Comprehensive School Mathematics Program

CSMP IN ACTION

CEMREL, Inc.
Comprehensive School Mathematics Program
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The Comprehensive School Mathematics Program materials included herein are in the process of development. As a part of our continuing effort to evaluate and improve them, we ask that you comment in detail on the materials and on the way in which you used them.

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Comprehensive School Mathematics Program

CSMP IN ACTION

Foreword

Now used by more than 30,000 students, nationwide, CSMP's elementary school mathematics curriculum is an exciting, innovative program which proves---to students, teachers and parents, alike---that mathematics is fun! More importantly, it represents the best of basic mathematics education (in the truest sense of the word), taking a balanced approach to skill development, conceptualization and problem solving. Through its several distinctive features---"non-verbal" languages, "spiral" organization, and "pedagogy of situations"---the program succeeds in conveying the most important mathematical content through sound, well tested pedagogy.

Clearly, the CSMP program is also a somewhat radical departure from more traditional curricula. Perhaps for this reason, CSMP has found it difficult to even begin discussing the program, except through an in-depth, "hands on" exposure to the curriculum as it is actually used in the classroom . . . usually in the form of a two- or three-day awareness workshop.

While this in-depth exposure has proved to be an extremely effective approach (according to those who have participated), it is clearly not a very practical one, financially. It is hoped, however, that this booklet will be both effective and practical in providing substantive information about CSMP, nationally, to those who request it.

Even without seeing the performance of a play, one can learn and experience quite a bit through a careful reading of the script. In the same way, this booklet attempts to capture, on paper, the excitement and spontaniety of the workshop experience; while preserving the wealth of solid information such a workshop can convey.

In point of fact, the events described on the following pages have been distilled from a variety of actual workshops. While the descriptions here have been condensed somewhat for clarity and brevity, they are otherwise faithful representations of what actually occurred.

Not surprisingly, the organization of this booklet parallels the format of a typical workshop, session by session.

The initial activity conducted in most sessions is the demonstration class: a CSMP staff member teaches a fairly representative lesson to a class of local students (who have had no previous CSMP experience), with an audience of local teachers, parents and administrators looking on. In this way, the audience first encounters the program through the eyes of the students; no introductory explanations to the audience precede the demonstration class.

The class is usually followed by a question and answer period, and a discussion of the lesson just presented.

Finally---for what is called the group lesson and discussion---the members of the audience assume the role of students, as the CSMP staff member conducts a related, but more sophisticated lesson, answering questions as they arise.

The most notable departure from this format occurs in the final section of this booklet---called "Implementation Issues"---which is not concerned with program content so much as with questions often asked by schools considering adoption of the CSMP program.

One final note to the reader is in order. If, after reading this portrayal, you feel that your school or school district would benefit from a workshop similar to this one, you are urged to contact the Comprehensive School Mathematics Program. We will gladly discuss conducting such an awareness workshop in your area.

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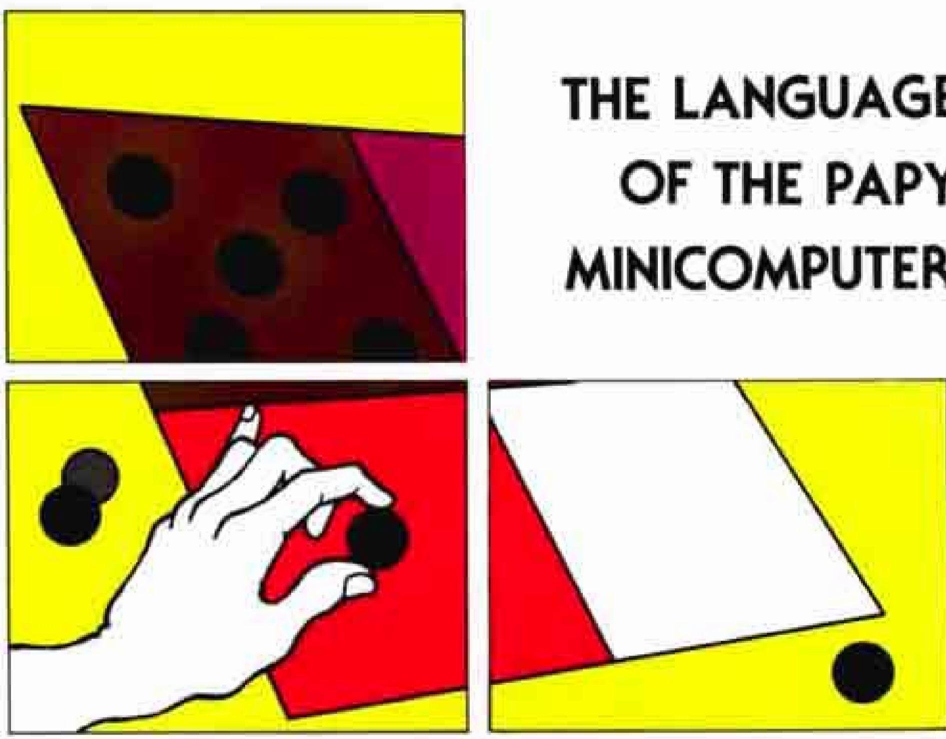
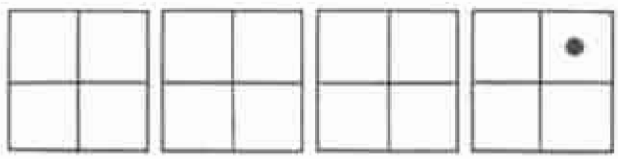
SESSION 1



THE LANGUAGE OF THE PAPY MINICOMPUTER

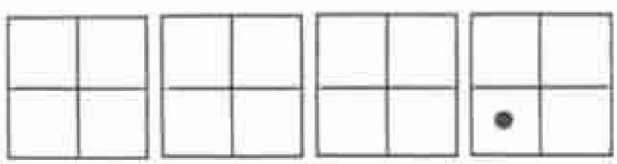
1a Demonstration Class: Introduction to the Minicomputer

The teacher stands quietly at the front of the room. Smiling, she gestures at the chalkboard to her right, from which four large, multi-colored squares[#] are hanging side-by-side. From the chalkboard ledge, she selects a disk the size of a half-dollar and touches it to one of the squares, where it sticks.



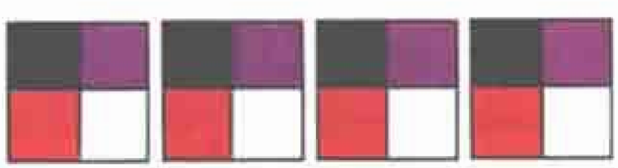
"When I put a checker here," she says slowly, "the number is 4."

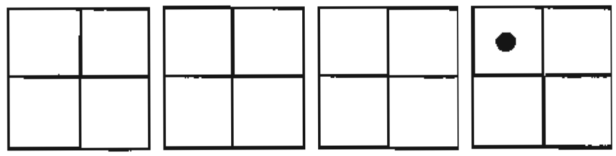
The twenty young children sitting cross-legged at her feet do not know quite what to make of this, and shift about restlessly on the carpeted floor.



"This . . . is the number 2," she continues, moving the checker as she speaks.

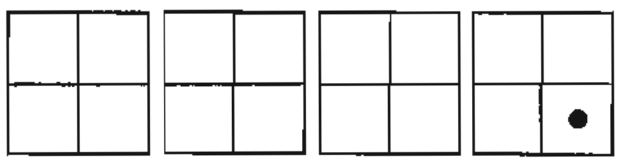
[#] Color is omitted from the illustrations in the description of this lesson. The four multi-colored squares look like this:





"And this is 8."

Near the back of the classroom, a small audience of teachers and administrators watches this intently, glancing alternately from the children to the soft-spoken teacher. Like the children, they are quiet and expectant.

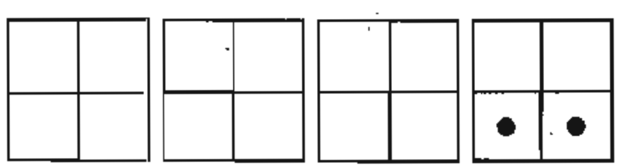


"Which is <u>this</u> number?" the teacher asks, leaning down to the students. They are uncertain; several peer at the chalkboard and frown.

"3?" one child ventures.

The teacher smiles again. "No, this is the number 1." She removes the checker. "Who can put the number 3 on the Minicomputer?" Offering a handful of checkers, she waves her other hand toward the four squares.

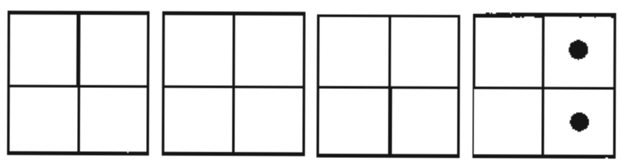
One girl from among the group stands, takes two checkers from the teacher and puts them on the Minicomputer.



"Very good," the teacher says as the child returns to her place. "2 + 1 = 3."

In the rear of the room, several observers exchange glances. One of them smiles brightly. The children are from her third grade class.

At the board, the game continues at a brisk pace. "Who would like to show 5 on the Minicomputer?" the teacher asks. Many hands poke up from the group and pointing, she selects one. A sandy-haired boy jumps up, takes a checker in each hand, and slaps them perfunctorily on the right-hand board.



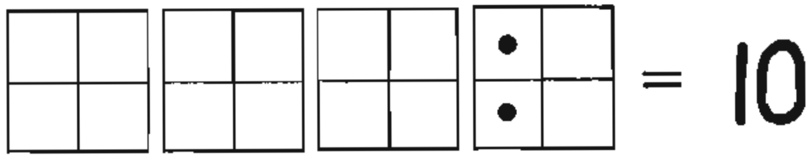
He is obviously very pleased with himself as he turns, smiling, and sits back down.

The teacher continues asking for volunteers to show, one at a time, the numbers 6, 9, and 7 on the Minicomputer. If the children were restless earlier, they are agitated now, jostling and leaning over each other in an effort to be chosen next.

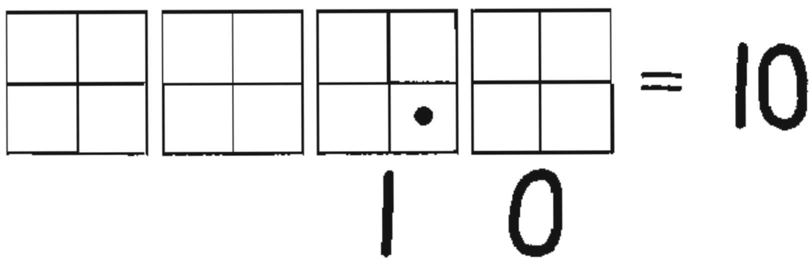
Most people in the audience mirror the students' excitement, tinged with more than a little wonder at how quickly the children have taken to the rules of this new game. A few are themselves still pondering the rules. "The red square stands for . . . how many?" one whispers hoarsely to her neighbor.

"It's for 2, I think," he whispers back. "But what's the value of the brown square?"

"Now, who can show me 10?" the teacher at the board is asking gently. A short boy has to stand up on tip-toe as he places the checkers.

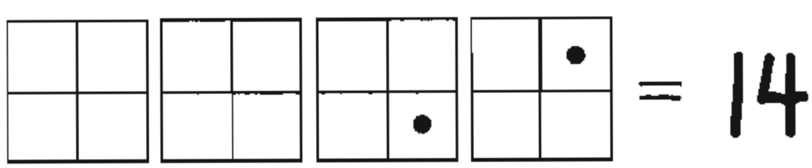


"Yes, very good. 8 + 2 = 10. Now . . ." The children become quiet as the teacher pauses and looks at each of them in turn. "I will show you a new way to show 10." Very dramatically, she removes both checkers from the Minicomputer, one in each hand. She hides one checker behind her back and moves the other to the adjacent Minicomputer board. She writes a numeral under each board.

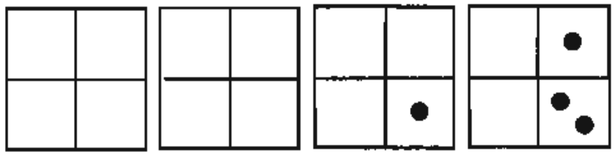


She gestures toward the Minicomputer. "10," she says simply, and then erases the numerals. Moving the single checker from the first board to the next and then, back, she continues. "All together: 1 -- 10 -- 1 -- 10." She pauses and blinks suddenly --- as if something has just occurred to her.

"Where is 14, I wonder?" Many hands shoot up.



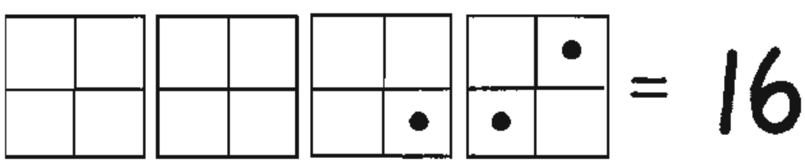
"Very good." She removes the checkers. "Now I would like 16." The child who comes up hesitates, and then puts the checkers up very deliberately, one-by-one.



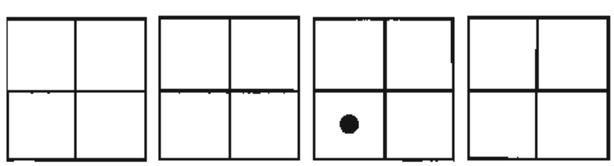
At this, the little group fairly explodes in protest. "No --- I can!" a small boy exclaims. At the back of the room, several teachers grimace; one shakes his head.

"No!" the teacher says to the audience, feigning surprise. "It is very good!" 10 + 4 + 1 + 1 = 16."

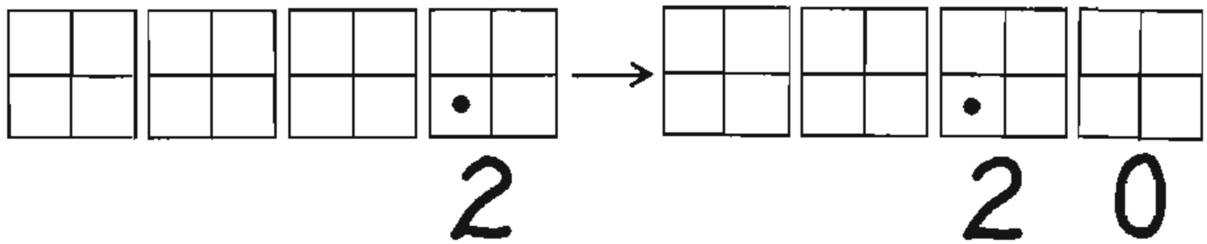
Everyone is momentarily stunned. "Of course," mumbles one teacher to herself. "Ah-hahh!" whispers the group of children. With the checkers still on the Mini-computer, the teacher continues. "And who can show this same number with <u>one less</u> checker?" A student does so.



"Good. Who can show 20?"

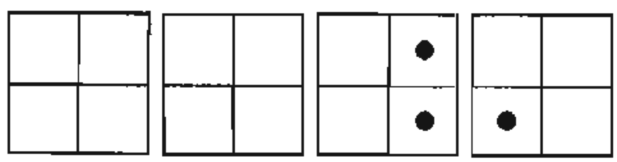


The observers in the rear lean forward, engrossed, to hear the students name each number correctly, as the teacher moves the checker from the first board to the next.



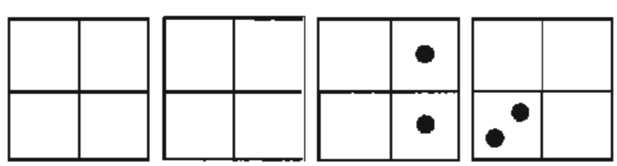
"2 . . . 20." Together, they repeat this for 4 and 40, 8 and 80, and 1 and 10.

"Who can show me 54?" the teacher asks. A slim, blue-eyed girl with red hair steps gingerly over her classmates to take several checkers in her hand.



"One moment," the teacher says gently when she turns to go back to her place. "I was asking for 54. Let us see which number this is." Pointing to each checker in turn, the teacher says, "40 . . . plus . . . 10 . . . plus 2 equals . . . ?" She looks to the girl expectantly.

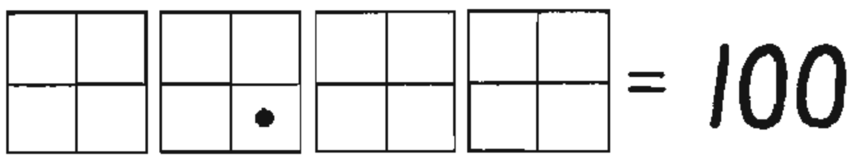
"52," the girl replies, and adds a checker.



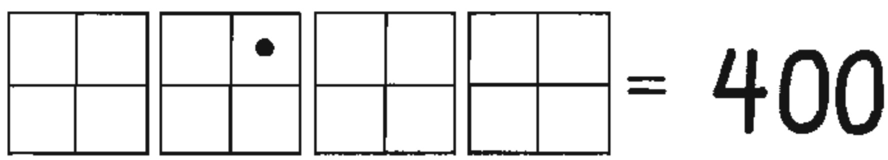
"54!" she says triumphantly. The observers smile as the girl resumes her place, and the group proceeds through several other two-digit numbers. Then, the teacher once again pauses. She leans down to the children. "I wonder . . . where is

100?" she whispers.

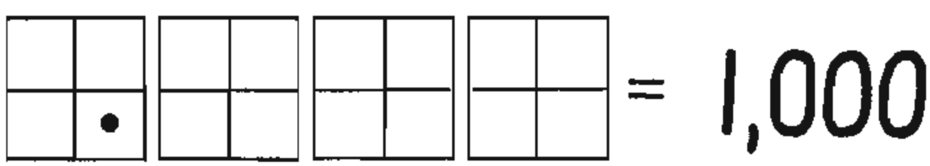
As she straightens, the group erupts in a flurry of waving hands and pleas for recognition. Two or three boys can no longer sit and are now up on their knees. The teacher nods silently to one of them and steps back as he bounds to the board.



"Yes. Where is 400?" The clamor grows louder.

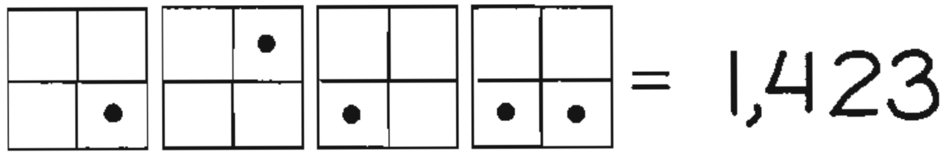


"Very good. Where is . . ." She hesitates. ". . . 1,000?" In the uproar that follows, several children are no longer satisfied with a single raised hand, and are now waving two.



"Yes . . . thank you," the teacher answers. "Now, I would like 1,423. Yes . . . ?"

A student comes up and shows this number.

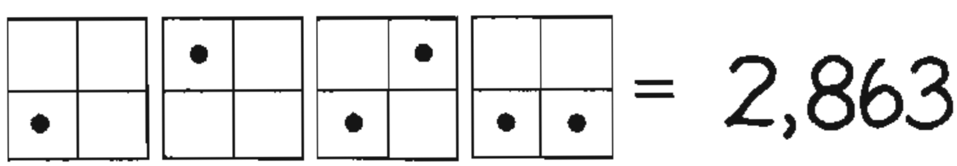


Fascinated during all of this is the regular classroom teacher of the students. "They have never seen these numbers," she tells one of her colleagues in disbelief. She blinks, and turns back to the board. The class has moved on to a new number.

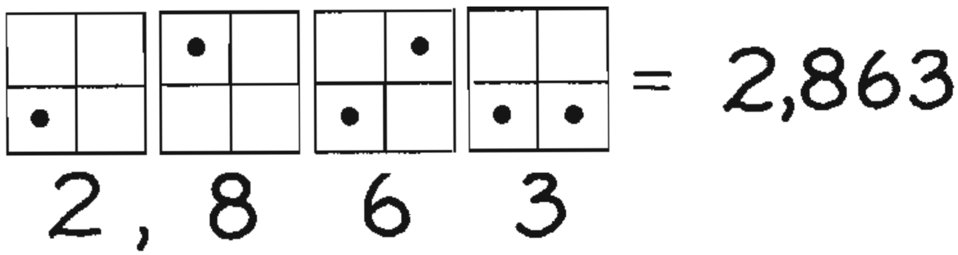
"How about 2,863?" The class offers anxiously to put the number on the Minicomputer, but the teacher demurs. "Which number did I ask for? Who can repeat the number?" she asks, and half of the raised hands drop suddenly from sight. In the excitement, some have forgotten this large number!

"Um . . . four hundred . . . ," one of the braver students begins uncertainly. Around the room, several good-natured chuckles are heard from those in sympathy with the student.

"No, that is not quite right," the teacher says smiling. "Listen carefully." She repeats the number and writes it on the board. A student comes up to put it on the Minicomputer.



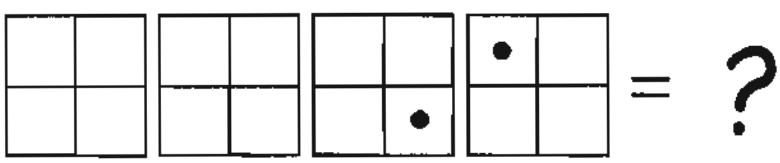
"Very good." Under each Minicomputer board, she writes the appropriate numeral.

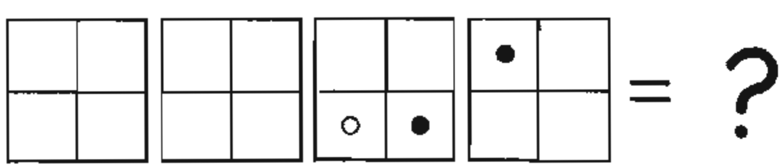


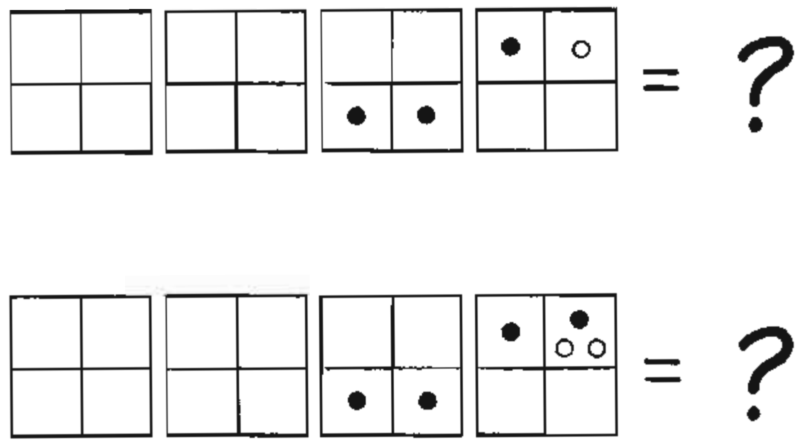
"Fascinating," breathes one of the teachers, as others grin in bewilderment and shake their heads. Unaware of this, the children are quite absorbed in the task of showing, one by one, a host of new four-digit numbers.

