose to survey the school grounds and make a scale drawing of the school.

Number and geometric relations are our bag. We want to dig into more of the investigations in the last part of this classroom section.
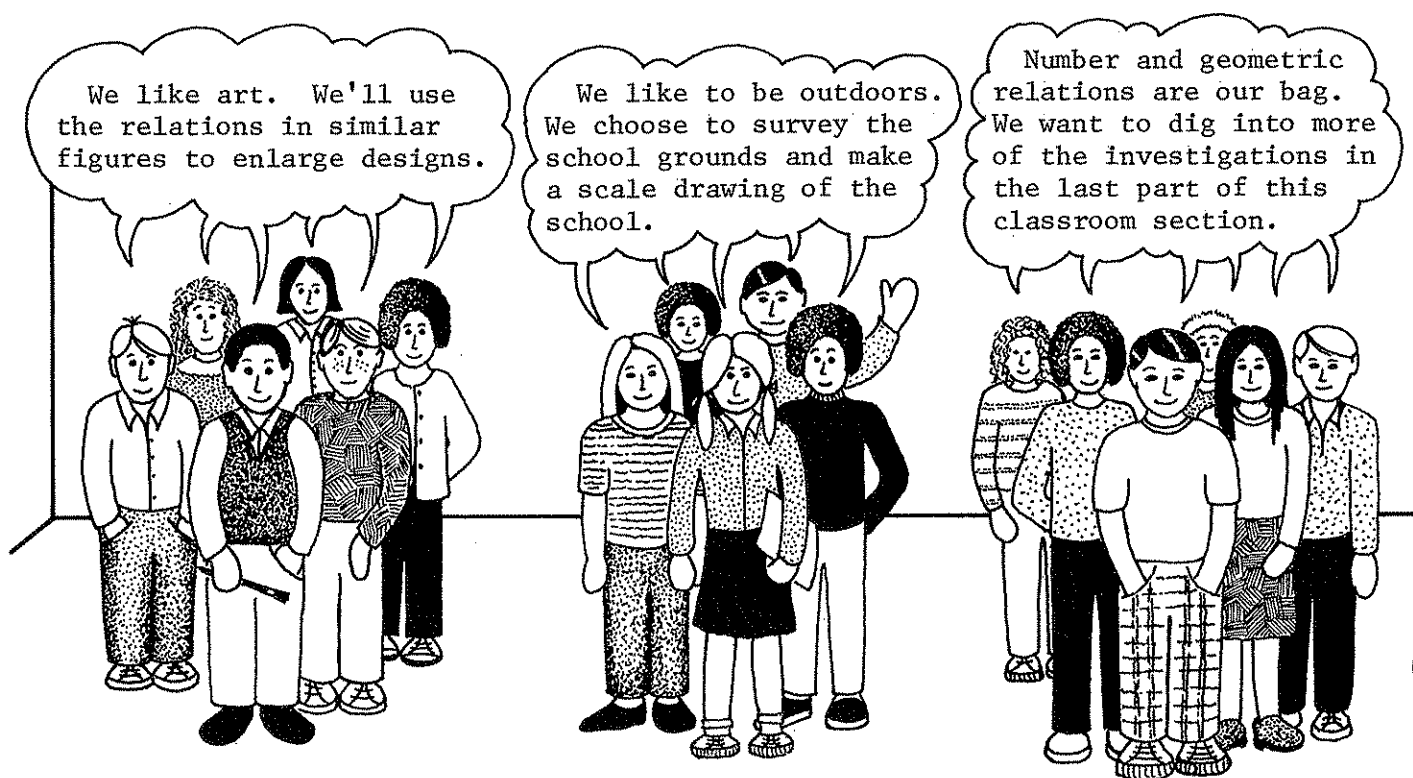
LOOK ALIKES

TOYS

IN EVERYDAY LIFE WE CAN FIND THINGS THAT LOOK ALIKE. WE CALL THESE THINGS SIMILAR.

HEY DAD, MY CAR LOOKS LIKE YOURS.

PHOTOGRAPHS AND DRAWINGS

IN NATURE

FRUIT

QUARTZ CRYSTALS

BIG PEOPLE - LITTLE PEOPLE

MY, LITTLE JOE IS A SPITTING IMAGE OF HIS DADDY.

FLOWERS

CAN YOU THINK OF OTHER THINGS THAT HAVE THE SAME SHAPE?

SHADOW STUMPERS

Have students decide which of the following shadows can be cast by a soft drink cup.

A Circle   A Trapezoid   A Triangle   An Ellipse

Shadows can be effectively used in studying similarity.

Two seemingly unrelated objects may have similar shadows. For example, a soft drink cup and a toy football can both cast circular shadows.

Show students two solids and let them decide if similar shadows can be cast. Have them sketch what the shadows will look like.

A good warm-up activity is to have students draw all the images they think are possible from a given solid. Let them compare these with friends. To resolve any question you can look at shadows.

A beginning list of objects that will help is:

Any rectangular box   Pencils
Toy football   Dowels
Soft drink cups   Coins
Nuts and bolts   Books
Toy balls   Ice cream cones
Buttons   Geoblocks
Straws   Soma cube pieces
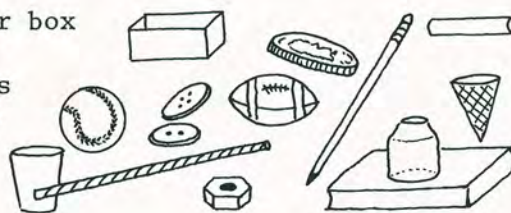
There are several ways to produce good shadows.

A)   Probably the easiest is to use an overhead projector.

B)   Cover one end of a cardboard box with either a light colored cloth or translucent paper. Place the object in the box and project the shadow onto the cloth or paper. Look at the shadow from the outside of the box.

C)   Project an image onto a piece of paper with a flash-light. The shadow can be traced and cut out if desired.

586

4 TO MAKE 1

IN

OUT

4 TO MAKE 1
FACTORY

HLUMPH!
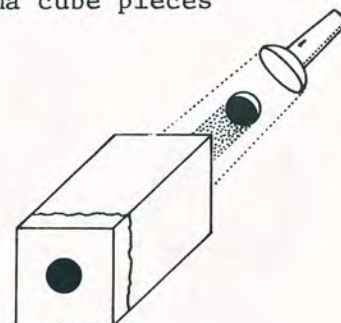
Enlargements of certain figures can be produced by using copies of the figure as building blocks. This can provide us with a good "hands on" activity for similarity. Any of the patterns below will provide students with such a figure.

Have students trace or construct four copies of the pattern they choose. When cut these four pieces can be assembled to produce a larger similar replica.

EXAMPLE :

1 2 3
4

PATTERNS :

x

x

2x

x x
2x   x
3x

587
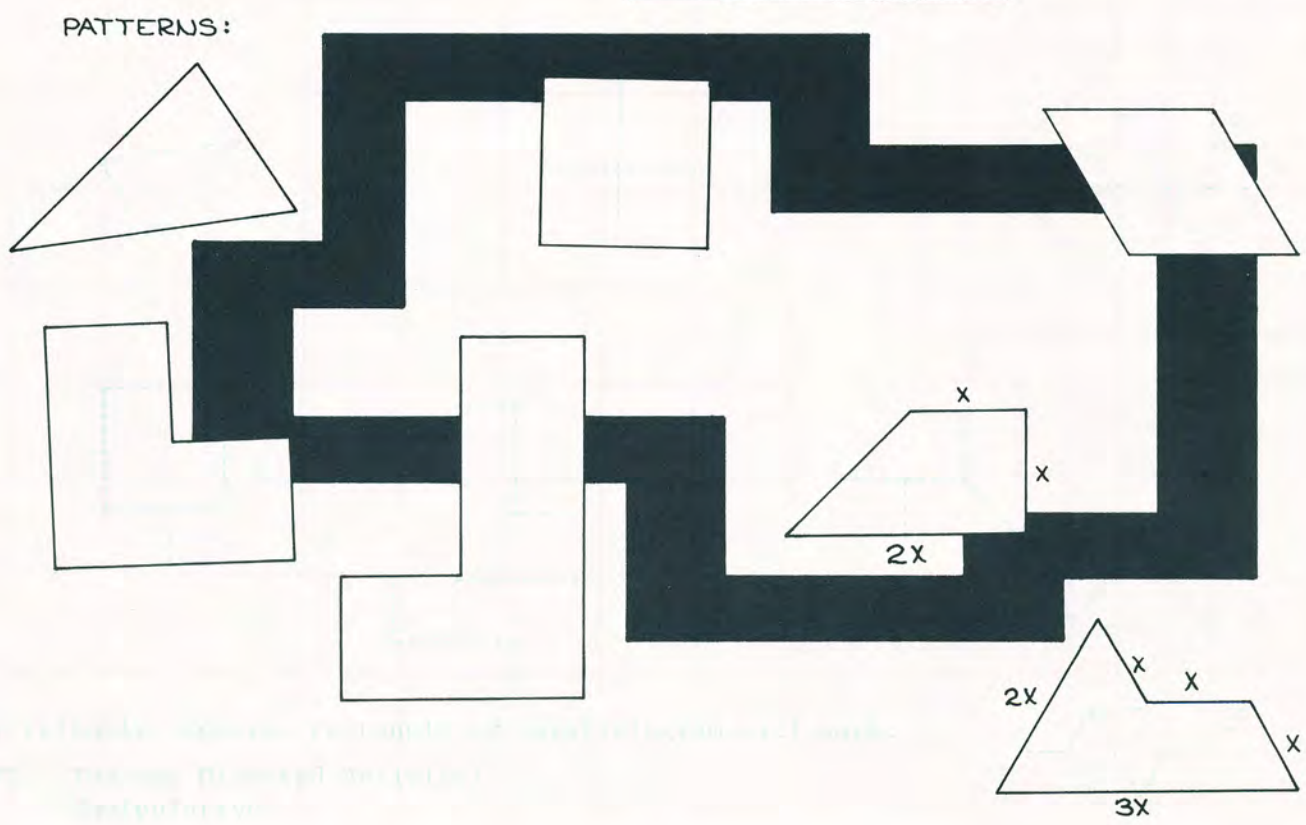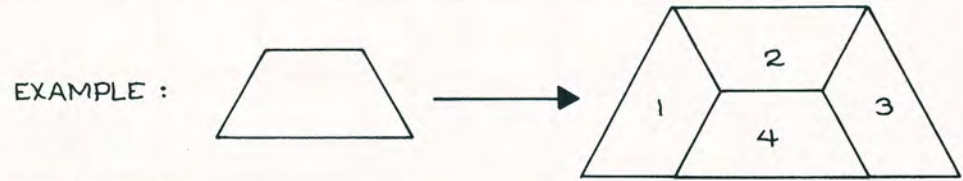
NAVAJO BLANKET DESIGN

Draw an enlarged picture of the design below. It has been started to the right. Color it if you want.
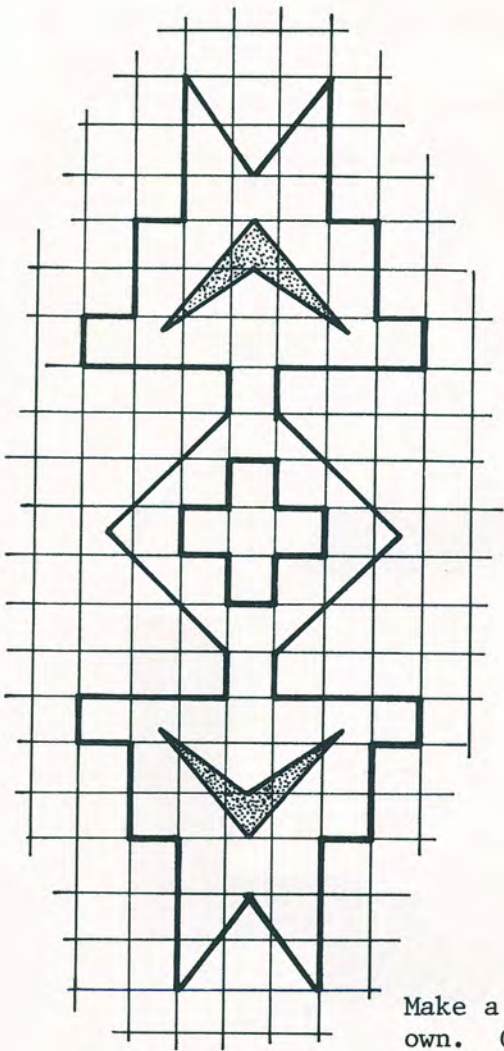
Make a design of your own. Give it to a friend to copy on another grid.
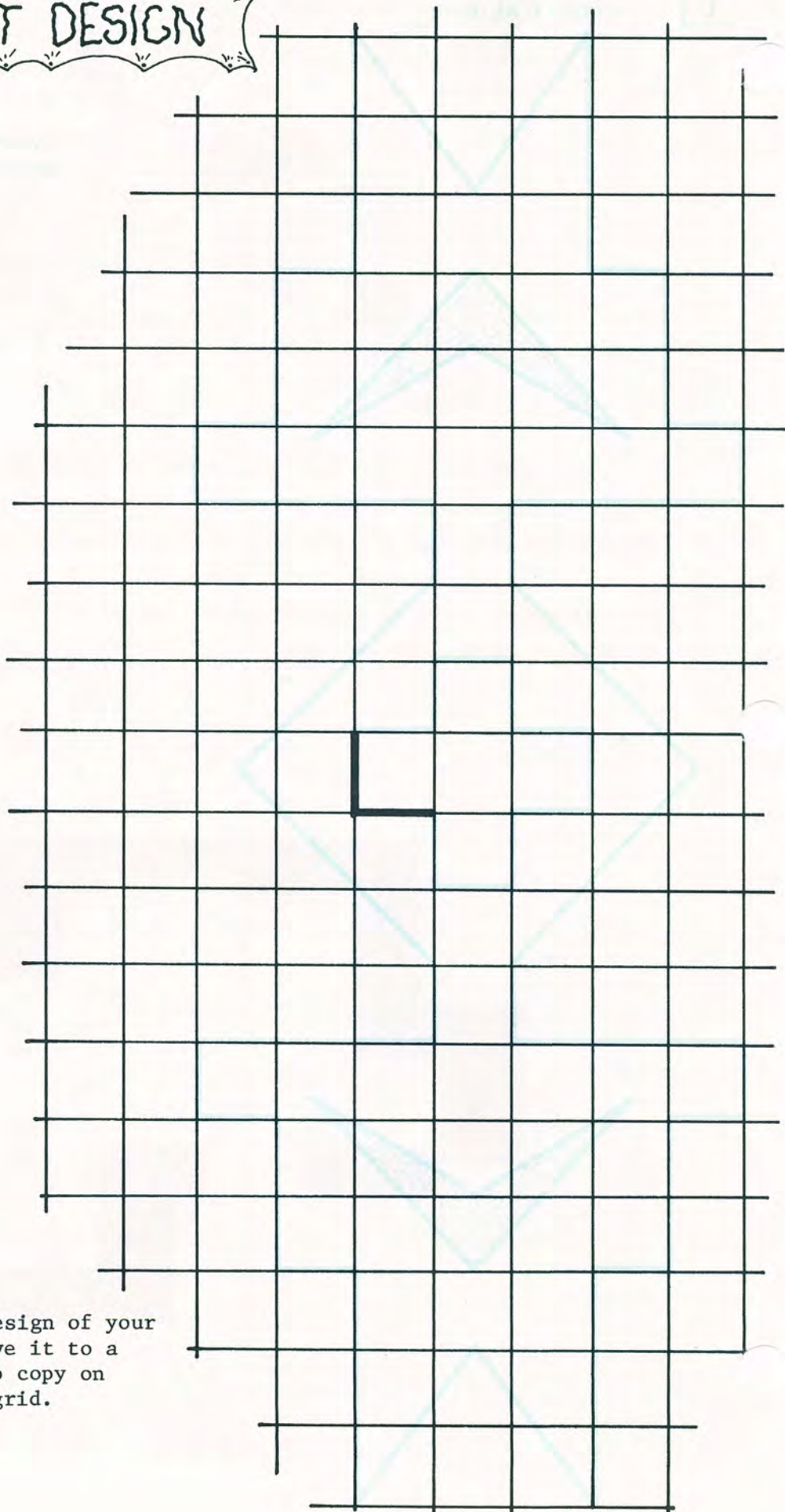
PATTERNS FOR GRIDS

TRANSFER THESE PAT-
TERNS TO A DIFFERENT
SIZED GRID TO CREATE
A SIMILAR DESIGN.
THE NEW FIGURE WILL
BE A REDUCTION OR AN
ENLARGEMENT.

IDEA FROM: *Graph Paper Designs*
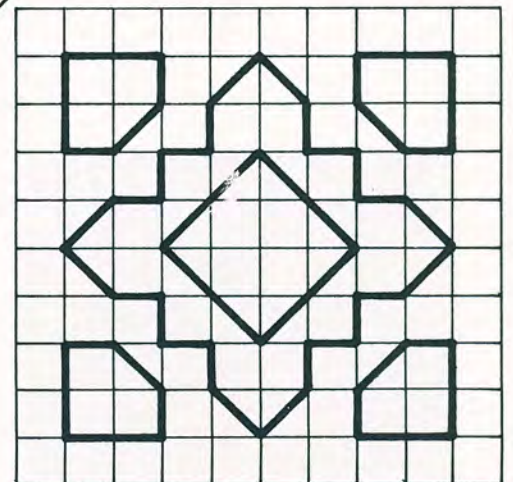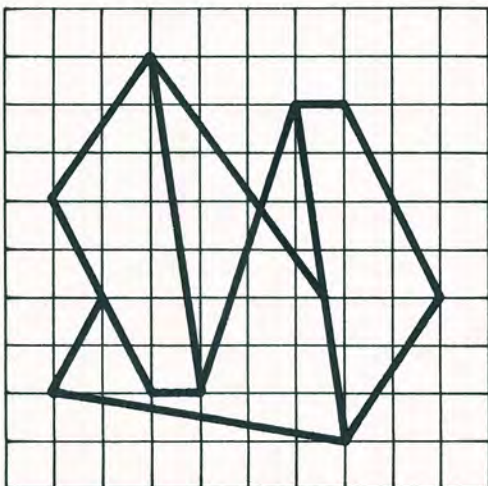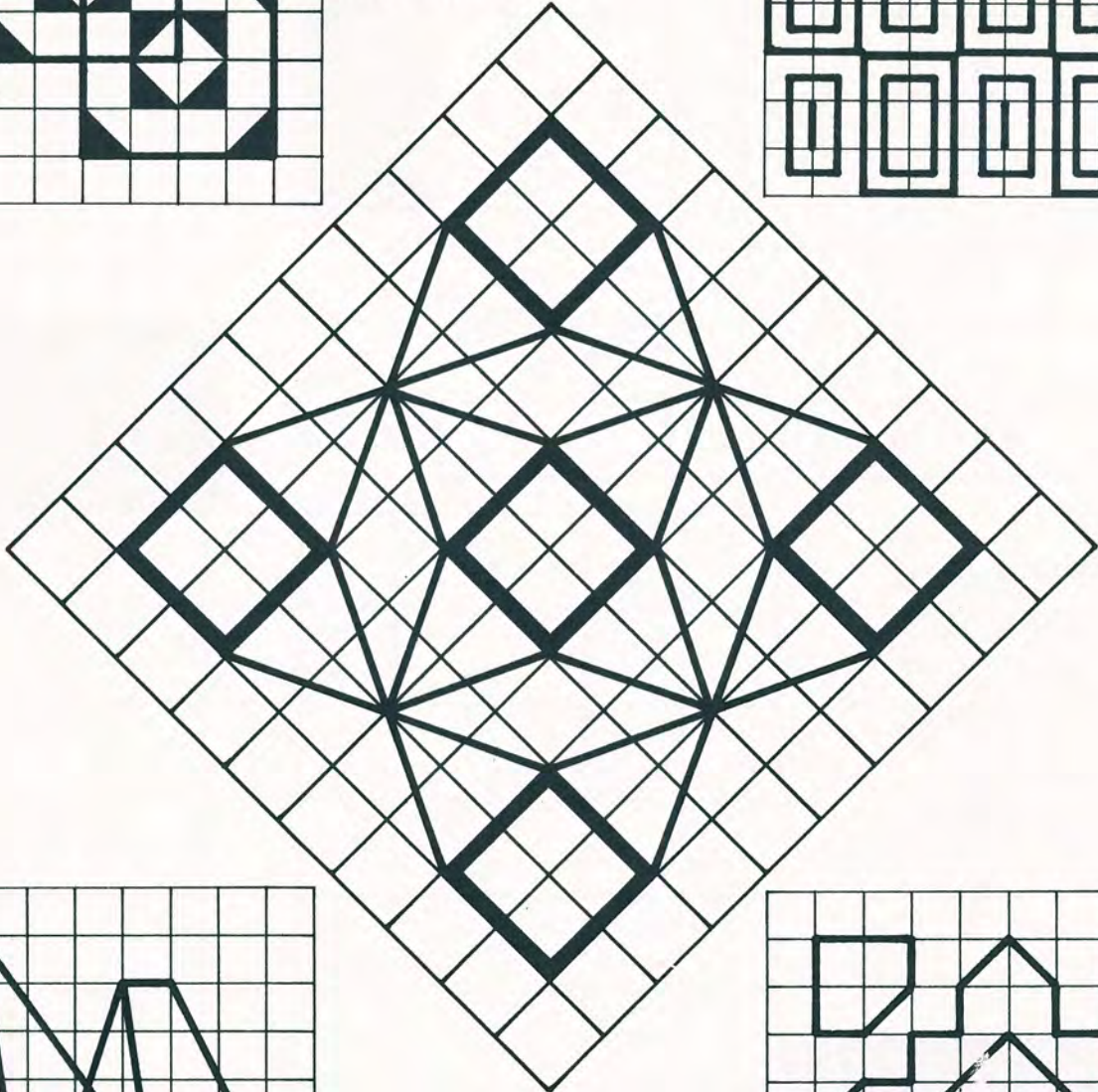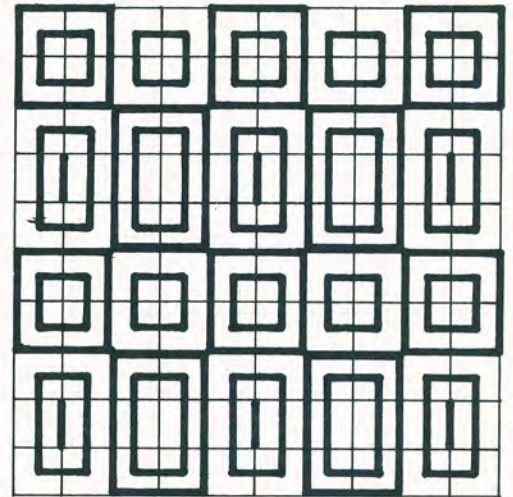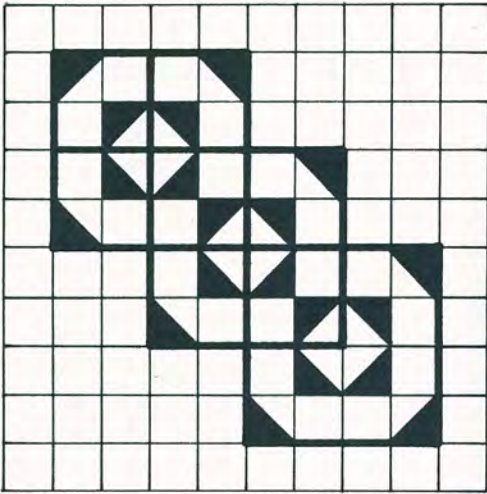
PATTERNS FOR
GRIDS (CONTINUED)
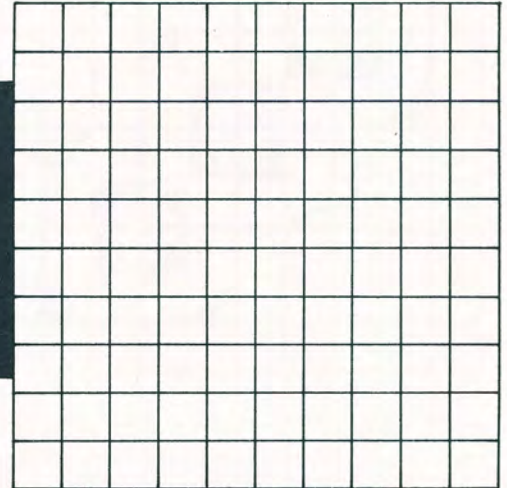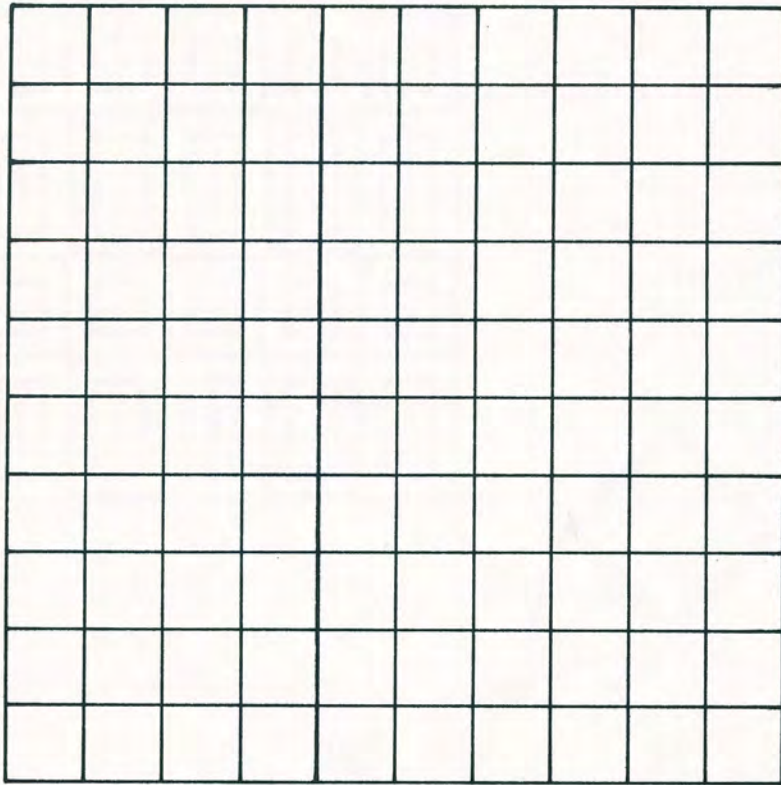
USE THESE GRIDS
TO CREATE YOUR
OWN DESIGNS OR
TO COPY, ENLARGE
OR REDUCE A
GIVEN PATTERN.

GRID ME UP!

On each of the grids draw a figure that is similar to the one shown to the right.
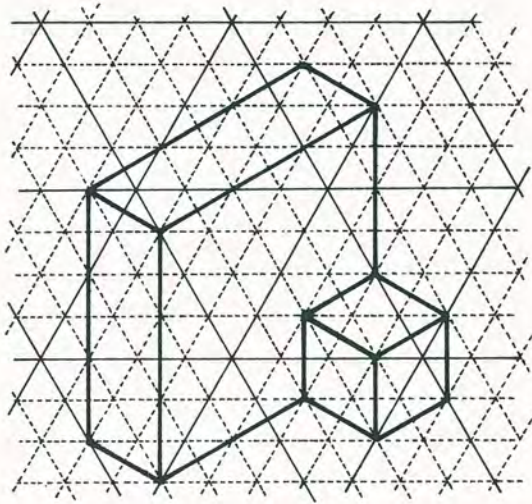
①
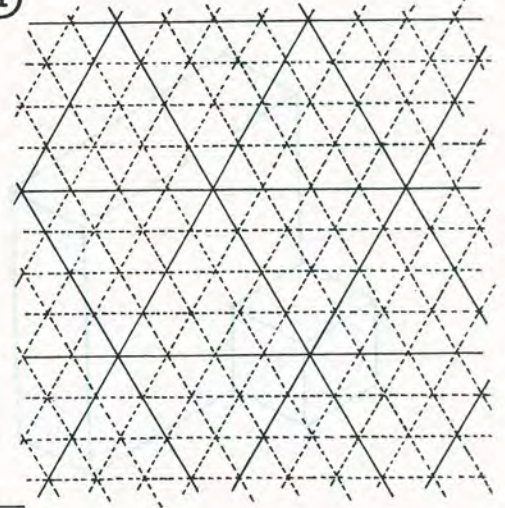
②
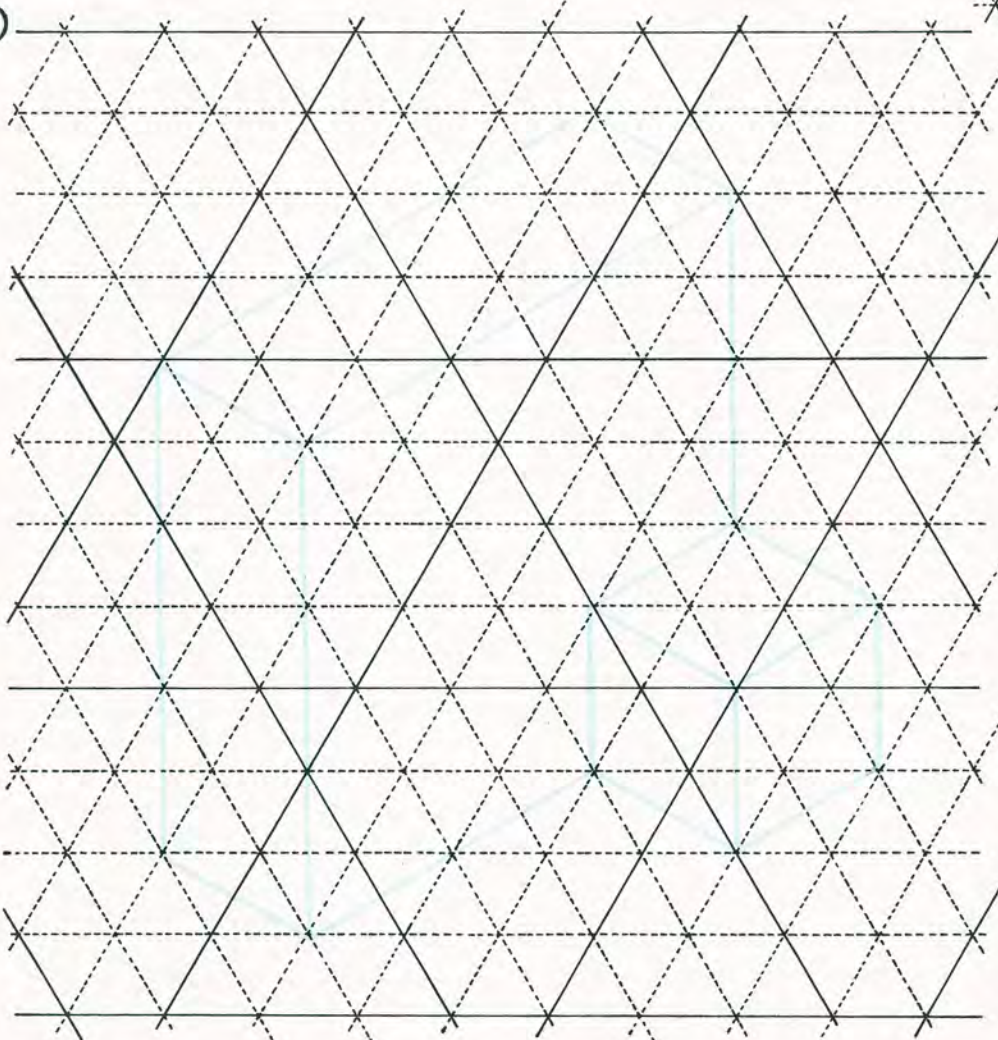
How do the lengths of the edges compare between each pair of the similar figures?

③
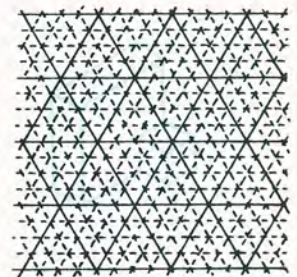
When you are done turn the page around so the small cube is on top.

GETTIN' BIGGER I

Here is our original.

This is similar to the original. Each edge is <u>twice</u> as long.

What did I do to the original to make this similar figure?

If you missed the last one you get one more chance. Try this figure to the right.

If you said each edge is three times as long you are a smart kid.

Get it? Each edge is four times as long.

Using the above as an example draw the next two similar figures on the grids to the right.

Compare your squares with a friend.

Follow the arrows across the page and make the next few similar figures.
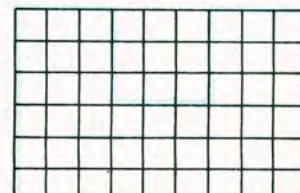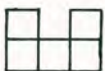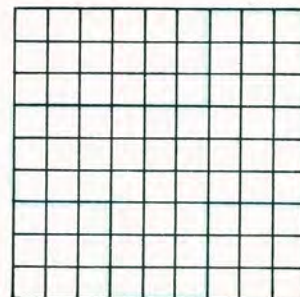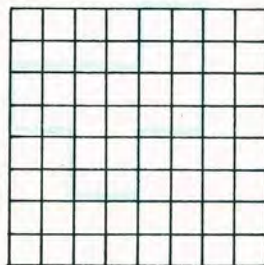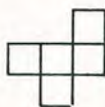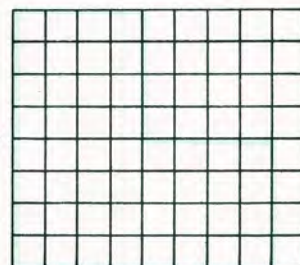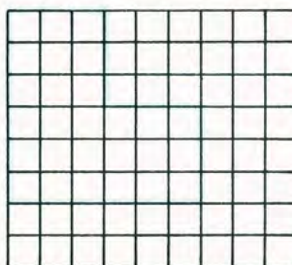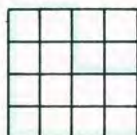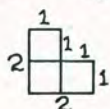
GETTIN' BIGGER I
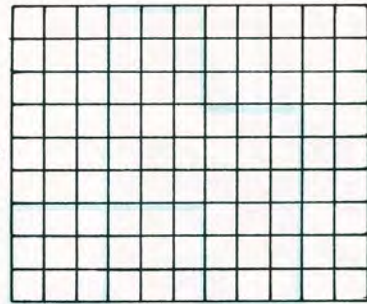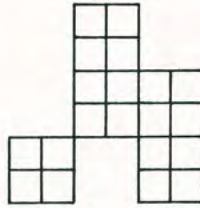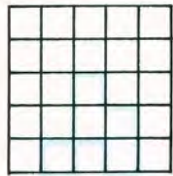
You have to work both ways. Make one similar figure that is smaller.

Make one that is larger.
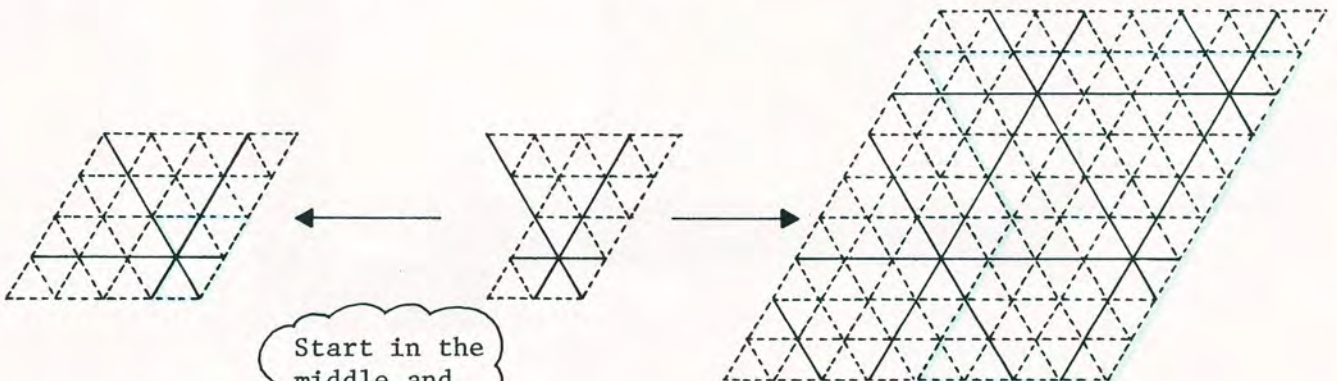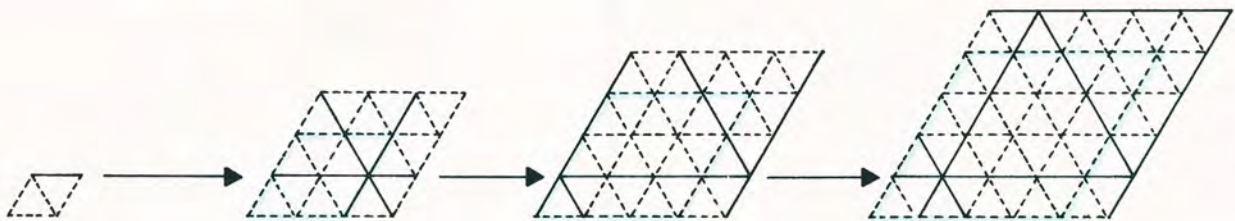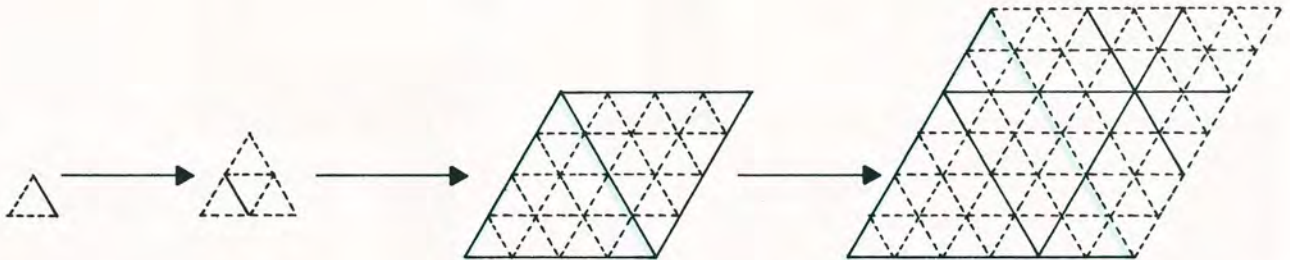
Start in the middle and work both directions.

I HAVE DESIGNS ON YOU
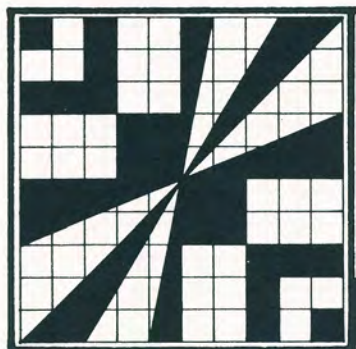
1. Copy the design on the grid below.
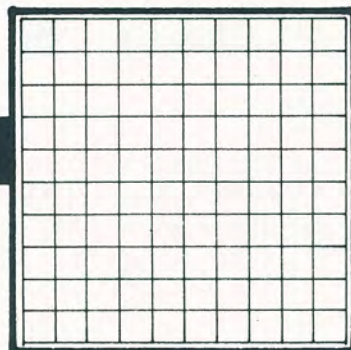
2. Make an enlargement of the original design on the grid below. Each length on the enlargement will be twice as long.
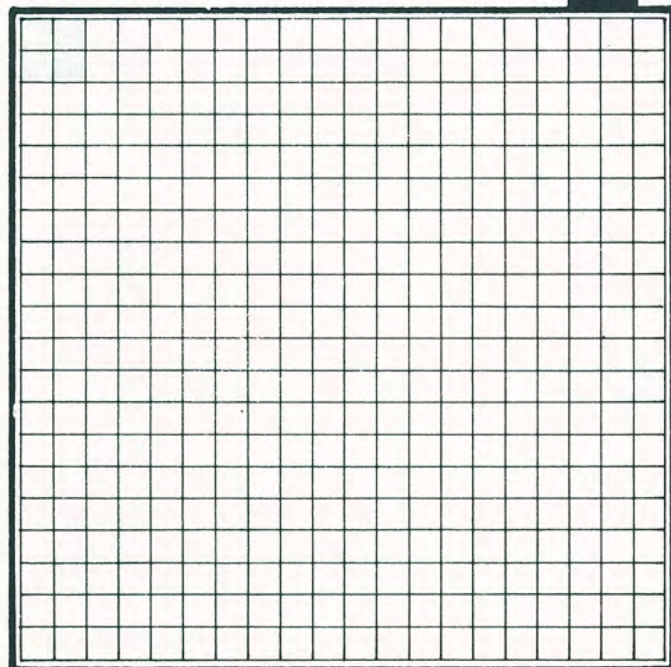
3. Make a reduction of the original design. Each length on the reduction will be half as long.

4. Make a design of your own on graph paper. Have a friend make a 2 to 1 enlargement of the design.

HOW DOES YOUR BILLIARD TABLE SIZE UP ?

The figures to the right represent the same billiard table. A ball has been hit from the lower left hand corner so that it travels at a 45° angle from the sides of the table (Figure 1). When the ball strikes the cushion of the table it rebounds from it at the same angle (Figure 2). If the ball continues, it will reach the upper left hand corner (Figure 3).

A BILLIARD TABLE HAS NO POCKETS

FIGURE 1   FIGURE 2   FIGURE 3
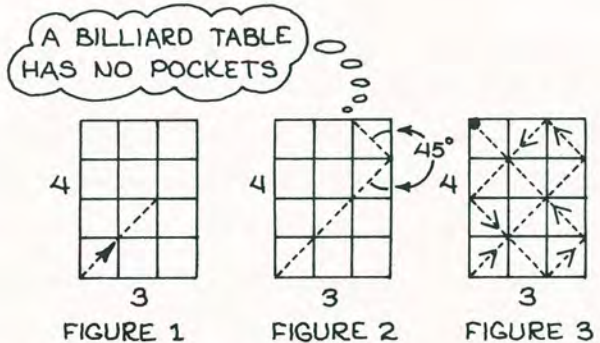
1) For each table below draw the path of the ball until it strikes a corner. Always shoot the ball along a 45° path from the lower left hand corner.

a)   6   3   LENGTH   WIDTH

b)   10   6

c)   5   2

d)   8   6

e)   7   3

f)   10   5

g)   6   4

h)   6   5

i)   5   3

j)   12   8
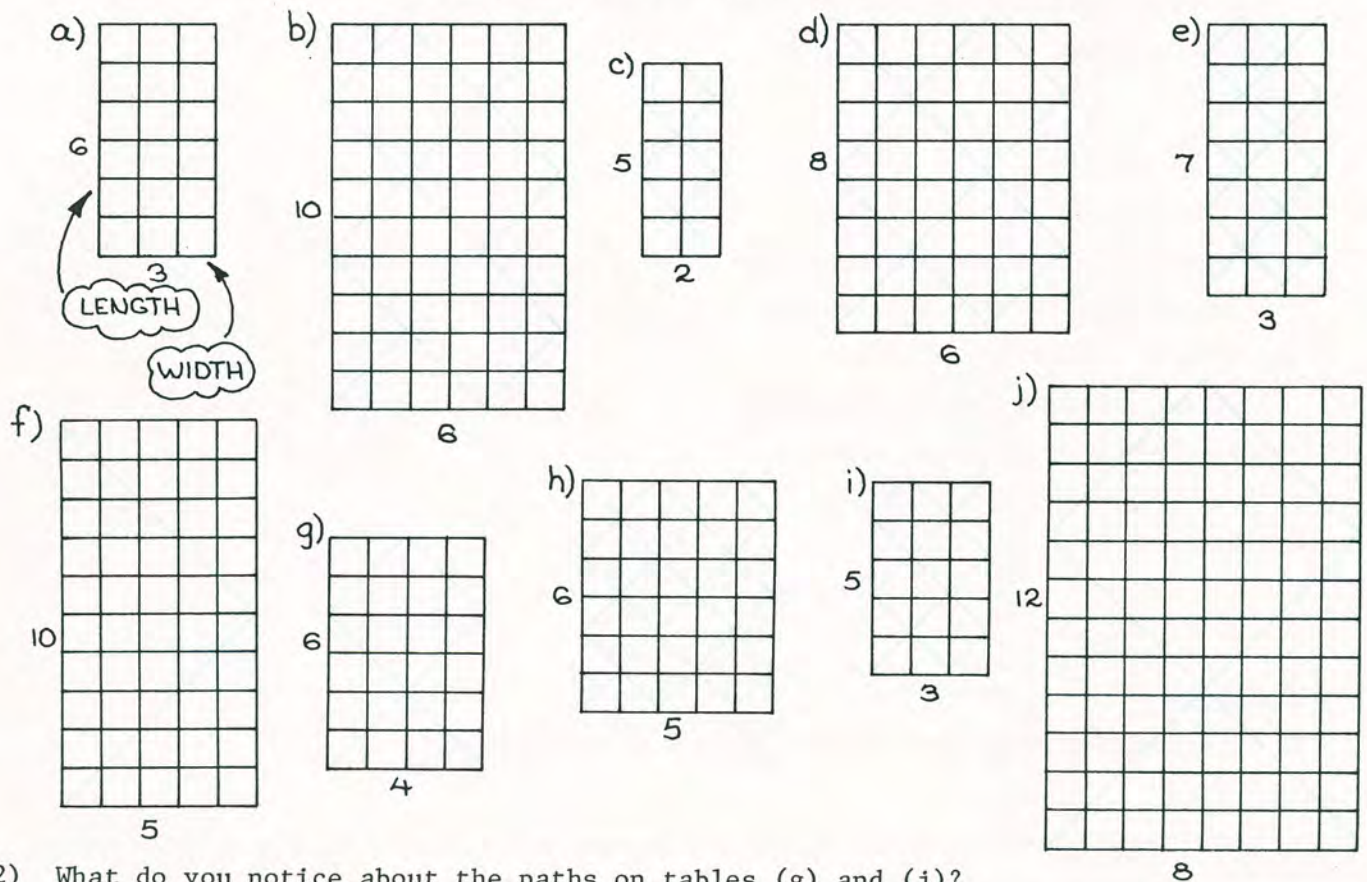
2) What do you notice about the paths on tables (g) and (j)?

3) Name another pair of tables that have similar paths. _____

4) Table (c) has a width of 2, a length of 5. Predict the dimensions of a table on which a ball will travel a similar path: width ____; length ____. Test your prediction by drawing the table and the path. Be sure to place your table on graph paper so that it is taller than it is wide. Experiment until you find a table that works. Use the examples above to help you.

5) Will the paths on these tables be the same?   10   6   20   15

595
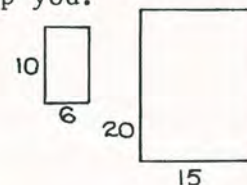
1) Question #4 stresses placement of the table on graph paper so that it is taller than it is wide. Students could investigate the paths on the tables to the right. The tables are similar (in fact congruent, if they are rotated the tables look identical) so the paths are similar. Since the paths begin at the lower left hand corner on both tables but end in different corners, students may not perceive them as similar paths.

a)

b)

PATHS MUST START AT THE LOWER LEFT HAND CORNER.
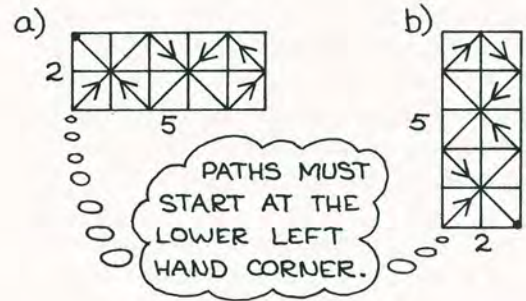
2) On the preceding page students discover that if the paths of the ball on two tables are similar, then the tables are similar. Some students may need additional exploration or guidance before they can answer question #5.

LENGTH

WIDTH

RECTANGLES ARE SIMILAR IF THE RATIOS OF LENGTH TO WIDTH ARE THE SAME:

$$\frac{3}{2} = \frac{6}{4}$$
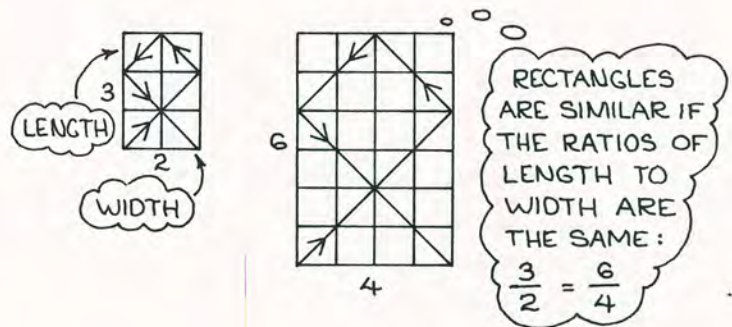
Extensions

Can students use the dimensions of the table to predict:

1) when the ball will travel over every square before reaching the final corner?

2) the corner where the ball will end up?

3) how many times the ball will strike the cushions if the starting position and the final corner are counted as hits?

HOW DOES YOUR BILLIARD TABLE SIZE UP ? (PAGE 3)

A procedure for guided discovery:

A) To help students with prediction #1 have them examine the tables on the student page. You may need to suggest that they look at the lengths and width. Some students may need a hint to look at factors (divisors) of the dimensions for each table.

Rule: If the length and width are relatively prime (1 is the only common factor) the ball travels over every square.

B) Since the path of the ball is the same on tables that are similar students should use tables whose dimensions are the simplest (length and width are relatively prime) to help them with prediction #2. A rectangle with simplest dimensions and having the same path as tables (a) and (f) would be one with length 2 and width 1. Suggest that students make a chart listing the simplest dimensions for each table with a different path and the final corner of the ball for each table.

You may also need to suggest that students notice which dimensions are odd and even to help them with the prediction.

LENGTH	WIDTH	FINAL CORNER	TOTAL HITS
2	1	UPPER LEFT	3

Rule: Given that the dimensions are relatively prime and if length and width are both odd, the ball ends in the upper right corner; if the length is odd and the width even, the ball ends in lower left corner; if the length is even and the width odd, the ball ends in upper left corner. (For these results tables are positioned so that they are taller than they are wide.)

Challenge: Could the final corner be the lower left corner?

C) *Rule: The number of total hits is the sum of the length and width. Again the dimensions must be relatively prime. To determine total hits count the initial position, the number of rebounds and the final position.*

Students should be encouraged to draw more tables on graph paper to check their predictions.

GETTIN' BIGGER II

This cube measures 1 unit on an edge. It is called a unit cube.

1
1
1
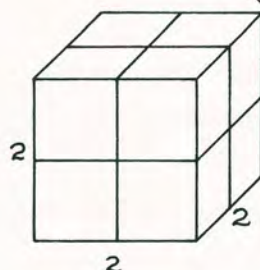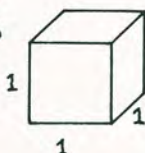
Unit cubes have been used to build this new cube. It is similar to the original. The length of each of the edges is 2 times as long as the edge of the unit cube's.

2
2
2

2
2
1
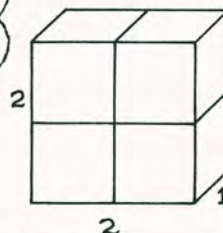
This is not similar to our original because some of the edges have not been increased.

Use unit cubes to make the next two similar solids for the figures below. In your first solid all edges should be twice as long and in your next solid all edges should be 3 times as long.

①
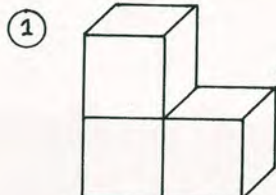
DID YOU USE 24 CUBES FOR YOUR FIRST SOLID?

②

③

④

⑤
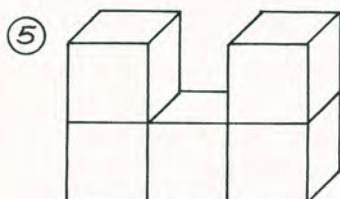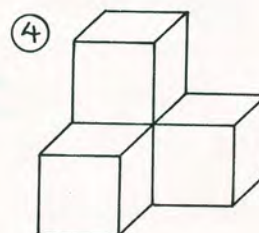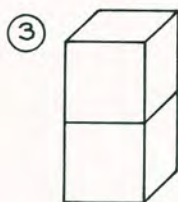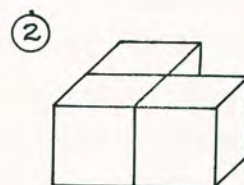
Use a ratio of 2:1 to make a similar figure.

GRID PAIRS

A) 1) Write the coordinates of points A, B and C shown in the triangle to the right.

   A _____   B _____   C _____

2) Double each of the above coordinates.

   A' _____   B' _____   C' _____

3) Graph A', B' and C' and △A'B'C'.

4) How do the sides of △A'B'C' compare in length to the sides of △ABC? _____

_____

5) Measure the distances from (0,0) to each point.

   A' _____   B' _____   C' _____   A _____   B _____   C _____

6) How do the distances from (0,0) to A', B' and C' compare to the distances from (0,0) to A, B and C? _____
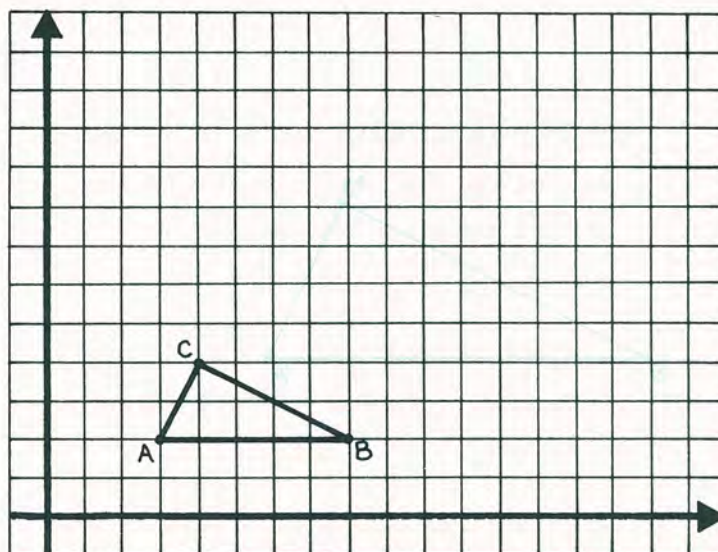
Triangle A'B'C' is similar (same shape) to triangle ABC. △ A'B'C' is an enlargement by a scale factor of 2 (each length is twice as long).

Use another piece of grid paper. Draw a pair of axes.

B) 1) Draw the rectangle whose vertices are at (2,1) (6,1) (6,3) (2,3).

2) Write the coordinates of the vertices of the enlargement of the rectangle using a scale factor of 3. (   ) (   ) (   ) (   )

3) Connect the new points to make an enlargement of the rectangle.

C) 1) Draw the triangle whose vertices are at (2,1) (5,2) (1,3).

2) Write the coordinates of the vertices of the enlargement of the triangle using a scale factor of 4. (   ) (   ) (   )

3) Connect the new points to make an enlargement of the triangle.

D) 1) Draw the square whose vertices are at (4,12) (8,12) (8,16) (4,16).

2) Write the coordinates of the vertices of the reduction of the square using a scale factor of 1/2. (   ) (   ) (   ) (   )

3) Connect the new points to make a reduction of the square.

E) 1) Draw a design of your own on grid paper.

2) Write the coordinates of the vertices of your design.

3) Have a friend choose a scale factor.

4) Write the coordinates of the vertices of the enlargement using the scale factor.

5) Have your friend draw the enlargement.

LOOK ALIKE FIGURES

1) Place tracing paper over this grid and trace as many differently sized triangles that are similar to the shaded triangle as you can.

   You may include only those triangles that can be traced directly from this grid.

2) Find another familiar figure on the grid and repeat #1.

---
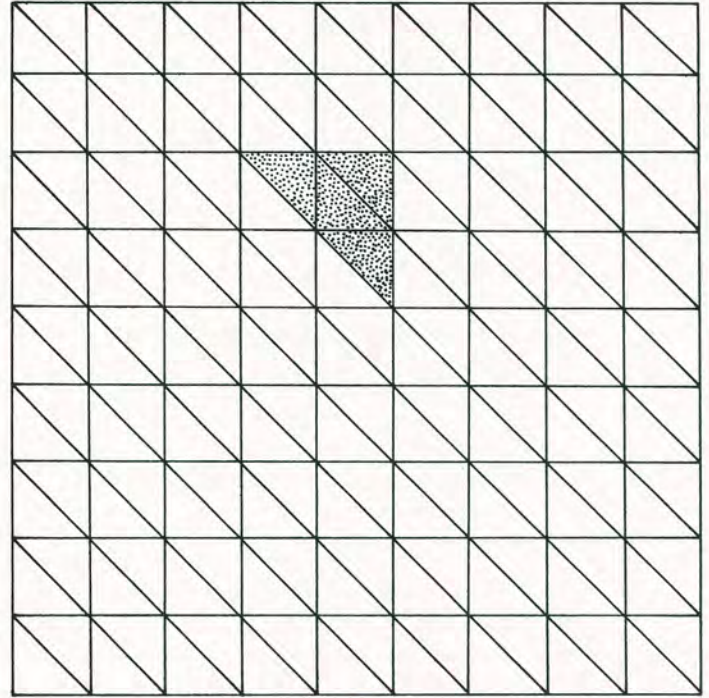
1) Draw a large equilateral triangle.

2) Find the midpoints of the sides of the triangle. Join them to make a new triangle.

3) Find the midpoints of the sides of this new triangle. Join them to make a third triangle. Repeat.

4) What do you notice about the triangles?

5) Will this method of drawing similar triangles work on a triangle that is not equilateral?
   Check your guess by drawing a triangle and repeating steps #2 and #3.

6) Will this method work on other figures? Investigate some quadrilaterals.
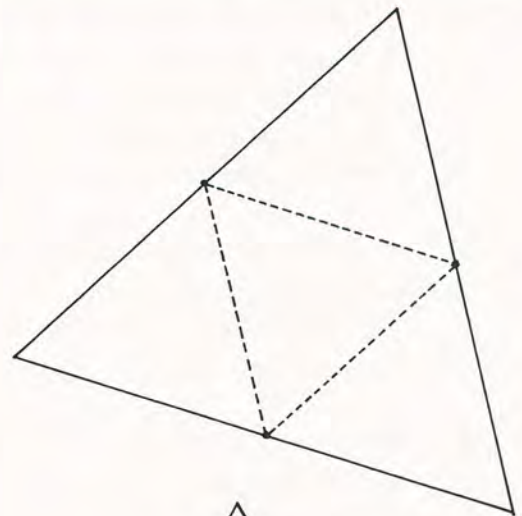
ARCHIE TEXS' RULER

Activity:

(1) Complete the ruler by marking sides B and C to show the given scale.

(2) Cut out this chart, fold on the lines, and paste the flap under to make your architect's ruler.
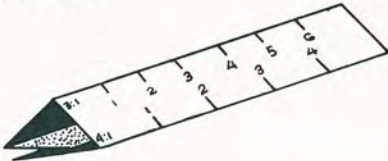
(3) On another paper use the 1:1 scale to draw a rectangle 2 units wide and 4 units long.

(4) Now use the 2:1 scale to make a rectangle that is similar by making it 2 times as wide and 2 times as long.

(5) Use the 1:1 scale and draw a different rectangle any size you want. Now make a rectangle that is similar by using the 3:1 scale.

(6) Use the 1:1 scale and draw a square 3 units on a side. Make a square that is similar by using the 6:1 scale.

Challenge: Use the 4:1 scale to make a drawing of Jennifer's doghouse that is similar to the one shown. Hint: Measure angles, too.

JENNIFER

A   B   C
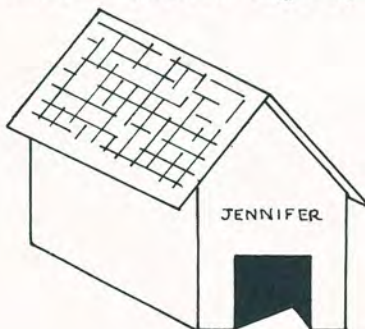
Fold Under And Paste To Back Of A

1:1   2:1   3:1   4:1   5:1   6:1

CONSTRUCTING CARTONS

TAB

A

TAB

B
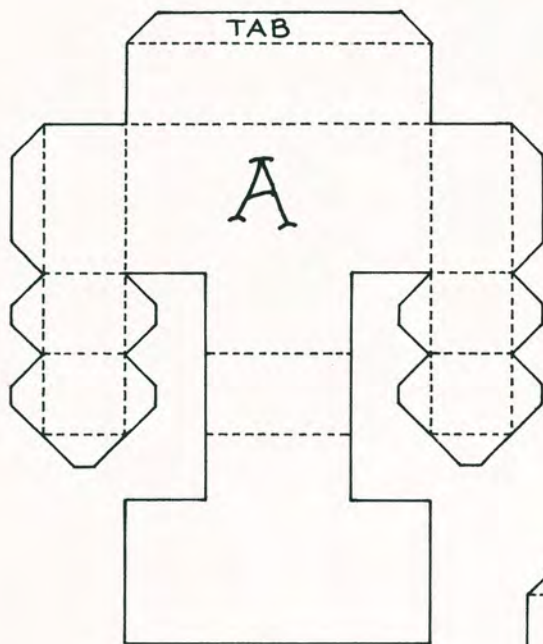
1) Use Archie Texs' ruler to make similar patterns that have each length twice as long as the ones shown.

2) Cut out the patterns on the solid lines and fold on the dotted lines. Use glue or paste on the tabs to make the models.

TAB

C
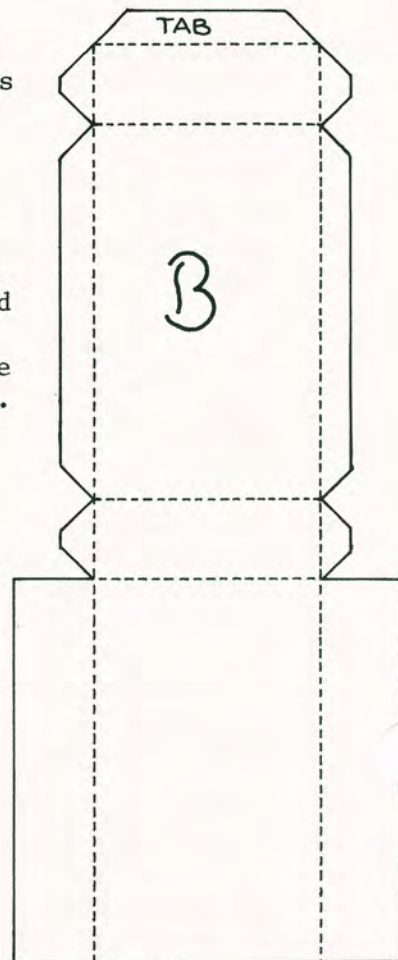
3) Use eight assembled models of each pattern to build a similar (but larger) model. Two of the models work. Which one does not work? A, B or C?

IDEA FROM: *Open-Ended Task Cards*

Permission to use granted by Teachers Exchange of San Francisco
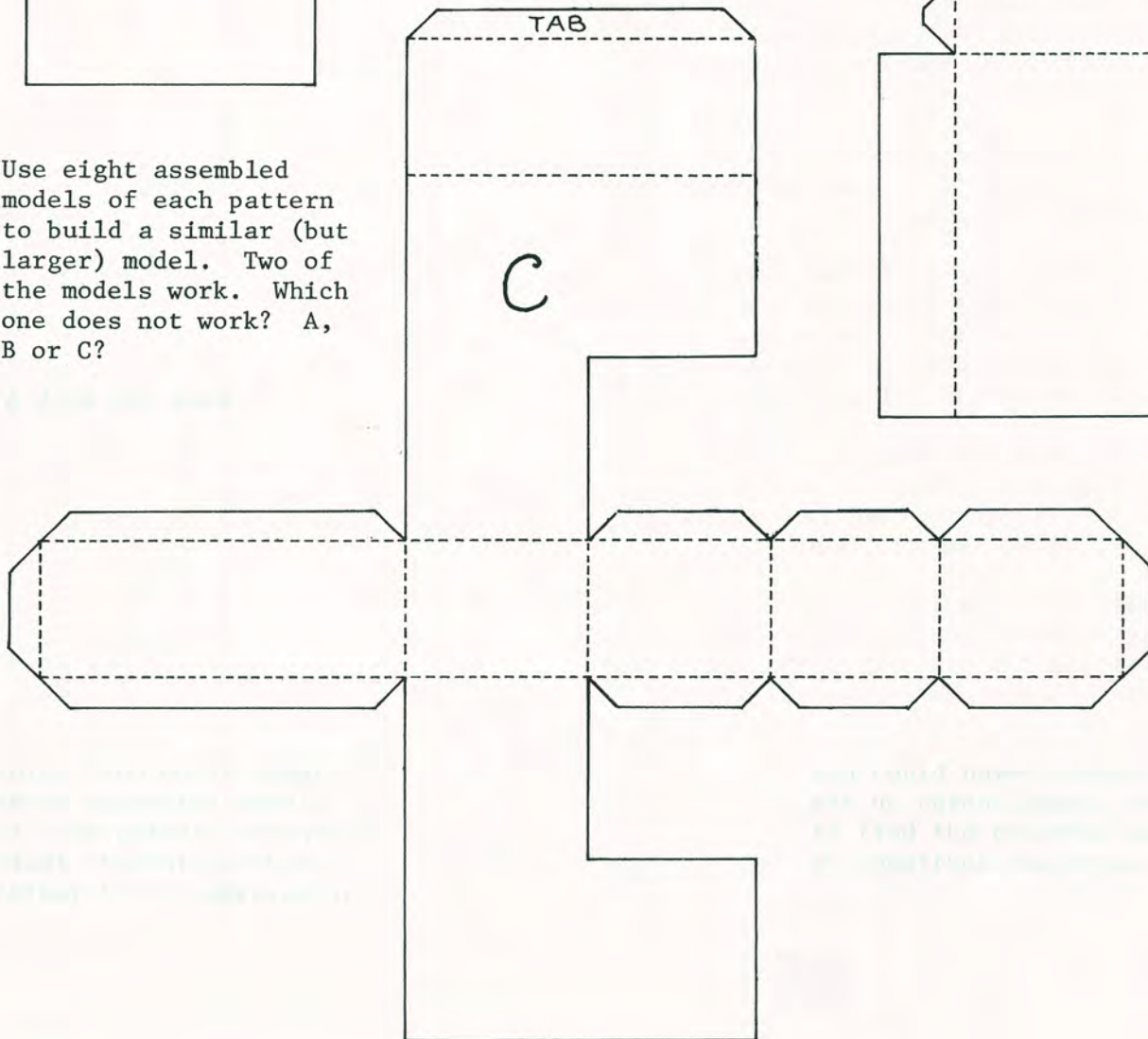
BIGGER THAN LIFE

1) Draw rays PA, PB and PC.
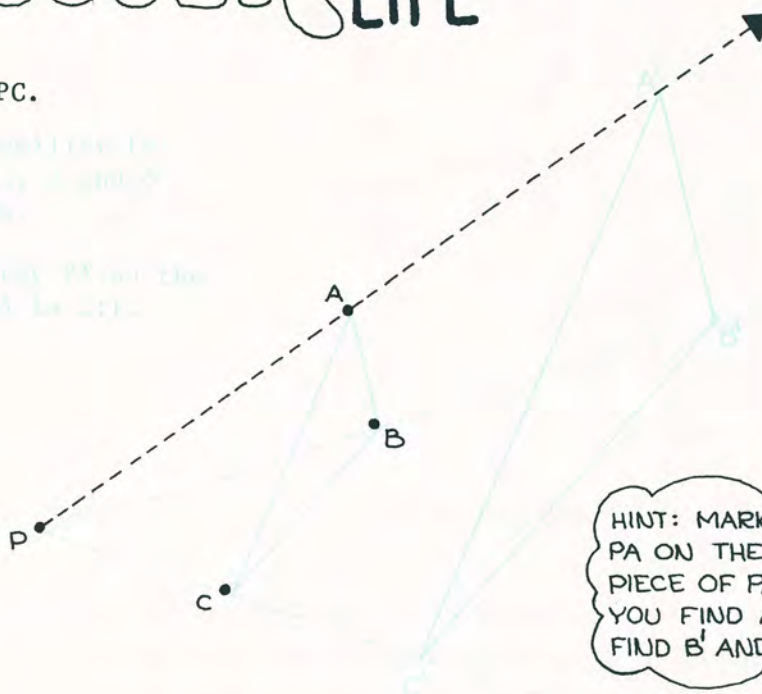
> HINT: MARK THE LENGTH PA ON THE EDGE OF A PIECE OF PAPER TO HELP YOU FIND A'. REPEAT TO FIND B' AND C'.

2) a) Mark point A' on ray PA so $\overline{PA'}$ is twice as long as $\overline{PA}$.

   b) Mark point B' on ray PB so $\overline{PB'}$ is twice as long as $\overline{PB}$.

   c) Mark point C' on ray PC so $\overline{PC'}$ is twice as long as $\overline{PC}$.

3) a) Draw line segments to connect A to B, B to C and C to A.

   b) Draw line segments to connect A' to B', B' to C' and C' to A'.

   c) What figures do you get? _____

   d) What do you notice about the figures? _____

4) Find the lengths in millimetres of these line segments.

   $\overline{AB}$ _____ $\overline{BC}$ _____ $\overline{CA}$ _____ $\overline{A'B'}$ _____ $\overline{B'C'}$ _____ $\overline{C'A'}$ _____

5) How do these lengths compare? A'B' and AB _____
   B'C' and BC _____ C'A' and CA _____
   In each case is one about twice as long as the other? _____

> Triangle A'B'C' is similar to triangle ABC. Point P is the starting point.
> Triangle A'B'C' is an enlargement of triangle ABC with a scale factor of 2.

A SHRINK
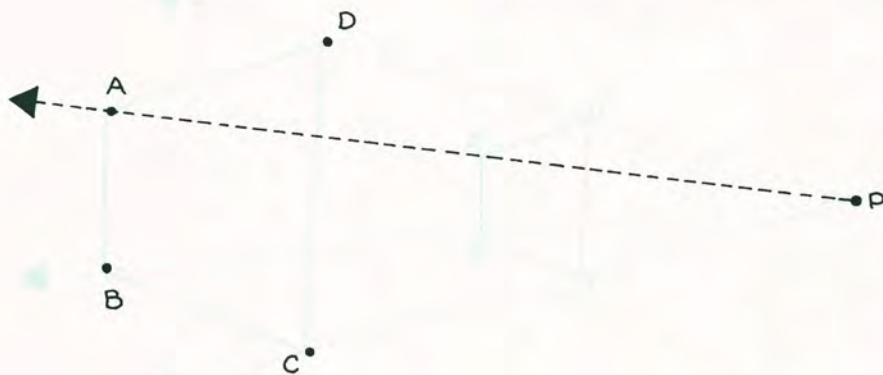
1) Draw rays PA, PB, PC and PD.

2) a) Mark point A' on ray PA so PA' is half of PA.
   b) Mark point B' on ray PB so PB' is half of PB.
   c) Mark point C' on ray PC so PC' is half of PC.
   d) Mark point D' on ray PD so PD' is half of PD.

> HINT: MARK THE LENGTH PA ON THE EDGE OF A PIECE OF PAPER. FIND THE MIDPOINT BY FOLDING AND USE THE MIDPOINT TO FIND A! REPEAT FOR B', C' AND D'.

3) a) Connect A to B, B to C, C to D and D to A with line segments.
   b) Connect A' to B', B' to C', C' to D' and D' to A' with line segments.
   c) What figures do you get? _____
   d) What do you notice about the figures? _____

4) Find the lengths in millimetres of these line segments.

   $\overline{AB}$ _____ $\overline{BC}$ _____ $\overline{CD}$ _____ $\overline{DA}$ _____

   $\overline{A'B'}$ _____ $\overline{B'C'}$ _____ $\overline{C'D'}$ _____ $\overline{D'A'}$ _____

5) How do these lengths compare? A'B' and AB _____
   B'C' and BC _____   C'D' and CD _____
   D'A' and DA _____
   In each case is one length about half of the other length? _____

> Trapezoid A'B'C'D' is similar to trapezoid ABCD. Point P is the starting point.
> Trapezoid A'B'C'D' is a reduction of trapezoid ABCD with a scale factor of $\frac{1}{2}$.

WHERE'S THE POINT?

1) a) Do the two triangles at the right look similar? _____

   b) Draw rays A'A, B'B and C'C.

   c) What do you notice? _____
      _____

   d) Label this point P.
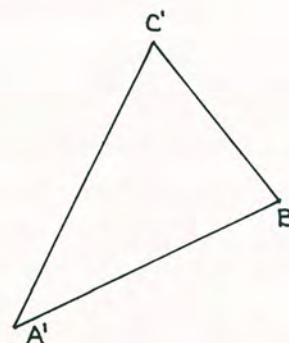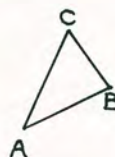
2) Find the length in millimetres of each of these line segments.

   $\overline{PA}$ _____ $\overline{PB}$ _____ $\overline{PC}$ _____ $\overline{PA'}$ _____ $\overline{PB'}$ _____ $\overline{PC'}$ _____

3) How do these lengths compare? PA' and PA _____
   PB' and PB _____   PC' and PC _____

4) a) Is triangle A'B'C' an enlargement or reduction of triangle ABC? _____

   b) What is the scale factor? _____

   c) Is triangle ABC an enlargement or reduction of A'B'C'? _____

   d) What is the scale factor? _____

5) Find the starting point for each pair of similar figures. Test whether the three starting points lie on the same straight line.

ENLARGEMENTS
AND
REDUCTIONS

Students could investigate placing the starting point P outside, inside or on the original figure. An interesting activity is to draw enlargements of the same figure with a starting point in three different positions. The following suggestions could be used on worksheets. You can think of many more.

a) LETTERS OF THE ALPHABET
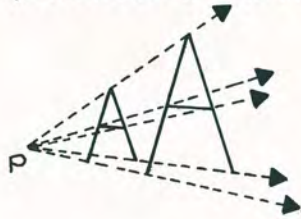
b) POLYGONS
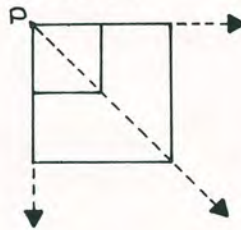
c) DESIGNS
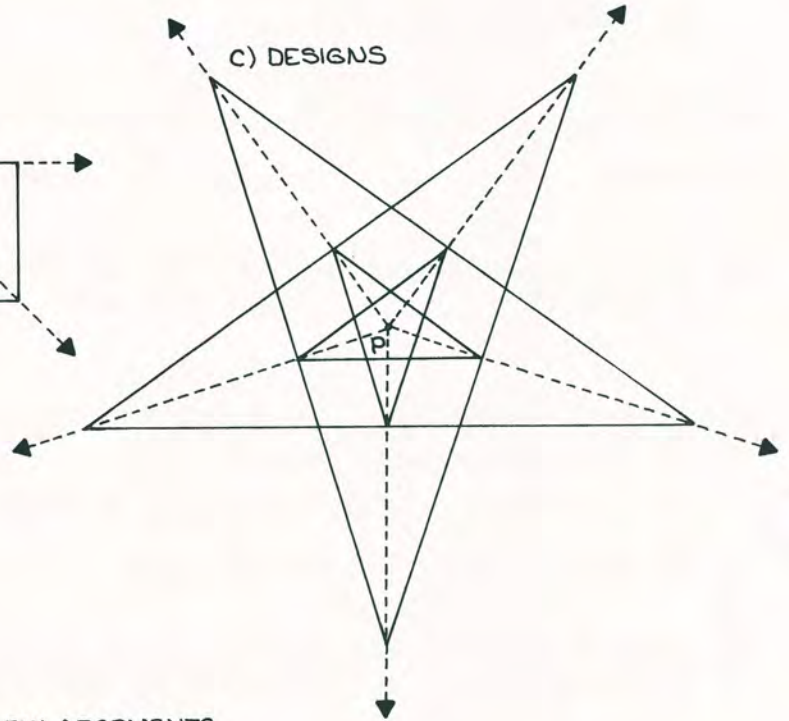
d) SEVERAL ENLARGEMENTS FROM THE SAME STARTING POINT
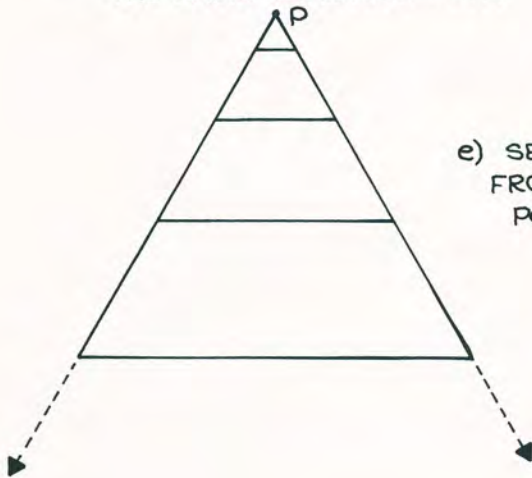
e) SEVERAL ENLARGEMENTS FROM DIFFERENT STARTING POINTS.

A NEGATIVE FEELING

I•   •K

E•   P

M
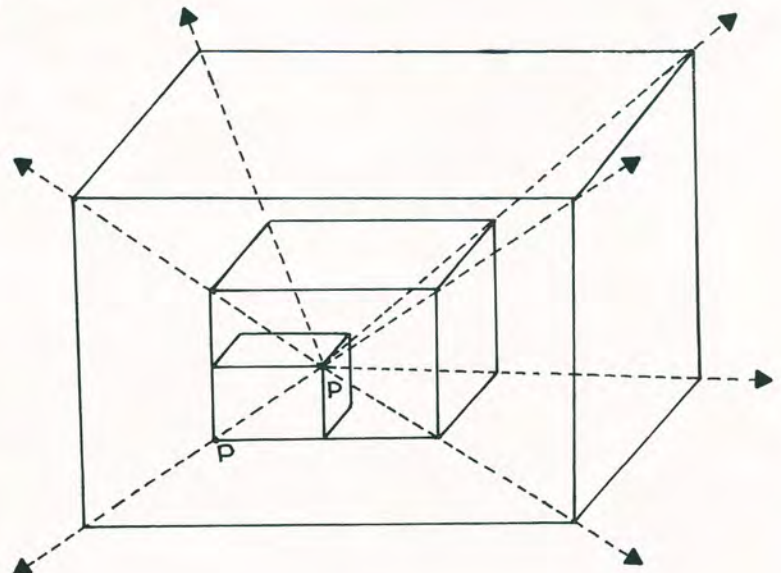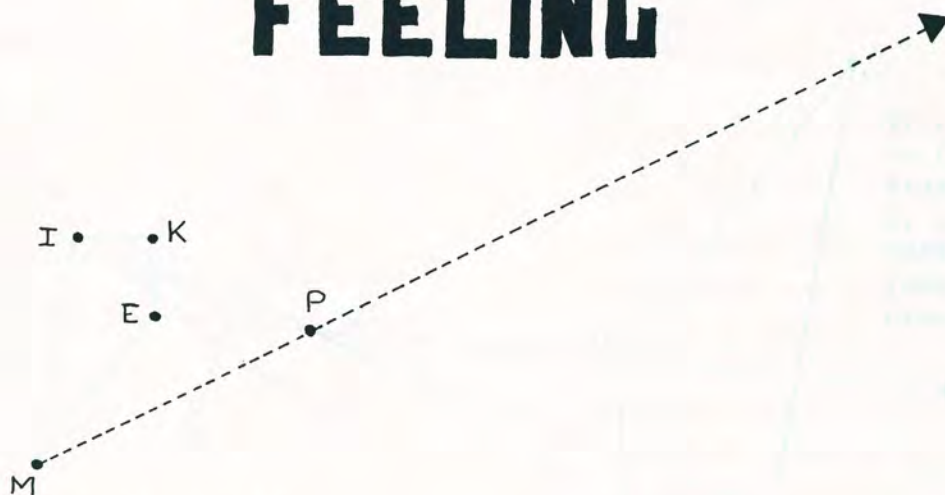
1) Draw rays MP, IP, KP, EP.

2) a) Mark point M' on ray MP so P is
        between M and M' and PM' is twice
        as long as PM.

    b) Mark point I' on ray IP so P is
        between I and I' and PI' is twice
        as long as PI.

    c) Mark point K' on ray KP so P is
        between K and K' and PK' is twice
        as long as PK.

    d) Mark point E' on ray EP so P is
        between E and E' and PE' is twice
        as long as PE.

HINT: MARK THE LENGTH MP ON THE EDGE OF A PIECE OF PAPER TO HELP YOU FIND M'. USE THE SAME METHOD TO LOCATE I', K' AND E'

3) a) Connect M to I, I to K, K to E and E to M with line segments.

    b) Connect M' to I', I' to K', K' to E' and E' to M' with line segments.

    c) What do you notice about figures MIKE and M'I'K'E'? _____
    _____

4) Find the length in millimetres of these line segments.

    $\overline{MI}$ _____ $\overline{IK}$ _____ $\overline{KE}$ _____ $\overline{EM}$ _____

    $\overline{M'I'}$ _____ $\overline{I'K'}$ _____ $\overrightarrow{K'E'}$ _____ $\overline{E'M'}$ _____

5) How do these lengths compare? M'I' and MI _____

    I'K' and IK _____   K'E' and KE _____   E'M' and EM _____

    In each case is one length about twice as long as the other? _____

Figure M'I'K'E' is similar to figure MIKE. Point P is the starting point. Since point P is between two figures the enlargement is called a <u>negative enlargement</u>. Figure M'I'K'E' is an enlargement with a scale factor of -2. When you make a negative enlargement, one figure will appear upside down.

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ENLARGEMENTS OF ENLARGEMENTS

A) 1) Trace this star onto the <u>middle</u> of another sheet of paper.

2) Using P as the starting point, make an enlargement of the star using a scale factor of 2.

3) Using P as the starting point, make an enlargement of the enlarged star using a scale factor of 3.

4) Trace the original star onto another piece of paper. First make an enlargement of the star using a scale factor of 3. Then make an enlargement of the enlarged star using a scale factor of 2.

5) How do the final enlargements in (3) and (4) compare? _____

6) What is the scale factor that compares the final enlargement to the original star? _____ Is it what you expected?

B) 1) Draw a small rectangle on grid paper. Mark P on one of the corners of the rectangle.

2) Make an enlargement of the rectangle using a scale factor of 3.

3) Make an enlargement of the enlarged rectangle using a scale factor of 4.

4) With a different color make an enlargement of the original rectangle using a scale factor of 2.

5) Make an enlargement of this enlarged rectangle using a scale factor of 6.

6) How do the final enlargements of the rectangle compare? _____

7) What is the scale factor that compares the final enlargement to the original rectangle? _____ Is it what you expected?

Challenge: Suppose you use the same steps as above but use different starting points. Would anything about the final enlargements be different?

A SNAPPY SOLUTION TO
SIMILAR FIGURES

Materials Needed: Several identical rubber bands, a thumbtack, a centimetre ruler, butcher paper, large table.

Activity: Loop two identical rubber bands together to form a knot in the middle.

I   To make an enlargement of triangle ABC on the butcher paper:

1) Pick a point P so the distance from P to A is longer than the length of a rubber band.
2) Hold one end of the rubber band on point P with your thumb or use a thumbtack. Be sure to place tape over the thumbtack to secure it.
3) With a pencil in the other end, stretch the rubber bands until the knot is over A. Mark a dot with the pencil and label the dot A'.
4) Repeat step 2 with the knot over B to find B', then over C to find C'.
5) Connect A', B' and C'.

6) Measure the lengths of the sides of the two triangles.

   A'B' _____   B'C' _____   C'A' _____

   AB _____   BC _____   CA _____

7) How do these lengths compare?

   A'B' and AB _____

   B'C' and BC _____

   C'A' and CA _____

8) In each case is the first length about twice as long as the second length? _____

   The rubber bands have helped you to make an enlargement of the triangle using a scale factor of 2.

9) Draw another figure and make an enlargement of it using a scale factor of 2.

A SNAPPY SOLUTION TO SIMILAR FIGURES
(CONTINUED)

II) How many rubber bands would you use to make an enlargement using a scale factor of 3?
Could you make a reduction using 3 rubber bands?
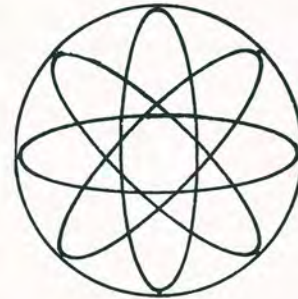
III) Designs with curved lines can be enlarged by watching just the knot and moving the pencil so the knot traces over the design.

IV) Find a design that you like and make an enlargement using a scale factor of 3. A large, simple design is easier to enlarge than a small, complicated one.
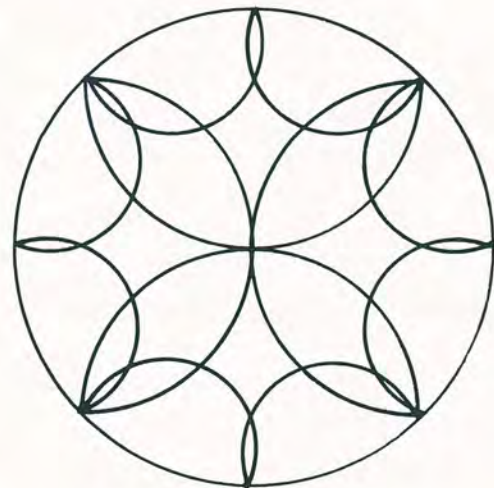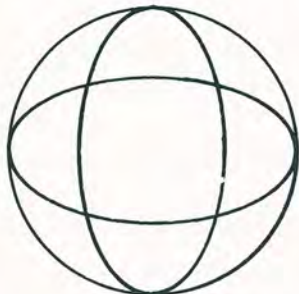
V) Use a classmate's enlargement and make a reduction of the design. Compare your reduction to the original design.