"Now, we will do something new," begins the teacher after some moments. She gestures to the now-empty Minicomputer. "I will put checkers on the Minicomputer very slowly. You must watch carefully, and think to yourself which number is there."

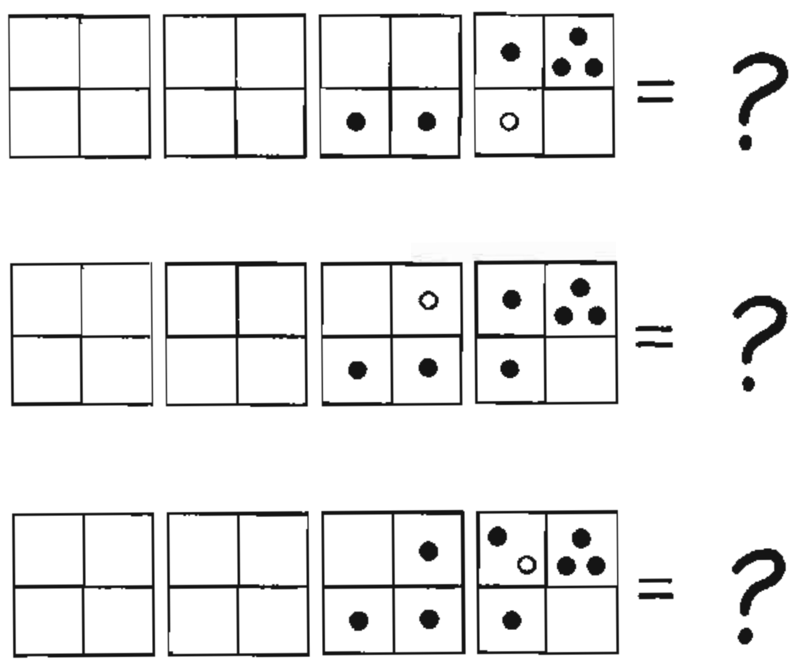
As the room builds in silence and concentration, the teacher poses several numbers, one at a time, by adding checkers to the colored boards.







"What number is this?" she asks. A curly-haired boy near the back of the group responds that it is 50. "Yes; now watch carefully." One at a time, she adds three more checkers.

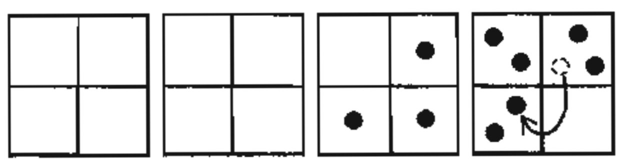


"Now what number is it?" Many students raise their hands, but she holds a finger to her lips. "Whisper the number to me," she says, and leans forward to the nearest child. "Yes, that is correct," the teacher tells the girl, patting her lightly on the head. The girl smiles smugly and leans back. The boy to her right knows the number, too; he scoots forward quickly to whisper it next. "Yes, very good," replies the teacher. Several whispers later, she stands. "Now aloud . . . ?"

The students nearly shout. "100!"

"Yes." The teacher returns to the Minicomputer. "I will move one checker. Tell me whether the new number is bigger, smaller, or the same."

Dramatically, she moves a checker from the 4-square to the 2-square.



She nods to a child, who replies that the new number is smaller.

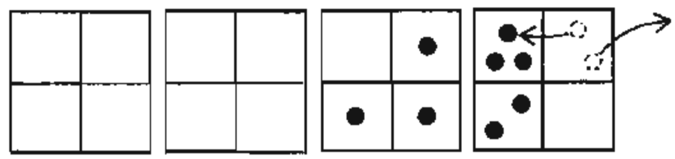
"From 4 to 2 --- how much smaller?"

"2," answers another child.

"What is the number now?" asks the teacher.

"98," replies the child.

Using both hands simultaneously, the teacher removes two checkers from the 4-square, placing one of them on the 8-square and laying the other one aside. She is careful to exaggerate her actions, making the move obvious and easy to observe.



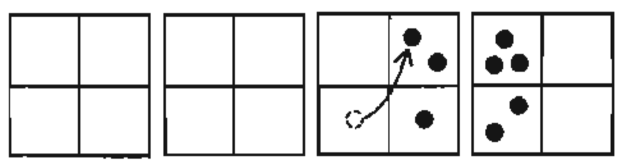
A student responds that the number is now larger. "Why?" asks the teacher.

"Because you went from 4 to 8."

"Ahhh . . . I think perhaps you did not watch closely enough," she replies, and repeats the movement.

"Oh!" the child says quickly. "The number is the <u>same</u> . . . because 4 + 4 = 8!"

The teacher moves a checker from the 20-square to the 40-square.



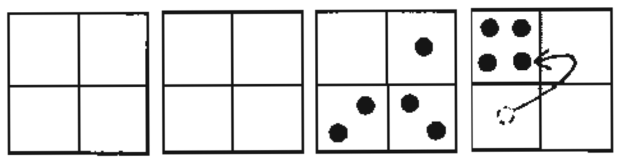
"The number is larger . . . by 20," says a student.

The teacher agrees. "And now what is the number? We had 98; plus 20 is . . . ?

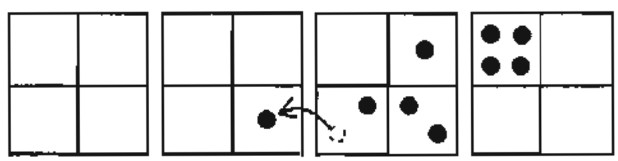
"118!" a student cries.

The audience of teachers, mostly quiet up until now, is fairly buzzing. "Shhh!" one of them urges. Another, near the rear of the group, is poised on the edge of her chair; the young man in front of her is quite tall, and so she strains forward to watch the game unfold. Meanwhile, other moves have been made.

"Now we have 126 on the Minicomputer," the teacher is saying. "Who can make my number six bigger, by moving just <u>one</u> checker?" A little green-eyed girl pushes her hair back soberly as she comes up to try this. She moves a checker from the 2-square to the 8-square.



"Very good. Now . . . a very difficult one. Who can make it 80 larger by moving just one checker?" she asks, peering about the little group. A student stands quickly and walks to the front. He moves a checker from the 20-square to the 100-square.

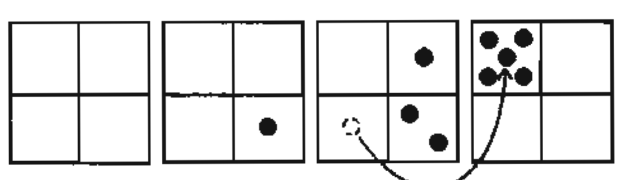


"Yes; very good," the teacher says. "What is our new number?" She writes the standard addition calculation on the chalkboard.

$$\begin{array}{r}132\\+80\\\hline\end{array}$$

"212," a student responds.

"Yes. Now, I would like to end the lesson with 200 on our Minicomputer. How much less to get 200?" she asks. A young boy says that 200 is 12 less. "Who can make the number 12 smaller, by moving just one checker?" After a little while, the child moves a checker from the 20-square to the 8-square.



"Yes," the teacher agrees. "From 20 to 8 makes the number 12 smaller; we now have 200 on the Minicomputer. But, to be sure we did not make any mistakes, let us check." Together, they count as she points to each of the checkers in turn.

"8 . . . + 8 is 16 . . . + 8 is 24 . . . + 8 is 32 . . . + 8 is 40 . . . + 10 is 50 . . . + 10 is 60 . . . + 40 is 100 . . . + 100 is 200."

"Yes; it is 200," the teacher says, turning to smile at the students. "Now it is time to stop." In the back of the room, the teachers applaud spontaneously.

1b Questions and Answers About the Minicomputer

"We watched a class of third graders, who took to the Minicomputer very quickly. It looked as though it could be used with even younger students. When is it introduced in the CSMP curriculum?"

"In our curriculum the children have some experience with the Minicomputer even at the Kindergarten level. Of course, the lessons are not as sophisticated as the one you have just seen. For Kindergarten students the Minicomputer provides an alternate way of representing numbers -- a way which does not tax the child's motor skills as writing numerals often does. With the Minicomputer, poor dexterity cannot prevent the young child from exploring the world of numbers as deeply as his or her intellectural capability allows."

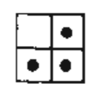
"I noticed that the Minicomputer has a ones' board, a tens' board and so on. Do you use this aspect of the Minicomputer to teach place value?"

"Place value is so integral to the Minicomputer that specific lessons on place value are not needed. Each time a number is read from the Minicomputer or put on it, the concept of place value is reinforced. In addition to that, there is a positional system within each Minicomputer board: a checker on the white square has a different value from a checker on the purple square. With the Minicomputer, the child is introduced to the fundamental idea of a positional system very early and with small numbers. The Minicomputer relies totally on such a positional system whereas positional notation or place value does not seem significant in our conventional decimal numeration system until the children begin to work with relatively large numbers."

"That was one of the things that impressed me in the lesson we watched. The students seemed to be at ease even with a number in the thousands. Do you find this is generally the case?"

"The Minicomputer allows young children to become familiar with relatively large numbers and to do calculations with them earlier than they would normally be able to in the conventional decimal representation. All of the representations and manipulations on the first board have analogs on the remaining boards and the extension is quite natural for the children."

"You mentioned calculations. Do you mean the mental calculation involved in reading a number or something else?"

"First there are activities involving mental calculation. These exercises help us make apparent some basic arithmetic facts. For instance, the symbol '7' tells us nothing about the number seven. The Minicomputer representation '  ' tells us that 7 = 4 + 2 + 1 and, by using checkers of different colors, could show that 7 = 6 + 1 or 7 = 5 + 2 or 7 = 4 + 3. By extension, similar facts are discovered for 70, 700 and so on. The Minicomputer may also be used to perform all four basic arithmetic operations. We can add, subtract, multiply, and divide on the Minicomputer and can do so with large numbers. This allows children to experience interesting problems which would usually be impossible at this stage in their growth, since the calculations involved would be too complex."

"Won't the use of the Minicomputer in such situations lead to dependence on it?"

"We find that not to be the case. Each child reaches a point where he or she no longer needs the Minicomputer to do calculations and switches to the usual paper and pencil algorithm. The real beauty of the Minicomputer is that its use leads naturally to this transition by preparing the student for paper and pencil algorithms. For example, in addition problems on the Minicomputer, a checker is physically moved from the ones' board to the tens' board in exactly the same situations in which we 'carry' in a paper and pencil algorithm. Presented in this way, the algorithm is easily understood and used. Similarly, the use of the Minicomputer lays the groundwork to help students understand the other arithmetic operations and leads to the usual algorithms."

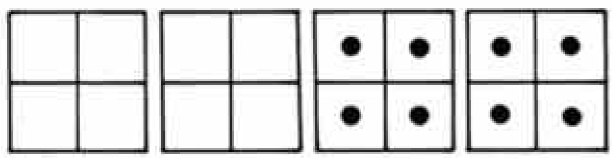
"Do you abandon the Minicomputer as the children switch to paper and pencil?"

"No, but the emphasis changes. It is used to set up situations which lead to new concepts or which stress mental arithmetic skills. As an example, the Minicomputer may be used to motivate the introduction of decimal numbers and then as a vehicle for working with them. During this morning's lesson, you saw an example of the mental arithmetic involved in calculating the increase or decrease resulting from moving checkers from square to square on the Minicomputer. During the next session of the workshop, you will see another example, a Minicomputer game, in which both mental arithmetic and strategic thinking are required."

1c Group Lesson and Discussion: The Minicomputer Golf Game

After observing a lesson and discussing it, the workshop participants are given the chance to learn by doing. Now assuming the role of students, the adults prepare for a lesson using the Minicomputer in a more sophisticated way.

The workshop leader places eight checkers on the Minicomputer.

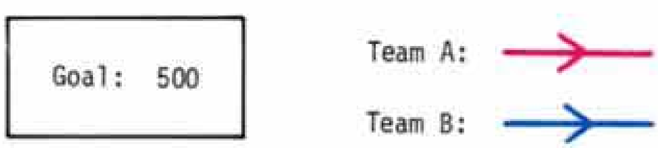


"Which number is this?" she asks.

"165," one teacher answers quickly, while others seem slightly less sure of themselves.

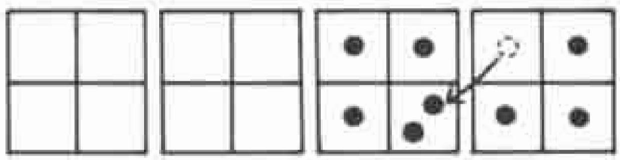
Sensing this, the leader points to the two left most checkers and the audience responds "80 + 20 = 100". She points to the remaining checkers, two at a time, as the group keeps an accummulative total, finally arriving at 165.

The participants are grouped into two teams, and the following information is written on the board.



The leader explains: "We are starting from 165 and our goal is 500. Players from Team A and from Team B will play in turn. Each time, they can move one checker from any square to any other square. The number may be increased or decreased. The first team to reach the goal will be the winner."

The first player from Team A cautiously moves a checker from the 8-square to the 10-square.

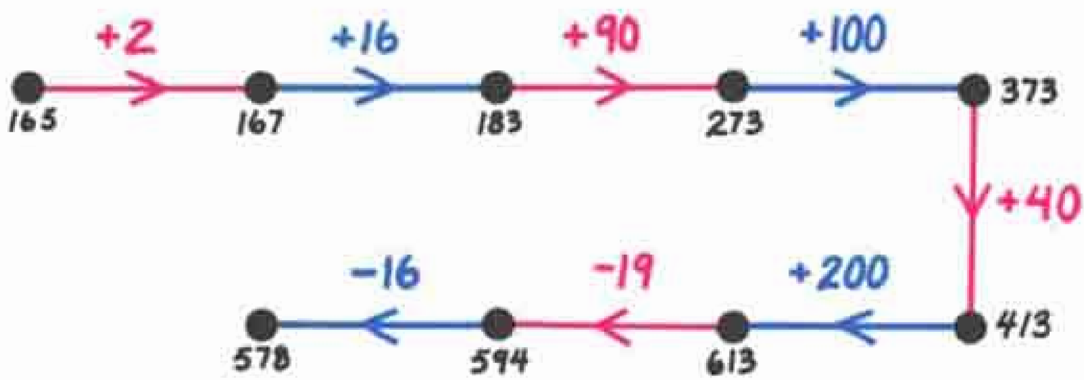


"Did you increase or decrease the number?" asks the leader.

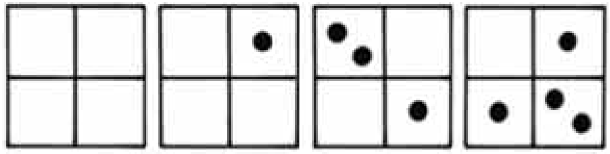
"I increased it by two," answers the player.

The leader draws a +2 red arrow starting from 165 and ending at 167.

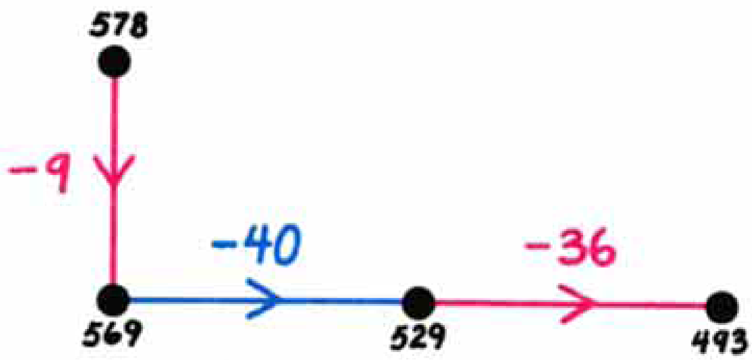
Play continues as the teams' members take turns --- some boldly, others cautiously --- until the arrow picture looks like this:



And the Minicomputer boards look like this :



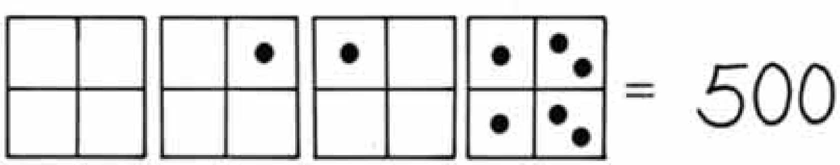
It is Team A's turn and the faces of several team members light up ; the team's next move could reach the goal and secure a victory. Unfortunately, their next player does not recognize this move, and the game must continue.



This time, Team B's player sees the winning move : she moves a checker from the 1-square to the 8-square.

"Seven larger --- that's 500," announces the leader. "Team B is the winner."

The final configuration on the Minicomputer is shown below.



One of the participants (whose anticipated victory did not materialize) says, "Team A had a winning move earlier, but our player didn't make it. Did you notice this and if so, why didn't you mention it?"

The leader replies, "Yes, I was aware of that. However, here, as in the classroom, games are useful only if the players are not afraid of exploring the game situation. When we criticize and impose a 'best way' of solving the problem, we stifle the freedom and enjoyment of the game and subtly encourage the students to limit their creative thinking. As they play again and again, the students naturally find ways of improving their moves and progressively develop strategies."

Another teacher in the group comments, "I'm sure my fourth graders would enjoy the game, but I have one reservation: the brighter students in my class would surely become frustrated if their moves were often cancelled out by the moves of the less skillful players. Haven't you had this experience?"

"Yes, and this is why we use the game collectively only until the students know how it is played," the leader responds. "Then, children of similar numerical ability play it together in small groups. We have also used the game extensively with slow learners, in a teacher-supervised situation. We find it is especially effective in helping them to increase their numerical skills. If you would like to know more about this way of using the Minicomputer Golf game, you may want to read the booklet Math Play Therapy[*].

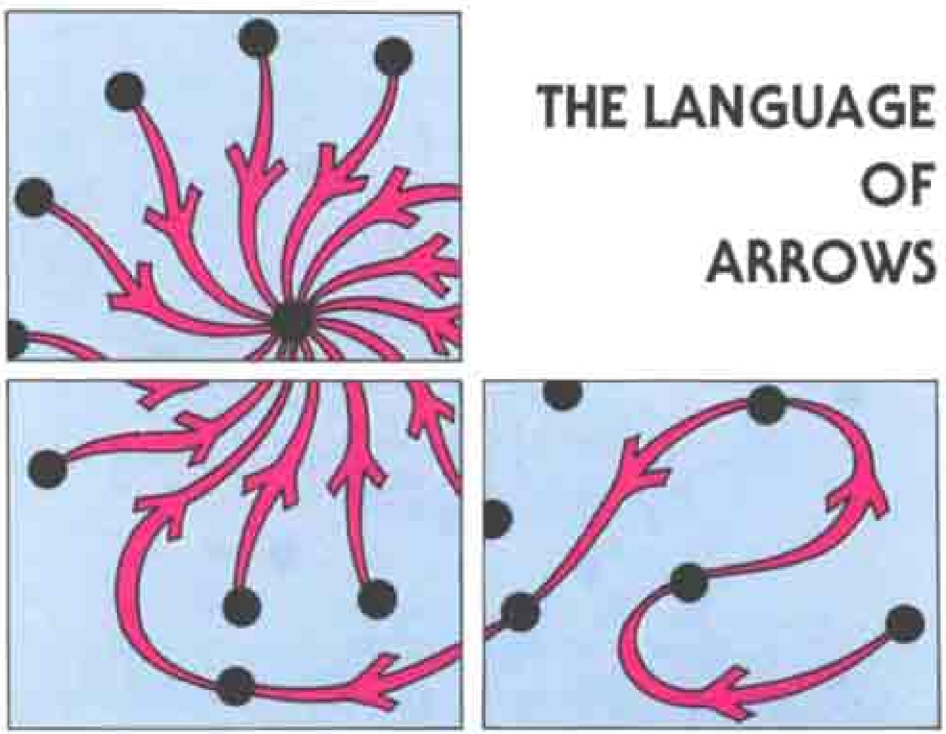
"Just recently, we conducted a series of classes and workshops similar to this one. I'd like to show you one of the 'letters' we received from the class of

[*] Available from CEMREL, Inc. - CSMP

students who had just 'met' the Minicomputer for the first time." She carefully unfolds a large sheet of paper, and holds it up before the group.


I liked your games.
I liked your Mini-Computer
It isn't every day you see
a man walking around with
a neat thing like a Mini-Comp-
uter
Greg

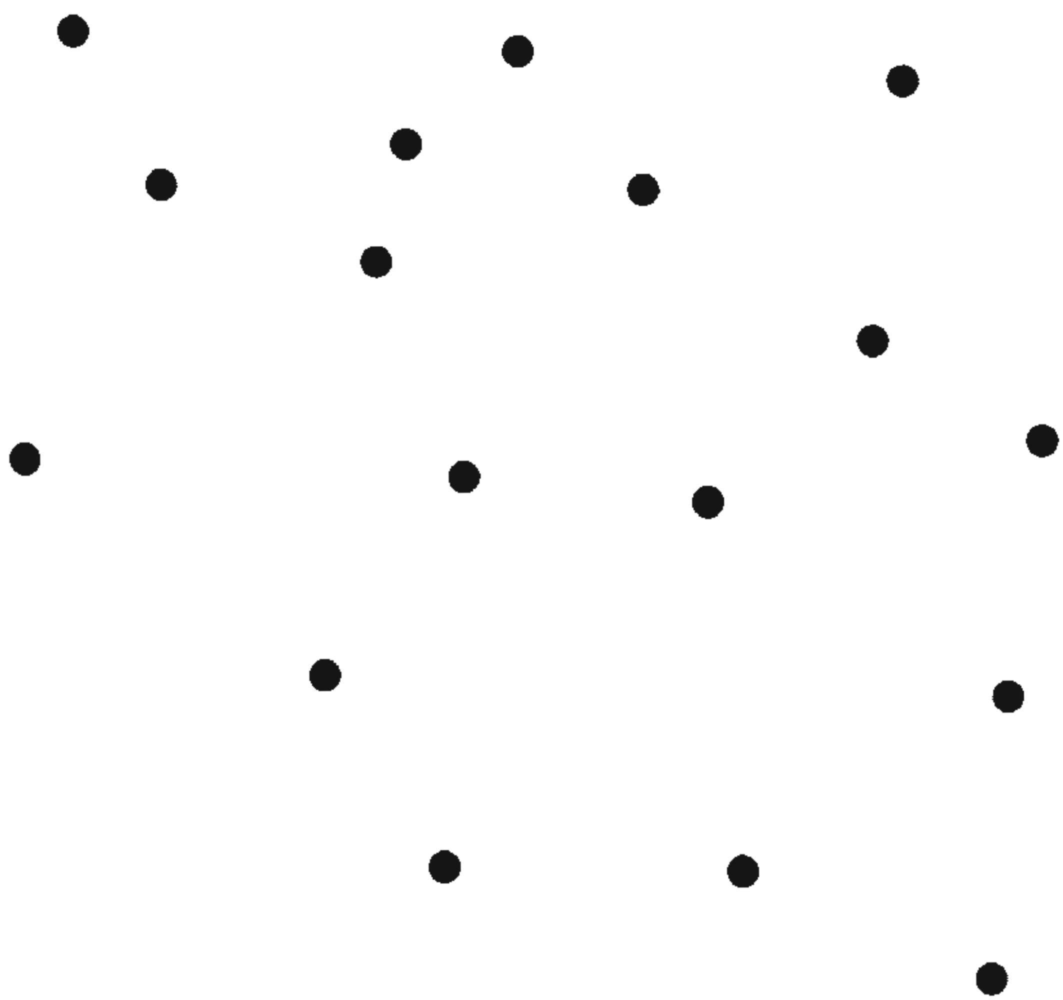
SESSION 2



THE LANGUAGE OF ARROWS

2a Demonstration Class: The Sister-Brother Relations

The class of young children becomes quiet as the teacher deliberately draws large, impressive dots spread out across the board. Every few moments, she glances secretively over her shoulder at the little group.



"Here are some children on their playground at school," she announces. "Can you tell me which ones are girls?"

One red headed girl is sure that she sees several. "This one, and this one, and --- "

"Ahh, but are you <u>sure</u> that those are girls?" the teacher interrupts gently.

"Yes," the little girl insists, nodding. She is quite positive.

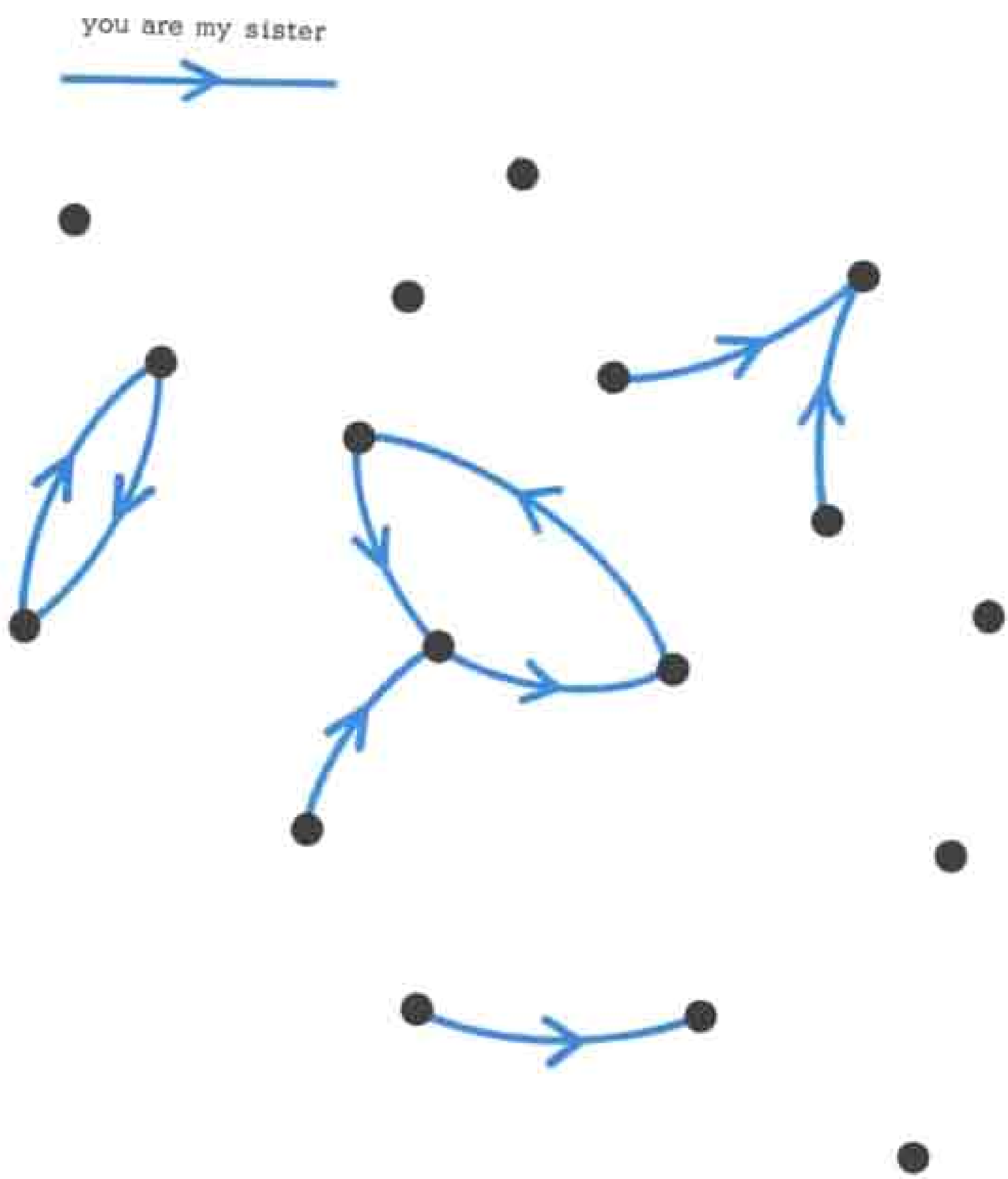
"How can you tell?" the teacher asks, looking to the dots and pretending amazement. The members of the audience are smiling.

"Oh . . ." The student's voice trails off as she decides. "I guess you can't <u>really</u> know for sure; they all look alike!" Several students seem quite disappointed at this development.

"Yes, unfortunately, that is the case," the teacher continues sympathetically. "<u>However</u> . . . the children are playing a game which may help us know which are boys and which are girls. The game is called "you are my sister". A child who has a sister on the playground points to her and says, 'You are my sister.'" As she describes the game, the teacher draws a blue arrow in the top left-hand corner of the picture.

you are my sister

The teacher continues, "Let me show you how it looks when the children play this game." Very deliberately, she draws some bright blue arrows and connects dots in the picture, as the students wait anxiously. Some try to guess where the next arrow will be drawn.



"Now," she says, leaning down toward the class, "who can show where there is <u>surely</u> a girl?"

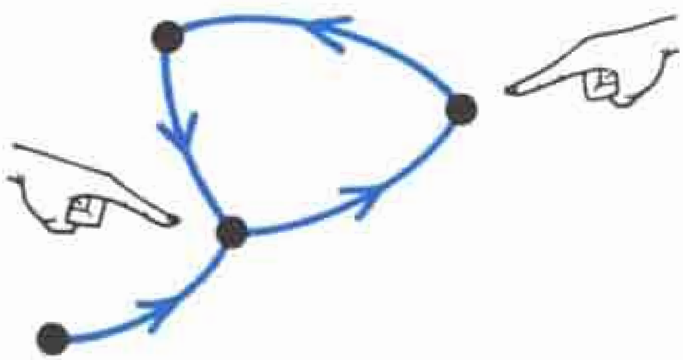
Looking quite relieved by this new development, the red headed girl points to a dot, "This is a girl."

"You are sure? How do you know?"

"Because she is the sister of this other child," the student responds confidently.

The students are staring at the picture and one by one their hands go up. Unable to wait recognition, they all talk at once: "I see a girl!" The audience is amazed and pleased by the excitement of the class.

The teacher allows several students to identify girls, when one boy announces in a very concerned voice, "There is something wrong. These two are girls," he points to two dots simultaneously,



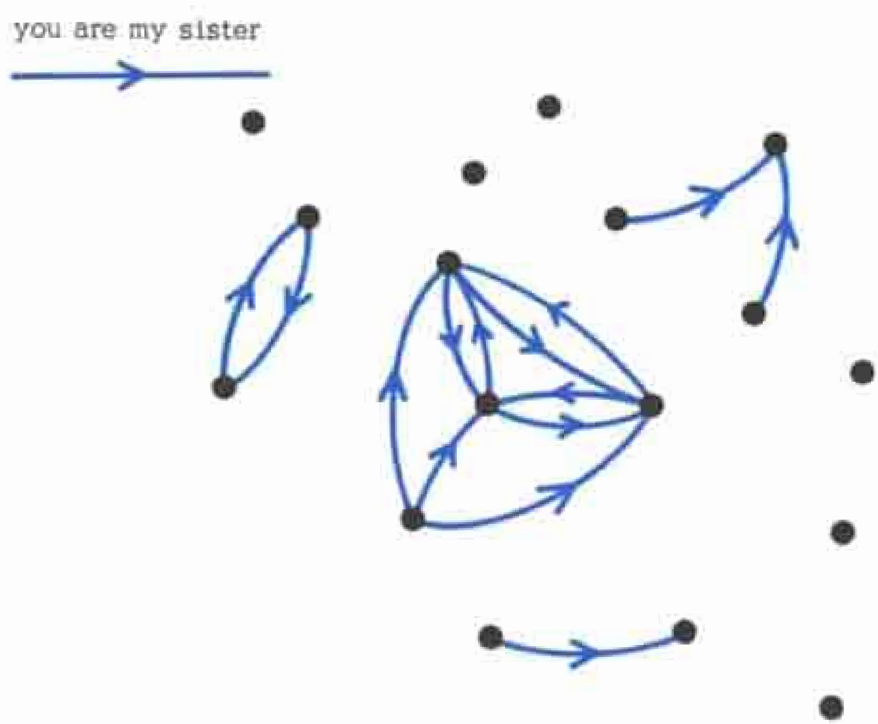
"and one is saying to the other 'You are my sister', but the other one is not pointing back. The two girls should be pointing to each other."

"You are right," the teacher assures him, "perhaps that girl was not listening when the game was played. Would you draw an arrow, to show that she points to her sister?"

The boy carefully draws an arrow, trying to imitate the beautiful thick arrows drawn by the teacher. The audience is buzzing; this discovery is bringing relief to some and amazing others.

"Are there <u>other</u> blue arrows that we are <u>sure</u> can be drawn?" the teacher asks.

Hearing this, the students begin to re-evaluate the picture and very shortly, have discovered several missing arrows. The teacher allows each of these to be drawn in by the student who discovered it. Mistakes are made, but the other students are quick to correct them.



"I believe that all the blue arrows which can be drawn are now in our picture; no blue arrow is missing." The teacher points to a dot, "Is this a girl or a boy?"

"You can't tell," a student answers.

"Yes you can!" shouts another. "He _must_ be a boy!" The teachers in the rear of the room lean forward, startled.

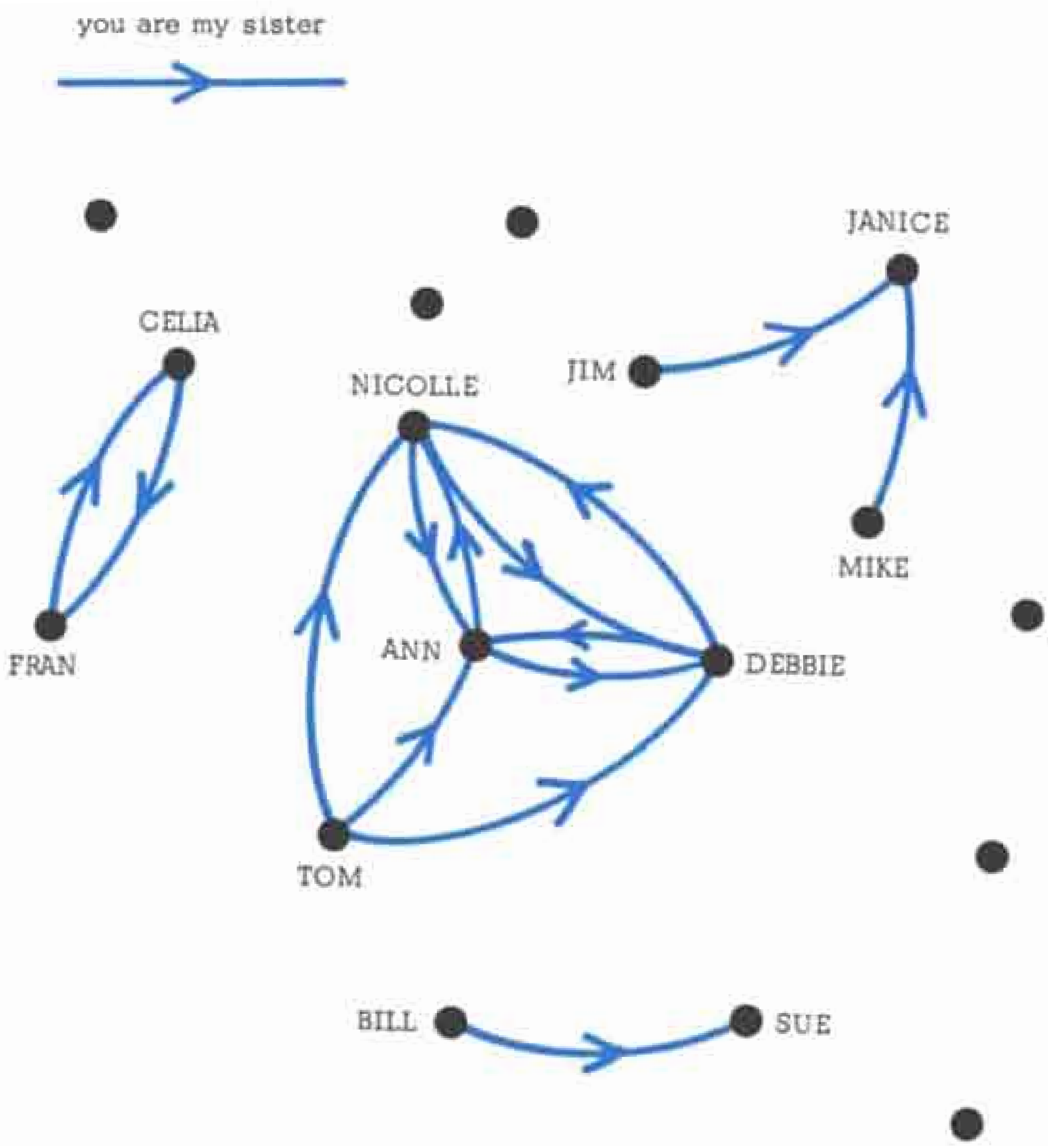
"How do you know?" the teacher persists.

"If that were a girl," the student begins, "then both children would be sisters . . . and there would be _two_ blue arrows, instead of just one!" The student smiles triumphantly.

"You are correct!" the teacher says. "It is a boy. What might this boy's name be?"

"Bill," offers a boy named "Bill". The teacher writes this name next to the appropriate dot, then points to the dot at the other end of the arrow. "And this girl . . . what is _her_ name?"

"Sue." One-by-one, the students identify the children which they know must be boys or girls; and they suggest names for each.

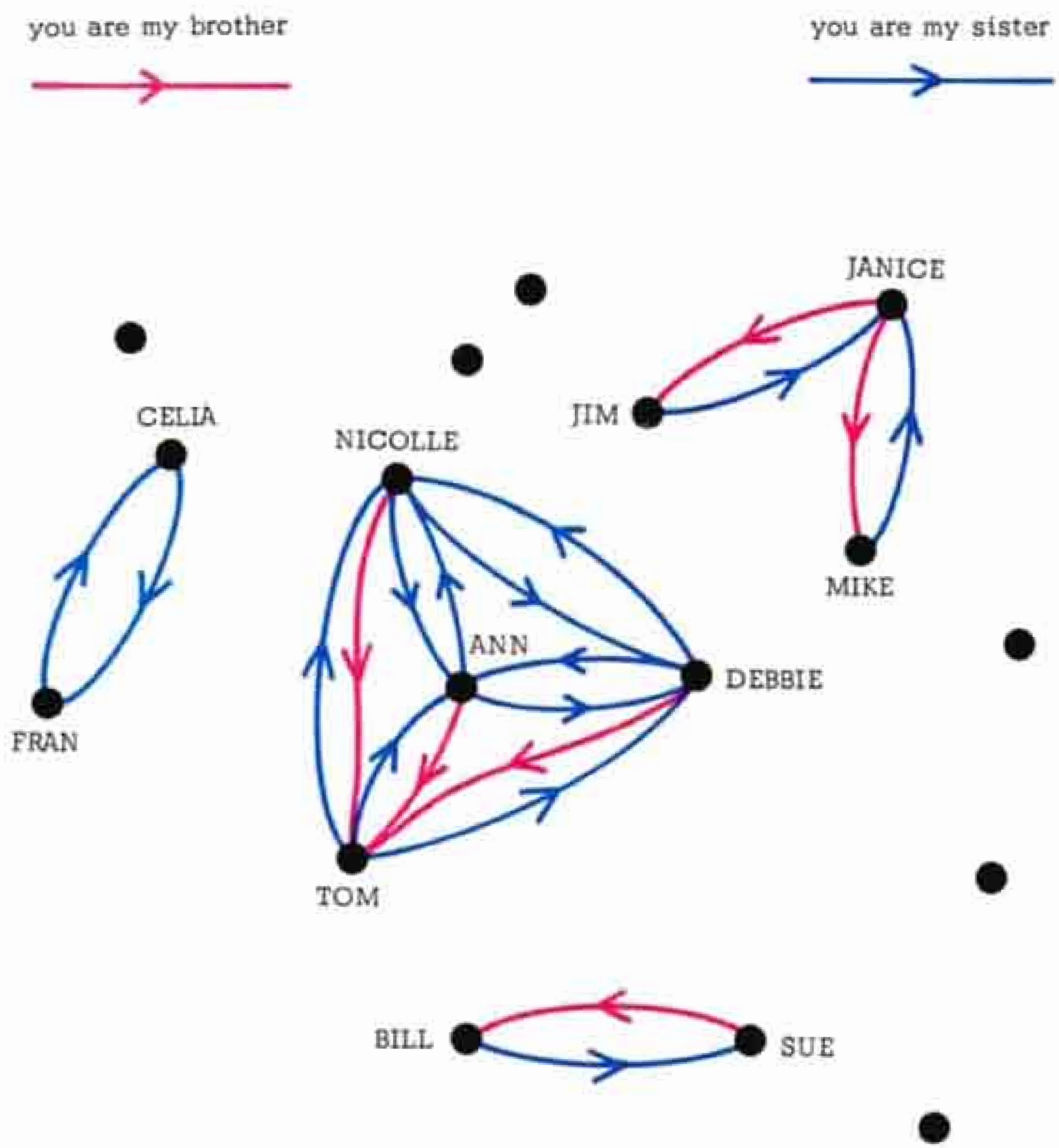


"Very good," the teacher says at last. "Now . . . the children on the playground are playing a new game. Do you know what this new game is called?" She looks expectantly from one thoughtful face to another.

One young face near the front suddenly brightens. "You are my brother?"

"Yes!" the teacher answers. "We can show them playing this new game with red arrows. Where can we draw red arrows in the picture?"

In turn, the students come up to do this, until the chalkboard is literally filled with colored arrows.

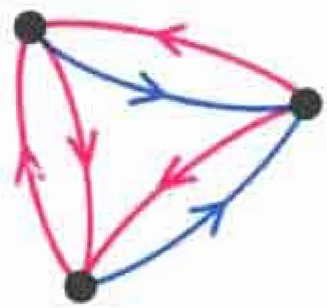


"Have we drawn all of the red arrows that we know can be drawn?"

The students and the audience nod in agreement.

"I'm not so sure we are finished." The teacher gently prods the class to re-examine the situation.

"No, I see two more red arrows!" a child exclaims. "Between Jim and Mike." The teacher nods to her and she comes up to add these two red arrows.



"How do we know this is correct?"

"Jim and Mike are both Janice's brothers; so they must also be each other's brothers."

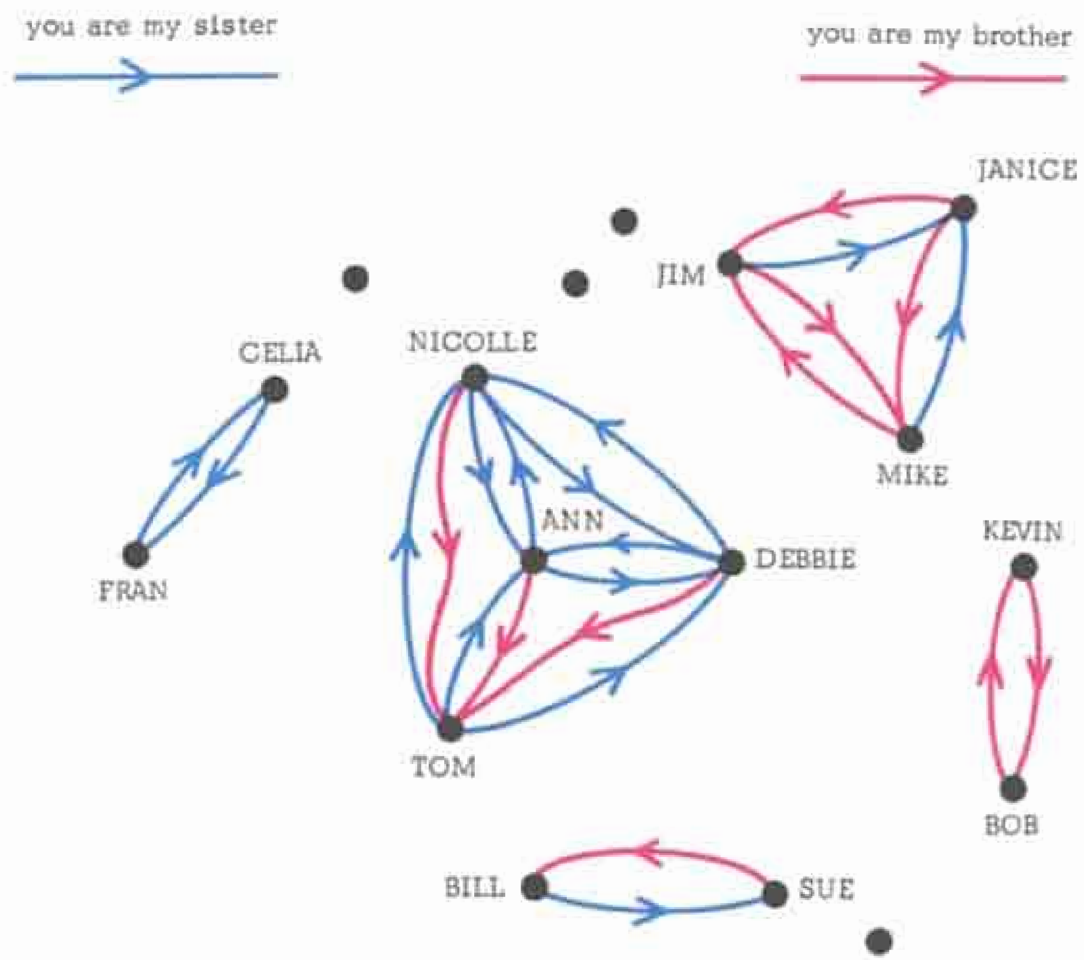
"This is true. Now, I haven't yet told you anything about these two children . . . ." She points to two "lonely" dots at the right side of the picture. "As it happens, we can draw this red arrow." She does so.