Challenge: Make an enlargement of a design of your choice using a scale factor of $2\frac{1}{2}$.

SIMILAR FIGURES AND THE PANTOGRAPH

I. A projection point and similarity ratio (scale factor) can be used to draw a new figure similar to an original. In the diagram to the right P is the projection point, STUR is the original figure and the similarity ratio is 2:1. The points S', T', U' and R' are located on the appropriate rays by making PS':PS = 2:1, PT':PT = 2:1, PU':PU = 2:1 and PR':PR = 2:1.
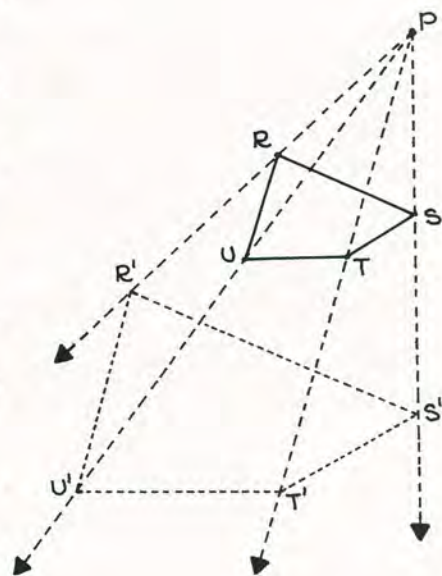
Why is figure S'T'U'R' similar to figure STUR? Similar triangles can be used to establish the relationship. Consider △PS'T' and △PST. Both triangles contain ∠P. Since PS':PS = PT':PT = 2:1, we can say that △PS'T' is similar to △PST. So T'S':TS = 2:1. Using the same argument on these pairs of triangles: △PT'U' and △PTU, △PU'R' and △PUR, △PS'R' and △PSR, we can show that T'U':TU = 2:1, U'R':UR = 2:1, R'S':RS = 2:1. Thus the ratios of corresponding sides are equal. But to show similarity between the two figures it is also necessary to show that corresponding angles are congruent. △PT'S' is similar to △PTS, so ∠PT'S' ≅ ∠PTS. Likewise △PT'U' is similar to △PTU, so ∠PT'U' ≅ ∠PTU. Therefore ∠U'T'S' ≅ ∠UTS. In the same manner ∠U' ≅ ∠U, ∠R' ≅ ∠R and ∠S' ≅ ∠S. So figure S'T'U'R' is similar to figure STUR.

II. A pantograph is a device which is used to draw a new figure which is similar to an original figure. It consists of four bars hinged together at B, C, D, E so that BD = CE and BC = DE (BCED is a parallelogram). The points A, D and P are necessarily collinear. Why does the pantograph draw similar figures? What determines the similarity ratio of the pantograph.
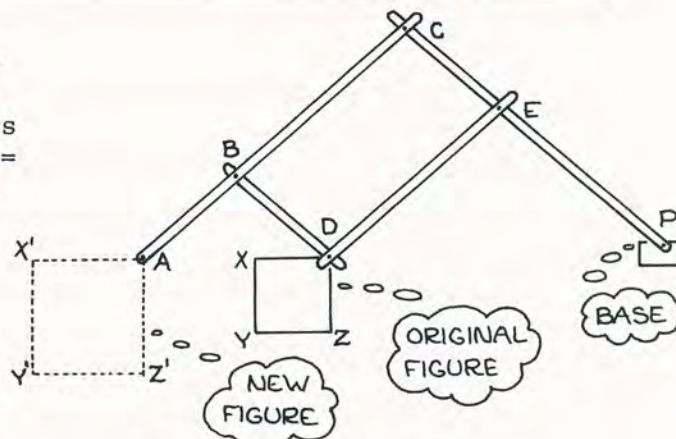
1) The pantograph contains three similar triangles: △ABD, △ACP and △DEP.

   a) △ABD is similar to △ACP since both contain ∠A and ∠ABD ≅ ∠C from the parallel lines $\overleftrightarrow{BD}$ and $\overleftrightarrow{CP}$.

   b) △DEP is similar to △ACP since both contain ∠P and ∠DEP ≅ ∠C from the parallel lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{DE}$.

2) Because △ACP is similar to △DEP, AC:DE = AP:DP. The lengths AC and DE are constant since they are the lengths of two bars on the pantograph. The lengths AP and DP change as the pantograph moves from point to point on the original to draw the new figure. However their ratio, AP:DP, is constant since it equals AC:DE. This constant ratio makes the new figure similar to the original (see the discussion in I). Also the ratio AC:DE is the similarity ratio for the pantograph.

BUDDIES

The two figures to the right have exactly the same shape. Figure VWXYZ is similar to figure ABCDE.

Each angle of figure VWXYZ can be matched with a congruent angle of figure ABCDE as shown.

∠V and ∠A are <u>corresponding</u> angles.

$\overline{VZ}$ and $\overline{AE}$ are <u>corresponding</u> sides since they match when the angles are paired.

Four more pairs of similar figures are drawn to the right. Use them to answer the questions below.

> The letters in the blanks should answer the riddle below.

WHAT DID THE FOUR 90° ANGLES HAVE IN COMMON?

1) Angle D corresponds to angle ___.

2) Side X'Y' corresponds to side B___.

3) Side D'A' corresponds to side G___.
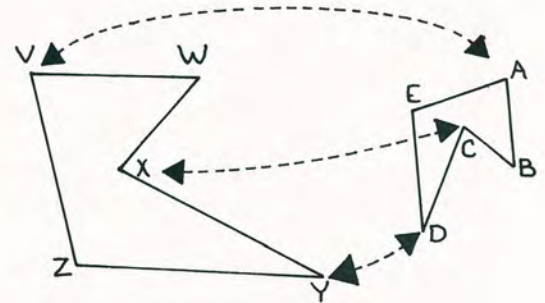
4) Angle P corresponds to angle ___.

5) Side B'C' corresponds to side K___.

6) Angle O corresponds to angle ___.

7) Side JN corresponds to side C___.

8) Angle D' corresponds to angle ___.

9) Angle T' corresponds to angle ___.

10) Side EF corresponds to side D___.

11) Angle M corresponds to angle ___.

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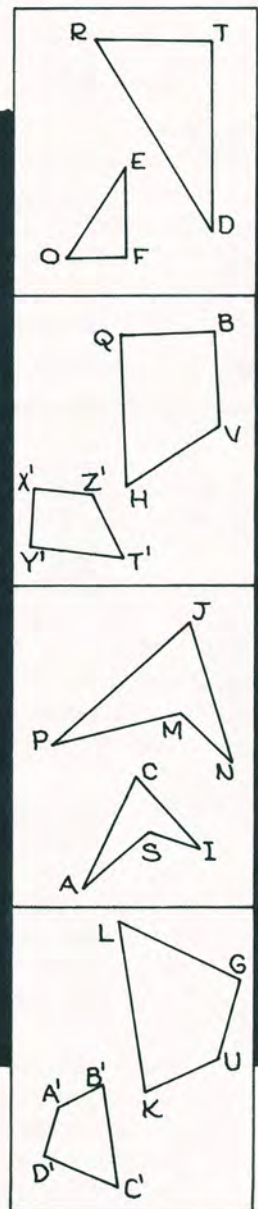
GEOBOARD SHAPES I

Use only 4 or 5 rubber bands on the geoboard.

1) Make each design on your geoboard.

   a.
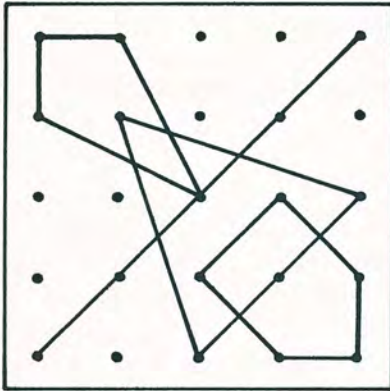
   b.
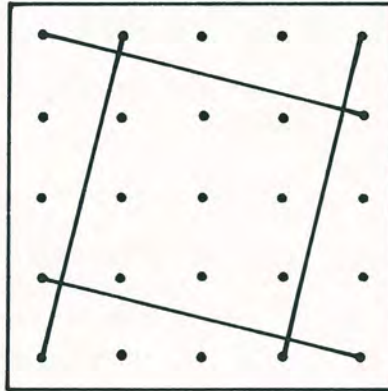
   c.

2) Make a design of your own on the geoboard. Copy your design on dot paper.

3) Make a stop sign on your geoboard. Copy the sign on dot paper.

4) Make a house, boat or an airplane on the geoboard. Copy each design on dot paper.
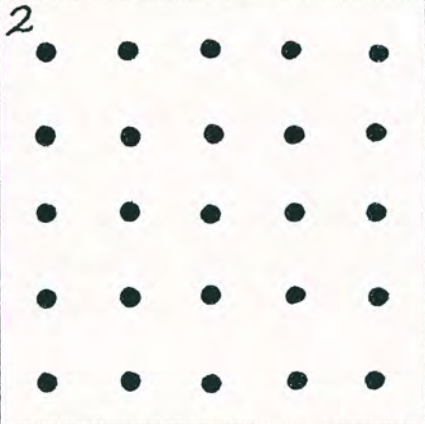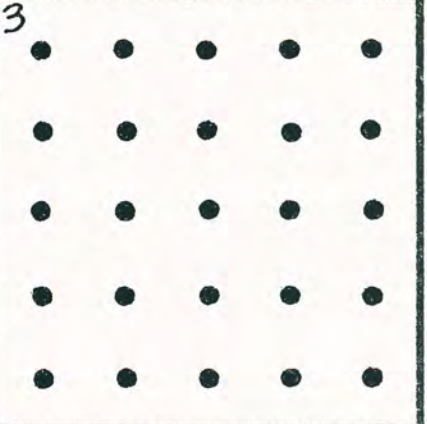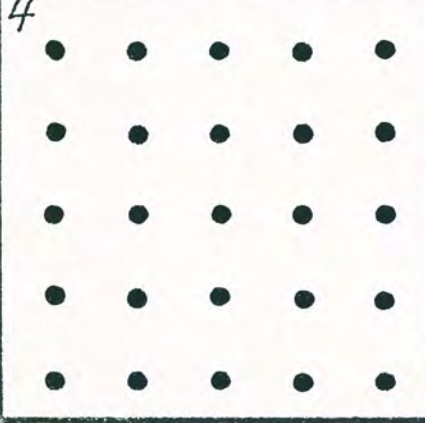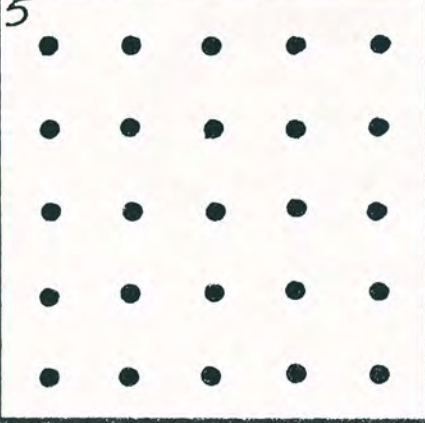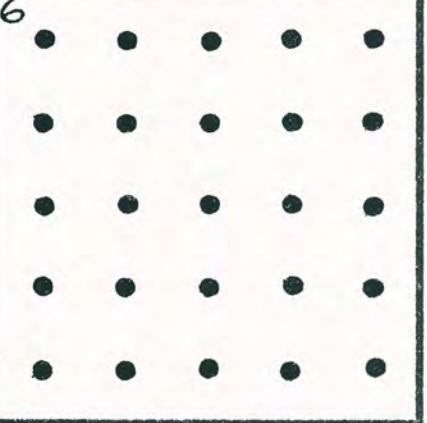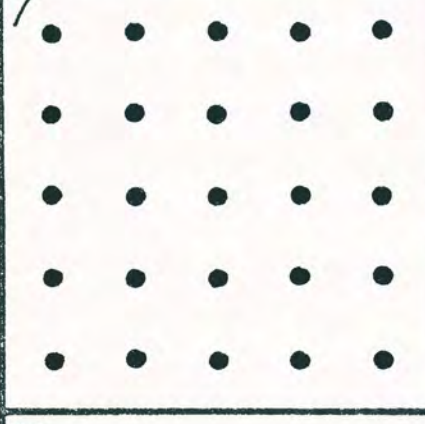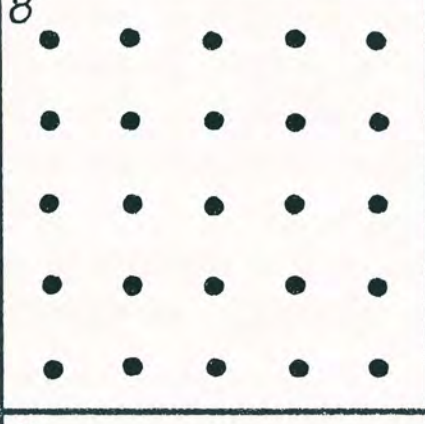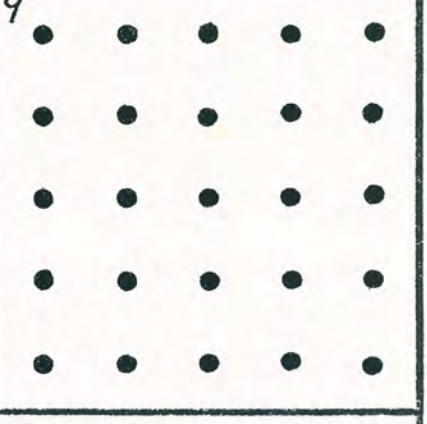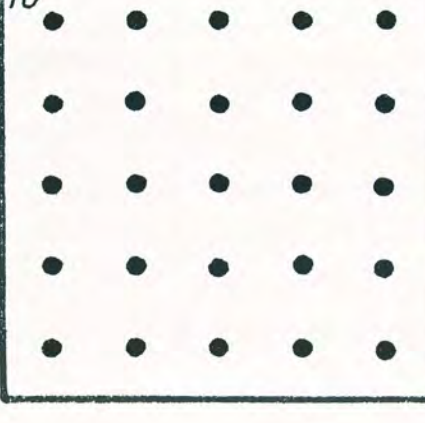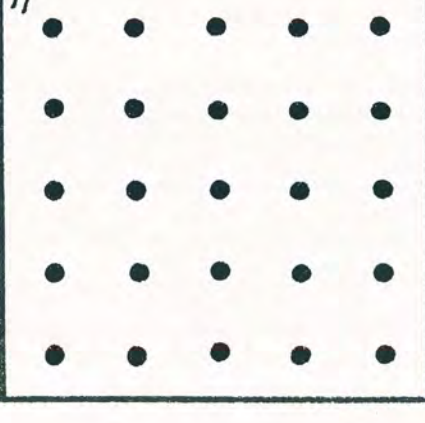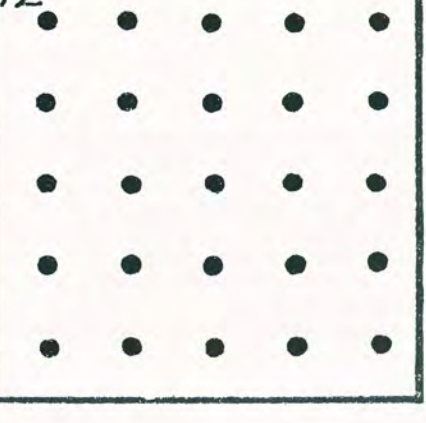
5) Make a triangle on the geoboard. Make a larger or smaller triangle having the same shape as the first triangle. Copy the triangles on dot paper. Is your triangle the same shape as your neighbor's? How many differently shaped triangles can you make?

6) Make your name on the geoboard. Copy each letter on dot paper.

GEOBOARD SHAPES II

I Copy the shapes below onto your geoboard.

A) How do the lengths of the shortest sides of the shapes compare?

B) How do the lengths of the longest sides compare?

C) How do the lengths of corresponding diagonals compare?

A DIAGONAL IS A SEGMENT JOINING THE VERTICES WHICH ARE NOT ON THE SAME SIDE.

II Copy the following designs and in each case make a similar figure that is either larger or smaller. For each design compare the length of any side or diagonal on the new figure to the length of the corresponding side or diagonal on the original figure.

A
LENGTH (NEW) IS
_____ TIMES
AS LONG AS
LENGTH (ORIGINAL).

B
LENGTH (NEW) IS
_____ TIMES
AS LONG AS
LENGTH (ORIGINAL).

C
LENGTH (NEW) IS
_____ TIMES
AS LONG AS
LENGTH (ORIGINAL).

D
LENGTH (NEW) IS
_____ TIMES
AS LONG AS
LENGTH (ORIGINAL).

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GEOBOARD SHAPES II (CONTINUED)

GEOBOARD RECORD SHEET

COMPARING SIMILAR SOLIDS

This activity involves measuring and comparing corresponding parts of similar polyhedra.

Students will need to make 3 or 4 similar polyhedra. The nets of a cube and tetrahedron are given here. Alternate ways of making polyhedra can be found on the pages *Folding Fun*   the Polyhedra subsection.

CUT ON THE SOLID LINES AND FOLD UP ON THE DOTTED LINES.

TETRAHEDRON

CUBE
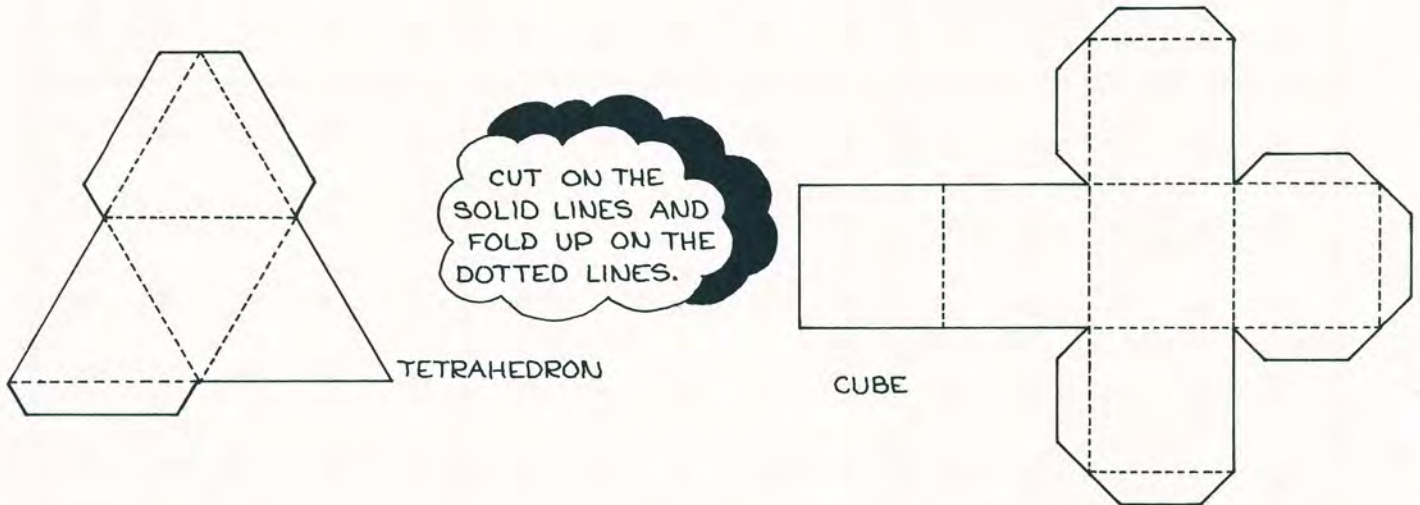
To make measurements easier, the edges of one polyhedron can be multiples of the edges of another polyhedron. For example, students could make one set of three tetrahedra with edges of 2 cm, 4 cm and 8 cm. When the polyhedra have edges 2 cm or smaller and less it becomes increasingly difficult to measure the edges and face angles.

The process of making models may be shortened by assigning each student a differently sized model to make. Circulating all models gives each student many models to measure.

Students can measure and compare corresponding parts of the similar polyhedra. A ready-made table may make recording easier for them. For example:

	SMALLER POLYHEDRON	LARGER POLYHEDRON	COMPARISON (RATIO)
LENGTH OF AN EDGE			
MEASURE OF AN ANGLE			
PERIMETER OF BASE			
YOUR CHOICE			

MAKE ALL MEASUREMENTS TO THE NEAREST ONE HALF CENTIMETRE.

REFLECT ON THAT IMAGE

Materials Needed: Concave mirror, candle, 4 cm x 4 cm pieces of glass or plastic, clay

The type of mirror needed for your investigation is a concave mirror.   An ordinary magnifying cosmetic or shaving mirror will work.

1) Find the focal length of the mirror. If the sun is out you can reflect the sun's rays to a fine point on a piece of paper. Use caution! If you have a good mirror it will

THE POINT WHERE ALL THE SUN'S RAYS ARE FOCUSED IS THE FOCAL POINT. THE DISTANCE BETWEEN THIS POINT AND THE MIRROR IS THE FOCAL LENGTH.

have the same effect as a magnifying glass and may give a very hot focal point. If the sun is not out use any object that is a long way off and project its image onto a piece of paper.   The focal length is _____ cm.

2) Describe images.

   a) Put a candle in front of a mirror at a distance slightly greater than the focal length. Using a piece of paper as a screen find the image (see the figure to the right). Describe the image using general terms (bigger, smaller, same shape, etc.)

   IMAGE OF CANDLE FLAME ON A PIECE OF PAPER

   THE OBJECT MUST BE AT A DISTANCE FROM THE MIRROR GREATER THAN THE FOCAL LENGTH.

   b) Move the candle about 2 metres from the mirror. Find the image with the paper. Again describe the image in general terms.

REFLECT ON THAT IMAGE
REFLECT ON THAT IMAGE

(CONTINUED)

II Some exact measurements.

Geometric shapes marked on clear material (glass or plastic about 4 cm x 4 cm)
make nice shapes to project. A flashlight can act as a light source. The
4 cm x 4 cm glass can be mounted on a piece of clay.

AN OBJECT MARKED ON A PIECE OF GLASS. MOUNTED ON A PIECE OF CLAY.

IMAGE IS PROJECTED ON A PIECE OF PAPER.
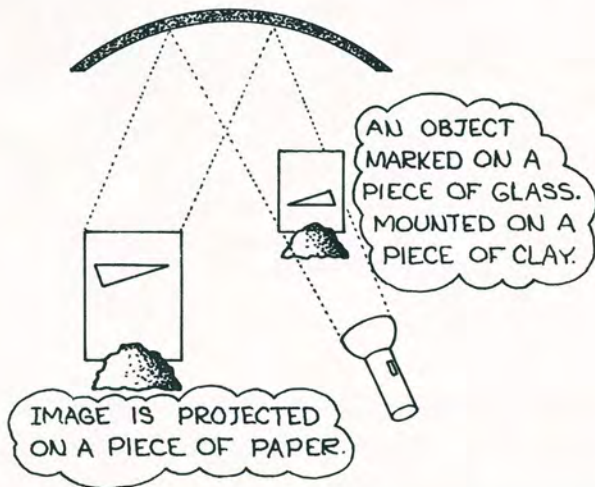
Set up your mirror and 4 cm x
4 cm glass piece as shown.

Measure the sides of your
figure to the nearest half
centimetre and record in a
table like the following.

	1ST SIDE (cm)	RATIO OBJECT : IMAGE	2 ND SIDE (cm)	RATIO OBJECT : IMAGE	3 RD SIDE (cm)	RATIO OBJECT : IMAGE
OBJECT						
IMAGE						

Use another object placed at a different spot. Make a table similar to the one
above that would be suitable for your new object. Measure and compare corre-
sponding parts of the object and image.

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LITTLE ONE BIG ONE
1 1

Similar figures have been described as having exactly the same shape. Oftentimes it is not easy to decide whether two shapes are actually similar. A method to measure and compare is needed.
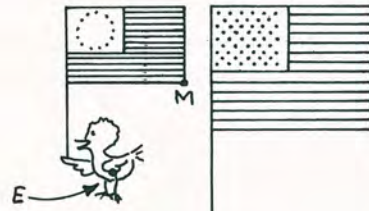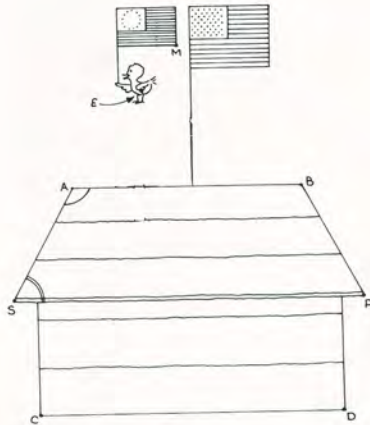
Parts that are in the same place in both pictures are called corresponding parts.

Measure these corresponding segments to the nearest one-half centimetre.

	LARGE FIGURE	SMALL FIGURE	RATIO LARGE : SMALL	SIMPLIFIED FORM OF RATIO
1776 FLAGSTAFF	2 cm	1 cm	2 : 1	2 : 1
TOP OF DOG HOUSE ($\overline{AB}$)			:	:
LINE SEGMENT AD			:	:
LINE SEGMENT ED			:	:
LINE SEGMENT AC			:	:

Do you notice anything about the simplified ratios?

Measure each of the following angles to the nearest whole degree.

	ANGLE MEASURE AT A	ANGLE MEASURE AT S	ANGLE MEASURE AT C
LARGER FIGURE			
SMALLER FIGURE			

Does the same relationship hold for corresponding angles as you found for corresponding sides?

Try to answer these without measuring any angles or sides. One value is given. Find the measure of the corresponding part.

	$\overline{MP}$	$\overline{DP}$	$\overline{SA}$	$\overline{MD}$	m∠P
LARGER FIGURE			3.4 cm	10.4 cm	
SMALLER FIGURE	4.4 cm	1.6 cm			61°

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The following pairs of figures are similar. Use corresponding parts and equivalent ratios to find the measures of the missing parts.

Directions: Find the correct answers at the bottom of the page. Connect the dots in the same order as the problems are lettered.
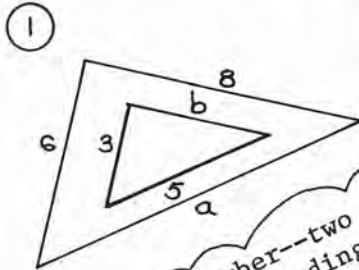
Remember—two figures are similar if corresponding angles are congruent and lengths of corresponding sides are in the same ratio.

△ ABC ~ △ DEF

△ABC ~ △DEF

16
•

18
•

• 20

57°
•

• 67°
STOP

• 74°

• 39°

• 7½

$10\frac{2}{3}$ •

• $6\frac{2}{3}$

• 83°

$5\frac{1}{4}$ •

• 26

• 21

40° STOP

$11\frac{1}{4}$

• 12

4 •

$13\frac{1}{3}$ STOP

• 16

• 30 STOP

• 10
START

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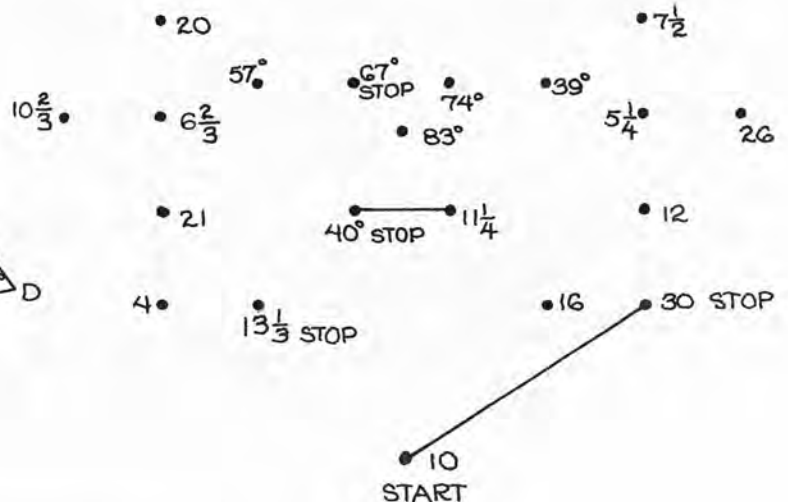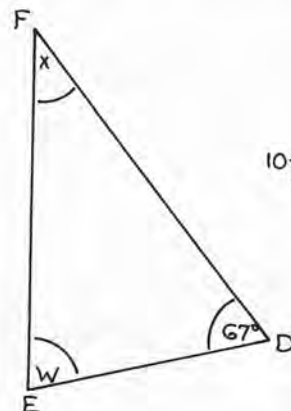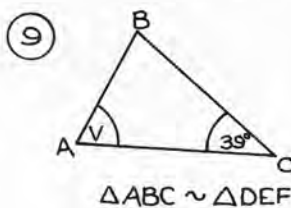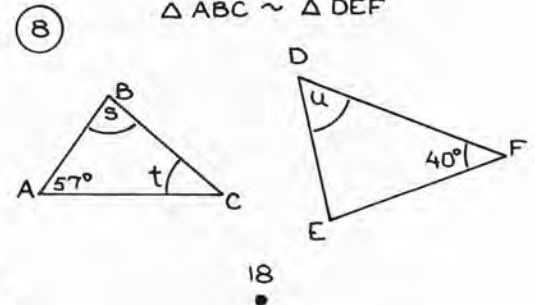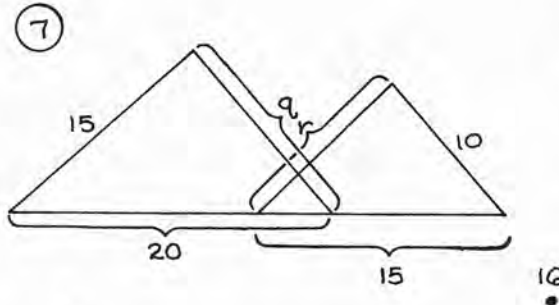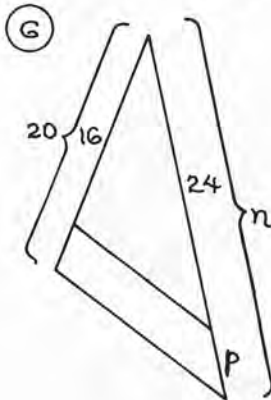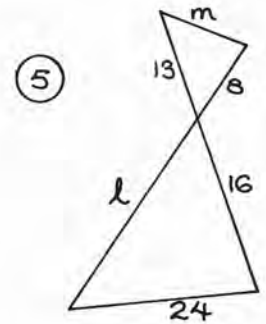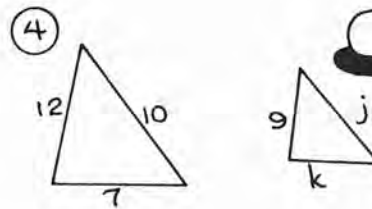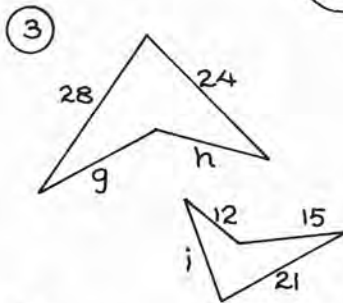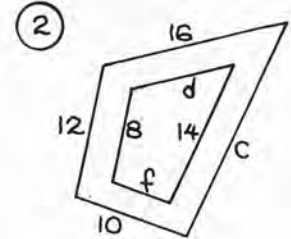
MEASURE UP!

Mr. Stevens needs to buy a ladder. He is going to paint his barn. The barn has a high peak on one side, and he wants to be sure the ladder is long enough. He found this chart in an old mail-order catalog.
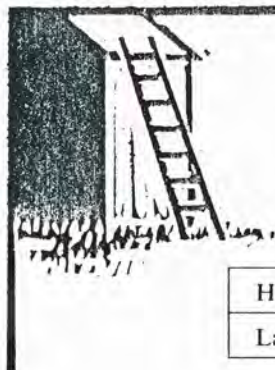
How to Order a Ladder

Measure the height of the wall.
Chart shows size to order.

Height (feet)	16	20	24	28	32	36
Ladder (feet)	20	24	28	32	36	40

Class Exercises

1. Mr. Stevens thought about using a direct method to measure the height of the barn. What direct procedures could he use?

2. Mr. Stevens found no direct method that would work. His next thought was to use the shadow method.

   a. What measurements are required for this method?
   b. The shadow method was not satisfactory. Give some possible reasons.

3. Mr. Stevens used this method. He located a point 25 feet from the barn. Then he measured $\angle CAB$, the *angle of elevation*.
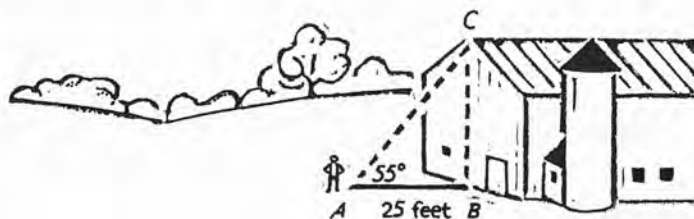
   a. Is this enough information to make a scale drawing of $\triangle ABC$?
   b. Make a scale drawing of the triangle.
   c. How can you use your scale drawing to find the approximate height of the barn?
   d. According to the chart, what ladder should Mr. Stevens buy?
   e. What similar triangles are used to solve this problem? Explain.

SOURCE: *Synchro-Math/Experiences*
Permission to use granted by Action Math Associates, Inc.

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FAR OUT!

1) The sketch shows measurements a Boy Scout troop made to find the distance across Clear Lake. Make a scale drawing to find the distance.

2) Find the distance across the gulch. Use the information in the figure; make a scale drawing.

3) The sketch illustrates how Miss South's class measured the height of a flagpole. Make a scale drawing to find the height of the pole. Hint: Notice that the transit is 1.5 m tall.
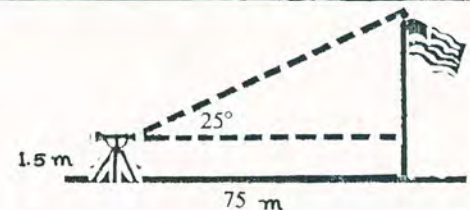
4) Chip and O'Duffer are on the school's golf team. Hole 5 of their course is shown here. The boys often wondered if they could play successfully across the stream rather than around it. One day they made the measurements.

   a) Chip can drive a ball 250 yards. Can he reach the green in one shot? Show your work.
   b) O'Duffer can drive a ball 200 yards. Can he reach the green in one shot?
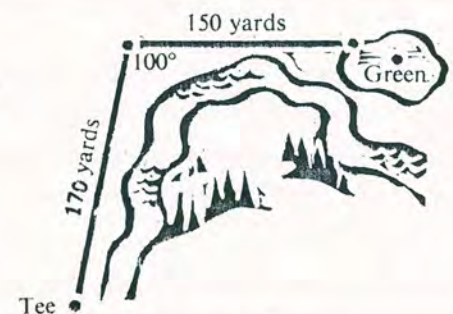
5) Two lookout towers are 25 km apart. Forest rangers can find the location of the fire using the data shown.

   a) Find the distance from A to C. Show your work.
   b) Find the distance from B to C.
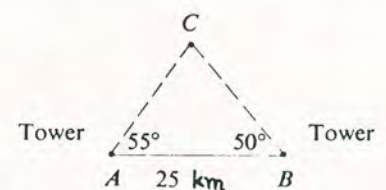
6) Al Pine wants to know how high Big Auntie Mountain is. He knows the angle of elevation from two different points.

   a) Use the information shown. Make a scale drawing.
   b) How high is Big Auntie?
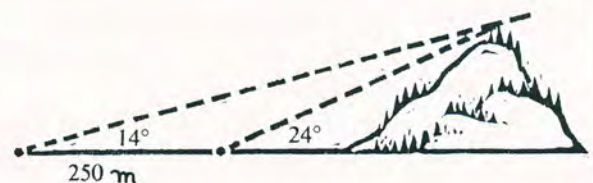
RIGHT-ON RATIOS

I'M ROBERTA. TO SOLVE THIS PROBLEM I DREW THIS PICTURE.

I ALWAYS LIKE TO SAVE SPACE ON MY PAPER.

Draw four right triangles. One of the acute angles in each triangle must be 35°. The four triangles must be different sizes.
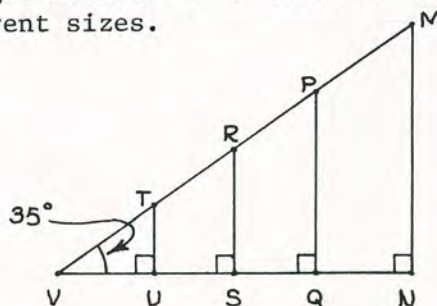
Class Exercises

1) Examine Roberta's drawing.

   a) Are there four right triangles in the drawing?

   b) Does each triangle have an acute angle of 35°?

   c) Are the triangles different in size?

2) Use Roberta's drawing. Find the ratios. Express each ratio as a decimal. Hint: You will find it easier if you measure the segments to the nearest millimetre.

   a) MN to VN   b) PQ to VQ
   c) RS to VS   d) TU to VU

3) All the ratios in 2a-2d are about .7. Will the corresponding result probably occur in another 35° right triangle? Explain.

4) Roberta was asked to draw a 70° right triangle and to find the ratio of a to b.

   a) Is this experiment necessary?

   b) Predict the ratio. Should it be 2 times the ratio for a 35° right triangle?
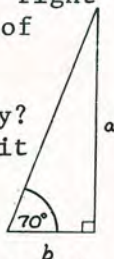
   c) Draw a 70° right triangle to test your prediction.

5) What is the ratio of a to b in a 45° right triangle?

6) The ratio of a to b for every 35° right triangle is approximately .7. Use this information to find the height of the flagpole. Note: b is 15 m.

7) The ratio of a to b for every 70° right triangle is approximately 2.7. Find the height of the building. Note: b is 12 m.
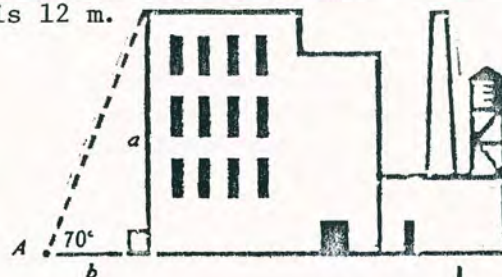
RIGHT-ON RATIOS

(CONTINUED)

8) Find a in each triangle. Use the ratio you know for a 35° right triangle.

a)

a
35°
47

b)

a
35°
32

c)

a
35°
63
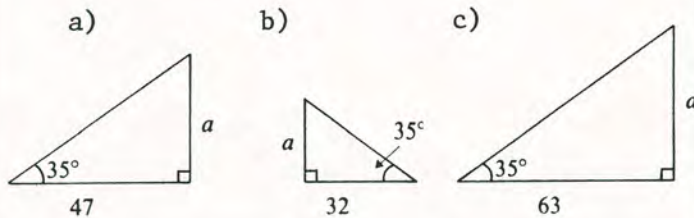
9) Find a in each triangle.

a)

a
70°
91

b)

a
70°
117

c)

a
70°
79
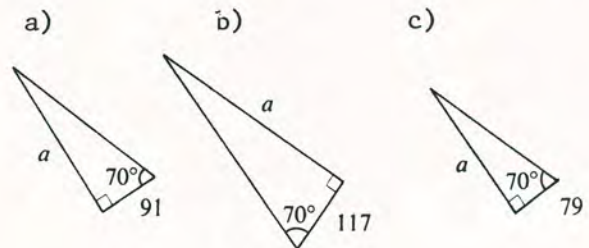
10) Find the height of the monument.

70°
15 m
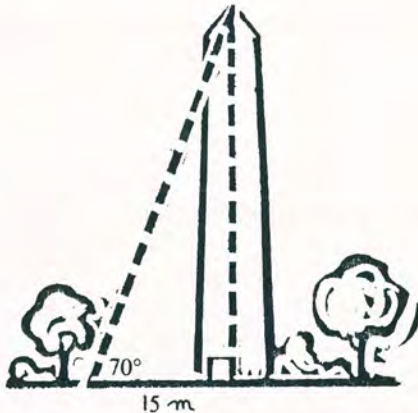
11) Find the distance across Mr. Millard's pond.

35°
13 m
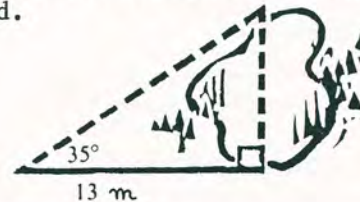
12) Bryan is standing 11 m from a tree. The angle of elevation is 45°. What is the height of the tree?

45°
11 m
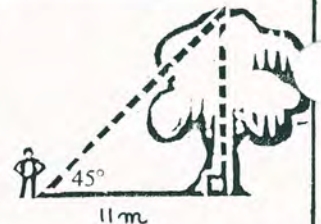
13) Draw three 20° right triangles of increasing size. Note: Roberta's method for drawing is all right.

a) For each triangle, find the ratio of a to b. Express each ratio as a decimal. Hint: Measure the segments to the nearest millimetre.

b) Study the results. What seems to be a reasonable value to use for this ratio?

14) Use the ratio you found for a 20° right triangle to find the approximate height of the cliff.

a
20°
129 m
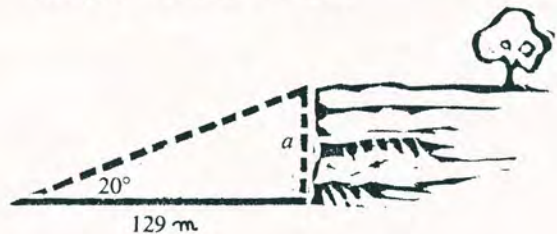
15) Figure out the ratio for a 40° right triangle. Use your ratio to determine the distance across the Crimea River.

40°
26 m
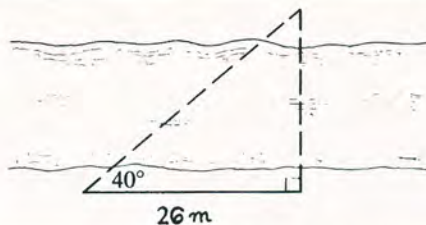
MAKE IT EASY ON YOURSELF

Activity I

Materials: Tagboard weight paper

1) Draw and cut out a triangle from the tagboard. Label the angles A, B, C. Trace around the triangular region to make a copy of triangle ABC. Now cut off the three corners.

2) Use the three corners as tracing patterns to draw a triangle. Make the length of each side longer than those in the original triangle ABC. Is the new triangle similar (the same shape) to triangle ABC? _____

3) Use the three corners again as tracing patterns to make another triangle. This time make the sides shorter than those in the original triangle ABC. Is this triangle similar to triangle ABC or the triangle you made in set #2? _____

4) Can you draw a triangle whose angles are congruent to the angles in these three triangles which would not be similar to triangle ABC? Use the three angle patterns to help you experiment.

5) Use any two of the three angle patterns to construct a triangle. Is this triangle similar to the original triangle ABC? _____   Try again using a different pair of angles. Is this triangle also similar to the original triangle ABC? _____   Can you draw a triangle having two angles congruent to angles in triangle ABC which would not be similar to triangle ABC?

6) Extension: If only one angle pattern is used to draw a triangle can you draw a triangle that is not similar to triangle ABC?

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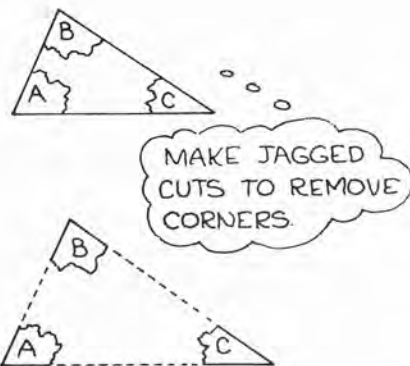
MAKE IT EASY
ON YOURSELF
(PAGE 2)

Activity II

Materials: Protractor and centimetre ruler

1) On your paper draw four segments with lengths 10 cm, 6 cm, 4 cm and 2 cm and label as shown.

A •————————————————————• B
10 cm

C •————————————————• D
6 cm

E •————————• F   G •———• H
4 cm   2 cm
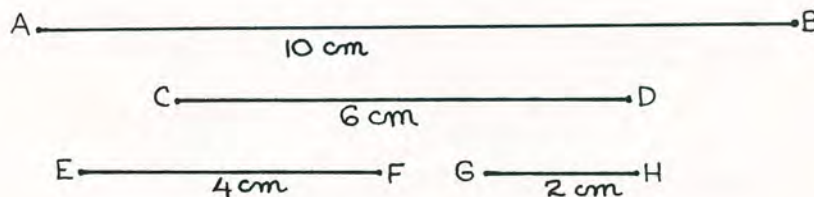
2) At points A, C, E and G use your protractor to draw a 50° angle. At points B, D, F, H draw a 30° angle.
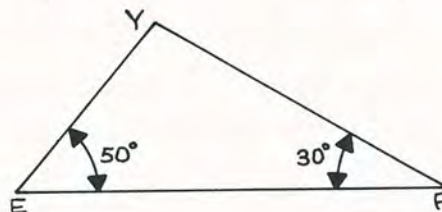
3) Label the triangles △ ABW, △CDX, △ EFY, △GHZ and measure the third angles:

m ∠ W = _____; m ∠ X = _____; m ∠ Y = _____; m ∠ Z = _____

Do the triangles appear to be similar? _____

4) Would it have been possible to calculate the measures of angles W, X, Y and Z without drawing the triangles? _____

If the given angles were 30° and 70° instead of 50° and 30° what would the third angle have measured? _____

5) Repeat step #1. Construct four triangles using 30° and 70° instead of 50° and 30°. Do these four triangles appear to be similar? _____ Does the third angle appear to be about 80°? _____

6) Based on the eight triangles you just drew would you say that

   a) the length of the side you start with is important in making the triangles similar? _____

   b) the size of the angles is important in making the triangles similar? _____

7) To construct similar triangles given a side in each triangle what is the <u>least</u> number of angles you must be given? _____

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MAKE IT EASY ON YOURSELF
(PAGE 3)

Activity III

Materials: 3 pieces of florists wire or pipe cleaners cut in lengths of 18 cm,
12 cm and 10 cm.

1) Use the three pieces of wire to make a triangle.
   On your paper carefully trace the triangle.

2) Fold each piece of wire in half as shown.

3) Use the folded pieces to make a triangle. Compare the new triangle to your
   tracing. Do the triangles appear to be similar? _____

4) Is each angle in the traced triangle congruent to
   an angle in the new triangle? _____   Check by
   placing each angle of the new triangle over an
   angle on the traced triangle.
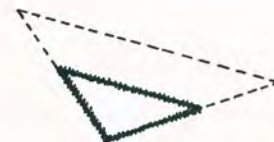
5) Unfold the wires and refold them so that each
   pair is two-thirds the original length.

6) Use these pieces to make a triangle. Compare the triangle to your tracing.
   Do the triangles appear to be similar? _____

7) Unfold the wires and refold them so that each
   piece is three-fourths the original length.

8) Repeat step #6.

9) If each side of △ ABC is four-fifths the length of the corresponding side in
   △ DEF what can you say about the triangles? _____

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MAKE IT EASY ON YOURSELF
(PAGE 4)

Activity IV

Materials: Compass, protractor and centimetre ruler

1) Construct a triangle whose sides are equal
    in length to those shown.

_____ 8 cm _____

_____ 6 cm _____

_____ 4 cm _____
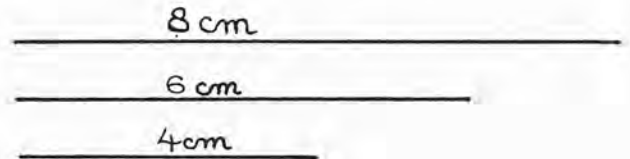
2) Now construct a triangle whose sides are
    half as long as each of these line segments.

3) Do the triangles appear to be similar? _____ Use your protractor to measure
    the angles in both triangles. Is each angle in the larger triangle congruent
    to an angle in the smaller triangle? _____

4) Construct a triangle whose sides are each $\frac{3}{2}$ as long as each segment in step #1.
    Is this triangle similar to those you have already made?
    Check by measuring the number of degrees in each angle.

5) Are these triangles similar? _____
    The corresponding sides are $\overline{PQ}$ and
    $\overline{ST}$, $\overline{RQ}$ and $\overline{UT}$, $\overline{PR}$ and $\overline{SU}$.
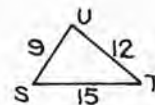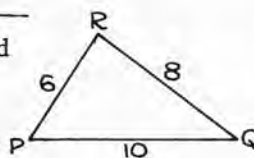
CHECK BY WRITING
THE RATIOS OF THE
CORRESPONDING
SIDES IN SIMPLEST
FORMS. ARE THE
RATIOS EQUAL?

6) Are these triangles similar? _____
    The corresponding sides are $\overline{AB}$ and
    $\overline{CD}$, $\overline{BX}$ and $\overline{DY}$, $\overline{AX}$ and $\overline{CY}$.
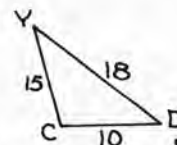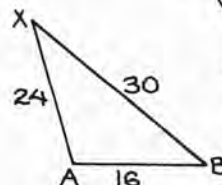
TELL IT LIKE IT IS

Use the angle-angle (AA) or side-side-side (SSS) similarity property to help you tell whether each pair of triangles is similar or not.

Write yes or no below each exercise.

a) 85° 85° 32° 63°

Answer: _____

b) IN WRITING RATIOS BE SURE TO COMPARE LONGEST SIDE TO LONGEST SIDE, SHORTEST SIDE TO SHORTEST SIDE.

6 9 12   8 12 16

Answer: _____

c) ☐

Answer: _____

d) 35°

Answer: _____

e) 18 24 27 32 14 21

Answer: _____

f) C 48° A C°°°°° $\overline{AB}$ IS PARALLEL TO $\overline{CD}$. 61° B D

Answer: _____

g) 15 8 6 10 8 12

Answer: _____

h) A 21° C 38° D B $\overline{AB}$ IS PARALLEL TO $\overline{CD}$.

Answer: _____

i) 4 5 3 4 5 6

Answer: _____

CHALLENGE:

1) B D A E C

IS △ ADE SIMILAR TO △ ACB? _____

WHY? _____

2) A 25° B F C 25° D E

CAN YOU FIND ANY SIMILAR TRIANGLES? _____

SURVEYING THE SITUATION

Often a surveyor finds it impractical to measure distances directly with a tape measure. They find these distances indirectly by using a transit and similar triangles.

To measure the distance across a wide river a surveyor sets up a base line along the river bank, marking the end points with stakes A and B. The transit (an instrument equipped with a telescope and protractor) is placed over stake A and the surveyor turns the telescope to see stake B. By rotating the telescope again to sight a landmark (a tree or rock) across the river, the surveyor measures the angle at A. The transit is then placed over stake B and a similar measurement is made.

On paper a triangle similar to the one above is drawn. The length of A'B' determines the similarity ratio or scale factor of the drawing. A convenient scale to use is 1 cm:10 m. Since $\angle A' \cong \angle A$ and $\angle B' \cong \angle B$ the scale drawing is similar to the original triangle.

By measuring TR on the scale drawing and setting up a proportion the corresponding measurement in the original triangle can be found. TR = 7 cm so 7 cm:distance across the river = 10 cm:100 metres. Thus the distance across the river is 70 metres.

Students can use the method outlined above to measure distances indirectly or to make a scale drawing of a field or playground.

If a commercial transit is not available students can assemble the homemade transit pictured to the right.

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SIDES AND ANGLE

1. (a) In the space to the right draw a
   triangle ABC so that AB = 6 cm and
   AC = 4 cm. Angle A can be any
   size. Label the vertices of your triangle.

   (b) Draw triangle DEF so that $\angle D \cong \angle A$,
   DE:AB = 1:2 and DF:AC = 1:2. Label
   the vertices of triangle DEF.

   (c) Use your protractor to measure $\angle B$
   and $\angle E$.

   ```
   CHECK YOUR
   DRAWINGS
   IF ∠B AND ∠E
   ARE NOT
   CONGRUENT.
   ```

   (d) What can you say about $\triangle ABC$ and $\triangle DEF$?

   _____

   (e) Without measuring what is the ratio
   EF;BC? _____

---

2. (a) Draw triangle XYZ in the space
   below so that XY = 5 cm and XZ =
   7 cm.   X can be any size. Label
   the vertices of the triangle.

   (b) Mark a point J on XY so XJ:XY = 1:2.

   (c) Mark a point K on XZ so XK:XZ = 1:2.

   (d) Draw $\overline{JK}$.

   (e) Use your protractor to measure $\angle Z$
   and $\angle XKJ$.

   (f) What can you say about $\triangle XYZ$ and
   $\triangle XJK$? _____

   (g) Without measuring what is the ratio
   KJ:ZY? _____

3. Without measuring write the missing
   lengths next to each segment.

   (a)

   (b)

   (c)

631
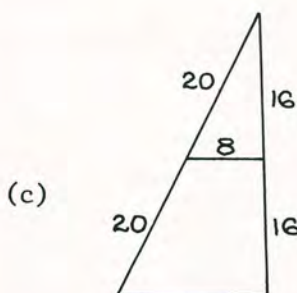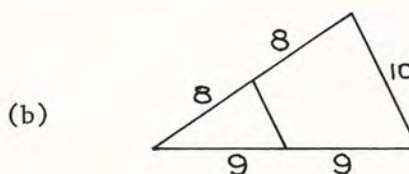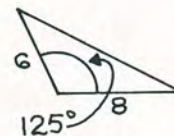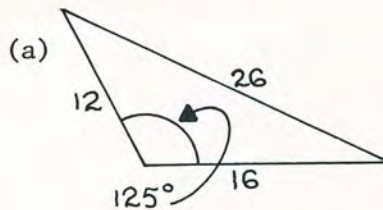
SIDES AND ANGLE

(CONTINUED)

I   The experiments on the student page are special cases of the Side-Angle-Side
    (SAS) similarity property for triangles.

SAS Similarity Property

If the ratios of two pairs of corresponding sides
of two triangles are equal and if the angles
included by the sides are congruent, then the
triangles are similar.

IF $\angle A \cong \angle D$
AND
$AB:DE = AC:DF$
THEN $\triangle ABC$ IS
SIMILAR TO $\triangle DEF$.
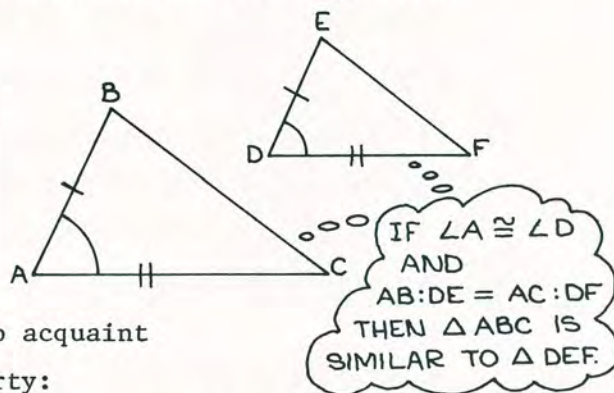
A guided discovery activity could be written to acquaint
students with the general SAS similarity property:

1. Draw and label $\triangle ABC$ so that AC = 9 cm, AB = 6 cm. $\angle A$ can be any size.

2. Draw and label $\triangle DEF$ so that $\angle D \cong \angle A$, DF:AC = 2:3 and DE:AB = 2:3.

3. Measure $\angle E$ and $\angle B$. What can you say about $\triangle ABC$ and $\triangle DEF$?

Different ratios could be used in step #2 and the experiment could be repeated
to convince students that the property always holds.

II Experiment #2 on the student page can be extended to establish some surprising
    results.

1.

$XJ:XY = 1:2$
AND
$XK:XZ = 1:2$
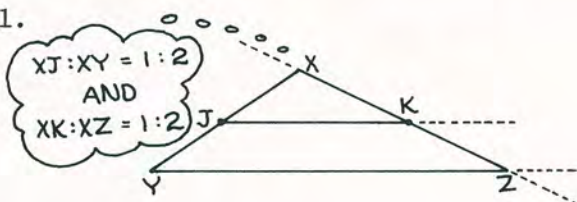
Since $\triangle XJK$ is similar to $\triangle XYZ$, $\angle XKJ \cong$
$\angle Z$. $\overleftrightarrow{XZ}$ is a transversal for lines JK
and YZ, making $\angle XKJ$ and $\angle Z$ correspond-
ing angles. Because $\angle XKJ$ and $\angle Z$ are
congruent, $\overleftrightarrow{JK}$ is parallel to $\overleftrightarrow{YZ}$.

2. In quadrilateral ABCD J, K, L and M are midpoints of the sides.
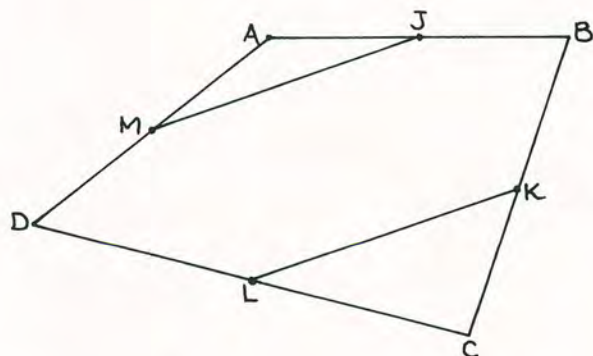
a) What can you say about $\overline{MJ}$ and $\overline{LK}$?
    (Hint: Draw $\overline{DB}$.)

b) Draw $\overline{JK}$ and $\overline{ML}$. What can you say
    about $\overline{JK}$ and $\overline{ML}$: (Hint: Draw $\overline{AC}$.)

c) What is special about quadrilateral
    MJKL?

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SPLITTING SIDES

Materials: Ruler, scissors, paper

1) Draw any large triangle on a separate paper and cut out two copies. Label the vertices of <u>one</u> triangle as shown.

2) Make a cut across △ABC parallel to side $\overline{AC}$ as shown by the dotted line and label points X and Y.

    a) What can you say about ∠X and ∠A? ∠Y and ∠C?

    b) Compare △XYB to the unlabeled original. Do they appear to be similar? _____

3) Now make a cut across △XYB parallel to side $\overline{YB}$ as shown. Compare the angles of the new triangle with the unlabeled original. Can you find congruent pairs of angles? _____ Do the triangles appear to be similar? _____

4) If you make a cut parallel to one side of the triangle would you expect the new triangle to be similar to ABC? _____

5)
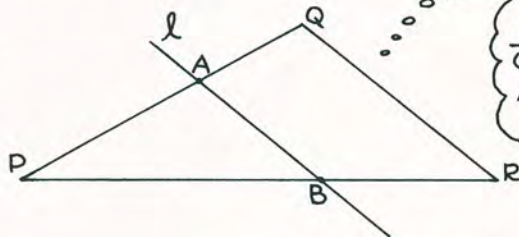
LINE ℓ IS PARALLEL TO $\overline{QR}$. WHAT CAN YOU SAY ABOUT △ABP AND △QRP?

6) How many triangles are similar to △CJD? _____ List them.

THE LINE SEGMENTS THAT APPEAR TO BE PARALLEL ARE PARALLEL.

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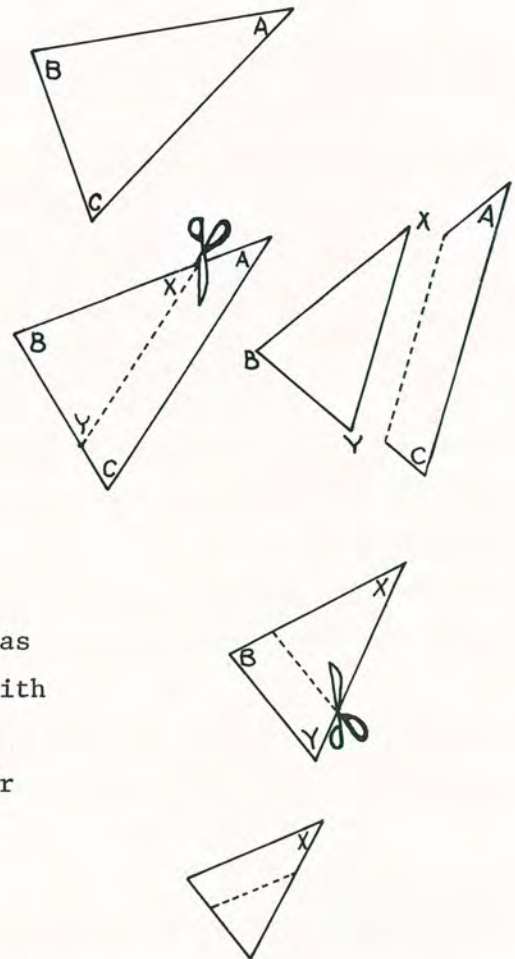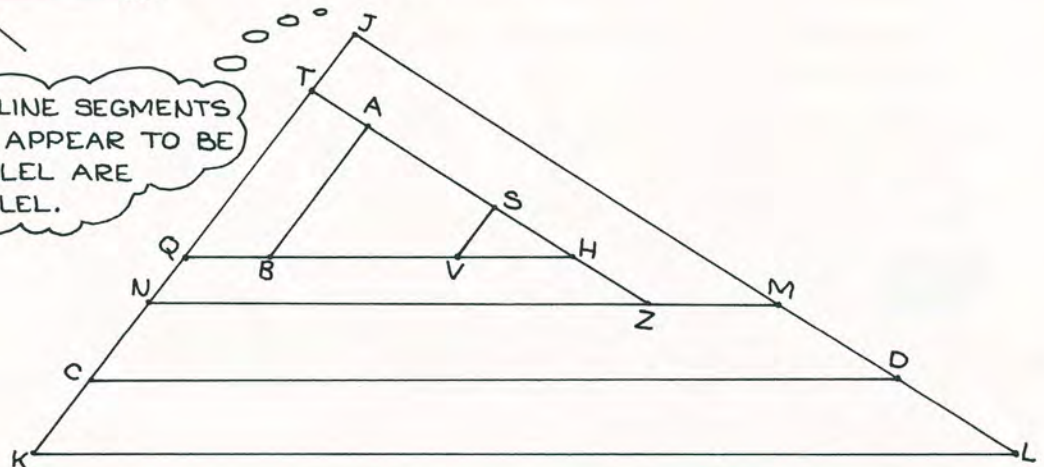
LINING IT UP

1) Take out a clean sheet of lined notebook paper.

2) Draw a line segment on your paper as shown.

> THE LINES ON THE NOTEBOOK PAPER DIVIDE YOUR LINE SEGMENT INTO SMALL LINE SEGMENTS. TWO OF THEM HAVE BEEN LABELED IN THE FIGURE, $\overline{AB}$ AND $\overline{CD}$.
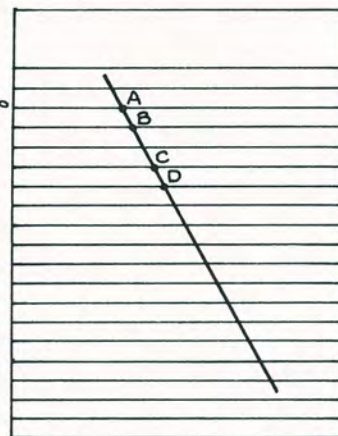
3) Pick any two of these small segments (like $\overline{AB}$ and $\overline{CD}$) on your paper and measure them to the nearest millimetre.

   What do you notice? _____

   Check two more small segments.

4) Devise a plan to use your notebook paper to divide $\overline{EF}$ into five equal lengths.

   E ——————————————— F

5) Use the notebook paper and a pencil to "divide" your finger into six parts of the same length.

6) Use your plan from #4 to divide $\overline{AB}$ into seven equal lengths.

   Now divide $\overline{AC}$ into seven equal lengths.

   Starting from A number the division points on $\overline{AC}$ from #1 to #6. Do the same on $\overline{AB}$.

   Draw line segments to join:

       C to B
       #6 on $\overline{CA}$ to #6 on $\overline{BA}$
       #5   to #5
           .
           .
       #1   to #1
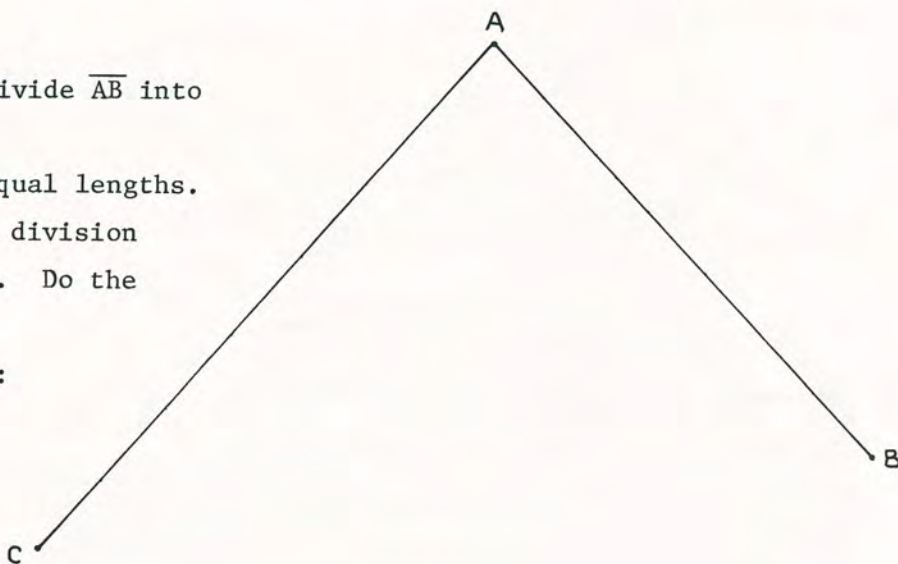
   What can you say about these segments?

   _____

DIV/IDE ME UP

To divide a line segment into five congruent parts using only a compass and straightedge:

1) On your paper draw a line segment about twice as long as the segment to the right. Label the end points A and B.

2) Draw ray $\overrightarrow{AC}$ forming $\angle BAC$. The construction is easier if $\angle BAC$ is less than 90°.

3) Make the opening of your compass a convenient size and mark five equal lengths on ray $\overrightarrow{AC}$ starting at point A. Label the last point D; the next-to-the-last point E. Draw $\overline{DB}$.

4) Make the opening of your compass equal to $\overline{DB}$ and from each division point on $\overline{AD}$ draw arcs below $\overline{AB}$.

5) Set your compass opening equal to DE. From B draw an arc that crosses the first arc you drew in #4. From this point draw an arc that crosses the second arc in #4. Repeat.

6) Draw line segments to connect the division points on $\overline{AD}$ with the arc intersections as shown in the diagram.

7) Check your construction by measuring the five segments on $\overline{AB}$ to see if they are congruent.
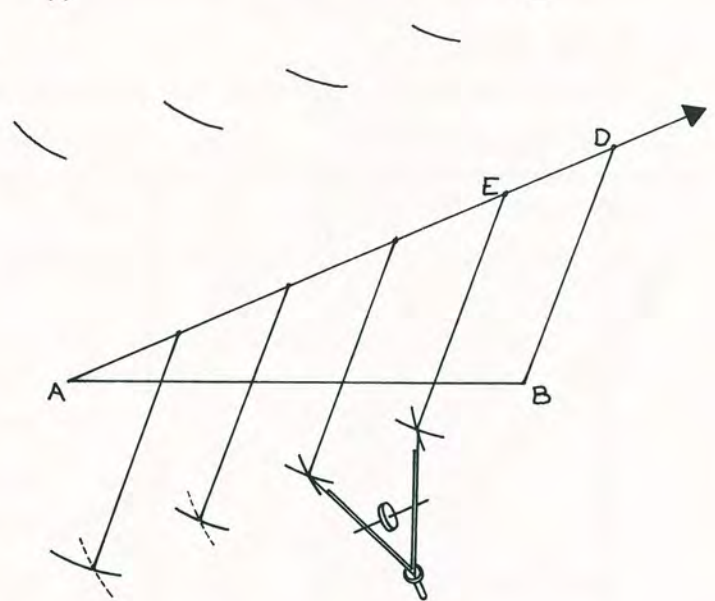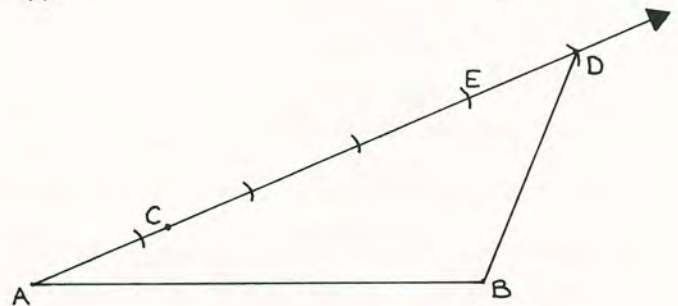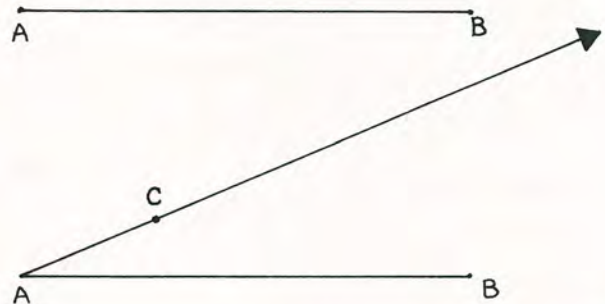
Try the method on some different line segments. Copy $\overline{XY}$ on your paper. Divide it into seven congruent parts.

PROPORTION - ALL

1) Use your ruler to measure these segments to the nearest half centimetre:

$\overline{AB}$ _____ $\overline{EF}$ _____

$\overline{BC}$ _____ $\overline{FG}$ _____

$\overline{CD}$ _____ $\overline{GH}$ _____

LINES $\overleftrightarrow{AE}$, $\overleftrightarrow{BF}$, $\overleftrightarrow{CG}$ AND $\overleftrightarrow{DH}$ ARE PARALLEL.

2) Write these ratios:

AB:EF = _____:_____

BC:FG = _____:_____

CA:GE = _____:_____

DA:HE = _____:_____

What do you notice? _____

_____
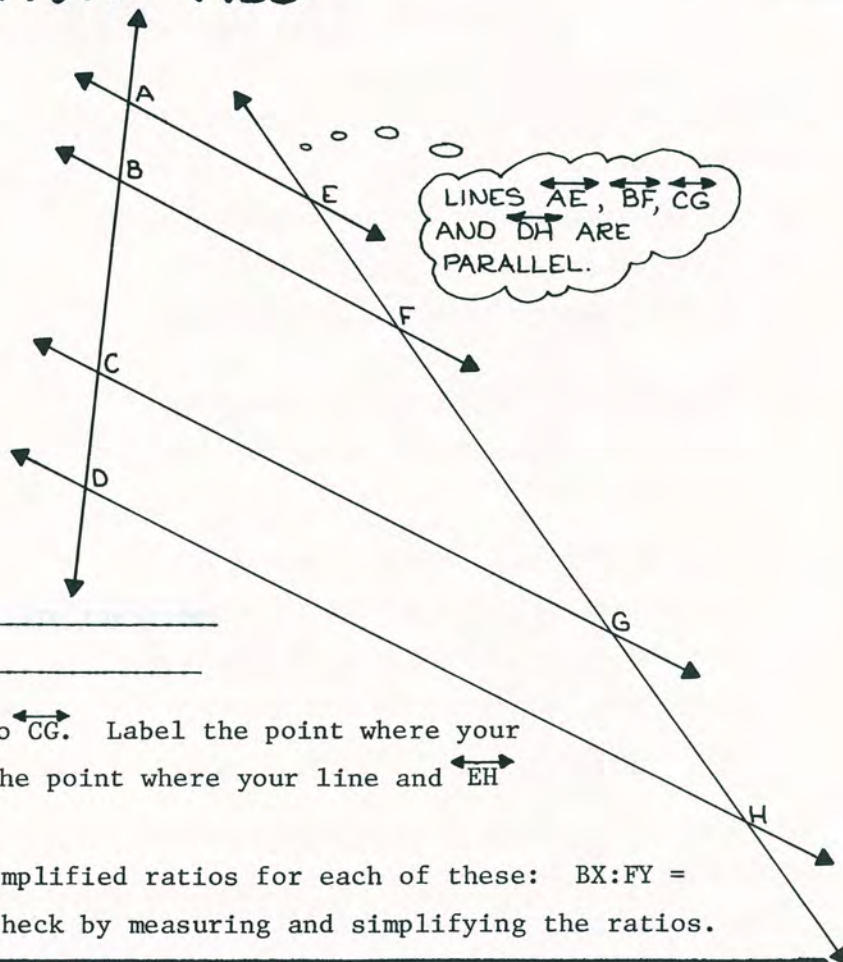
3) Carefully draw a line parallel to $\overleftrightarrow{CG}$. Label the point where your line and $\overleftrightarrow{AD}$ cross as X; label the point where your line and $\overleftrightarrow{EH}$ cross as Y.

Without measuring predict the simplified ratios for each of these: BX:FY = _____:_____, AX:EY = _____:_____ Check by measuring and simplifying the ratios.

LINES $\overleftrightarrow{AE}$, $\overleftrightarrow{BF}$, $\overleftrightarrow{CG}$ AND $\overleftrightarrow{DH}$ ARE PARALLEL.
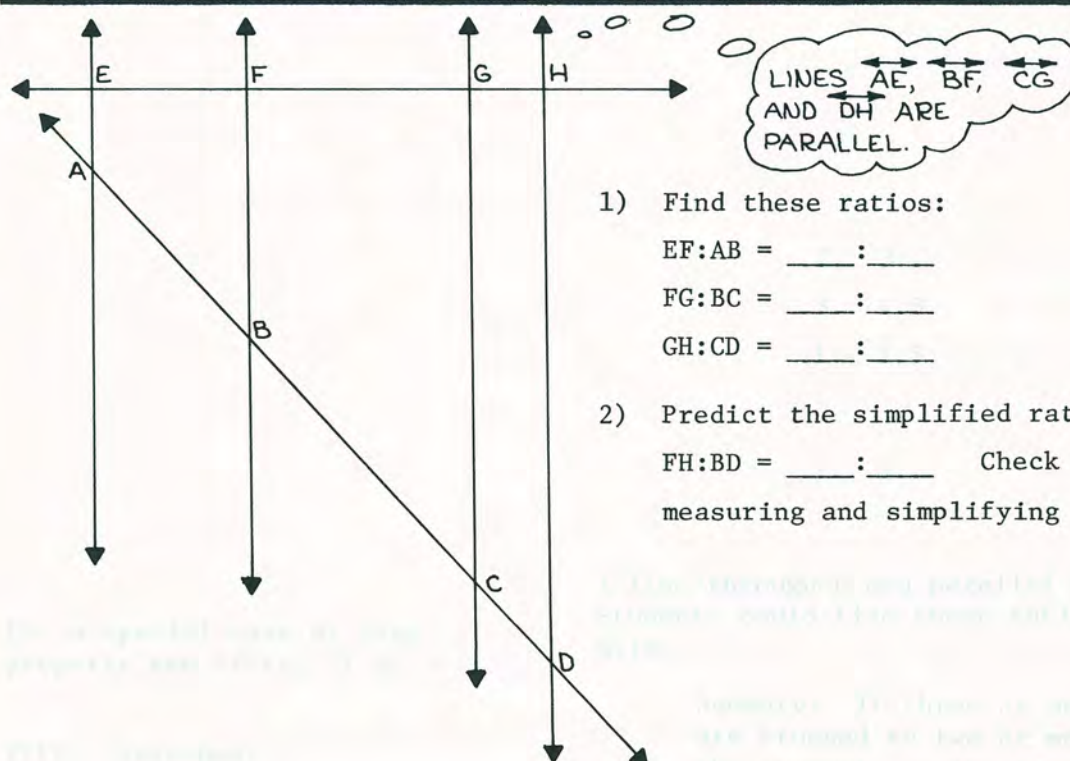
1) Find these ratios:

EF:AB = _____:_____

FG:BC = _____:_____

GH:CD = _____:_____

2) Predict the simplified ratios for

FH:BD = _____:_____ Check by measuring and simplifying the ratio.

SPLITTING ANGLES

I Construct the angle bisector of ∠A in each triangle. Each angle bisector meets $\overline{BC}$ at a point. Label the point X.

1)

2)

3)

HOW DO THE RATIOS
AB:AC AND BX:CX
COMPARE FOR EACH
TRIANGLE? _____
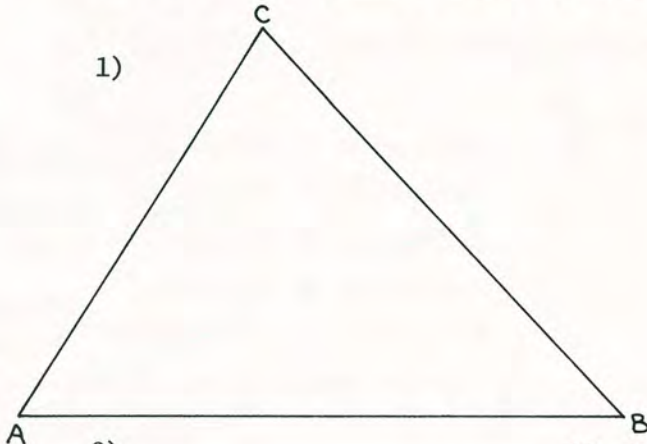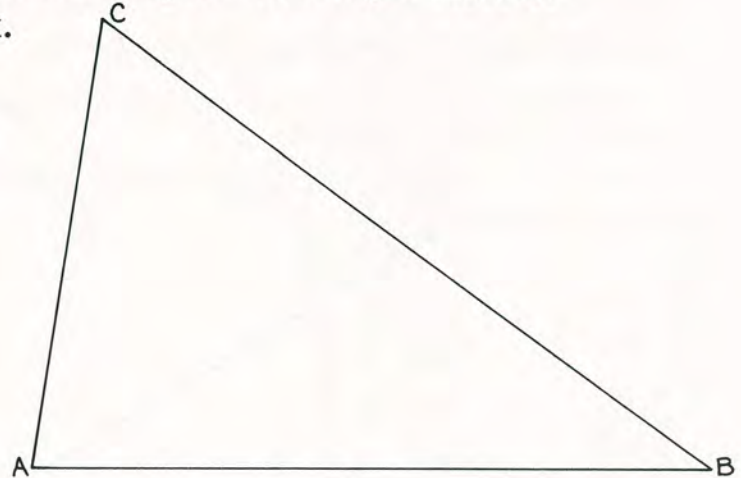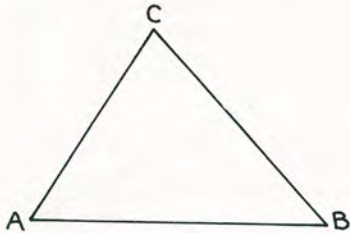
II Complete the table. Measure each segment to the nearest half centimetre.

TRIANGLE	AB	AC	BX	XC	AB:AC	BX:XC
1						
2						
3						

III Extension:

$m\angle D = m\angle F = 72°$
$m\angle E = 36°$
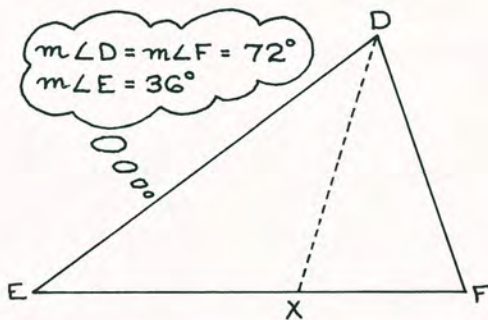
△DEF is a golden triangle so DF:DE ≈ .618:1.
If $\overline{DX}$ is the angle bisector of
∠D then FX:XE = ___ : ___

CHALLENGE: FIND
THE POINT T ON $\overline{PQ}$
SO THAT QT:TP = .618:1
T IS CALLED THE GOLDEN
SECTION OF $\overline{PQ}$

PAIRING UP

Materials: Centimetre ruler, string

There are three pairs of similar figures below. Measure the sides of figures I, II, III, IV to the nearest half-centimetre and write the length next to each segment. Then find the perimeter of each figure.

Use the string to help you measure the circumference of figures V and VI to the nearest half-centimetre.
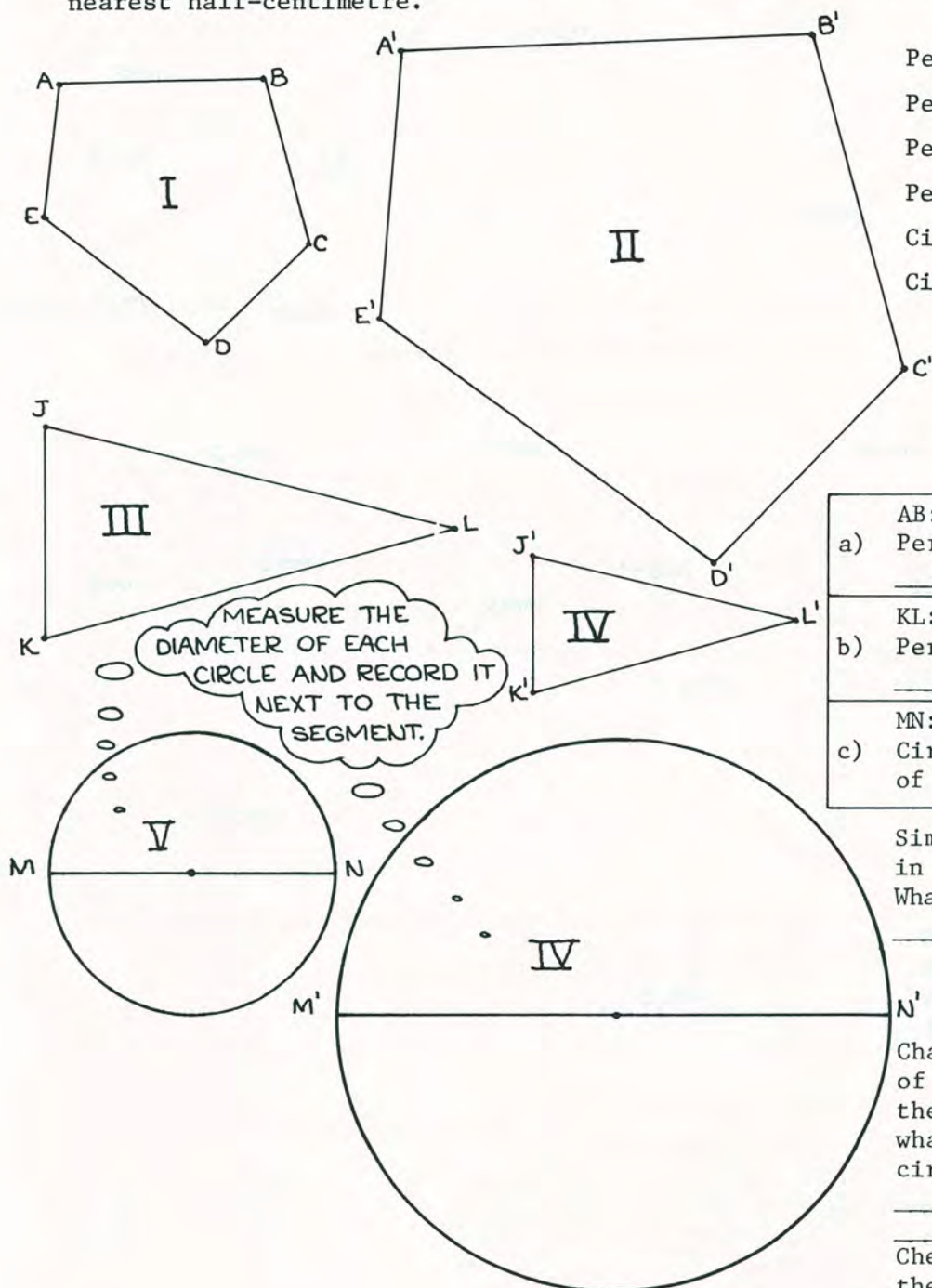
Perimeter of figure I   _____
Perimeter of figure II   _____
Perimeter of figure III   _____
Perimeter of figure IV   _____
Circumference of figure V _____
Circumference of figure VI _____

COMPLETE THE TABLE.

MEASURE THE DIAMETER OF EACH CIRCLE AND RECORD IT NEXT TO THE SEGMENT.

a)	AB:A'B' = _____ : _____ Perimeter of I:Perimeter of II = _____ : _____
b)	KL:K'L' = _____ : _____ Perimeter of III:Perimeter of IV = _____ : _____
c)	MN:M'N' = _____ : _____ Circumference of V:Circumference of VI = _____ : _____

Simplify each pair of ratios in the table.
What do you notice? _____

Challenge: If the diameter of circle A is three times the diameter of circle B what can you say about their circumferences? _____

Check you answer by drawing the circles.