" But then you must also draw a red arrow in the other direction," a boy in the back of the class protests.

" Can you explain why? "

The boy comes forward to give his explanation. He points to the lower dot. " If this were a girl, then there would be a blue arrow because she would be a sister of this boy. But all the blue arrows are drawn. So they are both boys; they are brothers."

The teacher silently hands him the red chalk and he draws the missing red arrow. She then asks him to label the dots with any names he likes.



"What about these other 'lonely' dots, where there are no arrows beginning or ending?" the teacher asks.

"They have no sisters or brothers," one student explains.

"No; we're not sure," a second one contests. "They have no brothers or sisters on the playground . . . but they could still have brothers or sisters somewhere else."

"Very good," concludes the teacher. "Now we are finished." With a short nod of the head, she replaces the chalk on the tray and heads back towards her seat.

2b Group Lesson and Discussion: The Language of Arrows

"It was fun to watch the children. They were so involved and were really thinking; but what do the sister-brother relations have to do with mathematics?"

"The unifying idea here is that of relation. 'Nicolle is a sister of Tom' describes a relation between two people, just as '7 is three less than 10' describes a relation between numbers, or 'line k is parallel to line m' describes a relation between geometric figures. In mathematics the study of relations is central. With the children we explore relations between numbers in ways which are similar to those you observed in this lesson."

"Could you give us some examples of numerical situations which use the language of arrows for problem solving?"

"Sure. Let's suppose you are second grade teachers and each of you has given your class this problem:

> Mark has 50¢ and wants to spend it all on pencils and erasers. Pencils cost 6¢ each and erasers cost 4¢ each. How many pencils and how many erasers could he buy?

How might your students solve this problem?"

One teacher grins. "Most likely by trial and error. Some might draw pencils and erasers."

"I might have a student who would start adding 6's to see how many pencils he could get; or another who would start adding 4's because she likes erasers," a second teacher comments.

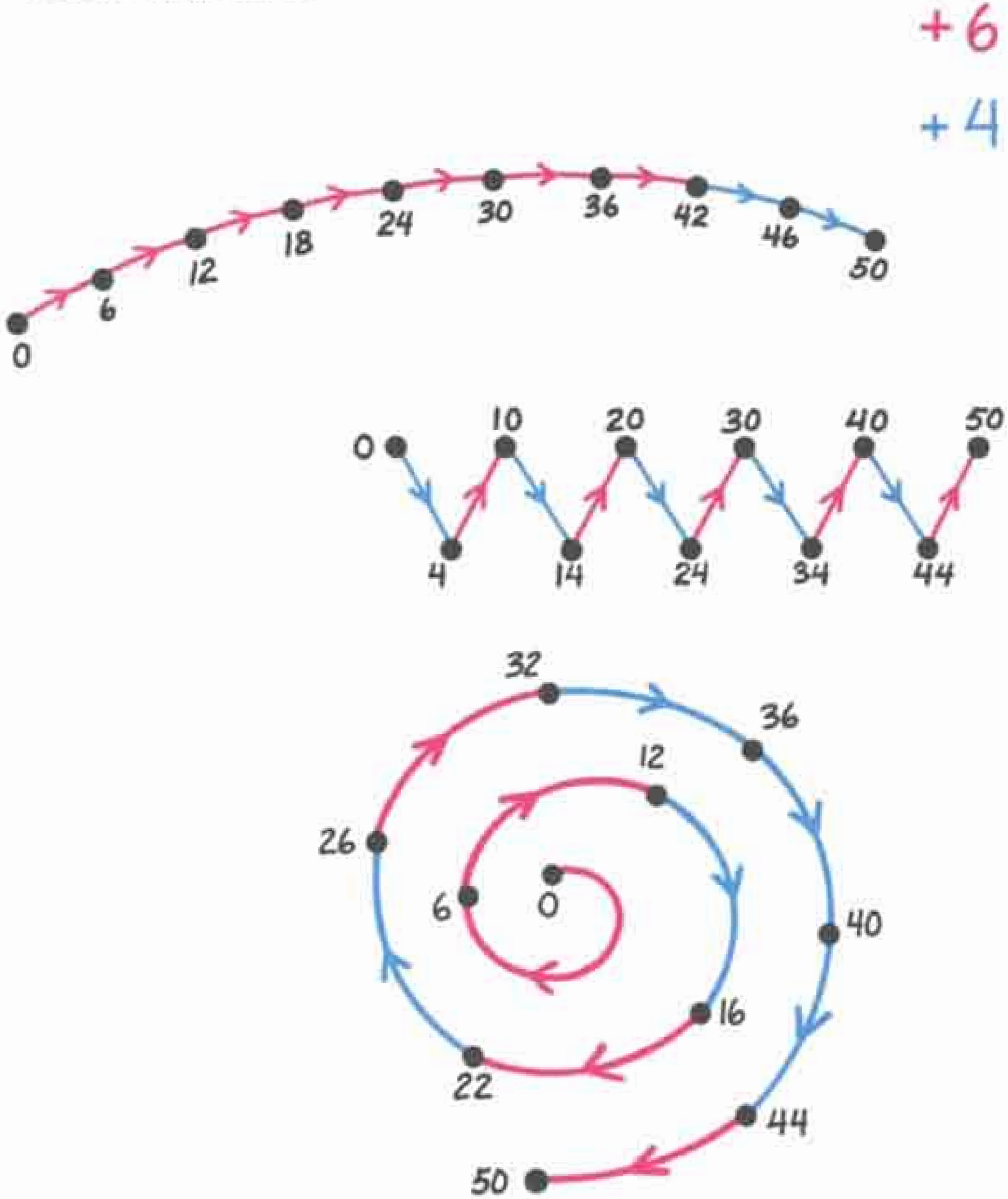
"One of my better students would most likely notice that one pencil and one eraser cost 10¢, and would conclude (since there are five dimes in 50¢) that he or she could buy five of each. But I also think some students would draw pictures," another person adds.

"I think we would all agree that some of our students would need help with the problem," one teacher mentions with concern in his eyes.

"I will tell you how most of our students would approach the problem after one year of CSMP experience." The leader moves toward the board and selects some colored chalk. "A natural solution would be to build an 'arrow road' from 0 to 50, using +6 and +4 arrows." Two large dots are drawn and labeled on the board.

"I would like each of you to try it," she continues, as paper and colored pencils are distributed to all.

After observing the teachers' pictures, the leader asks three people to draw their pictures on the board.



"How many pencils did this person (pointing to the upper picture) buy? How many erasers . . . ?"

"Seven pencils and two erasers."

"And in the other two pictures . . . ?"

"Five pencils and five erasers in each."

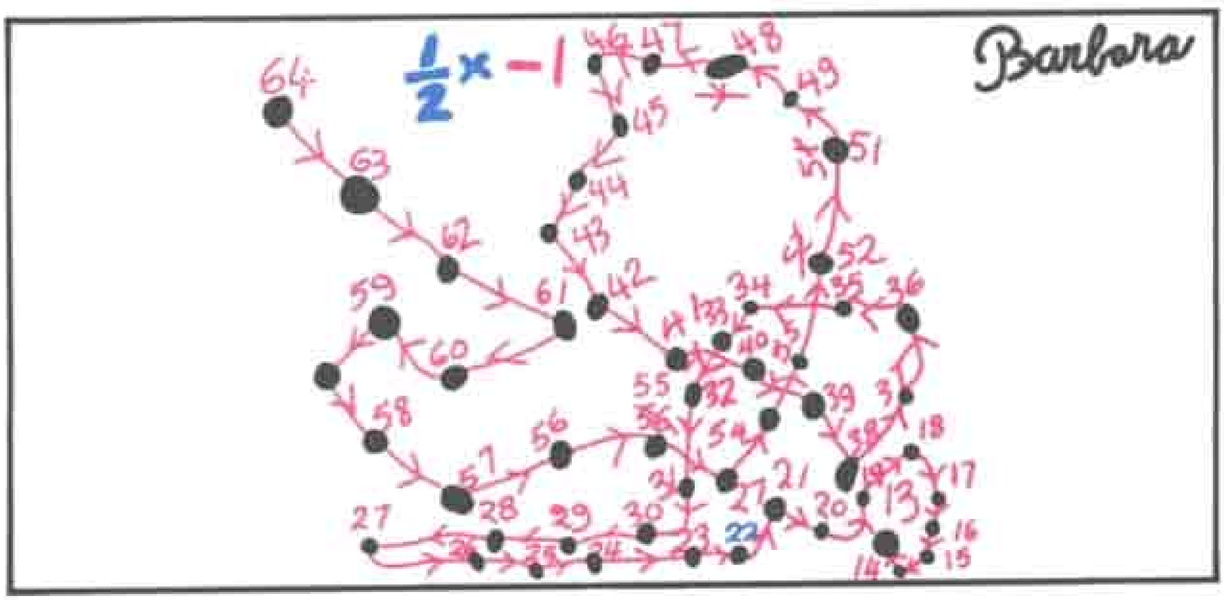
"Each picture portrays a correct solution to the problem, and yet each has its own style. With a class of thirty children you can imagine the variety of answers you would receive. These pictures give you clues about the students' thinking and help you to understand where their difficulties lie."

The leader erases the board and suggests a more challenging problem. "This time, try to build a road from 64 to 13 using $\frac{1}{2}$x and - 1 arrows." The information is displayed on the board.

The room is vibrant with conversation and then becomes quiet as the participants connect their last arrows to the dot for 13.

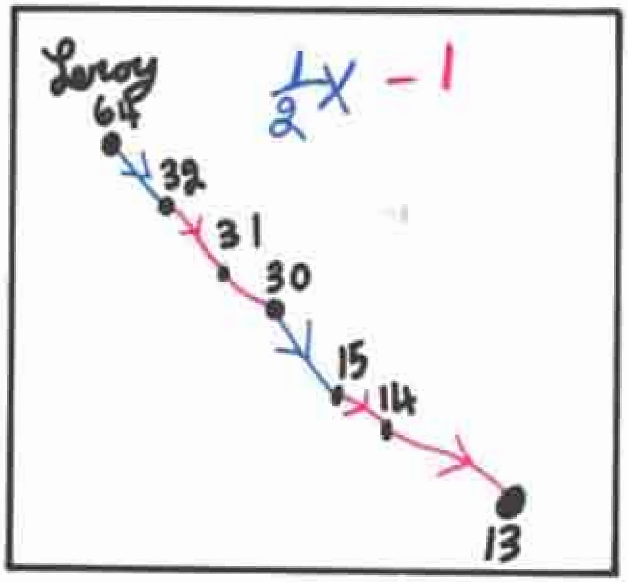
"If you look at the pictures around you, you will probably see a wide variety of solutions. I brought some pictures of this road our students have drawn."

The leader holds up a picture in which red is prominent. "This is Barbara's road."



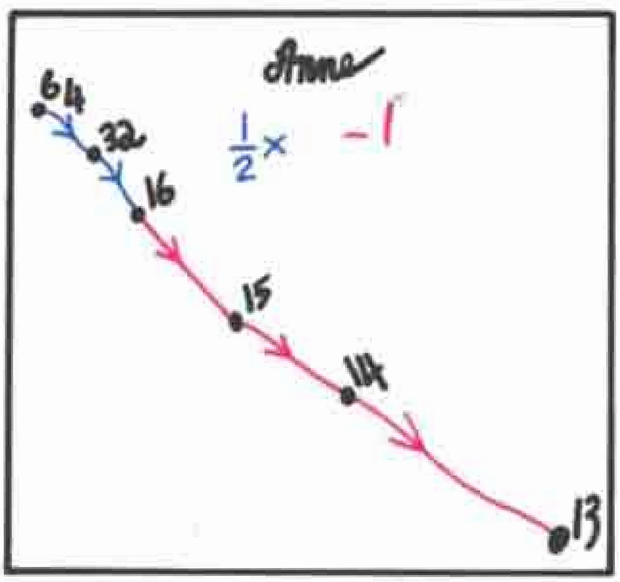
"Barbara is not concerned with finding the shortest solution, but takes pleasure in building a road from 64 to 13 by using just − 1 arrows. Although it is naive, it's a perfectly acceptable solution.

She holds up a second picture.



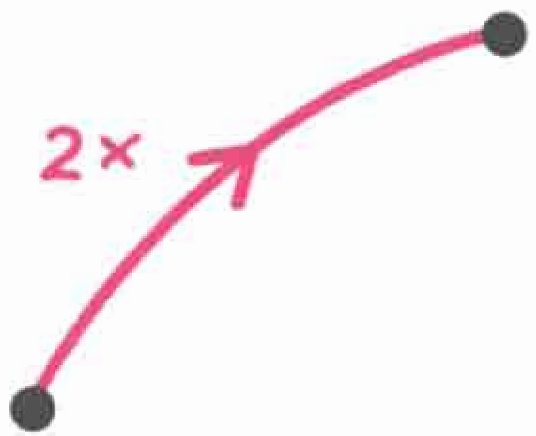
"Leroy prefers a pattern: one blue, two red, one blue, two red. Fortunately, it works!"

A third picture is displayed.

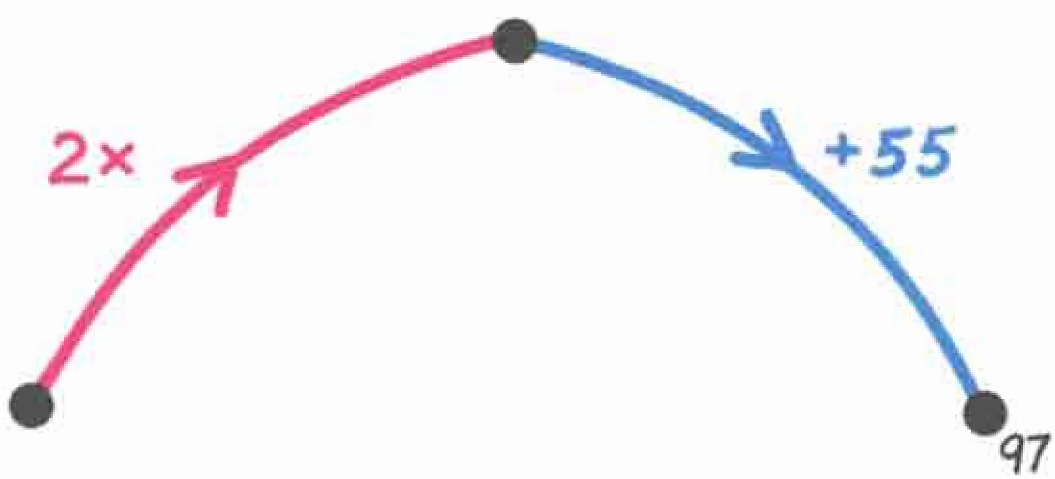


"Anne, on the contrary, finds a minimal solution immediately, taking one half of 64, one half of 32, and then subtracing three 1's."

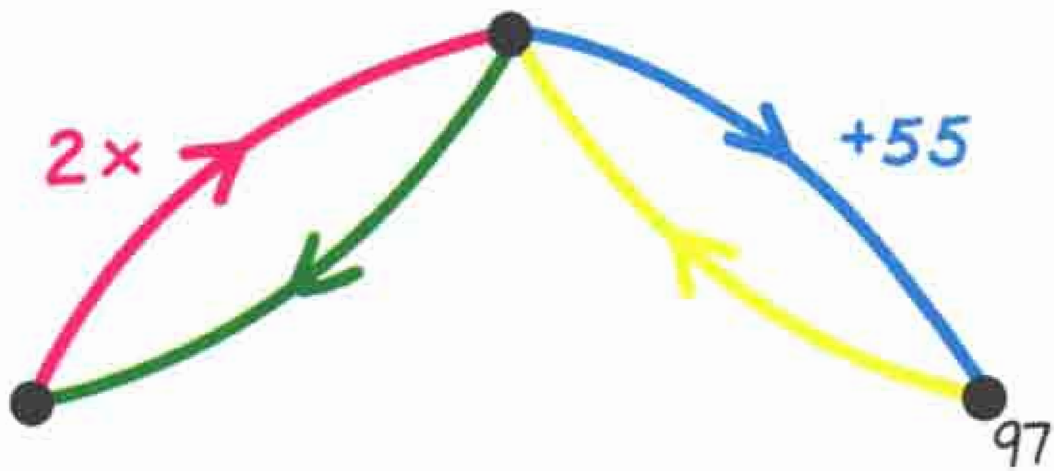
The board is cleared for a new problem. "I will tell you a story about Philip, who went to a fair. Philip bought two bubble pipes. I'm not going to tell you how much he paid for each one. But I will draw a dot for the price of one pipe and by using a 2x arrow, will arrive at the dot for the price of two pipes."



"Next Philip bought a Teddy Bear for 55¢. He spent 97¢ all together." The picture is extended.



"Confronted with this situation," continues the workshop leader, "some of our students suggest drawing return arrows."



"What could the yellow arrow be for?" she asks.

"For - 55," someone volunteers.

"So, the price of two pipes is . . . ?"

"97 - 55 = 42."

The leader labels the appropriate dot "42". "What could the green arrow be for?"

"One-half of."

3a Group Lesson: The String Game

Holding up a small cardboard container, the workshop leader shakes it dramatically. "In this box," she says to the audience, "I have some blocks in different shapes, colors and sizes." She points to someone near the front of the group. "What shapes could my blocks be?" she asks him.

"Squares . . . triangles . . . ?" The teacher nods and draws these two shapes on the chalkboard.

"Circles?" someone else offers.

"Yes. You know all of the different shapes, now. What colors could my blocks be?" Several different colors are suggested, and the leader adds the relevant ones to the list on the board. "The shapes come in just two sizes: 'Big' and 'Little'," she says. "So I will add these to our list, also."

RED	
BLUE	BIG
GREEN	LITTLE
YELLOW	

"You now know all of the colors, shapes, and sizes of my blocks; there is exactly one block of each type. How many different pieces do you think I have?" She shakes the container once more.

"24," is the reply.

"Yes, I have 24 different pieces," continues the leader, removing the lid from the container, and drawing a simple table on the board.

TEAM A	TEAM B

"We are going to play a game with these blocks, a game between two teams. But first, we must share the 24 blocks equally between the two teams." She divides the group into two teams. "Those of you on this side will be 'Team A'," she says, "and over here, 'Team B'."

"Describe a block you would like me to remove from this box, and I will place it on your team's side of this table. Which piece would you like me to put up for your team?" She points to a member of Team A.

"A small triangle."

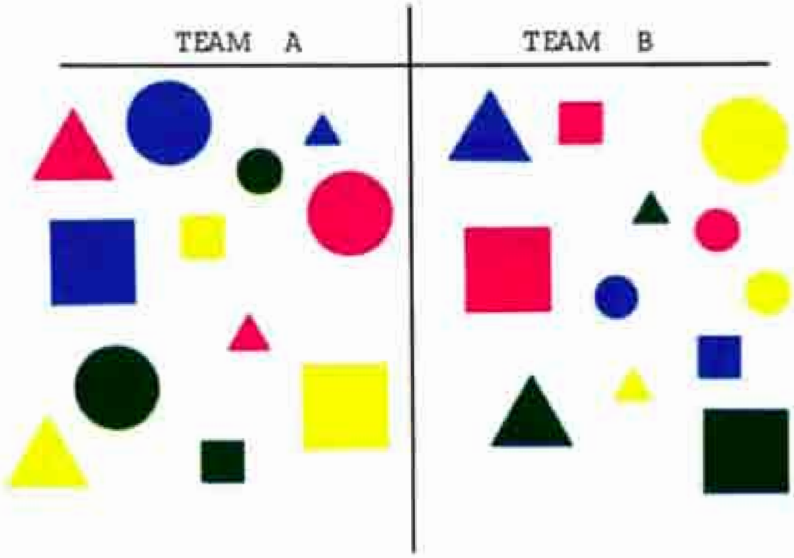
"Yes, but I have several 'small triangles'."

"A small, red triangle," the team member corrects.

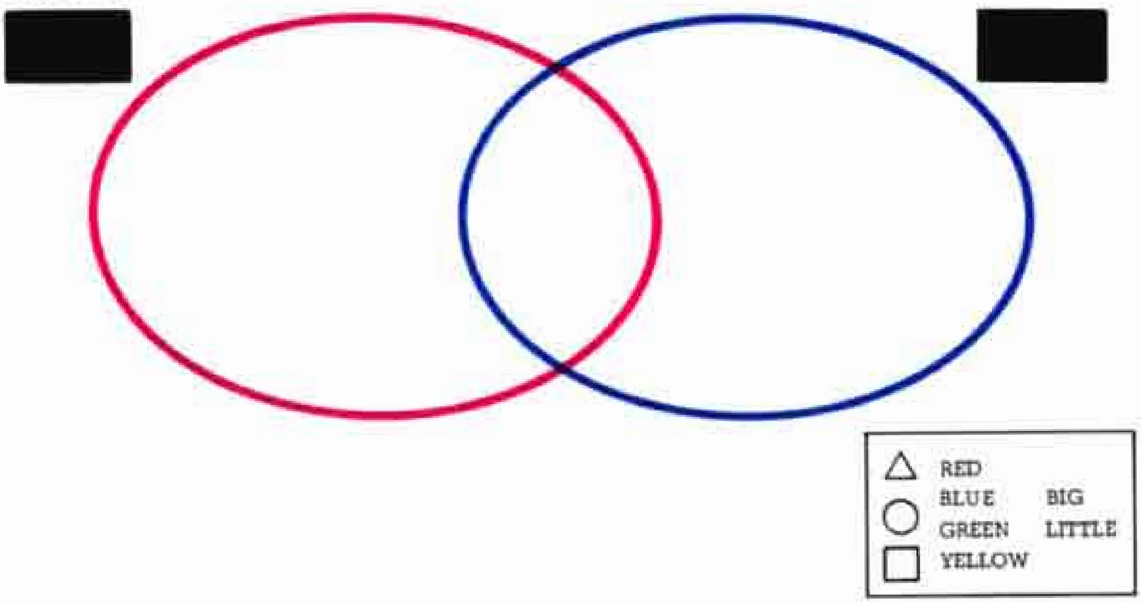
"Yes; that completely describes this piece," answers the leader, placing the piece on the board, where it sticks. "Now, someone from Team B choose a piece."

"A big, yellow circle."

Very shortly, the container is empty and the table on the board is full.

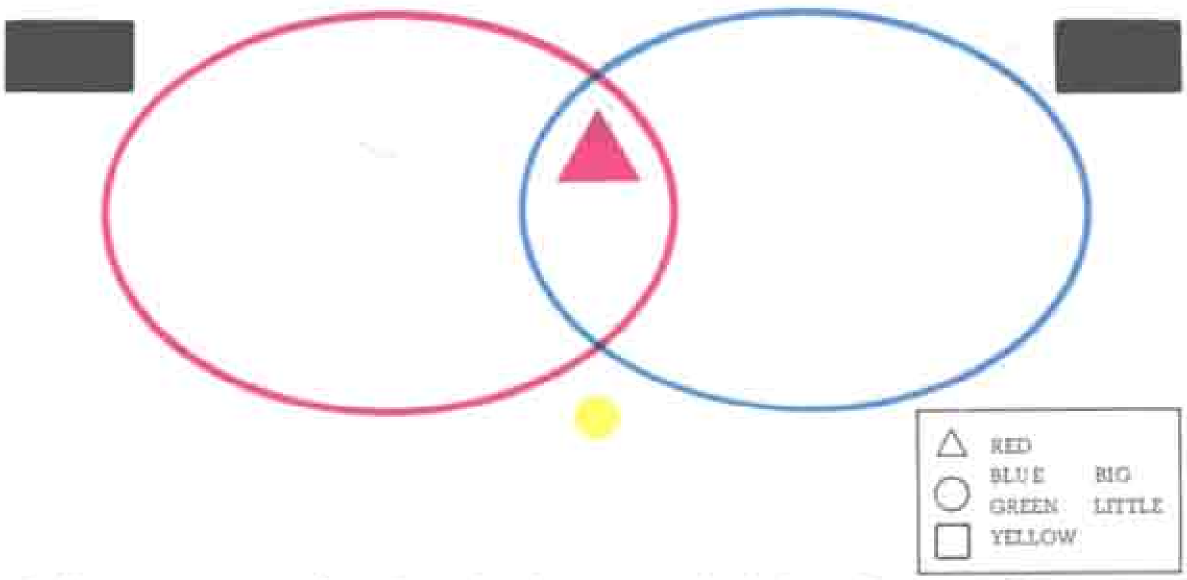


" We are ready to play the game now, I think." Moving to the left side of the chalkboard, the leader draws two large, colored loops, and places a card next to each one.

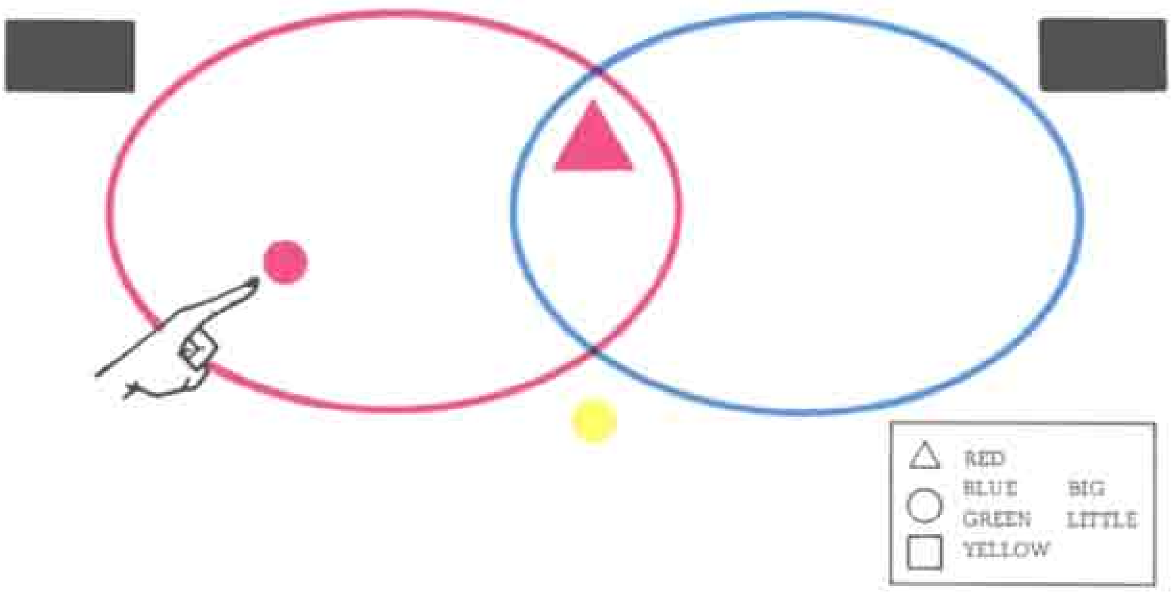


"On each card is written what the string is for," begins the leader. "But for now, this is a secret, so I have put the cards face down. I can only tell you that each card contains exactly one of the nine attributes in our list here; and that each piece belongs in exactly one of the four regions of the picture. Don't forget the region outside both strings."

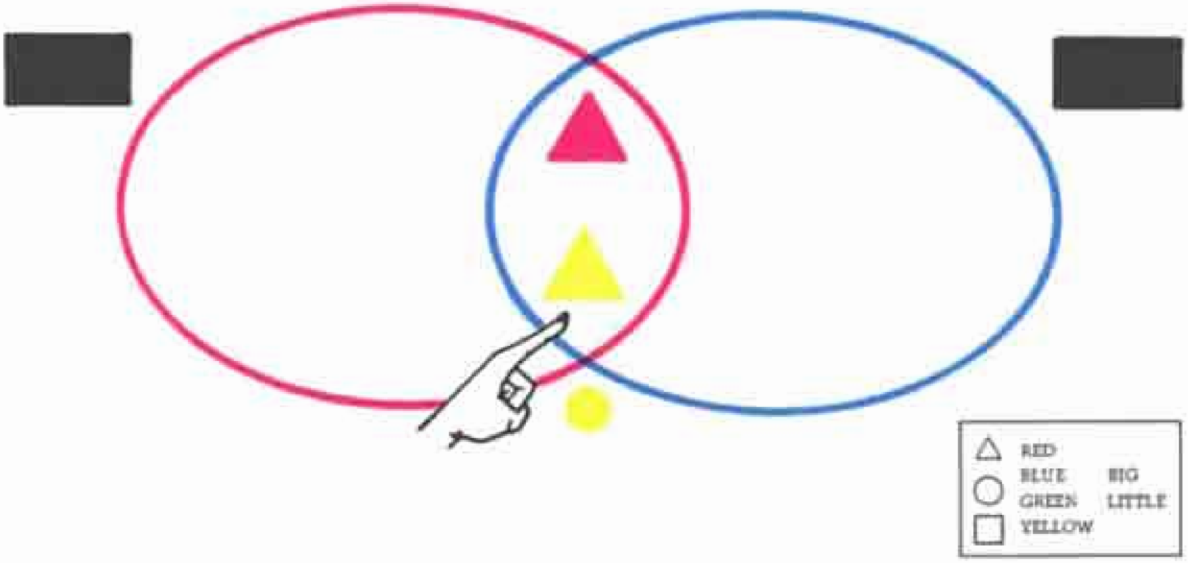
"The object of the game is to identify what is written on each of the two cards," she continues, "but first, you must properly place all of your team's pieces. The first team to correctly place all of its pieces in the picture and identify both string labels will be the winner. To help you get started, I will offer two clues." Taking one piece from each team's set of pieces, she places them in the picture.



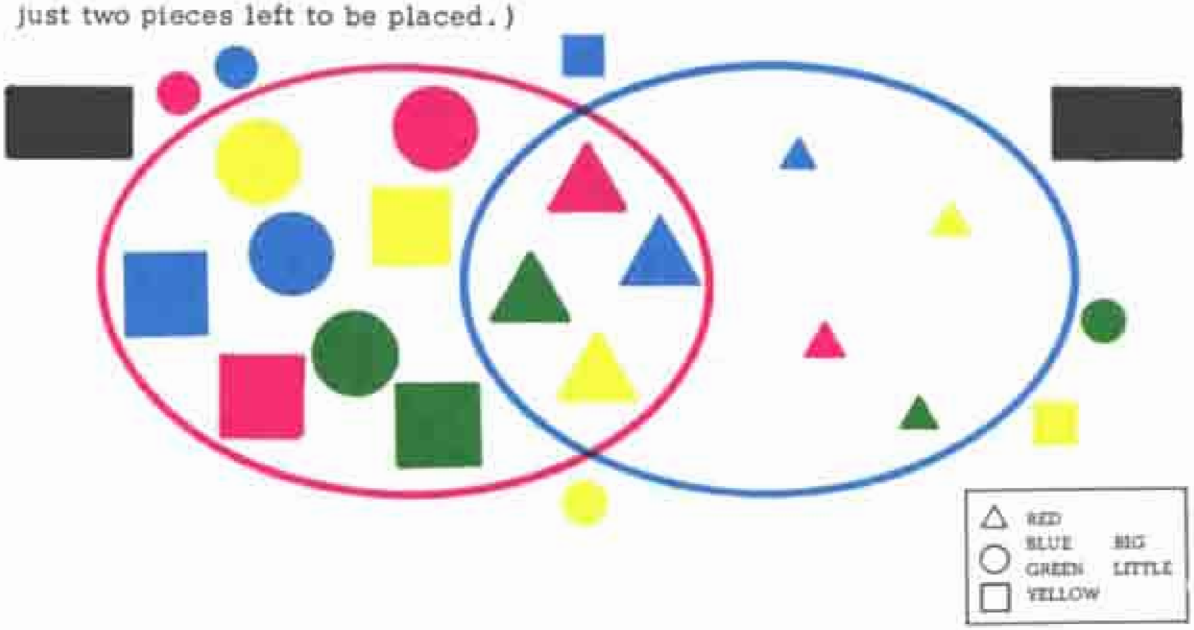
Calling on a person from Team B, she says, "Let's have Team B go first . . . choose any piece from among your team's pieces, and place it where you think it might belong." The team member does so.



"No, that is not correct." As the player replaces the piece in its original position among Team B's pieces, the leader calls on a person from Team A. "Now it is Team A's turn to make a play."



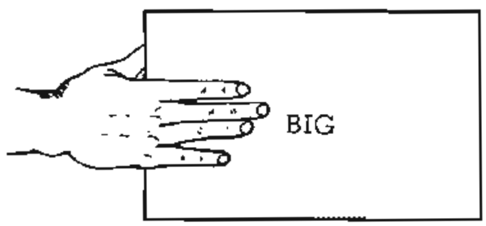
"Yes, that's correct. You may try a second piece; but then, it will be Team B's turn again." Members of each team continue to take turns, until all of Team B's pieces have been correctly placed in the picture. (Team A is not far behind, with just two pieces left to be placed.)

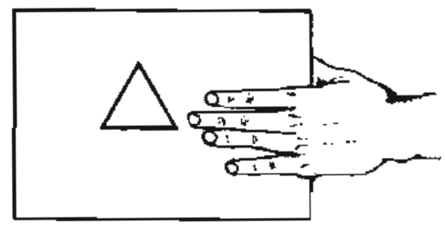


"You have played the last piece for your team," the workshop leader says to the young woman at the board. "Do you know what the strings are for . . . what is written on the cards?"

"The red string is for big pieces," she replies. "And the blue string is for triangles."

"That's correct," says the teacher, reaching to lift the two cards from their places on the board. Smiling, she holds them so they are visible to the group.





"Team B is the winner!" she says simply, removing all of the pieces from the string picture and returning them to the table.

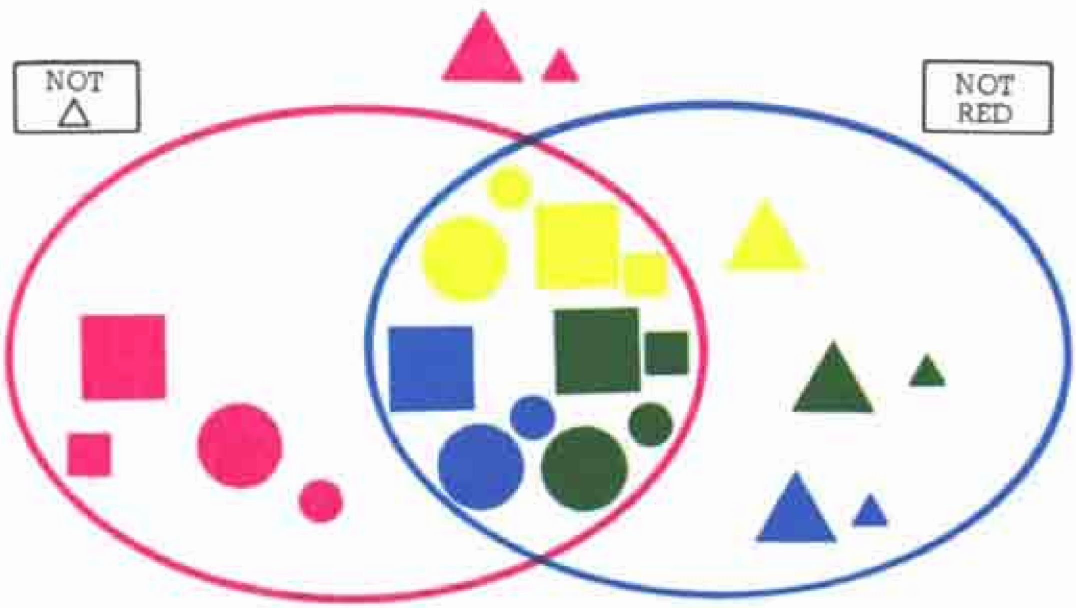
"Let's play this game again and give Team A a chance to even the score. This time, I will make the game a little more difficult." Members of the audience agree eagerly and the leader unfolds a paper poster, taping it to the board.

BIG	GREEN	□	NOT YELLOW
LITTLE	YELLOW	NOT □	NOT GREEN
RED	△	NOT ○	NOT RED
BLUE	○	NOT △	NOT BLUE

"This time, each card may have exactly one of these sixteen attributes written on it. Someone from Team A may go first."

There is a brief discussion of the new possibilities: 'NOT RED', 'NOT △', and so on. Then the leader again gives two starting clues and calls on a member of Team A to make the first play.

Slightly less sure of themselves than before, the members of each team take their turns putting pieces into the picture. Finally, a player from Team A places its last piece and identifies the two string labels correctly.



3b Group Lesson and Discussion: The Language of Strings

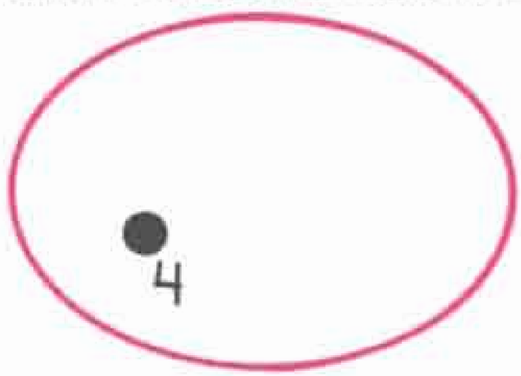
"I enjoyed the game very much! The children must really like it also. Do you play this game often throughout your program?"

"The String Game has a wealth of extensions and variations. Add a third string to the picture and a new level of intricacy emerges in the game. One may also change the string labels to categories such as: 'larger than 50', 'odd numbers', 'multiples of 3', and 'divisors of 27', and play the game with numbers instead of shapes. These variations, of course, are introduced very gradually in our program. For example, students have practice putting numbers in a two-string picture, with the string labels face-up, before they ever play a numerical version of the game itself. This gives them a chance to become more familiar with the ideas of multiples and divisors, and to get a better feeling for the ordering of negative numbers with positive numbers. Later on, the game may become a situation about which to pose questions."

"What sort of question, for example, would you ask students about the game?"

"After the students have played the string game with numbers many times, the teacher begins asking questions about the information to be gained from a single number clue. Let me give you an example."

The leader draws a dot and a red string on the board and labels the dot "4".

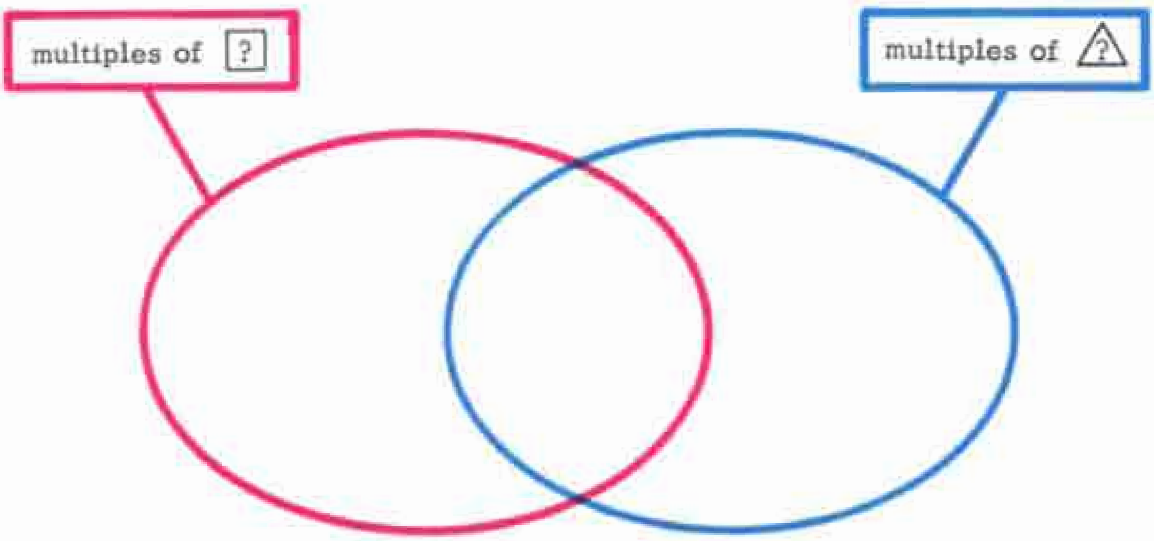


"If 4 is here, could the red string be for the odd numbers?" (No) "For the divisors of 27?" (No) "For the multiples of 2?" (Yes)

The board is erased as the leader continues.

"The language of strings goes beyond the String Game itself to provide a setting in which to ask many interesting questions about numbers. For example, let me pose a problem to you that we might give to our students."

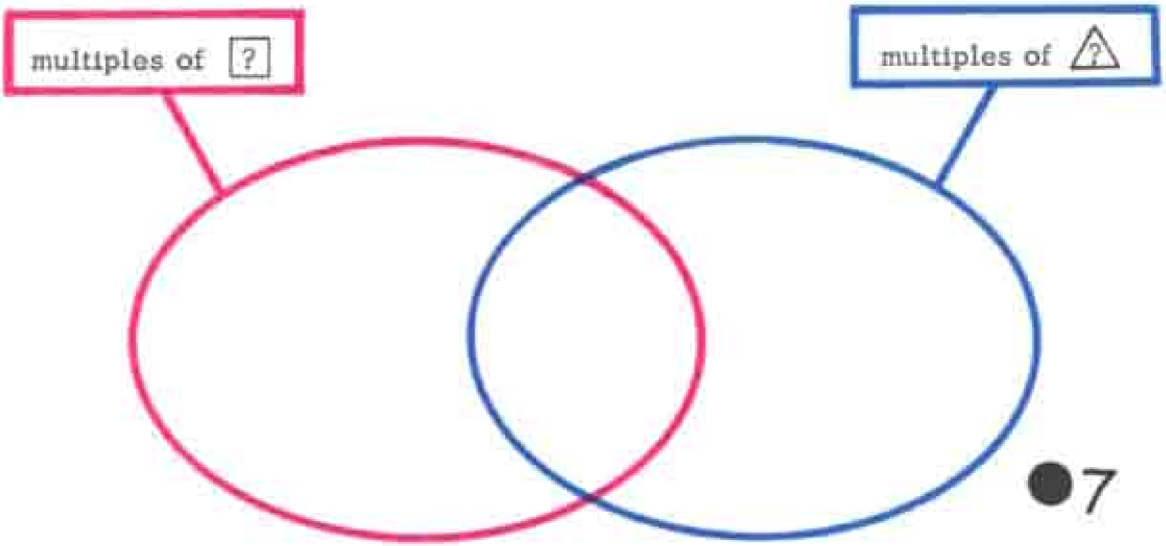
Two strings are drawn and labeled on the board.



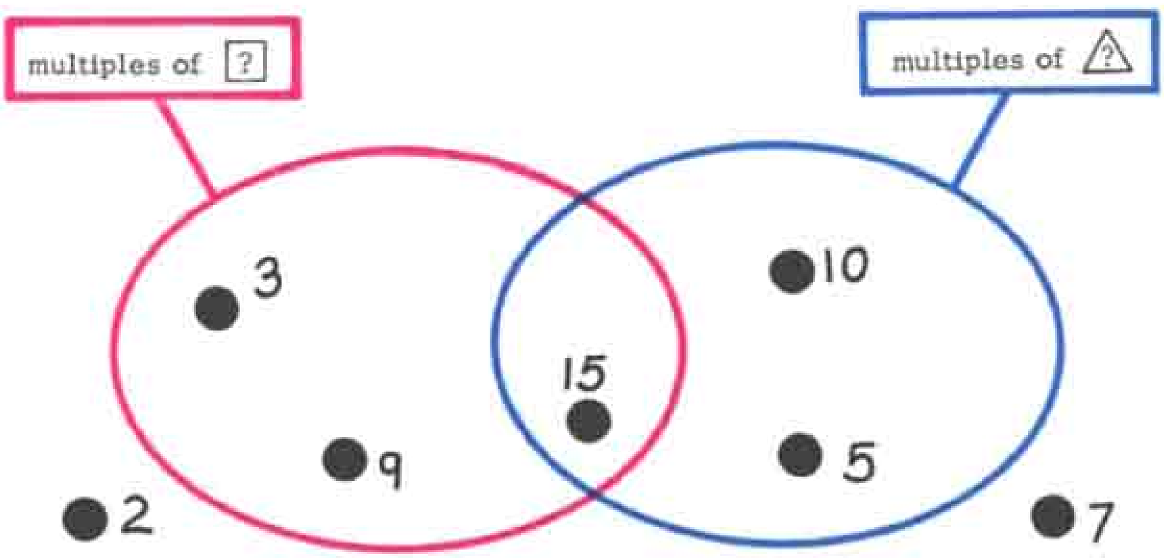
"The red string is for the multiples of some number -- the number hidden in this ' □ ' (box). The blue string is also for the multiples of some number -- the number hidden in the ' △ ' (triangle). Give me some numbers, and I'll put them in the string picture. See how quickly you can guess which number is in the box and which number is in the triangle."

"7."