A Very Special Ratio

DIAGRAM A
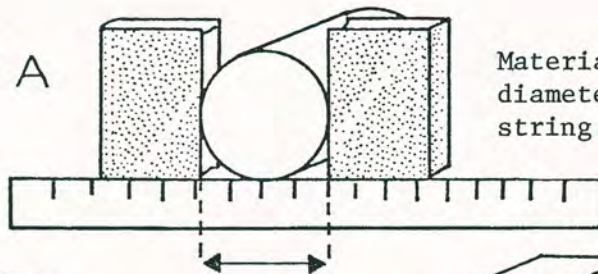
Materials: Metre stick, cans of varying diameters, rolls of tape, small wheels, string or paper to wrap around objects.
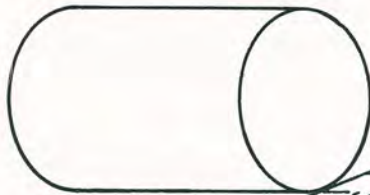
DIAGRAM B

Procedure:

1   Place the metre stick on a level surface.
2. Measure the diameter of a can by placing it on the metre stick. Record in the chart. (See diagram A.)
3. Wrap string or paper around the can one time and measure the length of the string or paper.
4. Record the measurement in the chart.
5. Carefully roll a can along the metre stick for one complete turn to check for accuracy in step 3. (See diagram B.)
6. Complete the chart. Use a calculator to find the values correct to one-decimal place.

DIAMETER OF CAN	LENGTH (CIRCUMFERENCE)	CIRCUMFERENCE ÷ DIAMETER

If you were careful in carrying out your experiments, the numbers in the last column are 3.1 or 3.2. This can be expressed as the ratio, circumference : diameter which is approximately 3:1.

To represent this ratio we use the Greek letter π (pi). π is pronounced "pie."

π cannot be exactly expressed as a decimal, no matter how many decimal places are used.

π is approximately

3.141592653589793238462643383279502884097169399375l0

SOME RECTANGLES ARE GOLDEN

Have you looked at different rectangles to see which shape most pleases your eye? Some look long and narrow, others look wide and stubby. There is one shape which appears to give the most satisfaction, the Golden Rectangle.

1) To make a Golden Rectangle:

   a) On your paper carefully draw an eight centimetre square.
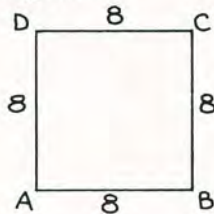
   b) Locate point E, the midpoint of segment AB. Set your compass opening equal to EC.
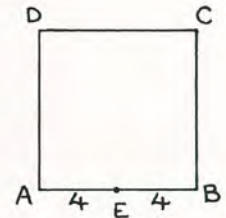
   c) Extend side AB and use your compass to mark a point F so that EF = EC.
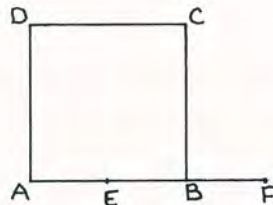
   d) Complete the rectangle. Label point M.
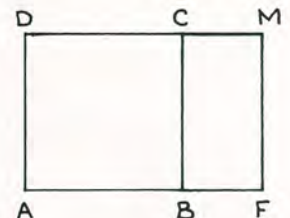
2) Carefully measure the length of rectangle AFMD. AF = _____

   Write the ratio $\frac{AD}{AF}$ and express it as a two place decimal. _____

   > In a Golden Rectangle the ratio of width to length is represented by the Greek letter Ø (phi). Ø is pronounced "fī"   Ø is approximately .62.

3) Consider your rectangle BFMC. Carefully measure the width and the length.

   BF = _____   FM = _____   Write the ratio $\frac{BF}{FM}$ and express it as a two place decimal. _____   What can you say about rectangle BFMC? _____

   _____

4) In rectangle BFMC make square BFPQ. (Hint: Set your compass opening equal to BF and mark points P and Q.) What would you guess about rectangle QPMC? _____

   _____ Check your guess by finding the ratio of PM to QP.
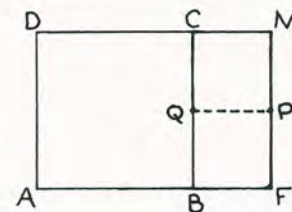
5) Do you see a way to make smaller and smaller Golden Rectangles? Experiment by drawing a Golden Rectangle inside rectangle QPMC.

6) Make a copy of rectangle AFMD. Use it to make a larger Golden Rectangle whose width measures AF.

7) Look for some examples of Golden Rectangles around you. Measure a postcard.

MEANS ARE MEANINGFUL

In $\triangle ABC$ the length of $\overline{BE}$ is called the mean proportional of the lengths of $\overline{AE}$ and $\overline{EC}$ because $\dfrac{AE}{BE} = \dfrac{BE}{EC}$.
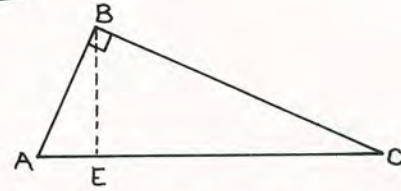
Students can discover this relationship by drawing and measuring the appropriate segments in several examples.
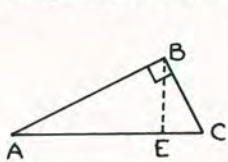
$\dfrac{AE}{BE} = \dfrac{4}{2}$ $\dfrac{BE}{EC} = \dfrac{2}{1}$
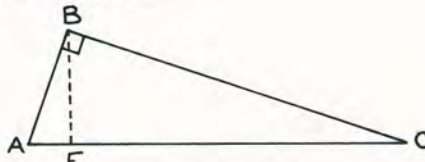
$\dfrac{AE}{BE} = \dfrac{1}{3}$ $\dfrac{BE}{EC} = \dfrac{3}{9}$
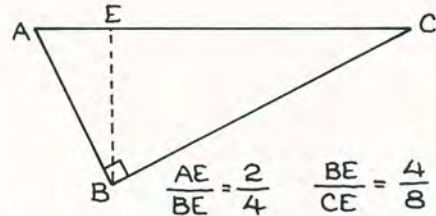
$\dfrac{AE}{BE} = \dfrac{2}{4}$ $\dfrac{BE}{CE} = \dfrac{4}{8}$

Given any two segments it is possible to construct a segment whose length is the mean proportional to the lengths of the two given segments.

As an example use these line segments.

1) Draw a line. Mark off $\overline{AE}$ and $\overline{DC}$ so E and D coincide. Bisect $\overline{AC}$ to find the midpoint, P.
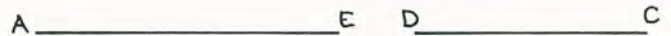
2) Use PA as a radius and draw a semi-circle through A and C.

3) Construct $\overline{BE}$ so $\overline{BE}$ is perpendicular to $\overline{AC}$. $\overline{BE}$ is the required segment.
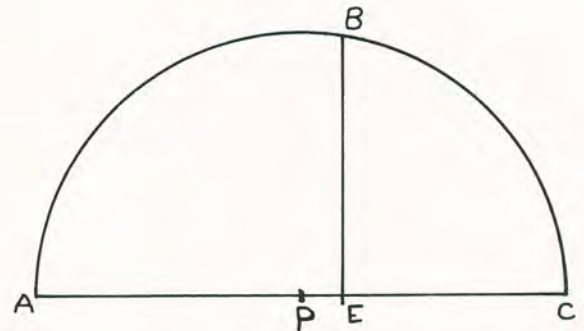
4) BE can be checked as the mean proportional by measuring or by the use of similar triangles.

   a) Draw $\overline{AB}$ and $\overline{BC}$. $\angle ABC$ is a right angle because it is inscribed in a semicircle with $\overline{AC}$ as a diameter.

   b) $\triangle ABC \sim \triangle AEB$ because both contain $\angle A$ and both contain a right angle, $\angle ABC$ and $\angle AEB$.

   c) $\triangle ABC \sim \triangle BEC$ because both contain $\angle C$ and both contain a right angle, $\angle ABC$ and $\angle BEC$.

   d) Therefore $\triangle AEB \sim \triangle BEC$. So the corresponding sides AE and BE and BE and EC have equivalent ratios, that is $\dfrac{AE}{BE} = \dfrac{BE}{EC}$.

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GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

AREA AND VOLUME

DO STUDENTS UNDERSTAND PERIMETER, AREA AND VOLUME?

No doubt you can think of students who do understand these concepts and others who do not understand them at all. More information on student understanding of perimeter, area and volume can be gleaned from recent national and state assessment tests discussed below.

Some Results of Assessment Tests

Recent test results related to perimeter, area and volume of the National Assessment of Educational Progress (NAEP) and of various state assessments have been summarized and interpreted by the NCTM Project for Interpretive Reports on National Assessment in the articles "Notes from National Assessment: Basic Concepts of Area and Volume" in the October 1975 issue and "Notes from National Assessment: Perimeter and Area" in the November 1975 issue of The Arithmetic Teacher. Cited below are some test questions and results.

The 1973 Michigan State Assessment results indicated that even by seventh-grade many students might not fully connect area to the concept of covering a region with congruent units. In a test item similar to that to the right less than half of the Michigan seventh-graders answered correctly with 19 percent giving 6 as the answer. It would seem these latter students counted the 6 pieces of the figure without regard to the shape or size of the pieces.

Estimate the area of the given figure in square units.

643
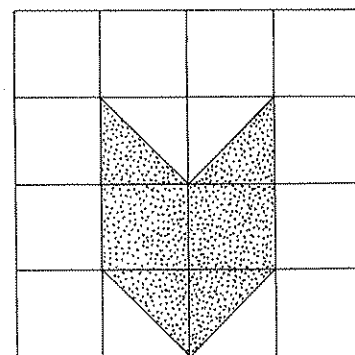
The table to the right gives the results of a question from the recent National Assessment. Only 7 percent of the 13-year-olds tested could calculate the area of a square when its perimeter was given. The table exposes the use of rote methods for obtaining area. The incorrect response of 144 sq. in. probably results from incorrectly equating area with length x width. This error became _more_ frequent with an _increase_ in age. The answer of 48 sq. in. might come from associating the perimeter of a square with 4 times the length of an edge. In a related item in the Michigan Assessment only one-fourth of the tested seventh graders could calculate the length of a side of a square when given its perimeter.

Another question and its results from NAEP are given to the right. About one-third of the 17-year-olds and young adults applied the correct procedure for calculating the area of a rectangle with an interior rectangle removed. The table gives no statistics for middle school age students. You might want to try this question with your own students who have studied area. How do the percentages compare?

A square has a perimeter of 12 inches. What is its AREA in square inches?

	13-year-olds	17-year-olds	Young adults
Correct response	7%	28%	27%
144 sq. in., 12 × 12, etc.	12%	19%	25%
48 sq. in., 4 × 12, etc.	20%	10%	10%
12	4%	3%	3%
24	6%	4%	2%
6	4%	2%	1%
3	3%	2%	2%
Other unacceptable response	16%	13%	10%
No response or "I don't know"	28%	19%	20%

Table from The Arithmetic Teacher, November 1975, p. 587.
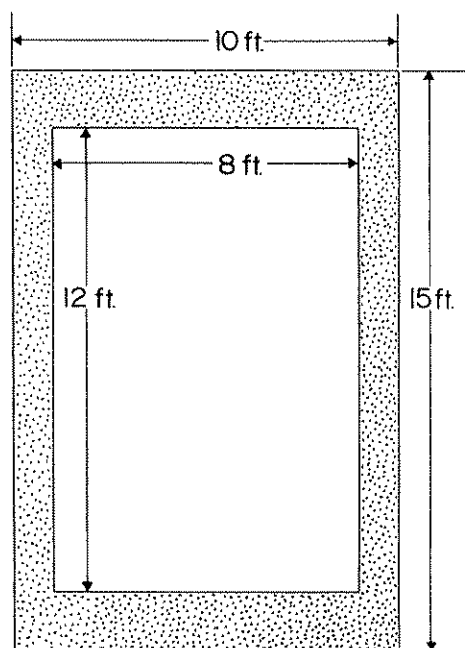
What is the area of the shaded portion of the figure?

	17-year-olds	Young adults
Correct response	18%	27%
Correct procedure-computing error	12%	7%
150 sq. ft., 10 × 15, etc.	10%	9%
96 sq. ft., 8 × 12, etc.	4%	2%
6 sq. ft., 2 × 3, etc.	5%	8%
Other unacceptable response	35%	31%
No response or "I don't know"	16%	16%

Table from The Arithmetic Teacher, November 1975, p. 588.

A volume-related question of the NAEP was similar to the question shown to the right. Notice that a high percent of 9-year-olds counted faces. These students would likely give an answer of 20 instead of the correct 16 for the example shown. Of the 17-year-olds tested, <u>less than half</u> answered correctly. How do your students perform on such a question?

How many unit cubes are contained in this solid?

Percent of respondents giving various responses on NAEP volume exercise

Response categories	Age groups		
	9	13	17
Correct response	6	21	43
The number of unit squares on the 3 pictured faces	48	27	13
Two times the number of unit squares on the three pictured faces	2	13	13
Other unacceptable response	39	36	27
No response or "I don't know"	5	3	3

Table from The Arithmetic Teacher, October 1975, p. 504.

Results from Florida, Michigan and Wisconsin assessments indicated that most students by seventh grade could find the perimeter of a simple geometric figure if the problem were straightforward. If extra information were included--the length of a diagonal, area, etc.--the problem was more difficult. "The results of exercises from the state assessments and from NAEP indicate that between one-third and one-half of all seventh graders cannot find the area of a rectangle given the length of two adjacent sides. Figures requiring more complex calculations are significantly more difficult." (Carpenter, et al., 1975)

From the examples given above it appears that many students do not understand perimeter, area and volume. Not only do many students apply formulas in a haphazard manner, but some also do not understand the basic concepts of measurement. There may be several reasons for this. It might be that students are not given enough experience with geometric ideas in the early grades or junior high to form a background for area and volume. It might be that formulas are introduced before students understand the concepts involved. Perhaps the concepts themselves are more difficult than is often assumed. A later part of this commentary gives two suggestions for helping students understand area and volume; however, there are no certain answers which will fit every class, every teacher or every student.

Determining Your Own Students' Understanding of Perimeter, Area and Volume

Your students probably are at several different levels of understanding perimeter, area and volume. Some might not understand when length, area and volume are conserved, that congruent units must be used, how to interpret drawings of three-

dimensional objects, or when to apply formulas. It is worth some time to find out
how well your students understand the above items. Here are some ideas which might
be used in an individual or class diagnostic quiz.

- Show a loop of string lying flat on a
  surface. Point out the oval shape en-
  closed by the string. Change the loop
  so it encloses an irregular shape.
  Ask students which shape has the great-
  er distance around (or perimeter). Do
  they realize the length of the string
  is preserved under this change?

- Show students a set of squares or
  a polygonal region. Rearrange the
  squares or parts of the region as
  they watch. Ask which arrangement
  covers more surface (or has the
  greater area). Do they understand
  that the area of a region is con-
  served when its parts are rearranged?

- The question to the right is from the
  NAEP. Of the 9-year-olds tested, 38%
  chose the 5-by-3 rectangle. 6% chose
  the 3-by-6 rectangle and 44% gave the
  correct answer. Perhaps the 5-by-3
  rectangle was popular because its
  shape was most like the 4-by-4 rect-
  angle. Some students might have cho-
  sen it because $5 + 3 = 4 + 4$. How
  well would your students do on this
  problem or a similar more difficult
  one? (Try comparing a 6-by-6 square
  with 5-by-7, 3-by-12 and 4-by-8
  rectangles.)
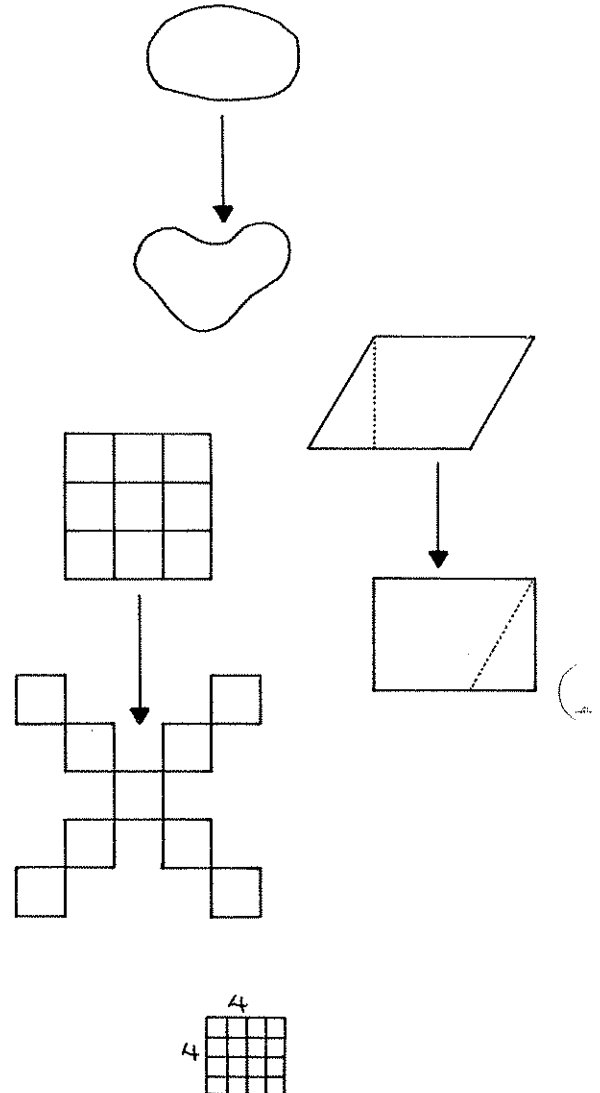
WHICH OF THE FIGURES BELOW
HAS THE SAME AREA AS THE FIGURE
ABOVE ?

646
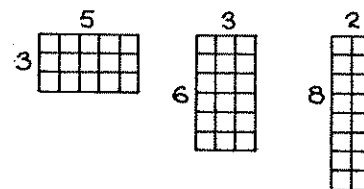
- Show students a tower built of cubes
  with a 3-by-2 base. Ask students to
  build a tower of cubes on a 2-by-2
  base so the new tower fills up the
  same amount of space as the original.
  Do students realize that both towers
  will have the same number of cubes?
  (Page 3 of *Lake and Island Board* in
  the Volume subsection gives a varia-
  tion of this question.)
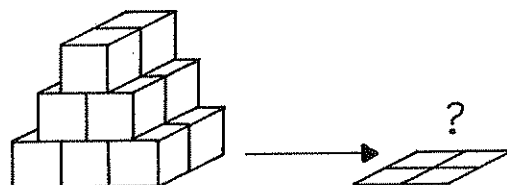
- Ask how many unit cubes make up
  these solids. Can your students
  interpret these drawings of three-
  dimensional solids? Do they give
  the correct answers of 24 and 12 or
  do they count faces and give answers
  of 26 and 20?
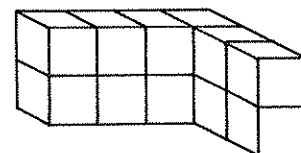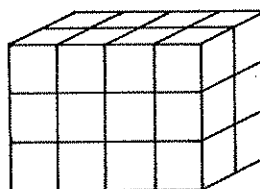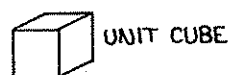
You might also try the items from the assessment tests given in the first part
of this commentary. How does the performance of your students compare to that
given in the tables?

HOW CAN WE HELP STUDENTS UNDERSTAND PERIMETER, AREA AND VOLUME?

Two suggestions for helping students understand perimeter, area and volume are
given below. You will find more ideas in the classroom materials and commentaries
of this section.

Provide Readiness Activities and Hands-On Experiences

Students might have a better under-
standing of perimeter, area and volume
if they are introduced with concrete
materials rather than through printed
pages. Concrete materials can also give
meaning to the formulas which some stu-
dents apply indiscriminately. Some
basic partitioning, covering and filling
activities are discussed below.
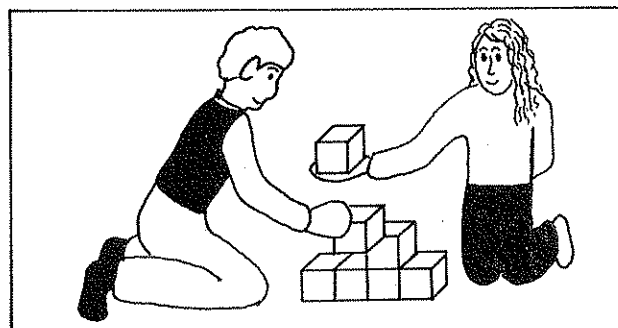
Students can place straws to find the distance
around a well-planned shape taped or drawn on the floor.
They can find the perimeter of a desk top by using a
pencil, Cuisenaire rod or handspan as a unit. String,
paper strip, trundle wheels and tape measures can be
used to find perimeter. These ideas and others are
developed in more detail in the classroom materials and
the commentary of the Perimeter subsection.

PERIMETER:
7 STRAWS
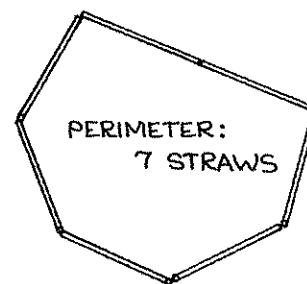
The amount of surface of a desk top can be approximated by covering it with books
of the same size, square tiles or triangular regions. When asked to determine the
number of mathematics books necessary to cover the classroom floor, students usually
try a variety of successful approximation methods other than the traditional length
times width formula.

Students can determine the number
of triangular regions needed to cover a
special parallelogram by drawing around
one triangular region to make a grid as
shown. Square grids can be made by
having students trace around a square
region. The grids can be used to deter-
mine areas of figures drawn on them.
(See the Area subsection.)

HOW MANY
ARE
NEEDED
TO COVER
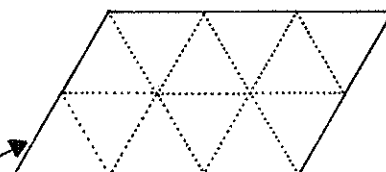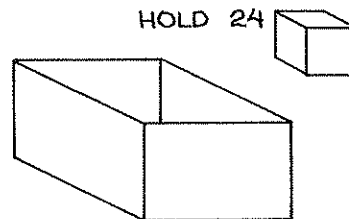
The capacity of a box can be determined by filling
it with unit cubes. The capacity of a bottle can be
found by counting the number of times a tablespoon, cup
or beaker of water can be emptied into the bottle. The
amount of space occupied by a rectangular prism can be
approximated by having students use cubes to build a
shape of the same size and shape. After students have
had many experiences of using cubes to build solids with
a given volume, they should learn to interpret drawings
of three-dimensional shapes. The performance of students
on the volume question cited on page 3 from the NAEP was
low. Perhaps students who counted faces had not learned

THIS BOX
WILL
HOLD 24
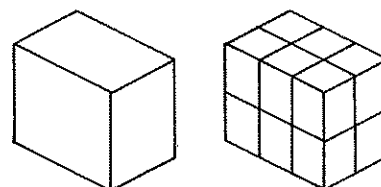
THIS OBJECT OCCUPIES
12   OF SPACE.

to interpret these drawings. Having
students build solids to correspond to
pictures can help them see there are
hidden cubes not represented by the
picture. Students should be allowed to
handle prisms, pyramids, cones . . .
before being asked to determine their
volumes.
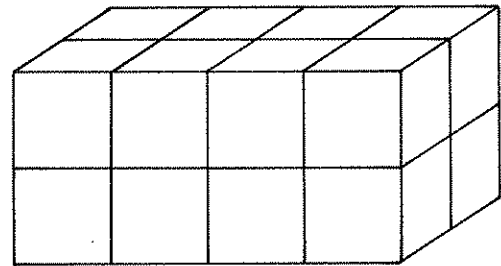
HOW MANY UNIT CUBES ARE
CONTAINED IN THIS SOLID?

Stress Problem-Solving Methods

"In order to overcome students' inflexibility in dealing with measurement situ-
ations, measurement problems should be presented in a way that emphasizes problem-
solving skills rather then as a collection of formulas with well-established patterns
of solution." (Carpenter, et al., 1975) Polya (1971) outlines four phases of problem
solving: (1) understanding the problem, (2) devising a plan, (3) carrying out the
plan, and (4) looking back. (*Problem Solving* in the TEACHING EMPHASES section gives
more background in problem solving.) Here are some suggestions for focusing the
attention of students on steps 1, 2 and 4—the most neglected phases. Most of these
suggestions were derived from the National Assessment article in The Arithmetic
Teacher, November 1975. (Carpenter, et al., 1975)

(1) Understanding the Problem.

    A. Encourage students to draw a picture or diagram and restate the problem
in terms of the diagram.

        a. A fence is to be put around a rectangular playground that is 21
metres long. How many metres of fencing are needed? Drawing a
rectangle and labeling the sides with their lengths could prevent
automatic addition or multiplication of the two given numbers.

        b. A cube has a volume of 27 cm^3. How long is each edge of the cube?
In this case students might prefer to make a three-dimensional
model. Making a cube from 27 centimetre cubes should help stu-
dents solve the problem.

    B. Give problems which have too much information or not enough information.
Ask students to identify information which is not needed or which must
be added to solve the problem.

        a. A rectangle has a width of 6 cm, a length of 8 cm and a diagonal
that is 10 cm long. What is the area of the rectangle? (Students
can identify 10 cm as information not needed to solve the problem.)

b. Find the number of 1-square foot tiles that are needed to cover the floor of a 15-foot wide rectangular room. (The length of the room is needed.)

c. Find the number of cubical cartons that will fit in a truck whose dimensions are 3 m by 3 m by 10 m. (The size of the cubical cartons is needed.)
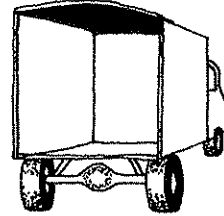
(2) Devising a Plan.

A. Give students some problems in which they describe how they would solve a problem, but <u>do not require</u> them to calculate the answers.

a. Find the area of the shaded region. ("I would subtract the area of the small rectangle from that of the bigger rectangle.")
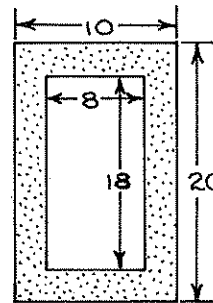
b. Find the perimeter of a square if its area is 25 cm^2. "First find the length of the sides by figuring what number times itself gives 25. Then add the four sides to get the perimeter."

B. Vary the problems. Instead of giving only area problems on a page, try combining perimeter and area problems. This should prevent students from adopting an "always multiply" pattern. (See the classroom materials and commentaries of the Area and Volume subsections for specific suggestions on combining perimeter and area, and surface area and volume.)

(4) Looking Back.

A. Encourage students to examine their answers to see if they are reasonable. Is 9 cm^2 a reasonable area for this triangle? No, because the whole square would have area 9 cm^2.
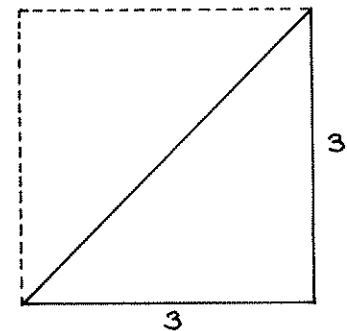
B. Encourage students to look back for another
way to solve the problem. "I added the
areas of the parts, but I could have sub-
tracted the area of the part missing from
the rectangle."
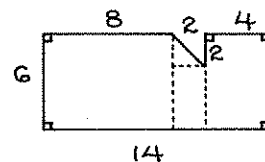
$(6 \times 14) - \frac{1}{2}(2 \times 2)$ instead of $(6 \times 8) + (2 \times 4) + \frac{1}{2}(2 \times 2) + (4 \times 6)$

Emphasizing problem-solving methods can contribute to a more meaningful study
of perimeter, area and volume. Students will also be exposed to methods of solving
problems which can be applied to many other problems.

Further Readings and References

Carpenter, Thomas; Coburn, Terrance; Reys, Robert; and Wilson, James. "Notes from
National Assessment: basic concepts of area and volume." The Arithmetic Tea-
cher, Vol. 22 (October, 1975), pp. 501-505.

_____. "Notes from National Assessment: perimeter and area." The Arithmetic
Teacher, Vol. 22 (November, 1975), pp. 586-590.

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

CONTENTS

PERIMETER

When the word "perimeter" appears in a problem, students might not remember its meaning. On the other hand, students who have learned what "Find the perimeter" means might not be able to solve problems which involve the concept of perimeter but not the word itself.

Often, we first teach students how to find the perimeter of geometric shapes which are drawn on paper. We then expect this learning to transfer to situations where they need to know "the distance around." For some students the learning transfers—for others it does not. Here is a suggestion which reverses that teaching order.

Set up a series of activity cards together with actual objects to be measured. Suggestions are given below. Organize the students into small groups or pairs and have them rotate through the activities. As the activities are being worked try to talk with the pairs or groups to see that they understand the concepts involved. Students could check their answers with a prepared answer sheet before moving to the next activity.

I. Materials: Small wooden box, a piece of copper stripping, ruler or tape measure.
   Question: How many centimetres of copper stripping would be needed to edge the top of this box? What would be the cost of the required stripping if the price is 20 cents for 10 cm?

II. Materials: One dress or robe, a piece of lace, tape measure.
   Question: How many metres of lace would be needed to edge the neck, hem and sleeves of this dress if the lace is sewn on flat? How many metres of lace would be needed if the lace is gathered so 4 cm of lace fits into 2 cm? At 50 cents a metre what is the cost for enough lace in each case?

III. <u>Materials</u>: A piece of edging, tape measure or trundle wheel, access to an irregular flower bed.

 <u>Question</u>: If garden edging costs 50 cents a metre, how much would it cost to buy enough edging to go all the way around the flower bed?

IV. <u>Materials</u>: Tape measure or trundle wheel, access to gym.

 <u>Question</u>: A black strip is to be painted around the gym floor so it forms a rectangle whose sides are 2 metres from the edge of the floor. If it takes 5 minutes to paint 1 metre of stripping, about how long will it take to paint the entire strip?
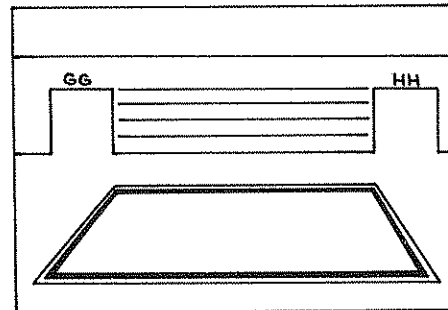
V. <u>Materials</u>: Centimetre tape measure and string or grid paper for making a scale drawing.

 <u>Question</u>: A farmer knows he put 100 metres of fencing around a rectangular garden that is 20 metres wide, but he has forgotten the garden's length. How long is it?

The problems above all involve the concept of perimeter. The <u>word</u> "perimeter" can be introduced when the problems are discussed after they have been solved. For example: "What did these problems have in common? They all involved the distance around some shape. The distance around a polygon or other flat shape is called its perimeter. The word pe<u>rim</u>eter has 'RIM' in it. This should help you remember that perimeter means distance around."

Students should learn that there are a variety of ways to determine the perimeter of planar shapes. If shapes with no oblique sides are drawn on grid or dot paper, the units can be counted to find the perimeter. (See *Perimeter on the Geoboard* and *Perimeter*.) The use of string, strips of paper and trundle wheels to find perimeters of curved or irregular shapes is discussed in *More Than One Way*. A limiting process can also be used to approximate a perimeter. Two such processes are described in *Around the Blob* and *Inside and Outside a Circle*.

Sometimes a tape measure can be placed all the way around a shape to determine its perimeter, but often a perimeter is calculated by adding the lengths of the edges of a shape. The idea of a perimeter being the sum of the lengths of the edges of a shape is one worth emphasizing. The problems on the next page involve this idea. Solving these problems requires the student to understand the geometric vocabulary. You might compare these with the activities described above—shouldn't students be able to solve <u>both</u> types of problems?

655

a) A square has perimeter 20 cm.
    How long is each side of the
    square?

b) A rectangle has a width of 15 cm
    and a perimeter of 70 cm. What is
    its length?

c) What is wrong with the numbers
    given below?

PERIMETER IS 44 UNITS
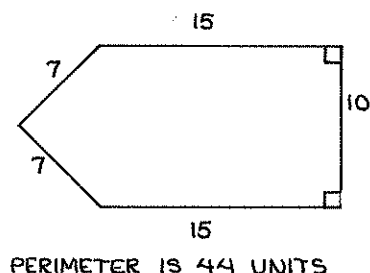
d) What is wrong with the numbers in
    this diagram? Can you put in cor-
    rect numbers and find the perime-
    ter? (Change the 15 to 17
    and the 16 to 15, or make
    any changes so the lengths
    of opposite sides agree.
    Perimeters might vary with
    the corrections made.)
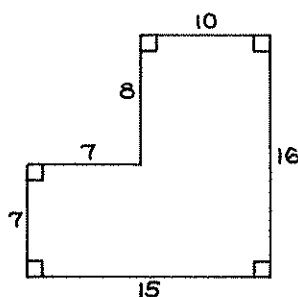
Problems similar to those above are given in *Perplexing Perimeters* and *Perimeter Problems to Pursue*.

Some students might give 13 units instead of the
correct 10 units for the perimeter of the figure to the
right. A similar mistake might occur when students are
asked to arrange 3 or more square tiles and then deter-
mine the perimeter of the arrangement. The perimeter of

one square is 4 units. Why isn't the perimeter of 3 squares in any arrangement
3 x 4 = 12? Why does the perimeter vary with the arrangement of the squares? This
might seem paradoxical to students since area is conserved when planar regions are
rearranged. One aid would be to dot the inside segment in drawings of figures to
remind students that inside segments are not counted in perimeter; this is done in
Perimeter Patterns of Polygons and *Arranging Squares*. Again, emphasize the RIM in
perimeter.
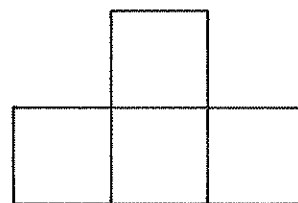
Arranging shapes to vary the perimeter suggests several interesting problems.
Some of these are given in *Arranging Squares*. Two more are given on the next page.
Activities relating perimeter and area are given in the commentary and classroom
pages of the Area subsection.

EXAMPLES:

● How can six unit squares be arranged on a 25-nail geoboard to obtain the maximum perimeter? Minimum perimeter? Can a geoboard arrangement of six unit squares have a perimeter with an odd number of units? (No--can you see why?) Can six squares be drawn on paper so the perimeter of the total figure is an odd number of units? (Yes--see illustration to the right.)
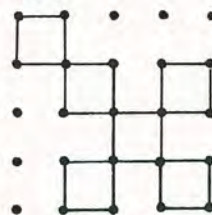
MAXIMUM PERIMETER
IS 24 UNITS
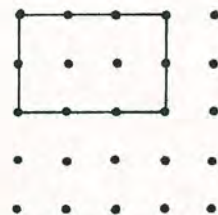
MINIMUM PERIMETER
IS TEN UNITS
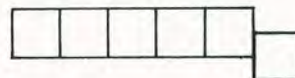
To have "odd" perimeters at least one square must share only part of an edge with another square.

● Which square tiles should be removed from a 3 x 3 square grid to maximize the perimeter of the remaining figure? What would be the answer if the original figure were a 4 by 4 grid?

P = 12
UNITS

P = 20
UNITS
(MAXIMUM)
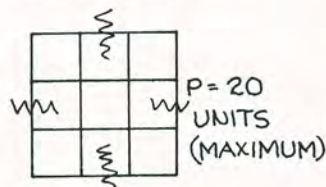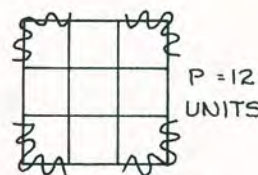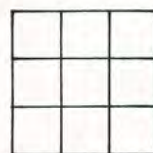
After students are thoroughly familiar with the circumferences of circles, they might enjoy exploring this paradox. Suppose we have two circular regions which are joined together at their centers so the circular regions _must_ turn together. The large circle is rolled along a straight line until it has made one revolution, so the distance from A to A' is the circumference of the larger circle. During this time, the smaller circle has also made one revolution, "so" the distance from B to B' must be the circumference of the smaller circle. But $\overline{AA'}$ has the same length as $\overline{BB'}$, so the two circles must have the same circumference!
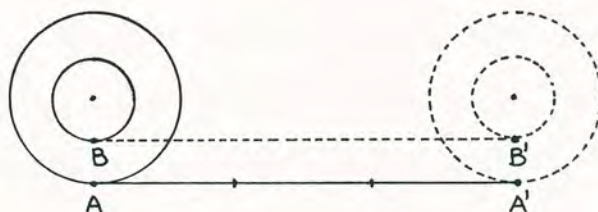
Students can explore this paradox with pulleys or gears. If string is wrapped around the pulleys, students will find the string around the larger pulley will unroll loosely if the string around the small pulley is kept taut. There is no way to keep both strings taut as is suggested by line segments AA' and BB' shown above.

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PERSONALIZED MEASURES

CUBIT

THUMB

PALM

SPAN
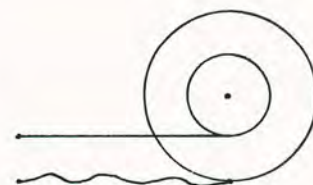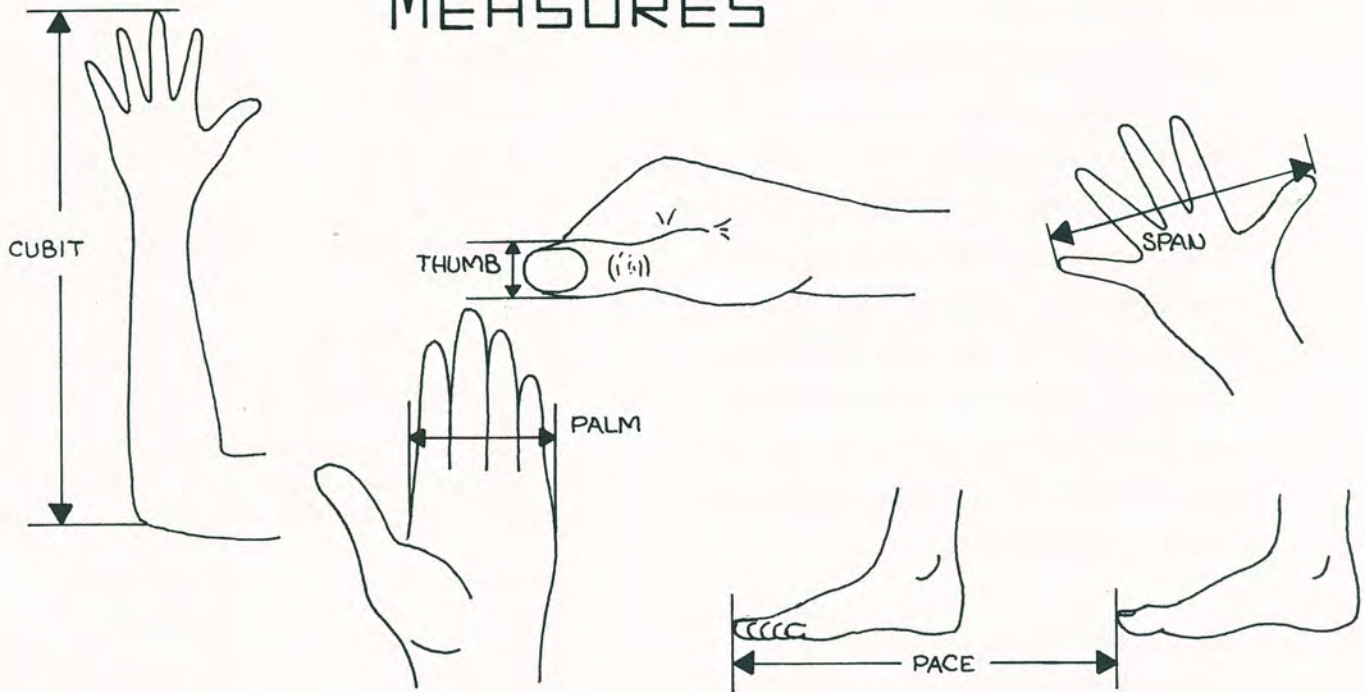
PACE

Body parts can be used to measure length. Some examples are shown above.

I First estimate each distance below and then measure to check your guess.

		Estimate	Measure
1)	The width of your classroom in paces	_____	_____
2)	The length of your classroom in paces	_____	_____
3)	The length of your teacher's desk in cubits	_____	_____
4)	The height of a door in spans	_____	_____
5)	The length of this sheet in palms	_____	_____
6)	The width of this sheet in thumbs	_____	_____

II Which one of your own units would you use to estimate the length or height of:

1) a pencil _____   4) a garage _____

2) a football _____   5) a fence around your yard _____

3) a tree _____   6) a baseball bat _____

MORE THAN ONE WAY
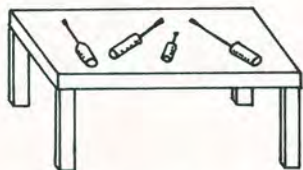
Any carpenter can tell you that to set a screw you must have a screwdriver of the correct size. Just like carpenters, your students may be able to do better if they have a wide range of tools (methods) to choose from. Here is a collection of methods which can be used to measure perimeters.

I To introduce perimeter you may use a story type approach.
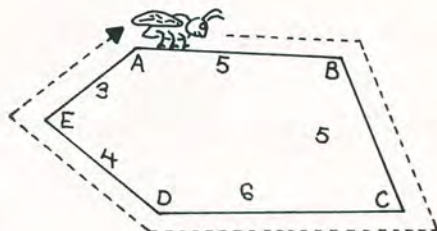
A bug crawls along the sides of a polygon until he returns to his starting position.

1) How far does the bug walk from point A to point B?
2) How far does the bug walk from B to C?
3) Starting at A how far does he walk to get all the way back around to A?
4) What is the perimeter of the polygon?

II Polygons which are convex can be rolled along a ruler. It helps to mark each edge as you rotate it, so if it slips you do not have to start over.
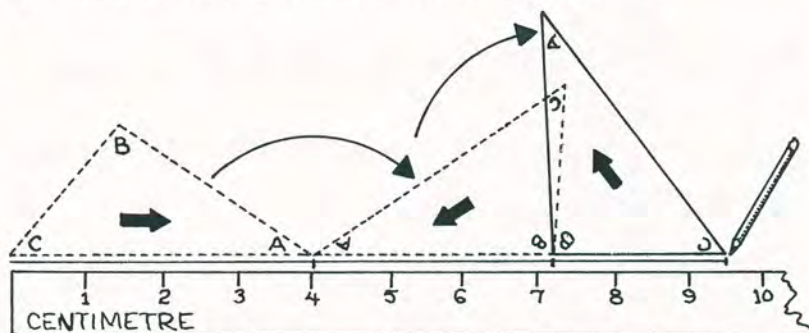
III Using a paper's edge move around the outside of a polygon marking each vertex as you go.
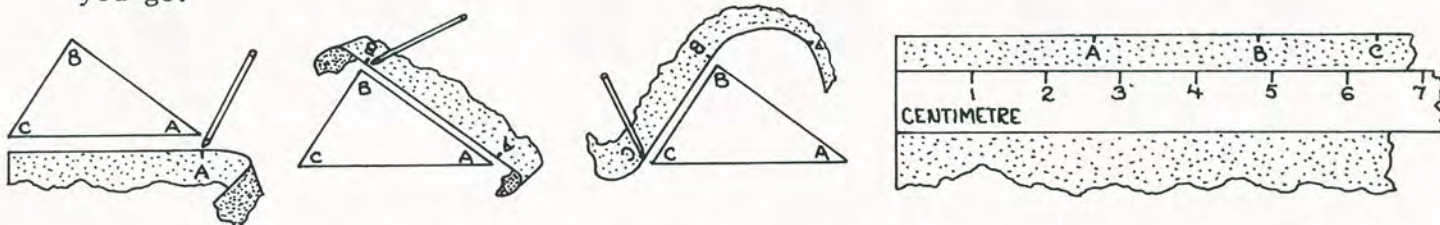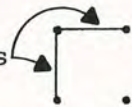
IV Carefully lay a string around the edge of shapes having curved edges, irregular edges or indentations. Measure the string's length to find the perimeter.
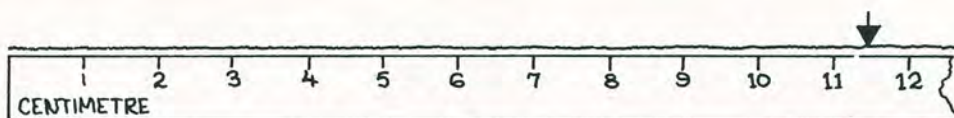
A complaint we commonly hear is that students cannot apply measuring skills to tasks outside the mathematics classroom. Keeping this in mind it may be wise to include measuring such things as rooms, fields, balls, etc., besides using just worksheets of mimeographed patterns.

PERIMETERS ON THE GEOBOARD

Materials Needed: Geoboard, dot paper

Activity: On a geoboard either of these distances ⌐ is called a unit length.

1) a) Make the smallest square you can with nails at its corners. What is its perimeter? _____

   b) Make the largest square you can with nails at its corners. What is its perimeter? _____

   c) Make a square that has a perimeter of 12 units. Record the square on dot paper.

2) a) Make the smallest rectangle that is not a square. What is its perimeter?
   _____

   b) Make the largest rectangle that is not a square. What is its perimeter?
   _____

   c) Make a rectangle that has a perimeter of 8 units. Record the rectangle on dot paper.

3) Make each of these figures on your geoboard. What is the perimeter of each?
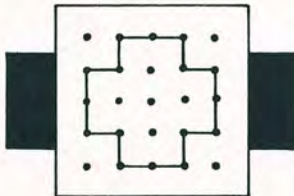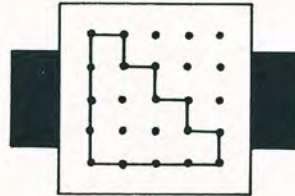
   a)   b)   c)

   _____   _____   _____
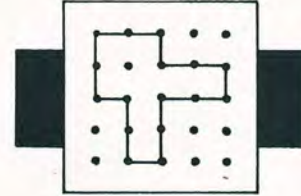
4) a) Make this figure. What is its perimeter?
      _____
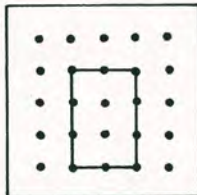
   b) Change it to this figure. What happens to the perimeter?
      _____
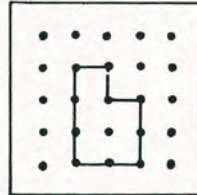
   c) Change it to this figure. What happens to the perimeter?
      _____
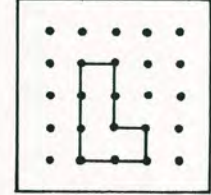
5) On your geoboard make as many figures as you can that have a perimeter of 14 units. Record the figures on dot paper. The figures in (4) may give you an idea for getting some of the answers.

IDEA FROM: *Geoboard Activity Card Kit*
Permission to use granted by the Cuisenaire Company of America, Inc.

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MORE PERIMETERS
ON THE GEOBOARD

On the geoboard to the right the approximate lengths of some segments are shown.

Use these lengths to find the approximate perimeter of the polygons shown below.

1.0   3.6   1.4
4.1   3.2
2.3

PERIMETER

A.

B.

C.

D.

E.
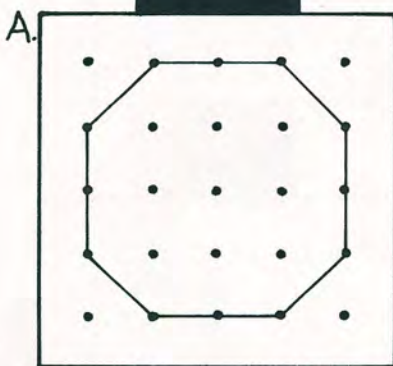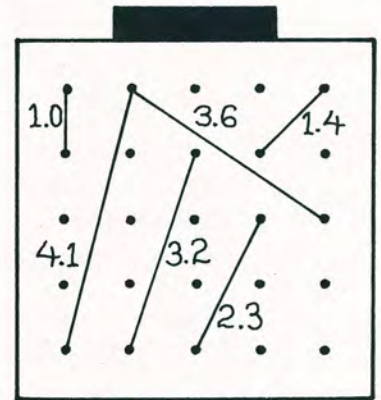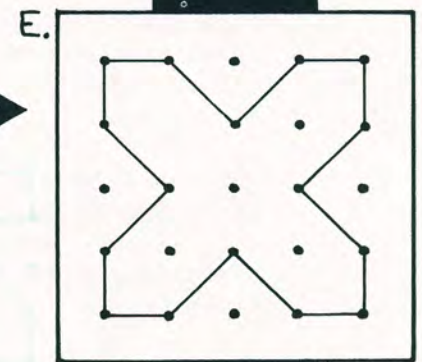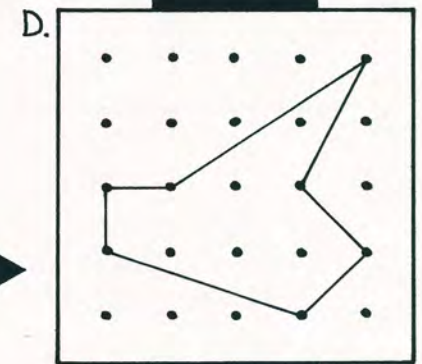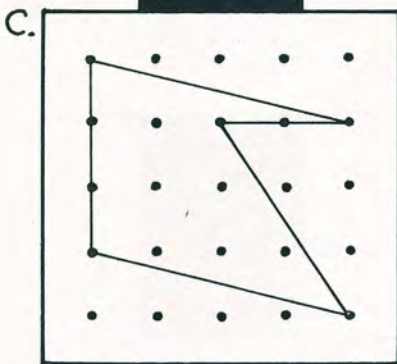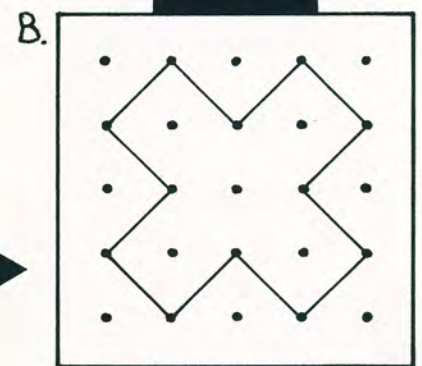
Polygon E has 16 sides. Try to make a polygon with more than 16 sides that has a perimeter
(a) greater than the perimeter of polygon E, (b) less than the perimeter of polygon E.

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Stringing Along
ON THE GEOBOARD

1) a) Have students tie a piece of string to form a closed loop.

   b) Measure the loop to find its length.

   c) Place it around as many nails as possible. Record the shape on dot paper.

   d) Find more shapes that can be formed using the same loop of string. Record on dot paper.

   e) What can be said about the perimeters of these shapes?
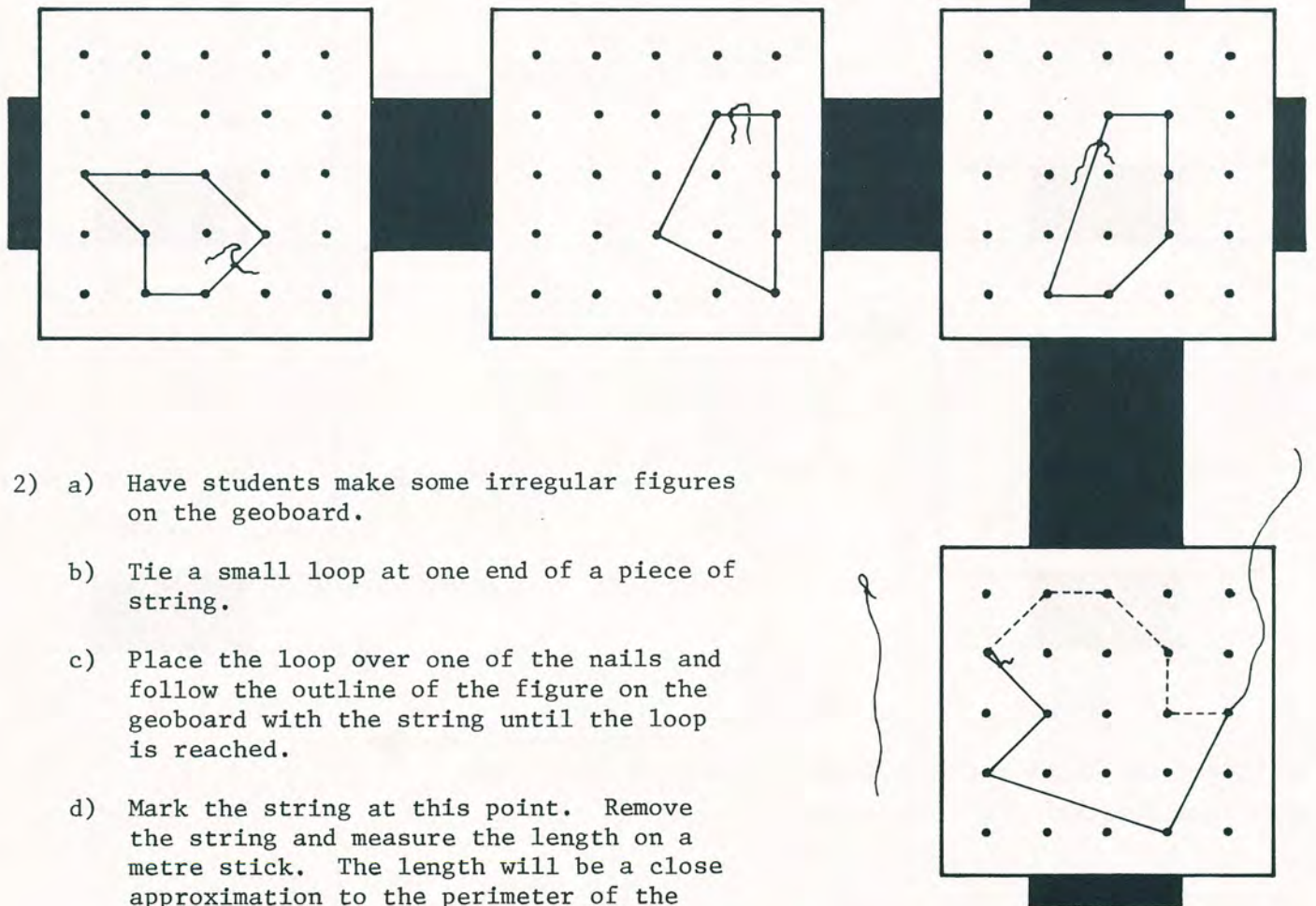
2) a) Have students make some irregular figures on the geoboard.

   b) Tie a small loop at one end of a piece of string.

   c) Place the loop over one of the nails and follow the outline of the figure on the geoboard with the string until the loop is reached.

   d) Mark the string at this point. Remove the string and measure the length on a metre stick. The length will be a close approximation to the perimeter of the figure.

PE RIM ETER

Find the perimeter of each figure below and look for the answer in the code at the bottom of the page. For each answer in the code, write the letter of the figure above it.

ONE UNIT
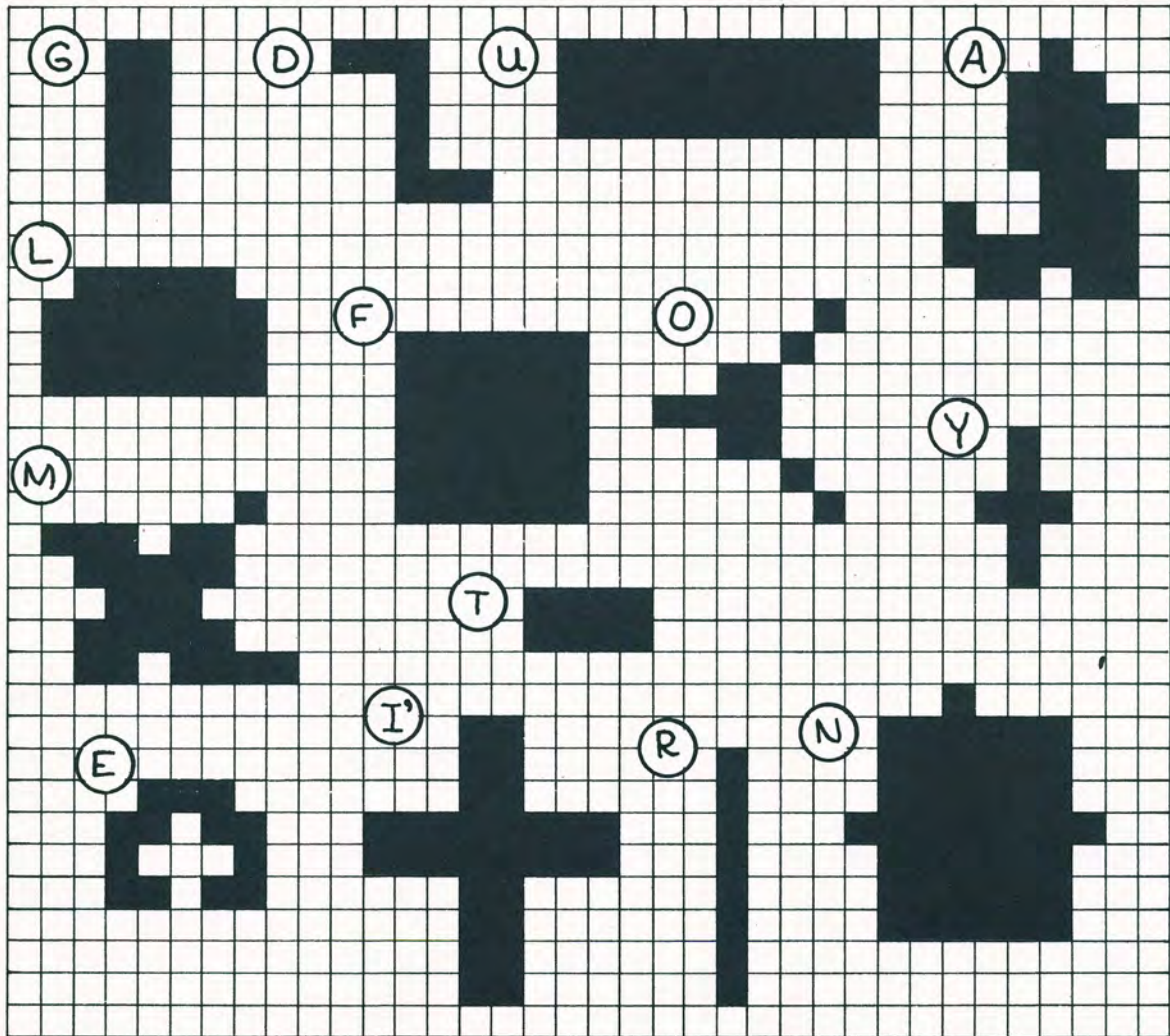
WHAT DID THE 179° ANGLE SAY TO ITS TWO RAYS ?   (Joke by Allan A. Miller as submitted to The Mathematics Student, February, 1976)

$\overline{30}$ $\overline{32}$ $\overline{28}$ $\overline{38}$ $\overline{30}$ $\overline{18}$ $\overline{28}$ $\overline{20}$ $\overline{28}$ $\overline{14}$ $\overline{18}$ $\overline{28}$ $\overline{28}$ $\overline{36}$ $\overline{32}$ $\overline{20}$ $\overline{34}$ $\overline{22}$ $\overline{22}$

$\overline{24}$ $\overline{22}$ $\overline{36}$ $\overline{12}$ $\overline{12}$ $\overline{28}$ $\overline{32}$ $\overline{16}$ $\overline{30}$ $\overline{26}$

PERPLEXING PERIMETERS

I Count to find the perimeter of each polygon.

ONE UNIT
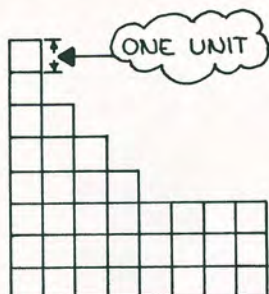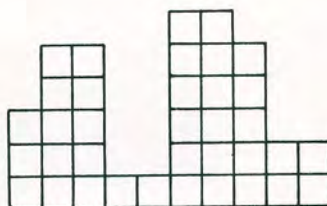
Perimeter: _____

Perimeter: _____

Perimeter: _____

II Find and write the length of the unmarked sides next to each segment. Find the perimeter of each polygon.

a)

7

3

1

2

7 - 2 = _____

3 - 1 = _____

Perimeter: _____

b)

8

7

6

12

Perimeter: _____

c)

6

12

11

8

4

21

20

9

Perimeter: _____

d)

9

7

4

8

6

7

10

26
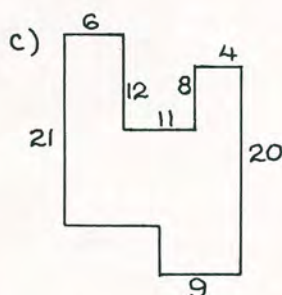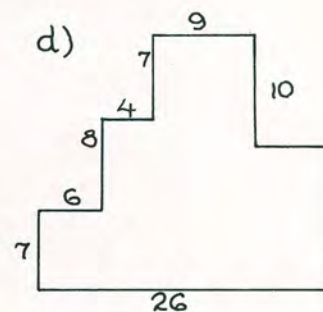
Perimeter: _____

III Find each perimeter.

a)

2

3

3

3

3

4

2

Perimeter: _____

b)

30

24

Perimeter: _____

c)

9

3

4

4

3

Perimeter: _____

d)

15

10

18
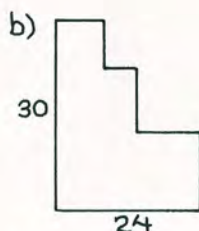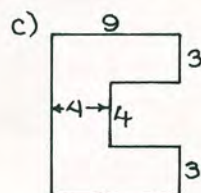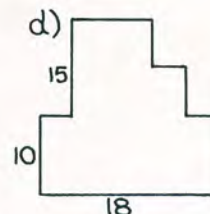
Perimeter: _____

Challenge: Susan and her sister Sara walk to school each day. Sara knows a different way to school. If the girls start at the same time and walk at the same speed, who will get to .school first? _____

Explain your answer. _____

_____

HOME

SUSAN'S PATH

SARA'S ROUTE

SCHOOL

PERIMETER PATTERNS OF POLYGONS

EXAMPLE:

1 TRIANGLE PERIMETER OF 3

2 TRIANGLES PERIMETER OF 4

3 TRIANGLES PERIMETER OF 5

4 TRIANGLES PERIMETER OF 6

# OF POLYGONS	1	2	3	4	5	10	23		93		n
PERIMETER	3	4	5	6	7	12		58		108	

1) DRAW MORE POLYGONS TO HELP YOU.

# OF POLYGONS	1	2	3	4	5	10	20		53		n
PERIMETER	4							68		202	

2)

# OF POLYGONS	1	2	3	4	5	10	20		54		n
PERIMETER								140		784	

3)

# OF POLYGONS	1	2	3	4	5	10	20	39			n
PERIMETER									143	452	

4)
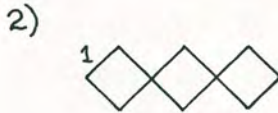
# OF POLYGONS	1	2	3	4	5	10	20		44		n
PERIMETER								210		462	

5)
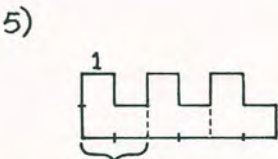
# OF POLYGONS	1	2	3	4	5	10	20		51		n
PERIMETER								290		440	

6)
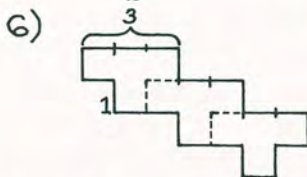
# OF POLYGONS	1	2	3	4	5	10	20		63		n
PERIMETER								250		556	

7) MAKE UP A POLYGON PERIMETER PATTERN OF YOUR OWN. EXCHANGE WITH A FRIEND AND TRY TO COMPLETE EACH OTHER'S TABLE.

# OF POLYGONS	1	2	3	4	5	10	20				n
PERIMETER											

ARRANGING SQUARES

Each activity below can be done independently. Activity 4 is more involved and is less structured than the previous activities. Supply students with square tiles (2 cm on a side is a convenient size) to use in the investigations.

I

EACH SQUARE MEASURES ONE UNIT ON A SIDE.
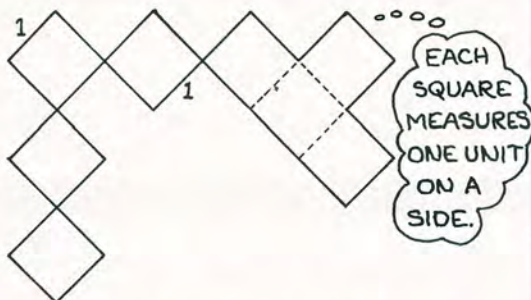
1) What is the perimeter of this figure with 8 squares? _____

2) What is the smallest perimeter you can get by arranging these 8 squares? _____

   Draw a figure to show your answer.

II

EACH SQUARE MEASURES ONE UNIT ON A SIDE.
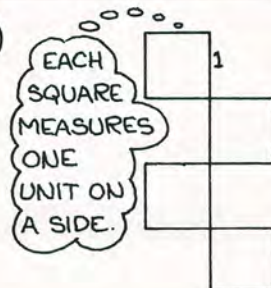
1) What is the perimeter of this figure? _____

2) Arrange the squares to get a perimeter of

   a) 14 units   b) 12 units

   c) 10 units   d) 8 units

   e) 9 units   f) 15 units

   Sketch a figure for each arrangement.

III Give each student five square tiles and a sheet of 2 cm grid paper.

1) Arrange the five tiles to make a figure so that the tiles touch either by a common side or by a corner.

2) Sketch your arrangements on grid paper and record the perimeter for each figure. (The side of each square measures one unit.)
   a) What is the maximum perimeter? _____
   b) What is the minimum perimeter? _____
   c) If we allow only whole number perimeters, can you make a figure for each perimeter between the maximum and minimum perimeters? _____

IV Give each student or group of students twenty square tiles. Each tile represents one table top.

Suppose you have 20 tables and are to arrange them for a banquet.

1) Determine the maximum number of people you can seat using 1, 2, 3, 4, ..., 20 tables if the tables used must be joined edge to edge. Also each table seats one person to an edge.

2) Determine how many people you can seat for each number of tables if you must allow all possible table arrangements. (The tables can be joined by a corner.)

3) If 30 people are to attend a banquet, how would you arrange the tables and how many would you use? (The tables do not have to be in one group.)

PERIMETER PROBLEMS
TO PURSUE

① A bug travels 24 metres in walking around a rectangle one time. If the rectangle is twice as long as it is wide, how long is each side?

② Jake used 40 posts to build a fence around his rectangular shaped garden. He used 14 posts on each long side of the garden. How many posts did he use on each short side?

③ The sum of the lengths of 3 sides of a regular octagon is 57 cm. What is the perimeter of the octagon?

④ Squares ABEF and BCDE are the same size. If you went around the shaded region once would
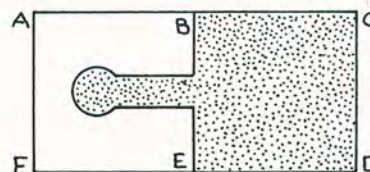
the distance be less than, equal to or greater than the distance around the unshaded region?

⑤ It is 90 feet from one base to the next on a regulation baseball diamond. On a Little League diamond the distances measure 60 feet. How much farther is it around the larger diamond?
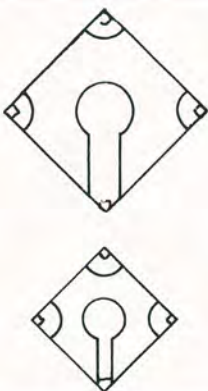
⑥ Find the number of sides in a regular polygon if the perimeter and the length of one side are given.

a) perimeter 88 cm; length 22 cm
b) perimeter 91 cm; length 7 cm
c) perimeter 1 m; length 20 cm

⑦ Suppose that each time you measure the side of a polygon your error is 2 mm. What would be the greatest (least) possible error in measuring the perimeter of

a) a quadrilateral?
b) a pentagon?
c) a hexagon?
d) an octagon?

⑧ An old theater prop frame measures 24 decimetres by 33 decimetres. What <u>one length</u> can be cut from each dimension so that the ratio of the new width to new length is 2 to 3?

667
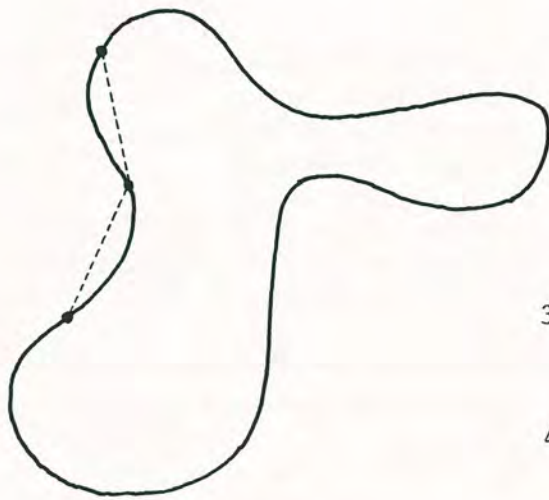
AROUND THE BLOB

1) Guess, in centimetres, the pe-
   rimeter of the blob to the right.

   _____ cm

2) Draw line segments connecting points
   on the blob. Two are done for you
   with dotted lines.

3) Measure each of these segments and find their
   total length. _____ cm

4) How does this length compare to the actual
   perimeter around the blob? _____

   _____

5) How can you get a better approximation of the actual perimeter of the blob? _____

   _____

6) Guess the perimeter of each of these curved figures. Then draw and measure line
   segments to get an approximation of the perimeter.
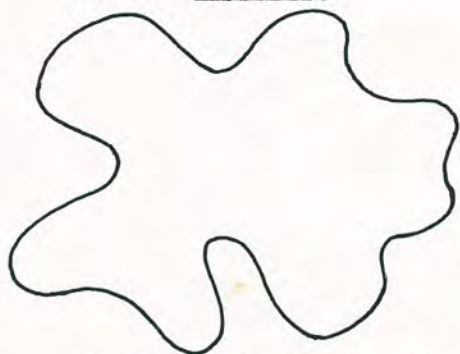
   Guess _____ cm   Guess _____ cm   Guess _____ cm

   Approximation _____ cm   Approximation _____ cm   Approximation _____ cm

7) Compare your approximations with a friend. Are they the same? _____ Why?

   _____

8) Measure the amount of string needed to lay around each figure. How do these
   lengths compare to the approximations in (6)? _____

9) Which method, line segments or string, gives the best approximation of each
   curved figure? _____

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INSIDE AND OUTSIDE A CIRCLE

1) Draw a circle with a radius of at least 6 centimetres. Mark 10 points on the circle. They do not have to be equally spaced. Label the points with lower case letters. (See diagram I.)

2) Connect consecutive points with line segments. Measure each inside segment in millimetres and record the lengths in a table like the one below. Find the total length and record.

3) Outside the circle draw line segments that touch the circle only at the points you marked in part (1) (a, b, c, etc.). Label the points where these outside segments cross with upper case letters. (See diagram II.)

4) Measure the outside segments (just between the points) in millimetres and record the lengths. Find the total and record.

		TOTAL
INSIDE SEGMENTS	— — — — —	
	— — — — —	
OUTSIDE SEGMENTS	— — — — —	
	— — — — —	
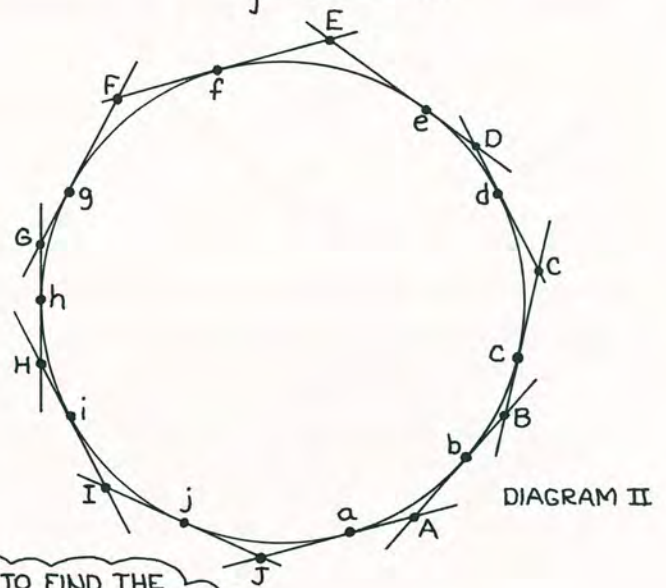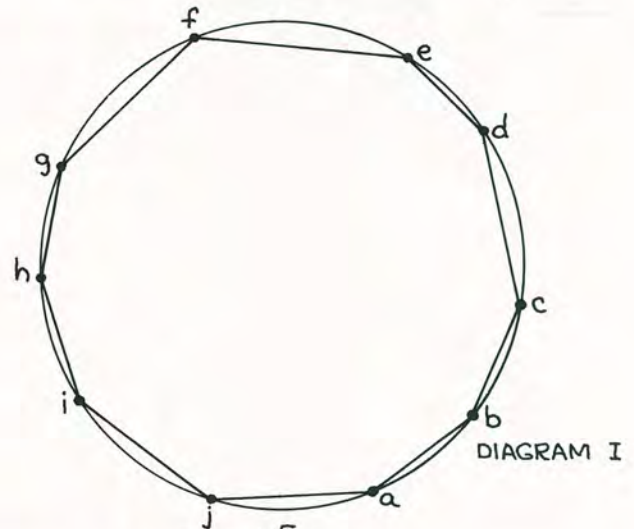
HINT: TO FIND THE AVERAGE ADD THE TWO TOTALS AND DIVIDE BY 2.

5) Find the average of the two totals.

6) Draw circles the same size as you drew before and repeat steps 1-5 using 13 points on the circle. Then use 16 points. As the number of points increases, what happens to the average of the two totals?

7) Lay a string around the circle. Measure its length. What do you notice?

8) How many points do you think should be used to get a very close approximation of the circumference (distance around) the circle?

9) You can repeat the activity using a larger circle and more points to get a close approximation.

GOING AROUND IN CIRCLES

I The diameter of one circle is twice as long as the diameter of another circle. How many times as long is the circumference? _____
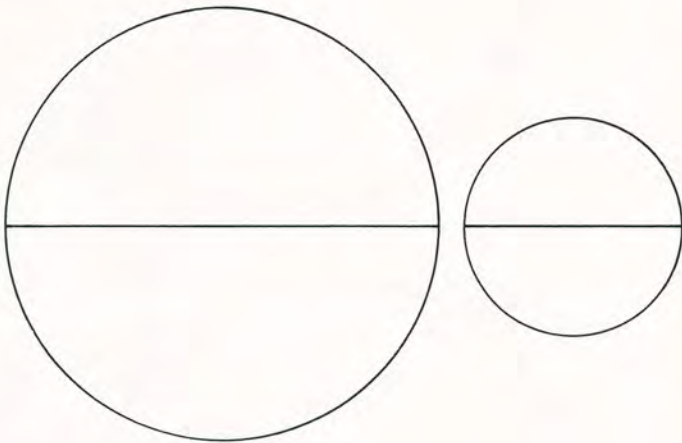
II The diameter of this circle has been divided into five parts. A circle has been drawn using each part as a diameter. If the circumferences of the five small circles are combined, will the total length be more than, equal to or smaller than the circumference of the large circle?

_____
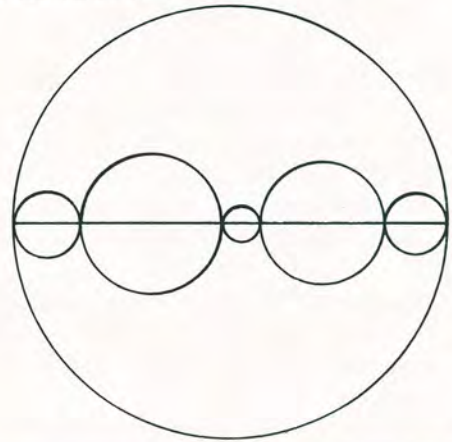
III Imagine a piece of string tightly wrapped around the equator. Now add a 12 metre piece of string so a larger circle that remains the same distance away from the equator can be formed. (See the figure below.) Could you crawl under this large circle? _____

LARGER CIRCLE WITH 12 METRES OF EXTRA STRING.

SOUTH POLE
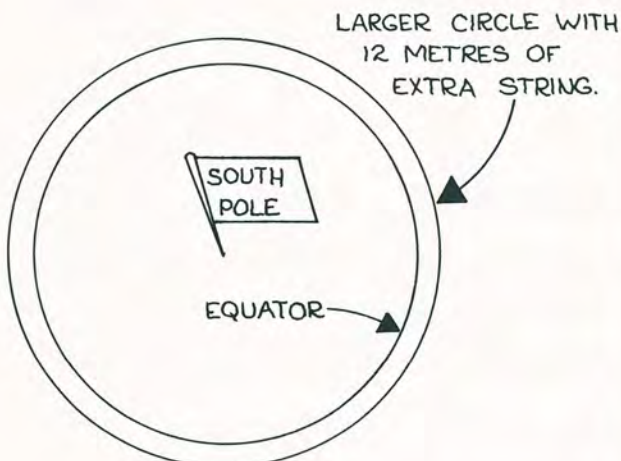
EQUATOR

IV If you are 170 cm tall, how much farther would your head travel than your feet if you were to run around the equator? _____

170 cm

NORTH POLE

IDEA FROM: *Geometry*

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GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

CONTENTS

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AREA

Since area formulas sometimes receive more than their share of emphasis, some students conclude that "area _is_ length times width." If two dimensions are given in an area word problem, these students might multiply the two numbers regardless of the figure's shape. If you are introducing or reviewing area, perhaps some of the activities below will help students realize that area is more than just a formula or set of formulas.

SOME INTRODUCTORY ACTIVITIES

● Have students find the areas of their desks by tiling (covering without leaving spaces) the desks with square tiles, equilateral triangles, regular hexagons, books, sheets of paper, etc. A discussion might be started with some of these questions: "Would circles work? Why not? If it takes 10 □'s to cover your desk, how many △'s would be needed? What shapes seem to work best for covering rectangular regions? Do you see why it is necessary to give the unit along with the number when you give the area of a region? Do you see why it is often useful to have standard units?"

DO YOUR STUDENTS THINK LIKE THIS?

Question: What is the area of this figure?

5

3

Area = length x width, so the area is 5 x 3 or 15.

The area of my desk is about 6 math books.

Mine is about 28 of these triangles.

- If the floor and ceiling of the classroom are tiled, ask students to determine the number of floor tiles needed to cover the entire floor and the number of ceiling tiles to cover the ceiling. Are the floor and ceiling the same size? (Probably.) Are there the same number of floor tiles as ceiling tiles? (Probably not if 9-inch floor tiles and 12-inch ceiling tiles are used.)

- Give students a set of 2 cm by 2 cm squares and a worksheet of conveniently sized and shaped figures. Have students find the area of each figure by covering it with squares. This could be demonstrated on an overhead.

HOW MANY OF THESE SQUARES ARE NEEDED TO EXACTLY COVER THE SHAPE TO THE LEFT?
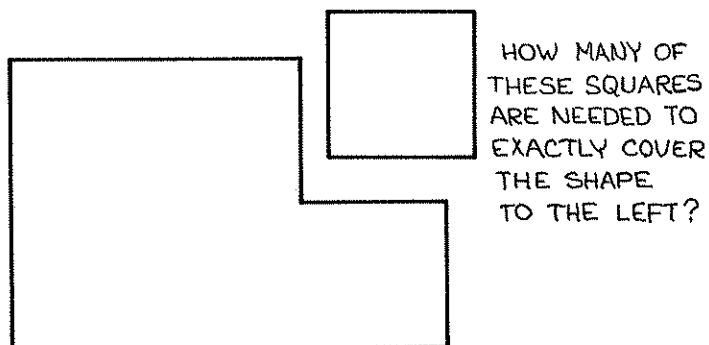
- Hand out another worksheet of shapes that will fit nicely on a centimetre grid. Have students determine the area of each shape in square centimetres by placing a centimetre grid under the page or a transparent grid over the figures. This also makes an excellent demonstration on the overhead.

HOW MANY SQUARE CENTIMETRES ARE NEEDED TO EXACTLY COVER THE FIGURE BELOW?
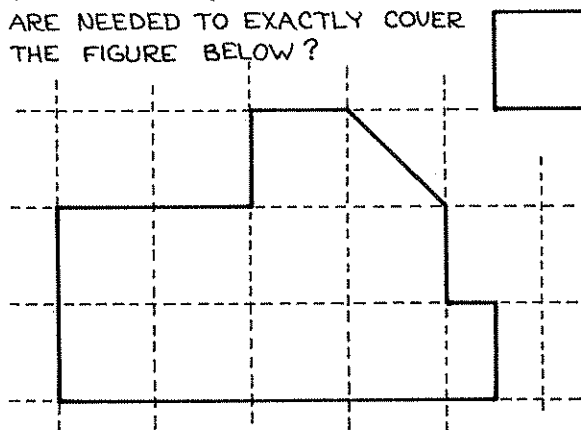
- Display an irregular shape on the overhead. How can the area of such a shape be found? Someone might suggest placing a transparent centimetre grid over it and counting the squares (see diagrams below). Which squares should be counted? There are 4 squares completely inside the figure, but that doesn't count a lot of pieces of squares. A $\frac{1}{2}$ cm grid has 25 squares inside or $6\frac{1}{4}$ cm^2. A $\frac{1}{4}$ cm grid would give a closer approximation of about $7\frac{1}{2}$ cm^2.
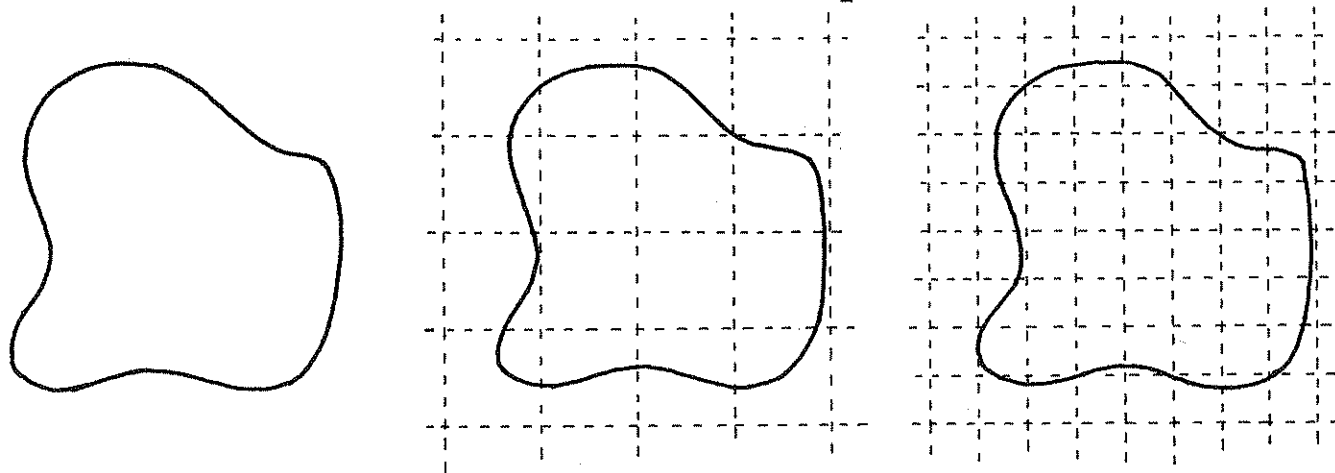
Another way to approximate the area of an irregular figure is given in *Area of Blobs*. Students are asked to average the number of squares completely inside the figure and the number of squares which have any part within the figure. For the centimetre grid on the previous page, the approximation would be $\frac{1}{2}(3 + 15)$ or 9 cm^2. The averaging process usually gives a much closer approximation than the process of counting only the inside squares.

ADDING, SUBTRACTING AND CONSERVING AREA

Some students might not realize that the area of a planar region is conserved when the figure is rotated, turned, translated or "cut up." It is important that students accept and understand conservation of area before they are given area activities and formula developments which <u>assume</u> this understanding. Here is an idea you can use to reinforce the understanding of conservation of area and to discover which students are having difficulty with this concept.

● Show a rectangular piece of transparent grid paper on the overhead. Ask students to determine the area of the rectangle. Cut the rectangle on its diagonal and rearrange the two pieces. What is the area of the new figure? Do students count the squares or do they see that the area is the same as before? Have students cut rectangles of the same size from grid paper. Have them cut up the rectangles to show many figures with the same area as the original rectangles.
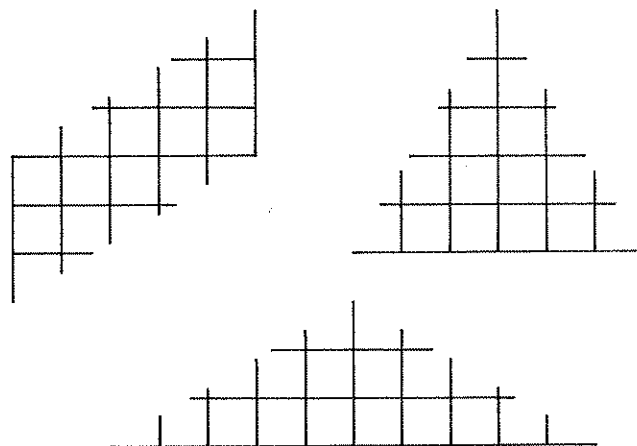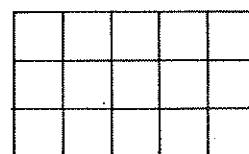
Many area problems require that students understand that the area of a planar region is the sum of the areas of its parts. The area of the figure to the right is the sum of the areas of the triangle, rectangle and semicircle. Some related activities are given on the next page.
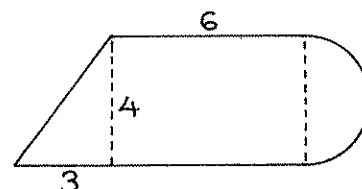
- Have each student cut a simple fig-
ure out of grid paper and label it
with the number of square units it
contains. Have small groups of stu-
dents put their figures together and
give the area of the composite fig-
ure. Some students will count the
squares again, but most will realize
the areas can be added.