" 7 goes here. "



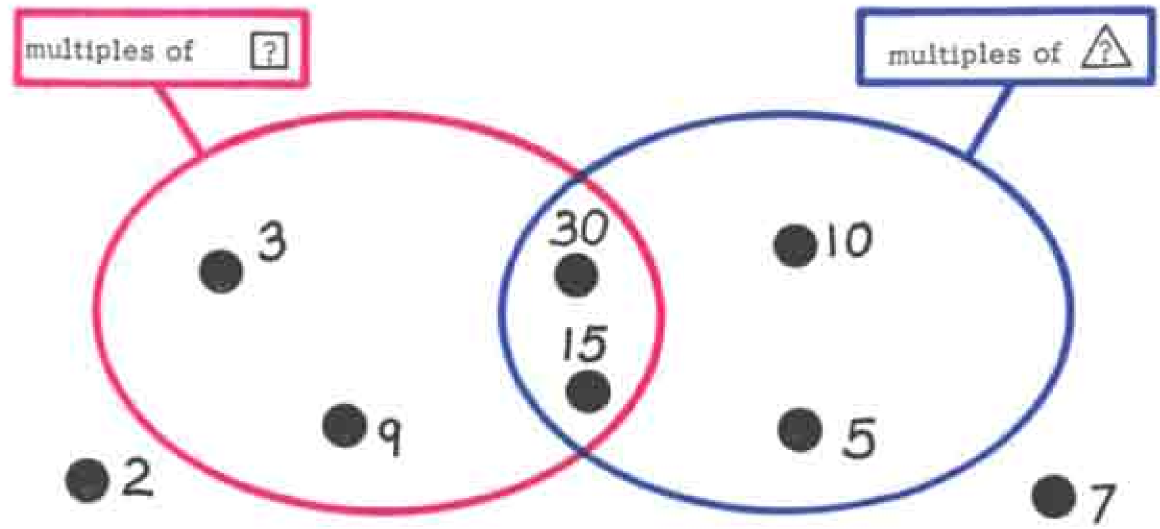
Several numbers are suggested and put in the picture.



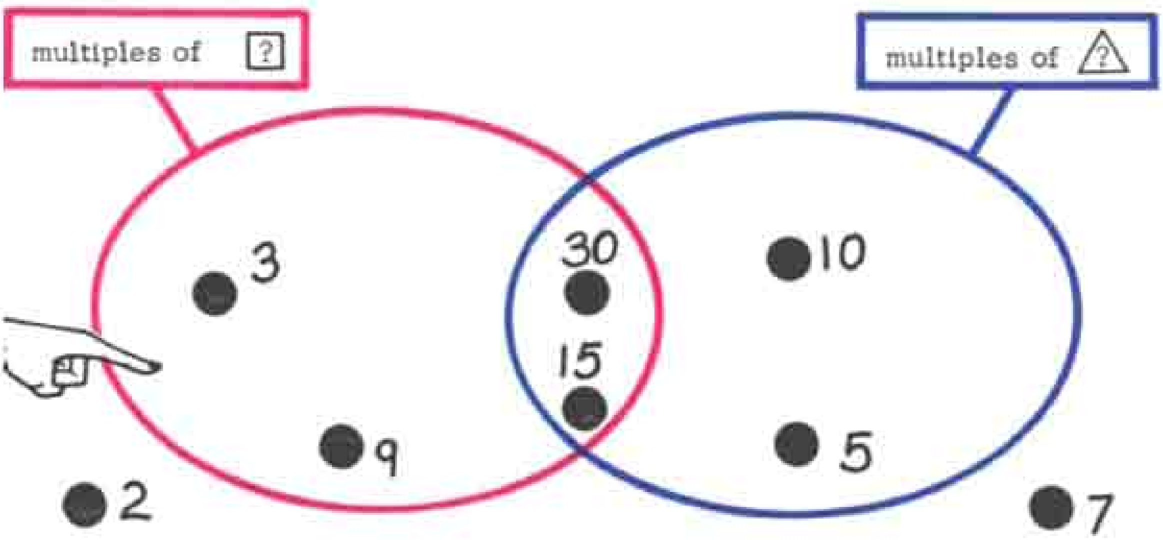
Most of the participants are indicating that they know which numbers are in the box and the triangle.

" Who can put 30 in the picture? "

Someone volunteers, putting 30 correctly in the center region.



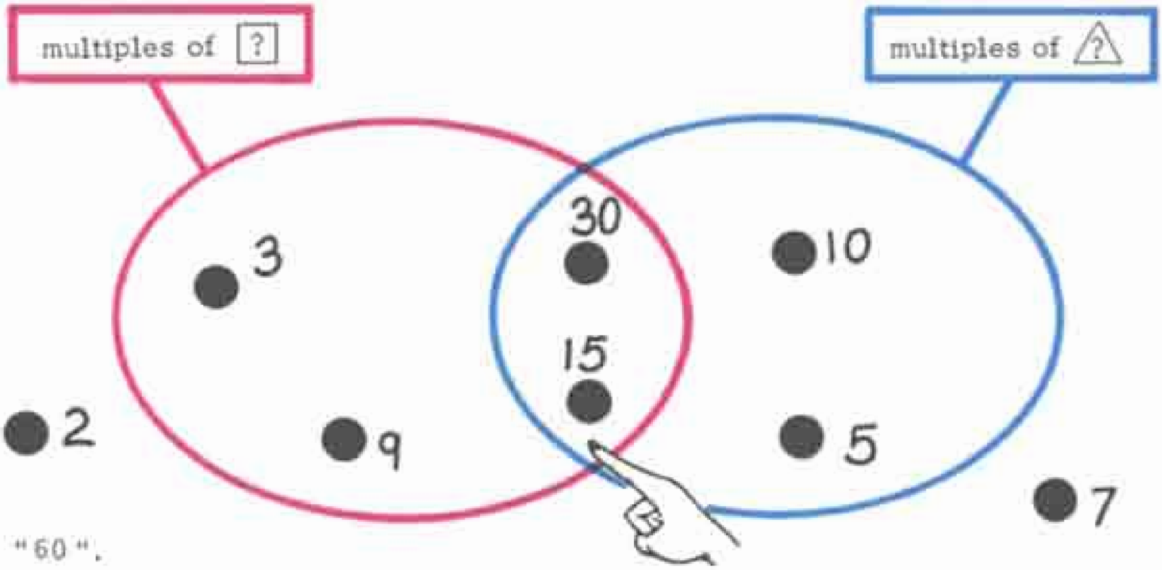
"That's right. Who can tell me another number which belongs here?"



"12".

Several participants are indicating approval.

"Give me another number which belongs here."

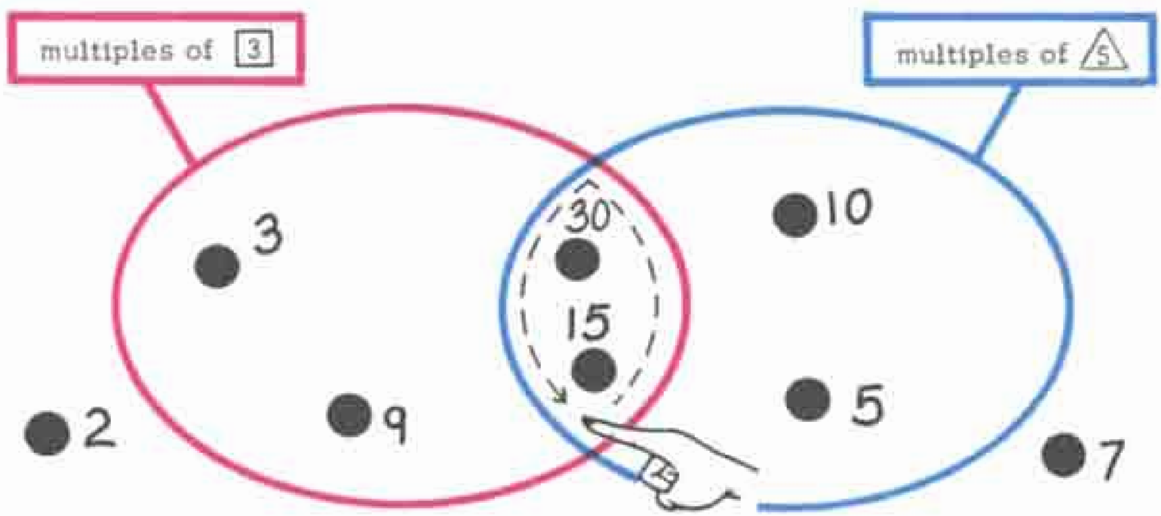


"60".

"What are the strings for?"

"The red string is for multiples of 3 and the blue string is for multiples of 5."

The leader writes "3" and "5" in the box and triangle, respectively, and then traces the middle region with her finger.



"Does anyone know what we could label this region?" she asks.

A volunteer thinks aloud: "45 is a multiple of 3 and a multiple of 5. So 45 would belong in the middle -- is that right?"

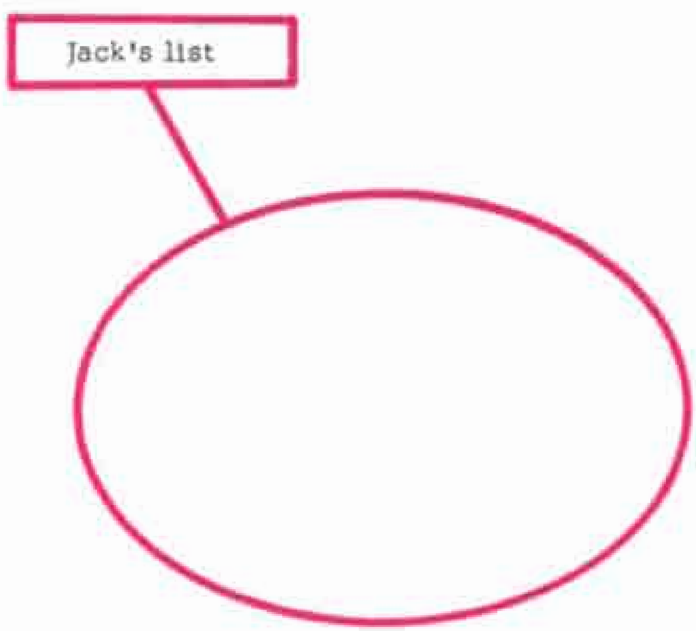
"Yes."

He continues. "The middle region has 15, 30, 45, 60, and . . . " he pauses. "Any multiple of 15."

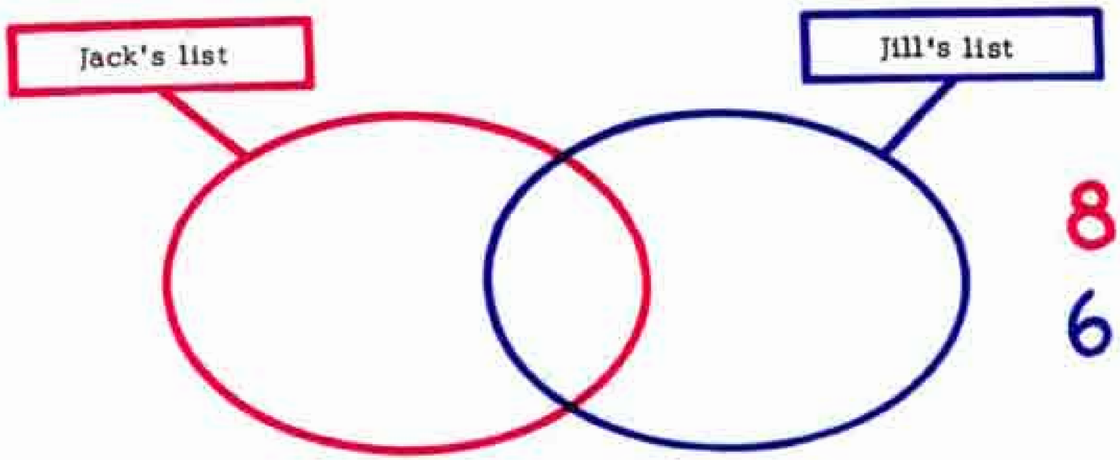
Some participants light up, "Ah!" Others are nodding, "Of course!"

The leader quickly erases the board and asks someone to distribute paper.

"There are other situations where the strings are not for number properties and yet there are numerical problems to be solved. Let me give you an example. Jack and Jill are twins. Their mother asked each of them to write the names of several friends whom they would like invited to their birthday party. Jack invited eight people . . . " (a red string is drawn and labeled on the board)



" . . . and Jill invited six people," (a blue string is drawn and labeled).

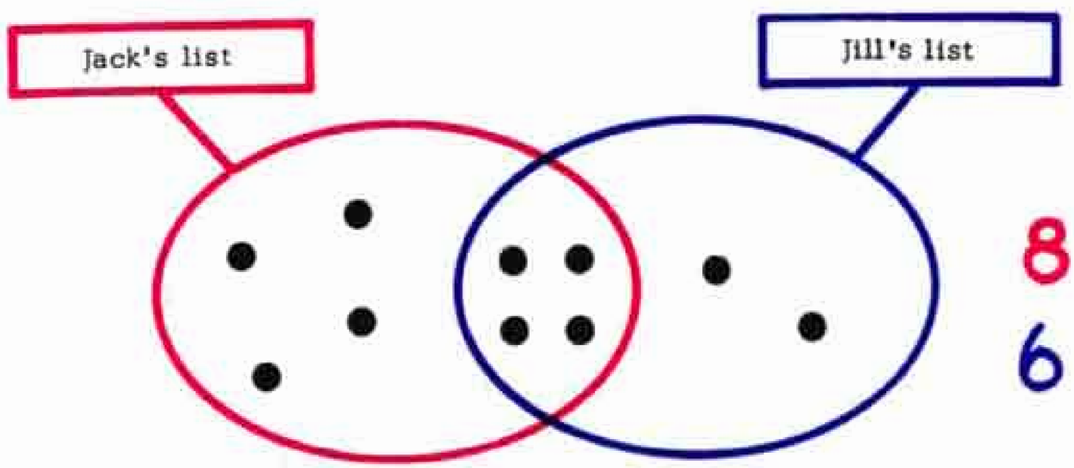


"Everyone who was invited to the party came. There were exactly ten people at the party. Could you draw this string picture on your paper and put dots in the picture for the ten people?"

Participants quickly begin drawing dots, then erasing a lot of dots, and , finally drawing some more.

"The trick is in the middle," one proud teacher shares with those around her.

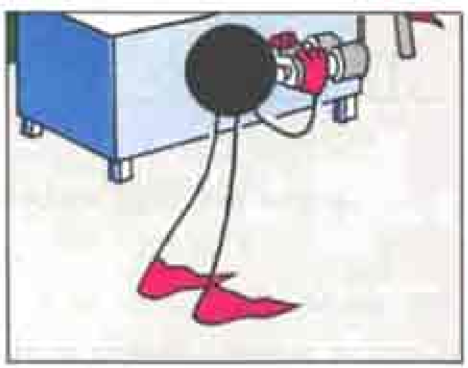
Most of the group solves the problem and the solution is displayed on the board.



"You will see the language of strings again in this workshop in the context of a 'detective story'."

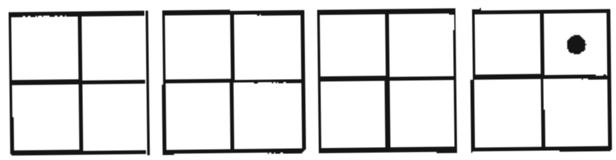
SESSION 4

DETECTIVE STORIES



4a Demonstration Class: Who Is Pif?

"Do you remember the Minicomputer from yesterday?" the teacher asks her young class quietly. "When I put a checker here . . . "



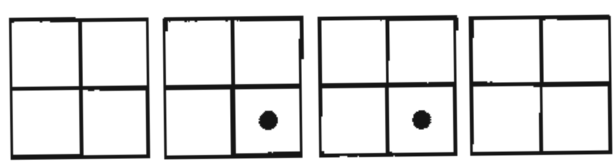
" . . . the number is . . . ?"

"4," replies a young girl.

"And a checker here . . . ?"

"40."

The teacher continues in this vein through a whole series of numbers. "And this number is . . . ?" she says finally.



"110," answers a small boy without hesitation.

"Very good. Now . . . " She pulls a small spiral tablet of paper from her skirt pocket. "I have written a secret number in this little notebook, and all I can tell you for now is that the secret number's name is 'Pif'." She looks from face to face with an expression of mischief. "Which number is Pif, do you suppose?"

A multitude of hands appears waving in front of her. She points to one.

"Pif is 36," declares a freckle-faced youngster, pushing a shock of blond hair from his eyes.

The teacher smiles at him. "It could be 36," she says simply.

"It's 14," a second student says.

"Perhaps it is 14," answers the teacher. "But do you know for sure?"

The student looks away shyly. "No . . . " The rest of the students are undaunted; they each believe they know the secret number.

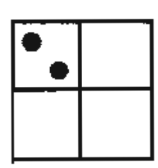
"Ahhh . . . we can guess many different numbers for Pif. But we cannot be sure," the teacher admonishes. Reluctantly, the hands are lowered. "So, I will have to give you a clue to Pif's identity."

She pauses momentarily and then, receiving the students' full attention, she announces the first clue in a stage whisper: "Pif can be put on the Minicomputer using exactly two checkers on the ones' board." Casually, she removes the other three Minicomputer boards, leaving only the ones' board.

"Now which number could Pif be?"

"16?" a student ventures.

"Come show us 16 with two checkers on the ones' board."

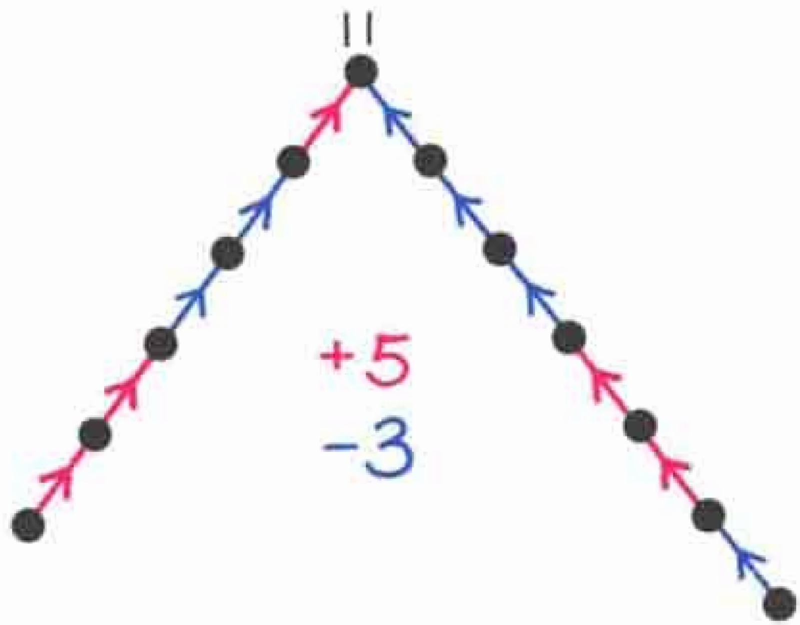


"Yes; very good. It might be 16." She writes this numeral on the chalkboard. "Which other numbers could Pif be?" In about three minutes, a whole list of possibilities has been discovered.

16, 12, 10, 9, 8, 6, 5, 4, 3, 2

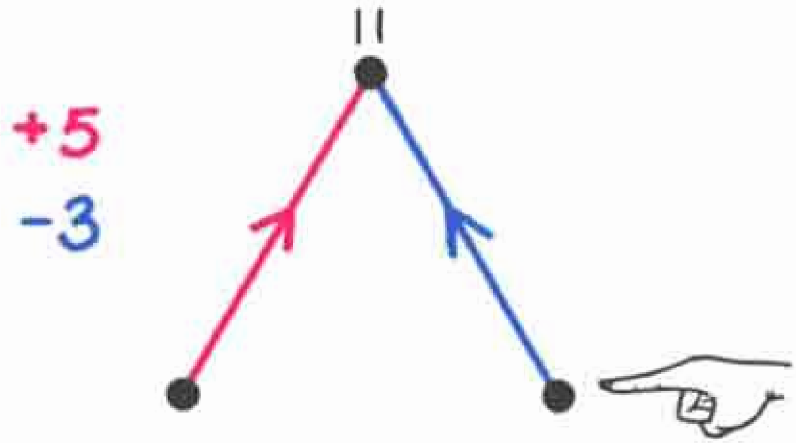
"Now we know that Pif is one of these numbers . . . " the teacher muses. "But which one, I wonder?"

As before, many students appear to be quite certain about Pif's identity . . . much to the audience's amusement. "But do you know for sure?" the teacher presses each time. "Perhaps I should give you another clue." She turns and begins to draw a picture on the board.



"This arrow picture is much like the one you saw yesterday," she explains. "But today, the dots are not for children; they are for numbers. The red arrows are for + 5 and the blue arrows are for - 3."

She leans toward the class. "Now . . . here is the clue: Pif is one of the numbers in this arrow picture." She straightens. "But first, we must discover which numbers are here. Who knows which number this is?" she asks, pointing to a dot near the top of the picture.



The students frown in concentration. "8?" one suggests.

"The blue arrow is for - 3," the teacher says gently. "Does 8 minus 3 equal 11?" The student shakes his head. "Then it cannot be 8," she concludes. "Yes?"

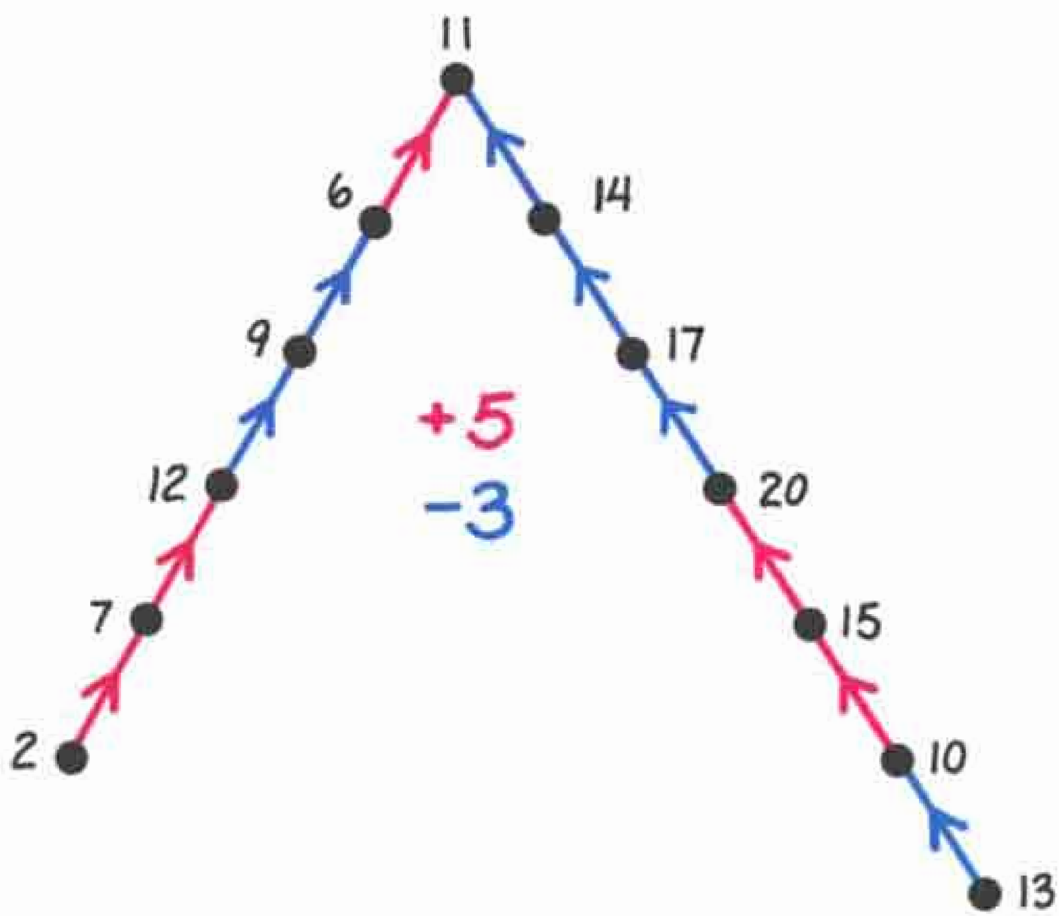
"14," a second child suggests.