AREA IS 14 SQUARE UNITS
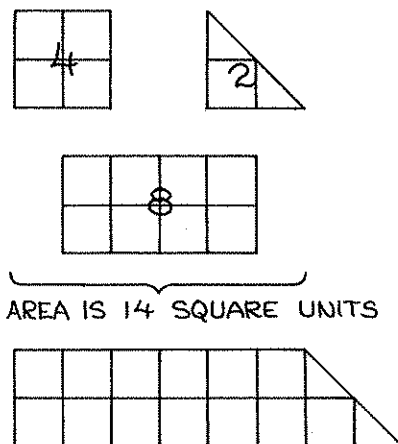
- Show students a figure and tell them
the area of the figure. Next show
the same figure divided into congru-
ent parts. What is the area of one
of the parts? These drawings are
similar to those used to represent
fractions. You can find more ideas
for figures like this in the FRACTIONS
section of the Number Sense and
Arithmetic Skills resource. You
might want to make a transparency or
worksheet including these problems
and others like them.

AREA IS 10 SQ. UNITS

AREA IS 18 SQ. UNITS

AREA IS 6 SQ. UNITS

AREA IS 32 SQ. UNITS
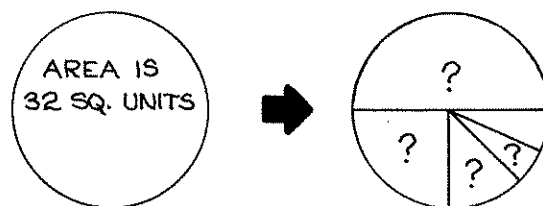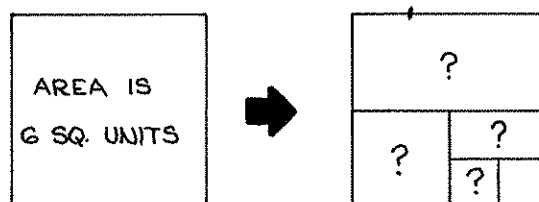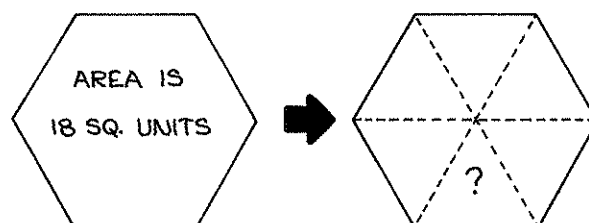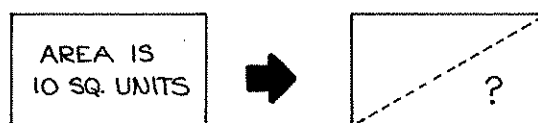
- Display a rectangle on the overhead.
Show that it can be covered with six
congruent blue rectangles or six
congruent red rectangles--be sure the
red and blue rectangles are of dif-
ferent shapes. Ask students to com-
pare the area of one blue rectangle
to the area of one red rectangle. Do
they see that the areas would be equal?

RED RECTANGLES   BLUE RECTANGLES

Tell students that the area of the rectangle is 96 cm^2 (use an 8 cm by 12 cm
rectangle). What is the area of three blue rectangles? Of three red rectangles?
One blue rectangle? One red rectangle?

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● Draw a smooth curve from one edge of of a rectangular piece of transparency to the opposite edge—be sure the curve always moves away from the first edge. Move a strip of paper parallel to the first edge so one end of the strip follows the curve. With another pencil, trace the path of the other end of the strip to make a curve "parallel" to the first curve. Ask students if they can think of a way to find the area of the region between the two curves. Some will probably suggest the grid method, but there is an easier way. Cut the transparency along the curves and remove the curved region. The two remaining pieces will fit together to form a rectangle. The area of the curved region is the difference of the areas of the rectangles—in this case 24 sq. units.
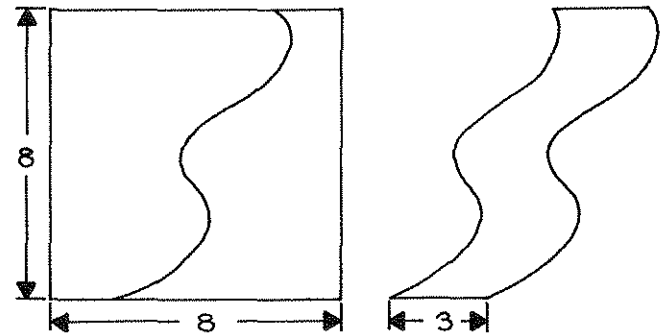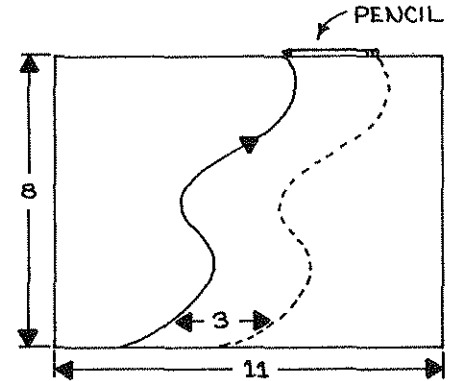
● Have each student choose a suitable width and make paths on rectangles as described in the previous activity. They can peel a color crayon, lay it flat and move it parallel to an edge so the width of the crayon colors the path. Be sure the paths stay on the paper and do not loop back down. What is the area of each path? (They should be the same for each student, if they have drawn the paths on rectangles of the same size and from the same length edge.) Are the rectangles formed after removing the paths congruent? Do these rectangles have the same area? Must the path always have the same area? ("Yes" to all three!)

677
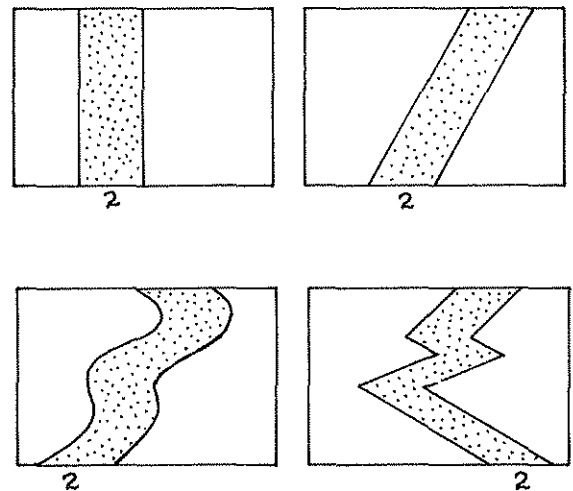
● After introducing the area of geo-
board figures as suggested in *Small
Squares, Half Squares Also* and *The
Rectangle Method,* students can use
a "subtraction method" to find the
area of an irregular shape on the
geoboard. The figure to the right
has been enclosed in a rectangle
whose area is 9 square units. The
areas of the right triangles outside
the figure can be determined by
methods discussed in *The Rectangle
Method.* The area of the figure must
be $9 - (3 + 1 + 1\frac{1}{2})$ or $3\frac{1}{2}$ square
units.
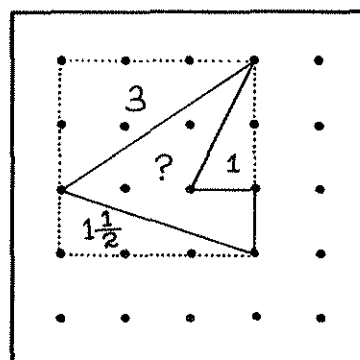
Many of the classroom materials in this subsection involve adding, subtracting
and conserving area. *Something for Nothing* presents paradoxes where area seems to
have disappeared. Since we know this is impossible, what did happen? *The Secret Area*
suggests that students find the area of a figure by finding the area of its parts.
The Rectangle Method has students find the area of an irregular shape on a geoboard
by a subtraction method similar to that discussed in the last activity above. The
development of formulas discussed below relies on students understanding these prin-
ciples of adding, subtracting and conserving area.

PERIMETER AND AREA--HOW ARE THEY RELATED?

Some students might not know that
two figures can have the same area but
different perimeters or that two figures
can have the same perimeter but differ-
ent areas. Here are some activities
which can help students understand that
area and perimeter are closely related
but do not determine each other.
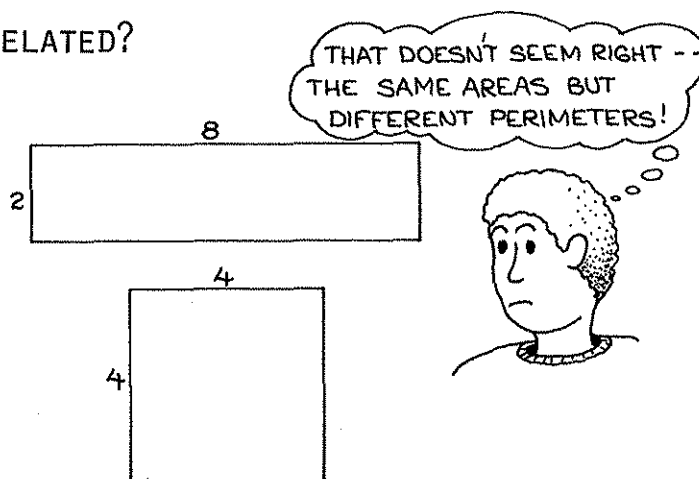
● Take a piece of string and tie it in
  a loop. Spread the loop over a trans-
  parent grid on the overhead and hold
  it in a rectangular shape. Have stu-
  dents count the number of squares
  enclosed. Now form a rectangle of a
  different shape. How many squares
  are enclosed? Are the areas of the
  two rectangles different? Are the
  perimeters different? Why?

● Join eight identical strips in a loop. Cardboard
  strips, erector set strips, D-stix or geostrips
  will work. Make different shapes with the strips.
  Do students see that the perimeters of the shapes
  are the same but the areas are different? What
  shape would have the least area?

.AREA ≈ O

● Arrange four square tiles in various ways on the
  overhead. Are the areas of the arrangements dif-
  ferent? Are the perimeters of the arrangements
  different? Which arrangement has the greatest
  perimeter? The least perimeter?
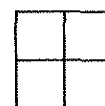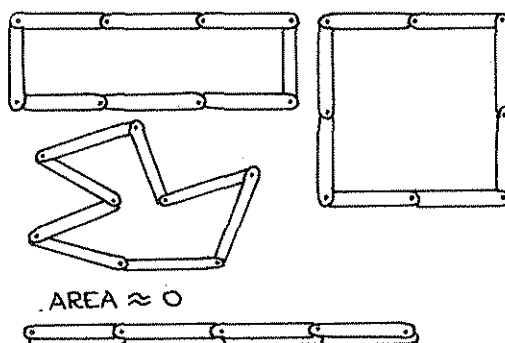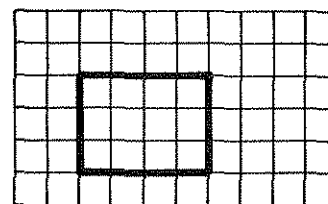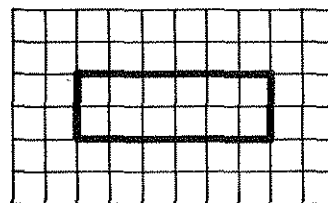
LEAST PERIMETER
8 UNITS
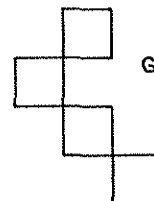
GREATEST PERIMETER
16 UNITS

    When students are convinced that area does not
determine perimeter and perimeter does not deter-
mine area, they can explore the perimeters and areas of shapes on a higher level.
Some slightly more advanced activities are given below. See also *Big Foot* and
A Sheepish Problem in the classroom materials.

● The area of a rectangle is 36 square units. Fill in
  the table of possible lengths, widths and perimeters.
  What are the dimensions of the rectangle with the
  least perimeter? (6 x 6) Can you determine the
  dimensions of the rectangle with the greatest peri-
  meter? (Did you try a fraction like $\frac{1}{2}$ for the
  width?--there is no greatest perimeter in this case!)

AREA OF RECTANGLE : 36 cm²		
W	L	P
1	36	74
2	18	40
3	12	30
4		
$\frac{1}{2}$	72	145

● Draw some squares whose edges are measured in centimetres. When will the number of square centimetres needed to cover a square be less than the number of centimetres to go around the square? When will they be the same number? Make a table and a graph to "compare" area and perimeter. (Note: If you have students use ratios to compare area and perimeter, be sure to have them record the units. A 10 cm by 10 cm square has an area of 100 cm^2 and a perimeter of 40 cm. The ratio of area to perimeter is 100 cm^2 to 40 cm, not 100 to 40. The same square has an area of 1 dm^2 and a perimeter of 4 dm for a ratio of 1 dm^2 to 4 dm, not 1 to 4. Notice that 100 cm^2 to 40 cm = 1 dm^2 to 4 dm, but 100 to 40 ≠ 1 to 4.)

EDGE	AREA	P
1 cm	1 cm^2	4 cm
2 cm	4 cm^2	8 cm
3 cm	9 cm^2	12 cm
4 cm	16 cm^2	16 cm
5 cm	25 cm^2	20 cm
6 cm	36 cm^2	24 cm
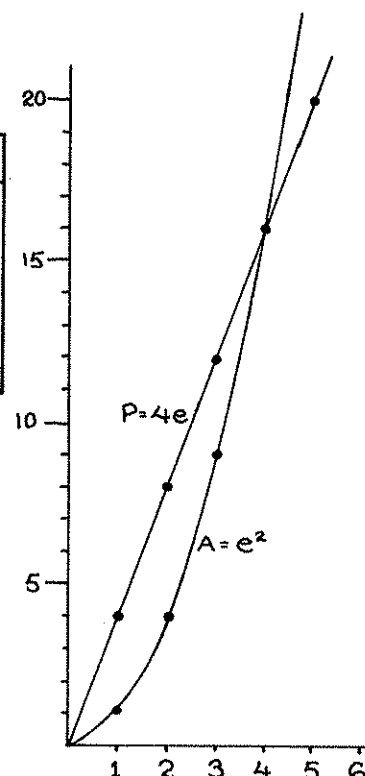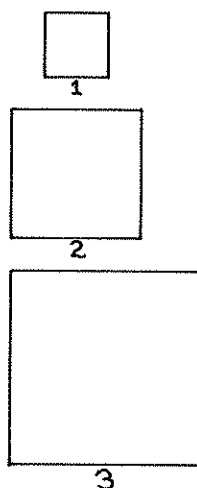
● This activity will demonstrate that a circle has the maximum area for a given perimeter. Pour a small bottle of bubble solution into a shallow pan. Make a large wire hoop that will fit into the pan. Tie a thread about 5 to 7 cm longer than the diameter of the hoop across the hoop. Tie the ends of an extra 7 cm piece of thread to the first string. Dip the hoop into the solution. The thread should float freely in the resulting film. Break the film between the double thread. A circular hole appears. Why? The soap film tries to pull together--to have the least possible soap film surface. This causes the hole to have the greatest possible area. The hole has the perimeter of the double thread's length. This perimeter becomes the circumference of a circle, so the thread will enclose the greatest area.
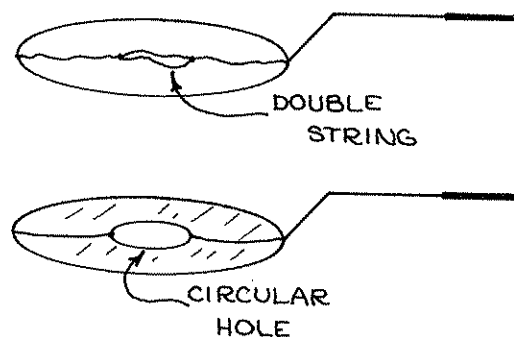
Challenge: Ask students if they can design a figure with a small area but a very large perimeter. They can try to keep the figure within a circle to be sure it has an area less than that of the circle. Can they design a planar figure with a small perimeter but a very large area? (No solution. The maximum area will be obtained when a given perimeter is the circumference of a circle.)
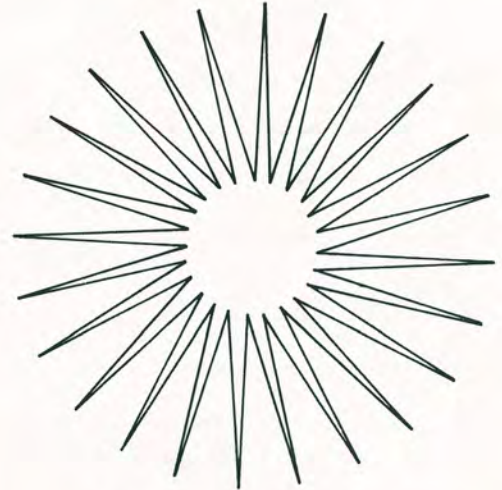
THE PERIMETER CAN BE INCREASED WITHOUT LIMIT BY INCREASING THE NUMBER OF POINTS.

Further Readings

Many articles in The Arithmetic Teacher, The Mathematics Teacher, Mathematics Teaching, School Science and Mathematics and NCTM yearbooks are devoted to the understanding and teaching of area, perimeter and volume. These articles are a rich source of ideas for the middle school teacher.

FRAMED

Using four Cuisenaire rods, two blue and two yellow, make the picture frame shown below.
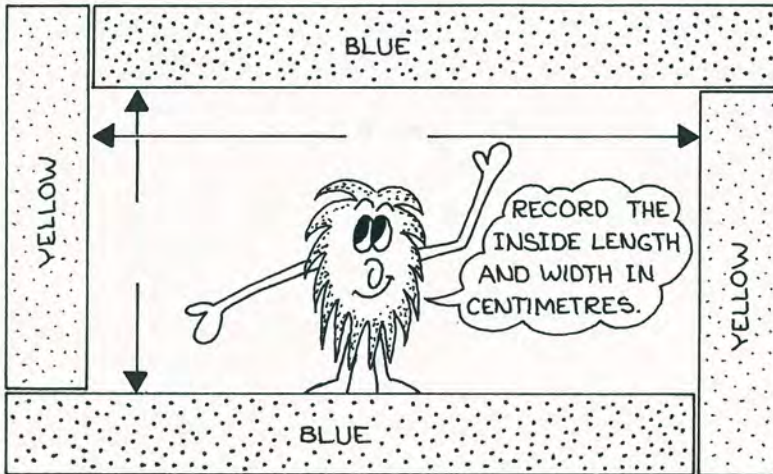
1) How many white rods does it take to completely cover the inside of this picture? _____ white rods

2) Can you cover this picture using just red rods? _____

3) How many red rods will it take? _____

4) Which can be used to completely cover the picture: light green rods? purple rods? yellow rods? dark green rods?

5) Re-arrange the four rods to make as many different picture frames as you can. Make a drawing of each on your paper.

6) With each new frame record the inside length and width, the number of white rods needed to fill the picture.

YOU CANNOT BUILD PICTURE FRAMES THAT HAVE AN OVERHANG OR THAT JUST MEET AT THE CORNERS OF EACH SIDE.
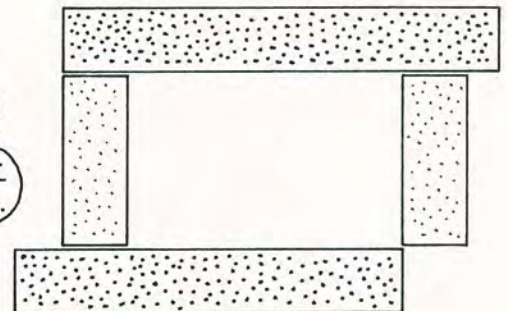
7) Special Challenge: Using two orange rods and two blue-yellow trains make and record the picture frame that encloses the largest picture.
Which rods (white through orange) when used alone can completely cover this picture?

CUISENAIRE RODS

Cuisenaire rods are colored wooden rods one centimetre high, one centimetre thick, and one to ten centimetres long. Each length is coded by a different color. Cuisenaire rods are useful for work with fractions, decimals, and whole number operations and geometric relationships.

Cuisenaire rods can be purchased from the Cuisenaire Company of America.

White
Red
Light Green
Purple
Yellow
Dark Green
Black
Brown
Blue
Orange

This is the Cuisenaire color code for the rods.
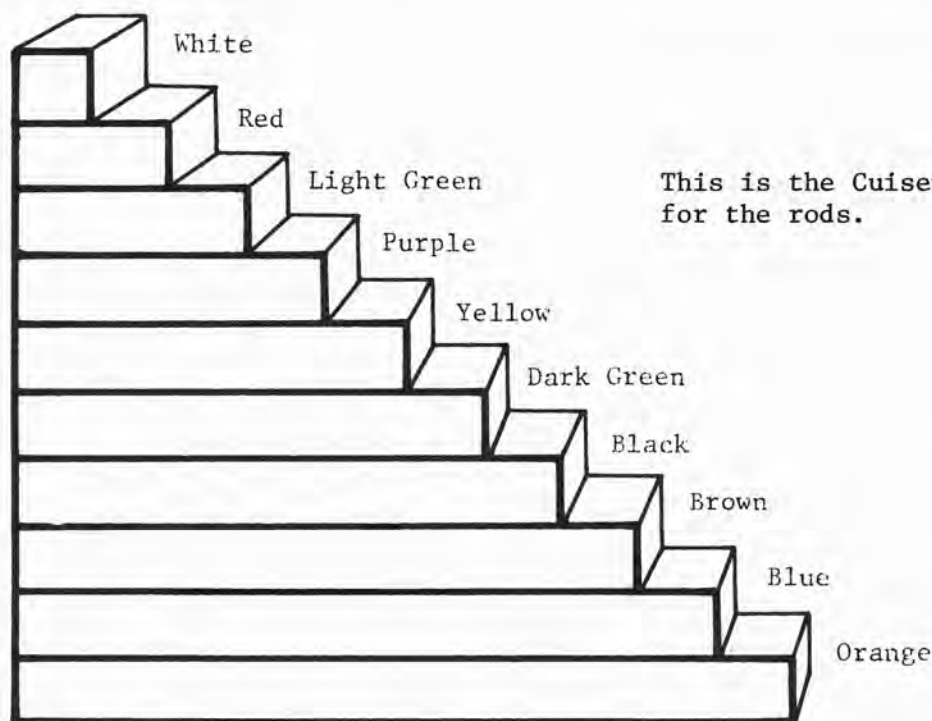
Readiness Activities:

1) Find the shortest rod and the longest rod.

2) Find a rod longer than the red and shorter than the purple.

3) Make a staircase using a rod of each color.

4) Name the rods by color; agree on names. (Example: light green ⟶ lime.)

5) Put the white, yellow, light green, purple and yellow rods in a box and shake. Without looking, find the white rod. How did you do it?

6) Put one light green and one red rod end to end like a train. Find one rod as long as your "train."

7) Take a blue rod. Place a yellow rod on top. What rod fits next to the yellow rod to make a train the same size as the blue rod?

8) If a red rod = 1, which rod = 2?

9) Take the light green rod. How many trains can you make as long as the light green rod?

■SMALL■
▪:SQUARES▪:

Materials Needed: Geoboard, rubber bands, dot paper

Call the smallest square on the geoboard one unit of area. □

It is also called 1 square unit.

Activity:

1) Make these figures on your geoboard and find the number of square units in the area of each figure.
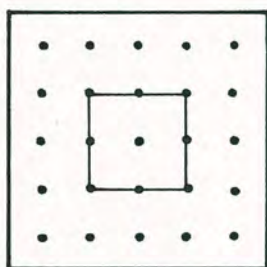
a)

_____ square units

b)
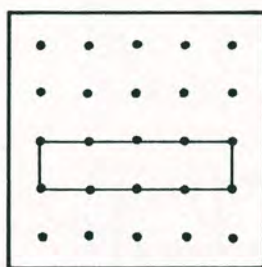
_____ square units

c)

_____ square units

d)

_____ square units

e)

_____ square units

f)

_____ square units

g)
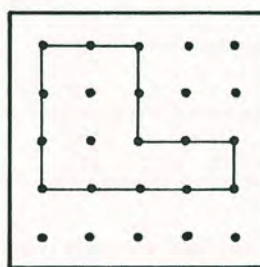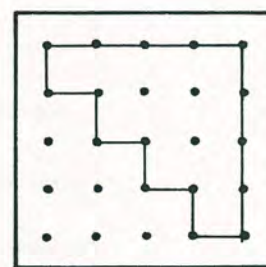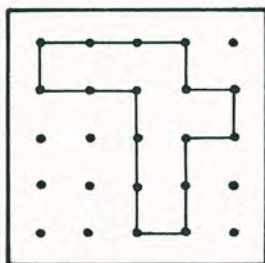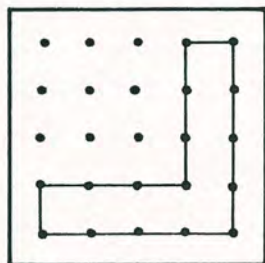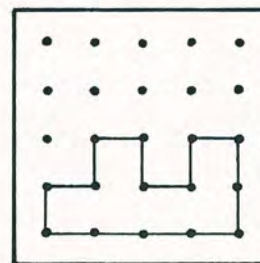
_____ square units

h)
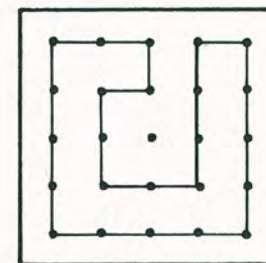
_____ square units

2) Make these figures on your geoboard. Record your answers on dot paper.

a) a square with an area of 9 square units

b) a rectangle with an area of 8 square units

c) a square with an area of 16 square units

d) a 6-sided figure with an area of 10 square units

e) an 8-sided figure with an area of 7 square units

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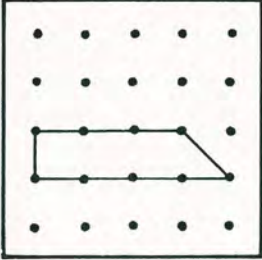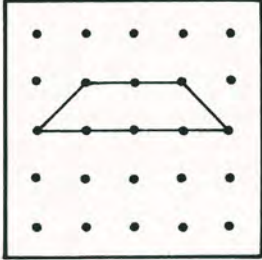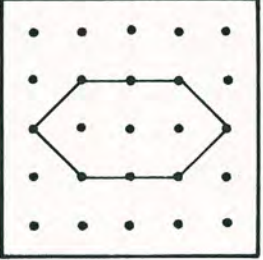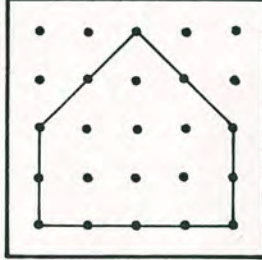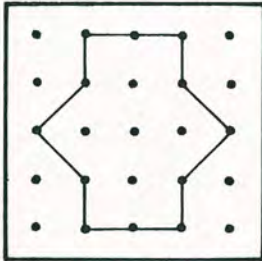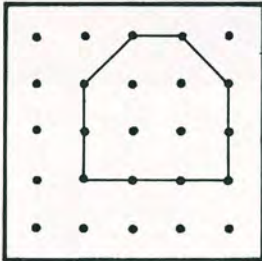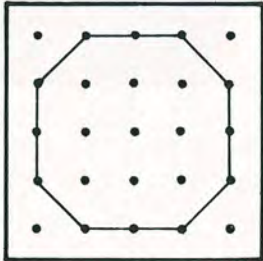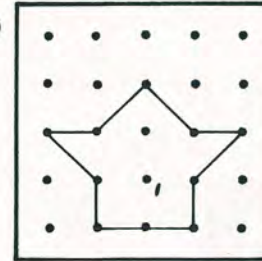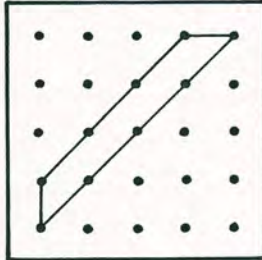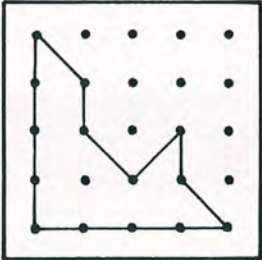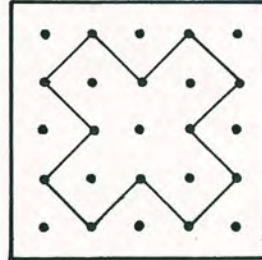
HALF SQUARES ALSO

Materials Needed: Geoboard, rubber bands, dot paper

If ☐ is 1 square unit, how many squares units is △ ? _____ square unit.

Activity:

1) Make these figures on your geoboard and find the area of each in terms of square units.

a)

_____ square units

b)

_____ square units

c)

_____ square units

d)

_____ square units

e)

_____ square units

f)

_____ square units

g)

_____ square units

h)

_____ square units

i)

_____ square units

j)

_____ square units

k)

_____ square units

l)

_____ square units

2) Make these figures on your geoboard. Record your answers on dot paper.

a) a square with an area of 2 square units

b) a rectangle with an area of 4 square units

c) an isosceles triangle with an area of 4 square units

TANGRAMS AND AREA

Materials Needed: Tangram pieces

Activity:

Use your Tangram pieces to find the areas of these figures. Fill in each of the charts. The unit of area is shown for each chart.
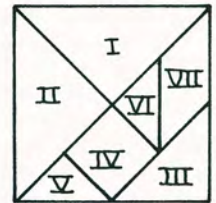
①

1 UNIT OF AREA					

②

1 UNIT OF AREA					

③

1 UNIT OF AREA					

1) Give each student a sheet of grid paper with a reference point marked near the lower left corner. Have them position the paper on the desk so the mark is toward the bottom.

2) Each student starts at the reference point and draws a figure from your directions.

   Some sample directions and the figure they describe are given below.

AT THE END OF EACH DIRECTION THE STUDENT MARKS A POINT AND DRAWS A LINE SEGMENT.

a) Up 4 spaces

b) Right 4 spaces

c) Up 3 spaces

d) Right 6 spaces

e) Down 4 spaces and right 4 spaces

f) Down 3 spaces

g) Down 3 spaces and left 3 spaces

h) Up 3 spaces

i) Left to starting point
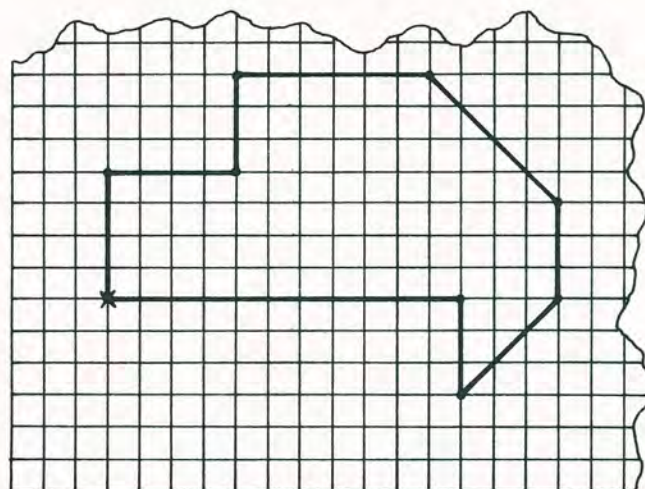
3) At first draw the figure with the class on a transparent grid on the overhead as you give the directions. Later you can draw the figure with the class but with the overhead light out or you can have it drawn ahead of time to share with the class at the end of the activity.

4) After the figure is drawn each student finds the area by counting squares and/or partitioning the figure into triangles and rectangles. If partitioned the area of each portion should be marked and then the amounts are added to get the total area. The first student to correctly find the area may share the solution with the class on the overhead. Since each figure can be partitioned in several ways other students can share their methods.

TOTAL AREA : 82½ SQUARE UNITS
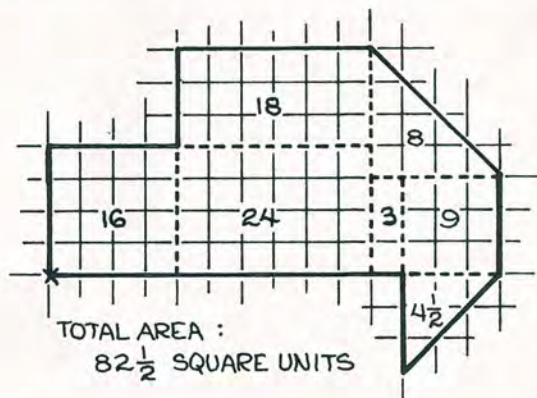
5) The directions for each figure can be simple or more involved to challenge your class.

6) Students could make their own picture and take turns giving the class directions for making the "mystery shape."

7) You could also give an area and ask students to make a figure with that area.

DON'T GET BUGGED BY THIS

Which of these four shapes has the longest "bug distance"? _____

THE "BUG DISTANCE" FROM POINT A TO POINT B IS THE LENGTH OF THE PATH THE BUG TRAVELS.

1) A _____

2) A _____

3) A _____

4) A _____

B
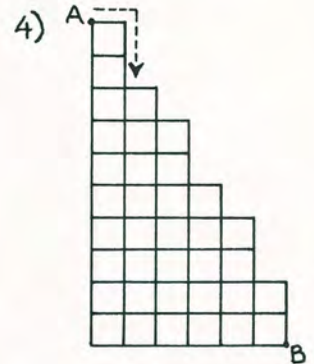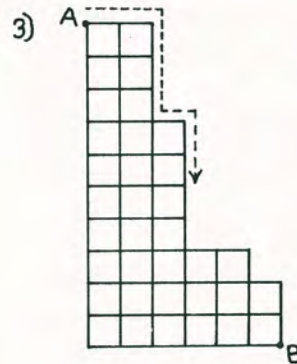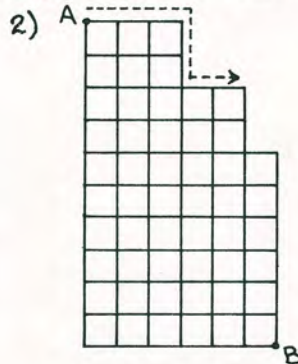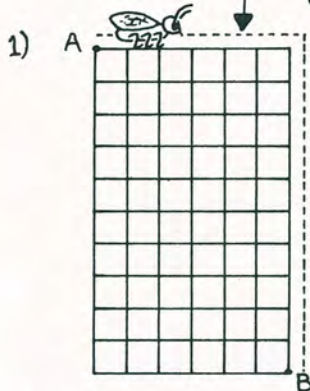
Find the area and perimeter of each figure.

What do you find that is the same for all four of these figures? _____

_____

What is different about each of these four? _____

_____

_____
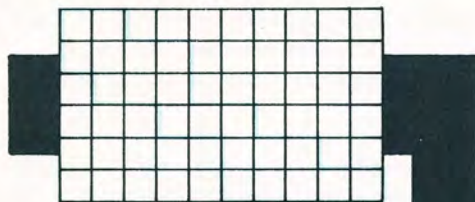
Area: 20 square units

Mark each grid to make a figure with a perimeter of 32 units but having the indicated area.
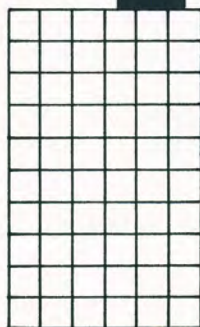
Area: 32 square units

Area: 27 square units

HOW MANY CAN YOU FIND?

Materials: Grid paper

YOU CAN COPY THE POLYGONS ON GRID PAPER TO HELP YOU.

1) Find the area and the perimeter of these polygons.
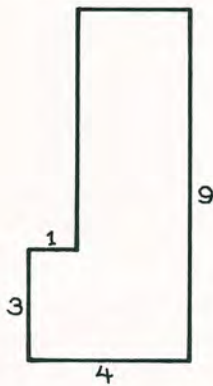
AREA: _____ SQ. UNITS   AREA: _____ SQ. UNITS   AREA: _____ SQ. UNITS   AREA: _____ SQ. UNITS

PERIMETER: _____ UNITS   PERIMETER: _____ UNITS   PERIMETER: _____ UNITS   PERIMETER: _____ UNITS

2) Use the grid paper to help you draw other polygons having a perimter of 26 units and an area of 30 square units. (The side of the smallest square on your grid paper measures one unit.)

a) How many differently shaped 6-sided figures can you find?

b) How many differently shaped 8-sided figures can you find?

c) Can you draw a 16-sided polygon with a perimeter of 26 units and an area of 30 square units?

d) Is there a maximum number of sides that such polygons can have? _____ If so, how many? _____

For each new polygon you discover you are awarded the following points:

4 points for a 4-sided figure
3 points for a 6-sided figure
2 points for an 8-sided figure
1 point for other polygons

What is your score? _____

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AREAS OF BLOBS

Materials: 1 cm and 1/2 cm grid paper
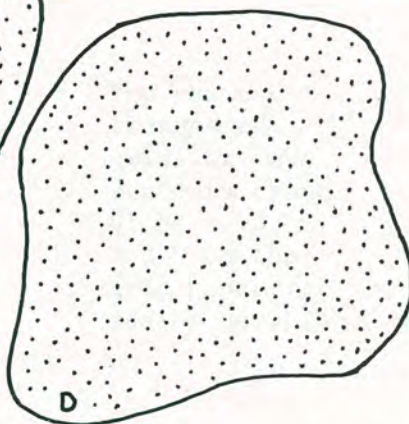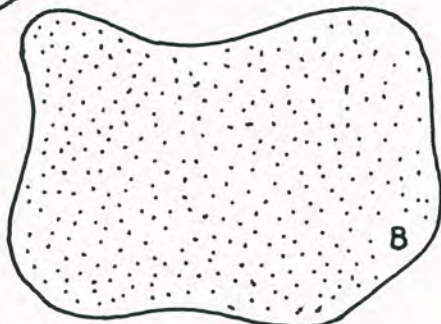
1) Estimate, then find the area of each blob in square centimetres. Use the centimetre grid to count the number of squares that lie completely inside each blob. Write the number in the table. Then use the centimetre grid to count the squares that lie inside or contain part of the blob. Write this number in the table.

BLOB	I ESTIMATE OF AREA IN cm²	II NUMBER OF SQUARES INSIDE BLOB	III NUMBER OF SQUARES INSIDE OR CONTAINING BLOB	IV AVERAGE OF II AND III
A				
B				
C				
D				

ADD THE NUMBER FROM II TO THE NUMBER FROM III AND DIVIDE BY 2.

2) Use the 1/2 cm grid paper to count the number of squares that lie completely inside each blob. Write the number in the table. Then use the 1/2 cm grid to count the squares that lie inside or contain part of the blob. Write this number in the table.

BLOB	I NUMBER OF SQUARES INSIDE BLOB	II NUMBER OF SQUARES INSIDE OR CONTAINING BLOB	III AVERAGE OF I AND II	IV AVERAGE AREA IN cm²
A				
B				
C				
D				

DIVIDE EACH NUMBER IN III BY 4.
FOUR SQUARES ½ cm ON A SIDE HAVE AN AREA OF 1cm²
1cm

Compare your numbers in column IV of problem #1 with those in column IV of #2. Are they equal? _____   If not, can you explain? _____

Which column of numbers gives the "best" square-centimetre area approximation?

_____

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BALANCE THE AREA

The area of a few "nice" polygons can be found using a formula but many shapes are irregular and their areas must be found in other ways, perhaps by superimposing a grid or approximating. This activity develops a method of determining the area of an irregular shape.

The method consists of comparing the mass of a flat shape with unknown area to a mass of a flat shape of the same material with known area.

Materials Needed: A fairly sensitive balance (the type found in most science classes), heavy tagboard or cardboard and scissors.

A) Transfer this irregular shape onto and tagboard. Cut it out.

B) Cut out carefully measured squares from the same material (four 5 cm x 5 cm, ten 2 cm x 2 cm, ten 1 cm x 1 cm).

C) Use the balance to determine the number of centimetre squares needed to balance the irregular shape.

The area of the irregular shape is _____ cm^2.
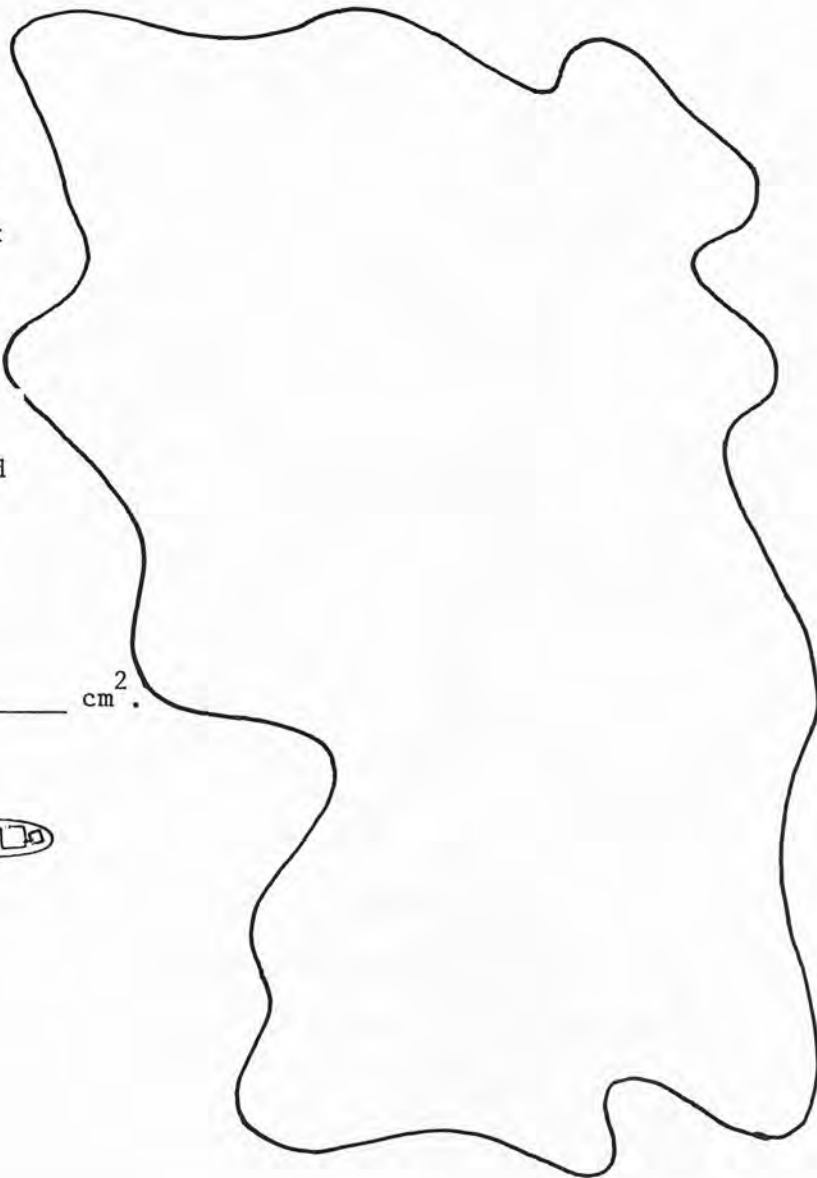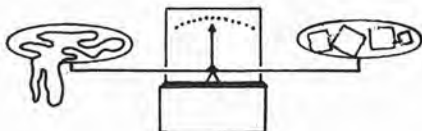
ARE SQUARES LARGER?

The geometric puzzles below could be enlarged and cut out of tagboard for a permanent puzzle. Place them in separate envelopes with instructions to arrange the pieces into a rectangle (parallelogram, trapezoid, etc.) and then into a square. Making each puzzle a different color will simplify putting the pieces back. Students enjoy signing their name on the back of the envelope of each puzzle they complete.

IDEA FROM: *Fun and Games with Geometry*
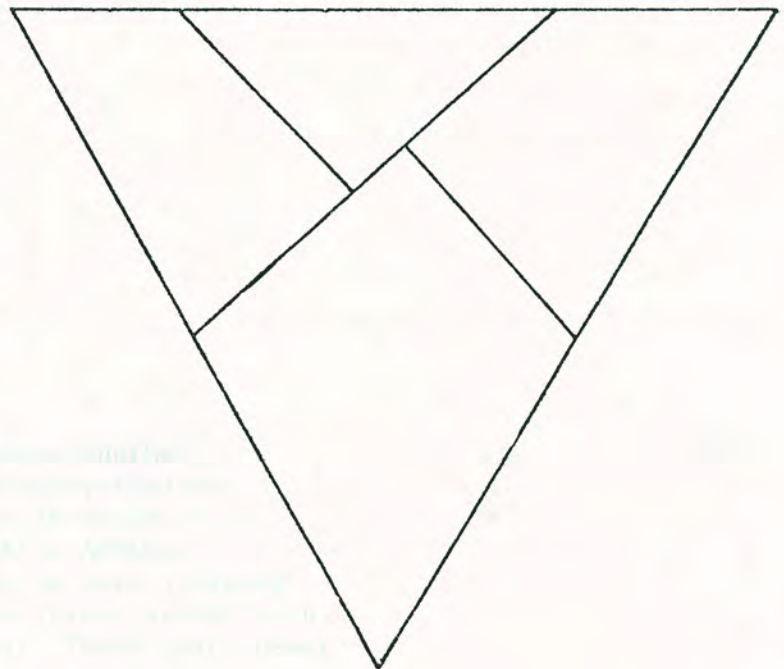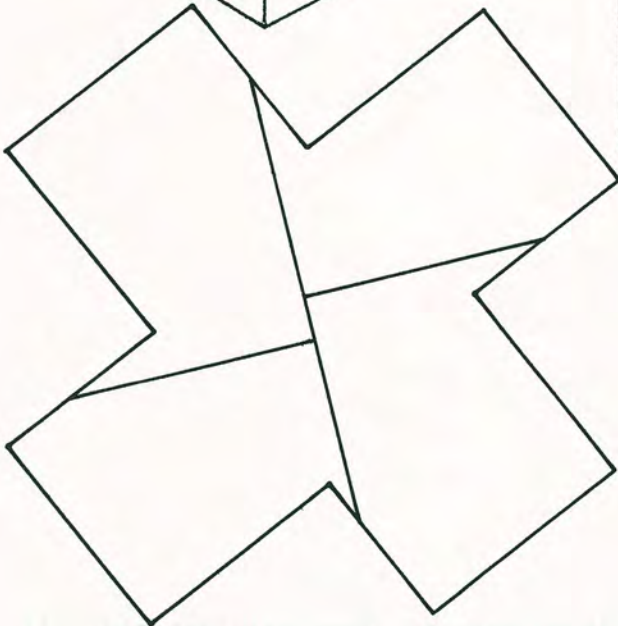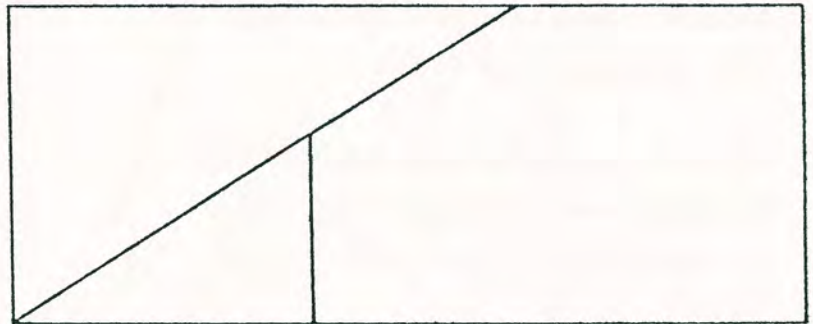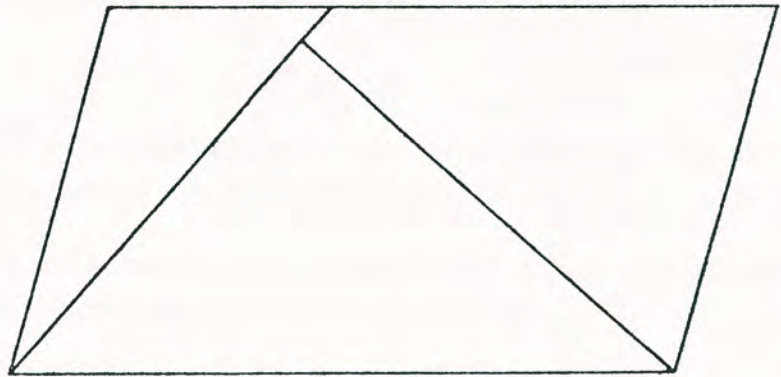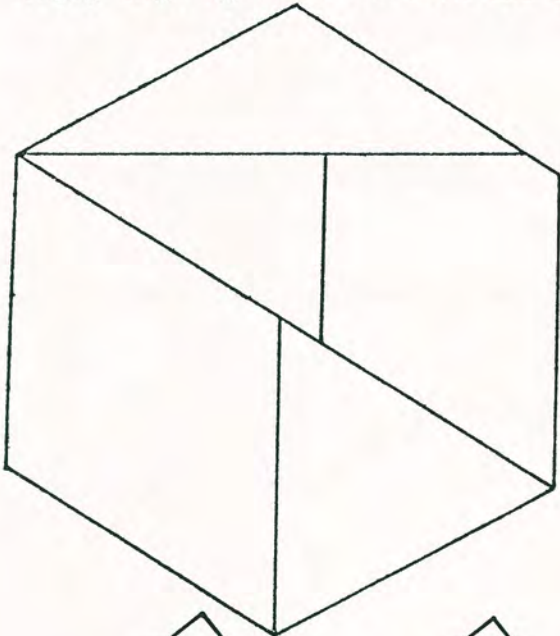
SOMETHING FOR NOTHING

I.

a) Copy the pattern to the right onto centimetre or larger grid paper.

b) Carefully cut out the six pieces, discarding the two shaded rectangles. What is the total area of the remaining four shapes? _____ square units.

c) Rearrange the remaining four shapes to make two different rectangles. One is a long rectangle having a length of 13 units, the other is a square 8 units on a side. Make a drawing of each rectangle.

d) The area of the long rectangle is _____ square units. The area of the square is _____ square units.

e) Examine each of the figures very carefully to find where the extra square units came from. Can you really get something for nothing?

THE AREA OF THE FIGURE TO THE LEFT IS 143 SQ. UNITS. THE AREA OF EACH OF THE SMALL RECTANGLES IS 40 SQ. UNITS.
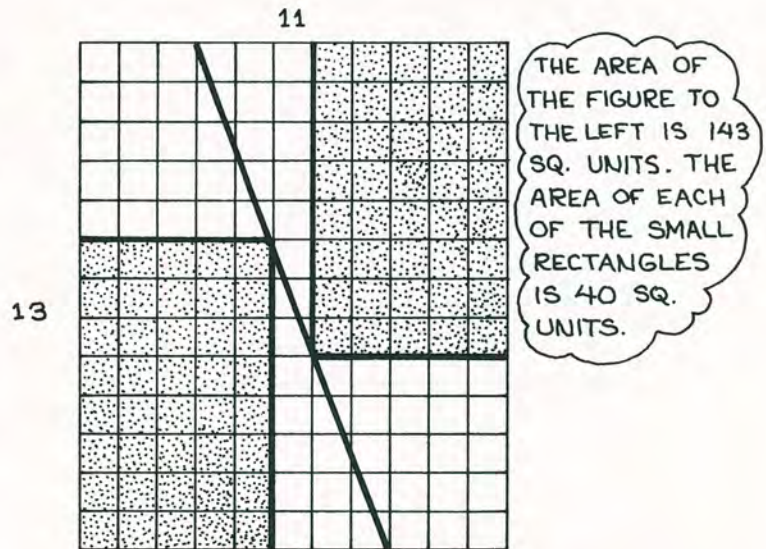
II.
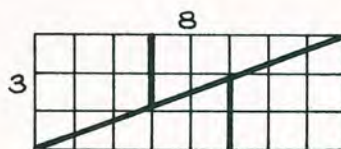
a) Copy the pattern to the left onto centimetre or larger grid paper.

b) The area of the 3 unit x 8 unit rectangle is _____ square units.

c) Cut out all four pieces.

d) Arrange these four shapes into a square. Make a drawing of your result.

e) Either by counting individual squares or using a formula find the area of this square. _____ square units.

f) Is it possible to increase the area just be rearranging the pieces?

SOMETHING FOR NOTHING (PAGE 2)

III.

a) Neatly copy the large triangle to the left onto centimetre or larger grid paper.
Color the back of the large triangle.

b) Find and record the following information about the large triangle.

Length of the base is _____ units.

Length of altitude is _____ units.

Area of the large triangle is _____ square units.

c) Carefully cut along the lines to make six pieces.

d) Arrange your six pieces to form the figure to the right.

e) Discover the following information about the figure.

Area of the hole in the middle is _____ square units.

Area of the shaded part. (The hole in the middle is not part of the shaded part.)_____ square units

f) Do you have the same area you started with? _____

TURN ALL PIECES OVER TO MAKE THIS FIGURE.
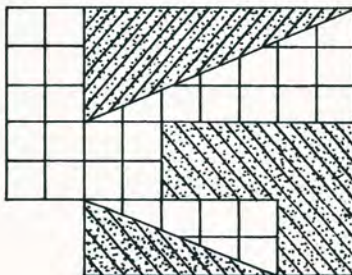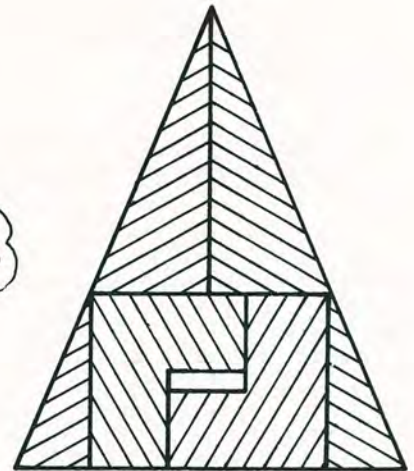
g) Re-arrange the same six pieces to form the figure at the left. Note some pieces are turned over and others are not.

h) Either by using a formula or by counting the individual squares find the area of this figure. _____ square units

IDEA FROM : *Mathematics A Human Endeavor* and "Mathematical Games," by Martin Gardner, *Scientific American*, January, 1958

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SOMETHING FOR NOTHING (PAGE 3)

Most students probably think area should be conserved, but may have a hard time seeing or explaining where the extra square units came from in the problems on the previous pages.

The following development can help explain the apparent conflict.

Numbers generated by the Fibonacci sequence can be used as side lengths of rectangles which work in these conservation of area problems.

> TO GET A NUMBER IN THE FIBONACCI SEQUENCE YOU ADD THE PREVIOUS TWO NUMBERS
>
> FIBONACCI SEQUENCE:
> 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Choose any four consecutive numbers from the Fibonacci sequence, say 2, 3, 5, 8. Arrange them in the following manner.

Cut the four pieces and re-arrange into a square.
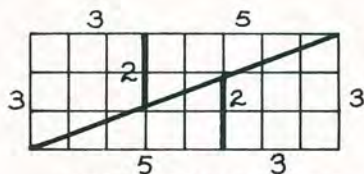
> THIS 3×8 RECTANGLE HAS AN AREA OF 24 SQ. UNITS.
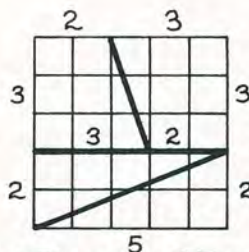
> THIS 5×5 SQUARE HAS AN APPARENT AREA OF 25 SQ. UNITS.

> Note there is a small thin gap between the quadrilaterals and the triangles. These two gaps make up the area for the 1 additional square that seems to come from nothing.

By moving one term to the right in the Fibonacci sequence and using 3, 5, 8, 13 instead of 2, 3, 5, 8 we will find the square having an apparent area one less than the rectangle.

Rearranging the four pieces yields this 8 x 8 square having an apparent area one less, that is, 64 square units.
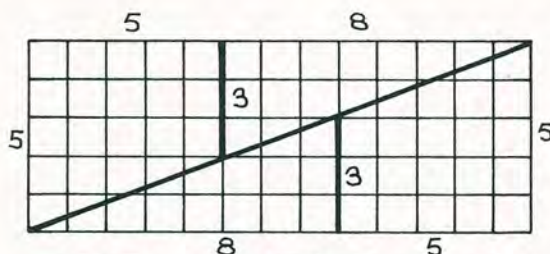
> THIS 5 × 13 RECTANGLE HAS AN AREA OF 65 SQ. UNITS.
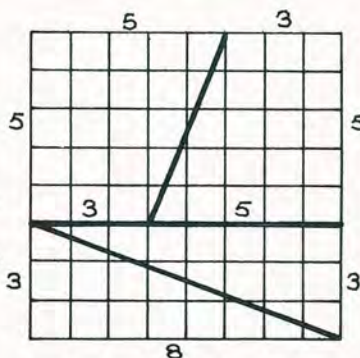
> Note the overlap between the quadrilaterals and triangles, which accounts for the loss of one square unit.

By continuing to move one term to the right to select the four consecutive Fibonacci numbers, an alternating pattern of the square having one more square unit or one less square unit than the rectangle occurs. For example, using 5, 8, 13, 21 will show an apparent gain of one square unit.

THE RECTANGLE METHOD

Materials Needed: Geoboard, rubber bands, dot paper

The smallest square on the geoboard represents the area unit for this activity. ☐

Activity:

1) a) The area of the rectangle is _____ square units.

   b) The area of the triangle is _____ of the area of the rectangle.

   c) The area of the triangle is _____ square units.

   d) Steps a, b and c are called the rectangle method for finding the area of the triangle.
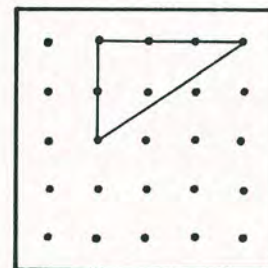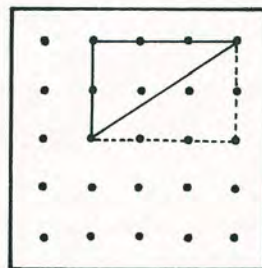
2) Look at these two diagrams.

   a) The area of the square is _____ square units.

   b) The area of the triangle is _____ of the area of the square.

   c) The area of the triangle is _____ square units.
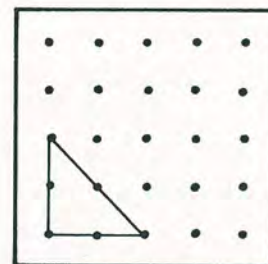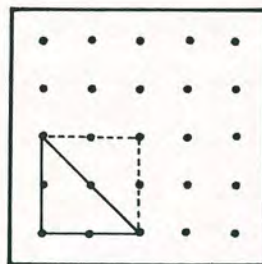
3) Make each of these triangles on your geoboard. Use a rubber band to make a square or rectangle from the triangle. Find the area of each triangle.
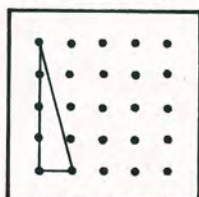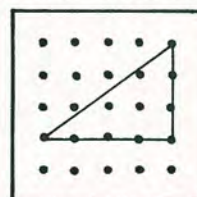
   a) b) c) d)
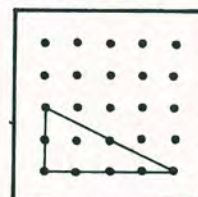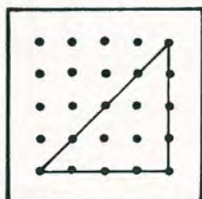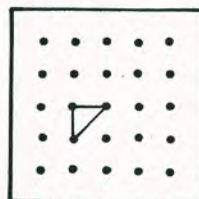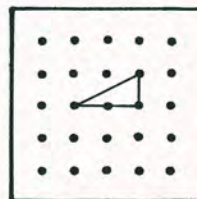
   e) f) g) h)
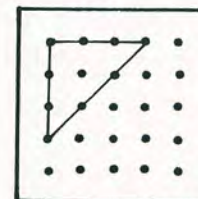
4) Use the rectangle method to find the area of each figure. A hint is given in the first two problems.
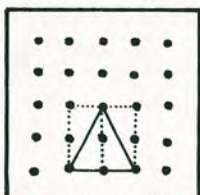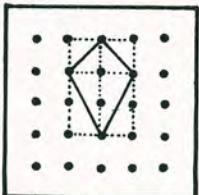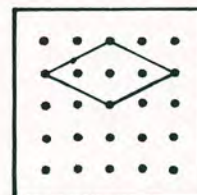
   a) b) c) d)
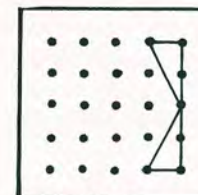
MORE ON THE
RECTANGLE
METHOD

Materials Needed: Geoboard, rubber bands, dot paper

The smallest square on the geoboard represents the area unit for this activity. ☐

Activity:

1) What is the area of triangle BCF?
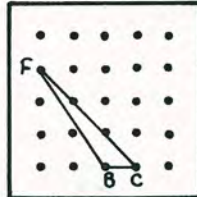
The area of triangle BCF can be found by finding the area of square ACDF and subtracting the areas of triangle CDF and triangle ABF.
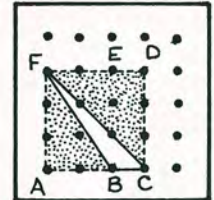
Step 1
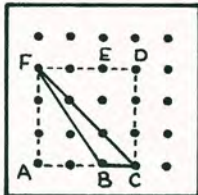
Make square ACDF. The area of square ACDF is _____ square units.

Step 2
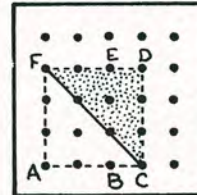
The area of triangle CDF is _____ square units.

Step 3
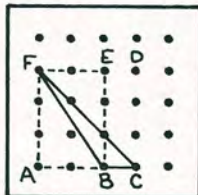
Make rectangle ABEF. The area of rectangle ABEF is _____ square units

Step 4
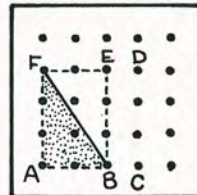
The area of triangle ABF is _____ square units.

The area of triangle BCF is _____ square units.

2) Find the area of each of these figures.

a)

b)

c)

d)

e)

f)

IDEA FROM: *Geoboard Activity Cards*—Intermediate

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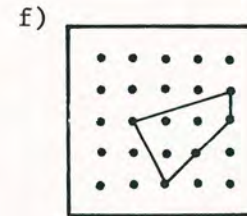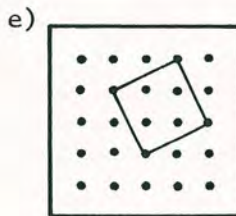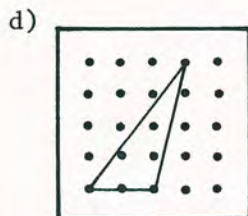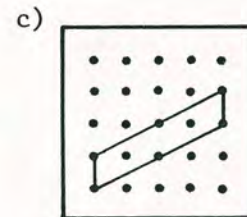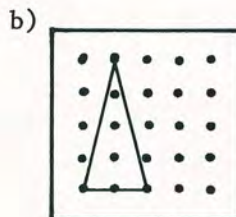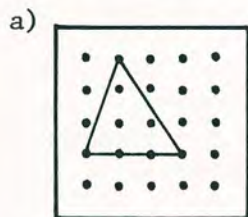
AREA FORMULAS

ON THE **GEOBOARD**

The formulas for the areas of various polygons can be developed using a geoboard. The ideas that follow use **┃** as the unit of length and **□** as the unit of area.

You may want to use these ideas as a teacher directed activity or to develop a series of activity cards so students can discover the formulas.

1) Area of a square

a)
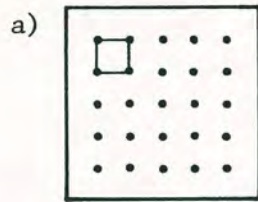
length of side = 1 unit
area = 1 square unit

b)
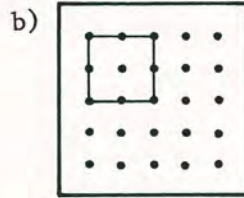
length of side = 2 units
area = 4 square units

c)
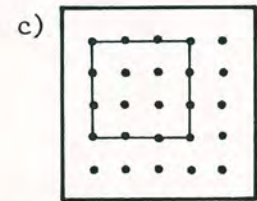
AREA = S × S

length of side = 3 units
area = 9 square units

2) Area of a rectangle

a)
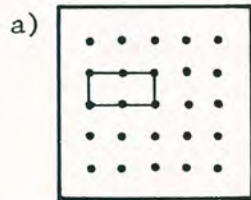
length of sides = 1 unit, 2 units
area = 2 square units

b)
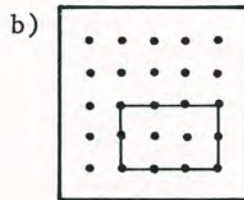
length of sides = 2 units, 3 units
area = 6 square units

c)
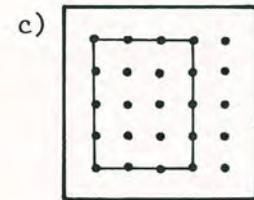
AREA = $\ell \times w$

length of sides = 3 units, 4 units
area = 12 square units

3) Area of right triangle

a)

length of legs = 1 unit, 3 units
area = $1\frac{1}{2}$ square units

b)
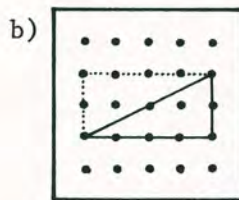
length of legs = 2 units, 4 units
area = 4 square units

c)
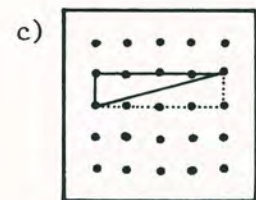
AREA = $\frac{1}{2}$ bh

length of legs = 1 unit, 4 units
area = 2 square units

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AREA FORMULAS

ON THE GEOBOARD

(CONTINUED)

4) Area of a triangle (the height is shown along a dotted line)

a)
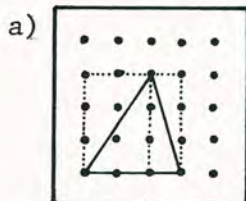
base = 3 units
height = 3 units
area = $4\frac{1}{2}$ square units

b)
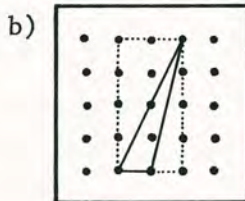
base = 1 unit
height = 4 units
area = 2 square units

c)
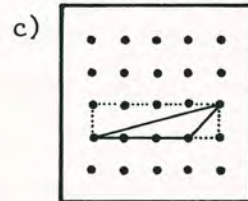
base = 3 units
height = 1 unit
area = $1\frac{1}{2}$ square units

AREA = $\frac{1}{2}$ bh

5) Area of a parallelogram (the height is shown along a dotted line)

a)
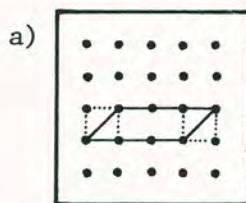
base = 3 units
height = 1 unit
area = 3 square units

b)
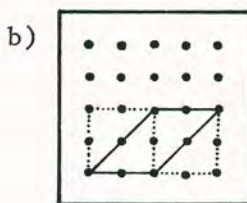
base = 2 units
height = 2 units
area = 4 square units

c)
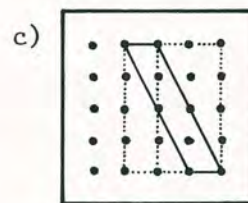
base = 1 unit
height = 4 units
area = 4 square units

AREA = bh

6) Area of a trapezoid (the height is shown along a dotted line)
   Have students make a "rotated" copy of the given trapezoid.

a)
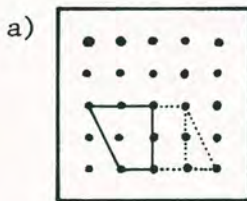
combined bases = 3 units
height = 2 units
area = 3 square units

b)
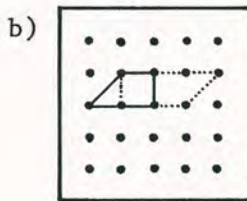
combined bases = 3 units
height = 1 unit
area = $1\frac{1}{2}$ square units

c)
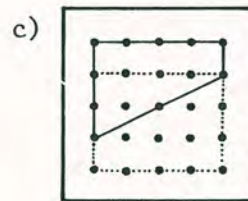
combined bases = 4 units
height = 4 units
area = 8 square units

AREA = $\frac{1}{2}$ h $(b_1 + b_2)$

Students may need some preliminary work on terminology such as legs, base or height.

BE PICKY ABOUT YOUR GEOBOARD

On a geoboard the area of a figure with its vertices located at nails can be determined by Pick's formula (shown to the right). B represents the number of nails on the boundary of a figure and I represents the number of nails in the interior of the figure. You will probably want to try the formula with several simple figures.

> PICK'S FORMULA
>
> $$AREA = \frac{B}{2} - 1 + I$$
>
> $B = 4, I = 1,$
> AREA = 2 SQ. UNITS

By using carefully designed activities (see pages 3 and 4 of this lesson) it is possible for students to discover Pick's formula. Lessons can be found in almost any material that includes the geoboard. See especially the May, 1974, issue of The Mathematics Teacher and Math Workshop, Level F, a textbook published by Encyclopedia Britannica Press, Inc.

As shown on page 3 one approach is to first have students discover that all 3-nail figures, 4-nail figures, etc., with no interior nails, have constant areas.

NAILS ON BOUNDRY	3	4	5	6	7	8	9	10
AREA	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

You may want to leave the discovery completely open, guide the students by providing hints on a chart (see #6 on page 3), or give the formula and have students use several figures to verify the formula.

On page 4 students now investigate the effect of including interior nails in the figure. The chart below shows the results of making 3-nail figures with varying numbers of interior nails.

NAILS IN INTERIOR	0	1	2	3	4	5
AREA	$\frac{1}{2}$	$1\frac{1}{2}$	$2\frac{1}{2}$	$3\frac{1}{2}$	$4\frac{1}{2}$	$5\frac{1}{2}$

For 4 nails on the boundary the chart looks like this.

NAILS IN INTERIOR	0	1	2	3	4	5
AREA	1	2	3	4	5	6

BE PICKY ABOUT YOUR GEOBOARD

(PAGE 2)

Using the formula from the previous activity, students should see that the area of a figure is increased by the number of nails in the interior.

Given figures with holes in the interior students can find the area by subtracting the area of the hole from the area of the outside figure.

Another method for determining the area of a figure with a hole(s) in it is given by the formula $\frac{B}{2} + I + (H - 1)$ where B represents the nails on the boundary of the figure and the boundary of the hole(s). H represents the number of holes. I represents the nails that are inside the figure but not inside a hole. For the example shown, B = 23, I = 1 (since there is only 1 nail that is not on a boundary or inside a hole) and H = 2. The area is $13\frac{1}{2}$ units.

All of the above relationships also hold for a matrix of dots of any size.

IDEA FROM: *Math Workshop,* Level F, and *Mathematics on the Geoboard*
Permission to use granted by the Cuisenaire Company of America, Inc.
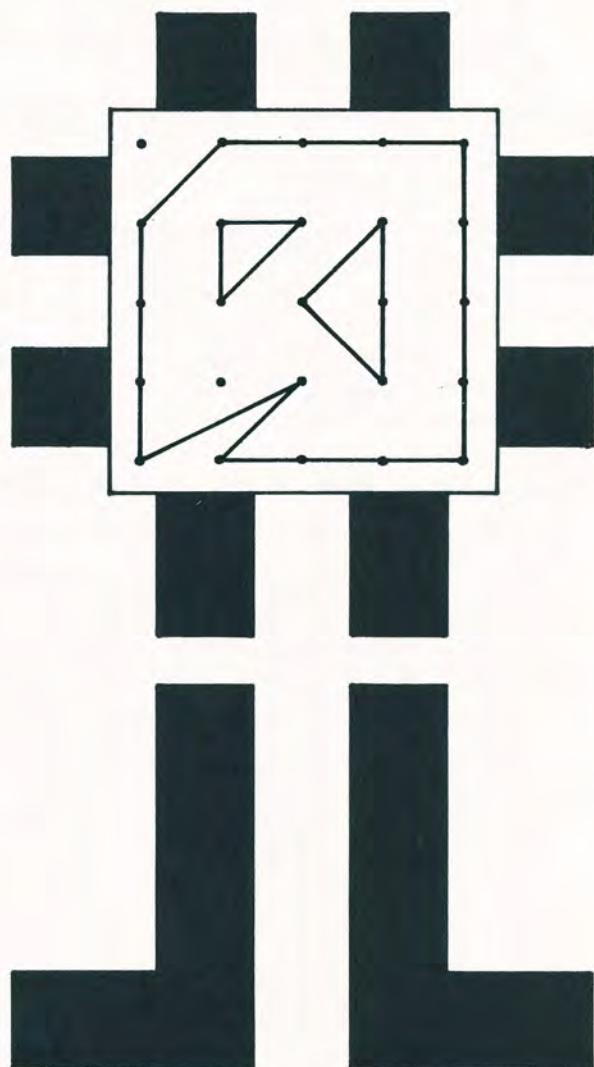
BE PICKY ABOUT YOUR GEOBOARD

(PAGE 3)

Materials Needed: Geoboard, rubber bands, dot paper

Activity:

1) Make these figures on your geoboard. Each is made by enclosing 3 nails with no nails in the interior. Find the area of each figure and record in the table below.

a) b) c) d)
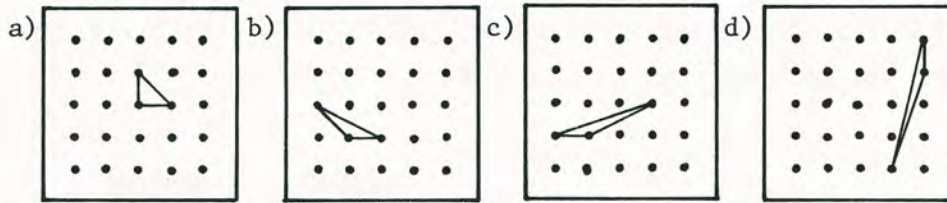
2) Make these figures on your geoboard. Each is made with 4 nails on the boundary and no nails in the interior. Find the area of each and record in the table.

a) b) c) d)
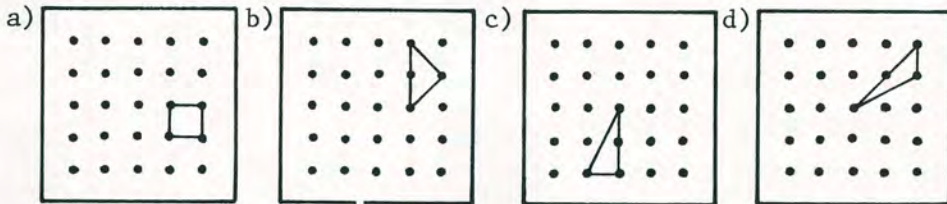
e) Use your dot paper to find 3 more figures like these. What is the area of each figure? _____

_____

3) Make these figures on your geoboard. Each has 5 nails on the boundary and no nails in the interior. Find the area of each and record in the table.
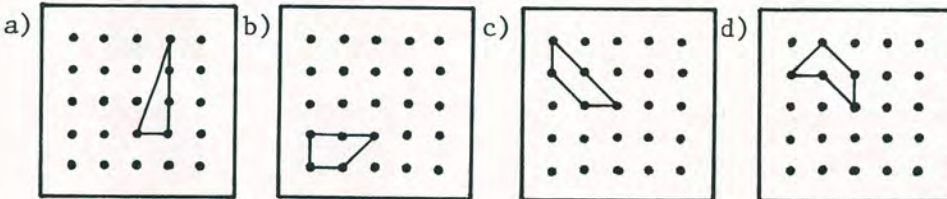
a) b) c) d)

e) Use your dot paper to find 3 more figures like these. What is the area of each figure? _____

_____

4)

AREA IN SQUARE UNITS

		a	b	c	d
NAILS IN BOUNDARY	3				
	4				
	5				

5) Predict the area for 6 nail, 7 nail, and 8 nail boundaries with no nails in the interior. Check your prediction on your geoboard.

BE PICKY ABOUT YOUR GEOBOARD

(PAGE 4)

Materials Needed: Geoboard, rubber bands, dot paper

Activity:

1) Make these figures on your geoboard. Find the area of each and record in the table below.

    a) a figure with 3 boundary nails and 1 nail inside

    b) a figure with 3 boundary nails and 2 interior nails

    c) a figure with 3 boundary nails and 3 interior nails (See the diagram to the right.)
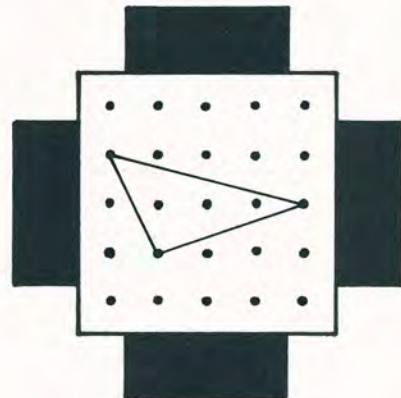
2) Make these figures on your geoboard. Find the area of each and record in the table.

    a) a figure with 4 boundary nails and 1 nail inside

    b) a figure with 4 boundary nails and 2 interior nails

    c) a figure with 4 boundary nails and 3 interior nails

3) Make on your geoboard and record on dot paper. Find the area of each and record.

    a) a figure with 5 boundary nails and 1 interior nail

    b) a figure with 5 boundary nails and 2 interior nails

    c) a figure with 5 boundary nails and 3 interior nails

4) Can you find a pattern to complete this chart?

INTERIOR NAILS

BOUNDARY NAILS	0	1	2	3	4
3	$\frac{1}{2}$				
4	1				
5	$1\frac{1}{2}$				
6	2				
7	$2\frac{1}{2}$				
8	3				

5) How do the interior nails change the formula you found on page 3?

_____

_____

_____

DOES THIS RING A BELL

| 64 SQ. UNITS | 81 SQ. UNITS |
| H | |

N

D

S

E

O

G

F
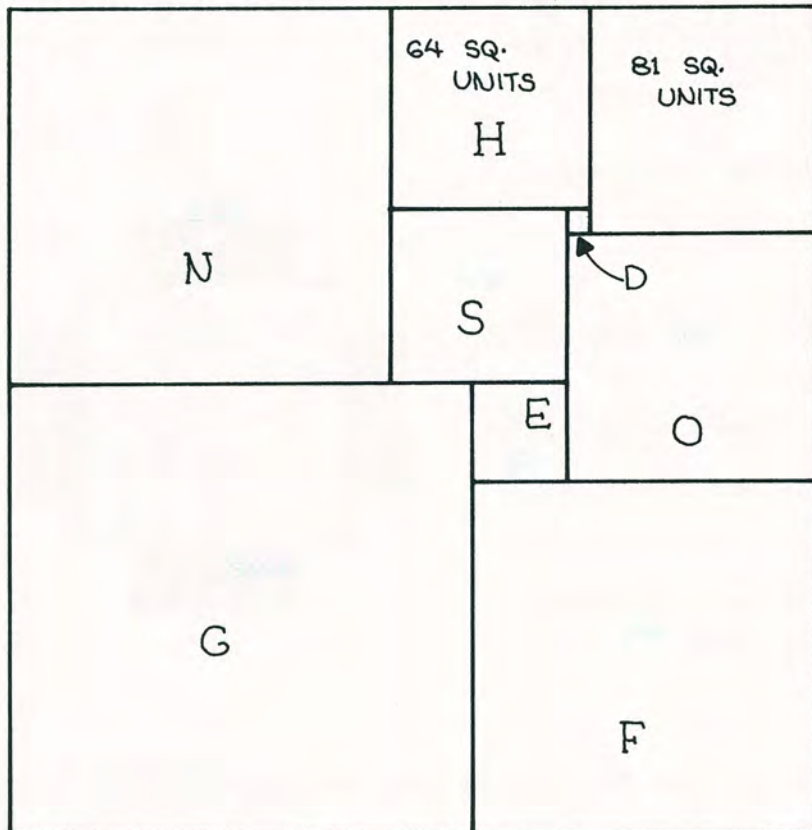
WHAT DID ALEXANDER GRAHAM BELL'S TEACHER SAY ABOUT HIM?

Use the two given areas in each figure to find the areas of the remaining squares. (No measuring is necessary.) Find your answer in the code below and put the letter in the blank above the correct answer.

$$\frac{H}{64} \quad \frac{}{16} \qquad \frac{}{4} \quad \frac{}{25} \quad \frac{}{14} \qquad \frac{}{100} \quad \frac{}{784} \quad \frac{}{625}$$

$$\frac{}{196} \quad \frac{}{256} \quad \frac{}{625} \quad \frac{}{49} \quad \frac{}{1296}$$

$$\frac{}{1} \quad \frac{}{256} \quad \frac{}{225} \quad \frac{}{324} \quad \overline{-} \quad \frac{}{25} \quad \frac{}{1089} \quad \frac{}{256} \quad \frac{}{225} \quad \frac{}{324}.$$
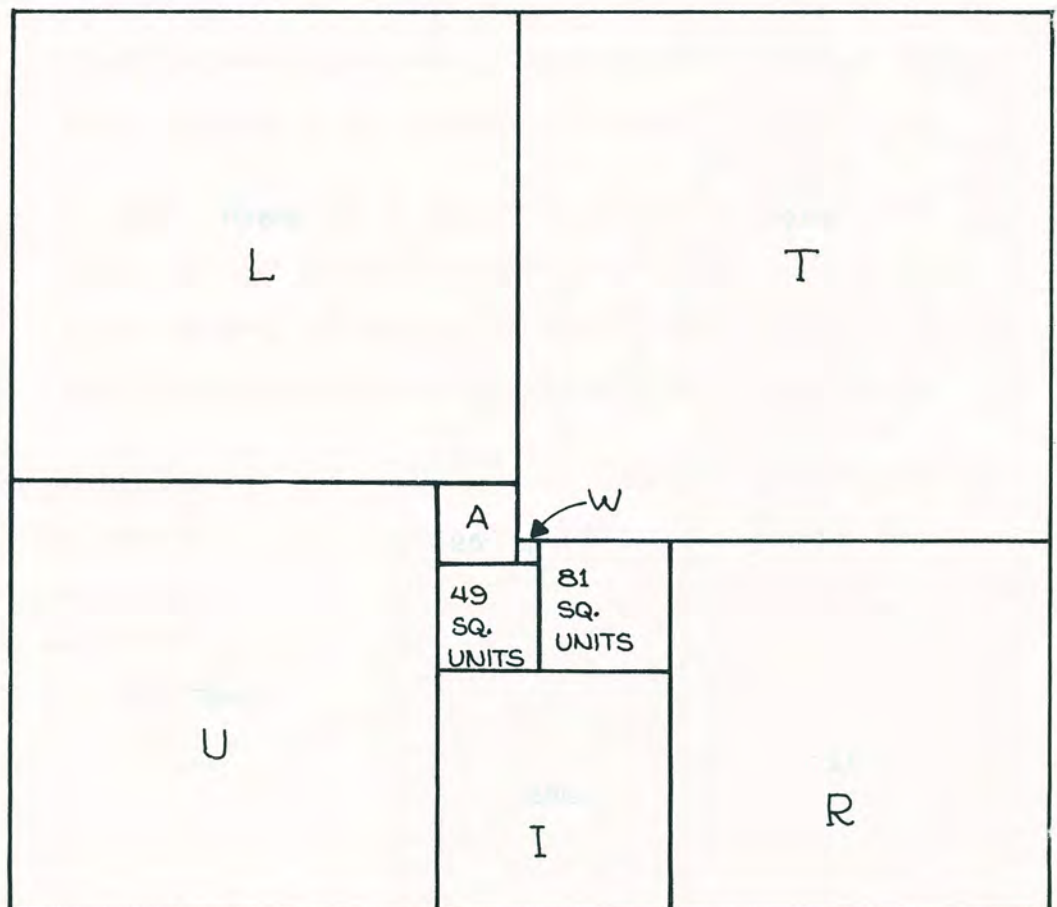
L

T

A

W

| 49 SQ. UNITS | 81 SQ. UNITS |

U

I

R

IDEA FROM: "Mathematical Games," by Martin Gardner, *Scientific American,* November, 1958

THE WIDE OPEN SPACES

1) Pete Piajay has six pictures, each with dimensions of 10 cm x 20 cm, that he wants to arrange in a 60 cm x 75 cm poster. Pete thinks arrangement (b) leaves the most area uncovered. Is he correct? _____ Explain. _____
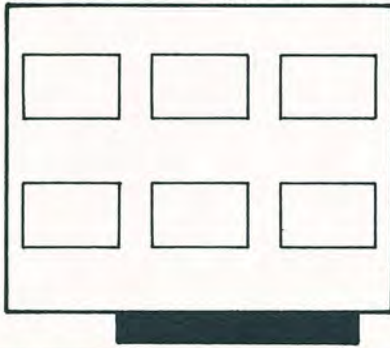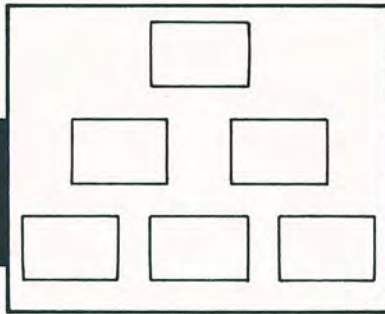
a)   b)   c)
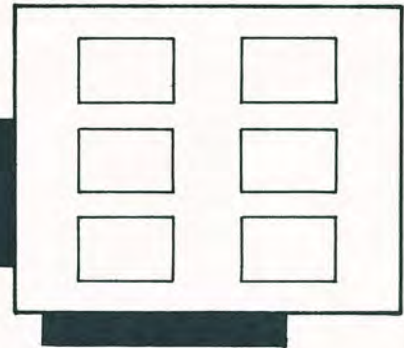
2) You may want to use grid paper to make a model for this.

a) You have a 17 x 21 picture mat and six 3 x 4 pictures

b) Place the pictures on the mat so the distance between the pictures and the distances from the edges of the mat are the same.

c) What is the area of the mat that is not covered by the pictures?
_____

d) What happens to the area of the uncovered surface if all six pictures are placed side by side?
_____
_____

e) If a border 1 unit wide is placed around the inside edge of the picture mat, how much of the mat will not be covered by the pictures or the border? _____
_____
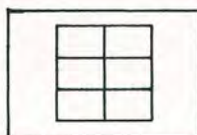
3) You may want to use grid paper to make a model for this.

a) You have a 20 x 25 picture mat and want to arrange five pictures on it.

b) The sizes of the pictures are 8 x 10, 6 x 5, 9 x 12, 10 x 15 and 5 x 20.

c) Could all five pictures be placed on the picture mat? _____

d) How much of the picture mat would be uncovered? _____

e) Use grid paper to show an arrangement of the five pictures.

f) Can an arrangement be made by placing the 6 x 5 picture in the exact center of the picture mat? _____

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2×2 = 4

Materials Needed: Metric ruler, tracing paper, scissors, grid paper

Activity:

1) Measure the sides of each figure below. Record the measurements on another piece of paper.
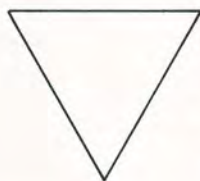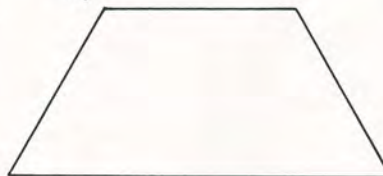
a)
b)
c)
d)
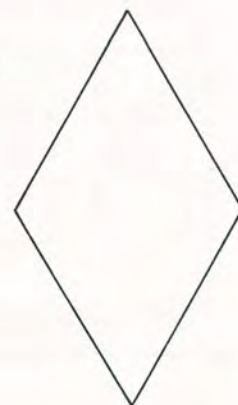
2) Use the tracing paper to make four exact copies of each figure. Arrange the four exact copies to make a larger similar figure that has the same shape as the ones above.

   a) How does the <u>length</u> of the sides of each larger figure compare to the length of the sides of the original figure? _____

   b) How does the <u>area</u> of each larger figure compare to the area of one of the original figures? _____

   c) Write a rule about what happens to the area when the lengths of the sides are doubled. _____

   d) Check your rule by drawing a rectangle. Make and cut out four exact copies and arrange them as you did for the figures above.

3) a) Draw a rectangle on the grid paper.

   b) Draw a similar rectangle whose length and width are twice as long as the rectangle in (a).

   c) Show that the area of the large rectangle is 4 times the area of the small rectangle.

   d) Repeat steps (a)-(c) three more times choosing a different rectangle each time.

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FLYING HIGH

THE WORLD IS FLAT

CLASS OF 1491

1) Using Cuisenaire rods build a frame around the picture to the left. How many white rods does it take to cover this picture? _____ How many white rods does it take to completely cover the top of the picture frame? _____

2) What is the area of the picture frame? _____

3) Make a frame around this picture that is two rods wide. Guess: will the area of the new frame be twice as great as the area of the single rod frame? _____

4) Find the area of the frame that is two rods wide. _____ Was your guess correct? _____

5) Continue to increase the width of the frame by one and complete the following table.

WIDTH OF THE FRAME	1 ROD	2 RODS	3 RODS	4 RODS	5 RODS
AREA OF THE FRAME					
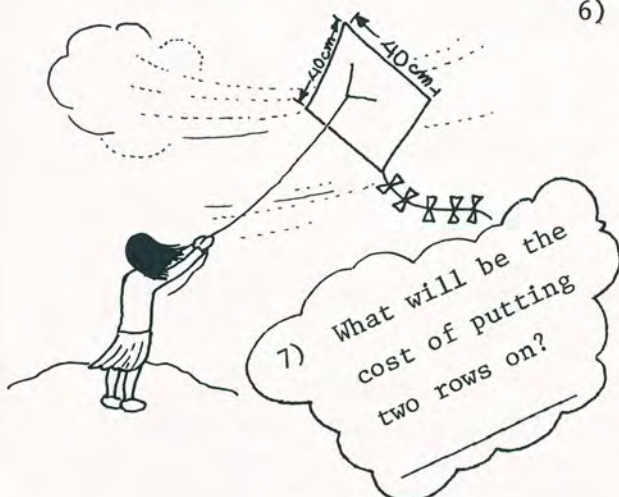
6) Teri wanted to decorate her kite. For 2¢ each she can buy 1 cm square decals to paste on. What will it cost her to paste one row around the edge of the kite? _____

7) What will be the cost of putting two rows on? _____

8) Teri learned the decals were designed by an astronaut and made her kite fly higher. How many decals does she need to cover her kite? _____ What will be the cost? _____

Big Foot

— String
Centimetre graph paper
Metric ruler

"Big Willie" has huge feet. He was curious as to just how big they actually were. He figured he might be able to find the area of his foot by using string.

- First, Willie cut a piece of string the same length as the perimeter (distance around) of his foot.
- Next, he used that length of string to make a square.
- Finally, he found the area enclosed by the square.

1. Do you think that Willie's method is reasonably accurate?
   a. Use your own foot. Follow the same steps that Willie used. Record your answer.
   b. Now, use squared paper. Trace around your foot and find its approximate area by counting squares.
   c. How do your two results compare?

2. Use the same length of string as you used in Exercise 1.
   a. Make a rectangle with a width of 2 cm. Find the length and the area of this rectangle.
   b. Copy the following chart. Complete the chart by making other rectangles with widths as indicated. Remember to use the same length of string as before.

Width	2	4	6	8	10	12	14
Length	?	?	?	?	?	?	?
Area	?	?	?	?	?	?	?

   c. Draw a graph which shows a comparison between width and area. Use the "width" and "area" numbers from your table.
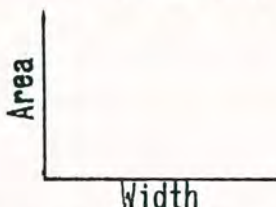
   d. Which rectangle seems to give the greatest area? What do you think of "Big Willie's" method?

EXTENSION: You have conducted an experiment with a fixed perimeter. Suppose, instead, that the area is fixed at 96 cm^2. (This means 96 square centimetres). Will the perimeter always be the same? Use squared paper to show your results.

A SHEEPISH PROBLEM

Materials Needed: Centimetre grid paper, 72 cm of string

I   A four-sided pen.

George Mutton, a sheep rancher, needs to fence in his newly born lambs
to protect them from coyotes at night. When he tries
to buy fencing he can only buy seventy-two metres.
He will use the fence to build a rectangular pen.
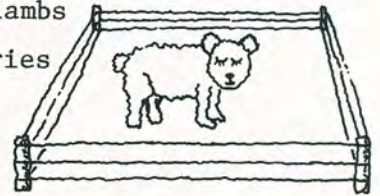
Let 1 cm of string represent 1 m of
fence. Using the 72 cm of string
as a fence and the grid paper as
Mr. Mutton's ranch, plan at
least 15 differently shaped
pens. Record the length of
both sides in this table.
You must use all 72 cm of
string each time.

IT HELPS TO ORGANIZE THIS FIRST COLUMN; SMALLEST AT THE TOP GETTING LARGER.

1st SIDE (METRES)	2nd SIDE (METRES)	AREA (SQ. METRES)
1	35	35
2		
4		
7		
35		

What are the dimensions of the
pen with the largest area? _____
_____   What is its area?
_____ m^2

II   A three-sided pen.

Mr. Mutton decides he can make
better use of his 72 metres of
fence if he uses a cliff wall
as one side of his pen. Now he
has to build 3 sides, the fourth
being the cliff. He still wants
to make a rectangular pen.

Make a chart similar to the one above.
What are the dimensions of the pen giving
the largest area? _____
What is the area? _____ m^2

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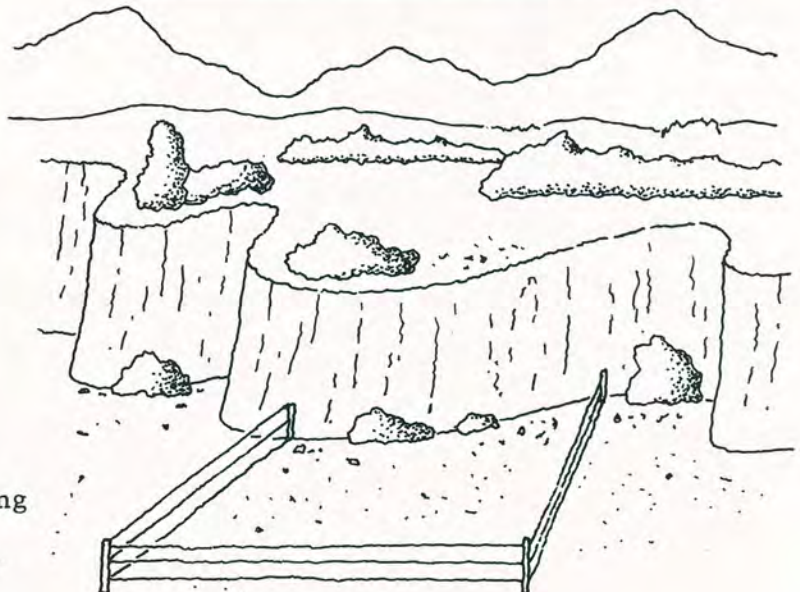
A
SHEEPISH
PROBLEM (CONTINUED)

There are many extensions with this problem.

I The solution to these problems may be found by using a graph.

Plot the length of one side on the horizontal axis and the area on the vertical axis.

MAXIMUM AREA WILL BE THE PEAK OF THE GRAPH.

GRAPH OF 4-SIDED PEN PROBLEM

AREA VS. LENGTH OF ONE SIDE

AREA

LENGTH OF ONE SIDE
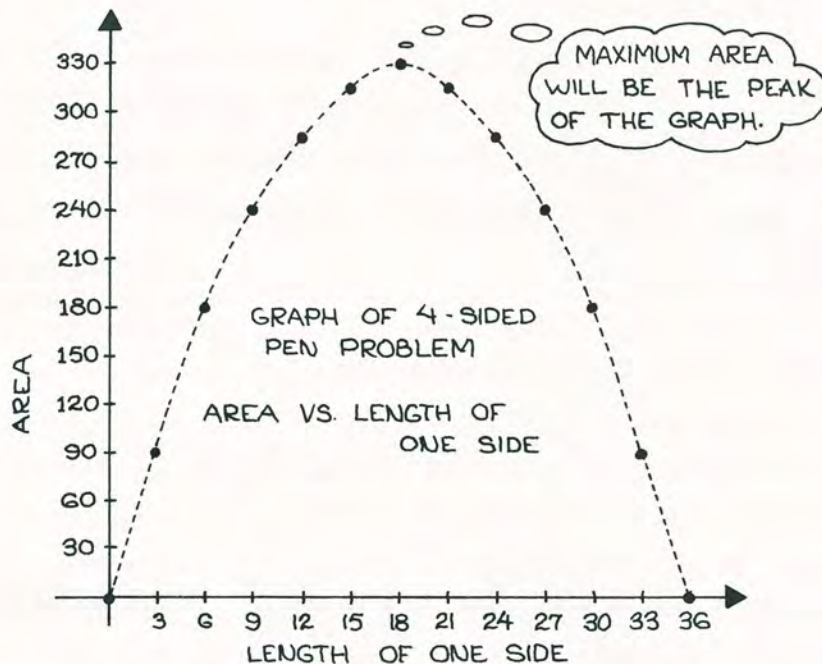
II Many sided pens.

Dropping the restriction that the pen must be a rectangle puts a nice twist into the problem. Students can then investigate what happens as more and more sides are added while retaining a constant perimeter of 72 metres.

What shape gives you a maximum area if you use only the 72 metres of fence?

What is the area of this shape?

III Special Investigation.

By using natural barriers as part of a wall, all of a wall or several walls, the problems can provide a challenge for better students of the class.

What would be the shape and area of the pen which yields maximum area using a canyon as one side of the pen?

PIE ARE ROUND, CORNBREAD ARE SQUARE

I   Cut a large circular disk from tagboard. A radius of 20 cm is convenient for a bulletin board display; a radius of 5 cm is handier for an overhead demonstration.

CIRCUMFERENCE EQUALS 2 × π × RADIUS
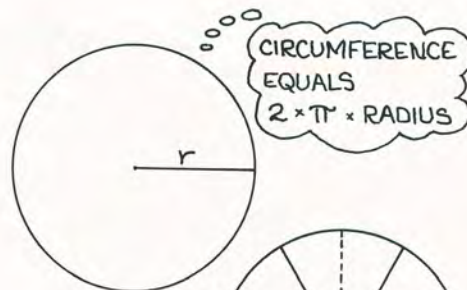
II Draw line segments to divide the disk into 12 congruent sectors. This can be done by drawing 30° central angles or by inscribing a regular 12-gon as described in *Inside the Circle I* in the Circles subsection.
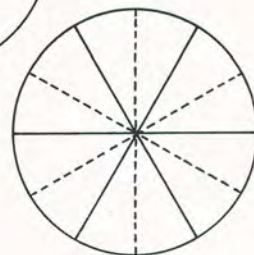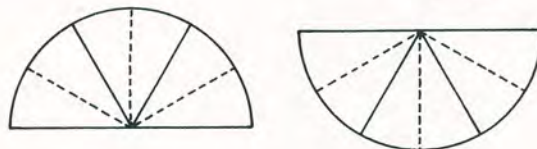
III Cut along one diameter to make two semicircular regions.

IV Make cuts along the line segments drawn in II, but be careful not to cut through to the original circle.
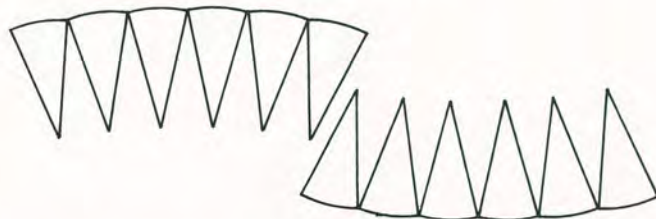
V   Fan out each semicircular region and fit them together as shown. This figure is very much like a parallelogram.

VI The area of a parallelogram is base times height.

VII The area of the "parallelogram" to the right is

$$AREA = b \times h$$

BASE × HEIGHT

$$\tfrac{1}{2} \times CIRCUMFERENCE \times RADIUS =$$

$$\tfrac{1}{2} \times 2 \times \pi \times RADIUS \times RADIUS =$$

$$\pi \times RADIUS \times RADIUS =$$

$$\pi \times r^2$$

THE HEIGHT IS THE RADIUS OF THE ORIGINAL CIRCLE.
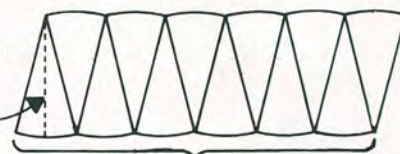
THE BASE IS ONE-HALF OF THE CIRCUMFERENCE OF THE CIRCLE.
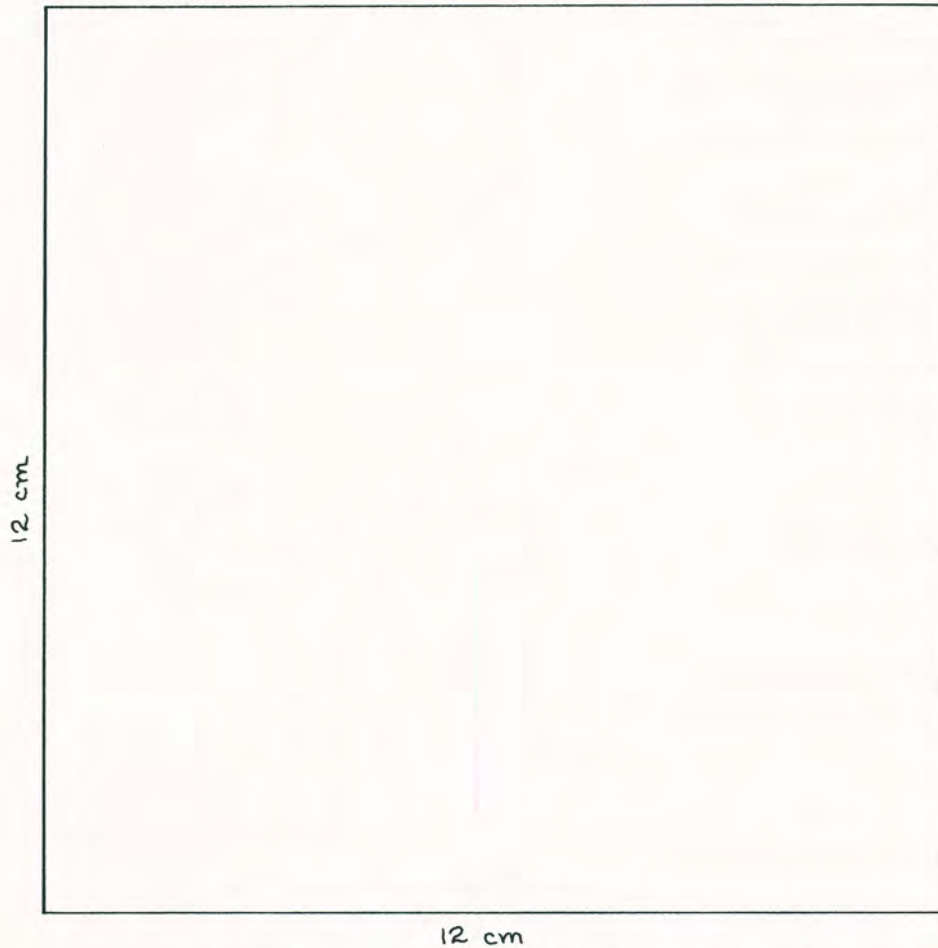
Inscribe a circle in the 12 cm x 12 cm square to the left.

TO INSCRIBE THE CIRCLE MEANS IT MUST TOUCH EACH SIDE OF THE SQUARE ONLY ONCE.

Record the following data about the circle.

A) Diameter = _____ cm

B) Radius = _____ cm

C) Area = _____ cm^2

12 cm

12 cm

Use horizontal and vertical lines to divide the large square into smaller squares. In each of the small squares a circle will be inscribed.

Use a calculator to help you complete this table.

	VERTICAL LINES	HORIZONTAL LINES	NUMBER OF CONGRUENT SQUARES	AREA OF ONE INSCRIBED CIRCLE	TOTAL AREA OF ALL CIRCLES
	1	1	4	$3^2\pi$	36π
	2	2	9	$2^2\pi$	36π
	3	3	16	$\left(\frac{3}{2}\right)^2\pi$	36π
	4	4	25	$\left(\frac{6}{5}\right)^2\pi$	36π
	5	5	36	$1^2\pi$	36π
	.	.			
	100	100	101^2	$\left(\frac{6}{101}\right)^2\pi$	36π

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Pizza Parlor

GET - Cardboard pizzas
      Pizza menu

1. Study the menu.
   a. Which kind of pizza would you like to order? How much does each size of this pizza cost?
   b. Look at the prices of some other pizzas. When you buy larger sizes, do their prices seem to increase in a reasonable way? Explain.

2. Find the most expensive pizza on the menu. Which pizza size do you think gives you the most for your money?
   a. Think of a way to solve this problem. Then talk with your teacher about your plan.
   b. Solve the problem. Show your steps so that they can be understood.

3. Find the least expensive pizza on the menu. Determine which pizza size is cheapest to buy.

4. A new pizza parlor is offering a Super-Duper-Giant pizza. What would be a reasonable price for the most expensive kind of pizza?

EXTENSION: Would the Super-Duper-Giant pizza be large enough for all the people in your class? What size pizza would you need to feed the entire school?

AREA PROBLEMS
TO ATTACK

①

The large square has an
area equal to the sum of
the areas of the five
smaller squares. The
length of the side of
the large square must be:

a) between 16 and 17 units
b) between 17 and 18 units
c) between 18 and 19 units
d) between 19 and 20 units
e) none of these

8

9   5

11   6
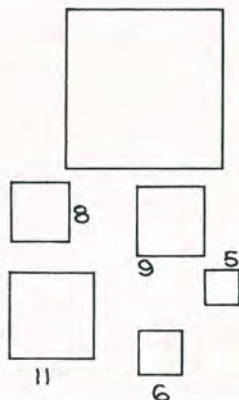
②

A man had a square window
one metre on a side that
let in too much light.
He blocked off half of
its area and still had a
square window which was
a metre high and a metre
wide.

How did he do this?

③ The figure to the right
contains a series of
squares. Each inside
square is formed by
connecting the mid points
of the next larger square.
Square #1 is the largest. A side of
square #3 measures 4 cm. Find the
area of square #8.

④

There are only two rectangles whose
dimensions are whole numbers and
whose area and perimeter are the
same number.

Can you find both?

⑤

Through a point on the
diagonal of a parallelo-
gram two segments are
drawn parallel to the
sides. What can you say
about the areas of the
two shaded parallelograms?
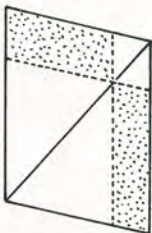
⑥

A 1 kilogram bag of grass
seed will seed a square
plot of land 10 metres on
a side. Will two 1-
kilogram bags of seed be
enough to seed a square
plot of ground 20 metres
on a side?

⑦

Cut the circular
region
into
2 parts
that have
the same
area but
are not
congruent.
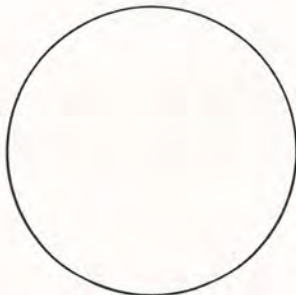
⑧

A square, a rectangle (not a square),
a parallelogram (not a rectangle) and
a circle each enclose regions measur-
ing 100 square centimetres. Arranged
in order of least to greatest
perimeter, they are

a) square, rectangle, parallelogram,
    circle
b) circle, rectangle, parallelogram,
    square
c) parallelogram, circle, rectangle,
    square
d) circle, square, rectangle,
    parallelogram
e) all perimeters are the same

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IT ALL "ADS" UP

Materials: Newspaper

1) Look for examples of large, medium and small size commercial advertisements in your newspaper.
   Select five of the most interesting. Find and record the width and length of each ad to the nearest half centimetre.

AD	WIDTH	LENGTH
# 1		
# 2		
# 3		
# 4		
# 5		

SAVE $2.00

10-gal. All-Glass Aquarium

5⁹⁹

Regular $7.99

IN YOUR NEWSPAPER YOU SHOULD FIND MANY ADVERTISEMENTS LIKE THIS.

2) Find the perimeter and area of each ad.

AD	PERIMETER (cm)	AREA (cm²)
# 1		
# 2		
# 3		
# 4		
# 5		

3) If you were to sell your five advertisements for $ .50 per square centimetre, how much money would you receive? _____

4) Pick a subsection of the newspaper and count the number of commercial ads. _____

5) Use the areas of the five ads in question #2 to help you approximate the total amount of advertising space in the subsection. _____

6) Figure out how much money you could make if you sold all the advertisements in the subsection for $ .50 per square centimetre. _____

YOUR OWN SQUARE METRE

Materials Needed: Metric measuring tape or metre stick, almanac, masking tape

Activity:

1) a) Find the area of your classroom floor in square metres.

    b) If the floor space is divided evenly, how many square metres are there
        for each person in your class (include your teacher)?

    c) Use chalk or tape and show on the floor the amount of floor space for
        each person.

2) Population density is the average number of people per square mile of land area.

    a) Find the population and area of your city and find the population density.

    b) Find the number of <u>square feet</u> for each person.

3) a) Use the almanac to find the population density of your state.

    b) Choose a state that you think has a very low population density. Is the
        population density higher or lower than your state?

    c) Find the population density of the United States, not including Hawaii and
        Alaska.

    d) If Hawaii and Alaska are included, what do you think will happen to the
        population density of the United States? Find the population density to
        check your guess.

4) a) Find the population density of China and India.

    b) Choose five other countries and find their population densities. Are the
        population densities higher or lower than those of the United States, China
        and India?

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

CONTENTS

PYTHAGOREAN THEOREM

For any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

          —The Pythagorean Theorem

$$c^2 = a^2 + b^2$$
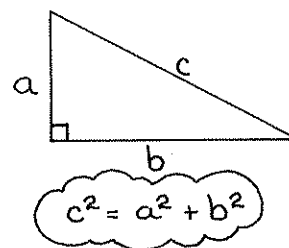
SOME HISTORY

The elaborate temples and pyramids of ancient people required exact planning. How did the Egyptians lay out the bases for pyramids? How did they keep things "square"? One theory which is not documented is that they used a rope with evenly spaced knots to form a right triangle with sides of length 3, 4 and 5. This right triangle could have been used to ensure that the base angles of a square pyramid were right angles. We do not know if the Egyptians actually used such a triangle or not, but there is documented evidence that a thousand years before Pythagoras the Babylonians were familiar with the relationship we call the Pythagorean theorem. A Babylonian clay tablet even lists some Pythagorean triples (sets of whole numbers a, b and c which satisfy $c^2 = a^2 + b^2$) and numbers which can be used to generate the triples as described later in this commentary and in the page *Pythagorean Triples*.
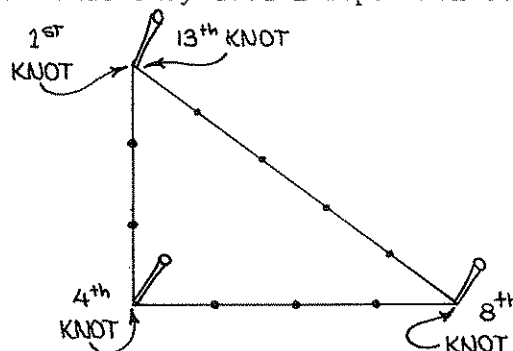
Pythagoras is usually credited with having discovered a general proof of the theorem. His proof might have been based on the following geometric relationship. Four copies of a right triangle are arranged inside a square with edge a + b in two different ways as shown to the right. The area of the region left uncovered must be the same, so $c^2 = a^2 + b^2$.
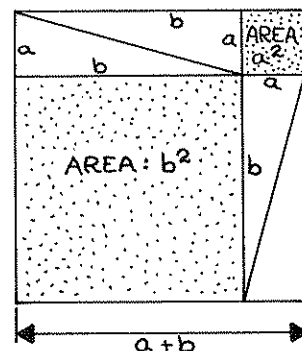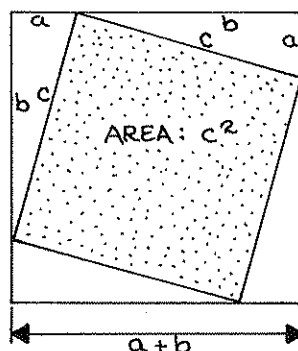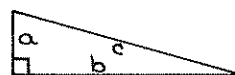
Since the time of Pythagoras, many new proofs or illustrations of the Pythagorean theorem have been discovered. E. S. Loomis has collected and classified 370 of these in his book <u>The Pythagorean Proposition.</u> Several different ways of illustrating this important theorem are given in the classroom pages of this subsection.

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DISCOVERING THE PYTHAGOREAN THEOREM

Students can explore this relation among the sides of a right triangle in several different ways. *Pythagoras Cubed* suggests that students use colored cubes, multibase blocks or Cuisenaire rods to fill in squares on the legs and hypotenuse of a right triangle. In the case of colored cubes, students can record the number of cubes used to fill in each square and look for a pattern. Perhaps they can discover that the number of cubes needed for the square on the hypotenuse is equal to the total number of cubes needed for the squares on the legs. You might prefer to have students first fill in only the large square, then use the same pieces to cover the two smaller squares. If your students like tangrams, geoboards or paper cutting, you might try some of the ideas in *Pythagoras and Tangrams*, *Pythagoras on the Geoboard* or *Cutting and Covering*.

One teacher found that a prepared discovery lesson led to a surprising result for everyone.[*] A worksheet with right triangles was prepared. Students were to measure the sides of the triangles, record the lengths in a table and search for a pattern. The completed table is shown to the right. Each set of three numbers is a Pythagorean triple and the teacher hoped students might discover $a^2 + b^2 = c^2$. The students saw a different pattern. One student declared that $a^2 = b + c$. After examining the table, it was seen to be true for the first four entries. Another student declared $a^2 = 2(b + c)$ for the next four
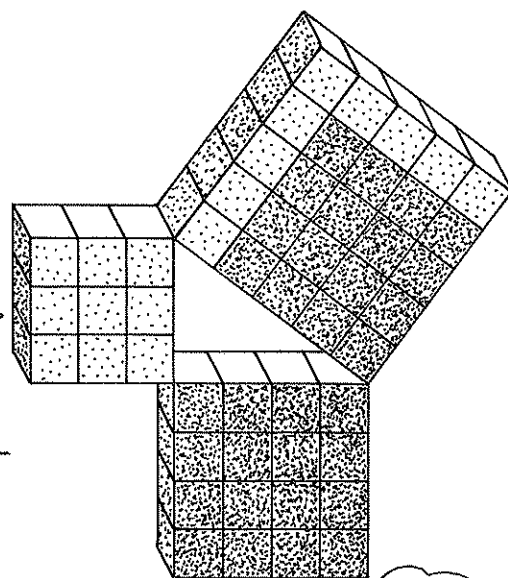
TRIANGLE	a	b	c	a^2	b^2	c^2
A	7	24	25			
B	3	4	5			
C	5	12	13			
D	9	40	41			
E	6	8	10			
F	8	15	17			
G	10	24	26			
H	12	35	37			
I	15	36	39			
J	9	12	15			
K	20	21	29			
L						
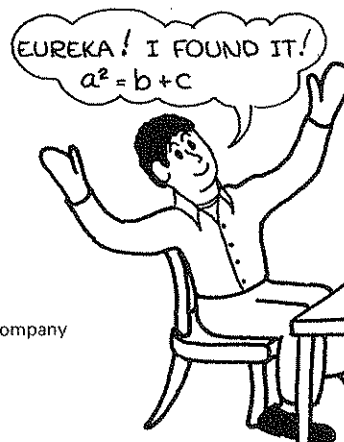
A CALCULATOR HELPS WITH THESE!

EUREKA! I FOUND IT!
$a^2 = b + c$

entries. Eventually, it was decided that $a^2 = b + c$ if b and c differ by 1, $a^2 = 2(b + c)$ if b and c differ by 2, $a^2 = 3(b + c)$ if b and c differ by 3 and so on. These were certainly accurate observations made by the students--but not necessarily the predefined pattern hoped for by the teacher. The question of why the student discovery works can be satisfied with a bit of algebra.

$$a^2 + b^2 = c^2 \text{ can be written } a^2 = c^2 - b^2$$
$$\text{or } a^2 = (c - b)(c + b)$$
$$\text{if } c - b \text{ is 1, we have } a^2 = c + b$$
$$\text{if } c - b \text{ is 2, we have } a^2 = 2(c + b), \text{ etc.}$$

PYTHAGOREAN TRIPLES

Pythagorean triples can be generated by several different methods. Some of these are given on the page *Pythagorean Triples*. All of the primitive Pythagorean triples (those which have no common divisors except 1--e.g., 3, 4, 5 or 5, 12, 13) can be obtained by the method described below.

(i) Pick whole number values for s and t so that s is greater than t, s and t have no common divisors except 1 and only one of s and t is odd.

(ii) Find values for a, b and c using these formulas:
$$a = 2st, \qquad b = s^2 - t^2, \qquad c = s^2 + t^2$$

example: let s = 4, t = 3

a = 24, b = 7, c = 25

Students can discover that multiplying all three terms of a Pythagorean triple by a whole number produces another Pythagorean triple. The right triangles associated with the multiples of a triple are similar.

APPLICATIONS

The Pythagorean theorem and its converse (If the square of the length of one side of a triangle is equal to the sum of the square of the length of the other two sides, the triangle is a right triangle) are valuable in solving many problems in geometry and applied mathematics. Some examples are given on the next few pages.

● How can a 6 m by 8 m rectangle be laid out on a flat playground using only heavy string, a metric measuring tape and some wooden stakes? If students realize that the diagonal of a rectangle is the hypotenuse of a right triangle, they can calculate its length $d^2 = 6^2 + 8^2 = 100$ or $d = 10$. After laying off an 8 m line segment, students can attach a 10 m and a 6 m piece of string to the end points of the 8 m segment and form a (6, 8, 10) right triangle. The rectangle can then be completed.
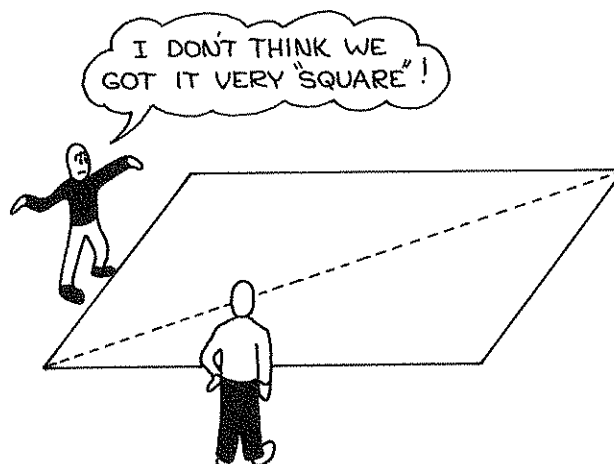
● How can the height of a right circular cone be determined? A right triangle can be formed by a radius, the altitude of the cone and a line segment from the cone's vertex to the circular base. If the radius and the slanted line segment are measured, the height of the cone can be found by applying the Pythagorean theorem. In the diagram $s^2 = h^2 + r^2$. Some background in solving simple equations and a calculator or square root table might be necessary to find an actual value for h. (There certainly are easier ways to find the height of an actual cone. The method shown to the right can be used for pyramids, triangular regions, etc.)

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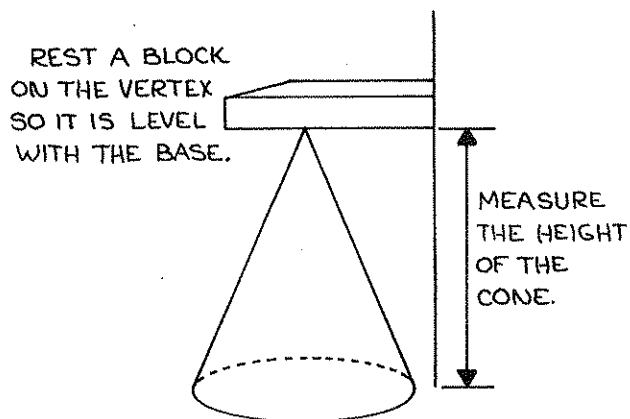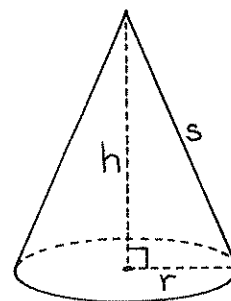
● Can a pair of 210 cm crosscountry skis be stored in a 60 cm by 90 cm by 170 cm rectangular closet?

The longest distance in the closet is on the diagonal.

$d^2 = 170^2 + t^2$ (see diagram)

$t^2 = 60^2 + 90^2$ (see second diagram)

so, $d^2 = 170^2 + 60^2 + 90^2$

$d = \sqrt{170^2 + 60^2 + 90^2}$

$d \approx 201.5$ (The skis won't fit!)

In general the length of a diagonal d of a rectangular box with dimension x, y and z can be found with this formula: $d^2 = x^2 + y^2 + z^2$

so, $d = \sqrt{x^2 + y^2 + z^2}$
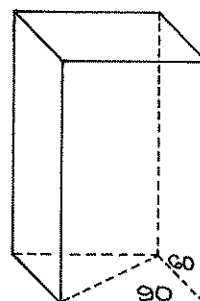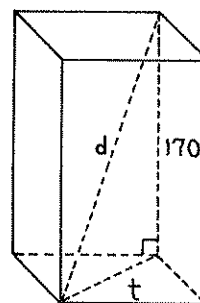
A spider and a fly are inside and at opposite ends of a 20 cm by 20 cm by 60 cm box. The spider is at point S, 1 cm from the bottom of the box midway between two vertical edges. The fly is at point F, 1 cm from the top of the box midway between two vertical edges. The spider wants to walk over to the fly. What path should he take to walk the shortest possible distance?
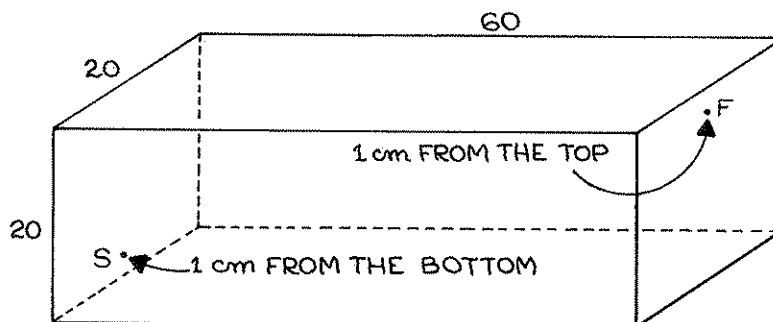
Students can mark and measure various paths on a scale model of the box. Do they have a guess for the shortest path? Remind students that the shortest path between two points on a plane is on a straight line. How can the box be made into a plane shape? Cut the box open and lay it flat. Now the two points can be connected by a straight line segment.

Four different ways of opening the box are shown below:

a.

$1 + 60 + 19 = 80$

b.

$\sqrt{80^2 + 18^2} \approx 82.6$

c.

$\sqrt{71^2 + 29^2} \approx 76.7$

d.

$\sqrt{62^2 + 40^2} = 74.0$
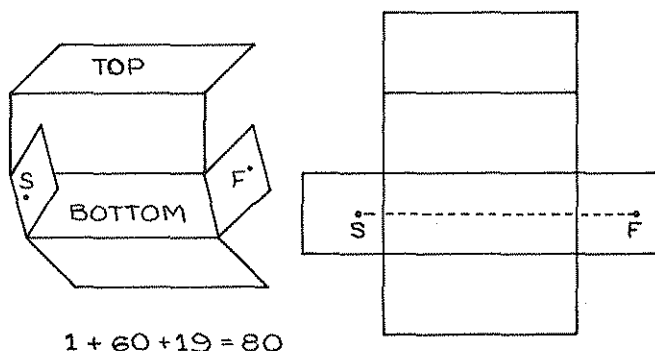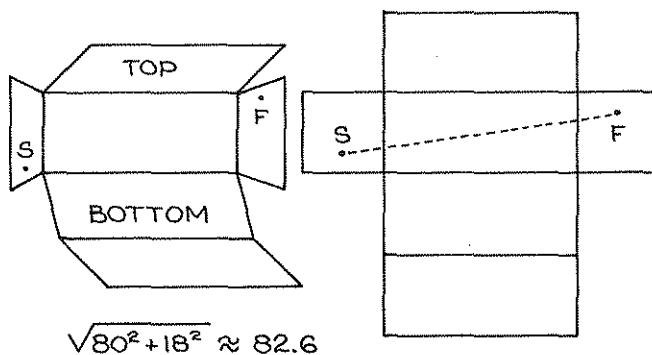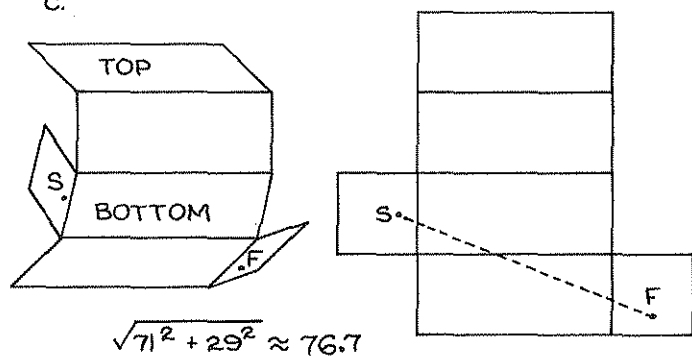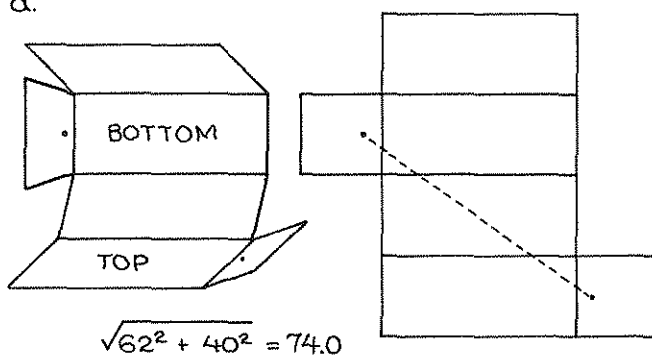
Students can use the Pythagorean theorem and a calculator to compute the distances. The shortest path (d) can be sketched on the model of the box and the model can be folded up again to see the path. Due to the symmetry of the box, there are two shortest paths—the spider could have headed toward the other side of the box.
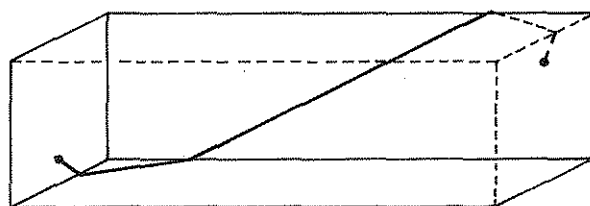
● How can the distance between two given points on a coordinate plane be determined? /Suppose the two points are (2,3) and (8,6). The two points can be connected by a line segment. By drawing appropriate horizontal and vertical line segments, a right triangle is formed. The distance between the

two points is the length of the hypotenuse of the right triangle. The base of the triangle is 6 units and the height of the triangle is 3 units—students can determine these measurements by eyeballing down or across to the coordinate axis and counting units. The distance between the two points $= \sqrt{6^2 + 3^2}$.
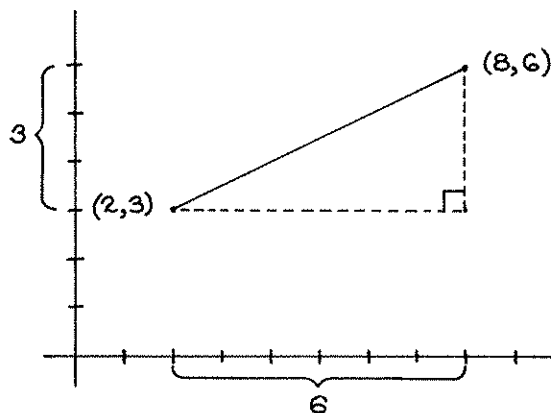
More than twenty word problems and applications for the Pythagorean theorem are given in <u>Exploring Mathematics on Your Own</u> by William H. Glenn and Donavon A. Johnson. The problems include the computation of distances, velocities and forces. The entire forty-page section on the Pythagorean theorem is interesting reading for teacher background.

Further Readings and References

Courant, Richard and Robbins, Herbert. <u>What is Mathematics?</u> New York: Oxford
        University Press, 1941, pp. 40–42.

Eves, Howard. <u>An Introduction to the History of Mathematics</u>. New York: Holt,
        Rinehart and Winston, Inc., 1953, pp. 47–76.

Gillings, Richard J. <u>Mathematics in the Time of the Pharaohs</u>. Cambridge, Massachu-
        setts: Massachusetts Institute of Technology, 1972, pp. 1, 238, 242.

Glenn, William H. and Johnson, Donovan A. <u>Exploring Mathematics on Your Own</u>. New
        York: Dover Publications, Inc., 1960, pp. 95–138.

PYTHAGORAS CUBED

Manipulatives like colored cubes, multibase blocks, Cuisenaire rods and interlocking centimetre cubes can be used to illustrate the Pythagorean theorem. The manipulatives can be used in a demonstration on the overhead projector or by individual students at their desks. Prepare a transparency (or worksheet) with the outline of a right triangle carefully drawn to fit the dimensions of the manipulative used. (The triangle should have dimensions (3,4,5), (6,8,10), (5,12,13), (7,12,13) or some other convenient set of Pythagorean triplets.) Make (or have students make) a square on each of the two legs using the chosen manipulative. Demonstrate (or have students show) that the manipulatives used in the two squares can be rearranged into one large square on the hypotenuse of the triangle. It may be necessary to draw the outlines of the squares for some students.

Three examples are shown below. Interlocking centimetre cubes have been used in the 3-4-5 case; Cuisenaire rods are used for the 5-12-13 case; and multibase blocks ($\frac{3}{8}$") are used with the 7-24-25 case to take advantage of the flats (10 x 10). Grid paper could be used. If so, students will need scissors to cut some squares to make an exact covering.

EXAMPLE 1

3-4-5 case using interlocking centimetre cubes

R = Red
Y = Yellow
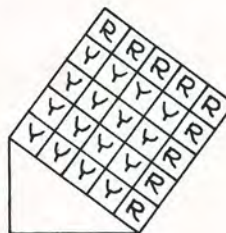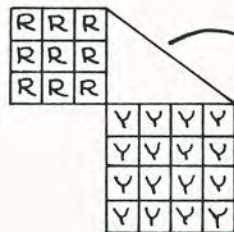
The name Cuisenaire and the color sequence of the rods are trademarks of the Cuisenaire Company of America, Inc.

EXAMPLE 2
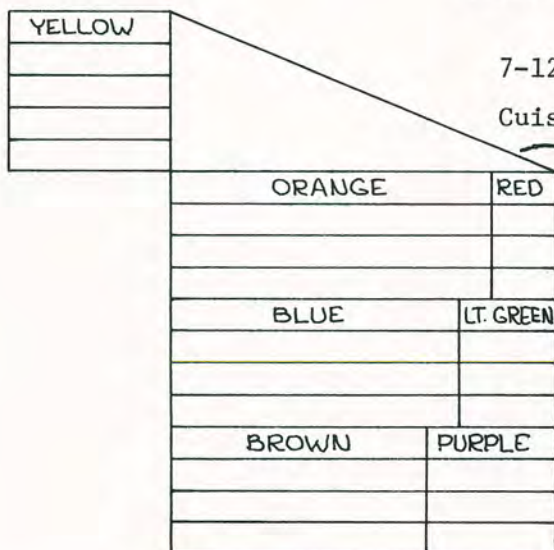
7-12-13 case using Cuisenaire rods
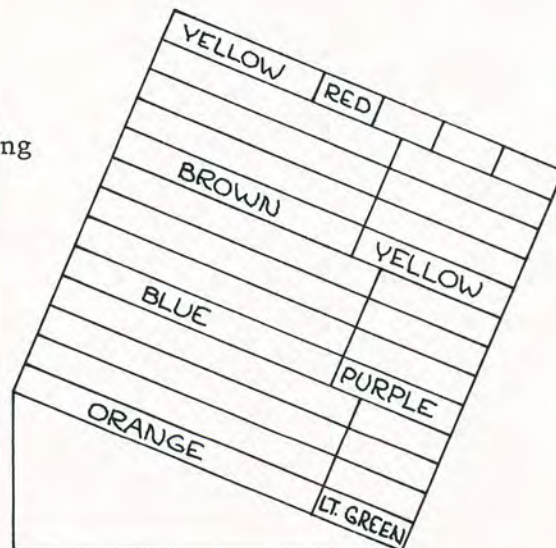
PYTHAGORAS CUBED

(CONTINUED)

EXAMPLE 3

7-24-25 case using
multibase blocks

LONG

U

FLAT
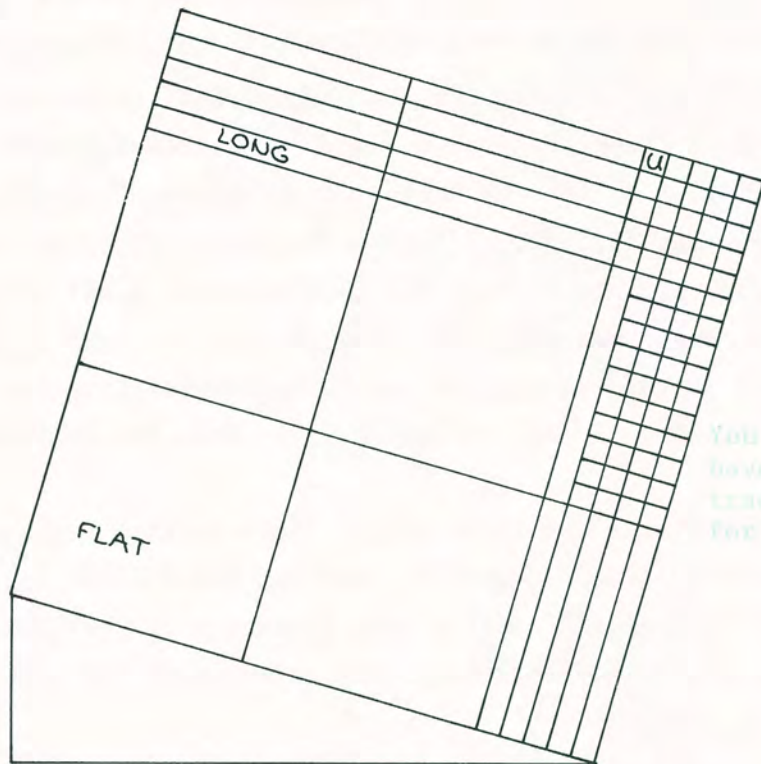
You may want to
have students
trade 40 units
for 4 longs.
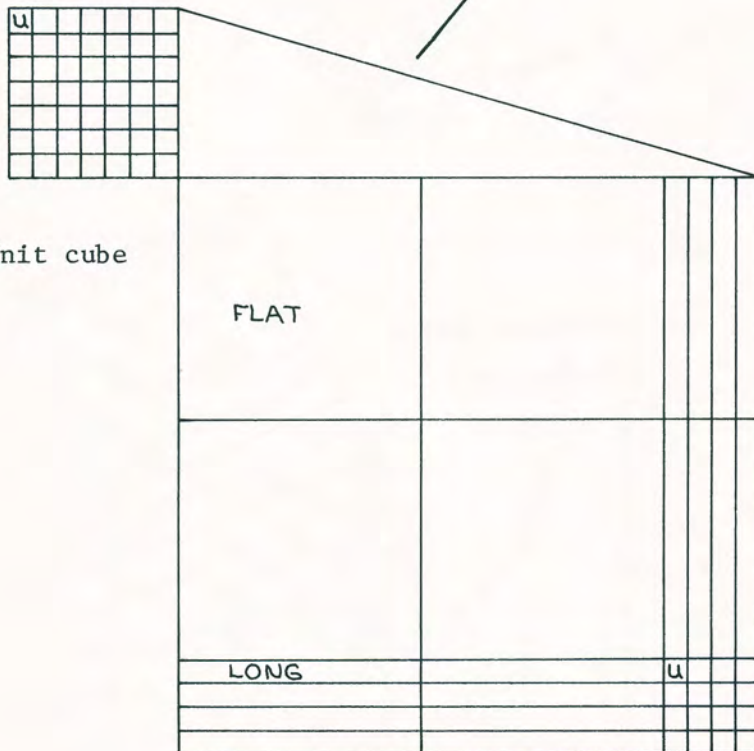
U = Unit cube

FLAT

LONG

u

PYTHAGORAS
-ALL FILLED UP-

Construct three containers 15 x 15 x 1, 20 x 20 x 1 and 25 x 25 x 1 using the following method.

From heavy construction paper or poster board make a 17 cm x 17 cm square. Mark a border of 1 cm around the square. Cut on the solid lines and fold on the dotted lines. Fold the borders up and attach the flaps with tape. Use 22 cm x 22 cm and 27 cm x 27 cm squares for the other two containers.

After all three containers have been made fill each of the two smaller ones with puffed rice (or a similar material). Then demonstrate how the rice in these two boxes will fill the larger box. The pouring shows that 25 x 25 x 1 = (20 x 20 x 1) + (15 x 15 x 1) or $25^2 = 20^2 + 15^2$.

The squares could be arranged to show that they will fit together to form the border of a right triangle.

The demonstration could be followed up by providing (or having students make) similar containers with sides of 1 cm, 2 cm, ..., 10 cm. Are there any two filled containers which can exactly fill a third container? What right triangles can be associated with these three containers?

Similar demonstrations could use containers which have bases that are other geometric shapes. The graphics below show some possibilities.

EQUILATERAL TRIANGLES
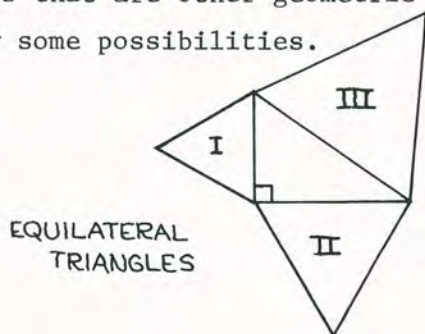
SEMICIRCLES
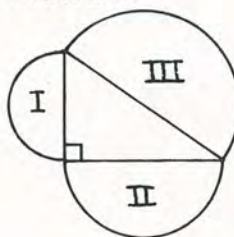
REGULAR PENTAGONS
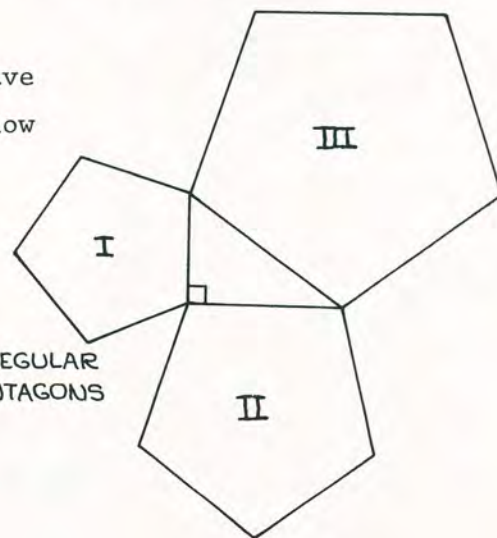
PYTHAGORAS AND TANGRAMS

Tangrams can be used as a model for the Pythagorean theorem. Combine the pieces from five sets (same size) of tangrams, preferably four sets the same color and one set a different color. Using the small triangle, the medium triangle or the large triangle, it is possible to build a square on all three sides such that the pieces in the squares on the two legs will exactly cover the pieces in the square on the hypotenuse.

Here are three suggestions for using tangrams to illustrate the Pythagorean theorem.

a) If students understand the theorem, ask them if they can discover a way to illustrate it using the small or large triangle.

b) Show students an illustration of the Pythagorean theorem using a medium triangle. Ask them to illustrate it using the small or large triangle.

c) Give students a worksheet showing a triangle the size of the small, medium or large triangle and the squares on the legs and hypotenuse of the triangle. Have them fit pieces on the two small squares and then rearrange the pieces to fit on the large square.
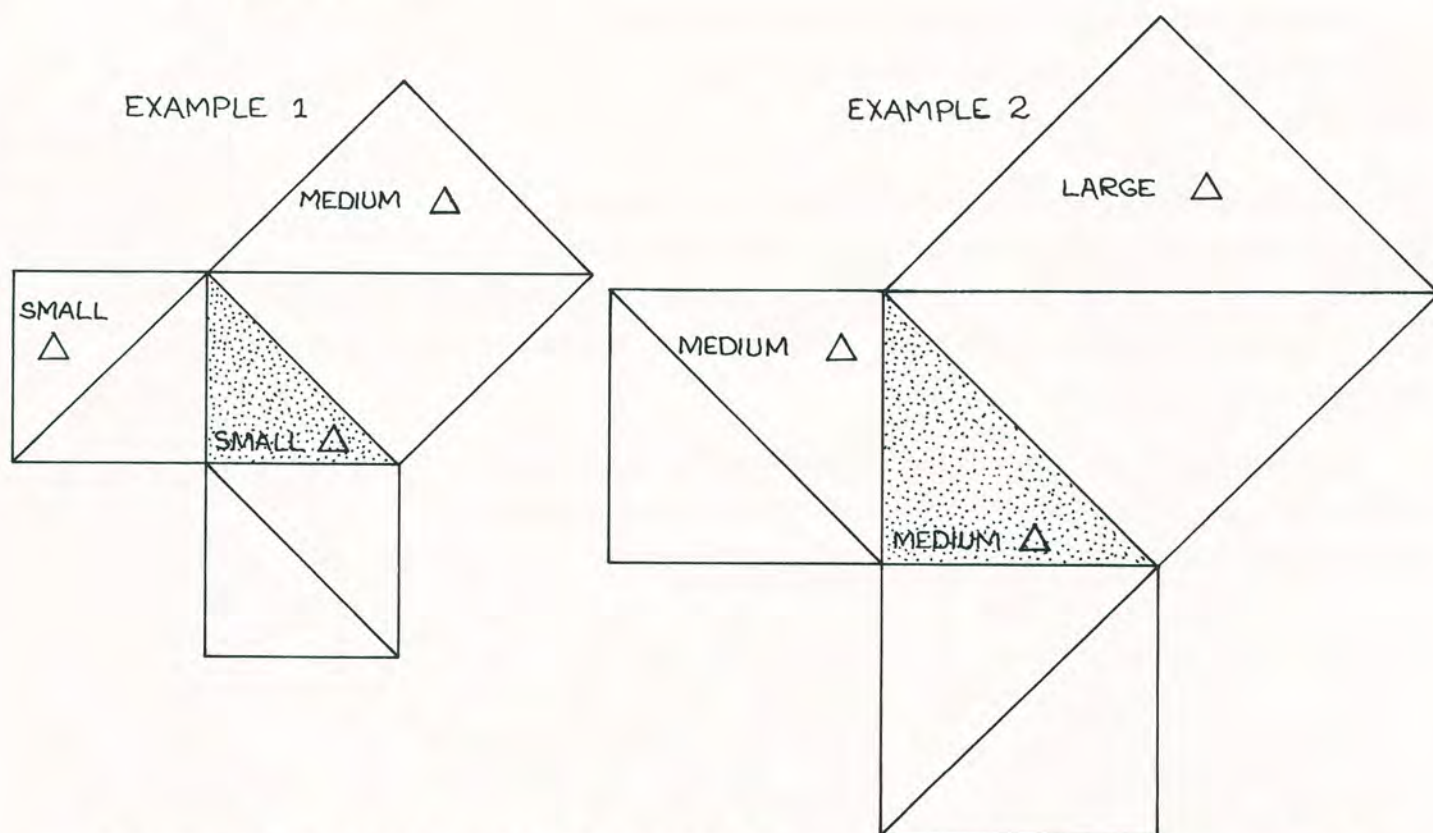
In Example 1 the four small triangles will exactly cover the two medium triangles.
In Example 2 the four medium triangles will exactly cover the two large triangles.

PYTHAGORAS AND TANGRAMS

(CONTINUED)

EXAMPLE 3

LARGE △

MEDIUM △
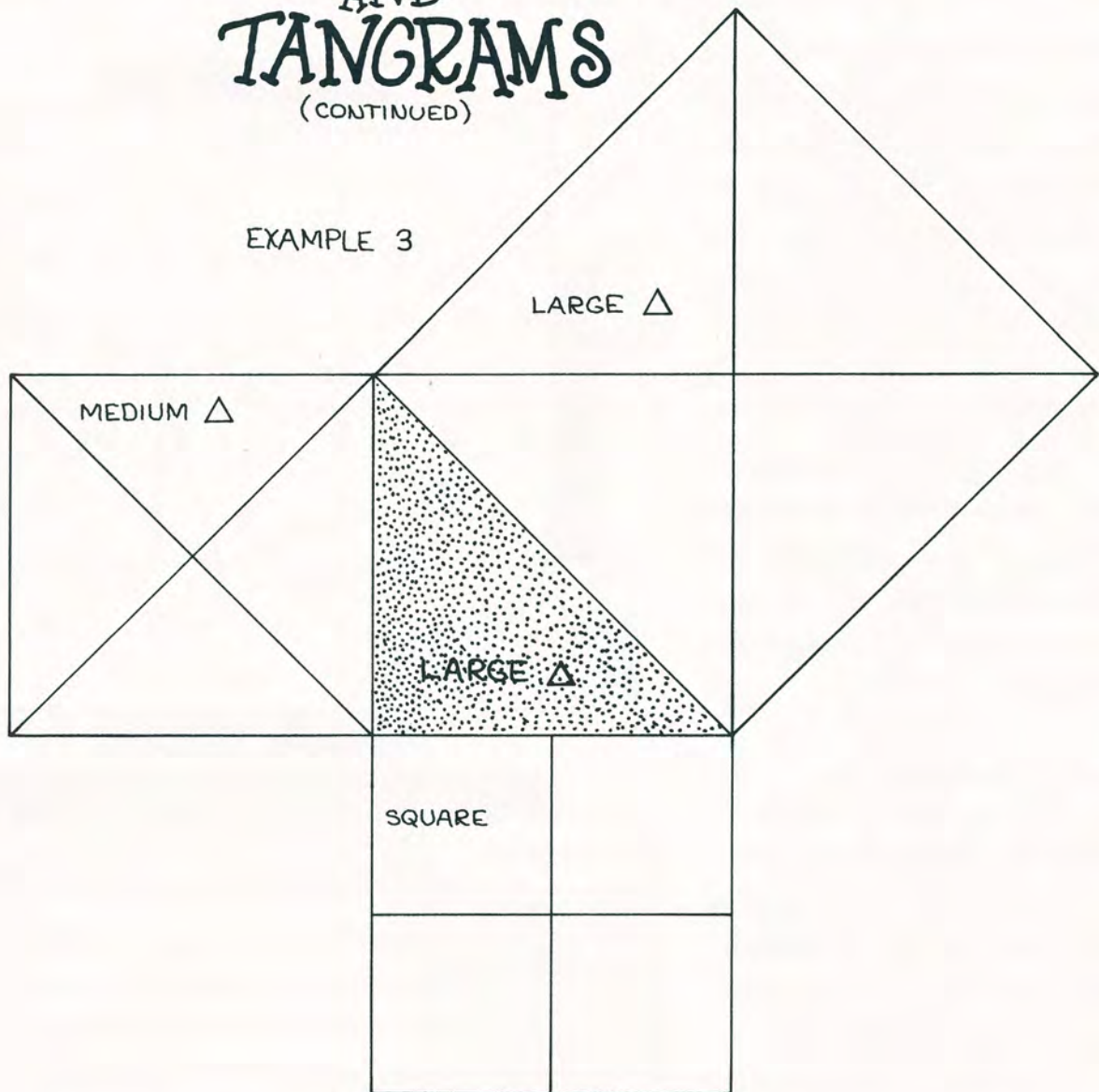
LARGE △

SQUARE

In Example 3 the four medium triangles and the four squares will exactly cover the four large triangles as shown in the diagram to the right.
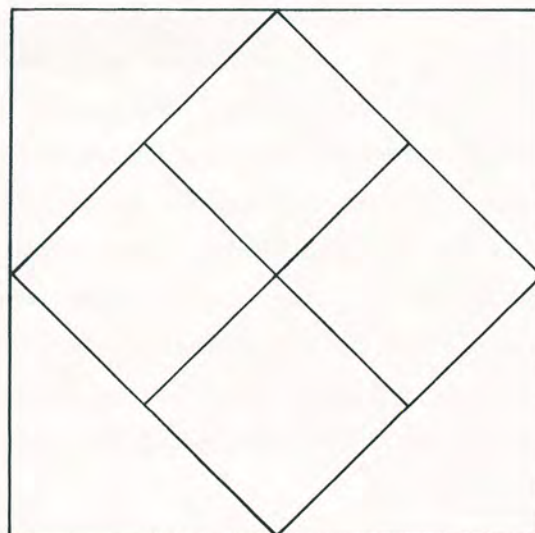
PYTHAGORAS
ON THE GEOBOARD

Four geoboards joined together as shown are needed to provide enough space to investigate the relationship of the sides of a right triangle.

Depending on your class you may want to make this activity

A) an open investigation,

Make a right triangle on your geoboard. Do you notice anything about the areas of the squares that can be constructed on the sides of the triangle.

B) a student discovery,

Make a right triangle using the vertices shown in the table. Make a square on each side of the triangle. Find and record the area of the three squares. What relationship do you see in the table?

C) or a teacher directed discovery.

Have students work in groups of four. Using a transparency of dot paper, show a right triangle and the squares on the legs and hypotenuse of the triangle. Have students copy the figures on their geoboards, compute the areas of the squares and record the areas in a table. Help them to discover the relationship.

A convenient method of finding the areas of the squares is to use Pick's formula which states that the area of a polygon equals $\frac{B}{2} + I - 1$ where B is the number of nails in the boundary of the polygon and I is the number of nails in the interior of the polygon. See *Be Picky About Your Geoboard* in the Area subsection for a development of Pick's formula.
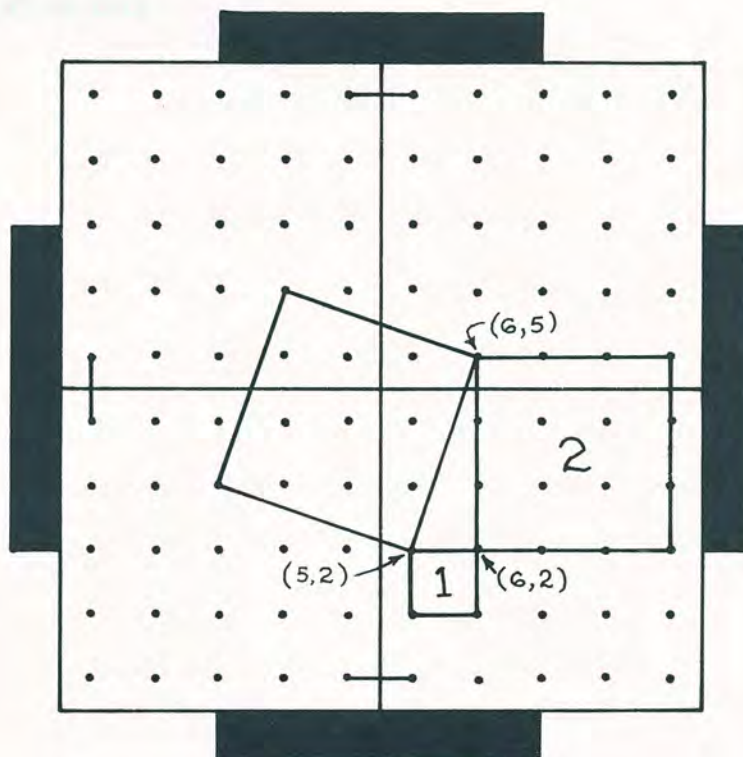
VERTICES	AREA OF SQUARE ON HYPOTENUSE	AREA OF SQUARE 1	AREA OF SQUARE 2
(1,1) (1,2) (2,1)			
(3,1) (3,4)(4,1)			
(7,3)(5,3)(7,5)			
(7,7)(7,4)(5,7)			
MAKE ONE OF YOUR OWN			

PYTHAGORAS SQUARED

You will need to make 8 congruent triangles, four red and four blue, with sides of 5 cm, 12 cm and 13 cm. Also make two 17 cm × 17 cm squares.

Materials Needed: Four red and four blue right triangles from your teacher

Two squares from your teacher

Metric ruler

Activity:

1) Measure a side of each square. ___ cm

Are both squares the same size?   Yes

2) Measure the sides of one of the triangles.   Short side ___ cm

Medium side ___ cm

Hypotenuse ___ cm

Are all the triangles the same size?   Yes

3) Cover one square as in Diagram 1.

Measure the sides of square I. ___ cm
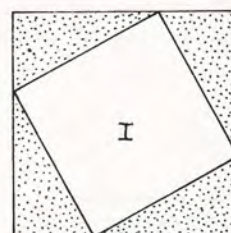
4) Cover the other square as in Diagram 2.

Measure the sides of square II. ___ cm

Measure the sides of square III. ___ cm
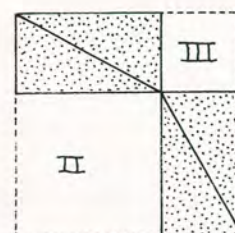
DIAGRAM 1   **DIAGRAM 2**

5) What can you say about the areas of squares I, II and III?

6) Does the area of square I = the area of square II + the
area of square III?   Yes

7) Side of square I ___ cm, hypotenuse of triangle ___ cm

Side of square II ___ cm, medium side of triangle ___ cm

Side of square III ___ cm, short side of triangle ___ cm

> The area of the square on the hypotenuse is equal to the
> sum of the areas of the squares on the other two sides.

Two congruent right triangles with sides of 5, 12 and 13 arranged as shown in the activity will produce squares with areas of 5², 12² and 13². The activity shows that 13² = 5² + 12².

PYTHAGORAS SQUARED

Similar activities could be done by arranging four congruent right triangles as shown in the diagram below. The algebra is shown as an explanation and would be used only with students who have had experience with the symbols.
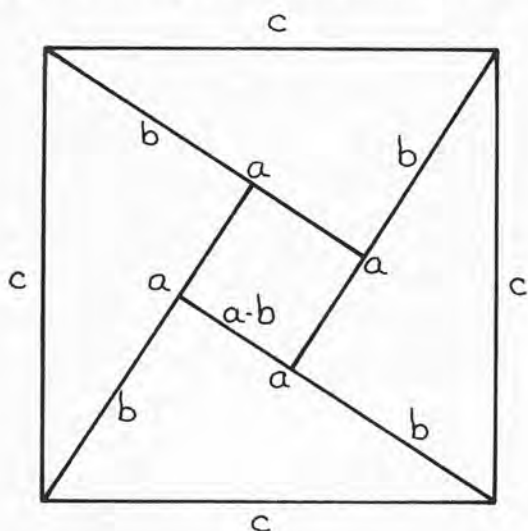
The large square has an area of c^2. This square region is divided into four triangles each with an area of $\frac{1}{2}ab$ and a square with an area of $(a - b)^2$.

$$c^2 = 4(\tfrac{1}{2}ab) + (a - b)^2$$
$$c^2 = 2ab + a^2 - 2ab + b^2$$
$$c^2 = a^2 + b^2$$
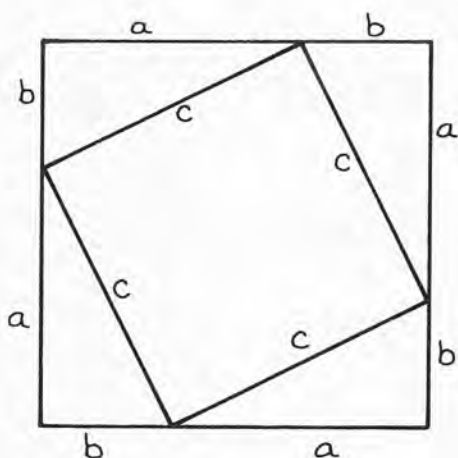
Diagram 1 on the previous page is illustrated here. The large square has an area of $(a + b)^2$. This square region is divided into four triangles each with an area of $\frac{1}{2}ab$ and a square with an area of c^2.

$$(a + b)^2 = 4(\tfrac{1}{2}ab) + c^2$$
$$a^2 + 2ab + b^2 = 2ab + c^2$$
$$a^2 + b^2 = c^2$$

Careful use of grid paper can be used to illustrate these examples. A simple way to arrive at the area of a triangle is to use a rectangle and then cut the rectangle along a diagonal. The area of each triangle formed is one-half the area of the rectangle.

CUTTING AND COVERING

The Pythagorean theorem can be demonstrated as a paper cutting and/or paper folding activity. The effect will be more pronounced if the square on the hypotenuse is a different color or different shading than the squares on the other two sides.
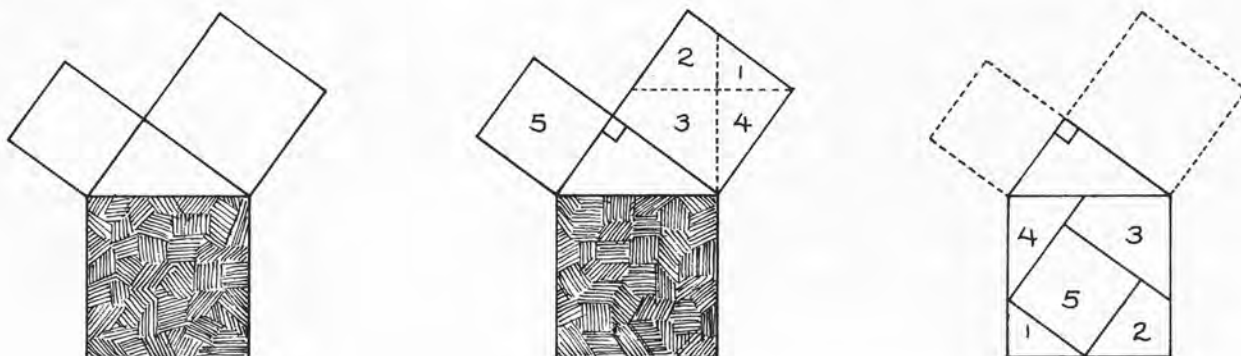
I. Section the medium-sized square using two perpendicular lines through the vertices as shown. Cut out sections 1 – 5 and cover the large square.
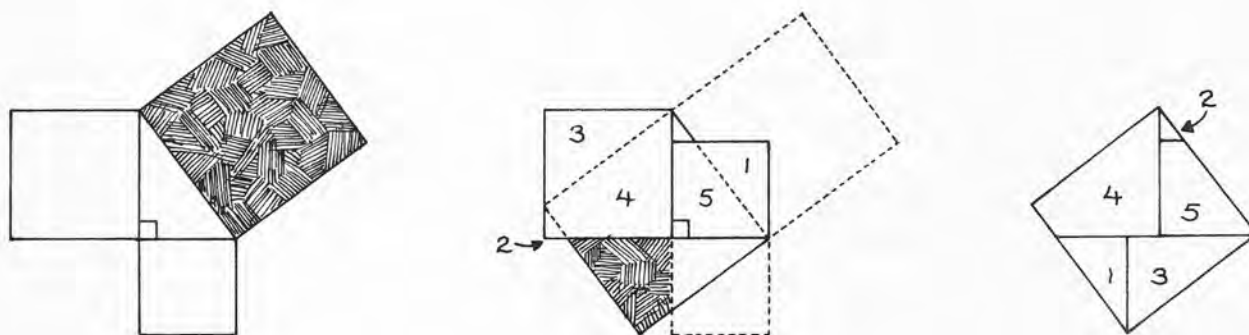
II. Fold the small square over the top of the triangle and the large square underneath the triangle. Cut off the sections 1 – 3 of the small and medium squares that extend beyond the large square. Position sections 1 – 3 so the large square is completely covered by the folded (4 – 5) and cut (1 – 3) sections.

SQUARING THE SQUARES

Making a third square
from any two given
squares
Pythagorean Theorem
AREA & VOLUME

3
5
1

Materials Needed: Paper, scissors, metric ruler

Activity:

1) Make and cut out <u>any</u> two squares.

2) Mark the midpoints of each side of each square.

3) Mark one corner of the small square.

4) Place the small square next to the large square so the midpoints match. Put a dot where the marked corner touches.

5) Turn the small square $\frac{1}{4}$ of a turn counterclockwise and move it to the top. Match the midpoints and mark.

6) Repeat for the other two sides of the large square.

7) Connect opposite dots with a line segment. Measure one of the line segments to the nearest millimetre.
   Cut the large square into the four pieces shown.

8) Arrange the four pieces with the small square in the middle to make a larger square. Measure one of the sides of the larger square to the nearest millimetre.

9) Make two different squares and repeat the activity. Can you find two squares that can't be put together into one large square using this method?
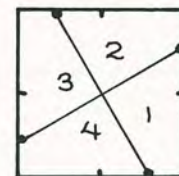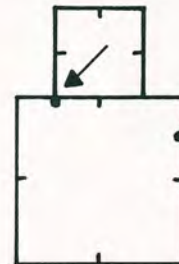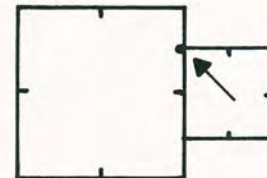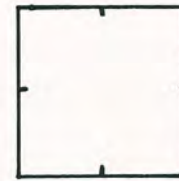
The purpose of putting the two squares together and marking is to create a right triangle whose legs are the same as the sides of the squares. The line joining opposite dots is the hypotenuse. Thus any two squares can be reassembled to make a larger square. The explanation would involve the Pythagorean theorem. The two squares represent the squares of the two sides and the constructed square represents the square of the hypotenuse.

TYPE: Activity Card/Manipulative

IDEA FROM: *Problems—Purple Set,* Nuffield Mathematics Project
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PYTHAGOREAN THEOREM

Using similar triangles
to prove the
Pythagorean theorem
Pythagorean Theorem
AREA & VOLUME

5

I

TEACHER PAGE

> **PYTHAGOREAN THEOREM:**
>
> In each right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

A formal proof of the Pythagorean theorem using similar triangles.

Given any right triangle ABC with the right angle at C, draw a perpendicular from C to the hypotenuse. Label the foot of the perpendicular D and the two segments e and f as shown.
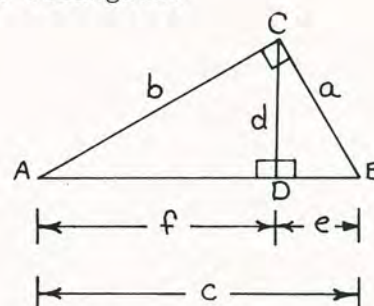
Consider triangles ABC and ACD. Both triangles have angle A in common. Angles ACB and D are congruent so the two triangles are similar. Similar triangles have corresponding sides which are proportional so $\frac{f}{b} = \frac{b}{c}$ or $f = \frac{b^2}{c}$.
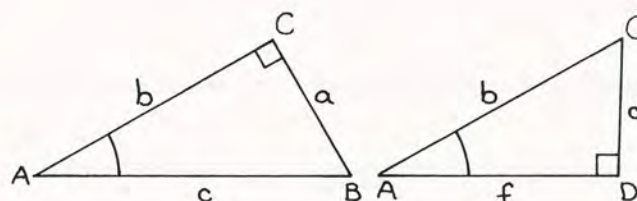
Consider triangles ABC and CBD. Both triangles have angle B in common. Angles ACB and D are congruent so the two triangles are similar. Similar triangles have corresponding sides which are proportional so $\frac{e}{a} = \frac{a}{c}$ or $e = \frac{a^2}{c}$.
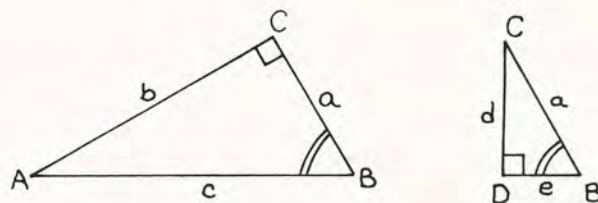
But $c = e + f$ so by substitution,

$$c = \frac{a^2}{c} + \frac{b^2}{c}.$$

Multiplying both sides by c yields

$$c^2 = a^2 + b^2.$$

735

The Egyptian Rope Trick

Materials Needed: 12 metres of clothesline rope

Activity:

1) Tie a small knot at each end of the rope. Then tie pieces of string to the rope at 1 metre intervals.

   —| METRE

2) Make as many differently sized triangles as you can with the rope. A vertex can only occur at a marked position on the rope. Use all the rope for each triangle. The triangle shown to the right is <u>not</u> allowed.

   Record the lengths of the sides of the triangles you can make.

   3 METRES
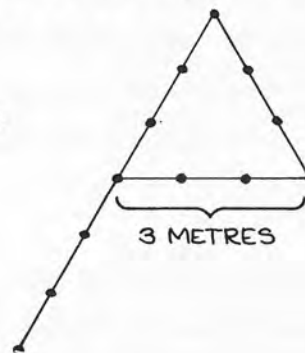
3) The diagram to the right represents the playing court for the game Four Square. Which of the triangles above would you use to help you make the Four Square court on the floor or playground? ____

   _____

   Right angles are needed at the corners.

   4 METRES

   4 METRES
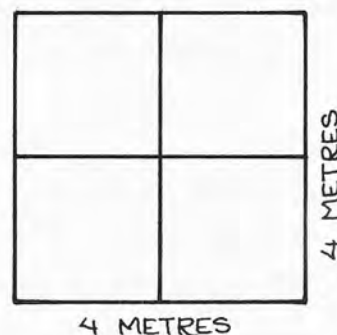
4) Use chalk, tape or paint to make the Four Square court. Look up the rules in a game book if you don't know them.

PYTHAGOREAN TRIPLES

Pythagoras and his followers were very interested in numbers and what they considered the mystical powers of numbers. One of their primary concerns was perfect squares--the product of a whole number with itself. The Pythagoreans (as their secret society was known) were interested in finding whole numbers a, b and c that satisfied the relationship $a^2 + b^2 = c^2$. We call these values a Pythagorean triple (or Pythagorean numbers). For example, 3, 4, 5 and 5, 12, 13 are Pythagorean triples since $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$.

A) Pythagoras found a rule for finding a Pythagorean triple. Let m be any <u>odd</u> whole number, then m, $\frac{1}{2}(m^2 - 1)$ and $\frac{1}{2}(m^2 + 1)$ will be a Pythagorean triple.

   1) Let m = 3. Use the rule to show that the other two numbers are 4 and 5.
   2) Let m = 5. Use the rule to show that the other two numbers are 12 and 13.
   3) Find some other Pythagorean triples.
   4) The rule can be verified by showing that $[m]^2 + [\frac{1}{2}(m^2 - 1)]^2 = [\frac{1}{2}(m^2 + 1)]^2$.
   5) What happens if m is an even whole number? Will the numbers you get still satisfy $a^2 + b^2 = c^2$? Why wouldn't these make the Pythagoreans happy?

B) Many Pythagorean triples have two consecutive numbers, e.g., 5, 12, 13. For any whole number n, the numbers 2n + 1, $2n^2 + 2n$ and $2n^2 + 2n + 1$ from a Pythagorean triple with two consecutive numbers.

   1) Use this rule to find Pythagorean triples for n = 4, then n = 5 and finally n = 8.
   2) Verify the rule.
   3) If n is a fraction, will the three numbers generated satisfy $a^2 + b^2 = c^2$? Try n = $\frac{1}{2}$.

C) For two whole numbers m and n with m $>$ n, $m^2 - n^2$, 2mn and $m^2 + n^2$ form a Pythagorean triple.

   1) Use several values of m and n to find some Pythagorean triples.
   2) What happens if m = n? Will the numbers satisfy $a^2 + b^2 = c^2$?
   3) What happens if m $<$ n? Will the numbers satisfy $a^2 + b^2 = c^2$?

A PYTHAGOREAN SPIRAL

TEACHER IDEA

The first triangle in the center of the design is an isosceles right triangle with legs of 1 unit. By the Pythagorean theorem the hypotenuse has length $\sqrt{2}$ units. This length and a side of 1 unit are then used to make a right triangle with a hypotenuse of length $\sqrt{3}$ units. This process is continued to generate the remainder of the design.