"Does 14 minus 3 equal 11?" the teacher asks. "Yes; very good." She labels the dot "14". "Now, what is this next number?"

"17," a youngster responds. One-by-one, they continue until all of the dots in the picture have been labeled.

"Remember, I said that Pif is in this arrow picture somewhere. Let's look at our list again. 16 is in our list . . . but is it in this arrow picture?"

16, 12, 10, 9, 8, 6, 5, 4, 3, 2



"No," answers a girl near the front of the class.

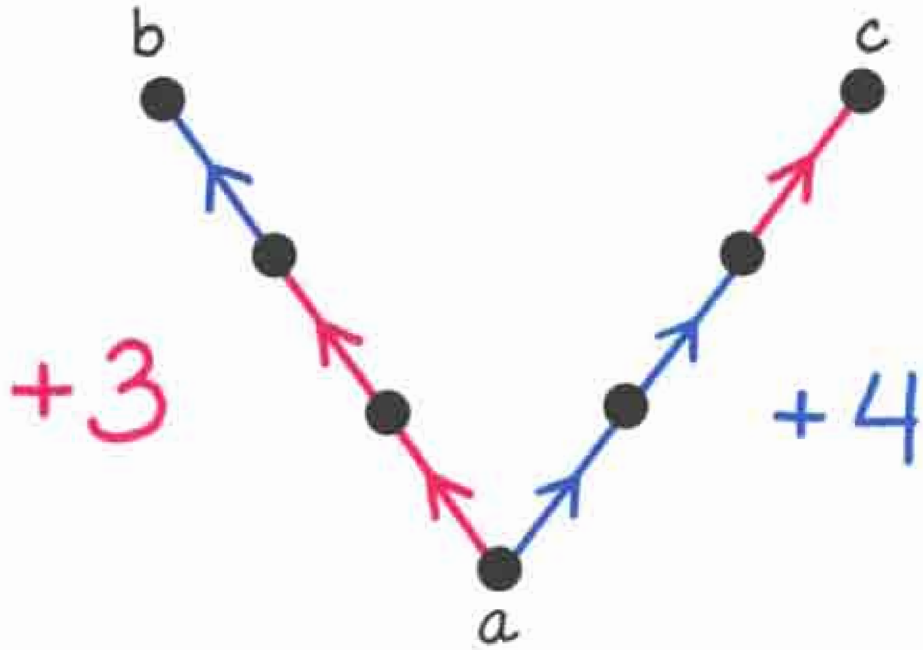
"Then Pif cannot be 16," the teacher says, crossing "16" off of the list. "Can Pif be 12?"

"Yes," is the reply. Within a few moments, all but five possibilities have been eliminated from the original list.

~~16~~, 12, 10, 9, ~~8~~, 6, ~~5~~, ~~4~~, ~~3~~, 2

"Now," the teacher says as she erases all but the list from the chalkboard, "we know that Pif is either 2 or 6 or 9 or 10 or 12. I will give you another clue to help you discover Pif's identity." Once again, she takes the colored chalk in hand and carefully draws an arrow picture.

(The teacher's drawing does not actually include the letters "a", "b", and "c"; they are included here for the reader's convenience in following the progress of the lesson.)



"Pif is also in this arrow picture," the teacher continues. "But this drawing is more mysterious . . . do you know any of the numbers in this arrow picture?" Frowning at the chalkboard, many of the students shake their heads. "Well then . . . can you tell me where the smallest number in the picture is?"

The students think quietly about the situation. Then, one of them comes up to point to a dot (labeled "a").

"Yes," replies the teacher. She draws the attention of the students to two other dots (labeled "b" and "c"). "Do you know which of these two numbers is the larger?" she asks. The students are thoughtful but do not answer. "Let us look carefully at this picture," the teacher continues after a moment. "How much larger is this number (b) than this one (a)?"

"Seven larger," a student volunteers.

"No, I do not think so," the teacher answers gently. Tracing the arrows (between a and b) in turn, she says, "plus 3 . . . plus 3 . . . plus 4 is . . . ?"

Many students wave their hands excitedly this time. "Plus 10 --- 10 larger," one of them answers, and the teacher agrees.

"Now, how much larger is this number (c) than this one (a)?" she asks.

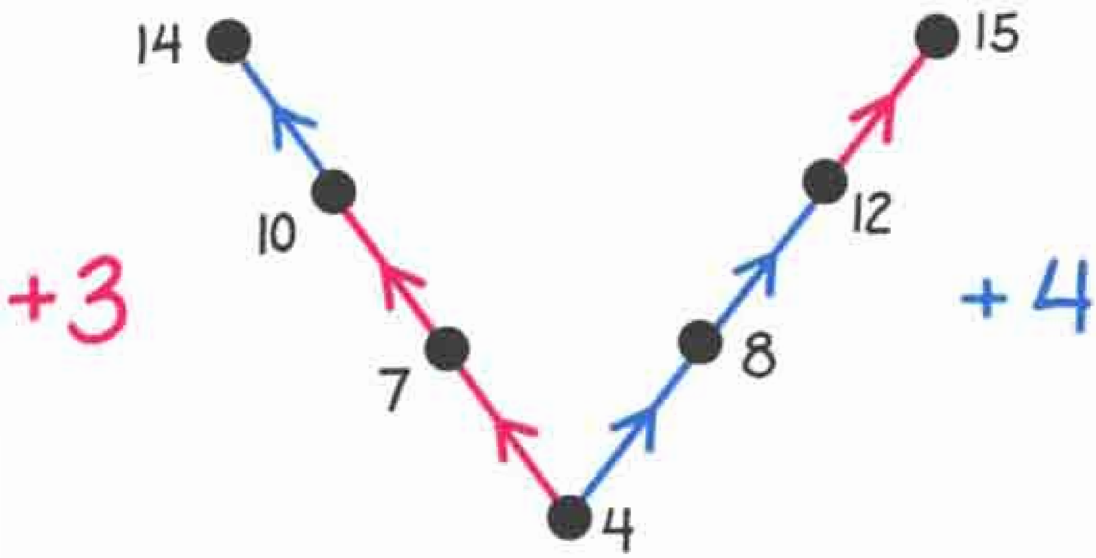
The students are silent for only a moment. She nods to one, who answers, "11 larger."

"How do you know?" the teacher asks.

"Because plus 4 and plus 4 and plus 3 is plus 11," the child answers.

"Yes, it is 11 bigger," the teacher agrees, and holds her finger to her lips for the group's attention; children and adults alike strain forward in quiet anticipation. "Here is the next clue. Pif and his friend, 15, are both in this arrow picture," she continues at last, "and 15 is the largest number in the arrow picture." She pauses for a moment to let this information be absorbed.

"Where is 15 in the arrow picture?" she inquires. A child skips up to point to 15's dot (labeled "c"). The teacher writes "15" next to this dot and, one by one, the students give labels for all of the dots.



"Remember, Pif is in this arrow picture somewhere," the teacher says. "The number 2 is in our list . . .

12, 10, 9, 6, 2

" . . . but is it also in this arrow picture?"

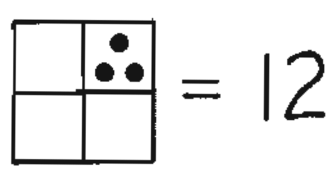
"No --- Pif cannot be 2," the students are quick to answer, and the teacher crosses "2" from the list. Together, they continue to check the other numbers until there are just two numbers remaining on the list.

12, 10, ~~9~~, ~~6~~, ~~2~~

"Now, I will give you one more clue, and perhaps you will know who Pif is." The teacher erases the arrow picture and turns to the ones' board of the Minicomputer. She picks up three checkers and holds them aloft. The group waits.

"Pif can be put on the ones' board of the Minicomputer with exactly three checkers, all on the same square. Who is Pif? Do you know my secret number, now?" she asks, eyes bright.

The class erupts with a flurry of hands. "Shh," she says, "whisper the secret number to me." She bends down, first to one student, then to the next. "Yes . . . yes . . . yes . . . " she says in turn. Pointing to one of the last students, she instructs, "Show us Pif on the Minicomputer --- with three checkers on the same square."



Some of the children begin to clap, but the teacher cautions them. "We know we can put 12 on the Minicomputer this way . . . but are you sure that Pif cannot be 10?"

"I tried all the squares in my mind," replies one student shyly, "but I couldn't show 10 on the Minicomputer that way."

"Ahh," the teacher says, "then Pif cannot be 10. But let us check to make sure that 12 is truly Pif, the secret number." She holds up the little notebook and looks slowly from face to face. With a flourish, she flips back the pages to reveal the number written therein.

"Hooray!" shout the children and the audience breaks into applause.

4b Group Lesson and Discussion: A Detective Story About Paf

"The students responded very enthusiastically to this morning's lesson. Are these detective stories a regular feature of your program?"

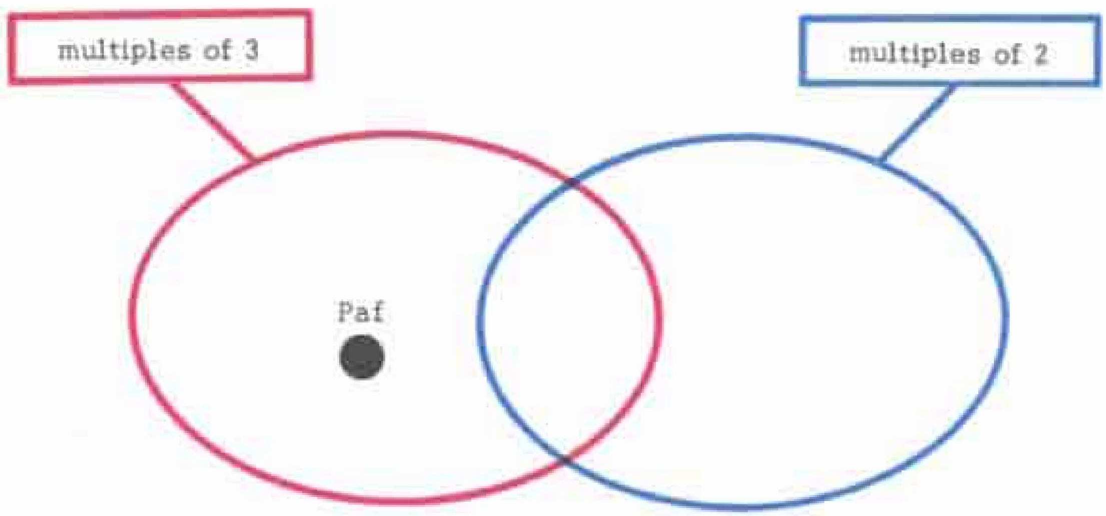
"Yes; the students' response to detective stories, in itself, would provide sufficient basis for their frequent use. But beyond this, the format is ideal for reviewing many concepts because one can employ any combination of the mathematical languages. And each story can present these concepts in a slightly different context. In the course of a detective story, a student is naturally confronted by a large number of calculations and by the necessity to analyze a mathematical situation carefully. Thus, each experience is not only new and unique; each is also a review and reinforcement of past insights."

"But do the students continue to respond eagerly to detective stories? What happens to the slower students?"

"During the course of several school years, we have observed a correlation between our use of these stories and the deeper involvement of students in mathematical thinking. This is markedly true for the "slower" student, from whom the detective stories generally elicit a greater response during collective lessons. The device of providing clues introduces an element of suspense which, in turn, arouses a powerful motivation to learn to use and be fluent in the mathematical languages. To support this motivation, we choose the clues very carefully. In each detective story, the clues and their pacing are designed to provide an opportunity for each student in the group to participate and to contribute to the solution.

"I would like to see another detective story so that I can measure it against what you have just said."

"Yes, of course, and you should have the experience of solving this kind of problem in order to appreciate the students' involvement. I have another detective story for just that purpose. My secret whole number is Paf, and Paf is in this string picture on the board. Do you know who Paf is?"



"What information does the picture give us about Paf?" hints the leader.

"Paf is a multiple of 3 and is not a multiple of 2," answers one of the participants.

"Paf could be 9," replies another.

"Or 27."

"Or 3."

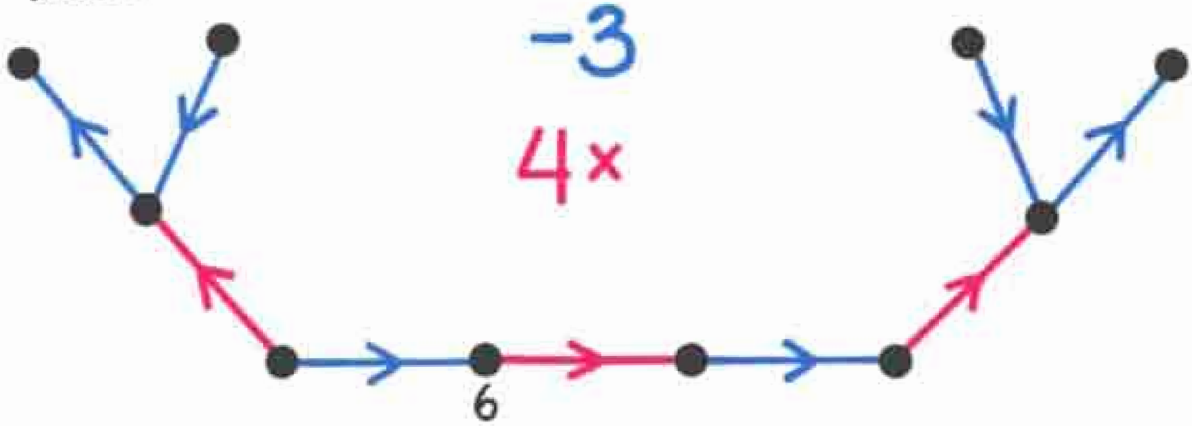
"You just add six each time. 3, 9, 15, 21, and so on."

"Paf could be any odd multiple of 3."

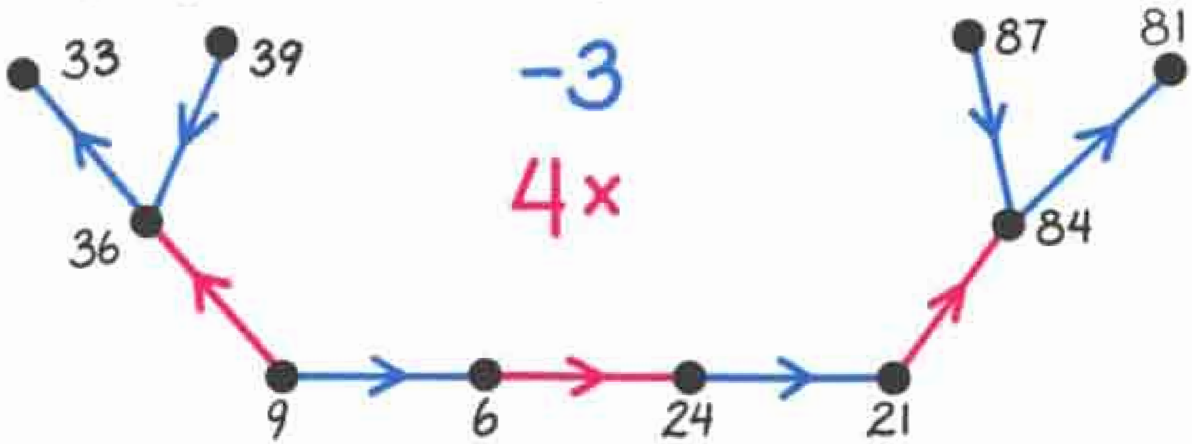
As these suggestions are made, the leader lists on the board the numbers that Paf could be.

3, 9, 15, 21, 27, 33, 39,
45, 51, 57, 63, 69, 75, 81, ...

"The second clue is given by this arrow picture. Paf is one of the numbers in this picture."

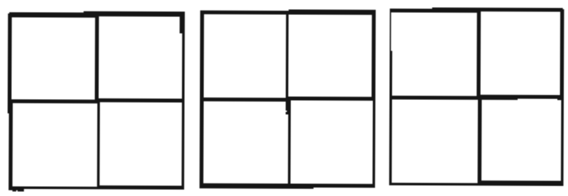


The participants collaborate in labeling the dots.



Comparing the list of numbers on the board with those in the picture, the group concludes that Paf could be 9, 21, 33, 39, 81, or 87.

Before the third clue is given, the workshop leader displays three Minicomputer boards.



"Paf cannot be put on the Minicomputer using exactly two regular checkers."

The participants quietly study the list of numbers that Paf could be. From the expressions on their faces, it is clear that most of them are mentally putting each number on the Minicomputer.

"Paf could be 87, because you need at least four checkers to put it on the Minicomputer," concludes one member of the group.

"Paf cannot be 9, because 9 can be put on the Minicomputer using two checkers," observes another.

"Paf could be 39."

"Or 33."

The participants agree that 9, 21, and 81 must be eliminated from the list, and that Paf could be 33, 39, or 87.

"And now for my last clue," announces the leader as she writes this information on the board.

$$3 \qquad + \qquad 6 \qquad \times \qquad 5 \qquad) \; ($$

"A name for Paf can be written using each of these symbols (and no others) exactly once.

One after another, participants come to the board and write names for six different numbers.

$$(3+6)\times 5 = 45 \qquad (3\times 6)+5=23$$

$$(3+5)\times 6 = 48 \qquad (3\times 5)+6=21$$

$$(6+5)\times 3 = 33 \qquad (6\times 5)+3=33$$

"Paf is 33," several participants proclaim simultaneously.

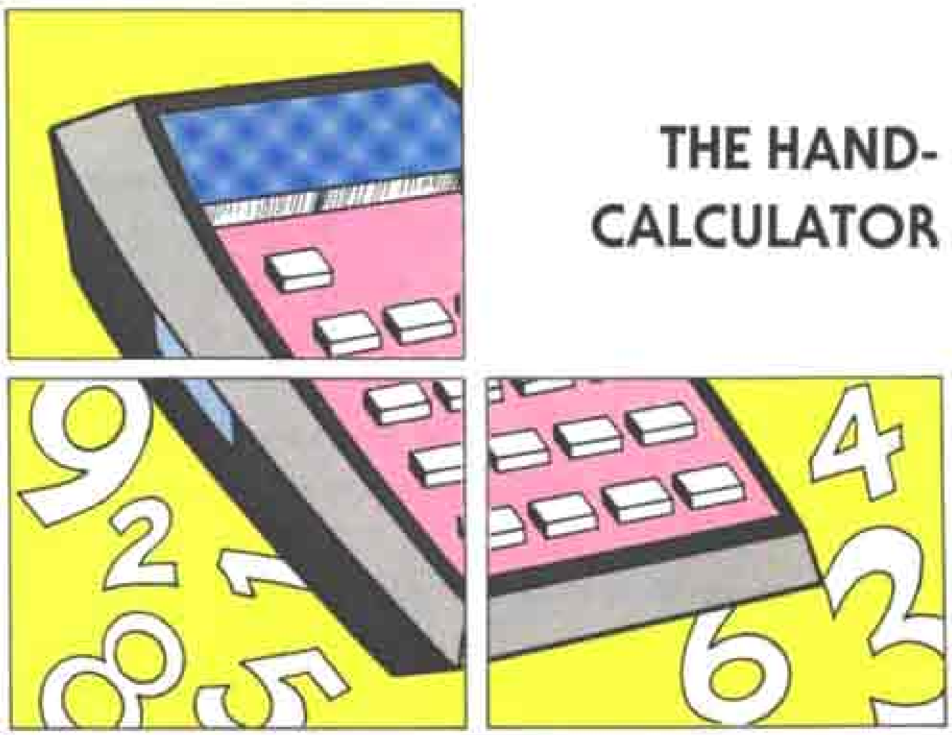
"This is a typical fourth grade detective story. The clues involve the students at several levels. The first clue requires them to analyze a string picture and test for divisibility by 2 and by 3. The clue can also call into play the students' ability to detect and utilize patterns.

"The second clue provides not only a chance to practice multiplication by 4 and subtraction by 3, but also an opportunity to use the return arrows for −3 and 4x.

"The third clue asks for analyzing the negation of a statement and reading numbers from the Minicomputer.

"The last clue calls on students' mental arithmetic ability and on their ability to think combinatorially to determine the complete list of possibilities."

SESSION 5



THE HAND-CALCULATOR

5a Introduction: The CSMP Viewpoint

It is now obvious that recent technological advances (along with dramatically lower costs) have combined to make the hand-calculator readily available to nearly everyone. Somewhat less obvious are the uses to which this potentially powerful tool might be put in the classroom setting, particularly in teaching elementary mathematics.

Clearly, a great deal of research and development must yet be done to provide answers to such questions; the surface has barely been scratched. We believe it worthwhile, however, to relate some of the program's efforts to employ the hand-calculator in teaching mathematics, if only because these efforts appear to be quite different from avenues currently being pursued by others.

In essence, CSMP has utilized the hand-calculator primarily as a pedagogical tool --- rather than as a computational device. Persons familiar with the program's other "non-verbal" languages (strings, arrows, etc.) will appreciate how the calculator itself may also become a "language" . . . through which to speak about interesting mathematical situations, to work with them, and to arrive at some level of insight into these situations. In fact, in developing lessons for its Intermediate Grades Program, CSMP explored a number of situations which would have been impossible to present clearly without the use of the hand-calculator.

This distinction in CSMP's approach to using the hand-calculator is not a trivial distinction, as will be seen through examples of specific activities which follow.

Regardless of approach, however, the increasing proliferation of the hand-calculator seems certain to have a dramatic impact on mathematics education, particularly in revealing the patent absurdity of rote drill and memorization of arithmetic facts (upon which has been placed a recently renewed emphasis); not to mention the current obsession with mastery of standard algorithms for long multiplication and long division.

In our opinion, lengthy repetitive calculation does not constitute a worthwhile occupation of intelligent minds . . . especially when this calculation is performed more quickly, more easily and more accurately by a tool : the calculator. One can only speculate on how mathematics education might have evolved, were such devices readily available centuries ago, freeing the human mind to pursue more valuable --- and more human --- avenues of reasoning.

In effect, we now have the opportunity to choose those mathematical activities which are truly deserving of our time and energy --- and of our children's. A few examples of such activities are portrayed on the following pages.

5b Group Lesson and Discussion: The Hand-Calculator

"Perhaps the most productive way to relate some of our experiences with children and the hand-calculator," begins the workshop leader, "is to let you experience them yourselves." Moving among the group, she distributes calculators, until every member of the group is holding one.

"As I describe some of the activities we have conducted," she continues, "it would be helpful for you to follow along on your own machines. For example, after turning on the calculators, and allowing five or six minutes for the students to explore the machines freely, and become familiar with them . . . I might say: 'Hide your displays!'" She demonstrates, covering the top of her calculator so that its display is not visible.

"Now," she says slowly. "Listen carefully. Press [5] [x] [7] [-] [6] [=]. What number should be on your display?"

"29," is the reply.

"Yes," she continues. "Let us check and see if that is what we have." The members of the group do so. "Now, everyone has 29? Good; you may watch your displays now. Press [+] [5] [=] [=] [=] [=]. What number is on your display?"

"49."

"Yes; now hide your displays. Press [=] [=]. What number should be on your display?"

"59."

"Yes; why? How do you know?"

"Because the calculator is counting by fives," a group member replies. "49; 54; 59."

"Very good," says the leader. "Check to see that you have 59, and then hide your displays once again. You may now press [=] as many times as you need to, to reach 79. When you think you have it, you may look." She pauses. "You have it? How many times did you press [=]?"

"Four times."

"Why?"

The person is thoughtful for just a moment. "Because 4 x 5 = 20, and 59 + 20 = 79."

"If you were to press [=] [=], etc., many times . . . would you ever meet the number 117?" A young man near the rear shakes his head. "Why not?" she asks him.

"Because as long as you are counting by fives, the numbers will always have either 4 or 9 for the last digit."

"Let's check that," the leader continues. "Starting with 79 on our displays, let's press [=] [=] · · · (and so on)." Soon, the group members see the numerals on their displays jump from "114", to "119".

"You were correct," says the leader. "We did not meet 117. Now, press [=] once more . . . " Everyone does so. "And we have 124 on our displays. Press any keys you like, as many times as you like, until you have 200 on your displays. Try to remember your method of solving this problem, so you can tell us how you did it."

"Yes? What was your method of reaching 200?" the leader asks after a pause.

"124 [+] [76] [=] 200."

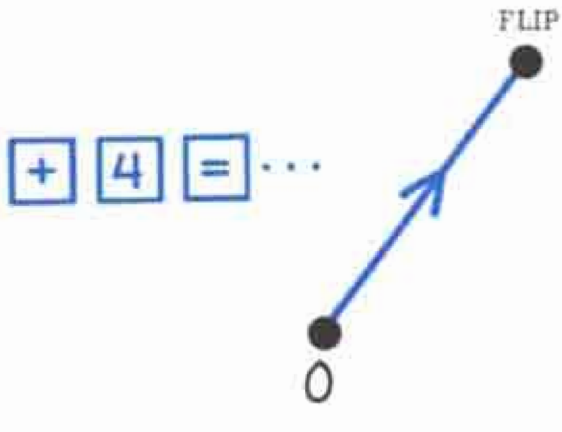
"Yes, of course. And for the students who are just beginning this activity, that is the most common response. But as I tell them," she says, "it is not necessarily the most <u>interesting</u> method of reaching 200. Another student might tell us that she pushed [+] [5] [=] [=] [=] . . . (15 times), which gave her 199; then, [+] [1] [=] 200."

"To encourage the children to be more creative in their strategies, I might suggest on one day, that they again put 124 on their displays, and find a new way of reaching 200 --- by pushing [x] at least once. On another day, I might suggest that they use [÷] at least once, in reaching the goal; or that they also use decimal numbers which are not whole numbers in finding a solution. In this way, the students do not restrict themselves and very quickly begin to discover very imaginative solutions, as you might expect."

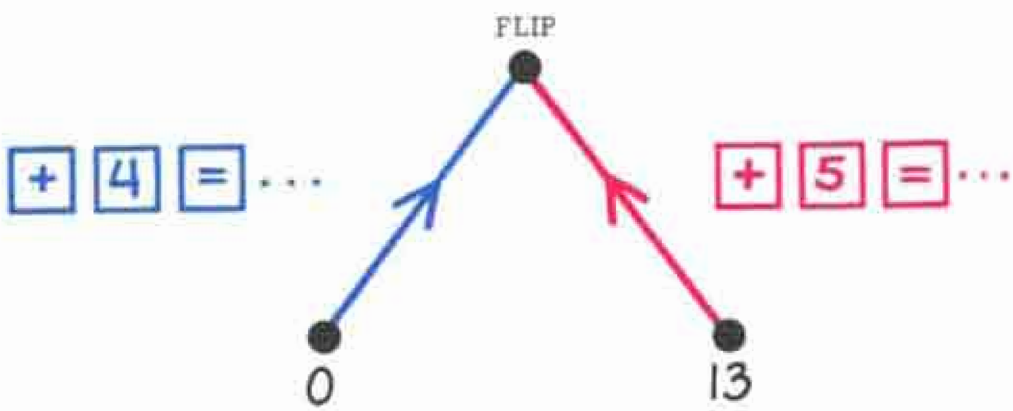
"For example, one student, starting with 124, pressed: [x] [2] [=] (248); [÷] [4] [=] (62); [+] [38] [=] (100); [x] [2] [=] (200)." She looks from face to face as if to emphasize her point.

"These activities we have been doing would constitute a very good warm-up for working with the hand-calculator. We might now continue with something new: a detective story, about a secret number called 'Flip'."

"To help you discover Flip's identity, I will give you these clues. Starting with 0 on my calculator, I press [+] [4] [=] [=] · · ·; and after pressing [=] a certain number of times, Flip appeared on my display." The workshop leader turns and draws this picture on the board.



"Also," she continues, "starting with 13 on my calculator, I press [+] [5] [=] . . . ; and Flip again appears on my display." She adds a second arrow to her picture.



"Which numbers could Flip be, do you think?"

"28," is one answer.

"48," is another. "And 68."

"88 . . . " offers someone else. "And 108."

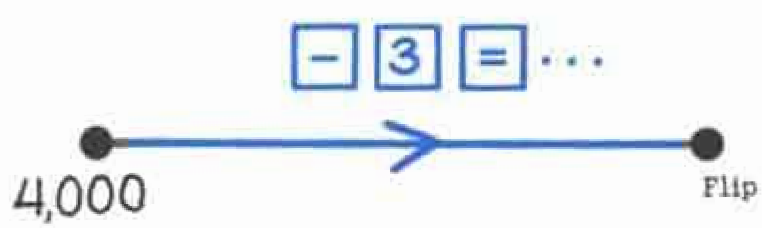
"And 128, 148, and so on," interjects the leader. "How quickly you find the pattern! And how methodical you are. In working with the students, however, the matter is entirely different. At some point in the discussion, a student might say, 'Flip could be 88.' Then, another student might suggest, 'It could be 64'; (he has forgotten the clue for the red arrow, which would not be satisfied by 64). Next: '1,408', and '28', and '228 and 2,002,228!'"

"In other words," the leader explains, "the students are not methodical in exploring the possibilities suggested by the clues; the numbers they suggest seem to change in size, at random. Occasionally, the suggested number is incorrect, as one or the other clue is momentarily forgotten. And the students seem determined to move quickly to the larger possibilities --- the little numbers are too 'babyish', they feel!"

"The understanding teacher allows this spontaniety and makes the most of it, only later forcing the group's thinking to converge on the pattern: 'What is the smallest possible number?' (28) and 'What is the next possible number?' (48), and so on, until the pattern is recognized.

The members of the audience nod in understanding, and the leader continues with the detective story. "So, we know that Flip could be 28, 48, 68, 88, and so on. The next clue we would usually offer is not presented in the language of the hand-calculator; for this reason, I will not introduce the clue here. Suffice it to say that the second clue limits the possibilities for Flip to 28 and 48."

"Here is the final clue, then, to help you discover which number is Flip; you may use your calculators, if you wish." She draws this picture on the board.



"Now," she says, facing the group. "Who is Flip?"

After a brief pause, several of the members of the group raise their hands. "Flip is 28," one of them answers.

"How do you know?"

"Because 4,000 is one more than 3,999 --- a multiple of 3. If you keep subtracting 3, you will always get a number which is one more than a multiple of 3. 28 is such a number, but 48 is not."

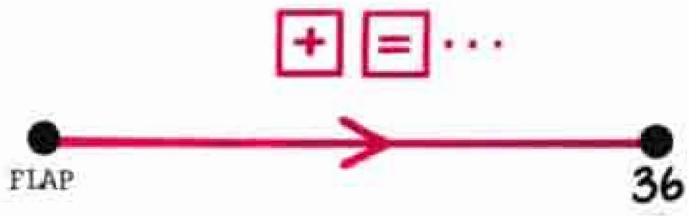
"A very good explanation," says the leader. "Did the rest of you use similar methods? Good."

"Perhaps you would find it interesting to learn how some of the students have used this clue to discover Flip. Their methods show, rather dramatically, how much differently they think about numbers, having had some experience with the hand-calculator."

"We presented this detective story in a small group, with four slower students (grade 5), and just after this final clue was given, one of the students announced, 'Flip is 28!' and he explained: 'I started with 4,000 and pressed [-] [54] [=] . . . , until I got 4; then I pressed [+] [3] [=] . . . and got 28, but not 48.' I asked him why he had chosen 54 and he replied that 'it is a multiple of 3 and doesn't take so long'!"

"Another student said that she started with 4,000, pressed [-] [3] [=] . . . , and got 3,928, but not 3,948; 'so I know I'll also get 28 and not 48,' she said."

The smile of the workshop leader reflects the amusement of the group. "Without exploring the entire detective story," she continues, "let me show you another clue in the language of the hand-calculator, which can be used as a first clue, because it generates a large number of possibilities. By the way, this is appropriate for students in grades 4 - 5." Erasing her previous drawings, she puts this new one on the board.



Starting with the secret number 'Flap' on my display, if I press [+] [=] . . . , '36' will appear on my display. Which numbers could Flap be?"

Very quickly, the group has offered this list of possibilities . . .

18, 12, 9, 6, 4, 3, 2, 1

. . . all of which, as someone points out, are positive divisors of 36.

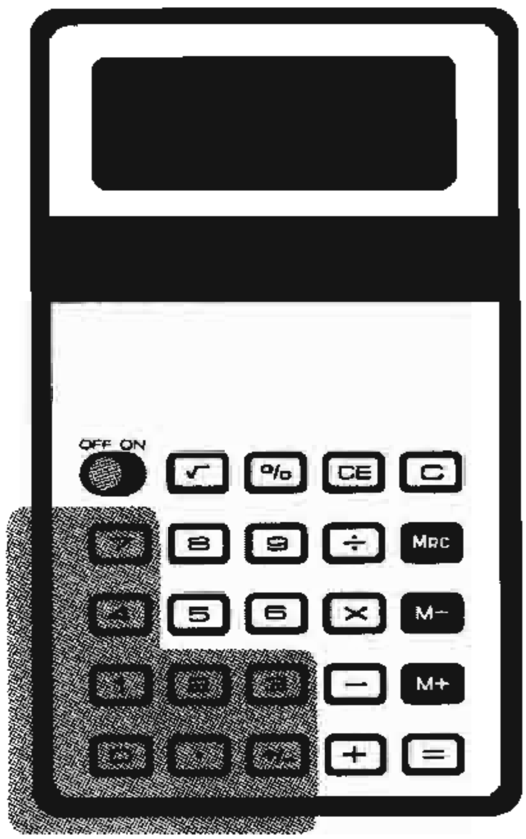
"Have we found all of the numbers that Flap could be?" asks the leader. "What about decimal numbers which are not whole numbers?"

"Of course," answers one of the group members. "Flap could be 0.2, or 0.4."

"Or 0.5, or 0.25" someone else offers. "Or 1.2 or 1.8."

"Yes," says the workshop leader. "As you see, there are quite a few possibilities here --- more than we could list!" Anticipating the group's question, she continues. "The chief advantage of using this clue is that it offers students a chance for experiencing the decimal numbers in an informal way. Even the student who is not skillful with decimals, through the support of the hand-calculator, can gain a great deal of valuable experience with decimal numbers."

The workshop leader turns to erase the board. "We will end the workshop with something different," she says. "Using strips of masking tape, we can modify our calculators, limiting the digits which are available to us." After distributing pieces of tape, she sketches this picture on the board.



"Simply tape your calculator like this," she says. "Now try this problem: put 400 on the display of your calculator by pushing any of the remaining buttons you wish."

A hush comes over the people as they begin; but they soon erupt in nervous laughter when they realize the surprising difficulty of this task. "Remember your method in obtaining the answer, so you can tell it to us," the leader instructs. "Yes? You have it?" she asks after a little while. "What was your method?"

"I pressed [999] [-] [599] [=]," is the answer.

"Very good; another?"

"[88] [-] [8], which gives you 80," begins one person. "And then [x] [5] [=], which gives you 400."

The group explores and discusses several other possibilities, until the leader suggests a second problem. "This time, put <u>0.5</u> on your displays. When you have found one solution, try for another; but pretend that each 'push' costs 1¢, and try not to spend more than 10¢."

After several moments of concentrated activity, a variety of solutions are suggested. "Press [5] [+] [=] [=] (15)," says one person, "[÷] [5] (3) [÷] [6] [=]. That costs 9¢."

"Press [9] [-] [5] (4)," offers another, "[÷] [8] [=]."

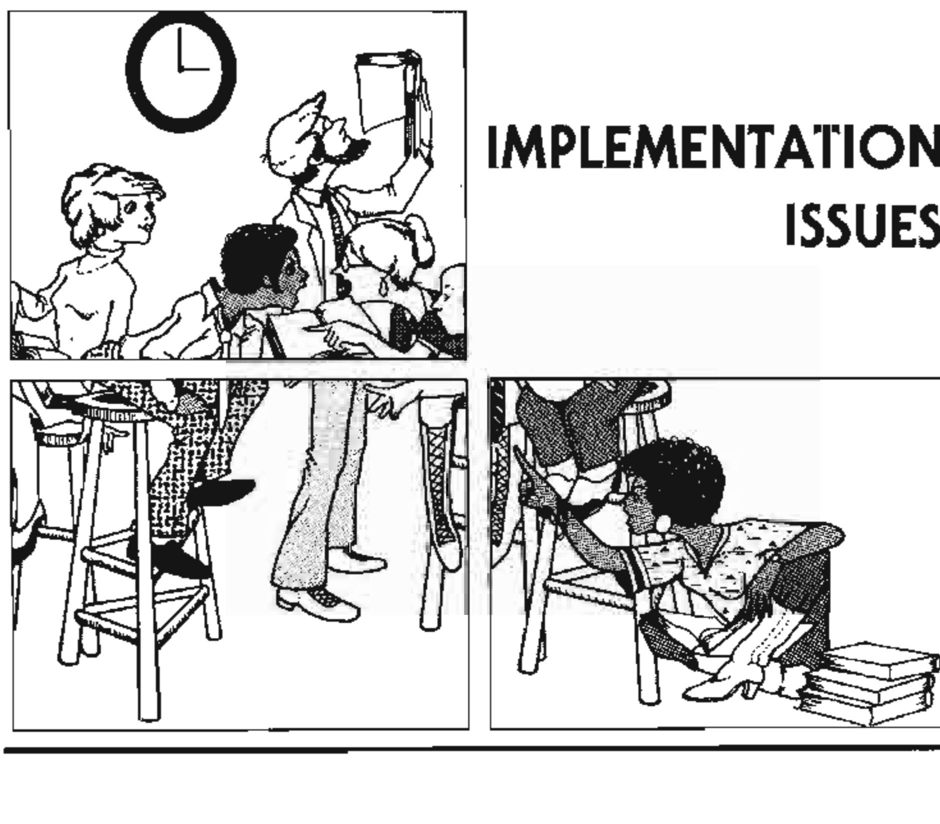
"How much does this solution cost?" asks the leader. "Yes; 6¢. Does anyone have a cheaper solution than this?" A young man in the center of the group raises his hand.

"Is it all right to use 'square root'?" he asks. The workshop leader nods, and he continues. "Well, this would cost 5¢. I pressed [9] [√] [÷] [6] [=]."

Everyone in the group agrees that this is an interesting solution, and for sure, the cheapest one.

"I will leave you with one last question," says the workshop leader. "I will not give you the solution to it, and we will not have time to discuss it, but you can think about it and solve it later on your own: Using this modified hand-calculator, is it possible to display <u>any number</u> which could be displayed on a regular hand-calculator?"

SESSION 6



IMPLEMENTATION ISSUES

6 Group Discussion: Implementing the CSMP Curriculum

"For the past three days," the principal begins, "we've been involved in a rather intensive workshop to familiarize us with the CSMP elementary school curriculum and help us decide whether to implement this program at our school. During this final session, everyone will have the opportunity to make comments or ask questions about aspects of the curriculum . . . including details of its implementation process. Let me begin with an observation and question.

"From watching the four demonstration classes, it appears to me that this program is appropriate for students of all ability levels. This class had several extremely bright students, plus six or seven who are several years below grade level; yet all of the students seemed to be deeply involved with the lessons and seemed to thoroughly enjoy them. Am I correct in my perception?"

"Definitely," the workshop leader replies. "The CSMP Elementary School Program is intended for all children in the schools. Each learning situation is posed so as to involve all of the students at their own levels, so that even the 'slowest' can succeed in dealing with some aspects of the situation that challenges him or her. In the lessons you've seen here, you may have noticed that the questions alternated — they were almost deliberately geared to first the brighter students and then to the slower ones, and so on. In this way every child has the opportunity to contribute something."

"So for these lessons to be effective, the teacher must be almost a creative artist," observes the principal.

"Exactly. I think that what we are getting at here, is that in the CSMP curriculum, it's not just 'possible' to take your own initiative as a teacher . . . it is imperative that you do. Your own experience and abilities are very important in this."

"Maybe that's what concerns me a little," one member of the group says, only half-joking. "What about teachers who can't translate your materials into exactly the sort of lessons we've seen during the workshop?"

"You are not expected to," the leader replies, looking quickly to the principal. "It is not necessary. But that becomes more apparent when you have examined the curriculum materials themselves."

"Speaking of materials . . . of what materials does the curriculum consist?"

"The materials will vary somewhat from grade level to grade level; but generally, several observations pertain across the board. First, of course, is the 'Teacher's Guide'; it is one to five volumes in length, contains important background information, lays out day-by-day lesson plans, and, of course, presents the lessons themselves. The lessons are often described in the format of a dialogue between teacher and student . . . with narrative comments interposed where they're needed.

"The student materials consist of workbooks, story-workbooks, worksheets, and storybooks; to minimize expense, all of these are printed on newsprint, as you see." She fans out several copies for the group's inspection.

"How are workbooks, worksheets and story-workbooks used?"

"A workbook provides extra experience in a given area, and its pages are arranged to increase in difficulty; the students proceed as far as they feel capable. In this way, all of the students are challenged, and the teacher receives important feedback on the students' grasp of the subject along the way. Usually, one day a week is set aside to use the workbooks.

"The story-workbooks are similar to the workbooks, except that the problems are incorporated into a story setting, as the name implies.

"And the worksheets," she continues, holding an example, "are tied to specific lessons; that is, after undertaking a particular lesson, you would distribute the appropriate worksheets to the class. The worksheets provide follow-up on the ideas of the lesson and perhaps extend them. The storybooks, I think, are self-explanatory; you should look at several, though, to appreciate their flavor and the way they exploit the non-verbal languages.

"We refer to these student materials as 'consumables'," she says, replacing the copies on the table. "Because each year they are consumed, and must be replaced for the following year."

"How expensive are CSMP curriculum materials?" asks a teacher.

"Generally, costs are comparable to commercial programs, such as those you may be using now," answers the leader. "I would also like to say that until a commercial publisher decides to produce the curriculum materials, CSMP and our governing institution, CEMREL, will continue to underwrite and publish them. At the present time, costs are comparable to commercial programs, but we hope for costs to go steadily downward . . ." she glances about the room, ". . . as the ranks of the program's users swell. Our first printing runs were for 1,000 copies of the guides. If this can be increased to 5,000 or 10,000 copies, the cost for everyone will be dramatically lowered."

A young woman continues with the question, "How do CSMP students do on standardized tests?"

"While CSMP is opposed to the philosophy underlying standardized testing," begins the leader slowly, "test results are available to us from several sources. The results indicate that, on tests, CSMP classes generally do as well as or better than do non-CSMP classes. It should be pointed out that standardized testing and traditional programs are correlated and that the CSMP student has been exposed to a great deal of mathematics that is not taught in more traditional programs nor considered in the construction of standardized tests.

"This might be an appropriate time to give you an overview of what would happen, should you decide to adopt the program." She looks to the principal, who nods.

"You would first appoint someone to fill the role of 'coordinator', as we refer to this person. The coordinator oversees the operation of the program throughout the school district or, in this case, in the school.

"Your principal, or someone in authority at the district level, would sign a 'Memorandum of Understanding' with CEMREL . . . naming the coordinator, agreeing to provide the appropriate training, and specifying other details of the arrangement.

"This summer, the coordinator from your school would come to our offices in St. Louis for training and would then return here to conduct the same training program for those of you who will be using the curriculum in the fall.

"All teachers would begin their training at the Kindergarten level, regardless of the grade level at which they will eventually teach. For Kindergarten and first grade teachers, we recommend a minimum of 16 hours of such training. Teachers at the level of the Upper Primary Grades (generally, second and third grade teachers) would begin at the same level, but would continue their training for 16 additional hours, receiving a total of 32 hours. Teachers of the Intermediate Grades (grades 4-6, generally) would receive up to 56 hours of training. Coordinators, naturally, receive as much training as they will need to duplicate the workshop for the teachers in their home district.

"This training syllabus has been formalized and incorporated in the booklet A Manual for Coordinators* and the necessary training materials for each grade level have been packaged into kits."

She looks around expectantly. "I would imagine that many of you still have questions you'd like to have answered, either in the group here, or individually; we've arranged to leave the next hour free for that purpose. In the meantime, I've enjoyed myself very much these past few days, as I hope you have. Thank you all."

*Available from CEMREL, Inc. - CSMP.