This activity could be done for the aesthetic appeal of the spiral-like design, as an approximation exercise to actually measure the length of each hypotenuse to the nearest millimetre, or as an application of the Pythagorean theorem with the length of each hypotenuse written as a radical, e.g., $\sqrt{5}$. Calculators could be used to find these lengths. A student with background in programming could write a program to generate the length of each hypotenuse in the sequence.
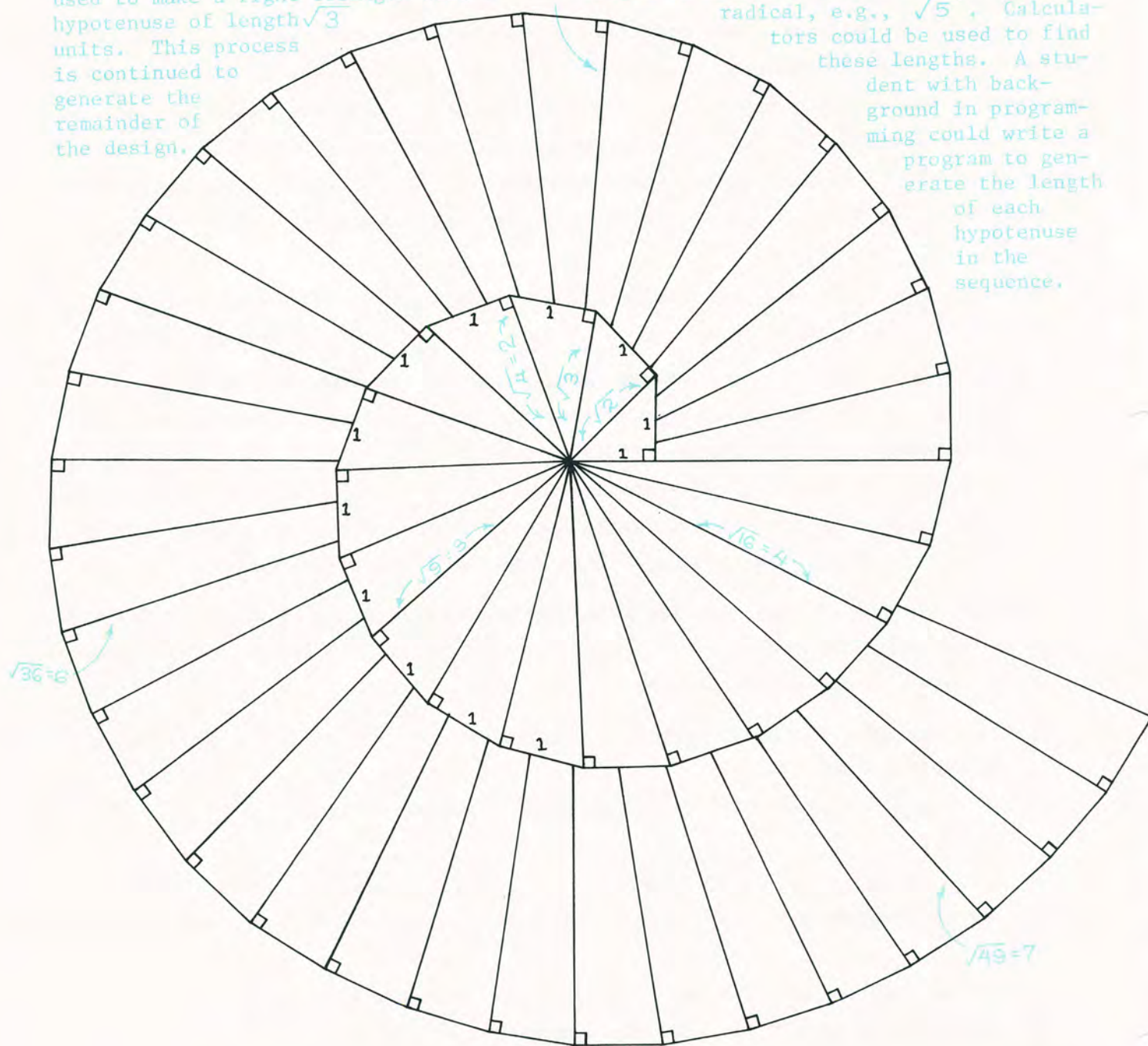
The scale of this design is 1 unit:1.5 centimetres. If students are going to measure the lengths, it would be better to use a scale of 1 unit:1 centimetre or 1 unit:1 inch.

TYPE: Teacher Idea:Bulletin Board/Transparency

738

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

CONTENTS

SURFACE AREA

The meaning of the words "surface
area" can be difficult for students to
retain. If the RIM in perimeter helps
students remember that perimeter is
the distance around a planar shape,
perhaps they can associate the SUR in
surface area with the word "surround."
This word asoociation, along with some
of the wrapping and unfolding activi-
ties suggested below, could help give
meaning to the words "surface area."

> I have my students think of _surround_ when they see the word _surface area_ —it helps them remember to add in the areas of all the faces.
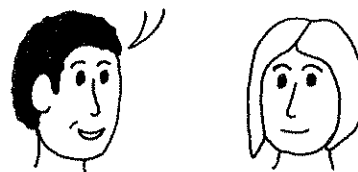
Here are some ideas for introductory activities involving the concept of sur-
face area. These activities should help students realize that the area of the
"bottom" is also included in the surface area of a solid.

● Objects can be covered with grid
paper to help determine their surface
area. Metric geoblocks can be covered
with centimetre grid paper fairly
accurately. (Metric geoblocks are des-
cribed in _Geoblocks I_ in the Polyhedra
subsection.)
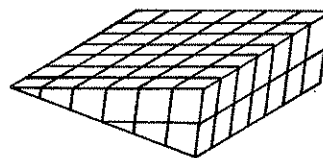
● Small boxes or paper fold-up models
of polyhedra can be unfolded. The area
of the resulting net can be approximated
by placing the net on grid paper. Stu-
dents can also use formulas to find the
area of each part of the net and then
compute the surface area of the poly-
hedral model.

● Small boxes, blocks, pyramids or
other shapes can be wrapped in tin foil.
By slitting enough of the edges of the
wrapper, the foil can be laid flat and
the area determined by one of the
methods discussed above. (See _Geo-
blocks II_ in the Polyhedra subsection
for a related foil activity.)
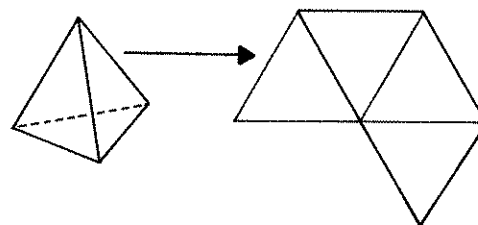
The surface area of this tetrahedron
is 4 x the area of one triangle.
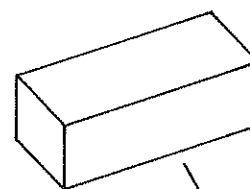
Surface area =
   2 x area of top
+ 2 x area of end
+ 2 x area of side

1
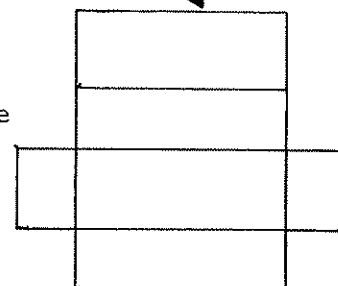
● The outlines of faces of a poly-
hedron can be traced onto grid paper
and the sum of the areas computed to
find the surface area of the polyhe-
dron. If the outlines are traced on
plain paper, formulas can be used to
find the individual areas.
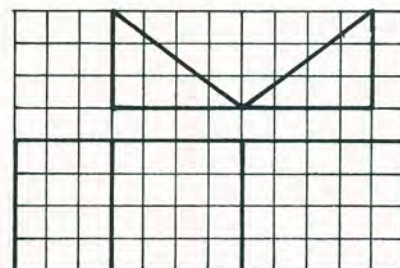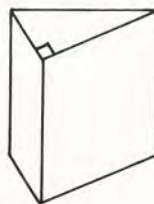
● The surface areas of cones or
cylinders can also be found by the
methods given above. Solid cones
or cylinders could be wrapped or
paper nets could be measured. A
method for finding the surface area
of a sphere is given in

Surface Area of a Sphere.

SURFACE AREA:
ABOUT 52
SQUARE UNITS
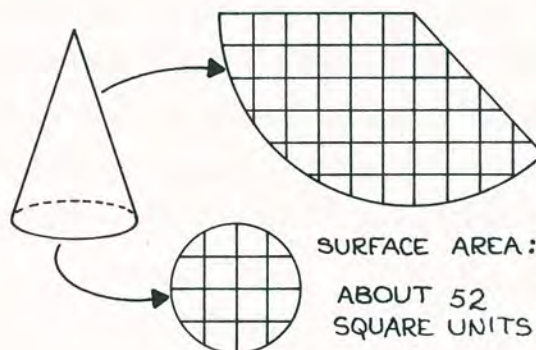
Word problems involving practical applications of total surface area are not easy
to find or invent. Perhaps this is because only the walls of houses are painted and
formica is applied only to the tops and sides of tables. In everyday life, the total
surface area of an object is rarely computed. Even when a box is wrapped or a cush-
ion is recovered, the amount of paper or material is usually estimated and the excess
is folded under or cut off--a far cry from finding a number measurement to represent
the surface area. One source for applications of surface area is in the relation
between surface area and volume. "A closable box-shaped carton can be designed so
its capacity is 1000 cm^3. Out of the three possible boxes shown below, which uses
the most cardboard? Which uses the least amount of cardboard? Can you design a car-
ton to hold 1000 cm^3 that uses less cardboard than any of these?"
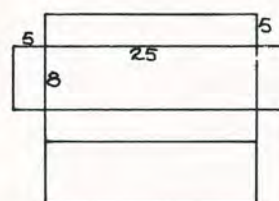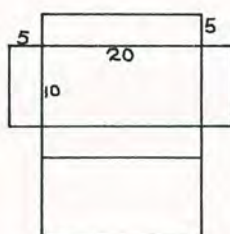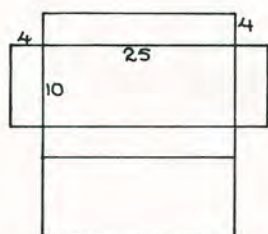
More ideas on relating surface area and volume are given in the classroom materials
and the commentary to the Volume subsection.

STAIRCASES

Materials: 36 cubes

THIS STAIRCASE IS ONE UNIT HIGH
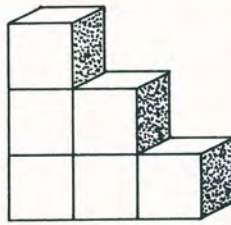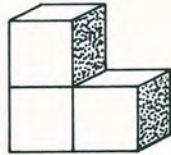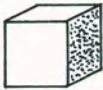
THIS STAIRCASE IS FOUR UNITS HIGH.

1) How many cubes are needed to make each staircase above? Record your answer in the table below. You can build each staircase with cubes to check your answers.

2) Find the surface area of each staircase above. Be sure to count the faces on the bottom of the staircase. Record your answers in the table below. Again you can build the staircases with cubes to check your answers.

3) Complete the table. Use cubes to build each staircase. Do you see any patterns?

4) Challenge: If a staircase is 20 units high, how many cubes will it contain?

_____

_____

What is the surface area of this staircase?

_____

_____

HEIGHT OF STAIRCASE IN UNITS	NUMBER OF CUBES	SURFACE AREA IN SQUARE UNITS
1	1	6
2	3	14
3		
4		
5		
6		
7		
8		
9		
10		

SPACE STATIONS

Materials: 36 cubes
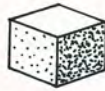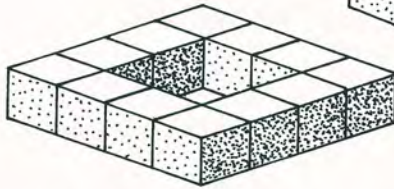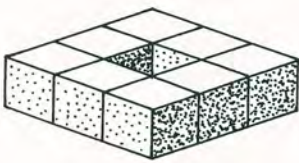
1 UNIT
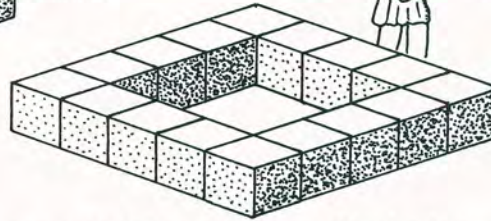
THIS STATION HAS FIVE CUBES ON A SIDE.

1) Count the number of cubes in each square space station above. Find the surface area of each space station and record in the table below. You may wish to build the space stations with cubes to help you.

2) Use your cubes to build larger square space stations, 6 cubes on a side, 7 cubes on a side. Record the total number of cubes in each space station and the surface area of the space station in the table.

Do you see any patterns?

If so, explain. _____

_____

_____

_____

TOTAL NUMBER OF CUBES IN SPACE STATION	SURFACE AREA OF THE SPACE STATION IN SQUARE UNITS
8	32
12	

10	

3) Investigate rectangular space stations. Count the total number of cubes in each space station and find the surface area of the space station.
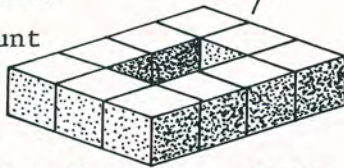
Do you see a pattern? _____

_____

_____

4) Investigate double-layered space stations.
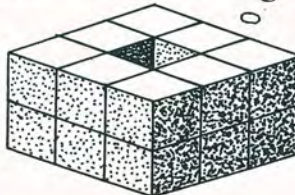
Can you find a relationship between the number of cubes in each double-layered space station and the surface area of the space station? _____

_____

_____

INK YOUR PAD

Materials Needed: Cuisenaire rods

Build a two-story house with the white rod and the red rod.

Use an imaginary ink pad and stamp that covers one square centimetre (1 cm^2) to "paint" this two-story house.
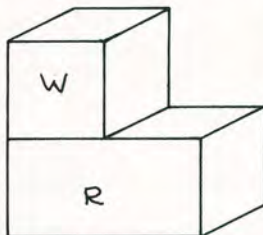
I   You do not have to paint underneath the house. How many stamps does it take to cover the house? _____ stamps

THE END OF ANY ROD HAS AN AREA OF 1 cm^2. IT COULD BE USED AS THE STAMP.

II   Cover the next three larger houses shown below.
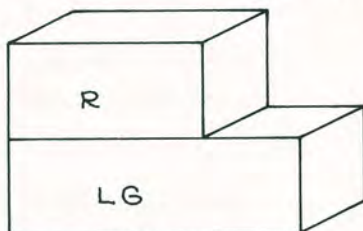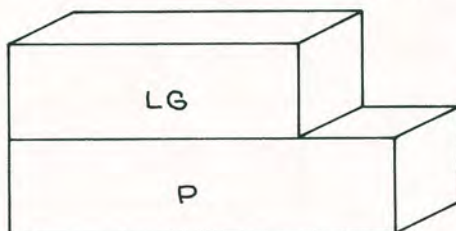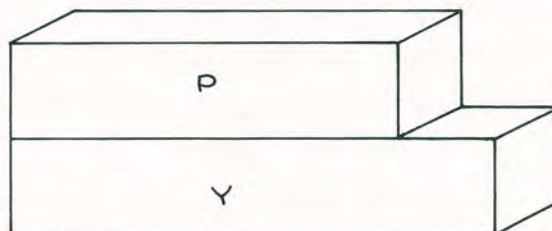
_____ stamps   _____ stamps   _____ stamps

Can you find a pattern in these numbers?   _____

III Build and "paint" a house using a blue rod and an orange rod. Use the pattern to help you find the number of rods needed. _____

High water houses: Two rods of one length and one of the next larger length. The first house is constructed from two red rods and one light green rod.

IV   How many stamps are needed to cover this house? _____ stamps
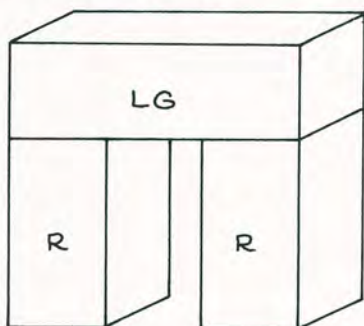
BUILD AND COVER THE NEXT TWO HIGH WATER HOUSES.

Design your own house and ask a friend to determine how many stamps are needed to paint it.

IDEA FROM: *Student activity Cards for Cuisenaire Rods*

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SURFACE AREA OF A SPHERE

Materials: Tennis ball, basketball, globe, etc.,
             grid paper, felt pen, tracing paper,
             ruler, 2 sheets of tagboard (file folders)

The formula for the surface area of a sphere is $4 \times \pi \times r^2$
        where r represents the radius of a sphere.

The radius of a sphere can be determined by
placing the sphere on a ruler and using two
sheets of tag board or bookends to find the
diameter as shown to the right.

THE RADIUS IS 6 cm.

TAG BOARD   TAG BOARD

16 cm   28 cm
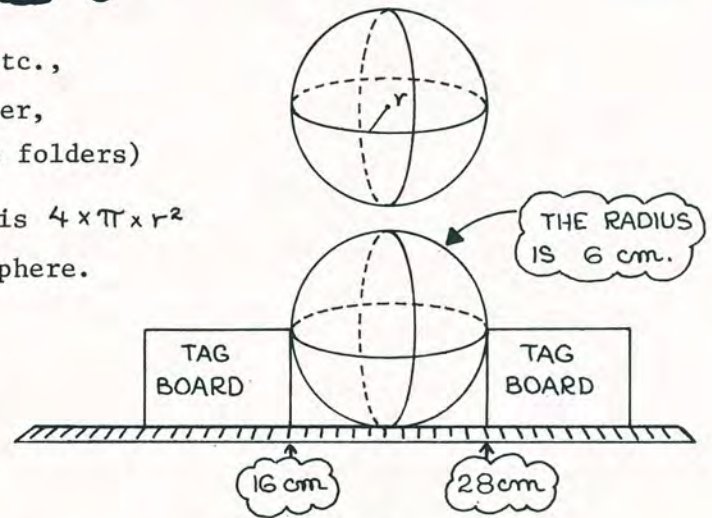
The method outlined below will give a rough approximation of the surface area of a
sphere.   The accuracy can be checked by the formula.

1)   Use a felt pen on the ball to:

    a)   Draw a great circle around the middle of the
         ball.

    b)   Draw small circles close to the top and bottom
         as shown.

    c)   Draw equally spaced arcs from the top circle
         to the bottom circle. The sections formed
         will have the same area. If many arcs are
         drawn each little section is nearly a trapezoid.

TOP

MIDDLE

BOTTOM

NEARLY A TRAPEZOID
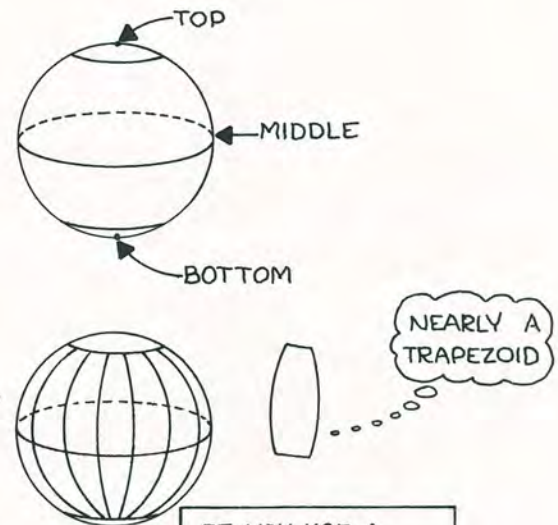
2)   Hold tracing paper against the ball and trace one
     trapezoid. Cut out the tracing and lay it on grid
     paper to find the area. Multiply by the number of
     trapezoidal shapes to find the total area of these
     shapes.

IF YOU USE A
BASKETBALL OR
SOCCER BALL YOU
MAY CHOOSE TO
USE THE SECTIONS
ALREADY MARKED

3)   Trace the upper and lower sections. Cut them out and use grid paper to find their
     areas.

4)   Add the results in #2 and #3 to find the approximate surface area of the ball.

5)   If the ball is old and heavy duty scissors are available, cut the ball into sec-
     tions. These sections can be traced on grid paper to determine their area.

WHAT'S ON THE OUTSIDE?

Materials Needed: 15 cubes

Activity: Using the three views, build each model. Find the surface area of each model.

EXAMPLE:

TOP   FRONT   SIDE

BUILD THE MODEL FIRST IF YOU WANT.

COUNT THE NUMBER OF SQUARES ON THE OUTSIDE TO FIND THE SURFACE AREA.
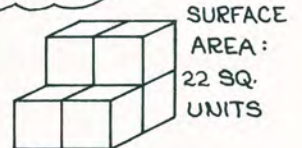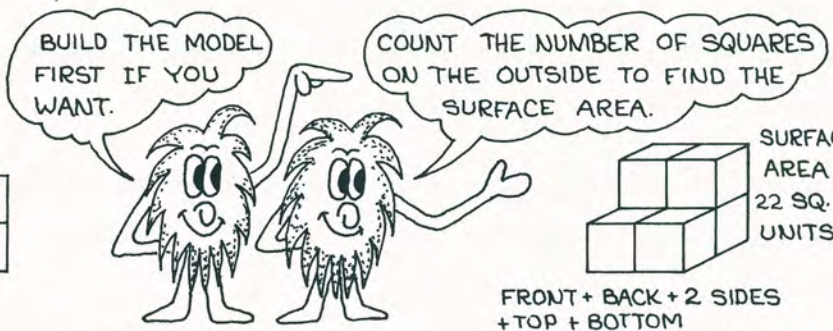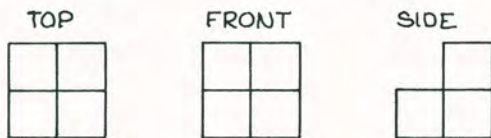
SURFACE AREA: 22 SQ. UNITS

FRONT + BACK + 2 SIDES + TOP + BOTTOM

TOP   FRONT   SIDE

①

TOP   FRONT   SIDE

⑤

②

⑥

③

⑦

④
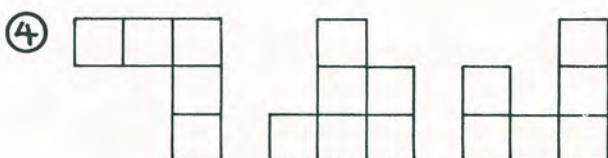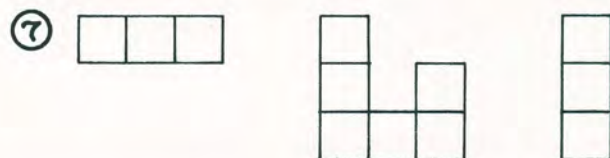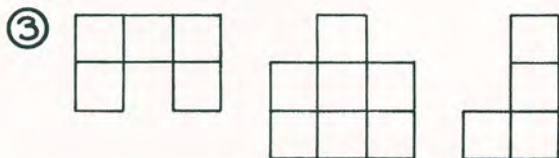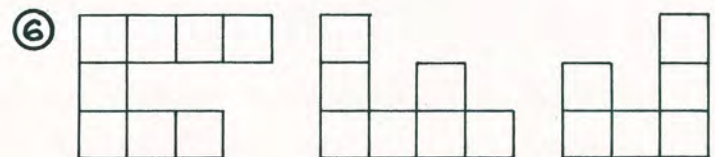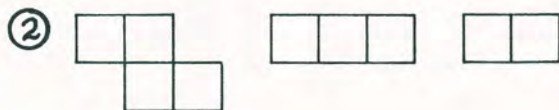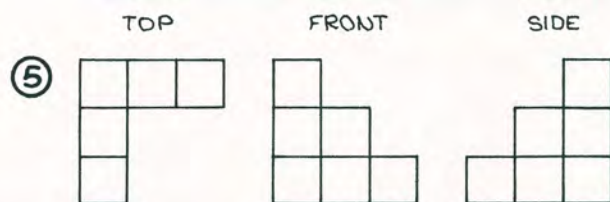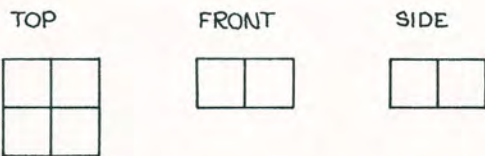
⑧ Make a model of your own and draw all three views. What is the surface area? Give your drawing to a friend to build.

THE CUBE PAINTER

Materials: 100 cubes and a calculator

Activity:

1) Build a 2 x 2 x 1 model with the cubes.

   a) If the entire model is painted, how
      many cubes will have
      4 faces painted? _____
      3 faces painted? _____
      2 faces painted? _____

WHAT IS THE TOTAL NUMBER OF CUBE FACES PAINTED? _____

2) Build these models.

3 x 3 x 1 MODEL

	NUMBER OF CUBES	NUMBER OF CUBE FACES PAINTED
4 FACES PAINTED	4	4 × 4 = 16
3 FACES PAINTED	4	4 × 3 =
2 FACES PAINTED		1 × 2 =
TOTAL		

4 x 4 x 1 MODEL

	NUMBER OF CUBES	# OF CUBE FACES PAINTED
4 FACES	4	4 × 4 =
3 FACES		
2 FACES		
TOTAL		

5 x 5 x 1 MODEL

	NUMBER OF CUBES	# OF CUBE FACES PAINTED
4		
3		
2		
TOTAL		

6 x 6 x 1 MODEL

	NUMBER OF CUBES	# OF CUBE FACES PAINTED
4		
3		
2		
TOTAL		

7 x 7 x 1 MODEL

	NUMBER OF CUBES	# OF CUBE FACES PAINTED
4		
3		
2		
TOTAL		

8 x 8 x 1 MODEL

	NUMBER OF CUBES	# OF CUBE FACES PAINTED
4		
3		
2		
TOTAL		

9 x 9 x 1 MODEL

	NUMBER OF CUBES	# OF CUBE FACES PAINTED
4		
3		
2		
TOTAL		

10 x 10 x 1 MODEL

	NUMBER OF CUBES	# OF CUBE FACES PAINTED
4		
3		
2		
TOTAL		

3) For each model above how many faces on the cubes are not painted?

    2 x 2 x 1 model _____   3 x 3 x 1 model _____   4 x 4 x 1 model _____
    5 x 5 x 1 model _____   6 x 6 x 1 model _____   7 x 7 x 1 model _____
    8 x 8 x 1 model _____   9 x 9 x 1 model _____   10 x 10 x 1 model _____

THE CUBE PAINTER RETURNS

Get 100 cubes to make each of these models or answer the questions by looking at the diagrams.

Suppose the cube painter was able to paint the entire surface, including the bottom, of the model. Fill in the table for each of the models that you make.

Make a model of your own.

How many cubes would have:

WHICH MODEL HAS THE SMALLEST NUMBER OF CUBE FACES PAINTED?_____

MODEL	A	B	C	D	E	yours
6 faces painted	___	___	___	___	___	___
5 faces painted	___	___	___	___	___	___
4 faces painted	___	___	___	___	___	___
3 faces painted	___	___	___	___	___	___
2 faces painted	___	___	___	___	___	___
1 face painted	___	___	___	___	___	___
0 faces painted	___	___	___	___	___	___

MODEL A

100 CUBES LONG

YOU PROBABLY WILL ONLY WANT TO MAKE PART OF THIS ONE.
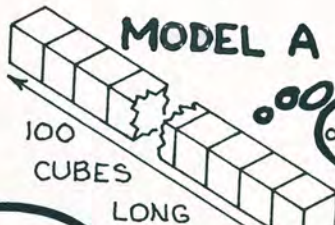
BUILD THIS ONE FLAT ON YOUR TABLE.

MODEL B

25 CUBES LONG

MODEL C

10 CUBES LONG

10 CUBES WIDE

MODEL D

10 CUBES LONG

5 CUBES HIGH

MODEL E
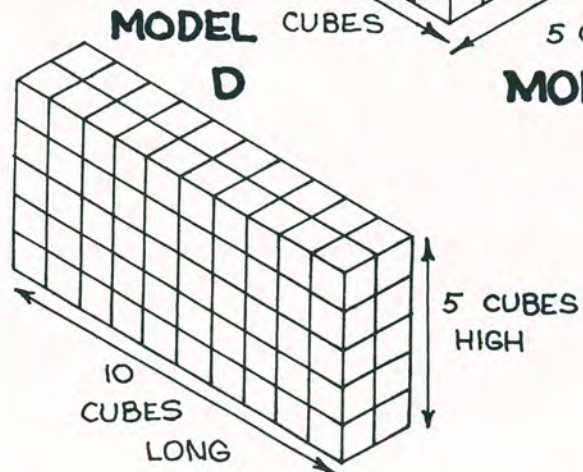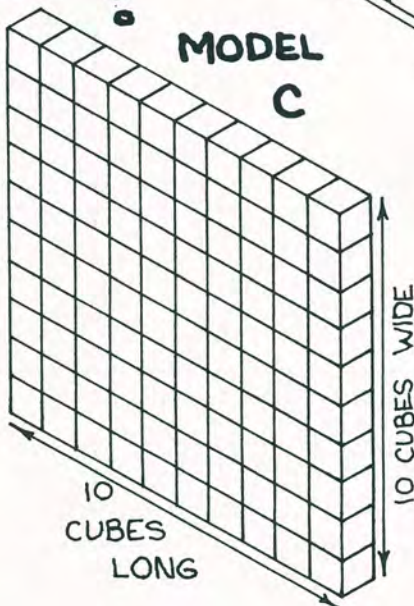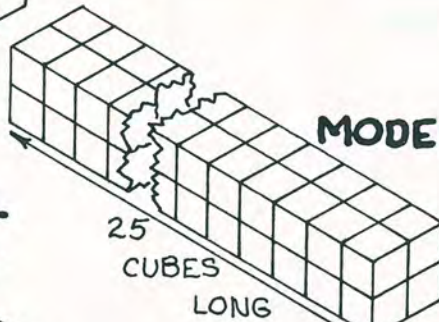
5 CUBES

5 CUBES

4 CUBES

PAINT-LESS

Materials: 36 cubes   }1 UNIT

1) Use 12 cubes to build as many differently shaped rectangular prisms (box-like shapes) as you can. Record the dimensions of each prism in the table. Find and record the surface area and the total length of all the edges of each prism.

Which prism would be the cheapest to paint? _____
_____ (Write the dimensions.)

4 UNITS
3 UNITS
1 UNIT
4 UNITS
1 UNIT
3 UNITS

THESE PRISMS ARE NOT DIFFERENTLY SHAPED BECAUSE THEIR SIDES ARE THE SAME LENGTH.

Dimensions	Surface area	Total length of edges
4 units x 1 unit x 3 units	38 square units	32 units
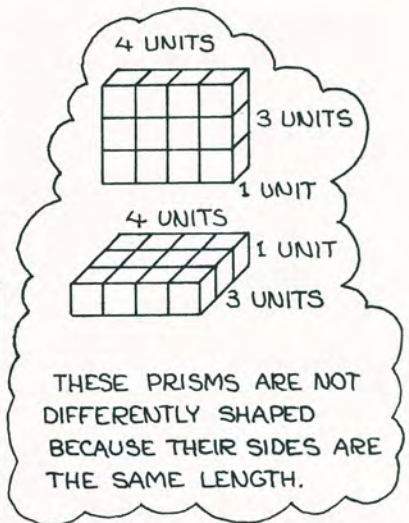
2) Repeat investigation #1 with 24 cubes. Record your answers in the table below.

Which 24 cube prism would be the cheapest to paint? _____

YOU MAY WISH TO COMPARE YOUR PRISMS WITH A FRIEND'S.

Dimensions	Surface area	Total length of edges
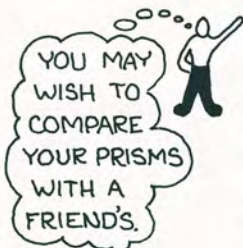
3) Repeat investigation #1 with 36 cubes. Draw a table like the one above and record your answers.

Which 36 cube prism would be the cheapest to paint? _____

4) Challenge: Write a general rule for finding the total surface area of any prism of this type. _____
_____

Write a general rule for finding the total length of the edges of any prism of this type. _____

749

CHANGING SURFACE AREAS 1

Materials Needed: 125 centimetre cubes

Activity:
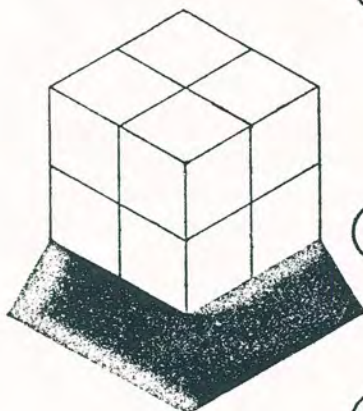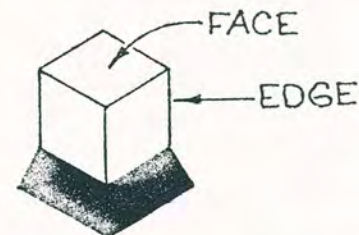
FACE
EDGE

1) Use 1 cube. Call it the unit cube.
a) How many faces on the cube? _____
Each edge is 1 cm, so each face is 1 cm by 1 cm and has an area of 1 cm^2.
b) The unit cube has a total surface area of _____ x 1 cm^2 or _____ cm^2.

2) Build a model of a cube, with each edge 2 cm. You should have used 8 cubes.
a) What is the area of each face? _____
b) What is the surface area of this cube? 6 x _____ or _____

3) Continue to build larger cubes with edges of 3 cm, then 4 cm, then 5 cm . . . until you run out of cubes. Find the surface area of each cube and record it in this chart. See if you can finish the chart up to a cube that has an edge of 10 cm.

EDGE OF CUBE (cm)	TOTAL CUBES USED	AREA OF 1 FACE (cm²)	SURFACE AREA OF CUBE (cm²)	RATIO OF SURFACE AREAS OF LARGE CUBE TO UNIT CUBE	SIMPLIFIED RATIO
1	1	1	6	6 : 6	1 : 1
2	8	4	24	24 : 6	4 : 1
3	27				
4	64				
5					
6					
7					
8					
9					
10					

4) Predict the simplified ratio of the surface area of a large cube to the surface area of the unit cube if the large cube has an edge of:

a) 20 cm _____ : _____   b) 12 cm _____ : _____   c) 1.5 cm _____ : _____

CHANGING SURFACE AREAS 2

Materials: A set of centimetre cubes (at least 200)

Activity:

MODEL 1
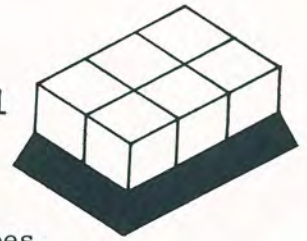
① Make a 3 x 2 x 1 box-shaped model from the cubes.
Find the surface area of the model. _____ cm^2

MODEL 2
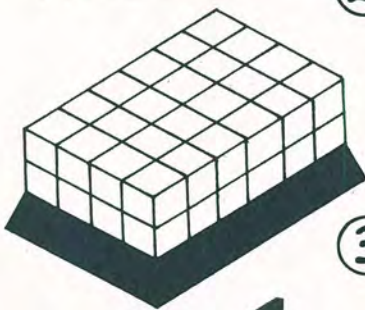
② Make a box-shaped model twice as long, twice as wide and twice as high as the model in exercise #1.
a) The surface area of Model 2 is _____ cm^2.
b) The surface area of Model 2 is _____ times the surface area of Model 1.

③ Make a box-shaped model three times as long, three times as wide and three times as high as the model in exercise #1.
a) The surface area of Model 3 is _____ cm^2.
b) The surface area of Model 3 is _____ times the surface area of Model 1.

④ Complete the chart below for models that have dimensions 4 times the dimensions of Model 1; 5 times; 6 times.

MODEL	SIZE	SURFACE AREA (cm^2)
1	3 × 2 × 1	
2	6 × 4 × 2	
3	9 × 6 × 3	
4		
5		
6		

a) The surface area of Model 4 is _____ times the surface area of Model 1.
b) The surface area of Model 5 is _____ times the surface area of Model 1.
c) The surface area of Model 6 is _____ times the surface area of Model 1.

⑤ You predict: If a model has dimensions that are 10 times the dimensions of Model 1, its surface area is _____ times the surface area of Model 1.

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

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VOLUME

The volume of a geometric figure refers to the amount of space the figure occupies or encloses. A rectangular solid might <u>occupy</u> the same amount of space as 24 centimetre cubes. An empty box might <u>enclose</u> a space which could be occupied by 24 centimetre cubes. Both have a volume of 24 cubic centimetres or 24 cm^3. The word volume is also used to refer to the capacity of an open container like a bottle, swimming pool or paper cup. In each case giving the volume of an object involves stating a number and a unit.

SOME HANDS-ON ACTIVITIES

Providing concrete experiences with volume (before abstracting to paper and pencil activities or formulas) might give students a better understanding of volume. Here are two suggestions.

- Have students find the volume of open boxes by filling them with unit cubes. Have students approximate the volume of small sealed boxes by building congruent (or nearly congruent) shapes with unit cubes. (This works well with geoblocks--see *Volume with Geoblocks*.)
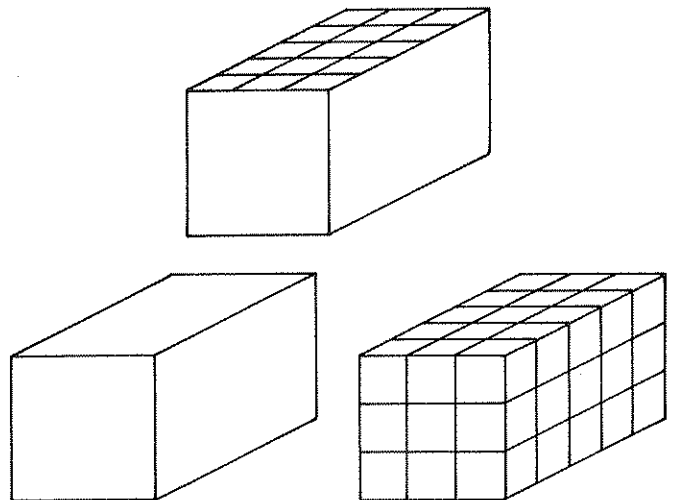
- Glue or tape unit cubes together in various shapes. Let students handle the shapes and determine the volume by counting the cubes. Ask students to build shapes with a volume of 12 cubic centimetres, 10 cubic inches and 6 cubic decimetres. Do your students realize they must use cubes of different sizes in each case?
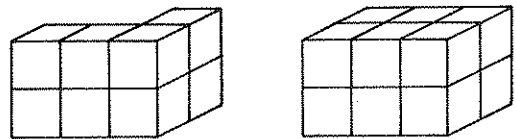
Many volume activities require students to look at drawings of three-dimensional objects and find the volumes. Often students do not interpret these drawings the way we intend. Some students might not realize that the drawing to the right represents anything more than side-by-side parallelograms. If asked to find the volume of this solid, some students will answer "8" because there are eight parallelograms.
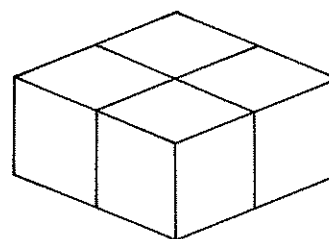
Stacking Cubes asks students to make a model with cubes to correspond to the picture shown. This important step should help students interpret the pictures correctly. They should realize that there are many hidden cubes and that these drawings represent "layered" figures. After building a model of cubes to correspond to a picture, students can count the cubes in their models. They can then check the picture again. What cubes are not shown in the picture? Can students use picture (2) from the page and explain why 14 cubes are needed?

Perhaps if more students had spent time learning to interpret drawings like those shown in *Stacking Cubes*, student performance on the seemingly simple volume question in the recent National Assessment might have been better. (See page 2 of the main commentary to AREA & VOLUME for more information on this National Assessment question.) Many of the classroom pages in the Volume, Surface Area and Polyhedra subsections ask students to build models from pictures. Let's be sure to include this important step of learning to correctly interpret drawings of three-dimensional objects.
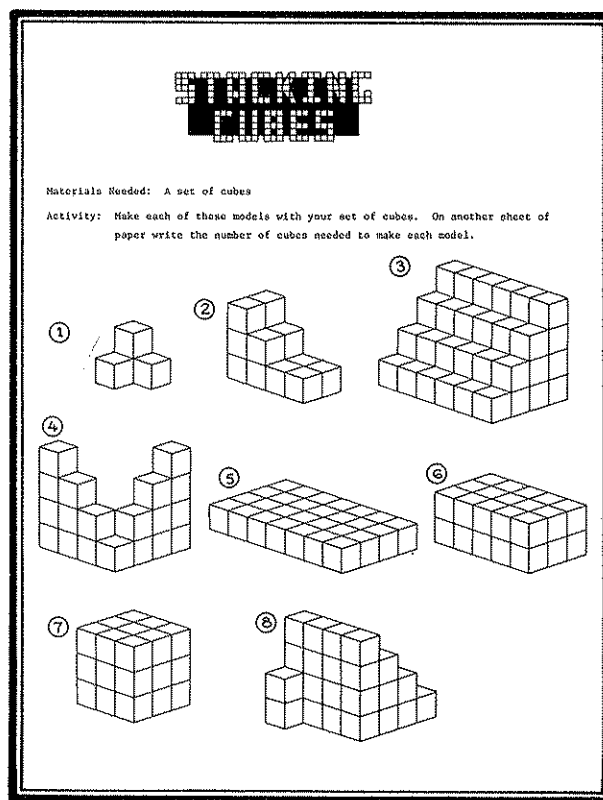
THE DISPLACEMENT METHOD OF FINDING VOLUME

The volume of an irregularly-shaped object can be found by submerging it in a full beaker of water and measuring the overflow or by finding the change in water level of a partially-full graduated cylinder. The volume of the rock in the beaker to the right is 25 millilitres

WATER LEVEL
AT 75 ml

WATER LEVEL
AT 100 ml

(or 25 cubic centimetres, since 1 millilitre and 1 cubic centimetre are measurements of space). *Methods for Finding Volume*--part C and *Eureka, I've Found It* use the displacement method.

Research seems to show that this displacement method is much more difficult for students to understand. You probably have students in your class who are not ready to comprehend displacement of volume. You might try some of the activities below to see if students realize the volume of objects is conserved "underwater." If students have trouble understanding the activities below, you might want to stay with more direct methods of finding volume--filling containers with blocks, water or sand.

- Show students a group of rocks submerged in a beaker of water. Ask if the water level will change if the rocks are spread out more on the bottom of the beaker. Do students realize that the rocks will displace the same amount of water in each arrangement?
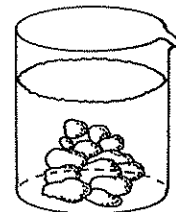
- Show students two objects of the same size and shape but of different mass. Ask which object ct will displace more water. Students might think t the heavier object will displace more water. They can check their guess by putting each object in a full beaker of water and measuring the overflow or by reading the change in level on a partially-full graduated cylinder. Ask if an object will displace more water in a fat container or a thin one--a tall one or a short one. Do students realize that the shape of the container does not affect the amount of water displaced?

STEEL BALL

MARBLE

OR   ?

OR   ?
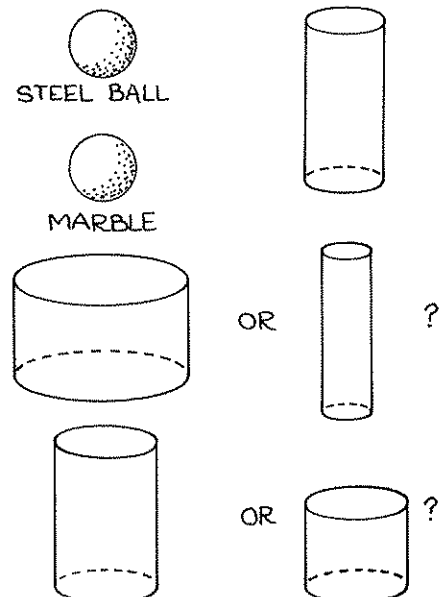
● Glue or tape together plastic or wooden cubes in
  the two arrangements shown. Tape one arrangement
  to the bottom of a large coffee can and fill the
  can with water. Measure the amount of water used
  to fill the can. Empty the can. Replace the first
  arrangement with the second. Ask if more or less
  water will be needed to fill the can this time. Do
  students realize the same amount of water will be
  needed in both cases?
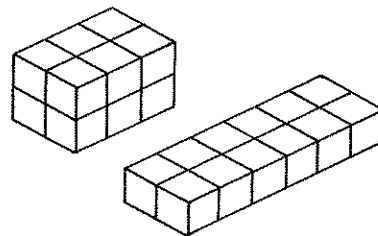
SOME VOLUME EXPLORATIONS

There are many volume investigations which can involve students in problem
solving, applications and laboratory activities.

● Have students mark irregular containers at the
  half-full (or half-empty) point. Do they mark the
  cone closer to the top than to the bottom? What
  methods can be used to determine where the contain-
  ers should be marked?

● Have students cut sectors from the same size cir-
  cles to form open cones. Is there a relationship
  between the volume of a cone and the percent used
  from the circle? Students can use rice to find
  the volumes and make a graph to decide.

● Ask students to devise methods for finding their
  own volumes. The water displacement method would
  work, but here is another way of approximating a
  person's volume. Most people float or almost
  float, so their bodies have about the same den-
  sity as water. A cubic foot of water weighs about
  62 pounds. A litre of water has a mass of almost
  exactly 1 kilogram. A person who weighs 124 pounds
  has a volume of about 2 cubic feet. A person who
  has a mass of 40 kg has a volume of 40 litres.
  (Hurrah for the ease of computing with metric mea-
  surements!)
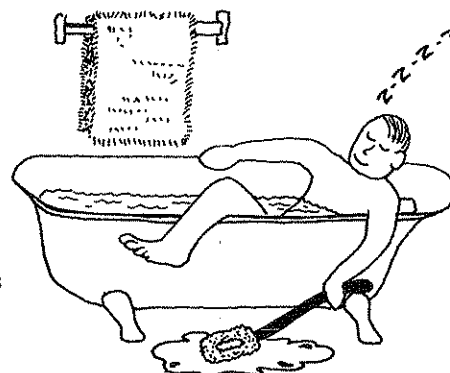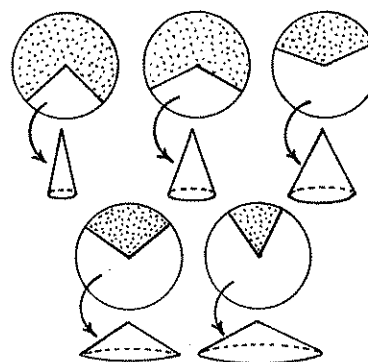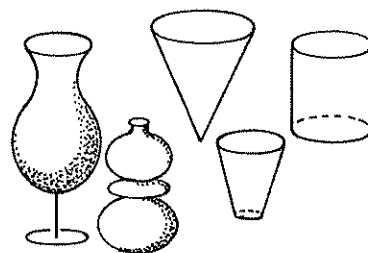
● Science Research Associates suggests students investigate the size of eggs. What are the volumes of small, medium, large, extra large and jumbo eggs? Which size of egg is the best buy? Students can use hard-boiled eggs and the displacement method to find volumes. A calculator would be useful for finding the cost per millilitre of egg. Students might like to do some research to find out how eggs are sized--is it by volume, mass, length, width or some other method?

EGG SIZE	VOLUME OF ONE EGG	COST FOR ONE EGG	COST PER ml
SMALL			
MEDIUM			
LARGE			
X-LARGE			
JUMBO			

● Students who can find the capacity of a container sometimes have difficulty finding the amount of material needed to make the container. The capacity of a mug might be $1\frac{1}{2}$ cups, but what is the volume of the clay of which the cup is composed? A more likely problem is this:

A concrete water tub is shown to the right. How much water will it hold if the concrete walls and bottom are exactly 1 decimetre thick? How much concrete would be needed to make such a tub?
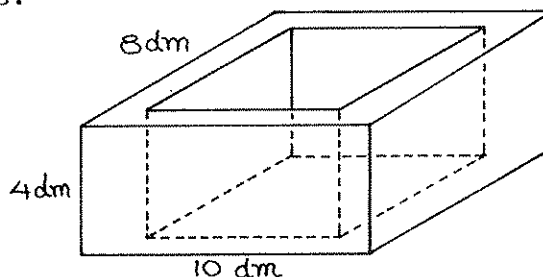
A class of college students who were studying to be elementary teachers had much difficulty with this problem until they used cubes to build a model of the tank. The answer was obtained by counting cubes, but the students also discovered they could find the amount of cement by subtracting: (10 x 4 x 8) - (8 x 3 x 6). Building a model does help in many cases!

SURFACE AREA AND VOLUME--ARE THEY RELATED?

Students can explore the relationship between surface area and volume much like they investigated perimeter and area (see the Area commentary). They can discover that two objects can have the same volume but different surface areas or the same surface areas and different volumes. Some suggested activities are given below.

759

- Have students arrange four cubes in various ways. Are the volumes of the arrangements different? Are the surface areas of the arrangements different? (Remind them to count bottom surfaces.) Which arrangement has the greatest surface area? The least surface area? If the four cubes represent apartment units, which arrangements might be the cheapest to build? The most economical to heat? Which arrangements would have the most wall space for windows? The smallest roof?
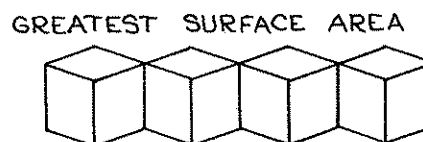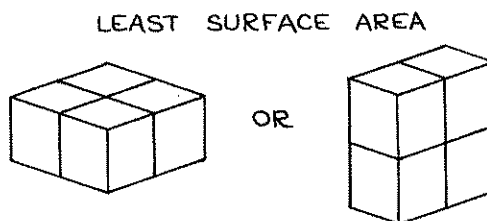
- Show students a cube built with 27 unit cubes. How would removing a corner unit cube affect the volume and the surface area? How would removing a unit cube from a different location affect volume and surface area?
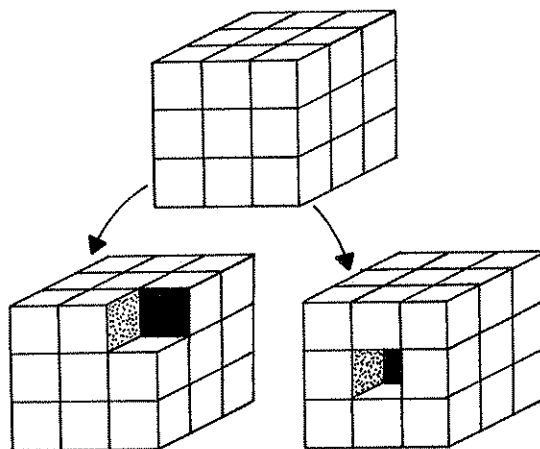
- The volume of a box is 64 cubic units. Fill in the table of possible dimensions and surface areas. What are the dimensions of the box with the least surface area? (4 x 4 x 4) Is it possible to determine the dimensions of the box with the greatest surface area? (No. To see why try fractions like $\frac{1}{2}$, $\frac{1}{4}$, ...)

LEAST SURFACE AREA

OR

GREATEST SURFACE AREA

VOLUME OF BOX : 64 cm³

WIDTH (cm)	LENGTH (cm)	HEIGHT (cm)	SURFACE AREA (cm²)
1	1	64	258
1	2	32	196
1	4	16	168
1	8	8	160
2	2	16	136
2	4	8	112
4	4	4	64
$\frac{1}{2}$	1	128	385

- Use centimetre cubes to build larger cubes. Make a table to show the length of an edge, the volume and the surface area of each cube. Students might benefit from comparing the graphs for the volumes ($v = e^3$) and the surface areas (S.A. $= 6e^2$).
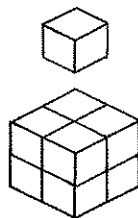
EDGE	VOLUME	SURFACE AREA
1 cm	1 cm^3	6 cm^2
2 cm	8 cm^3	24 cm^2
3 cm	27 cm^3	54 cm^2
4 cm	64 cm^3	96 cm^2
5 cm	125 cm^3	150 cm^2
6 cm	216 cm^3	216 cm^2

- Bring some bubble solution to class. Students can blow bubbles from frames of different shapes. Why is each bubble a sphere? Why not a cube or tetrahedron? (The soap film encloses a certain amount of air and a sphere has the least surface area for a given amount of volume.) Other soap film surfaces of least area can be demonstrated by dipping wire polyhedral frames into solution.
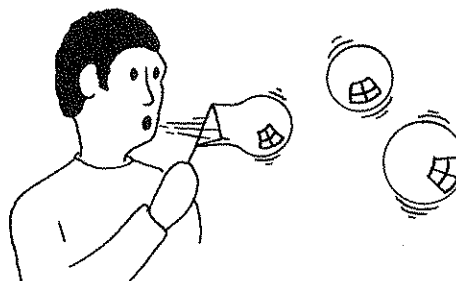
A cube with a volume of 1 cubic metre has more surface area than a sphere whose volume is 1 cubic metre.

Largest Container, Hot also relate surface area and volume. *Mathematics in Science and Society* gives a series of interesting activities on the relationship between the surface area and volume of living things.
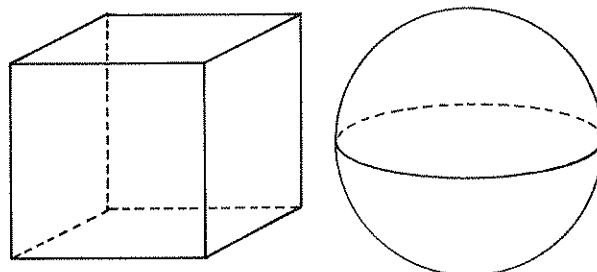
You Be the Judge and *When You're Hot, You're* MATHEMATICS AND BIOLOGY in the resource

DEVELOPING VOLUME FORMULAS

The students' understanding of volume in concrete settings can be the foundation on which shortcuts or formulas are developed. Some students will begin taking shortcuts when asked to find the number of cubes needed to fill a box. They might put one layer of cubes on the bottom of the box and a stack of cubes in a corner. If there are 12 cubes in a layer and 2 more cubes stacked in a corner to show 3 layers, 3 x 12 or 36 cubes must be needed. This idea is developed in *Layer Upon Layer* shown to the right. Here students have a chance to develop their own volume shortcut for box-like shapes.

The length x width x height formula for rectangular prisms might make sense to students when the dimensions are whole numbers, but does it work in other cases? *I. C. Parts Company* has students count the whole unit cubes and parts of unit cubes in figures like that shown to the right. They then compare the number of unit cubes to the product of the dimensions. Again the formula is tied to viewing volume as a number with a unit.

The development of several other volume formulas is discussed in *Volume with Geoblocks* and *Cylinders, Cones and Spheres.*

12 WHOLE CUBES
6 HALF CUBES
} 15 CUBIC UNITS

$2\frac{1}{2} \times 2 \times 3 = 15$

SOME ADDITIONAL NOTES

Many of the ideas in this resource can be used to teach several concepts. For example, the page *I. C. Parts Company* has students count whole unit cubes and part unit cubes to find the volume of a box-like shape. This same idea could be used for an area lesson. By counting, the area of the figure to the right is $15\frac{3}{4}$ square units. The product of the dimensions, $4\frac{1}{2}$ x $3\frac{1}{2}$ is also $15\frac{3}{4}$. You might want to follow the format of *I. C. Parts Company* to develop an area page.
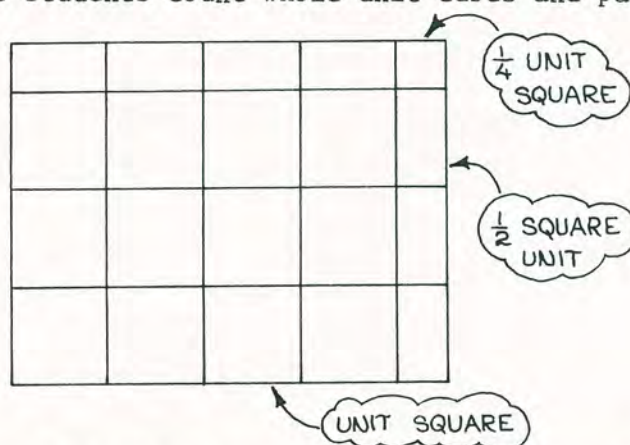
Ideas for teaching area can sometimes be adapted to volume lessons. Here are some volume adaptations of ideas discussed on page 4 of the Area commentary.

- Make a solid figure from clay or Play-Doh, or use oranges, styrofoam solids, etc. Tell students the volume of the solid (this can be an arbitrary number of cubic units). Cut the solid into various fractional parts. What is the volume of each part? If the volume of one part is given, can students determine the volume of the whole solid?
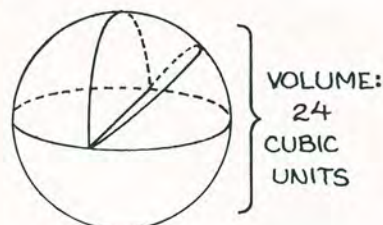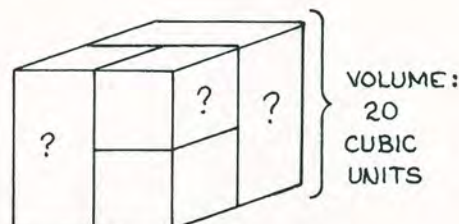
- Make a cardboard box and 6 congruent paper blocks which will fill the box as shown in the top diagram to the right. Make 6 more congruent blocks which will fill the box as shown in the bottom diagram. Ask students to compare the volume of one block with a block of a different shape. Do they realize both blocks have the same volume?
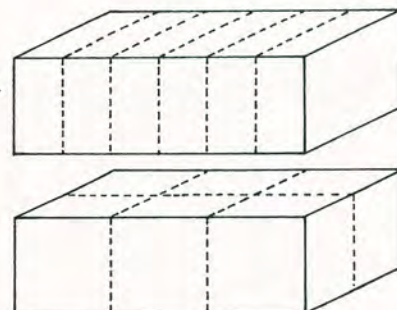
These few adaptations connect area, volume and fraction ideas. You can probably find many more ideas you will want to adapt throughout the resource.

TALKING CUBES

Materials Needed: A set of cubes

Activity: Make each of these models with your set of cubes. On another sheet of paper write the number of cubes needed to make each model.

FITTING THE PIECES TOGETHER

Materials Needed: A set of Soma® pieces

Activity:
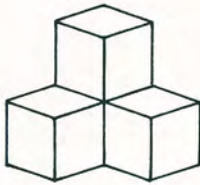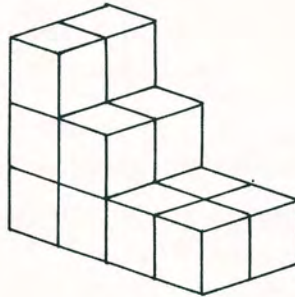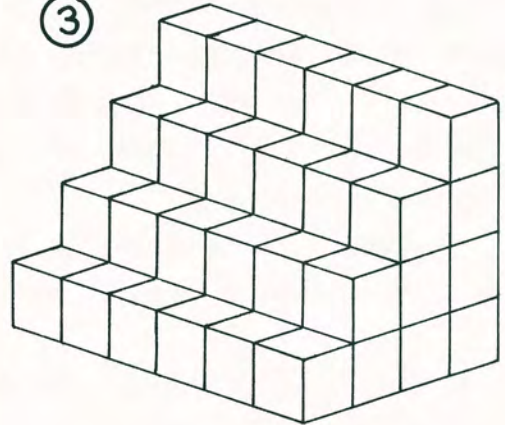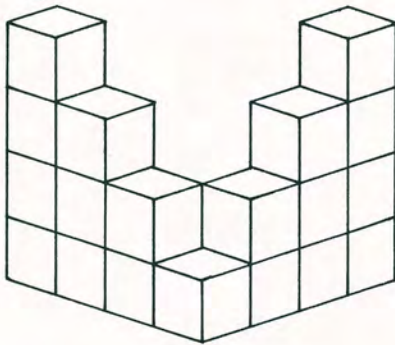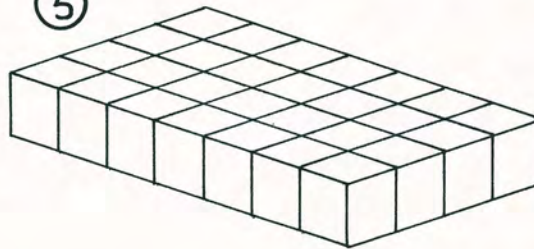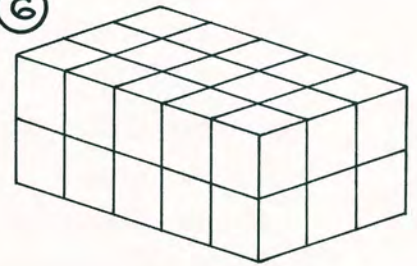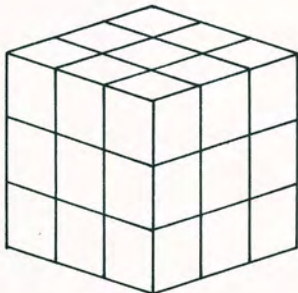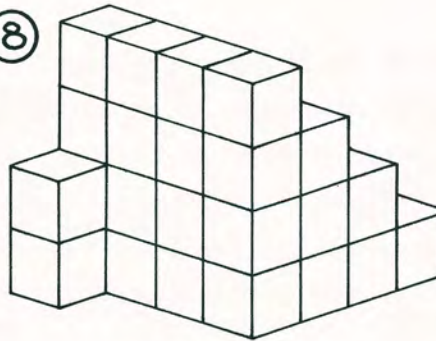
A) Write the number of cubes needed for each piece of the Soma® puzzle.
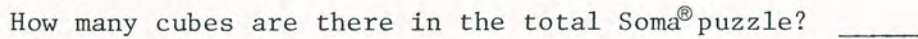
1)   2)   3)   4)   5)   6)   7)
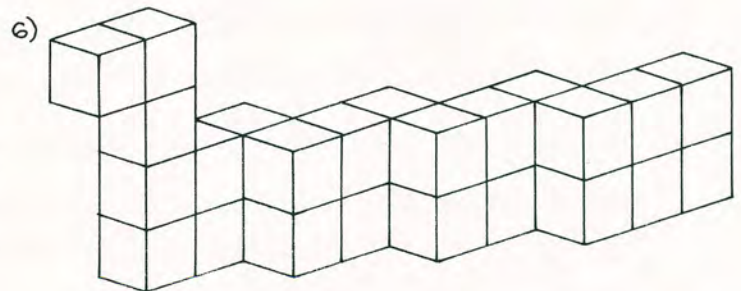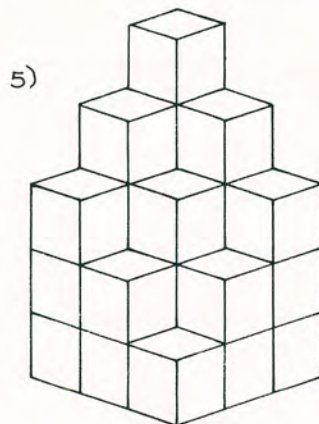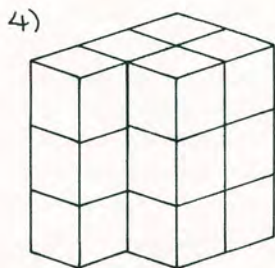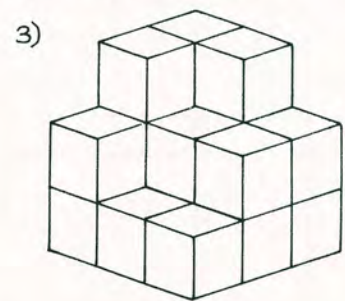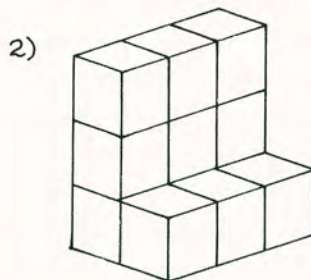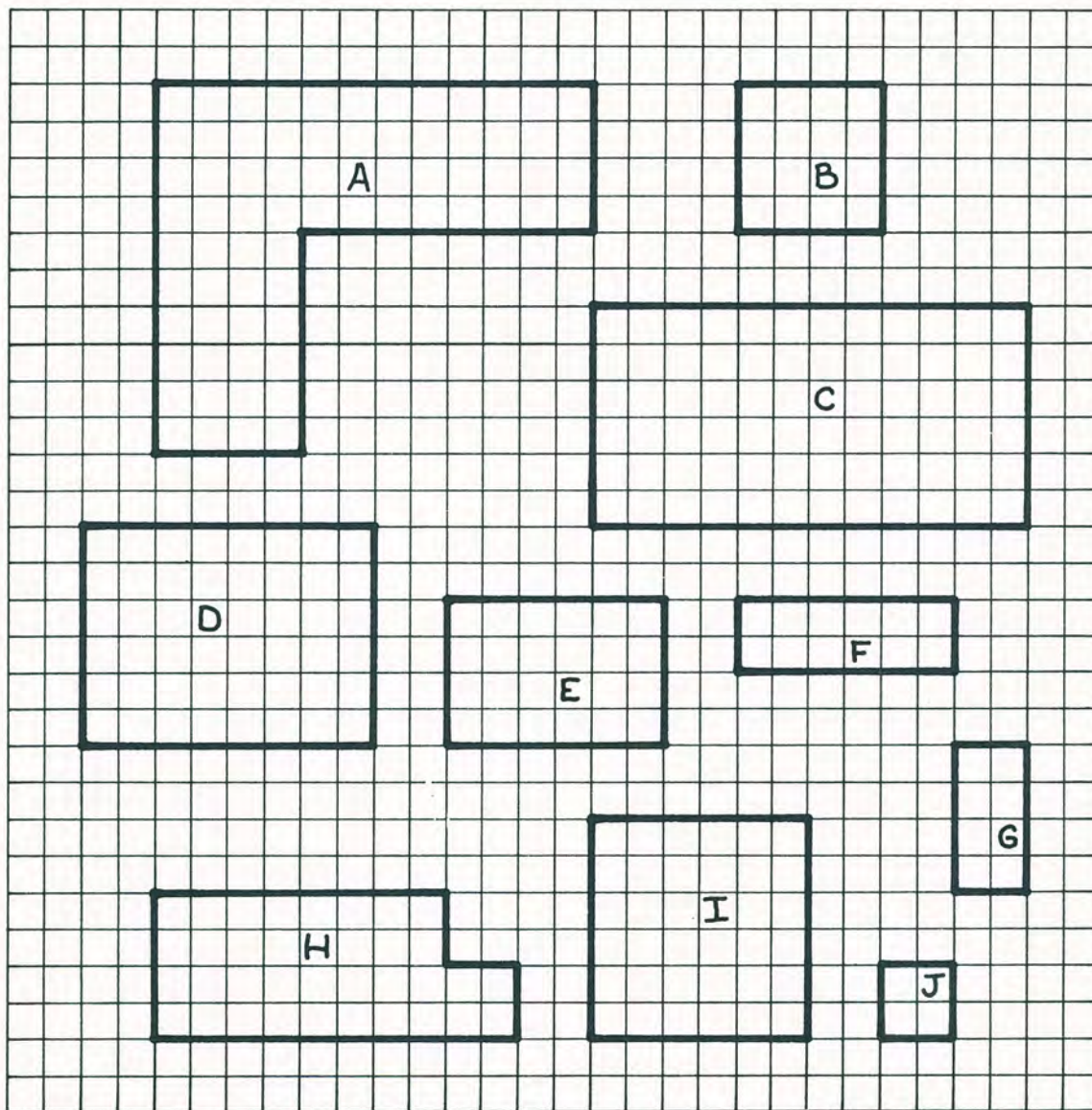
___   ___   ___   ___   ___   ___   ___

How many cubes are there in the total Soma® puzzle? ___

B) Each of the figures below was made with pieces of the Soma® puzzle. Write the
    number of cubes needed for each figure. There are no hidden holes in the figures.
    You can use your Soma® pieces to make the figures (some are hard to do).

1)   2)   3)

___   ___   ___

4)   5)   6)

___

LAKE & ISLAND BOARD

Sample questions for perimeter, area and volume investigations are written below. These can be developed into activity cards for use with the lake and island board.

Perimeter (Materials: centimetre ruler, string, Cuisenaire rods)

On your lake and island board 1 centimetre represents 1 kilometre.

1) Which island do you think is longer to walk around, island A or island C? Guess _____

2) Measure to check your guess.

   The perimeter of island A is _____ kilometres.

   The perimeter of island C is _____ kilometres.
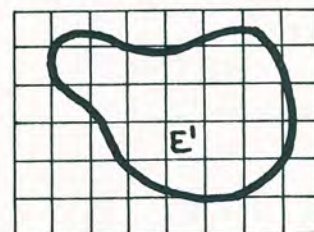
3) Do you think any islands have the same perimeter? _____
After you make your guess measure the perimeter of each island.

A _____   D _____   G _____   J _____

B _____   E _____   H _____

C _____   F _____   I _____

(If island E' is on the board have students suggest methods for measuring the perimeter.)

4) With Cuisenaire rods build bridges to link all the islands.
What is the minimum number of bridges needed? _____
What is the total length of the bridges? _____

5) Plan a route from island A to island J. How long is the route? _____

6) If you lived on island B, how far would you have to travel to reach island H?
_____   Explain your route.

Area (Materials: Centimetre transparent grid, ruler) On the lake and island
board one centimetre represents one kilometre.

1) If the islands were covered with grass, guess which would provide the largest
pasture? _____

After you make your guess find the area of each island. Be sure to estimate
before you measure.

A _____   D _____   G _____   J _____

B _____   E _____   H _____

C _____   F _____   I _____

2) Are there any islands which would produce the same amount of grass? _____
_____   _____

If the areas of two islands are the same are their perimeters also the same?
_____   Explain _____

3) If it is possible to graze 200 cattle on island J, how many cattle could you
graze on island D? _____

4) The owner of island E wishes to exchange it for island D. How much bigger in
area is island D than island E? _____

5) Suppose the owner of island E wishes to triple her land holdings. Which island should she buy? _____

6) You have just parachuted from a 747 over the islands. What is the probability that you will not land in the water? _____

7) You have inherited $5000 and have decided to purchase an island. Land on each island sells for $100 for each km^2. Each kilometre of water frontage costs an additional $50. Which islands could you buy? _____ _____

Volume (Materials: 100 centimetre cubes)

1) An old resort hotel stands on island E. The hotel completely covers the island and is two stories high. Use your cubes to build the hotel. If each cube represents one room, how many rooms are in the hotel? _____

2) You own island B and wish to build a hotel on it that has the same amount of room as the old resort hotel on island E. Use cubes to build the hotel. If the hotel must completely cover the island how many stories high is the building? _____

3) The owner of island F wishes to build a hotel that has twice the number of rooms as the old resort hotel on island E. If the hotel completely covers island F, how many stories high will the building be? _____

4) Suppose you wish to build a large hotel with 144 rooms. If building codes prohibit buildings more than 10 stories high, on which islands could you build? _____ ____

THE PAINTED CUBE

Materials: about 100 wooden cubes
Use the cubes to answer the questions following the story.
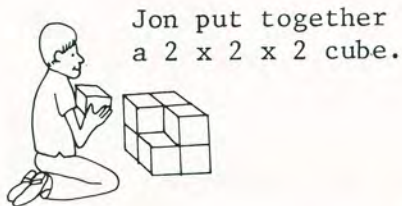
Jon put together a 2 x 2 x 2 cube.

Then he painted the six outside faces.
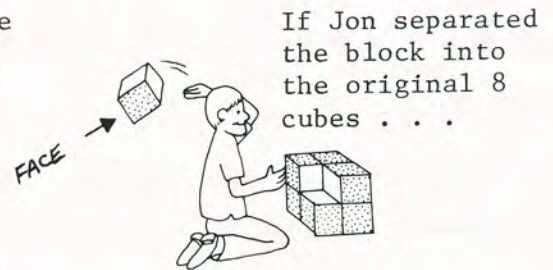
If Jon separated the block into the original 8 cubes . . .

FACE

I. How many **small** cubes would have exactly . . .
    4 faces painted? _____
    3 faces painted? _____
    2 faces painted? _____
    1 face painted? _____
    0 faces painted? _____
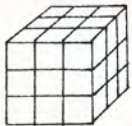
II. Repeat for a 3 x 3 x 3 cube.
    How many small cubes would have exactly . . .
    4 faces painted? _____

HOW MANY SMALL CUBES ARE THERE IN ALL? _____

    3 faces painted? _____
    2 faces painted? _____
    1 face painted? _____
    0 faces painted? _____

III. How many small cubes are <u>in the cube to the left?</u> _____

    A. How many of the smaller cubes are painted on four faces? _____
    B. How many on just three faces? _____
    C. How many on just two faces? _____
    D. How many on just one face? _____
    E. How many on zero faces? _____
    F. What is the sum of your answers to questions A, B, C, D, and E? _____
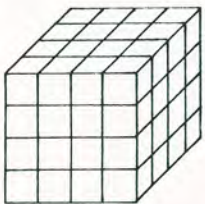
    How does this compare with your first answer to Part III?

IV. Extend your investigations to larger cubes. Put your results in a table like this one. Are there any patterns?

NO.OF CUBES ON A SIDE	NUMBER OF PAINTED FACES					TOTAL NO. OF CUBES
	0	1	2	3	4	
2						
3						
4						
5						
6						
8						
10						
20						

769

VOLUME WITH CUBES

Materials Needed: A set of at least 100 cubes

Activity:

1) Build each of these models. Record the number of cubes needed for each model.
   Record the length, width and height below each model.

a)   b)   c)   d)
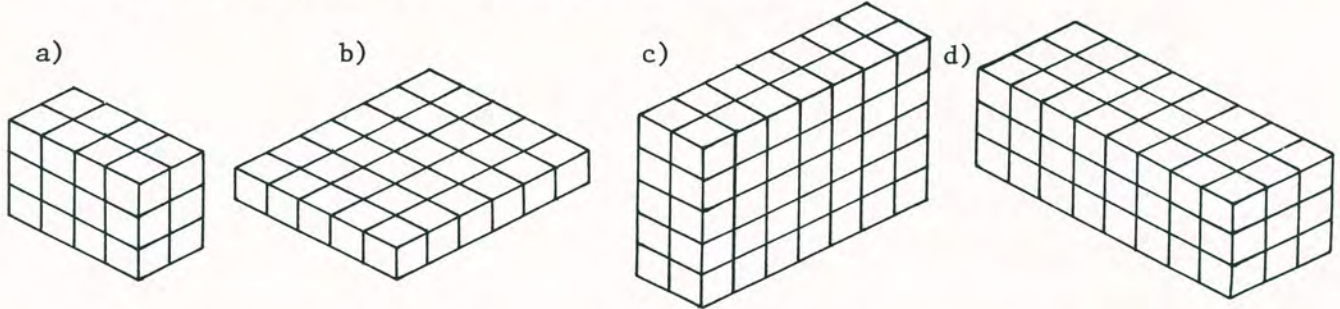
2) The dimensions of four models are
   given below. Build each model and
   record the number of cubes needed.

	LENGTH	WIDTH	HEIGHT	NUMBER
a	4	2	4	
b	3	5	2	
c	5	7	1	
d	8	2	3	

3) The number of cubes needed to
   build a box model are given below.
   Record possible dimensions for each
   model, then build the model.

	CUBES	LENGTH	WIDTH	HEIGHT
a	60			
b	36			
c	100			
d	78			

Compare your models with a friend's.
Are they the same or different?

4) The number of cubes needed to build a box model are given below. Use the clues to
   help you find the dimensions. No models can have a dimension of 1 unit. Build each
   model.

	Length	Width	Height
a) 64 cubes (all dimensions the same)	_____	_____	_____
b) 64 cubes (all dimensions different)	_____	_____	_____
c) 36 cubes (all dimensions different)	_____	_____	_____
d) 48 cubes (two dimensions the same)	_____	_____	_____

LAYER UPON LAYER
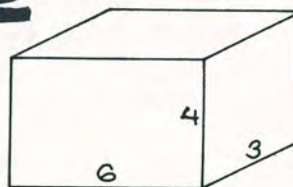
A) How many unit cubes would be
   needed to fill this box?

To find out:

1) How many unit cubes would
   be needed to cover the
   base? _____

2) How many layers would it
   take to fill the figure?

   _____

3) The figure has a volume of
   _____ cubic units.
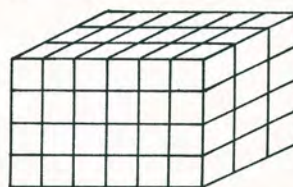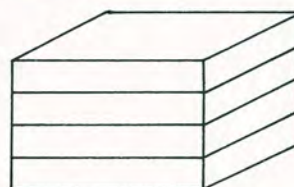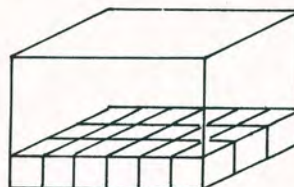
B) 1) How many unit cubes would
      cover the base? _____

   2) How many layers to fill
      the figure? _____

   3) The volume is _____
      _____
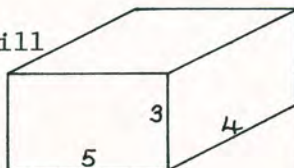
C) 1) Base _____

   2) Layers _____

   3) Volume _____
      _____
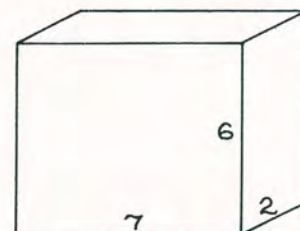
D) 1) Base _____

   2) Layers _____

   3) Volume _____
      _____
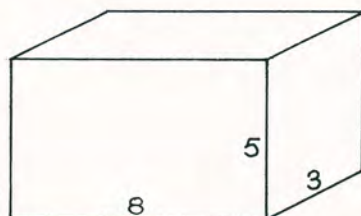
E) 1) Base _____

   2) Layers _____

   3) Volume _____
      _____
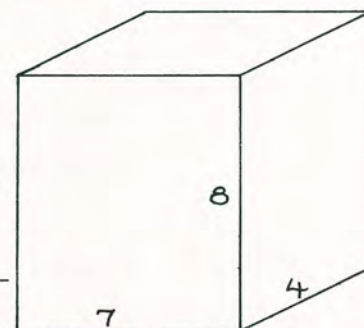
Did you discover a rule for finding the volume? _____ _____

_____

THE DARING YOUNG MAN

Find the volume of each figure. Write the letter above the correct answers in the code below.
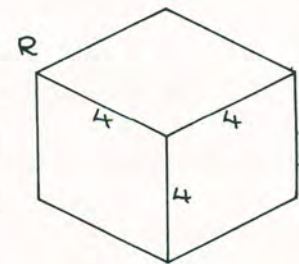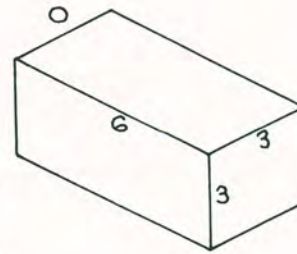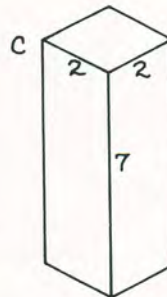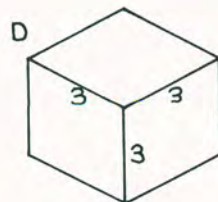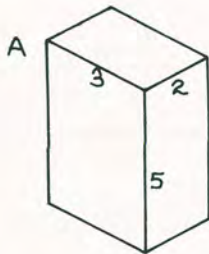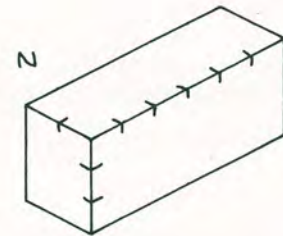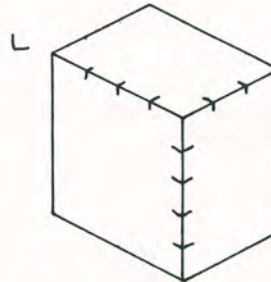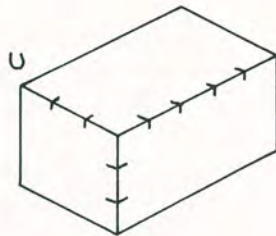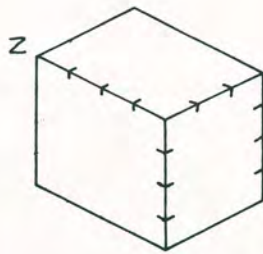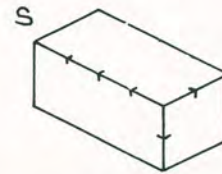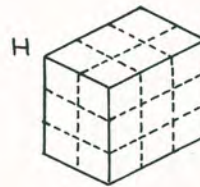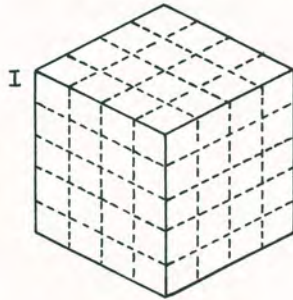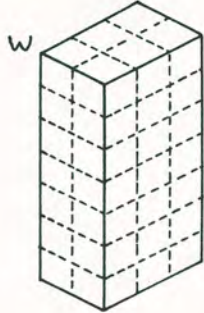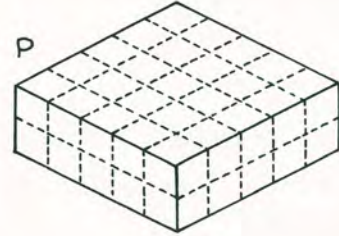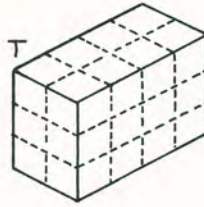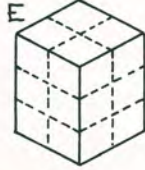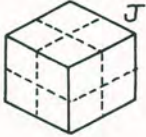
WHAT IMPORTANT EVENT HAPPENED IN 678 B.C. ?

$\overline{12}$ $\overline{45}$ $\overline{28}$ $\overline{60}$ $\overline{80}$ $\overline{27}$ $\quad$ $\overline{8}$ $\overline{54}$ $\overline{80}$ $\overline{36}$ $\overline{12}$ $\overline{27}$ $\quad$ $\overline{24}$ $\overline{18}$ $\overline{12}$

$\overline{28}$ $\overline{80}$ $\overline{64}$ $\overline{28}$ $\overline{45}$ $\overline{16}$ $\quad$ $\overline{42}$ $\overline{80}$ $\overline{24}$ $\overline{18}$ $\quad$ $\overline{18}$ $\overline{80}$ $\overline{16}$

$\overline{24}$ $\overline{64}$ $\overline{30}$ $\overline{50}$ $\overline{12}$ $\overline{48}$ $\overline{54}$ $\overline{80}$ $\overline{27}$ $\quad$ $\overline{30}$ $\overline{28}$ $\overline{24}$.

A BAKER'S DOZEN

Find the volume of each box even though part of the box is hidden. Write the letter above the correct answers in the code below.

U

T

N

S — ALL DIMENSIONS THE SAME.

H

O — 4 LAYERS

R

G — 5 LAYERS

I

C

K

E

W

M

B

D — 3 LAYERS

A — ALL DIMENSIONS THE SAME
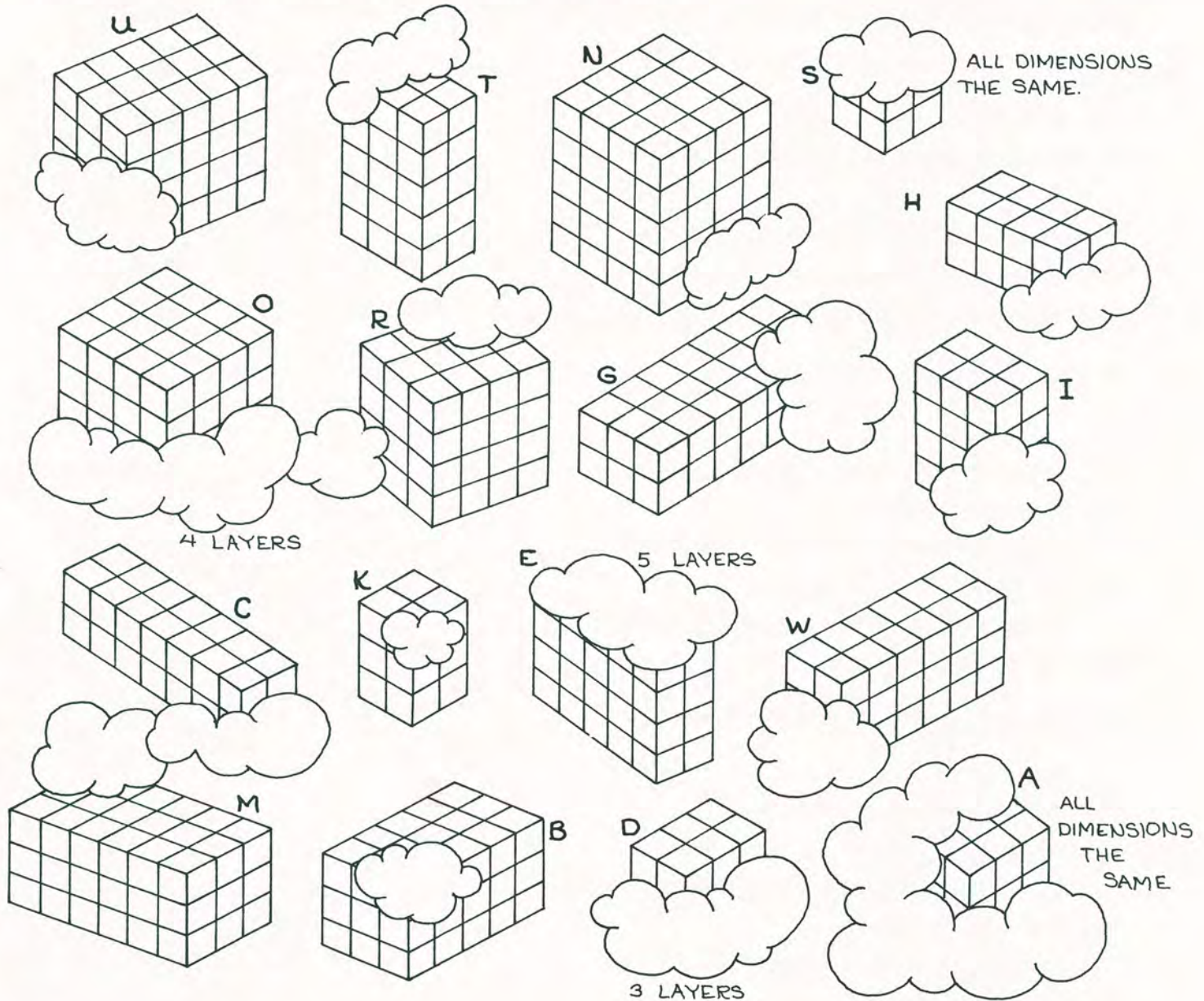
WHY DID THE BAKERS GO ON STRIKE?

THEY WANTED $\overline{}_{54}$ $\overline{}_{64}$ $\overline{}_{48}$ $\overline{}_{50}$ $\overline{}_{18}$ $\overline{}_{64}$ $\overline{}_{60}$ $\overline{}_{42}$ $\overline{}_{16}$ AND

$\overline{}_{45}$ $\overline{}_{27}$ $\overline{}_{30}$ $\overline{}_{30}$ $\overline{}_{50}$ $\overline{}_{48}$ $\overline{}_{36}$ $\overline{}_{64}$ $\overline{}_{48}$ $\overline{}_{12}$ $\overline{}_{24}$ $\overline{}_{80}$ $\overline{}_{42}$

$\overline{}_{28}$ $\overline{}_{64}$ $\overline{}_{80}$ $\overline{}_{18}$ $\overline{}_{24}$ $\overline{}_{30}$ $\overline{}_{24}$ $\overline{}_{64}$ $\overline{}_{80}$ $\overline{}_{8}$.

3.6. PARTS COMPANY

For this sheet the cube shown to the right will be the unit cube.
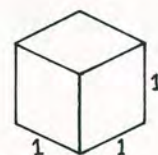
1) a) The dimensions of Figure 1 are

$2\frac{1}{2}$, _____ , _____

  b) How many uncut unit cubes in the figure?

_____

  c) How many $\frac{1}{2}$ cubes are in the figure? _____
  How many unit cubes will these make? _____

  d) The volume of Figure 1 is _____ .
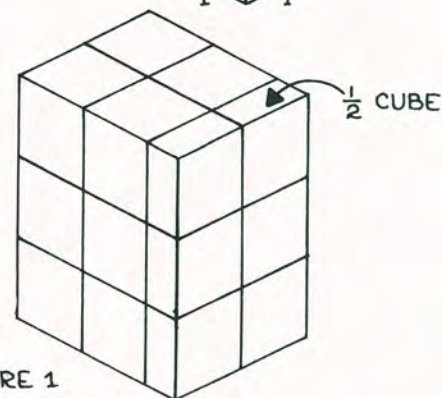
2) a) The dimensions of Figure 2 are

_____ , _____ , $2\frac{1}{2}$

  b) How many uncut unit cubes? _____

  c) How many $\frac{1}{4}$ cubes? _____
  These will make how many unit cubes? _____

  d) How many $\frac{1}{2}$ cubes? _____
  These will make how many unit cubes? _____

  e) The volume of Figure 2 is _____ .
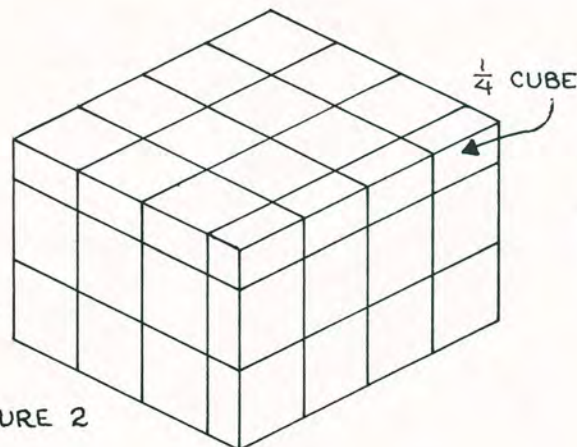
3) a) The dimensions of Figure 3 are

_____ , $3\frac{1}{2}$, _____

  b) How many uncut unit cubes? _____

  c) How many $\frac{1}{8}$ cubes? _____
  This will make what part of a unit cube? _____

  d) How many $\frac{1}{4}$ cubes? _____
  These will make how many unit cubes? _____

  e) How many $\frac{1}{2}$ cubes? _____
  These will make how many unit cubes? _____

  f) The volume of Figure 3 is _____ .
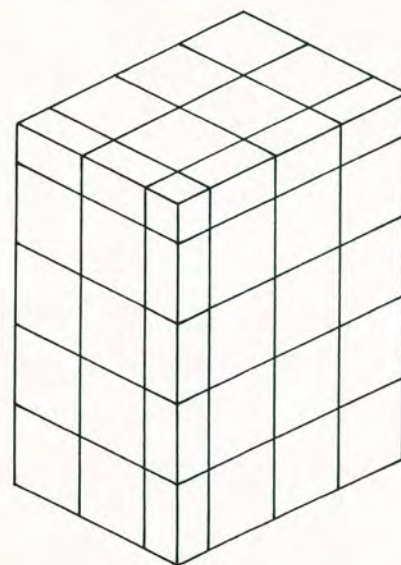
4) Multiply

  a) $2\frac{1}{2} \times 2 \times 3 =$ _____

  b) $3\frac{1}{2} \times 4 \times 2\frac{1}{2} =$ _____

  c) $2\frac{1}{2} \times 3\frac{1}{2} \times 4\frac{1}{2} =$ _____

  d) Compare these answers with the last answers in (1) - (3).

1

1   1

$\frac{1}{2}$ CUBE

FIGURE 1

$\frac{1}{4}$ CUBE

FIGURE 2

FIGURE 3

VOLUME WITH GEOBLOCKS

Geoblocks can be used to examine the concept of volume and to discover formulas relating to volume of solids. (See *Geoblocks I* in the Polyhedra subsection for a page describing the set of Geoblocks and providing some readiness activities.)

The volume of a polyhedron, measured in cubic units, is the measurement of the space enclosed by its faces. The standard volume unit in the Geoblock set is a cube 1 centimetre on a side. The rectangular prism (box) shown below has a volume of 64 cubic centimetres (cm^3) because it would take 64 of the standard units to make a congruent block.
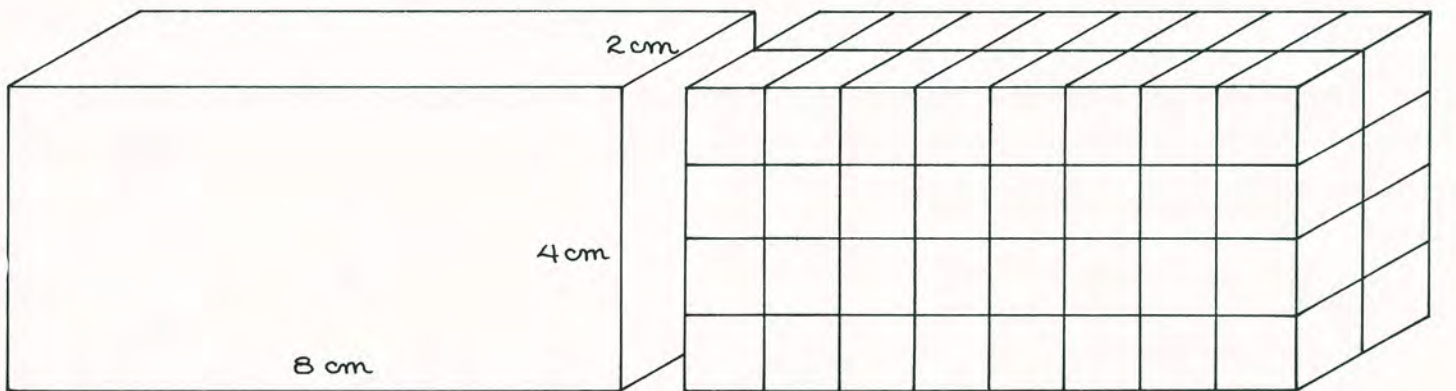
The volume could also be found by finding the number of volume units in one layer and then counting the number of layers.
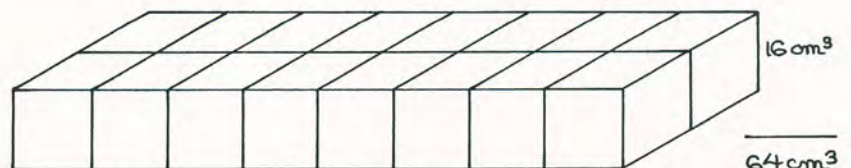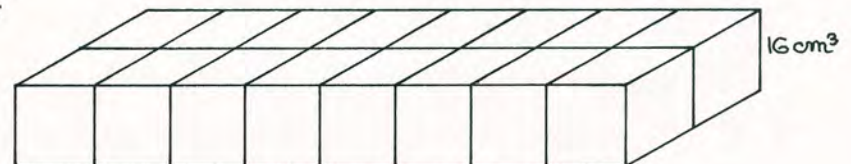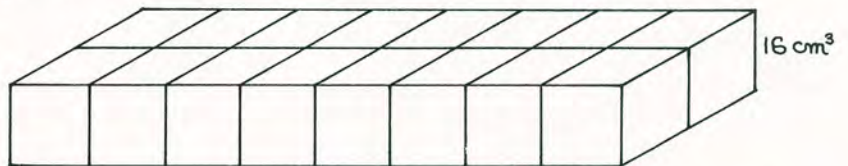
Each process can lead to the discovery that the volume of a rectangular prism is determined by multiplying the length, width and height.

$$V = \ell \times w \times h$$

VOLUME WITH GEOBLOCKS

(PAGE 2)

Students can discover that two congruent right triangular prisms can be placed together to form a rectangular prism. Hence the volume of each right triangular prism is half the volume of the rectangular prism. The example shows two right triangular prisms, each with a volume of 4 cm^3.
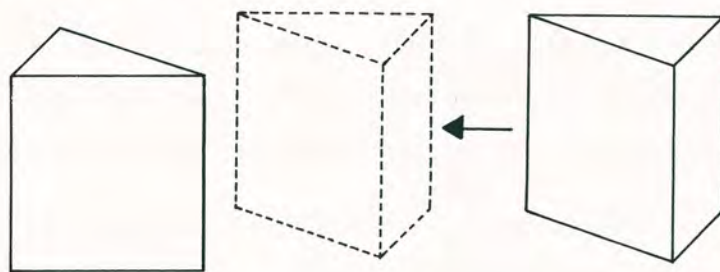
$$V = \tfrac{1}{2} \times l \times w \times h$$
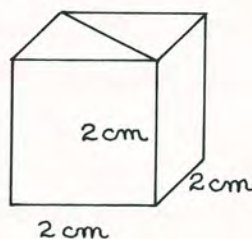
The volume of isosceles triangular prisms can be determined in two ways. First find two right triangular prisms which can be placed together to make a block congruent to the isosceles triangular prism.
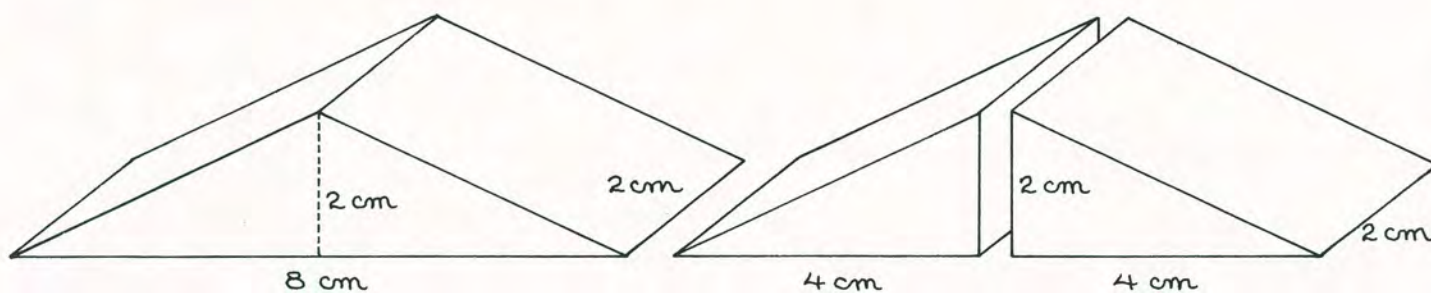
Method 1:

The extra two blocks can be placed to form a rectangular prism which has a volume of 32 cm^3, so the volume of the isosceles triangular prism is half as much or 16 cm^3.
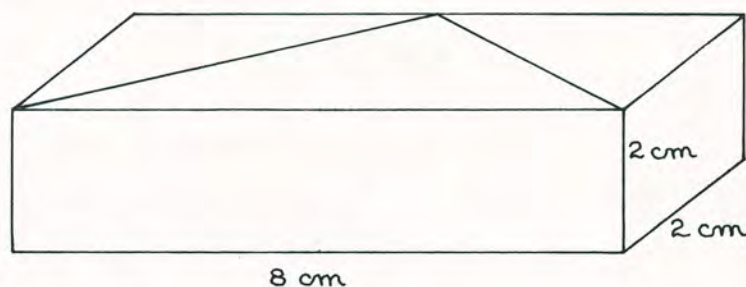
VOLUME WITH GEOBLOCKS

Method 2:

From the previous discovery about right triangular prisms each of the extra blocks has a volume of $\frac{1}{2} \times \ell \times w \times h$ or $\frac{1}{2} \times 4 \times 2 \times 2$ or 8 cm^3. So the isosceles triangular prism has a volume that is twice as much or 16 cm^3. Note that the length used here is the length of the right triangular prism.

For isosceles triangular prisms: $V = \frac{1}{2} \times \ell \times w \times h$

Note: These three formulas generalize to the standard formula for the volume of any prism, i.e. $V = Bh$ where B represents the area of the base. For the rectangular prism $\ell \times w$ represents the area of the base and for a triangular prism with one of the triangular faces as the base $\frac{1}{2} \times \ell \times w$ represents the area of the base.

To find the volume of the pyramid geoblock students will probably try to fit pyramids together to form some other shape. Three pyramids can be placed together to form a rectangular prism (actually a cube). So the volume of the pyramid is $\frac{1}{3} \times \ell \times w \times h$ or $21\frac{1}{3} \text{ cm}^3$. This formula also generalizes to the standard formula for the volume of a pyramid, i.e. $V = \frac{1}{3} Bh$ where B represents the area of the base.

COMPLETED CUBE
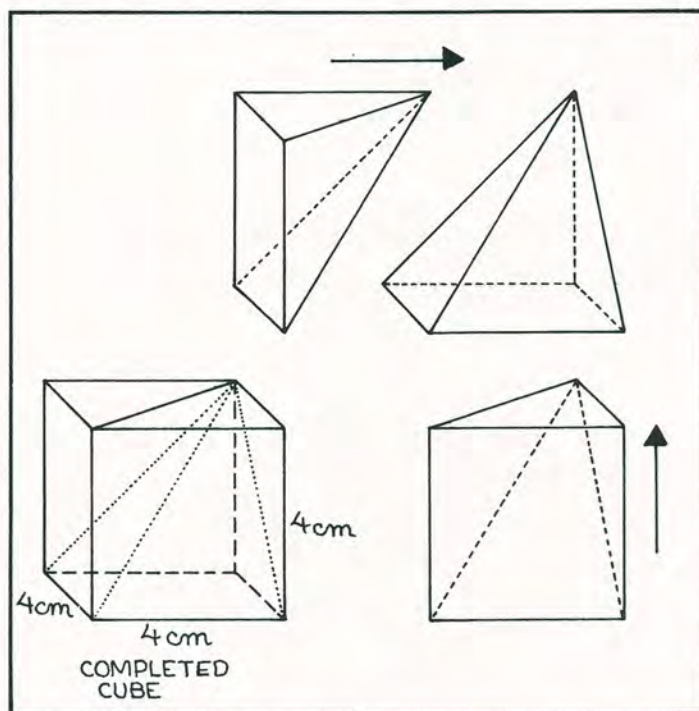
Each of these ideas could be developed into teacher directed activity cards or small group investigations to have students discover the formulas. Of course, some students may not make the desired discovery and will have to construct congruent models and count the cubes to determine the volume of a particular geoblock.

THE PIT AND THE POOL

Find the volume of each of these figures.

1)
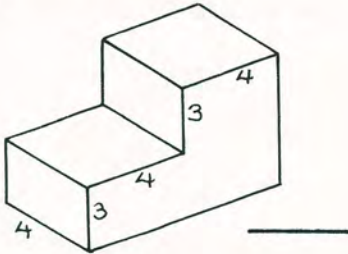
2)
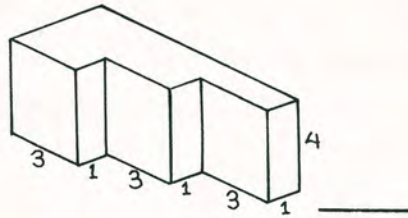
3)
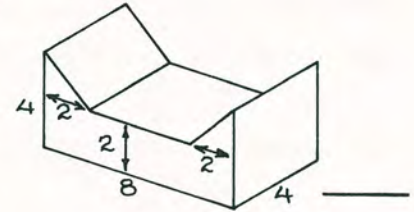
4)
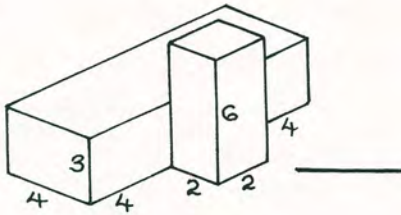
5)
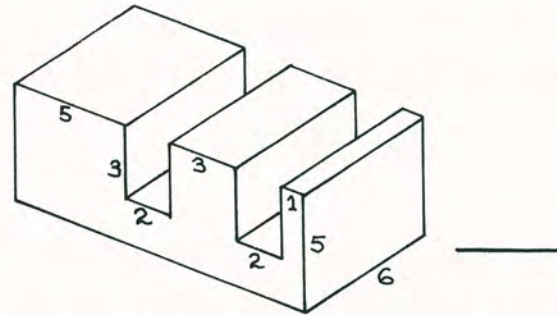
6) How much dirt has been taken out of this hole? _____ cubic units
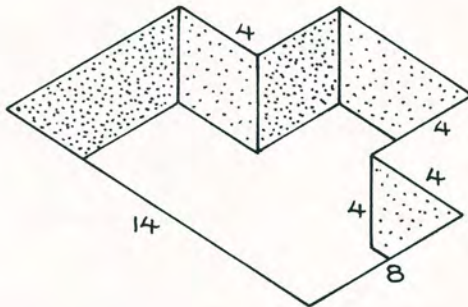
7) How much water would be needed to fill this swimming pool up to 1 unit from the top? _____ cubic units
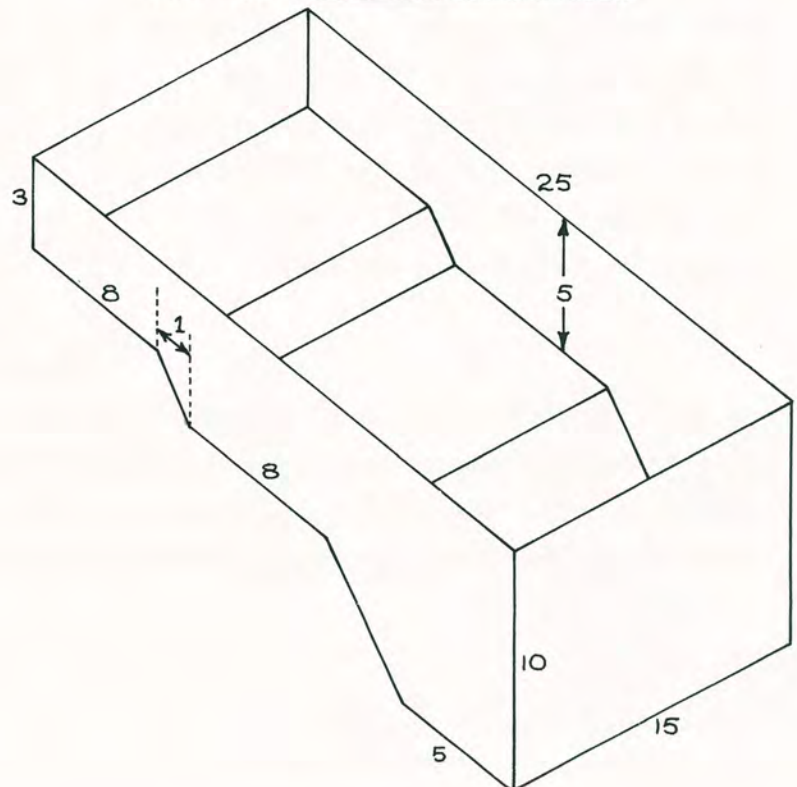
MAXIMUM VOLUME

Materials Needed: 1 cm grid paper

Cut an 18 cm x 18 cm square from the centimetre grid paper. From each corner remove a 6 cm x 6 cm square. Fold the remaining part into an open-topped box.
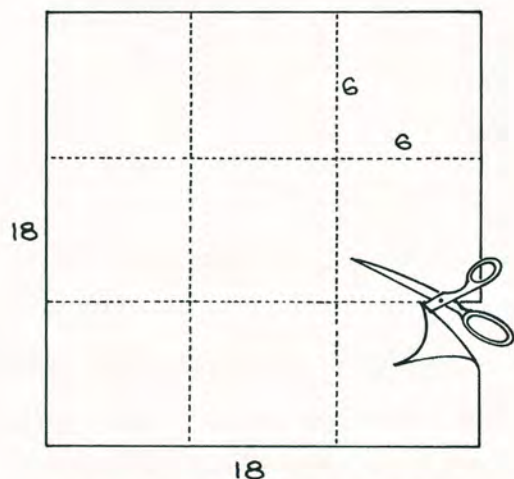
WHAT ARE THE DIMENSIONS OF THE BOX?

_____ cm x _____ cm x _____ cm

HOW MANY UNIT CUBES (1cm x 1cm x 1cm) WOULD IT TAKE TO FILL THE BOX? _____

Cut only along the lines on the grid paper. Use more 18 cm x 18 cm pieces. By cutting out corner squares of different sizes, find as many differently sized boxes as you can.

Record the length, width, height and volume of each box in the chart below. Is there a box with maximum volume?

DIMENSIONS OF BOX (cm)								
VOLUME (cm³)								

Start with a 10 cm x 10 cm piece of grid paper. Find the dimension of the open-topped box that would have the maximum volume. Squares of any size can be cut from the corners.

METHODS FOR FINDING VOLUMES

Materials: 8 to 10 commercial containers, some about the same approximate size; e.g.,
oatmeal and other dry cereal boxes, paper cups, empty cans and bottles.
Filler Material: Sand, cornmeal, puffed rice, etc.

I   Comparing Volumes

A) Have students guess which container holds the least; the most. Have them sort
their guesses from least to greatest volume.

B) Suggest that students use the filler material to check their guesses. Have them
decide on a method. One possibility is to fill the predicted largest container
and pour from one container to the next. Students can discuss their methods.

II Measuring Volumes

A) Non-standard unit of volume measure

1) Supply students with the net of an open cube duplicated on thin tagboard.
Students use scissors and tape to construct the open cubes.
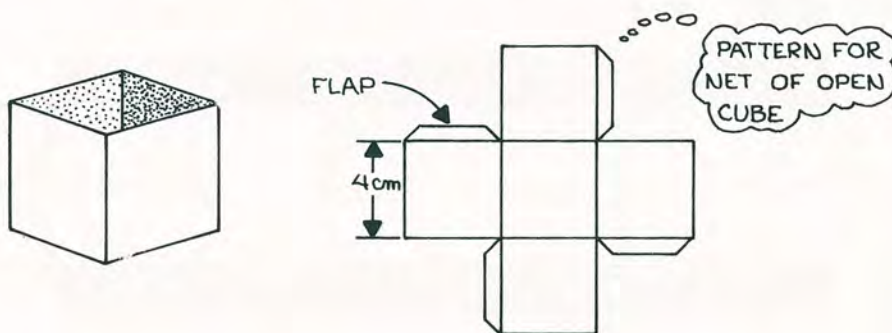
2) Tell students that the open cube has a volume measure of 1 cubic unit. Have
them use the cube and filler to measure the volume of each commercial con-
tainer to the nearest whole volume unit.

By noting that the volume of the open cube is also 64 cm^3 students could also
give the approximate volumes of each container in cubic centimetres.

METHODS FOR FINDING VOLUMES

(CONTINUED)
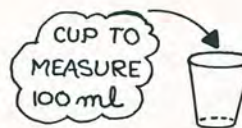
B) Calibrating a jar

1) Supply students with a 100 ml measure (a marked paper cup) and an empty wide-mouth canning jar. Students estimate the number of 100 ml measures in the jar and then measure the volume of the jar by dumping cupfuls of the filler into the jar. After each cupful mark the level of the filler on the outside of the jar to calibrate it.

2) Students can then use the calibrated jar to measure the volume of each commercial container to the nearest 100 ml.

If a graduated cylinder is available students can measure the volumes to greater precision.

C) Displacement

1) Supply students with the calibrated jar from part B and a pile of marbles all the same size. Have students estimate how many marbles will occupy as much space in the jar as one 100 ml of water.

2) Students can find out by filling the jar with water to the 3rd mark. Then add marbles until the water level rises to the 4th mark and count the marbles. Reinforce the concept of displacement by asking students the volume of these marbles.

3) Supply students with several small objects that do not float in water; e.g., rocks, dice, hard boiled egg. What is the volume of each object? That is, how much water does each displace?

4) Ask students how they might use a 2-cup measuring cup and water displacement to measure a half cup of butter.

MAKE A DIP STICK

This activity calibrates a dowel which can then be used as a dipstick to find the amount of liquid in a container.

Equipment: Eight to ten containers approximately the same height but having different shapes, e.g., detergent bottle, starch bottle, pop bottle. catsup bottle, milk carton, vase, bubble bath containers

Eight to ten thin wooden dowels
Eight to ten graduated cylinders that measure in ml (medicine cups from a hospital work nicely)

1. (a) Use an irregularly shaped bottle for a classroom demonstration. Let the students make conjectures about where the marks will appear. Pour 50 ml of water into the bottle. Carefully lower a thin dowel into the bottle until it touches the bottom. Lift the dowel out and mark the water level. Repeat the procedure until the bottle is full.

   (b) The dowel is now calibrated to measure the amount of liquid in the bottle to the nearest 50 ml.

   (c) Discuss how the spacing of the marks is related to the shape of the bottle.

2. (a) Divide the class into groups. Give each group a bottle and have them make a dipstick for their container.

   (b) Collect the bottles and the dipsticks. Have each group try to match the dipsticks with the appropriate containers.

3. (a) Ask students if they know any uses for dipsticks.

   (b) Suggest that each student check the oil and/or transmission fluid level in the family car.

   (c) How does the gas station operator measure the fuel in the station's tanks? Suggest that each student check at their neighborhood station. Perhaps the attendant will demonstrate the use of the dipstick.

Eureka, I've Found It!

GET - A bundle of orange rods and a white rod
Coffee can
Pie tin
Graduated cylinder (millilitres)
Several jars
Several objects
Water

1. A white rod is one cubic centimetre. What is the volume of one orange rod? What is the volume of your bundle of orange rods?

2. Completely fill the coffee can with water.
   - Place a pie tin underneath.
   - Immerse the bundle of rods in the water.
   - Collect the water that spills over and pour it into the graduated cylinder.
   - How many millilitres of water were displaced by the rods?

   Do the experiment again to check your result.

   TALK WITH YOUR TEACHER.

3. Use the graduated cylinder to help you determine the volume of the smallest jar.

4. Estimate and then determine the volume of each of the other jars.

5. Use the "water displacement" method to find the approximate volume of each of the other objects.

CLEAN UP YOUR MESS!

EXTENSION: What is the meaning of the title of this card? An encyclopaedia or a book on the history of mathematics might help. Look under "Archimedes."

BUILD A MODEL OF A CUBIC METRE

Building a frame showing a space of one cubic metre requires twelve dowels each one metre long, tape and pipe cleaner.

TAPE TWO PIPE CLEANERS TO EACH END OF THE TWELVE DOWELS AS SHOWN.

TWIST THE PIPE CLEANERS TOGETHER TO FORM THE CORNERS OF THE CUBE.
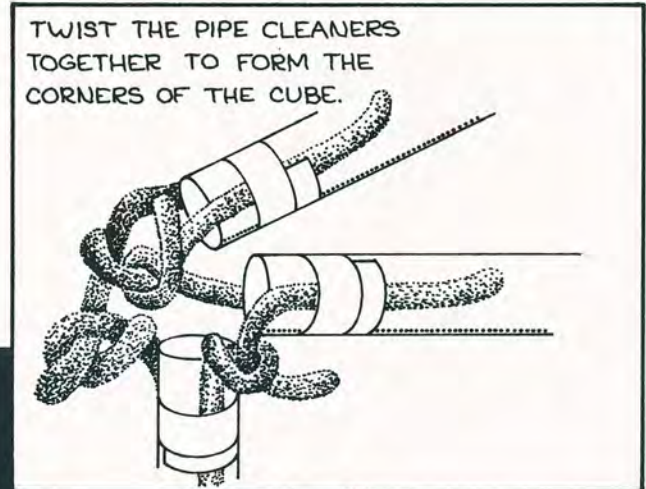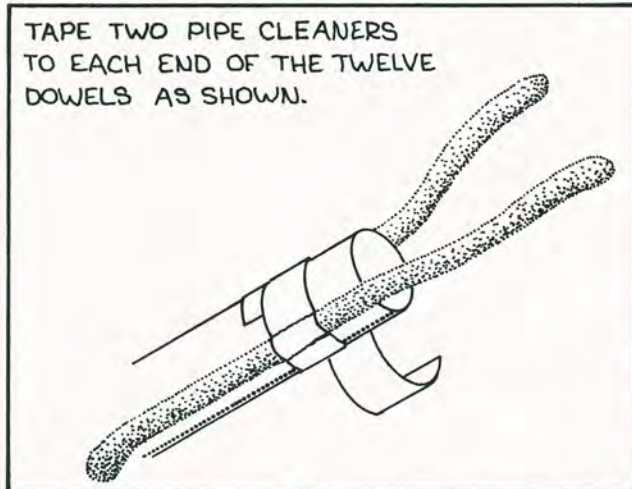
Have students assemble the cubic metre. Place it in a corner and tape it to the walls for stability. Activity cards using the cube can be placed next to the cube or attached directly to it. Examples of some questions are:

A) Estimate how many cubic metres of space are in this room. How many cubic metres of air does each person in the room have for his own?

B) If you crowd, how many people could you cram into a cubic metre?

C) What is the volume of a human body?

For comparison.

Build a cubic decimetre (10 cm on a side).

A) How does this compare to the cubic metre?

B) Exactly how many of these would it take to fill the cubic metre?

C) How many cubic centimetres does it take to fill a cubic decimetre?

Comparison of cubic decimetre and a litre.

Make a cubic decimetre from material that will hold a liquid. A tag board model lined with a small plastic bag will work. Using a measuring device calibrated in millimetres (100 ml, 200 ml, 250 ml, etc.) fill the cubic decimetre.

Students should discover that these four measurements are equivalent: 1 cubic decimetre, 1000 cubic centimetres, 1000 millilitres and 1 litre.

CYLINDERS, CONES AND SPHERES

I Volume of a cylinder

The layer approach as described in *Layer Upon Layer* for
rectangular prisms can be used to develop the formula
for the volume of a cylinder. Consider a layer of sand
1 cm deep in the bottom of a cylinder of radius 3 cm.
The volume of sand is $\pi \times (3\,cm)^2 \times 1\,cm = 9 \times \pi\,cm^3 \approx 28\,cm^3$.

Students could check this value by measuring the sand.
See *Methods for Finding Volumes*.   If four more layers
of sand are added to the cylinder so the height of the
sand is 5 cm, the volume of sand is

$$\pi \times (3\,cm)^2 \times 5\,cm \approx 140\,cm^3.$$

Students could also check this value by measuring the sand.

If the height of the cylinder is 20 cm, then when full the cylinder can hold
$\pi \times (3\,cm)^2 \times 20\,cm \approx 565\,cm^3$.
The general formula for the volume of a cylinder of radius r and height h is
$\pi \times r^2 \times h$ or $B \times h$ where B is the area of the base.
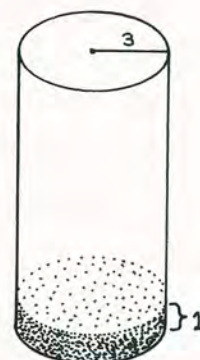
II Volume of a Cone

1) Procedure for making a cone of same radius and
   height as a cylinder:

   Find a can from which the top has been cut.
   Roll a piece of thin cardboard to make a cone.
   Put a small piece of tape at the tip of the
   cone to maintain the shape. Place the cone in
   the can so that the tip touches the base.
   Spread out the cone to fill out the can at the
   top and fasten on the inside with tape.

   Mark the cone with a pencil around the top of
   the can. Remove the cone. Tape the seam
   outside securely and cut along the mark.
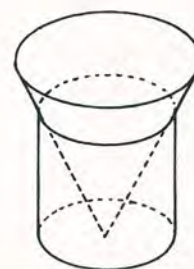
2) Have students guess how many conesful of sand, cornmeal, etc., would be
   needed to fill the can. Demonstrate by filling the can.
   Since three conesful are needed the volume of the cone is one-third the
   volume of the cylinder.

785

CYLINDERS, CONES AND SPHERES
(CONTINUED)

The general formula for the volume of a cone of radius r and height h is $\frac{1}{3} \times \pi \times r^2 \times h$ or $\frac{1}{3} \times B \times h$ where B is the area of the base.

(A similar relationship holds for pyramids and rectangular prisms. See *Volume with Geoblocks*.)
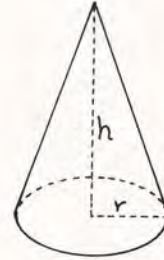
III Volume of a Sphere

Roll a double thickness of transparency plastic into a cylinder so that a slick surfaced ball will fit inside. Fasten the plastic with tape to maintain the shape.

Hold the base of the cylinder snugly against a table, place the ball inside and pour enough sand (or cornmeal) to just cover the ball. You may need to squeeze the cylinder so the sand fills the space between the sphere and the base of the cylinder.

BE SURE THIS SPACE IS FILLED WITH SAND.
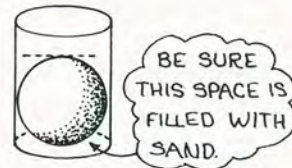
SAND AND BALL

SAND

Mark the height of the sand on the cylinder. Dump out the sand and ball and pour the sand back into the cylinder. Mark the height of the sand on the cylinder and compare the height of the two marks. The second mark should be about one-third as high as the first mark. Thus the volume of a sphere is two-thirds the volume of a cylinder that just contains it.
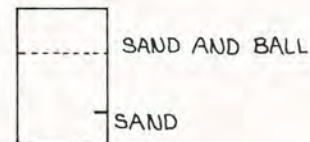
The generalized formula for the volume of a sphere of radius r is

THE HEIGHT OF THE CYLINDER IS THE DIAMETER OF THE SPHERE.

THE VOLUME OF THE SPHERE IS $\frac{2}{3}$ THE VOLUME OF THIS CYLINDER.

$$\frac{2}{3}\underbrace{\left[\pi \times r^2 \times 2r\right]}_{\substack{\text{the volume of} \\ \text{a cylinder just} \\ \text{containing it.}}} = \frac{4}{3} \times \pi \times r^3.$$
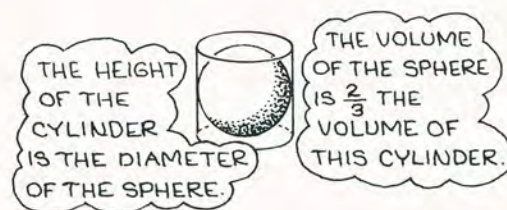
IV

These developments can be done efficiently with commercial models. Sources for these models are:

Creative Publications, Inc., P.O. Box 10328, Palo Alto, CA 94303

Educational Teaching Aids Division, 159 W. Kenzie St., Chicago, IL 60610

Gamco Industries, Inc., Box 1911, Big Spring, TX 79720

Geyer Instructional Aids Co., Inc., P.O. Box 7306, Fort Wayne, IN 46807

NESTED SPHERES

12 cm

12cm

12 cm

I. A sphere which is nested inside a cube has its center at the center of the cube and is tangent to all six faces.

What is the diameter of the sphere?
_____ cm

What is the radius of the sphere?
_____ cm

What is the volume of the sphere?
_____ cm^3

VOLUME OF A SPHERE CAN BE FOUND BY USING THE FORMULA $V = \frac{4}{3} \pi r^3$

II. Imagine making three slices with a knife as in the picture.

How many smaller congruent cubes are formed if the cuts pass through the midpoints of the edges? _____
As in part (I) spheres are to be nested in each of the smaller cubes. What is the diameter of one of the smaller nested spheres? _____ cm
Its radius? _____ cm Its volume? _____ cm^3 What is the total volume of the eight smaller spheres? _____ cm^3

III. Making more equally spaced slices produces more and smaller cubes. A sphere can be nested in each cube. Imagine making the number of cuts listed in the table. Complete this table. Do you see anything special in the table?

NUMBER OF EQUALLY SPACED CUTS		TOTAL CUTS	NUMBER OF CONGRUENT CUBES		VOLUME OF ONE NESTED SPHERE	TOTAL VOLUME OF ALL NESTED SPHERES
(PART I ABOVE)	0	0	(THE ORIGINAL)	1		
(PART II ABOVE)	1	3				
	2	6				
	3	9				

LARGEST CONTAINER

This activity will help students see that containers which have the same surface area do not necessarily have the same volume.

I   Tall containers – vs – short containers

Using two rectangular pieces of tag board (shown below) students can roll up two cylinders each having the same surface area (in this case 15 cm x 24 cm or 360 cm^2).

AN EXTRA cm ON THE END TO USE AS A TAB MAKES THE MODEL EASIER TO ASSEMBLE

Students can study, predict and record information concerning the cylinders. Some suggestions about the surface area and volume might key them in on critical information. Have students check their predictions. See *Methods for Finding Volume* in this section for a variety of techniques to use.

II Changing the number of sides.

To assemble the models below students will need four 15 cm x 25 cm pieces of tag board. The first piece is to be rolled into a right circular cylinder having a height of 15 cm and a circumference of 24 cm. The remaining three are marked in the appropriate places and folded into the patterns shown.

Students can arrange the containers from least to greatest volume. Have them check their guesses and measure the volume of each by using the methods in *Methods for Finding Volume*.

A VARIETY OF VOLUME VEXATIONS

1) A package is 3 cm x 4 cm x 6 cm. How many such packages can be placed inside a box with inside dimensions of 6 cm x 8 cm x 12 cm?
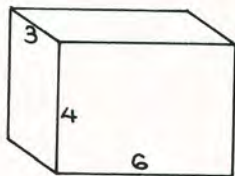
2) Which is the better buy?

a) Oranges 4 cm in radius that cost 20¢ each.

b) Oranges 5 cm in radius that cost 30¢ each.

SUNKIST   SUNKIST
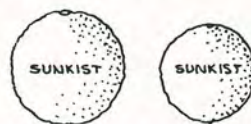
3) Two dozen small boxes each measuring 3 cm x 6 cm x 8 cm will exactly fit in a large box which has a base measuring 12 cm x 12 cm. What is the height of the box?
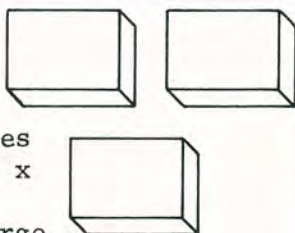
4) This drinking glass is a perfect cylinder.

How could you measure exactly half a glass of water with absolutely no measuring instruments whatever?

5) How can you measure out exactly 4 litres of water if you have only a 3 litre and a 5 litre container?

3 LITRE   5 LITRE

NEITHER CONTAINER IS MARKED.

6) A theater decides to change the shape of the popcorn container from a box to a pyramid and charge only half as much. Is this a bargain?

THE TOPS ARE THE SAME SIZE

BOTH CONTAINERS ARE THE SAME HEIGHT
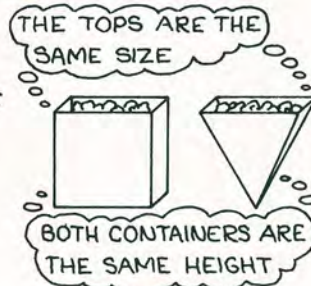
7) Which will carry the most water?

3   4   5
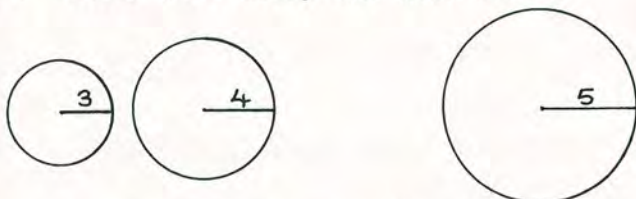
a) Two pipes; one with 3 dm radius, the other with 4 dm radius.

b) One pipe with 5 dm radius.

8) The Alaskan pipeline could easily be 5000 km long. If it has only a 20 cm inside diameter, how many litres of oil would it take to just fill the pipe?

1 LITRE EQUALS 1000 cm³

Cheapest Drink

GET - Box of assorted drinking cups
from local restaurants for
this year

1. Which restaurant sells the cheapest large size soft drink?
   Medium size soft drink? Small size soft drink?

2. Which size drink gives you the most for your money?
   Make a guess. Record your guess.

3. Now think of a way to actually find out which size soft
   drink gives you the most for your money.
   Use your procedure to find the answer.
   Record your data in a table.
   Which size was the best buy? Record.

4. Write up your conclusions or use a graph to display your
   findings.

EXTENSION: Get the box of cups for last year.
   Did the sizes change?
   Did the prices change?
   If the prices increased, find the percent of
   increase for one of the cups.
   At this same rate of increase, how much will the
   same size drink cost in 10 years?

CHANGING VOLUMES 1

Materials Needed: 100 centimetre cubes

Activity:

A centimetre cube has 1 unit of volume.

MODEL 1

1) Make 3 different models:

a) One twice as long as the unit cube.

b) One twice as long and twice as wide as the unit cube.

c) One twice as long, twice as wide and twice as high as the unit cube.
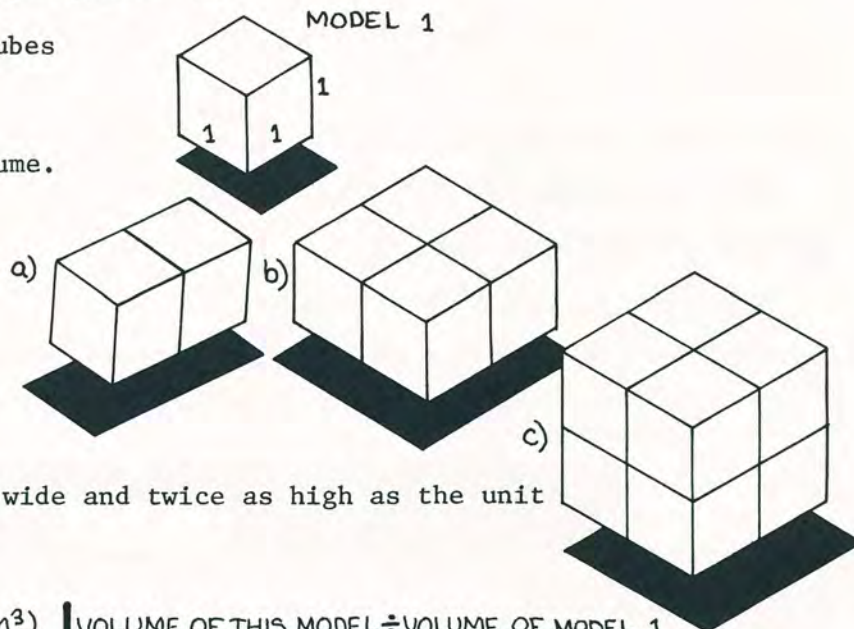
MODEL	DIMENSIONS	VOLUME (CM³)	VOLUME OF THIS MODEL ÷ VOLUME OF MODEL 1
a)	2 x 1 x 1		
b)	2 x 2 x 1		
c)	2 x 2 x 2		

2) Make 3 different models:

d) One three times as long as the unit cube.
e) One three times as long and three times as wide as the unit cube.
f) One three times as long, three times as wide and three times as high as the unit cube.

MODEL	DIMENSIONS	VOLUME (CM³)	VOLUME OF THIS MODEL ÷ VOLUME OF MODEL 1
d)			
e)			
f)			

3) Make 3 different models:

g) One four times as long as the unit cube.
h) One four times as long and four times as wide as the unit cube.
i) One four times as long, four times as wide and four times as high as the unit cube.

MODEL	DIMENSIONS	VOLUME (cm³)	VOLUME OF THIS MODEL ÷ VOLUME OF MODEL 1
g)			
h)			
i)			

The Jones have a swimming pool that is 2 metres deep, 4 metres wide and 7 metres long. Mr. Smith, who lives next door, wants to build a larger pool. How many times as much water will Mr. Smith need if he builds a pool twice as long and twice as wide but with the same depth? _____

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CHANGING VOLUMES 2

Materials Needed: 200 centimetre cubes

Activity:

1) a) Use the cubes to make the model to the right.

MODEL 1

    b) The volume (in cm³) of this model is _____
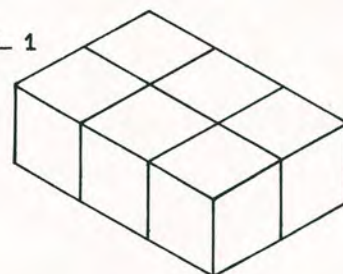
2) Make 3 boxshaped models:

    a) One twice as long as Model 1.

    b) One twice as long and twice as wide as Model 1.

    c) One twice as long, twice as wide and twice as high as Model 1.

MODEL	DIMENSION	VOLUME (cm³)	VOLUME OF THIS MODEL ÷ VOLUME OF MODEL 1
a	6 × 2 × 1	12	
b			
c			

Make 3 more models:

d) One three times as long as Model 1.

e) One three times as long and three times as wide as Model 1.

f) One three times as long, three times as wide and three times as high as Model 1.

MODEL	DIMENSION	VOLUME (cm³)	VOLUME OF THIS MODEL ÷ VOLUME OF MODEL 1
d	9 × 2 × 1	18	
e			
f			

3) If two dimensions of Model 1 are made to be seven times as long, what will be the volume of the new model? _____

4) If the volume of a model the same shape as Model 1 is 25 times as much as the volume of Model 1, what can you say about the dimensions of the new model? _____

_____

_____

YOU BE THE JUDGE

Materials: Heavy paper (like file folders), ruler, scissors, compass and rice, sand or another convenient substance for filling cylinders.

Make a cylinder from a 10 cm x 14 cm rectangle and a 2.3 cm radius circle as shown.

1) What is the area of the rectangular sheet? _____ cm^2

Roll the rectangle into a cylinder and tape the circle onto one end to make an open-ended cylinder.

TABS FOR TAPING THE BOTTOM ONTO THE CYLINDER.
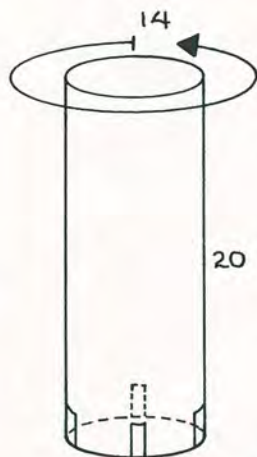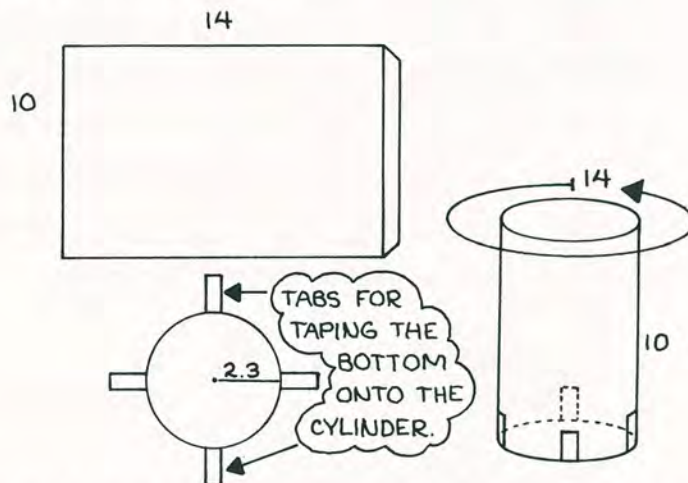
Make a second open-ended cylinder that is twice as high as the original. (Use a 20 cm x 14 cm rectangle.)

2) How does the area of this rectangular piece of paper compare to the area of the first one? _____

3) How many times as much do you think this new cylinder will hold? _____

4) Using the first cylinder as a scoop fill the second container with rice. How many scoops did it take? _____

5) Was your prediction correct? _____

Make a third and a fourth open-ended cylinder with the third having a radius twice the radius of the first cylinder and the fourth having both the height and radius doubled.

Judge and check the following:

6) The area of the rectangular piece is how many times the area of the rectangular piece of the first cylinder?

7) The volume is how many times the volume of the first cylinder?

8) Using the original cylinder as a scoop, fill these cylinders with rice to check your predictions.

Third Cylinder

Fourth Cylinder

_____   _____

_____   _____

_____   _____

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WHEN YOU'RE HOT YOU'RE HOT

All hot water heaters in Cubesville are built in the shape of cubes. Cid needs to buy a new one and wants to buy the most efficient size in order to save energy. The salesman told Cid that the surface area of the heater allows the heat from the water to escape and the most efficient size is one with the smallest surface area per litre of water. Complete the chart below to help Cid make a decision on the most efficient of the four hot water heaters to buy.

REMEMBER: $1000 \text{ cm}^3 = 1$ LITRE

	1	2	3	4
LENGTH OF EDGE (cm)	40 cm	60 cm	80 cm	100 cm
SURFACE AREA (cm²)				
VOLUME (cm³)				
VOLUME (LITRES)				
SIMPLIFIED RATIO OF SURFACE AREA (cm²) TO VOLUME (LITRES)	:	:	:	:

1) Which water heater is the most efficient? _____ Can you explain? _____
_____
_____

2) What other factors should Cid consider before he buys a water heater? _____
_____

3) Could there be a hot water heater with a simplified ratio of 1 cm²:1 litre? _____ What would be the length of an edge? _____

4) Water has a mass of about 1 kilogram per litre. How much mass will there be for the water in heater #4? _____

WHEN YOU'RE HOT YOU'RE HOT

(CONTINUED)

This calculator exercise will show that the cube is not the most efficient shape for a water heater. A cylinder is more efficient and the sphere is the most efficient of all. An able student, using a calculator and the data from the previous page, can discover this fact.

For the cylinder a height equal to the edge of the cube on the previous page has been used. The radius will have to be computed. The volume of a cylinder is $\pi r^2 h$ so the radius will be $\sqrt{\frac{V}{\pi h}}$. The surface area of a cylinder is $2\pi r (r + h)$.

CYLINDER	1	2	3	4
HEIGHT				
RADIUS				
SURFACE AREA				
VOLUME				
SIMPLIFIED RATIO OF SURFACE AREA (cm²) TO VOLUME (LITRES)				

For the sphere the volume is $\frac{4}{3}\pi r^3$ so the radius is $\sqrt[3]{\frac{3V}{4\pi}}$ and the surface area is $4\pi r^2$.

SPHERE	1	2	3	4
RADIUS				
SURFACE AREA				
VOLUME				
SIMPLIFIED RATIO OF SURFACE AREA (cm²) TO VOLUME (LITRES)				

The students should notice that a larger size of the same shape is more efficient and that for a given measurement of volume the sphere is the most efficient shape.

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

GEOGLOSSARY

These are informal definitions for middle school students and teachers. For more formal definitions consult the *Mathematics Dictionary*, 4th ed., edited by Robert C. James and Edwin F. Beckenbach, published by Van Nostrand Reinhold Co., 1976.

acute angle. An angle whose measure is greater than 0° and less than 90°.

acute triangle. A triangle in which each of the three angles measures less than 90°.

adjacent angles. Two angles in the same plane that have a common vertex and a common side "between" them.

∠1 and ∠2 are adjacent angles.
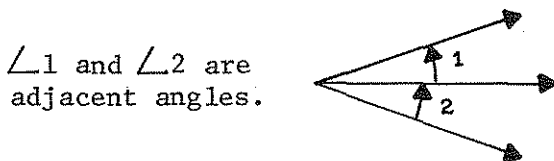
adjacent sides. Sides of a polygon having a common endpoint.

Side $\overline{BA}$ and side $\overline{AD}$ are adjacent sides.
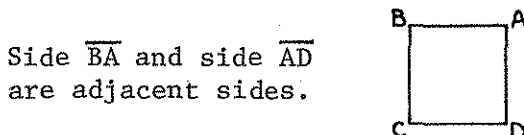
alternate exterior angles. In the sketch, angles 1 and 8 are alternate exterior angles; so are angles 2 and 7. (It is not necessary for two of the lines to be parallel.)
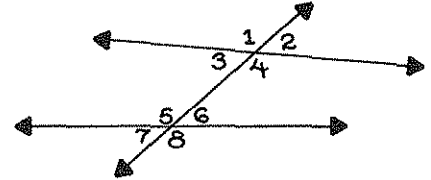
alternate interior angles. In the sketch above, angles 3 and 6 are alternate interior angles; so are angles 4 and 5.

altitude. 1) A line segment with one endpoint at a vertex of a polygon and the other on the line containing the side opposite the vertex. An altitude is perpendicular to the line containing the side opposite the vertex.
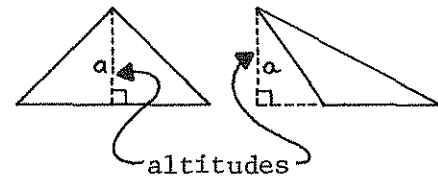
altitudes

2) A line segment in space figur as shown.
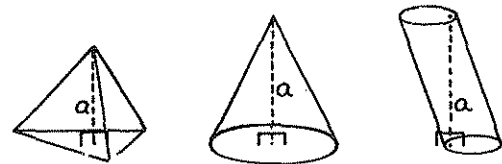
3) times used to mean length of altitude.

angle. Two rays having a common endpoint.
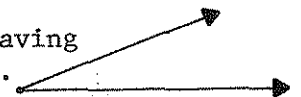
approximation. 1) An eaucated guess that involves arithmetic skills, visual perceptual skills, and/or measuring skills. 2) A process through which an educated guess is determined as a close, but not precise answer to a given problem.

arc.   1) A connected part of a circle. Points A and C are endpoints of minor arc ABC and major arc ADC. If the endpoints of an arc are on the diameter, the arc is called a semicircle. The symbol for an arc is ⌒ . For example, $\overarc{ABC}$. 2) In topology, an edge of a network is often called an arc.
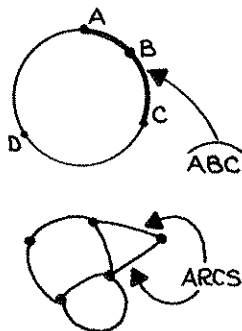
area.   The measure of a closed region in a plane. A unit of area is usually a square region.

axis.   1) A special line related to figures.

axis   axis   axis
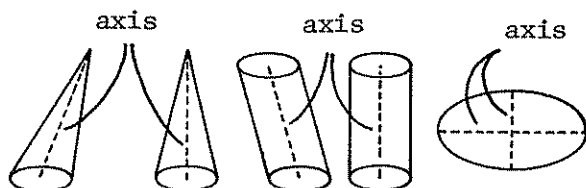
2) See coordinate axes.

axis of symmetry.   A line of symmetry of a figure.

axis of symmetry
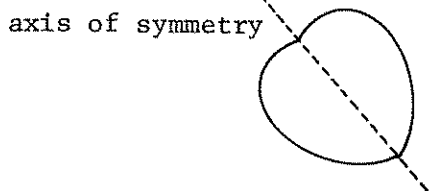
base.   1) Any side of a triangle can be called the base. In triangle ABC, if $\overline{BC}$ is called the base, angles B and C are the base angles. 2) The bases of a trapezoid are the two parallel sides. 3) A face of a space figure can be called a base. The bases of a prism or a cylinder are two parallel faces.

bases

base
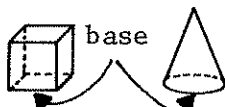
bearing.   The direction of one point with respect to another point. The angle of direction is determined by the position of the point relative to the north-south line.

The bearing of point A from point B is 315°. The bearing of point B from point A is 135°.
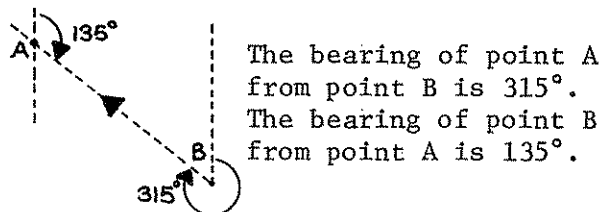
bisect.   To separate or divide into two parts of equal measure.

bisector.   A point, line or ray which separates an angle or line segment, or other geometrical figure into two parts of equal measure.

capacity.   The amount a container can hold. This is often called the volume of a container.

Cartesian coordinates.   See coordinates and coordinate axes.

center of a sphere.   A point equally distant from all points on a sphere.
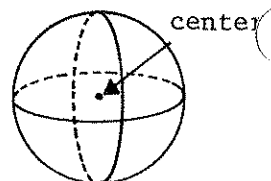
center

centimetre.   A unit of length in the Metric System of Measures. One centimetre (cm) is $\frac{1}{100}$ or .01 metre (m). Also, 2.54 cm is approximately equal to 1 inch.

central angle.   A central angle has its vertex at the center of a circle. In the figure, $\angle$ ANG is a central angle.
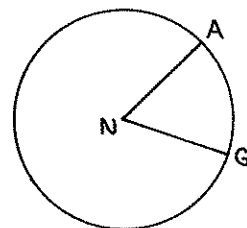
chord.   A line segment whose endpoints are on a circle.

chord

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circle.   A plane curve
consisting of all
points at a given
distance (radius)
from a fixed point
in the plane,
called the center.
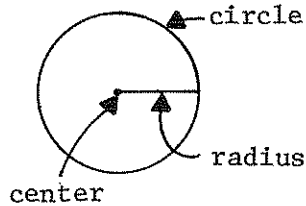
circular region.   A circle and its
interior.

circumference.   The distance around a
circle.

circumscribed circle.   A circle is
circumscribed about a polygon when
each vertex of the
polygon is a point
of the circle. In
the figure, the
circle is circum-
scribed about the
triangle.
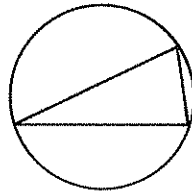
closed curve.   A figure that starts at
a point and comes back to that point.
A simple closed curve does not cross
itself.

Simple Closed
Figures
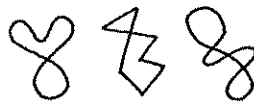
Closed Figures That
Are Not Simple

closed space figure.   A figure which
separates space into sets of points
inside the figure, outside the figure
and on the figure.
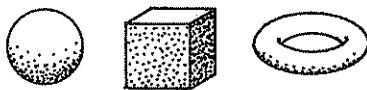
Closed Space Figures

collinear points.   Points
contained in the same
line.
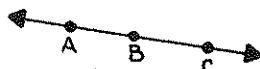
compass.   An instrument
for drawing circles or
for constructing con-
gruent line segments.
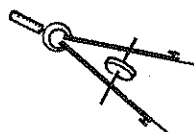
complementary angles.   Two angles, the
sum of whose measurements is 90°.
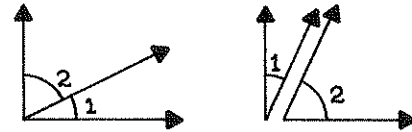
Angle $\angle 1$ is the complement of angle $\angle 2$.

concentric circles.   Two
or more circles in a
plane having the same
point for their center.

concurrent lines.   Two or
more lines with a point
in common.
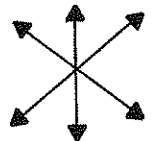
cone.   A solid having a circular or
elliptical base and a vertex. (See
illustrations.)
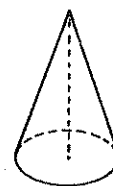
Right Circular Cone
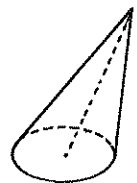
Oblique Circular
Cone

congruent figures.   Two figures are
congruent if they are alike in size
and shape. The symbol for congru-
ent is $\cong$.

reflection

translation

Two congruent figures can be made to
coincide by some combination of ro-
tations, reflections or translations.

conic sections.   Curves formed by the
intersection of a plane and a double
right circular cone. The conic sec-
tions are usually
thought of as the
circle, ellipse,
parabola, hyper-
bola, but also
include a straight
line, a point and a
pair of intersect-
ing lines.

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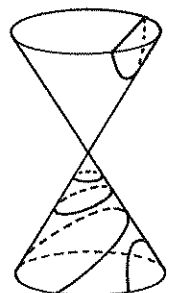
convex polygon. A polygon which lies entirely on one side of any one of its extended edges.
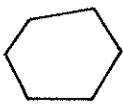
Convex   Not Convex

convex polyhedron. A polyhedron which lies entirely on one side of any plane containing one of its faces.

coordinate axes. The number lines used to name points on a plane with ordered pairs. The two number lines are usually perpendicular.
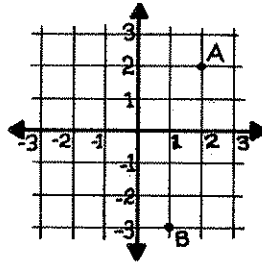
coordinates. 1) A number associated with a point on a line.

-1 0 1 2 3 4
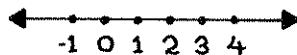
2) Ordered pairs used to name points on a plane. Point A has coordinates (2,2); point B has coordinates (1,-3). The x-coordinate is listed first; the y-coordinate is listed second.

corresponding parts. Any pair of angles or sides in two congruent or similar triangles having the same relative positions. Angle A corresponds to angle A'. Side BC corresponds to side DE.
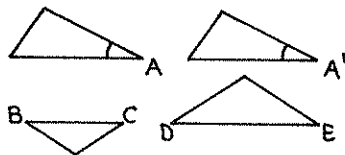
cube. A rectangular prism (box-like) where all six faces are squares.
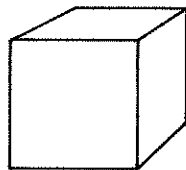
cycloid. A path made by a point on the rim of a wheel as the wheel is rolled in a plane along a straight line.
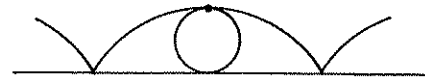
cylinder. A solid having two circular or elliptical bases and a lateral surface. (See illustrations.)
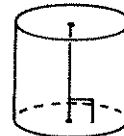
Right Circular   Oblique Circular
Cone   Cone

deca-. Ten, having ten.

decagon. A polygon with ten sides.

Decagon

deci-. One-tenth.

degree. Unit of angular measure. A right angle measures 90°.

diagonal. A line segment joining two nonadjacent vertices of a polygon. In the figure, segment AB is a diagonal.
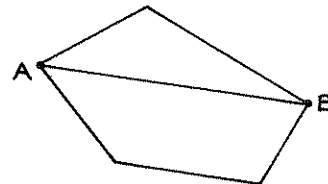
diameter. 1) A line segment (chord) that passes through the center of the circle and has its endpoints on the circle.
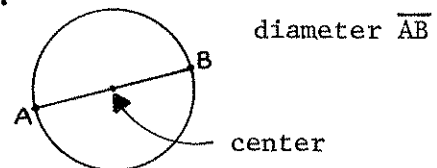
diameter $\overline{AB}$

center

2) The length of a chord which passes through the center of a circle.

dihedral angles. The space angle formed by two intersecting planes.

dodecagon. A 12-sided polygon.

dodecahedron. A polyhedron with 12 faces.
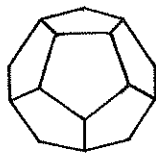
edge. 1) A line segment which is the intersection of two faces of a solid. 2) A portion of a network between two vertices. Sometimes called arcs.

edges

edges
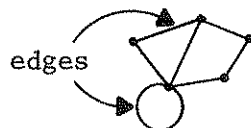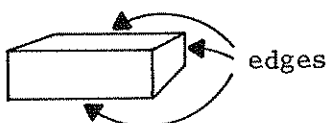
ellipse. A closed conic section. Circles are special ellipses.
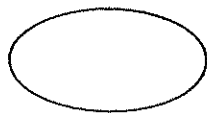
equiangular. Having all the angles congruent.

equilateral. Having all the sides congruent.

Euclidean plane. The set of points in a flat surface extending infinitely.

exterior angle. 1) In each figure, angle ABC is an exterior angle of the polygon.
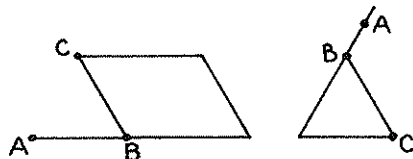
2) When two lines are cut by a transversal, the angles 1, 2, 3 and 4 are called exterior angles.
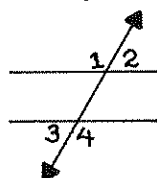
face. Any one of the plane regions that make up a 3-D solid. For example, in a cube each of the six square sides is a face of the cube.

frustrum. The portion of a pyramid or cone between its base and a plane section parallel to the base.
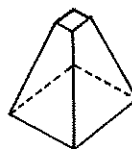
frustrum of a pyramid

frustrum of a cone

geodesic. The shortest path connecting two points on a surface.
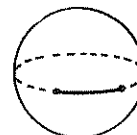
golden rectangle. A rectangle of pleasing proportion occurring in both nature and art. Its sides have the ratio

$$\frac{\ell}{w} = \frac{1 + \sqrt{5}}{2}$$

ℓ

w
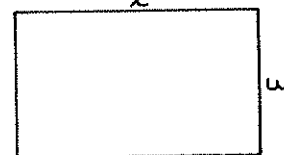
-gon. Angle or figure having many angles.

graph. 1) A picture used to illustrate a given collection of data. The data might be pictured in the form of a bar graph, a circle graph, a line graph, or a pictograph. 2) A set of points plotted on grid paper associated with an equation and x,y coordinates. 3) To draw the graph of.

height. 1) Of a polygon--the distance along the perpendicular from one vertex to the opposite side or base of the triangle.
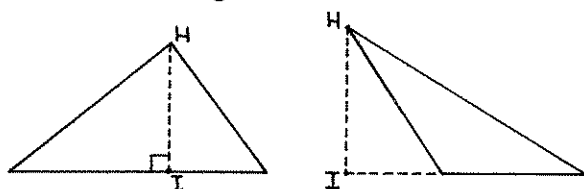
The length of $\overline{HI}$ is the height of these triangles. 2) Of a cone or pyramid--the distance along the perpendicular from the vertex to the plane of the base.
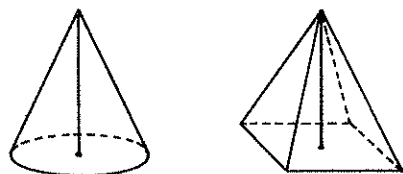
3) Of a cylinder or prism--the distance along a perpendicular between two opposite bases.
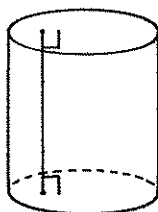
helix. A curve which lies on a cylinder or cone in a spiral-like manner.

helix
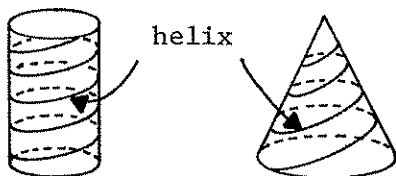
hemisphere. Half of a sphere.

hepta-. Seven, having seven.

heptagon. A seven-sided polygon.
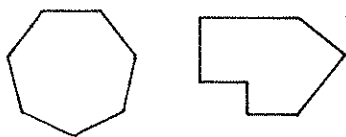
hexa-. Six, having six.

hexagon. A six-sided polygon.

regular hexagon

irregular hexagon

hexagonal. 1) Having six angles and six sides. 2) Having a hexagon for a base.

hexagonal pyramid
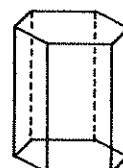
hexagonal prism

hexahedron. A polyhedron with 6 faces. A regular hexahedron is a cube.

hexiamonds. Polygonal figures made up of congruent equilateral triangles in which at least one side of a triangle must match up with another side.
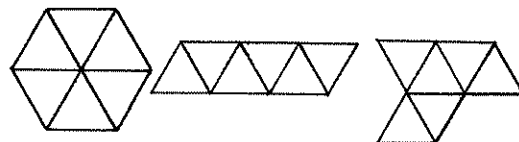
hyperbola. A conic section having two branches.

hyperbola
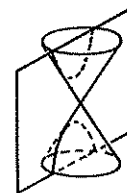
hypotenuse. The side opposite the right angle in a right triangle.

hypotenuse
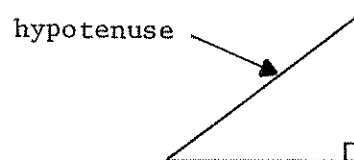
hypsometer. A tool for measuring heights indirectly.

icosahedron. A polyhedron with 20 faces.

inclined plane. A plane meeting a horizontal plane obliquely.

inclined plane
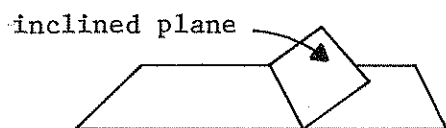
inscribed angle. An angle whose vertex is on a circle and whose sides cross the circle.

inscribed circle. A circle that lies within a polygon and each side of the polygon is tangent to the circle.
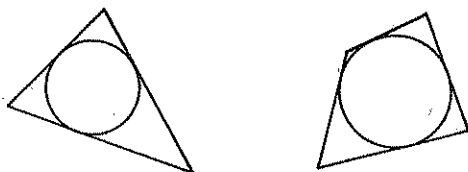
inscribed polygon. A polygon with each vertex on a circle.

interior angles. 1) Angles A, B, C and D are called interior angles of the polygon. 2) Angles 1, 2, 3 and 4 are the interior angles formed by a transversal and two lines.
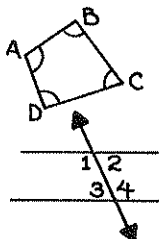
intersect. To share common points. 1) Two different lines can intersect in no points or 1 point. 2) Two different planes can intersect in a line or not at all. 3) A plane can intersect a cone in a circle, a point, an ellipse, etc.

invariant. A property that is not altered when certain changes take place.

isosceles triangle. A triangle with two congruent sides.
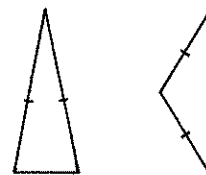
Klein bottle. A one-sided surface in topology which is sometimes represented by a bottle like that shown. A <u>true</u> Klein bottle is impossible to make in our space.
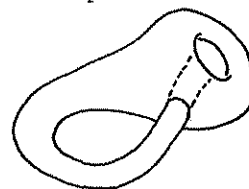
lateral surface. The curved surface of a cone or cylinder. Also may refer to a prism.

lateral surfaces
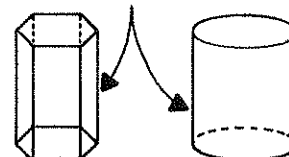
legs of a triangle. The sides of a right triangle which form the 90° angle.

legs
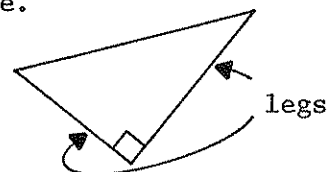
length. 1) The measure of one line segment or curve with respect to a <u>unit</u> of measure. 2) Sometimes used to denote the longer side of a rectangle.

line. An undefined term. A line is a set of points. There is only one line through any two points.

linear equation. An equation whose graph is a straight line. Example: $y = x - 1$.

803
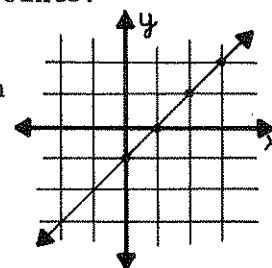
line segment.   Two points on a line
and all the points between them.
The two points are called the end-
points of the line segment. Line
segment AB is written $\overline{AB}$.

line segment $\overline{AB}$
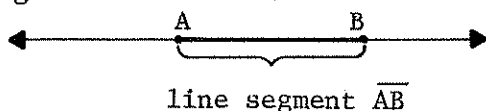
line symmetry.   A
figure which can
be reflected
across a line and
still fit on its
outline has line
symmetry.
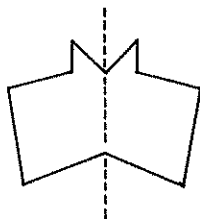
locus.   Any set of points satisfying
certain conditions. The locus of
points in space equidistant from a
given point is a sphere.

major arc of a circle.   An arc larger
than a semicircle. (See arc.)

measure.   1) A number relating a
given object to a suitable standard
unit. 2) The process of finding the
length, distance, weight, area, vol-
ume, etc., of a given object.

median.   1) Of a triangle--a line seg-
ment whose endpoints are a vertex of
the triangle and the midpoint of the
opposite side.

median

2) Of a trapezoid--the line segment
joining the midpoints of the non-
parallel sides of a trapezoid.

median

metre.   The basic unit of length in
the Metric System that is slightly
more than a yard. A metre is approx-
imately 39.37 inches.

midpoint.   A point that divides a line
segment into two segments of the
same length.   midpoint

minor arc of a circle.   An arc of a
circle less than a semicircle. (See
arc.)

Moebius strip.   A
one-sided surface
made by putting a
half-twist in a
strip of paper and
fastening the ends.
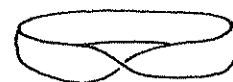
network.   A configuration of points and
arcs used to represent road systems,
relations, etc. Size and shape are
not considered
important in
networks. The
points are
called vertices
and the curved
and straight
segments are
called arcs or
edges.
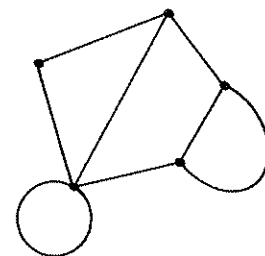
nona-.   Nine, having nine.

nonagon.   A nine-sided polygon.

noncollinear points.   Points not on the
same line.

oblique.   Slanting.

oblique
line

oblique
prism

oblique
circular
cone

oblique
circular
cylinder
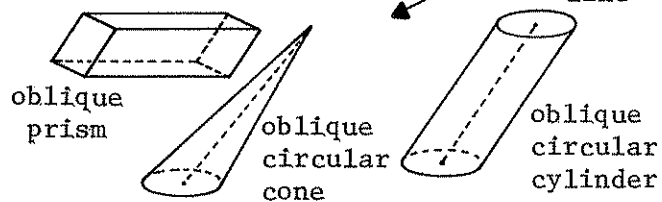
obtuse.   Lacking sharpness.

obtuse angle.   An angle whose degree
measure is greater than 90 and less
than 180.

obtuse triangle.   A triangle with one
angle that measures greater than 90°.
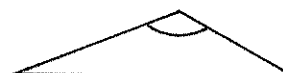
octa-.   Eight, having eight.

octagon.   An eight-sided polygon.
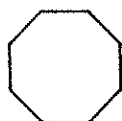
regular octagon

irregular octagon

octahedron.   A polyhedra having 8 faces.
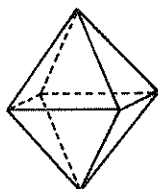
opposite angles.   1) For the pair of intersecting lines, angles 1 and 3 are a pair of opposite vertical angles. Angles 2 and 4 are another pair of opposite vertical angles.   2) For the quadrilateral MADE angles M and D are a pair of opposite angles.   So are angles A and E.
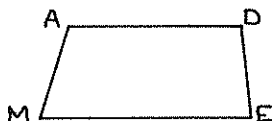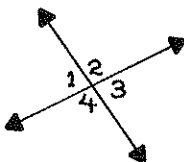
origin.   The point where coordinate axes intersect.   The origin has coordinates (0,0).

origin
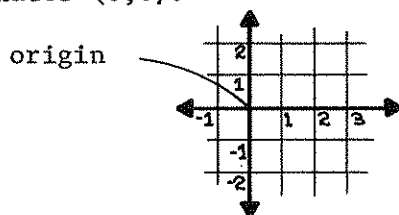
pantograph.   A tool for drawing enlargements or reductions of figures. It can be used to make scale drawings and similar figures.

parabola.   A conic curve which is the set of points equidistant from a fixed point and a fixed line.

parabola as a cross section of a cone

fixed line

fixed point (focus)
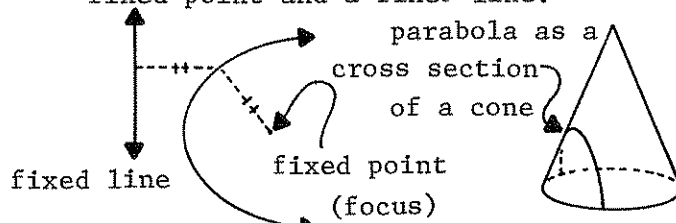
parallel lines.   Two lines which lie in the same plane and do not intersect.
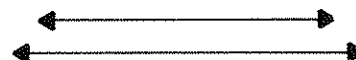
parallel postulate.   Through a point not on a given line, one and only one line may be drawn which is parallel to the given line.

parallelogram.   A quadrilateral with its opposite sides parallel.
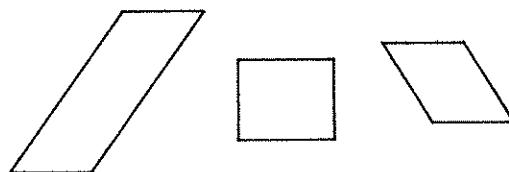
parallelopiped.   A polyhedra with 6 faces all of which are parallelograms.
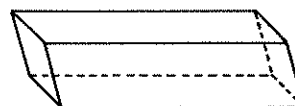
penta-.   Five, having five.

pentagon.   A five-sided polygon.

regular pentagon
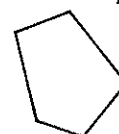
irregular pentagon

pentagram.   A five-pointed star
    formed from a regular pentagon.
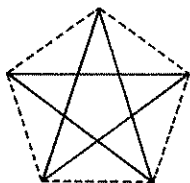
pentominoes.   Plane figures made by
    arranging five squares so that at
    least one edge of each square touches
    the edge of another square.

some pentominoes
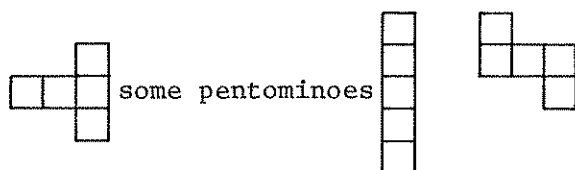
perimeter.   The sum of the lengths of
    the sides of a polygon.

3m   3m
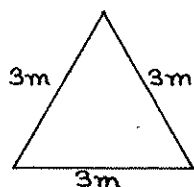
perimeter = 9 m

3m

perpendicular bisector.   A line which
    meets a line segment at a 90° angle
    and separates the line segment into
    two congruent segments.

A

the perpendicular

B
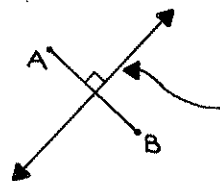
bisector of $\overline{AB}$

perpendicular lines.   Two lines that
    intersect at right angles to each
    other.
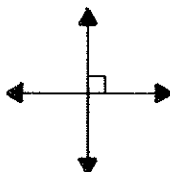
pi (π).   The ratio of the circumfer-
    ence to the diameter of a circle
    (approximately 3.1416).

plane.   A flat surface. Any straight
    line containing two points of a plane
    lies entirely in the plane. A
    parallelogram is often used to repre-
    sent a plane.
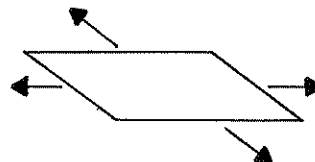
plane figure.   A set of points in a
    plane. Examples are angles, circles
    and polygons.

plane of symmetry.   A plane which
    separates a figure into two halves,
    one of which is a reflection of the
    other.

plane of symmetry
for a cube
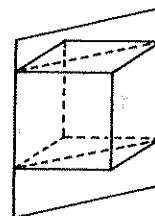
plane region.   The interior of a simple
    closed curve, not usually including
    the curve itself.

rectangular   circular
region   region

point.   An undefined term. A point has
    a position in space but no dimensions.
    Its position can be specified by
    coordinates.

point of tangency.   A special point
    shared by two or more figures.

point of tangency
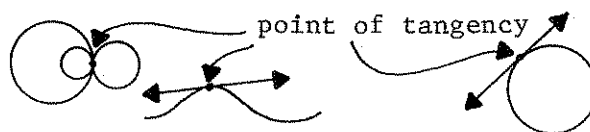
poly-. Many or several.

polygon. A simple closed plane figure made up of line segments.
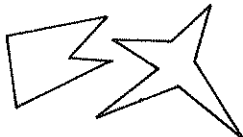
convex polygons   nonconvex polygons

polyhedron. A closed space figure whose faces are polygonal.
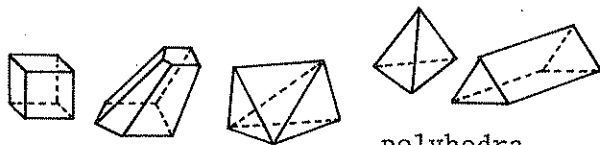
polyhedra

prism. A 3-D figure that has two polygonal regions as bases and lateral faces that are parallelogram-shaped.
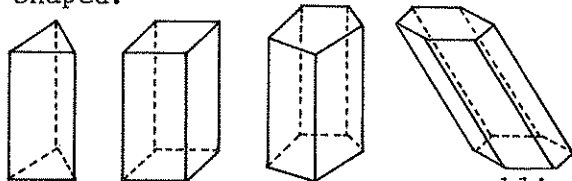
triangular prism   rectangular prism   pentagonal prism   oblique hexagonal prism

projection point. In the figure below point P is called the projection point. Other names for a projection point are center of enlargement, perspective point and center of projection.
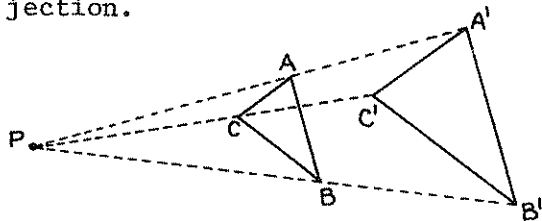
protractor. A tool used to measure angles.
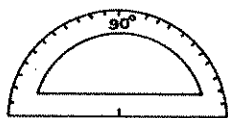
pyramid. A 3-D figure with one polygonal base and triangular lateral faces.
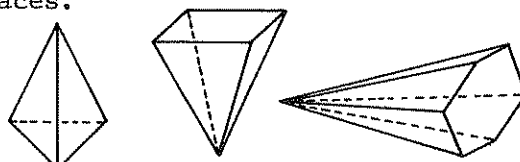
triangular pyramid   square pyramid   hexagonal pyramid

Pythagorean theorem. A theorem which says, "the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the legs."
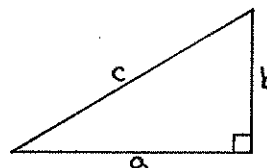
$$c^2 = a^2 + b^2$$

quadri-. Four or fourth.

quadrilateral. A four-sided polygon.

rectangle   rhombus
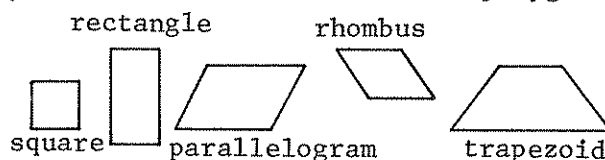
square   parallelogram   trapezoid

quadrilaterals

radian. Unit of angular measure. A central angle with measurement 1 radian will cut off an arc whose length is the same as the length of the radius of the circle. 360° corresponds to 2π radians. A right angle has radian measure $\frac{\pi}{2}$.

1 radian
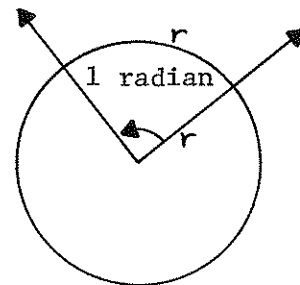
radius. 1) Any segment from a point on the circle or sphere to its center. 2) The distance from the center to any point on the circle or sphere.
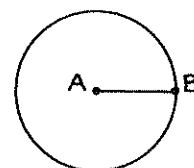
ray.   Part of a straight line (sometimes called a half-line) with an endpoint.

rectangle.   A four-sided polygon that has four right angles.

square

golden rectangle
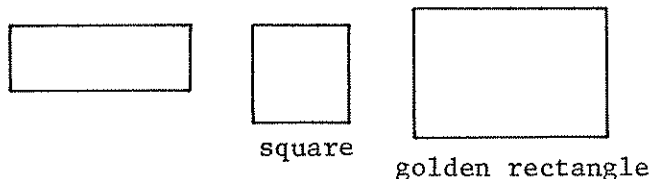
rectangular parallelopiped.   A polyhedron with six faces, each of which is a rectangle.
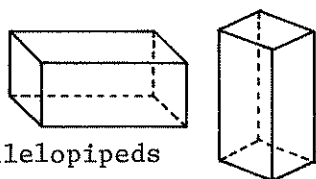
rectangular parallelopipeds

(also rigid rectangular prisms)

reflection.   A motion in which a figure is "flipped" over a line.
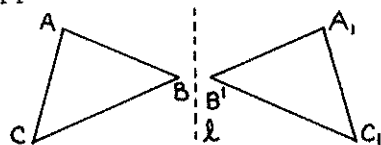
△A'B'C' is a reflection of △ABC across line ℓ.

reflex angle.   Any angle whose measure is more than 180° and less than 360°.

reflex angle
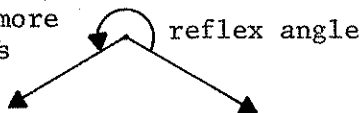
region of a plane.   Part of a plane. The interior of a simple closed curve is a region. The exterior of a simple closed curve is a region.

a region

a region
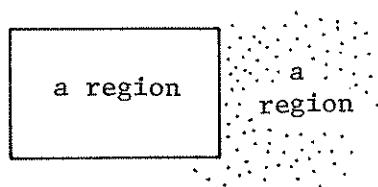
regular polygon.   A polygon having all sides congruent and all interior angles congruent.

regular polygons
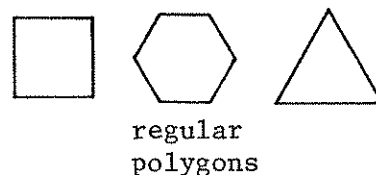
rhombus.   A parallelogram with four congruent sides.

square
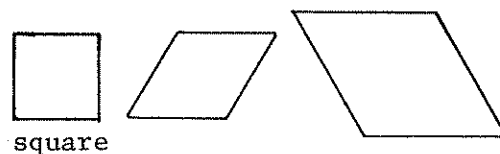
right angle.   An angle that has the measure of 90 degrees, often shown like ⌐→ to represent 90°. (There is no left angle and no wrong angle.)

90°

right triangle   rectangle   90°
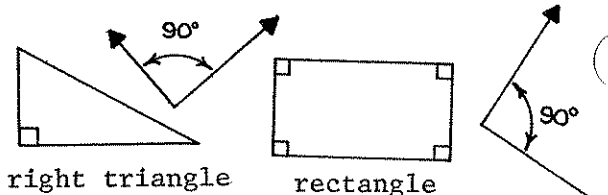
right triangle.   A triangle that has one right angle (90°).
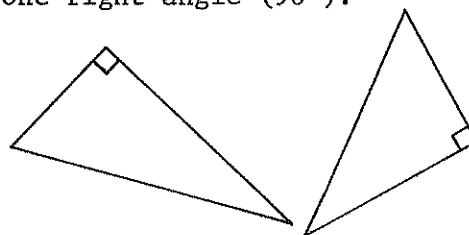
rotation.   A motion in which a given figure is "turned" about a fixed point.

fixed point
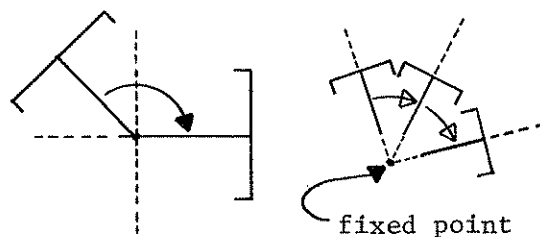
rotational symmetry. A figure which can be rotated a number of degrees about an interior point and still fits on its outline has rotational symmetry.

scale drawing. A drawing whose angles are congruent to and whose linear measurements are in a certain ratio to the measurements of the original. A map is a scale drawing.

These two figures are scale drawings of each other.
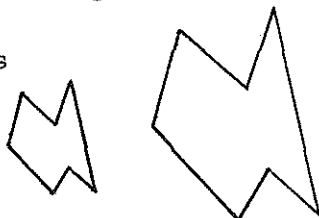
scalene triangle. A triangle in which each side is a different length.

secant of a circle. A line that intersects a circle in two points.

secant
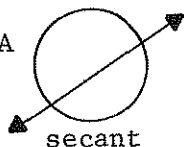
sector of a circle. A pie-shaped region of a circle bounded by 2 radii and an arc of the circle.

sector

segment. See line segment.

semicircle. An arc of the circle whose endpoints are on a diameter.

semicircle
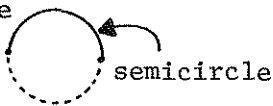
similar figures. Figures which have the same shape, but not necessarily the same size. The corresponding angles of similar figures are congruent. Corresponding sides are in a given ratio.

similar figures

simple curve. A curve that does not cross itself.

simple closed curves

simple curves (not closed)
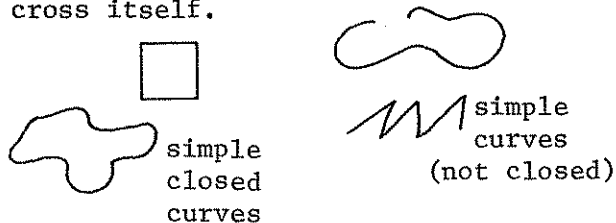
skew lines. Two lines that are NOT in the same plane.
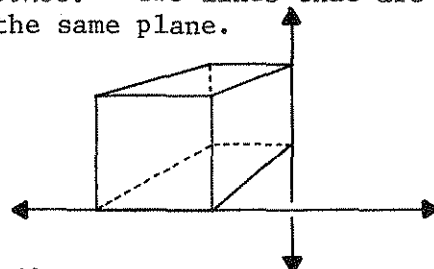
solid figure. A three-dimensional figure. Cubes, spheres, pyramids and cones are called solids.

space. The set of all points.

space figure. Any set of points in space. Often referring to three-dimensional figures or solids.

sphere. A set of points in space equidistant from a given point called the center.

center

sphere
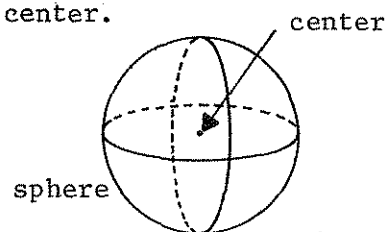
spiral. A coiled curve on a plane.

square. A four-sided polygon that has four right angles and four sides that are the same length. A square is also a rectangle, a parallelogram, a rhombus and a quadrilateral.
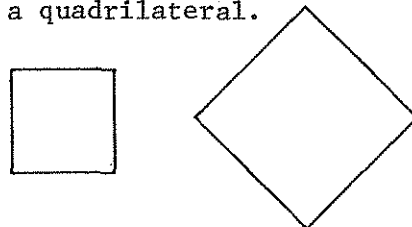
straight angle.   An angle with degree measure 180.
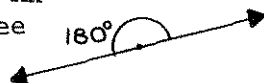
straightedge.   A tool for drawing pictures of straight lines. It has no markings like a ruler.
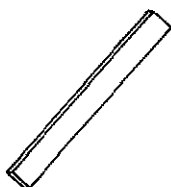
supplementary angles.   Two angles whose degree measure sum is 180. Angle 1 and 2 are supplementary angles.
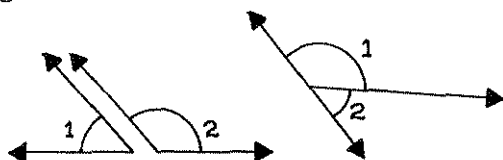
symmetric figure.   A figure which can be reflected across a line or rotated about an interior point to fit on its outline.
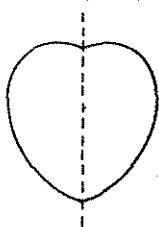
This figure has line symmetry.   This figure has rotational symmetry about point P.

tangent.   1) Loosely—a line that has exactly one point in common with a geometric figure.   2) To touch in only one point. A plane can be tangent to a sphere.
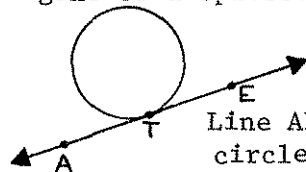
Line AE is tangent to the circle at point T.

tangram.   An old Chinese puzzle made of the seven pieces shown. The pieces can be arranged to form many figures.
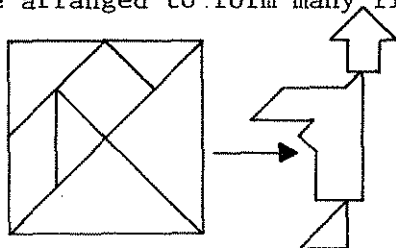
tessellation.   A repeated pattern of regions that can completely cover a plane.
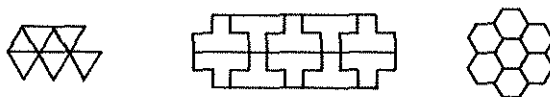
tetrahedron.   A polyhedron with four faces. A regular tetrahedron has four congruent triangular regions or faces.
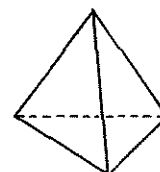
torus.   A doughnut-shaped space figure.

transit.   An instrument used in surveying to measure angles. It consists of a small telescope which rotates and scales which measure the angle through which it rotates.

translation.   A motion where each point of a figure "slides" the same direction and the same distance.

$\overrightarrow{AA'} = \overrightarrow{BB'}$
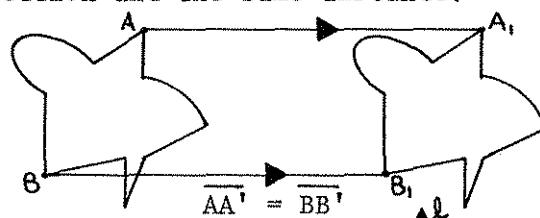
transversal.   A straight line intersecting two or more lines which may or may not be parallel.   Line ℓ is a transversal.
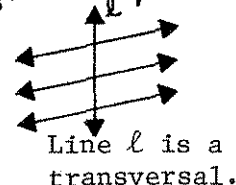
trapezoid.   A four-sided polygon with exactly one pair of parallel sides.
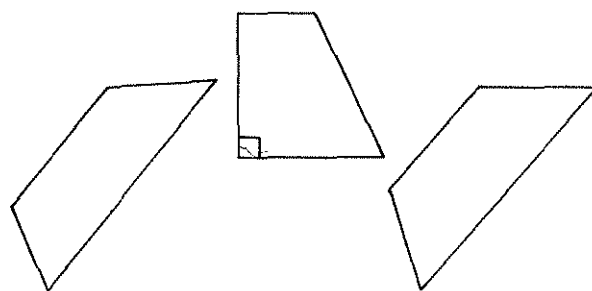
triangle.   A three-sided polygon.

acute triangle
obtuse triangle
right triangle

equilateral triangle
isosceles triangle
scalene triangle
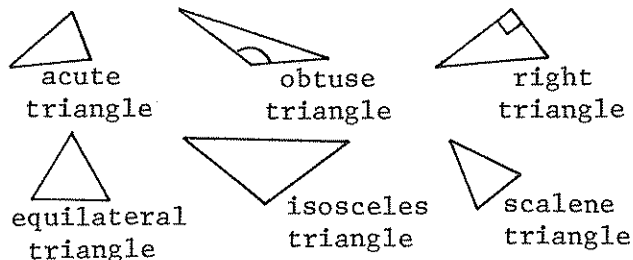
triangular pyramid.   A 3-D figure that has four triangular regions as faces. If the faces are all congruent, the triangular pyramid is a regular tetrahedron.

irregular   regular
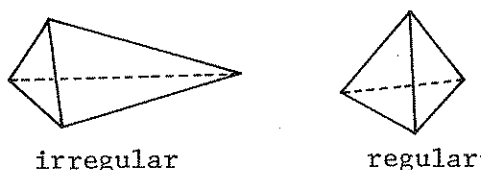
trisect.   Separate into three parts of equal measure.

truncate.   Cut off.

truncated pyramid

truncated cube
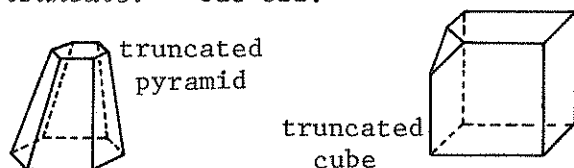
unit.   An amount or quantity adopted as a standard of measurement.
Examples:
Length--centimetres, inches, kilometres, miles, metres, light years.
Mass--grams, kilograms.
Area--centimetre^2, inches^2, yards^2, metres^2.
Volume--centimetre^3, inches^3, kilometres^3.
Time--minutes, seconds, hours.

vertex.   1) The point that two or more rays, sides or edges have in common.

vertex

vertex

vertex
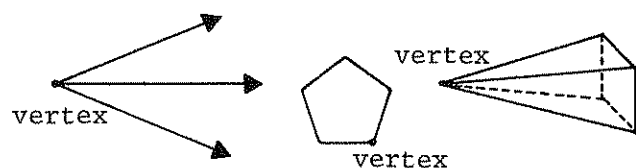
2) In a network, a point where two or more edges meet.
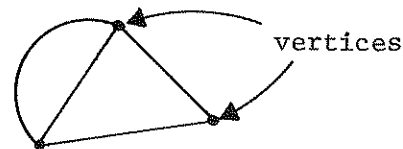
vertices

vertical angles.   In the diagram angles 1 and 3 are a pair of vertical angles. Angles 2 and 4 are also a pair of vertical angles. Vertical angles are congruent.
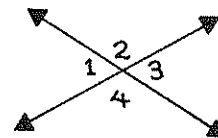
volume.   The measure of a space region. The measure is found by using an appropriate standard unit like the cubic centimetre (cm^3) or the cubic foot or cubic metre (m^3).

1 cm

1 cubic cm = 1 cm^3

1 cm

1 cm

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

ANNOTATED BIBLIOGRAPHY

The following is a list of sources used in the development of this resource. It is not a comprehensive listing of materials available. In some cases, good sources have not been included simply because the project did not receive permission to use the publisher's materials or a fee requirement prohibited its use by the project.

ACTIVITIES IN MATHEMATICS, Second Course. Johnson, Donovan A.; Hansen, Viggo P.; Peterson, Wayne H.; Rudnick, Jesse A.; Cleveland, Ray; and Bolster, L. Carey. Glenview, Illinois: Scott, Foresman and Company, 1973. (Scott, Foresman and Company, 1900 East Lake Ave., Glenview, IL 60025)

432 pp; cloth; teacher's guide; color

A collection of 85 activities covering the areas of graphs, statistics, proportions, and geometry are contained in this book.

ACTIVITIES WITH GEOBLOCKS. Seymour, Dale; and Greenes, Carole. Palo Alto, California: Creative Publications, Inc., 1975. (Creative Publications, Inc., P.O. Box 10328, Palo Alto, CA 94303)

95 pp; paper; b/w; teacher's guide

This book features teacher content and activities to be used with a set of Geoblocks. Emphasis is placed on linear area and volume measure as well as fractions and the Pythagorean theorem.

ACTIVITIES WITH SQUARES FOR WELL-ROUNDED MATH. Schreiner, Nikki Bryson. Palos Verdes Estates, California: Touch and See Educational Resources, 1973. (Touch and See Educational Resources, P.O. Box 794, Palos Verdes Estates, CA 90274)

119 pp; paper; b/w; activity cards; low reading level

This book consists of 94 activity cards in book form for students, each one involving squares and each one providing answers for teacher and students.

AN ADVENTURE IN GEOMETRY. Ravielli, Anthony. New York: The Viking Press, Inc., 1966. (The Viking Press, Inc., 625 Madison Ave., New York, NY 10022)

117 pp; cloth; color; student reference; medium reading level

This is a very attractive book in which illustrations are used to convey the meaning of many geometric terms.

AFTERMATH, VOLUMES I-IV. Seymour, Dale. Palo Alto, California: Creative Publica-
     tions, 1971. (Creative Publications, INc., P.O. Box 10328, Palo Alto, CA 94303)

     97 pp each; paper; b/w; teacher reference

APPLIED MATHEMATICS CARDS. Goddard, T.R. and Grattidge, A.W. Huddersfield, England:
     Schofield and Sims Limited, 1966. (Schofield and Sims Limited, 35 St. Johns
     Rd., Huddersfield, England HD1 5DT)

     150 cards; b/w; activity cards; medium reading level

          *These five sets of activity cards deal with lots of applications
     on many interesting topics and are language-oriented to British culture.*

THE ARITHMETIC TEACHER. Reston, Virginia: National Council of Teachers of
     Mathematics. (National Council of Teachers of Mathematics, 1906 Association
     Dr., Reston, VA 22091)

          *This magazine, published eight times a year, January through May
     and October through December, contains a wealth of ideas and activities
     for the elementary and middle school matheamtics teacher.*

ART FORMS IN NATURE. Haeckel, Ernst. New York: Dover Publications, Inc., 1974.
     (Dover Publications, Inc., 180 Varick St., New York, NY 10014)

     100 pp; paper; teacher reference; b/w

          *The book contains the collection of Haeckel's carefully drawn graphics
     of rare animals and plants. The drawings help to make the reader aware of
     the beauty, symmetry, and varied shapes that occur in nature.*

ART 'N' MATH. Billings, Karen; Campbell, Carol; and Schwandt, Alice. Eugene,
     Oregon: Action Math Associates, Inc., 1975. (Action Math Associates, Inc.,
     1358 Dalton Dr., Eugene, OR 97404)

     63 pp; paper; workbook; b/w; medium reading level

          *This is a collection of 19 activities that combine math skills with
     an art project.*

CIRCLES. (Topics from Mathematics). Mold, Josephine. London: The Syndics of
     the Cambridge University Press, 1967. (Cambridge University Press, 32 East
     57th St., New York, NY 10022)

     32 pp; paper; color; student reference

          *The book illustrates making a circle with a ruler, making
     curves like the cardioid using circles and packing circles.*

"Collapsoids." Pedersen, Jean J. *THE MATHEMATICAL GAZETTE,* Vol. LIX (June, 1975),
     pp. 81-94. London: G. Bell and Sons LTD.

        *The article contains instructions for constructing collapsoids and
     a discussion of the properties of collapsoids. (The MATHEMATICAL GAZETTE
     is a journal of the Mathematical Association, 259 London Rd., Leicester,
     LE2 3BE, England. The copyright of all material published in the MATHEMATICAL
     GAZETTE is vested solely in the Mathematical Association. The MATHEMATICAL
     GAZETTE is published by G. Bell & Sons, Ltd., of York House, Portugal Street,
     London WC2A 2HL.)*

"Conics from straight lines and circles: parabolas." Maletsky, Evan M. *THE
     MATHEMATICS TEACHER,* Vol. 66, No. 3 (March, 1973), pp. 243-246. Reston,
     Virginia: National Council of Teachers of Mathematics. (National Council
     of Teachers of Mathematics, 1906 Association Dr., Reston, VA 22091)

"Creatamath, or -- geometric ideas inspire young writers." *READINGS IN GEOMETRY
     FROM THE ARITHMETIC TEACHER.* Washington, D.C.: National Council of Teachers
     of Mathematics, 1970. (National Council of Teachers of Mathematics, 1906
     Association Dr., Reston, VA 22091)

CREATIVE CONSTRUCTIONS. Seymour, Dale; and Schadler, Reuben. Palo Alto, California:
     Creative Publications, Inc. 1974. (Creative Publications, Inc., Box 10328,
     Palo Alto, California 94303)

     62 pp; paper; b/w; workbook; medium reading level

        *This is a workbook containing many designs that can be
     created from inscribed triangles, squares, pentagons, hexagons
     and octagons.*

CRYSTALS AND CRYSTAL GROWING. Holden, Alan and Singer, Phyllis. New York: Anchor
     Books, Doubleday and Company, Inc. 1960. (Doubleday and Company, Inc., 245 Park
     Ave., New York, NY 10017)

CURVES AND THEIR PROPERTIES. Yates, Robert C. Washington, D.C.: The National Council
     of Teachers of Mathematics, 1952. (National Council of Teachers of Mathematics,
     1906 Association Dr., Reston, VA 22091)

     245 pp; cloth; teacher reference

        *This book is an indepth treatment of curves which includes
     descriptions and proofs of properties.*

"Decoding student names, or if Alan is 42, then Robyn must be 82." Bernstein,
     Robert; and Barson, Alan. *THE ARITHMETIC TEACHER,* Vol. 22, No. 7 (November,
     1975), pp. 591-592. Reston, Virginia: National Council of Teachers of Mathe-
     matics. (National Council of Teachers of Mathematics, 1906 Association Dr.,
     Reston, VA 22091)

"Discovery with cubes." Reys, Robert E. *THE MATHEMATICS TEACHER,* Vol. 67, No. 1
     (January, 1974), pp. 47-50. Reston, Virginia: National Council of Teachers
     of Mathematics. (National Council of Teachers of Mathematics, 1906 Association
     Dr., Reston, VA 22091)

THE DIVINE PROPORTION. Huntley, H.E. New York: Dover Publications, Inc., 1970.
     (Dover Publications, Inc., 180 Varick St., New York, NY 10014)

     186 pp; paper; teacher reference; b/w

          The golden ratio, ∅, is the basis for many seemingly unrelated
     topics in matheamtics. This book covers those topics in a sophisti-
     cated manner.
          Chapter XIII, Spira Mirabilis (pp. 164-176), emphasizes the golden
     triangle, (a 72°, 72°, 36° isosceles triangle), the golden ratios to
     be found, and the logarithmic spiral that can be drawn by connecting
     vertices of a nested sequence of similar golden triangles.

THE DOT AND THE LINE. Juster, Norton. New York: Random House, Inc., 1963 (Random
     House, Inc., 201 East 50th St., New York, NY 10022)

     unp; cloth; humorous book; b/w; medium reading level

          This book is written as a spoof on the romance of a dot and a
     line (with a squiggle making a romantic triangle) which brings out
     many characteristics of lines.

ESSENTIALS OF MATHEMATICS 1. Sobel, Max A.; Maletsky, Evan M.; and Hill, Thomas J.
     Lexington, Massachusetts: Ginn and Company (Xerox Corporation) 1970. (Ginn and
     Company (Xerox Corporation), 191 Spring St., Lexington, MA 02173)

     425 pp; hardback; textbook; b/w

          This is the first of a four-book series designed to provide a sound
     mathematical education for those students not *going into the formal*
     algebra-geometry sequence.

EUREKA. Seymour, Dale; and Gidley, Richard. Palo Alto, California: Creative
     Publications, Inc., 1967. (Creative Publications, Inc., Box 10328, Palo Alto,
     California 94303)

     128 pp; paper; color; workbook; medium reading level

          A very interesting collection of problems and drawings to
     enrich the mathematics classroom atmosphere is presented in
     this book.

EXPLORATIONS IN MATHEMATICS. Wiebe, Arthur J. and Goodfellow, James W. New York:
     Holt, Rinehart and Winston, Publishers, 1970. (Holt, Rinehart and Winston,
     Publishers, 383 Madison Ave., New York, NY 10017)

     504 pp; cloth; b/w; testbook; medium reading level

          A wide variety of activities, including topics from geometry,
     graphing, probability, statistics, and theory of numbers, is given
     in this book.

EXPLORING CUBES, SQUARES AND RODS. Trivett, John. New Rochelle, New York: Cuise-
naire Company of America, Inc., 1973. (Cuisenaire Company of America, Inc.,
12 Church St., New Rochelle, NY 10805).

68 cards; b/w; activity cards; medium reading level

*This is a set of 68 cards intended as a guide to the teacher,
which suggest activities and games that explore a wide variety
of mathematics topics: area, symmetry, volume, patterns, weight,
etc. These cards are to be used with a set of Cuisenaire Cubes,
Squares and Rods.*

EXPLORING MATHEMATICS, INSIGHTS AND SKILLS. Peters, Max; Schor, Harry; Meng, Gloria;
and Wayne, Alan. New York: Globe Book Company, Inc., 1974. (Globe Book
Company, Inc., 175 Fifth Ave., New York, NY 10010)

approx. 300 pp/book; paper and cloth; b/w; textbook

*This is one of a series of four books. The series is a standard
text.*

EXPLORING MATHEMATICS ON YOUR OWN. Glenn, William H. and Johnson, Donovan A. New
York: Dover Publications, Inc., 1960. (Dover Publications, Inc., 180 Varick
St., New York, NY 10014)(Copyright now held by Donovan A. Johnson)

303 pp; paper; teacher reference; b/w

*The book contains many interesting ideas and activities that can be
presented in the classroom. Special emphasis is given to numeration
systems, number patterns, the Pythagorean theorem, sets, sentences,
operations, topology and fun activities.*

FLAT STANLEY. Brown, Jeff. New York: Juvenile Books, Harper & Row, Publishers, Inc.,
1964. (Harper & Row Publishers, Inc., 10 E. 53rd St., New York, NY 10022)

FLATLAND. 5th rev. ed. Abbott, Edwin A. New York: Barnes and Noble, Inc., 1963.
(Barnes and Noble, Inc., Division of Harper & Row Publishers, Inc., 10 E. 53rd
St, New York, NY 10022)

FREEDOM TO LEARN. Biggs, Edith E. and MacLean, James R. Canada: Addison-Wesley
(Canada) Ltd., 1969. (Addison-Wesley (Canada) Ltd., P.O. Box 580, 36 Prince
Andrew Place, Don Mills, Ontario, M3C 2T8, Canada)·

202 pp; cloth; teacher reference; color

*This book emphasizes the active approach to learning elementary
mathematics and gives many suggestions for organizing lab lessons,
classrooms, materials, etc. to promote an active approach.*

FUN AND GAMES WITH GEOMETRY. Haugaard, James and Horlock, David. Los Gatos, Cali-
fornia: Contemporary Ideas, 1973. (Prentice-Hall Learning Systems, Inc., P.O.
Box 47 X, Englewood Cliffs, NJ 07632)

100 pp; paper; teacher reference; b/w

*A book of teacher pages and appropriate masters to be thermofaxed
for student use. Polygons, map coloring and coordinate graphing are
three of the book's emphases.*

FUN WITH NUMBERS. San Francisco: San Francisco Unified School District, n.d. (San
    Francisco Unified School District, 135 Van Ness Ave., San Francisco, CA 94102)

GAMES AND PUZZLES FOR ELEMENTARY AND MIDDLE SCHOOL MATHEMATICS. Reston, Virginia:
    National Council of Teachers of Mathematics, 1975. (National Council of Teachers
    of Mathematics, 1906 Association Dr., Reston, VA 22091)

    280 pp; paper; teacher reference

GEOBOARD ACTIVITY CARD KIT. Trivett, John. New Rochelle, New York: Cuisenaire
    Company of America, Inc., 1971. (Cuisenaire Company of America, Inc., 12
    Church St., New Rochelle NY 10805)

    142 cards; b/w; activity cards; medium-high reading level

        *This collection of activity cards to be used with either a
    square or circular geoboard covers such concepts as polygons,
    angles, symmetry, fractions, area, transformations, circles,
    and games.*

GEOBOARD ACTIVITY CARDS, Intermediate. Barson, Alan. Fort Collins, Colorado: Scott
    Resources Inc., 1971. (Scott Resources Inc., 1900 E. Lincoln, Box 2121, Fort
    Collins, CO 80522)

    121 cards; color; activity cards; low-medium reading level

        *The kit contains 121 activities that cover basic concepts of
    mathematics.*

GEOMETRIC EXCURSIONS, rev. ed. Oakland County Mathematics Project. Pontiac, Michigan:
    Oakland Schools, 1970. (Oakland Schools, 2100 Pontiac Lake Rd., Pontiac, MI 48054)

    92 pp; paper; b/w; teacher's guide

        *This book is one of a series written under the Oakland County
    Mathematics Project. Rather than a formal approach, students are
    here given the opportunity to work with geometry rather than absorb
    terminology.*

"Geometrical activities for the upper elementary school." *THE TEACHING OF GEOMETRY
    AT THE PRE-COLLEGE LEVEL.* Steiner, Hans-Georg, ed. Dordrecht, Holland: D.
    Reidel Publishing Company, 1971, pp. 77-118. (D. Reidel Publishing Company,
    306 Dartmouth St., Boston, MA 02116)

    387 pp; paper; b/w; teacher reference

        *The publication is a collection of papers prepared for or
    presented to the second Comprehensive School Mathematics Program
    (CSMP) International Conference. The article cited lists many
    activities, most of a problem-solving nature, that can be used
    with students.*

GEOMETRY. Jacobs, Harold R. San Francisco: W.H. Freeman and Company, Publishers, 1974. (W.H. Freeman and Company, Publishers, 660 Market St., San Francisco, CA 94104)

701 pp; cloth; textbook; b/w; high reading level

A high school textbook which emphasizes a transformational approach and presents many enrichment topics. A very light but interesting approach is used throughout the book.

GEOMETRY A TRANSFORMATION APPROACH. Coxford, Arthur F. and Usiskin, Zalman P. River Forest, Illinois: Laidlaw Brothers, A Division of Doubleday & Company, Inc., 1971. (Laidlaw Brothers, A Division of Doubleday & Company, Inc., Thatcher and Madison, River Forest, IL 60305)

612 pp; paper; teacher reference; b/w

A senior high text designed for average students, this book uses a transformational approach to teach the concepts of Euclidean geometry. This is an excellent resource for information on transformations.

GEOMETRY IN MODULES: AN INFORMAL COURSE, Books A-D with Teacher's Manual. Lange, Muriel. Menlo Park, California: Addison-Wesley Publishing Company, Inc., 1975. (Addison-Wesley Publishing Company, Inc., 2725 Sand Hill Rd., Menlo Park, CA 94025)

128 pp. ea.; paper; textbook

The text presents an informal treatment of geometry with an emphasis on discovery, problem solving and application. Modules A and B are largely devoted to measurement and are suitable as an introduction to geometry at the junior high level.

GEOMETRY IN THE MATHEMATICS CURRICULUM. Thirty-   Yearbook of the National Council of Teachers of Mathematics. Reston, Virginia: National Council of Teachers of Mathematics, 1973. (National Council of Teachers of Mathematics, 1906 Association Dr., Reston VA 22091)

472 pp; cloth; teacher reference; b/w

This 36th yearbook contains four sections, Informal Geometry, Formal Geometry in the Senior High School, Contemporary Views of Geometry, and The Education of Teachers, with articles written by mathematics educators.

GEO-RING POLYHEDRA. Silvey, Linda. Palo Alto, California: Creative Publications, Inc., 1972. (Creative Publications, Inc., Box 10328, Palo Alto, CA 94303)

32 pp; paper; b/w; teacher reference

This book contains instructions and graphics for using geo-rings (polygonal shapes with borders for stapling or pasting) to assemble the five Platonic solids and the thirteen Archimedian solids. A kit of geo-rings is also available from the publisher.

THE GEOSQUARE TEACHER'S MANUAL. Bradford, John and Bartram, Harlan. Fort Collins,
    Colorado: Scott Resources,Inc., 1967. (Scott Resources, Inc., 1900 E. Lincoln,
    Box 2121, Fort Collins, CO 80522)

    30 pp; paper; teacher's guide; b/w

        *This book contains activities, constructions and problems that can
    be used with Geosquares (a commercial version of the manipulative
    more commonly known as the geoboard).*

GIANT GOLDEN BOOK OF MATHEMATICS -- EXPLORING THE WORLD OF NUMBERS AND SPACE.
    Adler, Irving. New York: Golden Press, 1958, 1960. (Western Publishing
    Company, Inc., 1220 Mound Ave., Racine, WI 53404)

    56 pp; paper; student reference; medium reading level

        *The book contains easy-to-read sections on number concepts, geom-
    etry, mathematics in nature, navigation, probability, the slide rule, etc.*

GRAPH PAPER DESIGNS. Lettau, John H. Los Gatos, California: Contemporary Ideas,
    1972. (Lettau Educational Enterprises, Inc., P.O. Box 88, Santa Maria, CA 93454)

    64 pp; paper; teacher reference; b/w

        *This book illustrates how graph paper can be used to create
    designs. 31 sample designs are provided.*

THE GRAPHIC WORK OF M.C. ESCHER. Escher, M.C. New York: Ballantine Books, a division
    of Random House, Inc., 1971. (Ballantine Books, a division of Random House, Inc.,
    201 East 50th St., New York, NY 10022)

GRAPHITI. Hayward, California: Activity Resources Company, Inc., n.d. (Activity
    Resources Company, Inc., P.O. Box 4875, Hayward, CA 94540)

    25 pp; paper; teacher reference; b/w

        *There are twenty-five sheets of coordinate pictures for students
    to graph.*

GREATER CLEVELAND MATHEMATICS PROGRAM. Chicago: Science Research Associates, Inc.,
    1968. (Educational Research Council of America, Rockefeller Building,
    Cleveland, OH 44113)

    284 pp; cloth; textbook; b/w

        *The book is from a series of standard textbooks for grades one through
    six.*

GUIDE TO ADVENTURE IN MATHEMATICS FOR THE 70'S. Hewitt, B. Langley, Bucks, England:
    St. Paul Publications, 1970. (St. Paul's Publications, Middlegreen, Slough,
    England SL3 6BT)

    277 cards; color; activity cards; medium reading level

        *A set of 277 handwritten and laminated activity cards that introduces
    students to a variety of mathematics concepts, helps them discover new
    relationships, and provides some practice in the application of these
    concepts.*

A HANDBOOK OF AIDS FOR TEACHING JUNIOR-SENIOR HIGH SCHOOL MATHEMATICS. Krulik,
   Stephen. Philadelphia: W.B. Saunders Company, 1971. (W.B. Saunders Company,
   218 W. Washington Square, Philadelphia, PA 19105)

   120 pp; paper; teacher reference; b/w

        *This is a book of forty-one multi-sensory aids that can be made
   and used by teachers and students in junior and senior high schools.*

HOW TO SOLVE IT. Polya, G. Princeton, New Jersey: Princeton University Press,
   1945. (Princeton University Press, 41 William St., Princeton, NJ 08540)

IDEAS FOR MANIPULATIVE MATERIALS--ELEMENTARY MATHEMATICS. Shoemaker, Terry and
   Swadener, Dr. Marc. Longmont, Colorado: Northern Colorado Educational Board
   of Cooperative Services, 1972. (Northern Colorado Educational Board of
   Cooperative Services, 830 South Lincoln, Longmont, CO 80501)

   b/w activity cards; medium reading level

        *This is a series of file-folder sized cards with ideas for using
   manipulative materials in the areas of sets, numbers, geometry, mea-
   surement, probability, number theory and functions.*

INDEPENDENT ACTIVITIES. Maney, Ethel S. Elizabethtown, Pennsylvania; The
   Continental Press, Inc., 1958. (The Continental Press, Inc., Elizabethtown,
   PA 17022)

   24 pp; paper; workbook; b/w

        *This book is one of a series of reading readiness workbooks and
   concentrates on using visual perception to copy designs.*

INVESTIGATING SCHOOL MATHEMATICS, Second Edition. Eicholz, Robert E.; O'Daffer,
   Phares G.; and Fleenor, Charles R. Menlo Park, California: Addison-Wesley
   Publishing Company, Inc., 1976. (Addison-Wesley Publishing Company, Inc.,
   2725 Sand Hill Rd., Menlo Park, CA 94025)

   cloth; textbook series; color

        *This is a textbook series for grades 1-6. The books are colorfully
   illustrated and include the usual topics in mathematics. A spiral
   approach is used for most topics. The development of coordinate
   planes was used as a basis for activities in this resource.*

LAB GEOMETRY--Teacher's Edition, Final Draft. Bellevue, Washington: Bellevue
   Public Schools, Fall, 1972. (Bellevue Public Schools, District 405, 310 102nd
   Ave., N.E.,Bellevue, WA 98004)

   402 pp; paper; teacher reference

        *This is a book for a high school geometry course that uses a
   discovery approach and hands-on activities to introduce concepts.
   The text is not commercially available.*

THE LABORATORY APPROACH TO MATHEMATICS. Kidd, Kenneth; Myers, Shirley S; and
   Cilley, David M. Chicago: Science Research Associates, Inc., 1970. (Science
   Research Associates, Inc., 259 East Eric St., Chicago, IL 60611)

   282 pp; paper; teacher reference; b/w

       This is an excellent resource that details the organization,
   planning and facilities needed for using a mathematics laboratory.
   A series of activities concerning the use of ratio is presented.

LINE DESIGNS. Seymour, Dale; and Snider, Joyce. Palo Alto, California: Creative
   Publications, 1968. (Creative Publications Inc., Box 10328, Palo Alto, CA
   94303)

   58 pp; paper; b/w; teacher reference

       The pamphlet contains a description of the materials and
   techniques used in making line designs. Sample line designs
   are also included.

LOOKING AT SOLIDS. (Mathematics in the Making 3) Bell, Stuart E. New York:
   Houghton Mifflin Company, 1969. (Houghton Mifflin Company, Educational
   Division, 1 Beacon St., Boston, MA 02107)

   32 pp; paper; teacher reference; b/w

       This is a collection of introductory activities dealing with polygons
   and polyhedra. A variety of methods is listed for folding polyhedra.

MAGIC SQUARES. Fults, John Lee. LaSalle, Illinois: Open Court Publishing Company,
   1974. (Open Court Publishing Company, P.O. Box 599, LaSalle, IL 61301)

   104 pp; paper; teacher reference; b/w

       This book traces the history, description and construction of Magic
   Squares.

MARTIN GARDNER'S SIXTH BOOK OF MATHEMATICAL GAMES FROM SCIENTIFIC AMERICAN.
   Gardner, Martin. San Francisco: W.H. Freeman and Company, 1971. (The only
   edition of the book now in print is the 1975 paperback edition available from
   Charles Scribner's Sons, 597 Fifth Ave., New York, NY 10017)

   262 pp; cloth; teacher reference; b/w

       This is a book of entertaining mathematical subjects and puzzles
   written in layman's language.

MATH ACTIVITY CARDS. (Macmillan Elementary Mathematics) Clarkson, David M. New
   York: The Macmillan Publishing Company, Inc., 1969, 1970. (Macmillan
   Publishing Company, Inc., 866 Third Ave., New York, NY 10022)

   237 cards; color; activity cards; low reading level

       The five kits, A-E, contain 237 activity cards intended for
   students in grades 2-6. The cards can be used to supplement
   classroom lessons.

MATH ACTIVITY WORKSHEET MASTERS. Stokes, William T.; and Laycock, Mary. Palo Alto,
   California: Creative Publications, Inc., 1974. (Creative Publications, Inc.,
   Box 10328, Palo Alto, CA 94303)

THE MATH ENTHUSIAST, Issue 12 (September, 1974). Portland, Oregon: Oregon Council
      of Teachers of Mathematics. (Oregon Council of Teachers of Mathematics, 4015
      S.W. Canyon Rd., Portland, OR 97221)

> *This pamphlet, now included in the publication THE OREGON MATHEMATICS*
> *TEACHER, contains many classroom ideas for the primary and middle school*
> *teacher.*

MATH EXPERIMENTS WITH THE GEO STRIPS. Ewbank, William A. Troy, Michigan: Midwest
      Publications Company, Inc., 1972. (Midwest Publications Company, Inc., P.O.
      Box 129, Troy, MI 48084)

77 pp; paper; teacher reference

> *The book contains a collection of guided discovery activities, to*
> *be done with Geo-Strips, that explore such concepts of geometry as*
> *congruence, similarity, symmetry, rigidity, etc.*

MATH EXPERIMENTS WITH THE 1-INCH COLOR CUBE. Ewbank, William A. Troy, Michigan:
      Midwest Publications Company, Inc., 1972. (Midwest Publications Company, Inc.,
      P.O. Box 129, Troy, MI 48084)

60 pp; paper; teacher reference; color

> *This collection of enrichment and hands-on experiments uses cubes to*
> *explore such mathematics concepts as addition, subtraction, similarity,*
> *area, and volume.*

MATH IN NATURE (a series of posters with discussion). Newton, David E. Portland,
      Maine: J. Weston Walch, Publisher, Inc., 1970. (J. Weston Walch, Publisher,
      Inc., Box 658, Portland, ME 04104)

18 posters; b/w; medium reading level

> *These 18 black and white posters (11"x14") with discussion on*
> *each poster are good for bulletin boards, opaque projectors and*
> *a nature and mathematics unit.*

MATH PROJECTS: POLYHEDRAL SHAPES. Bassetti, Fred; Ruchlis, Hy; and Malameut,
      Daniel. Brooklyn, New York: Book-Lab, Inc., 1968. (Book-Lab, Inc., 1449
      37th St., Brooklyn, NY 11218)

48 pp; paper; teacher reference; b/w

> *This is a book concerning how to make geometric solids using*
> *cardboard panels that are connected with rubber bands.*

MATH WORKSHOP, Level F, rev. ed., Teacher's Guide. Wirtz, Robert W.; Botel, Morton;
Beberman, Max; and Sawyer, W.W. Chicago: Encyclopaedia Britannica Press, 1967.
(Encyclopaedia Britannica Educational Corporation, 425 N. Michigan Ave., Chicago,
IL 60611)

320 pp; cloth; color; textbook; medium reading level

*This is the sixth of a series of textbooks which use a spiral
approach to present materials dealing with the topics of structures,
sets, numbers and counting, numeration, addition and subtraction,
multiplication and division, functions and relations, geometry, and
measurement.*

MATHEMATICAL CARNIVAL. Gardner, Martin. New York: Alfred A. Knopf Publishers,
Inc., 1975. (Alfred A. Knopf Publishers, Inc., 201 E. 50th, New York, NY 10022)

274 pp; cloth; teacher reference; b/w

*This is an excellent collection of games, puzzles and brain teasers
that previously occurred in the author's column in SCIENTIFIC AMERICAN.*

"Mathematical Games." Gardner, Martin. *SCIENTIFIC AMERICAN*. New York:
Scientific American, Inc. (W.H. Freeman and Company, Publishers, 660 Market
St., San Francisco, CA 94104)

*Each issue of SCIENTIFIC AMERICAN contains a column (Mathematical
Games) by Martin Gardner dealing with some interesting mathematical
game, puzzle or paradox. Most ideas can be adapted for use with
middle school students.*

"Mathematical Games." Gardner, Martin. *SCIENTIFIC AMERICAN,* Vol. 198, No. 1
(January, 1958), pp. 92-96.

An interesting fallacy concerning areas is given in this article.

"Mathematical Games." Gardner, Martin. *SCIENTIFIC AMERICAN,* Vol. 199, No. 5
(November, 1958, pp. 136-142.

*A problem-solving activity for finding areas of squares is given
here.*

"Mathematical Games." Gardner, Martin. *SCIENTIFIC AMERICAN,* Vol. 217, No. 1
(July, 1967, pp. 112-116.

*An interesting game of connecting points with arcs is the subject
of this article.*

"Mathematical Games." Gardner, Martin. *SCIENTIFIC AMERICAN,* Vol. 228, No. 1
(January, 1973), pp. 108-115.

*This article presents a game for two involving the joining of
points to force the opponent to draw the line segment that completes
a triangle.*

MATHEMATICAL SNAPSHOTS, *2nd ed.* Steinhaus, Hugo. New York: Oxford University
Press, 1969. (Oxford University Press, 200 Madison Ave., New York, NY 10016)

MATHEMATICS. (Life Science Library Series) Bergamini, David. Morristown, New
Jersey: Time Inc., 1970. (Time, Inc., Time & Life Bldg., Rockefeller Center,
New York, NY 10020)

196 pp; cloth; student reference; b/w; color; medium reading level

*The colored and black and white photographs and graphics in this
historical survey of mathematics make this an excellent resource for
student and teacher alike. The book covers the development of mathe-
matics from counting to computers, probability and modern geometries.*

MATHEMATICS A HUMAN ENDEAVOR. Jacobs, Harold R. San Francisco: W.H. Freeman and
Company, Publishers, 1970. (W.H. Freeman, Publishers, 660 Market St., San
Francisco, CA 94104)

529 pp; cloth; textbook; b/w; high reading level

*This text for a liberal arts course is an excellent resource book.
Many of the ideas are suitable for or could be adapted for middle
school students.*

MATHEMATICS FOR SCHOOLS, *Level II, Books 5 and 6, Teacher's Resource Book.* Fletcher,
Harold and Howell, Arnold A. Reading, Massachusetts: Addison-Wesley Publish-
ing Company, Inc., 1972. (Addison-Wesley Publishing Company, Inc., South St.,
Reading MA 01867)

140 pp; paper; teacher reference

*The resource presents many discovery activities using a variety of
concrete materials to introduce mathematical concepts to students ages
7-13.*

A *MATHEMATICS LABORATORY HANDBOOK FOR SECONDARY SCHOOLS*. Krulik, Stephen. Philadel-
phia: W.B. Saunders Company, 1972. (W.B. Saunders Company, 218 W. Washington
Square, Philadelphia, PA 19105)

107 pp; paper; teacher reference; b/w

*Thirty-six activities for a mathematics laboratory are presented
here along with a listing of some commercially-made materials and
their uses.*

MATHEMATICS ON THE GEOBOARD. Niman, John and Postman, Robert. New Rochelle, New
York: Cuisenaire Company of America, 1974. (Cuisenaire Company of America,
Inc., 12 Church St., New Rochelle, NY 10805)

*Using the geoboard as a model, concepts of geometry, probability,
number theory, and topology are developed in this book.*

THE MATHEMATICS TEACHER. Reston, Virginia: The National Council of Teachers of
Mathematics. (National Council of Teachers of Mathematics, 1906 Association
Dr., Reston, VA 22081)

>   *This magazine, published eight times a year, January through*
>   *May and October through December, contains a wealth of ideas and*
>   *activities for the middle school and secondary mathematics teacher.*

MATHEMATICS, THE MAN-MADE UNIVERSE. Stein, Sherman, K. San Francisco: W.H.
Freeman and Company, 1969. (W.H. Freeman and Company, Publishers, 660
Market St., San Francisco, CA 94104)

415 pp; cloth; b/w; textbook; high reading level

>   *This textbook is intended for the nonmath-oriented student in*
>   *high school or college. This book covers topics from number*
>   *theory, topology, set theory, geometry, and algebra in an easy-*
>   *to-read theorem-proof style.*

MATHEX. Montreal, Canada: Encyclopaedia Britannica Publications LTD., 1970.
(Encyclopaedia Britannica Publications LTD., 2 Bloor Street West, Suite 1100,
Toronto, Ontario, M4W 3J1, Canada)

approx. 44 pp in each; paper; workbook; b/w; medium reading level

>   *MATHEX is a series of ten student workbooks with corresponding*
>   *Teacher Resource books. Books 1-5 are designed for primary grades*
>   *and books 6-10 are designed for grades 4-6. The topics covered*
>   *in books 1-5 are matching and graphing, numeration, operations,*
>   *geometry, measurement, and estimation and in books 6-10 are*
>   *graphing and probability, numeration, operations and problem*
>   *solving, geometry, and measurement.*

MATHIMAGINATION, Books A-F. Marcy, Steve and Marcy, Janis. Palo Alto, California:
Creative Publications, Inc., 1973. (Creative Publications, Inc., Box 10328,
Palo Alto, CA 94303)

48 pp in each; paper; workbooks; b/w

>   *Each book provides drill in the form of a puzzle on basic concepts*
>   *and skills of a mathematics topic; Book A: beginning multiplication*
>   *and division; Book B: operations with whole numbers; Book C: number*
>   *theory; Book D: fractions; Book E: decimals and percents; Book F:*
>   *geometry, measurement and cartesian coordinates.*

MATHLAB--JUNIOR HIGH. McFadden, Scott; Anderson, Keith; Schaaf, Oscar. Eugene,
Oregon: Action Math Associates, Inc., 1975. (Action Math Associates, Inc.,
1358 Dalton Dr., Eugene, OR 97404)

90 cards; paper; b/w; medium reading level

>   *This is a series of activity cards in book form and divided*
>   *into eight different areas of interest, some of which are: cal-*
>   *culator, perimeter and area, volume, science, and applications.*

MINNEMAST, UNIT 18, SCALING AND REPRESENTATION. Minnesota Mathematics and
    Science Teaching Project. Minneapolis: Minnesota Mathematics and Science
    Teaching Project, 1971. (Minnesota Mathematics and Science Teaching Project,
    720 Washington Ave., Minneapolis, MN 55414)

    80 pp; paper; b/w; teacher's guide

        *This title is one of 29 units for grades K-3. The unit covers
    activities and observations children can do to investigate the idea
    of scale representations.*

MIRA MATH FOR ELEMENTARY SCHOOL. (A Mira Math Co. Publication) Palo Alto,
    California: Creative Publications, Inc., 1973. (Creative Publications, Inc.,
    Box 10328, Palo Alto, CA 94303)

    88 pp; paper; b/w; teacher's guide

        *Using a reflective piece of plastic called a "Mira," students
    are introduced to properties of reflections, congruence, symmetry
    and construction techniques. The book is designed for grades 4-6
    and presents the activities in a very intuitive manner.*

MIRROR MAGIC. (The Franklin Mathematics Series) Abbott, Janet S. Chicago:
    Franklin Publications, Inc., 1970. (Lyons and Carnahan, Division of Rand
    McNally and Company, P.O. Box 7600, Chicago, IL 60680)

    120 pp; paper; textbook; color; low reading level

        *It includes many activities using mirrors to demonstrate symmetry.
    All activities are for students.*

MOVE IN ON MATHS 4. Whittaker, Dora. London: Longman Group, Limited, 1974.
    (Longman Group, Limited, 5 Bentinck St., London, England W1M 5RN)

    63 pp; paper; color; textbook; medium reading level

        *This book is a student text which motivates each mathematics
    concept with real-world examples or applications.*

NETWORKS. (Mathematics in the Making 8) Bell, Stuart E. Boston: Houghton Mifflin
    Company, 1969. (Houghton Mifflin Company, Educational Division, 1 Beacon St.,
    Boston, MA 02107)

    32 pp; paper; student reference; color; medium reading level

        *The phamplet introduces students to elementary topics in topology:
    simple closed curves, inside and outside of a region, map coloring and
    a Moebius strip.*

NEW OXFORD JUNIOR MATHEMATICS, Books 4 and 5. Williams, E.M. and James, E.J. Ely
   House, London: Oxford University Press, 1971. (Oxford University Press, 200
   Madison Ave., New York, NY 10016)

   150 pp; paper; color; textbook; medium reading level

      These student textbooks, which use a discovery approach to a
   variety of mathematics concepts, provide real-world applications,
   and include some drill and practice pages to check students basic
   computation.

"Notes From National Assessment: Basic Concepts of Area and Volume." Carpenter,
   Thomas P.; Coburn, Terrence G.; Reys, Robert E.; and Wilson, James W. *THE*
   ARITHMETIC TEACHER, Vol. 22, No. 6 (October, 1975) pp. 501-507. Reston,
   Virginia: National Council of Teachers of Mathematics. (National Council
   of Teachers of Mathematics, 1906 Association Dr., Reston, VA 22091)

"Notes From National Assessment: Perimeter and Area." Carpenter, Thomas P.;
   Coburn, Terrence G.; Reys, Robert E.; and Wilson, James W. *THE ARITHMETIC*
   TEACHER, Vol. 22, No. 7 (November, 1975) pp. 586-590. Reston, Virginia:
   National Council of Teachers of Mathematics. (National Council of Teachers
   of Mathematics, 1906 Association Dr., Reston, VA 22091)

NOTES ON MATHEMATICS IN PRIMARY SCHOOLS. Members of the Association of Teachers
   of Mathematics. London: Syndics of the Cambridge University Press, 1969.
   (Cambridge University Press, 32 East 57 St., New York, NY 10022)

   340 pp; paper; teacher reference

      A collection of ideas about the newer methods of mathematics
   teaching in primary school, this book includes reports from
   actual lessons and illustrations of children's work. It also
   encourages the reader to do the mathematics as she reads about it.

OPEN-ENDED TASK CARDS. Risk, J. San Francisco, California: J. Risk, 1971.
   (Teacher's Exchange of San Francisco, 600 35th Ave., San Francisco, CA 94121)

   21 cards; color; activity cards; low reading level

      This is a series of 18 teacher-written activities using supplies
   available in the classroom to supplement regular mathematics lessons.

PAPER AND PENCIL GEOMETRY. (The Franklin Mathematics Series) Roper, Susan.
   Chicago: Franklin Publications, Inc., 1970. (Lyons and Carnahan, Division
   of Rand McNally and Company, P.O. Box 7600, Chicago, IL 60680)

   104 pp; paper; color; workbook; medium reading level

      This is a workbook providing paper-pencil and laboratory
   activities on the basic concepts of geometry--points, lines,
   planes, polygons, angles, circles and curves, perimeter,
   area, and solids.

PAPER FOLDING FOR THE MATHEMATICS CLASS. Johnson, Donovan A. Washington, D.C.:
National Council of Teachers of Mathematics, 1957. (National Council of
Teachers of Mathematics, 1906 Association Dr., Reston, VA 22091)

32 pp; paper; teacher reference

*This book is a collection of exercises that use
paper folding to illustrate and help students discover relation-
ships of lines and angles. This technique can add realism and
interest to your mathematics teaching.*

PAPER PLUS. Ranucci, Ernest R., ed. Portland, Maine: J. Weston Wasch, Publisher,
Inc., 1973. (J. Weston Walch, Publisher, Inc., Box 658, Portland, ME 04104)
(Copyright now held by Mrs. Ernest R. Ranucci)

71 pp; paper, b/w; puzzle book

*This book, now out-of-print, is a collection of puzzles, some
involving paper-folding, some involving dissection, most of them
unusual.*

PATTERNS AND PUZZLES IN MATHEMATICS. (The Franklin Mathematics Series) Horne,
Sylvia. Chicago: Franklin Publications, Inc., 1970. (Lyons and Carnahan,
Division of Rand McNally and Company, P.O. Box 7600, Chicago, IL 60680)

96 pp; cloth; textbook; color; medium reading level

*A collection of puzzles and games--many of them geometric in
nature, and many of them providing hands-on activities--are given
in this book.*

PATTERNS IN SPACE. Beard, Colonel Robert S. Palo Alto, California: Creative
Publications, Inc., 1973. (Creative Publications, Inc., Box 10328, Palo Alto,
CA 94303)

237 pp; paper; teacher reference; b/w

*Hundreds of geometric designs and constructions complete with
instructions on how to make them are shown.*

PERCEPTION GAMES. Hoolim, Haar. Newton, Massachusetts: Selective Educational
Equipment (SEE), Inc., 1968. (Selective Educational Equipment (SEE), Inc.,
3 Bridge St., Newton, MA 02195)

game; medium reading level

*The set includes a deck of cards and a series of 15 games that
require a player to visualize the cards in varying combinations
in order to form patterns. The set is available from Selective
Educational Equipment (SEE), Inc.*

PHOLDIT. Goldberg, Steven A. San Jose: Billiken Publications, Inc., 1972. (Activity Resources Company, Inc., P.O. Box 4875, Hayward, CA 94540)

36 pp; paper; teacher reference; b/w

A collection of paper folding exercises is given here, and large, easy-to-follow graphics allow students to use the book.

PIC-A-PUZZLE. Schadler, Reuben A. and Seymour, Dale G. Palo Alto, California: Creative Publications, Inc., 1970. (Creative Publications, Inc., Box 10328, Palo Alto, CA 94303)

POINTS, LINES AND PLANES. Luce, Marnie. Minneapolis: Lerner Publications Company, 1969. (Lerner Publications Company, 241 First Ave., N., Minneapolis, MN 55401)

PROBLEM SOLVING IN GEOMETRY. Vol. IV of Soviet Studies in the Psychology of Learning and Teaching Mathematics. Kilpatrick, Jeremy and Wirszup, Izaak, ed. Chicago: University of Chicago, 1970. (University of Chicago Press, 5801 Ellis Ave., Chicago, IL 60637)

154 pp; paper; b/w; teacher reference

This volume contains a collection of translations of Soviet literature dealing with the psychology of mathematical instruction in geometry.

PROBLEMS--RED SET, --GREEN SET, --PURPLE SET. Nuffield Mathematics Project. London: Newgate Press Limited, 1971. (John Wiley and Sons, Inc., 605 Third Ave., New York, NY 10016)

approx 100 pp each book; paper; b/w; teacher's guide

The book contains a collection of problems centered around three areas: Computation and Structure, Shape and Size, and Graphs Leading to Algebra.

PYTHAGOREAN PROPOSITION. Loomis, E.S. Washington, D.C.: National Council of Teachers of Mathematics, 1940. (National Council of Teachers of Mathematics, 1906 Association Dr., Reston, VA 22091)

REFLECTIONS AND ROTATIONS. Oakland County Mathematics Project. Pontiac, Michigan: Oakland Schools, 1970. (Oakland Schools, 2100 Pontiac Lake Rd., Pontiac, MI 48054)

58 pp; paper; b/w; teacher's guide

One of a series written by the Oakland County Mathematics Project, the book has many student activities to develop the concept of reflection and rotation.

ROLLING. (Topics From Mathematics) Mold, Josephine. London: Cambridge University Press, 1973. (Cambridge University Press, 32 East 57 St., New York, NY 10022)

32 pp; paper; b/w; student reference

> *This book illustrates what happens when difference shapes are rolled. The pieces in the spirograph game are studied.*

SCHOOL MATHEMATICS I AND II. Eicholz, Robert E.; O'Daffer, Phares G.; Brumfiel, Charles F.; Shanks, Merrill E.; and Fleenor, Charles R. Menlo Park, California: Addison-Wesley Publishing Company, Inc., 1971. (Addison-Wesley Publishing Company, Inc., 2725 Sand Hill Rd., Menlo Park, CA 94025)

cloth; textbook series; b/w; medium reading level

> *SCHOOL MATHEMATICS I and SCHOOL MATHEMATICS II are the 7th and 8th grade texts that complement the ELEMENTARY SCHOOL MATHEMATICS series from the same publishers.*

THE SCHOOL MATHEMATICS PROJECT, Books A-H with Teacher's Guides [Metric]. Great Britain: Cambridge at the University Press, 1971. (Cambridge University Press, 32 East 57 St., New York, NY 10022)

approx. 300 pp/book; paper; teacher's guide

> *The series is intended for students from grades 7-10. The books present a spiral, activity-oriented approach that interweaves arithmetic, algebra and geometry with an attempt to emphasize the practical application of each topic.*

THE SCHOOL MATHEMATICS PROJECT, Books 1-5 [Metric]. Great Britain: Cambridge at the University Press, 1970. (Cambridge University Press, 32 East 57 St., New York, NY 10022)

295 pp; cloth; textbook; high reading level

> *Chapter 10 of this book is an excellent chapter on filling space with polyhedra.*

THE SCIENTIFIC AMERICAN BOOK OF MATHEMATICAL PUZZLES AND DIVERSIONS. Gardner, Martin. New York: Simon and Schuster, 1959. (Simon and Schuster, Inc., 630 Fifth Ave., New York, NY 10020)

178 pp; paper; teacher reference

> *This book is an excellent collection of games, puzzles, paradoxes and brain teasers accompanied by lively commentaries which are mathematically challenging and entertaining to read.*

THE SECOND SCIENTIFIC AMERICAN BOOK OF MATHEMATICAL PUZZLES AND DIVERSIONS. Gardner, Martin. New York: Simon and Schuster, 1961. (Simon and Schuster Inc., 630 Fifth Ave., New York, NY 10020)

253 pp; cloth; teacher reference; b/w

> *This book is another of Gardner's excellent collections of games, puzzles and interesting ideas.*

SEEING SHAPES. Ranucci, Ernest R., ed. Palo Alto, California: Creative Publica-
    tions, Inc., 1973. (Creative Publications, Inc., Box 10328, Palo Alto, CA
    94303)

    94 pp; paper; teacher reference

        *An excellent collection of materials designed for use from
    grades 7 to the junior college level, this book provides exer-
    cises to strengthen visual perception and the understanding of
    spatial relationships. Emphasis is on discovery and an attempt
    is made to provoke imaginative thinking.*

"SELECTED INVESTIGATIONS FOR MATH LAB." Kidd, Kenneth. Chicago: Science Research
    Associates, Inc., n.d. (Mimeographed) (Science Research Associates, Inc.,
    259 East Erie St., Chicago, IL 60611)

    10 pp; teacher reference

        *This is a selection of laboratory activities presented at a
    National Council of Teachers of Mathematics' annual meeting held in
    Atlantic City in April 1974. The emphasis was the use of ratios and
    proportions to solve problems.*

SHAPES, SPACE, AND SYMMETRY. Holden, Alan. New York: Columbia University Press,
    1971. (Columbia University Press, 562 W. 113th St., New York, NY 10025)

    200 pp; paper; teacher reference; b/w

        *Photographs of 3-D models made from many different materials and
    information about these models comprise the book. The photographs
    are extremely well done.*

SOLID MODELS. (Topics from Mathematics) Mold, Josephine. London: The Syndics of
    the Cambridge University Press, 1967. (Cambridge University Press, 32 East
    57 St., New York, NY 10022)

    32 pp; paper; color; student reference

        *Nets for making solid models are shown in this book, along
    with illustrations of the finished model. The principle of duality
    and several of the semi-regular solids are also presented.*

SPHERELAND. Burger, Dionys. New York: Thomas Y. Crowell Company, 1965. Translated
    by Cornelia J. Reinboldt. (Thomas Y. Crowell Company, 666 Fifth Ave., New York,
    NY 10003)

SPIRALS. Sitomer, Harry and Sitomer, Mindel. New York: Thomas Y. Crowell Company,
    1974. (Harper and Row, Publishers, Inc., 10 E. 53rd St., New York, NY 10022)

    33 pp; cloth; color; student reference; low reading level

        *This book gives illustrations of where spirals occur in
    nature, in weather, in tools, etc.,and also provides several
    methods for students to make spirals.*

SRA MATH APPLICATIONS KIT. Friebel, Allen C. and Gingrich, Carolyn Kay. Chicago:
   Science Research Associates, Inc., 1971. (Science Research Associates, Inc.,
   259 East Erie St., Chicago, IL 60611)

270 cards; color; activity cards; medium reading level

   This is an excellent collection of 270 activity cards that pre-
sents a large number of problems and activities in science, sports
and games, occupations, social studies, and everyday things that
students, using elementary school mathematics as the basic tool of
investigation, can explore.

STRAW POLYHEDRA. Lacock, Mary. Palo Alto, California: Creative Publications, Inc.,
   1970. (Creative Publications, Inc., Box 10328, Palo Alto, CA 94303)

STUDENT ACTIVITY CARDS FOR CUISENAIRE RODS. Davidson, Patricia: Fair, Arlene;
   and Galton, Grace. New Rochelle, New York: Cuisenaire Company of America,
   Inc., 1971. (Cuisenaire Company of America, Inc., 12 Church St., New Rochelle,
   NY 10805)

130 cards; b/w

   This is a kit consisting of 130 cards in ten sets with an
accompanying teacher's manual to be used with Cuisenaire rods.
It provides a wide variety of games, activities and problems
which develop mathematical concepts appropriate in Grades K-6.

SYNCHRO-MATH/EXPERIENCES. Schaaf, Oscar; McFadden, Scott; and Kramer, Klaas.
   Chicago: Lyons and Carnahan, 1972. (Copyright now with Action Math
   Associates, 1358 Dalton Dr., Eugene, OR 97404)

469 pp; cloth; b/w; textbook; medium reading level

   This is a textbook that emphasizes a discovery technique
of learning.

TARGET: MEETING MATHEMATICS. Kane, Robert B.; Hater, Mary Ann; Holtan, Boyd D.;
   Rudolph, William B.; and Smith, Stephen T. New York: American Book Company,
   1973. (American Book Company, Division of Litton Educational Publishers,
   450 West 33rd St., New York, NY 10001)

590 pp; cloth; textbook series; b/w; medium reading level

   The book is a standard textbook for grade eight.

TEACHING MATHEMATICS: A SOURCE BOOK OF AIDS, ACTIVITIES AND STRATEGIES. Sobel,
   Max A. and Maletsky, Evan M. Englewood Cliffs, New Jersey: Prentice-Hall,
   Inc., 1975. (Prentice-Hall, Inc. Englewood Cliffs, NJ 07632)

240 pp; paper; b/w; teacher reference

   As the title states, this book contains many aids, activities
and strategies that can be used in teaching mathematics.

THE THEMES OF GEOMETRY. (Mimeographed.) MacPherson, Eric D. For further infor-
    mation write: Eric D. MacPherson, Dean, Faculty of Education, University of
    Manitoba, Winnipeg, Manitoba, Canada R3T 2N2.

        *This is an unpublished collection of activities in which students
    discover various geometric properties and relationships.*

TRADEMARKS: A HANDBOOK OF INTERNATIONAL DESIGN. Wildbur, Peter. New York: Van
    Nostrand Reinhold Company, 1966. (Van Nostrand Reinhold Company, Division
    of Litton Educational Publishing, Inc., 450 W. 33rd St., New York, NY 10001)

TRADEMARKS AND SYMBOLS, Volume 2. Kuwayama, Yasaburo. New York: Van Nostrand
    Reinhold Company, 1973. (Van Nostrand Reinhold Company, Division of Litton
    Educational Publishing, Inc., 450 W. 33rd St., New York, NY 10001)

THE UNEXPECTED HANGING AND OTHER MATHEMATICAL DIVERSIONS. Gardner, Martin. New
    York: Simon and Schuster, 1969. (Simon and Schuster, Inc., 630 Fifth Ave.,
    New York, NY 10020)

    255 pp; cloth; teacher reference; b/w

        *This book contains a nice collection of puzzles, paradoxes and brain
    teasers--all mathematically challenging. The answers are included.*

"Using Mathematical Structures to Generate Artistic Designs." Forseth, Sonia and
    Troutman, Andria Price. *THE MATHEMATICS TEACHER,* Vol. 67, No. 5 (May, 1974),
    pp. 393-398. Reston, Virginia: National Council of Teachers of Mathematics.
    (National Council of Teachers of Mathematics, 1906 Association Dr., Reston,
    VA 22091)

VECTOR GAME DIRECTIONS. Brooklyn, New York: Plan B Corporation, 1971. (Plan B
    Corporation, 3140 Coney Island Ave., Brooklyn, NY 11235)

    game; medium reading level

        *This is a game where players manuever a playing piece about a
    game board using direction cards. Points are scored or subtracted
    according to instructions on the game board.*

VISUAL ILLUSIONS COLORING BOOK. Horemis, Spyros. New York: Dover Publications,
    Inc., 1973. (Dover Publications, Inc., 180 Varick St., New York, NY 10014)

VISUAL MOTOR SKILLS--LEVEL TWO. Maney, Ethel S. Elizabethtown, Pennsylvania:
    The Continental Press, Inc., 1958. (The Continental Press, Inc., Elizabethtown,
    PA 17022)

    24 pp; paper; workbook; b/w

        *This book is one of a series of reading readiness workbooks and
    concentrates on using visual perception to copy and complete designs.*

WEEPLE PEOPLE. Gillespie, Craig D. Scarsborough, Ontario: McGraw-Hill Ryerson
    Limited, 1971. (McGraw-Hill Ryerson Limited, 330 Progress Ave., Scarsborough,
    Ontario M1P 2Z5)

WONDER-FULL WORLD OF NUMBERS. Bezuszka, Father Stanley and Kenney, Margaret J.
    Boston: Boston College Mathematics Institute, 1971. (Boston College Mathe-
    matics Institute, Boston College, Chestnut Hill, MA 02167)

    96 pp; paper; teacher reference; b/w

        *This is a collection of activities designed to give students
    proficiency in the basic operations of arithmetic. An interesting
    puzzle and recreational format is used for the exercises.*

YES, MATH CAN BE FUN! Brandes, Louis Grant. Portland, Maine: J. Weston Walch,
    Publisher, Inc., 1960. (J. Weston Walch, Publisher, Inc., Box 658, Portland,
    ME 04104)

    263 pp; paper; workbook; b/w; medium reading level

        *The book is a collection of recreational activities in the form of
    puzzles, games, number oddities, illusions, problems and projects to
    stimulate students.*

GEOMETRY AND VISUALIZATION

Placement Guide for Tabbed Dividers

SELECTED ANSWERS

LINES, PLANES AND ANGLES

LINES

228 (A) Some possibilities: (1) rulers and yardsticks (2) pens and pencil
(3) table legs (4) flourescent light bulbs (5) edges of the room where
the walls intersect (B1) $\overline{HG}$ (B2) $\overline{FE}$ (B3) $\overline{HE}$ (B4) $\overline{HF}$ also $\overline{GF}$
(C) 1; 3; 5; 10; 15 (D) 21 (E) 45

231 (8) $n-1^2$ (if geoboard has n nails per row) (11) 4; 12; 6; 4

233 (3) no (6) 0; 1; 6; 10; 45 (7) GA; IC; GI; DF; AC; DC; HA; IB; FA;
DI; GB; HC (8) one possible answer set: AD; AG; AE; AH; AI (9) 2;
5; 9

235 (1) neither, they have the same length (3) straight (4) straight
(6) neither, they have the same length (7) 2nd from the left
(8) 3rd from top

238
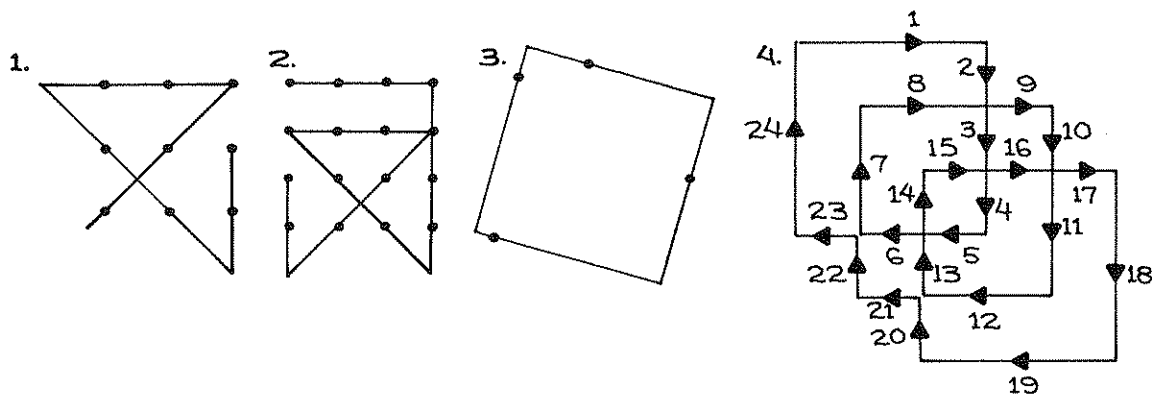
240 (A1) yes; no; no (C 1-4) answers will be from among: $\overrightarrow{RV}$; $\overrightarrow{RS}$; $\overrightarrow{RT}$;
$\overrightarrow{RW}$; $\overrightarrow{ST}$; $\overrightarrow{SW}$; $\overrightarrow{TW}$; $\overrightarrow{TU}$ (e) 42 (F) They are twice as large, since the
rays can begin at either of the two endpoints.

241 (1) infinitely many (2) 1 (3) $\overleftrightarrow{VR}$; $\overleftrightarrow{SV}$; $\overleftrightarrow{RS}$; $\overleftrightarrow{SR}$ (4) 6 (5) 6;
10; 15; 21 (7) 28, 36 (8) 10

242 (3) 3 (4) 6 (6) 15; 21 (3) 3 or 4; yes; The fold (lines) must
intersect in the interior of the paper (4) 4 (5) 7

<u>Page Numbers</u>

PLANES

258   (7,1)

259   (1) (5,8); (11, 2);   (2) (2,1); (8,1); possibilities: (3,1); (4,1)
(5,1); (6,1); (7,1); 1   (3) (10,6); (10,10); 10   (4) 4; 5
(5) (5,1); (10,8); (8,5)   (6) (10,2)   (7) (4,4)   (8) no; (1,3);
(2,6); (3,9); 3 times as much; no

ANGLES

272   (A) possible answers: intersecting wall edges; compasses; the letters
V, L; partially opened books; baseball diamond   (b) possible answers:
∠BAC, ∠BAD, ∠BAG, ∠CAD, ∠CAE, ∠DAE   (c) 21; 45

273   (1) 1→1; 2→4; 3→2; 4→3   (2) 1→3; 2→4; 3→1; 4→2

274   (1) d; b; a; c; e   (2a) d   (2b) a   (2c) c   (2d) d; e   (4)a;
d; b; c

276   (C1) 1; 3; 5; 7; 8   (C2) RPU

280   (1) A; C   (2) X; Y   (3) W; Z   (4) 1; n; none; 1, m or n

281   (1) d; j   (2) e; g   (3) f   (4) c; h; i   (5) b   (6) a   (7) c

283   (1) yes   (3) they are the same size   (4a) 3 & 6; 4 & 5; 7 & 10;
8 & 9;   (4B) 2 & 3; 5 & 8; 6 & 7   (4C) 5 & 8; 6 & 7   (4D) 1 & 3;
2 & 4

292   (2) 180°   (3) 270"   (5a) 45 °   (5b) 180°   (5c) 90°   (5d) 225°
(5e) 135°   (5f) 270°   (5g) 315°

SYMMETRY AND MOTION

308   (1) yes   (2) no   (3) no   (4) yes

309   (5) no   (6) yes   (7) no   (8) no   (9) yes   (10) yes   (11) no
(12) yes   (13) yes   (14) no

314   (1) C   (2) C   (3) C   (4) B   (5) A   (6) B   (7) C   (8) A

POLYGONS & POLYHEDRA

CURVES & CURVED SURFACES

CURVED SURFACES

479 (1a) sphere (1b) cone; frustrum of a cone; cylinder (1c) rectangular prism (2b) a point (2c) any direction (2e) it can move in any direction (2f) they can move in any direction (3b) a line segment (3c) just two directions (3d) a straight line (3e) they move in a straight line (4b) a line segment (4c) a circle (5b) a line segment (5c) a circle

480 (2) the maximum number of pieces is 8 (3) the maximum number of pieces is 15 (4) only circles

485 (2) B (3) A (5a) 4 (5b) 1 (5c) 2 (5d) 5 (5e) 3 (5f) 6

CIRCLES

505 (2) 7; 28

507 (2) 7 (3) 91

514 (1) (2) (3)

MYSTERY POLYGONS
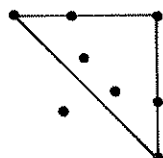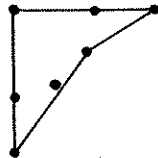
518 B; C; D; E; F; H; C; F; H

519 (4) hexagon (4a) sides are congruent (4b) angles are congruent (5a) triangle (5b) sides and angles are congruent (7) yes (8) square, octagon, polygons with 48, 96,...sides

520 (5) square (5a) sides are congruent (5b) angles are congruent (5c) diagonals (7) yes (8) polygons with 32, 64,...sides

521 (9) pentagon (10) 5-pointed star (12) polygons with 20, 40, 80,... sides

526 (1a) 3 (1b) 4 (1c) 2 (1d) 6 (2a) 4 (2b) 6 (2c) 3 (2d) 8 (3a) 10 (3b) 2 (3c) 5 (3d) 4 (4a) 2 cm (4b) 6 cm (4c) 3 cm (4d) 4 cm (5a) 5 cm (5b) 2 cm (5c) 2.5 cm (5d) 4 cm

527 (1a) $\angle$ XCY; $\angle$ ACB (1b) $\overset{\frown}{YX}$; $\overset{\frown}{AB}$ (1c) $\overset{\frown}{ABZ}$ (1d) $\overset{\frown}{AZY}$, $\overset{\frown}{ABY}$ (1e) $\angle$ ACB (1f) 60°; 30°; 30° (1g) add the measures of $\overset{\frown}{BX}$ and $\overset{\frown}{XY}$

527   together to get the measure of $\overset{\frown}{BY}$ (2a) 50   (2b) 130   (2d) 180
(2e) 310   (3a) 40   (3b) 60   (3c) 100   (3d) 260   (3e) 320
(3f) 300   (4a) 120   (4b) 90   (4c) 20

531   (I4) C   (II4) any 3 colinear points

OTHER CURVES

549   (I7) parabola   (II7) ellipse   (III5) hyperbola

550   (I) ellipse   (II) hyperbola

551   (3a) for each cyclist, the bridge and the shelter are equally close

SIMILAR FIGURES

609   (I7) twice as long; twice as long; twice as long

614   (IA) one is twice as long as the other   (IB) one is twice as long as
the other   (IC) one is twice as long as the other

623   (1a) yes   (1b) yes   (1c) yes   (3) yes   (4b) no   (5) 1; 1
(6) 10.5 m   (7) 32.4 m

624   (8a) 32.9   (8b) 22.4   (8c) 44.1   (9a) 245.7   (9b) 315.9
(9c) 213.3   (10) 50.4 m   (11) 9.1 m   (12) 11 m   (13b) .36
(14) 47 m   (15) 22 m

625   (2) yes   (3) yes   (5) yes; yes   (6) yes

637   (II) equal   (III) .618:1

AREA & VOLUME

PERIMETER

660   (1a) 4   (1b) 16   (2a) 6   (2b) 14   (3a) 16   (3b) 16   (3c) 16
(4a) 10   (4b) no change   (4c) no change

664   (I) 32; 40; 36   (IIa) 20   (IIb) 52   (IIc) 106   (IId)90   (IIIa) 42

<u>Page Numbers</u>

664   (IIIb) 108   (IIIc) 48   (IIId) 86   Challenge: they'll arrive at the same time. The routes are of equal length.

667   (1) 4 m, 8 m   (2) 17   (3) 152 cm   (4) same   (5) 120 ft.
(6a) 4   (6b) 13   (6c) 5   (7a) 8 mm   (7b) 10 mm   (7c) 12 mm
(7d) 16 mm   (8) 6 dm

668   (5) use a string to help measure   (9) string

670   (I) twice   (II) equal   (III) yes   (IV) about 10.7 m

AREA

682   (1) 32   (2) yes   (3) 16   (4) purple   (6) 5 x 7; 4 x 7; 3 x 9;
35; 32; 27   (7) white, red, light green, purple, yellow, dark green, orange

684   (1a) 4   (1b) 4   (1c) 8   (1d) 10   (1e) 7   (1f) 7   (1g) 6
(1h) 11

685   (1a) 3 1/2   (1b) 3   (1c) 6   (1d) 12   (1e) 10   (1f) 8
(1g) 14   (1h) 6   (1i) 3 1/2   (1j) 7   (1k) 8   (1ℓ) 10

686   (1) 2; 2; 2; 4; 6   (2) 1; 2; 1/2; 3; 1 1/2   (3) 1/2; 1/4; 1 1/2;
1 1/2; 1 1/2

689   (1) each polygon has area 30 sq. units and perimeter 26 units

690   (2) column IV in #2 gives the better approximation

696   (1a) 6   (1b) 1/2   (1c) 3   (2a) 4   (2b) 1/2   (2c) 2
(3a) 1 1/2   (3b) 2   (3c) 6   (3d) 4   (3e) 8   (3f) 1/2   (3g) 1
(3h) 4 1/2   (4a) 2   (4b) 3   (4c) 4   (4d) 2

702   (2e) 1 square unit   (3e) 1 1/2 square unit

705   (1) no; Each arrangement leaves an equal area uncovered.   (2c) 285
(2d) the area is unchanged   (2e) 211   (3c) yes   (3d) 32 cm^2
(3f) no

706   (2a) it is twice as long   (2b) it is four times as large   (2c) when side lengths are doubled, area is quadrupled

707   (1) 18; 22   (2) 22 cm^2 (4) 52 cm^2 (5) 22 cm^2; 52 cm^2; 90 cm^2
136 cm^2; 190 cm^2 (6) $3.12   (7) $6.08   (8) 1600; $32.00

709   (I) 18 m x 18 m; 324 m^2 (II) 36 m x 18 m; 648 m^2

710   (II) a circle; about 413 m^2 (III) a semi-circle; about 825 m^2

Page Numbers

712   (A) 12   (B) 6   (C) 36π

714   (1) C   (2) He covered half of it, but it is still there.   (3) 1/2 cm^2
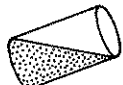   (4) 4 x 4; 3 x 6   (5) they are equal   (8) D

SURFACE AREA

742   (1) 1; 3; 6; 10

743   (2) total number of cubes = 4 (number of cubes per side -1); area =
         4 x total number of cubes   (3) area = 4 x number of cubes   (4) area =
         3 x number of cubes

744   (I) 12   (II) 17; 22; 27; stamps = volume of smaller cube x 5 + 7
         (III) 52   (IV) 28

746   (1) 16   (2) 18   (3) 34   (4) 39   (5) 34   (6) 46   (7) 26

747   (a) 4; 0; 0; 16   (2) 1; 9; 12; 2; 30; 8; 4; 16; 16; 24; 8; 48; 4; 12;
         9; 25; 16; 36; 18; 70; 4; 16; 16; 36; 16; 48; 32; 96; 4; 20; 25; 49; 16;
         60; 50; 126; 4; 24; 36; 64; 16; 72; 72; 160; 4; 28; 49; 81; 16; 84; 98;
         198; 4; 32; 64; 100; 16; 96; 128; 240;   (3) 8; 24; 48; 80; 120; 168;
         224; 288; 360

748   (A) 0; 2; 98; 0; 0; 0; 0;   (B) 0; 0; 0; 8; 9; 2; 0; 0;   (C) 0; 0;
         4; 32; 64; 0; 0;   (D) 0; 0; 0; 8; 44; 48; 0;   (E) 0; 0; 0; 8; 32;
         42; 18   (E has the smalles number of cube faces painted)

749   (1) 2 x 2 x 3   (2) 2 x 3 x 4   (3) 3 x 3 x 4   (4) area = 2 (length x
         width x length x heighth x width x heighth)   length of edges = 4 x
         (length + width + heighth)

750   (A) 6   (1b) 6; 6   (2a) 4 cm^2 (2b) 4 cm^2; 24 cm^2 (4a) 400:1
         (4b) 144:1   (4c) 2.25:1

VOLUME

764   (1) 4   (2) 14   (3) 60   (4) 19   (5) 28   (6) 30   (7) 27

765   (A1) 3   (A2) 4   (A3) 4   (A4) 4   (A5) 4   (A6) 4   (A7) 4; 27
         (B1) 8   (B2) 12   (B3) 19   (B4) 15   (B5) 27   (B6) 27

766   (2) 44; 36

767   (3A) 44 km   (3B) 16 km   (3C) 36 km   (3D) 28 km   (3E) 20 km
         (3F) 16 km   (3G) 12 km   (3H) 28 km   (3I) 24 km   (3J) 8 km
         (4) 9; 19 km   (1A) 72 km^2 (1B) 16 km^2 (1C) 72 km^2 (1D) 48 km^2
         (1E) 24 km^2 (1F) 12 km^2 (1G) 8 km^2 (1H) 36 km^2 (1I) 36 km^2
         (1J) 4 km^2 (a) yes; H and I, A and C; no; They may have different
         shapes and therefore different perimeters   (3) 2400   (4) 24 km^2

768   (5) A or C   (6) about .36   (7) B, E, F, G, H, I J   (1) 48
         (2) 3   (3) 8   (4) A, B, C, D, E, H, I

769   (I) 0; 8; 0; 0; 0;   (II) 0; 8; 12; 6; 1   (III) 64   (IIIA) 0
     (IIIB) 8   (IIIC) 24   (IIID) 24   (IIIE) 8   (IIIF) 64; equal
     (IV2) 0; 0; 0; 8; 0; 8   (IV3) 1; 6; 12; 8; 0; 27   (IV4) 8; 24; 24;
     8; 0; 64   (IV5) 27; 54; 36; 8; 0; 125   (IV6) 64; 96; 48; 8; 0; 216
     (IV8) 216; 216; 72; 8; 0; 512   (IV10) 512; 384; 96; 8; 0; 1000
     (IV20) 5832; 1944; 216; 8; 0; 8000; patterns: $(n-2)^3$; $6(n-2)^2$; $12(n-2)$
     8; 0; n^3

770   (1a) 24; 2 x 3 x 4   (1b) 30; 1 x 5 x 6   (1c) 70; 2 x 5 x 7
     (1d) 72; 3 x 3 x 8   (2) 32; 30; 35; 48   (4a) 4 x 4 x 4
     (4b) 2 x 4 x 8   (4c) 2 x 3 x 6   (4d) 3 x 4 x 4 or 2 x 2 x 12

771   (A1) 18   (A2) 4   (A3) 72   (B1) 20   (B2) 3   (B3) 60 cubic
     units   (C1) 14   (C2) 6   (C3) 84 cubic units   (D1) 30   (D2) 5
     (D3) 150 cubic units   (E1) 28   (E2) 8   (E3) 224 cubic units
     Volume = area of base x height

774   (1a) 2; 3   (1b) 12   (1c) 6; 3   (1d) 15   (2a) 4; 3 1/2   )2b) 24
     (2c) 4; 1   (2d) 20; 10   (2e) 35   (3a) 2 1/2; 4 1/2   (3b) 24
     (3c) 1, 1/8   (3d) 9; 2 1/4   (3e) 26; 13   (3f) 39 3/8   (4a) 15
     (4b) 35   (4c) 39 3/8

778   (1) 144   (2) 72   (3) 80   (4) 144   (5) 318   (6) 544
     (7) 1732.5

789   (1) 8   (2) B   (3) 24 cm   (4) pour until you reach .

     (5) Fill the 3-litre bag and pour it into the 5-litre bag. Again fill
the 3-litre bag and pour water from it into the 5-litre bag until the
5-litre bag is full. Dump the 5-litre bag, and pour the 1 litre in
the 3-litre bag into the 5-litre bag. Fill the 3-litre bag and pour
it into the 5-litre bag. The 5-litre bag now contains 4 litres of
water.   (6) no   (7) they will carry equal amounts   (8) about
157,000,000 litres

792   (1b) 6 cm^3 (3) 294 cm^3

793   (1) 140   (2) it is twice as large   (3) twice as much   (4) 2
     (6) 2; 4   (7) 4; 8

794   (1) 4; it has the least surface area per unit of volume   (3) yes;
     6000 cm   (4) about 1000 kg